Gravitational waves

Relativistic Astrophysics and Cosmology: Lecture 14

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Monday 6th November 2023

Pre-lecture question:

Why have we progressed so rapidly in gravitational wave detection?

Last time

- Binary systems
- Gravity in the weak field limit
- Geometry of a gravitational wave

This lecture

- Linearised gravity beyond the vacuum
- Emission of gravitational waves
- Detection of gravitational waves

Next lecture

- The general weak-field metric & gravitational lensing

Beyond vacuum: the general linearised solution

- We wish now to find the general solution to

$$\Box^2 \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}, \quad \partial_\mu \bar{h}^{\mu\nu} = 0.$$

- Using a standard "retarded Green's function"[1] approach the general solution is:

$$\bar{h}^{\mu\nu}(ct, \vec{x}) = -\frac{4G}{c^4} \int \frac{T^{\mu\nu}(ct - |\vec{x} - \vec{y}|, \vec{y})}{|\vec{x} - \vec{y}|} d^3\vec{y},$$

  where $(ct, \vec{x})$ is the spacetime location of the observer, the integral is performed over the stress energy tensor over a set of source locations $\vec{y}$ and retarded times $ct - |\vec{x} - \vec{y}|$.
- For our purposes however, we can make the compact source approximation

$$\boxed{\bar{h}^{\mu\nu}(ct, \vec{x}) = -\frac{4G}{c^4 r} \int T^{\mu\nu}(ct - r, \vec{y}) d^3\vec{y}.}$$

[1] The Green's function causally satisfies $\Box^2 G = \delta^{(4)}(x)$ with $G = \delta(ct - |\vec{x}|)\theta(ct)/4\pi|\vec{x}|$ (Heaviside θ).

- Now examining the components of the compact source approximation:

$$Mc^2 = \int T^{00}\, d^3\vec{y} \qquad \text{total energy of source particles} \equiv Mc^2.$$

$$P^i c = \int T^{0i}\, d^3\vec{y} \qquad c\times \text{ total } i\text{-momentum of source} \equiv P^i c.$$

$$S^{ij} = \int T^{ij}\, d^3\vec{y} \qquad \text{integrated internal stresses in the source.}$$

- We know that to first order $\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} = 0$.
- This means the first two of these are conserved: $\dot{M} = \dot{P}^i = 0$ (integrate both sides and apply the divergence theorem to the spatial derivative terms).
- Using similar manipulations of Gauss theorem and integration by parts we can obtain a very useful (and non-trivial) expression for the third term:

$$S^{ij} = \int T^{ij} d^3\vec{y} = \frac{1}{2c^2}\frac{d^2}{dt^2}\int T^{00} y^i y^j d^3\vec{y} = \frac{1}{2c^2}\ddot{\mathcal{I}}^{ij}, \qquad \text{where} \qquad \mathcal{I}^{ij} = c^2 \int \rho y^i y^j d^3\vec{y}.$$

- This expresses the integrated internal stresses in terms of acceleration of the quadrupole moment tensor.

- We note that we can without loss of generality choose the spatial coordinates in the centre of momentum frame of the source so that $P^i = 0$.
- Putting this all together, we find that

$$\boxed{\bar{h}^{00} = -\frac{4GM}{c^2 r}, \qquad \bar{h}^{i0} = \bar{h}^{0i} = -\frac{4GP^i}{c^3 r} = 0, \qquad \bar{h}^{ij} = -\frac{2G}{c^6 r}\ddot{\mathcal{I}}^{ij}.}$$

- We see that therefore the far-field of a compact source falls into two parts:
  1. A steady field from constant mass M of the source.
  2. A possibly time-varying field arising from integrated internal stresses/quadrupole moment (responsible for gravitational waves).
- An important observation is that the compact source approximation demonstrates that the far-field solution to linearised equations decays as $\sim \frac{1}{r}$.
- Gravitational wave detectors directly measure strain $h^{\mu\nu}$, in contrast to electromagnetism, where detectors measure $F \sim \partial A \sim \frac{1}{r^2}$.
- The upshot of this is that doubling the sensitivity, multiplies the volume by a factor of 8.
- However, since it is quadrupoles that are responsible for emission, combined with weak coupling of gravitation to matter, means gravitational radiation is incredibly weak (albeit much longer range).

