CLASSICAL DYNAMICS

(20 lectures) Prof M. C. Payne

- **Newtonian Mechanics, frames of reference:** ($\approx 2\frac{1}{2}$ lectures)
  including rotating frames, 'fictitious' forces (centrifugal and Coriolis forces), etc.
- **Orbits:** ($\approx 1\frac{1}{2}$ lectures this term, ≈ 2 lectures next term)
  circular and elliptical, Kepler's laws etc.
- **Rigid body dynamics:** ($\approx 2\frac{1}{2}$ lectures)
  moment of inertia (tensor), Euler's equations, gyroscope etc.
- **Introduction to Lagrangian mechanics:** (≈ 1 lectures)
  Hamilton's principle and Lagrange's equation, etc.
- **Normal Modes:** (≈ 2 lectures)
  for many-particle systems, degrees of freedom, etc.
- **Elasticity:** ($\approx 3\frac{1}{2}$ lectures)
  Hooke's law, Young's modulus, bending of beams, etc.
- **Fluid Dynamics:** (≈ 5 lectures)
  basic principles, Bernoulli's theorem, vortices, viscosity, etc.

NEWTONIAN MECHANICS

- **Newtonian Mechanics** is:
  - **Non-relativistic** i.e. velocities $v \ll c$ (speed of light $= 3 \times 10^8$ m s^{-1});
  - **Classical** i.e. Et, $etc \gg \hbar$, (Planck's constant $= 1.05 \times 10^{-34}$ J s).
- **Assumptions**:
  - mass independent of velocity, time or frame of reference;
  - measurements of length and time are independent of the frame of reference;
  - all parameters can be known precisely.
- **Mechanics**:

  = Statics (absence of motion);

  \+ Kinematics (description of motion, using vectors for position and velocity);

  \+ Dynamics (prediction of motion, and involves forces and/or energy).

Basic Principles of Newtonian Dynamics

- Essentially, the whole of this course involves applications of Newton's laws.

  Masses accelerate if a force is applied.

  (Or if there is no force applied, there is no acceleration, i.e. Newton's 1st law.)

- Newton's 3rd law: if one body exerts a force on a second body, the second body exerts an equal and opposite force on the first.
- The rate of change of momentum (mass $\times$ velocity) is equal to the applied force.
- Vectorially, momentum $\boxed{\boldsymbol{p} = m\boldsymbol{v}}$ so

$$\boxed{\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \frac{\mathrm{d}(m\boldsymbol{v})}{\mathrm{d}t} = \boldsymbol{F}\,.}$$

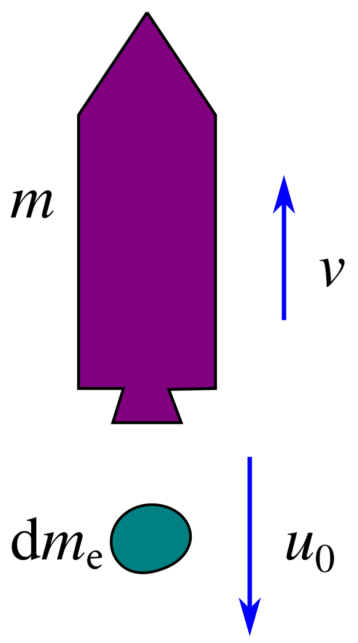
- Usually m is a constant so

$$m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = m\boldsymbol{a} = \boldsymbol{F}\,.$$

- The general case is

$$m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + \frac{\mathrm{d}m}{\mathrm{d}t}\boldsymbol{v} = \boldsymbol{F}\,.$$

Example: A rocket.



- Rocket of mass $m(t)$ moving with velocity $v(t)$ expels a mass dm_e of exhaust gases backwards at velocity $-u_0$ relative to the rocket.
- In the absence of gravity or other external forces -$u_0\, dm_e + m\, dv = 0$.
- (This is easiest to see in the instantaneous rest frame of the rocket)
- However, $dm_e = -dm$, so $u_0\, dm + m\, dv = 0$.
- Integrating, $v = u_0 \log(m_i/m_f)$, where $m_{i,f}$ are the initial and final masses.
- For a rocket accelerating upwards against gravity

$$m\frac{dv}{dt} + \frac{dm}{dt}u_0 + mg = 0.$$

SIMPLE HARMONIC OSCILLATOR (SHO)

- Mass m moving in one dimension with coordinate x on a spring with restoring force $F = -kx$.
  The constant k is known as the **spring constant**.
  Note: this is usually a good approximation close to *any* potential minimum (discussed later).
- Newtonian equation of motion: $m\ddot{x} = -kx$, where $\dot{x}$ denotes $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ etc.
- General solution: $x = A\cos\omega t + B\sin\omega t$ where $\omega^2 = k/m$.
  (Or can also write solution as $x = \mathrm{Re}(C\mathrm{e}^{\mathrm{i}\omega t})$, where C is complex.)
- Can integrate the equation of motion to get a conserved quantity — the **energy**.
- Multiplying the equation of motion by $\dot{x}$ gives

$$m\dot{x}\ddot{x} + k\dot{x}x = 0 \quad \text{so} \quad \boxed{\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 = E = \text{constant.}}$$

- Here the quantity $T \equiv \frac{1}{2}m\dot{x}^2$ is the **kinetic energy** of the mass and $V \equiv \frac{1}{2}kx^2$ is the **potential energy** stored in the spring.
- For many dynamical systems (such as the SHO) the time t does not appear explicitly in the equations of motion and the **total energy** $E = T + V$ is conserved. This conserved quantity is also known as the **Hamiltonian**.

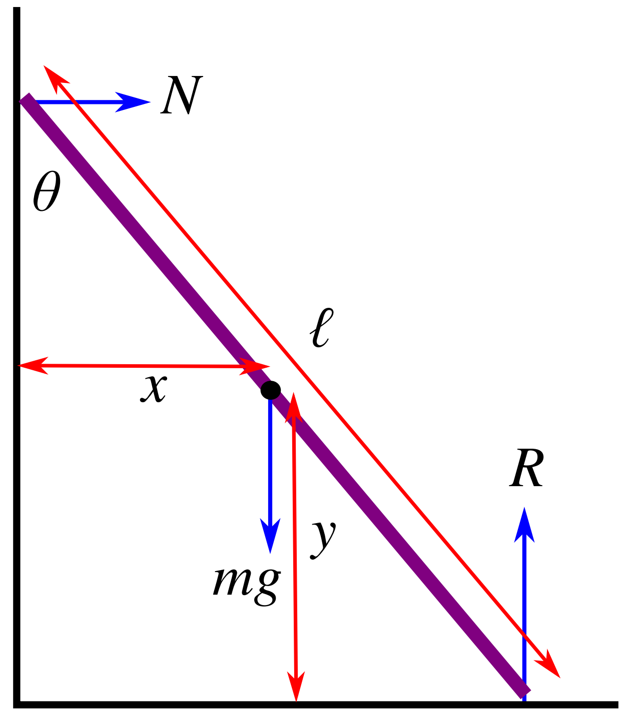
The Energy Method

- If, from physical grounds, energy is conserved, then can always derive the equations of motion of systems that only have *one degree of freedom* (such as the Simple Harmonic Oscillator): $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = E$ so $\dot{x}(m\ddot{x} + kx) = 0$ so $m\ddot{x} = -kx$.
- This works because $\dot{x}$ is generally non-zero. This is called the **energy method**.
- Sometimes the energy method can be used to derive the equations of motion of much more complicated systems with n degrees of freedom in a similar way. It is not rigorous, but works for most of the systems studied in this course. The theoretically more advanced methods of **Lagrangian** and **Hamiltonian** mechanics derive the equations of motion from a variational principle — see later. They are rigorous, but still use the kinetic energy T and potential energy V (actually in the combination $\mathcal{L} = T - V$).
- Later in the course the energy method will be used to derive the equation of motion of a particle at radius r in a central force: $\frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r) = E$.

  Differentiating with respect to time gives: $\dot{r}\left(m\ddot{r} + \frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}r}\right) = 0$.

  Which, by cancelling $\dot{r}$, gives the equation of motion.

Example: A ladder leaning against a smooth wall, resting on a smooth floor.



Approaches to solving this problem include:

1) Newtonian method using reaction forces N and R.
   Take moments to get the angular acceleration.

2) The energy method (using θ as *the* variable):
   (a) Work out the Potential energy.
   This is $V = \frac{1}{2}mg\ell\cos\theta$.
   (b) Work out the Kinetic energy.
   This the sum of the kinetic energy of the centre of mass plus the energy of rotation about the centre of mass:

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2,$$

where $I = \frac{1}{12}m\ell^2$ is the moment of inertia of a uniform rod about its centre.

The coordinates of centre of mass are

$$x = \tfrac{1}{2}\ell \sin\theta, \qquad y = \tfrac{1}{2}\ell\cos\theta.$$

The velocities are

$$\dot{x} = \tfrac{1}{2}\ell\cos\theta\dot{\theta}, \qquad \dot{y} = -\tfrac{1}{2}\ell\sin\theta\dot{\theta}.$$

Thus

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}I\dot{\theta}^2 = \cdots = \tfrac{1}{6}m\ell^2\dot{\theta}^2.$$

(c) Apply the energy method:

$$\frac{\mathrm{d}(T+V)}{\mathrm{d}t} = 0,$$

so

$$\dot{\theta}\left(\tfrac{1}{3}\ddot{\theta}m\ell^2 - \tfrac{1}{2}mg\ell\sin\theta\right) = 0.$$

(d) So the equation of motion is:

$$\ddot{\theta} = \frac{3g}{2\ell}\sin\theta.$$

REVISION: VECTOR CALCULUS

- In dynamics **vectors** are used to describe the positions, velocities and accelerations of particles and other bodies, as well as the forces and couples that act on them.

  Notation, e.g. for Cartesian coordinates:

  $$\boldsymbol{r} = x\widehat{\boldsymbol{e}}_x + y\widehat{\boldsymbol{e}}_y + z\widehat{\boldsymbol{e}}_z = x\widehat{\boldsymbol{i}} + y\widehat{\boldsymbol{j}} + z\widehat{\boldsymbol{k}} = (x, y, z).$$

  Will now do a brief revision of vectors, vector functions, vector identities and integral theorems.

- **Scalar product:**

  $$\boldsymbol{a} \cdot \boldsymbol{b} = a_x b_x + a_y b_y + a_z b_z = |a|\,|b| \cos\theta,$$

  (θ being the angle between the $\boldsymbol{a}$ and $\boldsymbol{b}$).

- **Vector product:**

  $$\boldsymbol{a} \times \boldsymbol{b} = (a_y b_z - a_z b_y, \ldots),$$

  of magnitude $|a|\,|b| \sin\theta$, in a direction perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$.

  Note: $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{a} \times \boldsymbol{b}$ form a *right handed set*.

