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Heston model

The physics of stochastic volatility models

Cambridge University – Fabio Maggioni, 08.02.2024

Heston model

The physics of stochastic volatility models

- Beyond Black-Scholes assumptions
- Definition
- Dynamics
- Partial Differential Equation
- Characteristic function

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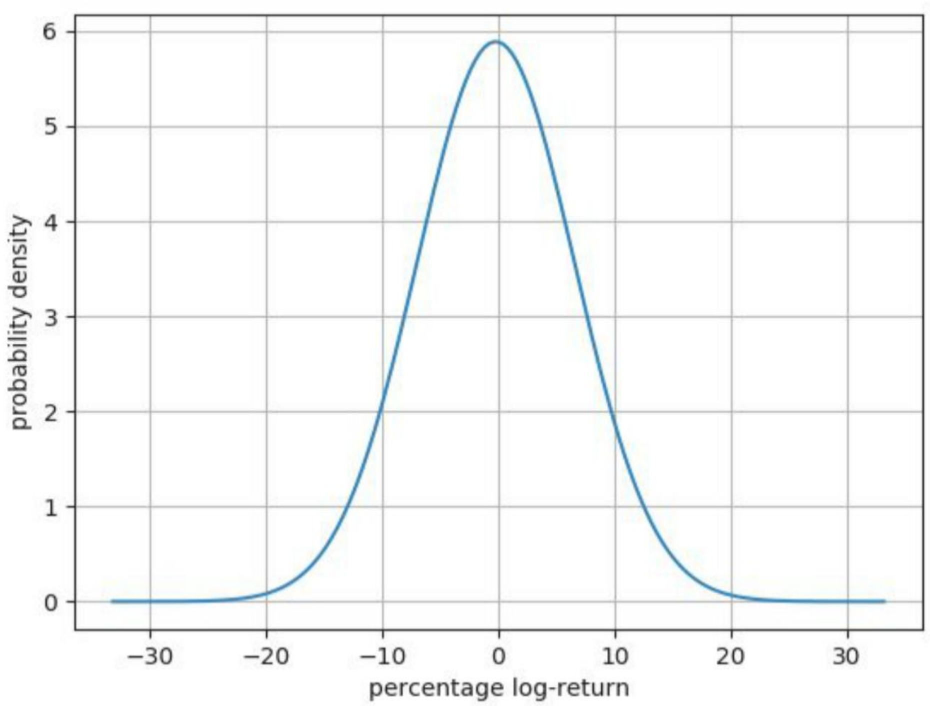
Beyond Black-Scholes

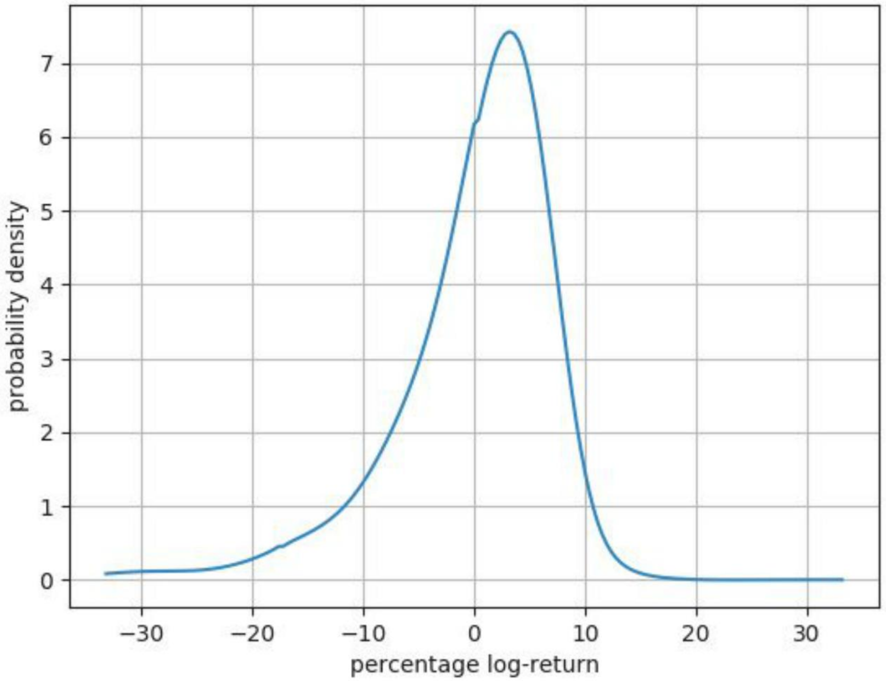
Pros	Cons	Market
• Closed form solution for option prices • Mathematical tractability • Great for communication	• Uncorrelated log-returns • Gaussian log-returns	• Persistence of variability (high vol and low vol periods) • Spot-vol correlation • Fat tailed log-return (kurtosis and skew)

