BOUND/UNBOUND ORBITS

A previously derived result, slide 48, is

$$r = \frac{r_0}{1 + e\cos\phi},$$

where e is the *eccentricity* of the orbit.

Also, results on slide 54, combine to give

$$e^2 = 1 + \frac{2EJ^2}{mA^2} \qquad \text{(with } A = GMm\text{)}.$$

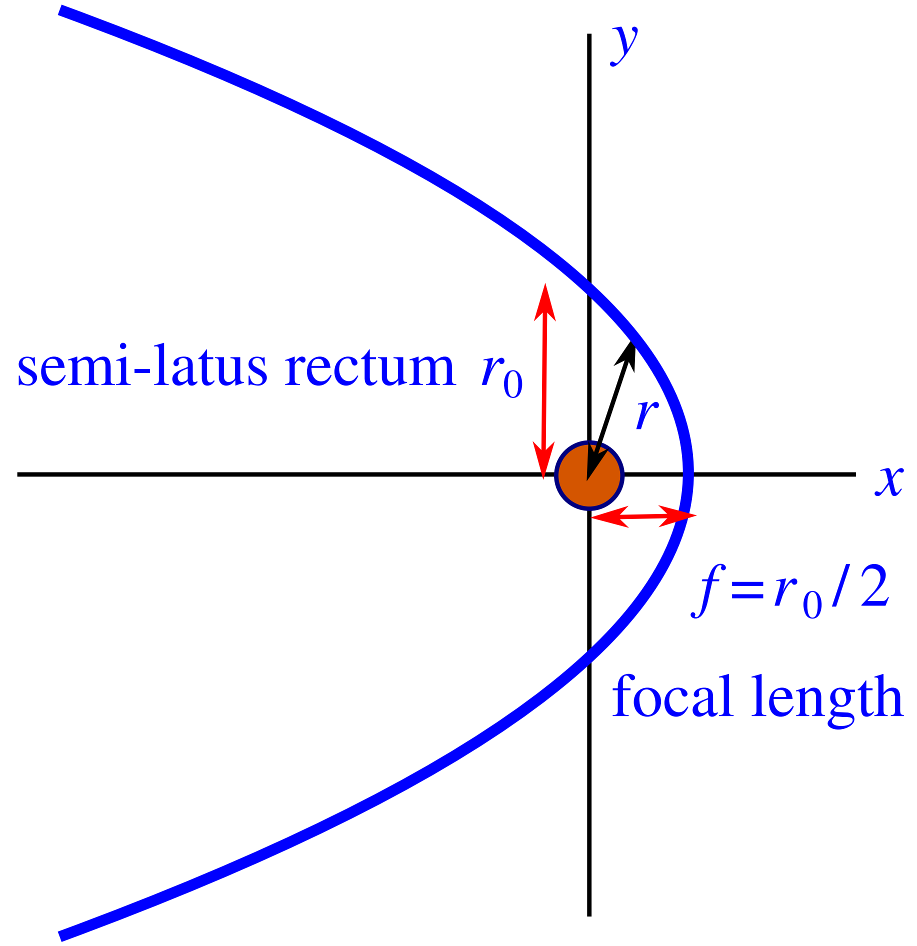
- If $e = 0$, it is a bound circular orbit;
  the total energy is a specific negative value.
- if $0 < e < 1$, it is bound elliptical orbit;
  the total energy E is negative.
- if $e = 1$, it is an unbound parabolic orbit;
  the total energy is $E = 0$.
- if $e > 1$, it is an unbound hyperbolic orbit;
  the total energy E is positive.

Parabolic orbits

That is $E = 0$, and $e = 1$ (so $r_{\text{min}} = \frac{1}{2}r_0$).

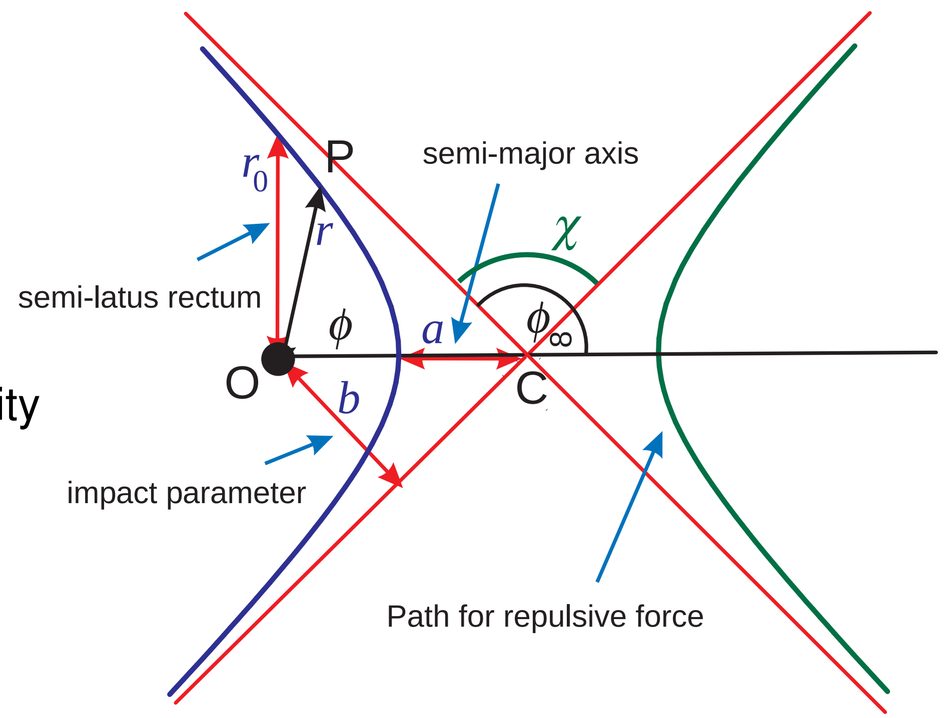
- So $r = r_0 - r\cos\phi = r_0 - x$.
- And $y^2 = r^2 - x^2 = r_0^2 - 2r_0 x$.
- Putting the 'focal length' $f = \frac{1}{2}r_0$, then

  $$y^2 = 4f(f - x).$$



Hyperbolic Orbits

- **Attractive potential**: all previous formulae still valid, but $e > 1$ so $a < 0$ and energy
$$E = -\frac{A}{2a} = \frac{(e^2-1)A}{2r_0} > 0.$$
- The **impact parameter** b and velocity at infinity v_∞, determine angular momentum $J = mbv_\infty$, and energy $E = \frac{1}{2}mv_\infty^2$.
- Most problems require the total angle of deflection $\chi = 2\phi_\infty - \pi$ (with χ positive).



- Asymptotes are at $\pm\phi_\infty$; from the equation of the conic section (valid for $e < 1$ or $e > 1$),
$$\cos\phi_\infty = -1/e \quad \text{so} \quad \sec\phi_\infty = -e \quad \text{so} \quad \tan^2\phi_\infty = e^2 - 1.$$

Note: that $\pi/2 < \phi_\infty < \pi$.

The asymptotes can also be obtained from the equation of the orbit in Cartesian coordinates.

- Re-arranging a previous result, slide 50, gives

$$(1-e^2)\left(x+\frac{er_0}{1-e^2}\right)^2+y^2=\frac{r_0^2}{1-e^2}.$$

- But now $e>1$, so re-write this as

$$(e^2-1)\left(x-\frac{er_0}{e^2-1}\right)^2-y^2=\frac{r_0^2}{e^2-1} \quad \text{or} \quad \frac{(x-x_c)^2}{X^2}-\frac{y^2}{Y^2}=1,$$

  where $X=r_0/(e^2-1)$ and $Y=r_0/\sqrt{e^2-1}$, and $x_c=eX$.

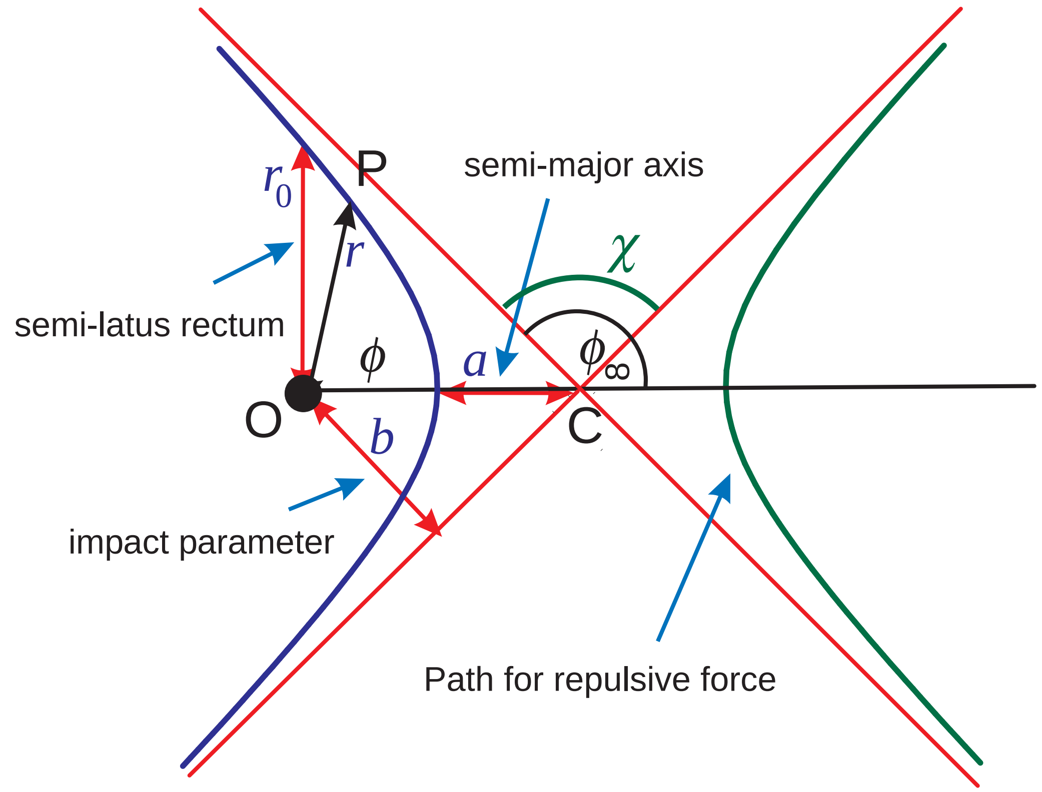
- The distance of closest approach is $x_c-X=r_0/(e+1)=a(1-e)=|a|(e-1)$, as expected.

- For large $|x|$, $|y|$, then $\dfrac{x^2}{X^2}\approx\dfrac{y^2}{Y^2}$, so

  $y=\pm(Y/X)x$, i.e. the gradient is $\pm(Y/X)=\pm\sqrt{e^2-1}$, as before.

Attractive potential

- The eccentricity can be found from the physical parameters E and J.
- From the definition of $r_0 = \dfrac{J^2}{mA}$, write $(e^2-1) = \dfrac{2r_0E}{A} = \dfrac{2J^2E}{mA^2}$.
- In terms of b and v_∞ this means
$$\tan^2\phi_\infty = e^2 - 1 = \frac{m^2v_\infty^4b^2}{A^2}.$$
- So $|\tan\phi_\infty| = \dfrac{mv_\infty^2 b}{A}$.



Repulsive potential:

For Rutherford scattering, for example.

- Change $A/r^2 \rightarrow -B/r^2$, define $J^2 = Bmr_0$ and use other branch $r_0 = r(e\cos\phi - 1)$.
- a is positive again and the total angle of deflection is now $\chi = \pi - 2\phi_\infty$ (χ negative).
- The distance of closest approach is $a(1+e)$.
- The asymptotes are still related to the physical parameters by $|\tan\phi_\infty| = \dfrac{mv_\infty^2 b}{B}$.

Note: $\frac{1}{2}\chi = \phi_\infty - \frac{1}{2}\pi$, so this can be written in terms of $\cot(\frac{1}{2}\chi) = \tan\phi_\infty$.

Or, you can more simply derive this result by considering the change of momentum along the initial direction of motion:

- This is $\Delta p = mv_\infty(\cos\chi - 1)$, which is due to the integrated effect of the repulsive force

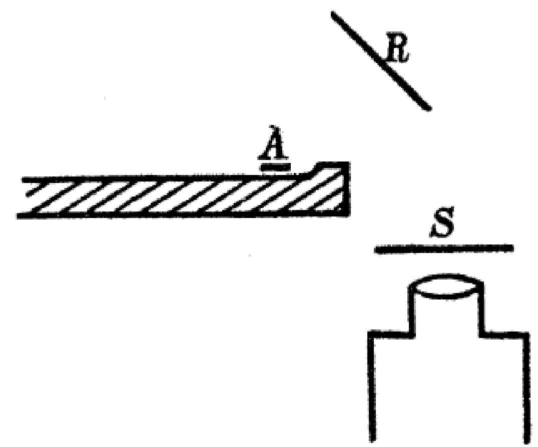
  i.e. $\displaystyle\int F\,\mathrm{d}t$

  where F is the appropriate *component* of the repulsive force.

$\boxed{\text{RUTHERFORD SCATTERING}}$ (New content in this section is NON-EXAMINABLE)

Evidence of ‘backscattering’ of α particles from various metals — Geiger & Marsden 1909, ‘On a Diffuse Reflection of the α-Particles’, Proceedings of the Royal Society of London, Series A, **82**, 495.

- ‘A’ is source of α-particles (shielded from detector by lead);
  ‘R’ is thin metal foil;
  ‘S’ is detector (a ‘scintillator’).
- Backscattering from A $\longrightarrow$ R $\longrightarrow$ S
  was observed with various metal foils.



1. Metal.	2. Atomic weight, A.	3. Number of scintillations per minute, Z.	4. A/Z.	$Z/A^{3/2}$ ×10^4
Lead	207	62	30	208
Gold	197	67	34	242
Platinum...........	195	63	33	232
Tin	119	34	28	226
Silver	108	27	25	241
Copper..............	64	14·5	23	225
Iron	56	10·2	18·5	250
Aluminium	27	3·4	12·5	243

Note: confusingly, column 4 is actually Z/A ×100.

Explained by Rutherford 1911, ‘The scattering of α and β particles by matter and the structure of the atom’, Philosophical Magazine, Series 6, **21**, 669.

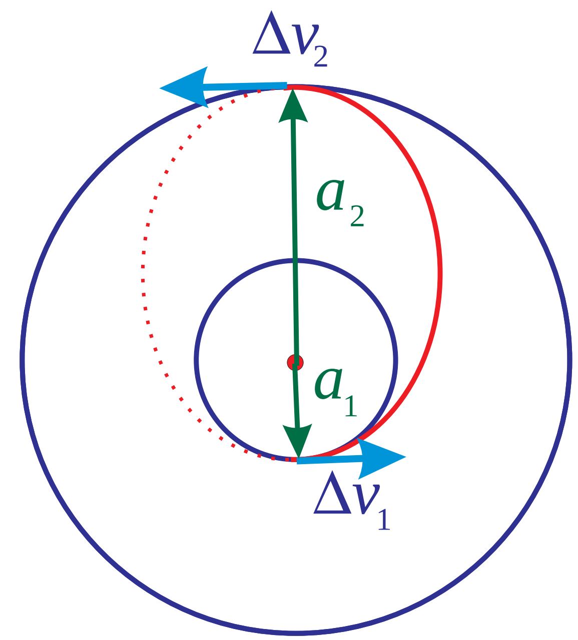
- For α particles of 2 MeV (i.e. speed of 2×10^{7} m s^{-1}), gold foil thickness 4×10^{-7} m, chance of deflection $>90^{\circ}$ was 1 in 20000.
- Forces required for large deflections only possible with a nuclear model of an atom, i.e. with all positive charge in a nucleus (the alternate ‘plum pudding’ model, with positive charges distributed throughout an atom does not work).
- Also, Rutherford’s theory predicted $Z/A^{3/2}$ should be constant, which is supported by the observations.
- Distance of closest approach, where theory still works $\approx 2.8\times10^{-14}$ m gives upper limit to size of the nucleus (note: the wavelength of α particles is about 0.5×10^{-14} m, so the classical model is just OK).

To quote Rutherford:

> “As if you fired a 15-inch naval shell at a piece of tissue paper and the shell came right back and hit you.”