The energy momentum in gravitational fields

- The problem of how to localise the "energy" of the gravitational field is long-standing.
- The equivalence principle allows us to remove local gravity.
- If you were to assign an energy to the field, by mass-energy equivalence this generates more gravity – Gravity gravitates, and hence is a nonlinear theory.
- In a general spacetime there is no symmetry, so we should not expect conserved quantities.
- However, we can make progress in the linearised theory by assigning the second order terms (i.e. the amount that the wave fails to satisfy the equations) to a stress energy tensor $t_{\mu\nu}$

$$G_{\mu\nu} = G^{(1)}_{\mu\nu} + G^{(2)}_{\mu\nu} + \ldots = -\frac{8\pi G}{c^4}(T_{\mu\nu} + t_{\mu\nu}) \quad \Rightarrow \quad t_{\mu\nu} \equiv \frac{c^4}{8\pi G}\left\langle G^{(2)}_{\mu\nu}\right\rangle .$$

  where angle brackets denote spacetime averaging over a small region to create a gauge-invariant quantity.
- After much effort, in the transverse traceless gauge, this can be found to be

$$t_{\mu\nu} = \frac{c^4}{32\pi G}\left\langle \partial_\mu h^{TT}_{ij} \partial_\nu h^{ij}_{TT} \right\rangle .$$

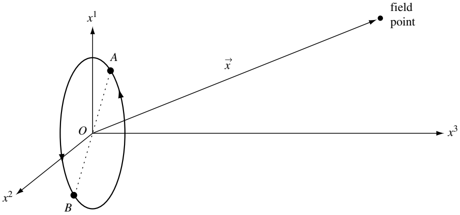
Energy loss due to gravitational wave emission

- Since $t \sim (\partial h)^2$, and $h \sim \ddot{\mathcal{I}}$, we can see that the energy transport is going to depend on third derivatives!
- Using expressions for the flux in direction n of $F = -ct^{0k} n_k$, and the luminosity $-r^2 \int F d\Omega$ after a little manipulation we find

$$\boxed{\frac{dE}{dt} = -L_{\mathrm{GW}} = -\frac{G}{5c^9} \left\langle \dddot{\mathcal{I}}_{ij} \dddot{\mathcal{I}}^{ij} \right\rangle .}$$

- One minor subtlety is that this is only true in the transverse traceless gauge, so more general to replace $\mathcal{I}^{ij}$ with its traceless equivalent $\mathcal{Q}^{ij} = \int T^{00}(y^i y^j - \frac{1}{3}\delta^{ij}) d^3\vec{y}$.

A binary illustration



- Consider a Newtonian circular binary system of equal masses $M_1 = M_2 = M/2$ with angular speed Ω and separation a:

$$\vec{x}_A = \left(\tfrac{a}{2}\cos\Omega t, \tfrac{a}{2}\sin\Omega t\right) = -\vec{x}_B, \quad \Omega^2 = \frac{GM}{a^3}.$$

- Taking $T^{00} = \rho c^2$ and substituting into $\mathcal{I}^{ij} = c^2 \int \rho x^i x^j d^3\vec{x}$, $\bar{h}^{ij} = -\frac{2G}{c^6 r}\ddot{\mathcal{I}}^{ij}$ gives

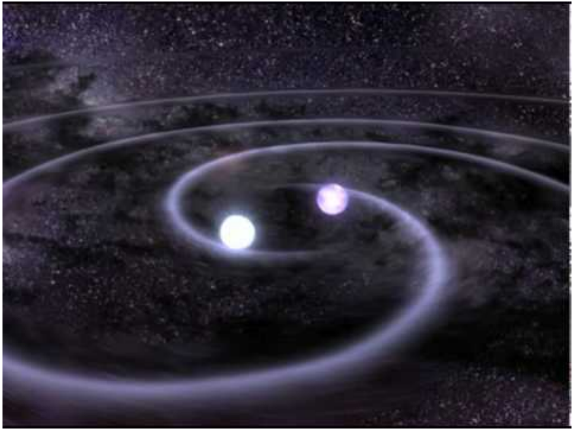
$$\mathcal{I}^{ij} = \frac{1}{8}Mc^2a^2 \begin{pmatrix} 1+\cos 2\Omega t & \sin 2\Omega t & 0 \\ \sin 2\Omega t & 1-\cos 2\Omega t & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \bar{h}^{\mu\nu}_{\text{rad}} = \frac{GMa^2\Omega^2}{c^4 r}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos 2\Omega t_r & \sin 2\Omega t_r & 0 \\ 0 & \sin 2\Omega t_r & -\cos 2\Omega t_r & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where t_r is the retarded time $t_r = t - \frac{r}{c}$, and we consider only the radiative parts of $\bar{h}^{\mu\nu}$.

