ELECTROMAGNETISM - Answers to Problem Sheet 1

Michaelmas Term, 2021

These worked answers to the Electromagnetism problems are offered to students and supervisors as a guide, and I hope they contain some generally useful hints. In some cases, the answers include a little research that goes beyond the stated problem. Where possible illustrative diagrams are included.

Health Warning

These solutions must be used with caution.

- The worked answers will be useless to you unless you have already made a very serious attempt to solve the problem yourself and have discussed it with your supervisor. If you consult the solutions earlier, they will just act as 'spoilers'.
- You must remember that there is often more than one way to solve a physics problem. I have sometimes suggested alternatives, but your supervisors will know many others.
- The most difficult part of a physics problem is knowing where to start. It is therefore quite possible that I have taken too much for granted at the beginning of some problems. If so, please let me know.
- I have given the numerical answers to several more decimal places than is really justified, so that you can check your calculations (and mine) carefully. I have used values of the physical constants taken from the formula book that you will use in the Examination.

Please let me know about any errors, typos and suggestions for improvement.

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1A) (a) Chop the line into symmetrically placed pairs at $\pm x$ so that the horizontal components of the two fields from the two segments cancel. The net field of the segment pair is:

$$d\boldsymbol{E} = 2\frac{1}{4\pi\epsilon_0}\left(\frac{\lambda dx}{r^2}\right)\cos\theta\hat{\mathbf{z}}\,, \tag{1}$$

where the conventions $\cos\theta = z/r$ as well as $r \equiv \sqrt{x^2+z^2}$ have been used. To find out the total field, integrate x from 0 to L:

$$\begin{aligned} E &= \frac{1}{4\pi\epsilon_0}\int_0^L \frac{2\lambda z}{(z^2+x^2)^{3/2}}dx & (2)\\ &= \frac{2\lambda z}{4\pi\epsilon_0}\left[\frac{x}{z^2\sqrt{z^2+x^2}}\right]\Big|_0^L & (3)\\ &= \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{z\sqrt{z^2+L^2}} & (4)\end{aligned}$$

and $\boldsymbol{E} = E\hat{\mathbf{z}}$.

(b) In the limit of $z \gg L$, the result simplifies:

$$E \simeq \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{z^2} \tag{5}$$

using the identity $q = 2\lambda L$, this is the same expression as for a point charge, i.e., $\frac{1}{4\pi\epsilon_0}\frac{1}{z^2}$. This is expected given that far away the exact distribution of the charge does not matter and the monopole term dominates.

(c) In the limit of $L \to \infty$, obtain the field of an infinite straight wire

$$E \simeq \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{z}\,. \tag{6}$$

This result is familiar from the lecture.

(d) See Sect. 2.17 of the lecture script. Gauss law is clearly very powerful!

2A) (a) Surface is equipotential, V is continuous at the surface. Let r_1 be radius of sphere, E_1 the electric field there, V_1 the potential, Q the charge on the sphere. Gauss $\Rightarrow Q = 4\pi\epsilon_0 r_1^2 E_1$. $E = -dV/dr$. So $V_1 = Q/4\pi\epsilon_0 r_1$, and since $V_1 = 5$ kV, $Q = 4\pi\epsilon_0 r_1 V_1 = 5.5606 \times 10^{-8}$ C. In this case, $E_1 = V_1/r_1 = 50,000$ Vm^{-1} (radially outwards). At $r_2 = 20$ cm, $V_2 = Q/4\pi\epsilon_0 r_2 = V_1 r_1/r_2 = 2.5$ kV, and $E_2 = E_1 r_1^2/r_2^2 = 12,500$ Vm^{-1}.

(b) Circular symmetry, so $\boldsymbol{E}$ radial. Charge Q on Earth (radius r) $= \epsilon_0\int \boldsymbol{E}\cdot d\mathbf{S} = 4\pi r^2 E\epsilon_0$. Hence $Q = 455500$ C.

(c) See figure 1. Note that equipotentials are perpendicular to electric field lines, which come out at right angles from each conductor, and start and end on charges. The density of field lines of course indicates the field strength.

