The perturbed universe

Relativistic Astrophysics and Cosmology: Lecture 23

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Monday 27th November 2023

Pre-lecture question:

Why does cosmology need dark matter?

Last time

- The primordial problems & solutions: horizons, flatness, monopoles & inflation

This lecture

- Perturbations in the universe
- Jeans analysis
- Spherical collapse analysis
- The need for dark matter
- Sketch of full-blown GR analysis

Next lecture

- The quantum universe & wrap-up of course

Historical evolution of perturbations

- The story of how perturbations develop in the Universe, as well as being important itself, is another interesting one in the History of Science.
- Newton, as evidenced by his correspondence with Richard Bentley, already knew in the late 17th Century that a slight overdensity in an otherwise infinite homogeneous medium, would attract material around around it, and so grow larger, leading ultimately to a run-away instability.
- This was quite against his view of what the natural world should be like, so he simply stored it away as a potential flaw in his theory of universal gravitation, rather than investigating further.
- James Jeans at the start of the 20th Century, produced the first treatment of the growth of inhomogeneities in a fluid medium. You may have met some of this already.
- Jeans showed that inhomogeneities grow exponentially fast.
- When an expanding universe is included in the analysis, instabilities grow algebraically fast.
- Interestingly, this result was first obtained by Lifshitz in 1946, using a full GR analysis, and not until the 1950's by Bonnor, using a Newtonian analysis!

Newtonian case: Jeans analysis

- Fluid dynamics under gravity is described by the Euler and Poisson equations:

$$\rho\frac{\partial\vec{v}}{\partial t} + \rho\vec{v}\cdot\nabla\vec{v} = -\vec{\nabla}P - \rho\nabla\Phi, \qquad \nabla^2\Phi = 4\pi G\rho,$$

  where $\vec{v}$, ρ and P are fluid velocity, density and pressure, and Φ is the Newtonian potential.
- Note that a static and homogeneous solution with ρ and P constant and $\vec{v} = 0$ is not a solution to this, since it requires $\nabla\Phi = 0 \Rightarrow \nabla^2\Phi = \rho = 0$, i.e. an empty universe.
- 'Newtonian Cosmology': Newtonian fluid dynamics when applied to the universe as a whole implies the fluid is expanding/contracting — exactly as in General Relativity!
- The "Jeans' swindle" is to ignore this problem, and proceed with simultaneous perturbation & Fourier expansions:

$$\rho = \rho_0 + \delta\rho e^{i\vec{k}\cdot\vec{x}-i\omega t}, \quad P = P_0 + \delta P e^{i\vec{k}\cdot\vec{x}-i\omega t}, \quad v = \delta v e^{i\vec{k}\cdot\vec{x}-i\omega t}, \quad \Phi = \Phi_0 + \delta\Phi e^{i\vec{k}\cdot\vec{x}-i\omega t}.$$

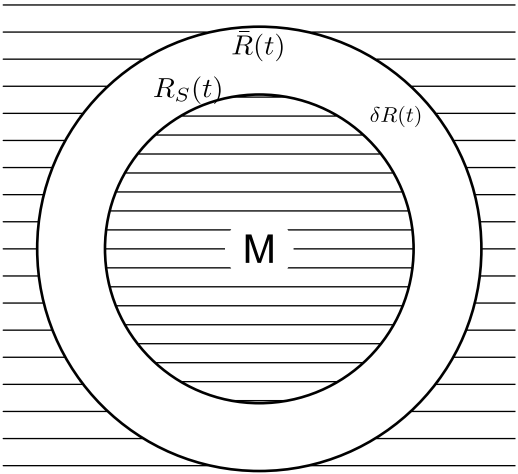
- Substituting these into the fluid equations & extracting first order terms (erroneously assuming the zeroth order equation is satisfied) one finds the dispersion relation:

$$\boxed{\omega^2 = c_s^2(k^2 - k_J^2), \qquad k_J^2 = \frac{4\pi G\rho_0}{c_s^2}, \qquad c_s^2 = \frac{P_0}{\rho_0}.}$$

