Natural Sciences Tripos Part III — Physics of the Earth as a Planet

Solutions: Examples Sheet 1: Plate tectonics, heat flow, and flexure — John Rudge

Please attempt the questions yourself before studying these solutions.

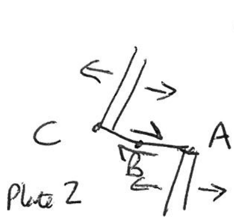
Recall that in spherical geometry, the Cartesian position vector $\mathbf{x}$ can be expressed as

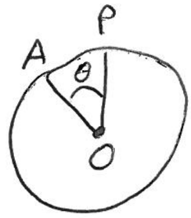
$$\mathbf{x} = (x, y, z) = r\left(\cos\phi\cos\lambda, \sin\phi\cos\lambda, \sin\lambda\right),$$

where r is radius, λ is latitude, and ϕ is longitude. The inverse transformation is

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2}, \\ \lambda &= \arcsin\left(z/r\right), \\ \phi &= \operatorname{arctan2}\left(y, x\right). \end{aligned}$$

1





The transform fault ABC is part of a small circle about the pole P. Hence

$$\mathbf{OA}\cdot\mathbf{OP} = \mathbf{OB}\cdot\mathbf{OP} = \mathbf{OC}\cdot\mathbf{OP} = r^2\cos\theta$$

where θ is the angle of ABC from the pole of rotation. For unit vectors we have

$$\widehat{\mathbf{OA}}\cdot\widehat{\mathbf{OP}} = \widehat{\mathbf{OB}}\cdot\widehat{\mathbf{OP}} = \widehat{\mathbf{OC}}\cdot\widehat{\mathbf{OP}} = \cos\theta. \qquad (*)$$

The corresponding unit vectors are

$$\begin{aligned} \widehat{\mathbf{OA}} &= (0°\mathrm{N}, 60°\mathrm{E}) = \left(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}, 0\right) \\ \widehat{\mathbf{OB}} &= (30°\mathrm{N}, 30°\mathrm{E}) = \left(\tfrac{3}{4}, \tfrac{\sqrt{3}}{4}, \tfrac{1}{2}\right) \\ \widehat{\mathbf{OC}} &= (45°\mathrm{N}, 0°\mathrm{E}) = \left(\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}\right) \end{aligned}$$

Let $\widehat{\mathbf{OP}} = (x, y, z)$ with $x^2 + y^2 + z^2 = 1$. Substitution into (*) gives

$$\tfrac{1}{2}x + \tfrac{\sqrt{3}}{2}y = \tfrac{3}{4}x + \tfrac{\sqrt{3}}{4}y + \tfrac{1}{2}z = \tfrac{1}{\sqrt{2}}x + \tfrac{1}{\sqrt{2}}z = \cos\theta$$

Elimination yields $\cos\theta = 0 \implies \theta = \pi/2$ and

$$\widehat{\mathbf{OP}} = \left(-\sqrt{\frac{3}{7}}, \sqrt{\frac{1}{7}}, \sqrt{\frac{3}{7}}\right) = \underline{(40.9°\mathrm{N}, 150°\mathrm{E})}$$

[It is equally valid to give the antipode of this vector, i.e. $\widehat{\mathbf{OP}} = (-40.9°\mathrm{N}, -30°\mathrm{E})$. Since $(\mathbf{OB} - \mathbf{OA})\cdot\mathbf{OP} = (\mathbf{OC} - \mathbf{OA})\cdot\mathbf{OP} = \mathbf{0}$, an alternative way of calculating $\mathbf{OP}$ is through a cross product, $\mathbf{OP} \propto (\mathbf{OB} - \mathbf{OA}) \times (\mathbf{OC} - \mathbf{OA})$, i.e. the pole of rotation is the normal to the plane through $\mathbf{OA}$, $\mathbf{OB}$, and $\mathbf{OC}$.]

$$|\mathbf{v}| = |\boldsymbol{\omega} \times \mathbf{r}| = |\boldsymbol{\omega}||\mathbf{r}| \sin\theta$$

Since $\theta = \frac{\pi}{2}$,

$$|\boldsymbol{\omega}| = \frac{|\mathbf{v}|}{|\mathbf{r}|} = \frac{100 \text{ mm yr}^{-1}}{6371 \text{ km}} = \underline{1.57 \times 10^{-8} \text{ rad yr}^{-1}},$$

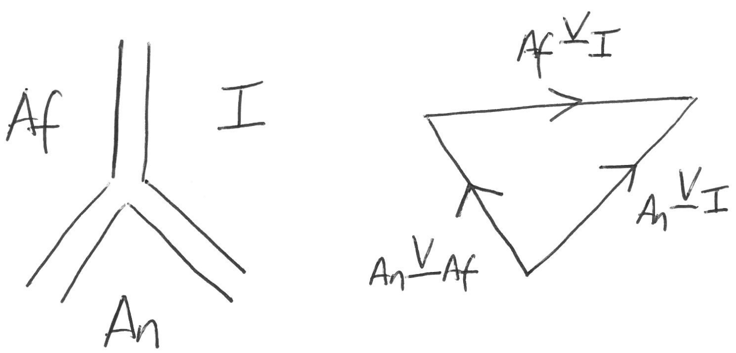
with $\boldsymbol{\omega}$ in the direction of the pole $\widehat{\mathbf{OP}}$.