- We can compute the third time derivative and use $L = -\frac{G}{5c^9}\langle \dddot{\mathcal{I}}_{ij}\dddot{\mathcal{I}}^{ij}\rangle$ to find

$$\dddot{\mathcal{I}}^{ij} = Mc^2a^2\Omega^3 \begin{pmatrix} \sin 2\Omega t & -\cos 2\Omega t & 0 \\ -\cos 2\Omega t & -\sin 2\Omega t & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \boxed{\frac{dE}{dt} = -L_{\mathrm{GW}} = -\frac{2G}{5c^5}M^2a^4\Omega^6.}$$

- Ω^6 is a pretty high loss rate!
- This is a reasonable model for the scaling of inspiralling neutrons stars, black holes or indeed a "lump" on the surface of a star.
- Exercise for the reader: Determine the polarisation of these gravitational waves, and the anisotropy of the flux.



Inspiral of binary pulsar of equal masses

- Taking $\Omega^2 = \frac{GM}{a^3}$ as it's Keplerian value, we can express purely in terms of M and a

$$\frac{dE}{dt} = -\frac{2G}{5c^5} M^2 a^4 \Omega^6 = -\frac{2G^4 M^5}{5a^5 c^5}.$$

- We may also write the total energy since $M_1 = M_2 = \frac{1}{2}M$, so $E_{\text{tot}} = -\dfrac{GM^2}{8a}$.
- Note that the total energy is negative, since it is bound, and therefore decreasing energy corresponds to decreasing a and increasing v.
- Eliminating a using Kepler's law, we may write $E = -\left(\dfrac{\pi^2 GM^5}{1024}\right)^{1/3} P^{-2/3}$, and arrive at

$$\dot{P} = -\frac{3}{10}\pi\left(\frac{16\pi GM}{Pc^3}\right)^{5/3}.$$

- This allows us to measure the mass of the system using measurements of P and $\dot{P}$, whilst also confirming the gravitational wave nature from the non-trivial $\dot{P} \propto P^{-5/3} \Rightarrow P \propto t^{3/8}$.

General binary system

- The calculations are identical to above, but fiddlier.
- One of the things which gravitational wave physicists often do is to use Kepler's law to remove a-dependence in place of Ω, which is measurable.

$$a = \frac{G^{1/3}M^{1/3}}{\Omega^{2/3}}, \quad E_{\text{tot}} = -\frac{1}{2}\frac{GM_1M_2}{a} = -\frac{1}{2}\frac{G^{2/3}M_1M_2}{M^{1/3}}\Omega^{2/3}, \quad \dot{E}_{\text{tot}} = -\frac{1}{3}\frac{G^{2/3}M_1M_2}{M^{1/3}}\Omega^{-1/3}\dot{\Omega}.$$

- The power of the gravitational waves is

$$I \sim \frac{M_1M_2}{M}a^2c^2\sin^2\Omega t \Rightarrow \qquad L_{\text{GW}} = \frac{32}{5}\frac{G(\frac{M_1M_2}{M})^2a^4\Omega^6}{c^5} = \frac{32}{5}\frac{G^{7/3}}{c^5}\frac{M_1^2M_2^2}{M^{2/3}}\Omega^{10/3}.$$

- Equating $\dot{E}_{\text{tot}}$ to L_{GW} leads to $\dot{\Omega} = \frac{96}{5}\frac{G^{5/3}}{c^5}\mathcal{M}^{5/3}\Omega^{11/3}$ with Chirp Mass $\boxed{\mathcal{M} = \frac{(M_1M_2)^{3/5}}{M^{1/5}}}$.