3A) Charges $\pm q$ at $z = \pm a/2$ have potentials

$$\pm\frac{q}{4\pi\epsilon_0}\left(x^2+y^2+(z\mp a/2)^2\right)^{-1/2}\,. \tag{7}$$

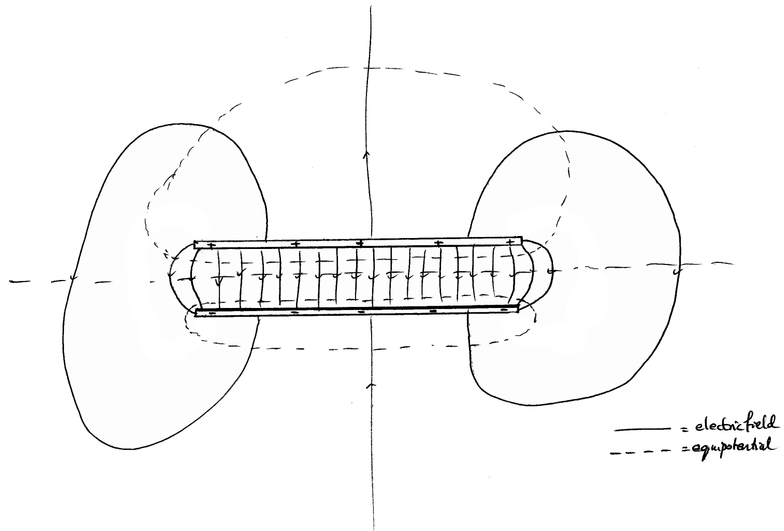

Figure 1: Electric field lines (solid) and equipotentials (dashed) for the parallel-plate capacitor in Q2c. For illustration, different equipotentials are shown at the top and bottom.

Consider the E_x component

$$E_x = -\frac{\partial V}{\partial x} = \frac{q}{4\pi\epsilon_0}\left(\frac{x}{(x^2+y^2+(z-a/2)^2)^{3/2}} - \frac{x}{(x^2+y^2+(z+a/2)^2)^{3/2}}\right) . \tag{8}$$

Define $r^2 \equiv x^2+y^2+z^2$ and expand the denominators for $a \ll r$

$$\frac{1}{(x^2+y^2+(z\pm a/2)^2)^{3/2}} \approx \frac{1}{r^3}\left(1 \mp 3az/(2r^2) + \ldots\right) , \tag{9}$$

so that

$$E_x \approx \frac{3qaxz}{4\pi\epsilon_0 r^5} , \quad E_y \approx \frac{3qayz}{4\pi\epsilon_0 r^5} . \tag{10}$$

The E_z component is

$$E_z = \frac{q}{4\pi\epsilon_0}\left(\frac{z-a/2}{(x^2+y^2+(z-a/2)^2)^{3/2}} - \frac{z+a/2}{(x^2+y^2+(z+a/2)^2)^{3/2}}\right) , \tag{11}$$

which generates another term $-qa/(4\pi\epsilon_0 r^3)$. Putting these together, we get

$$E_z \approx \frac{qa(3z^2-r^2)}{4\pi\epsilon_0 r^5} . \tag{12}$$

4B) Consider the field due to dipole 1 at the position of dipole 2. Recall the formula for the field of a dipole parallel to and perpendicular to its axis. The parallel and perpendicular components of $\boldsymbol{E}$ are

$$E_\parallel = \frac{2p_1\cos\theta_1}{4\pi\epsilon_0 d^3} , \quad E_\perp = \frac{-p_1\sin\theta_1}{4\pi\epsilon_0 d^3} . \tag{13}$$

Now consider the potential energy

$$U = -\boldsymbol{p}_2\cdot\boldsymbol{E} \;=\; -\frac{p_1p_2}{4\pi\epsilon_0 d^3}\;(2\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) , \tag{14}$$

