INTRODUCTION TO NORMAL MODES

- For a general Newtonian system, the equations of motion take the form

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\text{variables} = F\left(\text{variables}, \frac{\mathrm{d}}{\mathrm{d}t}\text{variables}\right)$$

- These equations are often very complicated (e.g. rigid body motion).

- However, we are often interested in **small displacements** of a system from **equilibrium**.

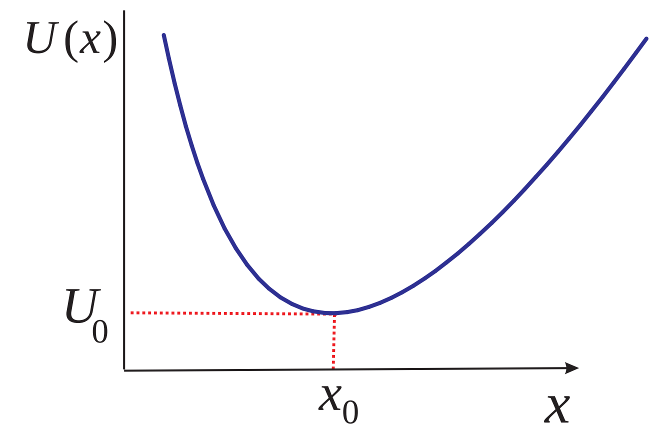
- If we expand the variables — conventionally $\{q_i\}$ — about their equilibrium values $\boldsymbol{q}_{\text{eq}}$: $\text{variables} \equiv \boldsymbol{q} \approx \boldsymbol{q}_{\text{eq}} + \delta\boldsymbol{q}$ we obtain the approximate equations

$$\ddot{\boldsymbol{q}} = \left.\frac{\partial F}{\partial \boldsymbol{q}}\right|_{\text{eq}} \cdot \delta\boldsymbol{q} + \left.\frac{\partial F}{\partial \dot{\boldsymbol{q}}}\right|_{\text{eq}} \cdot \delta\dot{\boldsymbol{q}} \qquad \text{i.e. } \textbf{linear equations}$$

- In general, **small displacements** about equilibrium lead to **linear equations**.

- This in turn leads to the **superposition of solutions** (i.e. we can analyse the problem in pieces and add them all up afterwards).

Revision for Normal Modes



- Consider a particle in a potential well $U(x)$.
- Total energy is conserved: $E = T + U = \frac{1}{2}m\dot{x}^2 + U(x)$.
- Use the energy method: $\dfrac{dE}{dt} = 0$, so $\dot{x}\left(m\ddot{x} + \dfrac{dU}{dx}\right) = 0$.
- The resulting equation of motion $m\ddot{x} + \dfrac{dU}{dx} = 0$ may well be nonlinear.
- Suppose there is an equilibrium position at x_0 where $\dfrac{dU}{dx} = 0$.
- To study small oscillations about x_0, expand $U(x)$ in a Taylor series:

$$U(x) = U_0 + \frac{1}{2}\left.\frac{d^2U}{dx^2}\right|_{x_0}(x - x_0)^2 + \ldots$$

- Defining the **small** displacement $\epsilon \equiv x - x_0$ and denoting $\left.\dfrac{d^2U}{dx^2}\right|_{x_0} \equiv U_0''$ (i.e. a constant) gives

  a linear equation of motion $\boxed{m\ddot{\epsilon} + U_0''\epsilon = 0,}$ which **is** SHM at angular frequency $\omega^2 = \dfrac{U_0''}{m}$.

NST IB Physics B (2021/22) — Dynamics: Handout III — Prof M. C. Payne

Definition of a Normal Mode

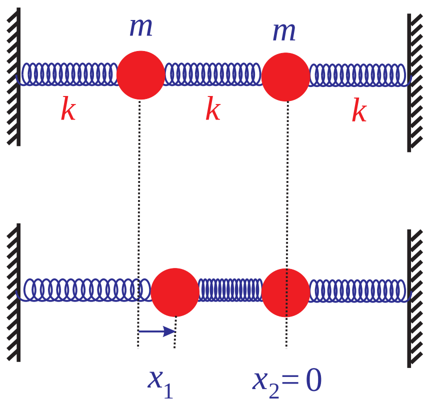
- We will study the free (unforced) small oscillations of dynamical systems about equilibrium. The system will oscillate, but a system with many dynamical variables can oscillate in many different ways. The important concept is that of the **normal mode** of oscillation.
- Definition: In a **normal mode** every element of the system oscillates at a **single frequency.**
- In a normal mode all parts of the system share the same periodic time dependence.
- For normal nodes, we are considering **free oscillations** in the absence of external forces.
- A general free oscillation of the system can be expressed in terms of a linear combination of the simple normal modes.
  - Boundary conditions at a particular time define the amplitude and phase of each normal mode.
  - Each normal mode evolves with time at its own frequency, and the normal modes can be recombined to find the solution at any later time.

Note: normal modes also useful when there is an external force.

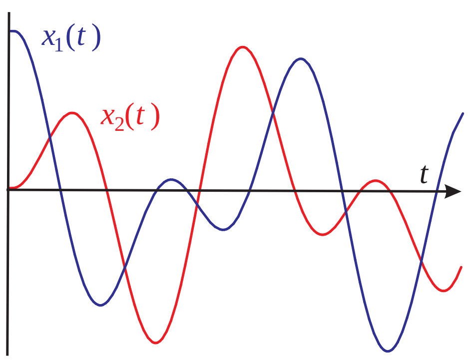
DYNAMICS OF A TWO-MASS SYSTEM

Consider a system of two masses and 3 ideal springs: (x_1, x_2) are the displacements of the masses from equilibrium. The masses are both m and the spring constants are all k.

- The general motion is rather complicated, but the system has a **symmetry** as the masses and spring constants are equal.
- Suppose the mass at x_1 is displaced from equilibrium at $t = 0$, but $x_2 = 0$. The masses are then released and allowed to oscillate freely.



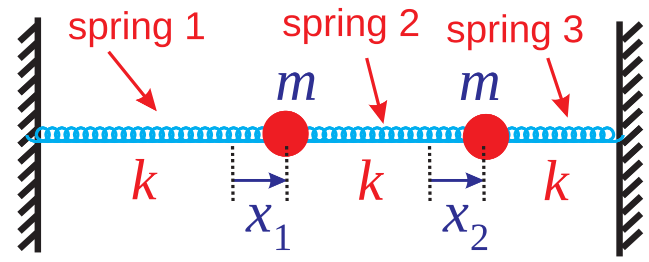
- The motion is as shown.



- The oscillation looks complicated, but there are only two frequencies present in the oscillation (because there are two coordinates — this is a general result).
- $\boxed{\textbf{NORMAL MODE THEOREM}}$

1) Considering forces

- Equations of motion for a two-mass system:



$$m\ddot{x}_1 = \underbrace{-kx_1}_{\text{spring 1}} + \underbrace{k(x_2 - x_1)}_{\text{spring 2}},$$

$$m\ddot{x}_2 = \underbrace{-k(x_2 - x_1)}_{\text{spring 2}} + \underbrace{(-kx_2)}_{\text{spring 3}}.$$

- We can rewrite this compactly using matrix notation $\begin{pmatrix} m\ddot{x}_1 \\ m\ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

- The equations of motion are coupled ODEs — we seek the Complementary Function by using a trial solution $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} e^{i\omega t}$ where X_1, X_2 are constants.

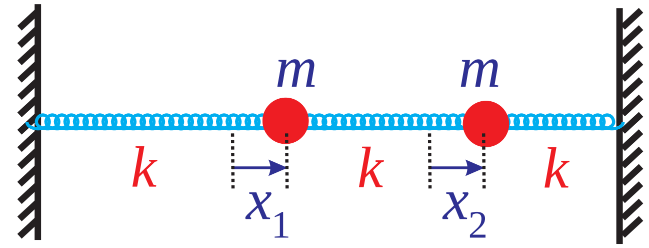
- The result is a set of **homogeneous** linear equations for X_1, X_2:

$$\begin{pmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

- Homogeneous equations only have non-trivial solutions (i.e. other than zero) if the determinant of the matrix is zero. So $(2k - m\omega^2)^2 - k^2 = 0$ or $(3k - m\omega^2)(k - m\omega^2) = 0$.

- There are two characteristic frequencies: $\omega^2 = 3k/m$ and $\omega^2 = k/m$.

2) Using Euler–Lagrange equations



- Total kinetic energy is $T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2)$.
- Potential energy is $V = \frac{1}{2}k(x_1^2 + (x_1 - x_2)^2 + x_2^2)$
- So the Lagrangian is $\mathcal{L} = \frac{1}{2}\left[m(\dot{x}_1^2 + \dot{x}_2^2) - k(x_1^2 + (x_1 - x_2)^2 + x_2^2)\right]$.
- The conjugate momenta $p_{x_i} = \dfrac{\partial \mathcal{L}}{\partial \dot{x}_i}$, are $p_{x_1} = m\dot{x}_1$ and $p_{x_2} = m\dot{x}_2$. Also

$$\frac{\partial \mathcal{L}}{\partial x_1} = -k\left[x_1 + (x_1 - x_2)\right] = k(x_2 - 2x_1),$$ and

$$\frac{\partial \mathcal{L}}{\partial x_2} = -k\left[-(x_1 - x_2) + x_2\right] = k(x_1 - 2x_2).$$

- So the equations of motion, $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) = \dfrac{\partial \mathcal{L}}{\partial x_i}$ are

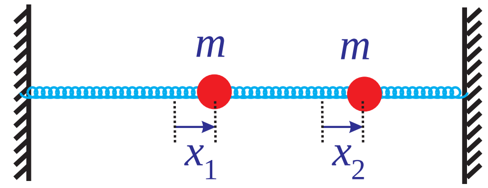
$$m\ddot{x}_1 = -k(2x_1 - x_2), \qquad m\ddot{x}_2 = -k(2x_2 - x_1),$$

as obtained considering the forces.

Normal Coordinates of a Two-Mass System

- $\begin{vmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{vmatrix} = 0$ gives characteristic frequencies $\omega^2 = \dfrac{3k}{m}$ and $\omega^2 = \dfrac{k}{m}$.

- The amplitudes (X_1, X_2) are the **normal modes** of the system.



1) $\omega^2 = \dfrac{3k}{m}$ so $\begin{pmatrix} -k & -k \\ -k & -k \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ so $X_1 + X_2 = 0$. The normal mode is $\propto \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

2) $\omega^2 = \dfrac{k}{m}$ so $\begin{pmatrix} k & -k \\ -k & k \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ so $X_1 - X_2 = 0$. The normal mode is $\propto \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

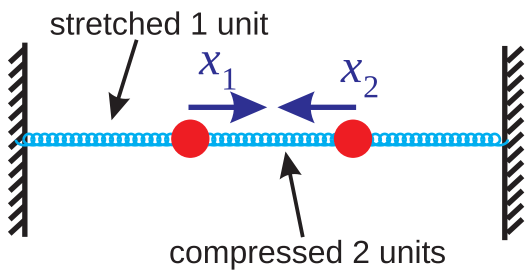
- Equations of motion: $\begin{array}{l} m\ddot{x}_1 = -2kx_1 + kx_2 \\ m\ddot{x}_2 = kx_1 - 2kx_2 \end{array}$ so $\begin{array}{l} m(\ddot{x}_1 - \ddot{x}_2) = -3k(x_1 - x_2) \\ m(\ddot{x}_1 + \ddot{x}_2) = -k(x_1 + x_2) \end{array}$

- We have simple (decoupled) equations for new variables $q_1 \equiv X_1 - X_2$ and $q_2 \equiv X_1 + X_2$.

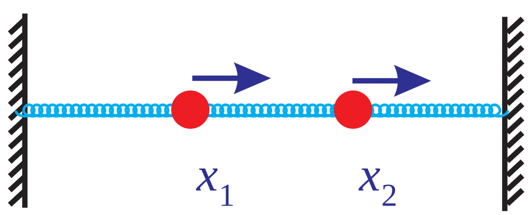
- The variables (q_1, q_2) (or any multiples) are the **normal coordinates of the system**.

NORMAL MODES OF A TWO-MASS SYSTEM

- The symmetry of the two-mass system makes the normal modes obvious.

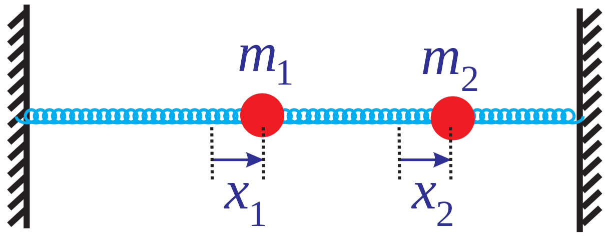


- **Symmetric mode**: has $X_2 = -X_1$, so that the central spring is compressed by twice as much as the outer springs are extended.
- We then have the equation of motion $m\ddot{x}_1 = -3kx_1$ (SHM as before).
- Knowing the shape (i.e. $X_2 = -X_1$) of the normal mode we can usually calculate the frequency using the **general rule** that: $\omega^2 = \dfrac{\text{restoring force per unit extension}}{\text{mass}}$.
- For the symmetric mode the restoring force on mass 1 per unit displacement is $3k$ so that $\omega^2 = \dfrac{3k}{m}$ as before (same for mass 2, so we have a normal mode).



- **Antisymmetric mode**: has $X_2 = X_1$, so that the central spring is unchanged. $m\ddot{x}_1 = -kx_1$, so $\omega^2 = \dfrac{k}{m}$.
- These are the normal modes of the two-mass system.

System with Two Unequal Masses



- Equations of motion:

$$\begin{aligned} m_1 \ddot{x}_1 &= -2kx_1 + kx_2, \\ m_2 \ddot{x}_2 &= kx_1 - 2kx_2. \end{aligned}$$

- We seek normal modes $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \mathrm{e}^{\mathrm{i}\omega t}$ as before.