- Useful vector identities include:

1) Scalar triple product

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$$

(which represents the volume of a parallelepiped with edges $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$).

Properties:

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}$$ (Interchange of dot and cross)

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=\boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a})=-\boldsymbol{b}\cdot(\boldsymbol{a}\times\boldsymbol{c})$$ (Permutations change sign)

2) Vector triple product

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}\,.$$

Note: vector outside bracket appears in both products, and outer pair takes the plus sign.

Gradient operator ∇

- In Cartesians coordinates this vector differential operator is

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right).$$

- Giving three operators:

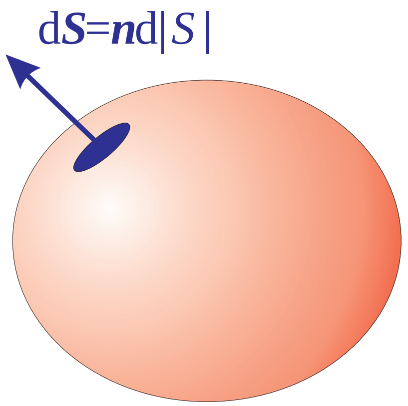
grad$(\varPhi)$	$\nabla\varPhi$	**vector** gradient of a **scalar** field $\varPhi$;
div$(\boldsymbol{E})$	$\nabla\cdot\boldsymbol{E}$	**scalar divergence** of a **vector** field $\boldsymbol{E}$;
curl$(\boldsymbol{E})$	$\nabla\times\boldsymbol{E}$	**vector curl** of a **vector** field $\boldsymbol{E}$.

Note: $\nabla\cdot(\nabla\times\boldsymbol{A}) = 0$ for *all* $\boldsymbol{A}$; and $\nabla\times(\nabla\varPhi) = 0$ for *all* $\varPhi$.

- A further useful identity is $\nabla\times(\nabla\times\boldsymbol{E}) = \nabla(\nabla\cdot\boldsymbol{E}) - \nabla^2\boldsymbol{E}$.

Note: in spherical and cylindrical coordinates the corresponding expressions are generally much more complicated so be careful to use the correct form of the differential operators — use the 'Mathematical Formula Handbook'.

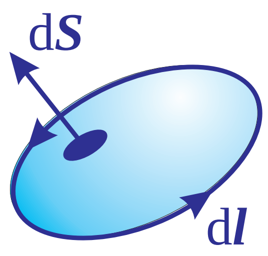
Divergence Theorem (Gauss' Theorem)



- Relates integral of flux vector $\boldsymbol{E}$ through a closed surface $\boldsymbol{S}$ ($\boldsymbol{n}$ outwards) to volume integral of $\nabla\cdot\boldsymbol{E}$ over the enclosed volume.

$$\boxed{\oint \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{S} = \int \nabla\cdot\boldsymbol{E}\,\mathrm{d}V.}$$

Stokes' Theorem



- Relates line integral vector $\boldsymbol{E}$ around closed loop $\boldsymbol{l}$ to surface integral of $\nabla\times\boldsymbol{E}$ over any surface bounded by the loop.

$$\boxed{\oint \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{l} = \int (\nabla\times\boldsymbol{E})\cdot \mathrm{d}\boldsymbol{S}.}$$

MECHANICS — REVIEW OF PART IA

Revision of dynamics of many-particle system.

- System of N particles. The ath particle of mass m_a is at position $\boldsymbol{r}_a$ and has velocity $\boldsymbol{v}_a$.
  It is acted on by an external force $\boldsymbol{F}_{a0}$ and internal forces $\boldsymbol{F}_{ab}$ from other particles.
- Total mass is $\boxed{M \equiv \sum_a m_a.}$
- The centre of mass $\boldsymbol{R}$ is defined by $\boxed{M\boldsymbol{R} \equiv \sum_a m_a \boldsymbol{r}_a.}$
- **Other concepts:**
  Total momentum $\boldsymbol{P}$. Total angular momentum $\boldsymbol{J}$.
  Total external force $\boldsymbol{F}_0$ and couple $\boldsymbol{G}_0$.
  Kinetic energy T, potential energy U, total energy E.
- **Total momentum** is changed by the **total external force**.
- **Total angular momentum** is changed by the **total external couple**.

- **Intrinsic angular momentum:** $\boldsymbol{J}'$ defined in the frame S' in which $\boldsymbol{P}' = 0$
  i.e. in the **zero momentum frame** (ZMF) (sometimes called the centre of mass (CoM) frame).
  The intrinsic angular momentum is independent of origin.

- The **ZMF** is thus special, and is often a good choice of reference frame.

- Galilean transformation from ZMF frame S' to frame S in which S' is moving at velocity $\boldsymbol{V}$ with the CoM at the origin in S' and, hence, at $\boldsymbol{R} = \boldsymbol{V}t$ in S.
  - **Momentum**: $\boldsymbol{P} = \boldsymbol{P}' + M\boldsymbol{V} = M\boldsymbol{V}$ (as $\boldsymbol{P}' = 0$).
  - **Angular Momentum**: $\boldsymbol{J} = \boldsymbol{J}' + M\boldsymbol{R}\times\boldsymbol{V} = \boldsymbol{J}'$ (as $\boldsymbol{R} = \boldsymbol{V}t$).
  - **Kinetic energy**: $T = T' + \frac{1}{2}MV^2$.

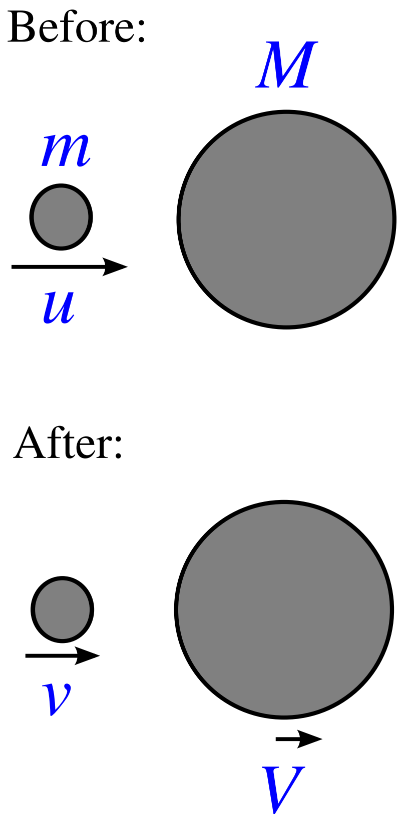
- **Impulse**: this is the integrated force over time, which is a change of linear momentum
  $$\int \boldsymbol{F}\,\mathrm{d}t = \Delta\boldsymbol{P}.$$
  Using impulses can often be very useful in collision problems.

  Note: if offset from the CoM, an impulse will also change the angular momentum.

OVERALL MOTION OF A BODY

- Development from Newton's Laws: $m_a \ddot{\boldsymbol{r}}_a = \boldsymbol{F}_a$,
  where $a = 1, N$ for the ath of N particles.
- **Overall motion** $\sum_a m_a \ddot{\boldsymbol{r}}_a = \sum_a \boldsymbol{F}_a = \sum_a \boldsymbol{F}_{a0} + \sum_a \sum_b \boldsymbol{F}_{ab}$,
  where $\boldsymbol{F}_{a0}$ is the **external** force on particle a and $\boldsymbol{F}_{ab}$ is the force on a due to b. Since $\boldsymbol{F}_{ab} = -\boldsymbol{F}_{ba}$ by Newton's 3rd Law, the $\sum_a \sum_b$ term above sums to zero.
- Define $M \equiv \sum_a m_a$ and $M\boldsymbol{R} \equiv \sum_a m_a \boldsymbol{r}_a$. $\boldsymbol{R}$ is the position of the Centre of Mass.
- With these definitions $M\ddot{\boldsymbol{R}} = \sum_a m_a \ddot{\boldsymbol{r}}_a = \sum_a \boldsymbol{F}_{a0} \equiv \boldsymbol{F}_0$.
- The Centre of Mass moves as if it were a particle of mass M acted upon by the total external force $\boldsymbol{F}_0$.
- In terms of momentum $\dot{\boldsymbol{p}}_a = \boldsymbol{F}_a$ and $\dot{\boldsymbol{P}} = \boldsymbol{F}_0$, where $\boldsymbol{P} = \sum_a m_a \dot{\boldsymbol{r}}_a$ is the total momentum.

Example: A simple elastic collision (note not in ZMF!).



- Consider a head-on collision of a small mass m with a stationary large mass M, with no external forces. There is an equal and opposite impulse acting on each mass.
- The impulse is $\Delta p = \int F\,\mathrm{d}t = MV$ and $\Delta p = \int F\,\mathrm{d}t = m(u-v)$.
- Eliminating the impulse, we see that linear momentum is conserved. $mu = mv + MV$, or $V = \left(\frac{m}{M}\right)(u-v)$.
- If the collision is perfectly elastic, then KE is also conserved. So $\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + \frac{1}{2}MV^2$, or $V^2 = \left(\frac{m}{M}\right)(u^2 - v^2)$.
- Eliminating V, then $\left(\frac{m}{M}\right)(u-v) = (u+v)$ (or $(u-v)=0$, i.e. no collision), so

$$v = -\left(\frac{M-m}{M+m}\right)u.$$

Note: (a) if $M = m$ then $v = 0$ (and $V = u$), (b) if $M \longrightarrow \infty$ then $v \longrightarrow -u$ (and $V \longrightarrow 0$).

Couple/Torque

- **Couple, torque**: $\boxed{\boldsymbol{G} \equiv \boldsymbol{r} \times \boldsymbol{F}\,.}$
- **Angular momentum:** $\boxed{\boldsymbol{J} \equiv \boldsymbol{r} \times \boldsymbol{p}\,.}$
- Since $\dot{\boldsymbol{p}}_a = \boldsymbol{F}_a$, then

$$\sum_a \boldsymbol{r}_a \times \dot{\boldsymbol{p}}_a = \sum_a \boldsymbol{r}_a \times \boldsymbol{F}_a .$$

- Expand RHS:

$$\text{RHS} = \sum_a \underbrace{\boldsymbol{r}_a \times \boldsymbol{F}_{a0}}_{\text{external force}} + \underbrace{\sum_a \sum_{b \neq a} \boldsymbol{r}_a \times \boldsymbol{F}_{ab},}_{\text{internal forces}}$$

and write the internal forces using Newton's 3rd Law, as

$$\sum_b \sum_{a<b} (\boldsymbol{r}_a - \boldsymbol{r}_b) \times \boldsymbol{F}_{ab} = \frac{1}{2} \sum_b \sum_a (\boldsymbol{r}_a - \boldsymbol{r}_b) \times \boldsymbol{F}_{ab}$$

which is 0, because $\boldsymbol{F}_{ab}$ is ***assumed*** to be along the line between particles a and b.