THE HOHMANN TRANSFER ORBIT

- The Hohmann transfer orbit is one half of an elliptic orbit that touches both the initial circular orbit and the desired circular orbit. Point 1 is at the minimum radius a_1, and point 2 is at the maximum radius a_2, as shown.



- For gravitational case put $A = GMm$.

- In circular orbits $T = -E = -\frac{1}{2}U$, for elliptical orbits $\langle T \rangle = -\langle E \rangle = -\frac{1}{2}\langle U \rangle$ (this relationship between the average KE and average PE is the 'Virial Theorem').

- The initial energy is $E_1 = -\dfrac{GMm}{2a_1}$, and $v_1 = \sqrt{\dfrac{GM}{a_1}}$ is the speed of the initial circular orbit, which has to be increased until the spacecraft has the energy of the transfer orbit

$$E_t = -\frac{GMm}{a_1 + a_2}.$$

- The important thing to know is Δv_1 – the increase in speed required at point 1 — since that determines the amount of fuel used.
- Working from the energies:

$$E_{\rm t} = -\frac{GMm}{a_1+a_2} = -\frac{GMm}{a_1} + \tfrac{1}{2}mv_{{\rm t}_1}^2,$$

where $v_{{\rm t}_1}$ is the speed in the transfer orbit at point 1, so

$$\tfrac{1}{2}v_{{\rm t}_1}^2 = \frac{GMa_2}{a_1(a_2+a_1)}.$$

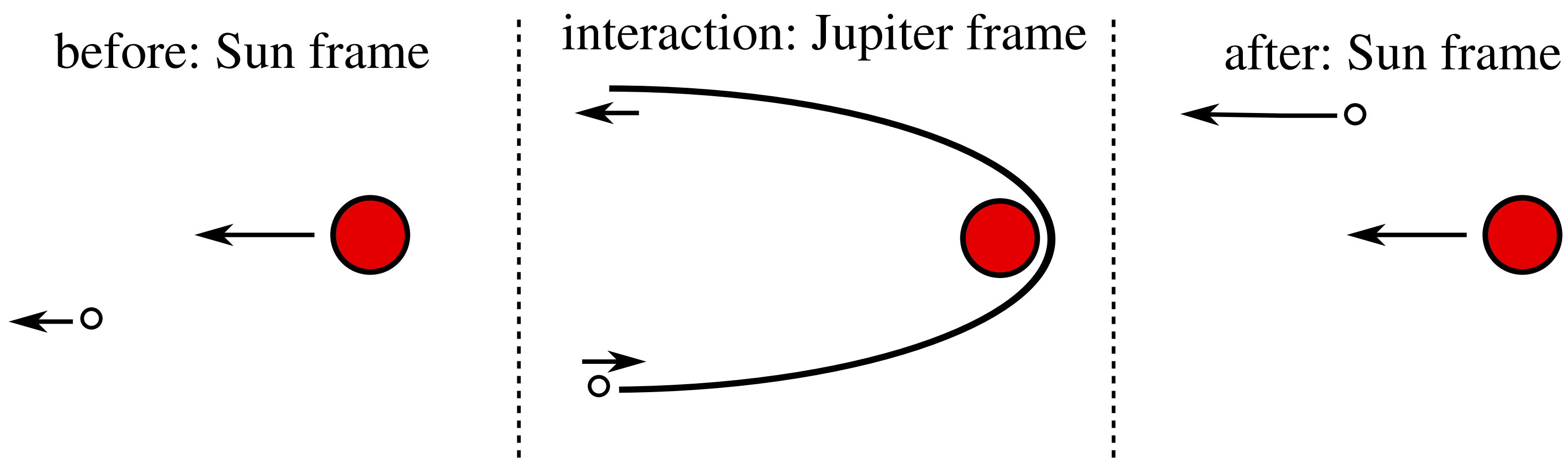
- So $\Delta v_1 = v_{{\rm t}_1} - v_1 = \sqrt{\dfrac{2GMa_2}{a_1(a_2+a_1)}} - \sqrt{\dfrac{GM}{a_1}}$.

To enter the larger circular orbit, a second burn will be required once the spacecraft reaches point 2 - this can be analysed in the same way.

The Hohmann transfer is the most fuel efficient orbit, unless there are other massive bodies in the vicinity, in which case you can use the gravitational 'slingshot'.

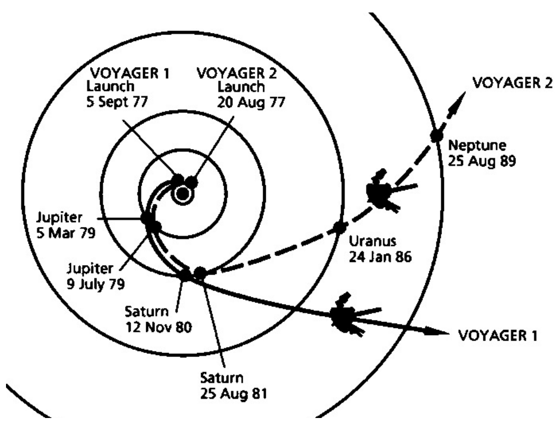
GRAVITATIONAL 'SLINGSHOT'

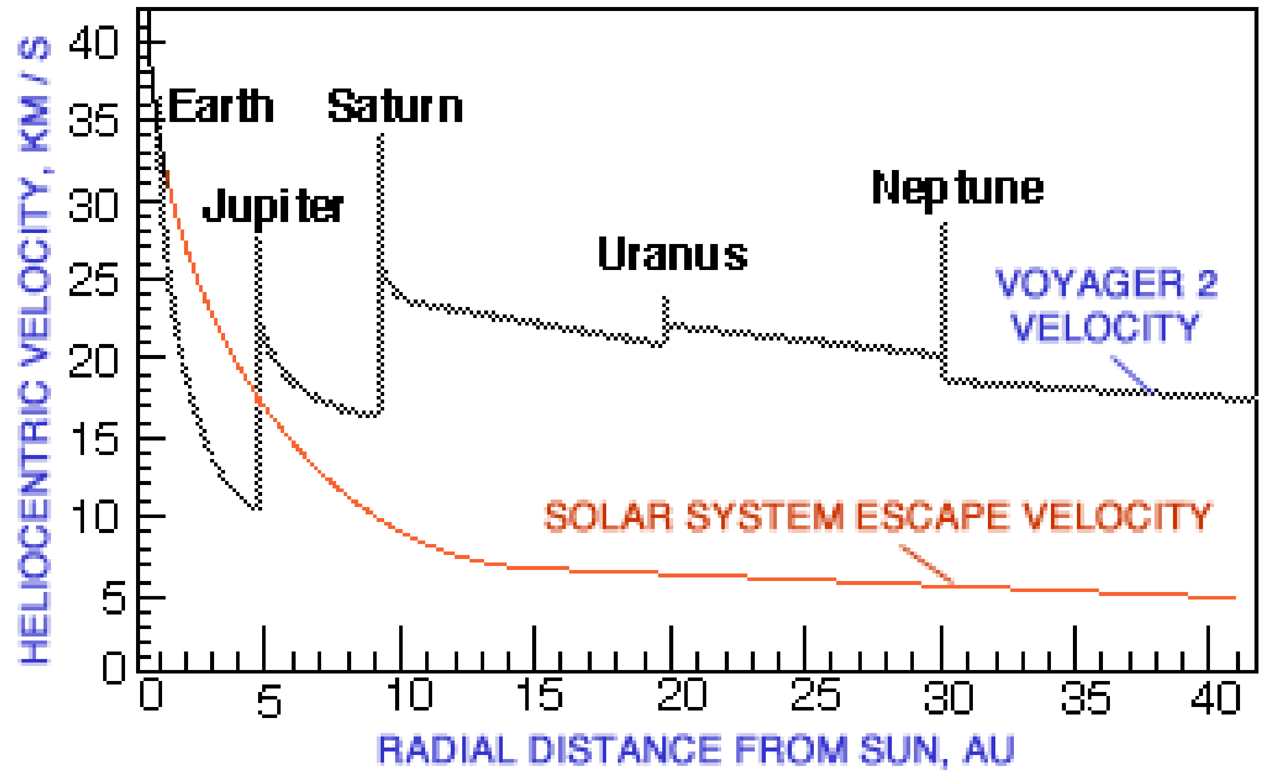
- The escape velocity of a spacecraft from the Solar system at the radius of the Earth's orbit is $\approx 42\ \mathrm{km\ s^{-1}}$, whereas the orbital velocity of the Earth is $\approx 30\ \mathrm{km\ s^{-1}}$.
- A gravitational 'slingshot' — or 'gravity assist' — around another planet can be used to increase kinetic energy and/or change direction of a probe in order to visit other bodies in the Solar system, or escape from the Solar system.
- For example, an interaction with Jupiter:



GRAVITATIONAL SLINGSHOT — VOYAGER 2

The Voyager 2 probe, launched in 1977, made a ‘grand tour’.



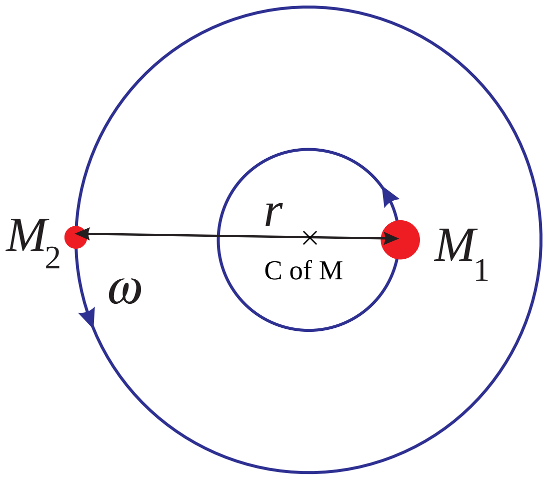


See: `http://www2.jpl.nasa.gov/basics/`

Note: the fly-by of Neptune was to provide a close view of the Neptune’s moon Triton, rather than to increase the speed.

The Two-Body Problem and 'Reduced Mass'

- In reality, the two masses M_1 and M_2 orbit their centre of mass.
- Each orbit is an ellipse in a common plane with the centre of mass at one focus.
  The ellipses have the same eccentricity and phase.
- One important case is circular motion.
  Mass M_1 is distance $\dfrac{M_2 r}{M_1+M_2}$ from the CoM.

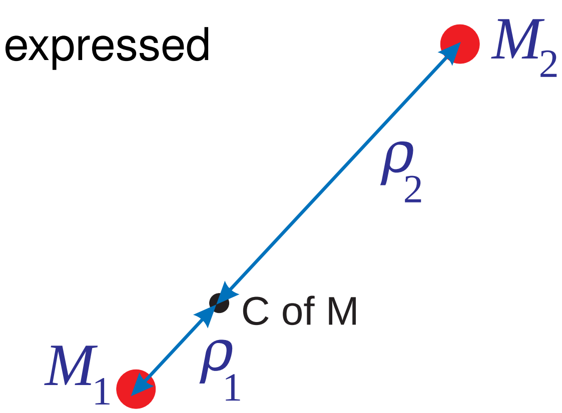


- Balance of forces for M_1: $\dfrac{GM_1M_2}{r^2} = M_1\omega^2\dfrac{M_2 r}{M_1+M_2}$
  so $\omega^2 = \dfrac{G(M_1+M_2)}{r^3}$.
- You get the same result by considering the balance of forces for M_2.
- The result can be rearranged as $\mu r\omega^2 = \dfrac{GM_1M_2}{r^2}$, which includes the true separation r and the actual force $\dfrac{GM_1M_2}{r^2}$, but a modified **reduced mass** term $\mu \equiv \dfrac{M_1M_2}{M_1+M_2}$.

THE TWO-BODY PROBLEM

- Two particles of masses M_1 and M_2 orbiting each other — positions $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$.

- The energy, angular momentum and equations of motion can be expressed in terms of the **reduced mass** $\mu \equiv \dfrac{M_1 M_2}{M_1 + M_2}$ and $\boldsymbol{r}_1 - \boldsymbol{r}_2$.



- The centre of mass is at $\boldsymbol{R}_0 = \dfrac{M_1 \boldsymbol{r}_1 + M_2 \boldsymbol{r}_2}{M_1 + M_2}$.

- Define

$$\begin{aligned}
\boldsymbol{\rho}_1 &\equiv \boldsymbol{r}_1 - \boldsymbol{R}_0 = \frac{M_2}{M_1 + M_2}(\boldsymbol{r}_1 - \boldsymbol{r}_2), \\
\boldsymbol{\rho}_2 &\equiv \boldsymbol{r}_2 - \boldsymbol{R}_0 = \frac{M_1}{M_1 + M_2}(\boldsymbol{r}_2 - \boldsymbol{r}_1).
\end{aligned}$$

- Kinetic energy in the centre of mass frame:

$$T = \tfrac{1}{2}M_1\dot{\boldsymbol{\rho}}_1^2 + \tfrac{1}{2}M_2\dot{\boldsymbol{\rho}}_2^2 = \frac{1}{2}\left(\frac{M_1M_2^2}{(M_1+M_2)^2} + \frac{M_1^2M_2}{(M_1+M_2)^2}\right)(\dot{\boldsymbol{r}}_1-\dot{\boldsymbol{r}}_2)^2 = \frac{\mu}{2}(\dot{\boldsymbol{r}}_1-\dot{\boldsymbol{r}}_2)^2.$$

- Angular momentum:

$$\boldsymbol{J} = M_1\boldsymbol{\rho}_1\times\dot{\boldsymbol{\rho}}_1 + M_2\boldsymbol{\rho}_2\times\dot{\boldsymbol{\rho}}_2 = \mu(\boldsymbol{r}_1-\boldsymbol{r}_2)\times(\dot{\boldsymbol{r}}_1-\dot{\boldsymbol{r}}_2).$$

- Equations of motion: $M_1\ddot{\boldsymbol{r}}_1 = \boldsymbol{F}_{12}$, $M_2\ddot{\boldsymbol{r}}_2 = \boldsymbol{F}_{21} = -\boldsymbol{F}_{12}$, and

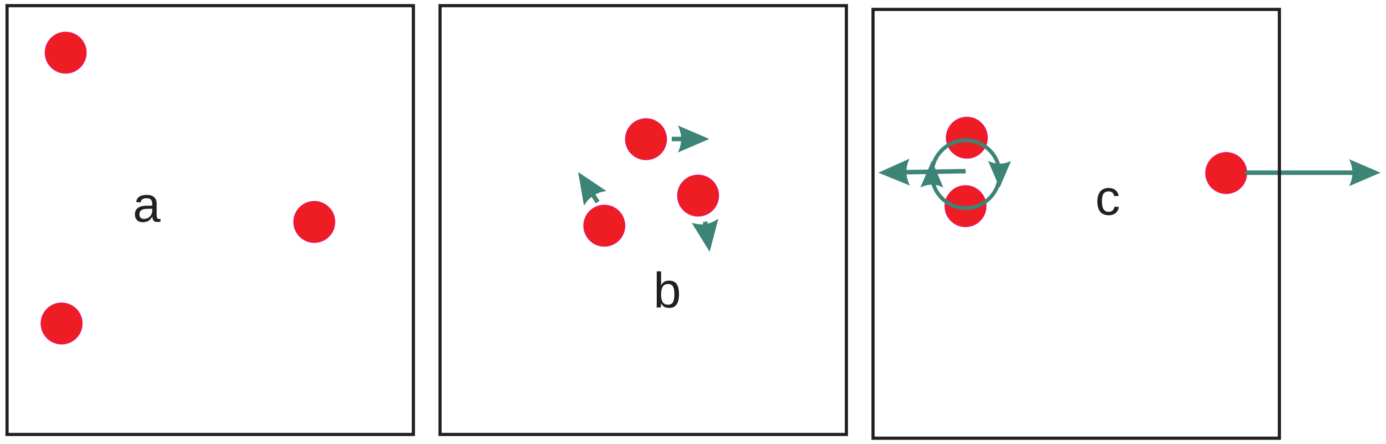
$$\ddot{\boldsymbol{r}}_1 - \ddot{\boldsymbol{r}}_2 = \left(\frac{1}{M_1}+\frac{1}{M_2}\right)\boldsymbol{F}_{12} = \frac{1}{\mu}\boldsymbol{F}_{12}, \qquad \ddot{\boldsymbol{R}}_0 = 0.$$

- The two-body problem has been reduced to the one-body problem in the centre of mass frame.

