Part III Physics Minor Topic
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Phase Transitions and Critical Phenomena

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Answers to problems

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Section I

Ginzburg–Landau theory & fluctuation corrections

Problem 1: Tricritical point

In this problem, we are only interested in the mean field solution, that is, the function $\mathbf{m}(\mathbf{r})$ which minimises the Ginzburg–Landau Hamiltonian,

$$\beta H = \int \mathrm{d}\mathbf{r} \left[\frac{t}{2}\mathbf{m}^2 + u\mathbf{m}^4 + v\mathbf{m}^6 + \frac{K}{2}(\nabla \mathbf{m})^2 - \mathbf{h}\cdot\mathbf{m} \right]. \tag{1.1}$$

It is easy to see that any fluctuations in $\mathbf{m}$ increase this functional, therefore, we can assume in this problem that $\mathbf{m}(\mathbf{r})$ is constant.

(a) Qualitative description. All in all, we have to minimise the function $F(\mathbf{m}) = \frac{t}{2}\mathbf{m}^2 + u\mathbf{m}^4 + v\mathbf{m}^6$ which only depends on the magnitude of $\mathbf{m}$. This function is plotted for fixed $v > 0$, $u < 0$ and several values of t in Fig. 1.1(a). For high enough reduced temperature, $F(m)$ has only one minimum at $m = 0$: this clearly corresponds to a disordered phase. At some $t > 0$, $F(m)$ develops minima at $|\mathbf{m}| = m^* \neq 0$: unlike in the quartic model discussed in lectures, these minima are separated from $m = 0$ by a maximum in between. Initially, this additional state is only metastable, but at $t^* > 0$, $F(m^*)$ reaches 0: at this point, the configuration with finite magnetisation becomes more stable and a first order phase transition occurs. The $m = 0$ state remains metastable for $0 < t < t^*$; below $t = 0$, only the ordered state exists.

This contrasts the situation for $u, v > 0$ discussed in lectures (see Fig. 1.1(b)). There, if $t > 0$, the only minimum is at $m = 0$: for negative t, this turns into a maximum and minima at finite m appear. Crucially, at infinitesimal negative temperatures, m^* is also infinitesimally different from 0, meaning there is no discontinuity in the order parameter. That is, for $u > 0$, the transition is second order: the point $t = u = 0$, the boundary between first and second order phase boundaries is expected to be special.

(b) First order phase boundary. The extrema of $F(\mathbf{m})$ are given by the zeroes of its derivative:

$$\nabla F = 2\mathbf{m}\left(3vm^4 + 2um^2 + \frac{t}{2}\right) = 0.$$

$\mathbf{m} = 0$ always solves this equation; the nonzero minima follow from solving the quadratic in m^2:

$$m_\pm^2 = \frac{-2u \pm \sqrt{4u^2 - 6tv}}{6v}.$$

If $t > 2u^2/3v$, the resulting m is not real. Below that, we get two different real values of $|\mathbf{m}|$: the smaller one corresponds to the maximum separating the minimum at zero and finite m, thus the position of the ordered phase minimum is

$$m^{*2} = \frac{-2u + \sqrt{4u^2 - 6tv}}{6v}. \tag{1.2}$$

Now, at the critical temperature the free energy of the two phases is equal: $F(m^*) = F(0) = 0$. That is, the critical temperature t^* and order parameter m^* satisfy both of these equations:

$$3vm^{*4} + 2um^{*2} + \frac{t^*}{2} = 0 \tag{1.3a}$$

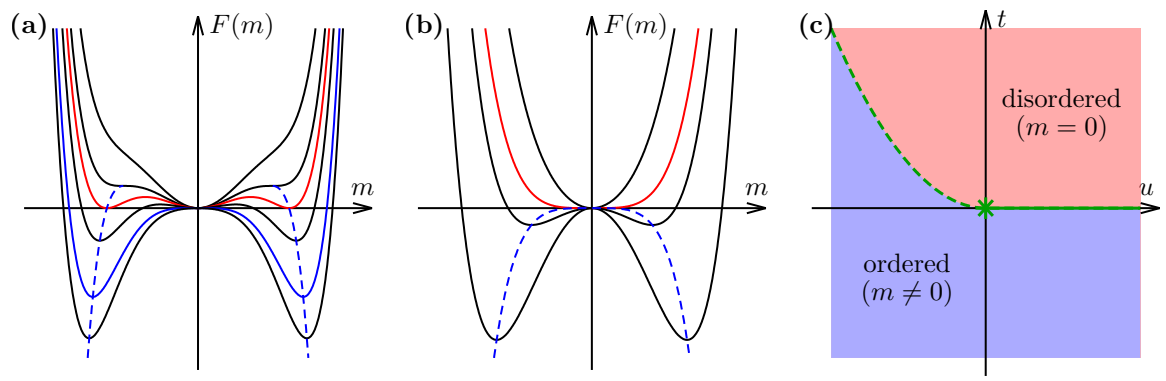

Figure 1.1. **(a)** Plots of $F(m)$ for several values of t with fixed $v > 0$ and $u < 0$. At high t, $F(m)$ has a single minimum at $m = 0$; as the temperature is decreased, other minima appear at finite m (locus in dashed blue). At some $t^* > 0$ (red), the free energy of this other minimum becomes 0, and thus a first-order phase transition happens. The disordered state remains metastable until $t = 0$ (blue) where the minimum turns into a maximum.
(b) Plots of $F(m)$ for several values of t with fixed $v > 0$ and $u > 0$. At $t > 0$, $F(m)$ has a single minimum at $m = 0$; below $t = 0$ (red), this turns into a maximum, and two minima (locus in dashed blue) appear away from it. Since the minima are infinitesimally close to $m = 0$ for small negative t, the transition is second order.
(c) Phase diagram of the mean field Hamiltonian for fixed v. For $u > 0$, a second order transition (continuous green line) occurs at $t = 0$ while for $u < 0$, there is a first order phase transition at $t^* = u^2/(2v)$ (dashed green line). The transitions separate a disordered ($m = 0$, red) and an ordered ($m \neq 0$, blue) from each other: as m evolves continuously within the ordered phase, there are no further phase transitions in that phase. The first and second order transition lines meet at $u = t = 0$: this tricritical point (green asterisk) behaves qualitatively differently from both transitions.

$$vm^{*4} + um^{*2} + \frac{t^*}{2} = 0 \qquad (\text{NB } m^* \neq 0) \tag{1.3b}$$

Subtracting the two equations from each other,[1] we get

$$2vm^{*4} + um^{*2} = m^{*2}(2vm^{*2} + u) = 0$$
$$m^* = \sqrt{\frac{-u}{2v}}. \tag{1.4}$$

The critical temperature now follows from substituting back into either of (1.3):

$$t^* = \frac{u^2}{2v}. \tag{1.5}$$

(c) Phase diagram. There are two different phases in the model: a high-temperature disordered ($m = 0$) and a low-temperature ordered ($m \neq 0$) one. The boundary between these phases is a second-order transition at $t = 0$ if $u > 0$ and a first-order transition at $t^* = u^2/(2v)$ if $u < 0$; from (1.2), one can see that m^* has no special features anywhere in the ordered phase. The overall phase diagram is shown in Fig. 1.1(c).

(d) Critical exponents at the tricritical point. For this part of the problem, we have to consider the $\mathbf{h} \cdot \mathbf{m}$ term, but ignore the quartic term:

$$F(\mathbf{m}) = \frac{t}{2}\mathbf{m}^2 + v\mathbf{m}^6 - \mathbf{h} \cdot \mathbf{m} \tag{1.6}$$
$$0 = \nabla F(\mathbf{m}^*) = \mathbf{m}^*(t + 6vm^{*4}) - \mathbf{h}.$$

[1] This is a practical method of searching for critical points. By subtracting equations from each other, it may be possible to avoid substituting the expression of m^*, or at least use only lower powers of it, which are easier to evaluate. One can also eliminate t^*, so an equation for m^* only can be found.

That is, the mean field order parameter will always be parallel to $\mathbf{h}$, and so both can be substituted with scalars: the mean-field equation is thus

$$m^*(t + 6vm^{*4}) - h = 0. \tag{1.7}$$

For $h = 0$, the appropriate solution of this equation is

$$m^* = \begin{cases} 0 & t > 0 \\ \sqrt[4]{\frac{-t}{6v}} & t < 0 \end{cases} ; \tag{1.8}$$

that is, in the ordered phase, $m \propto |t|^{1/4} \implies \beta = 1/4$.

Substituting (1.8) into (1.6) gives

$$F^* = \begin{cases} 0 & t > 0 \\ -\frac{(-t)^{3/2}}{3\sqrt{6v}} & t < 0 \end{cases} \implies C \propto -\frac{\mathrm{d}^2 F^*}{\mathrm{d}t^2} = \begin{cases} 0 & t > 0 \\ \frac{(-t)^{-1/2}}{4\sqrt{6v}} & t < 0 \end{cases} . \tag{1.9}$$

The heat capacity only depends on t in the ordered phase; there, $C \propto |t|^{-1/2} \implies \alpha = 1/2$.

Differentiating (1.7) with respect to h gives

$$\begin{aligned} \frac{\mathrm{d}m^*}{\mathrm{d}h}(t + 30vm^{*5}) - 1 = 0 \\ \chi = \frac{\mathrm{d}m^*}{\mathrm{d}h} = \frac{1}{t + 30vm^*5} = \begin{cases} \frac{1}{t} & t > 0 \\ \frac{1}{-4t} & t < 0 \end{cases} . \end{aligned} \tag{1.10}$$

Although the prefactors in the two cases differ, $\chi \propto |t|^{-1}$ on both sides of the transition $\implies \gamma = 1$.

Finally, if $t = 0$, (1.7) reduces to

$$6vm^{*5} - h = 0 \implies m^* = \left(\frac{h}{6v}\right)^{1/5} \implies \delta = 5. \tag{1.11}$$

As a final check, consider the scaling relations:

$$\begin{aligned} \alpha + 2\beta + \gamma = \frac{1}{2} + 2 \times \frac{1}{4} + 1 = 2 \quad \checkmark \\ \frac{\gamma}{\beta} = \frac{1}{1/4} = 4 = \delta - 1 \quad \checkmark \end{aligned}$$

Problem 2: Fluctuations around the mean field, the Ginzburg criterion

In this question, the vector nature of the order parameter and the gradient energy term K will become relevant, however, we omit the quartic term u and assume $\mathbf{h} = 0$:

$$\beta H = \int \mathrm{d}\mathbf{r} \left[\frac{t}{2}\mathbf{m}^2 + v\mathbf{m}^6 + \frac{K}{2}(\nabla \mathbf{m})^2 \right]. \tag{1.12}$$

(a) MF heat capacity. By (1.9),

$$C_{\mathrm{MF}} \propto -\frac{\mathrm{d}^2 f_{\mathrm{MF}}}{\mathrm{d}t^2} = \begin{cases} 0 & t > 0 \\ \frac{(-t)^{-1/2}}{4\sqrt{6v}} & t < 0 \end{cases}. \tag{1.13}$$

(b) Effective quadratic Hamiltonian of fluctuations.

$$\mathbf{m}^2 = (m^* + \phi_\ell)^2 + \sum_{\alpha=2}^{n} \phi_{t,\alpha}^2 = m^{*2} + \phi_\ell^2 + \sum_{\alpha=2}^{n} \phi_{t,\alpha}^2 + 2m^*\phi_\ell \tag{1.14a}$$

$$\begin{aligned} \mathbf{m}^6 = (\mathbf{m}^2)^3 &= m^{*6} + 3m^{*4}\left[\phi_\ell^2 + \sum_{\alpha=2}^{n} \phi_{t,\alpha}^2\right] + 6m^{*5}\phi_\ell + 12m^{*4}\phi_\ell^2 + \mathcal{O}(\phi^3) \\ &= m^{*6} + 15m^{*4}\phi_\ell^2 + 3m^{*4}\sum_{\alpha=2}^{n} \phi_{t,\alpha}^2 + 6m^{*5}\phi_\ell + \mathcal{O}(\phi^3) \end{aligned} \tag{1.14b}$$

$$(\nabla \mathbf{m})^2 = (\nabla \phi_\ell)^2 + \sum_{\alpha=2}^{n} (\nabla \phi_{t,\alpha})^2 \qquad (\text{NB } m^* \text{ is constant}) \tag{1.14c}$$

$$\begin{aligned} f &= \frac{t}{2}\mathbf{m}^2 + v\mathbf{m}^6 + \frac{K}{2}(\nabla \mathbf{m})^2 \\ &= f_{\mathrm{MF}} + \underbrace{(tm^* + 6vm^{*5})}_{0 \text{ by def. of } m^*,\ (1.7)}\ \phi_\ell + \left[\left(\frac{t}{2} + 15vm^{*4}\right)\phi_\ell^2 + \frac{K}{2}(\nabla\phi_\ell)^2\right] \\ &\quad + \sum_{\alpha=2}^{n}\left[\left(\frac{t}{2} + 3vm^{*4}\right)\phi_{t,\alpha}^2 + \frac{K}{2}(\nabla\phi_{t,\alpha})^2\right] \end{aligned} \tag{1.14d}$$

$$\beta H = (\beta H)_{\mathrm{MF}} + \frac{K}{2}\int \mathrm{d}\mathbf{r}\left\{ \left[\xi_\ell^{-2}\phi_\ell^2 + (\nabla\phi_\ell)^2\right] + \sum_{\alpha=2}^{n}\left[\xi_t^{-2}\phi_{t,\alpha}^2 + (\nabla\phi_{t,\alpha})^2\right]\right\} \tag{1.14e}$$

where the index MF denotes the mean-field values of each quantity, and the correlation lengths $\xi_{\ell,t}$ are given by

$$\xi_\ell^{-2} = \frac{t + 30vm^{*4}}{K} = \begin{cases} t/K & t > 0 \\ -4t/K & t < 0 \end{cases}; \tag{1.15a}$$

$$\xi_t^{-2} = \frac{t + 6vm^{*4}}{K} = \begin{cases} t/K & t > 0 \\ 0 & t < 0 \end{cases}. \tag{1.15b}$$

(c) Fourier analysis, correlators. Since the Hamiltonian (1.14e) is translation invariant, it can be diagonalised by introducing the Fourier modes

$$\phi_\alpha(\mathbf{q}) = \int \mathrm{d}\mathbf{r}\, \phi_\alpha(\mathbf{r}) e^{-i\mathbf{q}\cdot\mathbf{r}} \iff \phi_\alpha(\mathbf{r}) = \int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d}\, \phi_\alpha(\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{r}} \tag{1.16}$$

For example, the longitudinal term transforms as (note that $\phi(\mathbf{q}) = \phi(-\mathbf{q})^*$ as $\phi(\mathbf{r})$ is real)

$$\begin{aligned}
\int \mathrm{d}\mathbf{r}\, \phi_\ell^2 &= \int \mathrm{d}\mathbf{r} \int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d}\, \phi_\ell(\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{r}} \int \frac{\mathrm{d}\mathbf{q}'}{(2\pi)^d}\, \phi_\ell(\mathbf{q}') e^{i\mathbf{q}'\cdot\mathbf{r}} \\
&= \int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d} \int \frac{\mathrm{d}\mathbf{q}'}{(2\pi)^d}\, \phi_\ell(\mathbf{q})\phi_\ell(\mathbf{q}') \int \mathrm{d}\mathbf{r}\, e^{i(\mathbf{q}+\mathbf{q}')\cdot\mathbf{r}} \\
&= \int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d} \int \frac{\mathrm{d}\mathbf{q}'}{(2\pi)^d}\, \phi_\ell(\mathbf{q})\phi_\ell(\mathbf{q}')(2\pi)^d \delta(\mathbf{q}+\mathbf{q}') \\
&= \int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d}\, \phi_\ell(\mathbf{q})\phi_\ell(-\mathbf{q}) = \int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d}\, |\phi_\ell(\mathbf{q})|^2 \qquad (1.17a) \\
\int \mathrm{d}\mathbf{r}\, (\nabla\phi_\ell)^2 &= \int \mathrm{d}\mathbf{r}\, \nabla \left[\int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d}\, \phi_\ell(\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{r}} \right] \cdot \nabla \left[\int \frac{\mathrm{d}\mathbf{q}'}{(2\pi)^d}\, \phi_\ell(\mathbf{q}') e^{i\mathbf{q}'\cdot\mathbf{r}} \right] \\
&= \int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d} \int \frac{\mathrm{d}\mathbf{q}'}{(2\pi)^d} (i\mathbf{q})\cdot(i\mathbf{q}')\, \phi_\ell(\mathbf{q})\phi_\ell(\mathbf{q}') \int \mathrm{d}\mathbf{r}\, e^{i(\mathbf{q}+\mathbf{q}')\cdot\mathbf{r}} \\
&= -\int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d} \int \frac{\mathrm{d}\mathbf{q}'}{(2\pi)^d}\, \mathbf{q}\cdot\mathbf{q}'\, \phi_\ell(\mathbf{q})\phi_\ell(\mathbf{q}')(2\pi)^d \delta(\mathbf{q}+\mathbf{q}') \\
&= \int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d}\, \mathbf{q}^2\, \phi_\ell(\mathbf{q})\phi_\ell(-\mathbf{q}) = \int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d}\, \mathbf{q}^2\, |\phi_\ell(\mathbf{q})|^2; \qquad (1.17b) \\
\int \mathrm{d}\mathbf{r}\, \left[\xi_\ell^{-2}\phi_\ell^2 + (\nabla\phi_\ell)^2\right] &= \int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d} (\mathbf{q}^2 + \xi_\ell^{-2})\, |\phi_\ell(\mathbf{q})|^2. \qquad (1.17c)
\end{aligned}$$

All in all, the Hamiltonian becomes

$$\beta H = \beta H_{\mathrm{MF}} + \frac{K}{2} \int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d} \left[(\mathbf{q}^2 + \xi_\ell^{-2})\, |\phi_\ell(\mathbf{q})|^2 + (\mathbf{q}^2 + \xi_t^{-2}) \sum_{\alpha=2}^{n} |\phi_{t,\alpha}(\mathbf{q})|^2 \right]. \qquad (1.18)$$

In this form, the Hamiltonian is manifestly quadratic, and so the partition function is given by a Gaussian functional integral. Since the Hamiltonian is also diagonalised, the functional integral can be decomposed into the product of Gaussian integrals for each momentum space component: performing these integrals,[2] the momentum space correlator turns out to be

$$\langle \phi_\alpha(\mathbf{q})\phi_\beta(\mathbf{q}')\rangle = \frac{\delta_{\alpha\beta}}{K(\mathbf{q}^2 + \xi_\alpha^{-2})} (2\pi)^d \delta(\mathbf{q}+\mathbf{q}'). \qquad (1.19)$$

The real space correlator is now given by a Fourier transform:

$$\begin{aligned}
\langle \phi_\alpha(\mathbf{r})\phi_\beta(0)\rangle &= \int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d} \int \frac{\mathrm{d}\mathbf{q}'}{(2\pi)^d} \langle \phi_\alpha(\mathbf{q})\phi_\beta(\mathbf{q}')\rangle e^{i\mathbf{q}\cdot\mathbf{r}} = \delta_{\alpha\beta} \int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d} \int \frac{\mathrm{d}\mathbf{q}'}{(2\pi)^d} \frac{(2\pi)^d \delta(\mathbf{q}+\mathbf{q}')}{K(\mathbf{q}^2 + \xi_\alpha^{-2})} e^{i\mathbf{q}\cdot\mathbf{r}} \\
&= \frac{\delta_{\alpha\beta}}{K} \int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d} \frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{\mathbf{q}^2 + \xi_\alpha^{-2}}. \qquad (1.20)
\end{aligned}$$

Expressions for the last integral are given in the text of the question. The only point of complication is those of transverse fluctuations below the transition point. These (Goldstone) modes have an infinite correlation length, thus

$$\langle \phi_t(\mathbf{r})\phi_t(0)\rangle = \frac{r^{2-d}}{(2-d)S_d K}; \qquad (1.21)$$

for dimensions lower than 2, this expression diverges as $r \to \infty$, thus destroying long-range order. This is the statement of the Mermin–Wagner theorem.[3]

[2] Of course, continuum integrals of this sort are ill-defined: results like this can be quoted (*e. g.* in an exam) or derived in terms of an underlying lattice model (see below).

[3] The integral in turn is UV divergent for $d > 2$: due to the cutoff in $\mathbf{q}$-states at $|\mathbf{q}| \approx 1/a_0$, these do not give rise to physical problems.

A convention for lattice sums. In the next part of the question, we are invited to evaluate the partition function of the fluctuation modes. Since infinite products tend to be ill-defined, it is a good idea to perform any such calculations using a discrete, finite lattice. The continuous Fourier transform $\phi(\mathbf{q})$ is replaced by the Fourier series $\phi_{\mathbf{q}}$: to avoid confusion later, we fix a convention for these modes which corresponds well to the continuum Fourier transform (1.16). Consider

$$\phi^\alpha_{\mathbf{q}} = \frac{a_0^d}{\sqrt{V}} \sum_{\mathbf{r}} \phi_\alpha(\mathbf{r}) e^{-i\mathbf{q}\cdot\mathbf{r}} \iff \phi_\alpha(\mathbf{r}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{q}} \phi^\alpha_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}}. \tag{1.22}$$

where a_0^d is the volume of the unit cell and V is the volume of the system. In this convention, integrals of $\phi(\mathbf{r})^2$ &c. used in the expression of the free energy work out quite similarly to the continuum case. For example,

$$\begin{aligned}\int \mathrm{d}\mathbf{r}\, \phi_\ell^2 \equiv a_0^d \sum_{\mathbf{r}} \phi_\ell(\mathbf{r})^2 &= a_0^d \sum_{\mathbf{r}} \frac{1}{\sqrt{V}} \sum_{\mathbf{q}} \phi^\ell_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}} \frac{1}{\sqrt{V}} \sum_{\mathbf{q}'} \phi^\ell_{\mathbf{q}'} e^{i\mathbf{q}'\cdot\mathbf{r}} \\ &= \frac{a_0^d}{V} \sum_{\mathbf{q},\mathbf{q}'} \phi^\ell_{\mathbf{q}} \phi^\ell_{\mathbf{q}'} \sum_{\mathbf{r}} e^{i(\mathbf{q}+\mathbf{q}')\cdot\mathbf{r}} = \frac{a_0^d}{V} \sum_{\mathbf{q},\mathbf{q}'} \phi^\ell_{\mathbf{q}} \phi^\ell_{\mathbf{q}'} \mathcal{N} \delta_{\mathbf{q},-\mathbf{q}'} = \sum_{\mathbf{q}} |\phi^\ell_{\mathbf{q}}|^2\end{aligned} \tag{1.23}$$

where $\mathcal{N} = V/a_0^d$ is the number of lattice sites in the system. All in all, the Hamiltonian has the form

$$\beta H = \beta H_{\mathrm{MF}} + \frac{K}{2} \sum_{\mathbf{q}} \left[(\mathbf{q}^2 + \xi_\ell^{-2}) \, |\phi^\ell_{\mathbf{q}}|^2 + (\mathbf{q}^2 + \xi_t^{-2}) \sum_{\alpha=2}^{n} |\phi^{t,\alpha}_{\mathbf{q}}|^2 \right]. \tag{1.24}$$

From here, the momentum space correlator is given by

$$\langle \phi^\alpha_{\mathbf{q}} \phi^\beta_{\mathbf{q}'} \rangle = \frac{\delta^{\alpha\beta} \delta_{\mathbf{q},-\mathbf{q}'}}{K(\mathbf{q}^2 + \xi_\alpha^{-2})}; \tag{1.25}$$

the real space correlator follows from the Fourier transform

$$\begin{aligned}\langle \phi_\alpha(\mathbf{r}) \phi_\beta(0) \rangle &= \frac{1}{\sqrt{V}} \sum_{\mathbf{q}} \frac{1}{\sqrt{V}} \sum_{\mathbf{q}'} \langle \phi^\alpha_{\mathbf{q}} \phi^\beta_{\mathbf{q}'} \rangle e^{i\mathbf{q}\cdot\mathbf{r}} = \frac{1}{V} \sum_{\mathbf{q},\mathbf{q}'} \frac{\delta^{\alpha\beta} \delta_{\mathbf{q},-\mathbf{q}'}}{K(\mathbf{q}^2 + \xi_\alpha^{-2})} e^{i\mathbf{q}\cdot\mathbf{r}} \\ &= \frac{1}{V} \sum_{\mathbf{q}} \frac{\delta^{\alpha\beta}}{K(\mathbf{q}^2 + \xi_\alpha^{-2})} e^{i\mathbf{q}\cdot\mathbf{r}} \equiv \frac{\delta_{\alpha\beta}}{K} \int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d} \frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{\mathbf{q}^2 + \xi_\alpha^{-2}}\end{aligned} \tag{1.26}$$

since the density of reciprocal space modes is $(2\pi)^d/V$. The lattice calculation thus reproduces the continuum result quoted in (1.19).

