Lecture 3 Stationary Markov process(es)

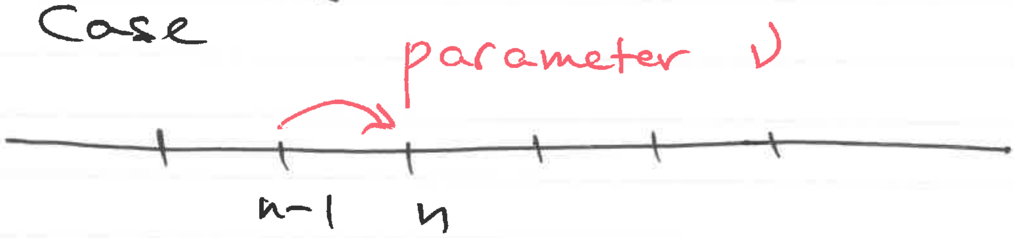
with the propagator

$$G(y_2\, y_1 \mid t_2 - t_1)$$

Two of these processes are especially important in Physics

Wiener process	Poisson process
independent steps	independent steps
≠ equal, Must step	only forward or not
Gaussian step distribution	"number of hits in a fixed target"

1D case



$\boxed{\nu : \text{rate of a single step}}$

probability to make a step in time dt

$$p_+ = \nu\, dt$$

The whole stochastic process: $n(t)$

- How far (n) did we get in time (t)?
- How long (t) should it take to reach (n)?

Still within a single step:

$(n-1) \longrightarrow (n)$

- What is the total (cumulative) probability to make this step in time t: $W(t)$
- Or equivalently, what is the "survival probability" to still not step in time t:

$$S(t) = 1 - W(t)$$

Then: $\underbrace{S(t+dt) = S(t) - \nu\, dt \cdot S(t)}$

$$\frac{dS}{dt} = -\nu S$$

$\boxed{\text{if } t=0,\ S(t)=1}$

$$\boxed{S(t) = e^{-\nu t}}$$

Now define probability density:

$$w = \frac{\partial W(t)}{\partial t} = -\frac{\partial S(t)}{\partial t} = \nu e^{-\nu t}$$

So we can evaluate integrals:

Average time $(n-1) \to (n)$ $\Big]$ $\langle t \rangle_{\text{step}} = \int_0^\infty t\, w(t)\, dt = 1/\nu$

Also $\langle t^2 \rangle - \langle t \rangle^2 = 1/\nu^2$, etc.

Exercise (simulation):

fix time of step Δt, and "flip a biased coin":

$p_+ = \nu \Delta t$ and $p_0 = 1 - \nu dt$

When do we reach a point (k) after a time $t = N\Delta t$. make k steps

This is a binomial distribution

$$P(k, N) = \frac{N!}{k!(N-k)!} p_+^k \, p_0^{N-k}$$

to reach (k) after (N) steps

At large N, with Np = constant λ

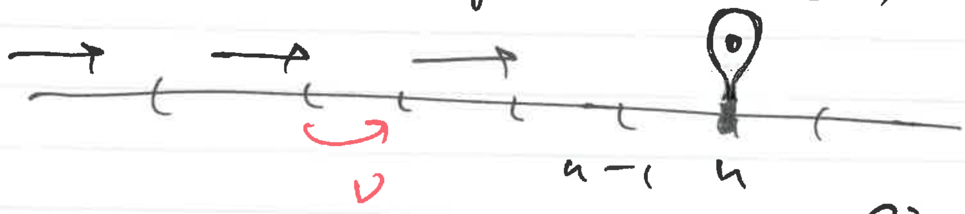
this reduces to

$$\boxed{p(k, \lambda) = \frac{\lambda^k}{k!} e^{-\lambda}}$$

this is "Poisson"

Now lets derive / understand it.

- What is the probability to reach a position (n) after time



(t), given step rate (ν)

Two external parameters: (n, t)

The process feature: (ν)

$$P(n, t+dt) = P(n,t)\cdot(1-\nu dt)$$

Stay at (n)

$$+\; P(n-1,t)\cdot \nu dt$$

Step from $(n-1)$

$$\frac{\partial P(n,t)}{\partial t} = \nu\left[P(n-1,t) - P(n,t)\right]$$

This is often called the "Master Equation"

$$\boxed{\dot{P} = \text{rate in} - \text{rate out}}$$

Recall we saw "characteristic function"

$$\phi_x(k) = \langle e^{ikx} \rangle_{P(x)}$$

Similar logic employs the "generating function"

$$\boxed{g(k,t) = \sum_{n=0}^{\infty} P(n,t)\, k^n}$$

Now: $$\frac{\partial g(k,t)}{\partial t} = \sum_{n=0}^{\infty} k^n \cdot \nu \left[P(n-1,t) - P(n,t) \right]$$

renumber the sum
$$= \sum_{n=0}^{\infty} \nu \left(k^{n+1} - k^n \right) P(n,t)$$

