Curvature

Relativistic Astrophysics and Cosmology: Lecture 3

Sandro Tacchella

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Pre-lecture question:

How do you know the earth isn't flat?

Last time

- Established Einstein's principle that real gravity is spacetime curvature, reducing local gravity to a fictitious force.
- Recapped the mathematics of calculating geodesics in curved spaces.

This lecture

- Meet the weak-field metric.
- Build an intuitive understanding of curvature from two dimensions.
- Understand the difference between intrinsic and embedded curvature.
- Motivate and discuss general radial metrics.
- Introduce and semi-derive the Schwarzschild metric.

Next lecture

- New material on the interior Schwarzschild solution.

When is a space curved?

- How could players in Civilization (left) and Planetary Annihilation (right) determine the shape/curvature of their world?
- Which of these metrics describe spaces that are curved?

$$ds^2 = dx^2 + dy^2 \qquad ds^2 = dx^2 + x^2\, dy^2$$

$$ds^2 = dx^2 - dy^2 \qquad ds^2 = dx^2 + \sin^2 x\, dy^2$$

$$ds^2 = dx^2 + dx\, dy + dy^2 \qquad ds^2 = dx^2 + \sinh^2 x\, dy^2$$

$$ds^2 = x^2\, dx^2 + y^2\, dy^2 \qquad ds^2 = y^2\, dx^2 + x^2\, dy^2$$

Why curvature in spacetime?

- Why do we discuss curvature in 4-dimensional space*time*, and not just space itself?
- Typical examples of curvature considered are surfaces embedded in 3-dimensional space, and it is not clear how or why time enters the picture.
- Several answers can be given to this, each instructive.

Mathematical

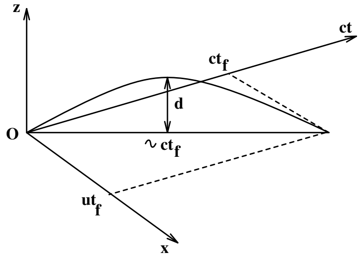
- Special relativity teaches us that space and time are inseparable parts of a larger entity — spacetime.
- A rotation of axes in spacetime (a Lorentz boost) can make the interval between events which are separated purely in time for one observer, appear to have a spatial component for another, and *vice versa*.
- Would therefore be odd if curvature applied just to space, and not to time as well.

Geometrical

- A very obvious thing about a geodesic on a 2-dimensional surface, e.g. that of a sphere, is that to specify it, we need only specify a *direction*.
- Choose a direction at a given point and we have picked out the *unique* geodesic passing through the point in that direction.
- For example, for an aircraft flying from Chicago to Delhi, the route it should take to get there most quickly is a great circle over the Arctic, and this is independent of the actual speed the aircraft flies at.
- In GR, this is not true in space (magnitude of 3-velocity matters as well as direction), but is true for space*time*.

- As a concrete example, let's consider the curvature of the path of a thrown ball.
- The curvature of the path of a ball through space obviously varies (e.g. lobbed vs direct)
- It however is constant in spacetime!
- Suppose we have a particle projected from the ground with initial velocity $(u, 0, v)$, so that there is motion only in the Oxz plane, with x horizontal, z vertical. The path in spacetime looks as follows.
- We have $z(t) = vt - \frac{1}{2}gt^2$; time of flight $= t_f = 2v/g$ and thus

$$\text{maximum height rise } d = z(t_f/2) = \frac{gt_f^2}{8}.$$



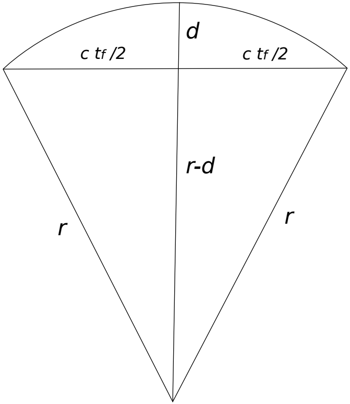
- Assuming non-relativistic velocities, we have $ut_f \ll ct_f$ and thus the length of the path as projected in the Oxt plane is $\sim ct_f$.