(d) Fluctuation corrections of the free energy. The partition function of the Hamiltonian (1.14e, 1.18) is given by the functional integral $\int \mathcal{D}\phi_\alpha(\mathbf{q}) \exp(-\beta H[\phi_\alpha(\mathbf{q})])$. Since in momentum space individual modes are decoupled from each other, the functional integral can be rewritten as a product of integrals over each mode. Starting off with discrete $\mathbf{q}$ modes,

$$\begin{aligned}\mathcal{Z} &= e^{-f_{\mathrm{MF}} V} \int \left(\prod_{\mathbf{r},\alpha} \mathrm{d}\phi_\alpha(\mathbf{r}) \right) \exp\left(-\frac{K}{2} \sum_{\mathbf{q}} (\mathbf{q}^2 + \xi_\alpha^{-2}) |\phi^\alpha_{\mathbf{q}}|^2 \right) \\ &\cong e^{-f_{\mathrm{MF}} V} \int \left(\prod_{\mathbf{q},\alpha} \frac{\mathrm{d}\phi^\alpha_{\mathbf{q}}}{a_0^{d/2}} \right) \exp\left(-\frac{K}{2} \sum_{\mathbf{q}} (\mathbf{q}^2 + \xi_\alpha^{-2}) |\phi^\alpha_{\mathbf{q}}|^2 \right) \\ &= e^{-f_{\mathrm{MF}} V} \prod_{\mathbf{q},\alpha}{}' \frac{1}{a_0^d} \int \mathrm{d}\phi^\alpha_{\mathbf{q}} \int \mathrm{d}\phi^\alpha_{-\mathbf{q}} \exp\left[-K(\mathbf{q}^2 + \xi_\alpha^{-2}) |\phi^\alpha_{\mathbf{q}}|^2 \right] \\ &\cong e^{-f_{\mathrm{MF}} V} \prod_{\mathbf{q},\alpha}{}' \frac{2}{a_0^d} \int \mathrm{d}\,\mathrm{Re}\,\phi^\alpha_{\mathbf{q}} \int \mathrm{d}\,\mathrm{Im}\,\phi^\alpha_{\mathbf{q}} \exp\left[-K(\mathbf{q}^2 + \xi_\alpha^{-2}) |\phi^\alpha_{\mathbf{q}}|^2 \right]\end{aligned}$$

$$= e^{-f_{\mathrm{MF}}V} \prod_{\mathbf{q},\alpha}{}' \frac{2}{a_0^d} \sqrt{\frac{\pi}{K(\mathbf{q}^2+\xi_\alpha^{-2})}} \sqrt{\frac{\pi}{K(\mathbf{q}^2+\xi_\alpha^{-2})}}$$
$$= e^{-f_{\mathrm{MF}}V} \prod_{\mathbf{q},\alpha} \sqrt{\frac{2\pi}{K a_0^d (\mathbf{q}^2+\xi_\alpha^{-2})}} \tag{1.27}$$

where the prime on the product requires to take only one of any $(+\mathbf{q}, -\mathbf{q})$ pair in the product and $\cong$ is taken to mean "equal in magnitude". At the first step, we include the $a_0^{d/2}$ to make the Fourier transform (1.22) unitary and thus its Jacobian of unit magitude. At the second step, we realise that $\phi_{\mathbf{q}}^\alpha$ are not independent since $\phi_{\mathbf{q}}^\alpha = \phi_{-\mathbf{q}}^{\alpha *}$: therefore, the integrals over them can be taken as a single integral over the complex plane spanned by either of them. At the third step, we make a change of variables from ϕ and ϕ^* to their real and imaginary parts. The Jacobian of this transformation is $-2i$ per pair of modes: the 2 is added as a prefactor, the $-i$ is absorbed by the $\cong$ sign.[4] Now we have two Gaussian integrals along the real and imaginary axes per mode pair: performing them and separating $(+\mathbf{q}, -\mathbf{q})$ pairs gives the final result.

Note the high level of pedantry in this derivation: most of the issues raised here can be conveniently swept under the carpet by using the identity $\log\det A = \operatorname{tr}\log A$ and fudging about what it means to take the trace of an infinitely large, almost diagonal matrix. Alternatively, it is usually fine to just quote (1.27) from the Hamiltonian (1.18).

The free energy density is given by

$$f = -\frac{\log \mathcal{Z}}{V} = f_{\mathrm{MF}} + \frac{1}{V}\sum_{\mathbf{q},\alpha} \log \sqrt{\frac{K a_0^d (\mathbf{q}^2+\xi_\alpha^{-2})}{2\pi}} \simeq f_{\mathrm{MF}} + \frac{1}{2}\sum_\alpha \int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d} \log\left[K(\mathbf{q}^2+\xi_\alpha^{-2})\right]$$
$$\simeq f_{\mathrm{MF}} + \begin{cases} \dfrac{n}{2}\displaystyle\int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d} \log(K\mathbf{q}^2 + t) & t > 0 \\ \dfrac{1}{2}\displaystyle\int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d} \log(K\mathbf{q}^2 - 4t) & t < 0 \end{cases}. \tag{1.28}$$

Note that we dropped the $a_0^d/(2\pi)$ under the square root. After taking the logarithm, such terms would only give rise to a constant shift in the free energy: since all physical quantities are given by derivatives of f, such constants are irrelevant. Similarly, for $t < 0$, $\xi_t^{-2} = 0$, therefore, the integral in (1.28) for transverse modes is constant: accordingly, they are dropped from the final form.

(e) Fluctuation corrections of the heat capacity & the upper critical dimension. The heat capacity is proportional to the second derivative of f: by (1.28), fluctuations give rise to a contribution

$$C_{\mathrm{fluct}} \propto -\frac{\mathrm{d}^2 f_{\mathrm{fluct}}}{\mathrm{d}t^2} = \begin{cases} \dfrac{n}{2}\displaystyle\int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d} \frac{1}{(K\mathbf{q}^2+t)^2} & t > 0 \\ 8\displaystyle\int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d} \frac{1}{(K\mathbf{q}^2-4t)^2} & t < 0 \end{cases}. \tag{1.29}$$

These are formally identical to the integrals discussed in lectures for standard Ginzburg–Landau theory.

For large wave numbers, the integral takes the form

$$C_{\mathrm{fluct}} \sim \int \frac{q^{d-1}\mathrm{d}q}{q^4} \sim q^{d-4}.$$

That is, the integral is divergent in the ultraviolet for $d > 4$. Since the lattice spacing provides a natural UV cutoff at $q \approx 1/a$, the integral will not diverge. Nevertheless, the greatest contribution to

[4] Alternatively, one could consider a real Fourier transform with sines and cosines, giving real and imaginary parts directly, as well as a real Jacobian in the first step.

the integral comes from the highest values of q where the dependence on t is negligible. That is, the mean field heat capacity always dominates fluctuation corrections for $d > 4$.

However, this doesn't mean the upper critical dimension is four! In the case discussed in lectures, the mean field heat capacity is a constant, so *any* infrared divergence in the fluctuations would dominate it. In our case, however, the mean field behaviour is also divergent as $t \to 0$, so it can dominate a divergent fluctuation contribution as well, provided it diverges slower than the mean field. Consider the change of variables $\mathbf{q} = \mathbf{u}\sqrt{|t|/K}$ in (1.29):

$$\int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d} \frac{1}{(K\mathbf{q}^2 + \alpha|t|)^2} = \int \frac{\mathrm{d}\mathbf{u}(|t|/K)^{d/2}}{(2\pi)^d} \frac{1}{|t|^2(\mathbf{u}^2+\alpha)^2} = \frac{|t|^{d/2-2}}{K^{d/2}} \int \frac{\mathrm{d}\mathbf{u}}{(2\pi)^d} \frac{1}{(\mathbf{u}^2+\alpha)^2} : \tag{1.30}$$

since the integrand is regular everywhere, the last integral will be a finite constant for $d < 4$. That is, the fluctuation contribution to the heat capacity for $d < 4$ diverges as $|t|^{d/2-2}$ which is less divergent than $C_{\mathrm{MF}} \propto |t|^{-1/2}$ if $d > 3$. The upper critical dimension is thus 3.

(f) Ginzburg criterion. The ratio of the fluctuation and mean field contributions to the heat capacity scale as

$$\frac{C_{\mathrm{fluct}}}{C_{\mathrm{MF}}} \propto \frac{|t|^{d/2-2}/K^{d/2}}{|t|^{-1/2}/v^{1/2}} = \frac{|t|^{(d-3)/2}}{\sqrt{K^d/v}} :$$

if $d \geq 3$, the fluctuation corrections vanish near the transition point, in line with $d_{\mathrm{u}} = 3$. Below the critical dimension, mean field theory is still valid sufficiently far from the transition point:

$$\frac{|t|^{(d-3)/2}}{\sqrt{K^d/v}} \ll 1 \implies |t| \gg \left(\frac{v}{K^d}\right)^{1/(3-d)}.$$

(g) General multicritical points. We now consider the Hamiltonian

$$\beta H = \int \mathrm{d}\mathbf{r} \left[\frac{t}{2}\mathbf{m}^2 + u_{2n}\mathbf{m}^{2n} + \frac{K}{2}(\nabla\mathbf{m})^2\right]. \tag{1.31}$$

The mean field contribution follows from ignoring the gradient term, and minimising the rest: for $t < 0$ (see Problem 1(d) for details of the derivation),

$$m^* \propto |t|^{1/(2n-2)} \implies f_{\mathrm{MF}} \propto |t|^{n/(n-1)} \implies C_{\mathrm{MF}} \propto |t|^{n/(n-1)-2} = |t|^{(2-n)/(n-1)}. \tag{1.32a}$$

As for the fluctuation corrections, it follows from dimensional analysis that $\xi^{-2} \propto t/K$ regardless of the exponent n. Therefore, the fluctuation contribution will be of a form similar to (1.29) which evaluates to (1.30). That is,

$$C_{\mathrm{fluct}} \propto |t|^{d/2-2} \tag{1.32b}$$

Comparing (1.32a) and (1.32b) shows that the mean field contribution is dominant if

$$\frac{n-2}{n-1} > 2 - \frac{d}{2} \implies d > 4 - 2\frac{n-2}{n-1} = \frac{2n}{n-1} = d_{\mathrm{u}}, \tag{1.33}$$

as given in the problem text.[5]

[5] Albeit much of this course is about performing complex calculations, this question is a reminder that simple power counting is a powerful tool when discussing critical phenomena where constant prefactors are often unimportant.

Problem 3: Spin waves

(a) Rewriting the partition function to different variables consists both of changing the integration measure and the integrand.
For the former, we have to sample the possible $\mathbf{S}_i$ (that is, 2d unit vectors) uniformly: introducing θ_i as the angle between $\mathbf{S}_i$ and some fixed axis, we have $\int \mathrm{d}\mathbf{S}_i \equiv \int \mathrm{d}\theta_i$.
For the latter, note that $\mathbf{S}_i \cdot \mathbf{S}_j = \cos(\theta_i - \theta_j)$ since the $\mathbf{S}_i$ are unit vectors: the partition function is thus

$$\mathcal{Z} = \int \prod_i \mathrm{d}\theta_i \exp\left[K \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)\right]. \tag{1.34}$$

(b) Quadratic approximation. Assuming the typical difference between neighbouring angles are small, the cosine can be Taylor expanded. Keeping terms up to second order gives

$$\mathcal{Z} = e^{KdL^d} \int \prod_i \mathrm{d}\theta_i \exp\left[-\frac{K}{2} \sum_{\langle ij \rangle} (\theta_i - \theta_j)^2\right] : \tag{1.35}$$

the prefactor is due to the zeroth-order term of the expansion (L^d is the number of lattice sites). By inspection of (1.35), the typical magnitude of $(\theta_i - \theta_j)^2$ will be $\mathcal{O}(1/K)$, that is, the quadratic approximation is appropriate as long as $K \gg 1$.

(c) Spin waves in a 1d chain. As discussed in the previous problem, the quadratic Hamiltonian (1.35) can be diagonalised by the Fourier transform (for simplicity, we follow the convention (1.22) and assume the lattice spacing is unity)

$$\theta_q = \frac{1}{\sqrt{L}} \sum_j \theta_j e^{-iqj} \iff \theta_j = \frac{1}{\sqrt{L}} \sum_q \theta_q e^{iqj} : \tag{1.36}$$

$$\begin{aligned} \beta H &= \frac{K}{2} \sum_j (\theta_j - \theta_{j+1})^2 = \frac{K}{2L} \sum_{j,q,q'} \theta_q e^{iqj} (1 - e^{iq}) \theta_{q'} e^{iq'j} (1 - e^{iq'}) \\ &= \frac{K}{2L} \sum_{q,q'} \theta_q \theta_{q'} (1 - e^{iq})(1 - e^{iq'}) \sum_j e^{i(q+q')j} = \frac{K}{2L} \sum_{q,q'} \theta_q \theta_{q'} (1 - e^{iq})(1 - e^{iq'}) L \delta_{q,-q'} \\ &= \sum_q \frac{K(q) |\theta_q|^2}{2} \end{aligned} \tag{1.37a}$$

where the dispersion relation $K(q)$ is

$$K(q) = K|1 - e^{iq}|^2 = 2K(1 - \cos q). \tag{1.37b}$$

(d) Spin waves in arbitrary dimensions. The derivation above generalises easily to $d > 1$ dimensional cubic (or other) lattices: the wave number q is replaced with a d-dimensional wave vector, and the Hamiltonian will have d terms per lattice site. All in all, we find

$$\beta H = \sum_{\mathbf{q}} \frac{K(\mathbf{q}) |\theta_{\mathbf{q}}|^2}{2} \tag{1.38a}$$

where the dispersion relation in a d-dimensional cubic lattice is

$$K(\mathbf{q}) = 2K \sum_{\alpha=1}^{d} (1 - \cos q_\alpha). \tag{1.38b}$$

Later, it will be important that $K(\mathbf{q}) \approx K\mathbf{q}^2$ for small $|\mathbf{q}|$.

(e) Specific heat. We first evaluate the partition function the same way as in Problem 2(d). By following the steps of the derivation of (1.27), we get

$$\mathcal{Z} = e^{KdL^d} \prod_{\mathbf{q}} \sqrt{\frac{2\pi}{K(\mathbf{q})}}. \tag{1.39}$$

The free energy follows from (1.39) as

$$F = -T \log \mathcal{Z} = T \left(-KdL^d + \sum_{\mathbf{q}} \frac{1}{2} \log \frac{K(\mathbf{q})}{2\pi} \right) \equiv -KTdL^d + \frac{1}{2} \int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d} \log \frac{K(\mathbf{q})}{2\pi}. \tag{1.40}$$

In a microscopic model, $K = \beta J$ where J is the energy cost associated with misaligned spins, and so

$$F = -JdL^d + \frac{TL^d}{2} \int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d} \left(\log \frac{J(\mathbf{q})}{2\pi} - \log T \right) = -L^d \left[\frac{T \log T}{2} + T \times \text{const.} + \text{const.} \right] \tag{1.41}$$

$$C = -T \frac{\partial^2 F}{\partial T^2} = \frac{L^d}{2} \tag{1.42}$$

which is indeed the equipartition heat capacity for L^d classical degrees of freedom.[6]

(f) Correlators, Mermin–Wagner. Since the theory governing $\theta_{\mathbf{r}} - \theta_0$ is Gaussian,

$$\langle \mathbf{S}_0 \cdot \mathbf{S}_{\mathbf{r}} \rangle = \langle \cos(\theta_{\mathbf{r}} - \theta_0) \rangle = \langle \mathrm{Re}\, e^{i(\theta_{\mathbf{r}} - \theta_0)} \rangle = \mathrm{Re} \langle e^{i(\theta_{\mathbf{r}} - \theta_0)} \rangle = \exp \left(-\frac{1}{2} \langle (\theta_{\mathbf{r}} - \theta_0)^2 \rangle \right). \tag{1.43}$$

To find the expectation value inside the exponential, we first calculate the momentum space correlators $\langle \theta_{\mathbf{q}} \theta_{\mathbf{q}'} \rangle$: since the Hamiltonian (1.37) is diagonal in these variables, we find

$$\langle \theta_{\mathbf{q}} \theta_{\mathbf{q}'} \rangle = \frac{\delta_{\mathbf{q},-\mathbf{q}'}}{K(\mathbf{q})} \tag{1.44}$$

$$\langle (\theta_{\mathbf{r}} - \theta_0)^2 \rangle = \frac{1}{L^d} \sum_{\mathbf{q},\mathbf{q}'} (e^{i\mathbf{q}\cdot\mathbf{r}} - 1)(e^{i\mathbf{q}'\cdot\mathbf{r}} - 1) \langle \theta_{\mathbf{q}} \theta_{\mathbf{q}'} \rangle = \frac{1}{L^d} \sum_{\mathbf{q}} \frac{(e^{i\mathbf{q}\cdot\mathbf{r}} - 1)(e^{i\mathbf{q}'\cdot\mathbf{r}} - 1)}{K(\mathbf{q})}$$

$$\equiv \int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d} \frac{2[1 - \cos(\mathbf{q}\cdot\mathbf{r})]}{K(\mathbf{q})} \tag{1.45}$$

In general, these integrals are hard to evaluate; however, if $|\mathbf{r}| \gg 1$, the cosine term in (1.45) will effectively average out the slow variations of $K(\mathbf{q})$, the only exception being $\mathbf{q} \approx 0$ where rate of change of the denominator, $|\nabla_{\mathbf{q}} K(\mathbf{q})|/K(\mathbf{q})$, diverges. Therefore, we can evaluate the integral focusing only the small $\mathbf{q}$ regime, where the approximation $K(\mathbf{q}) \approx K\mathbf{q}^2$ holds.

In one dimension, it is best to keep the two terms together:

$$\langle (\theta_r - \theta_0)^2 \rangle \approx \int_{-\pi}^{\pi} \frac{\mathrm{d}q}{2\pi} \frac{2[1 - \cos(qr)]}{K(q)} \approx \int_{-\pi|r|}^{\pi|r|} \frac{\mathrm{d}\xi/|r|}{2\pi} \frac{2(1 - \cos\xi)}{K \times (\xi/|r|)^2} \approx \frac{|r|}{\pi K} \underbrace{\int_{-\infty}^{\infty} \mathrm{d}\xi \frac{1 - \cos\xi}{\xi^2}}_{\pi} = \frac{|r|}{K} \tag{1.46a}$$

$$\langle \mathbf{S}_r \cdot \mathbf{S}_0 \rangle = e^{-|r|/2K}. \tag{1.46b}$$

That is, spin correlations decay with correlation length $2K$, so no ferromagnetism may survive at any finite temperature. This is a manifestation of the Mermin–Wagner theorem.

In more than two dimensions, the integral of the first term,

$$\int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d} \frac{2}{K(\mathbf{q})}$$

[6] In evaluating the $T \log T$ integral, we used that the volume of the Brillouin zone is $(2\pi)^d$.

is regular for small $\mathbf{q}$, therefore, it gives a finite additive term. The cosine in the second term can be replaced by $e^{i\mathbf{q}\cdot\mathbf{r}}$ as the denominator is symmetric in $\pm\mathbf{q}$: by applying the short wave vector approximation, we have to evaluate the Fourier transform

$$\int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d} \frac{2e^{i\mathbf{q}\cdot\mathbf{r}}}{K\mathbf{q}^2} = -\frac{2}{K} \frac{|\mathbf{r}|^{2-d}}{(2-d)S_d}$$

according to the hint in the problem sheet. In three dimensions in particular, this results in

$$\left\langle (\theta_{\mathbf{r}} - \theta_0)^2 \right\rangle = \text{const.} + \frac{2}{K} \frac{1}{4\pi|\mathbf{r}|} \tag{1.47a}$$

$$\lim_{|\mathbf{r}|\to\infty} \langle \mathbf{S}_{\mathbf{r}} \cdot \mathbf{S}_0 \rangle = \text{const.}, \tag{1.47b}$$

that is, the three-dimensional XY lattice is ferromagnetic on low enough temperatures where the quadratic approximation (1.35) holds.

The case of two dimensions can be treated as a limit of $d > 2$. In this limit, $|\mathbf{r}|^{2-d}/(2-d)$ will scale with $|\mathbf{r}|$ as $\log|\mathbf{r}|$:[7] therefore,

$$\left\langle (\theta_{\mathbf{r}} - \theta_0)^2 \right\rangle = \text{const.} + \frac{2}{K} \frac{\log \mathbf{r}}{S_2} = \text{const.} + \frac{\log|\mathbf{r}|}{K\pi} \tag{1.48a}$$

$$\langle \mathbf{S}_{\mathbf{r}} \cdot \mathbf{S}_0 \rangle \propto |\mathbf{r}|^{-1/2\pi K}. \tag{1.48b}$$

Thus there is no ferromagnetic order in two dimensions either, but at low temperatures, spin correlations decay as a power law rather than exponentially. This quasi-long range order is a special case of the Mermin–Wagner theorem, and has a central importance in the theory of topological phase transitions (see Chapter 4).