- This again gives a set of homogeneous linear (matrix) equations:

$$\begin{pmatrix} 2k - m_1\omega^2 & -k \\ -k & 2k - m_2\omega^2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

- The frequencies of the normal modes are found by setting the determinant to zero, so

$$3k^2 - 2k(m_1 + m_2)\omega^2 + m_1 m_2 \omega^4 = 0,$$

so that

$$\omega^2 = \frac{k}{m_1 m_2}\left(m_1 + m_2 \pm \sqrt{m_1^2 - m_1 m_2 + m_2^2}\right).$$

- Having determined the normal frequencies we use them to find the shape of the normal modes: $\dfrac{X_2}{X_1} = \dfrac{\left(m_2 - m_1 \mp \sqrt{m_1^2 - m_1 m_2 + m_2^2}\right)}{m_2}$.

- It is instructive to plot X_2/X_1 and the ratio of the two frequencies as a function of m_2/m_1.

- There are still **two normal modes**, in which the system oscillates at a single frequency.

- The general (free) oscillation is then an arbitrary sum of the two normal modes.

Example: Find the normal modes of the rotating double pendulum previously discussed, slide 127, for $\omega = 0$, and find when there are no stable normal modes if $\omega \neq 0$.

- The equations of motion can be written as

$$\begin{pmatrix} \ddot{\alpha}_1 \\ \ddot{\alpha}_2 \end{pmatrix} = \begin{pmatrix} \omega^2 - 2g/\ell & g/\ell \\ 2g/\ell & \omega^2 - 2g/\ell \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}.$$

- If $\omega = 0$, look for solutions with $\alpha_i = \mathrm{e}^{\mathrm{i}\Omega t}$. Then

$$\begin{pmatrix} \Omega^2 - 2g/\ell & g/\ell \\ 2g/\ell & \Omega^2 - 2g/\ell \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = 0.$$

  Put $\lambda = \Omega^2 \ell / g$ for convenience, then $\begin{vmatrix} \lambda - 2 & 1 \\ 2 & \lambda - 2 \end{vmatrix} = 0$,

  so $\lambda = 2 \pm \sqrt{2}$, or $\Omega^2 = (2 \pm \sqrt{2}) \dfrac{g}{\ell}$.

- The eigenvectors, i.e. modes, are $\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$ for lower frequency, and $\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$ for the higher frequency.
- If $\omega \neq 0$, then $\lambda = 2 \pm \sqrt{2} - \omega^2 \dfrac{\ell}{g}$, or

$$\Omega^2 = (2 \pm \sqrt{2})\frac{g}{\ell} - \omega^2,$$

so if $\omega^2 > (2 - \sqrt{2})\dfrac{g}{\ell}$ then it will be unstable.

NORMAL MODES OF A GENERAL SYSTEM

- Suppose the state of a system is specified by N coordinates $\{q_i\}$.

  Examples:

  1) system of M particles in 3-D — $\{\boldsymbol{r}_i\}$ $\quad (N = 3M)$;

  2) system of two masses and three springs in 1-D — $\{x_1, x_2\}$ $\quad (N = 2)$;

  3) gyroscope — $\{\theta, \phi, \chi\}$ $\quad (N = 3)$.

- The variables $\{q_i\}$ or $\underline{q}$ are usually called **generalised coordinates**.

- Suppose that the system has a position of equilibrium at $\underline{q} = \underline{0}$. Since the coordinates $\underline{q}$ specify the state of the system can write the kinetic energy as

$$T = \frac{1}{2} \sum_k m_k |\dot{\boldsymbol{r}}_{\boldsymbol{k}}|^2,$$

  where $\boldsymbol{r} = \boldsymbol{r}(\{q_i\})$ are the Cartesian coordinates of all of the parts of the system.

- Near equilibrium, we can make a Taylor expansion $\dot{\boldsymbol{r}} \approx \sum_i \dot{q}_i \left.\frac{\partial \boldsymbol{r}}{\partial q_i}\right|_{\text{eq}}$, so that $T = \frac{1}{2}\sum_{ij} M_{ij}\dot{q}_i\dot{q}_j \equiv \frac{1}{2}\underline{\dot{q}}^T \cdot \underline{\underline{\boldsymbol{M}}} \cdot \underline{\dot{q}}$.

- The 'mass matrix' $M_{ij} \equiv \sum_k m_k \left.\frac{\partial \boldsymbol{r}_k}{\partial q_i}\right|_{\text{eq}} \cdot \left.\frac{\partial \boldsymbol{r}_k}{\partial q_j}\right|_{\text{eq}}$ is a **constant**.

- Thus, for a system near equilibrium, the kinetic energy is a *quadratic* function of the $\underline{\dot{q}}$.

- The potential energy of a general system can also be written as a function of the generalised coordinates:

$$U = U(\underline{q}) \approx U_0 + \sum_i q_i \underbrace{\left.\frac{\partial U}{\partial q_i}\right|_{\text{eq}}}_{\substack{=0 \\ \text{at equilibrium}}} + \frac{1}{2}\sum_{ij} q_i q_j \underbrace{\left.\frac{\partial^2 U}{\partial q_i \partial q_j}\right|_{\text{eq}}}_{\substack{\equiv K_{ij} \\ \text{constant}}} + \dots$$

- The potential energy of a system near equilibrium is thus a *quadratic* function of the q.

- The total energy $E = U_0 + \frac{1}{2}\sum_{ij} M_{ij}\dot{q}_i\dot{q}_j + \frac{1}{2}\sum_{ij} K_{ij}q_iq_j$ is constant, so $\frac{\mathrm{d}E}{\mathrm{d}t} = 0$ or $\sum_{ij}\dot{q}_i\left(M_{ij}\ddot{q}_j + K_{ij}q_j\right) = 0$.
- The equations of motion for the system are then $\sum_j M_{ij}\ddot{q}_j + \sum_j K_{ij}q_j = 0$,

  or, equivalently $\boxed{\underline{\underline{\boldsymbol{M}}} \cdot \underline{\ddot{q}} + \underline{\underline{\boldsymbol{K}}} \cdot \underline{q} = \underline{0}.}$
- For a general system specified by a set of N coordinates q_i, the equations of motion for small oscillations about equilibrium at $\underline{q} = \underline{0}$ are $\underline{\underline{\boldsymbol{M}}} \cdot \underline{\ddot{q}} + \underline{\underline{\boldsymbol{K}}} \cdot \underline{q} = \underline{0}$.
- The properties of normal modes will be illustrated further by considering specific examples.

More Rigorous Method:

- If the state of a system is specified by a set of N coordinates q_i, the Lagrangian is in general

$$\mathcal{L} \equiv T - U = \mathcal{L}(\{q_i\},\{\dot{q}_i\},t).$$

- The **Lagrangian recipe** for mechanics (which is not emphasised in this course) then says that the equations of motion are derived from the Euler–Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i}.$$

- For our system in equilibrium at $\underline{q} = \underline{0}$, the Lagrangian approach confirms our earlier result.

Normal Modes in Practice

- To solve a problem involving normal modes we usually need to write down the 'mass matrix' $\underline{\underline{M}}$ and the 'spring constant' matrix $\underline{\underline{K}}$.

- **Example**: the two-mass system has coordinates $\underline{q} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, and the energies are

  $T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2)$ and $U = \frac{1}{2}k(x_1^2 + (x_1 - x_2)^2 + x_2^2)$.

- The matrix $\underline{\underline{M}}$ is straightforward since it contains only diagonal terms: The idea is to write

  $$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) = \frac{1}{2}\begin{pmatrix} \dot{x}_1 & \dot{x}_2 \end{pmatrix}\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}.$$

- The matrices $\underline{\underline{M}}$ and $\underline{\underline{K}}$ are **by construction** symmetric (e.g. $K_{ij} = K_{ji}$). This is because: 1) no antisymmetric part of K_{ij} could contribute to the quadratic form $\frac{1}{2}\sum_{ij} K_{ij} q_i q_j$, and 2) the symmetry of $\dfrac{\partial^2 U}{\partial q_i \partial q_j} = \dfrac{\partial^2 U}{\partial q_j \partial q_i}$.

- The potential energy is

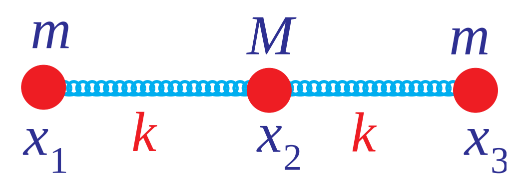
$$U = \tfrac{1}{2}(2kx_1^2 - 2kx_1x_2 + 2kx_2^2).$$

- When assembling the $\underline{\underline{K}}$ matrix we place the $2kx_1^2$ and $2kx_2^2$ on the diagonal, but have to split the off-diagonal term $-2kx_1x_2$ equally on either side: i.e.

$$U = \tfrac{1}{2}\begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

  In this way we get the correct result when the matrix products are evaluated (note that this problem does not occur when using forces - essentially due to Newton's 3rd Law)

TRIATOMIC MOLECULE



- We make a (classical) model of a triatomic molecule consisting of three masses and two springs:
- Consider one-dimensional motion only: coordinates (x_1, x_2, x_3).
- Kinetic energy: $T = \frac{1}{2}(m\dot{x}_1^2 + M\dot{x}_2^2 + m\dot{x}_3^2) = \frac{1}{2}\underline{\dot{x}}^T \cdot \underline{\underline{\boldsymbol{M}}} \cdot \underline{\dot{x}}$ so $\underline{\underline{\boldsymbol{M}}} = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix}$.
- Potential energy: $U = \frac{1}{2}(k(x_2 - x_1)^2 + k(x_3 - x_2)^2) = \frac{1}{2}\underline{x}^T \cdot \underline{\underline{\boldsymbol{K}}} \cdot \underline{x}$

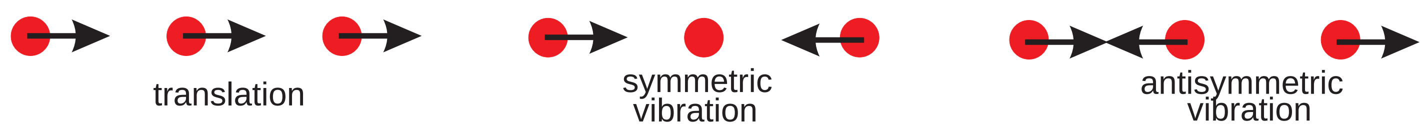
  so $\underline{\underline{\boldsymbol{K}}} = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}$.
- Seek normal modes $\underline{x}(t) = \underline{X}e^{i\omega t}$ so $(\underline{\underline{\boldsymbol{K}}} - \omega^2\underline{\underline{\boldsymbol{M}}}) \cdot \underline{X} = \underline{0}$.

- Non-trivial solution only if $\det(\underline{\underline{\boldsymbol{K}}} - \omega^2 \underline{\underline{\boldsymbol{M}}}) = 0$.
- That is

$$\begin{vmatrix} k - m\omega^2 & -k & 0 \\ -k & 2k - M\omega^2 & -k \\ 0 & -k & k - m\omega^2 \end{vmatrix} = 0 \quad \text{so} \quad \omega^2 (k - m\omega^2)\left(mM\omega^2 - (2m+M)k\right) = 0.$$

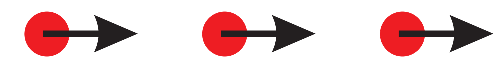
- The normal modes of the system have:

$$\omega^2 = 0; \qquad \omega^2 = \frac{k}{m}; \qquad \omega^2 = \frac{k(M+2m)}{Mm}.$$



- The normal modes of the classical model of a triatomic molecule.

  1) $\omega^2 = 0$ mode is $\propto \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

  This mode is free translation of the whole system along with the centre of mass.

  2) $\omega^2 = \dfrac{k}{m}$ mode is $\propto \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

  This is a **symmetric** vibration (w.r.t. inversion about central mass).

  There is no motion of the centre of mass or the central atom.

  3) $\omega^2 = \dfrac{k(M+2m)}{Mm}$ mode is $\propto \begin{pmatrix} 1 \\ -2m/M \\ 1 \end{pmatrix}$.

  This is a **antisymmetric** vibration.

  There is no motion of the centre of mass.

Alternative procedure: 1) ‘guess’ the shapes of the normal modes (all are obvious), then 2) substitute in to find the frequencies.

Orthogonality of Normal Modes

- Our equation for the normal modes $(\underline{\underline{K}} - \omega^2 \underline{\underline{M}}) \cdot \underline{Q} = \underline{0}$ is a simple generalisation of the eigenvalue equation $(\underline{\underline{A}} - \lambda \underline{\underline{I}}) \cdot \underline{x} = \underline{0}$ that you have already met.
- The matrices are symmetric, so the eigenvalues ω_i^2 are real, but the normal modes are not orthogonal in the usual sense.
- You can make the normal mode system look more like the usual eigenvalue equation by writing it as $(\underline{\underline{M}}^{-1} \underline{\underline{K}} - \omega^2 \underline{\underline{I}}) \cdot \underline{Q} = \underline{0}$, which is an eigenvalue equation as taught in mathematics, but it has the same (possibly non-orthogonal) eigenvectors, and is no longer necessarily symmetric (as for the triatomic system),
- The normal modes we have found $\underline{Q}_i$ are actually orthogonal in the sense that $\underline{Q}_i^T \cdot \underline{\underline{M}} \cdot \underline{Q}_j = 0$ for $i \neq j$.
- This means that the 'scaled' modes $\underline{\underline{M}}^{1/2} \cdot \underline{Q}_j$ **are** orthogonal in the usual sense.
- The correct eigenvalue problem for normal modes is actually $\left(\underline{\underline{M}}^{-1/2} \underline{\underline{K}} \underline{\underline{M}}^{-1/2} - \omega^2 \underline{\underline{I}}\right) \cdot (\underline{\underline{M}}^{1/2} \cdot \underline{Q}) = \underline{0}$, which retains all the helpful symmetry and orthogonality properties.