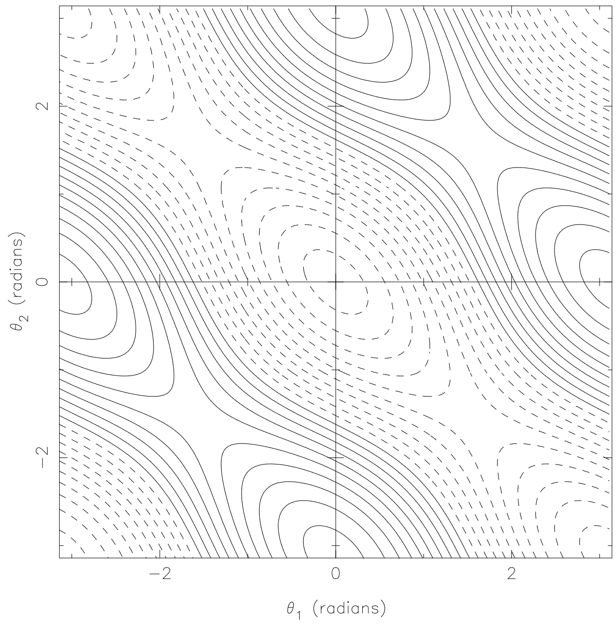

Figure 2: Potential energy surface plotted as a function of (θ_1, θ_2) for the two dipoles of Q4. The negative contours are dotted.

That's fine, but the answer needs the identity

$$3\cos(\theta_1 + \theta_2) + \cos(\theta_1 - \theta_2) = 4\cos\theta_1\cos\theta_2 - 2\sin\theta_1\sin\theta_2 \ . \tag{15}$$

5A) Gauss' law implies that the electric field is

$$E = \frac{Q}{4\pi\epsilon_0 r^2} \quad (r > a); \quad E = \frac{Qr}{4\pi\epsilon_0 a^3} \quad (r < a) \ . \tag{16}$$

Integrating, we get the potential

$$V = \frac{Q}{4\pi\epsilon_0 r} \quad (r > a); \quad V = \frac{Q(3a^2 - r^2)}{8\pi\epsilon_0 a^3} \quad (r < a) \ . \tag{17}$$

The constant of integration for $r < a$ is needed to make the potential continuous at $r = a$. We now form $U = \frac{1}{2}\int d\tau \rho V$, with $\rho = 3Q/(4\pi a^3)$:

$$U = \frac{3Q^2}{16\pi\epsilon_0 a^6}\int_0^a dr\ r^2(3a^2 - r^2) = \frac{3Q^2}{20\pi\epsilon_0 a} \ . \tag{18}$$

Solving $U = m_e c^2$ for a yields 1.69076×10^{-15} m.

6A) (a) To show there is no charge within a conductor, suppose otherwise. Then we can take a volume within the conductor, enclosing some of this charge, so that the net charge within it is non-zero. Then Gauss' law $\Rightarrow \boldsymbol{E} \neq \boldsymbol{0}$, which is not possible in a conductor. Hence contradiction.

(b) Take a pill-box in the surface of the conductor, of negligible height, and with top and bottom areas A. The field in the conductor is zero, so, by Gauss, $E_{\rm above}A - 0.A = \sigma A/\epsilon_0$.

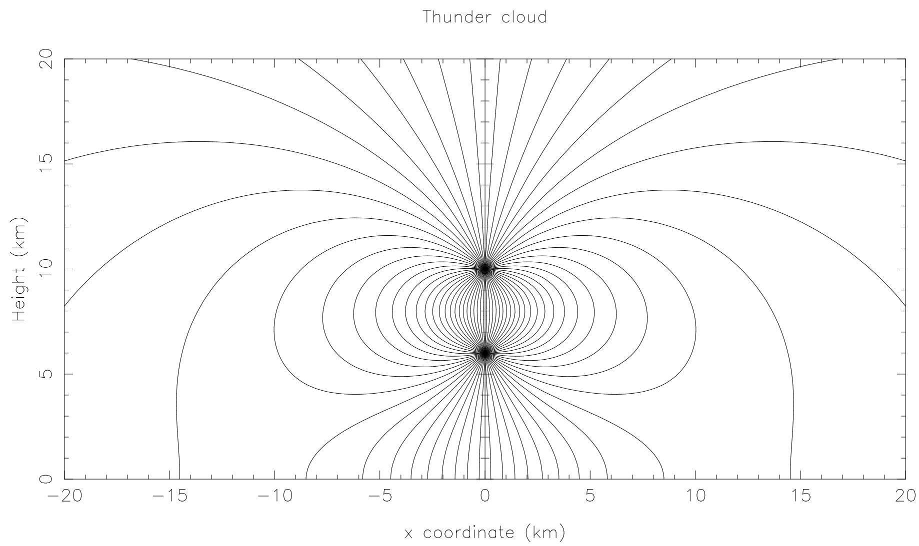

Figure 3: Field lines for the thundercloud of Q8. The reversal of the field at 11 km marks the boundary between field lines ending on the two different image charges.

