Seismology Problem Set 1 – Solutions

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1. Starting with the equations of motion

$$\rho \frac{\partial v_i}{\partial t} - \frac{\partial T_{ij}}{\partial x_j} = 0,$$

we integrate over U to obtain

$$\int_U \rho \frac{\partial v_i}{\partial t} \,\mathrm{d}^3\mathbf{x} - \int_U \frac{\partial T_{ij}}{\partial x_j} \,\mathrm{d}^3\mathbf{x} = 0.$$

Applying the divergence theorem to the second term given

$$\int_U \frac{\partial T_{ij}}{\partial x_j} \,\mathrm{d}^3\mathbf{x} = \int_{\partial U} T_{ij}\hat{n}_j \,\mathrm{d}S,$$

and hence we obtain the desired result

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_U \rho v_i \,\mathrm{d}^3\mathbf{x} = \int_{\partial U} T_{ij}\hat{n}_j \,\mathrm{d}S,$$

where we can bring the time derivative out of the first integrate because both ρ and U are time-independent.

The term $\int_U \rho v_i \,\mathrm{d}^3\mathbf{x}$ is clearly the linear momentum of the particles lying in U, and hence we have obtained Newton's second law for this sub-body. The total force acting on the sub-body is expressed as an integral of the traction $t_i = T_{ij}\hat{n}_j$ over the reference boundary ∂U. This represents contact forces acting on the sub-body due to its surroundings, with the traction being a force per unit area relative to the reference boundary. Here we also see that stress tensors relate the *orientation* of a surface to the *forces* per unit area acting on it. Here we could choose to work either with a referential surface or its instantaneous image in physical space. In the former case we end up with the first Piola-Kirchhoff stress, while in the latter leads to the *Cauchy stress*, this being the one generally used in fluid mechanics.

Differentiating the expression for the angular momentum, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \int_U \rho\epsilon_{ijk}\varphi_j v_k \,\mathrm{d}^3\mathbf{x} &= \int_U \rho\epsilon_{ijk}\varphi_j \frac{\partial v_k}{\partial t} \,\mathrm{d}^3\mathbf{x} + \int_U \rho\epsilon_{ijk} v_j v_k \,\mathrm{d}^3\mathbf{x} \\ &= \int_U \rho\epsilon_{ijk}\varphi_j \frac{\partial v_k}{\partial t} \,\mathrm{d}^3\mathbf{x},\end{aligned}$$

where we have used $\epsilon_{ijk} v_j v_k = 0$. Substituting in the equations motion (i.e. the conservation of linear momentum), we find

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_U \rho\epsilon_{ijk}\varphi_j v_k \,\mathrm{d}^3\mathbf{x} = \int_U \epsilon_{ijk}\varphi_j \frac{\partial T_{kl}}{\partial x_l} \,\mathrm{d}^3\mathbf{x},$$

and integrating by parts this becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_U \rho\epsilon_{ijk}\varphi_j v_k \,\mathrm{d}^3\mathbf{x} = \int_U \frac{\partial}{\partial x_l}\left(\epsilon_{ijk}\varphi_j T_{kl}\right)\,\mathrm{d}^3\mathbf{x} - \int_U \epsilon_{ijk}F_{jl}T_{kl}\,\mathrm{d}^3\mathbf{x}$$
$$= \int_{\partial U} \epsilon_{ijk}\varphi_j t_k \,\mathrm{d}S - \int_U \epsilon_{ijk}F_{jl}T_{kl}\,\mathrm{d}^3\mathbf{x},$$

where the first term on the right hand side is the torque on the sub-body due to contact forces acting on its boundary. If angular momentum is to be conserved, we must then have

$$\int_U \epsilon_{ijk}F_{jl}T_{kl}\,\mathrm{d}^3\mathbf{x} = 0,$$

and we U is arbitrary, this implies $\epsilon_{ijk}F_{jl}T_{kl} = 0$. This latter condition can only be met if $F_{jl}T_{kl}$ is a symmetric tensor (the sufficiency is clear, for necessity, act ϵ_{ipq} on the identity and use $\epsilon_{ipq}\epsilon_{ijk} = \delta_{pj}\delta_{qk} - \delta_{pk}\delta_{qj}$). We conclude that $\mathbf{F}\mathbf{T}^T$ is symmetric. If we then define $\mathbf{S}$ through $\mathbf{T} = \mathbf{F}\mathbf{S}$, we see that

$$\mathbf{F}\mathbf{S}^T\mathbf{F}^T = \left(\mathbf{F}\mathbf{S}^T\mathbf{F}^T\right)^T = \mathbf{F}\mathbf{S}\mathbf{F}^T \tag{1}$$

and hence $\mathbf{S}$ is indeed symmetric.