- If the wavenumber of a perturbation is less than the Jeans wavenumber $k < k_J$, i.e. the wavelength λ is greater than the Jeans wavelength $\lambda > \lambda_J = \frac{2\pi}{k_J}$ (or equivalently for perturbations above the Jeans mass $M_J \sim \frac{4\pi}{3}\lambda_J^3\rho_0$) , then we have an exponentially growing instability.
- Main difference in the cosmological case is that universal expansion causes the exponential rate of growth that the Jeans theory predicts for objects above their Jeans mass, to be reduced down to an algebraic rate of growth (i.e. $\delta\rho/\rho \propto t^n$ for some n), by the effects of the expansion.
- This is very important, since it tells us that much larger fluctuations are needed in the early universe to seed galaxy formation than would have been the case if their growth was exponential.

The cosmological case: collapse of spherical overdensities

- Imagine just one spherical overdense region: the equations governing the evolution of the radius of the sphere, R_S, must just be the dynamical equations for the scale factor.
- This follows since the symmetry of the problem is exactly the same in both cases, and a fundamental result (basically Gauss' Theorem) in cosmology is that in the spherically symmetric case material inside a sphere knows nothing about what is outside (See Lecture 17 on dynamical equations).
- Therefore it doesn't matter for the correctness of the equations whether the material ends at the sphere surface, or carries on homogeneously everywhere.



- So we imagine an overdensity of mass M gradually condensing out of the cosmic fluid.
- We let $R_S(t)$ be the radius of this perturbed sphere, and $\bar{R}(t)$ the radius of an unperturbed sphere encompassing the same mass M, and their difference $\delta R = R_S - R$.
- Both R_S and $\bar{R}$ separately satisfy the acceleration equation, which we take to be an EdS ($\Lambda = k = w = 0$) universe for simplicity:

$$\frac{\ddot{R}}{R} = -\frac{4\pi G\rho}{3} = -\frac{GM}{R^3}, \quad \Rightarrow \quad \delta\ddot{R} = \ddot{\bar{R}} - \ddot{R}_s = -GM\left(\frac{1}{\bar{R}^2} - \frac{1}{R_S^2}\right) \approx \frac{2GM}{\bar{R}^3}\delta R = \frac{8\pi G\bar{\rho}}{3}\delta R.$$

- Now in EdS, we have $\bar{\rho} = 3\bar{H}^2/(8\pi G)$, so the r.h.s. here is just $\bar{H}^2\delta R$, and we can write

$$\boxed{\text{for EdS:} \qquad \frac{d^2}{dt^2}(\delta R) \approx \bar{H}^2\delta R.}$$

- This argument can be straightforwardly extended to include pressure, Λ or curvature.

- What we have set up should give us the full exact evolution of the overdensity over time, until such point as internal pressure can no longer be ignored.
- $H = \frac{2}{3t}$ in matter dominated EdS and so in fact all we need to solve is

$$\delta \ddot{R} = \frac{4}{9t^2}\,\delta R.$$

- This equation is nice, since we can immediately see that a power law of the form $\delta R = At^n$ will be a good trial solution, since the $d^2/(dt^2)$ will pull down 2 powers of t, as required by the r.h.s.
- Substituting this trial form in, we immediately find we need $n(n-1) = 4/9$. This has two roots, one positive and one negative. The positive one is $4/3$, and represents the answer we are seeking here, since it corresponds to the 'growing mode' of the perturbation.
- Since in EdS, $\bar{R} \propto t^{2/3}$, we have for this growing mode