(c) When there is an electric field outside a conductor, charges move in the surface of the conductor to maintain $\boldsymbol{E} = \mathbf{0}$ inside. There is no net charge due to the external fields when integrated over the whole surface (some parts are positive and some negative). [By Gauss' law, a surface just inside the outer surface must enclose no net charge since $\boldsymbol{E} = \mathbf{0}$, and the conductor is assumed to be electrically neutral originally.] Thus external electric fields cannot reach a cavity inside the conductor. One can invoke the uniqueness theorem to say that nothing changes in the cavity when an external field is applied, since the boundary conditions (on the inner surface) stay the same.

Now consider the fields outside, caused by charges in the cavity. Gauss' law says the flux of $\boldsymbol{E}$ is non-zero outside, because of that enclosed charge. Thus that charge does affect the outside of the conductor. [The charge in the cavity produced image charges in the inner surface of the conductor, which have counterparts on the outer surface. The latter then generate the electric field outside.] Note that this assumes the conductor is isolated, so that charge builds up on the outside. To prevent this happening, just connect the conductor to ground (a large sink of charge), so that its potential is fixed. This enables sources of electrical noise (such as computers) to be prevented from radiating that noise into the environment.

7A) The electric field $\boldsymbol{E}_0$ at the sphere arising from the two charges is $2q/(4\pi\epsilon_0 d^2)$ from q to $-q$; the field is approximately uniform across the sphere as $a \ll d$. For a conducting sphere in a uniform field, we found an effective dipole moment $\boldsymbol{p} = 4\pi\epsilon_0 a^3 \boldsymbol{E}_0$ (pointing towards $-q$), so $p = 2qa^3/d^2$. The field of a dipole at distance d along the axis is $2p/(4\pi\epsilon_0 d^3)$, hence the additional force (attractive) is $4q^2a^3/(4\pi\epsilon_0 d^5)$. The original force was $q^2/(16\pi\epsilon_0 d^2)$, hence result.

8A) Electric field due to single charge Q at height z_1 and its image is

$$E_z = -\frac{Q}{4\pi\epsilon_0} \frac{2z_1}{(x^2 + y^2 + z_1^2)^{3/2}} \,. \tag{19}$$

The total is therefore

$$E_z = \frac{Q}{4\pi\epsilon_0} \left(\frac{2z_2}{(x^2 + y^2 + z_2^2)^{3/2}} - \frac{2z_1}{(x^2 + y^2 + z_1^2)^{3/2}} \right) \,. \tag{20}$$

Putting $z_1 = 10$ km, $z_2 = 6$ km, $Q = 40$ C gives $E_z = +12,782$ V m^{-1}, i.e. upwards. The field lines are plotted in Figure 3.

The next part asks us to solve

$$\frac{z_1}{(z_1^2 + r^2)^{3/2}} = \frac{z_2}{(z_2^2 + r^2)^{3/2}} \tag{21}$$

for $r^2 = x^2 + y^2$. Taking the two-thirds power of (21) and rearranging gives

$$r^2 = \frac{z_1^2 z_2^{2/3} - z_2^2 z_1^{2/3}}{z_1^{2/3} - z_2^{2/3}} \ . \tag{22}$$

For $z_1 = 10$ km, $z_2 = 6$ km this gives $r = 11.034$ km.

9A) Geometry: source at $(x, 0)$; $-q$ image at $(-x, 0)$ $-q$ images at $(x/2, \pm\sqrt{3}x/2)$; $+q$ images at $(-x/2, \pm\sqrt{3}x/2)$. Force is sum of 5 terms:

$$-2 \times \frac{q^2}{4\pi\epsilon_0 x^2} \times \frac{1}{2} \times \frac{1}{1^3} \ ; \quad +2 \times \frac{q^2}{4\pi\epsilon_0 x^2} \times \frac{3}{2} \times \frac{1}{\sqrt{3}^3} \ ; \quad -\frac{q^2}{4\pi\epsilon_0 x^2} \times \frac{1}{2^2} \ ; \tag{23}$$

