ELECTROMAGNETISM - PROBLEM SHEET 2

Michaelmas Term, 2021

Questions are marked easy ('A'), moderate difficulty (type 'B') or moderately demanding ('C')— do not shy away from those labelled 'C', they are just slightly more involved, and not aimed only at enthusiasts!

Varying electromagnetic fields, inductance and energy

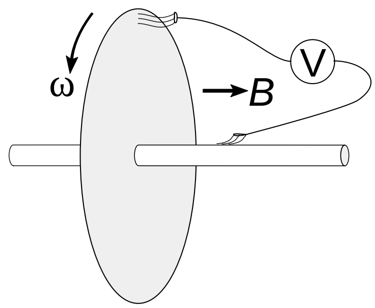

Figure 1: Faraday's disk.

23B) "Faraday's disk" consists of a metallic disk of radius a, and on its axis a conducting axle that is free to rotate relative to its supports (see Fig. 1). There are sliding electrical contacts on the outer rim of the disk and on the axle. A uniform magnetic field B is applied parallel to the axis. Calculate the e.m.f. generated between the rim and the axle when the disk rotates at angular frequency ω, (a) by considering the Lorentz force on the charge in an element of disk, and (b) by considering the rate of cutting flux.

In order to use the Faraday disk as a motor, a battery (of E.M.F. V_0) is now connected between the two contacts, and it drives a current I around the circuit so formed. Show that the torque exerted on the disk is $\Phi I/2\pi$, where Φ is the flux crossing the whole disk. Find I if the resistance of the complete circuit is R and account for the power consumed by the battery.

24C) A pendulum consists of a conducting sphere of radius r and density ρ suspended by a fine (conducting) wire of length l ($\gg r$) from an earthed pivot. The pendulum swings with a small amplitude in a vertical plane. A uniform horizontal magnetic field B is applied at right angles to this plane. Use Faraday's law to show that the potential on the sphere (relative to the pivot) is $-\frac{1}{2}l^2 B d\theta/dt$ (where θ is the angular displacement at time t). *[Think how you would form a loop if you measured the potential difference. The capacitance of a sphere was calculated earlier!]* What is the instantaneous couple on the pendulum due to the resulting current, and in what direction? Write down the new equation of motion for the pendulum, and show that the period of the pendulum changes by one part in $8\rho r^2/3\epsilon_0 l^2 B^2$. Does the period increase or decrease? Try some sensible numbers to see if the effect would be detectable.

25C) A circular loop of wire of density ρ and conductivity σ rotates freely about a diameter that is perpendicular to a magnetic field B. Show that the speed of rotation [averaged over one cycle] decays with time constant $\tau = 4\rho/B^2\sigma$.
Hint: Remember that the couple exerted on a magnetic dipole $\boldsymbol{m}$ *by a field* $\boldsymbol{B}$ *is* $\boldsymbol{m}\times\boldsymbol{B}$.

A spherical, hollow aluminium satellite in a geostationary orbit spins about an east-west axis in the earth's field of 120 nT. *Estimate* the fraction by which the speed of rotation decreases in a year because of the field. [$\rho = 2.7\times10^3$ kg m^{-3}; $\sigma = 3.8\times10^7\ \Omega^{-1}$ m^{-1}.]

26B) Calculate the mutual inductance of two coaxial single-turn circular loops of wire of radii r and $r'(\ll r)$ when their centres are a distance x apart. Hence find the flux linking a circular loop of radius r when a small magnetic dipole of moment m is aligned with, and on, its axis at a distance x from its centre.

If the dipole approaches from a large distance and passes through the loop at a constant speed u, show that the instantaneous voltage V induced in the loop is given by $V = 3\mu_0 mur^2x/2(x^2+r^2)^{5/2}$. Sketch the form of this function of x.

It is suggested that individual neutrons might be detected by observing the voltage induced in a small loop of wire by their magnetic fields as they pass through. Discuss the feasibility of this technique. [*The magnetic moment of a neutron is* 9.7×10^{-27} *J T*$^{-1}$.]