$$= \nu (k-1) \sum_{n=0}^{\infty} k^n P(n,t)$$

but this is the definition of $g(k,t)$

$$\boxed{\frac{\partial g}{\partial t} = \nu (k-1) \cdot g}$$

At $t=0$
$$P(n,0) = \delta_{n,0}$$
$$g(k,0) = 1$$

$$g(k) = e^{\nu (k-1) t}$$

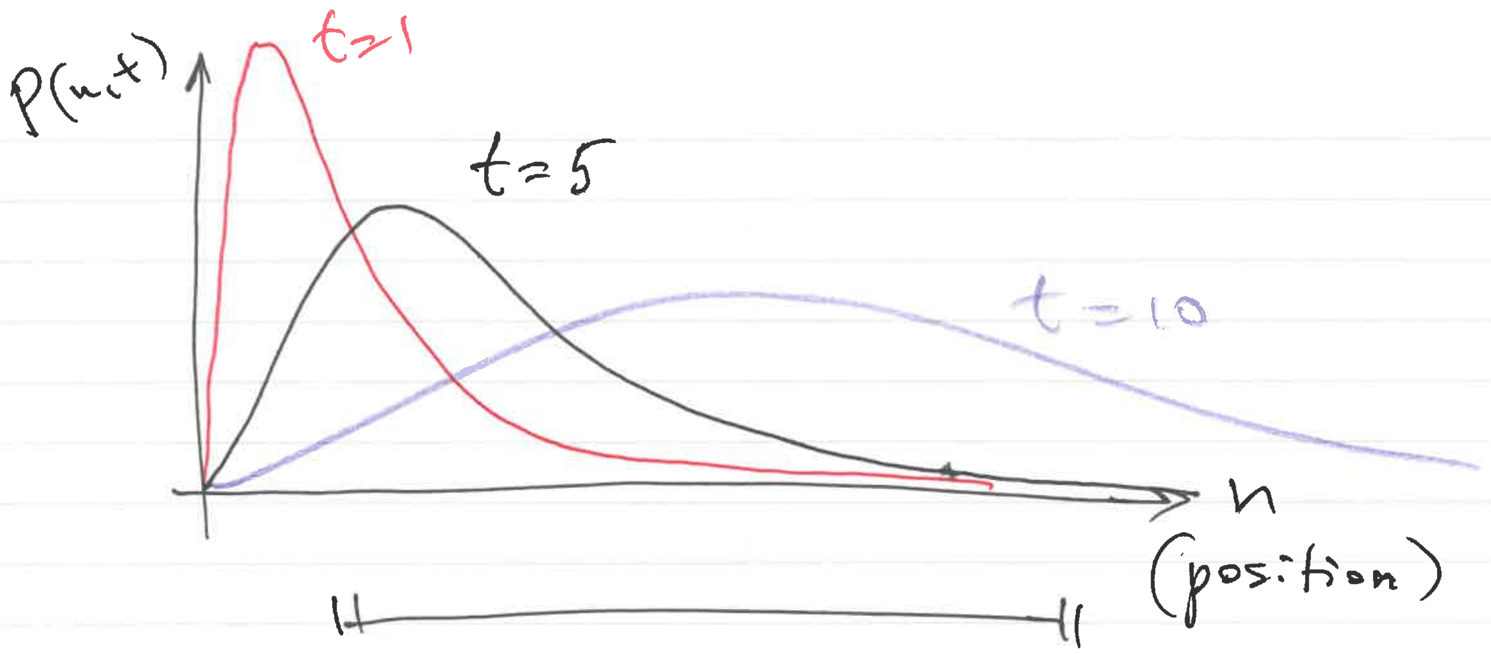
Now expand this exponential: (to isolate $\sum_n k^n (\dots)$)

$$g(k,t) = e^{-\nu t} \sum_{n=0}^{\infty} \frac{(\nu k t)^n}{n!} = \sum_{n=0}^{\infty} k^n \left(\frac{(\nu t)^n}{n!} e^{-\nu t} \right)$$

this is the Poisson distribution

$$\boxed{P(n,t) = \frac{1}{n!} (\nu t)^n e^{-\nu t}}$$

Before we had $\lambda = Np$, now use (νt) as the "time" parameter



⊙ Average time of the 10^{th} step

$n=10$

each step is independent!

$$\langle t_{10} \rangle = 10 \langle t_1 \rangle = 10/\nu$$

⊙ Probability that 10^{th} step $(n=10)$ occurs after time t.

$P(t_{10} > t)$ or $P(n(t) < 10)$

$\downarrow$

9 events at t', then step

$$\int_t^{\infty} \underbrace{\frac{(\nu t')^9}{9!} e^{-\nu t'}} \cdot \nu\, dt'$$

(the step #10 occurs: $t' > t$)

$\downarrow$

Sum of $0 \div 9$ steps at t:

$$\sum_{n=0}^{9} P(n,t)$$

$$= \sum_{n=0}^{9} \frac{(\nu t)^n}{n!} e^{-\nu t}$$

They better be equal!

A comment about Master Equation

⊙ Evolution:

$$P(n, t+\Delta t) = \sum_m G(n, t+\Delta t \mid m, t)\, P(m, t)$$

Subtract $P(n,t)$ from both sides

$$\frac{\partial P(n,t)}{\partial t} = \frac{1}{\Delta t}\Big[\sum_m G(n, t+\Delta t \mid m, t)\, P(m,t) - P(n,t)\cdot \underbrace{\sum_m G(m, t+\Delta t \mid n, t)}_{=1}\Big]$$

$$= \sum_m \frac{G(n, t+\Delta t \mid m, t)}{\Delta t}\cdot P(m,t)$$

"flux in" ($\forall m \to n$)

$$- \sum_m \frac{G(m, t+\Delta t \mid n, t)}{\Delta t}\cdot P(n,t)$$

"flux out" ($n \to \forall m$)

So: $$\frac{\partial P(n,t)}{\partial t} = \sum_m \big[w_{nm} P(m,t) - w_{mn} P(n,t) \big]$$

transition probabilities....

~~Not necessary to have $n > m$...~~

It is so for Poisson process, but the Evolution relation is more general.