Hence result

$$F = -\frac{q^2}{4\pi\epsilon_0 x^2}\left(\frac{5}{4} - \frac{1}{\sqrt{3}}\right) \equiv -\frac{A}{x^2} \ . \tag{24}$$

10A) The potential of a conducting sphere of radius r carrying a charge q is $V = q/(4\pi\epsilon_0 r)$ and its potential energy $U = qV/2$. The total charge of the system is conserved, so

$$Q = 4\pi\epsilon_0 \left(V_1 r_1 + V_2 r_2\right) \ = 4\pi\epsilon_0 V \left(r_1 + r_2\right) \ , \tag{25}$$

where V is the final potential of both spheres. The energy dissipated is thus

$$\Delta U = 2\pi\epsilon_0 \left(V_1^2 r_1 + V_2^2 r_2 \ - V^2\left(r_1 + r_2\right)\right) \ , \tag{26}$$

This simplifies to

$$\Delta U = 2\pi\epsilon_0 \frac{(V_1 - V_2)^2 r_1 r_2}{r_1 + r_2} \tag{27}$$

and evaluates to 2.8164×10^{-6} J.

If the wire is not sufficiently resistive, the system oscillates, and the energy will eventually be radiated away.

11A) Consider constant charge Q on plates area A and distance x apart. The field is $Q/(A\epsilon_0)$ and the potential difference is $V = Qx/(A\epsilon_0)$. The potential energy is therefore

$$U = \frac{Q^2 x}{2\epsilon_0 A} \ . \tag{28}$$

The force between the plates can be evaluated as $F = -dU/dx = -Q^2/(2\epsilon_0 A)$. The force per unit area (pressure) on a surface charge $\sigma = Q/A$ is therefore $p = \sigma^2/2\epsilon_0$ inwards.

Now do this at constant voltage, when the energy stored is

$$U = \frac{V^2 A\epsilon_0}{2x} \ . \tag{29}$$

Thus the work done *by* the external supports in moving from x to $x + \mathrm{d}x = -\mathrm{d}U/\mathrm{d}x\ \mathrm{d}x$. This term by itself would give the opposite sign of the force, but we need to consider the work done by an external battery needed to keep the potential constant as we move the plates apart. This is $dW = dQV = -V^2 A\epsilon_0 dx/x^2$, which is twice as large as the $-dU$ term and has the opposite sign. Changing the sign again to get back to the work done by the force between the plates, we again obtain $p = \sigma^2/2\epsilon_0$ inwards. [See Duffin p.117-118.]

12B) The (attractive) electrostatic force between the plates a distance x apart is $V^2 A\epsilon_0/(2x^2)$. Consider force balance $k(a-b) = V^2 A\epsilon_0/(2b^2)$. Then displace the upper disc slightly by dx, so the new upward force is

$$-kdx + 2V^2 A\epsilon_0 dx/(2b^3) = kdx(2a - 3b)/b \ . \tag{30}$$

This is a restoring force only if $b > 2a/3$.

Here is an alternative way, which considers the weight of the upper disc as well, and does it using an effective potential (as in the Dynamics course). Let the spring constant be k, and work from the equilibrium point ($x = a$) so that the weight can be ignored. The force is attractive so the stored electrostatic energy is *negative*, and the effective potential is

$$V_{\text{eff}}(x) = k(a-x)^2/2 - V^2\pi r^2\epsilon_0/(2x) \ . \tag{31}$$

Differentiating to find the new equilibrium gives

$$0 = \frac{dV_{\text{eff}}}{dx} = -k(a-x) + \frac{V^2\pi r^2\epsilon_0}{2x^2} \ . \tag{32}$$

This new equilibrium at $x = b$ will be stable if the second derivative of the potential is positive.

$$\frac{d^2V_{\text{eff}}}{dx^2} = k - \frac{V^2\pi r^2\epsilon_0}{x^3} = k\left(1 - \frac{2(a-x)}{x}\right) \ . \tag{33}$$

This evaluates to $k(3b-2a)/b$. So the equilibrium is stable if $b > 2a/3$.