Log-return distribution

Black-Scholes



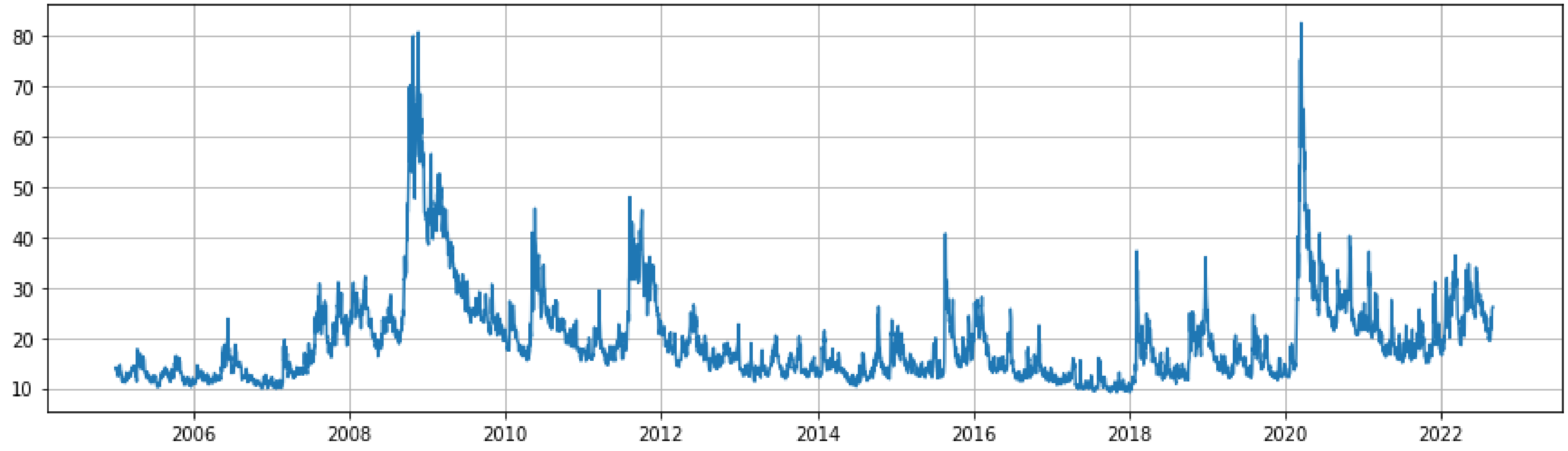
Financial markets



More complex models

$$dS_t = rS_t\, dt + \sigma_t S_t\, dW_t$$

- Deterministic time dependent volatility does not improve over normality of log returns
- Looking at Volatility Index we can guess that volatility is a good place to start: stochastic volatility models!



Stochastic volatility models

- Setting:

$$dS_t = rS_t\,dt + \sigma_t S_t\,dW_t^1$$
$$d\sigma_t = a(t,\sigma_t)dt + b(t,\sigma_t)\,dW_t^2$$

- With dW^1 and dW^2 being possibly correlated (ρ) random increments and a and b being “some functions”.

- According to the structure of functions a and b we can produce all sort of dynamics for the vol.

- Which one is appropriate?

Heston

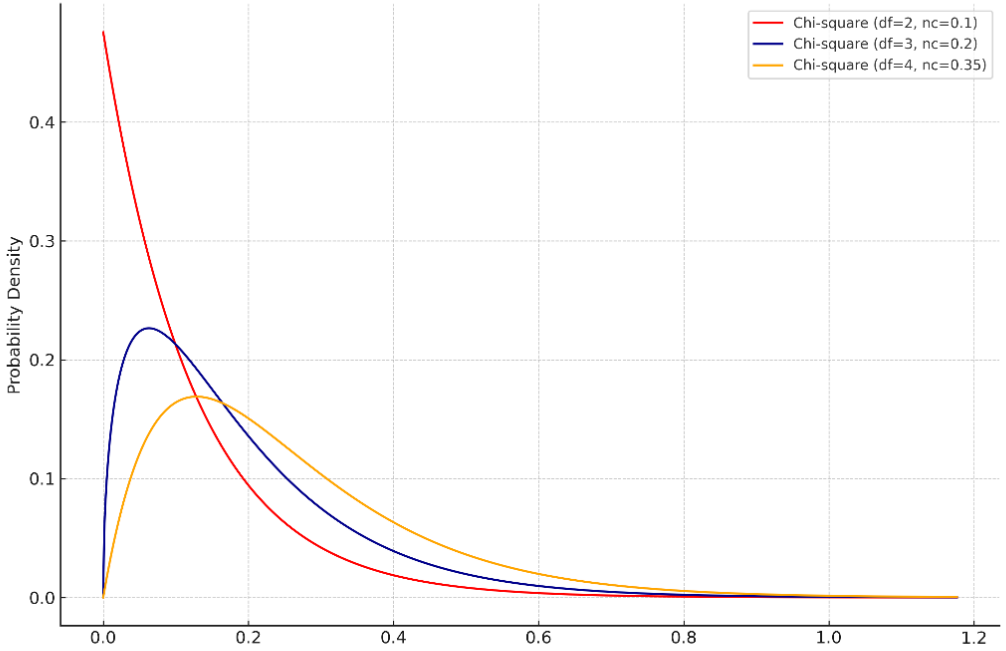
$$
\begin{cases}
dS_t = rS_t\, dt + S_t\sqrt{v_t}\, dW_t^1 \\
dv_t = \kappa(\bar{v} - v_t)\, dt + \gamma\sqrt{v_t}\, dW_t^2 \\
\mathrm{S}_{\mathrm{t}_0} = \mathrm{S}_0 \\
\mathrm{v}_{\mathrm{t}_0} = \mathrm{v}_0 \\
\mathrm{E}\left[dW_t^1\, dW_t^2\right] = \rho\, dt
\end{cases}
$$

- $\kappa = speed\ of\ mean\ reversion\ in\ years$
- $\bar{v} = long\ term\ variance$
- $v_0 = short\ term\ variance$
- $\gamma = vol\ of\ vol$
- $\rho = spot - variance\ correlation$

Heston

$$dS_t = rS_t\,dt + S_t\sqrt{v_t}\,dW_t^1$$
$$dv_t = \kappa(\bar{v} - v_t)\,dt + \gamma\sqrt{v_t}\,dW_t^2$$

- Instantaneous variance v_t is mean reverting
- Instantaneous volatility $\sqrt{v_t}$ has 2 sources of randomness:
  - Spot level via correlation ρ
  - Exogenous variance process guided by vol of vol γ
- Variance is distributed as a scaled Non central Chi square