2. We are given the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\rho\frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t} - \frac{1}{2}A_{ijkl}\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l} + \bar{S}_{ij}\frac{\partial u_i}{\partial x_j},$$

and can assume the standard form of the Euler Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial u_i} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial v_i} - \frac{\partial}{\partial x_j}\frac{\partial \mathcal{L}}{\partial F_{ij}} = 0,$$

where we have written $v_i = \frac{\partial u_i}{\partial t}$ and $F_{ij} = \frac{\partial u_i}{\partial x_j}$ for convenience. Differentiating the Lagrangian, we find

$$\frac{\partial \mathcal{L}}{\partial u_i} = 0, \quad \frac{\partial \mathcal{L}}{\partial v_i} = \rho\frac{\partial u_i}{\partial t}, \quad \frac{\partial \mathcal{L}}{\partial F_{ij}} = -A_{ijkl}\frac{\partial u_k}{\partial x_l} + \bar{S}_{ij},$$

and hence the equations of motion read

$$\rho\frac{\partial^2 u_i}{\partial t^2} - \frac{\partial}{\partial x_j}\left(A_{ijkl}\frac{\partial u_k}{\partial x_l}\right) = -\frac{\partial \bar{S}_{ij}}{\partial x_j},$$

which is what we obtained in lectures by linearisation of the exact equations. The natural boundary conditions take the general form

$$\frac{\partial \mathcal{L}}{\partial F_{ij}}\hat{n}_j = 0,$$

on ∂M, and hence reduce to

$$A_{ijkl}\frac{\partial u_k}{\partial x_l}\hat{n}_j = \bar{S}_{ij}\hat{n}_j.$$

For the next part, we have the energy density

$$E = \frac{1}{2}\rho\frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t} + \frac{1}{2}A_{ijkl}\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l},$$

which we differentiate to obtain

$$\frac{\partial E}{\partial t} = \rho\frac{\partial u_i}{\partial t}\frac{\partial^2 u_i}{\partial t^2} + A_{ijkl}\frac{\partial u_k}{\partial x_l}\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t},$$

where we have used the hyperelastic symmetry $A_{ijkl} = A_{klij}$. Applying the product rule to the second term on the right hand side, we have

$$A_{ijkl}\frac{\partial u_k}{\partial x_l}\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t} = \frac{\partial}{\partial x_j}\left(A_{ijkl}\frac{\partial u_k}{\partial x_l}\frac{\partial u_i}{\partial t}\right) - \frac{\partial}{\partial x_j}\left(A_{ijkl}\frac{\partial u_k}{\partial x_l}\right)\frac{\partial u_i}{\partial t},$$

from which we obtain

$$\frac{\partial E}{\partial t} - \frac{\partial}{\partial x_j}\left(A_{ijkl}\frac{\partial u_k}{\partial x_l}\frac{\partial u_i}{\partial t}\right) = 0,$$

on use of the equations of motion in the absence of a stress glut. Here we have a conservation law of the correct form, and identify

$$s_j = -A_{ijkl}\frac{\partial u_k}{\partial x_l}\frac{\partial u_i}{\partial t}.$$

Physically, this vector represents the flux of energy associated with the displacement field. Indeed, integrating the conservation law over an arbitrary volume, U, and applying the divergence theorem we obtain the integral statement

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_U E\,\mathrm{d}^3\mathbf{x} = \int_{\partial U} s_j\hat{n}_j\,\mathrm{d}S,$$

which makes the physical interpretation obvious.