2 The basis vectors can be obtained from partial differentiation of the position vector $\mathbf{x} = r(\cos\phi\cos\lambda, \sin\phi\cos\lambda, \sin\lambda)$,

$$\begin{aligned}
\mathbf{e}_r &= \frac{\partial \mathbf{x}}{\partial r} = (\cos\phi\cos\lambda, \sin\phi\cos\lambda, \sin\lambda), \\
\mathbf{e}_\lambda &= \frac{\partial \mathbf{x}}{\partial \lambda} = r(-\cos\phi\sin\lambda, -\sin\phi\sin\lambda, \cos\lambda), \\
\mathbf{e}_\phi &= \frac{\partial \mathbf{x}}{\partial \phi} = r\cos\lambda(-\sin\phi, \cos\phi, 0).
\end{aligned}$$

These produce unit vectors

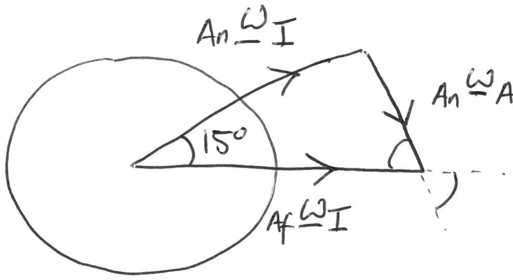
$$\begin{aligned}
&\widehat{\mathbf{e}}_r = (\cos\phi\cos\lambda, \sin\phi\cos\lambda, \sin\lambda) = \mathbf{a}_r \quad \text{“Up”} \\
&\widehat{\mathbf{e}}_\lambda = (-\cos\phi\sin\lambda, -\sin\phi\sin\lambda, \cos\lambda) = \mathbf{a}_N \quad \text{“North”} \\
&\widehat{\mathbf{e}}_\phi = (-\sin\phi, \cos\phi, 0) = \mathbf{a}_E \quad \text{“East”}
\end{aligned}$$

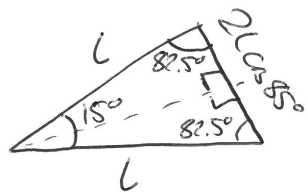


Given angular velocities:

$$\begin{aligned}
{}_{\mathrm{An}}\boldsymbol{\omega}_{\mathrm{I}} &= \left(0°\mathrm{N}, 45°\mathrm{E}, 6 \times 10^{-7}\ °\mathrm{yr}^{-1}\right) \\
{}_{\mathrm{Af}}\boldsymbol{\omega}_{\mathrm{I}} &= \left(15°\mathrm{N}, 45°\mathrm{E}, 6 \times 10^{-7}\ °\mathrm{yr}^{-1}\right)
\end{aligned}$$

Both these vectors lie in 45°E. Therefore the angle between the two vectors is 15°. Moreover, the vectors are both of the same magnitude, and so the angular velocity triangle is an isosceles triangle.





$$|{}_{\text{An}}\boldsymbol{\omega}_{\text{Af}}| = 2 \times 6 \times 10^{-7}\ {}^\circ\text{yr}^{-1} \times \cos 82.5^\circ = \underline{1.57 \times 10^{-7}\ {}^\circ\text{yr}^{-1}},$$

$${}_{\text{An}}\widehat{\boldsymbol{\omega}}_{\text{Af}} = \underline{(-82.5^\circ\text{N}, 45^\circ\text{E})}$$

What follows is a useful trick – the East and North components of the velocity can be obtained from the North and East components of the angular velocity using the following relation,

$$\begin{aligned}
\mathbf{v} &= \boldsymbol{\omega} \times \mathbf{r} \\
&= \boldsymbol{\omega} \times (r\mathbf{a}_r) \\
&= \boldsymbol{\omega} \times (r\mathbf{a}_{\text{E}} \times \mathbf{a}_{\text{N}}) \\
&= r|\boldsymbol{\omega}| \left((\widehat{\boldsymbol{\omega}} \cdot \mathbf{a}_{\text{N}})\, \mathbf{a}_{\text{E}} - (\widehat{\boldsymbol{\omega}} \cdot \mathbf{a}_{\text{E}})\, \mathbf{a}_{\text{N}} \right)
\end{aligned}$$

The velocities can be written as

$$\mathbf{v} = |\mathbf{v}| \left(\sin\theta \mathbf{a}_{\text{E}} + \cos\theta \mathbf{a}_{\text{N}} \right)$$

where θ is the *bearing* (clockwise angle from North), and $|\mathbf{v}|$ is the magnitude. Thus

$$\begin{aligned}
|\mathbf{v}| &= r|\boldsymbol{\omega}| \sqrt{(\widehat{\boldsymbol{\omega}} \cdot \mathbf{a}_{\text{N}})^2 + (\widehat{\boldsymbol{\omega}} \cdot \mathbf{a}_{\text{E}})^2}, \\
\theta &= \operatorname{arctan2}\left(\widehat{\boldsymbol{\omega}} \cdot \mathbf{a}_{\text{N}}, -\widehat{\boldsymbol{\omega}} \cdot \mathbf{a}_{\text{E}}\right)
\end{aligned} \tag{$\dagger$}$$

The triple junction is located at $(-20^\circ\text{N}, 70^\circ\text{E})$ and thus

$$\begin{aligned}
\mathbf{a}_{\text{E}} &= (-0.939693, 0.34202, 0) \\
\mathbf{a}_{\text{N}} &= (0.116978, 0.321394, 0.939693)
\end{aligned}$$

The unit vectors corresponding to the angular velocities are

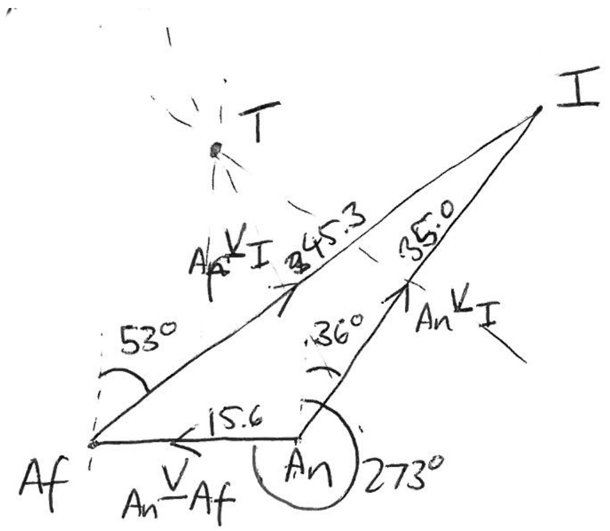
$$\begin{aligned}
{}_{\text{An}}\widehat{\boldsymbol{\omega}}_{\text{I}} &= (0^\circ, 45^\circ\text{E}) = (0.707107, 0.707107, 0) \\
{}_{\text{Af}}\widehat{\boldsymbol{\omega}}_{\text{I}} &= (15^\circ, 45^\circ\text{E}) = (0.683013, 0.683013, 0.258819) \\
{}_{\text{An}}\widehat{\boldsymbol{\omega}}_{\text{Af}} &= (-82.5^\circ, 45^\circ\text{E}) = (0.092296, 0.092296, -0.991445)
\end{aligned}$$