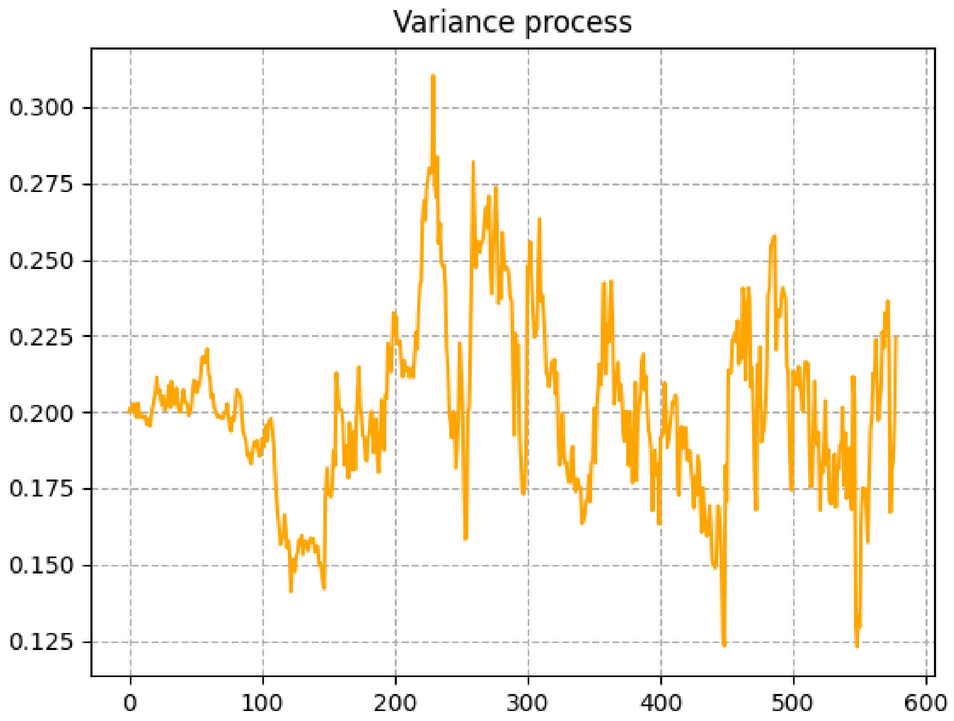
Dynamics at the limit

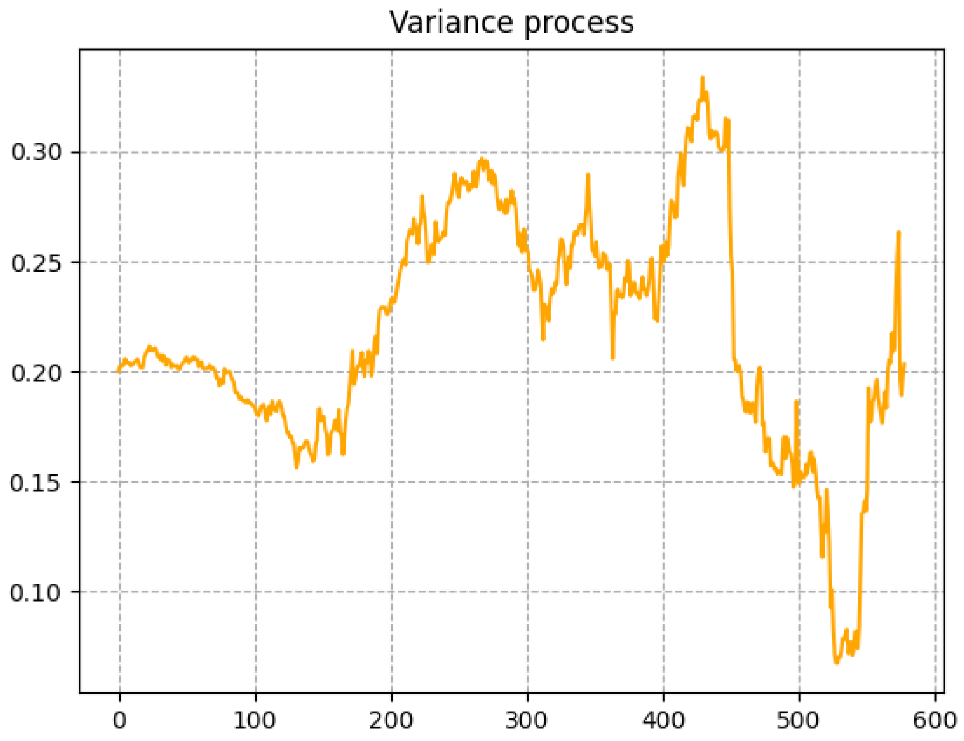
Let's look at some key quantities of the Heston model for an expiry time T.

- $E[v_T|v_0] = v_0 e^{-\kappa T} + \bar{v}\left(1 - e^{-\kappa T}\right)$
- $Var[v_T|v_0] = v_0 \frac{\gamma^2}{\kappa}\left(e^{-\kappa T} - e^{-2\kappa T}\right) + \frac{\bar{v}\gamma^2}{2\kappa}\left(1 - e^{-\kappa T}\right)^2$

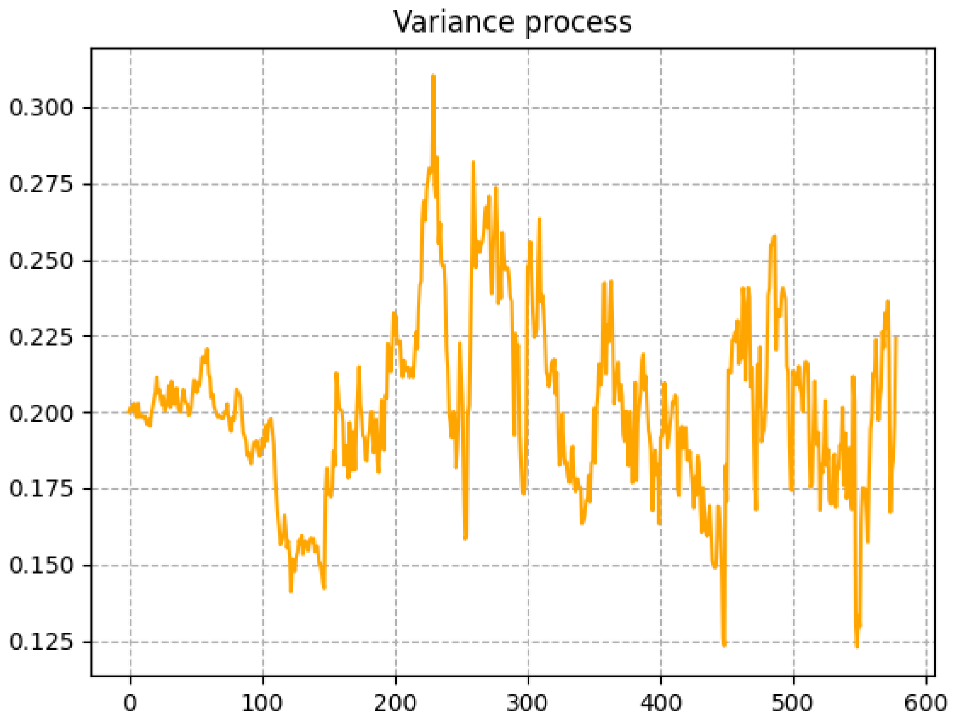
- For $\kappa \to \infty$, $E[v_t|v_0] \to \bar{v}$, $Var[v_t|v_0] \to 0$, Heston → Black&Scholes with $\sigma^2 = \bar{v}$
- For $\gamma \to 0$, $E[v_t|v_0] = v_0 e^{-\kappa t} + \bar{v}(1 - e^{-\kappa t})$, $Var[v_t|v_0] \to 0$, Heston → Black&Scholes with deterministic term structure

