Measuring the Universe

Relativistic Astrophysics and Cosmology: Lecture 19

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Pre-lecture question:

Why in an Einstein-de Sitter universe do galaxies look smaller the further away they are until redshift 1.25, when they start getting bigger again?

Last time

- Evolution of the universe: solving the cosmological equations of motion

This lecture

- Moving between observables and cosmological coordinates
- Angular diameter distance
- Luminosity distance
- Ages of the universe

Next lecture

- Constituents of the universe

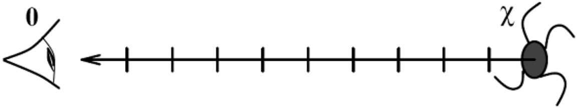
What on Earth can we measure?

- Note that in cosmology, for any given object we can measure :
  - Its sky location (θ, ϕ),
  - Its redshift z (via spectrometers),
  - Its flux $F(\nu)$.
- We can convert z into χ (proper distance from us) or t (cosmic epoch of emission), but this requires us to choose a cosmology.
- This explains why we report an object's redshift when speaking about either its age or distance (redshift is cosmology invariant).
- Note also the "four-dimensional" epoch-shell view of the Universe that cosmologists are expected to think in – concentric regions at different distances, volumes and times.

Proper distance, velocities and the Hubble Law

- We have seen the Hubble parameter $H = \frac{\dot{R}}{R}$ in both the velocity and continuity equations.
- The same quantity when evaluated today H_0 appears in Hubble's law – why?.
- Let us look at the Friedmann-Robertson-Walker metric again $\boxed{ds^2 = c^2 dt^2 - R^2(t) dX^2}$.
- Now consider arranging for measuring rods laid end to end to be simultaneously present at cosmic epoch t_0 ('now') at all points between us and a distant galaxy with comoving radial coordinate χ:

$$d_{\mathrm{proper}}(t_0, \chi) = R(t_0) \int_0^{\chi} d\chi' = R_0 \chi.$$



- Note however, that this requires a gigantic 'cosmic conspiracy' to carry out the measurement.
- We are not measuring back into the past to where we *see* the galaxy, but arranging for measurements which span great distances at a single instant of cosmic time.

- Now consider

$$d_{\text{proper}} = R\chi.$$

- Then since χ does not change for a comoving galaxy, we have

$$\dot{d}_{\text{proper}} = \dot{R}\chi.$$

- Noting $\dot{R} = HR$

$$\begin{aligned} \dot{d}_{\text{proper}} &= H(t)R(t)\chi \\ &= H(t)d_{\text{proper}}, \end{aligned}$$

- Evaluated now we get Hubble's law

$$\dot{d}_{\text{proper0}} = H_0 d_{\text{proper0}}$$

- although not between measurable quantities!

- Later we'll look further at how we can work with types of distance which are possible to measure.
- Note H tells us about velocities, in contrast to the dimensionless deceleration parameter

$$q(t) \equiv -\frac{\ddot{R}R}{\dot{R}^2}$$

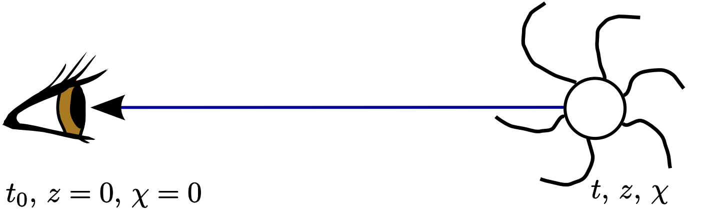
- Returning to our velocity equation

$$H^2 = \frac{8\pi G\rho}{3} + \frac{\Lambda c^2}{3} - \frac{kc^2}{R^2} \quad (B)$$

this relates the Hubble constant at a given time, to the density and Gaussian curvature (k/R^2) at that time, as well as the cosmological constant.