- This applies to all two-body problems (in constant external potentials).

THE THREE-BODY PROBLEM

- Some hierarchical systems can be stable indefinitely e.g. Sun, Earth and the Moon.
- A general 3-body encounter can be very complicated, but a general feature emerges.



- If 3 bodies are allowed to attract each other from a distance (a), they will speed up and interact strongly (b). Eventually the interaction is likely to form a close binary (negative gravitational binding energy) releasing kinetic energy, which may be enough for one of the bodies to escape to infinity (c).
- This mechanism is responsible for 'evaporation' of stars from star clusters.

GRAVITATIONAL POTENTIAL, FIELD AND TIDAL FORCES

There are three important aspects to gravitation — both Newtonian and General Relativity (GR).

1) Gravitational potential $\boxed{\phi(\boldsymbol{r}).}$ This determines energies, and is always relative — cannot have an absolute value.

- For a point mass:

$$\boxed{\phi = -\frac{GM}{R}.}$$

[Newtonian potential is one part of the metric of GR.]

2) Gravitational field $\boxed{\boldsymbol{g}(\boldsymbol{r}) = -\nabla\phi.}$ This determines accelerations and orbits. The field is also relative; e.g. nothing would change if our Galaxy (and other ‘Local Group’ galaxies) were all accelerating towards a ‘Great Attractor’.

- For point mass:

$$\boxed{|\boldsymbol{g}| = \frac{GM}{R^2}.}$$

[Gravitational field is one part of the ‘connection’ in GR].

3) Gravitational tidal field $\boxed{\boldsymbol{T}(\boldsymbol{a}) = \boldsymbol{a}\cdot\nabla\boldsymbol{g}}$ is the difference between the gravitational field offset by $\boldsymbol{a}$ from that at the reference point.

This is all one can feel and measure locally — it describes how the gravitational field varies in space.

$\boldsymbol{T}(\boldsymbol{a}) = \boldsymbol{a}\cdot\nabla\boldsymbol{g}$ [or equivalently $(\boldsymbol{a}\cdot\nabla)\boldsymbol{g}$] — is the scalar product of the change in position (i.e. $\boldsymbol{a}$) and the rate of change of the gravitational field with position.

The operator $\boldsymbol{a}\cdot\nabla$ is

$$a_x\frac{\partial}{\partial x} + a_y\frac{\partial}{\partial y} + a_z\frac{\partial}{\partial z}.$$

So, for the x-component

$$[\boldsymbol{T}(\boldsymbol{a})]_x = a_x\frac{\partial g_x}{\partial x} + a_y\frac{\partial g_x}{\partial y} + a_z\frac{\partial g_x}{\partial z},$$

or using the summation convention $[\boldsymbol{T}(\boldsymbol{a})]_i = T_{ij}a_j$ where $T_{ij} \equiv \dfrac{\partial g_i}{\partial x_j}$.

Since the gravitational field $\boldsymbol{g}$ varies with distance R as $1/R^2$ then the tidal field varies as $1/R^3$.

- For the gravitational field generated by a point mass M:

  a) Consider a reference position O, at a position $\boldsymbol{R}$ away from the mass, then

$$\boldsymbol{g}_{\mathrm{O}} = \frac{-GM}{R^2}\widehat{\boldsymbol{e}}_r.$$

Consider a point A offset by a small distance a from O along $\widehat{\boldsymbol{e}}_r$, then

$$\boldsymbol{g}_{\mathrm{A}} = \frac{-GM}{(R+a)^2}\widehat{\boldsymbol{e}}_r \approx \frac{-GM}{R^2}\left(1-2\frac{a}{R}\dots\right)\widehat{\boldsymbol{e}}_r,$$

(provided $a \ll R$).

So, the difference between $\boldsymbol{g}_{\mathrm{A}}$ and $\boldsymbol{g}_{\mathrm{O}}$ is

$$\frac{2aGM}{R^3}\widehat{\boldsymbol{e}}_r.$$

Or, for unit radial change in position,

$$\boldsymbol{T}(\widehat{\boldsymbol{e}}_r) = \frac{2GM}{R^3}\widehat{\boldsymbol{e}}_r.$$

b) Now consider a point B offset a small distance b along $\widehat{\boldsymbol{u}}_\theta$ (this corresponds to increasing θ

by $\Delta\theta = b/R$), then $|\boldsymbol{g}_\mathrm{B}| = |\boldsymbol{g}_\mathrm{O}|$ (for small $|b|$) with components

$$\begin{array}{rcll} -|\boldsymbol{g}_\mathrm{O}|\cos\Delta\theta & \approx & -|\boldsymbol{g}_\mathrm{O}| & \text{along } \widehat{\boldsymbol{e}}_r, \\ -|\boldsymbol{g}_\mathrm{O}|\sin\Delta\theta & \approx & -|\boldsymbol{g}_\mathrm{O}|\Delta\theta & \text{along } \widehat{\boldsymbol{e}}_\theta. \end{array}$$

So, the difference between $\boldsymbol{g}_\mathrm{B}$ and $\boldsymbol{g}_\mathrm{O}$ is

$$-|\boldsymbol{g}_\mathrm{O}|\Delta\theta\,\widehat{\boldsymbol{e}}_\theta = -\frac{GM}{R^2}\frac{b}{R}\widehat{\boldsymbol{e}}_\theta = -\frac{bGM}{R^3}\widehat{\boldsymbol{e}}_\theta.$$

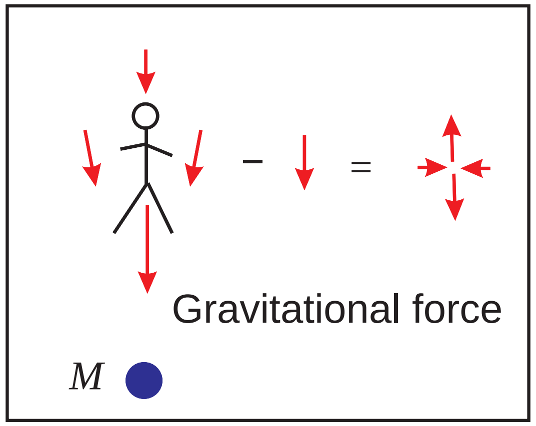
Or, for unit change in position along $\widehat{\boldsymbol{e}}_\theta$,

$$\boldsymbol{T}(\widehat{\boldsymbol{e}}_\theta) = \frac{-GM}{R^3}\widehat{\boldsymbol{e}}_\theta.$$

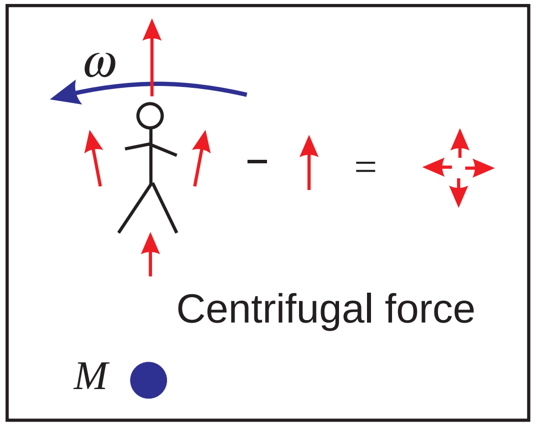
Similarly for a unit change in position along $\widehat{\boldsymbol{e}}_\phi$

$$\boldsymbol{T}(\widehat{\boldsymbol{e}}_\phi) = \frac{-GM}{R^3}\widehat{\boldsymbol{e}}_\phi.$$

- The gravitational field near a point mass is directed radially and is proportional to $1/r^2$. The tidal forces consist of a radial stretching proportional to $2GM/r^3$ per unit length and a sideways compression $-GM/r^3$.



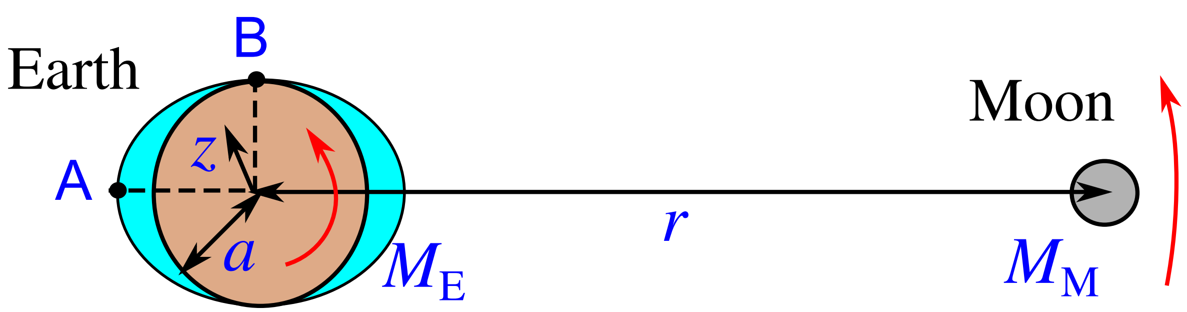
- Assuming the stick figure is corotating with the orbit (i.e. keeping the same relative orientation with respect to the mass M), then we must also add the contribution from the centrifugal force — this is a stretching in the two directions in the plane of rotation, and no contribution parallel to the rotation axis.
  - In this case, the sum of the tidal and centrifugal forces is a stretch proportional to $3GM/R^3$ per unit length along the radial direction, no contribution in the orbital plane and compression $-GM/R^3$ perpendicular to the plane.



- Tidal forces are weak on Earth and generally not very strong in the Solar system (except Jupiter and its moons), but tidal forces can be very large near compact objects such as neutron stars.

ORIGIN OF THE TIDES

Consider a simple model of tides from the Moon on an Earth, completely covered in water.



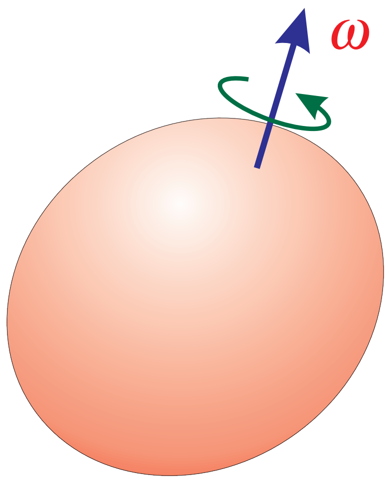
- In the orbital plane, the Earth rotates relative to the two bulges of water produced by the Moon's tidal forces. This gives two tides a day (which occur $\approx$ 1 hr later per day).
- The difference in the vertical tidal acceration a distance z from the Earth's centre is $2GM_\mathrm{M}z/r^3$ in the direction towards A, and $-GM_\mathrm{M}z/r^3$ towards B.
- Integrating gives the tidal potential $\phi_\mathrm{tide} = -\dfrac{3GM_\mathrm{M}a^2}{2r^3}$ at point A compared with B.
- So the height h of the tide at A compared with B is when $\phi_\mathrm{tide} = -gh$, where the surface gravity of the Earth is $g = GM_\mathrm{E}/a^2$.
- Eliminating g, then the height of tides is $h = \dfrac{3M_\mathrm{M}a^4}{2M_\mathrm{E}r^3} \approx 0.5$ m.

In reality things are complicated by various factors.

- The addition of tides from the Sun, which are about half as large those from the Moon:
  - 'spring' tides are when the Sun and Moon tides are in phase, two per month (when the Earth, Sun and Moon are in a line — a 'syzygy');
  - 'neap' tides are when the Sun and Moon tides are out of phase, and partially cancel, again two per month.
- Geography may mean:
  - tides are smaller due to limited flow (e.g. Mediterranean);
  - tides are larger (e.g. the Bay of Fundy has a tidal range ~ 15 m) – and some places have more than 2 tides per day – due to resonances/harmonics.

Notes: the rotation of the Earth is slowing down due to the friction from the tides; the Moon is receding from the Earth which conserves the angular momentum of the system (the current rate is *measured* using laser range finding as 38.08 ± 0.04 mm year^{-1}); the Moon now keeps the same face towards us, as its initial additional rotation has been dissipated by Earth tides.

Rigid Body Dynamics



- A **rigid body** is a special case of the many-particle system which was discussed earlier, in which all the inter-particle distances are fixed.
- The **location** of all the particles in the body can therefore be described by 6 coordinates: the position $\boldsymbol{R}$ of the centre of mass, and 3 angles (θ, ϕ, χ) (the Euler angles, defined later), which give the **attitude** of the body with respect to the spatial (x, y, z)-axes.
- More importantly, the **velocity** of any particle in the body is determined by the velocity $\boldsymbol{v}$ of the CoM and a single angular velocity $\boldsymbol{\omega}$.
- The **dynamics** of the rigid body is then determined by its total mass and the **inertia tensor** that relates the angular momentum $\boldsymbol{J}$ to the angular velocity $\boldsymbol{\omega}$.
- This tensor is the generalisation of the **moment of inertia** of rotation about a fixed axis.
- For a simple body spinning about a **fixed** high symmetry axis (say $\widehat{\boldsymbol{e}}_z$):
  - the moment of inertia $I = \sum m(x^2 + y^2)$ relates the angular momentum J to the angular velocity ω via $J = I\omega$;
  - the kinetic energy is also related to the angular velocity via $T = \frac{1}{2} I\omega^2 = \frac{1}{2} J\omega$.

Basic equations of motion

1) $M\ddot{\boldsymbol{R}} = \boldsymbol{F}_0$; centre of mass moves like particle of mass M under action of the resultant external force $\boldsymbol{F}_0$. Here consider only motion in the zero-momentum frame.

2) $\dot{\boldsymbol{J}} = \boldsymbol{G}_0$; the rate of change of angular momentum is equal to the total external couple $\boldsymbol{G}_0$.

- The angular velocity $\boldsymbol{\omega}$ determines the velocity of a particle at position $\boldsymbol{r}$ in the CoM frame (note that the origin in the CoM frame must lie on the rotation axis): $\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$.

- The angular momentum $\boldsymbol{J}$ is:

$$\boldsymbol{J} = \sum \boldsymbol{r} \times \boldsymbol{p} = \sum \boldsymbol{r} \times m(\boldsymbol{\omega} \times \boldsymbol{r}) = \sum m r^2 \boldsymbol{\omega} - \sum m\boldsymbol{r}(\boldsymbol{\omega} \cdot \boldsymbol{r}).$$

  Therefore $\boldsymbol{J}$ *is* proportional to $\boldsymbol{\omega}$ when the angular velocity is in a particular direction but it is *not* necessarily parallel to $\boldsymbol{\omega}$.

- This is a **tensor** relationship $\boxed{\boldsymbol{J} = \underline{\underline{\boldsymbol{I}}} \cdot \boldsymbol{\omega}}$ (or $J_i = I_{ij}\omega_j$ using the summation convention).

- Using matrix notation, in detail this is:

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \underbrace{\begin{pmatrix} \sum m(y^2+z^2) & -\sum mxy & -\sum mxz \\ -\sum mxy & \sum m(x^2+z^2) & -\sum myz \\ -\sum mxz & -\sum myz & \sum m(x^2+y^2) \end{pmatrix}}_{\underline{\underline{\boldsymbol{I}}}} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}.$$

- $\underline{\underline{\boldsymbol{I}}}$ is the **moment of inertia tensor** — here written as a matrix.

  Note: this matrix is symmetric (and real).

- The energy can also be written in terms of the Inertia tensor

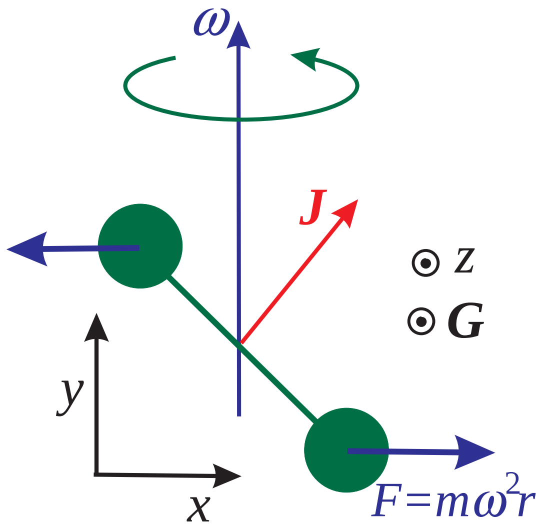
$$T = \sum \tfrac{1}{2} m(\boldsymbol{\omega}\times\boldsymbol{r})\cdot(\boldsymbol{\omega}\times\boldsymbol{r}).$$

- Using the vector identity $(\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C} = \boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})$ then

$$T = \sum \tfrac{1}{2} m\boldsymbol{\omega}\cdot\underbrace{(\boldsymbol{r}\times(\boldsymbol{\omega}\times\boldsymbol{r}))}_{\boldsymbol{J}/m}$$

or $\boxed{T = \tfrac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{J}.}$

Example: A Dumbbell.



