The quantum universe

Relativistic Astrophysics and Cosmology: Lecture 24

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Pre-lecture question:

Where do galaxies, planets and people originate?

Last time

- Perturbation theory (Jeans, spherical collapse and GR)

This lecture

- Primordial perturbations
- Radiation perturbations
- Quantum fluctuations in the early universe
- Wrap up and summary of course

Scalar perturbations in the Newtonian gauge

- We have as usual the cosmological field equations, which we write

$$H^2 = \frac{8\pi G}{3}\rho, \qquad \dot{H} + H^2 = -\frac{4\pi G}{3}(\rho + 3P), \qquad \dot{\rho} = -3H(\rho + P).$$

- We finished last lecture having sketched the derivation of scalar perturbations for a general perfect fluid in the Newtonian gauge $(c = 1)$ so that $ds^2 = (1 + 2\Phi)dt^2 - R^2(1 - 2\Phi)d\vec{x}^2$ and

$$3H(\dot{\Phi}+H\Phi)+\frac{k^2}{R^2}\Phi = -4\pi G\delta\rho, \qquad \dot{\Phi}+H\Phi = -4\pi G\delta q, \qquad \ddot{\Phi}+4H\dot{\Phi}+(3H^2+2\dot{H})\Phi = 4\pi G\delta P.$$

- This however requires us to specify the equation-of-state linking P and ρ, which usually defines $\delta\rho$, δP and δq.
- In this lecture we are going to do this for scalar fields for the primordial universe and radiation fluids for the pre-recombination universe, and then discuss how the two are linked by inflation.

Primordial perturbations

- In the primordial universe during inflation, we have a scalar field which can be modelled as a perfect fluid with
$$\rho = \tfrac{1}{2}\dot{\phi}^2 + V(\phi) \quad P = \tfrac{1}{2}\dot{\phi}^2 - V(\phi).$$
- Perturbing the field $\phi + \delta\phi$ we can identify from the stress energy tensor
$$\delta\rho = \dot{\phi}\delta\dot{\phi} + \frac{dV}{d\phi}\delta\phi - \dot{\phi}^2\Phi, \qquad \delta P = \dot{\phi}\delta\dot{\phi} - \frac{dV}{d\phi}\delta\phi - \dot{\phi}^2\Phi, \qquad \delta q = -\dot{\phi}\delta\phi.$$
- Using the Einstein equations we can relatively straightforwardly eliminate $\delta\dot{\phi}$ and $\delta\phi$
$$\ddot{\Phi} + \left(H - 2\frac{\ddot{\phi}}{\dot{\phi}}\right)\dot{\Phi} + \left(2\dot{H} - \frac{2H\ddot{\phi}}{\dot{\phi}} + \frac{k^2}{R^2}\right)\Phi = 0.$$
- This is an equation we can solve, from which we can then compute $\delta\rho$, δq and δP from the Einstein equations.
- We can also derive an equation in $\delta\phi$ alone, although this is not very easy to interpret.

Gauge considerations

- However, this would only give solutions in the Newtonian gauge!
- In order to isolate the physics, we consider quantities invariant under the transformation

$$t \to t + \delta t, \quad x^i \to x^i + \delta x^i = x^i + \partial^i \delta x \quad \Rightarrow \quad \Psi \to \Psi + H\delta t, \quad \delta q \to \delta q + (\rho + P)\delta t.$$

- The critical quantity to examine is the gauge-invariant comoving curvature perturbation

$$\boxed{\mathcal{R} = \Psi - \frac{H}{\rho + P}\delta q.}$$

- So-called because in a gauge co-moving with the fluid, i.e. the zero momentum frame $\delta q = 0$, then $\mathcal{R} = \Psi$ is just the curvature perturbation.
- There are several other independent gauge-invariant combinations of the variables which can be useful.
- For the fluids we consider (scalar fields and radiation) there is only one independent gauge invariant quantity.

