NST Part IB Physics B – 2021/22

Classical Dynamics – Examples 3

There will be, on average, about 2 questions for each lecture.

These are labelled:

A) relatively straightforward problems that do not need a lot of algebra;
B) problems that require some algebraic formulation and manipulation, often with some calculation also;
C) problems which are either harder or longer than B-problems.

In additional, some questions have optional parts at the end indicated by '*' for possible further investigation, and there are some other questions at the end for discussion.

Elasticity

28A) A straight tube 1 m long, of radius 10 mm and wall thickness 100 μm, is closed at both ends. It is found that when the pressure inside is increased from 10^5 Pa to 10^6 Pa the tube lengthens by 100 μm. What is the bulk modulus of the material of the tube?

29B) A pillar has length H when lying horizontally; it is of uniform isotropic material and has uniform diameter $\ll H$. When it was set up vertically on a valley floor its height became $H - h_1$. At a later date the valley was flooded; when the water reached the top of the pillar, its height was $H - h_2$. Find h_1 and h_2 in terms of Young's modulus E, Poisson's ratio σ and the density ρ of the pillar, the density ρ_w of water, and the acceleration of gravity.

By considering the case $\rho = \rho_\mathrm{w}$, find the relation between E, σ and the bulk modulus.

30B) A light horizontal cantilever of uniform cross section is such that, when a weight W is hung from its free end, a point half way along the cantilever is displaced downwards by d. Show by direct calculation that, if the same weight W is hung from the point half way along, the displacement of the free end is also d.

Re-examine the question, but for any two points on a cantilever whose cross section varies arbitrarily along its length. Use an energy argument to show that the stored energy when loads are applied at A and B must be the same whether the load at B is applied before or after that at A. Hence show that the deflection at A due to unit load at B is always equal to the deflection at B due to unit load at A.

31A) A light horizontal beam of rectangular section 2×1 has one end built into a wall, a diagonal of the rectangle being vertical. When a weight is hung on the free end, in what direction will the beam deflect?

32B) A light uniform horizontal beam is loaded at its mid-point. Determine the relative maximum deflections for the cases when the beam is

a) freely supported at both ends;
b) rigidly clamped at both ends;
c) rigidly clamped at one end, with the other end unsupported.

State clearly the difference in boundary conditions in the three cases. Where, in each case, is the beam most likely to break if it is overloaded?

[*Find the deflection of a heavy uniform cantilever, supporting a load at the far end.]

33B) A long straight cylindrical wire is fitted with 'universal couplings' at its two ends. (A universal coupling allows a torque to be transferred from one axis to another, with the axes not necessarily parallel.) Through these couplings a torque is applied, and their design is such that the wire can remain straight but twisted, or bend into a flat spring. Show that the energy stored in these two configurations for a given initial torque are in the ratio $1/G : 2/E$, and hence that the wire will bend rather than twist if $\sigma > 0$.

34C) An isotropic elastic medium of infinite extent contains a spherical hole of volume V and gas is pumped into the hole until a pressure P_0 has been built up. At a radial distance r the medium is displaced by $\mathit{\Delta}$. Show the strains parallel and perpendicular to the radial direction are, respectively

$$e_{\|} = \frac{\mathrm{d}(\mathit{\Delta})}{\mathrm{d}r} \qquad \text{and} \qquad e_{\perp} = \frac{\mathit{\Delta}}{r}.$$

Consider the relationships between the stresses and strains, and the equilibrium of a thin hemispherical shell in the medium to show

$$\tau_{\perp} = \tau_{\|} + \frac{r}{2}\frac{\mathrm{d}\tau_{\|}}{\mathrm{d}r},$$

(where $\tau_{\|}$ and $\tau_{\perp}$ are the stresses in directions parallel and perpendicular to the radial direction respectively). Hence show that as a function of radius r, the radial stress within the medium obeys the differential equation $4P' + rP'' = 0$ and that $P \propto 1/r^3$ in equilibrium. Show that the gas dilates the hole by an amount $\Delta V = 3P_0V/4G$.

Fluid Dynamics

35B) Two cylindrical jets of incompressible fluid, which have the same radius a and velocity components $(0, 0, v)$ and $(0, 0, -v)$ respectively, meet head-on at the origin and spread out to form a sheet in the $z = 0$ plane. Show that the thickness d of this sheet at a large distance r from the origin is a^2/r.