3. We need to evaluate

$$A_{ijkl} = \frac{\partial^2 W}{\partial F_{ij}\partial F_{kl}},$$

using the relation $W(\mathbf{F}) = U(\mathbf{C})$ with $\mathbf{C} = \mathbf{F}^T\mathbf{F}$. For the first derivative we repeat the calculation from Lecture 12

$$\frac{\partial W}{\partial F_{ij}} = \frac{\partial U}{\partial C_{kl}}\frac{\partial C_{kl}}{\partial F_{ij}} = \frac{\partial U}{\partial C_{kl}}\frac{\partial}{\partial F_{ij}}(F_{mk}F_{ml}) = 2F_{ik}\frac{\partial U}{\partial C_{kj}},$$

where we have used the identity

$$\frac{\partial C_{kl}}{\partial F_{ij}} = \delta_{jk}F_{il} + \delta_{jl}F_{ik}.$$

We can then differentiate again in the same manner to obtain

$$\begin{aligned}\frac{\partial W}{\partial F_{ij}\partial F_{kl}} &= \frac{\partial}{\partial F_{kl}}\left(2F_{im}\frac{\partial U}{\partial C_{mj}}\right)\\ &= 2F_{im}\frac{\partial^2 U}{\partial C_{mj}\partial C_{pq}}\frac{\partial C_{pq}}{\partial F_{kl}} + 2\delta_{ik}\frac{\partial U}{\partial C_{lj}}\\ &= 4F_{im}F_{kq}\frac{\partial^2 U}{\partial C_{mj}\partial C_{lq}} + 2\delta_{ik}\frac{\partial U}{\partial C_{lj}},\end{aligned}$$

where we have used the symmetry of C_{ij}. Evaluating this result at $F_{ij} = C_{ij} = \delta_{ij}$, we finally obtain

$$A_{ijkl} = 4\frac{\partial^2 U}{\partial C_{ij}\partial C_{kl}},$$

where we note that the term proportional to $\frac{\partial U}{\partial C_{lj}}$ vanishes as the equilibrium has been assumed to be stress free. Using this expression, the symmetry of C_{ij} then trivially implies that

$$A_{ijkl} = A_{jikl} = A_{ijlk}.$$

To determine the number of independent components of the elastic tensor, the easiest method is to first note that A_{ijkl} maps symmetric second-order tensors into symmetric second order tensors. Such second-order tensors are equivalent to six-dimensional vectors, and hence the elastic tensor can be identified with a 6×6 matrix, and hence has at most 36 independent components. Next, the hyperelastic symmetry $A_{ijkl} = A_{klij}$ implies that such a matrix is actually symmetric, and hence the number of independent components is equal to $6 + 5 + 4 + 3 + 2 + 1 = 21$.

If the initial stress does not vanish, the elastic tensor takes the form

$$A_{ijkl} = 4\frac{\partial^2 U}{\partial C_{ij}\partial C_{kl}} + \delta_{ik}\sigma_{lj},$$

where $\sigma_{ij} = 2\frac{\partial U}{\partial C_{ij}}$ is the (symmetric) equilibrium stress. Here the first term on the right has the full elastic symmetries, but the term depending on the equilibrium stress only obeys the hyperelastic symmetry. Thus, in a pre-stressed medium the elastic tensor looses some of its symmetries. Moreover, it can be shown that for a non-hydrostatic equilibrium stress, the second term is associated with anisotropy.

4. For this question we need to recall that the Christoffel matrix has components

$$\Gamma_{ik}(\mathbf{p}) = \frac{1}{\rho}A_{ijkl}p_j p_l,$$

along with the Christoffel equation

$$\mathbf{\Gamma}(\hat{\mathbf{p}})\mathbf{a} = c^2\mathbf{a},$$

which determines the phase speeds, c, and polarisation vectors, $\mathbf{a}$, for plane waves propagating in the direction $\hat{\mathbf{p}}$.

