Part IB Physics A : Lent 2022

QUANTUM PHYSICS EXAMPLES IV

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1. Denote the eigenfunctions of $\widehat{L}^2$ and $\widehat{L}_z$ with eigenvalues $l = 1$ and $m_l = -1, 0, 1$ by $|\phi_{-1}\rangle, |\phi_0\rangle, |\phi_1\rangle$. Use the ladder operators $\widehat{L}_+$ and $\widehat{L}_-$ to find the eigenfunctions of $\widehat{L}_x$ in terms of those of $\widehat{L}_z$.

A beam of atoms with zero spin and in the state $l = 1$ is traveling along the y-axis and passes through an x-Stern-Gerlach apparatus. The emerging beam with $m_l = 1$ is passed through a z-Stern-Gerlach apparatus. Into how many beams is this beam further split and what are the relative numbers of atoms in them?

What happens if the other two beams from the first Stern-Gerlach apparatus are treated in the same way?

2. For a system involving two particles of spin $s_1 = \frac{1}{2}$ and $s_2 = \frac{1}{2}$, find the eigenvalues of $\widehat{\boldsymbol{s}}_1 \cdot \widehat{\boldsymbol{s}}_2$ for the "anti-parallel" spin singlet state and the "parallel" spin triplet states, and comment on your results.

3. Derive the eigenfunctions of the operators $\widehat{J}^2$ and $\widehat{J}_z$ in terms of the eigenfunctions of $\widehat{L}^2$, $\widehat{S}^2$, $\widehat{L}_z$ and $\widehat{S}_z$ for the case $l = 1, s = \frac{1}{2}$.

4. A spin-$\frac{1}{2}$ particle is in state $|\chi\rangle$, having its spin aligned (as far as possible) along a unit vector $\boldsymbol{n}$ in the (θ, ϕ) direction in spherical polar coordinates. This state $|\chi\rangle$ will be such that

$$\boldsymbol{n}.\widehat{\boldsymbol{S}}\,|\chi\rangle = (n_x\widehat{S}_x + n_y\widehat{S}_y + n_z\widehat{S}_z)\,|\chi\rangle = +\frac{1}{2}\hbar\,|\chi\rangle\,.$$

Express $|\chi\rangle$ in terms of the spin eigenstates $|\chi_\uparrow\rangle$ and $|\chi_\downarrow\rangle$ corresponding to the z-axis, and hence find the relative intensities of the two beams produced when a beam of particles in the state $|\chi\rangle$ is passed through a z-Stern-Gerlach apparatus.

5. Given that neutrons, protons and electrons are all fermions, why is ^{4}He a boson? What is ^{3}He?

6. Three non-interacting identical spin-$\frac{1}{2}$ fermions are confined in a rectangular box with edges a, a and d. Find, for the ground state of the system, how: (i) its degeneracy; (ii) its energy; and (iii) its parity with respect to the centre of the box; behave as d varies in the range $0 < d < 2a$.

7. Two identical particles are in an isotropic 3D simple harmonic potential. Show that, if the particles do not interact and there are no spin-orbit forces, the degeneracies of the three lowest energy values are 1, 12, 39 if the particles have spin $\frac{1}{2}$, and 6, 27, 99 if the particles have spin 1.

8. Write brief notes on 'indistinguishability' in quantum mechanics. Comment on its consequences, and list a number of ways in which it reveals itself in experiments.

ANSWERS:

1. $\dfrac{1}{\sqrt{2}}(|\phi_1\rangle - |\phi_{-1}\rangle)$; $\dfrac{1}{2}(|\phi_1\rangle + |\phi_{-1}\rangle \pm \sqrt{2}\,|\phi_0\rangle)$. Split into 3 beams of relative intensities 1:2:1.

2. Singlet: $-\frac{3}{4}\hbar^2$. Triplet: $\frac{1}{4}\hbar^2$.

2. With the notation $\left|\Psi_{l,s,j,m_j}\right\rangle$, $|\phi_{l,m_l}\rangle$, $|\chi_{s,m_s}\rangle$:

$$\left|\Psi_{1,\frac{1}{2},\frac{3}{2},\frac{3}{2}}\right\rangle = |\phi_{1,1}\rangle \left|\chi_{\frac{1}{2},\frac{1}{2}}\right\rangle$$
$$\left|\Psi_{1,\frac{1}{2},\frac{3}{2},\frac{1}{2}}\right\rangle = \frac{1}{\sqrt{3}}\left[\sqrt{2}\,|\phi_{1,0}\rangle \left|\chi_{\frac{1}{2},\frac{1}{2}}\right\rangle + |\phi_{1,1}\rangle \left|\chi_{\frac{1}{2},-\frac{1}{2}}\right\rangle\right]$$
$$\left|\Psi_{1,\frac{1}{2},\frac{3}{2},-\frac{1}{2}}\right\rangle = \frac{1}{\sqrt{3}}\left[|\phi_{1,-1}\rangle \left|\chi_{\frac{1}{2},\frac{1}{2}}\right\rangle + \sqrt{2}\,|\phi_{1,0}\rangle \left|\chi_{\frac{1}{2},-\frac{1}{2}}\right\rangle\right]$$
$$\left|\Psi_{1,\frac{1}{2},\frac{3}{2},-\frac{3}{2}}\right\rangle = |\phi_{1,-1}\rangle \left|\chi_{\frac{1}{2},-\frac{1}{2}}\right\rangle$$
$$\left|\Psi_{1,\frac{1}{2},\frac{1}{2},\frac{1}{2}}\right\rangle = \frac{1}{\sqrt{3}}\left[\sqrt{2}\,|\phi_{1,1}\rangle \left|\chi_{\frac{1}{2},-\frac{1}{2}}\right\rangle - |\phi_{1,0}\rangle \left|\chi_{\frac{1}{2},\frac{1}{2}}\right\rangle\right]$$
$$\left|\Psi_{1,\frac{1}{2},\frac{1}{2},-\frac{1}{2}}\right\rangle = \frac{1}{\sqrt{3}}\left[|\phi_{1,0}\rangle \left|\chi_{\frac{1}{2},-\frac{1}{2}}\right\rangle - \sqrt{2}\,|\phi_{1,-1}\rangle \left|\chi_{\frac{1}{2},\frac{1}{2}}\right\rangle\right].$$

4. $|\chi\rangle = \cos(\theta/2)\,\mathrm{e}^{-i\phi/2}\,|\chi_\uparrow\rangle \ + \ \sin(\theta/2)\,\mathrm{e}^{i\phi/2}\,|\chi_\downarrow\rangle$; intensities $\cos^2(\theta/2)$, $\sin^2(\theta/2)$. [Note that this is not inconsistent with earlier statements about the wavefunction being unchanged by a rotation $\phi \to \phi + 2\pi$ since this referred to the particle's co-ordinates; here ϕ refers to the co-ordinates of the SG experimental set-up.]

6. For $0 < d < a:\ E = (\hbar^2\pi^2/2m)(9/a^2 + 3/d^2)$, fourfold degenerate, parity odd.

For $a < d < 2a:\ E = (\hbar^2\pi^2/2m)(6/a^2 + 6/d^2)$, twofold degenerate, parity odd.

For $d = a:\ E = (\hbar^2\pi^2/2m)(12/a^2)$, sixfold degenerate, parity odd.