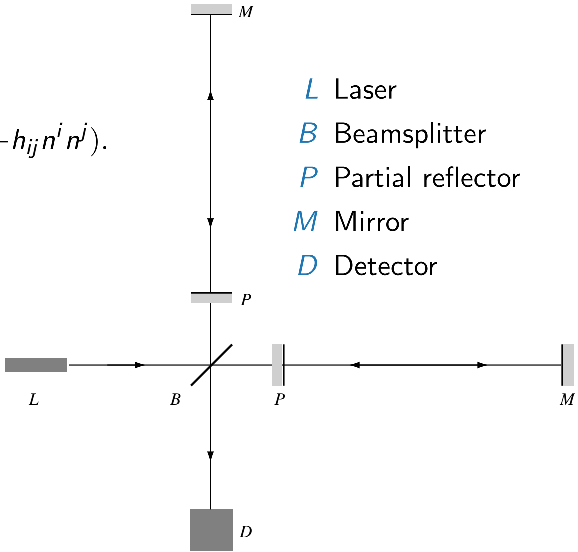
Detecting gravitational waves

- Gravitational wave affects the physical separation of two free particles:

$$l^2 = -g_{\mu\nu}\Delta x^i \Delta x^j = (\delta_{ij} - h_{ij})\Delta x^i \Delta x^j = l_0^2(1 - h_{ij} n^i n^j).$$

$$\Rightarrow \boxed{\frac{\delta l}{l_0} = -\frac{1}{2} h_{ij} n^i n^j} \text{ (to first order).}$$

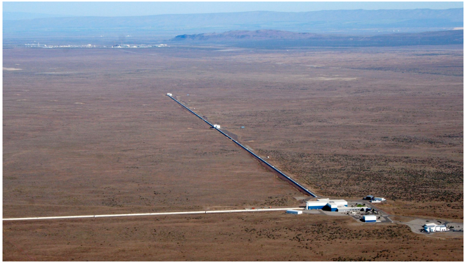
- This strain can be detected with a Michaelson interferometer.
- Second approach using "resonant" detection historically trialled over decades by Weber.
- Cavendish laboratory has long-term plans for gravitational wave observatory using atomic interferometers: AION [arxiv:1911.11755].



L Laser
B Beamsplitter
P Partial reflector
M Mirror
D Detector

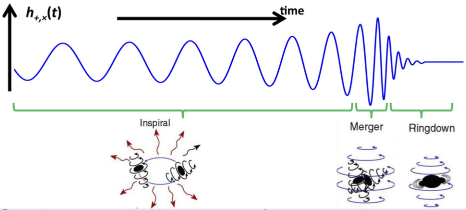
LIGO: Laser interferometer gravitational wave observatory

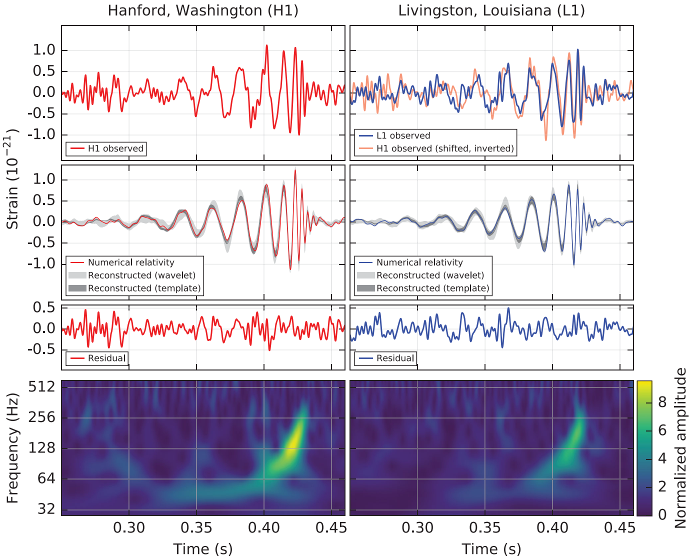
- The challenge is that for typical astrophysical sources at 100 Hz strain rate is $\frac{\delta l}{l_0} \sim 10^{-21}$.
- Requires a kilometre interferometer to be able to measure an attometre strain $(10^{-6}a_0)$.



- So we built one.
- Cavity resonance enables an effective arm length of $\sim 40km$.
- Suspended masses enable frequency-dependent isolation.
- Two detectors on either side of the US.
- Time-delay measurements localise source.
- Allows systematic error detection (Machines sensitive to shoreline ocean waves).
- $\sim \mathcal{O}(30)$ new detectors planned.

