ELECTROMAGNETISM - Problem Sheet 1

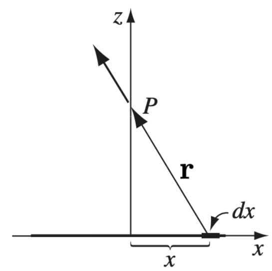
Michaelmas Term, 2021

The problems are marked as moderate difficulty (type 'A') or moderately demanding ('B')—do not shy away from those labelled 'B', they are just slightly more involved, and not aimed only at enthusiasts!

Electrostatics

1A) A line segment of length $2L$ carries a uniform charge per unit length λ.

(a) Find the electric field a distance z above the midpoint of the line segment, as shown in the diagram.



Hint: Use the symmetry of the system to argue that the $\boldsymbol{E}$ *field is oriented along* $\hat{\mathbf{z}}$*, write down the contribution to* $\boldsymbol{E}$ *from the segment* $\mathrm{d}x$ *as shown, and integrate along* x*.*

(b) Consider a point far away from the line, i.e., in the limit $z \gg L$. Find the simplified expression for the electric field in this limit. Compare the result to a point charge and comment on your findings.

(c) Starting from the expression in 1a, find the electric field in the limit of $L \to \infty$.

(d) Find E in the limit of $L \to \infty$ using Gauss' law. Compare your results to 1c. Which of the two approaches do you find more elegant?

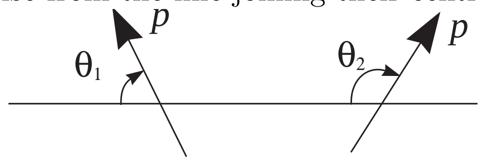
2A) (a) A conducting sphere of radius 10 cm is charged to 5 kV above earth potential and isolated. Find the charge on the sphere, the $\boldsymbol{E}$-field just outside it, and the field and potential 20 cm from its centre.

(b) Estimate the total charge on the Earth if the mean atmospheric $\boldsymbol{E}$-field near the surface is 100 V m^{-1}. [The radius of the Earth is about 6400 km.]

(c) Draw a simple sketch of the electric field lines and equipotentials around an isolated parallel-plate capacitor with equal and opposite charge on the two plates. Take the plates to be infinite strips about 5 times wider than their separation.

3A) A small dipole centred at the origin of coordinates has Cartesian vector components $(0, 0, p)$, where $p = qa$, q is the charge, and a the separation. By differentiating the expression for the potential at point (x, y, z), find the x, y and z components of the electric field when $(x^2 + y^2 + z^2) \gg a^2$.

4B) Two coplanar identical small electric dipoles of moment p are supported on pivots a large distance d apart. Each dipole can rotate only in the plane. Their angles of twist, θ_1 and θ_2, are measured clockwise from the line joining their centres as shown in the diagram.



Show, by treating each dipole as the sum of two components at right angles, that the potential energy of one in the field of the other is

$$-\frac{p^2}{8\pi\epsilon_0 d^3}\left(3\cos(\theta_1+\theta_2)+\cos(\theta_1-\theta_2)\right).$$

5A) Calculate the electrostatic energy $U = \frac{1}{2}\int \mathrm{d}\tau\rho V$ of a charge Q uniformly distributed throughout a sphere of radius a. Hence calculate a "classical radius" of the electron on the dubious assumption that its rest mass is due to the electrostatic energy.

6A) Use Gauss' Law to show the following:

(a) Any excess charge placed on a conductor must lie entirely on its surface.

(b) The electric field at the surface of a conductor is normal to the surface and has a magnitude σ/ϵ_0, where σ is the charge density (per unit area) on the surface.

(c) A closed, hollow conductor shields its interior from fields due to charges outside, but does not shield its exterior from fields due to charges placed inside it.

[*The electric field inside a conductor is, of course, always zero.*]

7A) Two point charges q and $-q$ are a distance $2d$ apart. Show that, if a conducting sphere of small radius a $(a \ll 2d)$ is placed midway between the two charges, the force on each is increased by a factor of approximately $(1 + 16(a^3/d^3))$.
Hint: you may use what you know about the electric field a long distance away from a dipole.

8A) The electrical system of a typical thundercloud can be represented by a vertical dipole consisting of a charge of $+40\,\mathrm{C}$ at a height of $10\,\mathrm{km}$ and a charge of $-40\,\mathrm{C}$ vertically below it at a height of $6\,\mathrm{km}$. What is the electric field at the ground immediately below the thundercloud and at what distance from there is the field at the ground zero?
Hint: treat the ground as a perfect conductor.

9A) Two infinite conducting plane sheets meet at an angle of $60°$. A particle carrying a charge q is constrained to move on the plane bisecting this angle. Obtain an expression for the force on the particle when it is a distance x from the line on which the sheets meet.