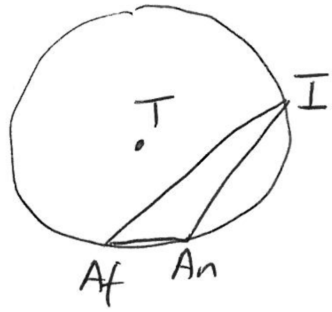
The appropriate dot products are

$$\begin{aligned}
{}_{\text{An}}\widehat{\boldsymbol{\omega}}_{\text{I}} \cdot \mathbf{a}_E &= -0.422618, & {}_{\text{An}}\widehat{\boldsymbol{\omega}}_{\text{I}} \cdot \mathbf{a}_N &= 0.309976, \\
{}_{\text{Af}}\widehat{\boldsymbol{\omega}}_{\text{I}} \cdot \mathbf{a}_E &= -0.408218, & {}_{\text{Af}}\widehat{\boldsymbol{\omega}}_{\text{I}} \cdot \mathbf{a}_N &= 0.542624, \\
{}_{\text{An}}\widehat{\boldsymbol{\omega}}_{\text{Af}} \cdot \mathbf{a}_E &= -0.055163, & {}_{\text{An}}\widehat{\boldsymbol{\omega}}_{\text{Af}} \cdot \mathbf{a}_N &= -0.891193.
\end{aligned}$$

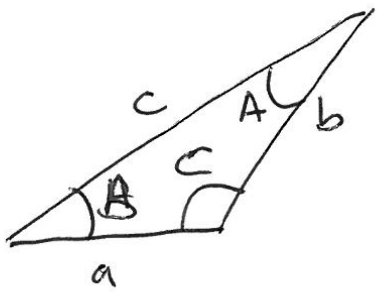
Application of ($\dagger$) with $r = 6371$ km and the given angular velocity magnitudes (remembering to convert from $^\circ\text{yr}^{-1}$ to rad yr^{-1}) yields

$$\begin{aligned}
{}_{\text{An}}\mathbf{v}_{\text{I}} &= \underline{34.97\ \text{mm yr}^{-1},\ \text{bearing } 036.36^\circ}, \\
{}_{\text{Af}}\mathbf{v}_{\text{I}} &= \underline{45.30\ \text{mm yr}^{-1},\ \text{bearing } 053.04^\circ}, \\
{}_{\text{An}}\mathbf{v}_{\text{Af}} &= \underline{15.55\ \text{mm yr}^{-1},\ \text{bearing } 273.54^\circ}.
\end{aligned}$$





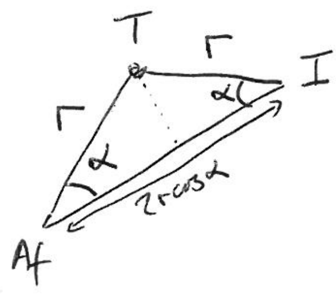
The triple junction is stable (as with any RRR). The perpendicular bisectors intersect at T. T will lie at the centre of the circumcircle going through Af, An, I.



$$
\begin{aligned}
a &= 15.55 \text{ mm yr}^{-1} & A &= 16.78°, \\
b &= 34.97 \text{ mm yr}^{-1} & B &= 40.50°, \\
c &= 45.30 \text{ mm yr}^{-1} & C &= 122.72°,
\end{aligned}
$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 53.84 \text{ mm yr}^{-1} = \text{diameter of circumcircle}$$

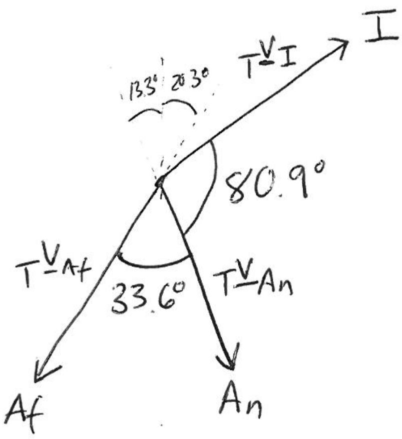
$$\text{radius of circumcircle} = 26.9 \text{ mm yr}^{-1} = \text{magnitude of triple junction velocity}$$