- Furthermore, it will be a good approximation to take this parabola as being the arc of a circle, with radius of curvature r given by the Pythagoras relation

$$r^2 = \left(\frac{ct_f}{2}\right)^2 + (r-d)^2$$

For $d \ll ct_f$ and $d \ll r$ (both satisfied!), this yields $r = c^2 t_f^2/(8d)$, and then combining with $d = gt_f^2/8$ we find for the radius of curvature of the path

$$\boxed{r = c^2/g = 9.162 \times 10^{15}\,\mathrm{m} = 0.968\,\mathrm{ly}} \qquad (\mathrm{ly}= \text{lightyears})$$



- This is independent of u and v (and of course of the mass), and we deduce that the curvature of world lines of freely falling bodies in spacetime is constant (for a given gravitational field).
- One should remember that we have worked it out in an embedding picture — one might prefer to say that it is the world lines that are straight, and the spacetime which is curved!

The Newtonian limit: The weak-field (slow moving) metric

- Last lecture we reviewed the mathematics of geodesics: Space tells matter how to move
- To make further progress with particle motion, we need to have available metrics to start from, and this requires finding solutions to Einstein's field equations.
- This in turn means getting to grips with curvature. So let's move towards this.
- By construction, description of gravity as spacetime curvature reduces to special relativity in local inertial frames. Must also check it reduces to Newtonian gravity in appropriate limits.
- In absence of gravity: spacetime has Minkowski geometry
  $\Rightarrow$ weak gravitational field corresponds to region of spacetime that is only 'slightly' curved
  $\Rightarrow$ in such a region there exist coordinates x^μ such that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad \text{where } |h_{\mu\nu}| \ll 1 \tag{1}$$

where $\eta_{\mu\nu}$ is the SR metric discussed earlier.

We carry this out for the gravitational field at the Earth's surface in the (non-examinable) Appendix, reaching the following result for the particle acceleration in ordinary time:

$$\frac{d^2\vec{x}}{dt^2} = -\tfrac{1}{2}c^2\vec{\nabla}h_{00}.$$

Compare with usual Newtonian equation of motion for a particle in a gravitational field $\Rightarrow$ identical if $h_{00} = 2\Phi/c^2$. Hence, gravity as spacetime curvature $\Rightarrow$ Newtonian gravity if particle slowly moving and metric such that, in weak field limit,

$$\boxed{g_{00} = \left(1 + \frac{2\Phi}{c^2}\right).}$$

How big is the correction to the Minkowski metric? Some values of Φ/c^2 for various systems are:

$$\frac{\Phi}{c^2} = -\frac{GM}{c^2 r} \sim \begin{cases} -10^{-9} & \text{at the surface of the Earth} \\ -10^{-6} & \text{at the surface of the Sun} \\ -10^{-4} & \text{at the surface of a white dwarf.} \end{cases}$$

Thus, even at the surface of a dense white dwarf $\Phi/c^2 \ll 1 \Rightarrow$ weak field limit is an excellent approximation.

Note: gravity as spacetime curvature has another immediate consequence. Since $g_{00} \neq 1 \Rightarrow$ coordinate time t does not measure proper time Consider a clock at rest in our coordinate system (i.e. $dx^i/dt = 0$) $\Rightarrow$ proper time interval $d\tau$ between two 'clicks' of the clock is

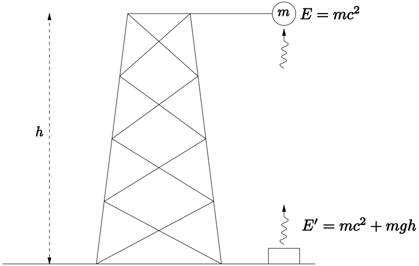
$$c^2 d\tau^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = g_{00} c^2\, dt^2,$$

from which we find that

$$\boxed{d\tau = \left(1 + \frac{2\Phi}{c^2}\right)^{1/2} dt.}$$

- This gives the interval $d\tau$ of proper time for a stationary observer near a massive object, corresponding to an interval dt of coordinate time, in a position where the gravitational potential is Φ.

- Since Φ is negative $\Rightarrow$ proper time interval is shorter than the corresponding interval for a stationary observer at a large distance from the object, where $\Phi \to 0$ and we would have $d\tau = dt$.



- Thus, as a bonus, our analysis has also yielded the formula for time dilation in a weak gravitational field.
- Now we think we know this (to first order at least) from prediction for Einstein Tower experiment.