Warning. Nothing in principle stops us from evaluating the expectation value of the direction of an individual spin: by the same machinery,

$$\langle S_{0,x} + iS_{0,y} \rangle = \langle e^{i\theta_0} \rangle = \exp\left(-\frac{1}{2}\langle\theta_0^2\rangle\right) = \exp\left(-\frac{1}{2}\int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d} \frac{1}{K(\mathbf{q})}\right)$$

which is a finite positive number in dimensions greater than two. This suggests that each individual spin is preferentially oriented along the $\theta = 0$ axis: this is clearly wrong because the Hamiltonian is invariant under rotating all spins at once! One could explain this away by the fact that $K(\mathbf{q} = 0) = 0$ and thus the corresponding term in the discrete lattice sum is infinite, but that's not quite right either. In fact, we were already wrong in assuming that the angles θ are themselves Gaussian distributed: as only their difference enters any Hamiltonian discussed in this problem, this is not true! The marginal distribution of each $\theta_{\mathbf{r}}$ is uniform (and so is that of objects like $\theta_0 + \theta_{\mathbf{r}}$, &c.), it is only relative angles (e.g. $\theta_{\mathbf{r}} - \theta_0$) that have a nontrivial distribution. This is a subtle point that can easily lead to errors when dealing with such Hamiltonians.

[7] In fact, the limit also contains a part diverging as $1/(2-d)$: it is absorbed by the corresponding limit of the integral of $1/(K\mathbf{q}^2)$ up to $|\mathbf{q}| \approx \pi$ which has the same divergence as $d \to 2$.

Section II

Scaling and Renormalisation Group

Problem 1: Migdal–Kadanoff method

In this problem, we carry out a real space RG procedure on the 1d Ising model

$$\mathcal{Z} = \sum_{\{\sigma_i=\pm1\}} e^{-\beta H[\sigma_i]}, \qquad \beta H = -\sum_{\langle ij\rangle}\left[J\sigma_i\sigma_j + \frac{h}{2}(\sigma_i+\sigma_j) + g\right]. \tag{2.1}$$

We are going to proceed by explicitly summing over every other σ_i and express $\mathcal{Z}$ solely in terms of the remaining spins. We will find that the new, decimated partition function is still of the form (2.1), but the parameters J, h, g are renormalised.

(a) Effective interaction between remaining spins. For the sake of simplicity, we denote spins on even-numbered sites by τ_i and spins on odd-numbered sites by σ'_i. In the 1d chain, each nearest-neighbour bond connects a τ spin to a σ' spin, therefore, we can associate these bonds with their τ endpoint. Since τ spins do not have bonds among each other, we can factorise the partition function for each configuration of σ'_i spins as

$$\begin{aligned}
\mathcal{Z} &= \sum_{\{\sigma'_i=\pm1\}} \sum_{\{\tau_i=\pm1\}} \exp\left[\sum_i \sum_{d=\leftrightarrows}\left(J\tau_i\sigma'_{i,d} + \frac{h}{2}(\tau_i+\sigma'_{i,d}) + g\right)\right] \\
&= \sum_{\{\sigma'_i=\pm1\}} \prod_i \left[\sum_{\tau_i=\pm1} \exp\left(J\tau_i\sigma'_{i\leftarrow} + \frac{h}{2}(\tau_i+\sigma'_{i\leftarrow}) + g + J\tau_i\sigma'_{i\rightarrow} + \frac{h}{2}(\tau_i+\sigma'_{i\rightarrow}) + g\right)\right] \\
&= \sum_{\{\sigma'_i=\pm1\}} \prod_{\langle ij\rangle} \left[\sum_{\tau_{ij}=\pm1} \exp\left(J\tau(\sigma'_i+\sigma'_j) + \frac{h}{2}(\sigma'_i+\sigma'_j) + h\tau + 2g\right)\right].
\end{aligned} \tag{2.2}$$

In the first step, we note that the Hamiltonian can be partitioned by τ spins ($\sigma'_{i\leftrightarrows}$ denote the σ' spins to the left and right of τ_i). Since these partitions simply add without any further interaction, the sum over all τ's can be reduced to a product of single τ sums. In the last step, we reinterpret these sums as belonging to effective bonds between σ' spins: the rationale is that each τ is between two neighbouring σ', and so one can write down a form of the partition function containing only σ'_i, and local variables τ which can be integrated out (or in this case, summed over).

The partition function (2.2) can be written in the form (2.1) provided there are parameters J', h', g' such that

$$\exp\left(J'\sigma'_i\sigma'_j + \frac{h'}{2}(\sigma'_i+\sigma'_j) + g'\right) = \sum_{\tau=\pm1} \exp\left(J\tau(\sigma'_i+\sigma'_j) + \frac{h}{2}(\sigma'_i+\sigma'_j) + h\tau + 2g\right): \tag{2.3}$$

for all possible values of $\sigma'_{i,j}$. In this case, the sum in brackets in (2.2) is replaced by a single exponential, and the product of exponentials turns into the exponential of a sum. We now demonstrate that such parameters indeed exist.

(b) Renormalising the parameters. (2.3) gives three distinct equations for J', h', g':

$$\sigma_i = \sigma_j = +1 \implies e^{J'+h'+g'} = e^{2J+2h+2g} + e^{-2J+2g} = 2e^{2g+h}\cosh(2J+h); \tag{2.4a}$$

$$\sigma_i \times \sigma_j = -1 \implies e^{-J'+g'} = e^{h+2g} + e^{-h+2g} = 2e^{2g}\cosh(h); \tag{2.4b}$$

$$\sigma_i = \sigma_j = -1 \implies e^{J'-h'+g'} = e^{-2J+2g} + e^{2J-2h+2g} = 2e^{2g-h}\cosh(2J-h). \tag{2.4c}$$

Dividing (2.4a) by (2.4c) gives

$$e^{2h'} = e^{2h}\frac{\cosh(2J+h)}{\cosh(2J-h)} \implies h' = h + \frac{1}{2}\log\left(\frac{\cosh(2J+h)}{\cosh(2J-h)}\right). \tag{2.5a}$$

Taking the combination (2.4a) × (2.4c)/(2.4b)^2 gives

$$e^{4J'} = \frac{\cosh(2J+h)\cosh(2J-h)}{\cosh^2(h)} \implies J' = \frac{1}{4}\log\left(\frac{\cosh(2J+h)\cosh(2J-h)}{\cosh^2(h)}\right). \tag{2.5b}$$

Finally, taking (2.4a) × (2.4c) × (2.4b)^2 gives

$$\begin{aligned} e^{4g'} &= 16e^{4g}\cosh(2J+h)\cosh(2J-h)\cosh^2(h) \\ \implies g' &= g + \frac{1}{4}\log\big(16\cosh(2J+h)\cosh(2J-h)\cosh^2(h)\big). \end{aligned} \tag{2.5c}$$

These are all of the form required by the problem text.

Note. It is easy to see that the same protocol could be carried out retaining only every bth spin for arbitrary integer b: σ' nearest neighbours would be separated by $(b-1)$ τ spins to be integrated out, but still, the nearest neighbour interaction structure would be retained. Furthermore, the two $(+1,-1)$ cases would still be equivalent by the space reversal symmetry of the chain, therefore, one would in general end up with three equations for the three renormalised parameters, which could then be solved in a similar fashion to the above derivation. The only advantage of using $b=2$ is that the sum over τ's in (2.3) only contains two terms rather than 2^{b-1} which would soon become cumbersome to enumerate.

(c) RG flow at zero field. If $h = 0$, $h' = \delta h = \frac{1}{2}\log[\cosh(2J)/\cosh(2J)] = 0$, that is, no magnetic field arises from the RG transformation. (This is reassuring, since h would break the inversion symmetry of the problem, not something we expect from RG.)

From (2.5b), we then find

$$J' = \frac{1}{4}\log\left(\frac{\cosh(2J)\cosh(2J)}{1^2}\right) = \frac{1}{4}\log\left[\cosh(2J)^2\right] = \frac{1}{2}\log\cosh(2J), \tag{2.6}$$

as stated in the problem text. Assuming the model is ferromagnetic (that is, $J>0$), $e^{-2J} < e^{2J}$, and so

$$J' = \frac{1}{2}\log\cosh(2J) = \frac{1}{2}\log\frac{e^{2J}+e^{-2J}}{2} < \frac{1}{2}\log e^{2J} = J.$$

That is, J always decreases upon the RG transformation. This leaves two possible fixed points:

- $J=0$ is a fixed point as $\log\cosh 0 = \log 1 = 0$. It is also stable as a small but finite J reduces by the RG transformation, so the RG flow is towards $J=0$.
- $J=\infty$ can also be regarded as fixed point. It is, however, unstable, since the RG flow for large finite J flows away from it.

The structure of the RG flow implies that unless J is infinite (that is, $T=0$), the 1d Ising model is never in ferromagnetic order.

Note. It is interesting to consider the RG flow for negative J, that is, antiferromagnetic coupling. By (2.6), J turns positive in the first RG step and then follows the same RG flow as the ferromagnetic model. This is not surprising: if the chain were AF ordered, next-nearest neighbours would preferentially align, that is, the chain would appear ferromagnetic if we only considered every other site.

(d) Low temperature expansion. We now consider the case of large J and small h, that is, the neighbourhood of the unstable fixed point. Using that $\cosh(x) \approx e^x/2$ for large x and $\cosh x \approx 1$ for small x, we find

$$e^{-J'} \approx \left(\frac{e^{2J+h}}{2}\frac{e^{2J-h}}{2}\right)^{-1/4} = \sqrt{2}e^{-J} \tag{2.7a}$$

$$h' \approx h + \frac{1}{2}\log\frac{e^{2J+h}/2}{e^{2J-h}/2} = h + \frac{1}{2}\log e^{2h} = 2h, \tag{2.7b}$$

as stated in the problem text.

(e) Scaling of the correlation length. Upon renormalisation, we throw out every other spin, therefore, every distance along the chain is halved. Therefore, the renormalised correlation length is half that of the original one:

$$\begin{aligned} \xi' &= \xi/2 \\ \xi = \xi(e^{-J}, h) &= 2\xi' = 2\xi(\sqrt{2}e^{-J}, 2h). \end{aligned} \tag{2.8a}$$

By iterating this procedure ℓ times, we get

$$\xi(e^{-J}, h) = 2^\ell \xi(2^{\ell/2}e^{-J}, 2^\ell h). \tag{2.8b}$$

The rest of the problem text is a discussion of η in this problem, rather than a question to be solved.

Problem 2: Lifshitz point

In lectures, Gaussian and perturbative RG was discussed for the isotropic quartic Ginzburg–Landau Hamiltonian. This problem generalises the same method to the case of an anisotropic Hamiltonian: for now, we think of it as describing a smectic liquid crystal, but such Hamiltonians often show up in the context of quantum phase transitions (see Chapter 5 of the handout) where (imaginary) time generally behaves differently from spatial coordinates.

(a) Fourier transform of the quadratic Hamiltonian. Substituting the Fourier transform

$$m(q_\parallel, \mathbf{q}_\perp) = \int \mathrm{d}r_\parallel \int \mathrm{d}\mathbf{r}_\perp m(\mathbf{r}) e^{-i\mathbf{q}\cdot\mathbf{r}} \iff m(r_\parallel, \mathbf{r}_\perp) = \int \frac{\mathrm{d}q_\parallel}{2\pi} \int \frac{\mathrm{d}\mathbf{q}_\perp}{(2\pi)^{d-1}} m(\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{r}} \tag{2.9}$$

into the given Hamiltonian, it is straightforward to obtain (see the solution of Problem I/2(c) for details)

$$\beta H_0 = \frac{1}{2} \int \frac{\mathrm{d}q_\parallel}{2\pi} \int \frac{\mathrm{d}\mathbf{q}_\perp}{(2\pi)^{d-1}} \left[t + K q_\parallel^2 + L \mathbf{q}_\perp^4 \right] |m(\mathbf{q})|^2 - h m(0). \tag{2.10}$$

For brevity, we introduce the propagator $G^{-1}(\mathbf{q}) = t + K q_\parallel^2 + L \mathbf{q}_\perp^4$. It is important to note that $m(0)$ denotes the Fourier mode at $\mathbf{q} = 0$, that is, the integral of m over all space.

(b) Gaussian RG. Renormalising a Gaussian field theory consists of two steps: coarse-graining by integrating out high wave number modes, and renormalising distances such that the highest wave numbers retained map to the original cutoffs. Due to the anisotropy, the necessary renormalisation along the parallel and the perpendicular directions will, in general, be different.

Coarse-graining. Assume that the highest physical wave vectors in the system are Λ along the parallel direction and λ along the perpendicular directions. In the coarse-graining process, we separate Fourier modes $m(\mathbf{q})$ into slow and fast ones:

$$m(\mathbf{q}) = \begin{cases} m_<(\mathbf{q}): & |q_\parallel| < \Lambda/b \quad \text{and} \quad |\mathbf{q}_\perp| < \lambda/c \\ m_>(\mathbf{q}): & |q_\parallel| > \Lambda/b \quad \text{or} \quad |\mathbf{q}_\perp| > \lambda/c \end{cases} \tag{2.11}$$

where b and c are constants to be determined. Since the quadratic Hamiltonian is separable in $\mathbf{q}$ modes, integrating out the fast modes only gives a constant prefactor:

$$\mathcal{Z} = \mathcal{Z}_> \int \mathcal{D}m_<(\mathbf{q}) \exp\left[-\frac{1}{2} \int^{\Lambda/b} \frac{\mathrm{d}q_\parallel}{2\pi} \int^{\lambda/c} \frac{\mathrm{d}\mathbf{q}_\perp}{(2\pi)^{d-1}} G^{-1}(\mathbf{q}) |m(\mathbf{q})|^2 + h m(0) \right]. \tag{2.12}$$

Since $\mathcal{Z}_>$ is just a constant, we can ignore it in discussing the critical behaviour.

Renormalisation. In order to recover the original theory, we must rescale the wave vector $\mathbf{q}$ such that we restore the original cutoffs. This is achieved by writing $q'_\parallel = b q_\parallel$, $\mathbf{q}'_\perp = c \mathbf{q}_\perp$. We also renormalise the magnetic field as $m'(\mathbf{q}') = m_<(\mathbf{q})/z$. Plugging all this into the remaining functional integral in (2.12) gives

$$\mathcal{Z} = \mathcal{Z}_> \int \mathcal{D}m'(\mathbf{q}') \exp\left[-\frac{1}{2} \int^{\Lambda} \frac{\mathrm{d}q'_\parallel}{2\pi b} \int^{\lambda} \frac{\mathrm{d}\mathbf{q}'_\perp}{(2\pi c)^{d-1}} \left[t + K q_\parallel^2 b^{-2} + L \mathbf{q}_\perp^4 c^{-4} \right] z^2 |m'(\mathbf{q}')|^2 - h z m'(0) \right]. \tag{2.13}$$

The renormalised coefficients can be read off from comparing (2.13) to (2.10):

$$t' = t b^{-1} c^{1-d} z^2; \tag{2.14a}$$

$$K' = K b^{-3} c^{1-d} z^2; \tag{2.14b}$$

$$L' = L b^{-1} c^{-3-d} z^2; \tag{2.14c}$$

$$h' = h z. \tag{2.14d}$$

Pedantry break. In the previous derivation, we simply wrote $\int \mathcal{D}m_<(\mathbf{q}) = \int \mathcal{D}m'(\mathbf{q}')$ without worrying about the renormalisation of m or $\mathbf{q}'$ in the metric itself. While we usually fudge away everything about functional integration metrics (indeed, they cannot be defined rigorously for continuum fields), it is instructive to go back to the lattice picture once to see why it is OK to do so.

The RG step reduces the number of $\mathbf{q}$ modes to be integrated over by a factor of bc^{d-1}, but they are still spread over the original domain of $\mathbf{q}$. This is taken into account explicitly by writing the momentum integrals in (2.13) in terms of $q'_\parallel/b$, $\mathbf{q}'_\perp/c$: rewriting (2.13) as a lattice product would reproduce (2.12) mode for mode.

However, each $\mathbf{q}$ mode of the original lattice gives rise to its own $\mathrm{d}m(\mathbf{q})$ integral: these measures are renormalised by the renormalisation of m, collecting a factor of z per mode. These factors can be taken inside the exponential and the $\mathbf{q}$ sum in it, adding a constant $\log z$ to the $\mathbf{q}$ integral in the free energy. Such additive constants can in general be ignored.

(c) Gaussian critical exponents. In general, b, c and z could be chosen arbitrarily, giving rise to an arbitrary RG transformation. To choose these coefficients appropriately, we must insist that the coefficients governing long-range fluctuations of the order parameter are kept constant.[8] In our case, these coefficients are K and L. By requiring $K' = K$ and $L' = L$ in (2.14), we obtain

$$c = b^{1/2}; \qquad z = b^{(d+5)/4}. \tag{2.15}$$

Substituting these values into the renormalisation of t and h gives

$$\begin{aligned} t' &= tb^{-1}b^{(1-d)/2}b^{2(d+5)/4} = tb^2 & h' &= hb^{(d+5)/4} \\ y_t &= 2 & y_h &= \frac{d+5}{4}. \end{aligned} \tag{2.16}$$

(d) Scaling of the free energy, the critical exponents α and δ. The singular part of the free energy is unchanged upon renormalisation,[9] therefore, the free energy density scales as the inverse of the volume:

$$\begin{aligned} F = Vf(t,h) &= \frac{V}{bc^{d-1}} f(t',h') = F' \\ f(t,h) &= b^{-1}c^{1-d} f(t',h') = b^{-(d+1)/2} f(b^2 t, b^{(d+5)/4} h). \end{aligned} \tag{2.17}$$

To obtain a univariate scaling function for f, we choose b such that $b^2 t = 1$, and define $g_f(h) = f(1,h)$:

$$f(t,h) = t^{(d+1)/4} g_f(ht^{-(d+5)/8}) : \tag{2.18}$$

from this form, we can read off the critical exponents

$$2 - \alpha = \frac{d+1}{4}; \qquad \delta = \frac{d+5}{8}. \tag{2.19}$$

(e) Scaling of the susceptibility, the critical exponent γ. From the Hamiltonian (2.10), the magnetisation is given by $m = \partial f/\partial h$, while the susceptibility is defined as $\chi = \partial m/\partial h$. From (2.18),

$$\chi = \frac{\partial^2 f}{\partial h^2} = t^{(d+1)/4} \left(t^{-(d+5)/8}\right)^2 g''_f(ht^{-(d+5)/8}) = t^{-1} g_\chi(ht^{-(d+5)/8}). \tag{2.20}$$

At $h = 0$, $g_\chi(0)$ is just a constant, and so $\chi \propto t^{-1} \implies \gamma = 1$.

[8] Almost. On the longest length scales, $\mathbf{q} \to 0$, only t is relevant; however, we must insist that t be a relevant RG parameter, that is, small negative and positive t flow to different limits under the RG transformation. This ensures the qualitative change of behaviour at the phase transition.

[9] We do discard contributions at high wave vectors, but these don't contribute to the singular behaviour.

(f) Gaussian magnetisation correlations. The magnetisation correlator $\langle m(\mathbf{q})m(\mathbf{q}')\rangle$ can be read off directly from (2.10) (see solutions for Section 1):

$$\langle m(\mathbf{q})m(\mathbf{q}')\rangle = \frac{(2\pi)^d\delta(\mathbf{q}+\mathbf{q}')}{t + Kq_\parallel^2 + L\mathbf{q}_\perp^4}. \tag{2.21}$$

The generalised susceptibility $\chi(\mathbf{q})$ is the response of the system to an external magnetic field with wave vector $\mathbf{q}$:

$$\beta H = \beta H_0 - h(\mathbf{q})m(-\mathbf{q}) - h(-\mathbf{q})m(\mathbf{q}) \implies \chi(\mathbf{q}) = \frac{\langle \partial m(\mathbf{q})\rangle}{\partial h(\mathbf{q})}. \tag{2.22}$$

The form of the Hamiltonian follows from Fourier transforming $\int \mathrm{d}\mathbf{r} h\cos(\mathbf{q}\cdot\mathbf{r})m$. In our case, the unperturbed Hamiltonian contains the Fourier term $G^{-1}(\mathbf{q})m(\mathbf{q})m(-\mathbf{q})$ (we added the $\pm\mathbf{q}$ terms): together with the additional terms in (2.22), it can be written as $G^{-1}(\mathbf{q})[m(\mathbf{q}) - G(\mathbf{q})\chi(\mathbf{q})][m(-\mathbf{q}) - G(\mathbf{q})h(-\mathbf{q})]$ (this adds a Ghh term too, but as h is set externally, we can ignore that). That is, the only change to the physics is shifting $m(\pm\mathbf{q})$ by $G(\mathbf{q})h(\pm\mathbf{q})$: the susceptibility is therefore

$$\chi(\mathbf{q}) = G(\mathbf{q}) = \frac{1}{t + Kq_\parallel^2 + L\mathbf{q}_\perp^4}. \tag{2.23}$$

In statistical physics, this relation between fluctuation correlators and response functions is known as the *fluctuation–dissipation theorem*. Note that (2.23) is consistent with (2.20).

(g) Quartic perturbation in Fourier space.

$$\begin{aligned} U &= u\int \mathrm{d}\mathbf{r}\, m^4 \\ &= u\int \mathrm{d}\mathbf{r} \int \frac{\mathrm{d}\mathbf{q}_1}{(2\pi)^d} m(\mathbf{q}_1)e^{-i\mathbf{q}_1\cdot\mathbf{r}} \int \frac{\mathrm{d}\mathbf{q}_2}{(2\pi)^d} m(\mathbf{q}_2)e^{-i\mathbf{q}_2\cdot\mathbf{r}} \int \frac{\mathrm{d}\mathbf{q}_3}{(2\pi)^d} m(\mathbf{q}_3)e^{-i\mathbf{q}_3\cdot\mathbf{r}} \int \frac{\mathrm{d}\mathbf{q}_4}{(2\pi)^d} m(\mathbf{q}_4)e^{-i\mathbf{q}_4\cdot\mathbf{r}} \\ &= u\int (\mathrm{d}\mathbf{Q}) m(\mathbf{q}_1)m(\mathbf{q}_2)m(\mathbf{q}_3)m(\mathbf{q}_4) \int \mathrm{d}\mathbf{r} e^{-i(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3+\mathbf{q}_4)\cdot\mathbf{r}} \\ &= u\int (\mathrm{d}\mathbf{Q}) m(\mathbf{q}_1)m(\mathbf{q}_2)m(\mathbf{q}_3)m(\mathbf{q}_4) \times (2\pi)^d\delta(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3+\mathbf{q}_4) \end{aligned} \tag{2.24}$$

where $\int(\mathrm{d}\mathbf{Q})$ is a shorthand for the four momentum space integrals. In principle, we could integrate over one of the $\mathbf{q}$, however, this more symmetric form tends to be easier to deal with.