$$\cos\alpha = \frac{45.3}{2 \times 26.9} \implies \alpha = 32.71°$$

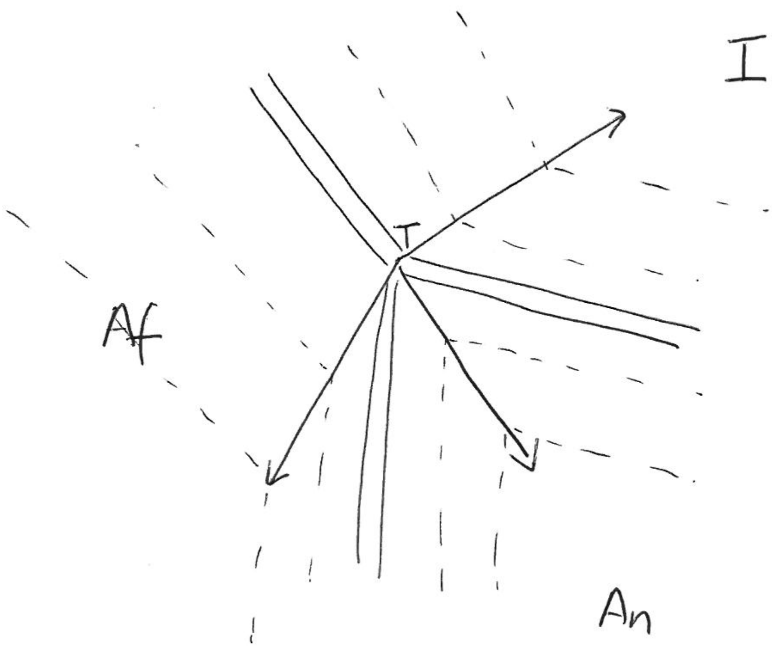
Hence,

$$
\begin{aligned}
{}_{\mathrm{Af}}\mathbf{v}_{\mathrm{T}} &= \underline{26.9 \text{ mm yr}^{-1},\ \text{bearing } 020.3°}, \\
{}_{\mathrm{An}}\mathbf{v}_{\mathrm{T}} &= \underline{26.9 \text{ mm yr}^{-1},\ \text{bearing } 346.7°}, \\
{}_{\mathrm{I}}\mathbf{v}_{\mathrm{T}} &= \underline{26.9 \text{ mm yr}^{-1},\ \text{bearing } 265.8°}.
\end{aligned}
$$



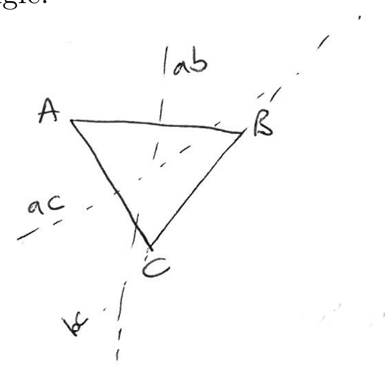
$$\text{Af-An ridge } \underline{\text{growing}} \text{ by } 26.9\cos\frac{33.6^\circ}{2} = \underline{25.8\ \text{mm yr}^{-1}}$$
$$\text{An-I ridge } \underline{\text{growing}} \text{ by } 26.9\cos\frac{80.9^\circ}{2} = \underline{20.5\ \text{mm yr}^{-1}}$$
$$\text{Af-I ridge } \underline{\text{shrinking}} \text{ by } 26.9\cos\frac{114.5^\circ}{2} = \underline{14.5\ \text{mm yr}^{-1}}$$

Magnetic anomalies shown by dashed lines below:



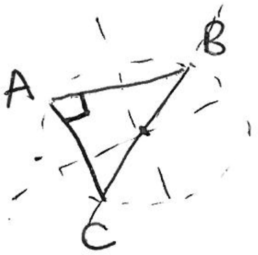
3 (a) **RRF**

Velocity triangle:

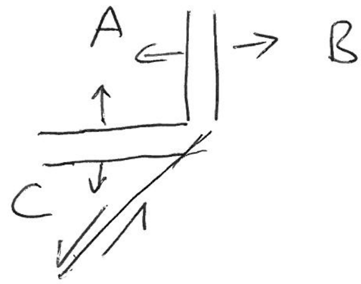


AB, AC ridges. BC transform.
Not stable in general.

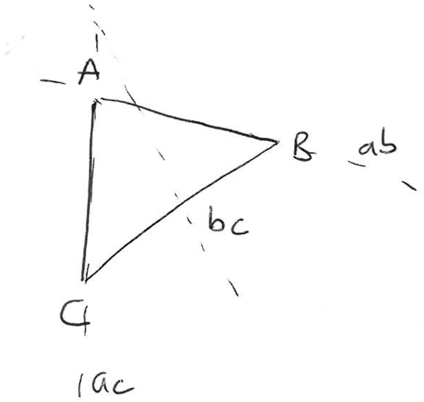
For a stable junction we need ab and ac to intersect on BC. The intersection of ab and ac is at the centre of the circumcircle of ΔABC. Hence, for a stable junction the centre of the circumcircle is at the midpoint of BC:



Therefore AB and AC meet at a right angle, so the ridges are perpendicular. An example stable configuration is shown below:

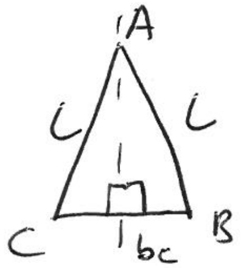


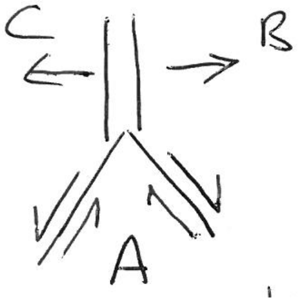
(b) **FFR**
Velocity triangle:



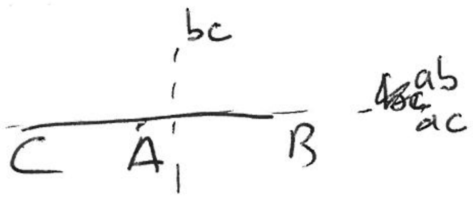
AB, AC transform. BC ridge.
Not stable in general.

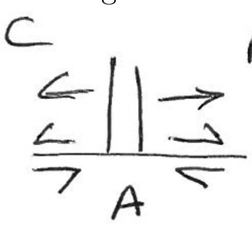
Can be stable if bc intersects A $\implies$ ABC is isosceles, e.g.



