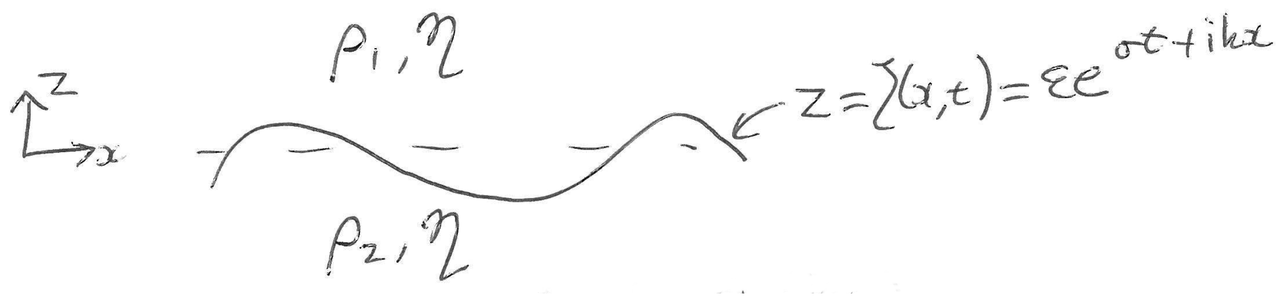
Natural Sciences Tripos Part III — Physics of the Earth as a Planet

Solutions: Examples Sheet 2: Mantle Convection — John Rudge

Please attempt the questions yourself before studying these solutions.

1



Assume $\psi \to 0$ as $z \to \pm\infty$ to write the streamfunctions in the two half spaces as

$$\psi_1 = (C + Dz)\mathrm{e}^{-kz}\mathrm{e}^{\sigma t + ikx},$$
$$\psi_2 = (A + Bz)\mathrm{e}^{kz}\mathrm{e}^{\sigma t + ikx}.$$

The velocities $\mathbf{v} = (u, 0, w)$ are then given by

$$\mathbf{v}_1 = \left(-\frac{\partial \psi_1}{\partial z}, 0, \frac{\partial \psi_1}{\partial x}\right) = \left(k(C + Dz) - D, 0, ik(C + Dz)\right)\mathrm{e}^{-kz}\mathrm{e}^{\sigma t + ikx},$$
$$\mathbf{v}_2 = \left(-\frac{\partial \psi_2}{\partial z}, 0, \frac{\partial \psi_2}{\partial x}\right) = \left(-k(A + Bz) - B, 0, ik(A + Bz)\right)\mathrm{e}^{kz}\mathrm{e}^{\sigma t + ikx}.$$

The linearised condition of continuity of velocity on the interface is

$$u_1 = u_2,\ w_1 = w_2 \text{ on } z = 0$$

which yields two equations relating the unknown constants,

$$kC - D = -kA - B, \tag{1}$$
$$C = A. \tag{2}$$

The horizontal component of traction is

$$\sigma_{xz} = \eta\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

and in the linearised equations is continuous on $z = 0$. We already have a condition imposing continuity of $\partial w/\partial x$ on the interface because w is continuous. Hence continuity of σ_{xz} implies

$$\frac{\partial u_1}{\partial z} = \frac{\partial u_2}{\partial z} \text{ on } z = 0,$$

from which one obtains another relationship between the unknown constants,

$$-kC + 2D = -kA - 2B. \tag{3}$$

The boundary condition on the normal component of the traction is a little more subtle. We have

$$[\mathbf{n} \cdot \underline{\underline{\boldsymbol{\sigma}}} \cdot \mathbf{n}]_{z=\zeta} = 0$$

where square brackets are used to represent the jump in the quantity across the interface $[\mathbf{n} \cdot \underline{\underline{\boldsymbol{\sigma}}} \cdot \mathbf{n}] = \mathbf{n} \cdot \underline{\underline{\boldsymbol{\sigma}_1}} \cdot \mathbf{n} - \mathbf{n} \cdot \underline{\underline{\boldsymbol{\sigma}_2}} \cdot \mathbf{n}$. We can decompose the stresses into a hydrostatic part and a perturbation, i.e. $\underline{\underline{\boldsymbol{\sigma}_1}} = \rho_1 g z \underline{\underline{\mathbf{I}}} + \underline{\underline{\widetilde{\boldsymbol{\sigma}_1}}}$ and $\underline{\underline{\boldsymbol{\sigma}_2}} = \rho_2 g z \underline{\underline{\mathbf{I}}} + \underline{\underline{\widetilde{\boldsymbol{\sigma}_2}}}$ to yield