27A) An ignition system for the spark plugs in a car engine consists of two coils, of 400 and 16000 turns, one tightly wrapped over the other so that little flux is lost. The coils are 10 cm long with a radius of 3 cm. A current of 3 A flows in the primary coil, and it is interrupted so that the current goes to zero in about 10^{-4} s. Estimate the maximum voltage produced across the secondary coil. Why is this useful? Which coil should be used as the primary, and why?

28B) An audio-frequency transformer, designed to operate at frequencies as low as 20 Hz, is used to transform the 5 Ω resistance of a secondary load into a reflected impedance, i.e., the impedance in an equivalent circuit in parallel with the primary coil. The reflected impedance equals the 75 Ω output resistance of the voltage generator attached to the primary coil. The magnetic core of the transformer is a toroid of average radius 5 cm, cross-sectional area 30 cm^2, and relative permeability $\mu = 500$. Calculate the minimum number of primary and secondary turns that must be wound on the core so as to achieve impedance matching, whilst having a primary inductance that presents an impedance of at least 150 Ω.
Hint: you may assume perfect coupling and zero coil resistance.

29C) A long paramagnetic circular rod of diameter 5 mm is suspended from a balance so that its lower end is between the poles of an electromagnet that produces a horizontal field. When a field of 1 T is switched on, the apparent mass of the rod increases by 1.5 g. Using an energy method, or otherwise, find the magnetic susceptibility of the material.
Hints: You may find the concept of virtual work useful to attack this problem by e.g. considering what happens if the rod is moved down a (small) amount. There is a general argument that shows that, if the currents are constant, the work done against the magnetic forces is the negative *of the change in the stored energy. The change in magnetic energy can be written as* $\Delta W = \int \mathrm{d}^3r \frac{1}{2}\mu_0 \boldsymbol{M}\cdot\boldsymbol{H}_0$, *which involves the unperturbed* $\boldsymbol{H}_0$ *field, and the magnetisation* $\boldsymbol{M}$. *As for the electric field in the case of a cylinder of dielectric, the internal field of the cylinder is related to the applied field by* $H_i = \frac{1}{1+\chi/2}H_0$.

30A) An alternating potential difference $V_0 \exp(-i\omega t)$ is applied to a parallel-plate capacitor of capacitance C whose electrodes are two circular disks of radius a. Integrate one of Maxwell's equations to find an expression for the amplitude of the magnetic field between the plates. Compare the field to that of the earth (50 μT) when $V_0 = 300$ V, $\omega = 10^4$ rad s^{-1}, $a = 300$ mm and the plates are 1 mm apart.

Electromagnetic waves

31B) (a) Estimate the electric field strength 100 km from a 100 kW television transmitter.

(b) Calculate the energy flux, energy density and r.m.s. electric field of sunlight

(i) at the surface of the Earth,

(ii) just above the surface of the Sun.
Hint: what do you need to average over to get the net flux?

Hint: assume that the Sun behaves as a black body with surface temperature 6000 *K. Its angular diameter as seen from the Earth is half a degree. The Stefan–Boltzmann constant* σ *is* 5.67×10^{-8} *W m*$^{-2}$ *K*$^{-4}$.

32B) Plane monochromatic waves are propagating in free space parallel to the x-axis in both positive and negative directions. At the origin the field strengths are given by

$$\begin{aligned} E_z &= E_0 \cos\omega t, & E_y &= 3E_0 \cos\omega t, \\ H_z &= -(E_0/Z_0)\cos\omega t, & H_y &= -(E_0/Z_0)\cos\omega t. \end{aligned}$$

Find the amplitudes and polarisations of the forward and backward travelling waves, and the net energy flux.

33A) Show that spherical dust grains within the Solar System which have a radius less than a critical radius r will be repelled by the Sun, whilst those with a radius greater than r will be attracted towards it. Calculate r given that the mass of the Sun is 2×10^{30} kg, its luminosity is 4×10^{26} W and the density of the dust grains is of order 10^3 kg m^{-3}.
[*Neglect the pressure exerted by the solar wind and assume that the dust grains perfectly absorb solar radiation.*]

34A) The electron density in the ionosphere reaches a maximum value of 1.5×10^{12} m^{-3}. What range of frequencies could be used for radio control of a satellite?