Thus check this is consistent. For the two observers, at heights x_1 and x_2 in the Earth's field, we have

$$\frac{\nu_2}{\nu_1} = \left(\frac{1 + 2\Phi(x_2)/c^2}{1 + 2\Phi(x_1)/c^2}\right)^{-1/2} \approx 1 - (\Phi(x_2) - \Phi(x_1))/c^2$$

and so $z \approx \Delta\Phi/c^2$, as we found before. Thus all ties in. (Might like to check sign is right as well.)

Beyond the weak field

- Last year you went through full apparatus of Riemann curvature tensor and the full Einstein equations
- However, for current purposes want to use a simpler apparatus that gives (most of) the results we need, and brings us quickly to discussion about the properties of the Schwarzschild metric and black holes, which is what we need for relativistic astrophysics.
- The point is that for a lot of what we do, we only need to consider 2-dimensional sub-surfaces
- E.g. for field around a spherically symmetric body (e.g. Earth or Black Hole) we have already found the time component:

$$\boxed{g_{00} = \left(1 + \frac{2\Phi}{c^2}\right) .}$$

- Then, due to the spherical symmetry, we can work in the equatorial plane $\theta = \pi/2$ with no loss of generality, and find the appropriate r part of the metric by considering curvature in the (r, ϕ) surface.
- (For cosmology, we have a t dependence as well, but since curvature is independent of spatial position, turns out to be enough to work in a (t, r) subsurface.)
- Now it's relatively easy to find the curvature of a 2d surface: for this we can use Gauss' Theorema Egregium, so will now work towards this
- In 2d, the most general $g_{\mu\nu}$ is

$$g_{\mu\nu} = \begin{pmatrix} g_{yy} & g_{yz} \\ g_{zy} & g_{zz} \end{pmatrix}$$

- However, the metric tensor can always be taken as symmetric, since in

$$ds^2 = g_{yy}\, dy^2 + g_{yz}\, dydz + g_{zy}\, dzdy + g_{zz}\, dz^2,$$

we can amalgamate the middle two terms to give a new $g_{yz} = g_{zy}$ equal to half the sum of the old g_{yz} and g_{zy} terms.
- Furthermore, since the $g_{\mu\nu}$ matrix is now a symmetric tensor, it is always possible to find a coordinate system in which it is locally diagonal.

- In fact, for all the metrics we need in our study of GR and cosmology, it turns out that with a sensible choice of coordinate system, $g_{\mu u}$ is diagonal everywhere, and we henceforth assume this, thereby reducing a great deal the possible complexity of the mathematics.
- So the most general $g_{\mu\nu}$ we need consider in two dimensions is thus

$$g_{\mu\nu} = \begin{pmatrix} g_{yy} & 0 \\ 0 & g_{zz} \end{pmatrix},$$

  where g_{yy} and g_{zz} are general functions of the coordinates (y, z).
- A standard example is the surface of a sphere of radius a.
- Taking the two coordinates as the usual polar angles θ and ϕ we have that the interval on the surface is

$$ds^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2),$$

  so that

$$g_{\mu\nu} = \mathrm{diag}(a^2, a^2 \sin^2\theta).$$

Gauss Theorema Egregium (beautiful theorem)

No matter what coordinate transformations we carry out (thereby changing the $g_{\mu\nu}$ of course), then in two dimensions the quantity

$$\boxed{K(y,z) = \frac{1}{2g_{yy}g_{zz}}\left\{-\partial_z^2 g_{yy} - \partial_y^2 g_{zz} + \frac{\partial_y g_{yy}\partial_y g_{zz} + {\partial_z g_{yy}}^2}{2g_{yy}} + \frac{\partial_z g_{yy}\partial_z g_{zz} + {\partial_y g_{zz}}^2}{2g_{zz}}\right\}}$$

is invariant! (It might be a function of position of course, but it is not a function of which coordinate system we happen to use (polar, rectangular, etc.).)

What the quantity K measures, invariantly, is intrinsic curvature.