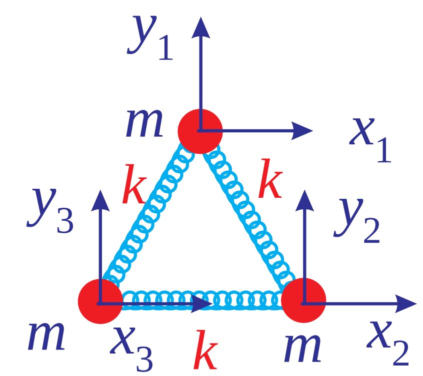
- It is possible to find suitable 'scaled' coordinates that diagonalise $\underline{\underline{M}}$ and $\underline{\underline{K}}$ simultaneously.
- This is a procedure for getting fully orthonormal modes:
  1) diagonalise $\underline{\underline{M}}$ - this is not always necessary as $\underline{\underline{M}}$ is often diagonal ;
  2) scale the coordinates so that $\underline{\underline{M}}$ becomes a unit matrix — this equivalent to taking the square root of $\underline{\underline{M}}$;
  3) diagonalise $\underline{\underline{K}}$ in the new coordinates.
- In this way the scaled $\underline{\underline{M}}$ remains diagonal during the final step as it is a unit matrix.

REMARKS ON NORMAL MODES

- The concept of a normal mode is very important, both theoretically and in practice.
- **Stability**: The normal frequencies ω^2 are real.
  1) If all the ω^2s are positive, the system is **stable.**
  2) If any of the ω^2s are negative ($\omega^2 = -\kappa^2$), then growing modes $\propto \mathrm{e}^{\kappa t}$ exist and the system is unstable.
  3) Zero-frequency modes can exist, and usually correspond to translation or rotation of the whole system. (Remember that the general solution to $\ddot{x} = 0$ is $x = At + B$).
- **Degeneracy**: two or more normal frequencies are equal. This often occurs because of some symmetry inherent in the system, but can also be **accidental** (i.e. not due to symmetry).
- You can use symmetry to guess normal modes: you can then usually find the frequencies fairly easily via elementary arguments - **NORMAL MODE THEOREM**.
- Group representation theory is a very powerful way of deducing the nature of normal modes. For example, applied to the triatomic molecule problem it says that the are two antisymmetric modes and one symmetric mode.

Classical Ozone Molecule

- Consider the classical model of an ozone molecule consisting of three masses and three springs in an equilateral triangle as shown. For normal modes **in the plane** of the molecule. Taking the generalised coordinates $\{q_i\} = (x_1, y_1, x_2, y_2, x_3, y_3)$

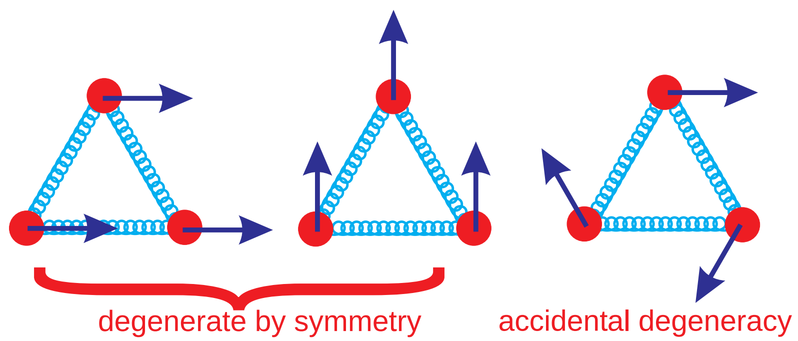


- The $\underline{\underline{M}}$ matrix is easy but the $\underline{\underline{K}}$ matrix is rather complicated:

$$\underline{\underline{M}} = m\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \underline{\underline{K}} = k\begin{pmatrix} 1/2 & 0 & -1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ 0 & 3/2 & \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & -3/4 \\ -1/4 & \sqrt{3}/4 & 5/4 & -\sqrt{3}/4 & -1 & 0 \\ \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 & 0 & 0 \\ -1/4 & -\sqrt{3}/4 & -1 & 0 & 5/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 & 0 & 0 & \sqrt{3}/4 & 3/4 \end{pmatrix}.$$

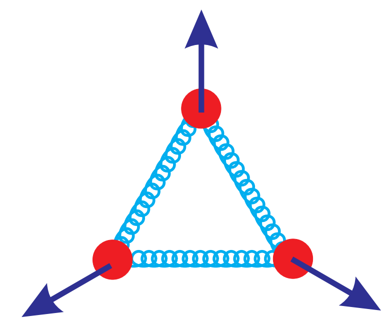
- We find the normal frequencies are given by: $\omega^2 = \left\{0, 0, 0, \frac{3k}{m}, \frac{3k}{2m}, \frac{3k}{2m}\right\}$.
- The system looks complicated, but most of the normal modes are obvious.

1) The three modes with $\omega^2 = 0$ correspond to translations and rotation.



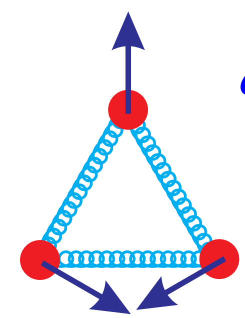
$\omega^2 = 0$

2) Plus there is a symmetric vibrational ('breathing') mode with $\omega^2 = \frac{3k}{m}$.



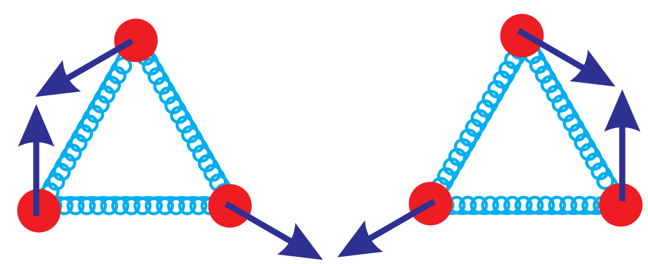
$$\omega^2 = \frac{3k}{m}$$

3) The remaining vibrational modes are **degenerate**, but must be orthogonal to the others, and so must have no net translation, no net rotation and no net outward motion. One possible mode is shown.



$$\omega^2 = \frac{3k}{2m}$$

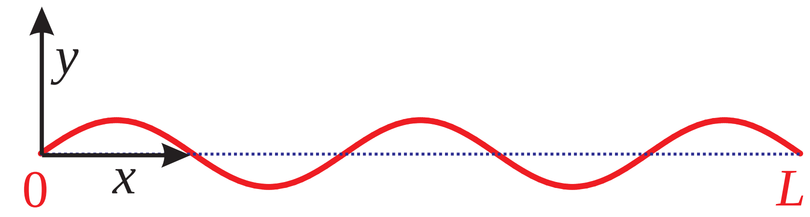
We can find other modes with the same frequency by rotating this mode by $\pm 120^\circ$.



Note: there are only two independent modes with $\omega^2 = 3k/2m$ since the sum of the three modes illustrated is zero and they are not linearly independent.

The symmetry properties of the normal modes (i.e. two different one-dimensional modes and two sets two-dimensional modes) of can be determined by group representation theory.

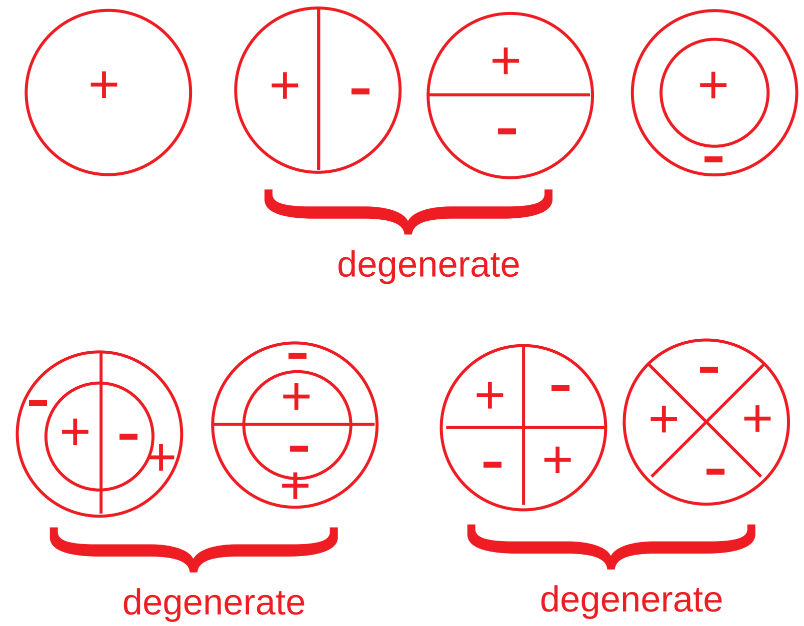
Example 1: Standing waves on a string.



- Fixed ($y=0$) at $x=0$ and $x=L$.
- Tension T and mass per unit length ρ, gives equation of motion $T\dfrac{\partial^2 y}{\partial x^2} = \rho\dfrac{\partial^2 y}{\partial t^2}$.
- Seek wave solution $\propto \exp(\mathrm{i}(\omega t - kx))$.
- Find relationship between k and ω: $\omega^2 = \dfrac{T}{\rho}k^2$.
- The boundary conditions fix the allowed k values: $k_n = \dfrac{n\pi}{L}$ (in this case they are **harmonically** related).
- The nth harmonic has n antinodes:

$$y(x,t) = \underset{\text{constant}}{A} \quad \underset{\substack{\text{shape of}\\ \text{normal mode}}}{\sin k_n x} \quad \underset{\substack{\text{time dependence}\\ \text{(same for whole system)}}}{\cos(\omega t + \chi)}.$$

Example 2: Waves on a circular membrane (i.e. drum) of radius a.



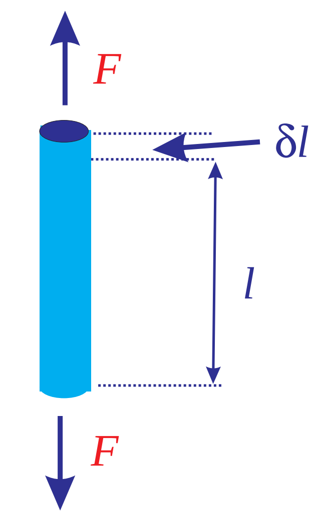
- Boundary conditions: $y(r=a)=0$.
- The (n,m)th mode has n nodes in r and m nodes in ϕ:

$$y(x,t) \;=\; \underset{\substack{\text{constant}\\ \text{amplitude}}}{A} \quad \underset{\substack{\text{shape of}\\ \text{normal mode}}}{J_m(k_{nm}r)\begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix}} \quad \underset{\substack{\text{time dependence}\\ \text{(same for whole system)}}}{\cos(\omega t+\chi)}$$

where $J_m(k_{nm}a)$ is the nth zero of the Bessel function of order m.

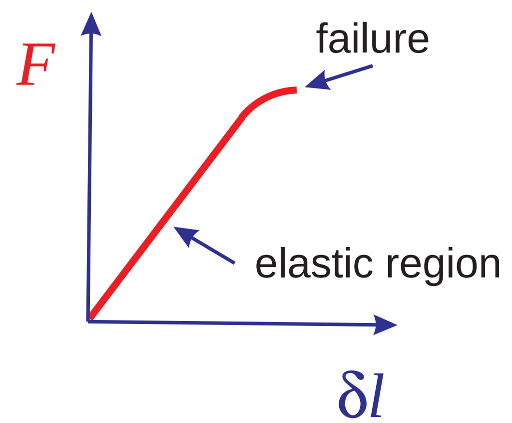
- The frequencies are $\omega_{nm}=k_{nm}\sqrt{T/\rho}$ and are **not** harmonically related.
- The lowest mode has $\omega_{01}=1.551\sqrt{T/\rho}\,\dfrac{1}{a}$.

Elasticity — Fundamentals

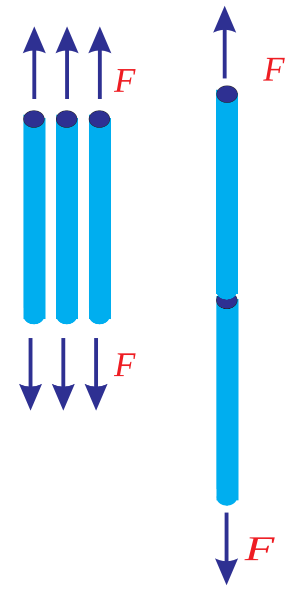


- **Revision**: An elastic wire of length l, cross-section A is subject to a stretching force F and extends by δl.
- In the **elastic region** the wire returns to its original length when the force is removed.
- Stretching beyond this point causes plastic deformation and fracture.
- In the **linear elastic region** the material obeys Hooke's Law (1678), i.e. **the extension is proportional to the force**.
- **Definition**: for small elastic displacements $\boxed{F = EA\frac{\delta l}{l},}$

  where E is Young's modulus.



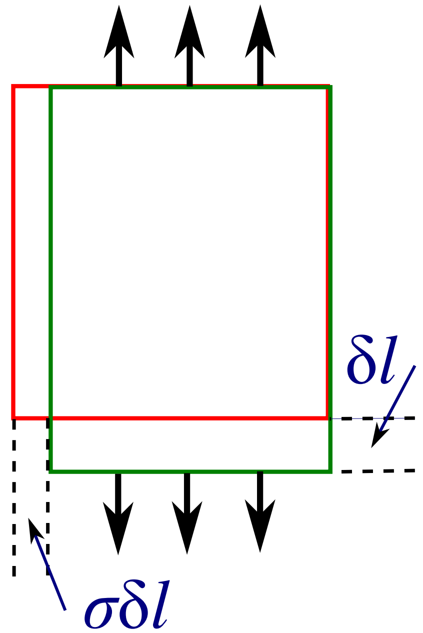
- For the same extension the force $\propto$ area.
- For the same force the extension $\delta l \propto l$.