Link between χ, z, t and H

- Note that we generally report the redshift of an object of an event, in place of it's emission epoch t, or comoving distance χ from us.
- This is because redshifts are measurable and therefore independent of choice of cosmology.
- A useful manipulation, which shows us how the χ coordinate can be worked out in terms of the Hubble parameter, comes from the following picture.



A galaxy at redshift z, & radial coordinate χ, emits a photon at time t, observed by us today (at $t = t_0$).

- Notice the difference between this diagram and that for 'proper distance' measurement.
- There we laid down metre sticks at a single cosmic instant (in fact t_0).
- Here we let a photon travel from the source to us.

- Get the relation between z, t, χ from the following manipulations.

$$dz = d(1+z) = d\left(\frac{R_0}{R}\right) = -\frac{R_0}{R^2}\dot{R}\,dt = -(1+z)H(z)dt.$$

- Noting for photons $ds^2 = c^2dt^2 - R^2d\chi^2 = 0$, and noting the relation $\boxed{R = \frac{R_0}{1+z}}$ we find

$$\boxed{\frac{d\chi}{c} = \frac{dt}{R} = -\frac{dz}{R_0H(z)}.}$$

- The χ coordinate of an object which emitted at time t received by us at t_0 is given by

$$\chi(t) = \int_t^{t_0}\frac{c\,dt'}{R(t')}, \quad \Rightarrow \quad \chi(z) = \int_0^z \frac{c\,dz'}{R_0H(z')}. \quad \Rightarrow \quad t_0 - t = \int_t^{t_0} dt = \int_0^z \frac{dz}{(1+z)H(z)}.$$

- So these expressions relate χ, t and z via the cosmology $H(z)$.

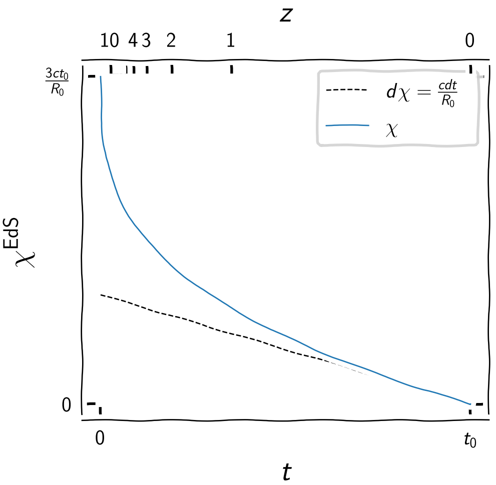
- Now let's calculate $\chi(z) = \int_0^z c\,dz/(R_0 H)$ in EdS, where $R \propto t^{2/3}$, so

$$H = \frac{\dot{R}}{R} \propto \frac{t^{-1/3}}{t^{2/3}} \propto \frac{1}{t} \propto \frac{1}{R^{3/2}} \propto (1+z)^{3/2}.$$

- Thus in EdS we have $H(z) = H_0(1+z)^{3/2}$.

$$\chi^{\mathrm{EdS}} = \frac{3ct_0}{R_0}\left(1 - \left(\frac{t}{t_0}\right)^{1/3}\right),$$
$$= \frac{3ct_0}{R_0}\left(1 - \frac{1}{\sqrt{1+z}}\right).$$

- N.B. have used $H = \frac{2}{3t} \Rightarrow H_0 = \frac{2}{3t_0}$.



- There are several noteworthy things (which we will revisit when we come to horizons):
  1. In an infinite EdS Universe, there is a maximum χ one can see out to (even at the big bang),
  2. Photons travel a "superluminal" proper distance $3ct_0$,
  3. This maximum comoving radius is at infinite redshift.