- The LHS for one particle can be written $\dot{\boldsymbol{J}}_a = \dfrac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}_a \times \boldsymbol{p}_a) = \underbrace{\dot{\boldsymbol{r}}_a \times \boldsymbol{p}_a}_{=0 \text{ since } m\dot{\boldsymbol{r}}=\boldsymbol{p}} + \boldsymbol{r}_a \times \dot{\boldsymbol{p}}_a$

- For the system of particles

$$\dot{\boldsymbol{J}} \equiv \sum_a \dot{\boldsymbol{J}}_a = \sum_a \boldsymbol{r}_a \times \dot{\boldsymbol{p}}_a = \text{RHS} = \sum_a \boldsymbol{r}_a \times \boldsymbol{F}_{a0} \equiv \boldsymbol{G}_0$$

i.e.

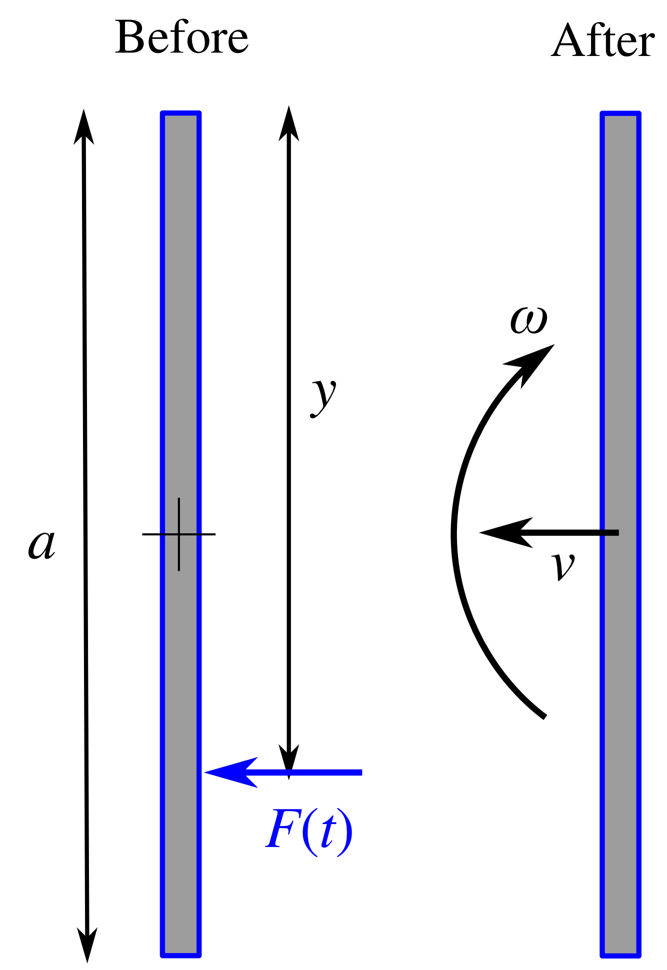
$$\boxed{\dot{\boldsymbol{J}} = \boldsymbol{G}_0,}$$

where $\boldsymbol{G}_0$ is the resultant couple from all **external** forces.

Example: The ‘centre of percussion’.

Consider a uniform rod of length a, mass m, initially at rest, suspended vertically on a smooth pivot. The rod is struck by a horizontal force $F(t)$ at a distance y below the pivot. $F(t)$ acts over a short time. Find y so that there is **no reaction force from the pivot**.

- The impulse provided by the force is $\int F\,\mathrm{d}t$.
- After the blow, the CofM of the rod moves at v, and it rotates at ω about its CofM.
- Since the rod pivots about its top end $\omega = 2v/a$.



- Consider the linear (horizontal) and angular (about CofM) momentum changes produced by the impulse:

$$\int F(t)\,\mathrm{d}t = mv, \qquad \text{and} \qquad \int F(t)(y - \tfrac{1}{2}a)\,\mathrm{d}t = \underbrace{I}_{=ma^2/12}\,\omega.$$

- So

$$\begin{aligned} mv(y - \tfrac{1}{2}a) &= \frac{ma^2}{12}\underbrace{\omega}_{=2v/a}, \\ (y - \tfrac{1}{2}a) &= \tfrac{1}{6}a, \end{aligned}$$

i.e. $y = \frac{2}{3}a$. This is the 'centre of percussion'.

Notes:

(a) If the impulse is applied at any other point there has to be a reaction force from the pivot to impose the relation $\omega = 2v/a$.

(b) If the impulse was *higher up*, there would be a reaction force from the pivot, to the *right*.

(c) If the impulse was *lower down*, there would be a reaction force from the pivot, to the *left*.

CHOICE OF ORIGIN

- Suppose the origin is displaced by a constant $\boldsymbol{a}$, giving new coordinates $\boldsymbol{r}$ with $\boldsymbol{r} = \boldsymbol{r}' + \boldsymbol{a}$.
- Then $\dot{\boldsymbol{r}} = \dot{\boldsymbol{r}}'$ and the overall motion is unaffected.
- What about the angular momentum $\boldsymbol{J}$? For one particle $\boldsymbol{J}_a = \boldsymbol{J}'_a + \boldsymbol{a}\times\boldsymbol{p}_a$, or for the system

$$\boldsymbol{J} = \boldsymbol{J}' + \sum_a \boldsymbol{a}\times\boldsymbol{p}_a = \boldsymbol{J}' + \boldsymbol{a}\times\boldsymbol{P},$$

  i.e. Thus, in general, $\boldsymbol{J}$ depends on the choice of origin *unless* $\boldsymbol{P} = 0$.
- **Intrinsic angular momentum:** $\boldsymbol{J}$ in the frame in which $\boldsymbol{P} = 0$ (i.e. ZMF). The Intrinsic angular momentum is independent of origin.
- The ZMF is thus special and is often a good choice for the reference frame.
- Similarly $\boldsymbol{G} = \boldsymbol{G}' + \boldsymbol{a}\times\boldsymbol{F}$.

KINETIC/POTENTIAL/TOTAL ENERGY

- **Work done:** force $\times$ (distance moved parallel to force).
- So change in **energy** $\boxed{= \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}\,.}$
- For a single particle

$$\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = m\ddot{\boldsymbol{r}} \cdot \mathrm{d}\boldsymbol{r} = m(\ddot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}})\mathrm{d}t\,.$$

But $\ddot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} = \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}}\right)$, so

$$\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \mathrm{d}(\tfrac{1}{2}mv^2).$$

- **Kinetic energy:** $\boxed{T \equiv \frac{1}{2}mv^2.}$
- Work done on particle $=$ change in kinetic energy.
- For a system of particles

$$\begin{aligned} \mathrm{d}T &= \sum_a \mathrm{d}T_a = \sum_a \boldsymbol{F}_a \cdot \mathrm{d}\boldsymbol{r}_a, \\ &= \sum_a \boldsymbol{F}_{a0} \cdot \mathrm{d}\boldsymbol{r}_a + \sum_b \sum_{a<b} \boldsymbol{F}_{ab} \cdot (\mathrm{d}\boldsymbol{r}_a - \mathrm{d}\boldsymbol{r}_b), \end{aligned}$$

where $\boldsymbol{F}_{ab} = -\boldsymbol{F}_{ba}$ (i.e. Newton's 3rd law) has been used.

- The ab-term can be written as $-\mathcal{F}_{ab}\,\mathrm{d}|\boldsymbol{r}_a - \boldsymbol{r}_b|$, where $\mathcal{F}_{ab}$ has magnitude $= |F_{ab}|$ and is positive if force is attractive, negative if repulsive.

- **Potential energy:** U is defined as
$$\mathrm{d}U = \sum_b \sum_{a<b} \mathcal{F}_{ab}\,\mathrm{d}|\boldsymbol{r}_a - \boldsymbol{r}_b|.$$

- Note the zero of $U = \displaystyle\int \mathrm{d}U$ is undefined. It is often chosen as $U = 0$ with particles at infinite separation, giving negative U for a system of particles with attractive forces.

- For a rigid body $\mathrm{d}U = 0$ since $|\boldsymbol{r}_a - \boldsymbol{r}_b|$ is fixed.

- **Total Energy:** $E = T + U$.

- As defined above
$$\mathrm{d}E = \mathrm{d}T + \mathrm{d}U = \sum_a \boldsymbol{F}_{a0} \cdot \mathrm{d}\boldsymbol{r}_a.$$

- The RHS term is the work done by external forces.

GALILEAN TRANSFORMATION

- Go from frame S' to S with $\boldsymbol{r} = \boldsymbol{r}' + \boldsymbol{V}t$; $\boldsymbol{V}$ constant; $t = t'$.

- **Momentum**

$$\boldsymbol{p} = \boldsymbol{p}' + m\boldsymbol{V}; \quad \boldsymbol{P} = \boldsymbol{P}' + M\boldsymbol{V},$$

i.e. $\boldsymbol{P}$ in S and $\boldsymbol{P}'$ in S' change equally (or remain unchanged in time if there is no external force). If $\boldsymbol{P}' = 0$, then S' is the **zero-momentum** or **Centre of Mass** frame.

- **Angular momentum**

$$\boldsymbol{J} = \sum_a (\boldsymbol{r}'_a + \boldsymbol{V}t) \times (\boldsymbol{p}'_a + m_a \boldsymbol{V}).$$

- There are 4 terms. The 4th is $\boldsymbol{V} \times \boldsymbol{V} = 0$. The others give

$$\boldsymbol{J} = \boldsymbol{J}' + \boldsymbol{V}t \times \boldsymbol{P}' + \sum_a \boldsymbol{r}'_a \times m_a \boldsymbol{V}.$$

The last term is

$$\sum_a (m_a \boldsymbol{r}'_a) \times \boldsymbol{V} = M\boldsymbol{R}' \times \boldsymbol{V}.$$

- Thus if S' is the zero-momentum frame, $\boldsymbol{P}' = \mathbf{0}$ and

$$\boxed{\underbrace{\boldsymbol{J}}_{\text{in } S} = \underbrace{\boldsymbol{J}'}_{\text{intrinsic}} + \underbrace{M\boldsymbol{R}' \times \boldsymbol{V}.}_{\text{motion of C of M in } S}}$$

- **Energy**

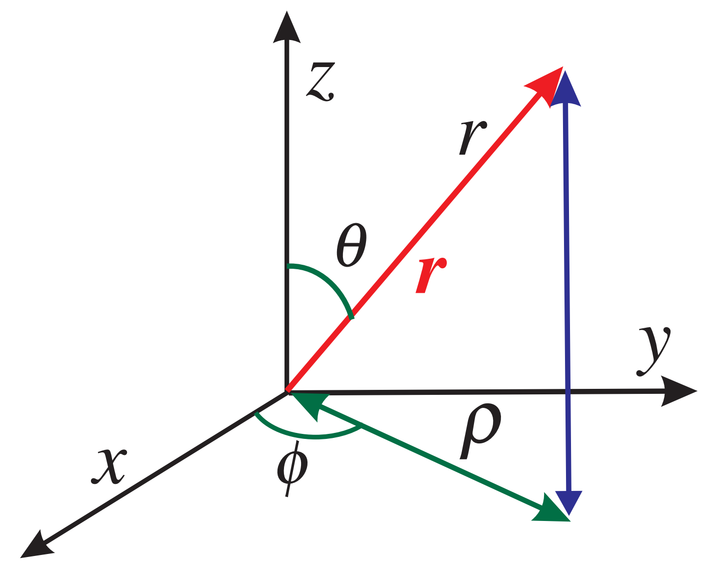
$$\begin{aligned} T &= \sum_a \tfrac{1}{2} m_a v_a^2 = \sum_a \tfrac{1}{2} m_a (\boldsymbol{v}_a' + \boldsymbol{V}) \cdot (\boldsymbol{v}_a' + \boldsymbol{V}) \\ &= T' + \underbrace{\sum_a m_a \boldsymbol{v}_a' \cdot \boldsymbol{V}}_{=0, \text{ if } S' = \text{zero-momentum frame}} + \tfrac{1}{2} M V^2. \end{aligned}$$

Or

$$\boxed{T = \text{KE in zero-momentum frame} + \tfrac{1}{2} M V^2.}$$

Note: the KE of a rotating rigid body is the KE of the linear motion of its CofM, plus the rotational KE about its CofM.