Wick's theorem and diagrammatical perturbation theory. The next part of the problem will ask us to evaluate the correlator $\langle m(\mathbf{q})m(\mathbf{q}')\rangle$ under the perturbed Hamiltonian $\beta H_0 + U$. To do so, we use Wick's theorem (also known as Isserlis' theorem): Given a number of (multivariate) Gaussian distributed variables with zero mean, the expectation value of a large product of them is given by the product of pairwise expectation values, summed over all possible pairings of the variables. These pairings can be conveniently represented by Feynman diagrams, where the Gaussian variables [in our case, $m(\mathbf{q})$] are represented by "legs", connected pairwise by lines in all possible ways. Additional terms introduced by interactions can be represented using vertices with associated Feynman rules.

In our case, every instance of U can be represented as a node with four outgoing edges.[10] When evaluating perturbations to an operator O, the $m(\mathbf{q})$ making it up appear as nodes with a single outgoing edge. All outgoing edges correspond to a particular $\mathbf{q}$-component of m: for single-edge

[10] If the interaction were rth order, U nodes would have r outgoing edges. If the order parameter were a vector, $\mathbf{m}^4$ would stand for $m^\alpha m^\alpha m^\beta m^\beta$: The contractions over components can be represented by two two-edge nodes, usually connected by a dashed line. In quantum field theory, one encounters vertices connecting several different fields with rather elaborate Feynman rules.

nodes, $\mathbf{q}$ is fixed by the definition of O, for U nodes, $\mathbf{q}_i$ are arbitrary as long as their sum is 0. This requirement can be enforced by associating the Feynman rule $u\int(\mathrm{d}\mathbf{Q})\,(2\pi)^d\delta(\Sigma\mathbf{q})$ with the node itself.

Wick's theorem is now equivalent to finding all possible ways of connecting edges in pairs. Each such diagram gives rise to a term in the expectation value as follows: multiply correlators corresponding to each connected pair of edges and any terms associated with nodes, and integrate over any free momenta $\mathbf{q}$ introduced by U nodes. Besides the obvious simplification of enumerating pairings graphically, this method also exploits the symmetries of the terms in the result. For example, permuting the labelling the outgoing edges of U nodes give rise to a number of equal terms which in principle all appear in the Wick's theorem expansion. Those can be represented by a single diagram together with a combinatorial factor giving the number of equivalent terms in the expansion; by way of example, the multiplicities of first and second order diagrams perturbing $\langle m(\mathbf{q})m(\mathbf{q}')\rangle$ are evaluated in Fig. 2.1.

Perturbed operator expectation values. In lectures, we were given the following expression for the expectation value of an operator O under the perturbed Hamiltonian $\beta H_0 + U$:

$$\langle O\rangle_{\beta H_0+U} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\langle OU^n\rangle_{0,c} \tag{2.25}$$

where the subscript 0 refers to the quadratic Hamiltonian βH_0 and the subscript c requires us to only include connected Feynman diagrams (that is, those which cannot be separated into two smaller diagrams not connected by an edge). While it is clear what is meant by this requirement, it is in order to give an argument for discarding disconnected diagrams. Consider the definition of $\langle O\rangle_{\beta H_0+U}$

$$\begin{aligned}\langle O\rangle_{\beta H_0+U} &= \frac{\int\mathcal{D}m(\mathbf{q})\,O[m]e^{-(\beta H_0+U)[m]}}{\int\mathcal{D}m(\mathbf{q})\,e^{-(\beta H_0+U)[m]}} = \frac{\int\mathcal{D}m(\mathbf{q})\,O(1-U+U^2/2!-\dots)e^{-\beta H_0}}{\int\mathcal{D}m(\mathbf{q})\,(1-U+U^2/2!-\dots)e^{-\beta H_0}}\\ &= \frac{\langle O\rangle-\langle OU\rangle+\langle OU^2\rangle/2!-\dots}{1-\langle U\rangle+\langle U^2\rangle/2!-\dots},\end{aligned} \tag{2.26}$$

where the expectation values in the last line are taken with respect to βH_0.

Each term of the numerator can be represented by a Feynman diagram which consists of the single-edge vertices corresponding to O and any number of interaction vertices due to U, and which may or may not be connected. In either case, the diagram can be separated into two components, one connected to O vertices and one without any O vertices (which may be empty). The first component will appear in the cumulant expansion in (2.25), while the second (the *vacuum bubble*) appears in the denominator of (2.26). However, if the connected and bubble components consist of n and m vertices, respectively, the corresponding diagram appears with a prefactor $1/(n+m)!$ in the numerator of (2.26) and with prefactor $1/(n!m!)$ in the denominator and the cumulant expansion. This difference is remedied by the increased multiplicity of the diagram in the numerator due to the distribution of the $n+m$ vertices between the two parts of the diagram, giving an additional combinatorial factor $\binom{n+m}{n} = (n+m)!/(n!m!)$. All in all,

$$\langle O\rangle-\langle OU\rangle+\langle OU^2\rangle/2!-\dots = \big(\langle O\rangle_c-\langle OU\rangle_c+\langle OU^2\rangle_c/2!-\dots\big)\big(1-\langle U\rangle+\langle U^2\rangle/2!-\dots\big);$$

together with (2.26), we indeed obtain (2.25).

A caveat for the connected diagram prescription arises when evaluating higher order correlators. These will contain diagrams in which not all O vertices are connected to each other: for example, two copies of the "fish" diagram in Fig. 2.1 are a valid contribution to $\langle mmmm\rangle$, and in principle, we should include them in the expansion in (2.25). However, such diagrams enforce additional momentum conservation constraints (e.g., the "double fish" diagram forces both pairs of connected momenta to sum to zero). In perturbation theory, it is usually safe to ignore special cases like this; in perturbative RG, they give rise to nonlocal couplings like $\left[\int\mathrm{d}\mathbf{r}\,m^2\right]^2$ which tend to be irrelevant. Therefore, we can safely focus on fully connected diagrams.

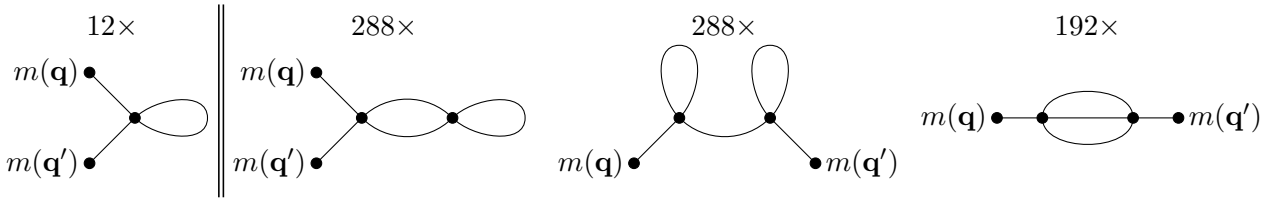

Figure 2.1. All connected diagrams of $\langle m(\mathbf{q})m(\mathbf{q}')\rangle$ in first and second order perturbation theory.
The only connected first-order diagram has both m nodes connected to the single U node: the edges connected to $m(\mathbf{q})$ and $m(\mathbf{q}')$ can be chosen of 4 and 3 respectively, thus there is a total of $4\times 3 = 12$ equivalent diagrams.
On the second diagram, the U node connected to the m nodes can be chosen out of 2; the connective edges can be chosen in 12 ways; by the same argument, the edges of the second node connecting it to the first can be chosen in 12 ways: a total of $2\times 12\times 12 = 288$ diagrams.
On the third diagram, there are 2 ways of matching m nodes to U nodes; on each U node, there are 4 ways of choosing the edge connecting to the m node, and 3 to the other U node: a total of $2\times(4\times 3)^2 = 288$ diagrams.
On the fourth diagram, there are 2 ways of matching m nodes to U nodes; on each U node, there are 4 ways of choosing the edge connecting to the m node; the remaining 3 nodes can be paired up in $3! = 6$ ways: a total of $2\times 4^2\times 6 = 192$ diagrams.

Perturbation to the free energy, vacuum bubbles. Later, in perturbative RG, we will have to evaluate the correction to the free energy due to the perturbation U. From the definition of the partition function, this is given by[11]

$$\mathcal{Z} = \int \mathcal{D}m\; e^{-(\beta H_0+U)[m]} = \mathcal{Z}_0 \frac{\int \mathcal{D}m\; e^{-U} e^{-\beta H_0}}{\int \mathcal{D}m\; e^{-\beta H_0}} = \mathcal{Z}_0 \times \langle e^{-U}\rangle_0$$
$$\beta F = \beta F_0 + \ln\langle e^{-U}\rangle_0. \tag{2.27}$$

In lectures, it was claimed that

$$\ln\langle e^{-U}\rangle_0 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\langle U^n\rangle_{0,c}. \tag{2.28}$$

To prove this, consider $\langle e^{-U}\rangle$ which is given by (2.28) without the subscripts c. In diagrammatic notation, this is equivalent all possible finite sets of connected bubble diagrams: consider one consisting of the diagrams C_i, each with n_i vertices and multiplicity m_i; clearly, $n = \sum_i n_i m_i$. Now, the number of ways to distribute the n interaction vertices to the diagrams is

$$\frac{n!}{\prod_i m_i!(n_i!)^{m_i}}:$$

without the $m_i!$, this is just a multinomial coefficient; the $m_i!$ are needed to properly count the identical diagrams which are not distinguishable. Combined with the prefactor $(-1)^n/n!$, the contribution of this diagram is

$$\prod_i \frac{1}{m_i!}\left(\frac{(-1)^{n_i}}{n_i!}C_i\right)^{m_i}, \tag{2.29}$$

where the product can be extended to connected diagrams with zero multiplicity. To obtain $\langle e^{-U}\rangle$ now, we have to sum (2.29) for all possible multiplicities m_i:

$$\langle e^{-U}\rangle = \prod_i \left[\sum_{m_i}\frac{1}{m_i!}\left(\frac{(-1)^{n_i}}{n_i!}C_i\right)^{m_i}\right] = \prod_i \exp\left(\frac{(-1)^{n_i}}{n_i!}C_i\right)$$
$$\ln\langle e^{-U}\rangle = \sum_i \frac{(-1)^{n_i}}{n_i!}C_i = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\langle U^n\rangle_{0,c}, \tag{2.30}$$

as required; in the last step, we simply group connected diagrams by their order.

[11] What happens in perturbative RG is slightly more elaborate due to the distinction between slow and fast modes; however, the idea and the argument for using connected diagrams is essentially the same.

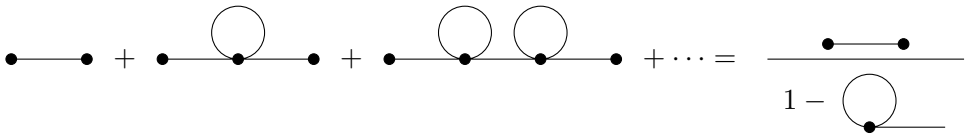

Figure 2.2. The random phase approximation.

(h) Perturbation corrections to correlators. We now evaluate the correlators $\langle m(\mathbf{q})m(\mathbf{q}')\rangle$ under the perturbed Hamiltonian $\beta H_0 + U$, using the perturbative expansion (2.25) to first order. The only such diagram is the leftmost "fish" diagram in Fig. 2.1: applying the Feynman rules discussed above, we get

$$\begin{aligned}
\delta\langle m(\mathbf{q})m(\mathbf{q}')\rangle &= -12\int(\mathrm{d}\mathbf{Q})\langle m(\mathbf{q})m(\mathbf{q}_1)\rangle_0 \times \langle m(\mathbf{q}')m(\mathbf{q}_2)\rangle_0 \times \langle m(\mathbf{q}_3)m(\mathbf{q}_4)\rangle_0 \times \\
&\qquad\qquad \times u(2\pi)^d\delta(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3+\mathbf{q}_4) \\
&= -12u\int(\mathrm{d}\mathbf{Q})G(\mathbf{q})(2\pi)^d\delta(\mathbf{q}+\mathbf{q}_1)\times G(\mathbf{q}')(2\pi)^d\delta(\mathbf{q}'+\mathbf{q}_2)\times G(\mathbf{q}_3)(2\pi)^d\delta(\mathbf{q}_3+\mathbf{q}_4)\times \\
&\qquad\qquad \times (2\pi)^d\delta(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3+\mathbf{q}_4) \\
&= -12u\,G(\mathbf{q})G(\mathbf{q}')(2\pi)^d\delta(\mathbf{q}+\mathbf{q}')\int\frac{\mathrm{d}\mathbf{q}_3}{(2\pi)^d}\,G(\mathbf{q}_3) \\
&= -12u\,G^2(\mathbf{q})\left[\int\frac{\mathrm{d}\mathbf{q}''}{(2\pi)^d}\,G(\mathbf{q}'')\right](2\pi)^d\delta(\mathbf{q}+\mathbf{q}')
\end{aligned} \tag{2.31}$$

where we have used that $G(\mathbf{q}) = G(-\mathbf{q})$.

(i) Renormalisation of χ and the transition point. From (2.23) and (2.31), the susceptibility $\chi(\mathbf{q})$ is given to first order by

$$\chi(\mathbf{q}) = G(\mathbf{q})\left[1 - 12uG(\mathbf{q})\int\frac{\mathrm{d}\mathbf{q}'}{(2\pi)^d}\,G(\mathbf{q}') + \dots\right] \tag{2.32}$$

Since this is a power series in u, it is straightforward to invert it: to first order, we find

$$\begin{aligned}
\chi^{-1}(\mathbf{q}) &= G^{-1}(\mathbf{q})\left[1 + 12uG(\mathbf{q})\int\frac{\mathrm{d}\mathbf{q}'}{(2\pi)^d}\,G(\mathbf{q}') + \dots\right] \\
&= G^{-1}(\mathbf{q}) + 12u\int\frac{\mathrm{d}\mathbf{q}'}{(2\pi)^d}\,G(\mathbf{q}') + \dots
\end{aligned} \tag{2.33}$$

Note. You are justified in finding this step a bit extreme. Indeed, if the expansion in (2.32) terminated after the first term, we wouldn't end up with a shift in transition point, whether or not we invert χ. However, the expansion (2.33) is a useful one: to get a feeling as to why, consider the series of diagrams in Fig. 2.2. It is easy to see that all horizontal lines correspond to the Gaussian correlator at the same wave vector, each loop introduces an integration over all $\mathbf{q}$, and each new loop introduces the same combinatorial factor; furthermore, the $1/n!$ associated with higher orders in (2.25) is cancelled by the number of ways the vertices can be ordered along the line. All in all, each diagram differs from the one before by a factor of $-12u\,G(\mathbf{q})\int(\mathrm{d}\mathbf{q}')G(\mathbf{q}')$ (this factor is represented by the partial Feynman diagram in the figure): the sum of the resulting geometrical series is precisely the first order result given in (2.33). Summing series of diagrams of the general structure in Fig. 2.2 is a common technique in field theory, where it is known as the *random-phase approximation (RPA)*.

Back to the problem. At the transition point, the long-wavelength susceptibility, $\chi(\mathbf{q}=0)$, is expected to diverge, that is, $\chi^{-1}(0) = 0$:

$$\chi^{-1}(0) = t + 12u\int\frac{\mathrm{d}\mathbf{q}'}{(2\pi)^d}\,G(\mathbf{q}') + \dots = 0.$$

$$t^* = 12u \int \frac{\mathrm{d}\mathbf{q}'}{(2\pi)^d} G(\mathbf{q}'); \tag{2.34}$$

more accurate estimates would follow from perturbative RG and/or higher-order perturbation theory.

(j) The upper critical dimension. In the previous parts of the problem, we treated the system as either Gaussian, or added interactions as a small perturbation. This picture is valid as long as the coefficient of the interaction, u, is an irrelevant parameter in the Gaussian RG: on large length scales then, u has no effect, and thus the Gaussian description is accurate. Writing the renormalisation rules into (2.24) gives

$$\begin{aligned} U_< &= u\left[\prod_{i=1}^{3}\int \frac{\mathrm{d}q_i'^{\parallel}}{2\pi b}\frac{\mathrm{d}\mathbf{q}_i'^{\perp}}{(2\pi c)^{d-1}}\right] z^4 m'(\mathbf{q}_1')m'(\mathbf{q}_2')m'(\mathbf{q}_3')m'(-\mathbf{q}_1'-\mathbf{q}_2'-\mathbf{q}_3') \\ \implies u' &= ub^{-3}c^{-3(d-1)}z^4 = ub^{(7-d)/2}: \end{aligned} \tag{2.35}$$

in the first line, we performed the integration over $\mathbf{q}_4$ explicitly to avoid dealing with the renormalisation of the δ function. Eq. 2.35 gives that if $d > 7$, u renormalises down, thus the Gaussian theory is valid. That is, the upper critical dimension of this model is $d^* = 7$.

Section III

Perturbative renormalisation group

Problem 1: Perturbative RG of the 2d XY model

In this problem, we revisit the low-temperature expansion of the two-dimensional XY model, first introduced in Problem I/3. In the expansion of the cosine dependence of the energy, we now keep the quartic as well as the quadratic term: the argument is still the difference in XY angles between neighbouring spins, $\phi_i - \phi_j$, which can be coarse-grained into $\nabla\phi$. All in all, we consider the following Hamiltonian:

$$\beta H[\phi(\mathbf{r})] = \int \mathrm{d}^2 r \left[\frac{K}{2} (\nabla\phi)^2 + u(\nabla\phi)^4 \right]. \tag{3.1}$$

Clearly, the first term of this Hamiltonian is Gaussian, whereas the second term must be treated as an interaction. The quadratic term can be diagonalised in reciprocal space (see (1.17b)):

$$\frac{K}{2} \int \mathrm{d}\mathbf{r}\, (\nabla\phi)^2 = \frac{K}{2} \int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^2}\, \mathbf{q}^2 |\phi(\mathbf{q})|^2, \tag{3.2}$$

while the quartic term becomes (see (2.24) for details)

$$\begin{aligned} u \int \mathrm{d}\mathbf{r}\, (\nabla\phi)^4 &= u \int \mathrm{d}\mathbf{r} \left[\int \frac{\mathrm{d}\mathbf{q}_1}{(2\pi)^2} \left\{ -i\mathbf{q}_1 \phi(\mathbf{q}_1) e^{-i\mathbf{q}_1\cdot\mathbf{r}} \right\} \right] \cdot \left[\int \frac{\mathrm{d}\mathbf{q}_2}{(2\pi)^2} \left\{ -i\mathbf{q}_2 \phi(\mathbf{q}_2) e^{-i\mathbf{q}_2\cdot\mathbf{r}} \right\} \right] \\ &\qquad \left[\int \frac{\mathrm{d}\mathbf{q}_3}{(2\pi)^2} \left\{ -i\mathbf{q}_3 \phi(\mathbf{q}_3) e^{-i\mathbf{q}_3\cdot\mathbf{r}} \right\} \right] \cdot \left[\int \frac{\mathrm{d}\mathbf{q}_4}{(2\pi)^2} \left\{ -i\mathbf{q}_4 \phi(\mathbf{q}_4) e^{-i\mathbf{q}_4\cdot\mathbf{r}} \right\} \right] \\ &= u \int (\mathrm{d}\mathbf{q})\ (\mathbf{q}_1\cdot\mathbf{q}_2)(\mathbf{q}_3\cdot\mathbf{q}_4)\ \phi(\mathbf{q}_1)\phi(\mathbf{q}_2)\phi(\mathbf{q}_3)\phi(\mathbf{q}_4) \times (2\pi)^2 \delta(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3+\mathbf{q}_4). \end{aligned} \tag{3.3}$$

(a) RG flow equations. The first part of the perturbative RG step is integrating out fast modes with wave vector between $\Lambda e^{-\ell}$ and Λ. Decomposing ϕ into slow and fast modes, the partition function can be written as

$$\begin{aligned} \mathcal{Z} &= \int \mathcal{D}\phi_< \mathcal{D}\phi_> e^{-\beta H_0[\phi_<]} e^{-\beta H_0[\phi_>]} e^{-U[\phi_<,\phi_>]} \\ &= \underbrace{\left[\int \mathcal{D}\phi_> e^{-\beta H_0[\phi_>]} \right]}_{\mathcal{Z}_0^>} \times \int \mathcal{D}\phi_< e^{-\beta H_0[h_<]} \frac{\int \mathcal{D}\phi_> e^{-\beta H_0[\phi_>]} e^{-U[\phi_<,\phi_>]}}{\int \mathcal{D}\phi_> e^{-\beta H_0[\phi_>]}} \\ &= \mathcal{Z}_0^> \int \mathcal{D}\phi_< e^{-\beta H_0[\phi_<]} \left\langle e^{-U} \right\rangle_{\phi_0^>} \end{aligned} \tag{3.4}$$

where H_0 and U stand for the quadratic and quartic components of the Hamiltonian, respectively, and we exploit that the unperturbed partition function is separable in $\phi_<$ and $\phi_>$ to take out $\mathcal{Z}_0^>$. As $\mathcal{Z}_0^>$ is a constant, it does not affect the thermodynamics of the renormalised model. The rest can be expressed through the renormalised Hamiltonian

$$\widetilde{\beta H}[\phi_<] = \beta H_0[\phi_<] - \ln \left\langle e^{-U} \right\rangle_{\phi_0^>} = \int_0^{\Lambda e^{-\ell}} \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d} \frac{K\mathbf{q}^2}{2} |\phi_<(\mathbf{q})|^2 + \langle U \rangle_{\phi_0^>} - \mathcal{O}(u^2) \tag{3.5}$$

using the explicit form of the unperturbed Hamiltonian and the cumulant expansion of the logarithm (2.28) to first order.

The expectation value $\langle U \rangle$ can now be expanded using Wick's theorem: we use diagrammatic perturbation theory to keep track of the terms of this expansion. The only interaction vertex is the

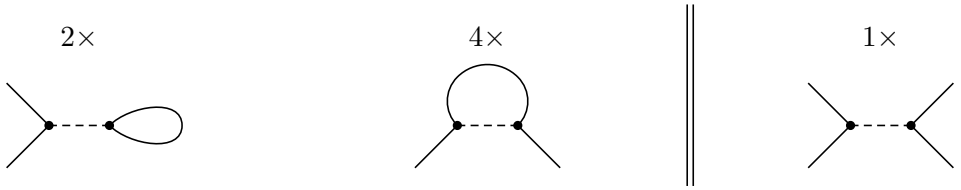

Figure 3.1. First order Feynman diagrams contributing to the renormalised quadratic and quartic terms, together with their combinatorial multiplicities.

four-legged one corresponding to the quartic interaction. The four ϕ fields entering (3.3) are not all on equal footing due to the dot products of wave vectors: to represent this, we break up each vertex into two two-legged ones, connected by a dashed line. Each vertex is thus taken to include the integral over $\mathbf{q}_{1,\dots,4}$ and the momentum-conserving delta-function as well as the dot products of the wave vectors of the two pairs of legs running into each end of the dashed line.