$$[\mathbf{n} \cdot \underline{\underline{\widetilde{\boldsymbol{\sigma}}}} \cdot \mathbf{n}]_{z=\zeta} = -(\rho_1 - \rho_2) g \zeta,$$

which can be linearised as

$$[\widetilde{\sigma}_{zz}]_{z=0} = -(\rho_1 - \rho_2) g \zeta.$$

The constitutive law yields

$$\left[-\mathcal{P} + 2\eta \frac{\partial w}{\partial z} \right]_{z=0} = -(\rho_1 - \rho_2) g \zeta.$$

We already have a condition imposing continuity of $\partial u / \partial x$ on $z = 0$ (and thus $\partial w / \partial z$ by incompressibility) because u is continuous. Thus the boundary condition reduces to

$$[\mathcal{P}]_{z=0} = (\rho_1 - \rho_2) g \zeta,$$

where $\mathcal{P}$ represents the perturbed (non-hydrostatic) pressure, given in each half-space as

$$\mathcal{P}_1 = 2\eta i k D \mathrm{e}^{-kz} \mathrm{e}^{\sigma t + ikx},$$
$$\mathcal{P}_2 = 2\eta i k B \mathrm{e}^{kz} \mathrm{e}^{\sigma t + ikx}.$$

Hence

$$2\eta i k (D - B) = (\rho_1 - \rho_2) g \epsilon. \tag{4}$$

The final boundary condition is the kinematic boundary condition, which in linearised form is

$$\frac{\partial \zeta}{\partial t} = w_1 = w_2 \text{ on } z = 0,$$

and yields

$$\epsilon \sigma = i k A. \tag{5}$$

Elimination of variables between (1-5) then yields

$$\sigma = \frac{(\rho_1 - \rho_2)\, g}{4 \eta k}.$$

Since $\tau = 1/\sigma$ and $k = 2\pi/\lambda$, we obtain the final result

$$\underline{\tau = \frac{8 \pi \eta}{(\rho_1 - \rho_2)\, g \lambda}}.$$

2 (i) The *Rayleigh number* for a convecting layer of fluid is defined by

$$\mathrm{Ra} = \frac{\rho g \alpha \Delta T a^3}{\kappa \eta}$$

where ρ is density, g is acceleration due to gravity, α is thermal expansivity, ΔT is temperature difference across the layer, a is layer depth, κ is thermal diffusivity, and η is dynamic viscosity. The Rayleigh number is a measure of the vigour of convection, with a higher Rayleigh number implying more vigorous convection. It can also be interpreted as a ratio of thermal diffusion time scale/ advection due to buoyancy time scale i.e. $\mathrm{Ra} = [a^2/\kappa] \,/\, [\eta / \rho g \alpha \Delta T a]$.

The *aspect ratio* is the ratio of the distance between rising and sinking regions to the depth of the layer.

The *planform* is the geometry of the convecting layer when viewed from above (e.g. squares, hexagons, rolls).

(ii) Pictures match as follows:

1.c) Symmetric top and bottom (equal sized upwellings and downwellings). Stable roll pattern.

2.a) Higher Rayleigh number – more vigorous convection. Appears time-dependent. Equal magnitude upwellings and downwellings. Thinner horizontal boundary layers and plumes.

3.d) Asymmetry top/bottom – narrow downwellings, broad, diffusive upwellings. No bottom boundary layer. Appears time-dependent.

4.b) Asymmetry top/bottom. Large upper thermal boundary layer, small bottom boundary layer. "Stagnant lid" at top.

(iii) As Ra is increased for a 3D layer go through

- No convection
- Rolls
- Bimodal flow
- Spokes

Shearing produces rolls aligned in the direction of shearing.