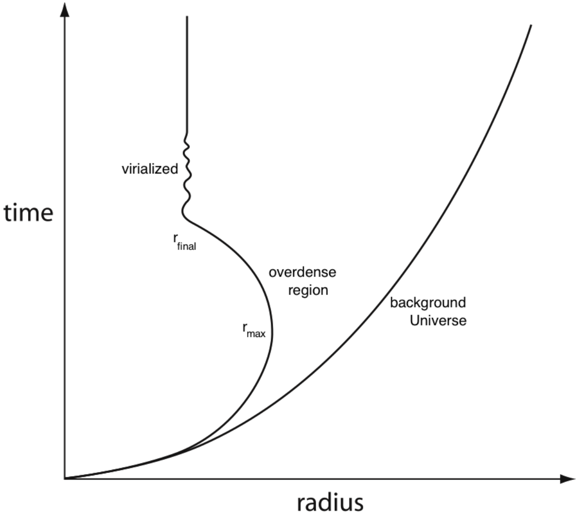
$$\boxed{\frac{\delta R}{\bar{R}} = -\frac{1}{3}\frac{\delta\rho}{\bar{\rho}} \propto \frac{t^{4/3}}{t^{2/3}} \propto t^{2/3} \propto \bar{R}(t).}$$

- So under the current assumptions, then a density perturbation grows in the same way as the scale factor.
- Exercise: find out the way $\delta\rho/\bar{\rho}$ depends on t for the other root of $n(n-1) = 4/9$. This is called the 'decaying mode'. (Should get $\propto t^{-1}$, i.e. indeed decaying.)
- These results are the same as we would get in Fourier analysis of Newtonian fluid perturbations, where we decompose into plane waves and look at zero pressure solutions.
- The fact that the growing mode grows only algebraically with time tells us we need much larger density perturbations in the early universe than the static-universe-based Jeans analysis would lead one to think.

Full set of stages of collapse

- Overdense regions initially evolve the same as the scale factor, but eventually "peel off".
- The approximation breaks down and we need a full nonlinear treatment.
- Collapse is halted by internal pressure and we can use the virial theorem.
- at $r_{\rm max}$ all energy in potential $E \sim -\frac{GM}{r_{\rm max}}$.
- at $r_{\rm final}$ the object is virialised, so $2T = -V$ and $E = T + V = \frac{V}{2}$ (see Lecture 7).
- Equating these $\frac{GM}{r_{\rm max}} = \frac{GM}{2r_{\rm final}}$ so the region equilibrates at half its maximum radius:

$$\boxed{r_{\rm final} = \frac{1}{2} r_{\rm max}.}$$



- The mean density is thus 8 times more than at maximum size.
- In practice, numerical simulation shows that at maximum size, it is typically ≈ 5 times denser than unperturbed surroundings, and the final density when the object has just virialised is about ≈ 200 times that of the background.
- We can use this to estimate the redshift z_f when galaxies might have formed.
- From the virial theorem again the typical velocity/dispersion of a galaxy/cluster is $v_\mathrm{c} = \sqrt{\frac{GM}{r}}$, and $\rho_b = \frac{3H_0^2}{8\pi G}\Omega_b(1+z_\mathrm{f})^3$ from the definition of Ω_b as baryon fraction today

$$\frac{\rho}{\rho_b} = \frac{2v_\mathrm{c}^2}{\Omega_b(H_0 r)^2(1+z_\mathrm{f})^3} >\approx 200.$$

- For clusters of size $r = 10, 30, 100\text{kpc}$, and a typical $v_c \sim 200\,\text{kms}^{-1}$, this gives $1 + z_\mathrm{f} < (7.6, 3, 1.5)\Omega_b^{-1/3}$, so we can link the size of a cluster to when it could have formed.