13A) The radius c of the inner dielectric evaluates nicely to 3 mm. The electric fields are

$$D = \frac{Q}{2\pi r}\ (a < r < b); \quad E_1 = \frac{Q}{2\pi\epsilon_0\epsilon_1 r}\ (a < r < c); \quad E_2 = \frac{Q}{2\pi\epsilon_0\epsilon_2 r}\ (c < r < b). \tag{34}$$

The total potential difference is

$$V = -\int_a^c dr\ E_1 - \int_c^b dr\ E_2, \tag{35}$$

so that the total capacity $C = Q/V$ is

$$C = \frac{2\pi\epsilon_0}{\log(c/a)/\epsilon_1 + \log(b/c)/\epsilon_2} \ . \tag{36}$$

The numerical value is 110.787 pF/m.

14A) The field E_0 is first decomposed into its components $E_\parallel$ and $E_\perp$ relative to the slab. Here $E_\parallel = E_\perp = E_0/\sqrt{2}$. The field in the slab $E'_\parallel = E_\parallel$ inside the slab, whereas $D'_\perp = D_\perp$ so that $\epsilon E'_\perp = E_\perp$. Resultant field in slab is $E_0(1/\sqrt{2}, 1/\sqrt{8})$, using $\epsilon = 2$.

Now resolve this relative to the cavity. The perpendicular component is $E_0(1/2 + 1/4)$ and the parallel component is $E_0(1/2 - 1/4)$, which become $3E_0/2$ and $E_0/4$ inside the cavity. Result is $E_1/E_0 = \sqrt{37}/4 = 1.5207$.

15A) We need the charge conservation equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{J} = 0 \tag{37}$$

and the constitutive relation for the conductivity

$$\boldsymbol{J} = \sigma \boldsymbol{E} \ . \tag{38}$$

For a uniform dielectric medium Maxwell 1 is

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho/\epsilon\epsilon_0 \ , \tag{39}$$

so that we have

$$\frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon\epsilon_0}\rho = 0 \ , \tag{40}$$

which has the general solution

$$\rho(\mathbf{x}, t) = \rho(\mathbf{x}, 0)\exp(-\sigma t/\epsilon\epsilon_0) \ . \tag{41}$$

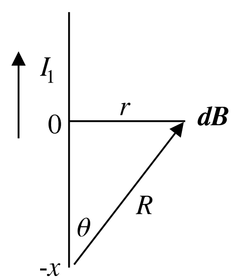

Figure 4: The wire of Q16.

16A) (a) See figure 4. B-S $\Rightarrow \int \mathbf{dB} = -\frac{\mu_0 I_1}{4\pi}\int_{-\infty}^{\infty}\frac{dx\ r}{(r^2+x^2)^{3/2}}$. But $r = R\sin\theta$ and $r = -x\tan\theta$, so $x = -r\cot\theta$ and $dx = r\mathrm{cosec}^2\theta d\theta$. Thus $B = \frac{\mu_0 I_1}{4\pi r}\int_0^{\pi}\sin\theta d\theta = \frac{\mu_0 I_1}{2\pi r}$. B is into the page at the point shown.

(b) The force on a region of the second wire from l_0 to $l_0 + l$, $F = \int dF = I_2\int_{l_0}^{l_0+l}\mathbf{dl}\times\mathbf{B} = I_2\int_{l_0}^{l_0+l}\mathbf{dl}\frac{\mu_0 I_1}{2\pi r} = \frac{\mu_0 I_1 I_2 l}{2\pi r}$, so $F/l = \frac{\mu_0 I_1 I_2}{2\pi r}$. This is consistent with the definition of the Amp, since $\mu_0 = 4\pi\times 10^{-7}$ Hm^{-1}.

17B) (a) See notes.

(b) The longitudinal magnetic field at x_0 along the axis of a solenoid with N turns per unit length is (Biot-Savart)

$$B_x = \frac{\mu_0 n I}{4\pi}\int_{x_0}^{\infty} dx\ \frac{2\pi a^2}{(a^2+x^2)^{3/2}} \ . \tag{42}$$