Note: the alternate definition for a spring is $F = k\,\delta l$. Here k is the constant of proportionality between force, F, and extension, δl for a **particular** spring. Clearly k depends on the material, length and cross-section of the spring.

YOUNG'S MODULUS AND POISSON'S RATIO

- Hooke's law: $\dfrac{F}{A} = E\dfrac{\delta l}{l}$.
- This says that the **stress** (force/unit area) is proportional to **strain** (fractional distortion).
- Consider a cube of elastic medium stretched by δl due to tension in the 1-direction.
- Response in other directions (2,3) is a (usually) **compression** by $\sigma\delta l$.
- σ is the **Poisson ratio** which is (usually) positive.
- For an **isotropic** solid (i.e. one with no preferred directions) the elastic properties are completely determined by the Young modulus E and the Poisson ratio σ.



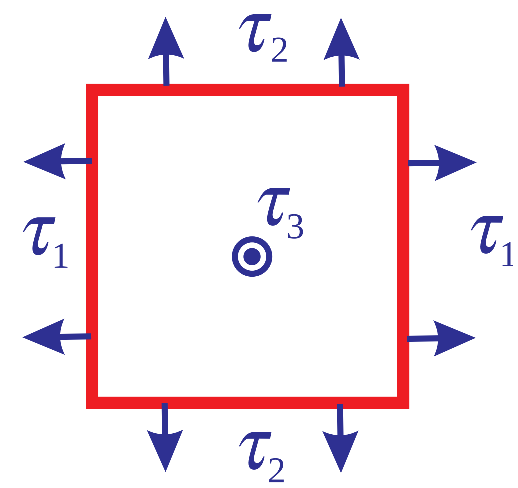
Note: later we will define two useful combinations of Young's modulus, E, and the Poisson ratio, σ, to give: (1) the shear modulus $G = \dfrac{E}{2(1+\sigma)}$, and (2) the bulk modulus $B = \dfrac{E}{3(1-2\sigma)}$.

STRESS AND STRAIN

- For an **isotropic** linear elastic medium

STRESS	$\propto$	STRAIN
(force/unit area trying to distort the medium)		(fractional distortions of the medium)

- For a unit cube of material apply tensions **(stresses)** (τ_1, τ_2, τ_3) along the x, y, z-axes. These produce distortions **(strains)** $\dfrac{\delta l}{l} \equiv (e_1, e_2, e_3)$.



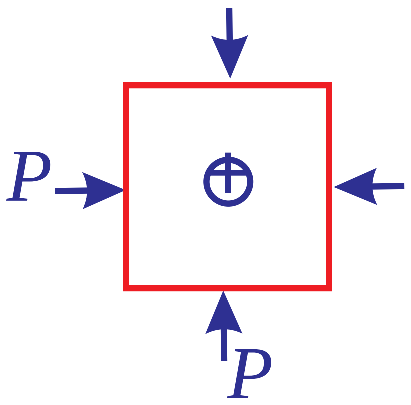
- Example: tension τ_1 along the x-axis (i.e. $\tau_2 = \tau_3 = 0$) produces strains $E(e_1, e_2, e_3) = \tau_1(1, -\sigma, -\sigma)$.

- The medium responds in a similar way if there are stresses τ_2 and τ_3. Since the strains are linear we have

$$\boxed{\begin{aligned} Ee_1 &= \tau_1 - \sigma\tau_2 - \sigma\tau_3, \\ Ee_2 &= -\sigma\tau_1 + \tau_2 - \sigma\tau_3, \\ Ee_3 &= -\sigma\tau_1 - \sigma\tau_2 + \tau_3. \end{aligned}}$$

- Note that things are much more complicated for non-isotropic materials.

BULK MODULUS



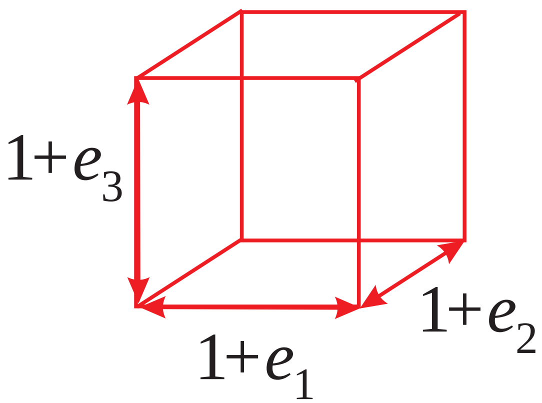
- Consider an isotropic medium under pressure P:

  $$\tau_1 = \tau_2 = \tau_3 = -P.$$

- Applying the relations

  $$E e_1 = \tau_1 - \sigma\tau_2 - \sigma\tau_3 = -P(1-2\sigma) \text{ etc.}$$

  then $e_1 = e_2 = e_3 = \dfrac{-P(1-2\sigma)}{E}$.



- The change of volume of the **unit** cube is

  $$(1+e_1)(1+e_2)(1+e_3) \approx 1 + (e_1+e_2+e_3)\ldots$$

  so that the pressure **decreases** the volume by

  $$|\delta V| = \frac{3P(1-2\sigma)}{E}$$

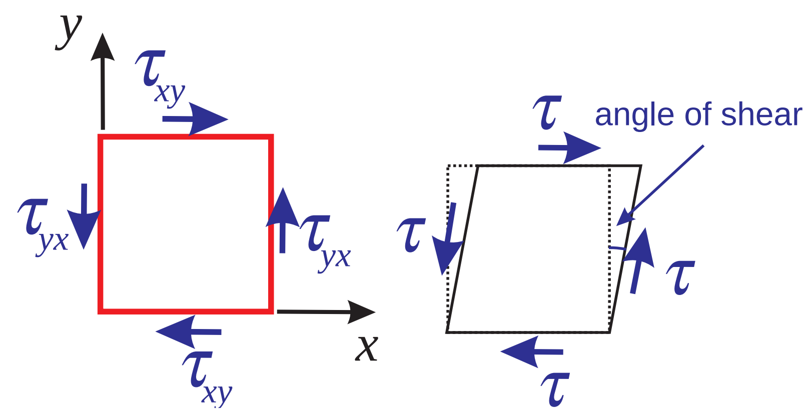
  (for this unit cube, with $V = 1$).

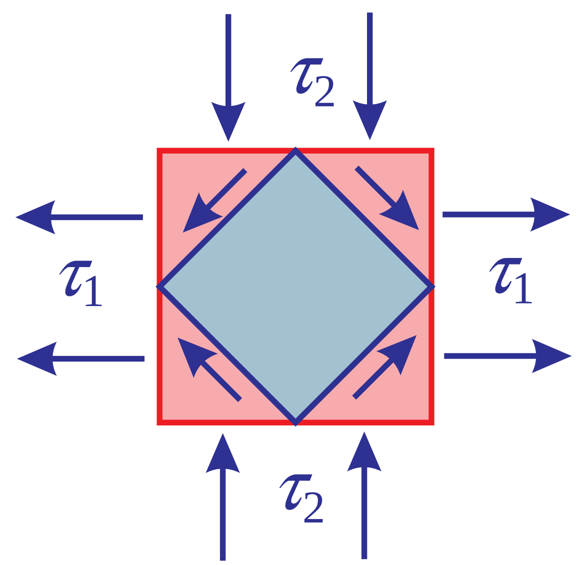
- The bulk modulus B is defined as $P \equiv -B\dfrac{\delta V}{V}$ so that $\boxed{B = \dfrac{E}{3(1-2\sigma)}}$.

- B has to be positive for a medium to be stable, so $\sigma < \frac{1}{2}$.

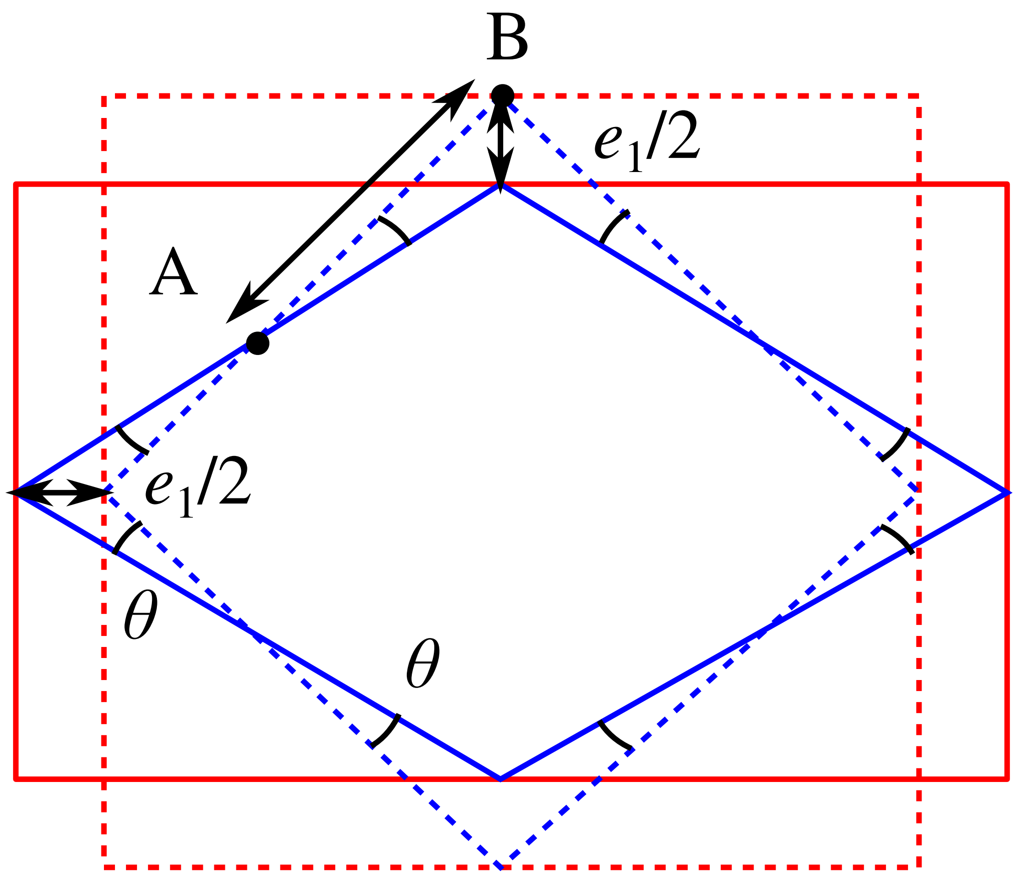
Shear Stress and Strain

- **Shear stress** τ_{xy} is the force/unit area in the x-direction transmitted across the y-planes.
- Shear stresses have to be symmetric $\tau_{yx} = \tau_{xy}$ as there can be no net couple in equilibrium.
- Shear stresses produce **shear strain**, which is defined by the **shear angle**.
- Shear can be produced by a combination of tension and compression: $\tau_1 = -\tau_2 = \tau$ (with $\tau_3 = 0$).
- Consider a unit cube. The shear **force** $\tau/\sqrt{2}$ is transmitted to inner square of side $1/\sqrt{2}$, so that the shear stress is τ.
- Strain given by $Ee_1 = \tau_1 - \sigma\tau_2 \equiv \tau(1+\sigma)$.





- For the relationship between θ and e_1, consider a unit cube is distorted as shown (greatly exaggerated).
- The distance AB is $1/(2\sqrt{2})$.
- The point B has moved by $e_1/2$ at angle $\pi/4$ to AB (it is $e_1/2$ because the distance from the centre to B is $1/2$).
- The angle θ is therefore
$$\theta \approx \frac{e_1/(2\sqrt{2})}{1/(2\sqrt{2})} = e_1$$
for small strains ($e_1 \ll 1$).



- The shear angle is 2θ = $2e_1$, which is the total change in the angle at the corner of the cube - see previous slide.

- The shear modulus is defined as $\boxed{G = \frac{\text{shear stress}}{\text{shear angle}}.}$
- Here the shear stress is $\tau = \frac{Ee_1}{1+\sigma}$, and the shear angle is $2e_1$,

  so $\boxed{G = \frac{E}{2(1+\sigma)}.}$
- For stability, we must have $\sigma > -1$, though this is not much of a limit in practice.

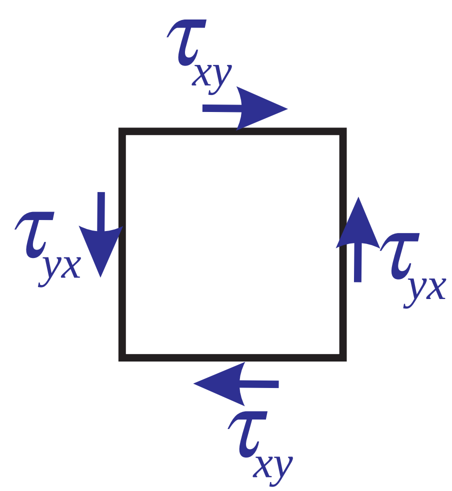
Formal Definition of Stress

- **Stress** is defined as the force/unit area transmitted across planes in the medium. It requires a matrix (tensor) notation to specify it completely. The force $\mathrm{d}\boldsymbol{F}$ transmitted across an element of vector area $\mathrm{d}\boldsymbol{S}$ is $\mathrm{d}\boldsymbol{F} = \underline{\underline{\tau}} \cdot \mathrm{d}\boldsymbol{S}$.

- As a matrix we have $\underline{\underline{\tau}} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$.

- τ_{yz} is the force/unit area in the y-direction transmitted across the z-plane (i.e. the plane perpendicular to the z-axis).