Consider a rotating dumbbell, with point masses m separated by a length $2a$, with its axis at 45° to the axis of rotation $\widehat{\boldsymbol{e}}_y$.

- Masses at $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ coordinates: $\begin{pmatrix} a/\sqrt{2} \\ -a/\sqrt{2} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -a/\sqrt{2} \\ a/\sqrt{2} \\ 0 \end{pmatrix}$,

  giving: $\underline{\underline{\boldsymbol{I}}} = \begin{pmatrix} ma^2 & ma^2 & 0 \\ ma^2 & ma^2 & 0 \\ 0 & 0 & 2ma^2 \end{pmatrix}$

- The angular momentum is then given by $\boldsymbol{J} = \underline{\underline{\boldsymbol{I}}} \cdot \boldsymbol{\omega} = \begin{pmatrix} ma^2 & ma^2 & 0 \\ ma^2 & ma^2 & 0 \\ 0 & 0 & 2ma^2 \end{pmatrix} \begin{pmatrix} 0 \\ \omega \\ 0 \end{pmatrix} = \begin{pmatrix} ma^2\omega \\ ma^2\omega \\ 0 \end{pmatrix}$.

Note: that $\boldsymbol{J}$ is **not** parallel to $\boldsymbol{\omega}$.

- As the dumbbell rotates, $\boldsymbol{J}$ precesses around with it, which means a couple must be applied to the dumbbell to enable it to rotate at constant angular velocity around the y-axis: $\boldsymbol{G} = \dot{\boldsymbol{J}} = \boldsymbol{\omega} \times \boldsymbol{J} = -ma^2\omega^2\widehat{\boldsymbol{e}}_z$.

- The same couple may be derived by viewing the dumbbell in a frame rotating at $\boldsymbol{\omega}$ with respect to the inertial centre of mass frame and calculating the effect of centrifugal forces.

- This couple precesses with angular velocity $\boldsymbol{\omega}$, just as $\boldsymbol{J}$ does. This gives a time-varying couple on the bearing that supports the dumbbell, which, if the bearings are not perfect, will cause ‘juddering’.

- Note that if the rotational axis does not pass through the centre of mass then a centripetal force $M\omega^2R$, where M is the total mass of the system and R is the distance of the CoM from the rotation axis, must also be applied to maintain the rotational motion.

THE MOMENT OF INERTIA TENSOR

- The Moment of Inertia tensor is symmetrical. There are therefore 3 real eigenvalues $\{I_1, I_2, I_3\}$ and 3 mutually perpendicular eigenvectors $\{\widehat{\boldsymbol{e}}_1, \widehat{\boldsymbol{e}}_2, \widehat{\boldsymbol{e}}_3\}$.

- With respect to these eigenvector axes, $\underline{\underline{\boldsymbol{I}}}' = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$.

- Write $\boldsymbol{J} = I_1\omega_1\widehat{\boldsymbol{e}}_1 + I_2\omega_2\widehat{\boldsymbol{e}}_2 + I_3\omega_3\widehat{\boldsymbol{e}}_3$, or in matrix form $\boldsymbol{J} = \begin{pmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{pmatrix}$.

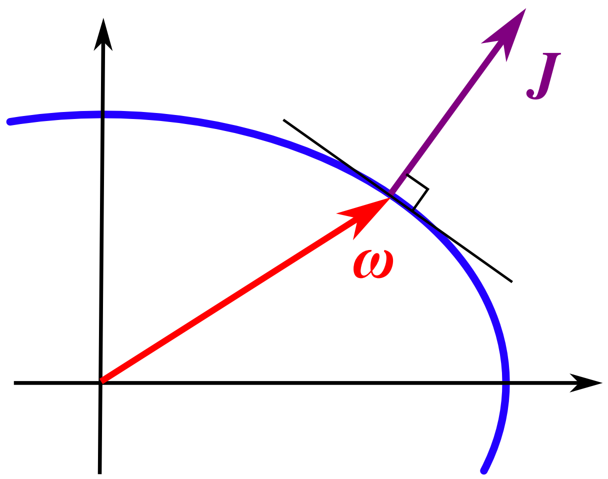
- The kinetic energy is $T = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$.

- $\{I_1, I_2, I_3\}$ are called the Principal Moments of Inertia and are the moments of inertia in the ordinary (or naive 1A) sense about the eigenvector axes; the eigenvector axes $\widehat{\boldsymbol{e}}_1$, $\widehat{\boldsymbol{e}}_2$ and $\widehat{\boldsymbol{e}}_3$ are called the Principal Axes.

- In $\boldsymbol{\omega}$-space, a surface of constant kinetic energy $T = T(\boldsymbol{\omega})$ is an ellipsoid, which is fixed to the body.
  This is the **Inertia Ellipsoid**.

- The axes of the ellipsoid have length $\propto 1/\sqrt{I_i}$, so the **smallest** I_i corresponds to **longest axis**.

- In $\boldsymbol{\omega}$-space, the gradient $\nabla_{\boldsymbol{\omega}} T = \left(\frac{\partial T}{\partial \omega_1}, \frac{\partial T}{\partial \omega_2}, \frac{\partial T}{\partial \omega_3}\right) = (I_1\omega_1, I_2\omega_2, I_3\omega_3) \equiv \boldsymbol{J}$,
  i.e. $\boldsymbol{J}$ is perpendicular to the surface of constant T at the position $\boldsymbol{\omega}$.



Example 1: Four particles of masses m, $2m$, $3m$ and $4m$ are connected by a light, rigid frame. They are at (x,y,z) positions of $(+a,+a,+a)$, $(+a,-a,-a)$, $(-a,+a,-a)$ and $(-a,-a,+a)$ respectively. Find the principal moments of inertia about the origin, and the principal axes.

- Then $\underline{\underline{\boldsymbol{I}}} = ma^2 \begin{pmatrix} 20 & 0 & 2 \\ 0 & 20 & 4 \\ 2 & 4 & 20 \end{pmatrix}$.

- If principal moments of inertia, the eigenvalues of the matrix, are λma^2, then
$$\begin{vmatrix} (20-\lambda) & 0 & 2 \\ 0 & (20-\lambda) & 4 \\ 2 & 4 & (20-\lambda) \end{vmatrix} = 0$$
which gives $(20-\lambda)[(20-\lambda)^2 - 20] = 0$, so $\lambda = 20$ or $\lambda = 20 \pm 2\sqrt{5}$.

- Thus the principal moments of inertia are
$(20-2\sqrt{5})ma^2, \qquad 20ma^2, \qquad (20+2\sqrt{5})ma^2$.

- The principal axes are the eigenvectors for these eigenvalues. i.e. for a vector $(a\widehat{\boldsymbol{x}}+b\widehat{\boldsymbol{y}}+c\widehat{\boldsymbol{z}})$ then

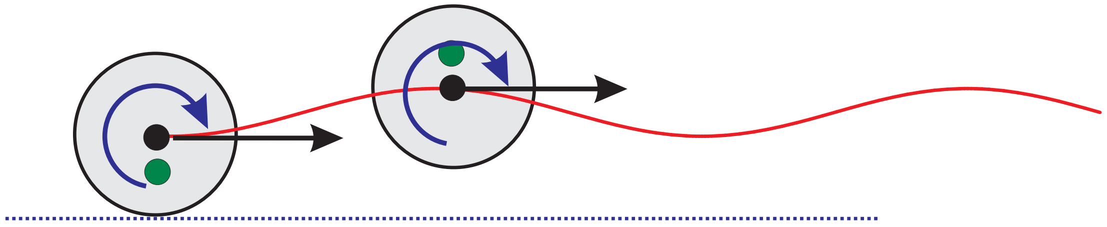
$$\begin{aligned}(20-\lambda)a+2c &= 0,\\ (20-\lambda)b+4c &= 0,\\ 2a+4b+(20-\lambda)c &= 0.\end{aligned}$$

- For $(20-\lambda)=0$, then $a:b:c=-2:1:0$, so the (normalised) principal axis is $(-2\widehat{\boldsymbol{x}}+1\widehat{\boldsymbol{y}})/\sqrt{5}$.
- For $(20-\lambda)=\pm 2\sqrt{5}$, then $a:b:c=1:2:\mp\sqrt{5}$, so the (normalised) principal axes are $(1\widehat{\boldsymbol{x}}+2\widehat{\boldsymbol{y}}\mp\sqrt{5}\widehat{\boldsymbol{z}})/\sqrt{10}$.

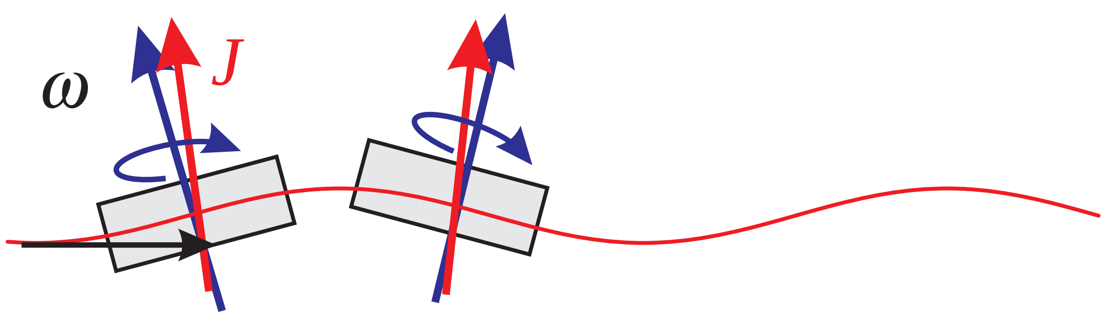
Note: these principal axes are orthogonal, as expected.

Example 2: Balancing of wheels.

- For an object to rotate smoothly about an axis, it must not only be ‘statically’ balanced (axis passes through the CoM), but also ‘dynamically’ balanced.
- This requires that it is set up so that $\boldsymbol{\omega}$ is parallel to $\boldsymbol{J}$.
  This occurs when the axis lies along a ‘principal axis’.
- **Case 1**. Static imbalance.
  Centre of mass is not on the axle, which causes **bouncing** of the wheel.



- **Case 2**. Dynamic imbalance.
  The axle is not a principal axis, which causes **wobbling** of the wheel.



Principal Axes of the Inertia Tensor

- The principal axes are fixed in the body and must be perpendicular to each other, **whatever the shape of the body**.

- As far as their rotational properties are concerned, bodies come in three types:

  1) Spherical Tops; all I_i's equal, e.g. sphere, cube.
     For a spherical top $\boldsymbol{J} = I\boldsymbol{\omega}$ with a **scalar** I, i.e. $\boldsymbol{J}$ is parallel to $\boldsymbol{\omega}$ in this case.
     Rotationally, the body is isotropic, with the same I about *any* axis.

  2) Symmetrical Tops; $I_1 = I_2 \neq I_3$, e.g. many simple molecules.
     Sub-types are **oblate** (lens or disc shaped) and **prolate** (cigar shaped).
     The $\widehat{\boldsymbol{e}}_3$ axis is unique, but $\widehat{\boldsymbol{e}}_1$ and $\widehat{\boldsymbol{e}}_2$ are anywhere in plane perpendicular to $\widehat{\boldsymbol{e}}_3$.

  3) Asymmetrical Tops; all three I's different. The principal axes are each unique.

- No one I_i can be larger than the sum of the other two. Thus, with respect to x, y, z along principal axes) $I_1 + I_2 = \sum m(y^2 + z^2 + x^2 + z^2) = I_3 + 2\sum mz^2 \geq I_3$.

- A special case is a flat body — a 'lamina' — with $z = 0$, for which $I_1 + I_2 = I_3$, which is known as the $\boxed{\text{Perpendicular Axes Theorem.}}$

Some Moments of Inertia

- Example of a lamina: Disc, mass M, radius a:

$$I_3 = \sum m(x^2 + y^2) = \int_0^a 2\pi r \frac{M}{\pi a^2} r^2 \mathrm{d}r = \tfrac{1}{2} M a^2.$$

$$I_1 = I_2 = \text{moment of inertia about a diameter} = \tfrac{1}{4} M a^2.$$

- Parallel Axes Theorem: for I about an axis **not** through centre of mass, say $\boldsymbol{a}$ away and parallel to a principal axis

$$I = \sum m(\boldsymbol{r} + \boldsymbol{a}) \cdot (\boldsymbol{r} + \boldsymbol{a}) = I_0 + M a^2 + 2\left(\sum m\boldsymbol{r}\right) \cdot \boldsymbol{a} = I_0 + M a^2$$

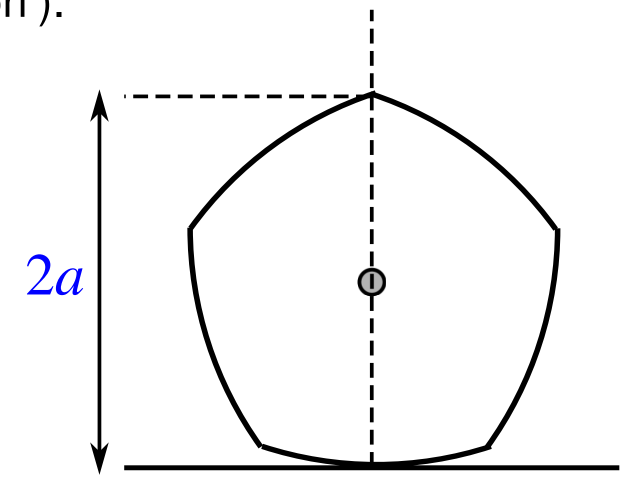
since $\sum m\boldsymbol{r}$ is zero when $\boldsymbol{r}$ is with respect to CoM.

(Here the vectors $\boldsymbol{r}$ etc. are all taken as 2-D projections in plane perpendicular to I axis.)

- Other useful moments of inertia.
  - Sphere, mass M, radius a: $I = \frac{2}{5} M a^2$.
  - Thin rod, mass M, length ℓ: $I = \frac{1}{12} M \ell^2$ (about axis through CoM, perpendicular to rod).
  - Rod, mass M, length ℓ, radius a: $I_1 = I_2 = M(\frac{1}{4} a^2 + \frac{1}{12} \ell^2)$, (about axes through CoM, perpendicular to rod), and $I_3 = \frac{1}{2} M a^2$ (about axes through CoM, along rod).

Example 1: Estimate the period of small oscillations of a vertical 50p coin on a flat surface (note: a 50p coin has a constant width – it is a 'Realeaux heptagon').

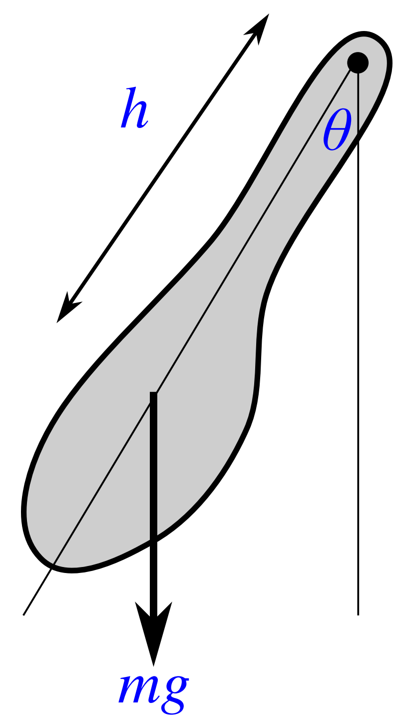
Use the 'energy method'.



