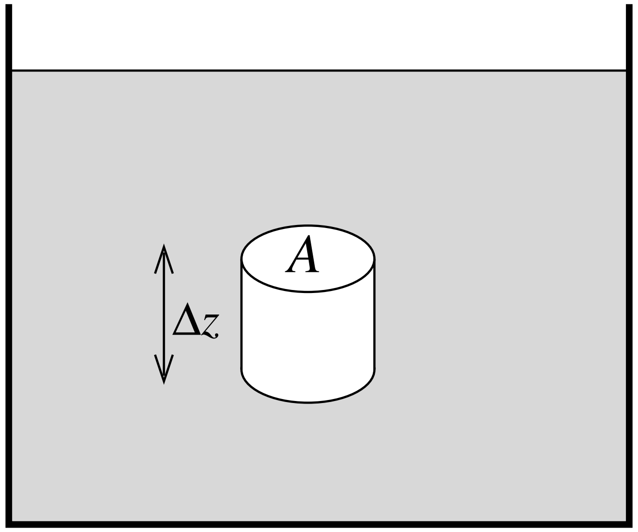
Archimedes Principle

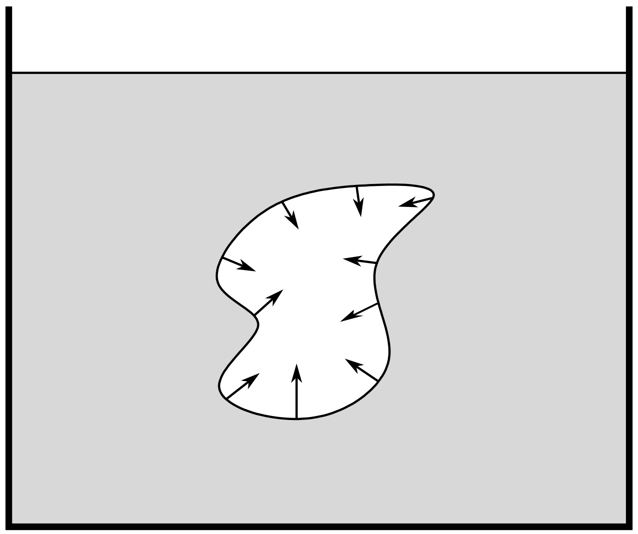
- In a fluid, the pressure increases with depth, due to the increasing weight of fluid supported.
- For depth z, for an incompressible fluid of density ρ, then $P(z) = \rho g z$.
- So a body immersed in the fluid will feel more pressure on its lower surfaces than on its upper surfaces, so feels an 'upthrust' (the net force from the pressure integrated over the whole surface of the body).



- Can easily work out the upthrust for a simple cylindrical element in the fluid. For cross-section of A, depth Δz, then (downward) pressure force on top is $P_0 A$ where P_0 is the pressure in the fluid at the top of the body.
- The (upward) pressure force on bottom is $(P_0 + \rho g \Delta z)A$.
- So the net upward force is $\rho g A \Delta z$, which is the mass of the displaced fluid $\left(\text{i.e. } \rho\underbrace{(A\Delta z)}_{\text{volume}}\right)$ times g, i.e. the weight of the displaced fluid.

Note: 10 m depth in water is a pressure difference of ≈ 1 atmosphere.

- A body of arbitrary shape can be built up of many simple elements to give a total upthrust of $\rho g V$.

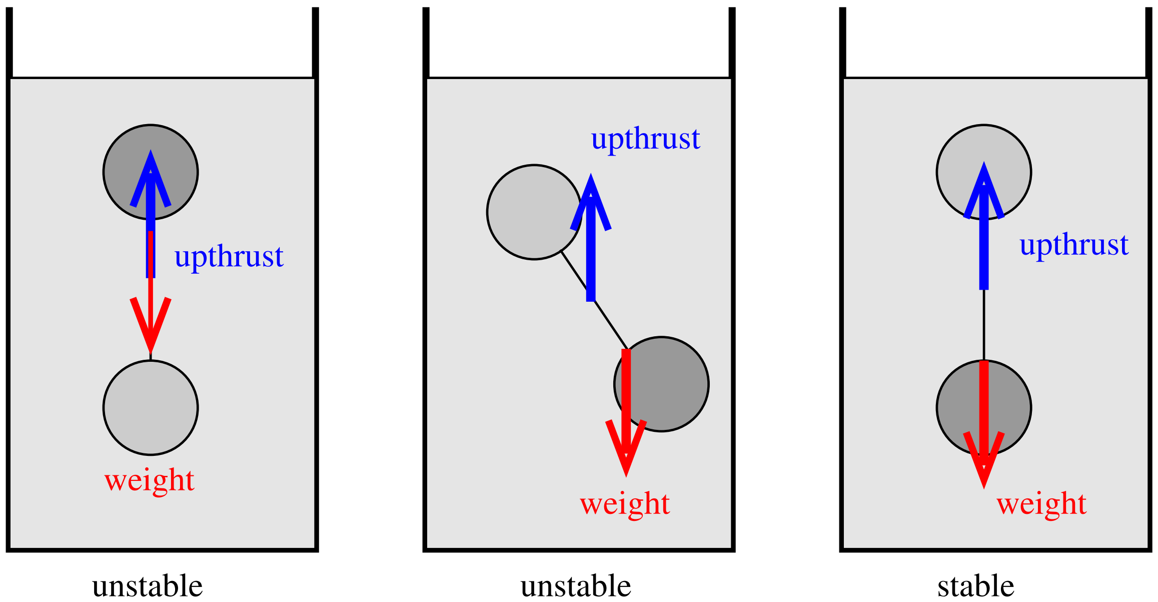


- Alternatively, consider an arbitrary shaped volume filled with the same fluid as the surroundings.
- Since the fluid in the arbitrary volume is in equilibrium, the vertical downward force due to its weight *must* be exactly balanced by the upthrust from the surrounding fluid.
- Consequently if we replace the arbitrary shaped volume of a fluid with a solid object of exactly the same shape, the object will feel an

  upthrust equal and opposite to the weight of the fluid it has displaced.
- This is **Archimedes Principle**.

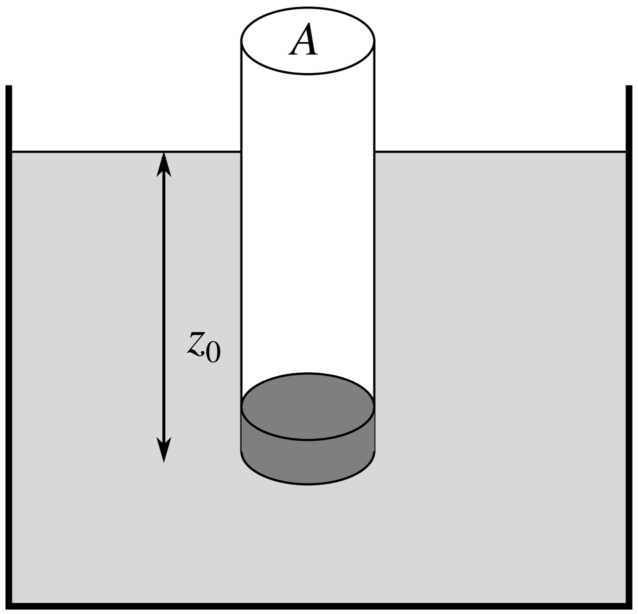
Note: the upthrust acts at the centre of mass of the displaced fluid.

Example 1: Consider a 'dumbbell' as shown, which has symmetrically shaped ends, but of different densities, with an average density equal to that of the fluid displaced.



- If the more dense end of the dumbbell is *directly above* the less dense end, the dumbbell is not stable, and will rotate until the dense end is directly *below* the less dense end.

Example 2: A 'hydrometer' is a weighted cylinder — mass M, cross-section A — with a scale to measure density of a fluid. If displaced vertically it will oscillate.



- At equilibrium, as shown, then

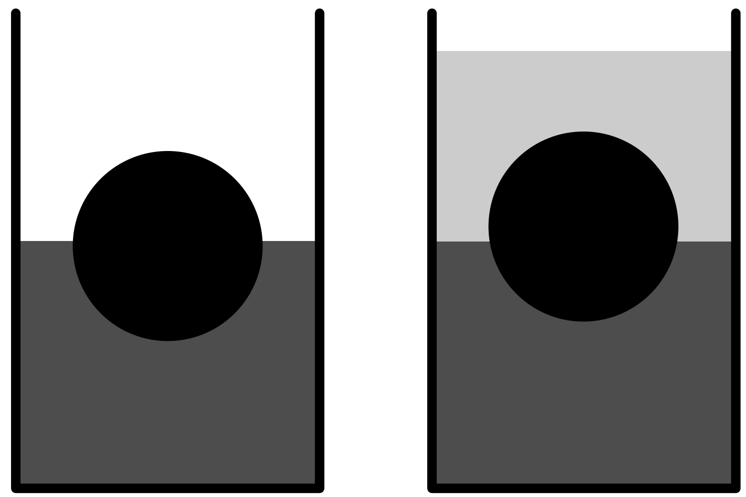
$$\underbrace{Mg}_{\text{weight}} = \underbrace{\rho(Az_0)g}_{\text{upthrust}}.$$

- If displaced downwards an additional z, there is an additional upthrust $\rho(Az)g$.

- So the equation of motion is $M\ddot{z} = -\rho A g z$ which is simple harmonic motion, with

$$\omega = \sqrt{\frac{\rho A g}{M}}.$$

Example 3: An iron cannonball floats in a mercury. If it is then covered in water, does the cannonball move up or down relative to the mercury or stay in the same place?

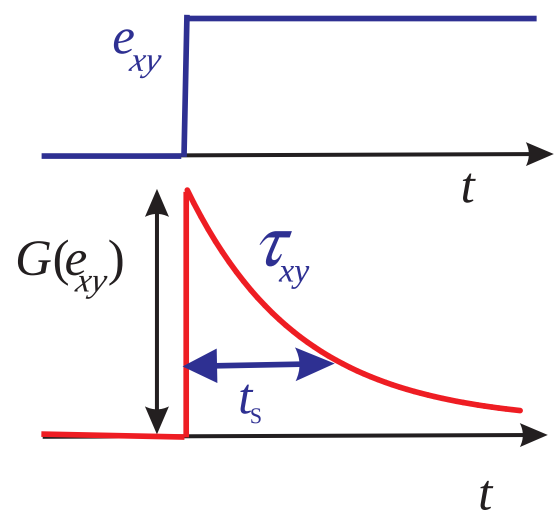


in mercury   in mercury and water

- In both cases the weight of the cannonball is balanced by the upthrust of the displaced fluid(s).
- In the first case, the cannonball displaces a particular volume of mercury to provide the required upthrust (neglecting the very small upthurst from the displaced air).
- In the second case, there is upthrust from the displaced water, so less mercury needs to be displaced to give the required (total) upthrust. So, the cannonball moves *upwards*.

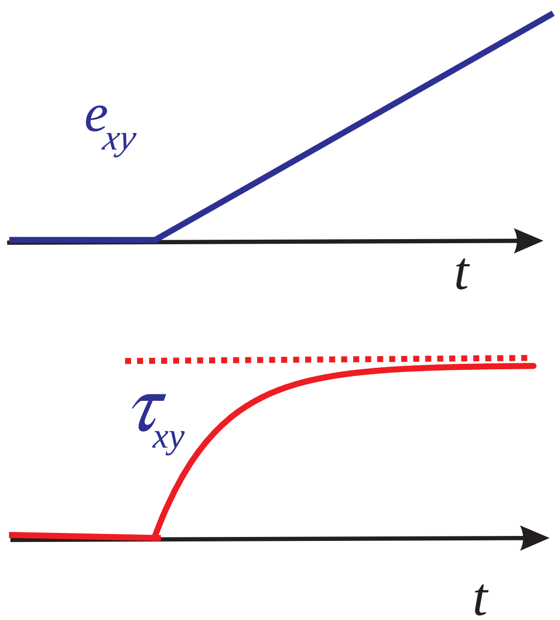
Stresses in Liquids and Gases

- Liquids and gases (**fluids**) cannot maintain a shear stress, because the molecules can move past each other over some timescale t_s.