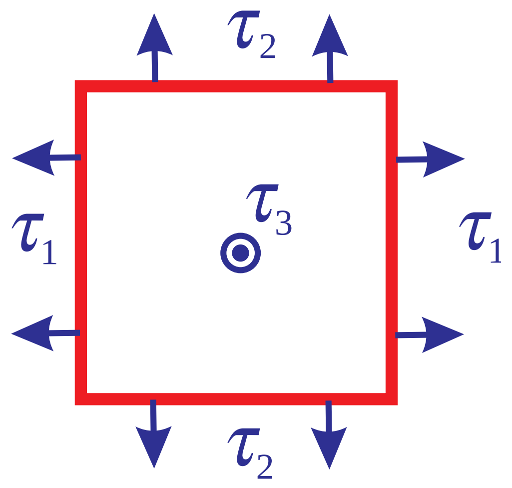
- The stress tensor is **symmetric**, i.e. $\tau_{yx} = \tau_{xy}$ etc. Any antisymmetric part represents a net couple that tends to rotate the medium, rather than distort it.



- The stress tensor can be made **diagonal** for a suitable choice of axes:

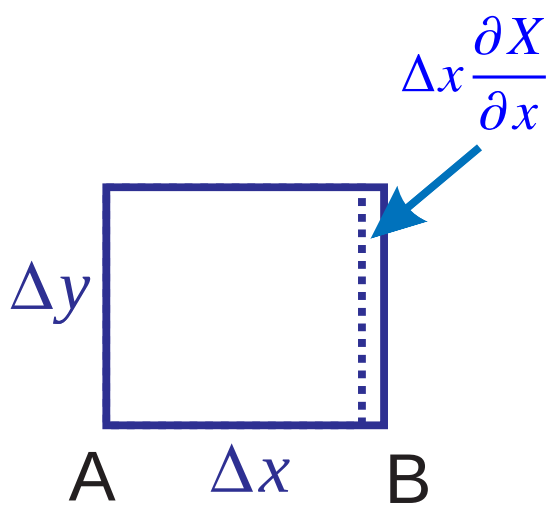
$$\underline{\underline{\tau}} = \begin{pmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & \tau_3 \end{pmatrix}.$$

- Any arbitrary stresses can be expressed in terms of three principal components (τ_1, τ_2, τ_3).



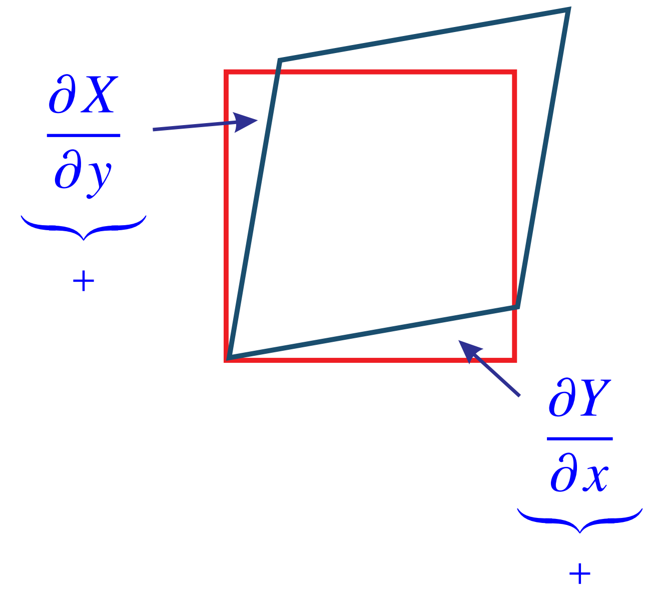
Formal Definition of Strain

- Local distortion of the medium takes a point at $\boldsymbol{x} = (x, y, z)$ to a new position $\boldsymbol{x} + \boldsymbol{X}(\boldsymbol{x}) = (x + X, y + Y, z + Z)$.
- The derivatives $\dfrac{\partial X}{\partial x}$, $\dfrac{\partial X}{\partial y}$, etc. contain information about the **strain**.
- Consider a small volume $(\Delta x, \Delta y, \Delta z)$. The distortion moves the point A at (x, y, z) by (X, Y, Z), but the point B at $(x + \Delta x, y, z)$ is moved by $\left(X + \Delta x \dfrac{\partial X}{\partial x}, Y + \Delta x \dfrac{\partial Y}{\partial x}, Z + \Delta x \dfrac{\partial Z}{\partial x}\right)$.
- The normal strains are
$$e_{xx} = \frac{\partial X}{\partial x},\ e_{yy} = \frac{\partial Y}{\partial y} \text{ and } e_{zz} = \frac{\partial Z}{\partial z}.$$

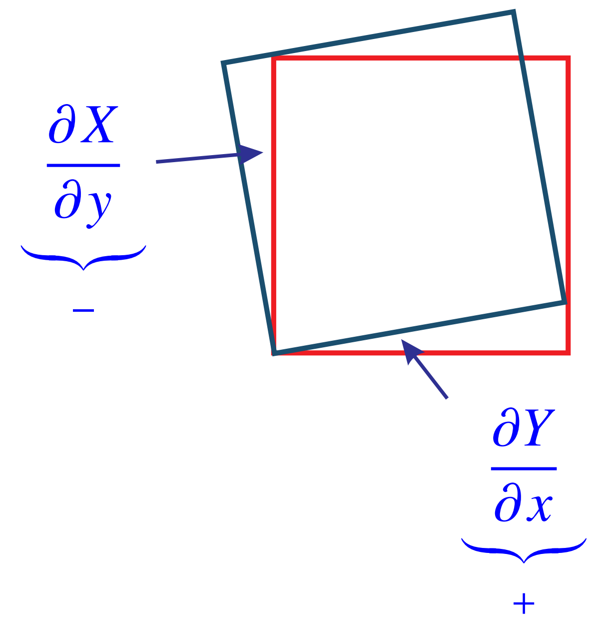


- The shear angle in the xy-plane is $\dfrac{\partial Y}{\partial x} + \dfrac{\partial X}{\partial y}$.

- Shear strain is defined as $e_{xy} = e_{yx} = \frac{1}{2}\left(\dfrac{\partial X}{\partial y} + \dfrac{\partial Y}{\partial x}\right)$.



- The quantity $\frac{1}{2}\left(\dfrac{\partial X}{\partial y} - \dfrac{\partial Y}{\partial x}\right)$ represents a **rotation** of the medium, not a distortion (it is a component of $\nabla\times\boldsymbol{X}$).



THE STRAIN TENSOR

- Since the quantity $\frac{1}{2}\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right)$ represents a **rotation** of the medium not a distortion there are only 6 independent components of strain $(e_{xx}, e_{yy}, e_{zz}, e_{xy}, e_{yz}, e_{xz})$.
- We write the **strain tensor** as $\underline{\underline{e}} = \begin{pmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{pmatrix}$.
- In suffix notation this is $e_{ij} = \frac{1}{2}\left(\frac{\partial X_i}{\partial x_j} + \frac{\partial X_j}{\partial x_i}\right)$.
- The distortion due to strain can be written $\delta \underline{X} = \underline{\underline{e}} \cdot \delta \underline{x}$.
- Because the strain tensor is symmetric we can **always** find a set of orthogonal principal axes which diagonalise the tensor.

  We can then write $\underline{\underline{e}} = \begin{pmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$.
- If the elastic medium is **isotropic**, then the axes that diagonalise the strain tensor **are the same** axes that diagonalise the stress tensor.

STRAIN TO STRESS

- Stresses and strains are related by $Ee_1 = \tau_1 - \sigma\tau_2 - \sigma\tau_3$, and by cyclic permutation:

  $Ee_2 = \tau_2 - \sigma\tau_3 - \sigma\tau_1$, and $Ee_3 = \tau_3 - \sigma\tau_1 - \sigma\tau_2$.

- Take $Ee_2 + \sigma Ee_3$ to eliminate τ_3, then $E(e_2 + \sigma e_3) = (\tau_2(1-\sigma) - \sigma\tau_1)(1+\sigma)$,

  so $\dfrac{E(e_2+\sigma e_3)}{(1+\sigma)} = (\tau_2(1-\sigma) - \sigma\tau_1)$, hence $\tau_2 = \dfrac{1}{(1-\sigma)}\left[\sigma\tau_1 + \dfrac{E(e_2+\sigma e_3)}{(1+\sigma)}\right]$.

- Similarly taking $\sigma Ee_2 + Ee_3$ to eliminate τ_2 gives $\tau_3 = \dfrac{1}{(1-\sigma)}\left[\sigma\tau_1 + \dfrac{E(e_3+\sigma e_2)}{(1+\sigma)}\right]$.

- Substituting back to the first stress/strain relationship gives

$$Ee_1 = \tau_1 - \frac{\sigma}{(1-\sigma)}[2\sigma\tau_1 + E(e_2+e_3)],$$

  so

$$E(e_1(1-\sigma) + \sigma(e_2+e_3)) = \tau_1 \underbrace{\left[(1-\sigma) - 2\sigma^2\right]}_{=(1+\sigma)(1-2\sigma)},$$

  giving the inverse relation from strain to stress of:

$$\tau_1 = \frac{E}{(1+\sigma)(1-2\sigma)}[(1-\sigma)e_1 + \sigma e_2 + \sigma e_3].$$

ELASTIC CONSTANTS

- We can tidy this up by defining $\lambda \equiv \dfrac{E\sigma}{(1+\sigma)(1-2\sigma)}$ (one of 'Lamé's constants')
  $\tau_1 = \lambda(e_1+e_2+e_3)+2Ge_1$ etc.

- There is a part of the stress that is proportional to the strain (the G term), plus an additional isotropic pressure proportional to the net change of volume $e_1+e_2+e_3 = \mathrm{Tr}(\underline{\underline{e}}) = \nabla\cdot\boldsymbol{X}$ (the λ term): $\underline{\underline{\tau}} = \lambda\,\mathrm{Tr}(\underline{\underline{e}})\underline{\underline{\boldsymbol{I}}} + 2G\underline{\underline{e}}$.

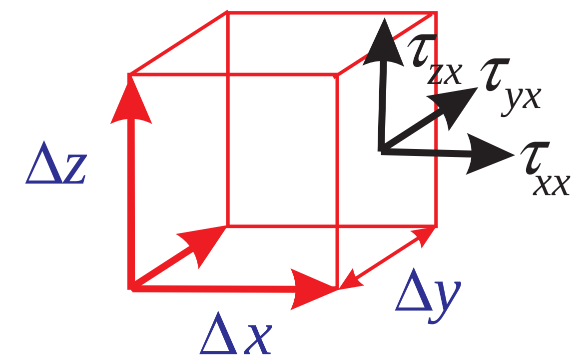
- If you need to work in axes that are not the principal axes of the stress and strain, the normal components are still related via $Ee_{xx} = \tau_{xx} - \sigma\tau_{yy} - \sigma\tau_{zz}$, etc.

- If there are off-diagonal shear strains, they are related to the shear stresses by
  $G(2e_{xy}) = \tau_{xy}$ etc, where the shear modulus $G = \dfrac{E}{2(1+\sigma)}$.

- Also, since $B = \dfrac{E}{3(1-2\sigma)}$, then $\lambda = B - \frac{2}{3}G$.

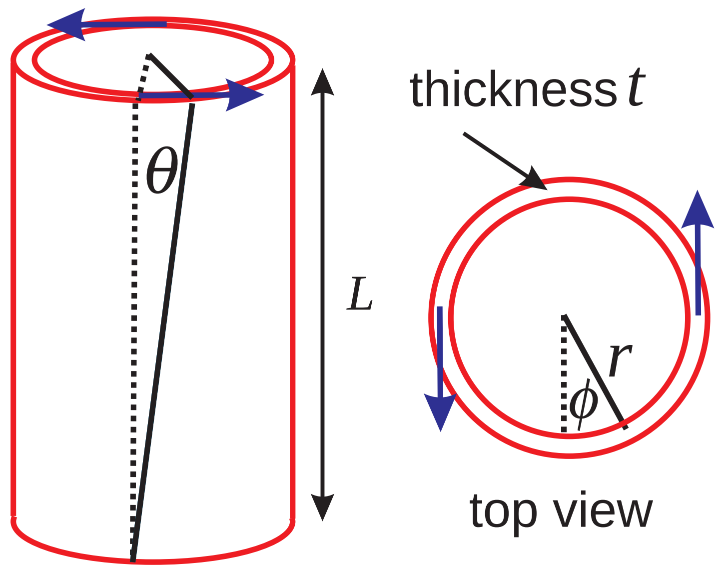
Energy stored in Elastic Strain

- As force $\propto$ extension $F = kx$, then work done $W = \displaystyle\int_0^x kx'\,\mathrm{d}x' = \tfrac{1}{2}kx^2 = \tfrac{1}{2}(kx)x$.
- Consider a small volume $(\Delta x, \Delta y, \Delta z)$.



- Net distortion along the side offset by Δx is $\Delta x(e_{xx}, e_{yx}, e_{zx})$.
- The forces on the faces are $\Delta y \Delta z(\tau_{xx}, \tau_{yx}, \tau_{zx})$.
- Stress $\propto$ strain throughout, so the energy stored by work done on the yz faces is $W = \frac{1}{2}\Delta x \Delta y \Delta z(e_{xx}\tau_{xx} + e_{yx}\tau_{yx} + e_{xz}\tau_{xz})$.
- The total energy stored, adding all three pairs of faces, is therefore
  $\frac{1}{2}(\text{Volume})\left(\tau_{xx}e_{xx} + \tau_{yy}e_{yy} + \tau_{zz}e_{zz} + 2\tau_{xy}e_{xy} + 2\tau_{yz}e_{yz} + 2\tau_{zx}e_{zx}\right)$.
- The energy stored in an isotropic elastic medium can only depend on the eigenvalues of the strain tensor, so using the principle axes, this tidies up to $\frac{1}{2}(\text{Volume})(\tau_1 e_1 + \tau_2 e_2 + \tau_3 e_3)$.
- The elastic energy per unit volume in a linear elastic medium is
  $U = \frac{1}{2}\left[(B - \frac{2}{3}G)(e_1 + e_2 + e_3)^2 + 2G(e_1^2 + e_2^2 + e_3^2)\right]$.
- We can also express this as $U(\underline{\underline{e}}) = \frac{1}{2}\left[(B - \frac{2}{3}G)(\mathrm{Tr}(\underline{\underline{e}}))^2 + 2G\,\mathrm{Tr}(\underline{\underline{e}}^2)\right]$.