- For example, we can plug in $g_{\theta\theta} = a^2$ and $g_{\phi\phi} = a^2 \sin^2\theta$, for the metric on the surface of a sphere.
- All derivatives of $g_{\theta\theta}$ vanish (since it is constant) and all ∂_ϕ derivatives vanish (since there is no ϕ dependence).
- This leaves

$$K(\theta,\phi) = \frac{1}{2a^4\sin^2\theta}\left\{-\partial_\theta^2(a^2\sin^2\theta) + \frac{\left(\partial_\theta(a^2\sin^2\theta)\right)^2}{2a^2\sin^2\theta}\right\} = \frac{1}{a^2} = \text{constant}$$

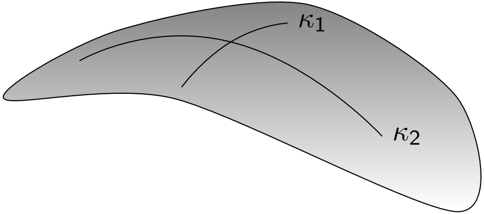
- K is called the **Gaussian curvature**, and we have just shown that the surface of a sphere has Gaussian curvature $1/a^2$

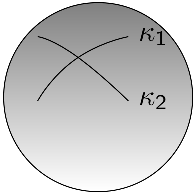
Meaning of K

- K is the Gaussian curvature, and has units $[L]^{-2}$.
- Gauss also introduced quantities κ_1 and κ_2, with dimensions $[L]^{-1}$, which are called the principal curvatures of the surface.
- Heuristic point of view is that any 2-d surface has two orthogonal directions in which it is changing most rapidly.
- κ_1 and κ_2 then measure the curvatures
- One finds that κ_1 and κ_2 cannot individually be found by measurements intrinsic to the surface (will discuss in a moment).

But Gauss found their product can be - in fact he proved:

$$K = \text{Gaussian curvature} = \kappa_1 \kappa_2 = \text{product of principal curvatures}$$





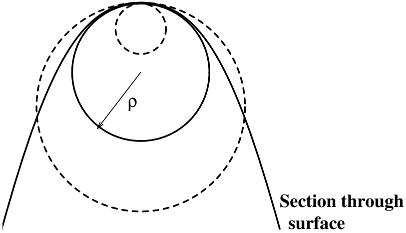


Figure: *The definition of principal curvature for a 1-dimensional section. The solid circle is the one that fits the section best at the point of contact, whilst the dotted circles are either too large or too small. The principal curvature is defined as $1/\rho$, where ρ is the radius of the best fitting circle.*

- Clearly for a sphere the curvatures in any two orthogonal directions are $1/a$, so $\kappa_1 = \kappa_2 = 1/a$ and we get $K = \kappa_1\kappa_2 = 1/a^2$ again for the sphere.
- More generally we have as definition of principal curvature:

- Now the fact that K can be expressed as the product of the two principal curvatures, though apparently harmless, leads to a surprising conclusion
- The surface of a cylinder is not curved!
- Let (z, ϕ) be coordinates on this surface (see Fig. 2), then it is clear that $\kappa_2 = 1/a$, but $\kappa_1 = 0$, so that $K = 0$.
- Should this worry us?
- The crucial point is that we must make a distinction between properties of a surface that are dependent on how it is *embedded* into a higher-dimensional space, and properties that are *intrinsic* to the surface.
- This is traditionally made clear by considering the viewpoint of some 2-dimensional creature, a 'bug' or player in a computer game, confined exclusively to the 2-dimensional surface and able to make measurements of distance, angle etc. only within the surface.

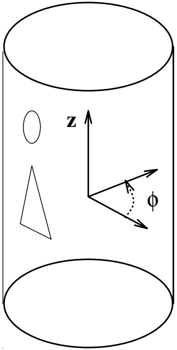
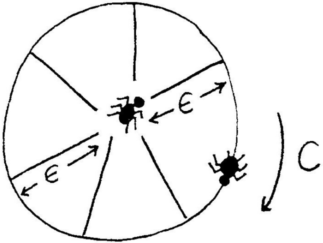


Figure: *Definition of coordinates on the surface of a cylinder of radius a. Also shown are a circle and triangle which despite being on a 'curved' surface, are still Euclidean.*

- E.g. suppose the player wishes to form a circle of radius ϵ.
- Could mark this out by radial movements as shown, then walk round the circumference, keeping count of the steps, to get the perimeter C say.
- Properties of the geometry that are accessible to this creature are called *intrinsic* whilst those that depend upon the viewpoint of a higher-dimensional creature, able to see how the surface is configured in the 3-dimensional space, are called *extrinsic*



- For example, suppose we draw various geometric figures upon a flat sheet of paper, for which $\kappa_1 = \kappa_2 = 0$, and then roll it up into a cylinder making $\kappa_2 \neq 0$.
- The player would not be able to detect any difference in the properties of the surface before and after it is rolled up. To them, the angles of a triangle still add up to 180°, the circumference of a circle is still $2\pi r$, etc., and the player correctly finds zero for the Gaussian curvature.

