ON THE CONVERGENCE OF ORDINARY INTEGRALS TO STOCHASTIC INTEGRALS

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1. Introduction. Let $y(t)$ be the real Wiener process Brownian motion process (with $E\{y(t)\} = 0$ and $E\{y^2(t)\} = |t|$) and let $y_n(t)$ be a sequence of approximations to $y(t)$ with the following properties. For each n, $y_n(t)$ is of bounded variation, continuous and converges a.s. to $y(t)$ as $n \to \infty$. Then $\lim_{n\to\infty} \int_0^a y_n(t)\, dy_n(t) = y^2(a)/2$; however, for the "corresponding" stochastic integral $\int_0^a y(t)\, dy(t) = (y^2(a) - a)/2$, ([1] p. 444). The reason for the difference between the two results is easily traced to P. Lèvy's theorem on the oscillation of the Brownian motion ([1] p. 395). Let $x(t)$, $a \leqq t \leqq b$ be the solution to the stochastic differential equation.

$$(1) \qquad \boxed{dx(t) = m(x(t), t)\, dt + \sigma(x(t), t)\, dy(t)}, \quad x(a) = x_a$$

where x_a is a random variable independent of $y(t) - y(a)$, $a \leqq t \leqq b$ and $m(\cdot,\cdot)$, $\sigma(\cdot,\cdot)$ satisfy the conditions for existence and uniqueness of $x(t)$ ([1] p. 288). Assuming that $y_n(t)$ has, a.s., a piecewise continuous derivative, let $x_n(t)$ be the sequence of integrals of the ordinary differential equations

$$(2) \qquad dx_n(t) = m(x_n(t), t)\, dt + \sigma(x_n(t), t)\, dy_n(t), \quad x_n(a) = x_a .$$

A direct calculation for the special case $dx_n(t) = x_n(t)\, dy_n(t)$, $x_n(0) = 1$ shows that $x_n(t)$ converges as $n \to \infty$ to a diffusion process which does *not* satisfy the same Kolmogoroff equations as the solution to the stochastic differential equation $dx(t) = x(t)\, dy(t)$, $x(0) = 1$. Certain problems lead to the question of existence, properties and the proper Kolmogoroff equation for the limit of $x_n(t)$. For example, physical systems driven by "white noise" lead directly to limits of $x_n(t)$ where $x_n(t)$ satisfies (2) (namely, Langevin equations). This follows from the fact that any physical experiment will correspond to (2) rather than (1) as Wiener process Brownian motion can only be approximated but not realized in the physical world. The object of this note is to derive relations between limits of integrals of the type $\int_a^b \psi(y_n(t), t)\, dy_n(t)$ and $\int_a^b \psi(y(t), t)\, dy(t)$ and corresponding relations between the solution of (1) and the solution of the Langevin equation (2) (the limit of the solutions of (2)).

The following types of approximations to the Brownian motion will be considered

A_1. For almost all ω, $y_n(t, \omega) \to y(t, \omega)$ for all t in $[a, b]$ and $y_n(t, \omega)$ are continuous and of bounded variation.

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A_2. A_1 and also: for almost all ω there exist $n_0(\omega)$, $k(\omega)$, both finite, such that for all $n > n_0$ and all t in $[a, b]$, $y_n(t, \omega) \leqq k(\omega)$.

A_3. A_2 and also: $y_n(t, \omega)$ has a piecewise continuous derivative.

A_4. A_3 and also: $y_n(t, \omega) \to y(t, \omega)$ uniformly in $[a, b]$.

In a recent paper [3], Stratonovich introduced a symmetrized definition of the stochastic integral and related the symmetrized definition of Itô's definition of stochastic integrals. It is shown in [3] that, under the symmetrized definition, the rules of transformation of equations become the same rules as in the ordinary calculus (rather than Itô's rules of transformation [2]). The results of this note are, very roughly, that $\int_a^b \psi(y_n(t), t)\, dy_n(t) \to (S) \int_a^b \psi(y(t), t)\, dy(t)$ (where the second integral is to be interpreted in the Stratonovich symmetrized sense) and a similar result for differential equations. All the stochastic integrals considered in the remainder of this note are in Itô's sense.