(iv) See detailed discussion in lecture notes. Can mention

- gravity and topography (long wavelength admittance indicates convective support)
- seismic tomography (slow and fast anomalies can be related to temperature)
- plate motions (the large scale circulation is that of the plates)
- (heat flow – but is it very difficult for a heat flow anomaly from a plume to been seen through plates with a thermal time constant of 60 Myr)

The circulation has narrow downwellings (subduction zones), passive upwellings (ridges) with a few active upwelling mantle plumes. Upwelling plumes indicate that at least some heat must come from below, but the plate scale circulation is generally more like that expected for pure internal heating. The magnitude of the plume heat flux is estimated to be $\sim 10\%$ that of the heat flux of sinking slabs, which would imply much of the heat source driving the convection is internal.

3 (i) We assume that the age t is a function of the parameters Q, κ, k, and T_0. These have units

$$\begin{aligned}
[Q] &= MT^{-3} \\
[\kappa] &= L^2T^{-1} \\
[k] &= MLT^{-3}\Theta^{-1} \\
[T_0] &= \Theta
\end{aligned}$$

There is only one combination of these parameters that can give a time scale. T_0 gives the temperature scale, $l = kT_0/Q$ gives the length scale, and the time scale is then l^2/κ. Thus

$$\underline{t \sim \frac{k^2 T_0^2}{\kappa Q^2}}$$

up to some dimensionless prefactor which would involve solving the relevant PDEs. Plugging the numbers in yields $t \sim 3.16 \times 10^{15}$ s $= \underline{100 \text{ Myr}}$.

(ii) We now assume age t is function of the parameters H, Q, κ, and k. These have units

$$\begin{aligned} [H] &= ML^{-1}T^{-3} \\ [Q] &= MT^{-3} \\ [\kappa] &= L^2T^{-1} \\ [k] &= MLT^{-3}\Theta^{-1} \end{aligned}$$

Once again, there is only one combination of these parameters that can give a time scale. The combination $l = Q/H$ provides a length scale, and l^2/κ gives a time scale. Thus

$$\underline{t \sim \frac{Q^2}{\kappa H^2}}$$

again up to some dimensionless prefactor. Plugging the numbers in yields $t \sim 1.6 \times 10^{19}$ s $= \underline{500 \text{ Gyr}}$ (longer than the age of the universe!). Radiogenic heating is not an important heat source if the heat transport is only by conduction. Kelvin did not know about *convection* in the Earth's mantle which allows heat to be removed more rapidly from the deep interior (and then the internal heating does become important).

4 (i) The integral heat balance is

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \rho c_p T \, \mathrm{d}V = \int_V \rho H \, \mathrm{d}V - \int_S \mathbf{q} \cdot \mathbf{n} \, \mathrm{d}S$$

This can be written in terms of mean quantities as

$$\rho c_p d \mathcal{A} \frac{\mathrm{d}\overline{T}}{\mathrm{d}t} = \rho d \mathcal{A} \overline{H} - \mathcal{A} \overline{q}$$

where $\mathcal{A}$ is the area of the top surface and d is the depth of the layer. This can be simplified to the desired form,

$$\underline{\frac{\mathrm{d}\overline{T}}{\mathrm{d}t} = \frac{\overline{H}}{c_p} - \frac{\overline{q}}{\rho c_p d}}, \qquad (*)$$

and the overbars will be dropped in what follows.

(ii) The Nusselt number Nu is the ratio of the heat loss by the convecting fluid to the heat loss expected were only conduction occurring. For a vigorously convecting fluid

$$\mathrm{Nu} = A \mathrm{Ra}^{1/3}$$

where A is a numerical prefactor ($A \approx 0.2$). The definition of the Nusselt number gives

$$\mathrm{Nu} = \frac{q}{k \dfrac{2T}{d}}$$

where the factor of 2 arises because the temperature difference across the layer is twice the mean temperature. The Rayleigh number is

$$\mathrm{Ra} = \frac{g\alpha(2T)d^3}{\nu\kappa}$$

with another factor of 2 for the same reason. Combining the three previous equations yields the desired relationship,

$$q = \underline{A\frac{2kT}{d}\left(\frac{2g\alpha T d^3}{\nu\kappa}\right)^{1/3}}.$$

(iii) Let $T = T_0 T'$ and $t = t_0 t'$ where T_0 and t_0 are temperature and time scales that will be chosen to put the equation into the desired form. T_0 and t_0 are determined from the balance of terms in (*),

$$\frac{T_0}{t_0} = \frac{H}{c_p} = \frac{1}{\rho c_p d} A \frac{2kT_0}{d}\left(\frac{2g\alpha T_0 d^3}{\nu\kappa}\right)^{1/3}.$$