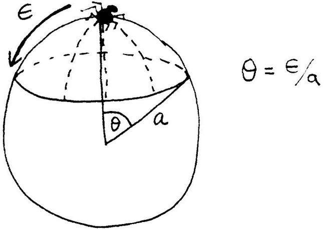
- The proof that the player would still find ordinary Euclidean properties for the figures drawn on it's surface is simple — the surface can be *unrolled* back to a flat surface without buckling tearing or otherwise distorting. This *cannot* be done for a sphere, for example, as is well known to cartographers [xkcd:977].
- To check Gauss's *Theorema Egregium*, and thus the assertion that the player will deduce $K = 0$ from it's metric determinations, note that the metric for the (z, ϕ) coordinate system is

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & a^2 \end{pmatrix},$$

  which since all the $g_{\mu\nu}$ are constant will certainly have $K = 0$.
- Above we introduced the idea that Euclidean notions get changed on a truly (intrinsically) curved surface. Thus by measuring departures from Euclideanity (again by intrinsic measurements only) can we use this as another way of deriving the Gaussian curvature K, rather than using Gauss's formula?

- E.g., for the player's 'circle', they can form

$$K = \frac{3}{\pi} \lim_{\epsilon \to 0} \left\{ \frac{2\pi\epsilon - C}{\epsilon^3} \right\}.$$



- This is difficult to prove for a general surface (though true) but we can prove it explicitly for a sphere of radius a:

$$C = 2\pi a \sin\theta = 2\pi a \sin(\epsilon/a) = 2\pi a \left(\frac{\epsilon}{a} - \frac{\epsilon^3}{3! a^3} + \dots \right) = 2\pi\epsilon \left(1 - \frac{\epsilon^2}{6a^2} \right)$$

to third order. Thus

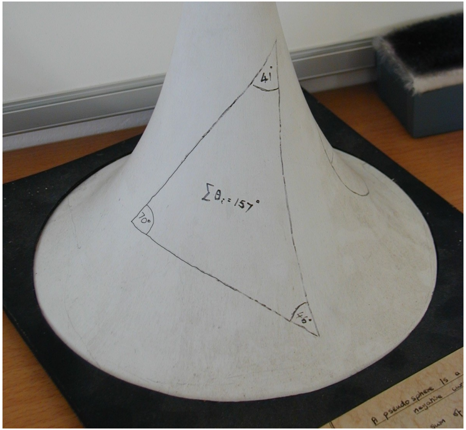
$$\frac{3}{\pi} \lim_{\epsilon \to 0} \left\{ \frac{2\pi\epsilon - C}{\epsilon^3} \right\} = \frac{3}{\pi} \lim_{\epsilon \to 0} \frac{2\pi\epsilon^3}{6a^2\epsilon^3} = \frac{1}{a^2} = K.$$

- This again shows that Gaussian curvature is intrinsic.
- A similar formula, this time for the area of the circle traced out, is $K = \frac{12}{\pi} \lim_{\epsilon \to 0} \left\{ \frac{\pi\epsilon^2 - A}{\epsilon^4} \right\}$.

ere.

Is negative curvature possible?

- Yes! For a two dimensional surface, a saddle shape will do it.
- This is because the principal curvatures in the two directions have opposite sign.
- Note using the methods of the previous section for a saddle surface one finds that one gets *larger* values for C and A than would be expected in Euclidean geometry, and thus indeed $K < 0$.



The pseudosphere with a 'hyperspherical' triangle shown. This is a 'saddle surface' (opposite sign principal curvatures) in which $\kappa_2 \propto 1/\kappa_1$ and thus has constant (negative) curvature. (The angles in the triangle add up to 157°.)

- But suppose we want a surface in 2-dimensions that is negatively curved *everywhere*, and does not have a boundary.
- It's clearly not possible to achieve this by embedding in 3-dimensional space, and we have no way of visualizing what such a surface would look like. However, we *can* set up coordinates and a metric in 2-dimensions which do have this property.
- Consider the metric, in polar coordinates (r, ϕ),

$$g_{\mu\nu} = \begin{pmatrix} f(r) & 0 \\ 0 & r^2 \end{pmatrix} .$$

- We see that distance determination in the radial direction has been modified by the function $f(r)$, while the azimuthal direction has the usual dependence upon r. Our aim is to plug this $g_{\mu\nu}$ into Gauss's *Theorema Egregium* and determine what functional form $f(r)$ must have to give a specified constant curvature K (i.e. independent of position (r, ϕ)).