Coordinate Systems



- Position vector $\boldsymbol{r}$ has
  Cartesian coordinates (x, y, z), or
  cylindrical polar coordinates (ρ, ϕ, z), or
  spherical polar coordinates (r, θ, ϕ).

- Standard relation between coordinate systems:

$$\boxed{\begin{aligned} x &= \rho\cos\phi &&= r\sin\theta\cos\phi, \\ y &= \rho\sin\phi &&= r\sin\theta\sin\phi, \\ z &= z &&= r\cos\theta, \end{aligned}}$$

and

$$\boxed{\begin{aligned} \rho &= \sqrt{x^2+y^2}, \\ r &= \sqrt{x^2+y^2+z^2}. \end{aligned}}$$

- Define unit vectors $(\widehat{\boldsymbol{e}}_x, \widehat{\boldsymbol{e}}_y, \widehat{\boldsymbol{e}}_z)$ along x, y, z axes.

- Similarly for cylindrical and spherical coordinates, define unit vectors along directions of increasing coordinates.

  $(\widehat{\boldsymbol{e}}_\rho, \widehat{\boldsymbol{e}}_\phi, \widehat{\boldsymbol{e}}_z)$ along directions of increasing (ρ, ϕ, z).

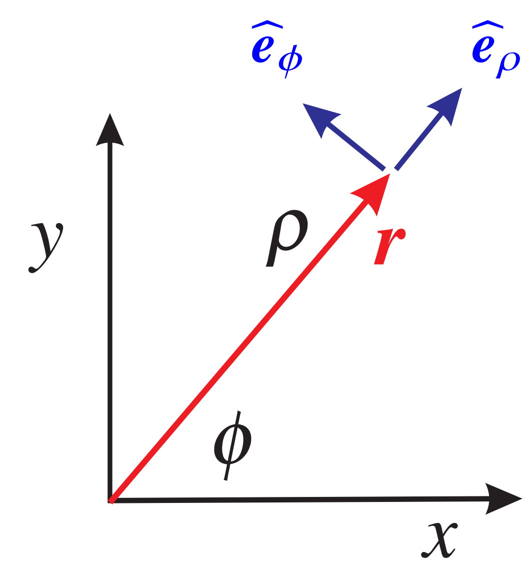
  $(\widehat{\boldsymbol{e}}_r, \widehat{\boldsymbol{e}}_\phi, \widehat{\boldsymbol{e}}_\theta)$ along directions of increasing (r, ϕ, θ).

- Position vectors are $\boldsymbol{r} = x\widehat{\boldsymbol{e}}_x + y\widehat{\boldsymbol{e}}_y + z\widehat{\boldsymbol{e}}_z = \rho\widehat{\boldsymbol{e}}_\rho + z\widehat{\boldsymbol{e}}_z = r\widehat{\boldsymbol{e}}_r$.

Notes: (a) the unit vectors are locally orthogonal in Cartesian, cylindrical and spherical polar coordinates, but only in the case of Cartesian coordinates are their directions constant. (b) Remember that the vector differential operators (i.e. grad, div, curl etc.) are complicated in cylindrical and spherical polar coordinates — look them up to be sure.

Dynamics in Cylindrical Polar Coordinates

- In Cartesians, the equation of motion of a particle is $m\ddot{\boldsymbol{r}} = \boldsymbol{F}$ or $m\ddot{x} = F_x$ etc. where the time derivatives $\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \equiv \dot{\boldsymbol{r}}, \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} \equiv \ddot{\boldsymbol{r}}$ etc.
- Consider cylindrical polars; ignore z-motion for the moment, $\boldsymbol{r} = \rho\,\widehat{\boldsymbol{e}}_\rho$ where $\widehat{\boldsymbol{e}}_\rho$, $\widehat{\boldsymbol{e}}_\phi$ and $\widehat{\boldsymbol{e}}_z$ are unit vectors in the directions of increasing ρ, ϕ, z.



- Note that the direction of the vectors $\widehat{\boldsymbol{e}}_\rho$ and $\widehat{\boldsymbol{e}}_\phi$ change as the particle moves: $\dot{\boldsymbol{r}} = \dot{\rho}\,\widehat{\boldsymbol{e}}_\rho + \rho\,\dot{\widehat{\boldsymbol{e}}}_\rho$
- As the particle moves from say P to P' in $\mathrm{d}t$, $\widehat{\boldsymbol{e}}_\rho$ and $\widehat{\boldsymbol{e}}_\phi$ rotate by $\mathrm{d}\phi$.
- Geometry gives $\mathrm{d}\widehat{\boldsymbol{e}}_\rho = \mathrm{d}\phi\,\widehat{\boldsymbol{e}}_\phi$ or $\dot{\widehat{\boldsymbol{e}}}_\rho = \dot{\phi}\,\widehat{\boldsymbol{e}}_\phi$ and similarly $\dot{\widehat{\boldsymbol{e}}}_\phi = -\dot{\phi}\,\widehat{\boldsymbol{e}}_\rho$ so

$$\boxed{\dot{\boldsymbol{r}} = \underbrace{\dot{\rho}\,\widehat{\boldsymbol{e}}_\rho}_{\text{radial}} + \underbrace{\rho\dot{\phi}\,\widehat{\boldsymbol{e}}_\phi}_{\text{transverse}}\ .}$$

- In cylindrical polar coordinates the radial velocity is $\dot{\rho}$ and the transverse velocity is $\rho\,\dot{\phi}$.

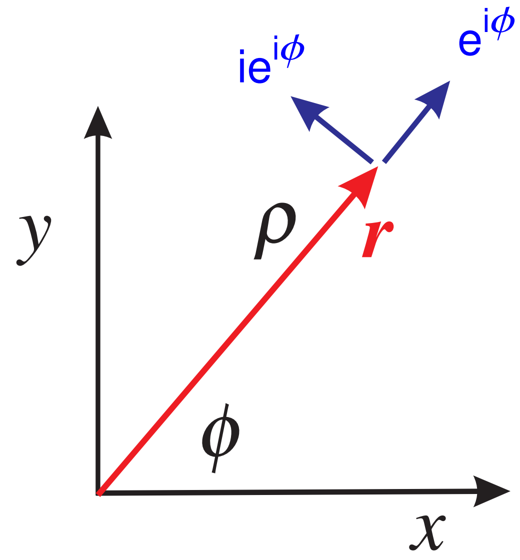
ACCELERATION IN POLAR COORDINATES

- Similarly, the rate of change of velocity:

$$\begin{aligned}\ddot{\boldsymbol{r}} &= \ddot{\rho}\,\widehat{\boldsymbol{e}}_\rho + \dot{\rho}\underbrace{\dot{\widehat{\boldsymbol{e}}}_\rho}_{\dot{\phi}\widehat{\boldsymbol{e}}_\phi} + \dot{\rho}\dot{\phi}\,\widehat{\boldsymbol{e}}_\phi + \rho\ddot{\phi}\,\widehat{\boldsymbol{e}}_\phi + \rho\dot{\phi}\underbrace{\dot{\widehat{\boldsymbol{e}}}_\phi}_{-\dot{\phi}\widehat{\boldsymbol{e}}_\rho} \\ &= \underbrace{(\ddot{\rho}-\rho\dot{\phi}^2)}_{\text{radial}}\widehat{\boldsymbol{e}}_\rho + \underbrace{(2\dot{\rho}\dot{\phi}+\rho\ddot{\phi})}_{\text{transverse}}\widehat{\boldsymbol{e}}_\phi .\end{aligned}$$

- The z-motion is independent: $(\ddot{\boldsymbol{r}})_z$ is just $\ddot{z}\,\widehat{\boldsymbol{e}}_z$ since $\dot{\widehat{\boldsymbol{e}}}_z = 0$.
- The radial acceleration is $\ddot{\rho}-\rho\dot{\phi}^2$, the second term being the **centripetal acceleration** required to keep a particle in an orbit of constant radius.
- The transverse acceleration is $2\dot{\rho}\dot{\phi}+\rho\ddot{\phi} = \dfrac{1}{\rho}\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\rho^2\dot{\phi}\right)$ which shows that it is related to the angular momentum per unit mass $\rho^2\dot{\phi}$.
- Spherical polars can be treated by putting $\boldsymbol{r} = r\,\widehat{\boldsymbol{e}}_r$, and expanding $\dot{\boldsymbol{r}}$ etc. with $\dot{\widehat{\boldsymbol{e}}}_r$ expressed in terms of $\widehat{\boldsymbol{e}}_r$, $\widehat{\boldsymbol{e}}_\theta$ and $\widehat{\boldsymbol{e}}_\phi$.