The primordial comoving curvature perturbation

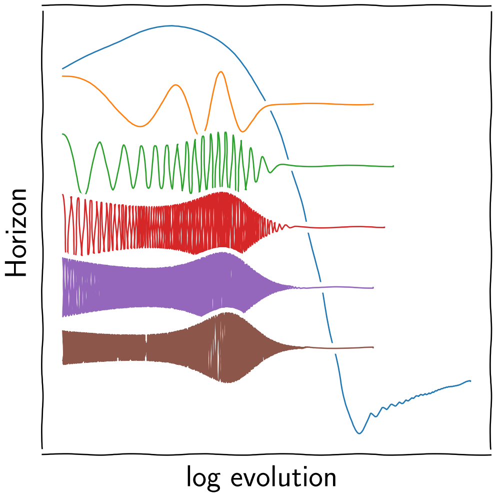
- For an inflaton universe, we have $\Psi = \Phi$ and $\delta q = -\dot\phi\delta\phi$ so $\boxed{\mathcal{R} = \Phi + \frac{H}{\dot\phi}\delta\phi.}$
- we can indeed see that up to a scaling factor, this is the only gauge invariant combination of the variables $\delta\phi$ and Φ.
- Re-phrasing the Φ evolution equation in terms of this we find

$$\ddot{\mathcal{R}} + \left(3H + 2\frac{\ddot\phi}{\dot\phi} - 2\frac{\dot H}{H}\right)\dot{\mathcal{R}} + \frac{k^2}{R^2}\mathcal{R} = 0 \quad \Rightarrow \quad \mathcal{R}'' + 2\frac{z'}{z}\mathcal{R}' + k^2\mathcal{R} = 0, \quad z = \frac{R\dot\phi}{H}.$$

  where in the second equation we have changed to conformal time η, and defined a new variable z (which is NOT redshift).
- In conformal time this is the Mukhanov-Sasaki equation – a simple harmonic oscillator with a friction $\frac{z'}{z}$, with wavevector k.
- It is conventional to also introduce the Mukhanov variable $v = z\mathcal{R}$. We do not here, as it is easier to see physics by not rescaling the curvature perturbation.

Solutions of the Mukhanov-Sasaki equation

- Plotted here are solutions at different k, as well as the comoving Hubble horizon.
- Before and after inflation, the horizon grows.
- As the universe inflates, the comoving horizon decreases, and the Mukhanov modes "freeze out".
- When $k \gg aH$, i.e. the mode is "within the horizon", and it oscillates.
- As the universe inflates, the friction term $\frac{z'}{z}$ acts to halt the evolution.



Pre-recombination radiation evolution

- We take the same scalar perturbation formalism in the Newtonian gauge, but this time for a perfect fluid, where the equation of state is much simpler:

$$P = w\rho, \qquad \delta P = w\delta\rho.$$

- We can define δq via the third Einstein equation, giving

$$\mathcal{R} = \frac{(\frac{5}{3} + w)}{1 + w}\Phi + \frac{2}{3H(1+w)}\dot{\Phi}, \quad \dot{\mathcal{R}} = -\frac{2w}{3H(1+w)}\frac{k^2}{R^2}\Phi.$$

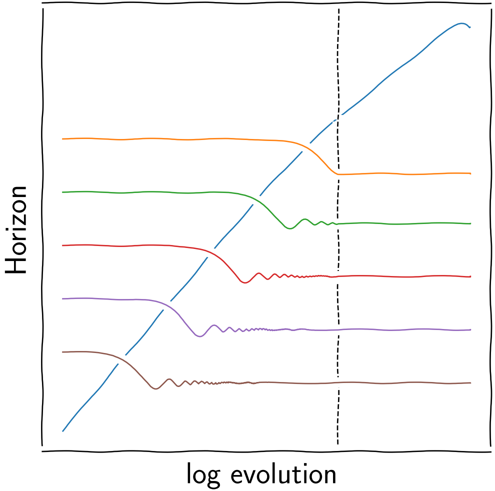
- Eliminating Φ we find

$$\ddot{\mathcal{R}} + 3H\dot{\mathcal{R}} + w\frac{k^2}{R^2}\mathcal{R} = 0 \quad \Rightarrow \quad \mathcal{R}'' + 2\mathcal{H}\mathcal{R}' + wk^2\mathcal{R} = 0.$$

- Note the similarity with the primordial evolution – the only change is $k \to \sqrt{w}k$ and $z \to R$, since we have defined the conformal Hubble parameter $\mathcal{H} = R'/R = \dot{R}/R^2 = H/R$.