The need for dark matter

- We now put together some of the above results, namely that $\frac{\delta\rho}{\rho} \propto R$ until $\frac{\delta\rho}{\rho} \sim 1$ and show that they lead to a major problem given we know from the cosmic microwave background anisotropies that $\frac{\delta T}{T} \sim 10^{-5}$ at recombination ($z \sim 1000$).
- In the early universe, when the perturbations are laid down, radiation is dominant and matter and radiation are strongly coupled and in thermal equilibrium.
- In this context, the type of perturbation we expect to form is called *'adiabatic'*, and have

$$\boxed{-\frac{1}{4}\frac{\delta\rho_r}{\rho_r} = \frac{\delta R}{R} = \frac{\delta T}{T} = -\frac{1}{3}\frac{\delta\rho_m}{\rho_m}.}$$

- We can see these are just differential versions of the $\rho_r \propto R^{-4}$ and $\rho_m \propto R^{-3}$ energy conservation relations.
- We know $\delta T/T \sim 10^{-5}$ at recombination, so $\delta\rho_m/\rho_m \sim 3 \times 10^{-5}$ at $z \sim 1000$.
- Tracking this forward today in proportion to R or $1 + z$, we find $\boxed{\delta\rho_m/\rho_m \sim 0.03}$.

- $\boxed{0.03 \ll 1}$, so indeed the linearised treatment *should* be valid, and the prediction is that objects would not have reached the non-linear regime as yet.
- But we have just said that by the time of formation, bound structures, which certainly do exist on many scales, should have achieved $\delta\rho/\bar{\rho} \gtrsim 200$!
- This is a real problem, and dark matter is the key to solving this.
- Dark matter decouples much earlier (well before recombination) than ordinary baryonic matter since it does not electromagnetically interact.
- The dark matter steadily forms potential wells into which the baryons later fall, rapidly catching up after decoupling at recombination.
- The radiation fluctuations are therefore a picture of the level of fluctuations in the ordinary baryonic matter at recombination ($\sim 10^{-5}$), whereas after recombination, when the photon pressure support which allowed the fluctuations to oscillate, rather than grow, disappears, the baryonic fluctuations can grow rapidly.
- They then catch up with the dark matter fluctuations, which have had a much longer time to grow and so can lead to bound non-linear structures developing in the universe.
- This forms the cosmological lynchpin of the formation of structure in the universe course.

Cosmological perturbation theory [non-examinable]

- What follows is a sketch of the key steps in the full general relativistic analysis of cosmological perturbation theory (note $c = 1$).
- Up until this lecture we have considered homogeneous and isotropic solutions (i.e. those that obey the cosmological principle).
- We need to lay out a formalism where we consider first order perturbations to
  - the metric $g_{\mu\nu}$,
  - the matter content ρ, P, u^μ,
  - the coordinates x^μ.
- Then examine the first order versions of the Einstein field equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ and continuity equations $\nabla_\mu T^{\mu\nu} = 0$.
- In addition we need to control for gauge freedoms and decompose into Fourier and Helicity modes.
- Until the end of the lecture this is non-examinable, but worth consideration as it draws together many of the threads we have been building up in different contexts over the course.

Perturbing the metric [non-examinable]

- The first step is to perturb the metric as before $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, but this time massaging it into a form which is suited to the cosmological symmetries of the background solution

$$ds^2 = (1+2\Phi)dt^2 - 2RB_i dx^i dt - R^2\left[(1-2\Psi)\delta_{ij} + 2E_{ij}\right]dx^i dx^j.$$

- Here as is conventional we denote spatial indices with Latin letters i, j, k, and have introduced the small quantities $h_{00} = 2\Phi$, $h_{0i}/R = B_i$, and $h_{ij}/R^2 = 2\Psi\delta_{ij} - 2E_{ij}$ where we have explicitly split the matrix h_{ij} into a trace term Ψ, and a traceless matrix E_{ij}.
- Φ is recognisably our Newtonian potential or lapse function, B is a 3-vector shift, Ψ is the spatial curvature perturbation and E_{ij} is the shear tensor.
- Note the time-space split – important in cosmology which has fundamental observers.
- Note the similarities with the weak field metric solutions previously – unlike then this solution is neither static nor stationary, although its large-scale time dependence is an overall scaling under the cosmological principle.
- With the δ_{ij} in the spatial portion, we have also assume a flat background. Background spatial curvature adds a lot more complexity!