10A) Two isolated spherical conductors of radii 30 and 90 mm are charged to 1.5 and $3\,\mathrm{kV}$ respectively. They are then connected by a fine resistive wire. How much heat will be generated in the wire in total, if its resistance is sufficient to overdamp the system?
Hint: Treat the problem as a reduced circuit of two capacitors and a resistor.

11A) Derive the expression for the pressure on a charged surface of a conductor by considering the energy of a charged parallel-plate capacitor when the capacitor is kept

(a) at constant charge,

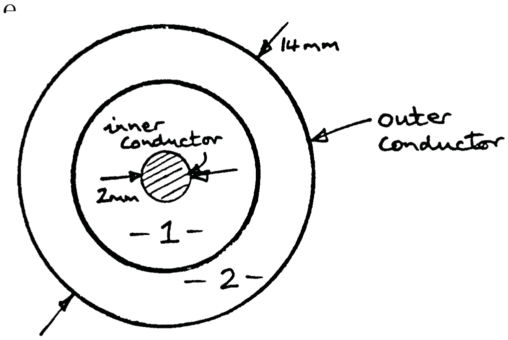
(b) at constant voltage.

Hint: ensure that you distinguish between the force acting between the plates, and the external force opposing it.

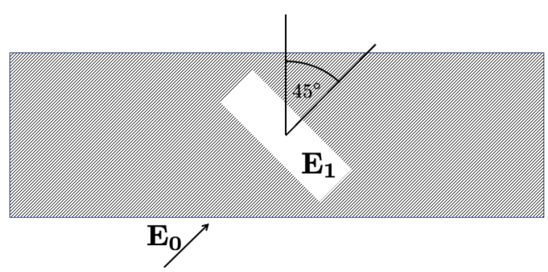
12A) Two conducting discs of radius r are arranged one above the other with their faces horizontal. The lower disc is fixed and the upper is suspended a distance $a \ll r$ above it by springs. When a potential difference V is applied, the equilibrium separation becomes b. Using the results of question 11, show that the (electrostatic) force between the discs is $V^2\pi r^2\epsilon_0/(2b^2)$.

Show that the system is stable only if $b > 2a/3$.

13A) A coaxial cable is constructed using two dielectrics as shown in the figure. Dielectric 1 occupies one sixth of the total area between the conductors, whose radii are 1 and 7 mm. The relative permittivities of the dielectrics are $\epsilon_1 = 5$ and $\epsilon_2 = 3$. Calculate the capacitance per unit length of the cable.



14A) A planar slab of a dielectric material for which $\epsilon = 2$ has a disc-shaped cavity in its interior, as shown from the side in the sketch below.



Both slab and cavity are much wider than they are thick, and the normal to the plane of the cavity is at 45° to that of the slab. The slab is placed in a uniform electric field E_0. If the field inside the cavity is E_1, what is the ratio E_1/E_0 when the direction of E_0 is parallel to the normal to the plane of the cavity?

15A) Show that an arbitrary charge distribution in a semiconductor of conductivity σ and permittivity ϵ decays with time constant $\tau = \epsilon\epsilon_0/\sigma$. [$\tau$ is known as the *dielectric relaxation time*.]

Hint: don't forget Ohm's Law, $\boldsymbol{J} = \sigma\boldsymbol{E}$, and charge conservation.

Magnetostatics

16A) (a) Use the Biot-Savart law to show that the magnetic field at a perpendicular distance r from an infinitely-long, thin wire, carrying a current I, is

$$B = \frac{\mu_0 I}{2\pi r} \; .$$

(b) Find the magnitude and direction of the force per unit length between two infinitely-long, parallel wires a distance r apart, if one carries a current I_1 and the other I_2. Is this consistent with the interim (1946-2019) definition of the ampere in the SI system of units?

17B) (a) Show, using the Biot-Savart law, that the magnetic field on the axis of a circular loop carrying a current I, and a distance x away from it, is

$$B = \frac{\mu_0 I a^2}{2(a^2 + x^2)^{3/2}} \; ,$$

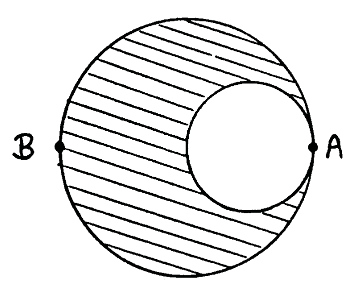
where a is the radius of the loop.

(b) A short magnetic dipole, of moment $\boldsymbol{m}$, is placed on, and aligned with, the axis of a long solenoid, of radius a, that has n turns per unit length.

If the current in the solenoid is I, calculate the force on the dipole when it is outside the solenoid and at a distance x_0 from the end of the solenoid.

18A) (a) A long wire of circular cross-section carries a steady current uniformly distributed over its cross-section. Show using Ampère's theorem that the magnetic field varies as r inside the wire, and as $1/r$ outside, where r is the distance from the centre of the wire.

(b) A long wire having the cross-section shown below carries a steady current. The radius of the cavity is half that of the wire. What is the ratio of the magnetic fields at A and B?