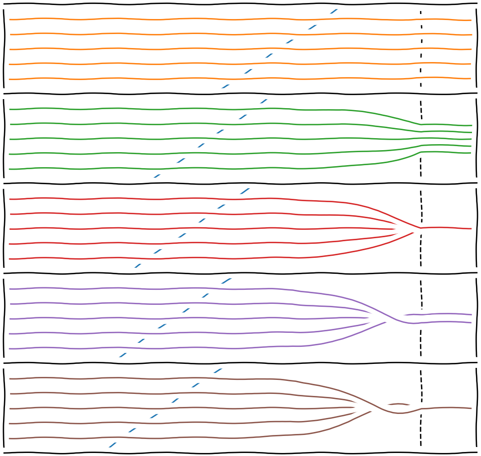
Solutions of the radiation evolution

- Outside the horizon, the modes remain frozen at their values at the end of inflation.
- As the horizon increases in a radiation dominated universe, the modes re-enter the horizon one at a time, highest frequency first.
- Note the subtle transition from radiation to matter domination, pre recombination, as well as the horizon shrinking at late times as dark energy begins to dominate.
- After recombination, photons and electrons decouple and free stream, reaching us as the CMB today.



Synchronised perturbations at recombination

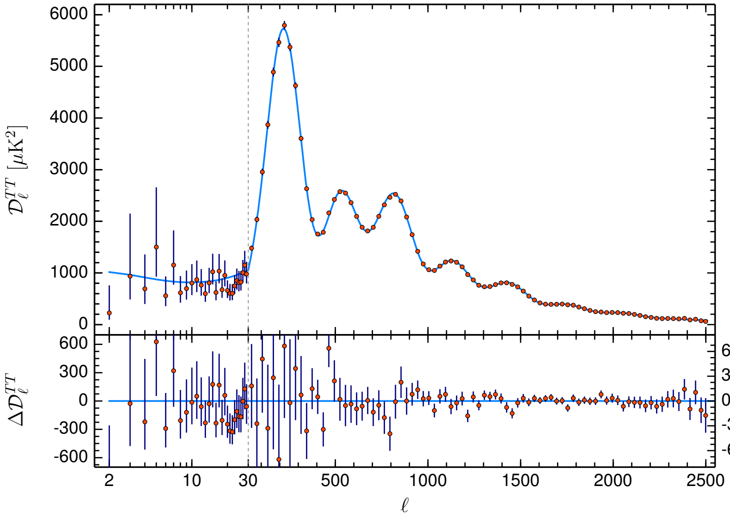
- In practice each wavevector k will have a distribution of values with spread given by a power spectrum $P(k)$.
- The perturbations re-enter the horizon, and arrive at recombination.
- We can now understand why the cosmic microwave background has a series of peaks.
- Those perturbations with wavelengths that reach recombination at an antinode will give a peak, and those at an node give a trough.
- The details of the plasma physics until recombination blurs and modulates this standing wave effect.



log evolution

The measured CMB power spectrum

- Different proportions of matter, photons & dark matter, the size of the universe, and lensing all adjust the heights and locations of the peaks.
- This gives rise to a "barcode" effect, allowing us to read the properties of the universe from the CMB power spectrum.



Quantum initial conditions for $\mathcal{R}$

- We nearly have all the pieces in place – the only remaining piece is initial conditions for $\mathcal{R}$.
- These come from the quantum mechanics of the early universe

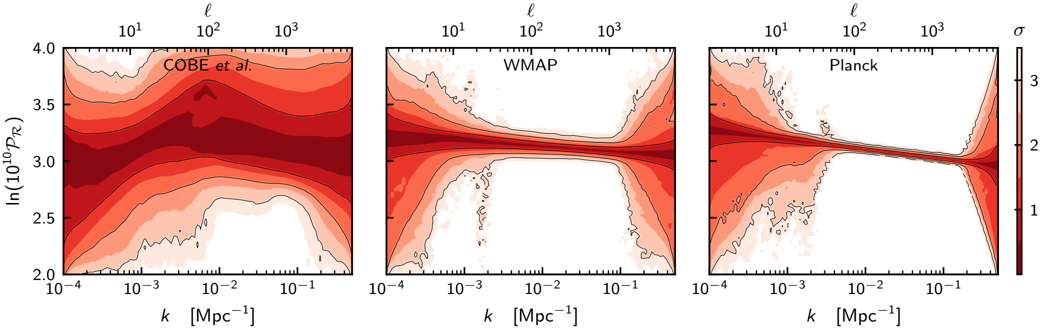
$$\hat{\mathcal{R}}(\eta, \vec{x}) = \int \left[\chi_k(\eta) \hat{a}_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} + \chi_k^*(\eta) \hat{a}_{\vec{k}}^\dagger e^{-i\vec{k}\cdot\vec{x}} \right] \frac{d^3\vec{k}}{(2\pi)^3}.$$