The Universe expressed in redshift

- Let's draw out more generally some key manipulations we just made in the EdS case.
- The redshift relation $(1+z) = \frac{R(t_o)}{R(t_e)}$ allows us to write $\boxed{R = \dfrac{R_0}{1+z}}$, or $a = (1+z)^{-1}$.
- The Friedmann equation simplifies

$$\frac{H^2}{H_0^2} = \Omega_{r,0}\left(\frac{R}{R_0}\right)^{-4} + \Omega_{m,0}\left(\frac{R}{R_0}\right)^{-3} + \Omega_{k,0}\left(\frac{R}{R_0}\right)^{-2} + \Omega_{\Lambda,0}$$

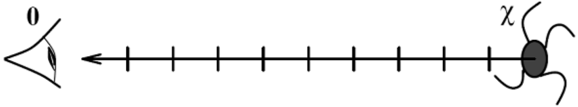
$$\Rightarrow \boxed{H = H_0\left(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}\right)^{1/2}.}$$

- In most cases of interest (including exams) only one or two of the above components are dominant/active.
- This equation for $H(z)$, combined with the link to χ and R gives you all you need.

Measuring distances (how not to do it)

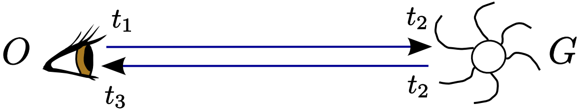
- On a cosmological scale we are interested in radial distances from us: several definitions.
- We have already defined the proper distance which is given by laying down meter sticks

$$d_{\text{proper}}(t_0, \chi) = R(t_0) \int_0^{\chi} d\chi' = R_0 \chi.$$



- Could equally define the radar distance (useful in the context of the solar system).
- Send out a pulse of EM radiation from our origin of coordinates, O, at time t_1, wait for it to reflect off a galaxy G at time t_2, pulse arrives back at O at time t_3.

$$d_{\text{radar}}(t_0) = \frac{1}{2} c(t_3 - t_1), \text{ where } t_0 = \frac{1}{2}(t_3 + t_1).$$



- This distance is manifestly *different* from the proper distance $d_{\text{proper}}(t_0, \chi)$.
- This is characteristic of distance determination in cosmology. The distance obtained depends upon the definition used.
- Both above methods hopelessly impractical – we have to assume something about the source in order to derive the luminosity distance and angular diameter distance.

Luminosity distance

- Suppose we have a source of absolute luminosity L at a distance d in an ordinary Euclidean static universe.
- The flux we receive in Wm^{-2} is
$$F = L/(4\pi d^2).$$
- Now consider when we are not in a Euclidean static universe, but that we know the source has a luminosity L and we observe a flux F.
- We call
$$d_L = \left(\frac{L}{4\pi F}\right)^{\frac{1}{2}},$$
the **luminosity distance**. This is an operational definition.
- Suppose the source emits at time t_1, with luminosity $L(t_1)$ and has comoving coordinate χ_1.

- There are three non-Euclidean spacetime effects which operate:

1. The area of the sphere over which the energy is spread is the proper area of a sphere at cosmic time t_0 (when we *receive* the radiation):

$$4\pi r^2 = 4\pi \left[\sigma R(t_0)\right]^2 = 4\pi \left[R_0 S(\chi_1)\right]^2 .$$

   Note implicit symmetry between us and the distant fundamental observer is being used here.

2. Individual photons, emitted with frequency ν_1 will arrive with a redshifted frequency ν_2:

$$\nu_2 = \nu_1 \frac{R(t_1)}{R(t_0)} = \frac{\nu_1}{(1+z)}.$$

   Apparent luminosity reduced by the same ratio.

3. Rates of arrival or photons are reduced by the same factor $1/(1+z)$.
   Again this lowers the energy.

- The total flux received at time t_0 is

$$F(t_0) = \frac{L(t_1)}{4\pi \left[R_0 S(\chi_1)\right]^2} \frac{1}{(1+z)^2}.$$

- Thus the luminosity distance is

$$d_L(t_0, \chi_1) = \left(\frac{L(t_1)}{4\pi F} \right)^{1/2} = R_0 S(\chi_1)(1+z).$$