Perturbing the matter [non-examinable]

- The second step is to perturb the matter, which in this instance we will assume at zero order is a perfect fluid parameterised by ρ, P, u^μ

$$\rho + \delta\rho, \quad P + \delta P, \quad u_0 = 1 + \Phi, \quad u_i = -\delta q_i/(\rho + P).$$

- The density and pressure terms are self-explanatory
- Note that the four velocity in this form satisfies $u^\mu u_\nu = 1$ to first order (hence the $1 + \Phi$).
- δq_i is the perturbation to the momentum density (hence the factors of $\rho + P$).
- The advantage of phrasing in terms of the momentum density is that for non-interacting fluids $\delta\rho$, δP and δq_i are all additive between components.
- Considerable additional complexity can be added to this step, e.g. interaction between components via Coulomb/Compton/Rayleigh scattering, anisotropic stress support e.g. free streaming photons and neutrinos, or an interacting dark sector.
- In the instance that the matter is a scalar field, we simply perturb this by $\phi + \delta\phi$.

Perturbing the coordinates (choosing the gauge) [non-examinable]

- In principle at this stage, we could dive right in and compute the first-order Einstein field equations and continuity equations

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu} \qquad \delta(\nabla_\mu T^{\mu\nu}) = 0.$$

- However, as when we considered gravitational waves, we can also perturb the coordinates

$$t + \delta t \qquad x^i + \delta x^i.$$

- Under this transformation, consider responding by perturbing a generic scalar field φ

$$\varphi \to \varphi - \dot{\varphi}\delta t - \partial_i \varphi \delta x^i.$$

- This cancels out the coordinate perturbation $\varphi(t+\delta t, x + \delta x^i) = \varphi(t, x^i) + \delta t \dot{\varphi} + \delta x^i \partial_i \varphi$.
- Similar considerations show that adjusting the other scalar variables via

$$\Phi - \delta\dot{t}, \quad \Psi + H\delta t, \quad B_i + \partial_i \delta t / R - R\delta\dot{x}_i, \quad E_{ij} - \partial_j \delta x_j, \quad \delta\rho - \dot{\rho}\delta t, \quad \delta P - \dot{P}\delta t, \quad \delta q_i + (\rho + P)\partial_i \delta t.$$

- Without care, this gauge freedom can obscure the underlying physics.
- However, it gives us an opportunity to choose a gauge in order to simplify/stabilise calculations:

  Newtonian gauge $B = E = 0$. Reduces to Newtonian gravity in small-scale limit. Popular for analytic work,

  Synchronous gauge $\Phi = B = 0$. Popular with numerical implementations such as `CAMB`,

  Uniform density gauge $\delta\rho = E = 0$. Useful for describing evolution of perturbations on super-horizon scales,

  Comoving gauge $\delta q = E = 0$ Useful for defining quantum initial conditions,

  Comoving orthogonal gauge $\delta q = B = 0$ as above,

  Spatially flat gauge $\psi = E = 0$ Useful for computing inflationary perturbations.
- Effectively with each of these we are choosing to without loss of generality turn off degrees of freedom by reconstructing coordinates, making sure we leave behind as many variables as we have equations.
- An alternative approach is to define Gauge independent variables, which we will see in the next lecture.

- The next step is to decompose the perturbations into Fourier modes so

$$\delta Q(t,\vec{x}) = \int \delta Q(t,\vec{k}) e^{i\vec{k}\cdot\vec{x}} \frac{d^3\vec{k}}{(2\pi)^3}.$$