- If you have been taking the quantum field theory course, this will look very similar to the usual Fourier synthesis of creation and annihilation operators.
- The only difference which the theory of quantum fields in curved spacetime makes is the introduction of general mode functions $\chi_k(\eta)$, which in the flat case are $\chi_k(t) = e^{-i\omega_k t}$.
- Passing the above expression through the equation of motion shows that the mode functions $\chi_k(\eta)$ also must satisfy the Mukhanov-Sasaki equation.
- Also require the usual canonical commutation relations $[a_{\vec{k}}, a_{\vec{k}'}^\dagger] = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}')$, which yields a Wronskian normalisation condition

$$\chi_k'' + 2\frac{z'}{z}\chi_k' + k^2 \chi_k = 0, \qquad \chi_k' \chi_k^* - \chi_k'^* \chi_k = -i/z^2.$$

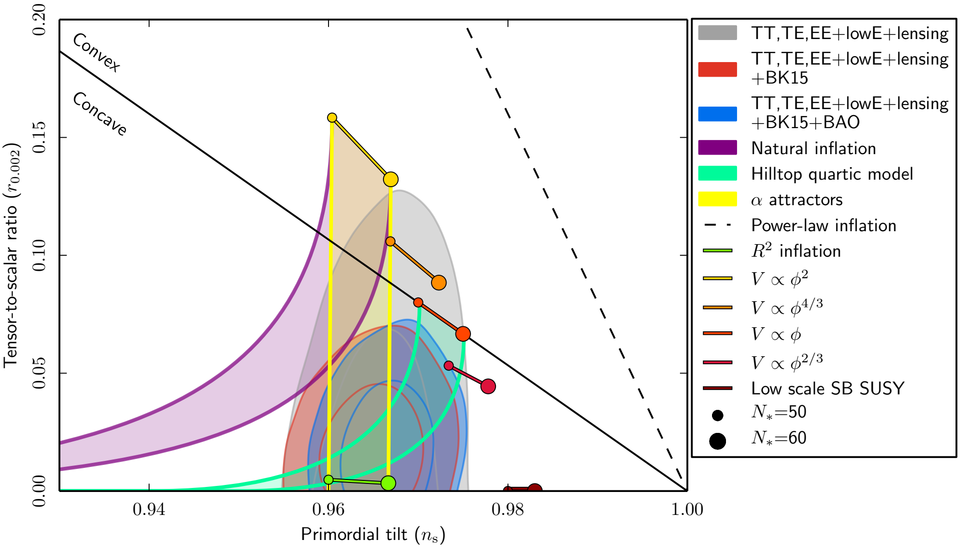
- In principle we have four degrees of freedom per complex mode.
- As we are generally interested in the final amplitude $|\chi_k|^2$, the overall phase is unimportant, and the Wronskian condition fixes another degree of freedom.
- To determine the modes completely, and therefore the initial conditions for our universe, we need to specify the quantum vacuum.
- This is much harder in curved spacetime, since the notion of "particle" becomes relative
- In de Sitter space one can defined the vacuum unambiguously, and "particle-less" = "diagonalised Hamiltonian" = "minimal energy" = "right-handed mode", and the answer is
$$\chi_k = \frac{e^{-ik\eta}}{z\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right).$$
- This set of initial conditions is called "Bunch-Davies". More detail in [arxiv:1607.04148].
- Note that this is a relatively minor modification to the usual flat space mode functions.
- If the universe is not expanding in a de Sitter state (e.g. at the beginning of inflation), then these concepts become distinct.
- This is exciting, since it means that theories of the quantum vacuum at the interface of gravity and QFT is testable in the night sky!

Predictions from inflation



- If inflation were perfectly de Sitter, this would predict a "flat" primordial power spectrum of perturbations $\mathcal{P}_\mathcal{R} = A_s \left(\frac{k}{k_*}\right)^{n_s-1}$ with $n_s = 1$.
- However, if the field "slowly rolls" to different values of H over the course of its evolution, then we would see a "tilt" in the power spectrum $n_s \neq 1$, which is what we see.
- One of the remaining unconfirmed predictions of inflation is $r > 0$, i.e. observations of tensor modes which manifest in B mode polarisation. We currently know $r < 0.036$.