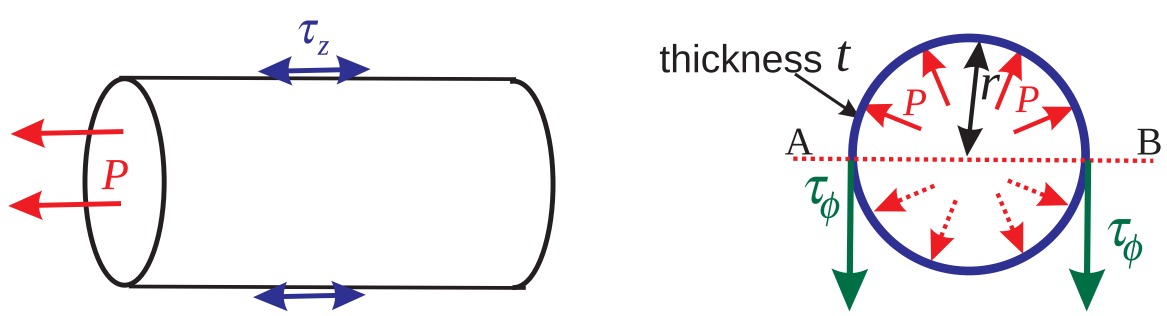
Example 1: Thin tube, radius r and thickness $t \ll r$ twisted by angle ϕ over its length L.



- The angle of shear is $\theta = \dfrac{r}{L}\phi$.
- Stress $\tau = G\theta$ and the force is $F = 2\pi r t G\theta$.
- The couple is $Fr = \dfrac{2\pi G r^3 t}{L}\phi$.

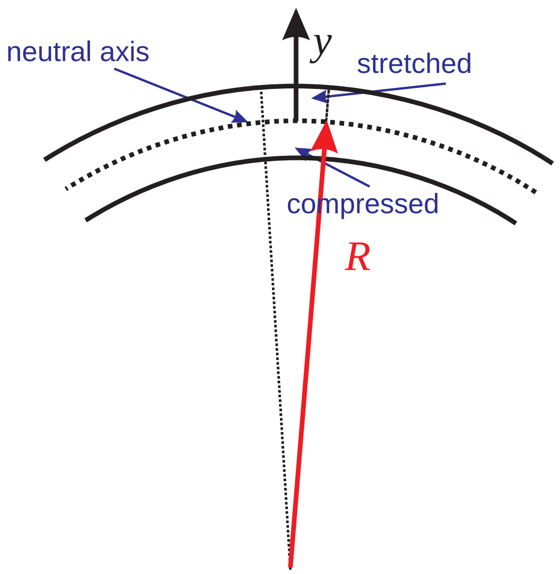
- For a solid cylinder of radius a, integrate over r: the couple is $\dfrac{\pi G a^4}{2L}\phi$.

Example 2: A thin tube radius r and thickness $t \ll r$ under pressure P.

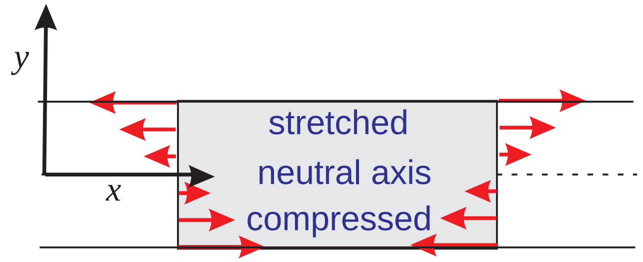


- Consider balance of forces in the plane AB: total upward (resolving components) force for length ℓ due to pressure is $2r\ell P$.
- This is balanced by 'hoop stresses' τ_ϕ with force $2t\ell\tau_\phi$.
- Hence $\tau_\phi = \dfrac{r}{t}P$, i.e. τ_ϕ is **very large** compared to P if the tube is thin.
- The transverse strain e_ϕ is given by $e_\phi = \dfrac{\delta r}{r}$.
- Balance of forces along tube: $2\pi r t\tau_z = \pi r^2 P$, so $\tau_z = \dfrac{rP}{2t}$, again much bigger than P.

BENDING OF BEAMS



- Consider an initially straight beam subjected to a (clockwise) **bending moment**. It distorts by bending into an arc with radius of curvature R, as shown.
- Consider only longitudinal stresses and strains (which is valid for small displacements of **thin** beams).
- There will be a **neutral axis** which is unstressed (i.e. no stretching or compression).
- Parts of the beam further from the centre are stretched; parts nearer the centre are compressed.
- The couple caused by bending the beam is the moment of the longitudinal forces on an element and has opposite signs on each face of the element.



- The **strain** at distance y from the neutral axis is $e_{xx} = \dfrac{y}{R}$. So the total couple across a cross-section of the beam (i.e. the bending moment) is $B = \displaystyle\int y \times \left(E\,\frac{y}{R}\right) \mathrm{dArea}$.

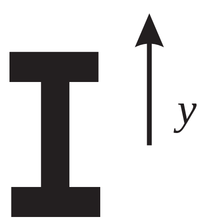
- Defining the 'moment of area' $I = \int y^2 \, \mathrm{dArea}$, then $B = \dfrac{EI}{R}$.
- The calculation of the moment of area is very similar to that of the moment of inertia.
- **Example:** For a beam $a \times b$ (in $x \times y$), then in y direction

$$I = \int_{-b/2}^{+b/2} y^2 a \, \mathrm{d}y = \left[\frac{y^3}{3} a \right]_{-b/2}^{+b/2} = 2 \left[\frac{(\frac{1}{2}b)^3}{3} a \right] = \frac{1}{12} \underbrace{(ab)}_{\text{area}} b^2.$$

Similarly, in the other direction $I = \frac{1}{12}(ab)a^2$.

- For a beam of circular cross-section with radius r, then it is $I = \frac{1}{4}(\pi r^2) r^2$.

Notes:

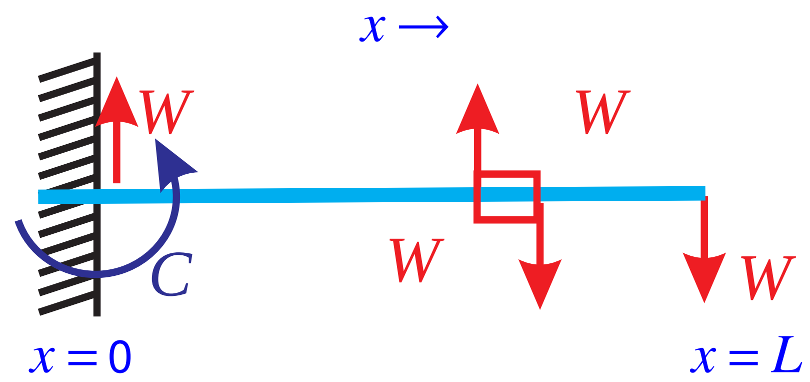


1) Stiffness of a beam is $\propto$ moment of area $I = \int y^2 \, \mathrm{dArea}$.

   This leads to the design of the 'I-beam'.

2) Stiffness varies in different directions, so the force and the deflection are not parallel in general. But the force and deflection **are parallel** for two orthogonal **principal axes**.

A SIMPLE BEAM



Consider a light, uniform beam fixed to a wall, and loaded with a weight W at $x = L$, as shown.

- Consider an short element of the beam, as shown. The forces on the element are:

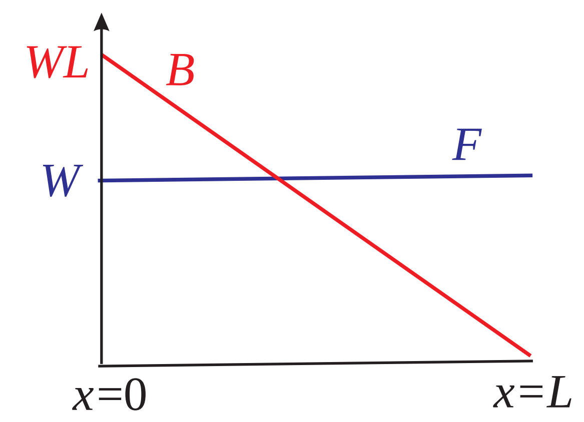
  1) W **downwards**, on the right hand side;

  2) W **upwards**, on the left hand side.

  3) These forces are generated by shear stresses (which we will ignore).

- The wall, must provide a force W **upwards** to the beam, and also an anti-clockwise couple $C = WL$.

- Along the beam, there is the same vertical force acting on the left hand face of each element of the beam, but the bending moment B decreases linearly from WL at $x = 0$ to zero at $x = L$.



- For a **small** vertical deflection y, measured **downwards**, then $\dfrac{1}{R} = y''$, where $' = \mathrm{d}/\mathrm{d}x$.
  (The full formula for the radius of a curve $y(x)$ is $1/R = y''/(1+(y')^2)^{3/2}$.)

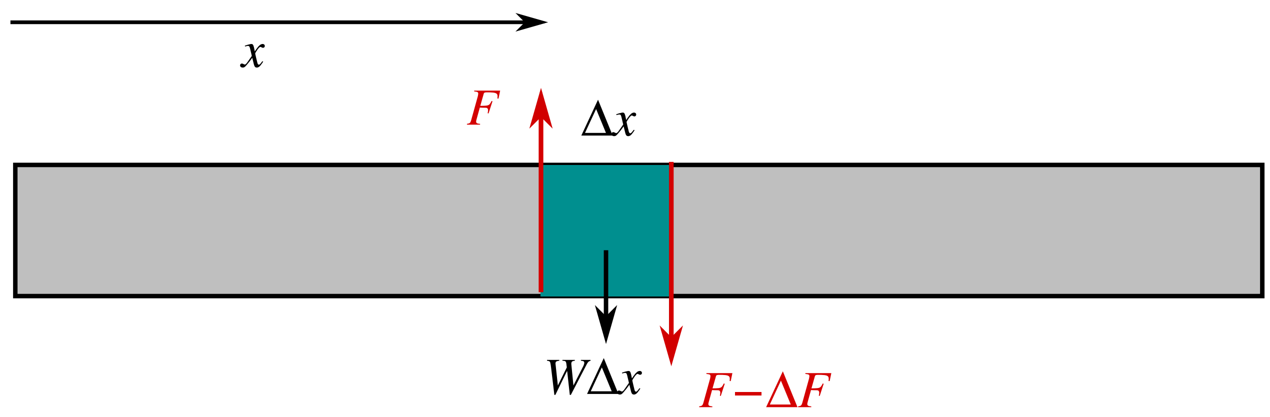
- Since the bending moment is $\boxed{B = EIy'',}$ then in this case
  $$EIy'' = W(L-x).$$

- Integrating and setting $y(0) = y'(0) = 0$ then
  $$y(x) = \frac{W}{EI}\left(\tfrac{1}{2}Lx^2 - \tfrac{1}{6}x^3\right).$$

- The beam is bent most strongly at the support — as expected.

- It is very easy to get signs wrong in these problems so do use common sense - if the applied force is downwards the deflection must be downwards.

GENERAL BEAM



- Consider the forces on an element of length Δx.
- If the vertical forces change from F on left (up), to $F - \Delta F$ on right (down), as shown.
- Balancing forces vertically, $\Delta F = W(x)\Delta x$, where $W(x)$ is the **load per unit length**.
- And the bending moment must change across the length Δx.
- The forces produce a couple $F\,\Delta x$ which must be balanced by a change in the bending moment B, with: $\Delta B = F\Delta x$.
- With the sign conventions used here $W = -\dfrac{\mathrm{d}F}{\mathrm{d}x}$ and $F = -\dfrac{\mathrm{d}B}{\mathrm{d}x}$.

Note: combining results, then $W = EIy''''$, which is used in a later example.

- To solve problems involving bent beams.

  1) **Draw a diagram** of the forces needed to support the beam at position x. This is called a **free-body diagram** because it shows the force necessary to support the portion of the beam (say) to the right of x.

  2) **Draw a diagram** of the bending moment in the beam by taking the moments of these forces.

  3) Then use $EIy'' = B(x)$, making sure the signs are correct so the displacement is in the correct direction, and apply the boundary conditions.

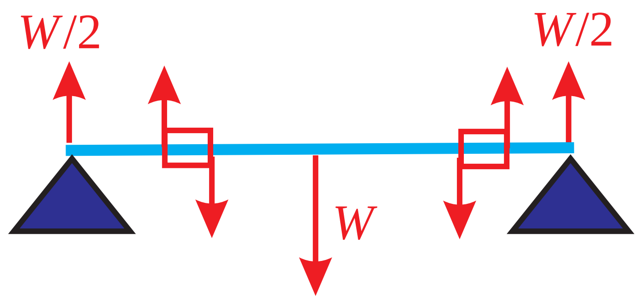
- Common boundary conditions are:

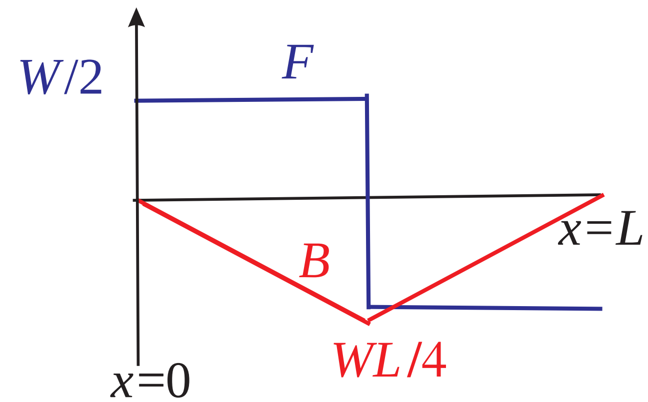
  1) **free end**: no force or couple, so $y'' = 0$, and $y''' = 0$;

  2) **cantilever**: force and couple, with y and y' given (usually zero);

  3) **hinge:** force, but no couple, $y' \neq 0$ but $y'' = 0$.