Variance dynamics

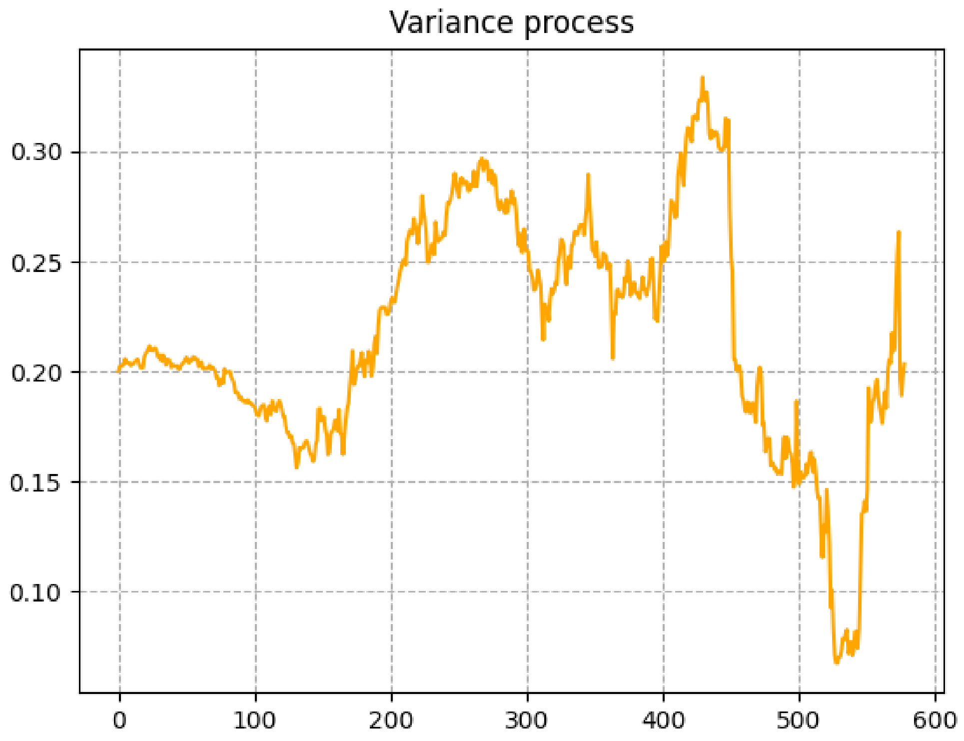




Variance dynamics

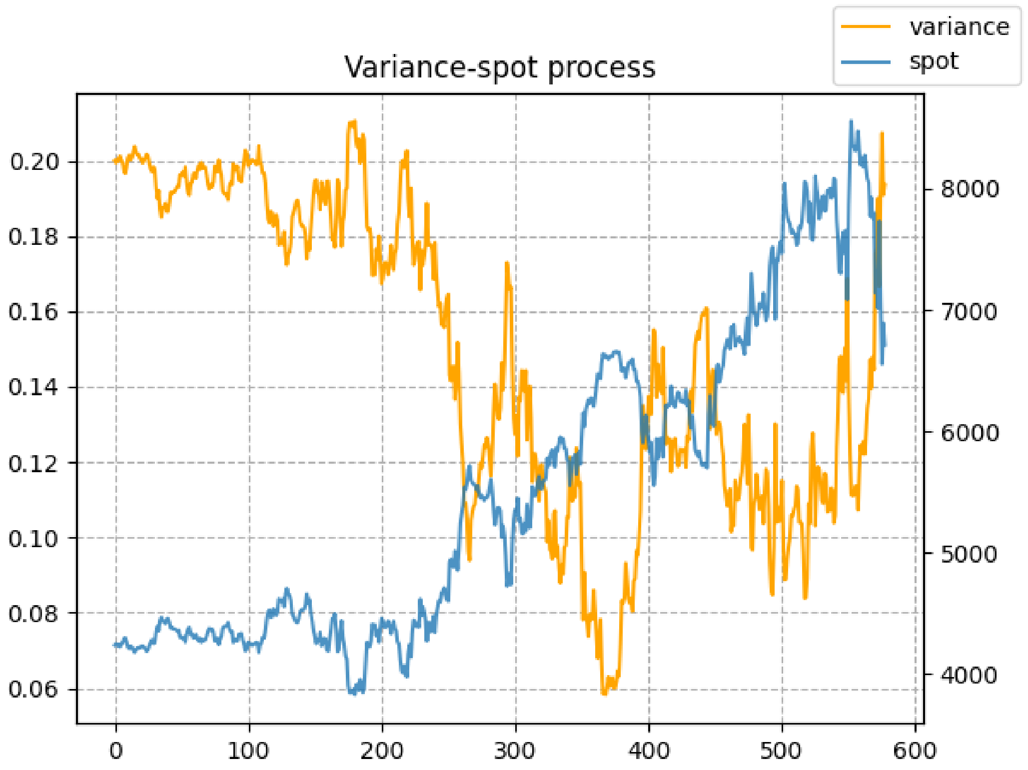


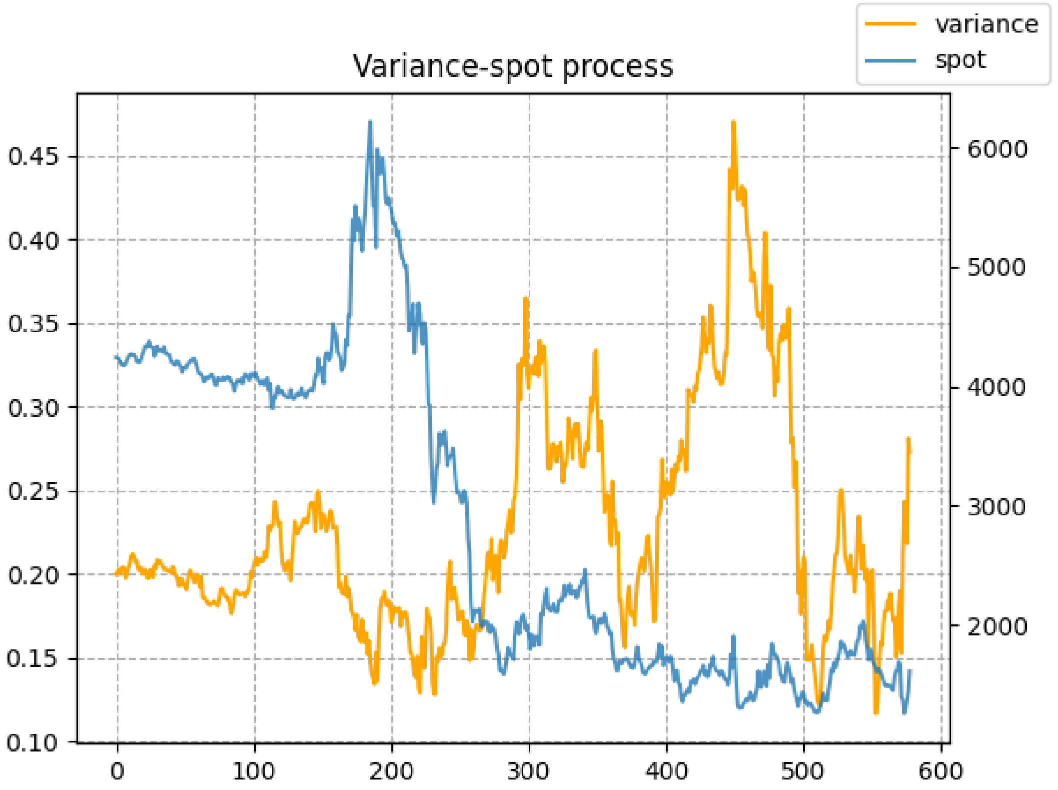
High mean reversion



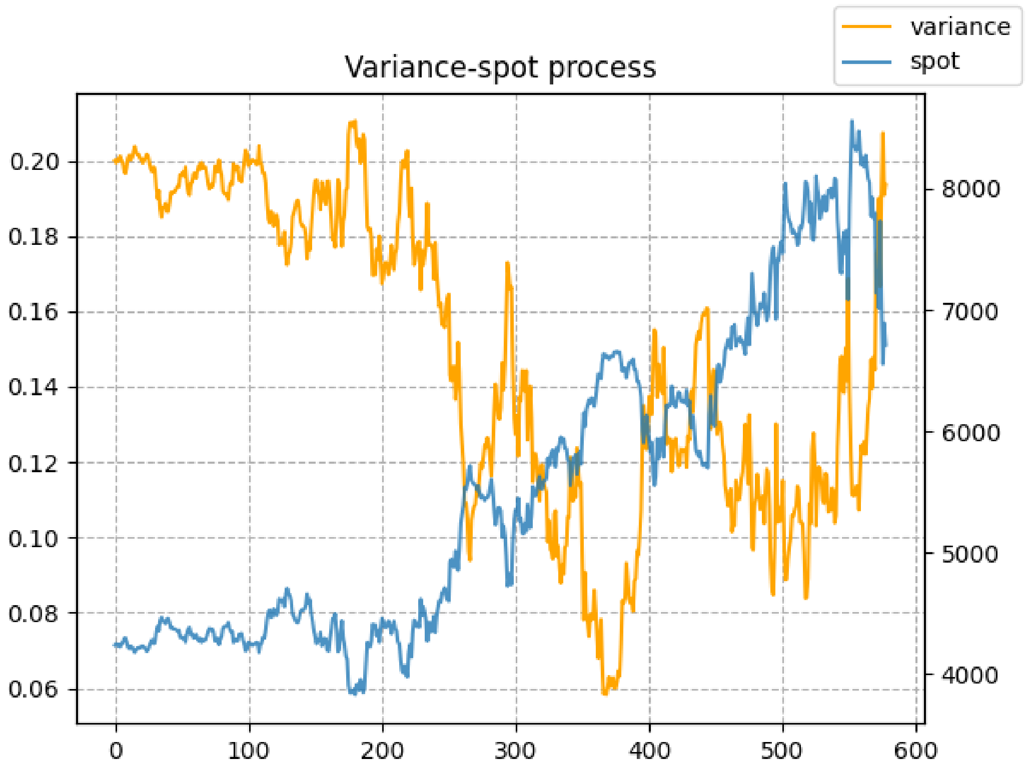
Low mean reversion

The Joint dynamics

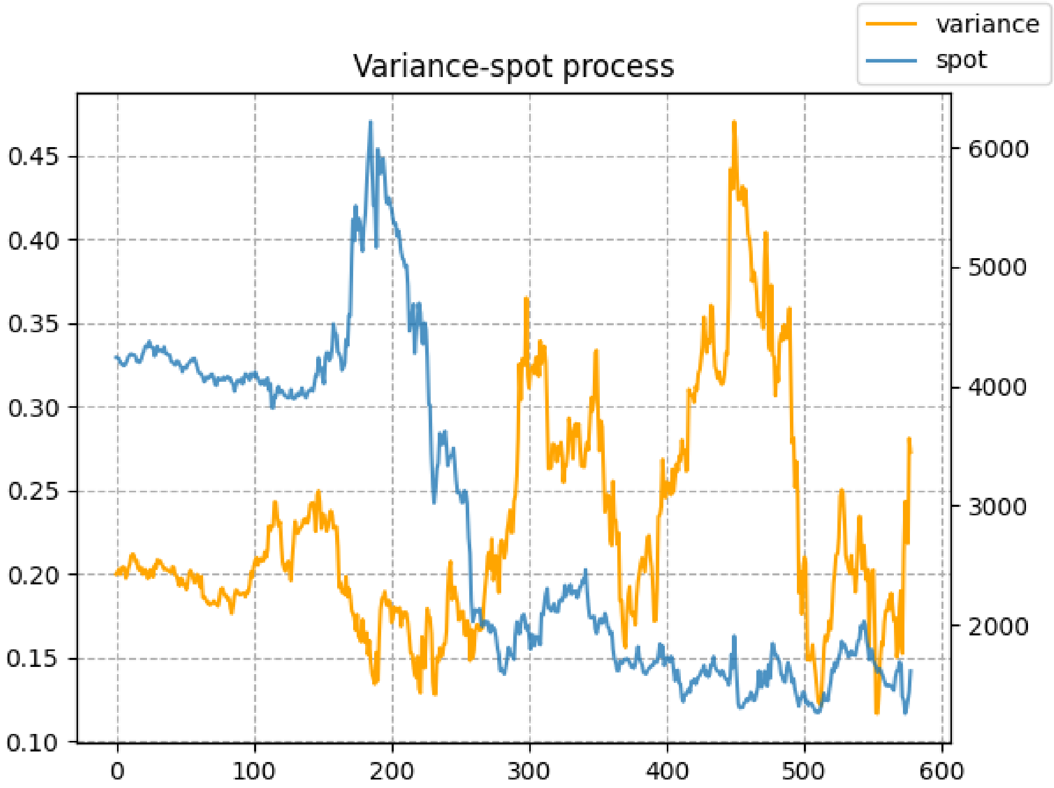




The Joint dynamics

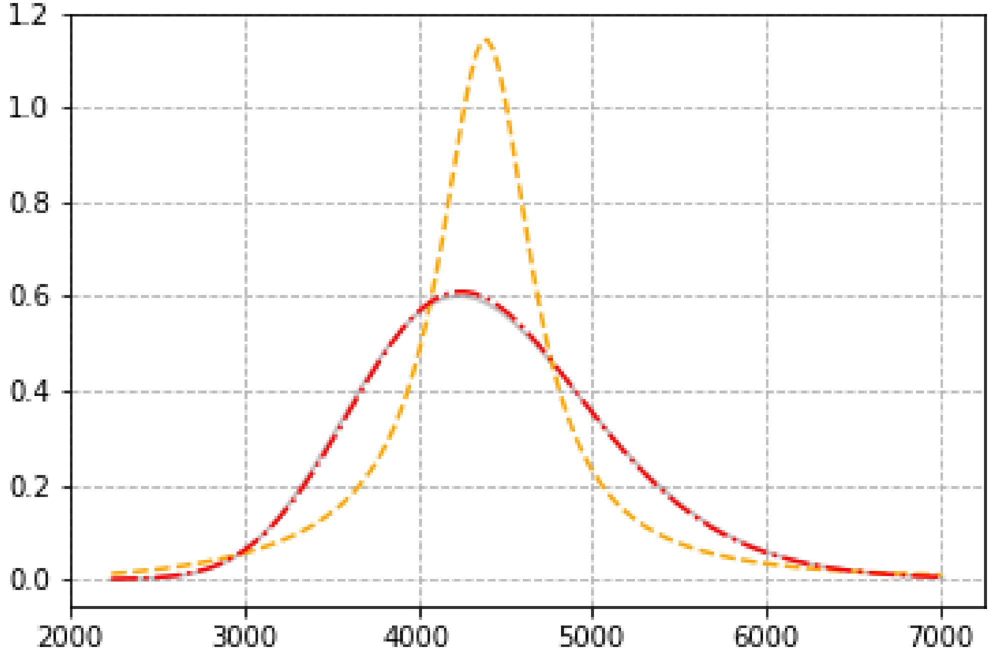


Variance-spot correlation

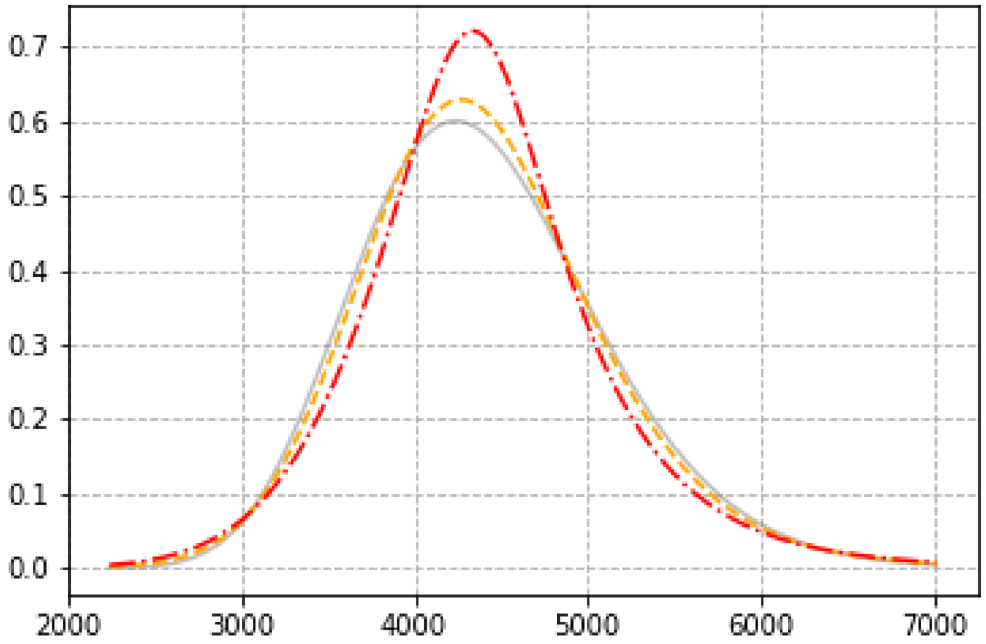


No variance-spot correlation

Heston stock distribution

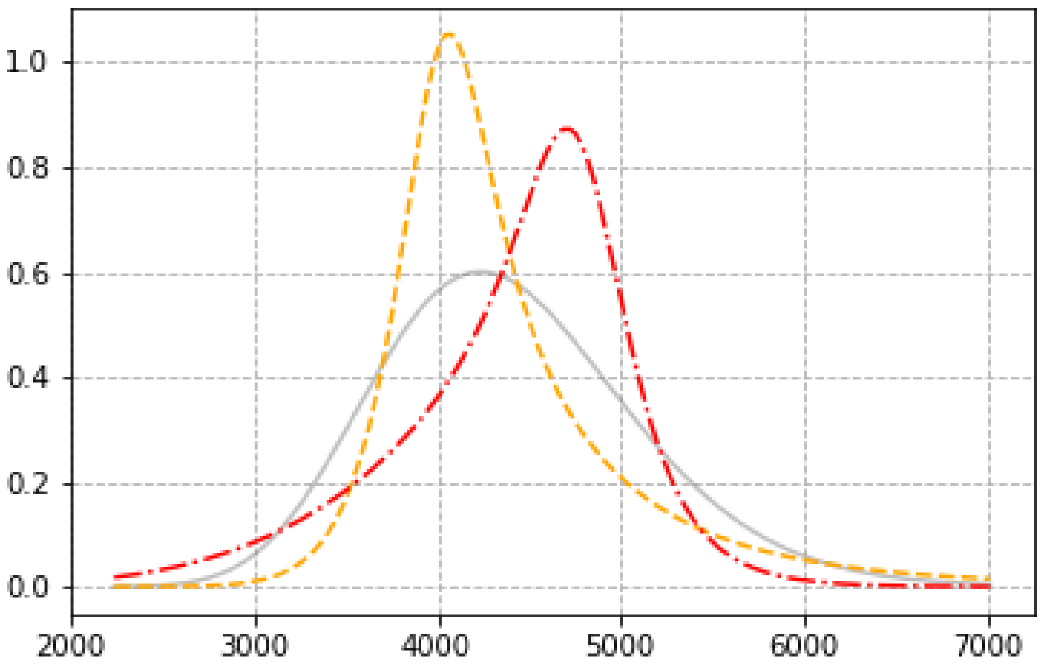