The Planck state-of-the-art [arxiv:1807.06211]



The full picture

1. Curvature perturbations are initialised during the exponentially expanding phase using quantum initial conditions.
2. These are evolved through inflation until they "freeze out" as the microscopic perturbations are inflated to cosmic scales.
3. Inflation finishes with the Universe in a flattened and almost homogeneous state, barring the quantum imprints.
4. The universe then "re-heats" through some unknown (arguably unimportant!) mechanism.
5. As the universe then expands more sedately in an EdS-style fashion, these quantum imprints come one-by-one "over the horizon", smallest scales first.
6. We observe this residual quantum patterning in the statistical properties CMB anisotropies.
7. Such anisotropies are the seeds of future large scale cosmic structure, as post-recombination matter perturbations collapse around these seeds.
8. In this manner, the quantum mechanics of the primordial universe is imprinted in the patterns of matter in the present day universe.

Summary of course

Intuitive GR: Lectures 1—3

- The equivalence principle: It *is* odd that heavier objects don't fall faster.
- Gravity as spacetime curvature: Gravity doesn't cause time to speed up and slow down, what we perceive as gravitation is in fact is the bending and stretching of time and space by matter.
- The weak field, slow moving metric $ds^2 = \left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 - d\vec{x}^2$.
- Gauss' theorema egregium for computing the curvature K of 2D curved surfaces, which can be applied to the Schwarzschild, & FRW cases to gain insight.
- The Schwarzschild metric $ds^2 = \left(1 - \frac{R_S}{r}\right) c^2 dt^2 - \left(1 - \frac{R_S}{r}\right)^{-1} dr^2 - r^2 d\Omega$.

Beyond Part II: Lectures 4—6

- The Schwarzschild interior solution & Oppenheimer Volkov equation.
- The stress energy tensor $T^{\mu\nu} = (\rho + \frac{P}{c^2})u^\mu u^\nu - Pg^{\mu\nu}$.
- Buchdahl's theorem $R_S < \frac{8}{9}R$ for any object.
- Conjugate momenta and orbits, general spherical metric $ds^2 = Ac^2dt^2 - Bdr^2 - r^2d\Omega$.
- Stability and ISCOs. 2μ Schwarzschild, 3μ circular photon orbits, 6μ ISCO.
- The Schwarzschild-de-Sitter metric $ds^2 = (1 - \frac{R_S}{r} - \frac{\Lambda r^2}{3})dt^2 - (1 - \frac{R_S}{r} - \frac{\Lambda r^2}{3})^{-1}dr^2 - r^2d\Omega$.
- The Kerr metric as a vacuum solution for a spinning black hole $ds^2 = (\cdots)dt^2 + (\cdots)dtd\phi - (\cdots)dr^2 - (\cdots)d\theta^2 - (\cdots)d\phi^2$ with lengthscales μ, $a = J/Mc$, $\Delta = r^2 - 2\mu r + a^2$ and $\rho^2 = r^2 + a^2\cos^2\theta$.
- The Reissner Nordström and Kerr-Newman metrics for charged & spinning black holes.
- The ergosphere & Thorne limit at $a = 0.998\mu$.
- Hawking black hole thermodynamics, and entropy budget of the universe.
- Gravity as a cosmic mechanism pumping disorder into SMBHs to create cosmic structure.

Compact objects: Lectures 6 – 12

- The virial theorem $2T = -V$.
- The Eddington-luminosity $L_{\text{edd}} = \frac{4\pi G M m_p c}{\sigma_T}$.
- The life cycle of stars & (non-)relativistic degeneracy pressure, Chandresekhar limit $1.4 M_\odot$.
- Many elements do not come from supernovae – whilst you are made of stardust, your smartphone is made of merging neutron stars.
- Astrophysics across the electromagnetic spectrum and beyond.
- Radiation process (line, Bremsstrahlung, synchrotron, (inverse) Compton, pair production).
- Accretion disks, radiative efficiency $L = \epsilon \dot{M} c^2$.
- Quasars and active galactic nuclei as engines of galaxies and probes of the deep Universe.
- Jets, superluminal motion & Doppler bias/"fingers of God".
- Gamma ray bursts as the creation of black holes from across the universe.
- Neutron stars as extreme relativistic objects.
- Pulsars as cosmic clocks for measuring gravitational waves and starquakes.