- So we are aiming to find

$$K(r,\phi) = \frac{1}{2g_{rr}g_{\phi\phi}}\left\{-\partial_\phi^2 g_{rr} - \partial_r^2 g_{\phi\phi} + \frac{\partial_r g_{rr}\partial_r g_{\phi\phi} + \partial_\phi g_{rr}{}^2}{2g_{rr}} + \frac{\partial_\phi g_{rr}\partial_\phi g_{\phi\phi} + \partial_r g_{\phi\phi}{}^2}{2g_{\phi\phi}}\right\}$$

where $g_{rr} = f(r)$ and $g_{\phi\phi} = r^2$. So

$$\frac{\partial g_{rr}}{\partial r} = f', \quad \frac{\partial g_{\phi\phi}}{\partial r} = 2r, \quad \frac{\partial^2 g_{\phi\phi}}{\partial r^2} = 2$$

- Thus

$$K(r,\phi) = \frac{1}{2fr^2}\left\{-2 + \frac{1}{2f}\left[2rf'\right] + \frac{1}{2r^2}\left[(2r)^2\right]\right\}$$

- i.e.

$$\boxed{K(r,\phi) = \frac{f'}{2f^2 r}}$$

where $f' = df/dr$.
- This is already independent of ϕ, and then since we want K constant we can integrate to obtain

$$-\frac{1}{f} = r^2 K + \text{const.}$$
- But we want the space to become ordinary flat space if $K = 0$ (zero curvature), and hence we set the constant to -1, obtaining

$$f(r) = \frac{1}{1 - Kr^2} \tag{2}$$
- Parenthetically, we can remark that in equation (2), we have gone most of the way to finding the Friedmann-Robertson-Walker metric, which is the metric describing the universe on large scales.
- Completing the present task though, we see that we can get a 2-dimensional surface, with constant negative Gaussian curvature everywhere, setting K to a negative constant in (2).

Derivations and properties of complete Schwarzschild metric

- We now derive the r component of the Schwarzschild metric.
- Our method is to employ the formula for the Gaussian curvature of a 2-surface with metric

$$g_{\mu\nu} = \begin{pmatrix} f(r) & 0 \\ 0 & r^2 \end{pmatrix}, \tag{3}$$

derived just now, which we found to be

$$K = \frac{f'}{2f^2 r}.$$

- In the empty space around a spherically symmetric body, mass M, we expect the spatial part of the metric to look like

$$ds^2 = f(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where the $f(r)$ function gives the radial distortion of distance due to the presence of the mass, but where in order to preserve complete isotropy we have an undistorted metric for the θ and ϕ directions.

- We can tie this metric in with (3) if we work only in the equatorial plane $\theta = \pi/2$, so that the coordinates in (3) are (r, ϕ).
- Now, how do we want $K(r)$ to vary with position? K can only be a function of r, M, and the constants G and c, and we will have succeeded in finding the correct r dependence if there is only one combination of these which gives K the correct units of 1/(distance squared). (Compare $K = 1/a^2$ for the curvature of a sphere.)
- Setting
$$K = \frac{1}{\text{length}^2} = [M][G]^l c^m r^{-n},$$
i.e. assuming curvature is linear in the mass, yields $l = 1$, $m = -2$ and $n = 3$. Thus $K = \text{const.} \times GM/(c^2 r^3)$ and we see that $K \propto r^{-3}$.
- (Note we'll call the constant in this α.)
- This fits in well with our conception of 'real gravity', which intrinsic curvature measures, manifesting itself via tidal forces, which indeed have a $1/r^3$ dependence.
- For the value of the constant, we can imagine an analogy for the distorted space produced by the gravitating body, in which the central mass creates a distortion in the same way as a sphere sitting on a rubber sheet would create a depression in the sheet.

