Lecture 5

we just did "Itoh Lemma"
and looked at
"GBM" $dS = \mu S dt + \sigma S dW$

Entry level of financial application:
"Black - Scholes equation"
→ Nobel Prize 1997 (also to Merton)

Analysis of dynamics and prediction of $\boxed{\text{option prices}}$ in a volatile market.

How to decide on a "portfolio" to buy: exchange $\boxed{\text{risk}}$ for $\boxed{\text{cash}}$

Crisis 1987
→ Traders colluded to punish the early adopters of B-S.

Crisis 2008
→ Too much trust into B-S analysis, incorrect use of it.

"Stock" $S(t)$: value of shares

$$dS = \mu S\,dt + \sigma S\,dW$$

"Option" is a contract when you (seller) agree to sell some stock $S(t)$ at a future time t, at a current price (S)

The buyer have to pay a fee for this : $V(S)$.

Obviously $V(S, t)$ and $V(t=0)=0$

- How to estimate the best $V(S)$

"Portfolio" of this transaction is what the buyer has :

cash
$$\Pi = -V(S) + \alpha S$$

(paid) ... (cash is some stock)

this is called "hedging"

In continuous trading, we have small increments

$$\boxed{\Delta\Pi = -\Delta V(S) + \alpha \Delta S}$$

Itoh Lemma Says:

$$dV(S) = \left(\frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2\right)$$

$$dV = \left(\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt$$

$$+ \sigma S \frac{\partial V}{\partial S} dW$$

Now put this back into $d\Pi$:

$$d\Pi = -\frac{\partial V}{\partial t}dt - \mu S\frac{\partial V}{\partial S}dt - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt$$

$$+ \alpha \mu S\, dt$$

$$\underbrace{+\left(\alpha \sigma S - \sigma S\frac{\partial V}{\partial S}\right)dW}_{\text{fluctuating part}}$$

We can choose the factor α such that the bracket $= 0$!

$$\boxed{\alpha = \frac{\partial V}{\partial S}}$$

☉ Then no volatility left in $d\Pi$

Now we wish to build the equation for $V(S)$, using some model Π.

Determine the l.h.s. $d\Pi$:

we need to make a decision on what we want to achieve:

$$d\Pi = r \cdot \Pi \, dt$$

we "dial in" the rate of growth.

What is the lowest risk-free rate of growth?

l.h.s. r.h.s.

$$r(\alpha S - V)\cancel{dt} = \left(-\frac{\partial V}{\partial t} - \cancel{\mu S \frac{\partial V}{\partial S}} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \cancel{\alpha \mu S}\right)\cancel{dt}$$

$$\left(\alpha = \frac{\partial V}{\partial S}\right)$$

Finally we have: with initial $V(t=0)=0$

$$\frac{\partial V(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = r\left(V - S\frac{\partial V}{\partial S}\right)$$

this is the B-S equation for $V(S,t)$. To solve it we need to know r and σ, but note that μ has disappeared ...

1) Consider the simplest: $r=0$

$$\frac{\partial V}{\partial t}+\frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2}=0$$

this is "Super diffusion" in space $S(t)$ — has known solutions

⊙ Are we certain we know σ?
All of the modern Science is about more accurate σ

2) Some standard Maths of B-S eq.

(A) Substitution: S – stock now ($t=0$)

E – stock at the end ($t=T$)

define $x=\ln(S/E)$

or $S=Ee^x$

(B) Re-name $V(S,t)$ into $\underbrace{V(Ee^x, T-\tau)}_{\text{Call it } Z(x,\tau)}$

$$\boxed{\frac{\partial Z}{\partial \tau}-\frac{\sigma^2}{2}\frac{\partial^2 Z}{\partial x^2}+\left(\frac{\sigma^2}{2}-r\right)\frac{\partial Z}{\partial x}+rZ=0}$$

Now it's homogeneous!

(c) Then call

$$u(x,t) = Z(x,\tau) \cdot e^{\beta x + \gamma \tau}$$

enter new parameters (β, γ)

then

$$\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} + \underline{A(\beta,\gamma)} \cdot \frac{\partial u}{\partial x} + \underline{B(\beta,\gamma)} \cdot u = 0$$

Choose β, γ such that $A = 0$, $B = 0$

$$\beta = r/\sigma^2 - 1/2$$

$$\gamma = r/2 + \frac{\sigma^2}{8} + r^2/2\sigma^2$$

then just simple diffusion eq.

$$\boxed{\frac{\partial u}{\partial \tau} = \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2}}$$

via Green's function

$$u(x,\tau) = \int \frac{1}{\sqrt{2\pi\sigma^2\tau}} e^{-\frac{(x-w)^2}{2\sigma^2\tau}} \cdot u(w,0)\, dw$$

when $x(\tau=0) = w$

then roll back substitutions!

first $Z(x,\tau)$ from $u(x,\tau)$,

then $V(S,t)$

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The Heston Model – The Physics of Stochastic Volatility Models

Date **Friday, 9 February**
Time **11:00 a.m. - 12:00 p.m.**
Room **Bragg Building, Small Lecture Theather**

Open to all Cambridge students

Join us for an exclusive lecture by Optiver on financial modeling. Dive into the complexities of the Heston model, a cornerstone in stochastic volatility modeling, and discover how it revolutionizes our understanding of financial markets.

Key agenda points

- Comprehensive overview of the Heston model's dynamic properties
- In-depth analysis of the affine structure inherent to the model
- Partial Differential Equations (PDEs) and Ordinary Differential equations (ODEs) of the model
- Practical application: Characteristic function in option pricing

About the lecturer

Fabio Maggioni is an Optiver quantitative researcher working on the development and improvement of new pricing models and techniques. The focus of his research is around the impact of rate and volatility term structure and dividend uncertainty on the early exercise of American options. He joined Optiver after earning a Master's Degree in Mathematical Engineering at Polytechnic University of Milan.