For the given elastic tensor we find

$$\begin{aligned}\Gamma_{ik}(\hat{\mathbf{p}}) = {} & \frac{\lambda+\mu}{\rho}\hat{p}_i\hat{p}_k + \frac{\mu}{\rho}\delta_{ik} + \frac{8\gamma}{\rho}\cos^2\theta\hat{\nu}_i\hat{\nu}_k + \frac{4\xi}{\rho}\cos\theta(\hat{\nu}_i\hat{p}_k + \hat{p}_i\hat{\nu}_k) \\ & - \frac{\zeta}{\rho}[\hat{\nu}_i\hat{\nu}_k + \cos\theta(\hat{\nu}_i\hat{p}_k + \hat{p}_i\hat{\nu}_k) + \cos^2\theta\delta_{ik}],\end{aligned}$$

where θ is the angle between the propagation direction and the symmetry axis. Note that the first two terms are those associated with an isotropic medium, while the remainder will be assumed to be small in the perturbation calculations. Considering the quasi P-wave, we can use non-degenerate perturbation theory, and hence simply apply the formula

$$\delta\alpha = \frac{1}{2\alpha}\hat{\mathbf{p}} \cdot \delta\boldsymbol{\Gamma}(\hat{\mathbf{p}})\hat{\mathbf{p}},$$

derived in Lecture 14, where $\delta\boldsymbol{\Gamma}$ is the anisotropic perturbation to the Christoffel matrix. In this case, we find that

$$\hat{\mathbf{p}} \cdot \delta\boldsymbol{\Gamma}(\hat{\mathbf{p}})\hat{\mathbf{p}} = \frac{8\gamma}{\rho}\cos^4\theta + \frac{8\xi - 4\zeta}{\rho}\cos^2\theta,$$

and hence for the quasi P-waves the phase speed perturbation is

$$\delta\alpha = \frac{1}{2\alpha}\left(\frac{8\gamma}{\rho}\cos^4\theta + \frac{8\xi - 4\zeta}{\rho}\cos^2\theta\right).$$

5. We are told to consider the wave-function

$$\psi(\mathbf{x}, t) = a\exp\left[\frac{-\mathrm{i}(E\,t - p_i x_i)}{\hbar}\right].$$

Putting this into the equation for a free particle, we find that the solutions works so long as we have the expected relation

$$E = \frac{1}{2m}\|\mathbf{p}\|^2.$$

We now look at the more complicated wave-function

$$\psi(\mathbf{x}, t) = a(\mathbf{x})\exp\left\{\frac{-\mathrm{i}[E\,t - \varphi(\mathbf{x})]}{\hbar}\right\},$$

where a and φ are functions of position. Differentiating this wave-function with respect to time we obtain

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = Ea\mathrm{e}^{-\mathrm{i}(Et-\varphi)/\hbar}.$$

Taking the spatial gradient of the wave-function, and retaining only leading-order terms in $1/\hbar$, we find that

$$\nabla\psi = \frac{\mathrm{i}}{\hbar}\nabla\varphi\, a\mathrm{e}^{-\mathrm{i}(Et-\varphi)/\hbar} + o(1).$$

In the same manner, we find that the Laplacian of ψ takes the form

$$\nabla^2\psi = -\frac{1}{\hbar^2}\|\nabla\varphi\|^2\, a\mathrm{e}^{-\mathrm{i}(Et-\varphi)/\hbar} + o(1/\hbar).$$

Putting these expressions into the time-dependent Schrödinger equation we find

$$\left(E - \frac{1}{2m}\|\nabla\varphi\|^2 - V\right)a\mathrm{e}^{-\mathrm{i}(Et-\varphi)/\hbar} = o(1/\hbar).$$

To leading order, we then obtain

$$E = \frac{1}{2m}\|\nabla\varphi\|^2 + V,$$

which does indeed take the form $E = H(\mathbf{x}, \nabla\varphi)$ with H the classical Hamiltonian. To solve for the phase function φ, we apply the method of characteristics, defining $\mathbf{p} = \nabla\varphi$ to be the *local momentum*. The eikonal equation restricts $(\mathbf{x}, \mathbf{p})$ to lie on a three-dimensional level surface on which the Hamiltonian is equal to the energy E. Given appropriate initial conditions, we can form this level surface through integration of Hamilton's equations

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\sigma} = \frac{\partial H}{\partial \mathbf{p}}, \quad \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\sigma} = -\frac{\partial H}{\partial \mathbf{x}},$$

with σ the generating parameter. Note that these equations take the expected classical form

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\sigma} = \frac{1}{m}\mathbf{p}, \quad \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\sigma} = -\frac{\partial V}{\partial \mathbf{x}}.$$

Moreover, once the classical path is known in phase space, we get

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\varphi[\mathbf{x}(\sigma)] = \nabla\sigma \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\sigma} = \frac{1}{m}\|\mathbf{p}\|^2 = 2(E - V),$$

and hence the phase function can be recovered through the integral

$$\varphi[\mathbf{x}(\sigma)] = \varphi_0 + 2\int_0^\sigma \{E - V[\mathbf{x}(\sigma')]\}\, \mathrm{d}\sigma',$$

taken along the classical trajectory.