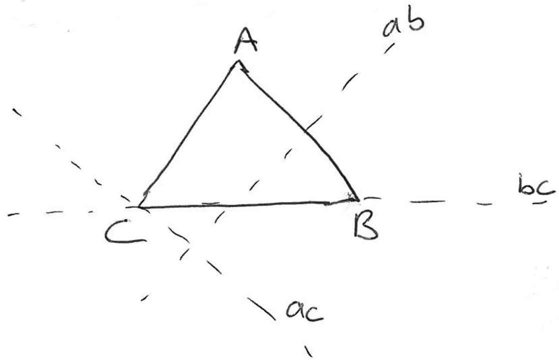


There is also a stable case where there is a straight line configuration:



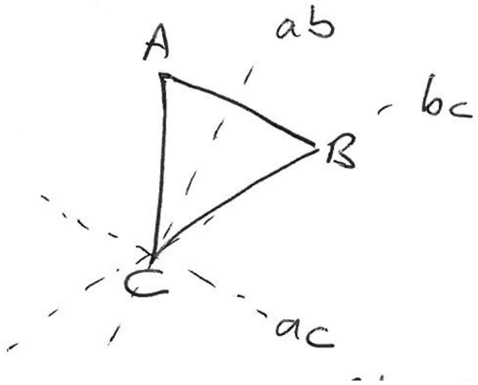


(b) **RTF** [case (i)]
Velocity triangle:

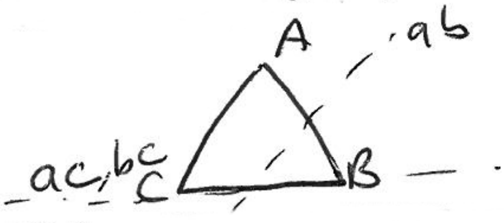


AB ridge, AC trench, BC transform.
Here C is the overriding plate.
Not stable in general.

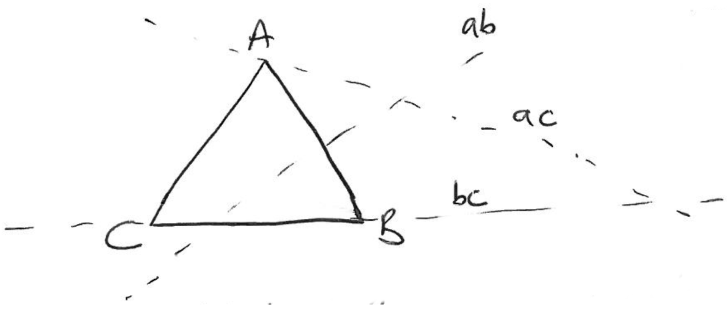
One possibility for a stable junction is if ab goes through C, i.e. we have an isosceles triangle with AC=BC:



Another possibility for a stable junction is to have ac and bc parallel:

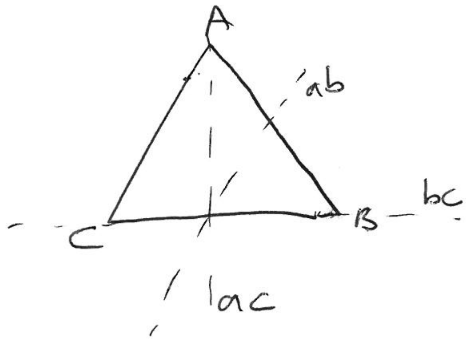


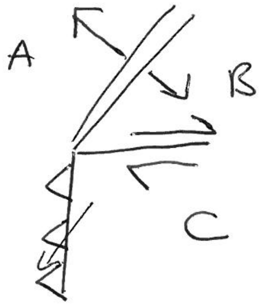
RTF [case (ii)]
Velocity triangle:



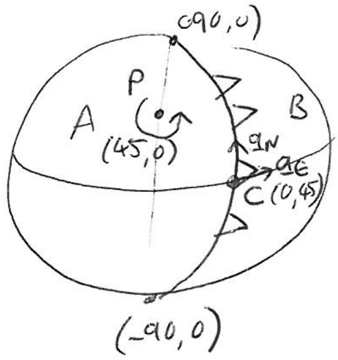
AB ridge, AC trench, BC transform.
Now A is the overriding plate.
Not stable in general.

For a stable junction one needs ab and ac to cross on bc, e.g.





4



$$ {}_{\mathrm{B}}\boldsymbol{\omega}_{\mathrm{A}} = \left(2\times10^{-8}\ \mathrm{yr}^{-1}, 45°\mathrm{N}, 0°\mathrm{E}\right) $$

The direction normal to the boundary is $\mathbf{a}_{\mathrm{E}}$. From "trick" in Q2, we can get the East component of velocity from North component of angular velocity:

$$ {}_{\mathrm{B}}\mathbf{v}_{\mathrm{A}}\cdot\mathbf{a}_{\mathrm{E}} = r|{}_{\mathrm{B}}\boldsymbol{\omega}_{\mathrm{A}}|{}_{\mathrm{B}}\widehat{\boldsymbol{\omega}}_{\mathrm{A}}\cdot\mathbf{a}_{\mathrm{N}} $$

Boundary position is $(\lambda, 45°\mathrm{E})$, thus

$$ \mathbf{a}_{\mathrm{N}} = \left(-\frac{1}{\sqrt{2}}\sin\lambda, -\frac{1}{\sqrt{2}}\sin\lambda, \cos\lambda\right) $$

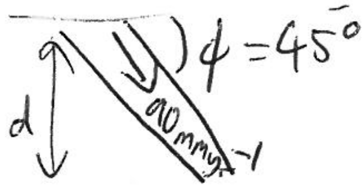
${}_{\mathrm{B}}\widehat{\boldsymbol{\omega}}_{\mathrm{A}} = (45°\mathrm{N}, 0°\mathrm{E}) = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ and hence

$$ {}_{\mathrm{B}}\widehat{\boldsymbol{\omega}}_{\mathrm{A}}\cdot\mathbf{a}_{\mathrm{N}} = -\frac{1}{2}\sin\lambda + \frac{1}{\sqrt{2}}\cos\lambda $$

Therefore the velocity normal to the boundary is

$$ \begin{aligned} {}_{\mathrm{B}}\mathbf{v}_{\mathrm{A}}\cdot\mathbf{a}_{E} &= 6371\ \mathrm{km}\times2\times10^{-8}\ \mathrm{yr}^{-1}\times\left(-\frac{1}{2}\sin\lambda+\frac{1}{\sqrt{2}}\cos\lambda\right) \\ &= \underline{63.71\ \mathrm{mm\ yr}^{-1}\times\left(\sqrt{2}\cos\lambda-\sin\lambda\right)} \end{aligned} $$

At the point C itself, $\lambda = 0$ and ${}_{\mathrm{B}}\mathbf{v}_{\mathrm{A}}\cdot\mathbf{a}_{E} = \underline{90.1\ \mathrm{mm\ yr}^{-1}}$.



