Seismology Problem Set 2

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1. From Lecture 18, we know that the least squares solution for the given problem is

$$\mathbf{m} = (\mathbf{A}^T \mathbf{C}^{-1} \mathbf{A} + \lambda \mathbf{B})^{-1} \mathbf{A}^T \mathbf{C}^{-1} \mathbf{d},$$

with $\mathbf{d}$ the data vector. Here we take $\mathbf{d} = \mathbf{A}\mathbf{m}_{\mathrm{in}}$, and hence find that

$$\mathbf{m}_{\mathrm{out}} = (\mathbf{A}^T \mathbf{C}^{-1} \mathbf{A} + \lambda \mathbf{B})^{-1} \mathbf{A}^T \mathbf{C}^{-1} \mathbf{A}\mathbf{m}_{\mathrm{in}}.$$

It follows that the resolution matrix takes the form

$$\mathbf{R} = (\mathbf{A}^T \mathbf{C}^{-1} \mathbf{A} + \lambda \mathbf{B})^{-1} \mathbf{A}^T \mathbf{C}^{-1} \mathbf{A}.$$

The resolution matrix relates the input and output of a synthetic inversion that does not include data errors. In an ideal case, we like the resulting model to be equal to that input, and hence we want $\mathbf{R}$ to be close to the identity.

2. A suitable Lagrangian for this problem can be defined by

$$L = J(\mathbf{u}) - \int_M \frac{\partial}{\partial x_j} \left(A_{ijkl} \frac{\partial u_k}{\partial x_l} \right) u'_i \, \mathrm{d}^3\mathbf{x} + \int_{\partial M} \left[\hat{n}_j \left(A_{ijkl} \frac{\partial u_k}{\partial x_l} \right) - \sigma g_i \right] w'_i \, \mathrm{d}S,$$

where we have used the elastic tensor $A_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, and u'_i and w'_i are Lagrange multipliers. As usual, variation w.r.t to the multipliers just gives the forward problem. Varying u_i, however, leads to the new condition

$$\int_{\partial M} h'_i \delta u_i \, \mathrm{d}S - \int_M \frac{\partial}{\partial x_j} \left(A_{ijkl} \frac{\partial \delta u_k}{\partial x_l} \right) u'_i \, \mathrm{d}^3\mathbf{x} + \int_{\partial M} \hat{n}_j \left(A_{ijkl} \frac{\partial \delta u_k}{\partial x_l} \right) w'_i \, \mathrm{d}S = 0,$$

where we have defined

$$\mathbf{h}'(\mathbf{x}) = \sum_{i=1}^m \frac{1}{\sigma_i^2} [\mathbf{u}(\mathbf{x}_i) - \mathbf{u}_i^{\mathrm{obs}}] \delta(\mathbf{x} - \mathbf{x}_i), \qquad (1)$$

to simplify the contribution associated with $J(\mathbf{u})$. To proceed, we need to integrate

by parts to isolate δu_i. The key step is

$$-\int_M \frac{\partial}{\partial x_j}\left(A_{ijkl}\frac{\partial \delta u_k}{\partial x_l}\right)u_i'\,\mathrm{d}^3\mathbf{x} = -\int_M \frac{\partial}{\partial x_j}\left(A_{ijkl}\frac{\partial \delta u_k}{\partial x_l}u_i'\right)\mathrm{d}^3\mathbf{x} + \int_M A_{ijkl}\frac{\partial \delta u_k}{\partial x_l}\frac{\partial u_i'}{\partial x_j}\,\mathrm{d}^3\mathbf{x}$$

$$= -\int_{\partial M}\hat{n}_j\left(A_{ijkl}\frac{\partial \delta u_k}{\partial x_l}\right)u_i'\,\mathrm{d}S + \int_M A_{ijkl}\frac{\partial \delta u_i}{\partial x_j}\frac{\partial u_k'}{\partial x_l}\,\mathrm{d}^3\mathbf{x}$$

$$= -\int_{\partial M}\hat{n}_j\left(A_{ijkl}\frac{\partial \delta u_k}{\partial x_l}\right)u_i'\,\mathrm{d}S + \int_M \frac{\partial}{\partial x_j}\left(A_{ijkl}\delta u_i\frac{\partial u_k'}{\partial x_l}\right)\mathrm{d}^3\mathbf{x}$$