Although we can evaluate this, we don't need to here, since we want the force on a dipole $\boldsymbol{F} = \boldsymbol{m}\cdot\boldsymbol{\nabla}\mathbf{B}$, which is just the value of the integrand of (42) at x_0. Hence

$$F = \frac{\mu_0 m n I a^2}{2(a^2+x_0^2)^{3/2}} \ . \tag{43}$$

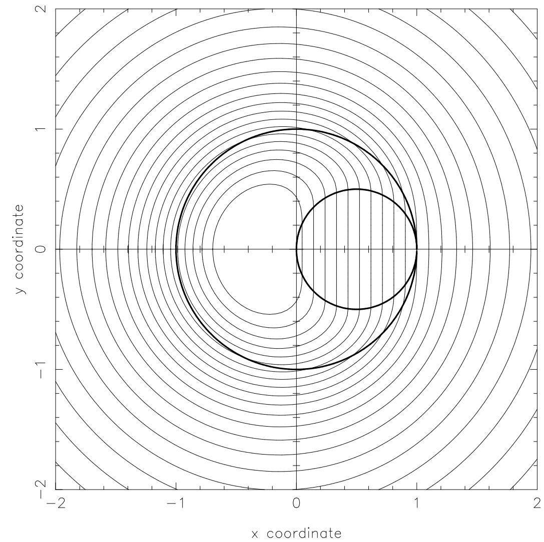

Figure 5: Field lines of **B** for the system of Q18.

18A) (a) Straightforward.

(b) Field is $\propto r$ inside uniform current and $\propto 1/r$ outside. Treat this case as two displaced cylindrical currents of opposite sign. The field from the centre to the point A is actually constant and equal to half the value it would be at A for a complete wire, whereas the field at B is 5/6 of its value. The answer is $B : A = 5 : 3$.

19B) The field $B(r)$ inside the column is

$$B(r) = \frac{\mu_0 I r}{2\pi a^2} \ . \tag{44}$$

The force per unit volume on the current is $\boldsymbol{j}\times\mathbf{B}$, which is radially inward. This is balanced by the pressure gradient

$$\frac{dp}{dr} = -\frac{\mu_0 I^2 r}{2\pi^2 a^4} \ , \tag{45}$$

which integrates to give

$$p(r) = \frac{\mu_0 I^2 (a^2 - r^2)}{4\pi^2 a^4} + \text{ constant}, \tag{46}$$

which is the given answer if the constant of integration is zero (so that $p(a) = 0$). For $a = 5$ mm and $I = 100$ A, we get $p(0) - p(a) =$12.732 Pa.

However, the magnetic pressure (not part of the electromagnetism course) in the column is $B^2/(2\mu_0)$, so

$$p_{\rm mag}(r) = \frac{\mu_0 I^2 r^2}{8\pi^2 a^4} \ , \tag{47}$$

which is *not* zero at the outside of the cylinder.

20B) Essentially the same as the derivation in the notes for a conducting or dielectric sphere. For external field $\mu_0 H_0 = B_0$, write the magnetic scalar potential

$$\phi_{\rm m} = \left(-H_0 r + \frac{m}{4\pi r^2}\right)\cos\theta \ \ (r > a); \quad \phi_{\rm m} = -H_1 r \cos\theta \ \ (r < a). \tag{48}$$

[Better to use m than the M given in the question.]

Match ϕ_{m} (or H_θ) and B_r at $r = a$:

$$
\begin{aligned}
H_1 &= H_0 - \frac{m}{4\pi a^3} \; ; \\
\mu H_1 &= H_0 + 2\frac{m}{4\pi a^3} \; .
\end{aligned} \tag{49}
$$

Eliminating H_1 gives the dipole moment

$$m = \frac{4\pi a^3(\mu - 1)}{\mu + 2} H_0 \; . \tag{50}$$

Field of dipole along z axis is

$$B_z = \frac{\mu_0 m}{2\pi z^3} \tag{51}$$

Force at $z = d$ is $\boldsymbol{\nabla}(\boldsymbol{m} \cdot \mathbf{B})$ (or $\boldsymbol{m}\cdot\boldsymbol{\nabla}\mathbf{B}$) $= -3\mu_0 m^2/(2\pi d^4)$, which is attractive and equal to

$$F_z = \frac{24\pi a^6 B_0^2}{\mu_0 d^4} \left(\frac{\mu - 1}{\mu + 2}\right)^2 \; . \tag{52}$$

21A) Magnetic circuit has

$$\oint d\boldsymbol{l}\cdot\boldsymbol{H} = nI \; . \tag{53}$$

High permeability electromagnet means that all this appears 'shorted out' in the air gap, so the field there has $H_{\mathrm{gap}}\, s = NI = B_{\mathrm{gap}}\, s/\mu_0$. The force on the pole pieces per unit area is $B_{\mathrm{gap}}^2/2\mu_0$, so the force is

$$F = \frac{B_{\mathrm{gap}}^2 b^2}{2\mu_0} = \frac{n^2 I^2 b^2 \mu_0}{2s^2} \; . \tag{54}$$