POLAR COORDINATES AND THE ARGAND DIAGRAM



- The complex plane $z \equiv x + \mathrm{i}y = \rho \mathrm{e}^{\mathrm{i}\phi}$ has the same structure as the two-dimensional plane
  $\boldsymbol{r} = x\widehat{\boldsymbol{e}}_x + y\widehat{\boldsymbol{e}}_y = \rho\widehat{\boldsymbol{e}}_\rho$.
- The unit vectors correspond to complex numbers:
  $\widehat{\boldsymbol{e}}_\rho \leftrightarrow \mathrm{e}^{\mathrm{i}\phi}, \qquad \widehat{\boldsymbol{e}}_\phi \leftrightarrow \mathrm{i}\mathrm{e}^{\mathrm{i}\phi}$.
- So the radial and transverse components can be derived.
  - **Velocity**:
  $$\frac{\mathrm{d}}{\mathrm{d}t}(\rho\mathrm{e}^{\mathrm{i}\phi}) = \dot{\rho}\mathrm{e}^{\mathrm{i}\phi} + \rho\dot{\phi}\,\mathrm{i}\mathrm{e}^{\mathrm{i}\phi}.$$
  - **Acceleration**:
  $$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\rho\mathrm{e}^{\mathrm{i}\phi}) &= \ddot{\rho}\mathrm{e}^{\mathrm{i}\phi} + 2\dot{\rho}\dot{\phi}\,\mathrm{i}\mathrm{e}^{\mathrm{i}\phi} + \rho\ddot{\phi}\mathrm{i}\mathrm{e}^{\mathrm{i}\phi} - \rho\dot{\phi}^2\,\mathrm{e}^{\mathrm{i}\phi}, \\ &= \underbrace{(\ddot{\rho} - \rho\dot{\phi}^2)}_{\text{radial}}\,\mathrm{e}^{\mathrm{i}\phi} + \underbrace{(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})}_{\text{transverse}}\,\mathrm{i}\mathrm{e}^{\mathrm{i}\phi}.\end{aligned}$$

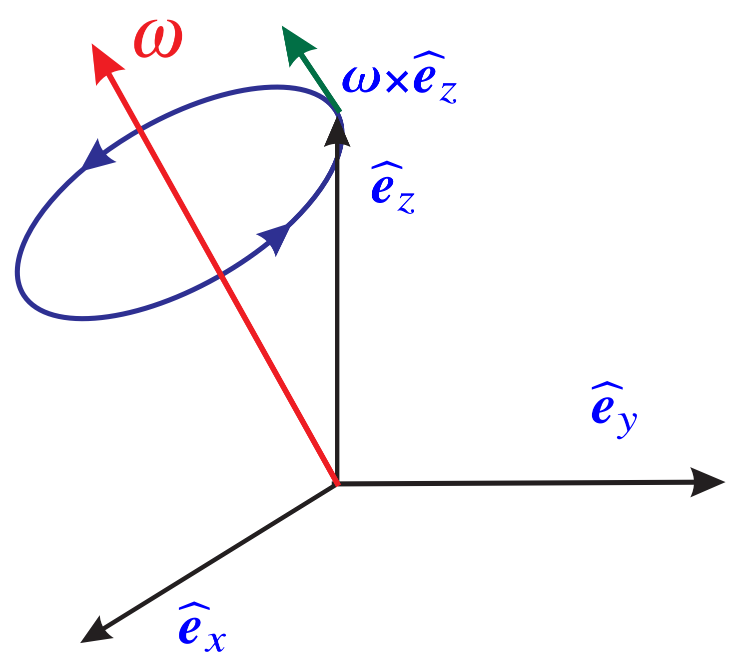
FRAMES IN RELATIVE MOTION

Suppose there is a frame S_0 in which $m\ddot{\boldsymbol{r}}_0 = \boldsymbol{F}$, with $\boldsymbol{F}$ generated by known physical causes. What is the apparent equation of motion in a moving frame S?

- **Case 1**: Suppose $\boldsymbol{r} = \boldsymbol{r}_0 - \boldsymbol{R}(t)$. Suppose the axes in S_0 and S remain parallel and $t = t_0$ (as always in classical physics): $\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{r}}_0 - \ddot{\boldsymbol{R}}$.
- For the special case $\ddot{\boldsymbol{R}} = 0$ (i.e. steady motion between frames), $m\ddot{\boldsymbol{r}} = m\ddot{\boldsymbol{r}}_0 = \boldsymbol{F}$, i.e. the **same** equation of motion applies in each frame (Galilean transformation).
- However, for general $\boldsymbol{R}(t)$: $m\ddot{\boldsymbol{r}} = m\ddot{\boldsymbol{r}}_0 - m\ddot{\boldsymbol{R}} = \boldsymbol{F} - m\ddot{\boldsymbol{R}}$.
- The **apparent force** in S includes both the actual force $m\ddot{\boldsymbol{r}}_0$ and a **fictitious force** of $-m\ddot{\boldsymbol{R}}$.
- Fictitious forces are: (a) associated with accelerated frames; (b) proportional to mass.

Note: According to general relativity, gravity is a fictitious force.

ROTATING FRAMES



- **Case 2**: Frame S rotates with angular velocity $\boldsymbol{\omega}$, so that the unit vectors rotate with respect to the inertial frame S_0.
- The rate of change is given by $\dot{\widehat{\boldsymbol{e}}}_z = \boldsymbol{\omega}\times\widehat{\boldsymbol{e}}_z$ etc.
- Let the frames coincide at $t=0$:

$$\boldsymbol{r}_0 = x\widehat{\boldsymbol{e}}_x + y\widehat{\boldsymbol{e}}_y + z\widehat{\boldsymbol{e}}_z = \boldsymbol{r},$$

and

$$\dot{\boldsymbol{r}}_0 = \dot{x}\widehat{\boldsymbol{e}}_x + x\dot{\widehat{\boldsymbol{e}}}_x + (y \text{ and } z \text{ terms }),$$

$$\dot{\boldsymbol{r}}_0 = \boldsymbol{v} + \boldsymbol{\omega}\times\boldsymbol{r}.$$

where $\boldsymbol{v} \equiv \dot{x}\widehat{\boldsymbol{e}}_x + \dot{y}\widehat{\boldsymbol{e}}_y + \dot{z}\widehat{\boldsymbol{e}}_z$ is the **apparent velocity** in S.

- The acceleration in S_0 is

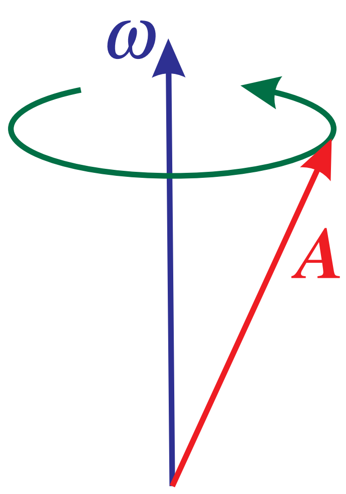
$$
\begin{aligned}
\ddot{\boldsymbol{r}}_0 &= \ddot{x}\widehat{\boldsymbol{e}}_x + 2\dot{x}\dot{\widehat{\boldsymbol{e}}}_x + x\ddot{\widehat{\boldsymbol{e}}}_x + y \text{ and } z \text{ terms} \\
&= \ddot{x}\widehat{\boldsymbol{e}}_x + 2(\boldsymbol{\omega}\times\widehat{\boldsymbol{e}}_x)\dot{x} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\widehat{\boldsymbol{e}}_x)x + y \text{ and } z \text{ terms} \\
&= \boldsymbol{a} + 2\boldsymbol{\omega}\times\boldsymbol{v} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}),
\end{aligned}
$$

  where $\boldsymbol{a} \equiv \ddot{x}\widehat{\boldsymbol{e}}_x + \ddot{y}\widehat{\boldsymbol{e}}_y + \ddot{z}\widehat{\boldsymbol{e}}_z$ is the **apparent acceleration** in S.

- Rewrite the momentum equation $m\ddot{\boldsymbol{r}}_0 = \boldsymbol{F}$ in terms of the apparent quantities $\boldsymbol{r}$, $\boldsymbol{v}$ and $\boldsymbol{a}$:

$$
\boxed{\underbrace{m\boldsymbol{a}}_{\text{apparent}} = \underbrace{\boldsymbol{F}}_{\text{real}} \quad \underbrace{-2m(\boldsymbol{\omega}\times\boldsymbol{v}) - m\,\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})}_{\text{fictitious}}.}
$$

- The observer in S has to add two **inertial** or **fictitious** forces to the eom:
  - Coriolis force: $\boxed{-2m(\boldsymbol{\omega}\times\boldsymbol{v}).}$
  - Centrifugal force: $\boxed{-m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}).}$



- There is an operator approach to rotating frames.

- For any vector $\boldsymbol{A}$ the rates of change in frame S_0 and in frame S are related by $\left[\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}\right]_{S_0} = \left[\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}\right]_S + \boldsymbol{\omega}\times\boldsymbol{A}$.

- Apply this operator relation twice to $\boldsymbol{r}$ ($\boldsymbol{r} = \boldsymbol{r}_0$ at $t = 0$):
$$\left[\frac{\mathrm{d}^2\boldsymbol{r}_0}{\mathrm{d}t^2}\right]_{S_0} = \left(\left[\frac{\mathrm{d}}{\mathrm{d}t}\right]_S + \boldsymbol{\omega}\times\right)\left(\left[\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right]_S + \boldsymbol{\omega}\times\boldsymbol{r}\right).$$

- Expanding and setting $\left[\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right]_S = \boldsymbol{v}$ and $\left[\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right]_S = \boldsymbol{a}$, and multiplying by m gives
$$m\ddot{\boldsymbol{r}}_0 = \boldsymbol{F} = m\boldsymbol{a} + 2m(\boldsymbol{\omega}\times\boldsymbol{v}) + m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}).$$

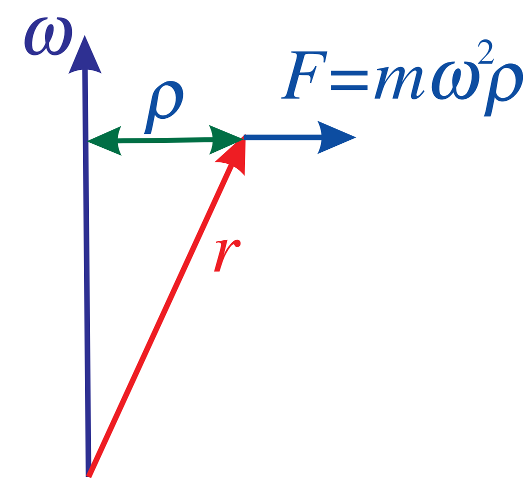
GENERAL ROTATING FRAME

- Observer moves on a path $\boldsymbol{R}(t)$ and uses a frame rotating at angular velocity $\boldsymbol{\omega}(t)$ which is *also* changing with time.
- From previous results and, because the time derivative now operates on $\boldsymbol{\omega}$, this gives the general formula for the acceleration measured in the observer's frame:

$$m\boldsymbol{a} = \boldsymbol{F} - 2m(\boldsymbol{\omega}\times\boldsymbol{v}) - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}) - m\ddot{\boldsymbol{R}} - m\dot{\boldsymbol{\omega}}\times\boldsymbol{r}.$$

- The extra $-m\dot{\boldsymbol{\omega}}\times\boldsymbol{r}$ term is called the 'Euler force'.

CENTRIFUGAL AND CORIOLIS FORCES



- **Centrifugal Force:** $-m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}) = m(\omega^2\boldsymbol{r} - (\boldsymbol{r}\cdot\boldsymbol{\omega})\boldsymbol{\omega})$,

  which can be written as

  $$= m\omega^2 \underbrace{(\boldsymbol{r} - (\boldsymbol{r}\cdot\widehat{\boldsymbol{\omega}})\widehat{\boldsymbol{\omega}})}_{\text{just } \rho\text{, outwards}}.$$

  So the Centrifugal Force is $m\omega^2\rho$ outwards.