- A sudden shear e_{xy} produces a stress τ_{xy} that rapidly decays (for a liquid at room temperature $t_s \sim 10^{-11}$ s).
- Fluids also cannot maintain a difference between their normal stresses. The differences in normal stress also decay (though the timescale t_p can be slightly different from t_s).
- Fluids can have many other interesting 'memories' of what happened to them in the past, but ones that are characterised by just these two decay times are called **Newtonian fluids**.

- To maintain a stress in a fluid, it has to be continuously sheared.



- The proportionality between the stress and the rate of shear is called the (dynamic) viscosity = η, with $\tau_{xy} = \eta \left(2 \frac{\mathrm{d}e_{xy}}{\mathrm{d}t} \right)$.
- It is related to G and t_s by $\eta = G t_\mathrm{s}$.
- (Even in very strong materials, such as steel, stresses will eventually decay. For example, piano wire will go out of tune (goes flat) after a period of years.)

FLOW OF IDEAL FLUIDS

- We shall look first at the flow of an **ideal fluid**, which is **incompressible** and has no **viscosity**, which is a useful first model for the flow of liquids and gases.
  (In Volume II of Feynman's 'Lectures on Physics', Feynman calls this the 'flow of dry water', in contrast to the 'flow of wet water' when viscosity is included.)
- After looking at the basic equations of fluid flow, we will derive the important **Bernoulli's equation** that relates the pressure to the flow velocity.
- For the case of **irrotational flow**, we will use velocity potential theory to find the flow patterns around obstacles in the fluid, and then use Bernoulli's equation to find the pressure.
- We shall then look at the effects of viscosity and its effects in simple 'laminar' flows — i.e. in locally parallel layers — when viscosity dominates.
- In real flows a very important consideration is the ratio of the inertial stresses to the viscous stresses. This characterised by the **Reynolds number**, which determines whether flows are laminar or turbulent.

FLUID DYNAMICS

- When the mean free path λ of particles in a liquid, gas or plasma is small compared to the scales of interest, we can treat the system as a **hydrodynamical fluid**.
- In an **ideal fluid** differences in normal stresses decay so fast that the only stresses present can be an **isotropic pressure** $\tau_1 = \tau_2 = \tau_3 = -P$.
- A fluid is composed of **fluid elements**, which are regions larger than λ — sufficiently large enough to have well-defined values of macroscopic properties.

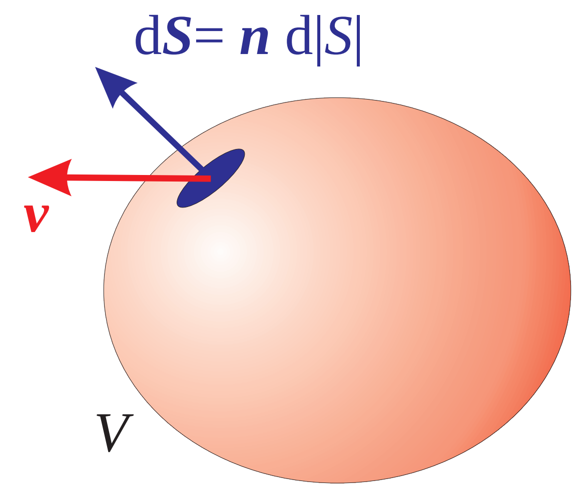
  For compressible fluids these properties are:
  - density $\rho(\boldsymbol{x},t)$;
  - velocity $\boldsymbol{v}(\boldsymbol{x},t)$;
  - pressure $P(\boldsymbol{x},t)$;
  - energy density $u(\boldsymbol{x},t)$.

  (Plus anything else that is relevant, e.g. magnetic field.)

- A fluid element is acted on by gravity (and other body forces), by pressure forces and other stresses on the surfaces.
- The fluid elements stay more or less well-defined (on scales $\gg \lambda$), but the surface of an element is moving at the local fluid velocity, so elements distort as they move around. The distortions can become very large — in the presence of turbulent motions the fluid elements can be dispersed.
- There are still some microscopic physical processes occurring on scales of λ that determine important properties of a fluid: **viscosity; heat conduction**.
- The macroscopic properties can also change discontinuously on scale of λ at **shock fronts** – not considered here.
- We will also neglect effects due to **surface tension**.

Conservation of Mass

- We shall derive the equations of compressible fluid dynamics and restrict them to the incompressible case afterwards.
- The density and velocity of a fluid are functions of position and time: $\rho(\boldsymbol{x},t)$ and $\boldsymbol{v}(\boldsymbol{x},t)$.
- Consider a volume V surrounded by a surface $\boldsymbol{S}$.
  The rate of mass leaving through area element $\mathrm{d}\boldsymbol{S}$ is $\rho\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S}$.



- The total mass leaving through the surface S is $\oint(\rho\boldsymbol{v})\cdot\mathrm{d}\boldsymbol{S}$.
  This is balanced by the rate of change of the mass inside V:
  $$\int\frac{\partial\rho}{\partial t}\,\mathrm{d}V+\oint(\rho\boldsymbol{v})\cdot\mathrm{d}\boldsymbol{S}=0,$$
  i.e. the change of mass inside V and the loss through surface $\boldsymbol{S}$.

- Using the **divergence theorem** we have

$$\int \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) \right) \mathrm{d}V = 0.$$

- This must be true for *any* volume V so we have the **continuity equation** $\boxed{\dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0.}$

- For **incompressible** flow the density ρ is a constant everywhere and we have

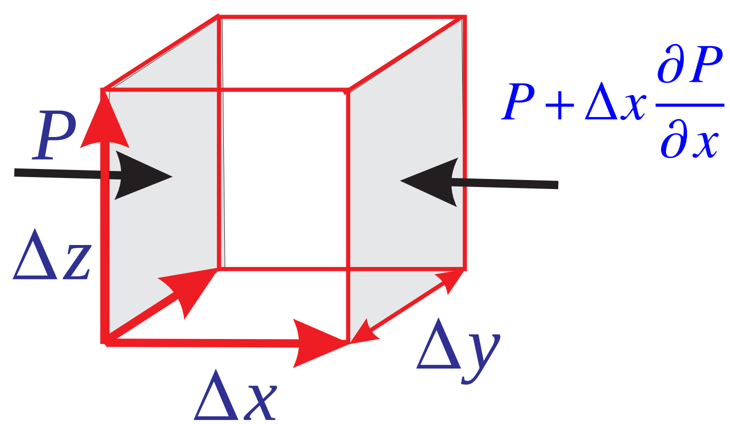
$$\boxed{\nabla \cdot \boldsymbol{v} = 0.}$$

  i.e. the $\boldsymbol{v}$ has zero divergence, so is a 'solenoidal' vector field.

EQUATION OF MOTION OF A FLUID

- Differences in normal stresses in an ideal fluid decay so fast that the only stresses present can be an **isotropic pressure** $\tau_1 = \tau_2 = \tau_3 = -P$.

- For a small element of fluid $(\Delta x, \Delta y, \Delta z)$, the variation of pressure causes acceleration.



- The net force on the x-faces is

$$F_x = (\Delta y \Delta z) \times \left(-\Delta x \frac{\partial P}{\partial x} \right) = -\Delta x \Delta y \Delta z \frac{\partial P}{\partial x},$$

and similarly for components along y and z.

- This force is proportional to the volume element $\Delta x \Delta y \Delta z$.

- In addition, there is the force due to gravity acting on the volume element, i.e. $\rho \boldsymbol{g} = -\rho \nabla \phi_{\mathrm{g}}$.

- So, we can generalise this to a vector equation of motion

$$\text{mass per unit volume} \times \text{acceleration} = -\nabla P + \rho \boldsymbol{g}\,.$$

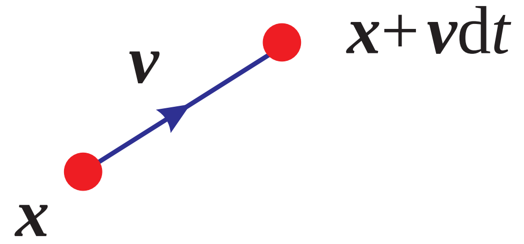
- This is written as $\boxed{\rho \frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\nabla P + \rho \boldsymbol{g},}$ which is **Euler's equation**.

  Note: $\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t}$ — the 'convective derivative' — is the **acceleration of the fluid element**.

  It is not equal to $\left(\frac{\partial \boldsymbol{v}}{\partial t}\right)$, **because the fluid element is moving**.

- This equation of motion is equally applicable to compressible or incompressible flow.

THE CONVECTIVE DERIVATIVE



- A material element moves with the velocity of the fluid, which is a function of space and time: $\boldsymbol{v}(\boldsymbol{x},t)$.
- For changes of $\mathrm{d}t, \mathrm{d}\boldsymbol{x}$, then the change in velocity is
$$\mathrm{d}\boldsymbol{v} = \mathrm{d}t\,\frac{\partial \boldsymbol{v}}{\partial t} + \mathrm{d}\boldsymbol{x}\cdot\nabla\boldsymbol{v} = \mathrm{d}t\,\frac{\partial \boldsymbol{v}}{\partial t} + \sum_i \mathrm{d}x_i \frac{\partial \boldsymbol{v}}{\partial x_i}.$$
- An element of fluid moves with velocity $\boldsymbol{v}$, so in time $\mathrm{d}t$, the position change is $\mathrm{d}\boldsymbol{x} = \boldsymbol{v}\,\mathrm{d}t$.
- Hence the changes following the fluid element is $\mathrm{d}\boldsymbol{v} = \mathrm{d}t\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v}\right)$.
- Hence define **convective derivative** $\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = \left.\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right|_{\text{moving with fluid}}$, i.e. $\boxed{\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v}.}$

VISUALISING FLUID FLOW

- The velocity of fluid $\boldsymbol{v}(\boldsymbol{x},t)$ is a function of position and time. It can be visualised it in various ways.

  1: **Streamlines**.

  A streamline follows the velocity vector $\boldsymbol{v}(\boldsymbol{x})$ **at a given time** t. It is analogous to the fieldlines of $\boldsymbol{E}$ and $\boldsymbol{B}$ you have already met in electromagnetism.

  2: **Particle paths**.

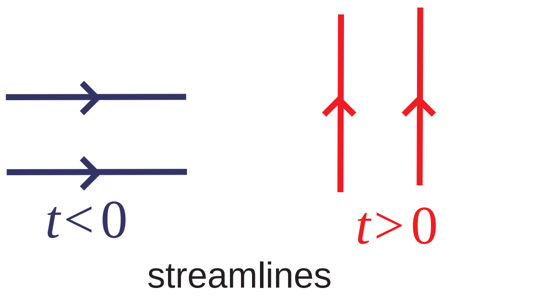
  A particle path is the path in space that a given fluid element follows over the course of time. The particle paths are important for following fluid elements.

  3: **Streaklines**.

  A streakline connects all points that go through a particular point in space over the course of time. For example, if a dye was released into a fluid at a particular point in space it would generate a streakline.

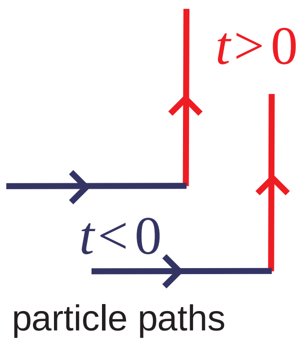
Example: Suppose $\boldsymbol{v}(\boldsymbol{x}) = V\widehat{\boldsymbol{e}}_x$ for $t < 0$ and changed to $\boldsymbol{v}(\boldsymbol{x}) = V\widehat{\boldsymbol{e}}_y$ for $t > 0$.

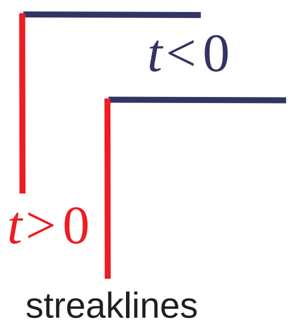
1. The streamlines are parallel to the x-axis for $t < 0$ and parallel to the y-axis for $t > 0$.