The question is then the same as Q12. The force balance condition is

$$\frac{bt^3(s_0 - s)E}{8l^3} = \frac{n^2 I^2 b^2 \mu_0}{2s^2} \; , \tag{55}$$

and considering a small displacement ds we get a change in force tending to separate the blades

$$dF = -\frac{bt^3 E ds}{8l^3} + \frac{n^2 I^2 b^2 \mu_0 ds}{s^3} = -\frac{bt^3(2s_0 - 3s)E ds}{8l^3 s} \; , \tag{56}$$

which is unstable if $s < 2s_0/3$. Putting $s = 2s_0/3$ in (55) gives

$$I_{\mathrm{c}}^2 = \frac{s_0^3 t^3 E}{27 l^3 \mu_0 n^2 b} \; . \tag{57}$$

22A) See Figure 6. For the magnetised cylinder, the surface current is $M\hat{\boldsymbol{u}}_\phi$ over the cylindrical face and zero on the ends. There are magnetisation poles $\boldsymbol{\nabla}\cdot\boldsymbol{H} = \boldsymbol{\nabla}\cdot\boldsymbol{M} = \pm M$ on the ends. Points to note:

- The field lines of $\mathbf{B}$ and $\boldsymbol{H}$ coincide outside the cylinder.
- Lines of $\mathbf{B}$ are continuous, but the surface current makes sharp kinks in the field lines.
- Lines of $\boldsymbol{H}$ are not kinked by the magnetisation currents, but they begin and end on the magnetisation poles.
- In the middle of a long solenoid, the $\boldsymbol{H}$ field is low — $\boldsymbol{H}$ is an 'end effect'.

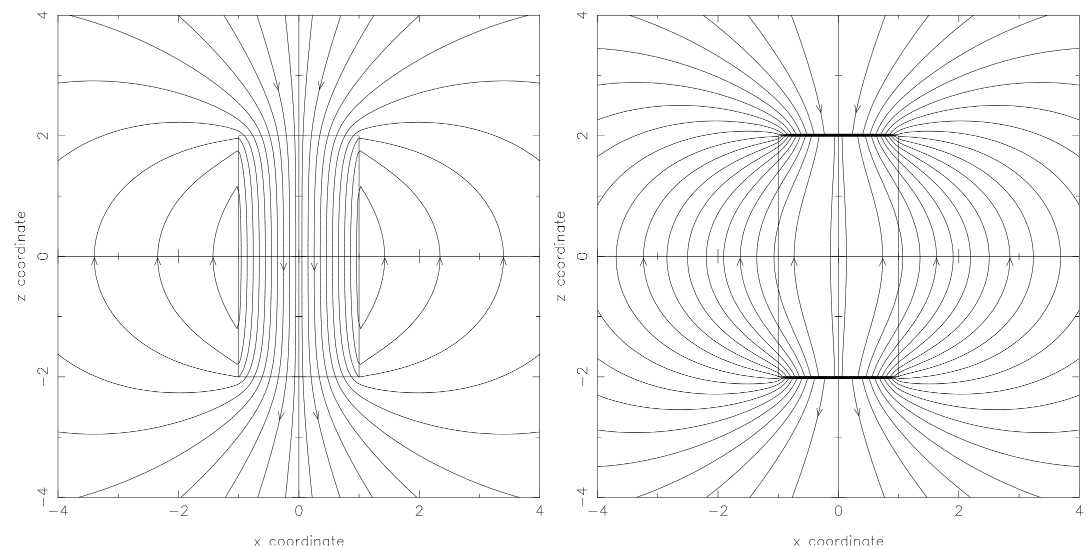

Figure 6: (left) Magnetic **B** field lines of a bar magnet (see Q22). They are identical to those of a short solenoid. (right) Magnetic ***H*** field lines of a bar magnet. They are the same as those that would be produced by two uniform magnetically charged discs, of opposite sign. [There seems to be a problem with the density of field lines drawn here outside the magnet, at top and bottom. For H they should be denser than outside at the sides, to match $B = \mu_0 H$ there.]