Diagrammatically expressing the cumulant expansion then consists of connecting legs corresponding to fast modes in all possible arrangements. These modes are replaced by the two-point correlator obtained from βH_0, $(2\pi)^2\delta(\mathbf{q}+\mathbf{q}')/(K\mathbf{q}^2)$. The remaining free legs represent slow modes.

All nontrivial first-order diagrams are listed in Fig. 3.1, together with their combinatorial multiplicities. (There are some diagrams with all legs connected: these, however, only give rise to a constant offset in $\widetilde{\beta H}$.)

Only one diagram is possible with four free legs. This gives rise to a renormalised quartic term that is equal to the original one: $\tilde{u} = u$.

Two different diagrams give rise to corrections to the quadratic coupling K. The first corresponds to the following perturbation:

$$\begin{aligned}
U_1^{(2)} &= 2u \int \frac{\mathrm{d}\mathbf{q}_{1\dots4}}{(2\pi)^8}\ (\mathbf{q}_1\cdot\mathbf{q}_2)(\mathbf{q}_3\cdot\mathbf{q}_4)\ \phi_<(\mathbf{q}_1)\phi_<(\mathbf{q}_2)\langle\phi_>(\mathbf{q}_3)\phi_>(\mathbf{q}_4)\rangle\ (2\pi)^2\delta(\mathbf{q}_1+\dots+\mathbf{q}_4) \\
&= 2u \int \frac{\mathrm{d}\mathbf{q}_{1\dots4}}{(2\pi)^8}\ (\mathbf{q}_1\cdot\mathbf{q}_2)(\mathbf{q}_3\cdot\mathbf{q}_4)\ \phi_<(\mathbf{q}_1)\phi_<(\mathbf{q}_2)\frac{(2\pi)^2\delta(\mathbf{q}_3+\mathbf{q}_4)}{K\mathbf{q}_3^2}(2\pi)^2\delta(\mathbf{q}_1+\dots+\mathbf{q}_4) \\
&= 2u\left[\int \frac{\mathrm{d}\mathbf{q}_1\,\mathrm{d}\mathbf{q}_2}{(2\pi)^4}\ \mathbf{q}_1\cdot\mathbf{q}_2\ \phi_<(\mathbf{q}_1)\phi_<(\mathbf{q}_2)(2\pi)^2\delta(\mathbf{q}_1+\mathbf{q}_2)\right]\left[\int \frac{\mathrm{d}\mathbf{q}_3\,\mathrm{d}\mathbf{q}_4}{(2\pi)^4}\mathbf{q}_3\cdot\mathbf{q}_4\ \frac{(2\pi)^2\delta(\mathbf{q}_3+\mathbf{q}_4)}{K\mathbf{q}_3^2}\right] \\
&= 2u\left[\int_0^{\Lambda e^{-\ell}} \frac{\mathrm{d}\mathbf{q}}{(2\pi)^2}\ \mathbf{q}^2|\phi_<(\mathbf{q})|^2\right]\left[\int_{\Lambda e^{-\ell}}^{\Lambda} \frac{\mathrm{d}\mathbf{q}'}{(2\pi)^2}\frac{1}{K}\right] = \frac{u\Lambda^2\ell}{K\pi}\int_0^{\Lambda e^{-\ell}} \frac{\mathrm{d}\mathbf{q}}{(2\pi)^2}\ \mathbf{q}^2|\phi_<(\mathbf{q})|^2;
\end{aligned} \tag{3.6a}$$

in the second line, we substitute the quadratic level correlator of the fast modes; in the third, we observe that both δ-function constraints are only satisfied if $\mathbf{q}_1+\mathbf{q}_2=0$, and rewrite one of them accordingly; furthermore, we note that the integral is now separable in the slow and fast modes. In the last line, we integrate over $\mathbf{q}_{2,4}$ to get rid of the δ-functions, and perform the trivial integral over fast modes. Likewise, the second diagram gives rise to the perturbation:

$$\begin{aligned}
U_2^{(2)} &= 4u \int \frac{\mathrm{d}\mathbf{q}_{1\dots4}}{(2\pi)^8}\ (\mathbf{q}_1\cdot\mathbf{q}_2)(\mathbf{q}_3\cdot\mathbf{q}_4)\ \phi_<(\mathbf{q}_1)\phi_<(\mathbf{q}_3)\langle\phi_>(\mathbf{q}_2)\phi_>(\mathbf{q}_4)\rangle\ (2\pi)^2\delta(\mathbf{q}_1+\dots+\mathbf{q}_4) \\
&= 4u \int \frac{\mathrm{d}\mathbf{q}_{1\dots4}}{(2\pi)^8}\ (\mathbf{q}_1\cdot\mathbf{q}_2)(\mathbf{q}_3\cdot\mathbf{q}_4)\ \phi_<(\mathbf{q}_1)\phi_<(\mathbf{q}_3)\ \frac{(2\pi)^4\delta(\mathbf{q}_1+\mathbf{q}_3)\delta(\mathbf{q}_2+\mathbf{q}_4)}{K\mathbf{q}_2^2} \\
&= \frac{4u}{K}\int \frac{\mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{q}'}{(2\pi)^4}\phi_<(\mathbf{q})\phi_<(-\mathbf{q})\frac{(\mathbf{q}\cdot\mathbf{q}')^2}{(\mathbf{q}')^2} = \frac{4u}{K}\int_0^{\Lambda e^{-\ell}}\left[\frac{\mathrm{d}\mathbf{q}}{(2\pi)^2}\ \mathbf{q}^2|\phi_<(\mathbf{q})|^2\int_{\Lambda e^{-\ell}}^{\Lambda}\frac{\mathrm{d}\mathbf{q}'}{(2\pi)^2}\cos^2\theta\right] \\
&= \frac{u\Lambda^2\ell}{K\pi}\int_0^{\Lambda e^{-\ell}} \frac{\mathrm{d}\mathbf{q}}{(2\pi)^2}\ \mathbf{q}^2|\phi_<(\mathbf{q})|^2,
\end{aligned} \tag{3.6b}$$

where θ is the angle the vectors $\mathbf{q}$ and $\mathbf{q}'$ make. In this case, the two integrals are not separable due to the dot product $\mathbf{q}\cdot\mathbf{q}'$: in the last step, we used that $\langle\cos^2\theta\rangle = 1/2$ is two dimensions. Summing

the contributions (3.6) thus gives rise to

$$\tilde{K} = K + \frac{4u\Lambda^2\ell}{K\pi}. \tag{3.7}$$

To complete the RG step, we also have to rescale the wave vector $\mathbf{q}$ and the field ϕ, that is, we write $\mathbf{q}' = \mathbf{q}e^{\ell}$ and $\phi'(\mathbf{q}') = \phi(\mathbf{q})/z$. Plugging these into the Hamiltonian (3.1–3.3) and collecting terms in the resulting forms gives the renormalised coefficients

$$K' = \tilde{K}e^{-4\ell}z^2; \qquad u' = ue^{-10\ell}z^4, \tag{3.8}$$

see (2.14) and (2.35) for a discussion. We are going to fix z by requiring that $K' = \tilde{K}$:[12] it follows that $z = e^{2\ell}$ and $u' = ue^{-2\ell}$. Reexpressing this and (3.7) in a differential form gives

$$\frac{\mathrm{d}K}{\mathrm{d}\ell} = \frac{4u\Lambda^2}{K\pi}; \qquad \frac{\mathrm{d}u}{\mathrm{d}\ell} = -2u, \tag{3.9}$$

as required.

(b) Spin correlator in the Gaussian theory. The reciprocal space correlators of ϕ are $\langle\phi(\mathbf{q})\phi(\mathbf{q}')\rangle = (2\pi)^2\delta(\mathbf{q}+\mathbf{q}')/(K\mathbf{q}^2)$; Fourier transforming this gives the real space correlators as (see (1.48) for details)

$$\left\langle[\phi(\mathbf{r}) - \phi(0)]^2\right\rangle = -\frac{1}{\pi K}\log\left(\frac{r}{a}\right), \tag{3.10}$$

where a is a normalisation parameter, expected to be $\mathcal{O}(\Lambda^{-1})$, that is, around the lattice spacing. (Assuming it *is* the lattice spacing is a bit much.) Now, since the theory is Gaussian,

$$G(r, K, u=0) = \left\langle\exp\left[i[\phi(\mathbf{r}) - \phi(0)]\right]\right\rangle = \exp\left[\left\langle[\phi(\mathbf{r}) - \phi(0)]^2\right\rangle\right] = (r/a)^{-1/2\pi K}, \tag{3.11}$$

as required.

(c) Endpoint of the RG flow. The RG flow equation for u is trivial to integrate to $u(\ell) = u_0e^{-2\ell}$: this tends to 0 as $\ell \to \infty$. As for K, (3.9) can be rearranged to

$$\begin{aligned}\frac{2K\mathrm{d}K}{\mathrm{d}\ell} &= \frac{8\Lambda^2}{\pi}u\\ K^2(\ell) - K_0^2 &= \frac{8\Lambda^2}{\pi}\int_0^\ell u(\ell')\mathrm{d}\ell' = \frac{4\Lambda^2u_0}{\pi}(1 - e^{-2\ell}).\end{aligned} \tag{3.12}$$

As $\ell \to \infty$, we obtain $K_* = \sqrt{K_0^2 + 4u_0\Lambda^2/\pi}$, as required.

(d) Renormalisation of the spin correlator in a single RG step. We assume that all fast modes are described well by the Gaussian theory, that is, we can ignore their interactions with each other: this is a good approximation if $K \gg 1/(u\Lambda^2)$, as specified in the problem. Now, we can write

$$\begin{aligned}G(r, K, u) &= \left\langle\exp\left[i[\phi(\mathbf{r}) - \phi(0)]\right]\right\rangle = \left\langle\exp\left(i\sum_{\mathbf{q}}\left(e^{i\mathbf{q}\cdot\mathbf{r}} - 1\right)\phi_{\mathbf{q}}\right)\right\rangle\\ &= \prod_{\mathbf{q}_>}\left\langle\exp\left[i(e^{i\mathbf{q}\cdot\mathbf{r}} - 1)\phi^>_{\mathbf{q}}\right]\right\rangle \times \left\langle\exp\left(i\sum_{\mathbf{q}_<}\left(e^{i\mathbf{q}\cdot\mathbf{r}} - 1\right)\phi^<_{\mathbf{q}}\right)\right\rangle_{\widetilde{\beta H}}.\end{aligned} \tag{3.13}$$

[12] The reason for this is that we do not expect a phase transition, so there is no need for relevant RG parameter. Instead, we are trying to show that the long-range behaviour of this model is well described by a quadratic theory. It is easiest to do so if the dominant quadratic coupling tends to something finite.

We use the lattice formalism here to keep the products over wave vectors well-defined. In the second line, the first product is over the fast modes: since they are described well by the quadratic theory, each $\phi^>_{\mathbf{q}}$ is an independent Gaussian variable, and so each term in the product can readily be evaluated:

$$\begin{aligned}\left\langle \exp\left[i(e^{i\mathbf{q}\cdot\mathbf{r}}-1)\phi^>_{\mathbf{q}}\right]\right\rangle &= \exp\left(-\frac{1}{2}\left\langle (e^{i\mathbf{q}\cdot\mathbf{r}}-1)\phi^>_{\mathbf{q}}(e^{-i\mathbf{q}\cdot\mathbf{r}}-1)\phi^>_{-\mathbf{q}}\right\rangle\right) = \exp\big(-[1-\cos(\mathbf{q}\cdot\mathbf{r})]\langle\phi^>_{\mathbf{q}}\phi^>_{-\mathbf{q}}\rangle\big)\\ &= \exp\left(-\frac{1-\cos(\mathbf{q}\cdot\mathbf{r})}{K\mathbf{q}^2}\right).\end{aligned} \tag{3.14}$$

Therefore, the first term of (3.13) is

$$\prod_{\mathbf{q}=\Lambda e^{-\ell}}^{\Lambda} \exp\left(-\frac{1-\cos(\mathbf{q}\cdot\mathbf{r})}{K\mathbf{q}^2}\right) = \exp\left(-\int_{\Lambda e^{-\ell}}^{\Lambda}\frac{\mathrm{d}^2q}{(2\pi)^2}\frac{1-\cos(\mathbf{q}\cdot\mathbf{r})}{K\mathbf{q}^2}\right) \simeq \exp\left(-\frac{2\pi\Lambda^2\ell}{(2\pi)^2K\Lambda^2}\right) = e^{-\ell/2\pi K}; \tag{3.15}$$

in the second step, we ignore $\cos(\mathbf{q}\cdot\mathbf{r})$ which fluctuates rapidly and averages to zero for $r \gg a$. As for the second term of (3.13), note that the expectation value is taken with respect to the renormalised Hamiltonian $\widetilde{\beta H}$ instead of the original βH: this is so because we have to keep track of the interactions of fast and slow modes which survive in the renormalised coupling $\tilde{K}$. To finish the RG step, we have to consider the rescaling of $\phi(\mathbf{r})$:

$$\phi'(\mathbf{r}') = \int\frac{\mathrm{d}^2q'}{(2\pi)^2}e^{i\mathbf{q}'\cdot\mathbf{r}'}\phi'(\mathbf{q}') = \int\frac{\mathrm{d}^2q}{(2\pi)^2}e^{2\ell}e^{i\mathbf{q}\cdot\mathbf{r}}\phi(\mathbf{q})e^{-2\ell} = \phi(\mathbf{r}), \tag{3.16}$$

that is, the magnitude of ϕ does not change in real space, and so the second term of (3.13) is equal to G under the fully renormalised Hamiltonian.[13] Putting it all together, we thus obtain

$$G(r,K,u) = e^{-\ell/2\pi K}\,G\Big(re^{-\ell},K(\ell),u(\ell)\Big) \tag{3.17}$$

for infinitesimal ℓ, as required.

(e) Renormalisation of the spin correlator – general case. We now have to iterate (3.17) over a large number of infinitesimal RG steps that take us from $\ell = 0$ to $\ell_* = \ln(r/r_0)$. We clearly end up with a factor of $G\big(r,K(\ell_*),u(\ell_*)\big)$ as well as the product of all the infinitesimal exponentials introduced by (3.17). The latter can be rewritten as the exponential of a sum which in turn becomes an exponential of an integral provided the RG steps are really infinitesimal:

$$G(r_0,K_0,u_0) = \exp\left(-\int_0^{\ell_*}\frac{\mathrm{d}\ell}{2\pi K(\ell)}\right)G\big(r,K(\ell_*),u(\ell_*)\big), \tag{3.18}$$

as required.

(f) Large-scale behaviour of the spin correlator. Consider two RG flows: the first is the flow we are interested in, starting at (K_0,u_0), while the second starts at the fixed endpoint of this flow, $(K_*,0)$. Let us write down (3.18) for both flows:

$$G(r_0,K_0,u_0) = \exp\left(-\int_0^{\ell_*}\frac{\mathrm{d}\ell}{2\pi K(\ell)}\right)G\big(r,K(\ell_*),u(\ell_*)\big) \tag{3.19a}$$

$$(r_0/a)^{-1/2\pi K_*} = G(r_0,K_*,0) = \exp\left(-\int_0^{\ell_*}\frac{\mathrm{d}\ell}{2\pi K_*}\right)G(r,K_*,0), \tag{3.19b}$$

[13] If we wanted to carry out the same protocol for $d \neq 2$, we would have to deal with the rescaling of $\phi(\mathbf{r})$ here, which would then enter the expectation value of an exponential, leading to a fairly intractable-looking calculation. Understanding the long-range physics (which is encoded in actual observables like G) would probably be easier with an RG flow in which $\phi(\mathbf{r})$ is kept constant. This would make K an irrelevant parameter for $d < 2$ and a relevant one for $d > 2$. We can convince ourselves (or work through the equivalent RG flow to show) that these correspond to disorder and long-range magnetic order, respectively.

where in the second line, we used the Gaussian result (3.11). We now divide these two equations and take the limit $\ell_* \to \infty$: there, $K(\ell) \to K_*$ and $u(\ell) \to 0$, so the correlators on the right hand side cancel, and we are left with

$$G(r_0, K_0, u_0) = \exp\left[-\frac{1}{2\pi} \int_0^\infty \mathrm{d}\ell \left(\frac{1}{K(\ell)} - \frac{1}{K_*} \right) \right] (r_0/a)^{-1/2\pi K_*}. \tag{3.20}$$

The statement of the problem now follows from the fact that $K(\ell)$ varies between K_0 and $K_* = K_0 + \mathcal{O}(u_0)$, therefore, the whole integral to be exponentiated is $\mathcal{O}(u_0)$. That is, the quadratic theory predicting power-law decay of spin correlations is also valid to the next order of the low-temperature expansion. On the other hand, the original XY model has a Kosterlitz–Thouless type phase transition which destroys these correlations: this is, however, a nonperturbative effect due to quantised vortices which would not be captured by a coarse-grained theory like this.

Note. Given the RG flow (3.12) of K, it is relatively easy to evaluate the integral in (3.20). Defining $w = 4\Lambda^2 u/\pi$ and noting that $\mathrm{d}\ell = -\mathrm{d}u/2u = -\mathrm{d}w/2w$, we have

$$\int_0^\infty \mathrm{d}\ell \left(\frac{1}{K(\ell)} - \frac{1}{K_*} \right) = \int_0^{w_0} \frac{\mathrm{d}w}{2w} \left(\frac{1}{\sqrt{K_*^2 - w}} - \frac{1}{K_*} \right) = -\frac{1}{K_*} \log\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{w_0}{K_*^2}} \right), \tag{3.21}$$

and so (3.20) becomes (note that $K_*^2 = K_0^2 + w_0$)

$$G(r_0, K_0, u_0) = \left(\frac{K_0 + K_*}{2K_*} \frac{1}{r_0/a} \right)^{1/2\pi K_*} \tag{3.22}$$

Problem 2: ε-expansion of the Ginzburg–Landau Hamiltonian

(a) General discussion of perturbative RG. This part is entirely bookwork. Decomposing the field $\mathbf{m}$ into slow and fast modes, the partition function can be written as

$$\begin{aligned}\mathcal{Z} &= \int \mathcal{D}\mathbf{m}_< \, \mathcal{D}\mathbf{m}_> \, e^{-\beta H_0[\mathbf{m}_<]} e^{-\beta H_0[\mathbf{m}_>]} e^{-U[\mathbf{m}_<,\mathbf{m}_>]} \\ &= \underbrace{\left[\int \mathcal{D}\mathbf{m}_> \, e^{-\beta H_0[\mathbf{m}_>]}\right]}_{\mathcal{Z}_0^>} \times \int \mathcal{D}\mathbf{m}_< \, e^{-\beta H_0[h_<]} \frac{\int \mathcal{D}\mathbf{m}_> \, e^{-\beta H_0[\mathbf{m}_>]} \, e^{-U[\mathbf{m}_<,\mathbf{m}_>]}}{\int \mathcal{D}\mathbf{m}_> \, e^{-\beta H_0[\mathbf{m}_>]}} \\ &= \mathcal{Z}_0^> \int \mathcal{D}\mathbf{m}_< \, e^{-\beta H_0[\mathbf{m}_<]} \left\langle e^{-U} \right\rangle_{\mathbf{m}_0^>} \end{aligned} \tag{3.23}$$

where we exploit that the unperturbed partition function is separable in $\mathbf{m}_<$ and $\mathbf{m}_>$ to take out $\mathcal{Z}_0^>$. As $\mathcal{Z}_0^>$ is a constant, it does not affect the thermodynamics of the renormalised model. The rest can be expressed through the renormalised Hamiltonian

$$\widetilde{\beta H}[\mathbf{m}_<] = \beta H_0[\mathbf{m}_<] - \ln \left\langle e^{-U} \right\rangle_{\mathbf{m}_0^>} = \int_0^{\Lambda/b} \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d} \frac{G^{-1}(\mathbf{q})}{2} |\mathbf{m}_<(\mathbf{q})|^2 - \ln \left\langle e^{-U} \right\rangle_{\mathbf{m}_0^>} \tag{3.24}$$

using the explicit form of the unperturbed Hamiltonian.

(b) Diagrammatical perturbation theory to second order. In Fourier space, the perturbation can be written as (see Eq. 2.24 for details)

$$\begin{aligned} U &= u \int \mathrm{d}\mathbf{r} \, (\mathbf{m}^2)^2 = u \int \mathrm{d}\mathbf{r} \, m^\alpha m^\alpha m^\beta m^\beta \\ &= u \int (\mathrm{d}\mathbf{q}) m^\alpha(\mathbf{q}_1) m^\alpha(\mathbf{q}_2) m^\beta(\mathbf{q}_3) m^\beta(\mathbf{q}_4) \times (2\pi)^d \delta(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3 + \mathbf{q}_4). \end{aligned} \tag{3.25}$$

The contractions over the indices α and β can be represented in a Feynman diagram by breaking up the four-legged vertices in Fig. 2.1 into two two-legged vertices, connected to each other (by a dashed line in these notes). Each vertex is thus taken to include the integral over $\mathbf{q}_{1,\dots,4}$ and the momentum-conserving delta-function as well as an index contraction over the two pairs of legs running into the same end of the dashed line.

To express the cumulant expansion of (3.24) diagrammatically, we connect legs corresponding to fast modes: those modes are replaced by the two-point correlators obtained from βH_0, $(2\pi)^d G(\mathbf{q}) \delta(\mathbf{q} + \mathbf{q}') \delta^{\alpha\beta}$. The remaining free legs correspond to slow modes entering the perturbed Hamiltonian.

Fig. 3.2 lists all connected first and second-order diagrams together with their combinatorial multiplicity (this will generally depend on the number of components of the order parameter due to index contractions over fast modes).

(c) Second order perturbations to parameters. We now calculate the renormalisation of u, t, and K coming from the second order diagrams in (3.24).

