NST Part IB Physics B – 2021/22

Classical Dynamics – Examples 2

There will be, on average, about 2 questions for each lecture.

These are labelled:

A) relatively straightforward problems that do not need a lot of algebra;
B) problems that require some algebraic formulation and manipulation, often with some calculation also;
C) problems which are either harder or longer than B-problems.

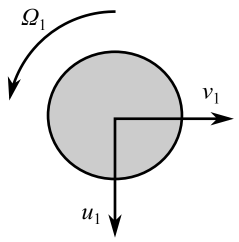
In addition, some questions have optional parts at the end indicated by '*' for possible further investigation, and there are some other questions at the end for discussion.

Rigid Bodies

15A) A uniform disc of radius 0.1 m and mass 0.4 kg is rotating with angular velocity 1 rad s^{-1} about an axis at 45° to its plane through its centre of mass. What is (a) its angular momentum, and (b) its kinetic energy? (You may assume the centre of mass is stationary.)

16B) Show that the moment of inertia of a uniform sphere of mass m and radius a about an axis through its centre is $\frac{2}{5}ma^2$.

A spherical 'super-ball' may be considered to be perfectly elastic, incompressible and rough, so that, when it collides with a surface, energy is conserved and the point of contact does not move. Such a super-ball is spinning about a horizontal axis at angular velocity Ω_1 as it falls towards a horizontal rough surface as shown below.



When it hits the rough surface it is travelling with components of velocity u_1 normal to the surface and v_1 parallel to the surface and perpendicular to Ω_1. Find the linear and angular velocities u_2, v_2 and Ω_2 with which it rebounds. (You may neglect gravity, but you will need to justify that the vertical component of the velocity is reversed after the collision.)

17B) A uniform rectangular tile drops without spinning until its corners reach positions $(0,0,0)$, $(2a,0,0)$, $(2a,2b,0)$, $(0,2b,0)$, when it strikes the top of a vertical pole at a point very close to the $(0,0,0)$ corner. Just before impact the velocity of the tile was $(0,0,-u)$. Assuming that the tile does not break, and that the impact is elastic (i.e. the kinetic energy of the tile is conserved), find immediately after impact:

a) the velocity of its centre;
b) the angular momentum about its centre;
c) its angular velocity.

Show that the velocity of the corner at $(0,0,0)$ becomes $(0,0,+u)$ immediately after impact.

18B) A uniform, smooth disc of mass m and radius a is initially at rest in space (i.e. there is no gravity). A small particle of the same mass m with an initial velocity u normal to the plane of the disc makes an elastic collision with the disc, striking it at a point midway between the centre and the rim. Find the velocity of the centre of mass of the disc and the angular momentum of the disc about its centre after the collision.

Suppose now that before the collision the disc was rotating about its own axis with angular velocity $\frac{2}{3}u/a$. Describe the motion after the collision as completely as you can, showing in particular that the plane of the disc returns to its initial orientation in the time taken for its centre to move through a distance $\pi a/\sqrt{2}$.

19B) A circular coin of radius a falls at speed u without rotating onto a smooth horizontal table. The perpendicular to its face makes an angle θ with the vertical. Determine the state of motion of the coin just after it strikes the table, assuming that the collision is elastic. Show that, when θ is small, the coin strikes the table a second time at an angle of $\frac{5}{11}\theta$.

20C) The moments of inertia of the Earth about polar and equatorial axes differ by 1 part in 300. Estimate the instantaneous rate of precession of the Earth's axis in mid-summer due to the couple exerted by the Sun alone, assuming the Earth's equatorial bulge to be concentrated in a ring round the equator. The Earth's axis is inclined at $23^\circ\!.5$ to the normal to its orbit.

[*Given that the Moon's tides are about twice as large as those due to the Sun, estimate the period of precession of the Earth's axis (the 'precession of the equinoxes').]

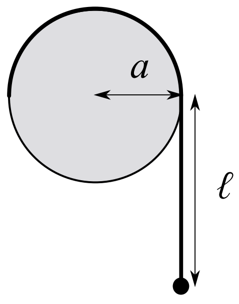
21B) A thin coin of radius a is spun on a perfectly rough table in such a way that its centre is stationary, while its axis precesses steadily about the vertical at a fixed inclination θ with angular velocity Ω. Show that $\omega_3 = 0$ (where the 3-axis is the rotational axis of symmetry of the coin) and hence that

$$\Omega^2 = \frac{4g}{a\sin\theta}.$$

Show also that, if θ is a small angle, the head of the coin, viewed from above, appears to rotate with angular velocity $\Omega(1-\cos\theta) \approx \sqrt{g\theta^3/a}$.

Lagrangian Dynamics

22B) A light string is wound around a fixed horizontal cylinder of radius a, with a mass m attached at the end. When the string is vertical, the distance from the point of contact of the string with the cylinder to the mass is ℓ. Find the equation of motion for θ, the angle the string makes with the vertical, and find the period for small oscillations.



23B) A bead of mass m runs freely without friction along a light circular wire which has a radius R, which is in a vertical plane. The circular wire is rotated at a constant angular speed ω about a vertical axis that passes through the centre of circle.

