Relativistic Astrophysics and Cosmology — Examples 2 — 2023

1. Using the results and notation of Handouts 4 and 5, and units with $c = G = 1$, verify:

   (i) that for a general spherically symmetric metric

   $$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2,$$

   the rapidity α in a circular orbit satisfies

   $$\tanh^2\alpha = \frac{rA'}{2A}$$

   (ii) that the circular orbit velocity $v = \tanh\alpha$ inside a spherically symmetric perfect fluid in hydrostatic equilibrium and with pressure $P(r)$, satisfies

   $$v^2 = \frac{G\left(m(r) + 4\pi r^3 P(r)/c^2\right)}{r - 2Gm(r)/c^2}.$$

   A new transport system is proposed *inside* the Earth in which a circular tunnel is built in the plane of the equator, concentric with the Earth's centre, and at a depth of 100 km from the surface. Approximating the Earth as a perfect fluid of the above type with constant density, estimate the percentage correction to v away from its Newtonian value due to (a) the term in $P(r)$ [answer: 1.6×10^{-9} %], and (b) the $-2m(r)$ in the denominator [answer: 6.7×10^{-8} %], for a circular orbit which lies along the tunnel. (Ignore all effects of rotation.)

2. Identifying the entropy of a Schwarzschild black hole as $S = k_{\rm B}\mathcal{A}/(4\ell_p^2)$, where $\mathcal{A}$ is its surface area and ℓ_p is the Planck length, calculate the entropy of a solar mass black hole [answer: 1.5×10^{-54} JK^{-1}].

   Compare this value with an estimate of the entropy of the Sun. (For the latter, you may like to use the results of Basu & Lynden-Bell, *Quarterly Journal of the Royal Astronomical Society*, **31**, 359 (1990), which showed that main sequence stars have an entropy per baryon of about $20k_{\rm B}$.)

3. Apply the uncertainty principle, as used in the lectures, to the hydrogen atom. Show by minimizing the sum of kinetic and potential energies, that the radius of a hydrogen atom is $a_0 = 4\pi\epsilon_0\hbar^2/m_e e^2$. By using the relativistic expression for K.E., show that if the effects of special relativity are included, the expression for the radius is altered to

   $$a^2 = a_0^2 - \lambda^2,$$

   where $\lambda = \hbar/m_e c$ is the Compton wavelength of an electron. Generalizing this expression to an atom with nuclear charge Z, show that the maximum charge on the nucleus of a stable atom is $4\pi\epsilon_0\hbar c/e^2$, the inverse of the fine structure constant, α_f. (Note $\alpha_f^{-1} \approx 137$.) Discuss.

4. Derive the Eddington limit and show how it changes between a pure ionized hydrogen gas and an ionized helium one. What would it be for partially ionized iron, where just the K and L shell electrons are retained. Is a human superEddington? A variable ultraluminous X-ray source is seen in the outskirts of a galaxy with a luminosity of 10^{34} W. What can you deduce about its likely nature – is it a black hole or neutron star, and what is its mass?

5. By treating it as Thomson scattering in the eletron's rest frame, show that for inverse Compton scattering the final photon energy $\overline{\epsilon}_f = \left(\frac{4}{3}\gamma^2 - \frac{1}{3}\right)\epsilon_0$, where ϵ_0 is the initial energy of the isotropic photon field and γ is the Lorentz factor of the electron (assumed large). [Hint: you can also get to this result by starting from the stress-energy tensor of a photon fluid]

6. If an accretion disc around a Schwarzschild black hole of mass M is viewed *face*-on, calculate the apparent redshift (due to gravitational and transverse-Doppler effects) of the innermost stable part at $r = 6GM/c^2$ [answer: 0.41]. If the same disc is viewed *edge*-on, compute the range of frequency shifts that will be observed [answer: 1.12 and -0.29]. Is any part of the disc *blue*shifted?

7. Calculate the radiation efficiency of a thin, Newtonian accretion disc that extends inward to $6GM/c^2$ around a black hole. Compare with the efficiency of a relativistic disc.

8. At what radius in an accretion disc around a Schwarzschild compact object does the temperature peak if the fraction of angular momentum absorbed by the object is β? From considerations of hydrostatic equilibrium, justify that $c_s \sim \Omega h$ where h is the thickness of a thin disk. Show that for large r, for a thin α-disk, the radial inflow velocity of the matter in the disc $v_r \approx \alpha c_s(h/r)$. [info: an α-disk has viscous stress $s_\phi = \alpha P$ and speed of sound $c_s^2 = P/\rho$, and from astrophysical fluid dynamics the equation of hydrostatic equilibrium is $\rho\nabla\Phi = -\nabla P$]

9. Using the expression for v_{app} for superluminal motion, how does it vary as a function of θ? What angle gives the maximum v_{app}?

10. The jet in M87 is probably inclined at 40° to the line of sight. Superluminal motion has been seen by radio astronomers within the core of the jet, with $v_{app} = 2.5$c. Estimate the bulk Lorentz factor of the jet, the velocity of the jet and the doppler factor (assume a spectral index $\alpha = 1$) of each side of the jet. Why is the counterjet not detected?