For the special case where $y_n(t)$ are polygonal approximations to $y(t)$, we obtained results which are similar to the results of this note but under considerably weaker conditions on $\sigma(x, t)$ (to be published).

2. Stochastic Integrals.

THEOREM 1a. *Let $\psi(\eta, t)$ have continuous partial derivatives $\partial\psi(\eta, t)/\partial\eta$, $\partial\psi(\eta, t)/\partial t$ in $-\infty < \eta < \infty$, $a \leqq t \leqq b$. Let $y_n(t)$ satisfy A_2. Then,* a.s.,

$$(3)\quad \lim_{n\to\infty} \int_a^b \psi(y_n(t), t)\, dy_n(t) = \int_a^b \psi(y(t), t)\, dy(t) + \tfrac{1}{2} \int_a^b [\partial\psi(y(t), t)/\partial y]\, dt.$$

If, in addition, $\psi(\eta, t)$ is independent of t ($\partial\psi/\partial t = 0$) then (3) holds with A^1 replacing A_2.

PROOF. Let $F(\lambda, t) = \int_a^\lambda \psi(\eta, t)\, d\eta$. Then

$$F(y_n(b), b) - F(y_n(a), a) = \int_a^b \psi(y_n(t), t)\, dy_n(t) + \int_a^b [\partial F(y_n(t), t)/\partial t]\, dt.$$

By the continuity of $\psi(\eta, t)$ and $\partial\psi(\eta, t)/\partial t$, A_2 and dominated convergence it follows that, a.s.

$$(4)\quad \lim_{n\to\infty} \int_a^b \psi(y_n(t), t)\, dy_n(t) = F(y(b), b) - F(y(a), a) - \int_a^b [\partial F(y(t), t)/\partial t]\, dt.$$

If $\partial\psi/\partial t = 0$ then (4) hold with A_2 replaced by A_1. A result of Itô [2] states that if $G(\xi, t)$ has a continuous first partial derivative with respect to t and a continuous second partial derivative with respect to ξ, $-\infty < \xi < \infty$, $a \leqq t \leqq b$, and if the random functions $f(t)$ and $g(t)$, $a \leqq t \leqq b$, are independent of the aggregate of differences $y(s) - y(t)$, $t \leqq s \leqq b$, $f(t) \in L_2[a, b]$, $f(t) \in L_1[a, b]$ a.s. and $dz(t) = g(t)\, dt + f(t)\, dy(t)$ then, a.s.

$$(5)\quad G(z(b), b) - G(z(a), a) = \int_a^b [\partial G(z(t), t)/\partial z] f(t)\, dy(t) + \int_a^b \{[\partial G(z(t), t)/\partial z] g(t) + [\partial G(z(t), t)/\partial t] + \tfrac{1}{2} f^2(t) [\partial^2 G(z(t), t)/\partial z^2]\}\, dt.$$

Applying this result to F:

$$(6)\quad F(y(b), b) - F(y(a), a) = \int_a^b \psi(y(t), t)\, dy(t) + \int_a^b \{[\partial F(y(t), t)/\partial t] + \tfrac{1}{2}[\partial\psi(y(t), t)/\partial y]\}\, dt, \quad \text{a.s.}$$

And (3) follows by substituting (6) into (4).