Rearrangement of the above yields

$$\underline{T_0 = B\frac{\rho H d^2}{k}\mathrm{Ra}_H^{-1/4}}, \quad \text{where } \mathrm{Ra}_H = \frac{\alpha\rho g H d^5}{k\kappa\nu} \quad \text{and } B = \frac{1}{2A^{3/4}} \approx 1.7,$$

and

$$\underline{t_0 = B\frac{d^2}{\kappa}\mathrm{Ra}_H^{-1/4}}.$$

The evolution equation can now be written in dimensionless form as

$$\frac{\mathrm{d}T'}{\mathrm{d}t'} = 1 - T'^{4/3}.$$

(iv) As $t' \to \infty$, $T' \to 1$ and thus $\underline{T \to T_0}$.

To determine the time constant for relaxation to steady-state, consider a small perturbation about steady state, $T' = 1 + \theta'$ where $\theta' \ll 1$. The evolution equation can then be linearised to

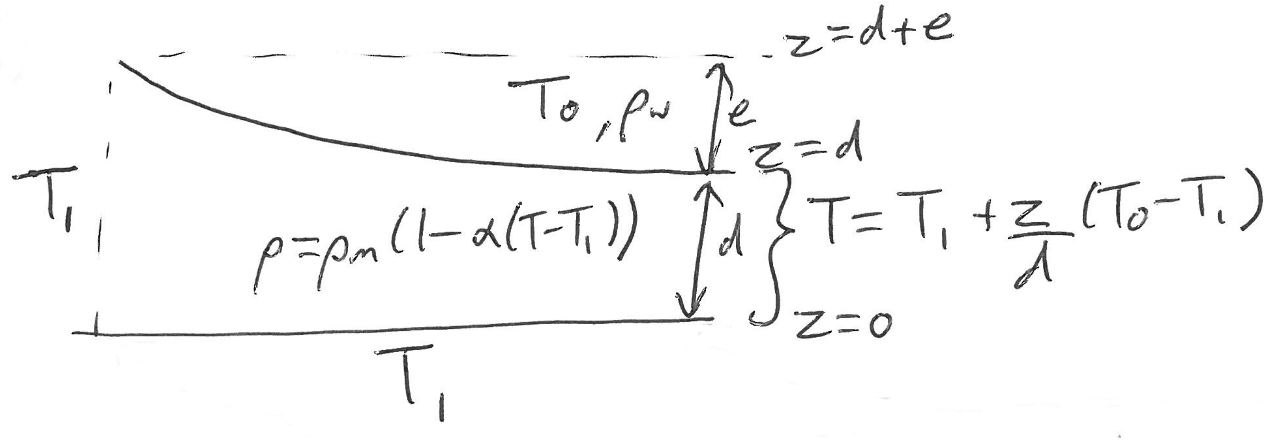
$$\frac{\mathrm{d}\theta'}{\mathrm{d}t'} = -\frac{4}{3}\theta',$$

which represents an exponential decay with dimensionless time constant $\tau' = 3/4$. In dimensional form this is

$$\tau = \frac{3t_0}{4} = \underline{\frac{3B}{4}\frac{d^2}{\kappa}\mathrm{Ra}_H^{-1/4}}.$$

(v) $\tau \sim 0.9$ Gyr.

5 (i)



The isostatic balance is

$$\rho_{\mathrm{m}}\,(d+e)=\rho_{\mathrm{m}}\left(d-\frac{d}{2}\alpha\,(T_0-T_1)\right)+\rho_{\mathrm{w}}e$$

where the left hand side represents the column at the ridge axis and the right hand side represents the column of old oceanic lithosphere. This can be rearranged to give the desired result,

$$e=\frac{\alpha(T_1-T_0)\rho_{\mathrm{m}}d}{2(\rho_{\mathrm{m}}-\rho_{\mathrm{w}})}.$$