$$\qquad -\int_M \delta u_i\frac{\partial}{\partial x_j}\left(A_{ijkl}\frac{\partial u_k'}{\partial x_l}\right)\mathrm{d}^3\mathbf{x}$$

$$= -\int_{\partial M}\hat{n}_j\left(A_{ijkl}\frac{\partial \delta u_k}{\partial x_l}\right)u_i'\,\mathrm{d}S + \int_{\partial M}\delta u_i\hat{n}_j\left(A_{ijkl}\frac{\partial u_k'}{\partial x_l}\right)\mathrm{d}S$$

$$\qquad -\int_M \delta u_i\frac{\partial}{\partial x_j}\left(A_{ijkl}\frac{\partial u_k'}{\partial x_l}\right)\mathrm{d}^3\mathbf{x},$$

where we note that the hyperelastic symmetry $A_{ijkl} = A_{klij}$ has been used. Using this result, the condition now reads

$$\int_{\partial M}\left[h_i' + \hat{n}_j\left(A_{ijkl}\frac{\partial u_k'}{\partial x_l}\right)\right]\delta u_i\,\mathrm{d}S - \int_M \delta u_i\frac{\partial}{\partial x_j}\left(A_{ijkl}\frac{\partial u_k'}{\partial x_l}\right)\mathrm{d}^3\mathbf{x}$$

$$+ \int_{\partial M}\hat{n}_j\left(A_{ijkl}\frac{\partial \delta u_k}{\partial x_l}\right)(w_i' - u_i')\,\mathrm{d}S = 0.$$

It follows that u_i' must satisfy

$$\frac{\partial}{\partial x_j}\left(A_{ijkl}\frac{\partial u_k'}{\partial x_l}\right) = 0,$$

subject to the boundary condition

$$\hat{n}_j\left(A_{ijkl}\frac{\partial u_k'}{\partial x_l}\right) = -h_i',$$

while also $w_i' = u_i'$ on ∂M. These are the adjoint equations for this problem. Having solved the forward and adjoint problems, the first-order perturbation to J can be written

$$\delta J = -\int_M \frac{\partial}{\partial x_j}\left(\delta A_{ijkl}\frac{\partial u_k}{\partial x_l}\right)u_i'\,\mathrm{d}^3\mathbf{x} + \int_{\partial M}\left[\hat{n}_j\left(\delta A_{ijkl}\frac{\partial u_k}{\partial x_l}\right) - \delta\sigma g_i\right]u_i'\,\mathrm{d}S.$$

Performing an integration by parts, this simplifies to

$$\delta J = \int_M \delta A_{ijkl}\frac{\partial u_k}{\partial x_l}\frac{\partial u_i'}{\partial x_j}\,\mathrm{d}^3\mathbf{x} - \int_{\partial M}\delta\sigma g_i u_i'\,\mathrm{d}S.$$

From this we see that the sensitivity kernel w.r.t σ is equal to $K_\sigma = -g_i u_i'$. To get the kernels for λ and μ, we note that

$$\delta A_{ijkl} = \delta\lambda\delta_{ij}\delta_{kl} + \delta\mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

from which we find

$$\delta J = \int_M \delta\lambda \frac{\partial u_i}{\partial x_i}\frac{\partial u'_j}{\partial x_j}\, \mathrm{d}^3\mathbf{x} + \int_M \delta\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i}\right)\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u'_i}{\partial x_j}\right)\,\mathrm{d}^3\mathbf{x},$$

where we have set $\delta\sigma = 0$ as that term has been dealt with. From this expression, we can now read off the desired sensitivity kernels

$$K_\lambda = \frac{\partial u_i}{\partial x_i}\frac{\partial u'_j}{\partial x_j}, \quad K_\mu = \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i}\right)\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u'_i}{\partial x_j}\right).$$

As typical with the adjoint method, we see that these sensitivity kernels depend on both the solution of the forward and adjoint problems. These two problems are identical in form, differing only in their force terms, and hence the overall cost of the calculations is that of two forward calculations.

