Natural Sciences Tripos Part III — Physics of the Earth as a Planet

Examples Sheet 2: Mantle Convection — John Rudge

Comments and corrections: e-mail to `jfr23@cam.ac.uk`.
The starred question at the end of the sheet is intended as an optional extra, and should not be done at the expense of the other questions on the sheet.

1 *The Rayleigh-Taylor instability.* In the lecture on postglacial uplift the time constant for the rebound of an interface between a viscous fluid (the mantle) and an inviscid fluid (the air) was derived. Here we consider a closely related problem, studying the behaviour of perturbations to an interface between two viscous fluids. Suppose fluid of density ρ_1 and viscosity η lies above a second fluid of density ρ_2 and identical viscosity η. The interface between the two fluids is initially horizontal ($z = 0$). Consider small sinusoidal perturbations to this interface, with interface position given by

$$z = \zeta(x,t) = \epsilon \mathrm{e}^{\sigma t + ikx},$$

where ϵ is the amplitude (assumed small), $k = 2\pi/\lambda$ is the wavenumber of the perturbations (λ is the wavelength), and σ is the growth rate (the time constant $\tau = 1/\sigma$). Show that

$$\tau = \frac{8\pi\eta}{(\rho_1 - \rho_2)g\lambda}.$$

Unsurprisingly, if $\rho_1 > \rho_2$ the interface is unstable, with long wavelengths growing fastest. This is known as the Rayleigh-Taylor instability and accounts for a diverse range of phenomena in the Earth Sciences (e.g. core formation, lithospheric deblobbing, and salt domes).
[Hint: You will need to solve the Stokes equations for each fluid,

$$\eta\nabla^2\mathbf{v}_1 = \rho_1 g\hat{\mathbf{z}} + \nabla p_1, \qquad \nabla\cdot\mathbf{v}_1 = 0,$$
$$\eta\nabla^2\mathbf{v}_2 = \rho_2 g\hat{\mathbf{z}} + \nabla p_2, \qquad \nabla\cdot\mathbf{v}_2 = 0,$$

treating each fluid as an infinite half space. This is most easily done using stream functions e.g. $\mathbf{v}_1 = (u_1, 0, w_1) = \nabla \times (0, \psi_1, 0)$, where

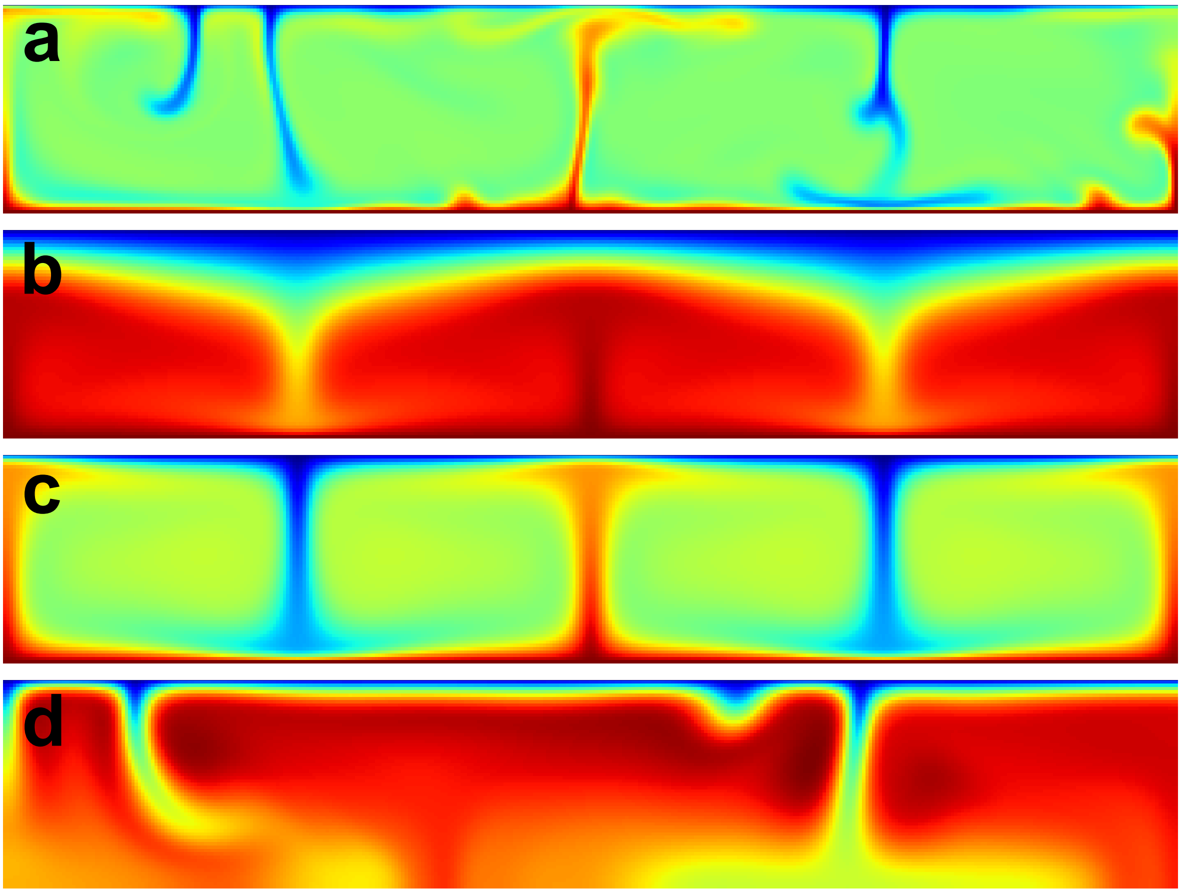
$$\psi_1 = \left[(A_1 + B_1 z)\mathrm{e}^{kz} + (C_1 + D_1 z)\mathrm{e}^{-kz}\right]\mathrm{e}^{\sigma t + ikx},$$
$$p_1 = -\rho_1 g z + 2\eta i k(B_1\mathrm{e}^{kz} + D_1\mathrm{e}^{-kz})\mathrm{e}^{\sigma t + ikx}.$$