2. The particle paths go along the $+x$-direction and suddenly turn into the $+y$-direction.

3. The streaklines go along the $-x$-direction and suddenly turn into the $-y$-direction.

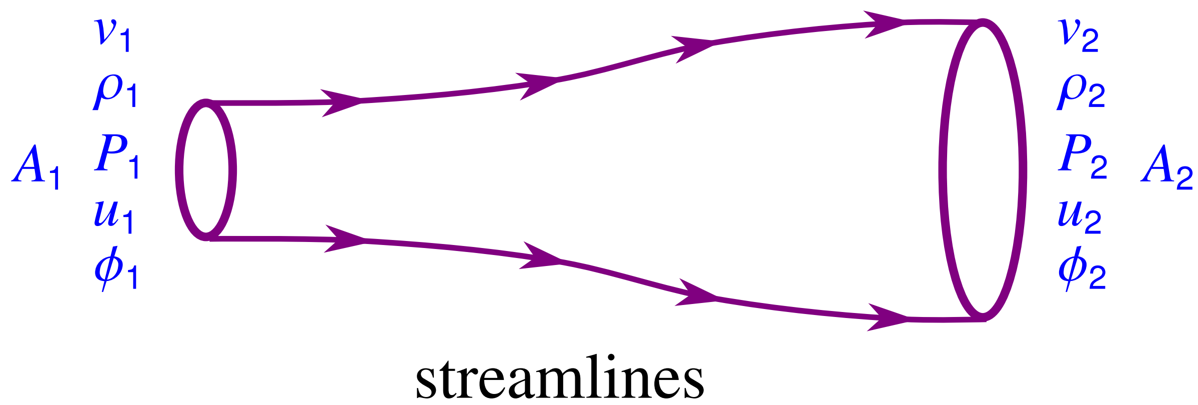




For the case of **steady flow** the streamlines, particle paths and streaklines all coincide.

BERNOULLI'S EQUATION FOR COMPRESSIBLE FLOW

- There is an important theorem which is often useful for cases involving steady flow, expressing the conservation of energy as it transported along a streamline: **Bernoulli's equation**.



- For steady flow, consider streamlines connecting areas A_1 and A_2.
- Energy flow rate in: $A_1 v_1(\rho_1\phi_1 + \frac{1}{2}\rho_1 v_1^2 + u_1)$ (ϕ is the gravitational potential and u is internal energy per unit volume), plus the pressure does work at rate $A_1 v_1 P_1$.
- So energy in is $A_1 v_1 \rho_1 \left[\left(\frac{u_1 + P_1}{\rho_1}\right) + \frac{v_1^2}{2} + \phi_1\right]$.
- Similarly for energy rate out.

- Since the mass flow is the same in and out, i.e. $A_1 v_1 \rho_1 = A_2 v_2 \rho_2$, this gives

  Bernoulli's equation which is $\boxed{\dfrac{u+P}{\rho} + \frac{1}{2}v^2 + \phi = \text{constant}}$ along a streamline.

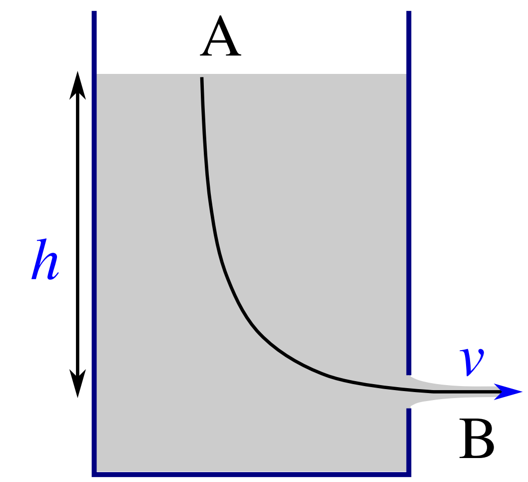
- For a perfect gas specific enthalpy $= u + P = \dfrac{\gamma}{\gamma - 1}P$.

- So for **incompressible flow**, when ρ is constant (i.e. $\gamma = \infty$, $u = 0$), then

  $\boxed{P + \frac{1}{2}\rho v^2 + \rho\phi = \text{constant}}$ along a streamline.

Note: if a streamline is/is not curved, then a pressure gradient perpendicular to the streamline is/is not needed to provide a centripetal force. (Vertically there may be a pressure gradient due to gravity, but that is not related to the curvature of the flow).

Example 1: Flow from water tank.



- Consider a flow line from the top of the tank at point A, to B outside the tank, after the water escapes.

  Note: the water jet flowing from the tank narrows as it exits the tank; at the hole the flow is **not** parallel.

- The pressure is the same at point A, where $v = 0$, and at point B outside the tank.

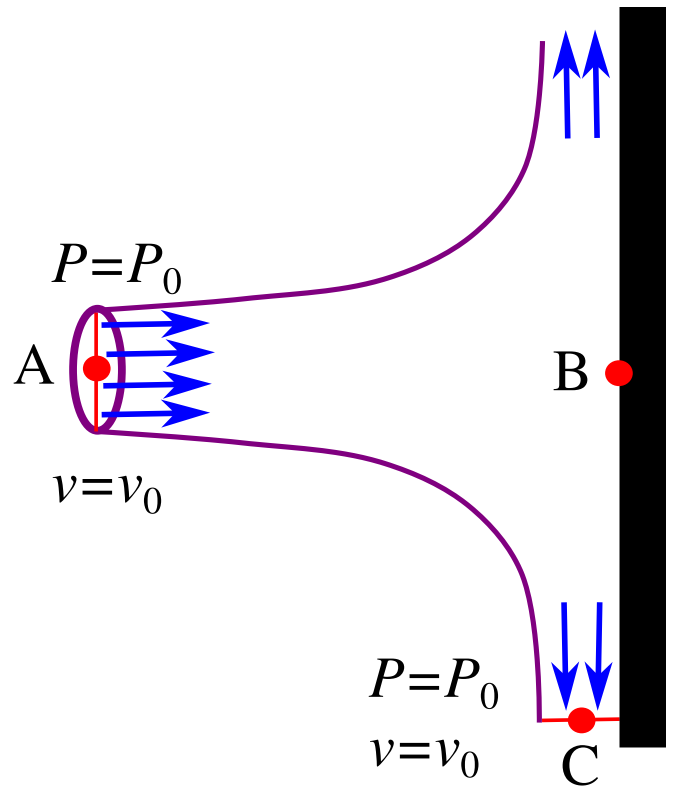
- Use Bernoulli: $P + \frac{1}{2}\rho v^2 + \rho g h = \text{constant}$.

- This gives the outflow velocity is $\sqrt{2gh}$.

- The actual draining rate is not necessarily equal to this velocity times the area of the hole, since the flow can still be converging as the water leaves the tank.

- The 'efflux coefficient' is the effective area of hole divided by its geometric area. This varies between 0.5 (for the Borda's mouthpiece, see discussion below) to 1. For a simple hole in the side of a tank it is, experimentally, ≈ 0.62.

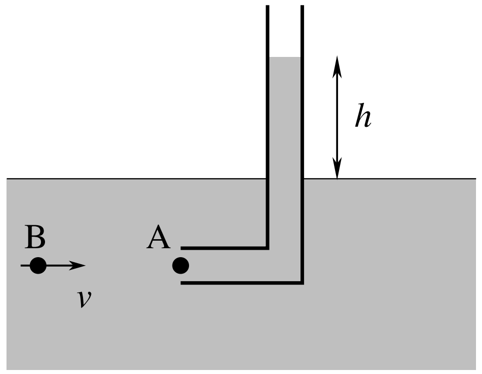
Example 2: A cylindrical jet hitting a wall (assuming that gravitational effects are small).



- The 'stagnation point' at B must have pressure $P_0 + \frac{1}{2}\rho v_0^2$.
- The pressure at C (at large radius compared to jet radius) must be P_0 again, so by Bernoulli, velocity is v_0 so the thickness of the film must decrease as the water spreads out.

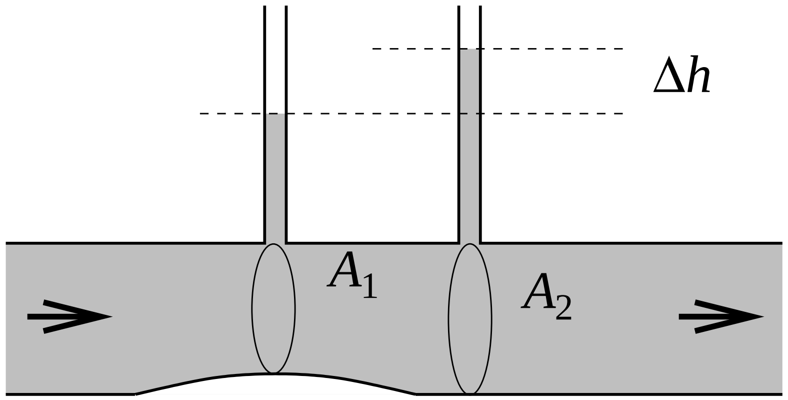
Example 3: A ‘Pitot’ tube. This is an ‘L’ shaped tube, which is used to measure the speed of fluid flow. (Originally used to measure the speed of the River Seine, in 1732. Still used to measure airspeed of modern aircraft.)

- Consider a point of stagnation (i.e. $v = 0$) at A, at the opening of the tube, and point B at the same depth, offset a long way from the tube in the moving fluid, at the same depth as A.



- For the streamline from A to B, using Bernoulli, then the pressure at A must be higher than the pressure at B by $\frac{1}{2}\rho v^2$.
- So, the height of the liquid in the tube above the surface of the surrounding liquid $h = \frac{1}{2}v^2/g$.

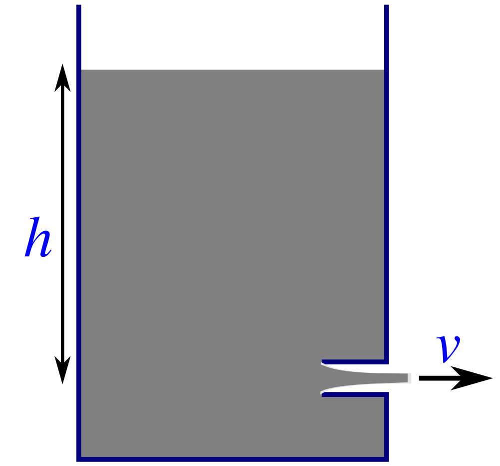
Example 4: A ‘Venturi’ meter is a constriction on a pipe, as shown, that allows the flow rate to be measured.



- For an incompressible fluid, flowing smoothly, if can be shown that the mass flow rate is

$$\rho A_1 A_2 \sqrt{\frac{2g\Delta h}{A_2^2 - A_1^2}}.$$

BORDA'S MOUTHPIECE



- There is a simple argument for the size of a jet from a re-entrant hole (Borda's mouthpiece).
- Consider a small hole, (far) away from the walls.
- The amount of momentum in the jet created per unit time is $(\rho v)\times(vA_{\text{jet}})$, where A_{jet} is the final cross-sectional area of the jet.
- This change of momentum must come from the forces on the walls.
- If the hole is small the fluid velocities near the walls will be very low, so the pressure is just the hydrostatic pressure $P_0+\rho gh$ over the entire surface **except at the hole**, where the pressure is P_0.
- The momentum created per unit time by the pressure is thus ρghA_{hole} which must be equal to $\rho v^2 A_{\text{jet}}$.
- Bernoulli gives $\frac{1}{2}\rho v^2=\rho gh$, so $A_{\text{jet}}=0.5A_{\text{hole}}$.
- In general, the efflux coefficient is in the range $0.5\longrightarrow 1$.

APPROXIMATIONS TO HYDRODYNAMICS

- The equations of compressible hydrodynamics are very complicated.
- They are difficult to integrate numerically — in 3-D computing time scales like (grid size)^4.
- But there are useful *approximations* that give insight.
  1) **Incompressible flow**: $\nabla \cdot \boldsymbol{v} = 0$.

     This a good model for liquids, but also a surprisingly good approximation for gases provided the flow is subsonic.
  2) **Irrotational flow**: this is the case when there is no **vorticity**, i.e. $\nabla \times \boldsymbol{v} = 0$.

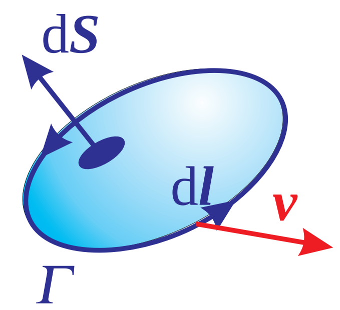
     The vorticity field $\boldsymbol{\omega} \equiv \nabla \times \boldsymbol{v}$ moves with the fluid, and vorticity is usually generated at boundaries, so often the bulk of a flow *is* irrotational.
- If $\nabla \times \boldsymbol{v} = 0$ the velocity field can be generated from a scalar potential $\boldsymbol{v} = \nabla \Phi$.