Perturbations to u**.** All second order diagrams resulting in a quartic correction have two free legs on both vertices: this ensures that none of the momenta on these free legs is constrained by the connecting fast modes, as it is the case for a single free leg on a diagram (see Fig. 3.2). By way of example, the first diagram corresponds to the following second order perturbation:

$$\begin{aligned} -8n \frac{u^2}{2!} \int \frac{\mathrm{d}\mathbf{q}_{1,\dots,8}}{(2\pi)^{8d}} \, & m_<^\alpha(\mathbf{q}_1) m_<^\alpha(\mathbf{q}_2) m_<^\beta(\mathbf{q}_7) m_<^\beta(\mathbf{q}_8) \times \langle m_>^\gamma(\mathbf{q}_3) m_>^\gamma(\mathbf{q}_5) \rangle \langle m_>^\gamma(\mathbf{q}_4) m_>^\gamma(\mathbf{q}_6) \rangle \times \\ & \times (2\pi)^d \delta(\mathbf{q}_1 + \dots + \mathbf{q}_4) \, (2\pi)^d \delta(\mathbf{q}_5 + \dots + \mathbf{q}_8) \qquad [\text{no sum over } \gamma] \end{aligned} \tag{3.26}$$

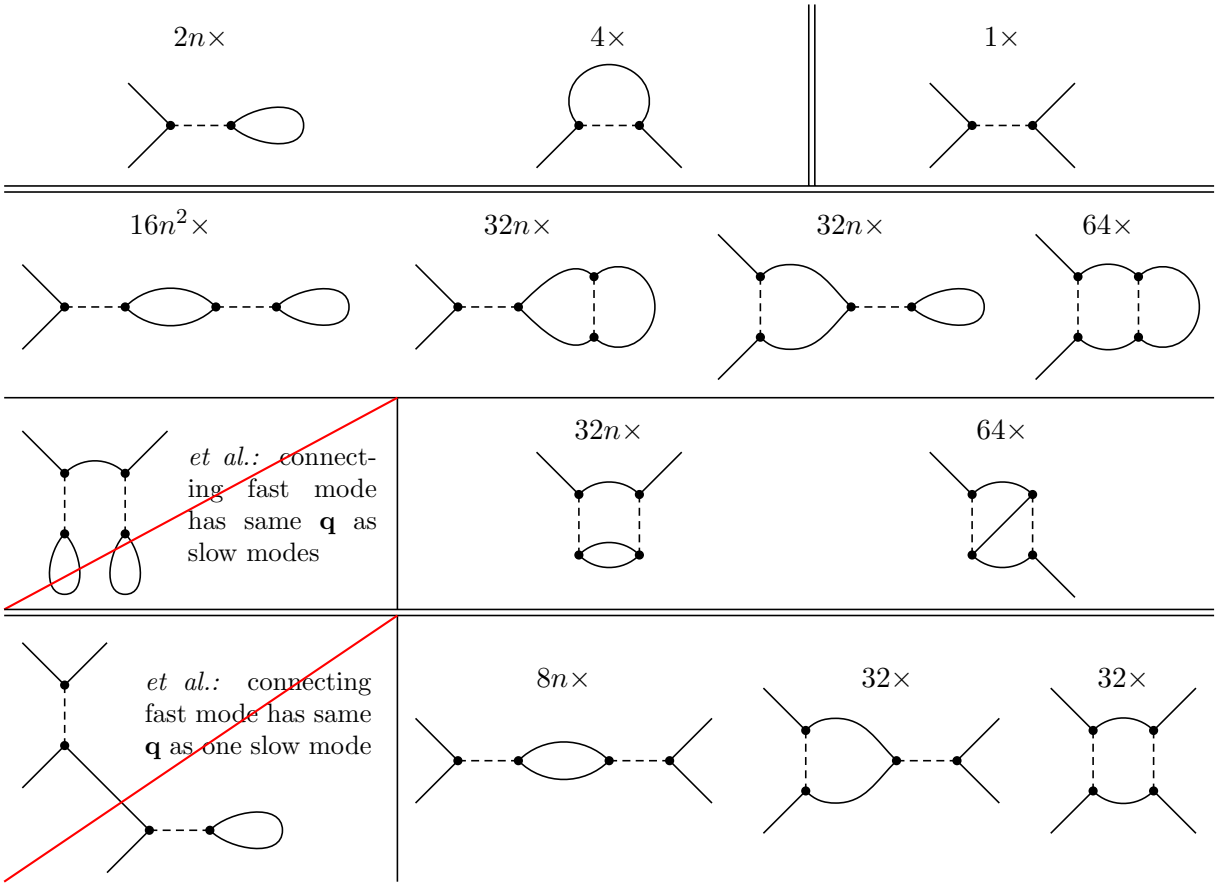

Figure 3.2. First and second order Feynman diagrams contributing to the renormalised quadratic and quartic terms, together with their combinatorial multiplicities. Diagrams are grouped by order, number of free legs, and distribution of free legs across diagrams (for second order ones). Some topologically possible diagrams do not correspond to any actual term in the Wick's theorem expansion due to kinematics, these are crossed out and some explanation is given in the figure.

where the minus sign and the division by 2! comes from the cumulant expansion (2.28) of $\ln\langle e^{-U}\rangle$. The form of the unperturbed correlators require that $\mathbf{q}_3+\mathbf{q}_5=\mathbf{q}_4+\mathbf{q}_6=0$: together with the delta-functions in the integral, this implies $\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_7+\mathbf{q}_8=0$, that is, the momentum conservation in (3.25) is observed by the renormalised quartic term too. Furthermore, the vertex delta functions imply that the sum of momenta on the two free legs of each vertex is equal to the sum of the connecting fast momenta: this is entirely feasible, however, it introduces a dependence of the correlators in (3.26) on $\mathbf{q}_1+\mathbf{q}_2$, not something present in the original term. In the first approximation, however, we assume that the connecting modes are equal and opposite, so they can be any of the fast modes.[14] In this approximation, (3.26) simplifies considerably:

$$-4nu^2\int\frac{\mathrm{d}\mathbf{q}_{1,\dots,4}}{(2\pi)^{4d}}\,m_<^\alpha(\mathbf{q}_1)m_<^\alpha(\mathbf{q}_2)m_<^\beta(\mathbf{q}_3)m_<^\beta(\mathbf{q}_4)\times(2\pi)^d\delta(\mathbf{q}_1+\dots+\mathbf{q}_4)\times\int_{\Lambda/b}^{\Lambda}\frac{\mathrm{d}\mathbf{q}}{(2\pi)^d}G(\mathbf{q})^2. \quad (3.27)$$

The contributions of the other two diagrams follow through very similarly: in the same approximation, their functional form will be identical to (3.27), resulting in the renormalised

$$\tilde{u}=u-4u^2(n+8)\int_{\Lambda/b}^{\Lambda}\frac{\mathrm{d}\mathbf{q}}{(2\pi)^d}G(\mathbf{q})^2. \quad (3.28)$$

[14] This is exact if $\mathbf{q}_1+\mathbf{q}_2=0$, otherwise, corrections depending on the magnitude of $\mathbf{q}_1+\mathbf{q}_2$ appear. These correspond to quartic terms involving derivatives of $\mathbf{m}$. Just below the upper critical dimension of the Ginzburg–Landau Hamiltonian (where the ε-expansion is valid), all such terms are irrelevant, and so we don't pursue them. Similarly, the renormalisation to t arising from second order diagrams is $\mathbf{q}$-dependent: this results in a renormalisation of K which must be carried through as K is a marginal RG parameter. We discuss this is more detail in the next part.

Perturbations to t and K. In this case we also have first order diagrams in Fig. 3.2 which give the contribution

$$\widetilde{\beta H} \supset (2n+4)u \int \frac{\mathrm{d}\mathbf{q}_{1,\dots,4}}{(2\pi)^{4d}}\, m^\alpha_<(\mathbf{q}_1) m^\alpha_<(\mathbf{q}_2) \times \langle m^\beta_>(\mathbf{q}_3) m^\beta_>(\mathbf{q}_4)\rangle \times (2\pi)^d \delta(\mathbf{q}_1 + \cdots + \mathbf{q}_4)$$

$$= (2n+4)u \int^{\Lambda/b} \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d}\, m^\alpha_<(\mathbf{q}) m^\alpha(-\mathbf{q}) \times \int_{\Lambda/b}^{\Lambda} \frac{\mathrm{d}\mathbf{q}'}{(2\pi)^d}\, G(\mathbf{q}') \tag{3.29a}$$

$$\tilde{t} \supset 4(n+2)u \int_{\Lambda/b}^{\Lambda} \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d}\, G(\mathbf{q}). \tag{3.29b}$$

Accounting for the second order perturbation is trickier. In the first class of diagrams (second row in Fig. 3.2), both loops must contain equal and opposite momenta, resulting in a $\mathbf{q}$-independent quadratic term, *i.e.* a renormalisation of t: determining the coefficients is an easy matter as the two loops can be integrated over separately. In the "1+1" diagrams (third row), however, the sum of the momenta of the three connecting fast modes must be equal to $\mathbf{q}$. This clearly introduces a $\mathbf{q}$-dependence to the coefficient of $|\mathbf{m}(\mathbf{q})|^2$: as the arising geometrical constraint is quite complicated, evaluating the integrals in the prefactor is hard. For now,[15] we simply write the total contribution of all second order diagrams as

$$\widetilde{\beta H} \supset -\frac{u^2}{2} \int^{\Lambda/b} \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d}\, m^\alpha_<(\mathbf{q}) m^\alpha(-\mathbf{q})\, \mathcal{A}(\mathbf{q})$$

$$\approx -\frac{u^2}{2} \int^{\Lambda/b} \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d}\, m^\alpha_<(\mathbf{q}) m^\alpha(-\mathbf{q}) \left[\mathcal{A}(0) + \frac{1}{2}\mathbf{q}^2 \left(\nabla^2_{\mathbf{q}} \mathcal{A}\right)_{\mathbf{q}=0}\right]; \tag{3.30a}$$

$$\tilde{t} \supset -u^2 \mathcal{A}(0); \tag{3.30b}$$

$$\tilde{K} \supset -\frac{u^2}{2} \left(\nabla^2_{\mathbf{q}} \mathcal{A}\right)_{\mathbf{q}=0}; \tag{3.30c}$$

the minus sign comes from the cumulant expansion of $\ln\langle e^{-U}\rangle$ ($\mathcal{A}(\mathbf{q})$ is positive); the division by 2 is included to simplify (3.30b). As well as $\mathbf{q}$, $\mathcal{A}$ clearly depends on Λ and b. In (3.30a), $\mathcal{A}(\mathbf{q})$ is Taylor expanded to second order in $\mathbf{q}$: there is no first order term as $\mathcal{A}$ is a smooth, rotationally symmetric function of $\mathbf{q}$; higher order terms would correspond to quadratic terms involving higher derivatives, all irrelevant just below the upper critical dimension.

Combining (3.29) and (3.30) gives finally

$$\tilde{t} = t + 4(n+2)u \int_{\Lambda/b}^{\Lambda} \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d}\, G(\mathbf{q}) - u^2 \mathcal{A}(0); \tag{3.31a}$$

$$\tilde{K} = K - \frac{u^2}{2} \left(\nabla^2_{\mathbf{q}} \mathcal{A}\right)_{\mathbf{q}=0}. \tag{3.31b}$$

(d) Rescaling the second order Hamiltonian. After the rescaling $\mathbf{q}' = \mathbf{q}b$ and $\mathbf{m}' = \mathbf{m}_</z$, the renormalised parameters are given by [see (2.14) and (2.35) for a discussion]

$$t' = \tilde{t} b^{-d} z^2; \qquad K' = \tilde{K} b^{-(d+2)} z^2; \qquad u' = \tilde{u} b^{-3d} z^4. \tag{3.32}$$

We want to ensure that $K' = K$: comparing to (3.31b) gives

$$z^2 = \frac{b^{d+2}}{1 - [u^2/(2K)] \left(\nabla^2_{\mathbf{q}} \mathcal{A}\right)_{\mathbf{q}=0}}. \tag{3.33}$$

As we shall see later, u is $\mathcal{O}(\varepsilon)$ in the region of interest: therefore, the correction introduced in the denominator is $\mathcal{O}(\varepsilon^2)$, that is, irrelevant in first order ε-expansion. We can thus write $z = b^{(d+2)/2}$, from which (3.32) becomes

$$t' = \tilde{t} b^2; \qquad u' = \tilde{u} b^{4-d}. \tag{3.34}$$

[15] It will turn out that the exact functional form doesn't matter in first order ε-expansion.

Writing an infinitesimal rescaling, $b = e^\ell$, into (3.31), (3.28), and (3.34) gives rise to the differential flow equations

$$\frac{\mathrm{d}t}{\mathrm{d}\ell} = 2t + 4(n+2)u\,G(\Lambda)\,\sigma_d\Lambda^d - u^2A(0) \tag{3.35a}$$

$$\frac{\mathrm{d}u}{\mathrm{d}\ell} = (4-d)u - 4u^2(n+8)\,G(\Lambda)^2\,\sigma_d\Lambda^d \tag{3.35b}$$

where $\sigma_d = S_d/(2\pi)^d$ is introduced by the integration over the infinitesimally thin spherical shell at $|\mathbf{q}| = \Lambda$.[16]

(e) RG flow below the upper critical dimension. For $d > 4$, (3.35b) shows that any positive u flows towards $u = 0$, therefore, the the only fixed point is the Gaussian one, $t = u = 0$. If, however, $d < 4$, (3.35b) provides for an other fixed point at

$$u^* = \frac{4-d}{4(n+8)G(\Lambda)^2\sigma_d\Lambda^d} = \frac{\varepsilon(t^*+K\Lambda^2)^2}{4(n+8)\sigma_d\Lambda^d} = \frac{K^2\Lambda^{4-d}}{4(n+8)\sigma_d}\varepsilon + \mathcal{O}(\varepsilon^2); \tag{3.36a}$$

$$t^* = -2(n+2)u^*G(\Lambda)\sigma_d\Lambda^d + \mathcal{O}(u^{*2}) = -\frac{2(n+2)\sigma_d\Lambda^d}{t^*+K\Lambda^2}u^* + \mathcal{O}(\varepsilon^2) = -\frac{n+2}{2(n+8)}K\Lambda^2\varepsilon + \mathcal{O}(\varepsilon^2). \tag{3.36b}$$

We can now linearise the flow equations (3.35) around this fixed point:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\mathrm{d}t}{\mathrm{d}\ell}\right)_{t^*,u^*} &= 2 + 4(n+2)u^*\sigma_d\Lambda^d\frac{\partial G(\Lambda)}{\partial t} + \mathcal{O}(\varepsilon^2)\\ &= 2 + 4(n+2)\frac{K^2\Lambda^{4-d}}{4(n+8)\sigma_d}\varepsilon\sigma_d\Lambda^d\frac{\partial}{\partial t}\left(\frac{1}{t+K\Lambda^2}\right) + \mathcal{O}(\varepsilon^2)\\ &= 2 - \frac{n+2}{n+8}\frac{K^2\Lambda^4\varepsilon}{(t^*+K\Lambda^2)^2} + \mathcal{O}(\varepsilon^2)\\ &= 2 - \frac{n+2}{n+8}\varepsilon + \mathcal{O}(\varepsilon^2);\end{aligned} \tag{3.37a}$$

$$\frac{\partial}{\partial u}\left(\frac{\mathrm{d}t}{\mathrm{d}\ell}\right)_{t^*,u^*} = 4(n+2)G(\Lambda)\sigma_d\Lambda^d + \mathcal{O}(\varepsilon); \tag{3.37b}$$

$$\frac{\partial}{\partial t}\left(\frac{\mathrm{d}u}{\mathrm{d}\ell}\right)_{t^*,u^*} = -4u^{*2}(n+8)\,\sigma_d\Lambda^d\,\frac{\partial[G(\Lambda)^2]}{\partial t} = \mathcal{O}(\varepsilon^2); \tag{3.37c}$$

$$\frac{\partial}{\partial u}\left(\frac{\mathrm{d}u}{\mathrm{d}\ell}\right)_{t^*,u^*} = (4-d) - 8u^*(n+8)G(\Lambda)^2\sigma_d\Lambda^d + \mathcal{O}(u^{*2}) = -\varepsilon + \mathcal{O}(\varepsilon^2); \tag{3.37d}$$

$$\therefore \frac{\mathrm{d}}{\mathrm{d}\ell}\begin{pmatrix}\delta t\\ \delta u\end{pmatrix} = \begin{pmatrix} 2 - \frac{n+2}{n+8}\varepsilon + \mathcal{O}(\varepsilon^2) & 4(n+2)G(\Lambda)\sigma_d\Lambda^d + \mathcal{O}(\varepsilon)\\ \mathcal{O}(\varepsilon^2) & -\varepsilon + \mathcal{O}(\varepsilon^2)\end{pmatrix}\begin{pmatrix}\delta t\\ \delta u\end{pmatrix}. \tag{3.37e}$$

The scaling dimensions of the fixed point are given by the eigenvalues of the 2×2 matrix. To first order in ε, it is upper triangular, therefore, its eigenvalues are given by the diagonal elements:

$$y_t = 2 - \frac{n+2}{n+8}\,\varepsilon + \mathcal{O}(\varepsilon^2); \qquad\qquad y_u = -\varepsilon + \mathcal{O}(\varepsilon^2). \tag{3.38}$$

We can identify the dimensions with t and u based on the eigenvectors corresponding to them: that of y_t only has a δt component. It is thus clear that δt is a relevant, while δu is an irrelevant coupling near the $O(N)$ fixed point, therefore, it is this one that will describe critical behaviour in $d < 4$.

The resulting RG flows for $d > 4$ (containing a single Gaussian fixed point) and $d < 4$ (two fixed points) are sketched in Fig. 3.3. The flow directions around the $O(N)$ fixed point take into account the eigenvectors of (3.37e).

[16] This number was denoted by K_d in the handout; I find it a very unfortunate choice given the other meaning of K in the problem.

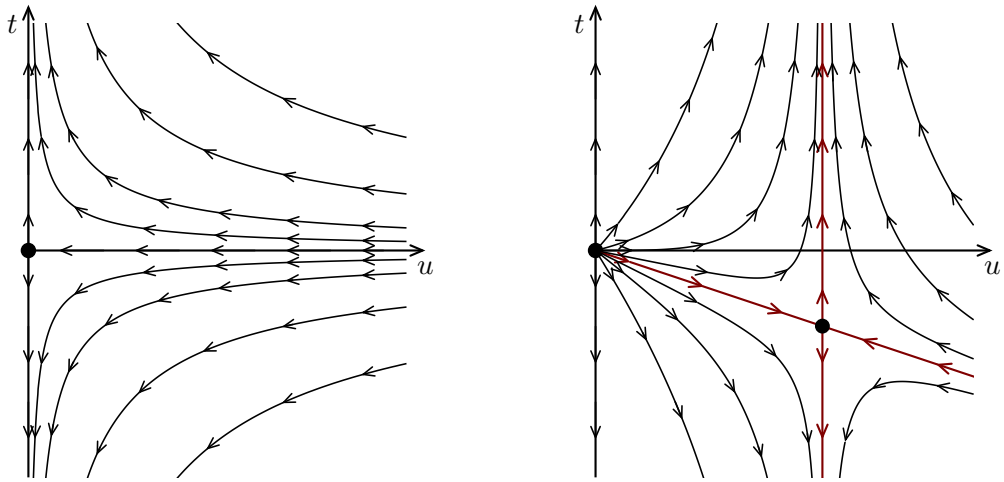

Figure 3.3. Sketch of the RG flow of the Ginzburg–Landau Hamiltonian in $d > 4$ (left) and $d < 4$ (right). Above the upper critical dimension, the only fixed point is the Gaussian one at $t = u = 0$: near this fixed point, u is irrelevant, that is, the transition is entirely described by t. Below $d = 4$, the Gaussian fixed point becomes unstable, and a new fixed point appears at $t^*, u^* = \mathcal{O}(\varepsilon)$: there, t is still a relevant RG coupling, but δu becomes irrelevant (dark red lines), therefore, it is this fixed point that describes the phase transition.

(f) Effect of magnetic field below the upper critical dimension. To take into account the effect of a magnetic field, we add the coupling term [see Problem II/2(a)]

$$(\beta H)_{\mathbf{h}} = -\int \mathrm{d}\mathbf{r}\, \mathbf{h}\cdot\mathbf{m}(\mathbf{r}) = -\mathbf{h}\cdot\mathbf{m}(\mathbf{q}=0). \tag{3.39}$$

The only effect of the renormalisation procedure on this term is the renormalisation of $\mathbf{m}(0)$ into $z\mathbf{m}'(0)$:

$$(\beta H)_{\mathbf{h}} = -\mathbf{h}\cdot\mathbf{m}(0) = -\mathbf{h}\cdot z\mathbf{m}'(0) = -\mathbf{h}'\cdot\mathbf{m}'(0) \tag{3.40}$$

where $\mathbf{h}' = z\mathbf{h}$. As found in (3.33), $z = b^{(d+2)/2} + \mathcal{O}(\varepsilon^2)$, therefore, the scaling dimension of the magnetic field is

$$y_h = \frac{d+2}{2} + \mathcal{O}(\varepsilon^2). \tag{3.41}$$

To first order in ε, this is the same as the Gaussian result: indeed, the relation of z and b depends on the renormalisation of K only, which we found to be unaffected in first order.

(g) Critical exponents in ε-expansion.

ν. Setting $h = 0$ and $b^{y_t}|\delta t| = 1$, the scaling relation of the correlation length gives

$$\xi(\delta t) = |\delta t|^{-1/y_t}\xi(\pm 1) \propto |\delta t|^{-1/y_t} \implies \nu = \frac{1}{y_t} = \frac{1}{2} + \frac{n+2}{4(n+8)}\varepsilon + \mathcal{O}(\varepsilon^2). \tag{3.42}$$

α. As discussed in Problem I/1(d), the heat capacity is proportional to $\partial^2 f/\partial(\delta t)^2$. The scaling relation of this can be derived from that of the free energy (for simplicity, we set $h = 0$):

$$f''(\delta t) = b^{-d}\frac{\mathrm{d}^2}{\mathrm{d}(\delta t)^2} f(b^{y_t}\delta t) = b^{2y_t-d} f''(b^{y_t}\delta t)$$
$$C(\delta t) = b^{2y_t-d} C(b^{y_t}\delta t)$$

Now, if we set $b^{y_t}|\delta t| = 1$ as before, we get

$$C(\delta t) = |\delta t|^{-(2y_t-d)/y_t} C(\pm 1) \propto |\delta t|^{-(2-d/y_t)} \implies \alpha = 2 - \frac{d}{y_t} = \frac{4-n}{2(n+8)}\varepsilon + \mathcal{O}(\varepsilon^2) \tag{3.43}$$

β. The magnetisation of the system can be obtained from the free energy as $m = \partial f/\partial h$: therefore, its scaling relation is given by

$$\frac{\partial f(\delta t, h)}{\partial h} = b^{-d}\frac{\partial f(b^{y_t}\delta t, b^{y_h}h)}{\partial h} = b^{y_h-d}f'(b^{y_t}\delta t, b^{y_h}h)$$
$$m(\delta t, h) = b^{y_h-d}m(b^{y_t}\delta t, b^{y_h}h)$$

where the prime denotes differentiation with respect to the second argument. Substituting $h = 0$ and $b^{y_t}\delta t = -1$, this gives

$$m(\delta t) = (-\delta t)^{(d-y_h)/y_t}m(-1) \propto (-\delta t)^{(d-y_h)/y_t} \implies \beta = \frac{d-y_h}{y_t} = \frac{1}{2} - \frac{3}{2(n+8)}\varepsilon + \mathcal{O}(\varepsilon^2) \quad (3.44)$$

γ. Susceptibility is given by $\chi = \partial m/\partial h = \partial^2 f/\partial h^2$:

$$\frac{\partial^2 f(\delta t, h)}{\partial h^2} = b^{-d}\frac{\partial^2 f(b^{y_t}\delta t, b^{y_h}h)}{\partial h^2} = b^{2y_h-d}f''(b^{y_t}\delta t, b^{y_h}h)$$
$$\chi(\delta t, h) = b^{2y_h-d}\chi(b^{y_t}\delta t, b^{y_h}h).$$

Substitute $h = 0$ and $b^{y_t}|\delta t| = 1$:

$$\chi(\delta t) = |\delta t|^{(d-2y_h)/y_t}m(-1) \propto |\delta t|^{-(2y_h-d)/y_t} \implies \gamma = \frac{2y_h-d}{y_t} = 1 + \frac{n+2}{2(n+8)}\varepsilon + \mathcal{O}(\varepsilon^2). \quad (3.45)$$

How accurate is the ε-expansion? It is instructive to compare the critical exponents predicted by first order ε-expansion with numerical results for a range of different models (*i.e.* different values of d and n):

Universality class		α		β		γ		ν	
d	n	Exact	ε	Exact	ε	Exact	ε	Exact	ε
3	1	0.110 07(7)	0.1667	0.326 53(10)	0.3333	1.2373(2)	1.1667	0.630 12(16)	0.5833
	2	−0.0146(8)	0.1	0.3485(2)	0.35	1.3177(5)	1.2	0.671 55(27)	0.6
	3	−0.12(1)	0.0455	0.366(2)	0.3636	1.395(5)	1.2273	0.707(3)	0.6136
2	1	0	0.3333	0.125	0.1667	1.75	1.3333	1	0.6667

Generally speaking, the results obtained from ε-expansion match up poorly with exact values; this is not surprising given that we used $\varepsilon = 4 - d = 1$ or 2 as a "small parameter". Performing the expansion to higher orders is increasingly complicated, and convergence is slow. The main merit of ε-expansion is giving a framework in which critical exponents can be worked out systematically in terms of a few parameters (such as d, n, and the anomalous dimension), thus underpinning the notion of universality between different systems beyond mean-field level.