At the interface it required that both components of the velocity and both components of the traction are continuous. Don't forget the kinematic boundary condition which relates the rate at which the interface moves to the normal velocity.]

2 (i) *High Rayleigh number convection.* Define the *Rayleigh number, aspect ratio* and *planform* of a convecting layer of fluid.

(ii) Shown below are snapshots of temperature from four simulations of convection in a 2D fluid layer. Match each picture to the correct description, giving detailed reasons for your answers.

1. Constant viscosity, heated from below, cooled from above, $\mathrm{Ra} = 10^5$.
2. Constant viscosity, heated from below, cooled from above, $\mathrm{Ra} = 10^6$.
3. Constant viscosity, heated internally, insulated from below, cooled from above, $\mathrm{Ra} = 10^6$.
4. Temperature-dependent viscosity (viscosity ratio $\eta_{\mathrm{top}}/\eta_{\mathrm{bottom}} = 10^4$), heated from below, cooled from above, $\mathrm{Ra}_{\mathrm{bottom}} = 10^6$.



(iii) How does the planform of a convecting 3D layer change as the Rayleigh number is increased? How does shear affect the 3D planform?

(iv) How may the planform of mantle convection on Earth be mapped, and what constraints does it impose on the nature of the heat sources driving the circulation?

3 (i) *The age of the Earth.* Kelvin estimated the age of the Earth t using estimates of the surface heat flux Q, the thermal diffusivity κ, the thermal conductivity k, and a guess at the Earth's initial temperature T_0. His calculation was based on a half space cooling model, with uniform initial temperature T_0 and a cooled ($T = 0$) top boundary. Using dimensional analysis, show that

$$t \sim \frac{k^2 T_0^2}{\kappa Q^2}.$$

Using $Q = 80$ mW m^{-2}, $\kappa = 10^{-6}$ m^2 s^{-1}, $k = 3$ W m^{-1} K^{-1}, $T_0 = 1500$ K give an order of magnitude estimate for t.

(ii) When Kelvin made his estimate, he was not aware that heat can be generated by radioactive decay. The current rate of heat production within the Earth is $H = 0.02$ μW m^{-3}. Suppose an infinite half space initially at uniform temperature $T = 0$ is heated internally, and cooled ($T = 0$) at the top boundary. Use dimensional analysis to give an order of magnitude estimate of "the age of the Earth" in such a model. Did Kelvin neglect an important heat source?

4 (i) *The Earth's thermal history.* Consider a convecting layer of fluid that is heated internally and cooled from above (zero temperature on the top boundary). Show that conservation of heat can be written as

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{H}{c_p} - \frac{q}{\rho c_p d},$$

where T is the mean temperature of the layer, c_p is the heat capacity, ρ is the density, d is the layer depth, H is the mean rate of internal heat production per unit mass, and q is the mean heat flux out of the layer.

(ii) Define the *Nusselt number* for a convecting layer of fluid. What is the relationship between Nusselt number and Rayleigh number for a vigorously convecting layer? Show that

$$q = A\frac{2kT}{d}\left(\frac{2g\alpha T d^3}{\nu\kappa}\right)^{1/3},$$

where A is a numerical prefactor from the Nu-Ra relation ($A \approx 0.2$).