- For a source at general coordinate χ

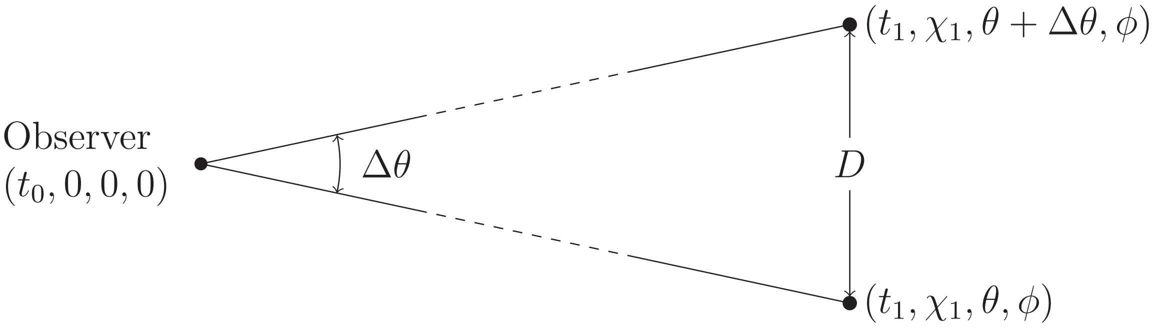
$$\boxed{d_L = \left(\frac{L}{4\pi F} \right)^{\frac{1}{2}} = R_0 S(\chi)(1+z).}$$

- This is an important quantity and it depends on the time history of the scale factor through the dependence on χ, since $\chi = \int c/R(t)\, dt$.
- i.e. if we somehow knew L via a standard candle (e.g. Type Ia supernovae), which allows us to calibrate L to other observable properties like colour if we know enough baryonic physics, then we can use the above relation to calculate χ from observables z, F, L for a given cosmology.
- Turning this around, you can use this to fit for a cosmology, i.e. determine the parameters or model of the universe given enough measurements of z, F, L.

Angular diameter distance

- Now suppose a source has proper diameter D. Then in Euclidean space, if it were at distance d it would subtend an angular diameter $\Delta\theta = D/d$.
- Define an operational **angular diameter distance**, in the non-Euclidean general case

$$d_\theta = \frac{D}{\Delta\theta}, \quad \text{i.e.} \quad \frac{\text{assumed proper size}}{\text{measured angular diameter}}.$$



- Consider two radial null geodesics meeting at the observer at angle $\Delta\theta$.
- Photons emitted at time t_1 from a source of proper diameter D at χ_1.
- From angular part of metric

$$D = R(t_1)S(\chi_1)\Delta\theta,$$

so that

$$d_\theta(t_0, \chi_1) = R(t_1)S(\chi_1) = R(t_0)\frac{R(t_1)}{R(t_0)}S(\chi_1) = \frac{R(t_0)S(\chi_1)}{(1+z)}.$$

- In general

$$\boxed{d_\theta = \frac{D}{\Delta\theta} = \frac{R_0 S(\chi)}{(1+z)}.}$$

- This is different from d_L by a factor $(1+z)^2$.
- Again, because of the χ dependence we need to know the time history of the scale factor $R(t)$ to evaluate it.
- i.e. if we somehow knew D via a standard ruler (e.g. Baryonic acoustic oscillations), which allow us calibrate or to measure a geometrical effect, then we can use the above relation to calculate χ from observables $z, \Delta\theta, D$ for a given cosmology.

Example: Angular Diameter Distances in a EdS and flat Λ

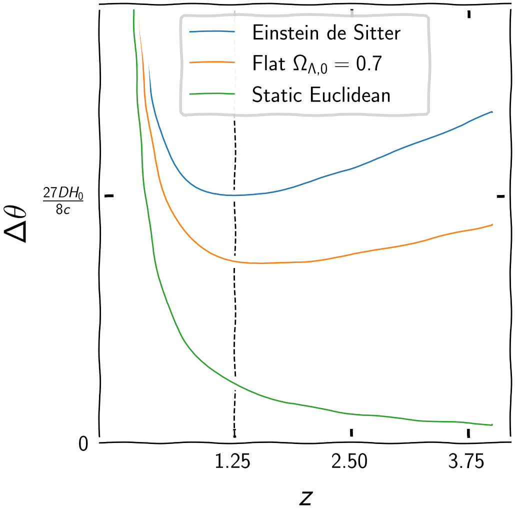
Show that, in an Einstein-de-Sitter universe, the angle subtended by an object has a minimum value for $z = 1.25$. What is this minimum angle for a typical galaxy?