Vol of vol = 0.1
Vol of vol = 0.7

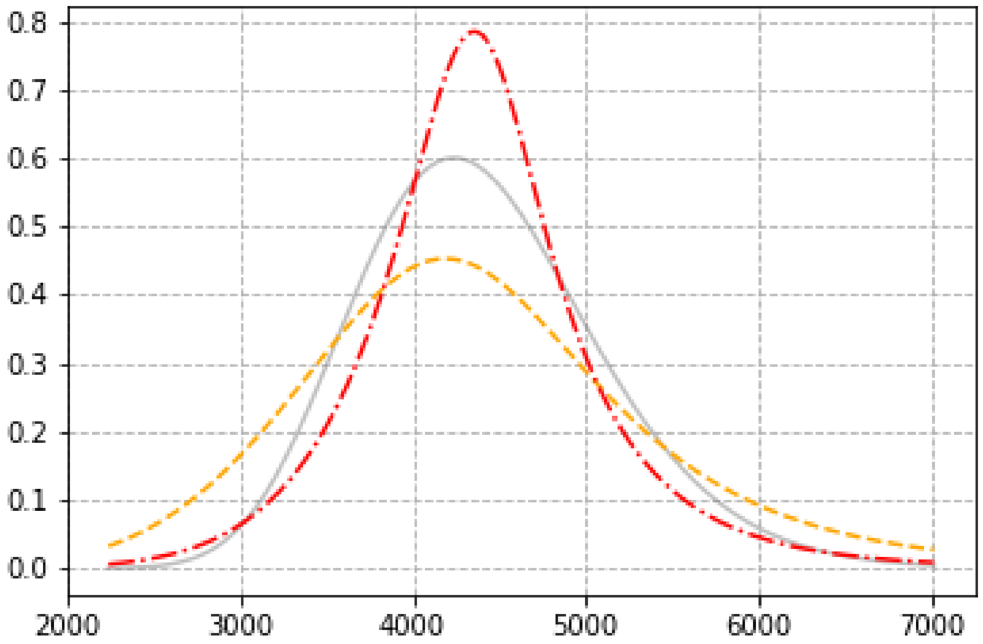


Mean reversion = 0.1
Mean reversion = 0.3

Heston stock distribution



$\rho = -0.7$ Left skew
$\rho = 0.7$ Right skew



$v_0 = 0.04$
$v_0 = 0.1$

Heston model – How to deal with it?

Process simulation
S(t), v(t)
→
Monte Carlo simulation
Simulate many paths (S, v) until expiry

Partial Differential Equation (PDE)
→
Finite difference model
Backpropagate the payoff from expiry

Characteristic Function
→
Carr-Madan formula
Fourier transform to compute prices fast

Heston model

