NST Part IB Physics B – 2021/22

Classical Dynamics – Examples 1

There will be, on average, about 2 questions for each lecture.

These are labelled:

A) relatively straightforward problems that do not need a lot of algebra;
B) problems that require some algebraic formulation and manipulation, often with some calculation also;
C) problems which are either harder or longer than B-problems.

In addition, some questions have optional parts at the end indicated by '*' for possible further investigation, and there are some other questions at the end for discussion.

Note: the first 9 questions are accessible from the 4 lectures given in the Michaelmas term.

Newtonian mechanics and the energy method

1A) A uniform solid cylinder is set rotating about its cylindrical axis and is then gently placed, with the axis horizontal, on a rough horizontal table. What fraction of its initial kinetic energy is dissipated in sliding friction before the cylinder eventually rolls smoothly along the plane?

2B) A ladder of length ℓ rests against a wall at an angle θ to the vertical. There is no friction between the top of the ladder and the wall and no friction between the bottom of the ladder and the ground. Derive the equation of motion of the ladder in terms of θ by two methods.

a) Using the energy method (from the kinetic and potential energies of the ladder as a function of θ and $\dot{\theta}$), as outlined in lectures.
b) By considering the forces and torques acting on the ladder (it is useful to differentiate the height and horizontal positions of the centre of mass as a function of θ with respect to time, to obtain the velocity of the centre of mass of the ladder).

[*The top of the ladder must lose contact with the wall at some stage. Show that this happens when $3\cos\theta = 2\cos\theta_0$, where θ_0 is the initial angle when the ladder is at rest. You will probably need to consider the forces/torques acting on the ladder.]

3B) A small solid cylinder of radius r lies inside a large fixed tube of internal radius R. If it rolls without slipping, show that the angular velocity of the cylinder ω is related to θ, the angle between the centre of mass of the solid cylinder, the centre of the large tube and the vertical by

$$\omega = \left(\frac{R-r}{r}\right)\dot{\theta}.$$

Find the period of small oscillations of the cylinder about its equilibrium position.

Rotating frames and fictitious forces

4A) Paraboloidal telescope mirrors can be made by 'spin casting', which involves rotating the molten glass and its container about a vertical axis as the glass solidifies. By considering the equilibrium of an element of the molten surface show that

$$\frac{\mathrm{d}z}{\mathrm{d}r} = \frac{\omega^2}{g}r,$$

where z is the height of the surface, r is the distance from the axis of rotation and ω is the angular velocity of rotation. For a mirror of focal length 2 m, what angular velocity is required? (The equation of a parabola is $z = r^2/4f$, where f is the focal length.)

[*Derive $z = r^2/4f$.]

5B) A train at latitude λ in the northern hemisphere is moving due north with a speed v along a straight and level track. Which rail experiences the larger vertical force? Show that the ratio R of the vertical forces on the rails is given approximately by

$$R = 1 + \frac{8\Omega v h \sin\lambda}{ga},$$

where h is the height of the centre of gravity of the train above the rails which are at a distance a apart, g is the acceleration due to gravity, and Ω is the angular velocity of the Earth's rotation.

[*Is it possible for the train to travel fast enough to tip over?]

6B) A stone is dropped from a stationary helicopter 500 m above the ground at the equator. How far from the point vertically beneath the helicopter does it land and in what direction?

You should solve this problem in two ways:

a) by consideration of the angular momentum of the stone;
b) by invoking the Coriolis force.

7C) A weather map shows a shallow, stationary depression centred over a point on the Earth's surface at latitude 50° N. The isobars corresponding to pressures of 998, 1000 and 1002 mbar are concentric circles of radius 50, 200 and 350 km respectively. Estimate the wind speed on the 1000 mbar isobar, neglecting the effects of friction between air and ground. You may assume that the wind direction follows the isobars (i.e. rotating anti-clockwise in circles around the depression as viewed from above).

[1 mbar = 100 Pa; you can take the density of air to be 1.2 kg m^{-3}.]

Orbits

8B) In a classical model of a multi-electron atom, electrons are assumed to move in a modified electrostatic potential $V(r)$, given by:

$$V(r) = -\frac{k}{r}\exp\left(-\frac{r}{a}\right),$$

where k and a are constants. Show that the *effective potential* is

$$V_{\text{eff}} = \frac{J^2}{2mr^2} - \frac{k}{r}\exp\left(-\frac{r}{a}\right).$$

Then show that, in a such a potential, circular orbits are unstable *unless*

$$\frac{r}{a} < \tfrac{1}{2}\left(1+\sqrt{5}\right).$$

[*Use MATLAB, or similar, to visualise how the effective potential varies with k.]