3. The cross correlation is defined by

$$C(\tau) = \int_{-\infty}^{\infty} s^{\mathrm{obs}}(t - \tau)s(\mathbf{x}_s, t)\,\mathrm{d}t,$$

and so the optimality condition is given explicitly by

$$\frac{\mathrm{d}C}{\mathrm{d}\tau}(\bar{\tau}) = -\int_{-\infty}^{\infty} \dot{s}^{\mathrm{obs}}(t - \bar{\tau})s(\mathbf{x}_s, t)\,\mathrm{d}t = 0.$$

If we perturb the synthetic seismogram s to $s + \delta s$ and write $\delta\bar{\tau}u$ for the change in $\bar{\tau}$, we then obtain to first-order accuracy that

$$-\delta\bar{\tau}\int_{-\infty}^{\infty} \ddot{s}^{\mathrm{obs}}(t - \bar{\tau})s(\mathbf{x}_s, t)\,\mathrm{d}t + \int_{-\infty}^{\infty} \dot{s}^{\mathrm{obs}}(t - \bar{\tau})\delta s(\mathbf{x}_s, t)\,\mathrm{d}t = 0,$$

which we can solve to obtain

$$\delta\bar{\tau} = \frac{\int_{-\infty}^{\infty} \dot{s}^{\mathrm{obs}}(t - \bar{\tau})\,\delta s(\mathbf{x}_s, t)\,\mathrm{d}t}{\int_{-\infty}^{\infty} \ddot{s}^{\mathrm{obs}}(t - \bar{\tau})s(\mathbf{x}_s, t)\,\mathrm{d}t}.$$

4. The part of the Lagrangian that depends explicitly on the motion is

$$\frac{1}{2}G\int_M \frac{\rho\rho'}{\{[\varphi_i - \varphi'_i][\varphi_i - \varphi'_i]\}^{\frac{1}{2}}}\,\mathrm{d}^3\mathbf{x}',$$

where it is understood that un-primed functions depend on $\mathbf{x}$ and primed ones on the dummy integration variable $\mathbf{x}'$. Perturbing the motion to $\boldsymbol{\varphi} + \delta\boldsymbol{\varphi}$ and expanding to first-order accuracy we find

$$\int_M \frac{\delta\mathcal{L}}{\delta\varphi_i}\delta\varphi_i\,\mathrm{d}^3\mathbf{x} = -\frac{1}{2}G\int_M\int_M \frac{\rho\rho'(\varphi_i - \varphi'_i)(\delta\varphi_i - \delta\varphi'_i)}{\{[\varphi_j - \varphi'_j][\varphi_j - \varphi'_j]\}^{\frac{3}{2}}}\,\mathrm{d}^3\mathbf{x}'\,\mathrm{d}^3\mathbf{x}.$$

Splitting this expression into two parts we find

$$\int_M \frac{\delta\mathcal{L}}{\delta\varphi_i}\delta\varphi_i\,\mathrm{d}^3\mathbf{x} = -\frac{1}{2}G\int_M\int_M \frac{\rho\rho'(\varphi_i - \varphi'_i)\delta\varphi_i}{\{[\varphi_j - \varphi'_j][\varphi_j - \varphi'_j]\}^{\frac{3}{2}}}\,\mathrm{d}^3\mathbf{x}'\,\mathrm{d}^3\mathbf{x}$$
$$+ \frac{1}{2}G\int_M\int_M \frac{\rho\rho'(\varphi_i - \varphi'_i)\delta\varphi'_i}{\{[\varphi_j - \varphi'_j][\varphi_j - \varphi'_j]\}^{\frac{3}{2}}}\,\mathrm{d}^3\mathbf{x}'\,\mathrm{d}^3\mathbf{x},$$

but by interchanging the order of integration we note that the two terms are actually equal, and hence

$$\int_M \frac{\delta \mathcal{L}}{\delta \varphi_i} \delta \varphi_i \, \mathrm{d}^3\mathbf{x} = -G \int_M \int_M \frac{\rho \rho'(\varphi_i - \varphi_i')\delta \varphi_i}{\{[\varphi_j - \varphi_j'][\varphi_j - \varphi_j']\}^{\frac{3}{2}}} \, \mathrm{d}^3\mathbf{x}' \, \mathrm{d}^3\mathbf{x},$$

which implies

$$\frac{\delta \mathcal{L}}{\delta \varphi_i} = -G \int_M \frac{\rho \rho'(\varphi_i - \varphi_i')}{\{[\varphi_j - \varphi_j'][\varphi_j - \varphi_j']\}^{\frac{3}{2}}} \, \mathrm{d}^3\mathbf{x}' = \rho \gamma_i,$$

as required.