- We've done most of the work for this already: recall $k = 0 \Rightarrow S(\chi) = \chi$ and

$$\chi^{\mathrm{EdS}}(z) = \frac{2c}{R_0 H_0}\left\{1 - \frac{1}{\sqrt{1+z}}\right\},$$

so

$$\Delta\theta = \frac{D(1+z)}{R_0\chi} = \frac{DH_0(1+z)}{2c[1-(1+z)^{-1/2}]}.$$

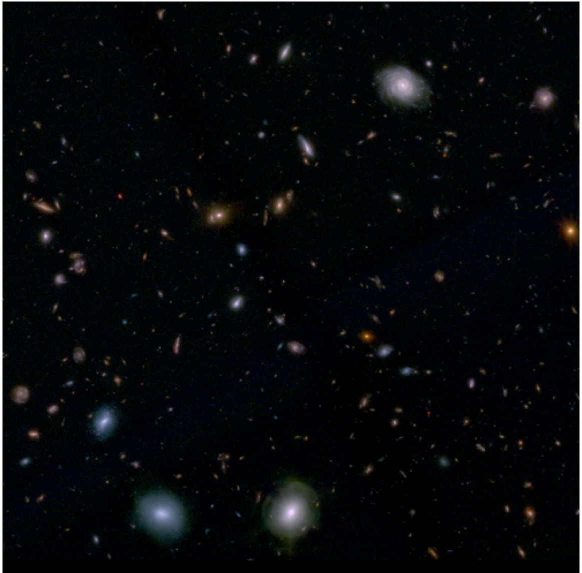


- Note for small z, $\Delta\theta \approx \frac{H_0 D}{c}\frac{1}{z}$ which is the 'naive expectation' for $\Delta\theta$ (also plotted).
- Can see that true function has a minimum with $\boxed{z = \frac{5}{4} = 1.25}$ and $\boxed{\Delta\theta_{\rm min} = \frac{27 D H_0}{8c}}$.
- Taking $D = 20\,{\rm kpc}$ as a typical galactic size, then yields $\Delta\theta_{\rm min} = 3.2$ arcsec for $H_0 = 70\,{\rm km s^{-1} Mpc^{-1}}$.
- So this would be (roughly) the minimum apparent size of galaxies in an Einstein-de Sitter universe - regardless of how far away they were!
- Of course, our Universe is not EdS, but is better modelled in it's "later" stages (i.e. back to the CMB at $z \sim 1000$) by a flat Λ universe with matter.
- On Examples Sheet 3, you will be asked to show

$$H(z) = H_0\left((1-\Omega_{\Lambda,0})(1+z)^3 + \Omega_{\Lambda,0}\right)^{1/2} \quad \Rightarrow \quad \Delta\theta = \frac{DH_0}{c}\frac{1+z}{\int_0^z \frac{dz}{\left((1-\Omega_{\Lambda,0})(1+z)^3+\Omega_{\Lambda,0}\right)^{1/2}}}$$

- Function has to be evaluated numerically (or can use elliptic functions), and is also plotted.
- Clear that as $\Omega_{\Lambda,0}$ is increased, there is still a minimum angular diameter, but the point of minimum moves further out in z and the curve is generally flatter at higher z.

- A place to look: Hubble Deep Field.
- Here, looking deep enough that virtually every object in picture is a galaxy rather than a star!
- The width of the picture is about 60 arcsec, so a minimum width of a galaxy around 2 to 3 arcsec is not ridiculous.
- However, galaxies have their own intrinsic distribution in size, down to quite small ones, and also this size distribution may **evolve**.
- Thus turns out that this test can't really be done (yet) with galaxies — more promising is with double radio sources.
- Width of these set by physics of interaction of interstellar medium, and may be a better standard ruler.