Section IV

Quantum phase transitions

Problem 1: Scaling at quantum critical points

In this problem, we determine the upper critical dimension of various Gaussian Hamiltonians perturbed by a quartic term. The procedure we employ is very similar to that in Problem II/2(j): we find the appropriate renormalisation of ω, k, and m such that the coefficients of the quadratic terms dominating long-range temporal and spatial correlations remain constant. (This doesn't include t: in order to have a phase transition, we must insist that t be a relevant parameter so that positive and negative t flow to different limits on the phase diagram.) Once these coefficients are identified, and the appropriate b, c, z found, we can find the renormalisation of u near the Gaussian critical point: at the upper critical dimension, u does not scale, while it is irrelevant above and relevant below.

(a) There are only two kinetic parameters, d and e, therefore, both must be kept constant. Similarly to (2.14), we get (note that the convention for b and c are swapped between the two problems)

$$\begin{aligned} d := d' &= dc^{-1}z^2 \times b^{1-d}c^{-1} \\ e := e' &= eb^{-2}z^2 \times b^{1-d}c^{-1}. \end{aligned}$$

Solving these two equations yields $c = b^2$, $z = b^{(3+d)/2}$. The renormalisation of u follows the exact same way as in (2.35):

$$u' = ub^{3(1-d)}c^{-3}z^4 = ub^{3-d} \implies d^* = 3 \text{ (\textit{i.e.} (2+1)d spacetime)}.$$

(b) Scaling is unaffected by ω being imaginary, so everything is the same as above. $d^* = 3$.

(c) Requiring that d and e be constant gives

$$\left.\begin{aligned} d := d' &= dbc^{-1}z^2 \times b^{1-d}c^{-1} \\ e := e' &= eb^{-2}z^2 \times b^{1-d}c^{-1}. \end{aligned}\right\} \implies \left\{\begin{aligned} c &= b^3 \\ z &= b^{(d+4)/2} \end{aligned}\right.$$
$$u' = ub^{3(1-d)}c^{-3}z^4 = ub^{2-d} \implies d^* = 2.$$

(d) Requiring that d and e be constant gives

$$\left.\begin{aligned} d := d' &= dc^{-2}z^2 \times b^{1-d}c^{-1} \\ e := e' &= eb^{-2}z^2 \times b^{1-d}c^{-1}. \end{aligned}\right\} \implies \left\{\begin{aligned} c &= b \\ z &= b^{(d+2)/2} \end{aligned}\right.$$
$$u' = ub^{3(1-d)}c^{-3}z^4 = ub^{4-d} \implies d^* = 4.$$

Note that in this case, ω and k could be treated on an equal footing, reproducing the ordinary Ginzburg–Landau Hamiltonian.

(e) In this case, we have three kinetic parameters, out of which only two can be kept constant. We want to keep those that are dominant at large time and length scales, *i.e.* at $\omega, k \to 0$. We note that ω^2 vanishes faster than $|\omega|$ as $\omega \to 0$, therefore, we ought to keep d and f constant. This reproduces case (a), therefore, $d^* = 3$. [It is instructive to determine the scaling of e:

$$e' = ec^{-2}z^2 \times b^{1-d}c^{-1} = eb^{-2},$$

that is, e renormalises down and so doesn't affect the long-range behaviour. By contrast, had we kept e and f constant, d would renormalise up, resulting in an untractable RG flow.]

Problem 2: $O(2)$ quantum rotors

In this problem, we consider an array of $O(2)$ quantum rotors described by the Hamiltonian

$$H = \sum_i \frac{\hat{L}_i^2}{2m} - \frac{g}{2} \sum_{\langle ij \rangle} \cos(\hat{\theta}_i - \hat{\theta}_j) \tag{4.1}$$

where m is the moment of inertia of the rotors, and $\hat{L}$ is the canonical momentum conjugate to $\hat{\theta}$:

$$\hat{L}_j = -i \frac{\partial}{\partial \theta_j} \implies [\hat{\theta}_j, \hat{L}_k] = i\delta_{jk}. \tag{4.2}$$

In the solution, we are going to rely on the fact that the eigenstates of an individual rotor are 2d angular momentum eigenstates between which the operators $e^{\pm i\hat{\theta}}$ act as ladder operators:

$$|n\rangle = \frac{1}{\sqrt{2\pi}} e^{in\theta}; \qquad H_1|n\rangle = \frac{n^2}{2m}|n\rangle; \qquad e^{\pm i\hat{\theta}}|n\rangle = |n \pm 1\rangle. \tag{4.3}$$

(a) Derivation of the Ginzburg–Landau action. As discussed in lectures, the thermal behaviour of the Hamiltonian (4.1) at inverse temperature β can be described in terms of an imaginary time Feynman path integral:

$$\mathcal{Z} = \int_{\theta_i(\beta) - \theta_i(0) = 2\pi n} \mathcal{D}\theta_i(\tau) \exp\left[- \int_0^\beta \mathrm{d}\tau \left(\sum_i \frac{m(\partial_\tau \theta_i)^2}{2} - g \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) \right) \right]. \tag{4.4}$$

The first term is due the Lagrangian equivalent of the kinetic energy term $\hat{L}^2/2m$; the boundary conditions ensure that the initial and final state is *equivalent* (note that angles are only defined up to 2π).

Introducing the order parameter. The interaction term can be Hubbard–Stratonovich decoupled by introducing an order parameter coupled to $e^{i\theta}$. This is also a useful choice because $e^{i\theta}$ gives the direction of the rotor on the complex plane, so a nonzero order parameter will signal a preferred direction along which the rotors are aligned. In particular, we write

$$\begin{aligned} \exp\left[g \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) \right] &= \exp\left[G_{ij} e^{-i\theta_i} e^{i\theta_j} \right] \\ &\propto \int \prod_i \mathrm{d}\Psi_i \, \mathrm{d}\Psi_i^* \exp\left[\sum_i (e^{i\theta_i} \Psi_i^* + e^{-i\theta_i} \Psi_i) - \sum_{ij} \Psi_i^* G_{ij}^{-1} \Psi_j \right] \end{aligned} \tag{4.5}$$

at any given imaginary time τ, where $G_{ij} = g/2$ for nearest neighbours and 0 otherwise. Hubbard–Stratonovich transforming the whole integral now amounts to splitting it up to small segments $\mathrm{d}\tau$, applying (4.5) to each of them, and turning the resulting product into the exponential of an integral again. All in all, (4.4) turns into

$$\mathcal{Z} = \int_{\Delta\theta = 2\pi n} \mathcal{D}\theta_i(\tau) \, \mathcal{D}\Psi_i(\tau) \, \mathcal{D}\Psi_i^*(\tau) e^{-S[\theta_i, \Psi_i]}; \tag{4.6}$$

$$S = \int_0^\beta \mathrm{d}\tau \left\{ \sum_i \left[\frac{m}{2} (\partial_\tau \theta_i)^2 + (e^{i\theta_i} \Psi_i^* + e^{-i\theta_i} \Psi_i) \right] + \sum_{ij} \Psi_i^* G_{ij}^{-1} \Psi_j \right\}. \tag{4.7}$$

We shall now use (4.7) to derive an effective action containing only the order parameter Ψ. We first note that the first term in the Hubbard–Stratonovich action simply gives the partition function of the

decoupled rotors, $\mathcal{Z}_0$. Since we know how to describe their behaviour, we integrate that part out, and treat the coupling between θ and Ψ as a perturbation:

$$\mathcal{Z} = \mathcal{Z}_0 \int \mathcal{D}\Psi_i(\tau)\,\mathcal{D}\Psi_i^*(\tau) \exp\Big(-\sum_{ij} \int_0^\beta \mathrm{d}\tau\, \Psi_i^* G_{ij}^{-1} \Psi_j \Big) \Big\langle \exp\Big[-\sum_i \int_0^\beta \mathrm{d}\tau \left(e^{i\theta_i}\Psi_i^* + e^{-i\theta_i}\Psi_i \right) \Big] \Big\rangle_{\mathcal{Z}_0} . \tag{4.8}$$

Power series expansion. Let us consider now an expansion of the last expectation value as a power series in its exponent. The zeroth order term is obviously 1.

To evaluate the first order term, consider the expression of imaginary time correlators given in the handout:

$$\langle A_1(\tau_1) \dots A_n(\tau_n) \rangle = \frac{1}{\mathcal{Z}_0} \sum_n \langle n | e^{-\beta H_0} \mathcal{T}[A_1(\tau_1) \dots A_n(\tau_n)] | n \rangle \tag{4.9}$$

where the A_i are arbitrary operators, $\mathcal{T}$ stands for imaginary time ordering and the Heisenberg picture operators are defined as $A(\tau) = e^{H\tau} A e^{-H\tau}$. While $|n\rangle$ can be an arbitrary orthonormal basis, it is convenient to use the eigenstates of H_0, that is, direct products of the angular momentum states of each rotor. The only operators appearing in the first order term are single $e^{\pm i\theta(\tau)}$ operators acting on some site. As discussed above, these act as ladder operators on the angular momentum eigenstates, that is, $e^{\pm i\theta(\tau)}|n\rangle \propto |n \pm 1\rangle$: this is orthogonal to $|n\rangle$, and since both are eigenstates of H_0, the matrix element of $e^{-\beta H_0}$ is 0. That is, the lowest nontrivial order of the expansion is the second.

It now follows from (4.9) that the only second order terms resulting in a nonzero expectation value are those in which the same site is acted on by an $e^{i\theta}$ and an $e^{-i\theta}$ operator, as only this combination returns an angular momentum eigenstate $|n\rangle$ to itself. Since we are interested in zero-temperature behaviour only, the second order term is given by

$$\begin{aligned}
\frac{1}{2}\Big\langle \Big[\sum_i \int_0^\beta \mathrm{d}\tau \left(e^{i\theta_i}\Psi_i^* + e^{-i\theta_i}\Psi_i \right) \Big]^2 \Big\rangle &= \sum_i \int \mathrm{d}\tau\, \mathrm{d}\tau'\, \Psi_i^*(\tau)\Psi_i(\tau') \left\langle e^{i[\theta_i(\tau)-\theta_i(\tau')]} \right\rangle \\
&= \sum_i \int \mathrm{d}\tau\, \mathrm{d}\tau'\, \Psi_i^*(\tau)\Psi_i(\tau') \times \begin{cases} \left\langle 0 \middle| e^{-\beta H} e^{\tau H} e^{i\theta_i} e^{-\tau H} e^{\tau' H} e^{-i\theta_i} e^{-\tau' H} \middle| 0 \right\rangle & \tau > \tau' \\ \left\langle 0 \middle| e^{-\beta H} e^{\tau' H} e^{-i\theta_i} e^{-\tau' H} e^{\tau H} e^{i\theta_i} e^{-\tau H} \middle| 0 \right\rangle & \tau < \tau' \end{cases} \\
&= \sum_i \int \mathrm{d}\tau\, \mathrm{d}\tau'\, \Psi_i^*(\tau)\Psi_i(\tau') e^{-|\tau-\tau'|/(2m)} :
\end{aligned} \tag{4.10}$$

in the second line, only the ground state contribution was kept from (4.9) as that is the only state relevant in the zero temperature thermal mix; since $H|0\rangle = 0$, the imaginary time evolution operators do not act on it. However, between τ and τ', the rotor i is in an $L = \pm 1$ state the energy of which is $1/(2m)$ and so the time evolution operator $e^{-|\tau-\tau'|H}$ results in a factor $e^{-|\tau-\tau'|/(2m)}$. The factor $1/2$ at the beginning of the expression disappears because $e^{i[\theta_i(\tau)-\theta_i(\tau')]}$ is a cross-term in the expansion of the square, and so it is multiplied by two.

By the same argument as for the first order term, the third (and generally all odd) orders in the expansion vanish: the operators $e^{\pm i\theta}$ cannot be paired up and the expectation value in (4.9) is zero for all angular momentum eigenstates.

Quartic terms, the origin of c.[17] The only relevant fourth order terms are again those where $e^{\pm i\theta}$

[17] The problem doesn't actually require you to go through this derivation, only to point out where the quartic c will come from and why it is positive. The short answer for the first part: if the two excitations created by $e^{\pm i\theta}$ operators live on the same site, they interact, and thus give rise to same-site four-point correlators in the effective Hamiltonian. The short answer for the second part: excitations (imagined to be independent) can interact in two ways. (i) The two excitations change L in different directions, thus annihilating each other in the overlap region: since there is no physical distinction between this scenario and having two non-overlapping excitations, this case disappears from the partition function. (ii) The two excitations change L in the same direction: in the physical partition function, this corresponds to a range of τ where the total excitation energy is four times that of a single excitation, rather than twice (as if the excitations were independent), therefore, the partition function decays faster in this region. In both cases, the interactions result in a smaller partition function than considering independent excitations: this corresponds to an increased effective energy, hence c must be positive.

terms can be paired up such that their overall action on an angular momentum eigenstate is to recover it. This can be done in two distinct ways: acting with one $e^{i\theta}$ and one $e^{-i\theta}$ on two different rotors; or by acting on one rotor with two of each. The first case is simple: the imaginary time evolution of the two rotors can be treated as independent, resulting in a term

$$\frac{1}{2}\sum_{i\neq j}\int \mathrm{d}\tau_1 \dots \mathrm{d}\tau_4\, \Psi_i^*(\tau_1)\Psi_i(\tau_2)\, \Psi_j^*(\tau_3)\Psi_j(\tau_4) e^{-(|\tau_1-\tau_2|+|\tau_3-\tau_4|)/(2m)}. \tag{4.11}$$

(The prefactor is obtained as follows: there are 4! ways of drawing the four different terms from the four copies of the exponent, cancelling out the $1/(4!)$ in the exponential; however, we take ij and ji as distinct, so the $1/2$ is necessary to avoid double counting.) It is easy to see that (4.11) simply gives half the cross-terms in the square of (4.10).

We now consider the case of four operators acting on the same rotor. Since there are $\binom{4}{2} = 6$ ways of choosing the two $e^{i\theta}$ terms out of the four, these result in a correction

$$\frac{1}{4}\sum_{i}\int \mathrm{d}\tau_1 \dots \mathrm{d}\tau_4\, \Psi_i^*(\tau_1)\Psi_i(\tau_2)\Psi_i^*(\tau_3)\Psi_i(\tau_4) \left\langle e^{i[\theta_i(\tau_1)-\theta_i(\tau_2)+\theta_i(\tau_3)-\theta_i(\tau_4)]}\right\rangle. \tag{4.12}$$

At zero temperature, the expectation value in (4.12) can be evaluated by applying (4.9) to the ground state. Consider first the case when the first two operators in time ordering change the ground state angular momentum by one and then return it to zero: restricting the integral to these configurations gives

$$\frac{1}{2}\sum_{i}\int \mathrm{d}\tau_1 \dots \mathrm{d}\tau_4\, \Psi_i^*(\tau_1)\Psi_i(\tau_2)\Psi_i^*(\tau_3)\Psi_i(\tau_4) e^{-(|\tau_1-\tau_2|+|\tau_3-\tau_4|)/(2m)} \tag{4.13}$$

restricted to the configurations where both $\tau_{1,2}$ are either before or after $\tau_{3,4}$. Note that in (4.13), we assume that $\tau_{1,2}$ and $\tau_{3,4}$ are paired up: of course, $\tau_{1,4}$ and $\tau_{2,3}$ is a valid pairing too, hence the doubled prefactor. Save for the restriction over the integration domain, (4.13) is of the same form as (4.11), meaning that together they generate the square of the second order term. We now turn to the terms excluded from both (4.12, 4.13) by restricting the integration domain.

The integral excluded from (4.12) is over those configurations where both $e^{i\theta}$ operators come before or after both $e^{-i\theta}$ operators. For this configuration, we can define the "lifetimes of excitations" $\Delta_{1,2}$, and the time of their overlap, Δ (see Fig. 4.1): the expectation value of $e^{i[\theta_i(\tau_1)-\theta_i(\tau_2)+\theta_i(\tau_3)-\theta_i(\tau_4)]}$ is now $e^{-(\Delta_1+\Delta_2+2\Delta)/(2m)}$. This implies that the only relevant parts of the integral are those where $\tau_{1,\dots,4}$ differ by $\mathcal{O}(m)$, and so, to leading order, all Ψ can be evaluated at the same average time τ[18]. Overall, we obtain the integral

$$\begin{aligned} &2\sum_i \int \mathrm{d}\tau\, \mathrm{d}\Delta_1\, \mathrm{d}\Delta_2\, \mathrm{d}\Delta\, |\Psi_i(\tau)|^4 e^{-(\Delta_1+\Delta_2+2\Delta)/(2m)} \\ &= 16m^3 \sum_i \int \mathrm{d}\tau\, |\Psi_i(\tau)|^4 \times \int_0^\infty \mathrm{d}u_1\, \mathrm{d}u_2 \int_0^{\min(u_1,u_2)} \mathrm{d}u\, e^{-u_1-u_2-2u}\,: \end{aligned} \tag{4.14}$$

for a given set of $\tau, \Delta_{1,2}, \Delta$, one can choose whether $\tau_1 < \tau_3$, $\tau_2 < \tau_4$ and $\tau_{1,3} < \tau_{2,4}$, a total of 8 equivalent terms, giving the first prefactor $8/4 = 2$.

The integral excluded from (4.13) is over those configurations where the imaginary time segments (τ_1, τ_2) and (τ_3, τ_4) overlap. Let the lengths of these segments, and the time difference between their starting points be Δ_1, Δ_2, and $-\Delta_2 < \Delta < \Delta_1$, respectively. Since the differences between $\tau_{1,\dots,4}$ will still be $\mathcal{O}(m)$, all Ψ can again be evaluated at the same time:

$$2\sum_i \int \mathrm{d}\tau\, |\Psi_i(\tau)|^4 \times \int_0^\infty \mathrm{d}\Delta_1\, \mathrm{d}\Delta_2\, (\Delta_1+\Delta_2) e^{-(\Delta_1+\Delta_2)/(2m)}$$

[18] Including differences in τ is not conceptually hard, even if inconvenient, see the treatment of the quadratic term below. The reason we ignore these terms is because they give rise to irrelevant couplings in the Ginzburg–Landau field theory we obtain in the end.

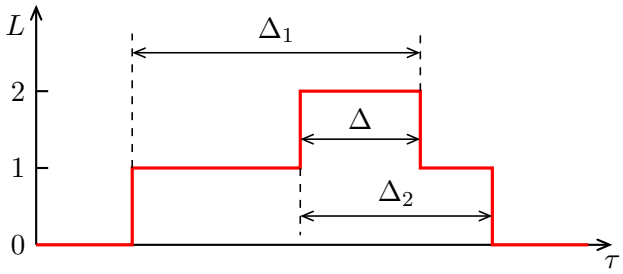

Figure 4.1. An example of two overlapping excitations in the partition function (4.12). The lifetimes of the two excitations are Δ_1 and Δ_2 in time order; their overlap is $\Delta < \Delta_{1,2}$. (Since the two excitations aren't physically distinct, a full overlap between the two could be reinterpreted in the way shown in the figure: including such configurations would result in double counting.) There are four sets of $\tau_{1,\dots,4}$ corresponding to this configuration, and four others correspond to going to -2 with the same steps: these account for the prefactor of 8 in (4.14).

$$= 16m^3 \sum_i \int \mathrm{d}\tau \, |\Psi_i(\tau)|^4 \times \int_0^\infty \mathrm{d}u_1 \, \mathrm{d}u_2 \, e^{-u_1-u_2}(u_1+u_2). \tag{4.15}$$

Here, for a given set of $\tau, \Delta_{1,2}, \Delta$, one can choose whether $\tau_1 < \tau_2$ and $\tau_3 < \tau_4$, a total of 4 equivalent terms, giving the first prefactor $4/2 = 2$. Since (4.13) does not depend on Δ, integrating over it is trivial. All in all, we find that replacing (4.12) with (4.13) increases the integral by

$$cm^3 \sum_i \int \mathrm{d}\tau |\Psi_i(\tau)|^4 \tag{4.16a}$$

where the constant c is given by

$$16 \int_0^\infty \mathrm{d}u_1 \, \mathrm{d}u_2 \, e^{-u_1-u_2} \left[u_1 + u_2 - \int_0^{\min(u_1,u_2)} e^{-2u} \, \mathrm{d}u \right] : \tag{4.16b}$$

the integral in the brackets can at most be $\min(u_1, u_2)$, so the integrand is always positive, therefore $c > 0$.