9B) A point mass m on the end of a light string of length ℓ is free to swing as a conical pendulum. Show that, in terms of the (constant) angular momentum J_z of the bob about a vertical axis through the fixed point of support and the inclination θ of the string to this axis, the energy of the pendulum may be written as

$$E = V(\theta) + \tfrac{1}{2}m\ell^2\dot{\theta}^2,$$

where

$$V(\theta) = mg\ell(1 - \cos\theta) + \frac{J_z^2}{2m\ell^2 \sin^2\theta}$$

is the *effective potential* that determines motion in θ.

By differentiating $V(\theta)$ twice with respect to θ, show

a) that the bob can move steadily round a circle, with $\theta = \theta_0$ and angular velocity Ω given by

$$\Omega^2 = \frac{g}{\ell\cos\theta};$$

b) that, if the pendulum is then given a little extra energy without changing its angular momentum, θ oscillates about θ_0 with angular frequency ω given by

$$\omega^2 = \Omega^2\left(1 + 3\cos^2\theta_0\right).$$

Use these results to discuss the precession of almost circular orbits of a conical pendulum, assuming $\theta_0 \ll 1$.

[*Use MATLAB, or similar, to visualise the orbit for different values of θ_0.]

[*Consider the conical pendulum in a frame of reference rotating about the vertical with angular velocity Ω. Write down the equations of motion, involving Coriolis force, which govern small displacements from its equilibrium at $\theta = \theta_0$, and hence verify that $\omega^2 = \Omega^2(1 + 3\cos^2\theta_0)$.]

10A) A satellite travelling round the Earth in a circular orbit with centre O experiences, at the point P, a sudden impulse which deflects it into a new orbit. By considering the changes in angular momentum and energy of the orbit, sketch the new orbits if the impulse acts:

a) in the direction of motion of the satellite;
b) in the reverse direction;
c) outwards along the line OP.

On each sketch you should show, in relation to O and P, the points A and B at which the new orbit is furthest from, and nearest to O.

11B) A spacecraft is in a circular orbit of radius $R = \alpha r$ around a planet, where r is the radius of the planet itself. A short 'burn' of the spacecraft's motor provides an impulse which halves its velocity without changing its direction, and this alters the orbit to one that just grazes the planet's surface. (You may assume that the effect of the burn on the mass of the satellite is negligible.) Sketch the new orbit. Deduce the value of α.

If the same impulse had been applied in a radial direction, i.e. inwards towards the planet's centre, would the new orbit then have reached the planet?

12B) Two stars of unequal mass are in circular orbit about one another. The more massive star suffers a spherically symmetrical loss of matter (it explodes). After explosion the masses of the stars are equal. In the ZMF frame, find condition that the stars will be gravitationally bound together after the explosion. Hence show that the binary system will be disrupted if $\alpha > 3$, where α is the ratio of the original masses.

13B) In an attempt to place a satellite into a geostationary orbit, the correct speed and radial distance are achieved but the direction of motion has an angular error θ in the plane of the orbit. Show that the maximum and minimum radii of the orbit are $a(1 \pm \sin\theta)$ where a is the radius of the geostationary orbit. A correction is made by a short burst of the satellite's booster while at maximum radius, to put it into a circular orbit. If the initial error θ is 1 minute of arc (i.e. 1/60th of a degree), calculate the period of this new orbit.

Why was making this correction not a good idea?

14C) α particles (atomic number 2) with energy E are scattered through an angle Φ by nuclei of atomic number Z. Assuming that the only interaction between the particles and the nuclei is the electrostatic force, show that the distance of closest approach is

$$\frac{Ze^2}{4\pi\epsilon_0 E}\left[1 + \operatorname{cosec}\left(\tfrac{1}{2}\Phi\right)\right].$$

α particles are scattered by lead ($Z = 82$) through an angle of 60°. As the energy of the α particles is increased the scattering is found to be in agreement with the Rutherford formula up to an energy $E_0 = 25$ MeV. Estimate an upper limit for the range of the nuclear force in lead.

Numerical Answers

Q1: 2/3.

Q4: ≈ 1.57 rad s^{-1}.

Q6. ≈ 24 cm to the East.

Q7: ≈ 28 km hr^{-1}.

Q11: $\alpha = 7$.

Q13: ≈ 1 day + 38 seconds.

Q14: $\approx 1.4 \times 10^{-14}$ m.

Additional Questions

1. Friction slows things down. So, why does atmospheric friction make an artificial satellite speed up?
2. Why is the Moon receding slowly from the Earth?
3. Why does the Moon always keep the same face towards the Earth?

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