Hubble Ultra Deep Field as viewed with JWST (JADES Survey)
NASA, ESA, CSA, M. Zamani (ESA/Webb), Brant Robertson (UC Santa Cruz), Sandro Tacchella (Cambridge), JADES Collaboration

(Counter)-intuition for the minimum angular size

- How can we understand the fact that in an EdS universe objects start looking bigger once they get further than $z = \frac{5}{4}$ (i.e. at comoving distance $\chi = \frac{ct_0}{R_0}$, time $t = \frac{8}{27}t_0$)?
- Let us consider the proper area of the spherical shell at comoving distance χ, at the time t that photons were emitted.

$$A = 4\pi R^2\chi^2 = 4\pi\left[R_0\left(\frac{t}{t_0}\right)^{2/3}\right]^2\left[\frac{3ct_0}{R_0}\left(1-\left(\frac{t}{t_0}\right)^{1/3}\right)\right]^2 = 36\pi c^2 t_0^2\alpha^4(1-\alpha)^2.$$

where we have defined $\alpha^3 = t/t_0$. There is a maximum volume – differentiating by α: $4\alpha^3(1-\alpha)^2 - 2\alpha^4(1-\alpha) \equiv 2(1-\alpha)\alpha^3(2-3\alpha)$, i.e. $\alpha = \frac{2}{3} \Rightarrow t = \alpha^3 t_0 = \frac{8}{27}t_0$.
- We can therefore understand that as you consider epoch shells further and further away, there are two competing effects:
  1. The shell is further, and therefore larger in comoving radius,
  2. The universe is younger and therefore smaller.
- At some point the second effect overtakes the first as objects of fixed size occupy larger and larger fractions of the universe [xkcd:2622].

Ages of real objects in the Universe

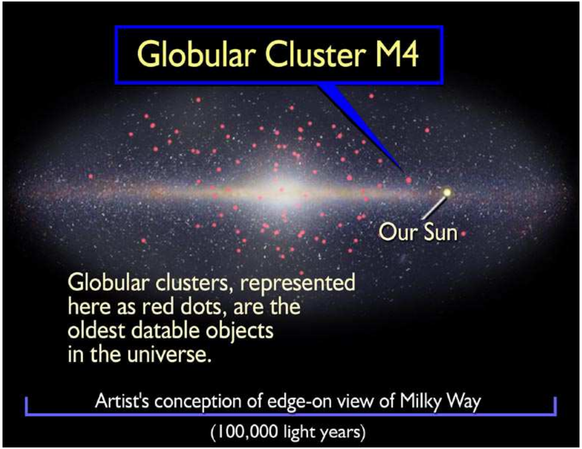
- We have encountered several ages of the Universe predicted by analytical models.
- (Friedmann models derived in appendix).
- Note these ages are all written in terms of present day quantities $H_0, q_0, \Omega_{\Lambda,0}$.
- Having obtained formulae for the ages in general models, we should make a comparison with the ages of actual objects in the universe.

Type ($w=0$)	Age t_0
Einstein de Sitter	$\frac{2}{3H_0}$
Flat Λ	$\frac{2\tanh^{-1}\sqrt{\Omega_{\Lambda,0}}}{3H_0\sqrt{\Omega_{\Lambda,0}}}$
de Sitter / Einstein	∞
Closed Friedmann	$\frac{q_0\cos^{-1}\left(\frac{1}{q_0}-1\right)-(2q_0-1)^{1/2}}{H_0(2q_0-1)^{3/2}}$
Open Friedmann	$\frac{(1-2q_0)^{1/2}-q_0\cosh^{-1}\left(\frac{1}{q_0}-1\right)}{H_0(1-2q_0)^{3/2}}$

- In the mid 1990's, this started causing real difficulties – results from the Hubble Space Telescope favoured 'high' ($80\,\mathrm{km s^{-1} Mpc^{-1}}$ and above) values of H_0.
- Theoretical prejudice was still that the universe was Einstein de Sitter, with $\Omega_{m,0}$ close to 1.

