Part IB Physics : Lent 2022

QUANTUM PHYSICS EXAMPLES II

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1. Consider the following operations, which act on $f(x)$ as described below, where c is a constant:

(a) $cf(x)$ – vertical scaling;
(b) $f(x) + c$ – vertical displacement;
(c) $f^2(x)$ – squaring;
(d) $\mathrm{d}f/\mathrm{d}x$ – differentiation;
(e) $g(x)f(x)$ – multiplication by a function;
(f) $f(\mathrm{d}f/\mathrm{d}x)$;
(g) $\mathrm{d}^2f/\mathrm{d}x^2$ – double differentiation;
(h) $f(cx)$ – horizontal scaling;
(i) $\sin f(x)$;
(j) $f(-x)$ – inversion.

Which of these operations are linear?

What are the eigenfunctions of the operations that are linear? (Note: some may not be normalizable.)

2. Which of the following operators are Hermitian, given that $\widehat{A}$ and $\widehat{B}$ are Hermitian?

$$\widehat{A} + \widehat{B} \qquad c\widehat{A} \qquad \widehat{A}\widehat{B} \qquad \widehat{A}\widehat{B} + \widehat{B}\widehat{A}$$

Show that in one dimension, for functions that tend to zero as $x \to \pm\infty$, the operator $\mathrm{d}/\mathrm{d}x$ is not Hermitian, but the operator $-i\hbar\mathrm{d}/\mathrm{d}x$ is Hermitian. Is the operator $\mathrm{d}^2/\mathrm{d}x^2$ Hermitian?

3. Show that any non-Hermitian operator $\widehat{A}$ can be written as a linear combination of two Hermitian operators.

4. Show that, in one dimension, the state functions e^{-x^2}, $x\mathrm{e}^{-x^2}$ and $(4x^2 - 1)\mathrm{e}^{-x^2}$ are mutually orthogonal.

5. ϕ_1 and ϕ_2 are normalised eigenfunctions of observable A which are degenerate, and hence not necessarily orthogonal. If $\langle\phi_1|\phi_2\rangle = c$ and c is real, find linear combinations of ϕ_1 and ϕ_2 which are normalised and orthogonal to: (a) ϕ_1; (b) $\phi_1 + \phi_2$.

6. A space-domain wave function $\psi(x)$ is shifted by x_0 to give a new wave function $\psi(x - x_0)$. Calculate the corresponding momentum-domain operator. Show that the momentum-domain wave function remains normalised even after the operator has been applied.

7. Write short notes on the following topics:

(a) The position of a particle is measured, and it is found to lie within a region having width Δx. The momentum is then measured, immediately afterwards, and it is found to lie within the range Δp. If the order of the measurements is changed, so that momentum is measured first and then position, do the results have to be the same?

(b) Suppose now that the position of a particle is measured, and it is found to lie within a region having width Δx, but then its position is measured again. What does quantum mechanics say about the positional uncertainty on the second measurement? For a free particle, find a lower bound estimate of the positional uncertainty as a function of time after the first measurement.

8. Observable A has eigenfunctions ψ_1 and ψ_2 with eigenvalues a_1 and a_2. Observable B has eigenfunctions χ_1 and χ_2 with eigenvalues b_1 and b_2, which can be expressed as

$$\chi_1 = (2\psi_1 + 3\psi_2)/\sqrt{13} \qquad\qquad \chi_2 = (3\psi_1 - 2\psi_2)/\sqrt{13}\,.$$

B is measured, and value b_1 is obtained. What would be the probabilities of getting a_1 and a_2 in a measurement of A immediately afterwards? After this measurement of A, B is again measured; what is the probability of getting b_1 again?