(iii) Show that the temperature equation can be written in non-dimensional form as

$$\frac{\mathrm{d}T'}{\mathrm{d}t'} = 1 - T'^{4/3}$$

if the rate of heat production is assumed constant.

(iv) The above system will relax to a steady state where the rate of internal heat production balances the rate of heat loss by convection. Show that the steady state temperature is

$$T_0 = B\frac{\rho H d^2}{k}\mathrm{Ra}_H^{-1/4}, \quad \text{where } \mathrm{Ra}_H = \frac{\alpha\rho g H d^5}{k\kappa\nu} \quad \text{and } B = \frac{1}{2A^{3/4}} \approx 1.7.$$

Show also that the time constant for relaxation to steady state is

$$\tau = \frac{3B}{4}\frac{d^2}{\kappa}\mathrm{Ra}_H^{-1/4}.$$

(v) For whole mantle convection, $\mathrm{Ra}_H \sim 10^9$, $d \sim 2000$ km and $\kappa \sim 10^{-6}$ m^2 s^{-1}. Estimate the time constant τ.

5 (i) *Ridge push.* The density of sea water is ρ_{w} and that of oceanic lithosphere is $\rho_{\mathrm{m}}(1 - \alpha(T - T_1))$, where T is the local temperature, ρ_{m} and T_1 are the density and temperature at the top of the upper mantle, and α is the coefficient of thermal expansion. In very old oceanic lithosphere the temperature profile is

$$T = T_1 + \frac{z}{d}(T_0 - T_1), \quad 0 \leq z \leq d,$$

where d is the plate thickness, z is the vertical distance above the base of the plate, and T_0 is the temperature of sea water. Beneath the ridge, the temperature is T_1 for all $z \geq 0$. Using isostasy, show that the elevation of the ridge above old oceanic lithosphere is

$$e = \frac{\alpha(T_1 - T_0)\rho_{\mathrm{m}} d}{2(\rho_{\mathrm{m}} - \rho_{\mathrm{w}})}.$$

By integrating the pressure difference between the ridge and old oceanic lithosphere, show that the total force exerted by the ridge per unit length of ridge is

$$F_{\mathrm{RP}} = (\rho_{\mathrm{m}} - \rho_{\mathrm{w}})ge\left(\frac{d}{3} + \frac{e}{2}\right) \approx \frac{1}{6}\alpha(T_1 - T_0)\rho_{\mathrm{m}} g d^2 \left(1 + \mathcal{O}(e/d)\right).$$

(ii) *Slab pull.* Assuming the average temperature of a sinking slab is $(T_1 + T_0)/2$, calculate the buoyancy force (slab pull) due to a slab of thickness d which penetrates to a depth D within mantle of temperature T_1. Assume the slab travels vertically. Show that the ratio of slab pull to ridge push is

$$\frac{F_{\mathrm{SP}}}{F_{\mathrm{RP}}} \approx \frac{3D}{d},$$

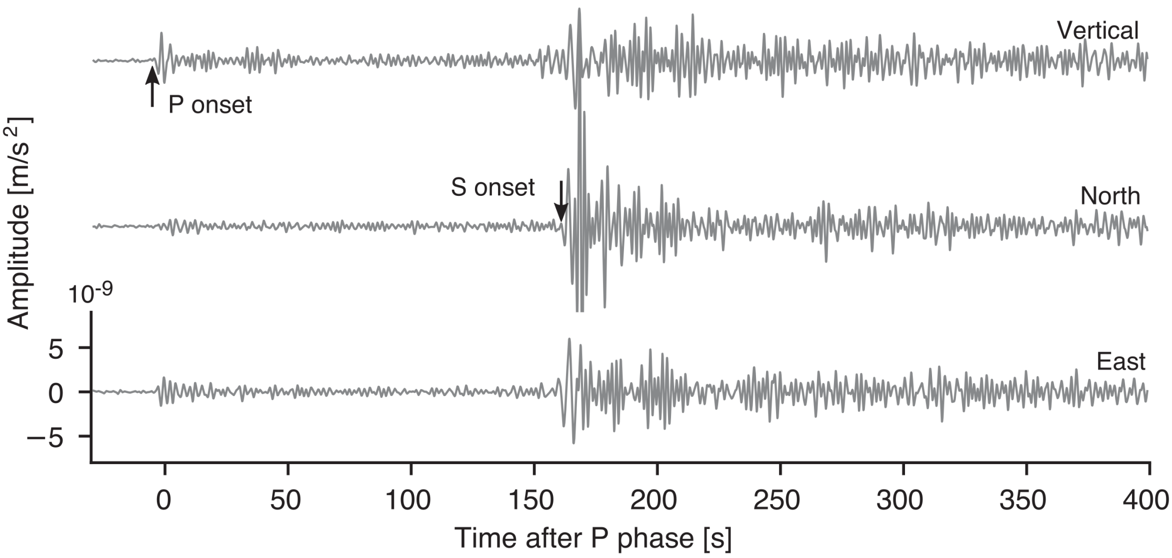
and comment on the relative magnitude of the two forces. How could this calculation be refined?