6. We are told that the travel time can be expressed in the form

$$T = \int_0^1 \frac{1}{\alpha(\mathbf{x})}\sqrt{\frac{\mathrm{d}x_i}{\mathrm{d}\gamma}\frac{\mathrm{d}x_i}{\mathrm{d}\gamma}}\, \mathrm{d}\gamma,$$

where γ parameterises the curve over the unit interval. Note that such a fixed-length parameterisation is necessary so that variations of this functional can be easily handled. We need to compute the first variation of this expression. This can be done from first-principles (and is a good exercise), but we will jump straight to the usual Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}\frac{\partial \mathscr{L}}{\partial \mathbf{x}'} - \frac{\partial \mathscr{L}}{\partial \mathbf{x}} = \mathbf{0},$$

where the Lagrangian here is

$$\mathscr{L}(\mathbf{x}, \mathbf{x}') = \frac{1}{\alpha(\mathbf{x})}\sqrt{\frac{\mathrm{d}x_i}{\mathrm{d}\gamma}\frac{\mathrm{d}x_i}{\mathrm{d}\gamma}}$$

and where we write primes for derivatives with respect to γ as we have already used dots for those with respect to σ. Simple calculations show that

$$\frac{\partial \mathscr{L}}{\partial \mathbf{x}} = -\frac{\|\mathbf{x}'\|}{\alpha^2}\nabla\alpha, \quad \frac{\partial \mathscr{L}}{\partial \mathbf{x}'} = \frac{1}{\alpha}\frac{\mathbf{x}'}{\|\mathbf{x}'\|},$$

and so we obtain the following second-order system of ODEs

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}\left(\frac{1}{\alpha}\frac{\mathbf{x}'}{\|\mathbf{x}'\|}\right) + \frac{\|\mathbf{x}'\|}{\alpha^2}\nabla\alpha = \mathbf{0}.$$

By contrast, the Hamiltonian form of the ray equations are obtained by starting from the p-wave Hamiltonian

$$H(\mathbf{x}, \mathbf{p}) = \frac{1}{2}\alpha(\mathbf{x})^2\|\mathbf{p}\|^2,$$

where α is the p-wave speed. The Hamiltonian ray equations are then

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{x}},$$

where we use dots to denote derivatives with respect to the ray generating parameter σ. From the given form of the Hamiltonian, we obtain

$$\dot{\mathbf{x}} = \alpha^2\mathbf{p}, \quad \dot{\mathbf{p}} = -\frac{1}{\alpha}\nabla\alpha,$$

where we have made use of the eikonal equation $H(\mathbf{x}, \mathbf{p}) = \frac{1}{2}$ which implies

$$\|\mathbf{p}\| = \alpha^{-1}.$$

These two sets of equations are defined with respect to different independent variables σ and γ, and so to compare them it will be necessary to perform suitable transformations. To do this, we note that the differential arc length along the ray satisfies by simple geometry

$$\mathrm{d}s = \|\mathbf{x}'\|\,\mathrm{d}\gamma = \alpha\,\mathrm{d}\sigma,$$

and that this leads to

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\gamma} = \frac{\|\mathbf{x}'\|}{\alpha}.$$

Using the chain rule, we then see that

$$\dot{\mathbf{x}} = \frac{\alpha}{\|\mathbf{x}'\|}\mathbf{x}', \quad \dot{\mathbf{p}} = \frac{\alpha}{\|\mathbf{x}'\|}\mathbf{p}',$$

and so Hamilton's equations become

$$\frac{\alpha}{\|\mathbf{x}'\|}\mathbf{x}' = \alpha^2\mathbf{p}, \quad \frac{\alpha}{\|\mathbf{x}'\|}\mathbf{p}' = -\frac{1}{\alpha}\nabla\alpha.$$

From the first of these equations we have

$$\mathbf{p} = \frac{1}{\alpha\|\mathbf{x}'\|}\mathbf{x}',$$

and substituting this into the second of the equations we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}\left(\frac{1}{\alpha}\frac{\mathbf{x}'}{\|\mathbf{x}'\|}\right) + \frac{\|\mathbf{x}'\|}{\alpha^2}\nabla\alpha = \mathbf{0},$$

which is precisely the Euler-Lagrange equation obtained from Fermat's principle.