- The reason this is a good idea is that since the background solutions are spatially homogeneous, Fourier modes decouple.
- Upon substituting the Fourier synthesis equation into the cosmological equations, the partial differential equations in $t, \vec{x}$ becomes a set of ordinary differential equations in t parameterised by $\vec{k}$.
- Less trivially, we can also decompose a vector field v_i and tensor field T_{ij} into helicity modes (v, v_i), (T, T_i, T_{ij})

$$\begin{aligned} v_i &\to \partial_i v + v_i, & T_{ij} &\to (\partial_i\partial_j - \tfrac{\delta_{ij}\partial^k\partial_k}{3})T + \tfrac{1}{2}(\partial_i T_j + \partial_j T_i) + T_{ij}, \\ (\partial_k v_k &= 0), & (\partial^k T_k &= \partial^k T_{ki} = T^k_k = 0). \end{aligned}$$

- This scalar-vector-tensor (SVT) decomposition is similarly a good idea since the background is rotationally invariant, and the helicity modes also decouple.

The scalar Einstein field equations in the Newtonian gauge

- As an example, if we focus on the scalar parts of Newtonian gauge $B = E = 0$ equations we find that in fact $\Phi = \Psi$ (in the absence of anisotropic stress) and

$$ds^2 = (1 + 2\Phi)dt^2 - R^2(1 - 2\Phi)d\vec{x}^2.$$

- The Einstein equations are

$$\begin{aligned} 3H(\dot{\Phi} + H\Phi) + \frac{k^2}{R^2}\Phi &= -4\pi G\delta\rho, \\ \dot{\Phi} + H\Phi &= -4\pi G\delta q, \\ \ddot{\Phi} + 4H\dot{\Phi} + (3H^2 + 2\dot{H})\Phi &= 4\pi G\delta P. \end{aligned}$$

- The continuity equations are

$$\begin{aligned} \delta\dot{\rho} + 3H(\delta\rho + \delta P) &= \frac{k^2}{R^2}\delta q + 3(\rho + P)\dot{\Phi}, \\ \delta\dot{q} + 3H\delta q &= -\delta P - (\rho + P)\Phi. \end{aligned}$$

…and this is pretty much the simplest version!

- Note that as for the background solutions, these equations are not independent.
- In fact there are really only three equations in the four variables $\Phi, \delta\rho, \delta P, \delta q$
- We must usually insert an equation of state linking $\delta\rho$ and δP, e.g. for matter $\delta P = 0$ is enough to use the first and third equations, but usually assume adiabaticity.

Where to from here?

- Needless to say that cosmological perturbation theory is generally something that one spends the better part of the first year of a PhD mastering.
- Nonetheless, armed with the concepts above, and a decent computer algebra package this is the bedrock of developing your own implementations which test theories of the early and late-time universe against the night sky.
- The major implementations are `CAMB` & `CLASS`, which add the full Boltzmann thermodynamic equations on top of this (which governs the interactions between the fluids).
- In the next lecture we will consider how these perturbations imprint themselves on the CMB, and the initial conditions for equations like these.
- Key resources for more (non-examinable) material are:
  - Scott Dodelson's "Modern Cosmology",
  - Ma and Bertschinger [arxiv:astro-ph/9506072],
  - Appendix A2 of [arxiv:0907.5424].

Summary

- Jeans analysis $\boxed{\omega^2 = c_s^2(k^2 - k_J^2), \quad k_J^2 = \frac{4\pi\rho_0}{c_s^2}, \quad c_s^2 = \frac{P_0}{\rho_0}}$ & exponential collapse.
- The derivation of cosmological algebraic collapse of spherical overdensities, e.g. for EdS

$$\boxed{\text{for EdS:} \qquad \frac{d^2}{dt^2}(\delta R) \approx \bar{H}^2 \delta R.}$$

- Adiabatic perturbations

$$-\boxed{\frac{1}{4}\frac{\delta\rho_r}{\rho_r} = \frac{\delta R}{R} = \frac{\delta T}{T} = -\frac{1}{3}\frac{\delta\rho_m}{\rho_m}.}$$

- The need for dark matter $\boxed{0.03 \ll 1}$.
- Where to look beyond key concepts of perturbation theory.

Next time

The quantum universe – initial conditions for inflation.