Ages of specific objects: Globular clusters

- Globular clusters are ‘metal’ poor and have a roughly spherical distribution, rather than flattened into plane of Galaxy.
- Suggests globular clusters are part of the oldest population in the Galaxy. Hence give lower bound on the age of the universe.
- They have $\sim 10^5$ stars born at the same time, presumably all with the same chemical composition.
- Thus their relative positions on the Hertzsprung-Russell diagram should be just due to differences in mass.



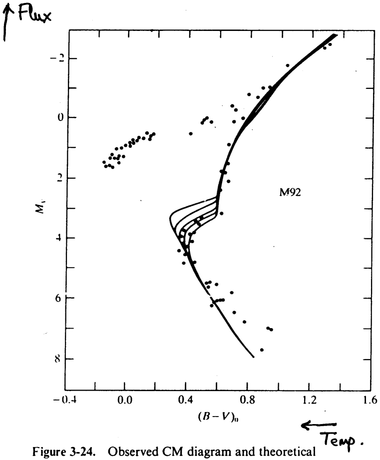


Figure 3-24. Observed CM diagram and theoretical isochrones for the globular cluster M92. The absolute-magnitude scale was fixed by setting $M_V(\mathrm{RR}) = +0.6$. The isochrones are for models with $(X, Y, Z) = (0.80, 0.20, 1 \times 10^{-4})$ at ages (from top to bottom) of 10, 12, 14, 16, and 18 ($\times 10^9$) years. The turnoff point indicates an age of $(14-16) \times 10^9$ years. [From (T2, 199).]

Age found = $14-16 \times 10^9$ yrs.

- Usual version of this is to plot 'colour' (indicating temperature) versus magnitude (basically luminosity, since all at same distance).
- They do indeed have a very tight locus, and the 'turn off point' from the main sequence, where the hydrogen has been exhausted, is a sensitive indicator of the age of the cluster.
- The lower down (in luminosity or flux) this turn off occurs, the older the cluster must be.
- In conjunction with models of stellar evolution, we can thus read off the age of a cluster.
- Example is shown for the cluster M92 — yields an age of 14 to 16 $\times 10^9$ y.
- A minimum age averaged over many clusters and taking into account the uncertainties in stellar evolution models, is thought to be about $12.5 \pm 1.5 \times 10^9$ years.

Radioactive dating

- Here we carry out dating of the elements themselves, rather than the rocks we find them in.
- As an example, we believe Uranium is formed in supernovae, with the isotopes being formed by a rapid process of neutron addition, called the ‘r-process’.
- This process is predicted to give an initial abundance ratio of

$$\left[\frac{U^{235}}{U^{238}}\right]_{\text{initial}} = 1.65 \pm 0.15,$$

  in the formation event which gives rise to the material.
- We say this happened in the very first generation of supernovae to give a lower bound on its age.

- Using the known decay rates of

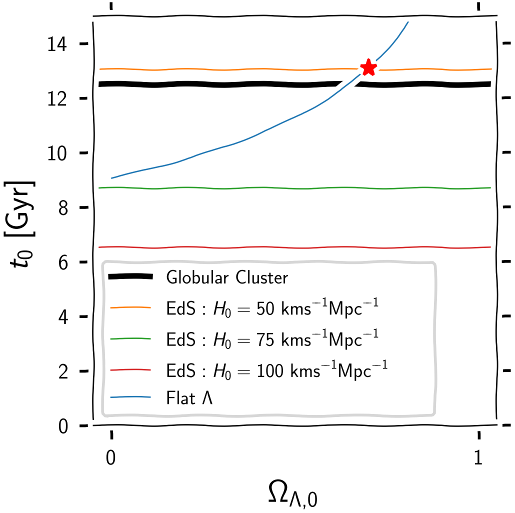
$$\begin{aligned} \lambda(U^{235}) &= 0.971 \times 10^{-9} \quad \text{per year} \\ \lambda(U^{238}) &= 0.154 \times 10^{-9} \quad \text{per year} \end{aligned}$$