- The shape of the depression would be like the saddle-shaped surface considered above, and suggests to us that K should be negative.
- This is only a heuristic argument of course. In order to obtain the actual value of the constant in the relation $K \propto r^{-3}$, one can either appeal to full GR, or alternatively demand that the results we are going to obtain below for particle motion should reduce to the standard Newtonian results in the weak field limit — this yields $\alpha = -1$ (confirming the sign of α expected above)

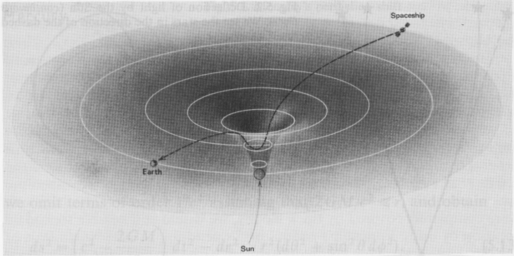


Fig. 5.3. A signal from a spaceship to the Earth arrives with delay because of the curvature of space within the gravitational influence of the Sun

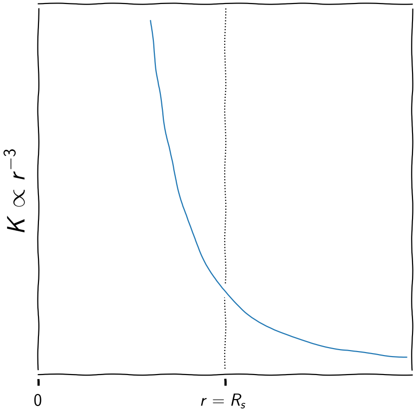
- The relation
$$K(r) = -\frac{GM}{c^2 r^3} = \frac{f'}{2f^2 r},$$
yields $-1/f = 2GM/(c^2 r)$ + another constant.
- This time however, we can get the constant immediately, since we know the space should be undistorted ($f = 1$) as $r \to \infty$. Thus
$$-\frac{1}{f} = \frac{2GM}{c^2 r} - 1 \quad \Longrightarrow \quad f(r) = \left(1 - \frac{2GM}{c^2 r}\right)^{-1}$$
and combining with our previous result with Newtonian $\phi = -GM/r$,
$$ds^2_{\text{time part}} = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2,$$
we recover the full Schwarzschild metric
$$\boxed{ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2.}$$
- Note sleight of hand – time component argument valid in weak field, but true in exact case.

- Can re-write using Schwarzschild radius and $d\Omega = d\theta^2 + \sin^2\theta d\phi^2$ as:

$$\boxed{ds^2 = \left(1 - \frac{R_S}{r}\right) c^2 dt^2 - \left(1 - \frac{R_S}{r}\right)^{-1} dr^2 - r^2 d\Omega.}$$

- Notice straightaway that the metric 'blows up' at the distance $r = \boxed{R_S = 2GM/c^2}$, which is known as the *Schwarzschild radius*, or *Schwarzschild horizon*.
- For a one solar mass object it evaluates to ~ 3 km and thus is only relevant for objects very much denser than the Sun.
- An object which lies entirely within its Schwarzschild radius is called a **black hole**.
- Within the radius R_S we note that an increment dr causes a *time*like increase in the interval ds (since its coefficient flips sign), while an increment of coordinate time dt causes a *space*like increase in ds.
- Thus inside the Schwarzschild radius, time and space effectively swap their roles, and in fact free motion in all directions in space is no longer possible, in the same way that backwards motion in time is not possible in ordinary space — all objects crossing the Schwarzschild radius from the outside can never escape again, but are inexorably drawn towards the centre, $r = 0$.

- Note that despite the coordinate singularity at $r = R_S$, and the strange behaviour for $r < R_S$, the gravity, as measured by the intrinsic curvature K, varies smoothly all the way to the origin, since by construction we have $K \propto r^{-3}$.
- Thus (on these grounds, though see Question 5, Examples 1 for another problem) it is expected that no particular harm would come to an astronaut crossing the Schwarzschild horizon from this cause



- However, it certainly would when they eventually reaches $r = 0$, where the gravity, as measured by the curvature, does become infinite.
- Thus falling through the surface and then into a black hole, is a possible, though eventually fatal, journey for an astronaut to undertake.

Including matter?