(iii) *Viscous drag.* Viscous drag forces $F_{DF} \sim \eta v$ where η is the mantle viscosity and v is the plate velocity. Suppose the sinking slab is part of an aspect ratio 1 convection cell. By considering the growth of the top thermal boundary layer, write down an expression for the thickness d of the subducting slab in terms of the thermal diffusivity κ, plate velocity v, and cell width D. By equating slab pull and viscous drag, show that

$$v \sim \frac{\kappa}{D}\left(\frac{\rho_{\mathrm{m}} g \alpha (T_1 - T_0) D^3}{\eta \kappa}\right)^{2/3}.$$

Give an order of magnitude estimate for this velocity using $\kappa = 10^{-6}$ m^2 s^{-1}, $D = 660$ km, $\rho_{\mathrm{m}} = 3300$ kg m^{-3}, $\alpha = 3 \times 10^{-5}$ K^{-1}, $g = 10$ m s^{-2}, $T_1 - T_0 = 1400$ K, $\eta = 10^{21}$ Pa s. Does your estimate of plate velocity seem reasonable?

6 *InSight.* Shown below is a seismogram from event S0235b recorded by the Mars InSight lander on June 26, 2019.



Estimate the epicentral distance from the marsquake source to the lander. Assume a typical P wave speed of 7.8 km s^{-1} and typical S wave speed of 4.5 km s^{-1}.

7* *The onset of convection.* The techniques described in the fluid dynamics of mantle convection lecture can be used to determine the onset of convection in a wide variety of different systems. In this example we consider the problem of convection in a porous medium (e.g. a sandstone) which is heated from below, cooled from above, and is impermeable at its top and bottom boundaries. The governing equations are

$$\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \kappa \nabla^2 T, \tag{1}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{2}$$

$$\mathbf{v} = -\frac{K}{\eta} \left(\nabla p - \rho_0 g \alpha T \hat{\mathbf{z}} \right). \tag{3}$$

(1) and (2) are the equations of conservation of heat and mass, and are the same as given in the lecture. (3) is known as *Darcy's law* and governs flow in a porous medium. K is the permeability of the porous medium (units m^2) and η is the dynamic viscosity (units Pa s). We consider the 2D problem, $\mathbf{v} = (u, 0, w)$. The boundary conditions are

$$T = \Delta T,\ w = 0, \quad \text{on } z = 0,$$
$$T = 0,\ w = 0, \quad \text{on } z = d.$$

(i) Show that the governing equations can be written in dimensionless form as

$$\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \nabla^2 T,$$
$$\nabla \cdot \mathbf{v} = 0,$$
$$\mathbf{v} + \nabla p = \mathrm{Ra} T \hat{\mathbf{z}},$$

with a single dimensionless parameter Ra (this is the Rayleigh number for porous convection).

(ii) Verify that the governing equations and boundary conditions are satisfied by the pure conductive solution

$$
\begin{aligned}
T &= 1 - z, \\
p &= \mathrm{Ra}\left(z - \frac{z^2}{2}\right), \\
\mathbf{v} &= \mathbf{0}.
\end{aligned}
$$

(iii) Show that the governing equations can be linearised about the pure conductive solution to give

$$
\begin{aligned}
\frac{\partial \theta}{\partial t} - \mathbf{u} \cdot \hat{\mathbf{z}} &= \nabla^2 \theta, \\
\nabla \cdot \mathbf{u} &= 0, \\
\mathbf{u} + \nabla q &= \mathrm{Ra}\theta \hat{\mathbf{z}},
\end{aligned}
$$

where θ, q and $\mathbf{u}$ are the perturbation temperature, pressure, and velocity.

(iv) Show that solutions of the form

$$
\begin{aligned}
\theta &= A \sin(n\pi z) \mathrm{e}^{ikx + \sigma t}, \\
q &= B \cos(n\pi z) \mathrm{e}^{ikx + \sigma t}
\end{aligned}
$$

are consistent with the boundary conditions and derive the dispersion relation

$$
\sigma = \frac{\mathrm{Ra} k^2}{k^2 + n^2 \pi^2} - (k^2 + n^2 \pi^2).
$$

(v) At what Rayleigh number does the layer first become unstable?