  Note: in fluid dynamics, this conventionally has a plus sign.
- If the fluid is both incompressible and irrotational, the velocity potential satisfies Laplace's equation $\nabla^2 \Phi = 0$.
  - In this important case we can use potential theory to find $\boldsymbol{v}$ and Bernoulli's equation to find the pressure.

KELVIN'S CIRCULATION THEOREM

- Consider the circulation around a loop Γ:

  $K = \oint_\Gamma \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{l}$ defines the amount of rotation around the loop.



- This is related to the **vorticity** $\boxed{\boldsymbol{\omega} \equiv \nabla \times \boldsymbol{v}}$ via Stokes' theorem. If $\boldsymbol{S}$ is a surface ending on Γ then the circulation is

$$K = \underbrace{\oint_\Gamma \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{l} = \int_S (\nabla \times \boldsymbol{v}) \cdot \mathrm{d}\boldsymbol{S}}_{\text{by Stokes}} = \int \boldsymbol{\omega} \cdot \mathrm{d}\boldsymbol{S}.$$

- Kelvin's circulation theorem states that the circulation around a loop moving with the fluid is constant, which we will now show.

- Applying the convective derivative

$$\frac{\mathrm{D}K}{\mathrm{D}t} = \oint_{\Gamma} \left(\underbrace{\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} \cdot \mathrm{d}\boldsymbol{l}}_{\text{1st term}} + \underbrace{\boldsymbol{v} \cdot \frac{\mathrm{D}(\mathrm{d}\boldsymbol{l})}{\mathrm{D}t}}_{\text{2nd term}} \right).$$

- Taking these two terms separately:

1st: Using the equation of motion previously derived, then

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} \cdot \mathrm{d}\boldsymbol{l} = \left(\frac{-\nabla P}{\rho} + \boldsymbol{g} \right) \cdot \mathrm{d}\boldsymbol{l} = \nabla \left(\frac{-P}{\rho} - \phi_g \right) \cdot \mathrm{d}\boldsymbol{l}$$

for an incompressible flow, with ϕ_g the gravitational potential.

2nd: $\frac{\mathrm{D}(\mathrm{d}\boldsymbol{l})}{\mathrm{D}t}$ is the difference in velocity between the ends of the element $\mathrm{d}\boldsymbol{l}$, i.e.

$$\frac{\mathrm{D}(\mathrm{d}\boldsymbol{l})}{\mathrm{D}t} = \nabla \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{l}.$$

- Consider the x-component, which is

$$\underbrace{(\nabla_x \boldsymbol{v})}_{\text{gradient}} \quad \text{times} \quad \underbrace{\mathrm{d}l_x}_{\text{change}}$$

which is a vector: $$\left(\frac{\partial v_x}{\partial x}\widehat{\boldsymbol{e}}_x + \frac{\partial v_y}{\partial x}\widehat{\boldsymbol{e}}_y + \frac{\partial v_z}{\partial x}\widehat{\boldsymbol{e}}_z\right)\mathrm{d}l_x$$

and similarly for the y- and z-components.

- So combining all components together gives

$$\left(\frac{\partial v_x}{\partial x}\mathrm{d}l_x + \frac{\partial v_x}{\partial y}\mathrm{d}l_y + \frac{\partial v_x}{\partial z}\mathrm{d}l_z\right)\widehat{\boldsymbol{e}}_x +$$
$$\left(\frac{\partial v_y}{\partial x}\mathrm{d}l_x + \frac{\partial v_y}{\partial y}\mathrm{d}l_y + \frac{\partial v_y}{\partial z}\mathrm{d}l_z\right)\widehat{\boldsymbol{e}}_y +$$
$$\left(\frac{\partial v_z}{\partial x}\mathrm{d}l_x + \frac{\partial v_z}{\partial y}\mathrm{d}l_y + \frac{\partial v_z}{\partial z}\mathrm{d}l_z\right)\widehat{\boldsymbol{e}}_z .$$

- Now take $\boldsymbol{v}\cdot$ of this, which is, reordering

$$\left(v_x \frac{\partial v_x}{\partial x}\mathrm{d}l_x + v_y \frac{\partial v_y}{\partial x}\mathrm{d}l_x + v_z \frac{\partial v_z}{\partial x}\mathrm{d}l_x\right) +$$
$$\left(v_x \frac{\partial v_x}{\partial y}\mathrm{d}l_y + v_y \frac{\partial v_y}{\partial y}\mathrm{d}l_y + v_z \frac{\partial v_z}{\partial y}\mathrm{d}l_y\right) +$$
$$\left(v_x \frac{\partial v_x}{\partial z}\mathrm{d}l_z + v_y \frac{\partial v_y}{\partial z}\mathrm{d}l_z + v_z \frac{\partial v_z}{\partial z}\mathrm{d}l_z\right).$$

- Writing $v_x \dfrac{\partial v_x}{\partial x} = \dfrac{\partial(\frac{1}{2}v_x^2)}{\partial x}$ etc, then this is

$$\frac{\partial}{\partial x}\left(\tfrac{1}{2}(v_x^2 + v_y^2 + v_z^2)\right)\mathrm{d}l_x +$$
$$\frac{\partial}{\partial y}\left(\tfrac{1}{2}(v_x^2 + v_y^2 + v_z^2)\right)\mathrm{d}l_y +$$
$$\frac{\partial}{\partial z}\left(\tfrac{1}{2}(v_x^2 + v_y^2 + v_z^2)\right)\mathrm{d}l_z$$

which can — at last — be written as $\nabla(\frac{1}{2}v^2)\cdot\mathrm{d}\boldsymbol{l}$.

- So, we now have

$$\frac{\mathrm{D}K}{\mathrm{D}t} = \oint_{\Gamma} \nabla\left(-\frac{P}{\rho} - \phi_{\mathrm{g}} + \tfrac{1}{2}v^2\right)\cdot \mathrm{d}\boldsymbol{l}\,.$$

- This is the integral around the loop Γ of the gradient of $\left(-\frac{P}{\rho} - \phi_{\mathrm{g}} + \frac{1}{2}v^2\right)$.

- But at the two ends of the loop, by Bernoulli's theorem, the quantity in brackets is the same so

$$\boxed{\frac{\mathrm{D}K}{\mathrm{D}t} = 0,}$$

  which implies that 'vortex lines are conserved and move with the fluid'.

Note: this is also true for 'isobaric' compressible flows — i.e. ones for which $P = P(\rho)$, so that the local gradients in pressure and density are parallel.

BERNOULLI'S EQUATION REVISITED

- Consider the gradient $\nabla(P+\rho\phi_g+\frac{1}{2}\rho v^2)$.
- We can rearrange $\nabla(\frac{1}{2}\rho v^2)$ using $\boldsymbol{v}\times(\nabla\times\boldsymbol{v})=\nabla(\frac{1}{2}v^2)-\boldsymbol{v}\cdot\nabla\boldsymbol{v}$.
- So $\nabla(P+\rho\phi_g+\frac{1}{2}\rho v^2)=\nabla P+\rho\nabla\phi_g+\rho\boldsymbol{v}\cdot\nabla\boldsymbol{v}+\rho\boldsymbol{v}\times(\nabla\times\boldsymbol{v})$.
- Previously (slides 211/212) we had
$$\rho\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t}=-\nabla P+\rho\boldsymbol{g}=-\nabla P-\rho\nabla\phi_g$$
and
$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t}=\frac{\partial\boldsymbol{v}}{\partial t}+\boldsymbol{v}\cdot\nabla\boldsymbol{v}.$$
- Combining these then $\nabla(P+\rho\phi_g)=-\rho\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t}=-\rho\frac{\partial\boldsymbol{v}}{\partial t}-\rho\boldsymbol{v}\cdot\nabla\boldsymbol{v}$, so
$$\nabla P+\rho\nabla\phi_g+\rho\boldsymbol{v}\cdot\nabla\boldsymbol{v}+\rho\boldsymbol{v}\times(\nabla\times\boldsymbol{v})=-\rho\frac{\partial\boldsymbol{v}}{\partial t}+\rho\boldsymbol{v}\times(\nabla\times\boldsymbol{v}),$$
or
$$\nabla(P+\rho\phi_g+\tfrac{1}{2}\rho v^2)=-\rho\frac{\partial\boldsymbol{v}}{\partial t}+\rho\boldsymbol{v}\times(\nabla\times\boldsymbol{v}).$$

- If the flow is **steady**, i.e. $\rho(\partial \boldsymbol{v}/\partial t) = 0$, then taking the dot product of each side of the previous equation with $\boldsymbol{v}$ gives
$$\boldsymbol{v} \cdot \nabla \left(P + \rho\phi_{\mathrm{g}} + \tfrac{1}{2}\rho v^2\right) = 0.$$
- This again proves that $P + \rho\phi_{\mathrm{g}} + \frac{1}{2}\rho v^2$ **is constant on a streamline** (Bernoulli's equation).
- If the flow is **steady** and **irrotational** (i.e. $\nabla \times \boldsymbol{v} = 0$), then $P + \rho\phi_{\mathrm{g}} + \frac{1}{2}\rho v^2$ **is a constant everywhere**.

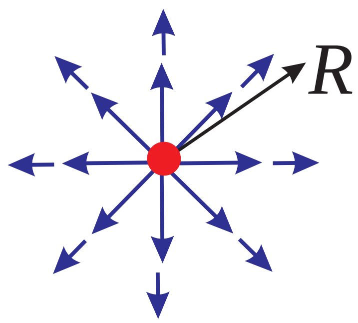
Note: this can be generalised to time-dependent flows. If the flow is irrotational and derived from a velocity potential, i.e. $\boldsymbol{v} = \nabla\Phi$, then there is an important generalisation:
$P + \rho\phi_{\mathrm{g}} + \frac{1}{2}\rho v^2 + \rho\dfrac{\partial \Phi}{\partial t}$ is constant everywhere, and at all times.

POTENTIAL FLOW OF IRROTATIONAL FLUID

- Circulation (vorticity) is conserved and moves with the fluid. It cannot be generated in the bulk of the fluid, but can only enter the fluid at boundaries. The regions close to boundaries that do contain vorticity are called **boundary layers**, which will be discussed in more detail later.
- If a fluid is initially irrotational (i.e. $\nabla \times \boldsymbol{v} = 0$ everywhere), it remains so for all time. Even with boundary layers, the bulk of the fluid is often irrotational to a very good approximation.
- If $\nabla \times \boldsymbol{v} = 0$, then we can find a **scalar potential** Φ such that the velocity is $\boldsymbol{v} = \nabla \Phi$.
- For incompressible, irrotational flow we have $\nabla \cdot \boldsymbol{v} = 0$ so the velocity potential satisfies Laplace's equation $\nabla^2 \Phi = 0$.
- Hence, we can use all the potential theory techniques learned in other physics topics to solve problems of fluid flow.

POTENTIAL FLOW — SOURCES AND SINKS

- Sources/sinks: analogues of charges in electrostatics ($\nabla \cdot \boldsymbol{v} \neq 0$, e.g. the end of the thin pipe).