- **Coriolis Force:** $-2m(\boldsymbol{\omega}\times\boldsymbol{v})$ appears if a body is moving with respect to a rotating frame.

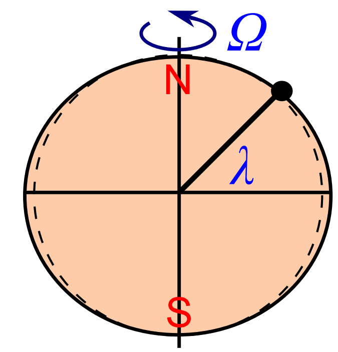
- Coriolis Force is a sideways force, perpendicular both to the rotation axis and to the velocity.

- Problems involving Coriolis Force can often be done by considering angular momentum.

Note: the minus signs, and the m in each term reminds us that these terms came from the 'other' (non-force) side of the equation, and the $\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})$ construction should remind you of the operator relation $[\mathrm{d}/\mathrm{d}t]_{S_0} = [\mathrm{d}/\mathrm{d}t]_S + \boldsymbol{\omega}\times$.

Fictitious Forces — Applications

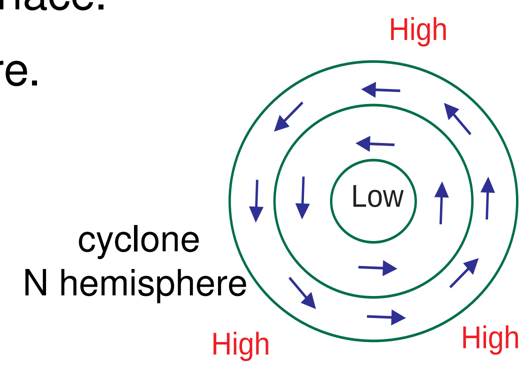
- **Centrifugal force** gives rise to the Earth's equatorial bulge:
  $$\sim \frac{\Omega^2 R}{g} \approx \frac{1}{300}.$$
  It can also be felt on roundabouts and other fairground rides/roller coasters.



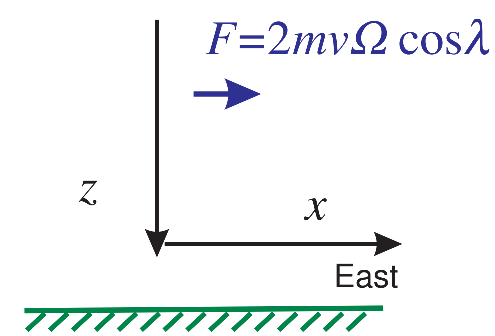
- **Coriolis force** due to motion on Earth's surface: $F = 2m\Omega v \sin\lambda$.

  This is the component of the force in the (local) plane of Earth's surface.
  Its direction is sideways and to the right in the Northern Hemisphere.
  (independent of direction of travel).

  Affects weather patterns (but *not* (disappointingly) bathtubs).

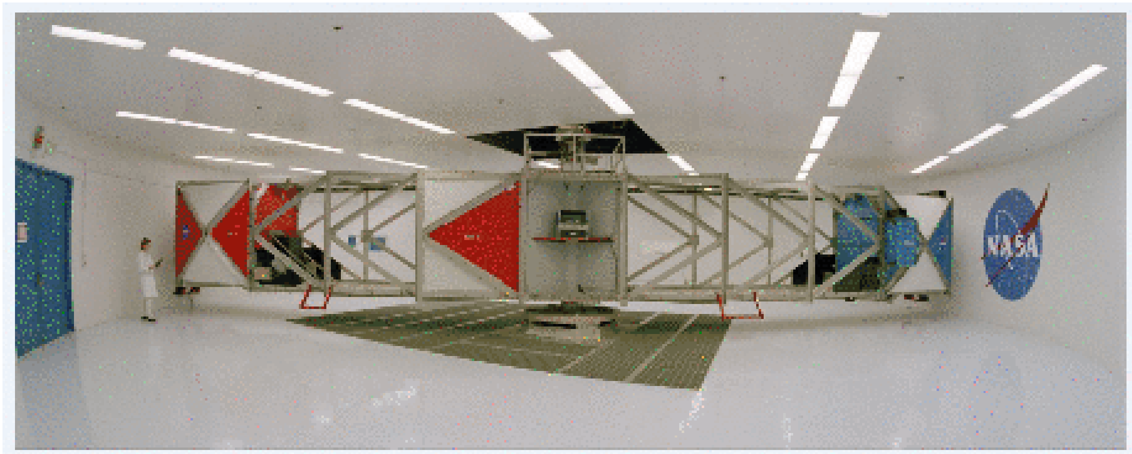


- **Coriolis force** on a falling body. If it starts from rest at time $t = 0$, then $v = gt$, and $m\ddot{x} = 2mgt\Omega\cos\lambda$ so $x = \frac{1}{3}g\Omega t^3\cos\lambda$.



- Also, Foucault pendulum, which precesses at $\Omega\sin\lambda$ due to the Coriolis force. At the North and South poles this is easily understood a result of the Earth rotating under the pendulum.

Example 1: The NASA Ames Research Center '20-G Centrifuge'.



Performance limits and specifications

Radius: 29 ft

Payload: 1,200 lbs

Max G: 20 *g* (human-rated to 12.5 *g*)

Max RPM: 50 RPM

`http://www.nasa.gov/ames/research/space-biosciences/20-g-centrifuge/`

- Apparent centrifugal acceleration at radius r, at angular speed ω is

$$a = \omega^2 r.$$

- So, for $r = 29$ ft $= 8.84$ m, and $a = 20g = 196.2$ m s^{-2}, the angular speed is

$$\omega = \sqrt{\frac{a}{r}} = \sqrt{\frac{196.2}{8.84}} = 4.71 \text{ rad s}^{-1}.$$

  or in terms of rpm (revolutions per minute), this is ≈ 45 rpm.

Example 2: A ballistic missile is fired from Oxford to Cambridge but without any correction for the Coriolis force. How far does it miss by?

The distance between Oxford and Cambridge is $D \approx 100$ km, and take the missile to have a constant (horizontal) speed, of $v \approx 1000$ km hr^{-1} (≈ 280 m s^{-1}) (i.e. air resistance is neglected).

- The sideways Coriolis acceleration is $a = 2\Omega v \sin\lambda$.
- If the time of flight is t, the sideways displacement will be $d = \frac{1}{2}at^2$.
- Then as $t = D/v$,

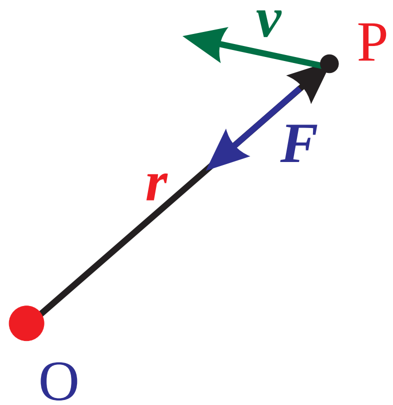
$$d = \frac{\Omega D^2 \sin\lambda}{v},$$

so

$$d \approx \frac{(2\pi/(24\times 3600))(10^5)^2 \sin(52^\circ)}{280} \approx 2 \text{ km}.$$

- Note that there is also a component of the Coriolis force normal to the Earth's surface which has been neglected here (as has the much larger force of gravity!).

ORBITS — CENTRAL FORCE FIELD



- Consider a particle moving in central force field.

  The potential $U(r)$ yields a radial force $\boldsymbol{F} = -\nabla U = -\frac{\mathrm{d}U}{\mathrm{d}r}\widehat{\boldsymbol{e}}_r$.

- There is no couple from central force, so that angular momentum is conserved:

$$\boxed{mr^2\dot{\phi} = J = \quad \text{constant (i.e. Kepler 2nd law).}}$$

- Therfore, the motion remains in the plane defined by position vector $\boldsymbol{r}$ and velocity $\boldsymbol{v}$.

- Total energy is conserved:

$$E = U(r) + \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) = \tfrac{1}{2}m\dot{r}^2 + U(r) + \frac{J^2}{2mr^2}.$$

- The **effective potential** $U_{\text{eff}}(r)$ has a contribution which arises from the angular velocity.

$$\boxed{U_{\text{eff}}(r) \equiv U(r) + \frac{J^2}{2mr^2}.}$$

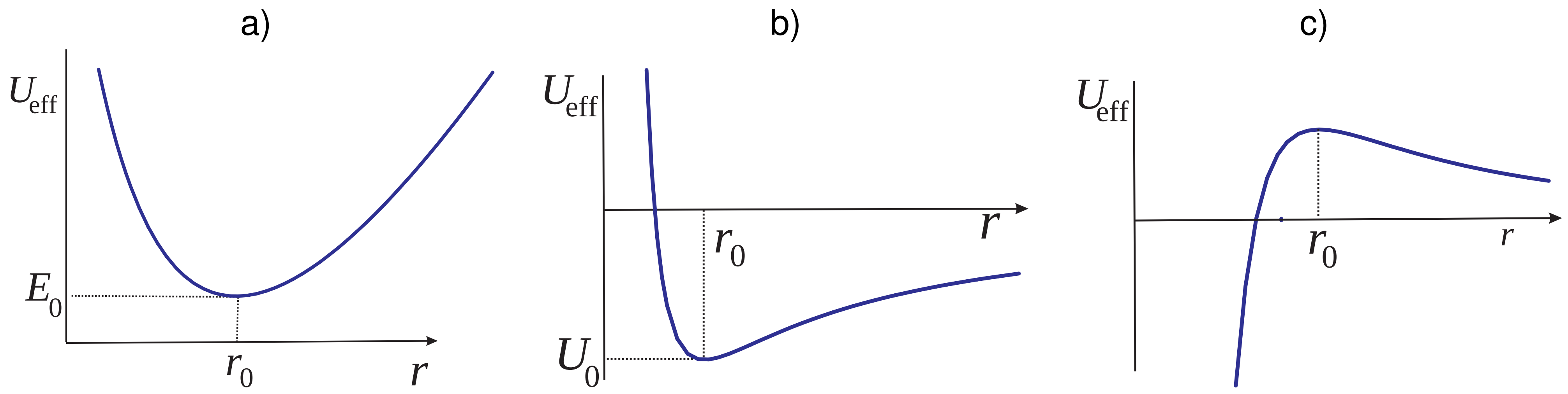
- The effective potential includes a centrifugal repulsive term $\propto 1/r^2$.