5. Considering first the continuity of the potential across the boundary, we require

$$[\phi]_-^+ = 0,$$

across $r = b + s\, h_1(\theta, \phi) + \cdots$, where $[\cdot]_-^+$ denotes the jump in a quantity across the relevant boundary in the direction of the outward unit normal. Expanding this to first order accuracy we obtain

$$\left[\phi_0 + s\, \frac{\mathrm{d}\phi_0}{\mathrm{d}r} h_1 + \phi_1 + \cdots \right]_-^+ = 0,$$

across the reference boundary $r = b$. Using the continuity of ϕ_0 and its normal derivative to the reference boundary, this reduces to

$$[\phi_1]_-^+ = 0,$$

for $r = b$. To deal with the continuity of the normal derivative, we first write

$$\hat{\mathbf{n}} = \hat{\mathbf{r}} + s\, \hat{\mathbf{n}}_1 + \cdot,$$

for the outward unit normal to the boundary. The term $\hat{\mathbf{n}}_1$ can be readily expressed in terms of h_1, but we will see that this is not required. Similar to above, the continuity of $\hat{\mathbf{n}} \cdot \nabla \phi$ across the aspherical boundary can be expanded to

$$[\hat{\mathbf{r}} \cdot \nabla \phi_0 + s\, \hat{\mathbf{r}} \cdot \nabla \nabla \phi_0 h_1 + s\, \hat{\mathbf{n}}_1 \cdot \nabla \phi_0 + s\, \hat{\mathbf{r}} \cdot \nabla \phi_1 + \cdots]_-^+ = 0,$$

across $r = b$. The zeroth-order term vanishes immediately, while we note that $\hat{\mathbf{n}}_1 \cdot \nabla \phi_0$ is zero as $\hat{\mathbf{n}}_1$ must be orthogonal to $\hat{\mathbf{r}}$ in order for perturbed normal to be a unit vector (and more generally, this term is proportional to the radial derivative of ϕ_0 which is continuous). The first-order condition therefore becomes

$$[\hat{\mathbf{r}} \cdot \nabla \nabla \phi_0 h_1 + \hat{\mathbf{r}} \cdot \nabla \phi_1]_-^+ = 0,$$

on $r = b$. To simplify the first term, we recall that ϕ_0 satisfies the spherically symmetric Poison equation

$$\frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} \left(r^2 \frac{\mathrm{d}\phi_0}{\mathrm{d}r} \right) = 4\pi G \rho_0.$$

Using this result, we see that

$$\hat{\mathbf{r}} \cdot \nabla\nabla\phi_0 = \frac{\partial^2\phi_0}{\partial r^2} = 4\pi G\rho_0 - \frac{2}{r}\frac{\partial\phi_0}{\partial r},$$

which can be substituted into the jump condition to obtain

$$[\hat{\mathbf{r}} \cdot \nabla\phi_1 + 4\pi G\rho_0 h_1]_-^+ = 0,$$

where again the continuity of the normal derivative of ϕ_0 has been applied.

6. We are told

$$\mathbf{u}(\mathbf{x},t) = \sum_{km}\left\{\int_0^t \frac{\sin[\omega_k(t-t')]}{\omega_k}\int_M \bar{S}_{ij}(\mathbf{x}',t')\frac{\partial\,|\,km\,\rangle^*}{\partial x_j}(\mathbf{x}')\,\mathrm{d}^3\mathbf{x}'\,\mathrm{d}t'\right\}|\,km\,\rangle\,(\mathbf{x}),$$

noting that arguments have now been included for clarity. In the case of a moment tensor point source, we see immediately that