- Have achieved a simple route through to Schwarzschild vacuum solution — this appropriate to Black Holes (possibly all the way to the centre)
- Also so far have assumed it is the appropriate metric in the vacuum outside a spherically symmetric body such as the Earth or Sun etc.
- So next will look at justifying this, and moreover getting metric inside such a body
- If can show this matches properly onto an exterior Schwarzschild metric, then indeed can apply latter in all spherically symmetric vacuum cases
- (Note we will later look at rotating black holes — the Kerr solution — and find there that no one has yet been able to match this to an interior solution!)
- Moreover, stars are an extremely important case — e.g. GR vital to understand neutron stars — so will next look at **static solutions with matter**

Beyond Gaussian curvature

- The Riemann curvature tensor R^d_{abc} (4-rank tensor) extends the Gaussian curvature into higher dimensions. It gives us the curvature for every geodesic.

$$R^d{}_{abc} \equiv \partial_b \Gamma^d{}_{ac} - \partial_c \Gamma^d{}_{ab} + \Gamma^e{}_{ac}\Gamma^d{}_{eb} - \Gamma^e{}_{ab}\Gamma^d{}_{ec}$$

- The Ricci curvature scalar gives us the average curvature for a volume

$$R \equiv g^{ab} R_{ab}$$

  where R_{ab} is the Ricci tensor.
- The Einstein tensor G^{ab} is the tensor that describes the curvature of spacetime in the field equations of GR

$$G^{ab} \equiv R^{ab} - \frac{1}{2} g^{ab} R.$$

Summary

- Our first metric, the weak field metric:

$$\boxed{ds^2 = \left(1 + \frac{2\phi}{c^2}\right) dt^2 - d\vec{x}^2.}$$

- Intrinsic vs entrinsic/embedded pictures.
- Gauss' Theorema Egregium / beautiful theorem.
- Spaces of constant curvature.
- The Schwarzschild metric & coordinate singularities

$$\boxed{ds^2 = \left(1 - \frac{R_S}{r}\right) c^2 dt^2 - \left(1 - \frac{R_S}{r}\right)^{-1} dr^2 - r^2 d\Omega.}$$

Next time

Including matter in the Schwarzschild spacetime

Appendix (non-examinable)

Reduction to Newtonian Gravity

We will work in a Cartesian frame on surface of (non-rotating) Earth.
Can assume in this coordinate system the metric is static $\Rightarrow$ all the derivatives $\partial_0 g_{\mu\nu}$ are zero (field not changing) In general, since the worldline of particle freely-falling under gravity is a geodesic:

$$\boxed{\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0.}$$

Equivalent to demanding

$$\frac{dx^i}{d\tau} \ll \frac{dx^0}{d\tau} \qquad \text{for } i = 1, 2, 3.$$

$\Rightarrow$ ignore 3-velocity terms in geodesic equation, giving

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{00}\, c^2 \left(\frac{dt}{d\tau}\right)^2 = 0. \tag{4}$$

Required connection coefficients $\Gamma^\mu{}_{00}$ are given to first order (last equality) by

$$\begin{aligned}\Gamma^\mu{}_{00} &= \tfrac{1}{2} g^{\kappa\mu}(\partial_0 g_{0\kappa} + \partial_0 g_{0\kappa} - \partial_\kappa g_{00}) \\ &= -\tfrac{1}{2} g^{\kappa\mu}\partial_\kappa g_{00} = -\tfrac{1}{2}\eta^{\kappa\mu}\partial_\kappa h_{00},\end{aligned}$$

Since the metric is assumed static, we thus have

$$\Gamma^0{}_{00} = 0 \qquad \text{and} \qquad \Gamma^i{}_{00} = \tfrac{1}{2}\delta^{ij}\partial_j h_{00},$$

where Latin index runs over $i = 1, 2, 3$.
Inserting these coefficients into (4) gives

$$\frac{d^2 t}{d\tau^2} = 0, \qquad \frac{d^2\vec{x}}{d\tau^2} = -\tfrac{1}{2}c^2\left(\frac{dt}{d\tau}\right)^2\vec{\nabla}h_{00}$$

First equation $\Rightarrow$ $dt/d\tau = \text{constant}$ $\Rightarrow$ can combine the two equations to yield the equation of motion for the particle

$$\frac{d^2\vec{x}}{dt^2} = -\tfrac{1}{2}c^2\vec{\nabla}h_{00}.$$

Compare with usual Newtonian equation of motion for a particle in a gravitational field $\Rightarrow$ identical if $h_{00} = 2\Phi/c^2$.
Hence, gravity as spacetime curvature $\Rightarrow$ Newtonian gravity if particle slowly moving and metric such that, in weak field limit,

$$\boxed{g_{00} = \left(1 + \frac{2\Phi}{c^2}\right).}$$