THEOREM 1b. *Let* $x(t) = \int_a^t g(s)\,ds + \int_a^t f(s)\,dy(s)$, $a \leqq t \leqq b$, *where* $g(s), f(s)$ *are random functions satisfying the following conditions*: $f(\cdot) \in L_2[a, b]$, $g(\cdot) \in L_1[a, b]$ a.s., *and* $f(t), g(t)$, $a \leqq t \leqq b$, *are independent of the aggregate of differences* $y(s) - y(t)$, $t \leqq s \leqq b$. *Let* $\psi(\eta, t)$ *satisfy the requirements of Theorem* 1a. *Let* $x_n(t)$ *be a sequence of approximations to* $x(t)$ *such that for almost all* ω, $x_n(t, \omega) \to x(t, \omega)$ *boundedly for all* t *in* $[a, b]$ *and* $x_n(t, \omega)$ *is continuous and of bounded variation. Then*

$$\begin{aligned}\lim_{n\to\infty} \int_a^b \psi(x_n(t), t)\,dx_n(t) &= \int_a^b \psi(x(t), t)\,dx(t) + \tfrac{1}{2}\int_a^b f^2(t)[\partial\psi(x(t), t)/\partial x]\,dt \\ &= \int_a^b \psi(x(t), t)g(t)\,dt + \int_a^b \psi(x(t), t)\,dy(t) \\ &\quad + \tfrac{1}{2}\int_a^b f^2(t)[\partial\psi(x(t), t)/\partial x]\,dt.\end{aligned}$$

The proof is the same as for 1a.

3. Stochastic differential equations. Let $y_n(t)$ have a piecewise continuous derivative. Let $m(x, t)$ and $\sigma(x, t)$ be continuous in $-\infty < x < \infty$, $a \leqq t \leqq b$ and satisfy the Lipschitz condition

$$|f(x, t) - f(\xi, t)| \leqq k\,|x - \xi|. \tag{7}$$

Then

$$dx_n(t) = m(x_n(t), t)\,dt + \sigma(x_n(t), t)\,dy_n(t),\ x(a) = x_a \tag{8}$$

has a unique solution which is continuous in $[a, b]$. If, in addition, $\sigma(x, t)$ has a first derivative with respect to x which is continuous in $-\infty < x < \infty$, $a \leqq t \leqq b$ and $\sigma(x, t)\cdot\partial\sigma(x, t)/\partial x$ satisfies (7), then the stochastic differential equation

Leads to Stratonovich F-P equation for P(x,t)

$$dx(t) = m(x(t), t)\,dt + \tfrac{1}{2}\sigma(x(t), t)[\partial\sigma(x(t), t)/\partial x]\,dt + \sigma(x(t), t)\,dy(t) \tag{9}$$

$x(a) = x_a$ (with x_a random variable independent of the differences $y(t) - y(a)$, $a \leqq t \leqq b$) has a.s. a unique continuous solution.

THEOREM 2. *If*

(i) $m(x, t)$, $\sigma(x, t)$, $\partial\sigma(x, t)/\partial x$, $\partial\sigma(x, t)/\partial t$ *are continuous in* $-\infty < x < \infty$, $a \leqq t \leqq b$.

(ii) $m(x, t)$, $\sigma(x, t)$, $\partial\sigma^2(x, t)/\partial x$, *satisfy the Lipschitz condition* (7).

(iii) $\sigma(x, t) \geqq \beta > 0$ (or $-\sigma(x, t) \geqq \beta > 0$) *and* $|\partial\sigma(x, t)/\partial t| \leqq k\sigma^2(x, t)$.

(iv) x_a *is a random variable independent of the aggregate of differences* $y(t) - y(a)$, $a \leqq t \leqq b$.

(v) $x_n(t)$ *and* $x(t)$ *satisfy*

$$dx_n(t) = m(x_n(t), t)\,dt + \sigma(x_n(t), t)\,dy_n(t);\quad x_n(a) = x_a \tag{8}$$

$$\begin{aligned}dx(t) &= m(x(t), t)\,dt \\ &\quad + \tfrac{1}{2}\sigma(x(t), t)[\partial\sigma(x(t), t)/\partial x]\,dt + \sigma(x(t), t)\,dy(t), \\ &\qquad x(a) = x_a\end{aligned} \tag{9}$$

Then for $y_n(t)$ *satisfying* A_3 *and for* t *in* $[a, b]$, $x_n(t) \to x(t)$ a.s. *as* $n \to \infty$; *and for* $y_n(t)$ *satisfying* A_4, $x_n(t) \to x(t)$ *uniformly in* $[a, b]$ a.s. *as* $n \to \infty$.