Lectures 13 – 15: Gravitational waves & lensing

- Binary systems & the generalisation of Kepler's laws $\Omega^2 = \frac{GM}{a^3}$, $E = -\frac{GM_1 M_2}{2a}$.
- Lagrange points & Roche lobe overflow.
- Gravity in the weak field limit $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.
- The gravitational wave metric in the transverse traceless gauge $h^{\mu\nu} = A^{\mu\nu} e^{ik\cdot x}$, $k \cdot k = 0$, a^+ and $a^\times$ polarisations.
- Gauge freedoms in perturbed gravity.
- LIGO as a gravitational wave observatory, the $1/r$ strain rate and why the first detection was so hard, but progress since has been so rapid.
- The compact source approximation and weak field relativistic metric $ds^2 = \left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 - 2\vec{A} \cdot d\vec{x} cdt - \left(1 - \frac{2\Phi}{c^2}\right) d\vec{x}^2$.
- Einstein's prediction for gravitational lensing $\alpha = 2\frac{R_S}{b}$.
- Strong, micro, weak, extended & time delay lensing.
- The event horizon telescope and our picture of a black hole.

Lectures 16 – 18: Cosmology Theory

- The geometry of the universe: Fundamental observers & the cosmic microwave background, cosmic & conformal time.
- The Friedmann Robertson Walker metric
  $ds^2 = c^2dt^2 - R^2d\vec{x}^2 = c^2dt^2 - R^2(d\chi^2 + S^2(\chi)d\Omega)$.
- Hubble's law due to the expansion of the universe, rather than galactic recession.
- The dynamics of the universe & the many and varied forms of the same cosmological equations $H^2 \sim \rho$, $\dot{H} + H^2 \sim \rho + 3P/c^2$.
- Ω formulation $H^2 = \sum_i \Omega_i a^{-3(1+w_i)}$.
- w-fluids, and the relevance of $\rho_m \sim R^{-3}$ and $\rho_r \sim R^{-4}$.
- The evolution of the universe: Einstein de Sitter ($\Lambda = k = 0$), Friedmann ($\Lambda = 0$), Flat Λ ($k = 0$) de Sitter and Einstein static universe.

Lectures 19 – 21: Measuring our Universe

- Earth measurements of sky location θ, ϕ, redshift z and flux F.
- Four-dimensional epoch-shell cosmology, mixing monotonic parameters z, t, R, χ, E, T.
- Luminosity $d_L = \sqrt{\frac{F}{4\pi L}} = R_0 S(\chi)(1+z)$ and angular diameter distances $d_\theta = \frac{D}{\Delta\theta} = \frac{R_0 S(\chi)}{1+z}$.
- Standard candles and standard rulers, ageing of objects in the universe.
- The constituents & thermal history of the universe.
- Matter, dark matter, neutrinos, photons, dark/vacuum energy/cosmological constant.
- Epochs of nucleosynthesis, recombination and reionisation.
- Dark matter as an explanation for rotation curves, cluster dynamics and accumulation of cosmic structure.
- Type Ia supernovae as standard candles, Baryon Acoustic Oscillations as standard rulers, the CMB anisotropy power spectrum as a barcode for our universe.

Lectures 22 – 24: The primordial universe

- Particle, event and Hubble horizons in cosmology.
- Early accelerated phase as explanation for the flatness, horizon and monopole problems.
- Scalar field cosmologies are “the simplest universes” and inflate without any coaxing.
- Growth of structure in the universe by gravitational amplification of small perturbations
- Why Jeans is more wrong, and a sketch of cosmological perturbation theory.
- Cosmic inflation as a link between primordial quantum mechanics and the large scale structure in the Universe.

Where next?

- The ΛCDM concordance model of the universe (cosmological constant, cold dark matter) represents a triumph of human understanding.
- With only six parameters $(\Omega_b, \Omega_c, H_0, \tau, A_s, n_s)$, it now gives a $> 1000\sigma$ fit [arxiv:1804.01318] to a wealth of diverse data.
- However, it is somewhat embarrassing to spell out the acronym, since it enumerates our two "fudge factors" – dark matter and dark energy .
- In addition, "tensions" between the inferences we make using different datasets have emerged and remained resilient to community fixes.
- A future onslaught of data is coming from machines like JWST, SKA, SO. . . , as well as multimessenger cosmology from LISA/Einstein telescopes and cosmic rays/neutrinos.
- Cosmology is arguably in an excellent position in comparison to other "big physics", in that we have (a) problems that need solutions with both our models and our data and (b) the promise of much, much more data in the near and far future.