Orbits in Power-Law Force

- Consider orbits resulting from a force law with $F = -Ar^n$ with A *positive*, so the force is *attractive*.
- The effective potential is then
$$U_{\text{eff}}(r) = \frac{Ar^{n+1}}{n+1} + \frac{J^2}{2mr^2}.$$
(the only exception is when $n = -1$, in which case U_{eff} then contains a $\log r$ term).
- Define r_0 so that $\left.\frac{\mathrm{d}U_{\text{eff}}}{\mathrm{d}r}\right|_{r=r_0} = 0$.
- The centrifugal potential is repulsive and $\propto r^{-2}$. A plot of $U_{\text{eff}}(r)$ shows which values of the index n lead to bound or unbound orbits, and which lead to stable or unstable orbits.

The potential is qualitatively different for different values of n.

a)	$n \geq -1$:	orbit at r_0 stable;	all orbits bound.
b)	$-3 < n < -1$:	orbit at r_0 stable;	unbound orbits for $E > 0$.
c)	$n < -3$:	orbit at r_0 unstable;	will go to $r = 0$ or $r = \infty$.

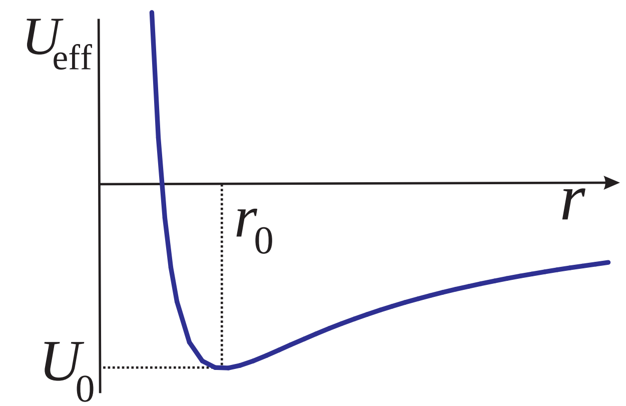


a) For $n \geq -1$ (including the $\log r$ potential), the potential increases as $r \longrightarrow \infty$ and the orbits are bound and stable.

b) For $-3 < n < -1$ the potential goes to zero at $r = \infty$ and the orbits can either be bound or unbound.

c) For $n < -3$ the attractive force as $r \longrightarrow 0$ overcomes the centrifugal repulsion and the orbits are not stable (this is the case for the central region of black holes in GR).

Nearly Circular Orbits in Power-law Force



- For $F = -Ar^n$, $n =$ index, with common cases $n = +1$ (2D, 3D SHM) and $n = -2$ (gravity, electrostatics).

$$U_{\rm eff} = \frac{Ar^{n+1}}{n+1} + \frac{J^2}{2mr^2}.$$

- **Nearly circular orbits** are oscillations/perturbations about r_0.

- Taylor expansion of $U_{\rm eff}$ gives

$$U_{\rm eff} = U_0 + (r - r_0) \left.\frac{\mathrm{d}U_{\rm eff}}{\mathrm{d}r}\right|_{r_0} + \tfrac{1}{2}(r - r_0)^2 \left.\frac{\mathrm{d}^2 U_{\rm eff}}{\mathrm{d}r^2}\right|_{r_0} + \ldots$$

- As $\frac{\mathrm{d}U_{\rm eff}}{\mathrm{d}r}$ is zero at $r = r_0$, then

$$\frac{\mathrm{d}U_{\rm eff}}{\mathrm{d}r} = Ar^n - \frac{J^2}{mr^3} = 0 \text{ at } r_0, \quad \text{i.e. } A = \frac{J^2}{mr_0^{n+3}}.$$

- The second derivative of U_{eff} is

$$\frac{d^2U_{\text{eff}}}{dr^2} = nAr^{n-1} + \frac{3J^2}{mr^4} \quad \text{which is} \quad \frac{(n+3)J^2}{mr_0^4} \text{ at } r_0.$$

so

$$U_{\text{eff}} \approx U_0 + \tfrac{1}{2}(r-r_0)^2 \left.\frac{d^2U_{\text{eff}}}{dr^2}\right|_{r=r_0} \dots \approx U_0 + \tfrac{1}{2}(r-r_0)^2 \underbrace{\frac{(n+3)J^2}{mr_0^4}}_{\text{a constant } U_0''} \dots$$

- Alternatively, using the energy method

$$\frac{d}{dt}\left(\tfrac{1}{2}m\dot{r}^2 + U_{\text{eff}}\right) = \dot{r}\left(m\ddot{r} + \frac{dU_{\text{eff}}}{dr}\right) = 0,$$

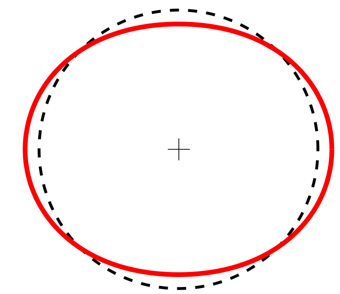
gives the SHM equation $m\ddot{r} + \dfrac{(n+3)J^2}{mr_0^4}(r-r_0) = 0$.

- Substituting $\epsilon = r - r_0$, this is $m\ddot{\epsilon} + U_0''\epsilon = 0$, i.e. simple harmonic motion about r_0 with angular frequency $\omega_p = \sqrt{\dfrac{U_0''}{m}}$, which is $\omega_p = \sqrt{n+3}\,\dfrac{J}{mr_0^2}$.

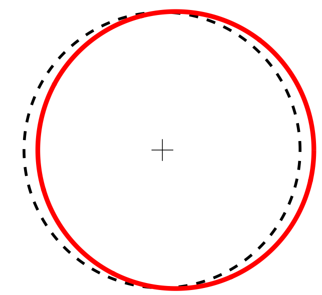
- It is useful to compare ω_p of the perturbation with ω_c of the circular orbit at r_0.
- Since $\omega_\text{c} = \dot{\phi} = J/mr_0^2$, then $\omega_\text{p} = \sqrt{n+3}\,\omega_\text{c}$.
- The simple cases are as follows.
  - $n = 1$. Force proportional to r,
    (i.e. simple harmonic motion),
    $\omega_\text{p} = 2\omega_\text{c}$, giving an ellipse centered on the origin.
  - $n = -2$. Inverse square force (e.g. planetary orbit),
    $\omega_\text{p} = \omega_\text{c}$, giving an ellipse with a focus at $r = 0$.
  - However, a general n gives non-commensurate ω_p and ω_c,
    with non-repeating and non-closed orbits.

    The case illustrated is n is a little less than 1. This gives a *nearly* elliptical orbit, which can be described as an elliptical orbit precessing slowly in the direction of orbital motion.

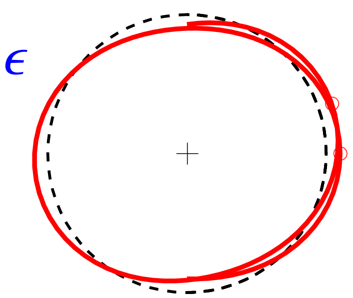
$n = 1$



$n = -2$



$n = 1 - \epsilon$



Solar System Models – Kepler's Laws

- Claudius Ptolemy (c. AD 100–170): the Sun and planets orbit around the Earth (with orbits described by a combination of circles).
- Nicolaus Copernicus (AD 1473–1543): planets orbit the (stationary) Sun.
- Tycho Brahe (AD 1546–1601): the Sun orbits the (stationary) Earth, and the planets orbit the Sun. Tycho made observations of planetary positions to an accuracy of 10 arcsec (the resolution of the eye is about 1 arcmin = 60 arcsec).
- Johannes Kepler (AD 1571–1630): tried to fit Tycho's accurate observations of Mars with circle-based orbits, but failed (with differences of ≈ 8 arcmin).

  After several years Kepler concluded that planetary orbits are *ellipses*.

Kepler's Laws:

1st Law: Planetary orbits are ellipses with the Sun at one focus.

2nd Law: The line joining the planets to the Sun sweeps out equal areas in equal times. (Implies conservation of angular momentum.)

3rd Law: The square of the period of a planet is proportional to the cube of its mean distance to the Sun (it is proportional to the cube of the length of the orbit's major axis).

ORBITS IN INVERSE SQUARE LAW FORCE

- Inverse square law force: $F = -\dfrac{A}{r^2}$ (with $A = GMm$ for gravity).
- **Angular momentum**: $J = mr^2\dot{\phi}$.
- **Energy**: $\frac{1}{2}m\dot{r}^2 + \dfrac{J^2}{2mr^2} - \dfrac{A}{r} = E$.
- It is slightly easier to work with $u = 1/r$: so $\dot{r} = \dfrac{\mathrm{d}r}{\mathrm{d}\phi}\dot{\phi} = -\dot{\phi}r^2\dfrac{\mathrm{d}u}{\mathrm{d}\phi} = -\dfrac{J}{m}\dfrac{\mathrm{d}u}{\mathrm{d}\phi}$.
- Substitute into the energy equation $\left(\dfrac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 + u^2 - \dfrac{2m}{J^2}(E + Au) = 0$.

  Consider the terms in u, i.e. $u^2 - \dfrac{2mA}{J^2}u$. By 'completing the square', this is

$$\left(u - \frac{mA}{J^2}\right)^2 - \left(\frac{mA}{J^2}\right)^2 = \left(u - \frac{1}{r_0}\right)^2 - \frac{1}{r_0^2},$$

  where $r_0 = \dfrac{J^2}{mA}$, the radius of the circular orbit with the same angular momentum J.

- So the energy equation is

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 + \left(u - \frac{1}{r_0}\right)^2 - \frac{1}{r_0^2} - \frac{2mE}{J^2} = 0$$

which can be written as

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 = \frac{e^2}{r_0^2} - \left(u - \frac{1}{r_0}\right)^2,$$

where $\dfrac{e^2}{r_0^2} \equiv \dfrac{2mE}{J^2} + \dfrac{1}{r_0^2}$, where $\boxed{e \text{ will be shown to be the } \textbf{eccentricity of the ellipse.}}$

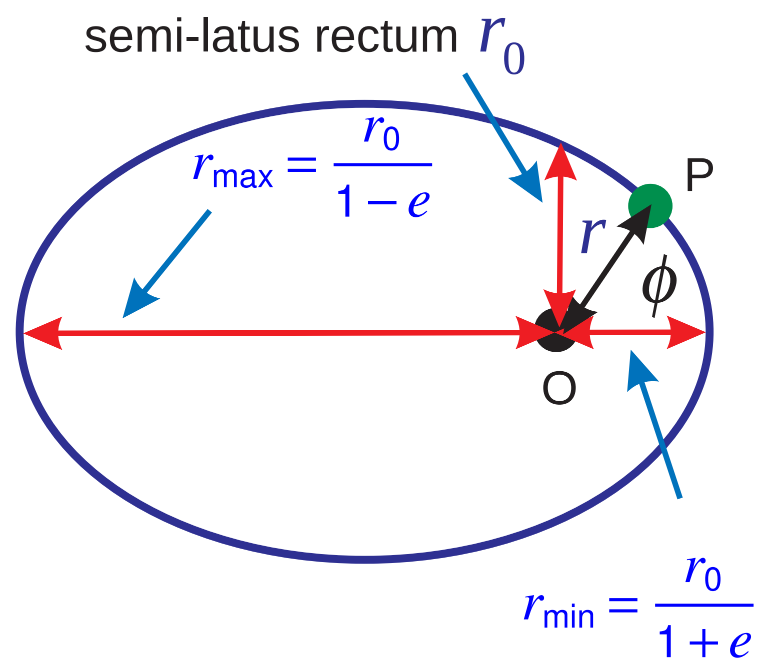
- Using a standard integral:

$$\int \frac{\mathrm{d}u}{\sqrt{\frac{e^2}{r_0^2} - \left(u - \frac{1}{r_0}\right)^2}} = \cos^{-1}\left(\frac{u - \frac{1}{r_0}}{\frac{e}{r_0}}\right) = \int \mathrm{d}\phi \quad \text{so} \quad u = \frac{1}{r_0}\left(1 + e\cos(\phi - \phi_0)\right).$$

- Equation of conic section: $\boxed{r_0 = r(1 + e\cos\phi).}$

Note: for a *repulsive* potential $r_0 = r(e\cos\phi - 1)$.