PROOF. Let $\Phi(\lambda, t) = \int_0^\lambda \sigma(u, t)^{-1}\, du$. Then:

$$\begin{aligned} &\Phi(x_n(t), t) \\ (10)\qquad &= \int_a^t [\partial\Phi(x_n(s), s)/\partial s]\, ds + \int_a^t [\partial\Phi(x_n(s), s)/\partial x_n]\, dx_n(s) + \Phi(x_a, a) \\ &= \int_a^t [\partial\Phi(x_n(s), s)/\partial s]\, ds + \int_a^t [m(x_n(s), s)/\sigma(x_n(s), s)]\, ds \\ &\qquad + y_n(t) - y_n(a) + \Phi(x_a, a). \end{aligned}$$

By Itô's result for stochastic differentials (Equation (5)) we have

$$\begin{aligned} &\Phi(x(t), t) \\ &= \int_a^t [\partial\Phi(x(s), s)/\partial s]\, ds + \tfrac{1}{2} \int_a^t [\partial^2\Phi(x(s), s)/\partial x^2]\sigma^2(x(s), s)\, ds \\ &\qquad + \int_a^t [\partial\Phi(x(s), s)/\partial x]\sigma(x(s), s)\, dy(s) + \Phi(x_a, a) \\ (11)\qquad &\qquad + \int_a^t [\partial\Phi(x(s), s)/\partial x][m(x(s), s) + \tfrac{1}{2}\sigma(x(s), s)[\partial\sigma(x(s), s)/\partial x]]\, dx \\ &= \int_a^t [\partial\Phi(x(s), s)/\partial s]\, ds + y(t) - y(a) + \int_a^t [m(x(s), s)/\sigma(x(s), s)]\, ds \\ &\qquad + \Phi(x_a, a). \end{aligned}$$

Since $m(x, t)$ is continuous and by (7) it follows that $|m(x, t)| \leqq K_1(1 + |x|)$ for some K_1, therefore

$$\begin{aligned} &|m(x, t)/\sigma(x, t) - m(\xi, t)/\sigma(\xi, t)| \\ (12)\qquad &\leqq |m(x, t)/\sigma(x, t) - m(\xi, t)/\sigma(x, t)| + |m(\xi, t)/\sigma(x, t) - m(\xi, t)/\sigma(\xi, t)| \\ &\leqq (K_2/\beta)(1 + |\xi|)\, |x - \xi|. \end{aligned}$$

Since $\sigma^{-2}(x, t) \cdot \partial\sigma(x, t)/\partial t$ is uniformly bounded, we have

$$(13)\qquad |\partial\Phi(x, t)/\partial t - \partial\Phi(\xi, t)/\partial t| \leqq K_3\, |x - \xi|.$$

We will now show that

$$(14)\qquad |\Phi(x, t) - \Phi(\xi, t)| \geqq K_4 \log(1 + |x - \xi|/(1 + |\xi|)).$$

By the continuity in t and by (7), $\sigma(x, t) \leqq C(1 + |x|)$. If x and ξ have the same sign, we can assume, without loss of generality that both are non-negative. Let $u = \max(x, \xi)$ $v = \min(x, \xi)$, then

$$\begin{aligned} &|\Phi(x, t) - \Phi(\xi, t)| \\ &\qquad \geqq (1/C) \int_v^u [1/(1 + w)]\, dw = (1/C) \log(1 + |u - v|/(1 + v)) \\ &\qquad \geqq (1/C) \log(1 + |x - \xi|/(1 + |\xi|)). \end{aligned}$$