Note that Fermat's principle can be established in a general anisotropic body, but here there is a bit more work involved, and it is actually simpler to obtain the result using a Legendre transformation to pass between the equations of Hamiltonian and Lagrangian mechanics. You do not have to know how to do this!

7. We are told the ray travels down into the half-space monotonically until the turning depth z_t. This means that along this segment of the ray, we can use z as a generating parameter, and hence the differential arc length can be written

$$\mathrm{d}s = \sqrt{1 + \left\|\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}z}\right\|^2}\,\mathrm{d}z.$$

Using this result, we obtain trivially

$$T = \int_0^{z_t} \frac{2}{\alpha}\sqrt{1 + \left\|\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}z}\right\|^2}\,\mathrm{d}z,$$

where the factor of two is due to the upwards going half of the ray.

Within the above expression for T, the horizontal position vector $\mathbf{x}$ as a function of z is not determined. To do this we can use Fermat's principle, and set the variation of T with respect to $\mathbf{x}$ to be zero. Note that the variation in $\mathbf{x}$ is taken to vanish at the start and end points. Let $\mathscr{L}(\mathbf{x}, \mathbf{x}')$ denote the integrand within the expression for T, where $\mathbf{x}' = \mathrm{d}\mathbf{x}/\mathrm{d}z$. The Euler-Lagrange equations for the problem are

$$\frac{\partial \mathscr{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\partial \mathscr{L}}{\partial x'}\right) = 0, \quad \frac{\partial \mathscr{L}}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\partial \mathscr{L}}{\partial y'}\right) = 0.$$

But here we see that the Lagrangian is independent of the horizontal co-ordinates, and so we get the conservation laws

$$\frac{\partial \mathscr{L}}{\partial x'} = c_x, \quad \frac{\partial \mathscr{L}}{\partial y'} = c_y,$$

where c_x and c_y are constants to be determined. A simple calculation shows that these conservation laws take the specific form

$$\frac{2}{\alpha}\frac{x'}{\sqrt{1 + (x')^2 + (y')^2}} = c_x, \quad \frac{2}{\alpha}\frac{y'}{\sqrt{1 + (x')^2 + (y')^2}} = c_y.$$

Considering the initial direction of the ray lies in the x-z plane, it follows that $c_y = 0$, and hence $y' = 0$, meaning that the ray stays within this initial plane. From simple geometry, we then see that

$$\frac{x'}{\sqrt{1 + (x')^2 + (y')^2}} = \sin\theta,$$

where θ is the angle between the vertical and the ray's tangent vector. This suggests we define a new constant $q = 2c_x$, and hence arrive at

$$\frac{\sin\theta}{\alpha} = q,$$

which is the familiar form of Snell's law. At the turning depth, we have $\theta = \pi/2$ by definition, and hence

$$\alpha[z_t(q)] = q^{-1}.$$

Because α increases monotonically, this equation has a unique solution for each initial ray orientation.

Along the down-going part of the ray path we have

$$\frac{\mathrm{d}x}{\mathrm{d}z} = \tan\theta = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}},$$

and using Snell's law we can eliminate θ from the right hand side to give

$$\frac{\mathrm{d}x}{\mathrm{d}z} = \frac{q\alpha}{\sqrt{1-q^2\alpha^2}},$$

and hence we obtain

$$X = \int_0^{z_t} \frac{2q\alpha\,\mathrm{d}z}{\sqrt{1-q^2\alpha^2}}.$$

Using the expression for x' in the original formula for T we then also obtain

$$T = \int_0^{z_t} \frac{2}{\alpha}\frac{\mathrm{d}z}{\sqrt{1-q^2\alpha^2}}.$$