9. For a certain system, the observable A has eigenvalues ± 1, with corresponding eigenfunctions u_+ and u_-. Another observable B also has eigenvalues ± 1, but the corresponding eigenfunctions are:

$$v_+ = (u_+ + u_-)/\sqrt{2} \qquad\qquad v_- = (u_+ - u_-)/\sqrt{2}$$

Show that $C \equiv A + B$ is an observable and find the possible results of a measurement of C.
Find the probability of obtaining each result when a measurement of C is performed on an atom in the state u_+, and express the corresponding eigenstates $w_\pm$ of the system immediately after the measurement in terms of u_+ and u_-.

10. By writing $\widehat{x}$ and $\widehat{p}$ in terms of the raising and lowering operators $\widehat{a}^\dagger$ and $\widehat{a}$, prove that, for the n^{th} excited state of a one-dimensional harmonic oscillator, $\Delta x \Delta p = (n+\frac{1}{2})\hbar$.

11. For a particle of mass m moving freely in one dimension, show that

$$\frac{\mathrm{d}\langle x^2\rangle}{\mathrm{d}t} = \frac{1}{m}\,\langle \widehat{x}\widehat{p} + \widehat{p}\widehat{x}\rangle \qquad \text{and} \qquad \frac{\mathrm{d}^2\langle x^2\rangle}{\mathrm{d}t^2} = \frac{2}{m^2}\,\langle \widehat{p}^2\rangle\,.$$

Show that, if $\mathrm{d}\left\langle x^2\right\rangle/\mathrm{d}t = 0$ at $t = 0$, then at later times t:

$$\left\langle x^2\right\rangle_t = \left\langle x^2\right\rangle_0 + \left\langle p^2\right\rangle_0 \frac{t^2}{m^2}\,.$$

12. For a certain system, A has eigenvalues a_1 and a_2 corresponding to eigenfunctions:

$$\psi_1 = (u_1 + u_2)/\sqrt{2} \qquad\qquad \psi_2 = (u_1 - u_2)/\sqrt{2}$$

where u_1 and u_2 are stationary states with energies E_1 and E_2. A is measured and found to have value a_1. Find how $\langle A \rangle$ subsequently varies with time.

13. Suppose that $\hat{H}$ is the Hamiltonian of a time-independent system. Using Dirac's bra-ket notation, and bearing in mind the definition of the function of an operator, show that $\hat{H}$ and $\exp\left[i\hat{H}t\right]$ commute.

14. Explain why, when using state vectors, the shift operator introduced in question 6 can be written $\exp\left[-i\hat{p}x_0/\hbar\right]$. Show that the operators corresponding to two different shifts x_{01} and x_{02} commute.

ANSWERS:

1. (a) any $f(x)$; (d) $e^{\alpha x}$; (e) $\delta(x - x_0)$; (g) $e^{\alpha x}$ or $\cos(kx + \phi)$; (h) constant or x^b; (j) $f(x) = \pm f(-x)$.

2. The following are Hermitian: $\widehat{A} + \widehat{B}$; $c\widehat{A}$ if c is real; $\widehat{A}\widehat{B}$ if $[\widehat{A}, \widehat{B}] = 0$; $\widehat{A}\widehat{B} + \widehat{B}\widehat{A}$; d^2/dx^2.

5. (a) $\dfrac{c\phi_1 - \phi_2}{\sqrt{1 - c^2}}$; (b) $\dfrac{\phi_1 - \phi_2}{\sqrt{2(1 - c)}}$.

8. 4/13; 9/13; 97/169.

9. $C = \pm\sqrt{2}$, with probabilities $\dfrac{(2 \pm \sqrt{2})}{4}$. And $w_\pm = \sqrt{\frac{1}{2}\left(1 \pm \frac{1}{\sqrt{2}}\right)}u_+ \pm \sqrt{\frac{1}{2}\left(1 \mp \frac{1}{\sqrt{2}}\right)}u_-$.

12. $\langle A \rangle = a_1 \cos^2 \omega t + a_2 \sin^2 \omega t$, where $\omega = (E_1 - E_2)/2\hbar$.