- For a mass m, and diameter $2a$, then the M of I of a 50p coin is: $\approx \frac{1}{2}ma^2$ about its centre, or about its edge $I \approx \frac{1}{2}ma^2 + ma^2 \approx \frac{3}{2}ma^2$.
- If the coin rotates by θ, the KE is $T = \frac{1}{2}I\dot{\theta}^2 \approx \frac{3}{4}ma^2\dot{\theta}^2$
- As the coin rotates, its centre of mass rises slightly, and the PE increases by $V = mga(1 - \cos\theta)$ (note - the top of the heptagon stays at the same height).
- So $E \approx ma\left(\frac{3}{4}a\dot{\theta}^2 + g\underbrace{(1-\cos\theta)}_{\approx\theta^2/2}\right)$ is a constant, and since $\frac{\mathrm{d}E}{\mathrm{d}t} = 0$ then $\frac{3}{2}a\dot{\theta}\ddot{\theta} + g\theta\dot{\theta} \approx 0$

  so $\ddot{\theta} \approx \dfrac{-2g\theta}{3a}$.
- This is SHM with $\omega \approx \sqrt{\dfrac{2g}{3a}}$. This has a period $T = \dfrac{2\pi}{\omega} \approx 2\pi\sqrt{\dfrac{3a}{2g}}$.
- For a 50p coin, $a \approx 1.4$ cm. So $T \approx 0.3$ s.

Example 2: For a 'compound pendulum' of mass m as shown, show that for two different distances of the pivot away from the centre of mass the period of oscillation is the same, and show how to measure g using such a pendulum.



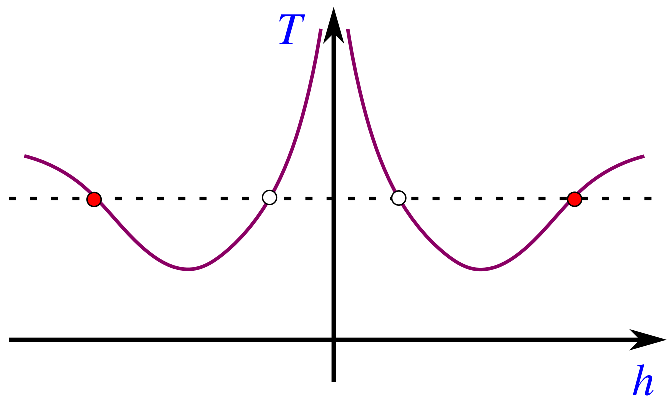
- Write the moment of inertia of the pendulum about its centre of mass as mk^2 (where k is the 'radius of gyration').
- The KE is $T = \frac{1}{2} I \dot{\theta}^2$, where $I = m(k^2 + h^2)$ is the moment of inertia about the pivot (a distance h from the centre of mass).
- The PE is $V = mgh(1 - \cos\theta) \approx mgh\theta^2/2$.
- Using the energy method $\mathrm{d}E(=T+V)/\mathrm{d}t = 0$ then

$$\ddot{\theta} \approx -\frac{gh}{k^2+h^2}\theta,$$

which is SHM with

$$T = \frac{2\pi}{\omega} \approx 2\pi\sqrt{\frac{k^2+h^2}{gh}}.$$

- For $h \ll k$ then $T \propto 1/\sqrt{h}$, and for $h \gg k$ then $T \propto \sqrt{h}$.



- So, there are two values of h that produce a given T, as shown (negative h on the plot means the other side of the centre of mass).

- If two values h_1 and h_2 give the same period T,
  $\dfrac{k^2+h_1^2}{gh_1} = \dfrac{k^2+h_2^2}{gh_2}$, so $k^2(h_2-h_1) = h_2^2 h_1 - h_1^2 h_2 = h_1 h_2 (h_2 - h_1)$.

- This means $k^2 = h_1 h_2$, and substituting back $\dfrac{T^2}{4\pi^2} = \dfrac{h_1 h_2 + h_1^2}{gh_1} = \dfrac{h_1+h_2}{g}$, or $g = \dfrac{4\pi^2}{T^2}(h_1+h_2)$.

Note: this is basis of 'Kater's pendulum'. This allows g to be determined accurately using 'knife edge' pivots on either side of the C of M (which does not depend on the M of I of the pendulum).

Moreover, as pointed out by Bessel, the periods do not to have to be exactly the same, and

$$\frac{4\pi^2}{g} = \frac{T_1^2+T_2^2}{2(h_1+h_2)} + \frac{T_1^2-T_2^2}{2(h_1-h_2)}.$$

In practice $h_1 - h_2$ in the second term is more difficult to measure than $h_1 + h_2$, but if T_1 is close enough to T_2, the overall accuracy is still very good.

Free Precession of Rigid Bodies

- Suppose $\boldsymbol{F} = 0$, $\boldsymbol{G} = 0$, i.e. an isolated body is spinning freely. The angular momentum $\boldsymbol{J}$ is constant. What about $\boldsymbol{\omega}$? The answer is easy if $\boldsymbol{J}$ and $\boldsymbol{\omega}$ are along a principal axis.

- If $\boldsymbol{J}$ and $\boldsymbol{\omega}$ do not lie along a principal axis the motion is much more complicated: the direction of $\boldsymbol{\omega}$ varies both in space and with respect to the body; for asymmetrical tops, the magnitude of $\boldsymbol{\omega}$ varies too. These variations are called **free precession**, to distinguish them from the **forced** precession produced by an external couple.

- The problem of handling free precession may be treated in three quite different ways.

  1) **Euler's equations**, which are the equations of motion in a coordinate frame moving with the body. This is the easiest approach to get some useful results, but it is slightly awkward to see what is going on in space axes.

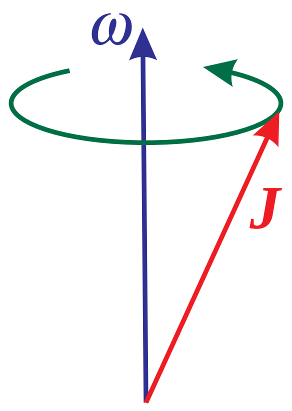
  2) **Poinsot's geometrical approach**, which can be insightful.

  3) **Lagrange's approach**, which gives the equations of motion with respect to fixed axes. This is straightforward once the Euler angles are defined.

- The first and last of these can be generalised to include forced motion, i.e. with external couples.

Euler's Equations for Rigid Body Motion

Euler's equations result from consideration of the change of angular momentum in the **body frame** S. The equations of motion are relatively simple in this frame, but the body axes are rotating with respect to the inertial frame S_0. Suppose there is an external couple $\boldsymbol{G}$.



1) In the inertial frame S_0, then $\left[\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t}\right]_{S_0} = \boldsymbol{G}$. The rates of change of $\boldsymbol{J}$ in frames S and S_0 are related by $\left[\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t}\right]_{S_0} = \left[\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t}\right]_{S} + \boldsymbol{\omega}\times\boldsymbol{J}$.

2) To see this note that:

   – if $\boldsymbol{J}$ is not changing in S, then in S_0 the $\boldsymbol{J}$ vector just rotates around $\boldsymbol{\omega}$;

   – if the body is not rotating, the rates of change of $\boldsymbol{J}$ in S and S_0 are the same.

   The two terms are linear in $\boldsymbol{J}$, hence the overall result.

- Therefore the vector equation of motion is

$$\boldsymbol{G} = \left[\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t}\right]_S + \boldsymbol{\omega}\times\boldsymbol{J}\,.$$

- $\boldsymbol{\omega}$ in frame S is $(\omega_1, \omega_2, \omega_3)$ and $\boldsymbol{J}$ is $(I_1\omega_1, I_2\omega_2, I_3\omega_3)$.

- Take one component

$$G_1 = I_1\dot{\omega}_1 + \omega_2 I_3\omega_3 - \omega_3 I_2\omega_2 = I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3$$

  and G_2, G_3 similarly.

- These are **Euler's equations** for the motion of a rigid body which, written out in full, are

$$\boxed{\begin{aligned} G_1 &= I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3, \\ G_2 &= I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1, \\ G_3 &= I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2. \end{aligned}}$$

- This result can also be derived directly by considering how the rotation of the body produces changes in the angular momentum in the inertial frame S_0 as the angular momentum along each of the Principal Axes is re-oriented in S_0.

Example: A lamina is free to move about a fixed point O in its plane. It is set in motion by an impulse Δp perpendicular to its plane, at a point $(x, y, 0)$ (using coordinates aligned with the principal axes, relative to O).

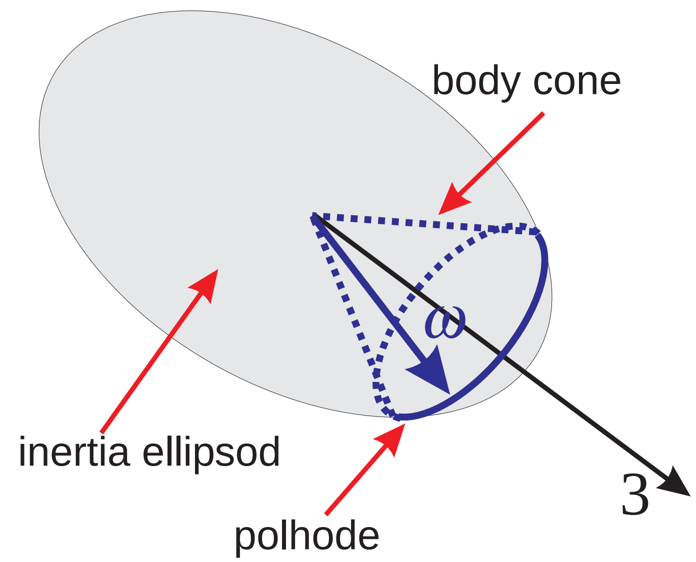
- Set the principal moments of inertia to be A, B and — by the perpendicular axis theorem — $A+B$ respectively.
- The initial angular speeds are given by: $|\omega_x(t=0)|A = (\Delta p)y$ and $|\omega_y(t=0)|B = (\Delta p)x$.
- Euler's equations give

$$\begin{aligned}
A\dot{\omega}_x + ((A+B)-B)\omega_y\omega_z &= 0,\\
B\dot{\omega}_y + (A-(A+B))\omega_z\omega_x &= 0,\\
(A+B)\dot{\omega}_z + (B-A)\omega_x\omega_y &= 0.
\end{aligned}$$

- The first of these gives $\omega_z = -\dot{\omega}_x/\omega_y$, and the second gives $\omega_z = +\dot{\omega}_y/\omega_x$.
- So $\dfrac{\dot{\omega}_x}{\omega_y} + \dfrac{\dot{\omega}_y}{\omega_x} = 0$ or $\dot{\omega}_x\omega_x + \dot{\omega}_y\omega_y = 0$.
- Integrating up, then $\omega_x^2 + \omega_y^2 = \text{constant} = \Delta p^2\left(\dfrac{x^2}{B^2} + \dfrac{y^2}{A^2}\right)$ (using the $t=0$ values).
- Hence the angular speed in the plane of the lamina is a constant.

Free Precession of the Symmetric Top

- Consider a simple case — the symmetric top with $I_1 = I_2 \neq I_3$.
- Euler's equations are $I_1\dot{\omega}_1 = (I_1 - I_3)\omega_2\omega_3; \quad I_1\dot{\omega}_2 = (I_3 - I_1)\omega_1\omega_3; \quad I_3\dot{\omega}_3 = 0.$ which implies that ω_3 is constant.
- Defining the **body frequency** $\Omega_b \equiv \dfrac{I_1 - I_3}{I_1}\omega_3$, then $\dot{\omega}_1 = \Omega_b\omega_2$ and $\dot{\omega}_2 = -\Omega_b\omega_1$, so $\ddot{\omega}_1 + \Omega_b^2\omega_1 = 0$ and $\ddot{\omega}_2 + \Omega_b^2\omega_2 = 0$.
- The general solution of these two coupled ODEs is $\omega_1 = A\sin(\Omega_b t + \phi_0)$, $\omega_2 = A\cos(\Omega_b t + \phi_0)$ so that, in the body frame, $\boldsymbol{\omega}$ precesses around the 3-axis with angular velocity Ω_b, i.e. at the **body frequency**.

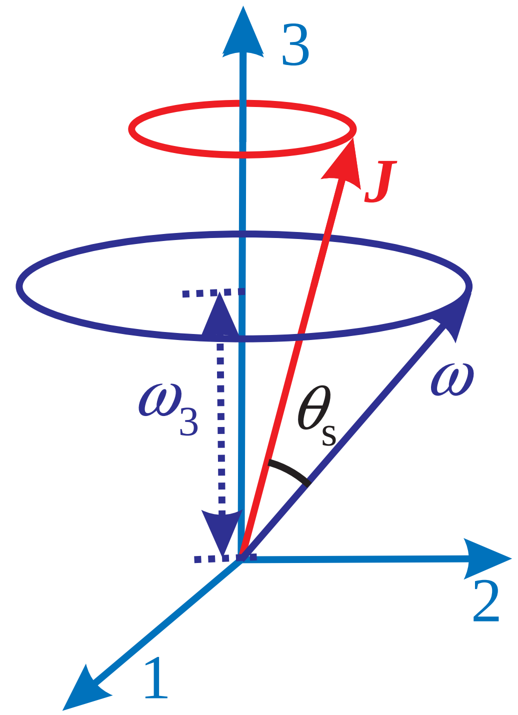


- The body frequency Ω_b can either be the same sign as ω_3 (prolate inertia ellipsoid, $I_3 < I_1$), or have the opposite sign (oblate inertia ellipsoid, $I_3 > I_1$).
- The surface traced out by the angular velocity vector is known as the **body cone**, and the curve traced out on the inertia ellipsoid by the $\boldsymbol{\omega}$ vector is known as the **polhode**.

Note: the case illustrated here is a **prolate** inertia ellipsoid.

1) In the 'body frame':

- Consider axes fixed in the body: ω precesses about the 3-axis at the body frequency $\Omega_\mathrm{b} = \dfrac{I_1 - I_3}{I_1}\omega_3$.
- J is not parallel to ω but still precesses about the 3-axis at the body frequency.

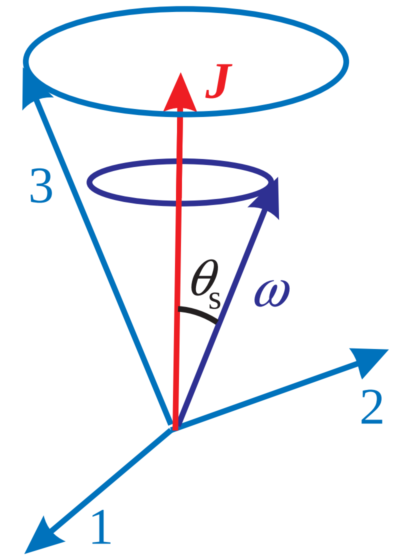


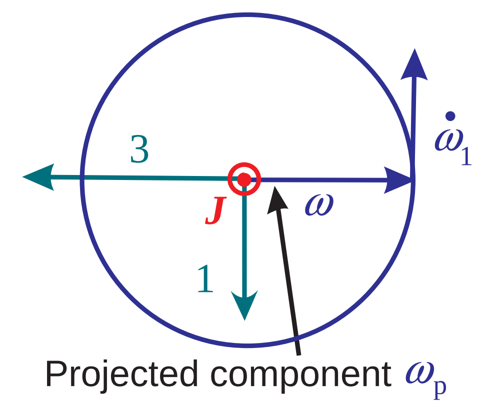
Note: the case illustrated here is **oblate**, i.e. $I_3 > I_1$ (so that the small axis of the inertia ellipsoid corresponds to the **largest** principal moment of inertia).

2) In the 'space frame':

- $\boldsymbol{J}$ is fixed, the body is rotating at $\boldsymbol{\omega}$, which rotates around $\boldsymbol{J}$ at the **space frequency**, Ω_{s}.
- To find the space frequency, look down the J vector when $\omega_1 = 0$.
- So $\boldsymbol{\omega} = (0, \omega_2, \omega_3)$, and $\boldsymbol{J} = (0, I_2\omega_2, I_3\omega_3)$.
- The projected component of ω shown is $\omega_{\mathrm{p}} \equiv |\boldsymbol{\omega}| \sin\theta_{\mathrm{s}} = \dfrac{|\boldsymbol{J} \times \boldsymbol{\omega}|}{|\boldsymbol{J}|}$.
- At this moment $\boldsymbol{J} \times \boldsymbol{\omega}$ is in the 1-direction and has magnitude
  $(I_2 - I_3)\omega_2\omega_3$
- Rate of precession, is $\Omega_{\mathrm{s}} = \dfrac{\dot{\omega}_1}{\omega_{\mathrm{p}}}$.
- But $I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3$, so the space frequency is

$$\Omega_{\mathrm{s}} = \underbrace{\frac{(I_2 - I_3)\omega_2\omega_3}{I_1}}_{\dot{\omega}_1} \cdot \underbrace{\frac{J}{(I_2 - I_3)\omega_2\omega_3}}_{1/\omega_{\mathrm{p}}}, \text{ or } \boxed{\Omega_{\mathrm{s}} = \frac{J}{I_1}.}$$





Space and Body Frequencies and Cones

- Two frequencies are important.