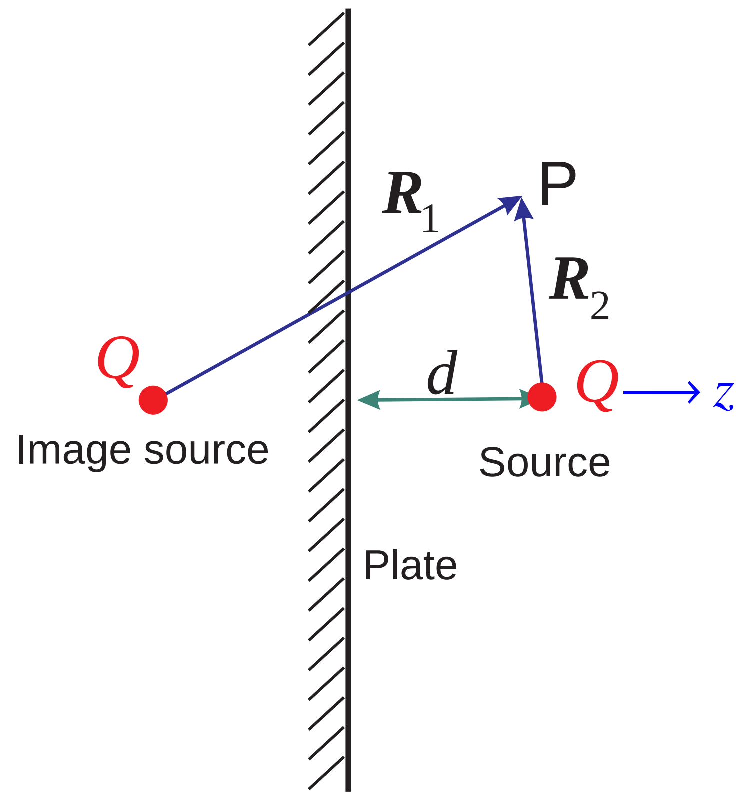
- For a flow rate Q from an isotropic nozzle the velocity potential is

$$\Phi = -\frac{Q}{4\pi R},$$

and the velocity flow is:

$$\boldsymbol{v} = \frac{Q}{4\pi R^2}\widehat{\boldsymbol{e}}_R .$$

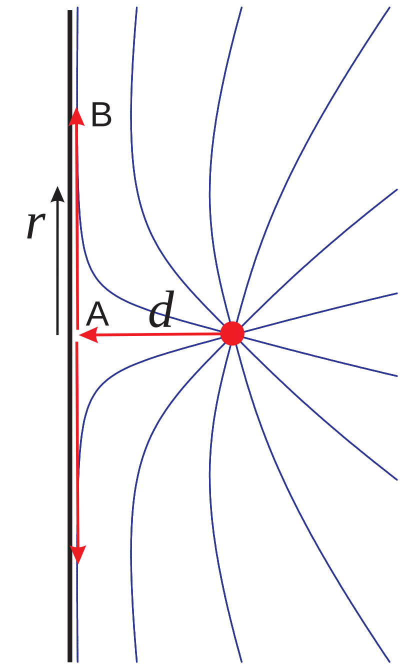
Example: Consider a source a distance d away from an infinite plate.



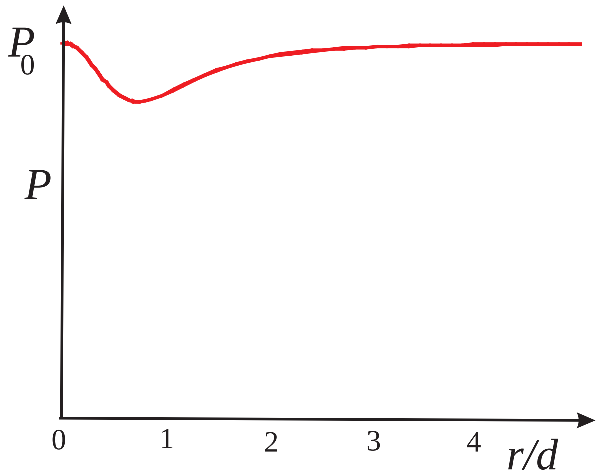
- Use 'method of images', as in electrostatics.
- Source Q at $(0,0,+d)$, and an image source also of Q at $(0,0,-d)$, to ensure there is no fluid flow across the plate in the x,y-plane.
- Potential at point P at x,y,z is
$$\Phi = -\frac{Q}{4\pi}\left(\frac{1}{|R_1|}+\frac{1}{|R_2|}\right).$$
- The velocity field is then given by $\boldsymbol{v} = \nabla\Phi$.
- We then use Bernoulli to find the pressure on the plate.
- At the plate the velocity potential
$$\Phi = -\frac{Q}{4\pi}\frac{2}{(d^2+r^2)^{1/2}}$$
where $r = \sqrt{x^2+y^2}$ is the cylindrical radial coordinate.

- The radial velocity at the plate is $\dfrac{Q}{2\pi}\dfrac{r}{(d^2+r^2)^{3/2}}$.
- Apply Bernoulli to streamline AB: $P = P_0 - \dfrac{Q^2\rho}{8\pi^2}\dfrac{r^2}{(d^2+r^2)^3}$, where P_0 is the pressure in the fluid far from the source.
- Integrate the pressure deficit $\Delta P = P - P_0$ to get the total force

$$F = \int_0^\infty \underbrace{\Delta P}_{\text{negative}} 2\pi r\,\mathrm{d}r = -\frac{Q^2\rho}{16\pi d^2}.$$

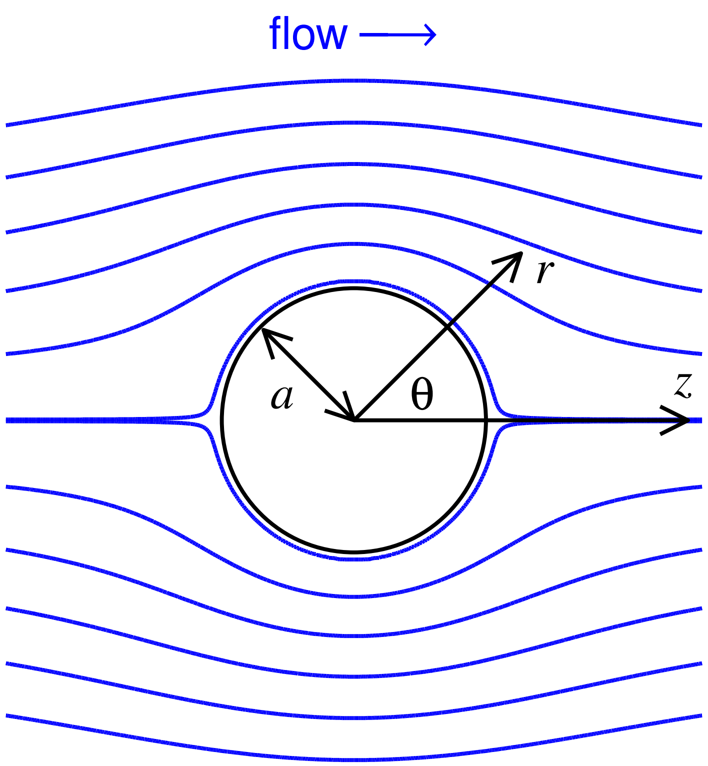


- Consequences:
  - Two sources (or two sinks) a distance D apart are attracted to each other by force $\dfrac{Q^2\rho}{4\pi D^2}$ (cf. electrostatics).
  - A source and a sink repel each other.



POTENTIAL FLOW PAST SPHERE

- Steady flow past sphere of radius a.
- At large distances flow is uniform velocity in z direction so the velocity potential is $\Phi = V_0 z = V_0 r\cos\theta$.
- Boundary condition: $v_r = 0$ at $r = a$.
- Cylindrically-symmetric solutions of $\nabla^2\Phi = 0$, which tend to zero at $r = \infty$ are
$$\Phi = \frac{A}{r} + \frac{B}{r^2}\cos\theta + \frac{C}{r^3}\tfrac{1}{2}(3\cos^2\theta - 1) + \ldots$$
- Present case needs a dipole at the centre of the sphere: $$\Phi = V_0 r\cos\theta + \frac{B}{r^2}\cos\theta.$$
- Radial velocity $v_r = \dfrac{\partial\Phi}{\partial r} = \cos\theta\left(V_0 - \dfrac{2B}{r^3}\right)$.
- Boundary conditions gives $B = \frac{1}{2}V_0 a^3$.



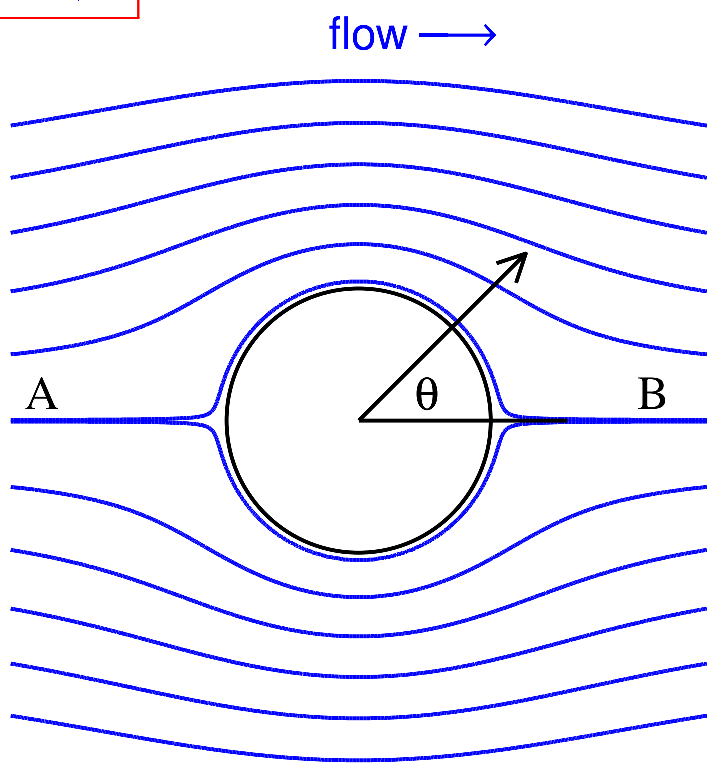
- Velocity potential for flow past sphere $\boxed{\Phi = V_0 \cos\theta \left(r + \frac{a^3}{2r^2} \right).}$

- Velocity $\boldsymbol{v} = \nabla \Phi$:

$$v_r = V_0 \cos\theta \left(1 - \frac{a^3}{r^3} \right);$$

$$v_\theta = -V_0 \sin\theta \left(1 + \frac{a^3}{2r^3} \right).$$

- At $r = a$ the velocity $v_\theta = -\frac{3}{2} V_0 \sin\theta$.

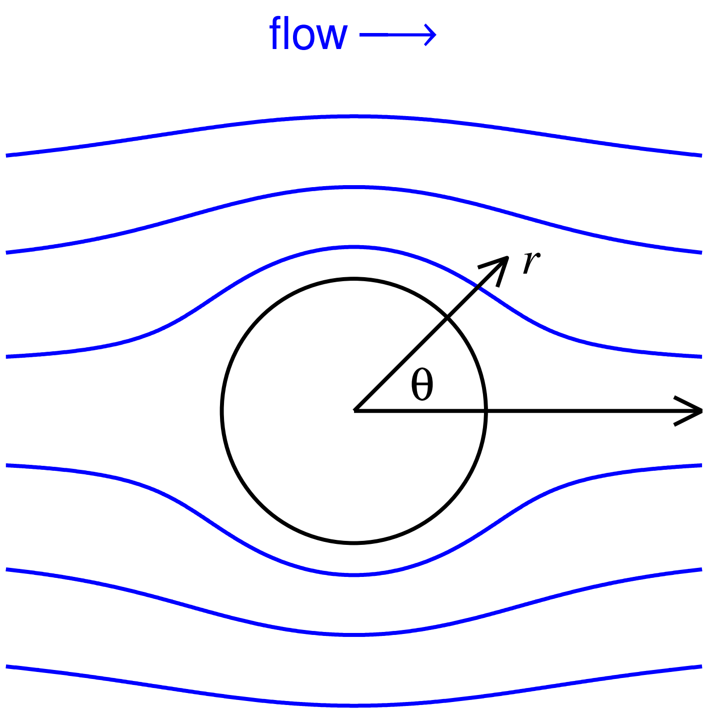


- Applying Bernoulli to the streamline AB, the pressure on sphere

$$P(\theta) = P_0 + \tfrac{1}{2}\rho V_0^2 - \tfrac{9}{8}\rho V_0^2 \sin^2\theta.$$

- There is high pressure at ‘stagnation points’ at front and rear of sphere on axis.
- The pressure is symmetrical front to back with this potential flow approximation, which implies there is no drag force.
- If the speed is high enough, at $\theta = \pm\pi/2$, then $P(\theta) < 0$, which is impossible, so the fluid undergoes ‘cavitation’ and bubbles form.
- For real flow, the fluid cannot slip past surface.
  - There is a ‘boundary shear layer’, which detaches from surface, causing a slow-moving wake.
  - The wake causes drag force (depends on value of viscosity via the dimensionless Reynolds number, $N_\mathrm{R} = (2a)v\rho/\eta$, see later discussion).
  - But the potential flow solution is quite good for rest of the system.