- Black-Scholes model – 2 parameters, 1d process:

$$dS = rS\,dt + \sigma S\,dW$$
$$S(t=0) = S_0$$

- Heston model dynamics – 6 parameters, 2d process:

$$dS = rS\,dt + \sqrt{v}S\,dW_S$$
$$dv = \kappa(\bar{v} - v)\,dt + \gamma\sqrt{v}\,dW_v$$
$$\mathrm{E}[dW_S dW_v] = \rho\,dt$$
$$S(t=0) = S_0$$
$$v(t=0) = v_0$$

Decorrelating correlated Brownian motions

- Let's start simple:

$$dW_S = dW_1$$

- Expectation and variance matches, so far so good

$$dW_v = a\, dW_1 + b\, dW_2$$

- Expectation:

$$E[dW_v] = E[a\, dW_1 + b\, dW_2] = aE[dW_1] + bE[dW_2] = 0 + 0 = 0$$

- Variance:

$$Var(dW_v) = Var(a\, dW_1 + b\, dW_2) = E\left[a^2 dW_1^2 + 2ab\, dW_1 dW_2 + b^2 dW_2^2\right] = a^2 dt + b^2 dt \Rightarrow a^2 + b^2 = 1$$

- Covariance:

$$Cov(dW_S, dW_v) = Cov(dW_1, a\, dW_1 + b\, dW_2) = E\left[a\, dW_1^2 + b\, dW_1 dW_2\right] = a\, dt \Rightarrow a = \rho \Rightarrow b = \sqrt{1 - \rho^2}$$

Decorrelating correlated Brownian motions

- Matrix notation:

$$\begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} dW_1 \\ dW_2 \end{pmatrix} = \begin{pmatrix} dW_S \\ dW_v \end{pmatrix}$$