Depth to deepest earthquakes:

$$ d = \frac{v\tau}{\sqrt{2}} \sim 600\ \mathrm{km\ if}\ \tau\sim10^{7}\ \mathrm{yr}. $$

5 The magnetic field due to the core is a dipole to a good approximation, with magnetic scalar potential

$$\phi^c = \frac{\mathbf{m}\cdot\mathbf{r}}{4\pi r^3}$$

where the dipole moment $\mathbf{m}$ points towards the South pole. The magnetic field is obtained by taking the gradient,

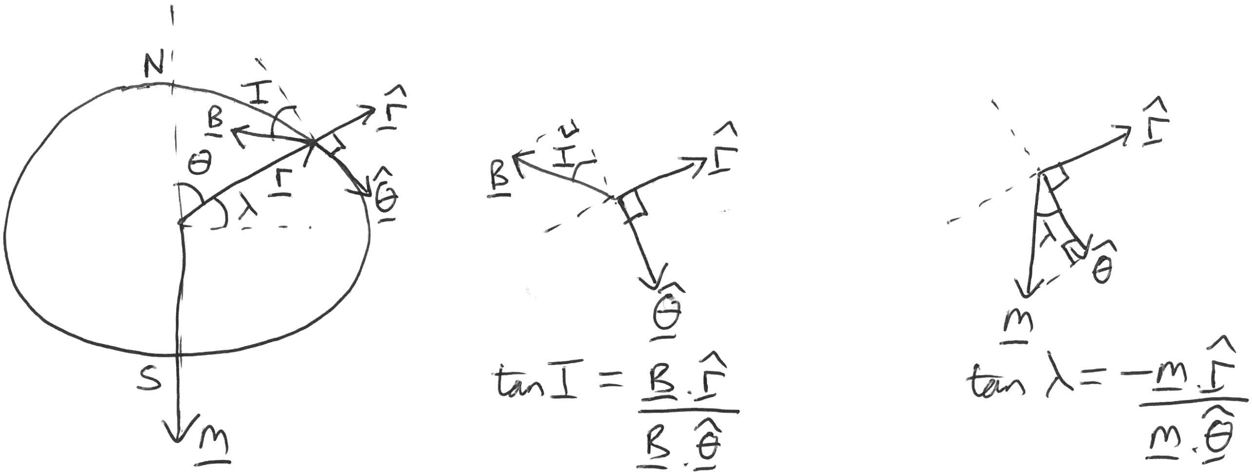
$$\mathbf{B}^c = -\mu_0 \nabla \phi^c = \frac{\mu_0}{4\pi}\left(\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{m}) - \mathbf{m}}{r^3}\right)$$

Splitting this into components gives

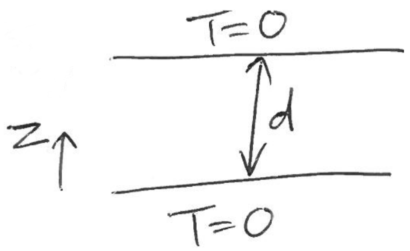
$$\mathbf{B}^c \cdot \hat{\mathbf{r}} = \frac{\mu_0}{2\pi}\mathbf{m}\cdot\hat{\mathbf{r}},$$
$$\mathbf{B}^c \cdot \hat{\boldsymbol{\theta}} = -\frac{\mu_0}{4\pi}\mathbf{m}\cdot\hat{\boldsymbol{\theta}},$$

with ratios

$$\tan I = \frac{\mathbf{B}^c\cdot\hat{\mathbf{r}}}{\mathbf{B}^c\cdot\hat{\boldsymbol{\theta}}} = -2\frac{\mathbf{m}\cdot\hat{\mathbf{r}}}{\mathbf{m}\cdot\hat{\boldsymbol{\theta}}} = 2\tan\lambda$$



6



$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial z^2}$$

where $T = 0$ at $z = 0, d$.

Scaling: $z = dz', \; t = \frac{d^2}{\kappa}t'$

$$\frac{\partial T}{\partial t'} = \frac{\partial^2 T}{\partial z'^2}$$

where $T = 0$ at $z' = 0, 1$.

Separation of variables. Seek solutions of the form $T = \mathcal{T}(t')\mathcal{Z}(z')$. Substitution yields

$$\frac{\dot{\mathcal{T}}}{\mathcal{T}} = \frac{\mathcal{Z}''}{\mathcal{Z}} = -\lambda$$

where λ is a constant. General solution of $\mathcal{Z}$ equation is

$$\mathcal{Z}(z') = A\cos\sqrt{\lambda}z' + B\sin\sqrt{\lambda}z'$$

b.c. at $z' = 0 \implies A = 0$, b.c. at $z' = 1 \implies \lambda = n^2\pi^2$. General solution of $\mathcal{T}$ equation is

$$\mathcal{T}(t') = C\mathrm{e}^{-\lambda t'}$$

Thus full solution is

$$T\left(z', t'\right) = \sum_{n=1}^{\infty} A_n \mathrm{e}^{-n^2\pi^2 t'} \sin n\pi z',$$

or, in dimensional form,

$$T\left(z, t\right) = \sum_{n=1}^{\infty} A_n \mathrm{e}^{-n^2\pi^2 \kappa t/d^2} \sin \frac{n\pi z}{d}.$$