  1) The 'Space Frequency' $\Omega_s \equiv$ the rotation speed of the plane of $\boldsymbol{\omega}$ and $\widehat{\boldsymbol{e}}_3$ about $\boldsymbol{J}$. It is the 'free precession' speed as seen by an inertial observer. From the above argument that it has the value $\boxed{\Omega_s = J/I_1,}$ so that it is always in the sense of $\boldsymbol{J}$ and is of the same order of magnitude as the rotation speed.

  2) The 'Body Frequency' $\Omega_b \equiv$ the rate at which $\boldsymbol{\omega}$ describes its cone about the 3-axis. From the Euler equations, then $\boxed{\Omega_b = \dfrac{I_1 - I_3}{I_1}\omega_3.}$ This can take either sign relative to Ω_s and can be very small if the body is nearly 'spherical' (in terms of its principal M of I).

- Two angles are important.

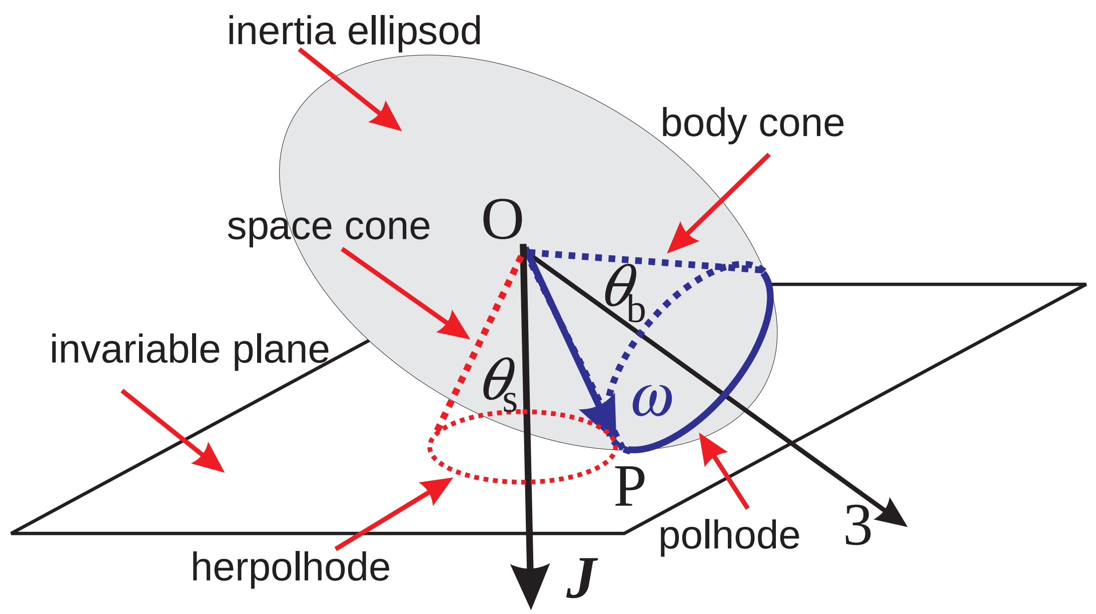
  1) The 'Space Cone' of half-angle $\theta_s \equiv$ the cone swept out by $\boldsymbol{\omega}$ as it precesses about $\boldsymbol{J}$. It is the angle of 'free precession' seen by an inertial observer.

  2) The 'Body Cone' $\theta_b \equiv$ the cone swept out by $\boldsymbol{\omega}$ as it precesses about $\widehat{\boldsymbol{e}}_3$.

- **Poinsot's construction** illustrates the relation between these angles/frequencies.

POINSOT'S TREATMENT FOR FREE PRECESSION

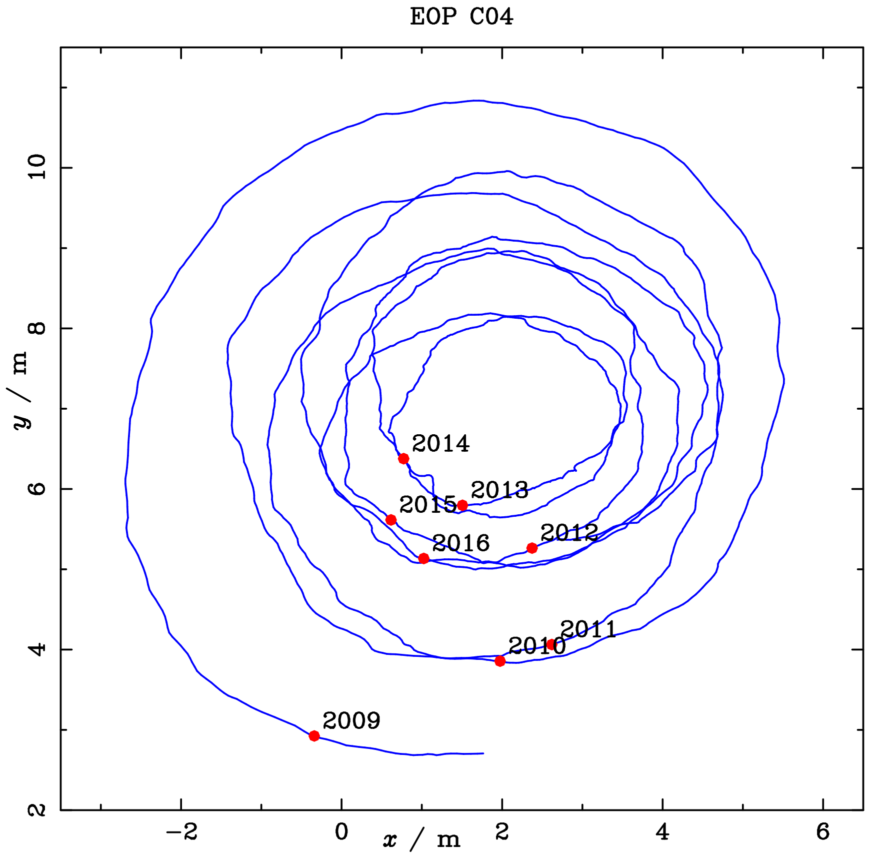
Here this is illustrated for a **prolate** inertia ellipsoid.



- For free precession (no external couple) only: constant $\boldsymbol{J}$ and $T = \frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{J}$ implies that the component of $\boldsymbol{\omega}$ in the $\boldsymbol{J}$-direction remains constant.
- As the $\boldsymbol{\omega}$-vector varies, it keeps its tip P in a plane perpendicular to $\boldsymbol{J}$; the plane is called the **invariable plane**.
- Since $\boldsymbol{J}$ is perpendicular to the surface of the ellipsoid at P, it follows that the ellipsoid is **tangential** to the invariable plane at P.

- The crucial step in Poinsot's argument is to say that, since the instantaneous motion is a rotation about OP, P is instantaneously at rest, and the ellipsoid therefore **rolls**, rather than slides, on the invariable plane.
- The locus of the tip of the $\boldsymbol{\omega}$ vector on the plane is called the **herpolhode**.
- The space and body frequencies can be calculated from geometric considerations, e.g. $\Omega_\mathrm{b}\sin\theta_\mathrm{b} = \Omega_\mathrm{s}\sin\theta_\mathrm{s}$ follows from the condition of no slipping between the space and body cones as they roll around $\boldsymbol{\omega}$.

Example: Free Precession — The Earth.



Data from: `http://hpiers.obspm.fr/`

- The Earth is in force-free motion, but is slightly oblate (flattened) $\boldsymbol{J}$ is slightly inclined to the symmetry axis.
- The Earth's oblateness: $\dfrac{I_3 - I_1}{I_1} \equiv \beta \approx \dfrac{\Omega^2 R}{g} \approx \dfrac{1}{300}$. The space-cone angle θ_s is negative for this case.
- Angles are very small: $\theta_\mathrm{s} \approx -\beta\theta_\mathrm{b} \approx -\theta_\mathrm{b}/300$.
- The space-cone is very small and is *inside* the body-cone, which swings round it each day.
- In ≈ 300 days, $\boldsymbol{\omega}$ moves in a cone round $\widehat{\boldsymbol{e}}_3$, giving a latitude change: **Polar wander**, or **Chandler wobble**. The $\boldsymbol{J}$, $\widehat{\boldsymbol{e}}_3$ angle is about 0.1 arcsec (a few metres at the surface).
- The period is actually 427-days, and irregular; the Earth is not rigid and its deformations affect the arguments.

FREE PRECESSION — TRIAXIAL BODIES

- Triaxial body has all principal moments of inertia different. Assume $I_1 < I_2 < I_3$.
- Free rotation is complicated, but using the conservation laws:

  $\boldsymbol{J}$ is conserved: $(I_1\omega_1, I_2\omega_2, I_3\omega_3)$;

  T is conserved: $\frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$.
- **If the body spins about the 1-axis** it cannot change $\boldsymbol{\omega}$ at constant $\boldsymbol{J}$ without **decreasing** its energy. It is therefore STABLE.
- **If the body spins about the 3-axis** it cannot change $\boldsymbol{\omega}$ at constant $\boldsymbol{J}$ without **increasing** its energy. It is therefore STABLE.
- **If the body spins about the 2-axis** it can change $\boldsymbol{\omega}$ at constant $\boldsymbol{J}$ in many ways while conserving energy. It is therefore UNSTABLE, and the polhode will make large excursions around the inertia ellipsoid.

- Consider Euler's equations for $\dot{\omega}_1, \dot{\omega}_2$:

$$\frac{\dot{\omega}_1}{\dot{\omega}_2} = \frac{\mathrm{d}\omega_1}{\mathrm{d}\omega_2} = \frac{I_2}{I_1}\frac{\omega_2(I_2 - I_3)}{\omega_1(I_3 - I_1)}.$$

- Integrating, the locus is a conic section: $\dfrac{\omega_1^2}{I_2(I_3 - I_2)} + \dfrac{\omega_2^2}{I_1(I_3 - I_1)} = \text{constant}.$
- The locus can be (as here) an ellipse (stable) or a hyperbola (unstable), depending on the signs of $I_i - I_j$.

THE MAJOR AXIS THEOREM FOR NON-RIGID BODIES

- Real objects are neither perfectly rigid nor perfectly elastic.
- During free precession, whilst the angular momentum is conserved, the centrifugal forces change as $\boldsymbol{\omega}$ axis moves. The object therefore deforms, and loses energy.
- $\boldsymbol{J}$ can move w.r.t. principal axes, but $\boldsymbol{J}^2$ is constant, so as KE is decreasing, the energy ellipsoid shrinks.
- To get the minimum energy for a certain J — align $\boldsymbol{J}$ with axis with largest I, i.e. the major axis.
- If $\boldsymbol{J}$ is not aligned with the major axis, energy can still be lost.
- **Major axis theorem** 'Any freely-rotating body that is not perfectly rigid, will lose kinetic energy and, whilst its angular momentum remains constant in magnitude and direction in space, its angular velocity vector moves with respect to the body coordinates until the body is rotating about its major axis'.

Examples:

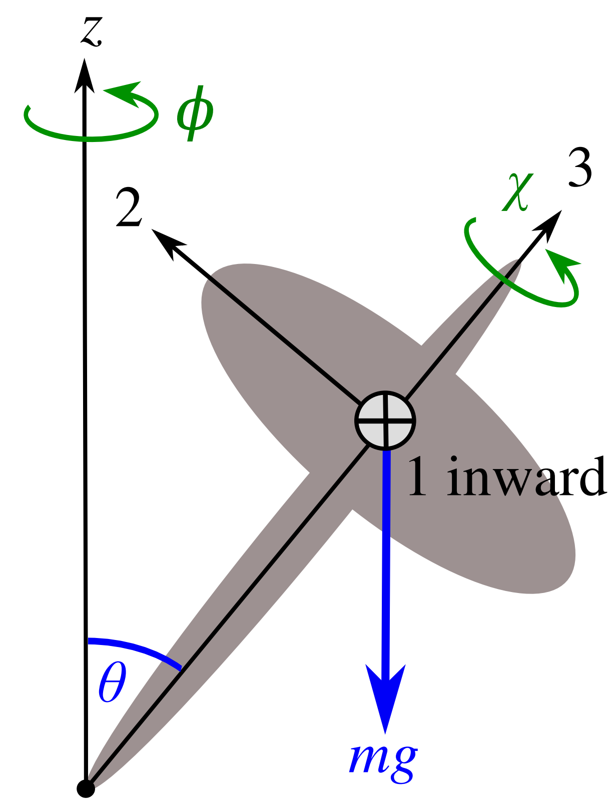
- **Celestial objects**. Freely-rotating celestial objects usually rotate about their major axes — asteroids, planets like the Earth, stars or spiral galaxies.

- **Chandler Wobble of the Earth**. The Earth is not rigid — free precession decays in about 68 years. The Earth's precession must be continuously excited/driven. The principal cause of the Chandler wobble is fluctuating pressure on the bottom of the ocean, caused by temperature and salinity changes and wind-driven changes in the circulation of the oceans.

- **Explorer 1 Satellite**. In 1958, a few months after Sputnik I, the US launched Explorer 1. It was a long cylindrical object, with flexible radio antennae protruding from the sides.
  - If the orientation of a satellite matters (e.g. for directing antennae or solar panels) it must be stabilised — a typical satellite with a solar panel will rotate 90° degrees in an hour from rest due to radiation pressure alone.
  - They 'stabilised' it by spinning about its length — the minor axis. During first orbit the angular momentum vector moved to the major axis (perpendicular to the middle of the satellite). It spent the rest of its mission cart-wheeling through space.

LAGRANGE'S APPROACH

- Consider the motion of a symmetrical top, at first isolated, and then supported at its base under gravity (take $I_1 = I_2 \neq I_3$).
- Let θ and ϕ be the spherical polar coordinates of the symmetry 3-axis. θ is the angle of the 3-axis from the vertical axis (i.e. along $\widehat{\boldsymbol{e}}_z$).
- Let χ be the angle of rotation of the top about the 3-axis.
- (θ, ϕ, χ) are the **Euler angles**, and are suitable coordinates to describe the motion of the body.



- If only ϕ is changing, the angular velocity is $\dot{\phi}\widehat{\boldsymbol{e}}_z$. Similarly, if only θ is changing, the angular velocity is $\dot{\theta}\widehat{\boldsymbol{e}}_1$, and if only χ is changing, the angular velocity is $\dot{\chi}\widehat{\boldsymbol{e}}_3$. So in general, the angular velocity is the sum of these, i.e.

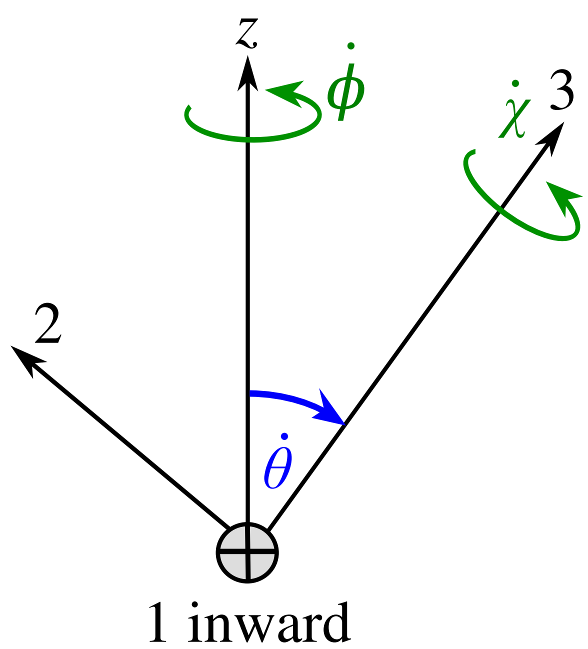
$$\boldsymbol{\omega} = \dot{\phi}\widehat{\boldsymbol{e}}_z + \dot{\theta}\widehat{\boldsymbol{e}}_1 + \dot{\chi}\widehat{\boldsymbol{e}}_3.$$

- For a symmetric top (i.e. $I_1 = I_2$), make a convenient choice of axes: let the 1-axis be (instantaneously) horizontal; the z-axis is in the 2–3 plane.