The vertical gradient of the horizontal pressure difference between the ridge and old oceanic lithosphere is

$$\frac{\mathrm{d}\Delta P}{\mathrm{d}z}=\begin{cases}\rho_{\mathrm{m}}\alpha\dfrac{z}{d}(T_1-T_0)g, & 0<z<d,\\ -\left(\rho_{\mathrm{m}}-\rho_{\mathrm{w}}\right)g, & d<z<d+e,\end{cases}$$

which can also be written using the elevation as

$$\frac{\mathrm{d}\Delta P}{\mathrm{d}z}=\left(\rho_{\mathrm{m}}-\rho_{\mathrm{w}}\right)g\begin{cases}\dfrac{2ze}{d^2}, & 0<z<d,\\ -1, & d<z<d+e.\end{cases}$$

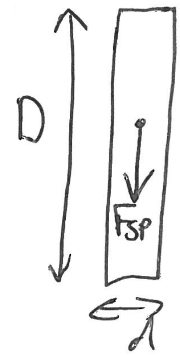
Integrating this once yields

$$\Delta P=\left(\rho_{\mathrm{m}}-\rho_{\mathrm{w}}\right)g\begin{cases}\dfrac{z^2e}{d^2}, & 0<z<d,\\ e+d-z, & d<z<d+e,\end{cases}$$

noting that the horizontal pressure difference is zero at the top and bottom of the columns. The ridge push force can be obtained by integrating again,

$$\begin{aligned}F_{\mathrm{RP}}&=\int_0^{d+e}\Delta P\;\mathrm{d}z\\&=\left(\rho_{\mathrm{m}}-\rho_{\mathrm{w}}\right)g\left(\int_0^d\frac{z^2e}{d^2}\;\mathrm{d}z+\int_d^{d+e}e+d-z\;\mathrm{d}z\right)\\&=(\rho_{\mathrm{m}}-\rho_{\mathrm{w}})ge\left(\frac{d}{3}+\frac{e}{2}\right).\end{aligned}$$

(ii)



Buoyancy force due to slab,

$$F_{\mathrm{SP}} = \frac{\rho_{\mathrm{m}} \alpha \left(T_1 - T_0\right)}{2} D d g$$

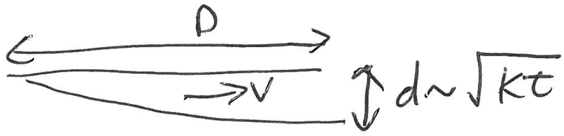
so ratio is,

$$\frac{F_{\mathrm{SP}}}{F_{\mathrm{RP}}} = \frac{Dd/2}{d^2/6} = \underline{\frac{3D}{d}}$$

Since $D \gg d$, typically $F_{\mathrm{SP}} \gg F_{\mathrm{RP}}$.

The calculation can be refined by taking into account the dip of the slab and the fact the slab heats up as it descends.

(iii)



The time t taken to travel a distance D is $t = D/v$. In this time a thermal boundary layer could grow by conduction to a thickness

$$d \sim \sqrt{\frac{\kappa D}{v}}$$

If the sinking slab has this thickness and descends to a depth D without warming up, the slab pull force will be

$$F_{\mathrm{SP}} \sim \frac{\rho_{\mathrm{m}} \alpha \left(T_1 - T_0\right)}{2} D \sqrt{\frac{\kappa D}{v}} g$$

If $F_{\mathrm{DF}} \sim \eta v$ and $F_{\mathrm{DF}} \sim F_{\mathrm{SP}}$ then

$$v \sim \frac{\kappa}{D} \left(\frac{\rho_{\mathrm{m}} g \alpha (T_1 - T_0) D^3}{\eta \kappa} \right)^{2/3} = \frac{\kappa}{D} Ra^{2/3}.$$

For numbers given $\mathrm{Ra} = 4.0 \times 10^5$, and $\underline{v \sim 26 \text{ cm yr}^{-1}}$. The fastest plates move at around ≈ 10 cm yr^{-1}, so this is the correct order of magnitude.

6 For a given epicentral distance d the time taken for a P wave to travel from source to receiver is $t_p = d/v_p$ where v_p is the P wave velocity. Similarly for the S wave, $t_s = d/v_s$. Thus the S-P differential travel time is

$$\Delta t = t_s - t_p = d \left(\frac{1}{v_s} - \frac{1}{v_p} \right)$$

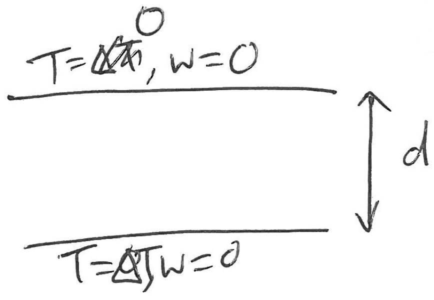
so

$$d = \frac{\Delta t}{\dfrac{1}{v_s} - \dfrac{1}{v_p}}$$

A differential travel time of 160 s can read off from the seismogram, yielding an epicentral distance of around <u>1700 km</u> (or around 29° given the 3,400 km radius of Mars).