- Monte Carlo simulation discretizing:

$$\left| \begin{array}{l} dS = rS\,dt + \sqrt{v}S\,dW_1 \\ dv = \kappa(\bar{v} - v)dt + \gamma\sqrt{v}(\rho\,dW_1 + \sqrt{1-\rho^2}\,dW_2) \end{array} \right.$$

Heston PDE derivation

- Heston model SDE:

$$\left| \begin{array}{l} dS = rS\,dt + \sqrt{v}S\,dW_1 \\ dv = \kappa(\bar{v} - v)dt + \gamma\sqrt{v}(\rho\,dW_1 + \sqrt{1-\rho^2}\,dW_2) \end{array} \right.$$

- Option price should change over time (on expectation) same as value of money – martingale (no-arbitrage):

$$e^{-rt}V(t, S(t), v(t)) = e^{-rT}E[V(T, S(T), v(T))]$$

- Let's analyze change to option value:

$$d(e^{-rt}V) = e^{-rt}dV - re^{-rt}Vdt$$

$$= e^{-rt}\left(\frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2 + \frac{\partial V}{\partial v}dv + \frac{1}{2}\frac{\partial^2 V}{\partial v^2}dv^2 + \frac{\partial^2 V}{\partial S \partial v}dSdv\right) - re^{-rt}Vdt$$

Heston PDE derivation

$$\begin{cases} dS = rS\,dt + \sqrt{v}S\,dW_1 \\ dv = \kappa(\bar{v} - v)\,dt + \gamma\sqrt{v}(\rho\,dW_1 + \sqrt{1-\rho^2}\,dW_2) \end{cases}$$

$$d(e^{-rt}V) = e^{-rt}\left(\frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2 + \frac{\partial V}{\partial v}dv + \frac{1}{2}\frac{\partial^2 V}{\partial v^2}dv^2 + \frac{\partial^2 V}{\partial S\partial v}dSdv\right) - re^{-rt}Vdt =$$

$$= e^{-rt}\left(\frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}\left(rS\,dt + \sqrt{v}S\,dW_1\right) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}vS^2\,dt + \frac{\partial V}{\partial v}(\kappa(\bar{v} - v)\,dt + \gamma\sqrt{v}(\rho\,dW_1 + \sqrt{1-\rho^2}\,dW_2))\right.$$
$$\left. + \frac{1}{2}\frac{\partial^2 V}{\partial v^2}\gamma^2 v\,dt + \frac{\partial^2 V}{\partial S\partial v}\gamma vS\rho\,dt\right) - re^{-rt}V\,dt$$

- On expectation this change should equal zero. $\mathrm{E}[dW] = 0$, so we only need to take care of dt terms:

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}rS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}vS^2 + \frac{\partial V}{\partial v}\kappa(\bar{v} - v) + \frac{1}{2}\frac{\partial^2 V}{\partial v^2}\gamma^2 v + \frac{\partial^2 V}{\partial S\partial v}\gamma vS\rho - rV = 0$$

Affine processes

- Given a d dimensional process X

$$dX = \mu(t,X)dt + \sigma(t,X)dW$$

- Characteristic function exponentially affine:

$$\phi(v,t,T) = E[e^{-\int_t^T r(s)ds + iu^T X(T)}] = e^{A(u,T-t)+B^T(u,T-t)X(T)}$$

- Necessary condition:

$$\mu(t,X) = a^0 + a^1 X$$

$$[\sigma(t,X)\sigma^T(t,X)]_{ij} = c_{ij}^0 + \sum_k c_{ij}^1{}_k X_k$$

- Drift and covariance matrix at most linear in all coefficients

Heston – affine process in log(S)

- $X = \log(S)$

$$d\begin{pmatrix} X \\ v \end{pmatrix} = \begin{pmatrix} r - \frac{1}{2}v \\ \kappa\bar{v} - \kappa v \end{pmatrix} dt + \begin{pmatrix} \sqrt{v} & 0 \\ \rho\gamma\sqrt{v} & \gamma\sqrt{1-\rho^2}\sqrt{v} \end{pmatrix} d\begin{pmatrix} W_1 \\ W_2 \end{pmatrix}$$

$$\sigma\sigma^T = \begin{pmatrix} v & \rho\gamma v \\ \rho\gamma v & \gamma^2 v \end{pmatrix}$$

- PDE in terms of X and $\tau = T - t$:

$$-\frac{\partial V}{\partial \tau} + (r - \frac{1}{2}v)\frac{\partial V}{\partial X} + \frac{1}{2}v\frac{\partial^2 V}{\partial X^2} + \kappa(\bar{v} - v)\frac{\partial V}{\partial v} + \frac{1}{2}\gamma^2 v\frac{\partial^2 V}{\partial v^2} + \gamma v\rho\frac{\partial^2 V}{\partial X \partial v} - rV = 0$$

Heston model characteristic function

- Chf:

$$e^{-rt}\phi(u,t,T) = e^{-rT}E\left[e^{iuX(T)}\right]$$

- Solution ansatz:

$$\phi(u,t,T) = e^{A(u,\tau)+B(u,\tau)X+C(u,\tau)v}$$

- Initial (final) conditions:

$$\phi(u,T,T) = e^{iuX(T)} \;\Rightarrow\; A(u,0) = C(u,0) = 0, B(u,0) = iu$$

- Inserting into PDE, results in a set of 3 ODE’s:

$$\frac{dB}{d\tau} = 0$$

$$\frac{dC}{d\tau} = \frac{1}{2}B(B-1) - (\kappa - \gamma\rho B)C + \frac{1}{2}\gamma^2 C^2$$

$$\frac{dA}{d\tau} = \kappa\bar{v}C + r(B-1)$$

Call option price and characteristic function

Call price:

$$C(K) = E[\max(S_T - K, 0)] = \int_0^{+\infty} \max(s - K)\, f(s) ds$$

We know the characteristic function in close form, but not the density. Using the Fourier Transform:

$$C(K) = F_\alpha^{-1}\big[F_\alpha[C]\big](k) = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{+\infty} e^{-ivk} \frac{\phi\big(v - i(\alpha + 1)\big)}{(\alpha + iv)(\alpha + iv + 1)} dv$$

Thank you!

Do you have any questions?

Fabio Maggioni
Quantitative researcher
Fabiomaggioni@optiver.com

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