- At this moment of time $\widehat{\boldsymbol{e}}_z = \widehat{\boldsymbol{e}}_3 \cos\theta + \widehat{\boldsymbol{e}}_2 \sin\theta$, so $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) = (\dot{\theta}, \dot{\phi}\sin\theta, \dot{\chi} + \dot{\phi}\cos\theta)$, with χ measured w.r.t. the moving z-3 plane.

- The angular momentum in body axes is given by

$$\boldsymbol{J} = (I_1\dot{\theta}, I_1\dot{\phi}\sin\theta, I_3(\dot{\chi} + \dot{\phi}\cos\theta)).$$



- The symmetry $I_1 = I_2$ provides a considerable simplification; in general need to resolve $\dot{\theta}$ and $\dot{\phi}\sin\theta$ onto the 1- and 2-axes which are rotated by the angle χ about the 3-axis.

- The moment of inertia I_1 is taken with respect to the stationary point within the body (i.e. the C of M for the isolated body, base if supported under gravity). The gravitational couple $\boldsymbol{G}$, if present, is in the 1-direction.

- There are three **constants of motion**, giving three equations.

  1) $J_3 = I_3(\dot{\chi} + \dot{\phi}\cos\theta)$ is constant.

     This follows from Euler's equation for $\dot{\omega}_3$ with $G_3 = 0$ and $I_1 = I_2$, implying $\dot{J}_3 = I_3\dot{\omega}_3 = 0$.

  2) $J_z = J_3\cos\theta + J_2\sin\theta = J_3\cos\theta + I_1\dot{\phi}\sin^2\theta$ is constant (since $G_z = 0$).

  3) The total energy $E = T + U$ is constant.

  The first two equations enable us to express $\dot{\phi}$ and $\dot{\chi}$ in terms of the angular momentum constants J_z and J_3 and the angle θ.

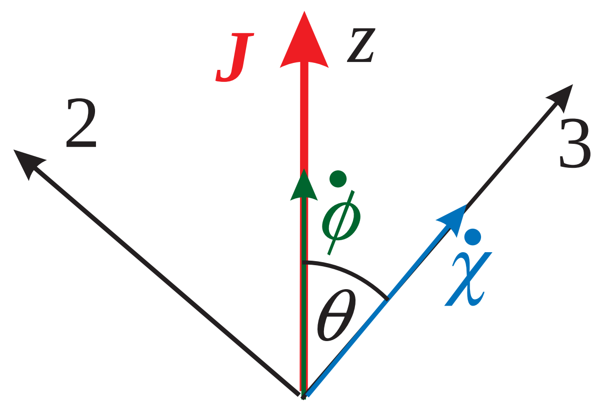
  From 2) above:

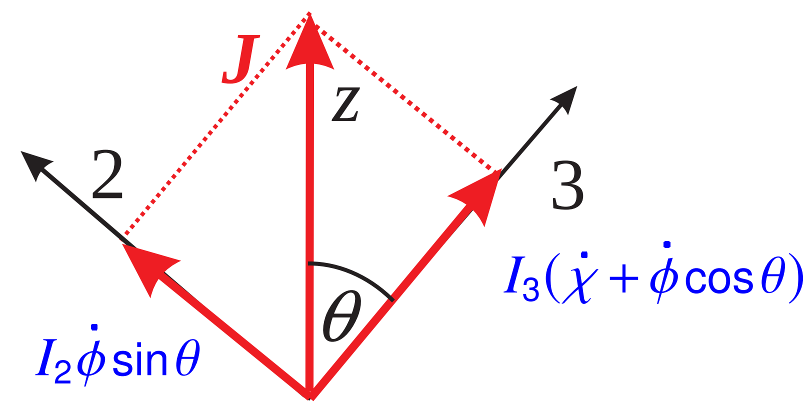
$$\dot{\phi} = \frac{J_z - J_3\cos\theta}{I_1\sin^2\theta}.$$

  From 1) above:

$$\dot{\chi} = \frac{J_3}{I_3} - \dot{\phi}\cos\theta = \frac{J_3}{I_3} + \frac{J_3\cos^2\theta - J_z\cos\theta}{I_1\sin^2\theta}.$$

FREE PRECESSION REVISITED





- Suppose that the body is isolated in free space. Take the $\boldsymbol{J}$ direction as defining the z-axis.
- Then $J_z = J$ since the total $\boldsymbol{J}$ is along z-axis and $J_3 = J\cos\theta$. Therefore

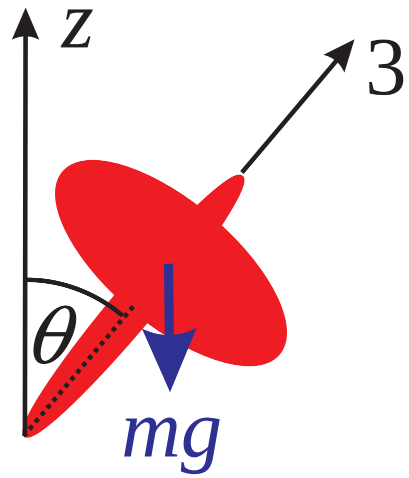
$$\dot\phi = \frac{J(1-\cos^2\theta)}{I_1\sin^2\theta} = \boxed{\frac{J}{I_1} = \Omega_s,}$$

$$\dot\chi = \frac{J\cos\theta}{I_3} - \frac{J}{I_1}\cos\theta = I_3\omega_3\left(\frac{1}{I_3}-\frac{1}{I_1}\right) = \boxed{\frac{I_1-I_3}{I_1}\omega_3 = \Omega_b.}$$

(using $J_3 = J\cos\theta$).

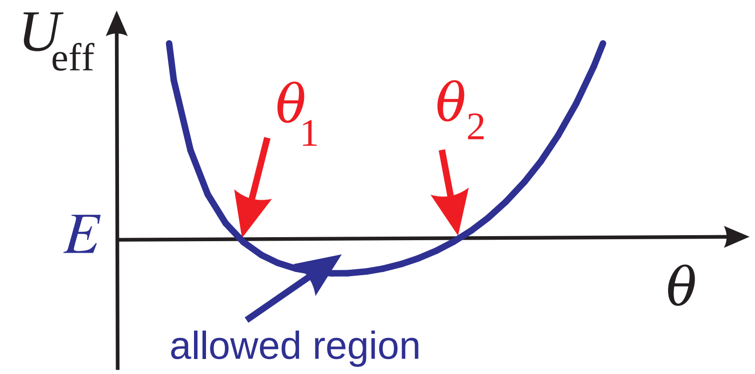
- The above constitutes another derivation of the space and body frequencies Ω_s and Ω_b.

Forced Precession — the Gyroscope



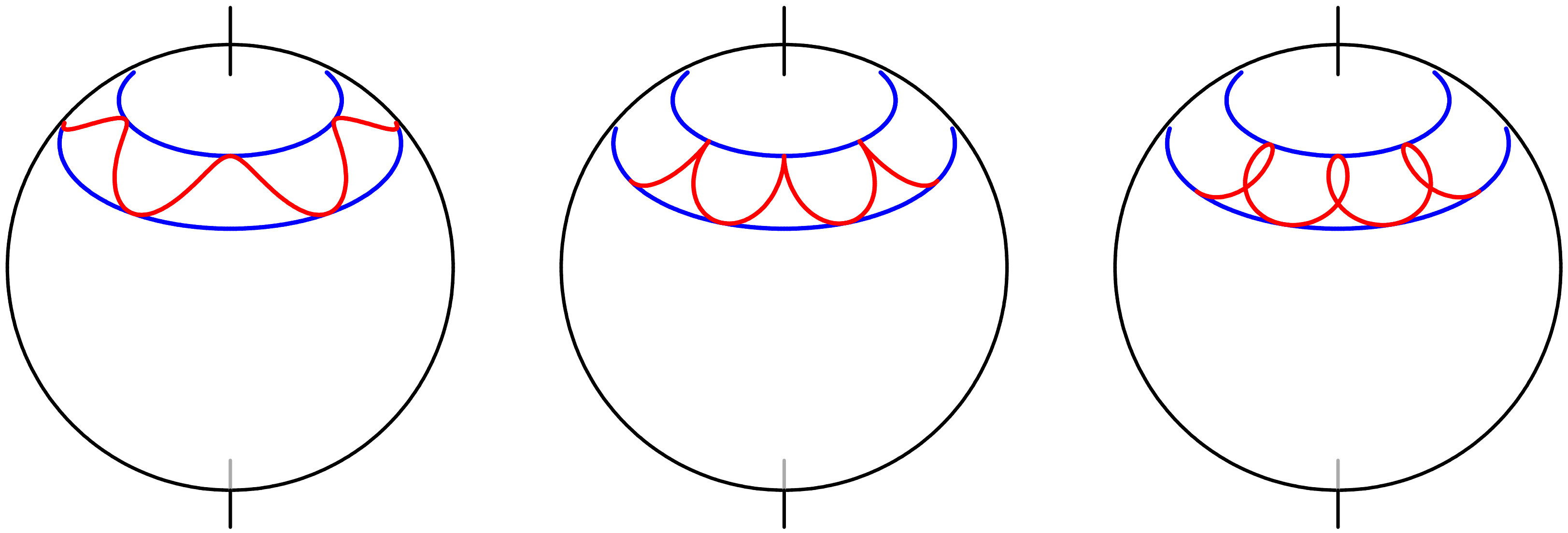
- Now consider the body supported at its base, so that there is a couple due to gravity. I_1 is now about the support, which is at h from the C of M.
- Conservation of J_z and J_3 give $\dot\phi$ and $\dot\chi$ as known functions of θ. Once θ is known as a function of time, ϕ and χ may in principle be found by integration.
- The angle θ may be found from the energy equation:
$$E = \tfrac{1}{2} I_1(\dot\theta^2 + \dot\phi^2 \sin^2\theta) + \tfrac{1}{2} I_3(\dot\chi + \dot\phi\cos\theta)^2 + mgh\cos\theta = \text{constant}.$$
- Substituting for $\dot\phi$ and $\dot\chi$ gives $E = \frac{1}{2} I_1 \dot\theta^2 + \underbrace{\frac{(J_z - J_3\cos\theta)^2}{2I_1\sin^2\theta} + mgh\cos\theta + \frac{J_3^2}{2I_3}}_{U_{\text{eff}}(\theta)}$.
- The problem is now solved in principle, since $\dot\theta$ is a known function of θ and may be integrated to give $\theta(t)$. Then ϕ and χ can be derived.
- Key physical results can be derived without recourse to the full mathematics by treating the θ motion as oscillation in an effective potential U_{eff} as in the treatment of orbits.

THE MOTION OF A GYROSCOPE



- The effective potential usually has the form shown (except in the special case $J_z = J_3$ – called a 'sleeping top' – with the axis vertical, spinning fast enough to stay vertical).
- For energies E above the minimum in U_{eff} there is an allowed region of θ where $\dot{\theta}^2 \geq 0$.
- In steady precession θ is at the 'equilibrium' position (minimum of U_{eff}) and constant ($E = U_0$). Conservation of the angular momenta imply that $\dot{\phi}$ and $\dot{\chi}$ are constant.
- The motion thus consists of steady rotation by $\dot{\chi}$ about the symmetry axis, with the axis itself steadily precessing by $\dot{\phi}$ at constant θ to the vertical.

- If E is slightly larger than the minimum of U_{eff}, then the θ motion can be treated as approximate SHM by Taylor expansion of U_{eff} about the minimum.
- The oscillations of θ give **nutation**, i.e. oscillations in $\dot{\chi}$ and $\dot{\phi}$ about the steady precession values.



- For energies substantially higher than U_0, the nutation can be quite complicated.

STEADY PRECESSION OF A GYROSCOPE

- The condition for steady precession is $\mathrm{d}U_{\mathrm{eff}}/\mathrm{d}\theta = 0$. After some algebra, and substituting for $\dot{\phi}$, this gives a quadratic equation:

$$\underbrace{\dot{\phi}^2 I_1 \cos\theta}_{\dot{\phi}^2 \text{ term}} - \underbrace{\dot{\phi} J_3}_{\dot{\phi} \text{ term}} + \underbrace{mgh}_{\text{couple}} = 0.$$

- This gives the steady precession speed $\dot{\phi}$ as a function of inclination θ, with solutions

$$\dot{\phi} = \frac{J_3 \pm \sqrt{J_3^2 - 4I_1 mgh\cos\theta}}{2I_1\cos\theta}.$$

- If $\cos\theta$ is positive (i.e. the top is above its base, not below), $\dot{\phi}$ is not physical unless $J_3^2 \geq 4I_1 mgh\cos\theta$, i.e. steady precession requires the top to be spinning fast enough.
- In the **gyroscopic limit** $\boldsymbol{J}$ is very large from rapid rotation about the symmetry axis and $J_3^2 \gg mghI_1$.

- In the gyroscopic limit the two possible precession frequencies $\dot{\phi}$ for a given θ are:

  1) 'slow precession' (i.e. neglect the $\dot{\phi}^2$ term), with
  $$\dot{\phi} \approx \frac{mgh}{J_3},$$
  which is independent of θ (this is the precession that is usually seen with a gyroscope);

  2) 'fast precession' (i.e. neglect the mgh couple term), with
  $$\dot{\phi} \approx \frac{J_3}{(I_1 \cos\theta)},$$
  which is the 'free precession' as derived previously, with $\dot{\phi} = \Omega_s$.

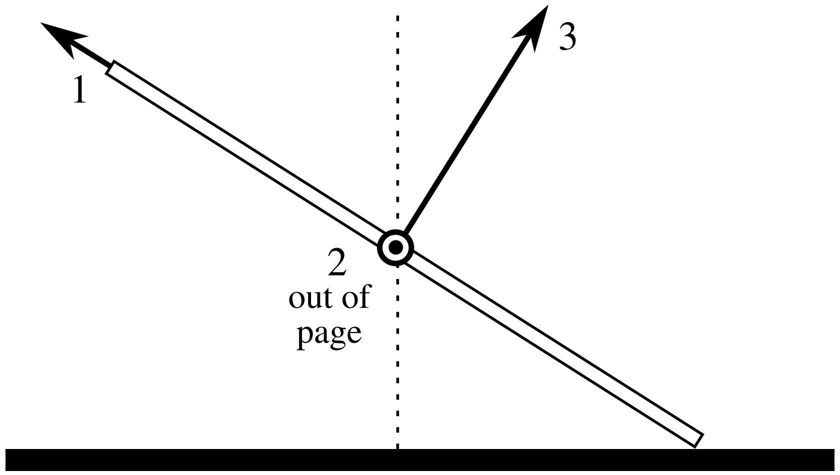
Note: for a horizontal gyroscope $\cos\theta = 0$, so the $\dot{\phi}^2$ term disappears in the quadratic equation, giving the 'slow precession' rate of $\dot{\phi} = mgh/J_3$, as derived in Part IA.