Flows Around a Cylinder



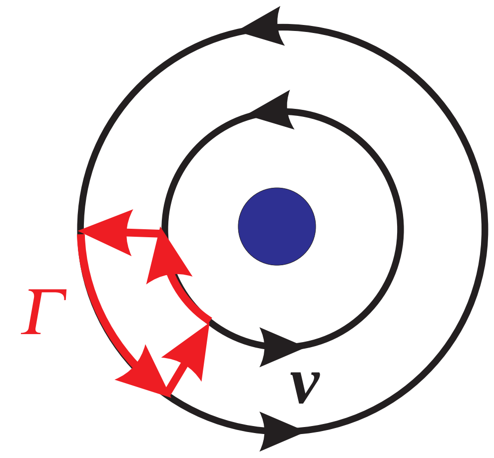
- Steady flow: $\Phi = V_0 r\cos\theta$ at large distance.
- Solution of $\nabla^2\Phi = 0$ that $\longrightarrow 0$ as $r \longrightarrow \infty$ are:

$$A\log r \quad + \quad \frac{B}{r}\cos\theta \quad + \quad \frac{C}{r^2}\cos 2\theta \ldots$$

line charge　　line dipole　　etc.

- Solution $\Phi = V_0 \cos\theta\left(r + \dfrac{a^2}{r}\right)$ satisfies

$$\frac{\partial\Phi}{\partial r} = 0 \text{ at } r = a\text{, i.e. } v_r = 0.$$

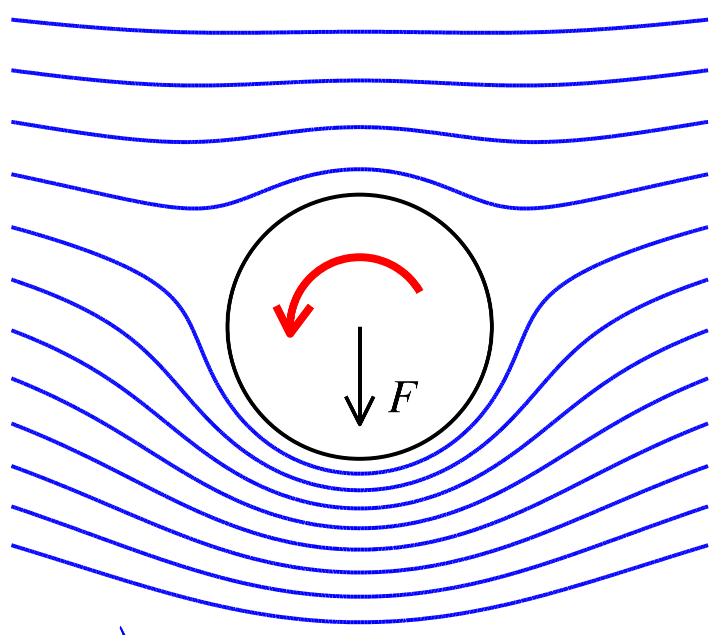


- A further irrotational flow around a cylinder is the **vortex** solution $\boldsymbol{v} = \dfrac{\kappa}{2\pi r}\widehat{\boldsymbol{e}}_\theta$.

- The circulation $\oint \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{l}$ around all loops Γ that **do not contain the cylinder** is zero. Hence $\nabla\times\boldsymbol{v} = 0$ everywhere in the fluid.

- The cylinder contains a rotating vortex of strength κ.

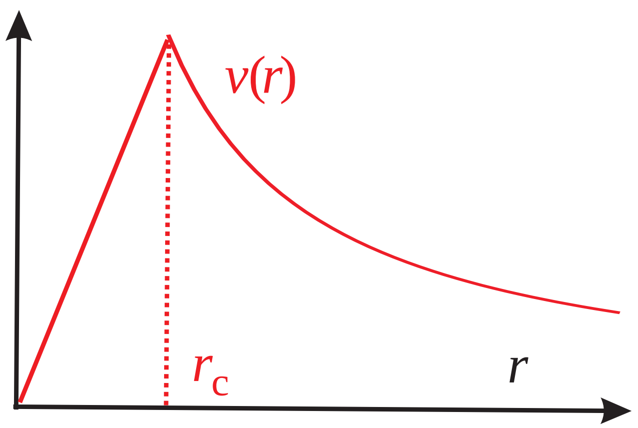
- The velocity potential for the flow is $\Phi = \dfrac{\kappa\theta}{2\pi}$, which is multi-valued ($\pm 2\pi n$, for integer n). It is a suitable potential for a rotating cylinder of radius a and angular velocity $\kappa/(2\pi a^2)$ when the velocity of the fluid is the same as the velocity of the surface of the cylinder.



- Add together steady flow and vortex:
  $$\Phi = V_0 \cos\theta \left(r + \frac{a^2}{r}\right) + \frac{\kappa\theta}{2\pi}.$$
- The velocity at $r = a$ is: $v_r = \dfrac{\partial \Phi}{\partial r} = 0$, and
  $$v_\theta = \frac{1}{r}\frac{\partial \Phi}{\partial \theta} = -2V_0 \sin\theta + \frac{\kappa}{2\pi a}.$$
- The pressure at the surface of the cylinder P is given by $\quad P + \tfrac{1}{2}\rho v_\theta^2 = P_0 + \tfrac{1}{2}\rho V_0^2,$
  so
  $$P = P_0 + \tfrac{1}{2}\rho V_0^2 - \tfrac{1}{2}\rho\left(4V_0^2 \sin^2\theta + \frac{\kappa^2}{4\pi^2 a^2} - \underbrace{\frac{2V_0\kappa\sin\theta}{\pi a}}_{\text{Asymmetric}}\right).$$
- This gives a net vertical force per unit length: $\displaystyle\int_0^{2\pi} \frac{\rho V_0 \kappa}{\pi a} a \sin^2\theta \, d\theta = \rho V_0 \kappa.$
- This is the $\boxed{\text{Magnus effect}}$ or $\boxed{\text{Magnus force}}$, which, using vectors, can be written $\boxed{\boldsymbol{F} = \rho \boldsymbol{V}_0 \times \boldsymbol{\kappa}.}$

Vortex Lines or Vortices

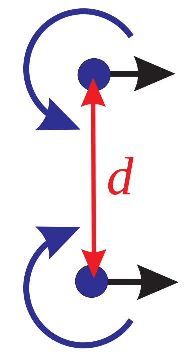
- It is also possible to have circulation in the pure fluid without a cylinder.
- However, if $\boldsymbol{v} = \dfrac{\kappa}{2\pi r}\widehat{\boldsymbol{e}}_\theta$, there would be a singularity at $r \longrightarrow 0$.
- Instead, the fluid can form a 'Helmholtz vortex'. This has a core of radius r_c which rotates as a solid body (with vorticity $\boldsymbol{\Omega} = \nabla \times \boldsymbol{v}$; note that the solid body angular speed of the core is $\omega_\mathrm{core} = \Omega/2$).



  1) for $r < r_\mathrm{c}$:
  $$\boldsymbol{v} = \frac{\kappa r}{2\pi r_\mathrm{c}^2}\widehat{\boldsymbol{e}}_\theta \text{ with } \kappa = \pi r_\mathrm{c}^2 \Omega.$$
  2) for $r > r_\mathrm{c}$:
  $$\boldsymbol{v} = \frac{\kappa}{2\pi r}\widehat{\boldsymbol{e}}_\theta.$$

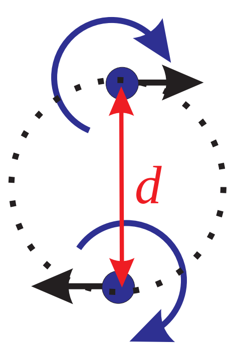
- This is analogous to the magnetic field around a thick wire carrying a uniform current.

- Kelvin's circulation theorem states: 'vortex lines are conserved and move with the fluid' so two parallel vortices should each move at the local velocity of the fluid created by the other vortex.



- Hence, two line vortices of opposite sign 'blow each other along' at $v_D = \dfrac{\kappa}{2\pi d}$.

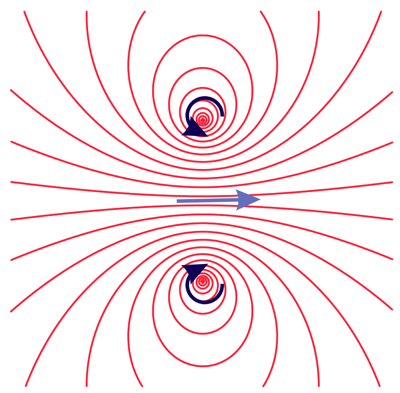
  In terms of forces, the Magnus force due to the drift of the vortices through the fluid is $\rho\boldsymbol{v}\times\boldsymbol{\kappa}$ (note that if the vortices are moving at $\boldsymbol{v}_D$ then the fluid velocity to use in this formula is $\boldsymbol{v} = -\boldsymbol{v}_D$). This force is balanced by the attraction between unlike vortices which is also due to a Magnus force in this case exerted on one of the vortices due to the velocity flow created by the other (cf. force between currents).



- Two vortices of same sign will circle around each other.

VORTEX RINGS

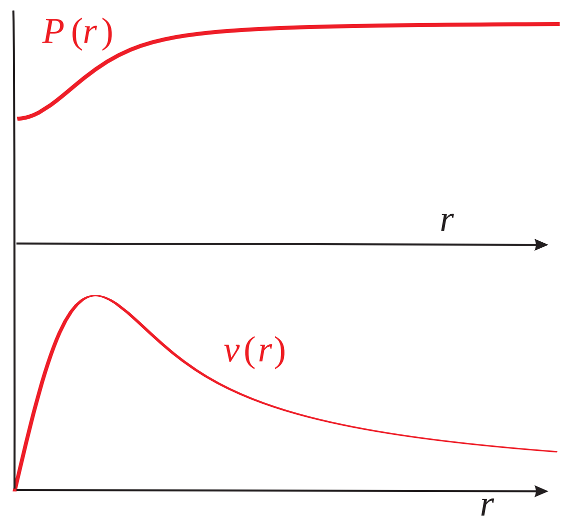
- It is also possible to make a **vortex ring**.
- A ring of of radius a drifts at $v_D = \dfrac{\kappa}{4\pi a} \log\left(\dfrac{a}{r_c}\right)$, where the core size r_c also determines how long the ring lasts before spreading.
- A vortex ring near a flat plate interacts with its image and its radius a increases.
- Can sometimes get vortex rings to pass through each other.



(**Note:** an exact solution of 'Navier–Stokes' equation for a vortex including viscosity gives:

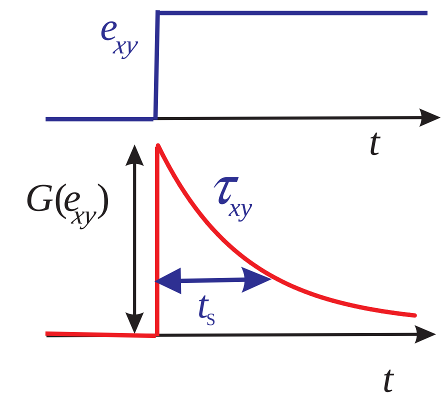
$$v_\theta(r) = \frac{K}{2\pi r}\left(1 - \exp\left(\frac{-r^2\rho}{4\eta t}\right)\right).$$

The core size increases $\propto t^{1/2}$ due to viscous diffusion.)

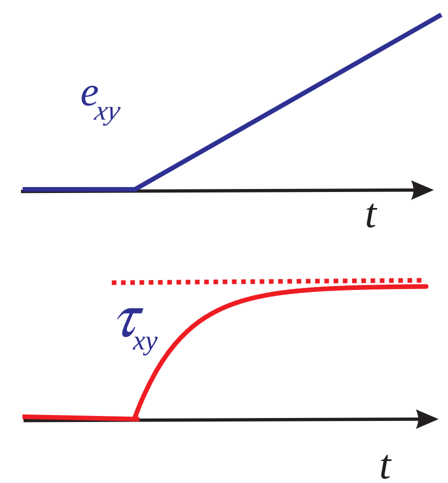


Real Fluids — Viscosity

- A sudden shear e_{xy} in a fluid produces a stress $\tau_{xy} = G(2e_{xy})$ that decays over a timescale t_s.



