Relativistic Astrophysics and Cosmology — Examples 3 — 2023

1. The orbital period of SMC X-1 is 3.892 days, the maximum pulse time delay is ± 53.46s and optical spectroscopy suggests its companion to be of $\approx 17 M_\odot$ and gives a radial velocity amplitude of $19\,\mathrm{kms}^{-1}$. Estimate the mass of the neutron star (Solution: $M = 1.08\ \mathrm{M}_\odot$).

2. Two neutron stars, each of a solar mass, are in 1 hr circular orbits about each other. How long do they take to merge (Solution: 12×10^6 yr)? What might be seen at that point?

3. The second gravitational wave event detected by LIGO (GW151226) lasted 1 s, during which the frequency rose from 35 Hz to 450 Hz at the peak amplitude. Estimate, using Newtonian approximations, (a) the chirp mass ($\approx 10\ \mathrm{M}_\odot$), (b) the masses of the primary and secondary black holes (46 $\mathrm{M}_\odot$ and 4 $\mathrm{M}_\odot$), and (c) the total gravitational wave energy emitted (0.9 $\mathrm{M}_\odot c^2$).

   The values of the two black hole masses in this event derived using full General Relativity were approximately $14 M_\odot$ and $7 M_\odot$. Show how the Newtonian approximations can be improved to get closer to these masses by increasing the BH separation corresponding to the final frequency (just before coalescence).

4. Show that the geodesic $\mathcal{L}$ function for the 'plus' polarisation state of the gravitational wave metric given in Lecture 13 has the form

$$\mathcal{L} = c^2\dot{t}^2 - \left(1 - a^+ e^{ik(ct-z)}\right)\dot{x}^2 - \left(1 + a^+ e^{ik(ct-z)}\right)\dot{y}^2 - \dot{z}^2$$

   Hence show that a massive test particle which starts at rest in the (x, y) plane remains at rest in this plane during the passage of the wave. How do you reconcile this with the diagrams, also given in Lecture 13, of relative displacement in the (x, y) plane for particles arranged initially in a circle?

5. Show that the Einstein radius, where a distant source appears as a perfect ring around a distant point mass, is given by $R_E = \sqrt{2R_S D}$, where $R_S = 2GM/c^2$ and $D = (D_s - D_l)D_l/D_s$. (D_s is the distance to the source and D_l the distance to the lens).

   Consider the situation where the distant lens and source are slightly misaligned by distance r at the lens and the light ray passes at distance R. Show that $R^2 + rR - R_E^2 = 0$, and thus that there are two solutions (images). Show that the total magnification of the two images

$$A = \left|\frac{R}{r}\frac{dR}{dr}\right| = \frac{u^2 + 2}{u\sqrt{u^2 + 4}},$$

   where $u = r/R_E$. Estimate the angular size of the Einstein ring produced by a Solar mass star (a) near the centre of our Galaxy; (b) in a distant galaxy. (See `http://arxiv.org/abs/astro-ph/9604011` if help required.)

6. Suppose the universe is infinite and static, and is filled uniformly with galaxies, each with intrinsic luminosity L. By considering the contributions to observed flux from successive spherical shells of thickness Δr, show the night sky should be infinitely bright. (This is known as Olbers' paradox.) What result do you get if you assume objects in front can block those behind?

7. The *Extended Copernican Principle* states that all positions in the universe should be equivalent from the point of view of cosmology. Show that the Hubble Law is compatible with this principle, by demonstrating that an observer on another galaxy will find the same law, with the same Hubble constant, and with all motion apparently centred on him/herself. Repeat this exercise for a universal rotation, with angular velocity $\boldsymbol{\omega}$, and show that this is also compatible with the Extended Copernican Principle.

8. For purely radial motions of a massive particle, show that the geodesic equations in the Friedmann-Robertson-Walker (FRW) metric are

$$\begin{aligned} R(t)^2\dot{\chi} &= \text{const.} \\ c^2\ddot{t} &= -R^2(t)H(t)\dot{\chi}^2 \end{aligned}$$

where $H = (1/R)(dR/dt)$ is the usual Hubble parameter, and $\dot{}$ denotes differentiation w.r.t. proper time for the moving particle. Use these results to establish the following:

   (a) Fundamental observers follow geodesics in the FRW metric;

   (b) The cosmic time t can be identified with their proper time;

   (c) Given that the ordinary velocity v of a particle w.r.t. a fundamental observer at time t is given by $v = R(t)(d\chi/dt)$, v satisfies

$$\frac{dv}{dt} = -Hv\left(1 - \frac{v^2}{c^2}\right)$$

(This result on the decay of peculiar velocities is important in understanding the development of perturbations in the expanding universe.)

9. Starting from the field equations for a flat matter-dominated universe with Λ, prove the result

$$H(z) = H_0\left((1-\Omega_{\Lambda 0})(1+z)^3 + \Omega_{\Lambda 0}\right)^{1/2}$$

where $\Omega_{\Lambda 0}$ is the value of $\Omega_\Lambda = \frac{\Lambda c^2}{3H^2}$ today.

10. Find expressions for the scale factor $R(t)$, Hubble parameter $H(t)$ and deceleration parameter $q(t)$ all as a function of time in

    (a) a matter dominated Einstein de Sitter ($\Lambda = k = 0$) universe, and

    (b) a radiation dominated Einstein de Sitter universe.