FORCED PRECESSION — GYROSCOPE NUTATION

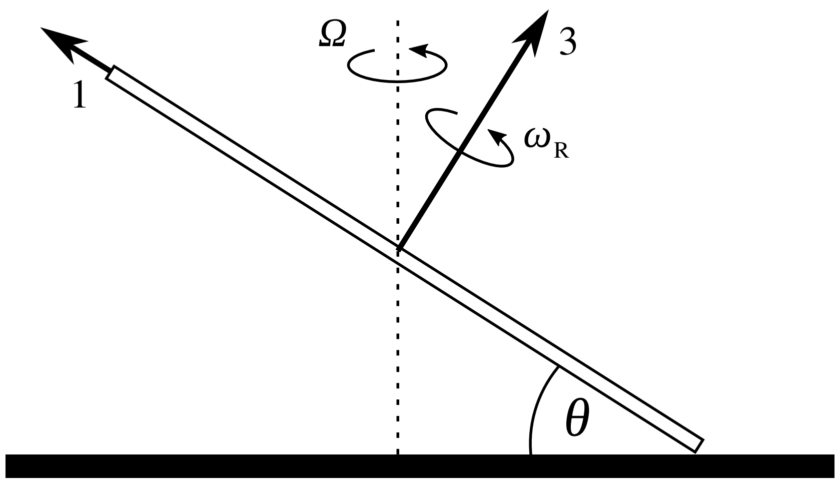
- The analysis of nutation about precession at general θ, even in the gyroscopic limit, is algebraically laborious. The case of nutation of a **horizontal** gyroscope is reasonably straightforward.
- Nutation of a gyroscope, with axis horizontal and supported at one end.
- Put $\theta = \pi/2 + \epsilon$. For small ϵ, $\cos\theta \approx -\epsilon$, $\sin\theta \approx 1 - \epsilon^2/2$.
- Then
$$U_{\text{eff}}(\theta) = \text{constant} + \epsilon\left(\frac{J_z J_3}{I_1} - mgh\right) + \epsilon^2\left(\frac{J_3^2}{2I_1} + \frac{J_z^2}{2I_1}\right) + \dots$$
for power series expansion in ϵ.
- The term $\propto \epsilon$ is zero at θ_0; therefore $J_z = \dfrac{mghI_1}{J_3}$, or $\dot{\phi} = \dfrac{mgh}{J_3}$.
- The gyroscope condition is $J_3^2 \gg mghI_1$, which means hence $J_3^2 \gg J_z^2$.
- The ϵ^2-term gives the 'restoring force' term in U_{eff} and hence the equation of motion is: $U_{\text{eff}} \approx \text{constant} + \epsilon^2 \dfrac{J_3^2}{2I_1}$, so $I_1\ddot{\epsilon} + \dfrac{J_3^2}{I_1}\epsilon \approx 0$.
- This gives SHM in ϵ at $\Omega \equiv \Omega_s = J_3/I_1$.

A SPINNING DISK

Consider a disk spinning on a flat surface, e.g. a coin.



Instantaneously, the motion of the disk relative to the principal axes shown must have $\omega_2 = 0$, and $\omega_3 = 0$ (if the point of contact does not slip). So, the rotation is just about the the 1-axis.



Alternatively, the motion can be thought of as the combination of a rotation (precession) at Ω about the vertical axis, plus a rotation at ω_R about the 3-axis, the axis of symmetry of the disk.

In this case, we must have

$$(\omega_1, 0, 0) = (\Omega \sin\theta, 0, \omega_\mathrm{R} + \Omega \cos\theta),$$

so $\omega_\mathrm{R} = -\Omega \cos\theta$.

THE ENERGY METHOD REVISITED

- If it is known from physical grounds that the energy is conserved, it is always possible to derive the equations of motion of systems that only have one degree of freedom (such as the SHM):

$$\tfrac{1}{2}m\dot{x}^2+\tfrac{1}{2}kx^2=E \quad \text{so} \quad \dot{x}(m\ddot{x}+kx)=0 \quad \text{so} \quad m\ddot{x}+kx=0.$$

- This works because $\dot{x}$ is in general non-zero.
- This is called the **energy method**.
- Often the equations of motion of more complicated systems with N variables can be derived in a similar way, which works for small oscillations near equilibrium. More advanced methods of Lagrangian and Hamiltonian mechanics derive the equations of motion of a system from a variational principle that involves the kinetic and potential energies in the form of the Lagrangian $\mathcal{L}=T-U$. This is rigorous, and always works.
- The advantage of Lagrangian and Hamiltonian methods is that only the total kinetic and potential energies of a system are needed. In Newtonian mechanics it is necessary to evaluate all the forces acting, in order to calculate the accelerations.

OUTLINE OF LAGRANGIAN MECHANICS

- Lagrangian mechanics starts with **Hamilton's Principle** that the **action** $S = \int \mathcal{L}(q_i, \dot{q}_i, t)\,\mathrm{d}t$ is stationary for *small* variations $\delta q_i(t)$ about the path $q_i(t)$ (for coordinates $q_i, i = 1, 2, \ldots$). (This is sometimes also called the 'Principle of Least Action', but note that in some circumstances the action is stationary, not necessarily a minimum.)

- Consider the variation δS as the path varies from $q_i(t)$ to $q_i(t) + \delta q_i(t)$, then

$$\delta S = \int_{t_1}^{t_2} \sum_i \left(\underbrace{\delta q_i \frac{\partial \mathcal{L}}{\partial q_i}}_{\text{A}} + \underbrace{\delta \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}}_{\text{B}} \right) \mathrm{d}t.$$

- However, the variations of $\delta \dot{q}_i(t)$ — i.e. the terms marked as B — are determined by $\delta q_i(t)$, so integrate by parts

$$\delta S = \sum_i \underbrace{\left[\delta q_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right]_{t_1}^{t_2}}_{\text{from B}} + \int_{t_1}^{t_2} \sum_i \delta q_i \left\{ \underbrace{\frac{\partial \mathcal{L}}{\partial q_i}}_{\text{from A}} - \underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right)}_{\text{from B}} \right\} \mathrm{d}t.$$

- The integrated parts vanish for fixed end points, so the condition that $\delta S = 0$ for all variations $\delta q_i(t)$ is

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i} \qquad \text{for all } i.}$$

- These are the **Euler–Lagrange** equations of motion. In classical mechanics the Lagrangian is $\mathcal{L} = T - V$, where T is the kinetic energy and V is the potential energy.

- The important quantities

$$\boxed{\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \equiv p_i}$$

  are known as the **conjugate momenta**.

- **If** the Lagrangian does not depend on one of the coordinates, say q_1 (i.e. $\partial \mathcal{L}/\partial q_1 = 0$), then from Lagrange's equations $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{q}_1}\right) = 0$ and the conjugate momentum p_1 is a **constant**.

- **Symmetries** (here $\mathcal{L}$ independent of q_1) thus lead to **conservation laws**.

Hamiltonian Dynamics and Conservation Laws

- Lagrangian dynamics uses $\mathcal{L}(q_i,\dot{q}_i,t)$ where there is a dependence between q_i and $\dot{q}_i$. The motivation for Hamiltonian dynamics is to find a function $H(q_i,p_i,t)$ that is **not** a function of the velocities $\dot{q}_i$. This is done by forming the **Hamiltonian**
$$H \equiv \sum_i p_i\dot{q}_i - \mathcal{L}(q_i,\dot{q}_i,t).$$
- The total variation of H is $\mathrm{d}H = \sum_i \left(\dot{q}_i\mathrm{d}p_i + p_i\mathrm{d}\dot{q}_i - \underbrace{\frac{\partial\mathcal{L}}{\partial q_i}}_{\dot{p}_i}\mathrm{d}q_i - \underbrace{\frac{\partial\mathcal{L}}{\partial \dot{q}_i}}_{p_i}\mathrm{d}\dot{q}_i\right) - \frac{\partial\mathcal{L}}{\partial t}\mathrm{d}t.$
- By the definition of p_i and Lagrange's equations this simplifies to
$$\mathrm{d}H = \sum_i \left(\dot{q}_i\mathrm{d}p_i - \dot{p}_i\mathrm{d}q_i\right) - \frac{\partial\mathcal{L}}{\partial t}\mathrm{d}t.$$
- This shows that H responds only to changes $\mathrm{d}q_i,\mathrm{d}p_i,\mathrm{d}t$ and is **not** a function of $\dot{q}_i$. It **is** a function $H(q_i,p_i,t)$ as required.

- Since $\mathrm{d}H = \sum_i \left(\frac{\partial H}{\partial q_i}\mathrm{d}q_i + \frac{\partial H}{\partial p_i}\mathrm{d}p_i \right) + \frac{\partial H}{\partial t}\mathrm{d}t$, then comparing terms, this gives **Hamilton's equations:**

$$\dot{q}_i = \frac{\partial H}{\partial p_i}; \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}; \quad -\frac{\partial \mathcal{L}}{\partial t} = \frac{\partial H}{\partial t}.$$

- Therefore

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \sum_i \underbrace{(\dot{q}_i \dot{p}_i - \dot{p}_i \dot{q}_i)}_{\text{zero}} - \frac{\partial \mathcal{L}}{\partial t},$$

or

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{\partial \mathcal{L}}{\partial t}.$$

- This leads to the important result that, if the Lagrangian does not depend on time explicitly (i.e. $\partial \mathcal{L}/\partial t = 0$), **the Hamiltonian is conserved**.

$\boxed{\text{Lagrangian and Hamiltonian dynamics are central to formulating the wave equations of quantum mechanics}}$

LAGRANGIAN DYNAMICS — EXAMPLES

- **Simple harmonic motion**: $\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$.

  $\mathcal{L}$ does not depend on t:

  so the energy (Hamiltonian) $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$ is conserved.

- **Orbits in central potential**: $\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - V(r)$.

  $\mathcal{L}$ does not depend on ϕ or t:

  so the angular momentum (p_ϕ) $J = mr^2\dot{\phi}$ is conserved;

  so the energy (Hamiltonian) $E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + V(r)$ is conserved.

- **Symmetric top**: $\mathcal{L} = \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{1}{2}I_3(\dot{\chi} + \dot{\phi}\cos\theta)^2 - mgh\cos\theta$.

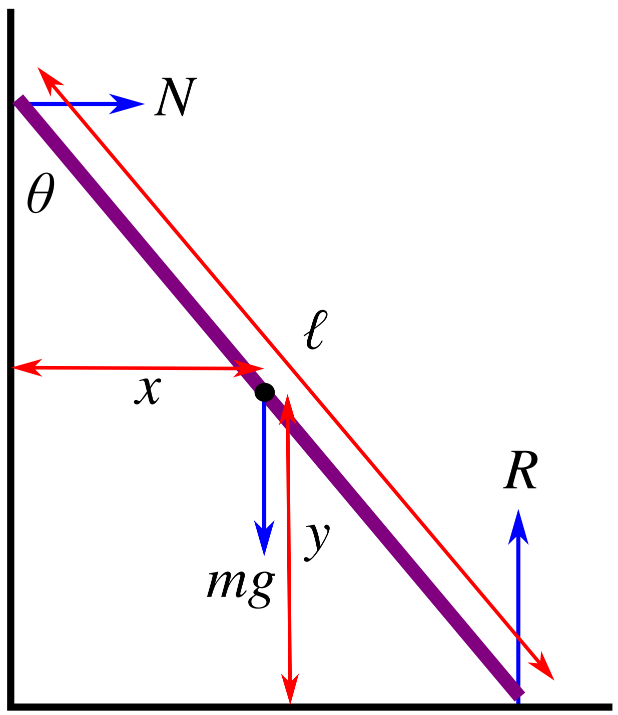
  $\mathcal{L}$ does not depend on ϕ, χ or t:

  so $J_z = I_3(\dot{\chi} + \dot{\phi}\cos\theta)\cos\theta + I_1\dot{\phi}\sin^2\theta$ is conserved;

  so the angular momentum (p_χ) $J_3 = I_3(\dot{\chi} + \dot{\phi}\cos\theta)$ is conserved;

  so the energy $E = \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{1}{2}I_3(\dot{\chi} + \dot{\phi}\cos\theta)^2 + mgh\cos\theta$ is conserved.

Example 1: The ladder from slide 7, problem 2.



- Total kinetic energy is $T = \frac{1}{6}m\ell^2\dot{\theta}^2$.
- Potential energy is $V = \frac{1}{2}mg\ell\cos\theta$.
- So the Lagrangian is $\mathcal{L} = \frac{1}{6}m\ell^2\dot{\theta}^2 - \frac{1}{2}mg\ell\cos\theta$.
- The conjugate momentum $p_\theta = \dfrac{\partial\mathcal{L}}{\partial\dot{\theta}}$ is

$$p_\theta = \tfrac{1}{3}m\ell^2\dot{\theta}.$$

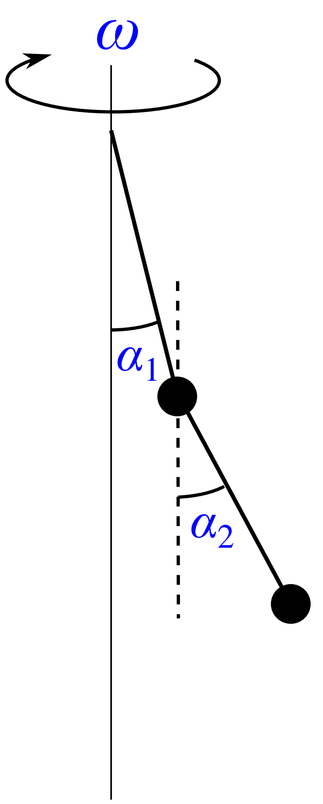
- So the equation of motion $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = \dfrac{\partial\mathcal{L}}{\partial\theta}$ is

$$\tfrac{1}{3}m\ell^2\ddot{\theta} = \tfrac{1}{2}mg\ell\sin\theta,$$

or

$$\ddot{\theta} = \frac{3g\sin\theta}{2\ell}.$$

Example 2: A rotating, planar double pendulum (each length ℓ), is as shown. Find the equations of motion for the small angles α_1 and α_2.



- The KE due to rotation about axis is
$\approx \frac{1}{2}m(\alpha_1\ell\omega)^2 + \frac{1}{2}m((\alpha_1\ell+\alpha_2\ell)\omega)^2 \approx \frac{1}{2}m\omega^2\ell^2\left(\alpha_1^2+(\alpha_1+\alpha_2)^2\right)$
(using a small angle approximation for sin).

- The KE in the plane of the masses is
$\approx \frac{1}{2}m(\ell\dot\alpha_1)^2 + \frac{1}{2}m(\ell\dot\alpha_1+\ell\dot\alpha_2)^2 \approx \frac{1}{2}m\ell^2\left(\dot\alpha_1^2+(\dot\alpha_1+\dot\alpha_2)^2\right)$.

- So the total KE is
$$T \approx \frac{1}{2}m\ell^2\left(\omega^2\left(\alpha_1^2+(\alpha_1+\alpha_2)^2\right)+\left(\dot\alpha_1^2+(\dot\alpha_1+\dot\alpha_2)^2\right)\right).$$

- The PE is
$mg\ell(1-\cos\alpha_1)+mg\left(\ell(1-\cos\alpha_1)+\ell(1-\cos\alpha_2)\right) \approx \frac{1}{2}mg\ell(2\alpha_1^2+\alpha_2^2)$.
(using small angle approximation for cos).

- So the Lagrangian is
$\mathcal{L} = \frac{1}{2}m\ell^2\left(\omega^2\left(\alpha_1^2+(\alpha_1+\alpha_2)^2\right)+\left(\dot\alpha_1^2+(\dot\alpha_1+\dot\alpha_2)^2\right)\right) - \frac{1}{2}mg\ell(2\alpha_1^2+\alpha_2^2)$
(strictly this is only approximate, given the small angle approximations made).

- The conjugate momenta are $p_1 = \dfrac{\partial \mathcal{L}}{\partial \dot{\alpha}_1} = m\ell^2(2\dot{\alpha}_1 + \dot{\alpha}_2)$ and $p_2 = \dfrac{\partial \mathcal{L}}{\partial \dot{\alpha}_2} = m\ell^2(\dot{\alpha}_1 + \dot{\alpha}_2)$.
- Using $\dfrac{\mathrm{d}}{\mathrm{d}t}(p_1) = \dfrac{\partial \mathcal{L}}{\partial \alpha_1}$, then $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(m\ell^2(2\dot{\alpha}_1 + \dot{\alpha}_2)\right) = \frac{1}{2}m\ell^2\omega^2(4\alpha_1 + 2\alpha_2) - 2mg\ell\alpha_1$, or
$$2\ddot{\alpha}_1 + \ddot{\alpha}_2 = \omega^2(2\alpha_1 + \alpha_2) - \frac{2g}{\ell}\alpha_1.$$
- Similarly
$$\ddot{\alpha}_1 + \ddot{\alpha}_2 = \omega^2(\alpha_1 + \alpha_2) - \frac{g}{\ell}\alpha_2.$$
- These can be combined to give the equations of motion for α_1 and α_2:
$$\ddot{\alpha}_1 = \omega^2\alpha_1 - \frac{2g}{\ell}\alpha_1 + \frac{g}{\ell}\alpha_2,$$
and
$$\ddot{\alpha}_2 = \omega^2\alpha_2 - \frac{2g}{\ell}\alpha_2 + \frac{2g}{\ell}\alpha_1.$$