- To maintain a shear stress in a fluid, it has to be continuously sheared.
- The proportionality between the stress and the rate of shear is called the **viscosity** $\tau_{xy} = \eta\left(2\dfrac{\mathrm{d}e_{xy}}{\mathrm{d}t}\right)$. It is related to G and t_s by $\eta = Gt_\mathrm{s}$.

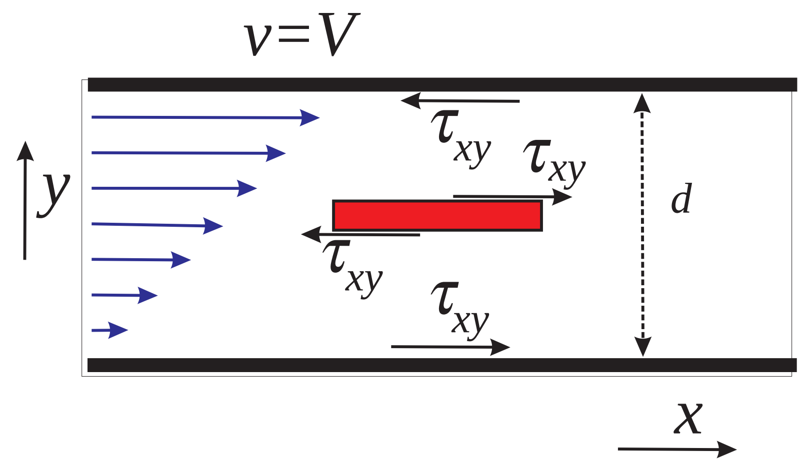


- Viscous stresses appear when there is a spatial variation of velocity:

$$2e_{xy} = \frac{\partial X}{\partial y} + \frac{\partial Y}{\partial x} \text{ so } 2\frac{\mathrm{d}e_{xy}}{\mathrm{d}t} = \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}.$$

- **Definition of viscosity**: for shear flow between two flat plates at $y = 0$ and $y = d$,

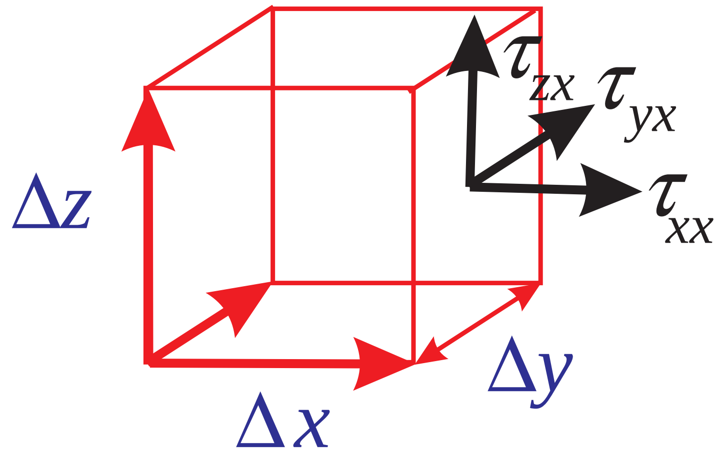
$$\boxed{\tau_{xy} = \frac{\text{Force}}{\text{Area}} = \eta \frac{\partial v_x}{\partial y}.}$$



- For a uniform velocity gradient, as shown, this gives a viscous stress $\tau = \eta \frac{V}{d}$.

- The viscosity is given by the force per unit area, per unit velocity gradient.

Viscous Forces and the Equation of Motion



- For **Newtonian fluids** the stress is linearly related to the rate of strain

$$\tau_{ij} = \eta \frac{\mathrm{d}(2e_{ij})}{\mathrm{d}t} = \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right).$$

- The variation of stresses across a volume element gives a force per unit volume

$$\sum_j \frac{\partial \tau_{ij}}{\partial x_j} = \eta \sum_j \left(\frac{\partial^2 v_i}{\partial x_j \partial x_j} + \frac{\partial^2 v_j}{\partial x_i \partial x_j} \right).$$

- In vector notation we have the equation of motion including viscosity:

$$\rho \frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\nabla P + \rho \boldsymbol{g} + \eta \left(\nabla^2 \boldsymbol{v} + \nabla(\nabla \cdot \boldsymbol{v}) \right).$$

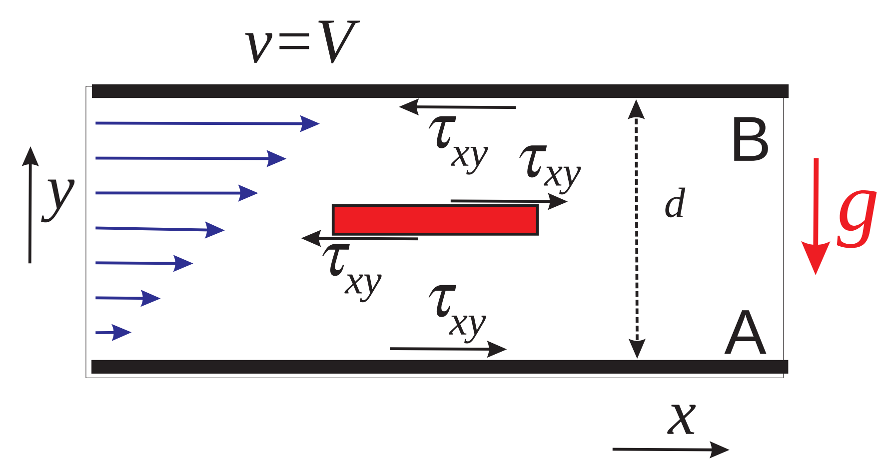
- For incompressible fluids $\nabla \cdot \boldsymbol{v} = 0$, so the additional viscous term is just $\eta \nabla^2 \boldsymbol{v}$.

(**Note:** compressible fluids also have a resistance to volume changes and have a bulk viscosity η', so the viscous force is $\eta \nabla^2 \boldsymbol{v} + (\eta + \eta')\nabla(\nabla \cdot \boldsymbol{v})$.)

VISCOUS SHEAR LAYER

- Equation of motion $\rho \dfrac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\nabla P + \rho \boldsymbol{g} + \eta \nabla^2 \boldsymbol{v}$.

- Simple case — shear flow between two flat plates at $y = 0$ and $y = d$.



- Steady flow: $0 = -\nabla P + \rho \boldsymbol{g} + \eta \nabla^2 \boldsymbol{v}$.

- The equation of motion is a **vector** equation:
  vertically $\dfrac{\mathrm{d}P}{\mathrm{d}y} = -\rho g$; horizontally $\eta \dfrac{\mathrm{d}^2 v_x}{\mathrm{d}y^2} = 0$.

- Vertical forces imply $P(y) = P(0) - \rho g y$ (as expected).

- Horizontal forces give: $v_x = A + By$, where A and B are constants.

- Boundary condition: no slip at either plate $v_x = \dfrac{Vy}{d}$.

- In steady flow the horizontal forces on a fluid element are zero, so the stresses on either side of the element must balance: $\tau_{xy} = \eta \dfrac{\mathrm{d}v_x}{\mathrm{d}y} = \eta V/d$.

- The stress $= \eta V/d$ is constant in y and is transmitted to both plates.

POISEUILLE FLOW — DRAINING PLATE

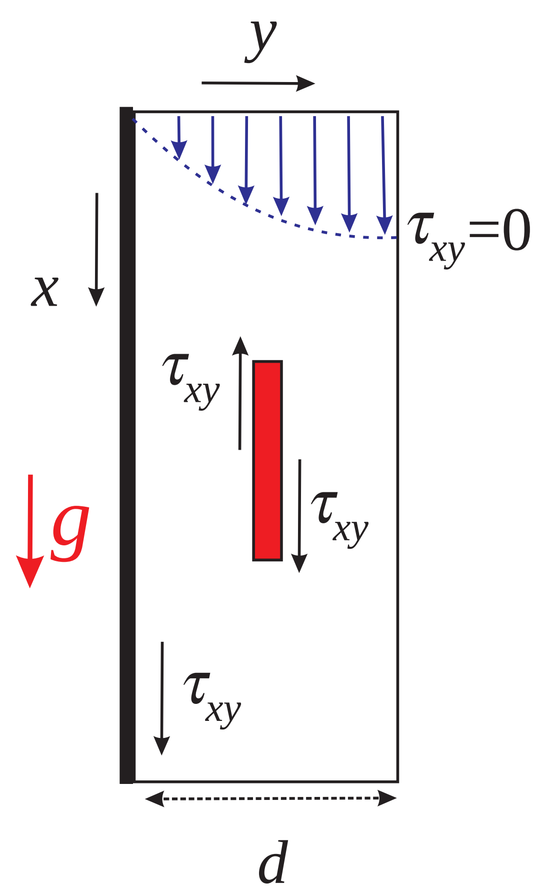
- Turn plate to vertical with free edge (keeping x longitudinal and y transverse). This situation models fluid draining from vertical surface.
- Pressure is equal to P_0 everywhere (ignoring density of air).
- The net viscous force on an element must now balance gravity:

  $$\mathrm{d}\tau_{xy} = -\rho g \, \mathrm{d}y, \text{ so that } \frac{\mathrm{d}\tau_{xy}}{\mathrm{d}y} = \eta \frac{\mathrm{d}^2 v_x}{\mathrm{d}y^2} = -\rho g.$$

- The solution is:

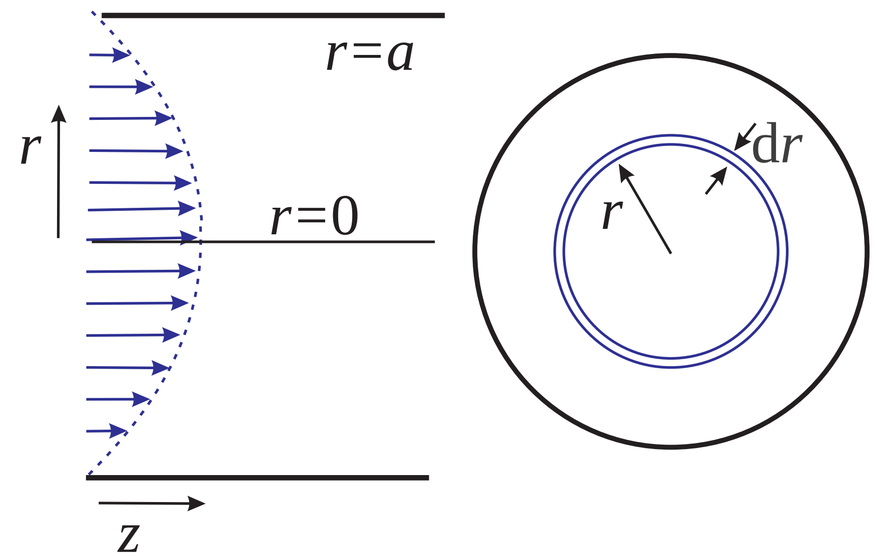
  $$\eta \frac{\mathrm{d}v_x}{\mathrm{d}y} = A - \rho g y,$$

  where A is a constant.
- Boundary condition: the stress must vanish at $y = d$, so $A = \rho g d$.



- Integrating and applying the boundary condition $v_x = 0$ at $y = 0$ gives
$$v_x = \frac{g\rho}{\eta}\left(yd - \tfrac{1}{2}y^2\right).$$
- This gives the characteristic parabolic (Poiseuille) flow profile.
- The speed at the surface is $\dfrac{g\rho d^2}{2\eta}$.
- The total volume flow rate per unit length along the surface is
$$\int_0^d v_x \,\mathrm{d}y = Q = \frac{g\rho d^3}{3\eta}.$$

Poiseuille flow in a pipe



- Flow in a pipe of circular cross-section with a longitudinal pressure gradient.
- Consider a fluid element between radii r and $r+\mathrm{d}r$ and length ℓ.
- The net pressure force on the ends is
$$-\frac{\mathrm{d}P}{\mathrm{d}z}(2\pi r\ell\,\mathrm{d}r)$$
towards $+z$ (but note, in this case $\mathrm{d}P/\mathrm{d}z$ is negative).
- For steady state flow this balanced by the viscous forces on the inside and outside of the element. The viscous force on the inside is $F_\mathrm{v} = \underbrace{(2\pi r\ell)}_{\text{area}}\eta\dfrac{\mathrm{d}v_z}{\mathrm{d}r}$.
- The net viscous force is $\left.\dfrac{\mathrm{d}F_\mathrm{v}}{\mathrm{d}r}\right|\mathrm{d}r$, towards $+z$, so $\dfrac{\mathrm{d}}{\mathrm{d}r}\left(2\pi r l\eta\dfrac{\mathrm{d}v_z}{\mathrm{d}r}\right) = 2\pi r l\dfrac{\mathrm{d}P}{\mathrm{d}z}$

(note that both $\mathrm{d}v_z/\mathrm{d}r$ and $\mathrm{d}P/\mathrm{d}z$ are negative).