Summarising (4.11, 4.12, 4.13, 4.16), the quartic term in the expansion is given to leading order by

$$\frac{1}{2} \left[\sum_i \int \mathrm{d}\tau \, \mathrm{d}\tau' \, \Psi_i^*(\tau) \Psi_i(\tau') e^{-|\tau-\tau'|/(2m)} \right]^2 - cm^3 \sum_i \int \mathrm{d}\tau |\Psi_i(\tau)|^4, \tag{4.17}$$

and so the whole expansion up to fourth order is

$$\begin{aligned} &\left\langle \exp\left[-\sum_i \int_0^\beta \mathrm{d}\tau \left(e^{i\theta_i} \Psi_i^* + e^{-i\theta_i} \Psi_i \right) \right] \right\rangle \\ &\qquad = 1 + \sum_i \int \mathrm{d}\tau \, \mathrm{d}\tau' \, \Psi_i^*(\tau) \Psi_i(\tau') e^{-|\tau-\tau'|/(2m)} + \frac{1}{2} \left[\sum_i \int \mathrm{d}\tau \, \mathrm{d}\tau' \, \Psi_i^*(\tau) \Psi_i(\tau') e^{-|\tau-\tau'|/(2m)} \right]^2 \\ &\qquad\quad - cm^3 \sum_i \int \mathrm{d}\tau |\Psi_i(\tau)|^4 + \dots . \end{aligned} \tag{4.18}$$

Re-exponentiation. (4.18) can be regarded as the first terms in the expansion of an exponential: the third term in particular is half the square of the second, meaning the only quartic term in the exponent will come from the last term. Therefore, we can rewrite (4.8) as

$$\begin{aligned} \mathcal{Z} = \mathcal{Z}_0 \int \mathcal{D}\Psi_i(\tau) \, \mathcal{D}\Psi_i^*(\tau) \exp\Bigg[&- \sum_{ij} \int_0^\beta \mathrm{d}\tau \, \Psi_i^* G_{ij}^{-1} \Psi_j + \sum_i \int \mathrm{d}\tau \, \mathrm{d}\tau' \, \Psi_i^*(\tau) \Psi_i(\tau') e^{-|\tau-\tau'|/(2m)} \\ &- cm^3 \sum_i \int \mathrm{d}\tau |\Psi_i(\tau)|^4 + \dots \Bigg]. \end{aligned} \tag{4.19}$$

Time derivatives. We now replace the double integral in (4.19) by a single integral using a similar idea as in the derivation of (4.16). The only difference is that now we have to keep track of terms containing up to two time derivatives, since those still give rise to a marginal RG coupling. Therefore, we introduce the variables $\tilde{\tau} = (\tau + \tau')/2$ and $u = (\tau - \tau')/(2m)$, and substitute them into the double integral to obtain

$$\begin{aligned}
\int \mathrm{d}\tau\, \mathrm{d}\tau'\, \Psi^*(\tau)\Psi(\tau') e^{-|\tau-\tau'|/(2m)} &= 2m \int \mathrm{d}\tilde{\tau}\, \mathrm{d}u \Psi^*(\tilde{\tau}+mu)\Psi(\tilde{\tau}-mu) e^{-|u|} \\
&= 2m \int \mathrm{d}\tilde{\tau}\, \mathrm{d}u \left[\Psi^* + mu\partial_\tau \Psi^* + \frac{m^2u^2}{2}\partial_\tau^2 \Psi^*\right] \left[\Psi - mu\partial_\tau \Psi + \frac{m^2u^2}{2}\partial_\tau^2 \Psi\right] e^{-|u|} + \mathcal{O}(\partial_\tau^3 \Psi^2) \\
&= 2m \int \mathrm{d}\tau\, |\Psi(\tau)|^2 \times \underbrace{\int_{-\infty}^{\infty} e^{-|u|}\, \mathrm{d}u}_{2} \\
&\quad - 2m^3 \int \mathrm{d}\tau \left[|\partial_\tau \Psi|^2 - \frac{\Psi^* \partial_\tau^2 \Psi + \Psi \partial_\tau^2 \Psi^*}{2}\right] \times \underbrace{\int_{-\infty}^{\infty} u^2 e^{-|u|}\, \mathrm{d}u}_{4} + \mathcal{O}(\partial_\tau^3 \Psi^2) \\
&= 4m \int \mathrm{d}\tau\, |\Psi(\tau)|^2 - 16m^3 \int \mathrm{d}\tau\, |\partial_\tau \Psi|^2 + \mathcal{O}(\partial_\tau^3 \Psi^2);
\end{aligned} \tag{4.20}$$

we did not list the integrals that are odd in u and hence vanish, and stopped at two time derivatives (higher-order terms would be irrelevant in the Ginzburg–Landau theory). In the last line, we integrated $\Psi^* \partial_\tau^2 \Psi + \Psi \partial_\tau^2 \Psi^*$ by parts. In summary, (4.19) becomes, to leading order,

$$\mathcal{Z} = \mathcal{Z}_0 \int \mathcal{D}\Psi_i(\tau)\, \mathcal{D}\Psi_i^*(\tau) \exp\left[-\int_0^\beta \mathrm{d}\tau\, L[\Psi^*, \Psi]\right]; \tag{4.21a}$$

$$L = \sum_{ij} \Psi_i^* G_{ij}^{-1} \Psi_j + \sum_i \left(-4m|\Psi_i|^2 + 16m^3 |\partial_\tau \Psi_i|^2 + cm^3 |\Psi_i|^4\right). \tag{4.21b}$$

Space derivatives. Finally, we coarse-grain (4.21b) by replacing the discrete Ψ_i with $\Psi(\mathbf{r})$ and expanding G^{-1} in the first term in spatial derivatives. The definition of G can be Fourier transformed to give

$$G(\mathbf{q}) = g \sum_{i=1}^{d} \cos(q_i a) \approx g\left(d - \frac{a^2 \mathbf{q}^2}{2}\right) \implies G^{-1}(\mathbf{q}) \approx \frac{1}{gd} + \frac{a^2 \mathbf{q}^2}{2gd^2}. \tag{4.22}$$

Coarse-graining comes from ignoring higher powers in $\mathbf{q}$, corresponding to higher derivatives in real space. That is, the first term in (4.21b) can be coarse-grained into

$$\sum_{ij} \Psi_i^* G_{ij}^{-1} \Psi_j = \sum_{\mathbf{q}} \Psi_{\mathbf{q}}^* G_{\mathbf{q}}^{-1} \Psi_{-\mathbf{q}} \approx \int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d} \left(\frac{1}{gd} + \frac{\mathbf{q}^2}{2gd^2}\right) \Psi_{\mathbf{q}}^* \Psi_{-\mathbf{q}} = \int \mathrm{d}\mathbf{r} \left(\frac{1}{gd} |\Psi(\mathbf{r})|^2 + \frac{1}{2gd^2} |\nabla \Psi|^2\right). \tag{4.23}$$

Writing this into (4.21) gives the required Ginzburg–Landau form:

$$\mathcal{Z} = \mathcal{Z}_0 \int \mathcal{D}\Psi(\mathbf{r}, \tau)\, \mathcal{D}\Psi^*(\mathbf{r}, \tau) \exp\left[-\int_0^\beta \mathrm{d}\tau \frac{\mathrm{d}\mathbf{r}}{a^d} \mathcal{L}[\Psi^*, \Psi]\right]; \tag{4.24a}$$

$$\mathcal{L} = t|\Psi(\mathbf{r}, \tau)|^2 + \frac{a^2}{2d^2 g} |\nabla \Psi|^2 + 16m^3 |\partial_\tau \Psi|^2 + cm^3 |\Psi|^4 \tag{4.24b}$$

where $t = 1/(dg) - 4m$ and c is given by (4.16).

(b) As discussed in the handout, the object appearing in the imaginary time action is the Hamiltonian expressed in terms of derivatives rather than canonical momenta. From this form, it is easy to see that the corresponding canonical momentum is $\pi = 32m^3 \partial_\tau \Psi$, and thus the Hamiltonian is

$$\mathcal{H} = \frac{|\pi|^2}{64m^3} + t|\Psi|^2 + \frac{a^2}{2d^2 g} |\nabla \Psi|^2 + cm^3 |\Psi|^4. \tag{4.25}$$

(c) Correlation length & energy gap near the QCP. If $d = 3$, the Lagrangian (4.24b) is the standard 4-dimensional Ginzburg–Landau one (the imaginary time and space derivatives enter the Hamiltonian the same way, so they can be treated on an equal footing, as discussed in Problem 1). Since this is the upper critical dimension of the Lagrangian, the mean field exponents hold, and so

$$\xi \sim \xi_\tau \sim |t|^{-\nu}; \qquad \nu = 1/2. \tag{4.26}$$

The energy gap is a typical energy scale of the system, and as such, it scales as $\xi_\tau^{-1} \sim t^{1/2}$.

(d) Behaviour and phase diagram in two dimensions. At zero temperature, there is still an order–disorder transition since the behaviour of the system at weak and strong interactions is qualitatively different:

- At very small mg, the ground state of the system is still the $L = 0$ eigenstate of each rotor, with a weak admixture of higher states by the perturbation: this state is clearly paramagnetic. The finite gap to the lowest excited state ($L = \pm 1$ on one site) persists.
- At very large mg, the Hamiltonian is dominated by the interactions: energy is minimised if $\theta_i - \theta_j = 0$ for all nearest neighbours. The ground state is ferromagnetic, low-lying excitations are gapless spin waves.

The quantum field Hamiltonian describing the zero-temperature behaviour can be derived in the exact same way as in part (a), and the result is also the same. The Hamiltonian is a 3d $n = 2$ Ginzburg–Landau Hamiltonian, which is also known to have an order–disorder transition. The corresponding critical exponents are not given by mean field theory, but they are listed in §2.7 of the handout. In particular,

$$\xi \sim \xi_\tau \sim |t|^{-\nu} \implies \Delta \sim |t|^{\nu} \qquad \nu \approx 0.7. \tag{4.27}$$

The phase diagram of the finite-temperature system is quite different from the 3d case. The classical limit of this model is the 2d XY model which is known not to have an ordered phase at $T > 0$. Accordingly, there is no classical critical point and so no critical line or classical critical region. The only feature of the phase diagram that persists is the quantum critical point and region [see Fig. 4.2(b)].

(e) Partition function of a single rotor. The eigenstates and energies of the single rotor Hamiltonian were discussed at the beginning of the solution. Substituting $m = 1$, the energy eigenvalues are $E_n = n^2/2$ for all integers n, and therefore, the partition function is

$$\mathcal{Z}_1 = \sum_{n=-\infty}^{\infty} \exp\left(-\beta \frac{n^2}{2}\right). \tag{4.28}$$

The correlator $\langle e^{i\hat{\theta}(\tau)} e^{-i\hat{\theta}(0)} \rangle$ can now be expressed using the eigenstate representation of the thermal density matrix, (4.9):

$$\begin{aligned}
\mathcal{Z}_1 \langle e^{i\hat{\theta}(\tau)} e^{-i\hat{\theta}(0)} \rangle &= \sum_n \langle n | e^{-\beta H} e^{i\hat{\theta}(\tau)} e^{-i\hat{\theta}(0)} | n \rangle = \sum_n \langle n | e^{-(\beta-\tau)H} e^{i\hat{\theta}} e^{-\tau H} e^{-i\hat{\theta}} | n \rangle \\
&= \sum_n \langle n | e^{-(\beta-\tau)H} e^{i\hat{\theta}} e^{-\tau H} | n-1 \rangle = \sum_n e^{-(\beta-\tau)n^2/2} e^{-\tau(n-1)^2/2} \langle n | e^{i\hat{\theta}} | n-1 \rangle \\
&= \sum_n e^{-\beta n^2/2 + \tau n - \tau/2}
\end{aligned} \tag{4.29}$$

using the ladder operator properties of $e^{\pm i\hat{\theta}}$.

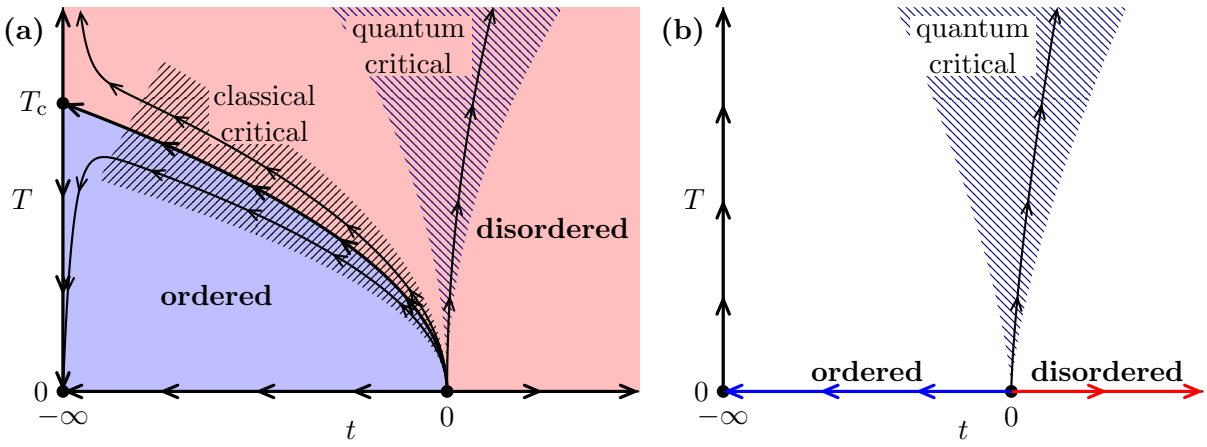

Figure 4.2. Phase diagram and RG flowlines of the quantum rotor model in three (a) and two (b) dimensions. In three dimensions, both the classical and the zero-temperature limit undergoes an order–disorder transition: these are connected by a critical RG flowline separating an ordered and a disordered phase. RG flowlines near the critical one pass near the classical fixed point, spend a long "RG time" there: as a result, their behaviour is governed by the classical fixed point, giving rise to a classical critical region. Away from this region, in the disordered phase, the RG flow is dominated by the finite-size scaling of temperature, resulting in a finite-temperature quantum critical region.
In two dimensions, the classical limit has no ordered phase due to the Mermin–Wagner theorem. Order does not survive at any finite temperature; however, a quantum critical region still appears, where the RG flow is dominated by finite-size scaling and thus critical exponents of the quantum critical point.

At low temperatures, the only relevant term in both (4.28) and (4.29) is the $n = 0$ one (the $n = \pm 1$ ones are suppressed as $e^{-1/2T}$, higher ones even more). In this approximation, $\mathcal{Z}_1 = 1$ and $\mathcal{Z}_1 \langle e^{i\hat{\theta}(\tau)} e^{-i\hat{\theta}(0)} \rangle = e^{-\tau/2}$, and so the imaginary time correlator is

$$\langle e^{i\hat{\theta}(\tau)} e^{-i\hat{\theta}(0)} \rangle = e^{-\tau/2} \tag{4.30}$$

for $\tau \ll \beta$ [if $\tau \lesssim \beta$, $n \neq 0$ terms also become relevant in (4.29)].

At high temperatures, many terms are relevant in both sums, and so they can be replaced by an integral:

$$\mathcal{Z}_1 \simeq \int_{-\infty}^{\infty} \mathrm{d}n \, \exp\left(-\beta \frac{n^2}{2}\right) = \sqrt{\frac{2\pi}{\beta}}. \tag{4.31}$$

$$\mathcal{Z}_1 \langle e^{i\hat{\theta}(\tau)} e^{-i\hat{\theta}(0)} \rangle \simeq \int_{-\infty}^{\infty} \mathrm{d}n \, \exp\left(-\beta \frac{n^2}{2} + \tau n - \frac{\tau}{2}\right) = e^{-\tau/2 + \tau^2/(2\beta)} \int_{-\infty}^{\infty} \mathrm{d}\tilde{n} \, e^{-\beta \tilde{n}^2/2} \tag{4.32}$$

$$\langle e^{i\hat{\theta}(\tau)} e^{-i\hat{\theta}(0)} \rangle \simeq e^{-\tau/2 + T\tau^2/2}. \tag{4.33}$$

(f) Partition function of a single rotor as a path integral. The partition function (4.28) is effectively a single-rotor version of (4.4) without interactions, and so we can use that to find the path integral form:

$$\mathcal{Z}_1 = \int_{\theta(\beta) - \theta(0) = 2\pi n} \mathcal{D}\theta(\tau) \exp\left[- \int_0^\beta \mathrm{d}\tau \frac{(\partial_\tau \theta)^2}{2} \right]. \tag{4.34}$$

(g) Expansion of the path integral around "classical trajectories". In reality, $\mathcal{Z}_1$ is a sum over possible values of the integer winding number n in (4.34). For each n, we can rewrite the trajectory as $\theta(\tau) = 2\pi n \tau/\beta + \theta_p(\tau)$ where θ_p now has the same value at $\tau = 0$ and β. Substituting this form into (4.34) gives

$$\mathcal{Z}_1 = \sum_{n=-\infty}^{\infty} \int_{\theta_p(0) = \theta_p(\beta)} \mathcal{D}\theta_p(\tau) \exp\left\{ - \int_0^\beta \mathrm{d}\tau \, \frac{1}{2} \left[\partial_\tau \left(\frac{2\pi n \tau}{\beta} + \theta_p \right) \right]^2 \right\}$$

$$= \sum_{n=-\infty}^{\infty} \int_{\theta_p(0)=\theta_p(\beta)} \mathcal{D}\theta_p(\tau) \exp\left[-\frac{\beta}{2}\left(\frac{2\pi n}{\beta}\right)^2 - \frac{2\pi n}{\beta} \underbrace{\int_0^\beta \mathrm{d}\tau\, \dot\theta_p}_{\theta_p(\beta)-\theta_p(0)=0} - \int_0^\beta \mathrm{d}\tau\, \frac{\dot\theta_p^2}{2}\right]$$

$$= \sum_{n=-\infty}^{\infty} \exp\left(-\frac{1}{2}\frac{(2\pi n)^2}{\beta}\right) \int_{\theta_p(0)=\theta_p(\beta)} \mathcal{D}\theta_p(\tau) \exp\left[-\frac{1}{2}\int_0^\beta \mathrm{d}\tau\, \dot\theta_p^2\right]. \tag{4.35}$$

Note. The general idea behind this calculation is separating potential trajectories into a corresponding "classical trajectory" (a solution of the corresponding Euler–Lagrange equations) and fluctuations around it. Since the classical trajectory extremises the action, any linear terms in fluctuations integrate to zero (like here), and so the integral over fluctuations reduces to a Gaussian integral (in our case, exactly; in general, as a saddle point approximation). The rationale in doing this is that the classical trajectory is usually a good first approximation of the physics, and fluctuations can be treated analytically using the saddle point approximation.

Eq. 4.35 can clearly be factorised into two terms that only contain n and θ_p, respectively. As $T \to 0$, the periodicity of the imaginary time dimension in the latter becomes irrelevant, so it can be treated as a 1d Gaussian path integral of the kind discussed in Problem I/3. In particular, (1.46) immediately gives

$$\left\langle e^{i\theta_p(\tau)} e^{-i\theta_p(0)} \right\rangle = e^{-\tau/2}. \tag{4.36}$$

Since β is large in our limit, many values of n contribute to the sum in the first term, therefore, it can be replaced with a (Gaussian) integral. It follows that the distribution of n is Gaussian with zero mean and variance $\beta/(2\pi)^2$; therefore,

$$\left\langle e^{2\pi i n\tau/\beta} \right\rangle = \exp\left[-\frac{1}{2}\frac{\beta}{(2\pi)^2}\left(\frac{2\pi\tau}{\beta}\right)^2\right] = e^{T\tau^2/2}$$

$$\left\langle e^{i\theta(\tau)} e^{-i\theta(0)} \right\rangle = e^{-\tau/2 + T\tau^2/2}. \tag{4.37}$$

This matches both the low and high temperature results found previously – the latter is somewhat surprising seeing we did not take the periodicity of τ into account.

(h) Real time correlators, thermal decoherence. Remembering that imaginary time was introduced by substituting $t = -i\tau$ into the evolution operator e^{-iHt}, imaginary time correlators can be turned into real time ones by undoing the substitution. Writing $\tau = it$ into (4.33), we obtain

$$\langle e^{i\hat\theta(t) - i\hat\theta(0)} \rangle_T = e^{-it/2 - Tt^2/2} \tag{4.38}$$

at high temperatures. The imaginary part of the exponent is easy to interpret. At zero temperature, the system will always be in the ground state: the effect of the two operators is to move the rotor to the $L = \pm 1$ state for time t. Since the energy difference between these two states is $1/2$, moving there for time t gives rise to a phase shift $t/2$. The real part of the exponent is the result of the different phase shifts incurred similarly by the different eigenstates making up the thermal mix: the higher the temperature, the faster this *thermal decoherence* is.

(i) Phase diagram in three dimensions. At zero temperature, there will be an order–disorder transition at $t = 0$. The "classical limit" of the quantum rotor model is the XY spin model where the the kinetic energy term of (4.1) disappears: this corresponds to $m = \infty \implies t = -\infty$. This model also has an order–disorder transition at some finite temperature T: it is logical to expect that these transition points are connected by a critical line in (t, T) plane, separating an ordered and a disordered phase.

The finite but low temperature behaviour of the system is still described by (4.24b), but τ is restricted to a segment of length β rather than the whole real line. It follows that this length is an irrelevant

parameter, therefore, its inverse, T is relevant. The finite-size scaling of the susceptibility χ is therefore given by the scaling hypothesis:

$$\chi = T^{-\gamma/\nu} X(T\xi). \tag{4.39}$$

On the aforementioned critical line, this susceptibility diverges: this is possible if X diverges at some value b of its argument. The critical temperature therefore scales as

$$T_c = \frac{b}{\xi} \propto t^{\nu} = t^{1/2}. \tag{4.40}$$

There are two distinct critical regions. One of them is near the transition line: RG flows originating near the $T = 0$ fixed point get near the classical fixed point, spend a long "RG time" near it, and thus their behaviour becomes dominated by the critical exponents of the classical fixed point. This region is called *classical critical*: it appears at $T\xi \approx b$ where X is dominated by the divergence at b. From this wording it is clear that this critical region is defined only approximately, and so it has no boundaries which could give rise to additional transitions.

The other critical region corresponds to RG flowlines dominated by finite-size scaling of the quantum field path integral. In this *quantum critical region*, $T\xi \gg b$: the RG flow never gets near the classical fixed point, and so the behaviour is controlled only by the quantum critical point.

This behaviour and a few representative RG flowlines are sketched in Fig. 4.2(a).

(j) Finite size scaling of the susceptibility. In the quantum critical region, $T\xi \gg b$, that is, finite-size effects are dominant. Accordingly, X is expected to be smooth function there, *i.e.* it does not contribute to the critical behaviour:

$$\chi \propto T^{-\gamma/\nu} = T^{-2} \tag{4.41}$$

in $3 + 1$ dimensions.

(k) Finite size scaling of the correlation length. This susceptibility is given by the fluctuation–dissipation theorem as

$$\chi = \frac{\partial M}{\partial h} = \frac{1}{k_{\mathrm{B}} T} \left(\langle M^2 \rangle - \langle M \rangle^2 \right) \tag{4.42}$$

where M is the mean magnetisation of the system, equivalent to the $q = 0$ Fourier component of the order parameter Ψ. Due to rotational symmetry, $\langle M \rangle = 0$, whereas $\langle M^2 \rangle = \langle \Psi(0)\Psi(0) \rangle \propto \xi^2$ using the Gaussian fluctuation correlators derived in §2.6 of the handout. All in all, $\chi \propto T^{-1}\xi^2$, but also $\chi \propto T^{-2}$: from these two, it follows that in the quantum critical regime, $\xi \propto T^{-1/2}$.