Example 1: A light, uniform beam freely supported at both ends, loaded in the middle.





- Bending moment is:
  $B = -\frac{1}{2}Wx$ for $0 < x < \frac{1}{2}L$, and $B = \frac{1}{2}W(x - L)$ for $\frac{1}{2}L < x < L$.
  The maximum bending moment is $\frac{1}{4}WL$.

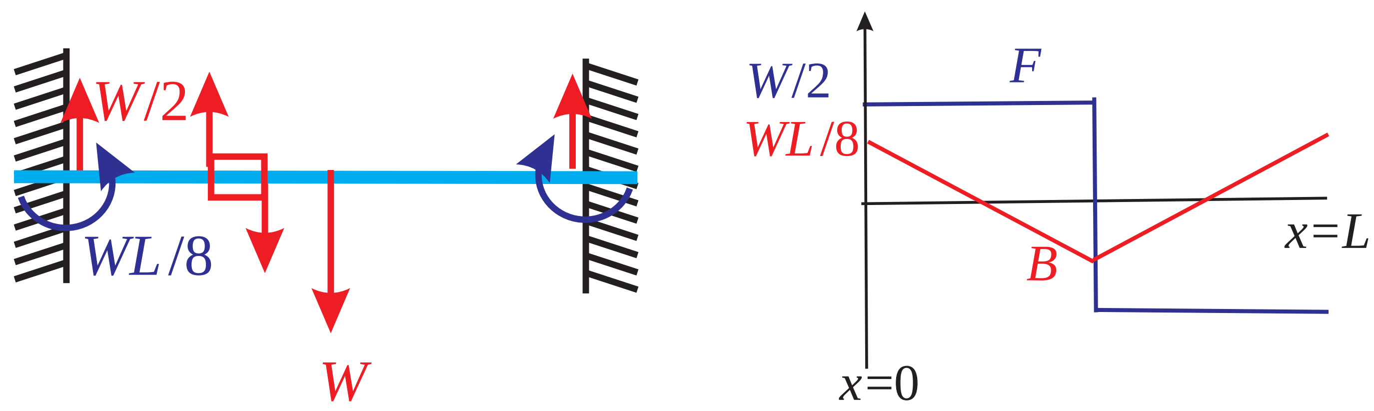
- Since the beam is symmetric about its centre, $y' = 0$ at $x = \frac{1}{2}L$.

- So, for $x < \frac{1}{2}L$, measuring y downwards again and setting $y = 0$ at $x = 0$

$$y(x) = \frac{W}{4EI}\left(\tfrac{1}{4}L^2 x - \tfrac{1}{3}x^3\right).$$

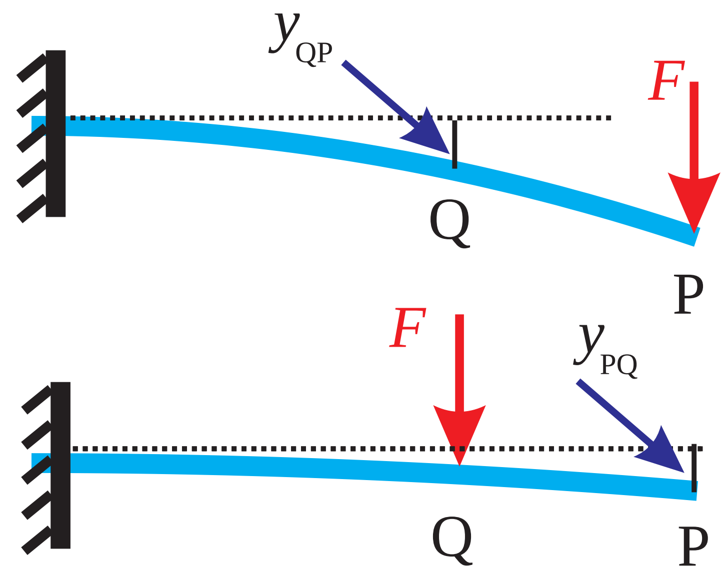
- The beam is bent most strongly in the middle.

Example 2: A light, uniform beam cantilevered at both ends.



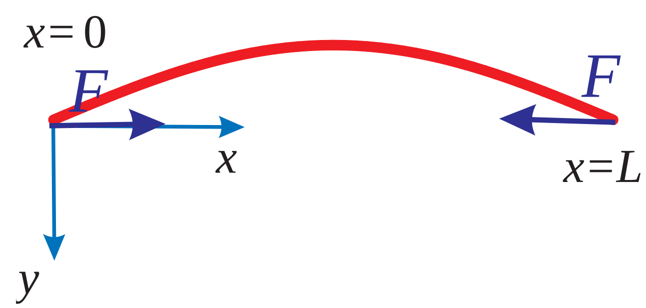
- Force diagram is the same as the previous example.
- But, the supports now apply couples also.
- It is possible to show the magnitude of the couples provided by the supports are $\frac{1}{8}WL$.
- Beam bent equally strongly in the middle and at both ends.

Reciprocity Theorem

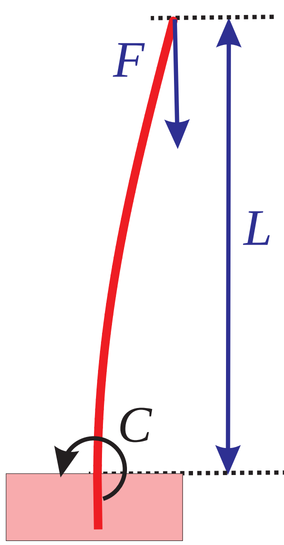


- Deflection at point Q due to a load F at point P is y_{QP}.
- Deflection at point P due to a load F at point Q is y_{PQ}.
- Apply both loads: first at P, then at Q.
- The energy stored in the beam as load at P is applied is $\frac{1}{2}F\,y_{PP}$ (load is increased from 0 to F as beam bends, hence factor of $\frac{1}{2}$).
- The energy stored in the beam as load at Q is applied is $\frac{1}{2}F\,y_{QQ} + F\,y_{PQ}$ (The load at Q is increased from 0 to F as beam bends, but the load at P is always F).
- The total energy is $F(\frac{1}{2}y_{PP} + \frac{1}{2}y_{QQ} + y_{PQ})$.
- The stored energy must be the same if the loads are applied in the order: first at Q, second at P. The energy is then $F(\frac{1}{2}y_{PP} + \frac{1}{2}y_{QQ} + y_{QP})$.
- Hence, it must be that $y_{PQ} = y_{QP}$.
- Note that this result relies on linearity

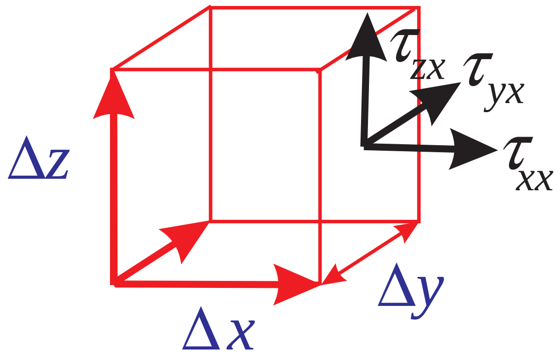
Euler Strut



- Consider a beam buckled, as show, with force applied F.
- The bending moment at position x is $B = -Fy(x)$.
- Setting $B = \dfrac{EI}{R} \approx EIy''$ gives $y'' + \dfrac{F}{EI}y = 0$.
- Since $y = 0$ at $x = 0$ the solution is $y = A\sin\sqrt{\dfrac{F}{EI}}x$, and the boundary condition $y = 0$ at $x = L$ sets $\sqrt{\dfrac{F}{EI}}L = \pi$. Hence the 'Euler force' is $\boxed{F_\mathrm{E} = \dfrac{\pi^2 EI}{L^2}.}$
- For small deflections the Euler force is independent of the displacement.
- If a beam is loaded longitudinally with a force $F < F_\mathrm{E}$ it will remain unbuckled ($y(x) = 0$), but under compression. But if the force is increased to $F \geq F_\mathrm{E}$ it will suddenly buckle.
- The Euler force depends on the method of support. For a beam cantilevered into the ground and loaded vertically the Euler force is $F_\mathrm{E} = \dfrac{\pi^2 EI}{4L^2}$.



Dynamics of Elastic Media



- Variation of stresses across a small volume $(\Delta x, \Delta y, \Delta z)$ causes acceleration.
- The net force in the x-direction on the x-faces is

$$F_x = \underbrace{(\Delta y \Delta z)}_{\text{area}} \times \underbrace{\Delta x \frac{\partial \tau_{xx}}{\partial x}}_{\text{stress}} = \text{Volume} \times \frac{\partial \tau_{xx}}{\partial x}.$$

- Summing over the other faces $F_x = \text{Volume} \times \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right)$.
- Equation of motion of an element is $\rho \frac{\partial^2 X}{\partial t^2} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}$ (where ρ is the density).
- The full equations of motion can be written in suffix notation as $\rho \frac{\partial^2 X_i}{\partial t^2} = \sum_j \frac{\partial \tau_{ij}}{\partial x_j}$ or as $\rho \frac{\partial^2 \mathbf{X}}{\partial t^2} = \nabla \cdot \underline{\underline{\tau}}$.

- Previously, we showed that the stress is related to the strain: $\underline{\underline{\tau}} = (B - \frac{2}{3}G)\,\mathrm{Tr}(\underline{\underline{e}})\underline{\underline{\boldsymbol{I}}} + 2G\underline{\underline{e}}$.

- Using $e_{ij} = \frac{1}{2}\left(\frac{\partial X_i}{\partial x_j} + \frac{\partial X_j}{\partial x_i}\right)$ and $\mathrm{Tr}(\underline{\underline{e}}) = \nabla \cdot \boldsymbol{X}$ then

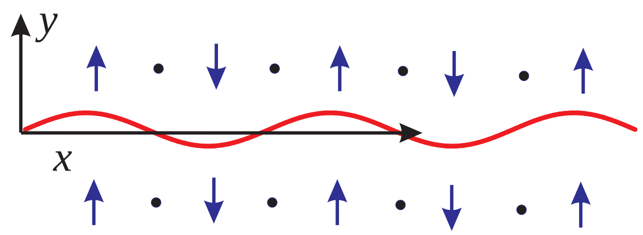
$$\begin{aligned}\sum_j \frac{\partial \tau_{ij}}{\partial x_j} &= (B - \tfrac{2}{3}G)\frac{\partial}{\partial x_i}\nabla \cdot \boldsymbol{X} + G\sum_j \frac{\partial}{\partial x_j}\left(\frac{\partial X_i}{\partial x_j} + \frac{\partial X_j}{\partial x_i}\right), \\ &= (B + \tfrac{1}{3}G)\frac{\partial}{\partial x_i}\nabla \cdot \boldsymbol{X} + G\nabla^2 X_i.\end{aligned}$$

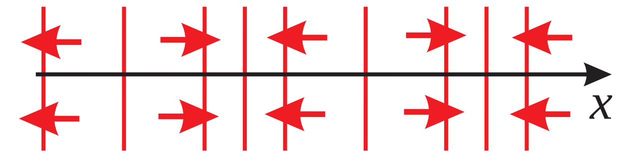
- We therefore get a **vector equation of motion** $\rho\dfrac{\partial^2 \boldsymbol{X}}{\partial t^2} = (B + \frac{1}{3}G)\nabla(\nabla \cdot \boldsymbol{X}) + G\nabla^2 \boldsymbol{X}$.

- This can also be written $\rho\dfrac{\partial^2 \boldsymbol{X}}{\partial t^2} = (B + \frac{4}{3}G)\nabla(\nabla \cdot \boldsymbol{X}) - G\nabla\times(\nabla\times\boldsymbol{X})$.

ELASTIC WAVES

- The vector wave equation for elastic waves: $\rho \dfrac{\partial^2 \boldsymbol{X}}{\partial t^2} = (B + \frac{1}{3}G)\nabla(\nabla \cdot \boldsymbol{X}) + G\nabla^2 \boldsymbol{X}$.
- We can find wave solutions (normal modes):
  e.g. a wave travelling the in x-direction $\boldsymbol{X} = (X_0, Y_0, Z_0)\mathrm{e}^{\mathrm{i}(\omega t - kx)}$.
- The divergence of $\boldsymbol{X}$ is $-\mathrm{i}kX_0\mathrm{e}^{\mathrm{i}(\omega t - kx)}$.
- The gradient of the divergence ($\nabla(\nabla \cdot \boldsymbol{X})$) is $(-k^2 X_0, 0, 0)\mathrm{e}^{\mathrm{i}(\omega t - kx)}$.
- The term $\nabla^2 \boldsymbol{X}$ is $(-k^2 X_0, -k^2 Y_0, -k^2 Z_0)\mathrm{e}^{\mathrm{i}(\omega t - kx)}$.
- The time derivative is $\dfrac{\partial^2 \boldsymbol{X}}{\partial t^2}$, i.e. $(-\omega^2 X_0, -\omega^2 Y_0, -\omega^2 Z_0)\mathrm{e}^{\mathrm{i}(\omega t - kx)}$.
- If the displacement is in the y-direction it is a transverse **shear**, or **S-wave**: and $\rho\omega^2 = Gk^2$, so the velocity $v_\mathrm{S}{}^2 = G/\rho$.
  It is the same if the displacement is in the z-direction.