- Integrating, then

$$r\eta\frac{\mathrm{d}v_z}{\mathrm{d}r} = \frac{r^2}{2}\frac{\mathrm{d}P}{\mathrm{d}z} + \underbrace{\text{constant}}_{=0}$$

(the constant of integration is zero since there is no stress at $r = 0$).

- Integrating again, and setting $v_z = 0$ at $r = a$ gives

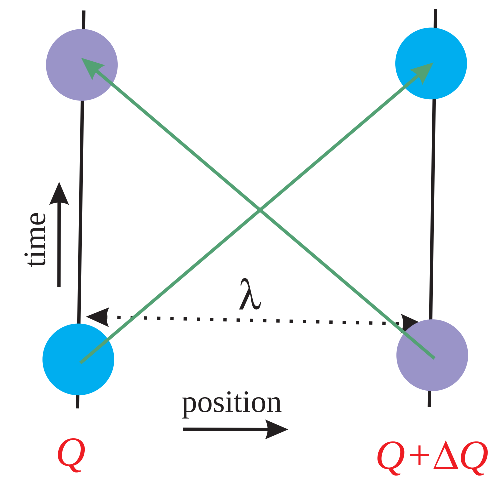
$$v_z = \frac{1}{4\eta}\left|\frac{\mathrm{d}P}{\mathrm{d}z}\right|\left(a^2 - r^2\right),$$

note the change to $|\mathrm{d}P/\mathrm{d}z|$.

- The total volume flow rate is $Q = \int_0^a 2\pi r v_z\,\mathrm{d}r = \frac{\pi a^4}{8\eta}\left|\frac{\mathrm{d}P}{\mathrm{d}z}\right|$.

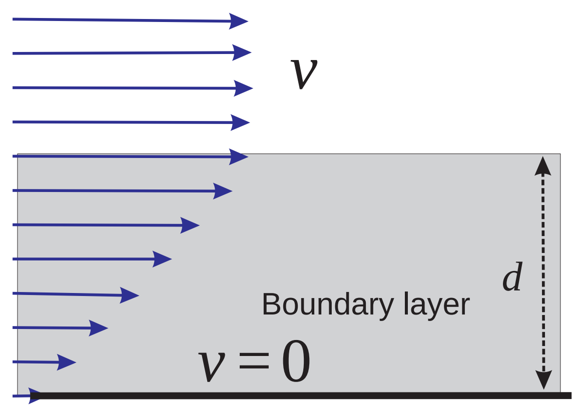
COLLISIONAL PROCESSES: VISCOSITY AND THERMAL CONDUCTIVITY

- Viscosity and thermal conductivity in fluids are determined by the extent to which the particle distributions at different spatial position can interpenetrate; i.e. it depends on the collisional mean free path λ_c.
- Consider the transport of a quantity Q which varies with position.
- Q might be the specific energy u (thermal conductivity) or a component of the momentum $\rho\boldsymbol{v}$ (viscosity).
- Transport of a Q then occurs by exchange of amount ΔQ, travelling a distance λ_c at velocity v_T.



- This random walk leads to a diffusion equation for property Q, in the frame moving with the fluid element: $\frac{1}{3}\lambda_c v_T \nabla^2 Q = \dfrac{DQ}{Dt}$.
- The factor of $\frac{1}{3}$ in the diffusion coefficient accounts for the fact that the blob is free to move in any of the three dimensions.
- From the form of the equation of motion of viscous fluids we conclude that the **kinematic viscosity** $\eta/\rho = \frac{1}{3}\lambda_c v_T$ (this is sometimes denoted by $\nu = \eta/\rho$).

BOUNDARY LAYERS



- Boundary condition for surface of solid body: no radial or tangential velocity (no slip).
- Flow past a solid body must develop a boundary layer with a velocity gradient. This is a region with vorticity ($\nabla \times \boldsymbol{v} \neq 0$), which can then enter the fluid flow.
- The ratio of the inertial stress ρv^2 to the viscous stress $\eta v / L$ is an important quantity, the **Reynolds number** $N_{\mathrm{R}} = \rho v L / \eta$ (v and L are characteristic velocity and length scales).
- In steady **laminar flow**, the kinematic viscosity is given by $\eta / \rho \sim \lambda_{\mathrm{c}} v_{\mathrm{T}}$, where λ_{c} is the mean free path and v_{T} is the thermal velocity.
- If the inertial stress is too high in a flow random transverse motions will cause turbulent mixing, which in turn increases the effective viscosity to include an 'eddy viscosity' $(\eta / \rho)_{\text{effective}} \sim \lambda_{\mathrm{c}} v_{\mathrm{T}} + L_{\text{eddy}} v_{\text{eddy}}$.
- Turbulence transports energy to smaller scales (and to larger scales) by splitting up of eddies on a timescale $L_{\text{eddy}} / v_{\text{eddy}}$. Energy is rapidly transported by nonlinear processes to the smallest scales, where viscous dissipation can occur.

Real Flow Past Sphere

- The flow of fluid past a sphere is a complicated phenomenon, displaying a great variety of behaviour, but dimensional analysis allows us to analyse the problem in terms of the dimensionless Reynolds number.
- With an assumption of an incompressible, viscous fluid, the flow around a sphere can only depend on the density of the fluid ρ, the viscosity η, the diameter of the sphere d and the upstream velocity of the flow v_0.
- Thus we expect that the dimensionless **Reynolds number** $N_\mathrm{R} = \dfrac{\rho v_0 d}{\eta}$ can be used to characterise the behaviour of the system.

- Consider the drag force F on the sphere.
- We can write this as any combination of the parameters $\{\rho, \eta, v_0, d\}$ that has the dimension of force, multiplied by a dimensionless function,
- It is conventional to use

$$F = \underbrace{\rho d^2 v_0^2}_{\text{a force}} \times \underbrace{C_\mathrm{D}(N_\mathrm{R})}_{\text{dimensionless}} ,$$

where C_D is the **drag coefficient**, which is a function of N_R, the Reynolds number.

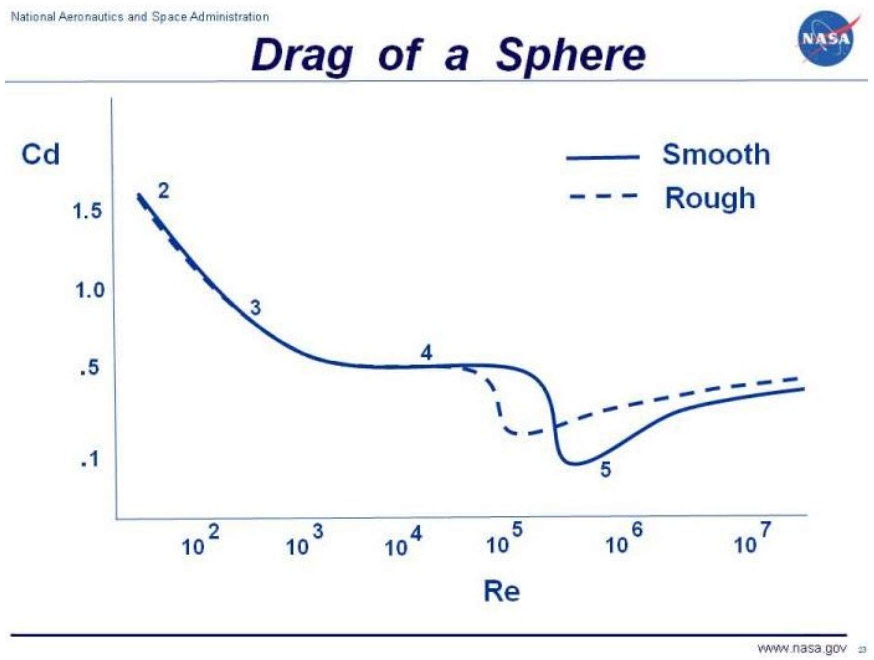
- Consider the extreme situations.

1) Low N_R — viscosity dominates — the drag force does not depend on ρ. We expect $F \propto \eta d v_0$, i.e. the drag coefficient $C_D \propto 1/N_\mathrm{R}$.

   Here the drag force is given by George Stokes' famous formula $F = 3\pi\eta d v_0$.

2) High $N_\mathrm{R} \gg 10^6$ — inertial effects dominate — the drag force does not depend on η. We expect $F \propto \rho d^2 v_0^2$, i.e. the drag coefficient is a constant.

EXPERIMENTAL FLOW PAST SPHERE



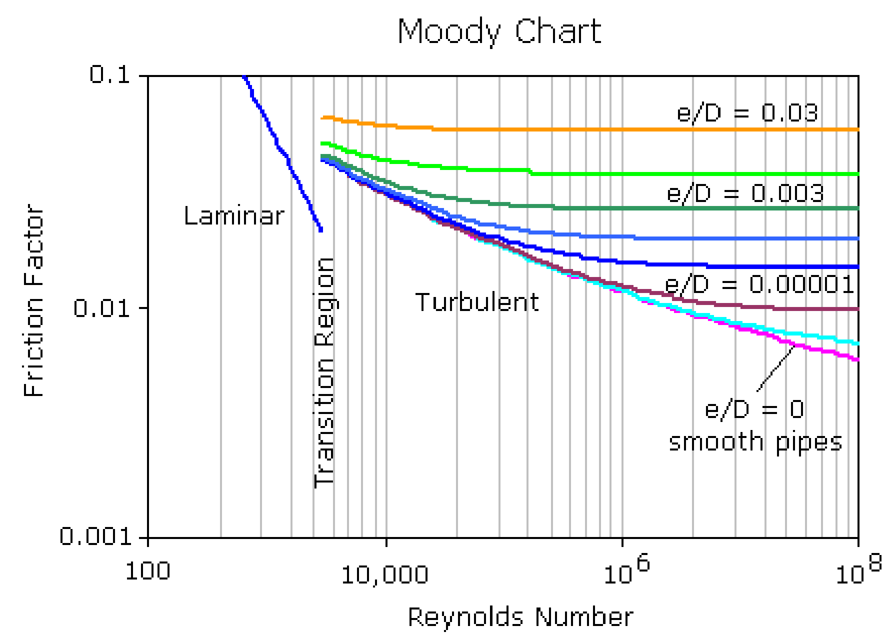
`http://www.grc.nasa.gov/WWW/k-12/airplane/dragsphere.html`

- The plot of the drag coefficient as a function of Reynolds number shows the effect of the transition from laminar flow to turbulent flow.
- For smooth sphere the drag coefficient suddenly decreases by a significant factor due to turbulence at a value around $N_{\mathrm{R}} \approx \text{few} \times 10^5$).
- Golf balls exploit this by having dimples that cause the transition to occur at slightly lower Reynolds number.

LAMINAR AND TURBULENT FLOW IN PIPES

- Poiseuille flow in pipe of circular cross-section, radius a, with longitudinal pressure gradient. The friction factor f defined by $\left|\frac{\mathrm{d}P}{\mathrm{d}z}\right| \equiv f\frac{\rho\overline{v}^2}{2a}$ (for mean flow $\overline{v} = \frac{Q}{\pi a^2}$). For laminar flow the flow rate is $Q = \frac{\pi a^4}{8\eta}\left|\frac{\mathrm{d}P}{\mathrm{d}z}\right|$, so $f = 32/N_{\mathrm{R}}$, where $N_{\mathrm{R}} = \rho(2a)\overline{v}/\eta$.

- Flow becomes turbulent at $N_{\mathrm{R}} \approx 2000$. After that $f \approx 0.03$ for turbulent flow. The friction factor also depends on the roughness of the walls of the pipe (denoted by e/D on plot).



http://www.efunda.com/formulae/fluids/calc_pipe_friction.cfm