35B) Show that the refractive index n of a plasma is given by

$$n^2 = 1 - \frac{Ne^2}{\epsilon_0 m_e \omega^2},$$

where N is the number density of free electrons, m_e the electron mass and ω the angular frequency of the electromagnetic radiation. Use this result to derive the dispersion relation for the radiation and show that the product of the phase and group velocities is equal to c^2.

A pulsar emits pulses of radio waves containing a wide range of frequencies. The interval between the times of arrival at the Earth of pulses at 400 MHz and 200 MHz is found to be 4 s. The mean electron density in interstellar space is known to be about 3×10^4 m^{-3}. What is the distance to the pulsar?

36B) Show that the wave impedance Z of a good conductor at low frequencies to an incident wave $E_0 \ \exp i(kx - \omega t)$ can be expressed as $Z = (1 - i)a$ with $a^2 = \omega\mu\mu_0/2\sigma$ where σ is the conductivity and μ is the relative permeability. Hence show that the power reflection coefficient is given approximately by

$$R \approx 1 - 2\sqrt{\frac{2\mu\epsilon_0\omega}{\sigma}} \ .$$

Calculate the electrical conductivity of aluminium given that the power reflection coefficient for 5 μm radiation incident normally is 97%. [*Assume* $\mu = 1$.]

37B) Radiation is incident normally at the plane surface of a good conductor. Using results from Question 36, derive an equation for the variation of the electric field with time and depth below the surface. Comment on the flow of energy in this situation.

It is possible to communicate with submerged submarines using very low frequency radio waves. If the frequency used is 16 kHz and the receiver has sufficient sensitivity to operate satisfactorily with electric fields whose amplitude is 100 times less than that just under the surface, to what depth can a submarine remain in radio contact?
[*Sea water has* $\sigma \approx 4\ \Omega^{-1}\ m^{-1}$, $\epsilon = 80$ *and* $\mu = 1$. *Assume the radiation is incident vertically.*]

38B) (a) A radio-frequency current flows in a wire of conductivity σ and radius $a \gg \delta$, the skin depth. Calculate the power dissipated in the wire and show that the effective resistance is as if all the current were flowing uniformly in a layer of thickness δ.

(b) A certain microwave oven operates at a frequency of 2.5 GHz, and includes a turntable which ensures that all parts of its volume are sampled equally by the food it is cooking. In a carefully controlled experiment, a series of spherical potatoes of diameter 2.5, 5, 7.5 and 10 cm are cooked in the oven. Each potato is cooked until its surface is soft to the touch. The potato is then removed and studied. It is found that the two smallest potatoes are cooked all the way through, but the two largest potatoes each have an uncooked region inside. Estimate the skin depth of radiation of this frequency in potatoes. Are the potatoes good conductors, good insulators, or in between? For the largest potato, the oven delivers 600 watts of power to the potato. Estimate the average (rms) electric field in the oven.

Transmission lines and waveguides

39B) A coaxial transmission line consists of an inner cylindrical conductor of radius 1 mm and a cylindrical outer conductor chosen to make the characteristic impedance 75 Ω. The space between the conductors is filled with a gas which can stand a maximum field of 10^5 V m^{-1} without dielectric breakdown. Estimate the maximum mean radio-frequency power that can be transmitted along this line into a matching load.

40B) An annular sheet (i.e. a flattened ring), of thickness $t = 1$ mm and made of a material of resistivity 0.5 Ω m connects the inner and outer conductors of an air-spaced coaxial transmission line at a point on the line. A low-frequency signal is fed into one end of the line and the other is terminated by its characteristic impedance. Calculate the relative amplitude of the wave reflected from the resistive sheet.

41A) An open-ended quarter-wavelength, air-spaced, parallel-wire transmission line is found to be in resonance with an oscillator when its length is 0.25 m. When a capacitance of 1 pF is connected across the open end, it is found that the length of the line must be reduced to 0.125 m to obtain resonance. Show that the characteristic impedance of the line is approximately 530 Ω.

Hint: remember that, if you choose to use the physicists' convention, $V \propto e^{i(kz-\omega t)}$, rather than $V \propto e^{-j(kz-\omega t)}$, the impedance of a capacitor is $i/\omega C$ not $1/j\omega C$.