INVERSE SQUARE LAW — ELLIPTICAL ORBITS ($E < 0$)



- Ellipse of eccentricity e ($0 < e < 1$).
  Centre of attraction at one focus.
- Polar equation: $r_0 = r(1 + e\cos\phi)$,
  r_0 is called the **'semi-latus rectum'**.
  Note: $r_{min} = r_0/(1+e)$ (for $\phi = 0$),
  and $r_{max} = r_0/(1-e)$ (for $\phi = \pi$).
- Cartesian equation (with O, one of the foci of the ellipse, as the origin for x, y), using $x = r\cos\phi$, then $r = r_0 - ex$.
- So, for general point P shown,

$$
\begin{aligned}
x^2 + y^2 &= r^2 = (r_0 - ex)^2, \\
x^2 + y^2 &= r_0^2 - 2er_0x + e^2x^2, \\
y^2 + x^2(1-e^2) + 2er_0x &= r_0^2.
\end{aligned}
$$

- Consider the terms on the LHS containing x and x^2

$$x^2(1-e^2)+2er_0x=(1-e^2)\left[x^2+\left(\frac{2er_0}{1-e^2}\right)x\right].$$

- Now 'complete the square'

$$(1-e^2)\left[\underbrace{\left(x+\frac{er_0}{1-e^2}\right)^2}_{\text{call this }(x')^2}-\frac{e^2r_0^2}{(1-e^2)^2}\right]=(1-e^2)\left[(x')^2-\frac{e^2r_0^2}{(1-e^2)^2}\right],$$

where $x'=x+\dfrac{r_0e}{1-e^2}$.

- Which gives

$$\begin{aligned}
y^2+(1-e^2)(x')^2 &= r_0^2\left(1+\frac{e^2}{1-e^2}\right),\\
&= r_0^2\left(\frac{(1-e^2)+e^2}{1-e^2}\right),\\
y^2+(1-e^2)(x')^2 &= \frac{r_0^2}{1-e^2}.
\end{aligned}$$

- Now re-arrange this to look like a standard ellipse equation

$$\begin{aligned}\frac{(1-e^2)(x')^2+y^2}{r_0^2/(1-e^2)} &= 1,\\ \frac{(1-e^2)^2(x')^2}{r_0^2}+\frac{(1-e^2)y^2}{r_0^2} &= 1.\end{aligned}$$

- This **is** an ellipse, in Cartesian coordinates

$$\frac{(x')^2}{a^2}+\frac{y^2}{b^2}=1$$

with

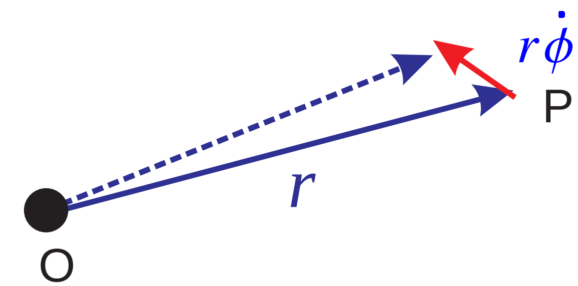
$$a=\frac{r_0}{1-e^2} \text{ and } b=\frac{r_0}{\sqrt{1-e^2}}.$$

- So, this gives the useful relations $r_{\mathrm{max}}=a(1+e)$ and $r_{\mathrm{min}}=a(1-e)$.
- And as a check, $2a=r_{\mathrm{max}}+r_{\mathrm{min}}$, as expected.

- Area of an ellipse is $\pi ab = \dfrac{\pi r_0^2}{(1-e^2)^{3/2}}$.
- Period T is $\dfrac{\text{Area}}{\text{Rate of sweeping out area}}$.
- Rate of sweeping out area: $\frac{1}{2} r^2 \dot{\phi} = \dfrac{J}{2m}$, hence the period is

$$T = \frac{2\pi r_0^2 m}{J(1-e^2)^{3/2}} = 2\pi\sqrt{\frac{ma^3}{A}},$$

i.e. Kepler's 3rd Law, with $T^2 \propto a^3$.



INVERSE SQUARE ORBITS — ALTERNATIVE DERIVATION

- **Shape of orbit**. Since the vectors $\boldsymbol{J}$, $\dot{\boldsymbol{v}}$ and $\dot{\widehat{\boldsymbol{e}}}_r$ have magnitudes $mr^2\dot{\phi}$, A/mr^2 and $\dot{\phi}$ respectively and are mutually perpendicular, so write
$$\boldsymbol{J}\times\dot{\boldsymbol{v}} = -A\dot{\widehat{\boldsymbol{e}}}_r.$$

- Since $\boldsymbol{J}$ is constant, the equation may be integrated to give $\boldsymbol{J}\times\boldsymbol{v} + A(\widehat{\boldsymbol{e}}_r + \boldsymbol{e}) = 0$, where $\boldsymbol{e}$ is a vector integration constant.

- Taking the dot-product of this equation with $\boldsymbol{r}$ gives
$$\underbrace{\boldsymbol{J}\times\boldsymbol{v}\cdot\boldsymbol{r}}_{=\boldsymbol{J}\cdot\boldsymbol{v}\times\boldsymbol{r}=-J^2/m} + A(r + \boldsymbol{e}\cdot\boldsymbol{r}) = 0.$$

- Therefore $r(1+\boldsymbol{e}\cdot\widehat{\boldsymbol{e}}_r) = r(1+e\cos\phi) = \dfrac{J^2}{mA} = r_0$,

  which is the polar equation of a conic with **focus** at $r = 0$ (Kepler's 1st Law).

- The major axis is in the direction of $\boldsymbol{e}$; e is the eccentricity of the orbit:

  $e = 0$ is circle; $e < 1$ is ellipse; $e = 1$ is parabola; $e > 1$ is hyperbola.

ALTERNATIVE DERIVATION— ENERGY OF THE ORBIT

- To get the energy take the scalar product of $A\boldsymbol{e} = -(\boldsymbol{J}\times\boldsymbol{v} + A\widehat{\boldsymbol{e}}_r)$ with itself (note that $\boldsymbol{J}$ and $\boldsymbol{v}$ are perpendicular):

$$A^2e^2 = J^2v^2 + 2\underbrace{\left(\boldsymbol{J}\times\boldsymbol{v}\cdot\widehat{\boldsymbol{e}}_r\right)}_{\boldsymbol{J}\cdot\boldsymbol{v}\times\widehat{\boldsymbol{e}}_r = -J^2/mr}A + A^2.$$

Therefore

$$A^2(e^2-1) = J^2\left(v^2 - \frac{2A}{mr}\right).$$

- Since the total energy is $E = \frac{1}{2}mv^2 - \dfrac{A}{r}$, and $r_0 = \dfrac{J^2}{mA}$, then

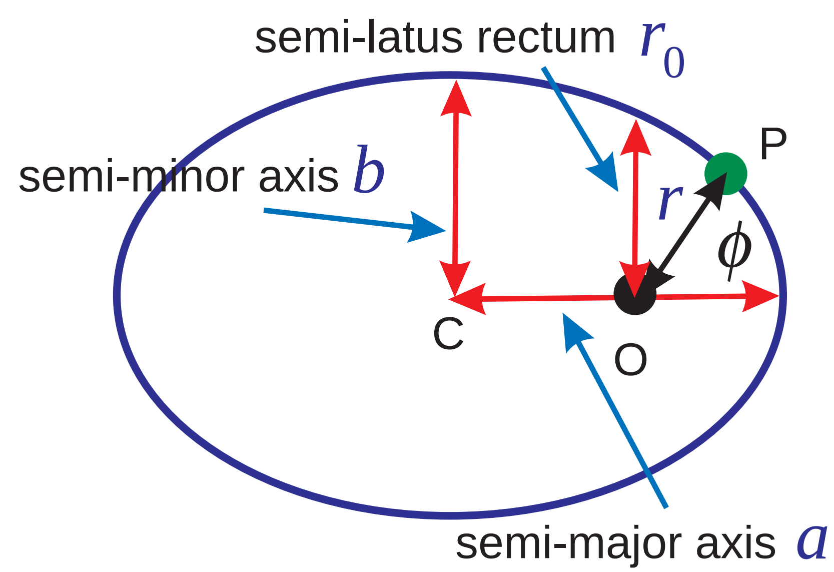
$$A^2(e^2-1) = \frac{2EJ^2}{m} = 2AEr_0.$$

- The major axis of the orbit is given by

$$2a = r_0\left(\frac{1}{1+e} + \frac{1}{1-e}\right) = \frac{2r_0}{1-e^2} = -\frac{A}{E}$$

i.e. $E = -A/2a$, independent of eccentricity.

Elliptical Orbits — Important Things to Remember



- The equation of an ellipse, eccentricity e, in polar coordinates is $\boxed{r_0 = r(1 + e\cos\phi)}$ ($e = 0$ for circle, $e < 1$ for ellipse).
- The distances of closest and furthest approach follow from this: $r_{\text{min}} = \dfrac{r_0}{1+e}$ and $r_{\text{max}} = \dfrac{r_0}{1-e}$.
- The semi-major axis a satisfies $2a = r_{\text{min}} + r_{\text{max}}$ so $\boxed{a = \dfrac{r_0}{1-e^2}}$ and $r_{\text{max,min}} = a(1 \pm e)$.

  Also $b = r_0/\sqrt{1-e^2}$.
- The semi-major axis a determines the **energy** and the **period** of the orbit

$$\boxed{E = -\frac{A}{2a}; \quad T = \frac{2\pi}{\omega}; \quad \omega^2 = \frac{A}{ma^3}.}$$

- The semi-latus rectum r_0 determines the **angular momentum** of the orbit: $\boxed{J^2 = Amr_0.}$

Note: for gravity, $A = GMm$.