together with the present abundance ratio of $\left[\frac{U^{235}}{U^{238}}\right]_{t_0} = 0.00723$ we get

$$t_{\mathrm{gal}} \geqslant \frac{\ln\left[U^{235}/U^{238}\right]_{\mathrm{initial}} - \ln\left[U^{235}/U^{238}\right]_{t_0}}{\lambda(U^{235}) - \lambda(U^{238})} \geqslant 6.6\,\mathrm{Gyr}.$$

- Can try this out with other ratios as well, and try to estimate over what period supernovae must have been going off and producing the elements incorporated in our rocks today.
- This yields estimates of 9 to 15 Gyr, similar to globular clusters.
- This work is not easy, and the results are still somewhat controversial.

Ages



- Can plot ages in a flat Λ universe, with different curves corresponding to different values of H_0.
- We have chosen the globular cluster $t_0 = 12.5\,\mathrm{Gyr}$ for our three reprentative curves with different values of H_0
- Taking e.g. $H_0 = 72\,\mathrm{kms^{-1}Mpc^{-1}}$ and $\Omega_{\Lambda,0} = 0.7$ yields $t_0 = 13.1\,\mathrm{Gyr}$, which while not very much above the ages of the globular clusters, does at least allow some time for them to form.

Summary

- Observables: sky location (θ, ϕ), redshift z, flux F, angular size $\Delta\theta$.
- Transfer to cosmological quantities using $\boxed{\dfrac{d\chi}{c} = \dfrac{dt}{R} = -\dfrac{dz}{R_0 H(z)}.}$
- (Yet another form of) the Friedmann equation

$$\boxed{H = H_0\left(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}\right)^{1/2}.}$$

- Luminosity and angular diameter distances (standard candles L and standard rulers D)

$$\boxed{d_L = \left(\frac{L}{4\pi F}\right)^{\frac{1}{2}} = R_0 S(\chi)(1+z),} \qquad \boxed{d_\theta = \frac{D}{\Delta\theta} = \frac{R_0 S(\chi)}{(1+z)}.}$$

Next time

Constituents of the Universe

Appendix: Ages in Friedmann models

- Consider the $k = +1$ matter-dominated model – evaluate R at the current time:

$$R_0 = R_0 \frac{a_m}{2}(1 - \cos\eta_0) \qquad \Rightarrow \qquad \cos\eta_0 = 1 - \frac{2}{a_m}$$

- But

$$a_m = \frac{\Omega_{m,0}}{\Omega_{m,0} - 1} \qquad (w = 0)$$

- Thus

$$\eta_0 = \cos^{-1}\left(1 - \frac{\Omega_{m,0} - 1}{\Omega_{m,0}/2}\right) = \cos^{-1}\left(\frac{1}{q_0} - 1\right)$$

using $q_0 = \Omega_{m,0}/2$ if $\Omega_{\Lambda,0} = 0$.

- Plugging in the expression for $t_0 = R_0 \frac{a_m}{2c}(\eta_0 - \sin\eta_0)$ and using $1 = \Omega_m + \Omega_k = \Omega_m - \frac{c^2}{R_0^2 H_0^2}$ for R_0:

$$t_0 = \frac{1}{H_0} \frac{q_0}{(2q_0 - 1)^{3/2}} \left[\cos^{-1}\left(\frac{1}{q_0} - 1\right) - \frac{1}{q_0}(2q_0 - 1)^{1/2} \right]$$

- we can deduce the age of a model universe just from observations which we can in principle make locally
- A very similar expression can be found in the open case – for the matter-dominated universe:

$$t_0 = \frac{1}{H_0} \frac{q_0}{(1 - 2q_0)^{3/2}} \left[\frac{1}{q_0}(1 - 2q_0)^{1/2} - \cosh^{-1}\left(\frac{1}{q_0} - 1\right) \right]$$