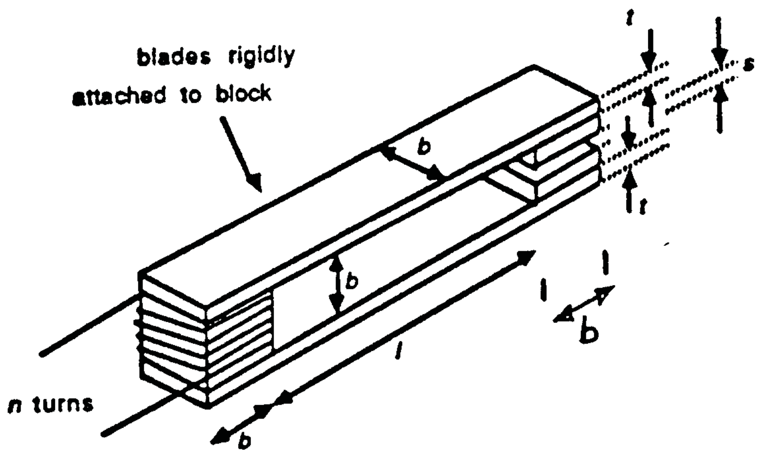
19B) A cylindrical column of mercury of radius a carries a current I, uniformly distributed over its cross-section. By considering the force per unit volume on the current, derive a formula for the pressure p as a function of radius r in the column ignoring the weight of the mercury. If $a = 5\,\text{mm}$ and $I = 100\,\text{A}$, what is the difference between the pressure at the centre and that on the edge of the column?

20B) A paramagnetic sphere of radius a and relative permeability μ is placed in a vacuum in a uniform magnetic field $\boldsymbol{B}$. Show that the total magnetic moment $\boldsymbol{m}$ induced in the sphere is given by

$$\mu_0 \boldsymbol{m} = \frac{4\pi a^3(\mu - 1)}{\mu + 2}\boldsymbol{B}.$$

If two such spheres are a distance d apart ($d \gg a$) along the direction of $\boldsymbol{B}$, what is the force between them?

21A) An electromagnet is constructed out of high permeability iron in the form shown in the figure.



The force F required between the pole-pieces to bend the blades and reduce the gap from s_0 to s is given by

$$F \approx \frac{bt^3(s_0 - s)E}{8l^3},$$

where E is Young's modulus. When the magnet is energised, the attractive magnetic force between the pole pieces is $B^2/2\mu_0$ per unit area, where B is the induction gap.

Show that:

(a) when a current I flows through the coil, the magnetic force on the pole-pieces is given by

$$F' \approx \frac{\mu_0 n^2 I^2 b^2}{2s^2},$$

where n is the number of turns in the coil;

(b) the system becomes unstable so that the pole-pieces snap together when $s = 2s_0/3$. *Hint: you may wish to consider the restoring force for small displacements from equilibrium.*

Hence find an expression for the critical current at which this occurs.

22A) Sketch, on separate diagrams and with qualitative justification, the field lines of $\boldsymbol{B}$ and $\boldsymbol{H}$ inside and outside a uniformly magnetised bar magnet.

Answers to problems

Worked solutions will be issued around lecture 15 and early next term.

1. (a) $\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{z\sqrt{z^2+L^2}}\hat{\mathbf{z}}$
2. (a) 5.5606 $\times 10^{-8}$ C, $50\,\mathrm{kV\,m^{-1}}$; 2.5 kV, $12.5\,\mathrm{kV\,m^{-1}}$; (b) About 4.6×10^5 C.
3. $\{x, y, z(1 - r^2/3z^2)\}(3pz/4\pi\epsilon_0 r^5)$, where $r^2 = x^2 + y^2 + z^2$.
5. $3Q^2/20\pi\epsilon_0 a$; 1.7×10^{-15} m.
8. 12.8 $\mathrm{kVm^{-1}}$ upwards; 11.0 km.
9. $-(5 - 4/\sqrt{3})q^2/16\pi\epsilon_0 x^2$.
10. 2.8×10^{-6} J.
11. $\sigma^2/2\epsilon_0$, where σ is the surface charge density.
13. $111\,\mathrm{pF\,m^{-1}}$.
14. 1.52.
16. $\mu_0 I_1 I_2/2\pi r$.
17. $\mu_0 NmIa^2/2(x_0^2 + a^2)^{3/2}$.
18. 3/5.
19. $P_0 + \mu_0 I^2(a^2 - r^2)/4\pi^2 a^4$; 12.7 Pa.
20. $24a^6\pi B^2(\mu - 1)^2/\mu_0(\mu + 2)^2 d^4$.
21. $I_c = \sqrt{s_0^3 t^3 E/27 l^3 \mu_0 n^2 b}$.

Please advise me of any errors, misprints or omissions in these answers – many thanks in advance!

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Michaelmas Term, 2021