$$\int_M \bar{S}_{ij}(\mathbf{x}',t')\frac{\partial\,|\,km\,\rangle^*}{\partial x_j}(\mathbf{x}')\,\mathrm{d}^3\mathbf{x}' = M_{ij}\frac{\partial\,|\,km\,\rangle^*}{\partial x_j}(\mathbf{x}_s)H(t'-t_s),$$

while the necessary time integral is readily evaluated for $t \geq t_s$:

$$\int_{t_s}^t \frac{\sin[\omega_k(t-t')]}{\omega_k}\,\mathrm{d}t' = \frac{1-\cos\omega_k(t-t_s)}{\omega_k^2},$$

and hence the eigenfunction expansion reduces to

$$\mathbf{u}(\mathbf{x},t) = \sum_{km}\frac{1-\cos\omega_k(t-t_s)}{\omega_k^2}M_{ij}\frac{\partial\,|\,km\,\rangle^*}{\partial x_j}(\mathbf{x}_s)|\,km\,\rangle\,(\mathbf{x}),$$

for $t \geq t_s$ and zero before this.

This final expression depends on ten source parameters, assuming that Earth structure is known. The dependence on the moment tensor is linear, but that on the source locations and source time is non-linear. There are several possible methods for solving the inverse problem. The most obvious is to define some sort of least-squares misfit between observed and predicted seismograms, and then optimise to find the best fitting source parameters. To do this gradient-based methods could be applied, and here the derivatives of with respect to the source parameters could be calculated directly with relative ease. Note that in this problem there are likely to be more data than model parameters, and hence it is likely to be over-determined. As a result, regularisation would not be needed. Because the problem is non-linear, it is not certain that the optimal solution obtained would be a global minimum. Due to the size of the inverse problem and the low cost of implementing mode summation (in a spherically symmetric earth at least) the application of Bayesian methods via something like MCMC would be feasible, and this could address the possible issue of local minima, while also providing a quantification of uncertainty.

7. Within the eigenvalue problem we take the test function equal to $\mathbf{s}$ and so obtain

$$-\omega^2 + \mathrm{i}\omega\,\langle\,\mathbf{s}\,|\,W\,|\,\mathbf{s}\,\rangle + \langle\,\mathbf{s}\,|\,H\,|\,\mathbf{s}\,\rangle = 0,$$

where we have used the normalisation condition. This is a quadratic equation for ω, and its solution yields

$$\omega = \frac{1}{2}\mathrm{i}\left\langle\,\mathbf{s}\,|\,W\,|\,\mathbf{s}\,\right\rangle \pm \sqrt{\left\langle\,\mathbf{s}\,|\,H\,|\,\mathbf{s}\,\right\rangle - \frac{1}{4}\left\langle\,\mathbf{s}\,|\,W\,|\,\mathbf{s}\,\right\rangle^2},$$

as required. Note that that the Coriolis form is anti-Hermitian, and hence $\left\langle\,\mathbf{s}\,|\,W\,|\,\mathbf{s}\,\right\rangle$ is an imaginary number. For this reason the expression for ω is real so long as

$$\left\langle\,\mathbf{s}\,|\,H\,|\,\mathbf{s}\,\right\rangle - \frac{1}{4}\left\langle\,\mathbf{s}\,|\,W\,|\,\mathbf{s}\,\right\rangle^2 > 0,$$

with the contribution from the Coriolis term always being *non-negative*. We see that in the rotating eigenvalue problem the Coriolis force always contributes to the *stability* of the earth model. As with the non-rotating case, the question of stability reduces to the properties of the potential energy form which here includes an additional contribution from centrifugal forces. This latter part can be written explicitly as

$$\int_M \rho \epsilon_{ijk}\epsilon_{klm} s_i^* \Omega_j \Omega_l s_m \, \mathrm{d}^3\mathbf{x},$$

and using the identity $\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$ this becomes

$$\int_M \rho \epsilon_{ijk}\epsilon_{klm} s_i^* \Omega_j \Omega_l s_m \, \mathrm{d}^3\mathbf{x} = \int_M \rho \left[(s_i\Omega_i)^*(s_j\Omega_j) - (s_i^* s_i)(\Omega_j\Omega_j)\right] \mathrm{d}^3\mathbf{x}.$$

Applying the Cauchy-Schwarz inequality we note that this term is always non-positive, and hence centrifugal forces are *destabilising*.