The slowest decaying mode ($n = 1$) dominates at long times,

$$T\left(z, t\right) \sim A_1 \mathrm{e}^{-\pi^2 \kappa t/d^2} \sin \frac{\pi z}{d},$$

and thus the time constant for cooling of the layer is

$$\tau = \frac{d^2}{\pi^2 \kappa}.$$

a) See lecture notes for full discussion. Key result is that for fast spreading ridges, away from the ridge axis,

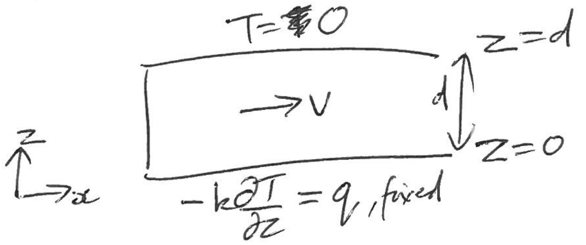
$$\text{elevation} \propto \mathrm{e}^{-t/\tau}$$

where t is plate age and τ is the thermal time constant above.
b) heat flow is given in lecture notes as

$$F = -k \left.\frac{\partial T}{\partial z}\right|_{\text{surface}} \approx \frac{k(T_1 - T_0)}{d}\left(1 + 2\mathrm{e}^{-t/\tau}\right)$$

Both heat flow and depth undergo exponential decay away from the ridge axis, decaying with age according to the thermal time constant.

7 To make the calculations cleaner, assume a large Péclet number situation (i.e. neglect horizontal conduction, which is appropriate for fast-spreading ridges).

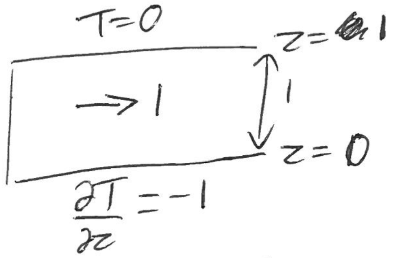


$$V\frac{\partial T}{\partial x} = \kappa\frac{\partial^2 T}{\partial z^2}$$

where $T = 0$ on $z = d$ and $-k\frac{\partial T}{\partial z} = q$ on $z = 0$.

Scaling: $z = dz'$, $x = \dfrac{d^2 V}{\kappa}x'$, $T = \dfrac{qd}{k}T'$

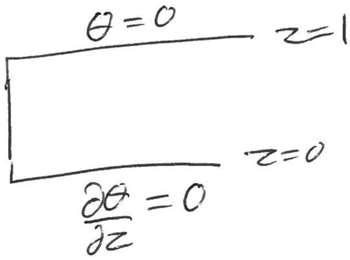
Scaled problem (dropping primes for ease of reading):



$$\frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial z^2}$$

where $T = 0$ on $z = 1$ and $-\frac{\partial T}{\partial z} = 1$ on $z = 0$.

The boundary conditions are not homogeneous, so first write $T = 1 - z + \theta$. The problem for θ is then:



$$\frac{\partial \theta}{\partial x} = \frac{\partial^2 \theta}{\partial z^2}$$

where $\theta = 0$ on $z = 1$ and $\frac{\partial \theta}{\partial z} = 0$ on $z = 0$.

Separation of variables. Seek solutions of the form $\theta = \mathcal{X}(x)\mathcal{Z}(z)$. Substitution yields

$$\frac{\mathcal{X}'}{\mathcal{X}} = \frac{\mathcal{Z}''}{\mathcal{Z}} = -\lambda$$

where λ is a constant. The general solution for $\mathcal{Z}$ is

$$\mathcal{Z}(z) = A\cos\sqrt{\lambda}z + B\sin\sqrt{\lambda}z,$$

with derivative

$$\mathcal{Z}'(z) = -\sqrt{\lambda}A\sin\sqrt{\lambda}z + B\sqrt{\lambda}\cos\sqrt{\lambda}z.$$

Applying the boundary conditions,

$$\mathcal{Z}'(0) = 0 \implies B = 0,$$
$$\mathcal{Z}(1) = 0 \implies \lambda = \left(\frac{\pi}{2} + n\pi\right)^2$$

The general solution for $\mathcal{X}$ is

$$\mathcal{X}(x) = C\mathrm{e}^{-\lambda x}.$$

Thus the full solution for T in dimensionless form is

$$T(x,z) = 1 - z + \sum_{n=0}^{\infty} A_n \mathrm{e}^{-\lambda_n x}\cos\sqrt{\lambda_n}z$$

where

$$\lambda_n = \left(\frac{\pi}{2} + n\pi\right)^2.$$

The slowest decaying mode is the $n = 0$ mode. This decays to 50% when $e^{-\lambda_0 \delta} = \frac{1}{2}$. Hence the dimensionless half-width is

$$\delta = \frac{1}{\lambda_0} \ln 2 = \frac{4}{\pi^2} \ln 2.$$

Returning to dimensional form, this is

$$\delta = \frac{4Vd^2}{\pi^2 \kappa} \ln 2.$$

The case considered in lectures had temperature fixed at the bottom. In this case, the relationship between half-width δ_T and plate thickness d_T is

$$\delta_T = \frac{Vd_T^2}{\pi^2 \kappa} \ln 2$$

whereas the fixed-flux case has the relationship between half-width δ_q and plate thickness d_q as

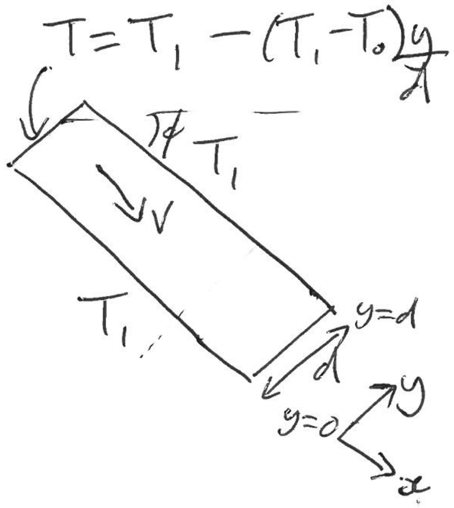
$$\delta_q = \frac{4Vd_q^2}{\pi^2 \kappa} \ln 2.$$

The surface observations constrain the half-width, so $\delta_T = \delta_q$. Thus $4d_q^2 = d_T^2$, and hence

$$d_q = \frac{d_T}{2},$$

i.e. the plate thickness inferred from a fixed-flux model is half that of a fixed-temperature model. $d_T \approx 125$ km (see lecture notes), and thus the estimated plate thickness for the fixed-flux model is $d_q \approx 63$ km. Vigorous convection beneath the base of the plate maintains the base of the plate to closer to a state of constant temperature rather than fixed flux.