- If the displacement is in the x-direction it is a longitudinal **compressional**, or **P-wave**. Both terms on the RHS now contribute so $\rho\omega^2 = (B + \frac{4}{3}G)k^2$.
  The velocity of this longitudinal wave is $v_{\mathrm{P}}^2 = (B + \frac{4}{3}G)/\rho$ (with $v_{\mathrm{P}} > v_{\mathrm{S}}$).

- For any $\boldsymbol{k}$-vector we can similarly find waves $\boldsymbol{X} = \boldsymbol{X}_0 \mathrm{e}^{\mathrm{i}(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})}$: there are three modes; two transverse ($v_{\mathrm{S}}^2 = G/\rho$) and one longitudinal ($v_{\mathrm{P}}^2 = (B + \frac{4}{3}G)/\rho$).

- These S-waves and P-wave modes describe waves propagating in a bulk medium (i.e. size $\gg$ wavelength). Waves near the surface of a medium have many interesting properties.

- Boundary conditions at a surface:

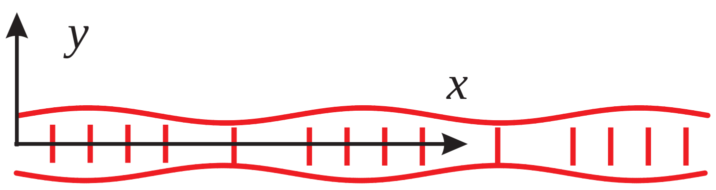
  1) 'free boundary': the normal component of the stress vanishes:
     $\underbrace{\boldsymbol{n}\cdot\underline{\underline{\tau}}}_{\boldsymbol{\tau}_n}\cdot\mathrm{d}\boldsymbol{S} = 0$, where $\mathrm{d}\boldsymbol{S} = \boldsymbol{n}|\mathrm{d}S|$;

  2) 'fixed boundary': normal component of the displacement vanishes: $\boldsymbol{n}\cdot\boldsymbol{X} = 0$.

Some simple examples.

- **Example 1:** longitudinal waves in x-direction along a thin rod (thickness $\ll$ wavelength). Then $\tau_{yy} = \tau_{zz} = 0$, so $E e_{xx} = \tau_{xx}$. The wave speed along the rod is $v^2 = E/\rho$.

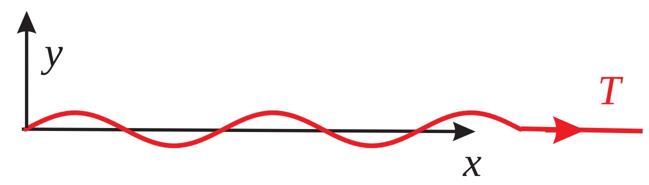
  The rod bulges sideways to relieve the stresses in the y- and z-directions.

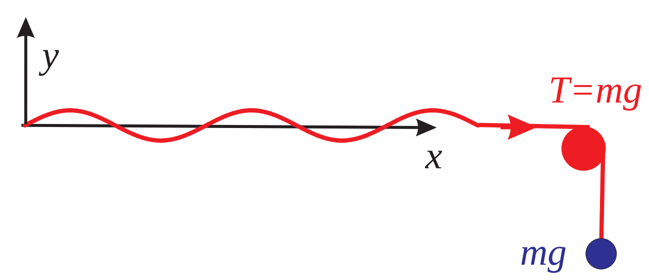
- **Example 2:** longitudinal waves in x-direction on a thin plate in the xy-plane(thickness $\ll$ wavelength). We have $\tau_{zz} = 0$, but the plate cannot move sideways in the y-direction so $e_{yy} = 0$. Thus $E e_{xx} = \tau_{xx} - \sigma \tau_{yy}$ and $0 = \tau_{yy} - \sigma \tau_{xx}$, and $E e_{xx} = (1-\sigma^2)\tau_{xx}$.
  The wave speed on a thin plate is $v^2 = \dfrac{E}{(1-\sigma^2)\rho}$.

- **Example 3:** longitudinal waves in x-direction in a bulk medium. The medium cannot move sideways ($e_{yy} = e_{zz} = 0$), so $\tau_{xx} = \dfrac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)} e_{xx} = (B + \frac{4}{3}G) e_{xx}$.
  This reproduces our earlier result for a P-wave $\left(v_{\mathrm{P}}^2 = (B + \frac{4}{3}G)/\rho\right)$.

ENERGY BALANCE IN WAVES



- Waves on a string in one dimension, uniform mass/length ρ, under tension T.
- Equation of motion for **small** transverse displacements $y(x,t)$ is $\rho\ddot{y} = Ty''$.



- Potential energy: total length $\int \sqrt{1+(y')^2}\,\mathrm{d}x$, so either 1) work done against elastic forces or 2) work done lifting weight, so potential energy $\approx \int \frac{1}{2}T(y')^2\,\mathrm{d}x$.
- Kinetic energy: $\approx \int \frac{1}{2}\rho\dot{y}^2\,\mathrm{d}x$.
- The total energy of the string between a and b is $E = \int_a^b \frac{1}{2}\left(\rho\dot{y}^2 + T(y')^2\right)\mathrm{d}x$.

- The rate of change $\frac{\mathrm{d}E}{\mathrm{d}t} = \int_a^b \left(\rho\ddot{y}\dot{y} + T\dot{y}'y'\right) \mathrm{d}x$, but $\rho\ddot{y} = Ty''$ so

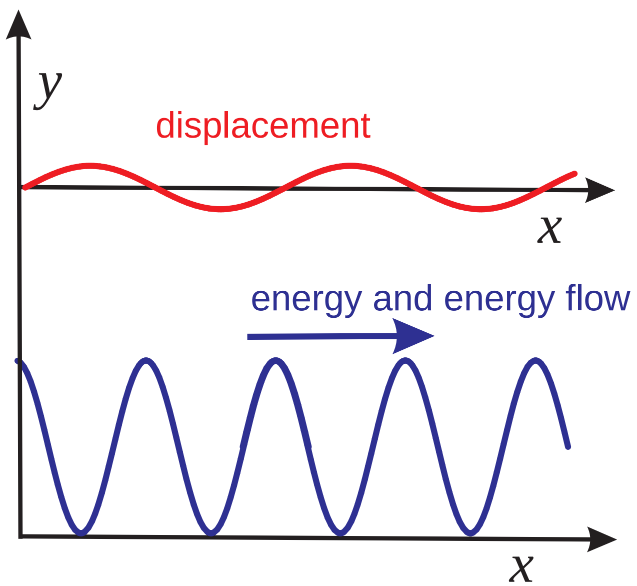
$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_a^b \left(Ty''\dot{y} + T\dot{y}'y'\right) \mathrm{d}x = \int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left(\dot{y}Ty'\right) \mathrm{d}x = [\dot{y}(Ty')]_a^b.$$

- $\frac{\mathrm{d}E}{\mathrm{d}t}$ must be equal to the net energy flowing into the interval, so the energy flow in the x-direction is $-\dot{y}(Ty')$.

- Energy flow in wave: $-$(velocity)$\times$(transverse force).

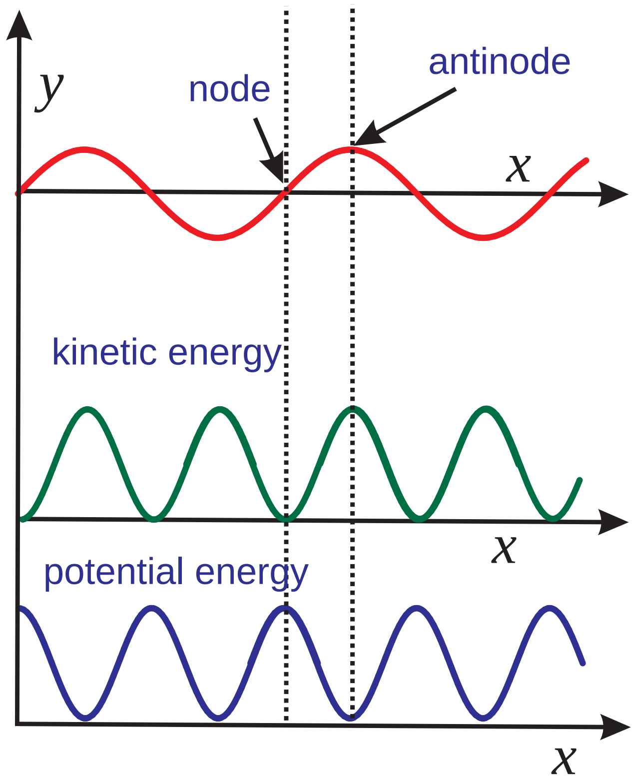
ENERGY FLOW IN TRAVELLING WAVES

- Consider a travelling wave $A\cos(\omega t - kx)$.
- Potential energy: $\frac{1}{2}T(y')^2 = \frac{1}{2}A^2Tk^2\sin^2(\omega t - kx)$.
- Kinetic energy: $\frac{1}{2}\rho\dot{y}^2 = \frac{1}{2}A^2\rho\omega^2\sin^2(\omega t - kx)$. This is equal to the potential energy $\frac{1}{2}A^2Tk^2\sin^2(\omega t - kx)$, since $\rho\omega^2 = Tk^2$.
- Energy flow: $-\dot{y}Ty' = A^2Tk\omega\sin^2(\omega t - kx)$.
- The energy flow is equal to $A^2Tk^2\sin^2(\omega t - kx) \times \dfrac{\omega}{k}$, the total energy density times the wave velocity.
- Both the kinetic and the potential energies are concentrated in the nodes, and the energy flows along with the wave velocity.



Energy Flow in Standing Waves

- Consider a Standing wave $\frac{1}{2}(A\cos(\omega t - kx) + A\cos(\omega t + kx)) = A\cos(kx)\cos(\omega t)$.
- Potential energy: $\frac{1}{2}A^2Tk^2\sin^2(kx)\cos^2(\omega t)$.
- Kinetic energy: $\frac{1}{2}A^2Tk^2\cos^2(kx)\sin^2(\omega t)$ (using $\rho\omega^2 = Tk^2$).
  This is in anti-phase with the potential energy.
- The kinetic energy is concentrated in the antinodes, and the potential energy is concentrated in the nodes.
- When the string is straight all the energy is kinetic.
- When the amplitude is a maximum all the energy is potential.
- The energy flows alternately between the nodes and the antinodes.
- Energy flow: $-\frac{1}{4}A^2Tk\omega\sin(2kx)\sin(2\omega t)$.



- These arguments generalise to all types of elastic wave:
  for a general elastic wave the energy flow rate is $-\underline{\underline{\tau}} \cdot \dot{\boldsymbol{X}}$.
- You have already met the energy flow in electromagnetic waves:
  the Poynting vector $\boldsymbol{E} \times \boldsymbol{H}$.
- The energy and energy flow comprise the **stress–energy(–momentum) tensor**.

NORMAL MODES OF AN ELASTIC BAR

For a uniform elastic bar, cantilevered at one end. Using the notation from slide 180.

- For a particular $y(x)$, then **if** the beam were at equilibrium, this would correspond to a weight per unit length distribution $W(x)$ along the length of the bar of
  $W = EIy''''$.
- So the force on the element due to the bending of the bar, which balances this, is $-EIy''''$.
- Thus, the equation of motion is: $\rho\ddot{y} = -EIy''''$
  (with ρ being the mass per unit length of the bar).
- Boundary conditions:
  1) cantilever at $x = 0$: so $y(0) = y'(0) = 0$;
  2) free at $x = L$: so $F(L) = B(L) = 0$, giving $y''(L) = y'''(L) = 0$.
- Look for normal modes with $y(x,t) = y(x)\mathrm{e}^{\mathrm{i}\omega t}$, i.e. solve:

$$EIy'''' - \omega^2 \rho y = 0.$$

- The general solution is: $y = A\mathrm{e}^{\mathrm{i}kx} + B\mathrm{e}^{-\mathrm{i}kx} + C\mathrm{e}^{kx} + D\mathrm{e}^{-kx}$, where $\omega^2 = \dfrac{EIk^4}{\rho}$.
- The boundary conditions at $x = 0$ restrict us to two constants
$$y = A[\cosh(kx) - \cos(kx)] + B[\sinh(kx) - \sin(kx)].$$
- Boundary conditions at $x = L$ then restrict the choice of k. With $y'' = y''' = 0$ then
$$\begin{aligned} 0 &= A(\cosh kL + \cos kL) + B(\sinh kL + \sin kL), \\ 0 &= A(\sinh kL - \sin kL) + B(\cosh kL + \cos kL), \end{aligned}$$
or in matrix notation
$$\begin{pmatrix} \cosh kL + \cos kL & \sinh kL + \sin kL \\ \sinh kL - \sin kL & \cosh kL + \cos kL \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0.$$
- The determinant of the matrix has to be zero, which simplifies to:
$$1 + \cosh kL \cos kL = 0.$$

- We have to solve $1+\cosh kL\cos kL=0$ numerically, but the allowed values of k very rapidly approach $k_n L \approx (n+\frac{1}{2})\pi$.

- When k_n is known, we substitute it to find the ratio B/A.

- The nth mode has n nodes. The first few modes have $kL=(1.875, 4.694, 7.854, 10.995, \ldots)$.

- The frequencies are $\omega_n = \left(\dfrac{EI}{\rho L^4}\right)^{1/2} \times (3.51, 22.0, 61.7, 120.9, \ldots)$.

- The modes are not harmonically related, and they rapidly increase in frequency (ω_n approximately proportional to n^2).