If x and ξ have opposite signs, we assume w.l.g. $u \geqq |v|$ and

$$|\Phi(x, t) - \Phi(\xi, t)| \geqq (1/C)\, [\log(1 + u) + \log(1 + |v|)]$$

$$\geqq (1/C) \log (1 + u + |v|) = (1/C) \log (1 + |x - \xi|)$$
$$\geqq (1/C) \log (1 + |x - \xi|/(1 + |\xi|)).$$

Let μ be the random variable $1 + \max_{a \leqq t \leqq b} x(t) = \mu$, then $\mu < \infty$ a.s. Subtracting (11) from (10) and using (12), (13), (14) we obtain

$$(15)\quad \log (1 + |x_n(t) - x(t)|/\mu)$$
$$\leqq K_5 |y(t) - y_n(t)| + K_5 |y(a) - y_n(a)| + K_5\mu \int_a^t |x_n(s) - x(s)| \, ds.$$

In order to deduce from (15) that $x_n(t) \to x(t)$ we will prove the following lemma. (Lemma 1 is similar to the Bellman-Gronwall lemma, which requires a condition stronger than (16). Thus, Lemma 1 is applicable whenever the latter is, but not conversely.)

LEMMA 1. *Let $f(t)$ be real, non-negative and continuous in $-\infty < a \leqq t \leqq b < \infty$. Let $0 < \mu < \infty$, $\rho > 0$ and let $\epsilon(t) \geqq 0$ and $\int_a^b \epsilon(t) \, ds < (\rho\mu e^{\mu\rho(b-a)})^{-1}$. Suppose that*

$$(16)\qquad \log (1 + f(t)/\mu) \leqq \log (1 + \epsilon(t)) + \rho \int_a^t f(s) \, ds.$$

Then

$$(17)\qquad f(t) \leqq \mu[\epsilon(t) + \rho\mu e^{\rho\mu(a-b)} \cdot \int_a^b \epsilon(t) \, dt]/[1 - \rho\mu e^{\rho\mu(a-b)} \int_a^b \epsilon(t) \, dt].$$

PROOF OF LEMMA. From (16)

$$(18)\qquad [1 + f(t)/\mu]/\exp (\rho \int_a^t f(s) \, ds) \leqq 1 + \epsilon(t)$$

or

$$[\rho\mu + \rho f(t)]/\exp (\rho \int_a^t f(s) \, ds + \rho\mu t) = -(d/dt) \exp (-\rho \int_a^t f(s) \, ds - \rho\mu t)$$
$$\leqq \rho\mu(1 + \epsilon(t))e^{-\rho\mu t}$$

Integrating from a to t:

$$e^{-\rho\mu a} - \exp (-\rho \int_a^t f(s) \, ds - \rho\mu t) \leqq e^{-\rho\mu a} - e^{-\rho\mu t} + \rho\mu e^{-\rho\mu a} \cdot \int_a^b \epsilon(t) \, dt.$$

For $\rho\mu e^{\rho\mu(b-a)} \int_a^t \epsilon(t) \, dt < 1$

$$\exp (\rho \int_a^t f(s) \, ds) \leqq 1/(1 - \rho\mu e^{\rho\mu(b-a)} \cdot \int_a^b e(t) \, dt)$$

and (17) is obtained by substituting the last inequality into (18).

Applying (17) to (15) with $\epsilon_n(t) = \epsilon(t) = \exp \{K_5[|y(t) - y_n(t)| + |y(a) - y_n(a)|]\} - 1$, $\epsilon_n(t) \to 0$ as $n \to \infty$ and by dominated convergence $\int_a^b \epsilon_n(t) \, dt \to 0$. Therefore, under A_3, $x_n(t) - x(t) \to 0$ a.s. and under A_4 for almost all samples $x(t) - x_n(t) \to 0$ uniformly in $[a, b]$.

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