42A) A film of magnesium fluoride (MgF_2) (refractive index = 1.38) one-quarter wavelength thick is deposited on crown glass (refractive index = 1.52). Show, using the expression for the reflection coefficient in terms of the impedance and in analogy to a terminated waveguide, that the reflected intensity of light is reduced to about 1%.

43B) An electromagnetic wave propagates in the x direction in a simple rectangular waveguide of width a and height b. The waveguide is operating in the TM_{11} mode, the lowest TM mode possible for standard waveguides, meaning that the x and y components of the magnetic field $\boldsymbol{B}$ are given by

$$B_x = 0; \qquad B_y = B_0\,\frac{\pi}{b}\cos\left(\frac{\pi z}{b}\right)\sin\left(\frac{\pi y}{a}\right)\;\exp i(k_x x - \omega t)$$

where

$$k_x = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{\pi}{a}\right)^2 - \left(\frac{\pi}{b}\right)^2}\,.$$

B_0 is a constant and the origin of the coordinate system is taken at an inside corner.

(a) Calculate all the components of $\boldsymbol{B}$ and $\boldsymbol{E}$, and demonstrate that they satisfy the proper boundary conditions.
(b) Show that the TM_{11} mode will not propagate if $\omega < c\sqrt{(\pi/a)^2 + (\pi/b)^2}$.
(c) Show that at least one TE mode in this waveguide has a lower cut-off frequency than for this TM mode.

44B) It is proposed to transmit television signals requiring a bandwidth of 8 MHz by means of a waveguide operating at 15 GHz. If the cut-off frequency of the guide is 12 GHz, what range of group velocities will be present in the signal?

Answers to problems

Worked solutions will be issued around lecture 15 and early next term.

23. $a^2\omega B/2$; $(V_0 - \Phi\omega/2\pi)/R$.
25. About 1 part in 10^3.
29. 1.9×10^{-3}.
26. Flux linked is $\mu_0 r^2 m/2(x^2+r^2)^{3/2}$.
27. 6.8 kV.
28. $n_1 = 446$; $n_2 = 115$.
30. $V_0\omega C/2\pi\epsilon_0 c^2 a$; ratio is 10^{-4}.
31. (a) About 50 mV m^{-1}, depending on your assumption about the radiation pattern of the antenna.
    (b) (i) 1.4 kW m^{-2}; 730 V m^{-1} r.m.s.; 4.7×10^{-6} Jm^{-3}.
    (ii) 74 MW m^{-2}; 235 kV m^{-1} r.m.s.; 0.49 J m^{-3}.
32. Forward wave $\sqrt{2}E_0$ at 45° to y axis; backward wave $2E_0$ parallel to y axis; mean flux density E_0^2/Z_0 along $-x$.
33. 600 nm.
34. > 11 MHz.
35. 5.3×10^{19} m.
36. $3\times10^7\,\Omega^{-1}\,\text{m}^{-1}$.
37. 9.2 m.
38. (b) $\approx$5 cm; $\approx$3000 Vm^{-1}.
39. Maximum mean power is 104 W.
40. 0.27.
43.

$$\{B_x, B_y, B_z\}/B_0 = \left\{0, \frac{\pi}{b}\cos\frac{\pi z}{b}\sin\frac{\pi y}{a},\ -\frac{\pi}{a}\sin\frac{\pi z}{b}\cos\frac{\pi y}{a}\right\}\ \exp i(k_x x - \omega t);$$

$$\begin{aligned}
E_x/B_0 &= i\frac{c^2}{\omega}\left(\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right)\sin\frac{\pi y}{a}\sin\frac{\pi z}{b}\ \exp i(k_x x - \omega t),\\
E_y/B_0 &= -\frac{\pi k_x c^2}{a\omega}\cos\frac{\pi y}{a}\sin\frac{\pi z}{b}\ \exp i(k_x x - \omega t),\\
E_z/B_0 &= -\frac{\pi k_x c^2}{b\omega}\sin\frac{\pi y}{a}\cos\frac{\pi z}{b}\ \exp i(k_x x - \omega t).
\end{aligned}$$

44. 1.7×10^5 ms^{-1}

Please advise me of any errors, misprints or omissions in these answers – many thanks in advance!

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