(a) Write down the Lagrangian for the system in term of the angle, θ, between the downward vertical from the centre of the circle and the position of the bead.
(b) Use the Euler–Lagrange equation to derive the equation of motion for the bead.

[*Find the period of small oscillation of θ about the stable value of θ if $\omega^2 > g/R$.]

Normal Modes

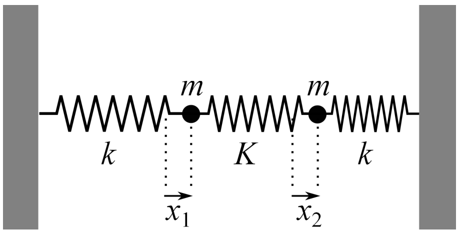
24B) When a diatomic molecule is 'adsorbed' onto the surface of a metal the frequency of its internal vibrational mode is changed. If we consider a horizontal surface with the axis of the molecule vertical, a simple model which might describe this phenomenon is as follows. The molecule consists of two point masses, m, separated by a light spring of spring constant k. The mass nearer to the metal is attached to a fixed point (the metal surface) with a light spring of constant K. The two springs are collinear and the motion of the masses is regarded as confined to the line of the springs. Find the normal modes and how their frequencies vary with the ratio of K to k. Sketch the results and comment on their physical significance for large and small K.

25B) A uniform rod of length a hangs vertically on the end of a light inelastic string of length a, the string being attached to the upper end of the rod. What are the frequencies, and shapes, of the normal modes of small oscillations in a vertical plane?

26C) A simple model of a jet engine comprises three identical thin rigid discs mounted equidistant on a uniform light torsional shaft. Describe the normal modes of rotational oscillations of the system.

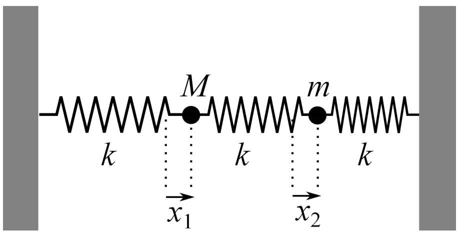
A small object entering the engine produces an abrupt change $\Delta\Omega$ in the angular velocity ω of the first disk. Obtain an expression for the maximum resultant angle of twist of the shaft between the discs, given that the angular frequency of the lowest vibrational normal mode is Ω.

27C) (a) Find the frequencies of the normal modes of the two-mass system shown below.



Sketch how the (frequencies)^2 vary with K/k, the ratio of the spring constant of the centre spring to the outer ones, for fixed k.

(b) Find the frequencies of the normal modes of the two-mass system shown below.



Sketch how the (frequencies)^2 vary with M/m (show the ratio M/m varying from 1 to ∞), for fixed m. Annotate your sketch with pictures showing the normal modes associated with each eigenvalue at $M \approx m$ and $M \gg m$.

(c) The two-mass system of part (b), with $M = m$, is driven by a force $F\sin(\omega t)$ applied to the first mass, where ω is not equal to either of the normal mode frequencies. As with a simple harmonic oscillator, the response can be represented as the sum of a steady-state response of frequency ω and a free response. Find the steady-state response of the system. Sketch the amplitude of the responses of x_1 and x_2 as a function of ω.

Numerical Answers

Q15: (a) $\left(1/\sqrt{2}, 0, \sqrt{2}\right) \times 10^{-3}$ kg m^2 s^{-1} w.r.t. obvious axes; (b) 0.75 mJ.

Q16: $u_2 = -u_1$, $v_2 = (3v_1 - 4a\Omega_1)/7$, $\Omega_2 = -(10v_1/a + 3\Omega_1)/7$.

Q20: $\sim 1.6 \times 10^{-4}$ rad yr^{-1}.

Q22: $a\dot{\theta}^2 + (\ell + a\theta)\ddot{\theta} + g\sin\theta = 0$ (for positive θ away from cylinder); $\omega \approx \sqrt{g/\ell}$.

Q23: (a) $\mathcal{L} = \frac{1}{2}m\left[R^2\left(\sin^2\theta\omega^2 + \dot{\theta}^2\right) + 2gR\cos\theta\right]$ + constant; (b) $\ddot{\theta} = \sin\theta(\cos\theta\omega^2 - g/R)$.

Q24: $m\omega^2/k = 1 + \frac{1}{2}K/k \pm \sqrt{1 + \frac{1}{4}K^2/k^2}$.

Q25: $\omega^2 = \left(5 \pm \sqrt{19}\right) g/a$.

Q26: $\leq \left(1 + 1/\sqrt{3}\right) \Delta\Omega/2\Omega$.

Additional Questions

1. If the UK changed from driving on the left to driving on the right, would this speed up or slow down the Earth's rotation? [*Estimate by how much.]
2. How many vibrational normal modes does an ammonia (NH_3) molecule have?
3. The ringing note produced by a tea-cup when it is tapped on the rim with a spoon is liable to vary in pitch depending on whereabouts in relation to the handle the cup is tapped. Predict this variation.