7 (i)

$$\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \kappa \nabla^2 T,$$
$$\nabla \cdot \mathbf{v} = 0,$$
$$\mathbf{v} = -\frac{K}{\eta} \left(\nabla p - \rho_0 g \alpha T \hat{\mathbf{z}} \right).$$



Scaling:

$$T = \Delta T T', \qquad x = dx', \qquad z = dz',$$
$$t = \frac{d^2}{\kappa} t', \qquad \mathbf{v} = \frac{\kappa}{d} \mathbf{v}', \qquad p = \frac{\eta \kappa}{K} p',$$

yields dimensionless equations (dropping primes)

$$\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \nabla^2 T,$$
$$\nabla \cdot \mathbf{v} = 0,$$
$$\mathbf{v} + \nabla p = \mathrm{Ra} T \hat{\mathbf{z}},$$

with

$$\mathrm{Ra} = \frac{\rho_0 g \alpha \Delta T d K}{\eta \kappa}.$$

(ii) Straightforward to substitute pure conductive solution to show satisfies governing equations.
$T = 1 - z$ satisfies $T = 1$ on $z = 0$ and $T = 0$ on $z = 1$.
Since $\mathbf{v} = \mathbf{0}$, $w = 0$ on $z = 0$ and $z = 1$.

(iii) To linearise, write

$$T = 1 - z + \theta,$$
$$p = \mathrm{Ra} \left(z - \frac{z^2}{2} \right) + q,$$
$$\mathbf{v} = \mathbf{u}$$

and expand to first order in θ, q, and $\mathbf{u}$ to obtain

$$\frac{\partial \theta}{\partial t} - \mathbf{u} \cdot \hat{\mathbf{z}} = \nabla^2 \theta,$$
$$\nabla \cdot \mathbf{u} = 0,$$
$$\mathbf{u} + \nabla q = \mathrm{Ra} \theta \hat{\mathbf{z}}.$$

(iv) With given solutions

$$\theta = A \sin(n\pi z) \mathrm{e}^{ikx + \sigma t},$$
$$q = B \cos(n\pi z) \mathrm{e}^{ikx + \sigma t}$$

$\theta = 0$ on $z = 0, 1$, and

$$w = -\frac{\partial q}{\partial z} + \mathrm{Ra}\theta \propto \sin n\pi z = 0 \text{ on } z = 0, 1,$$

so boundary conditions are satisfied. $\mathbf{u}$ can be eliminated from the governing equations to write

$$\begin{aligned} \frac{\partial \theta}{\partial t} + \frac{\partial q}{\partial z} - \mathrm{Ra}\theta &= \nabla^2 \theta, \\ -\nabla^2 q + \mathrm{Ra}\frac{\partial \theta}{\partial z} &= 0. \end{aligned}$$

Substituting the given solutions yields

$$\begin{aligned} \sigma A - n\pi B - \mathrm{Ra}A &= -\left(k^2 + n^2\pi^2\right) A, \\ \left(k^2 + n^2\pi^2\right) B + n\pi \mathrm{Ra}A &= 0 \end{aligned}$$

from which one obtains the dispersion relation

$$\sigma = \frac{\mathrm{Ra}k^2}{k^2 + n^2\pi^2} - (k^2 + n^2\pi^2).$$

(v) The layer first becomes unstable when $\sigma = 0$, which implies

$$\mathrm{Ra}_c = \frac{\left(k^2 + n^2\pi^2\right)^2}{k^2}$$

As a function of k, Ra_c has a minimum when

$$\frac{\mathrm{dRa}_c}{\mathrm{d}k} = 0 \implies k = n\pi$$

Thus $\mathrm{Ra}_c = 4n^2\pi^2$. The first mode to go unstable is the $n = 1$ mode, which corresponds to $\underline{\mathrm{Ra}_c = 4\pi^2}$.