Two jets of incompressible fluid have a rectangular cross-section, with thickness a in the x direction and width b ($\gg a$) in the y direction. Now suppose them to be tilted through angles $\pm\alpha$, so that the velocity components with which they meet become $(v\sin\alpha, 0, \pm v\cos\alpha)$. Show that the resultant sheet has thickness $a(1 + \sin\alpha)$ for $x > 0$ and $a(1 - \sin\alpha)$ for $x < 0$.

36B) A bath of cross-sectional area 1.5 m^2 is filled to a depth of 20 cm. Estimate the time taken for the water to drain away when the plug is pulled out, if the area of the plug hole is 15 cm^2.

37B) A small bubble is expanding at a constant volume rate Q a distance d away from a solid plane in a fluid of density ρ. Show that the pressure distribution on the plane is given by

$$P = P_0 - \frac{Q^2\rho}{8\pi^2}\frac{r^2}{(d^2+r^2)^3},$$

where r is the radial distance from the point of symmetry on the plane, and P_0 is the pressure at a large distance from the bubble. Hence, or otherwise, show that the force on the bubble is $\rho Q^2/16\pi d^2$.

Is the force repulsive or attractive? Explain why.

38B) Recall the potential for steady flow of incompressible fluid, with uniform speed v_0 at infinity, past a stationary sphere. Find an expression for the velocity at any point and sketch the streamlines. Where on the surface of the sphere is the pressure highest and the lowest and what are these values compared with the pressure at infinity?

If the pressure at infinity is 10^5 Pa, estimate how rapidly a sphere must travel in still water to risk cavitation.

39B) A river of frictionless incompressible water flows steadily over a flat circular sandbank where the water is half as deep as it is elsewhere. The depth of the river is very small compared with the diameter of the sandbank. By solving Laplace's equation in two dimensions for the flow potential making the (unjustified) assumption that the radial component of the fluid velocity doubles across the interface to the shallower region, show that above the sandbank the current is 4/3 times as fast as it is at large distances.

[*Use MATLAB, or similar, to solve this numerically to obtain a more realistic solution.]

40B) An incompressible fluid of density ρ and viscosity η flows along a pipe of length ℓ. Assuming the flow remains laminar, determine the volume flow rate Q if the pressure difference between the ends of the pipe is ΔP.

Calculate the total viscous force on the walls of the pipe.

What qualitative differences would you expect if instead the flow was turbulent?

41C) In a number of experiments on the terminal velocity of spheres falling through viscous fluids the following results were obtained:

a) Aluminium spheres (density = 2.7×10^3 kg m^{-3}) in propyl alcohol (density = 0.8×10^3 kg m^{-3}, viscosity = 4.50×10^{-3} Pa s).

Diameter of sphere / mm	1.5	3.0	6.0	12.0
Terminal velocity / m s^{-1}	0.167	0.33	0.58	0.88

b) Steel spheres (density = 7.83×10^3 kg m^{-3}) in olive oil (density = 0.93×10^3 kg m^{-3}, viscosity = 99×10^{-3} Pa s).

Diameter of sphere / mm	10.0	17.5	30.0	52.5
Terminal velocity / m s^{-1}	0.89	1.50	2.65	3.30

Examine these results critically in the light of dimensional analysis, and deduce the terminal velocity of a spherical hailstone (density = 0.9×10^3 kg m^{-3}) of diameter 2 mm in air (density = 1.3 kg m^{-3}, viscosity = 17×10^{-6} Pa s).

Numerical Answers

Q28: $\approx 1.5 \times 10^{11}$ Pa.

Q31: $\approx 37^\circ$ from the vertical.

Q36: about 5 min.

Q38: ≈ 12.6 m s^{-1}.

Q41: about 6 m s^{-2}.

Additional Questions

1. Explain the virtue of the ‘I’ cross-section used for steel beams/joists?
2. A sheet of thickness 1 mm is corrugated to a depth of 50 mm. Estimate the factor by which its stiffness to bending about an axis perpendicular to the corrugations exceeds that of the uncorrugated sheet. [Answer: about 2500.]
3. Why is it easier to blow out a candle than suck it out?

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