8



Again, neglect horizontal conduction.

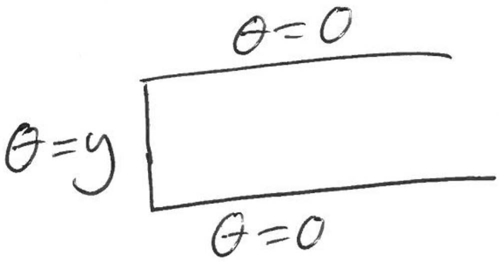
$$V\frac{\partial T}{\partial x} = \kappa \frac{\partial^2 T}{\partial y^2}$$

where $T = T_1$ on $y = 0, d$ and

$$T = T_1 - (T_1 - T_0)\frac{y}{d} \text{ on } x = 0$$

Scaling: $y = dy',\ x = \dfrac{d^2 V}{\kappa} x',\ T = T_1 - (T_1 - T_0)\theta.$

Scaled problem (dropping primes for ease of reading):



$$\frac{\partial \theta}{\partial x} = \frac{\partial^2 \theta}{\partial y^2}$$

where $\theta = 0$ on $y = 0, 1$ and $\theta = y$ on $x = 0$.

The general solution is

$$\theta(x, y) = \sum_{n=1}^{\infty} A_n \mathrm{e}^{-n^2\pi^2 x} \sin n\pi y.$$

The boundary condition at $x = 0$ yields

$$A_n = 2 \int_0^1 y \sin n\pi y \, \mathrm{d}y = \frac{2(-1)^{n+1}}{n\pi}.$$

Thus in dimensional form, the steady state temperature distribution is

$$T = T_1 - (T_1 - T_0) \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \exp\left(-\frac{n^2\pi^2\kappa x}{d^2 V}\right) \sin \frac{n\pi y}{d}.$$

The advective transport of heat in the dip-direction (per length of trench) is

$$\begin{aligned} Q_{\text{dip-direction}}(x) &= \int_0^d \rho C_p (T - T_1) V \, \mathrm{d}y \\ &= -\rho C_p (T_1 - T_0) V d \sum_{n \text{ odd}} \frac{4}{n^2\pi^2} \exp\left(-\frac{n^2\pi^2\kappa x}{d^2 V}\right) \end{aligned}$$

To get the vertical advective heat transport, we simply resolve the heat flux vector,

$$Q_{\text{vertical}}(x) = \underline{-\rho C_p (T_1 - T_0) V d \sin\phi \sum_{n \text{ odd}} \frac{4}{n^2\pi^2} \exp\left(-\frac{n^2\pi^2\kappa x}{d^2 V}\right)}.$$

The buoyancy force in the dip-direction (the slab pull force) is given (per length of trench) by

$$F_{\text{SP}} = \rho g \alpha \sin\phi \int_{x=0}^{L} \int_{y=0}^{d} (T_1 - T) \, \mathrm{d}y \, \mathrm{d}x$$

where L is the length of slab. Integration yields

$$F_{\text{SP}} = \underline{\rho g \alpha (T_1 - T_0) \sin\phi \sum_{n \text{ odd}} \frac{4}{n^4\pi^4} \frac{d^3 V}{\kappa} \left(1 - \exp\left(-\frac{n^2\pi^2\kappa L}{d^2 V}\right)\right)}.$$

9 From the lecture notes, the solution for the deflection is

$$w(x) = \mathrm{e}^{-x/\alpha}\left(A\cos\frac{x}{\alpha} + B\sin\frac{x}{\alpha}\right)$$

where

$$\alpha^4 = \frac{4D}{g\left(\rho_m - \rho_w\right)}.$$

Now

$$\frac{\mathrm{d}w}{\mathrm{d}x} = \frac{\mathrm{e}^{-x/\alpha}}{\alpha}\left((B-A)\cos\frac{x}{\alpha} - (A+B)\sin\frac{x}{\alpha}\right)$$

If plate remains unbroken, $\mathrm{d}w/\mathrm{d}x = 0$ at $x = 0 \implies A = B$. Hence

$$w(x) = A\mathrm{e}^{-x/\alpha}\left(\cos\frac{x}{\alpha} + \sin\frac{x}{\alpha}\right)$$

a) $w = 0$ where $\cos\frac{x}{\alpha} + \sin\frac{x}{\alpha} = 0 \implies \tan\frac{x}{\alpha} = -1 \implies \frac{x}{\alpha} = \frac{3\pi}{4} + n\pi$. The first point where $w = 0$ is at $x_0 = \underline{\frac{3\pi}{4}\alpha}$.
b) The forebulge is where $\mathrm{d}w/\mathrm{d}x = 0 \implies \sin\frac{x}{\alpha} = 0 \implies x_b = \underline{\pi\alpha}$.
From the Figure, $x_b \approx 250$ km.

$$T_e = \left(\frac{12D\left(1-\sigma^2\right)}{E}\right)^{1/3} = \left(\frac{3\alpha^4 g(\rho_m - \rho_w)\left(1-\sigma^2\right)}{E}\right)^{1/3} = \left(\frac{3x_b^4 g(\rho_m - \rho_w)\left(1-\sigma^2\right)}{\pi^4 E}\right)^{1/3} \approx \underline{32\ \text{km}}.$$