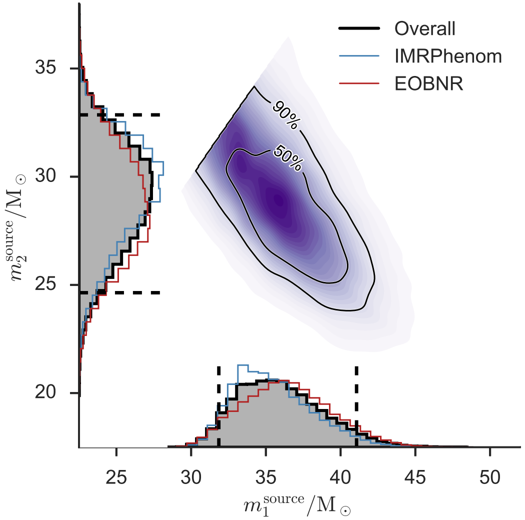
- Obviously great excitement in February eight years ago with announcement of first direct detection [arxiv:1602.03837].
- Can see classic merger waveform clearly by eye in both detectors.
- Steadily increasing frequency and amplitude (chirp), followed by "ringdown".





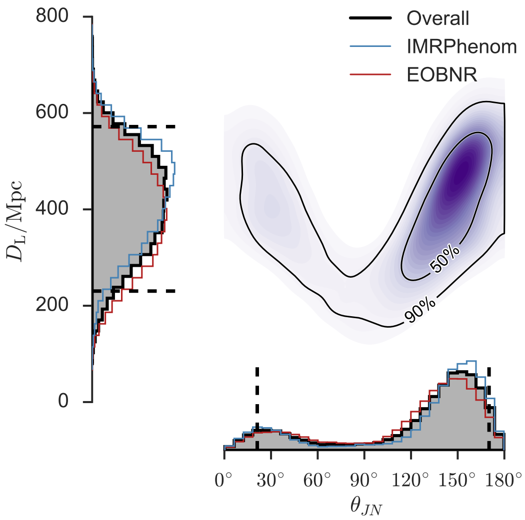
LIGO as a gravitational wave telescope

- What they detected was the emission from the final stages of coalescence of a binary black hole pair, each hole of mass $\sim 30M_\odot$ and with the total energy radiated away about $3M_\odot c^2$!
- By fitting models to the chirp, one can infer the parameters.
- Plot shows contours of probability of mass parameters $P(\text{parameters}|\text{data}, \text{model})$.
- NB: masses defined so that $m_1 > m_2$ (hence the unphysical "cut" in the distribution).
- NB: that LIGO is sensitive to the sum $m_1 + m_2$ more than the difference $m_1 - m_2$, hence correlation in posterior distribution.



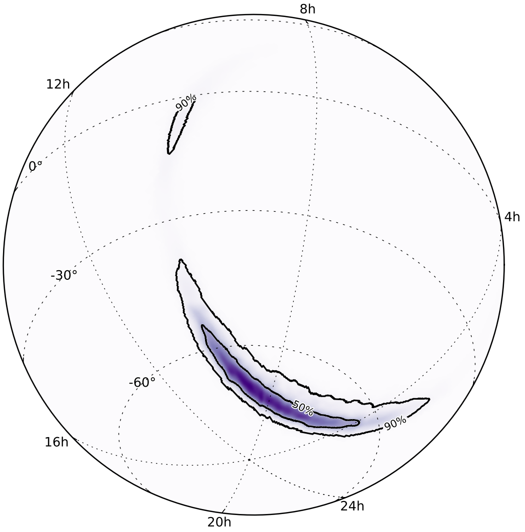
LIGO as a gravitational wave telescope

- Source signal in reality depends on:
  - m_1, m_2: mass of binary,
  - θ, ϕ: sky location,
  - D_L: distance,
  - Φ_c, t_c: phase and time of coalescence,
  - i, θ_{JN}: sky inclination and angle,
  - $\vec{S}_1$, $\vec{S}_2$ spins of objects.
- These posterior distributions are therefore two- and one-dimensional projections of the full representation of our knowledge.
- Here we can see luminosity distance against inclination – highly non-trivial correlation.
- Requires substantial Bayesian inference technology (Astrophysics is a driving force in the development of such techniques).



LIGO as a gravitational wave telescope

- One of the most useful sets of parameters is the sky location, as this allows multimessenger follow-up (see Lecture 8).
- Note that in the original event, LIGO is mostly sensitive to a "ring" on the sky.
- This is because you need three time delay measurements to localise in space.
- It is not a perfect ring due to the fact that Hanford and Livingston are not in the same plane (due to curvature of the earth).
- This is analogous to how our own ear geometry partially breaks the degeneracy (although strictly speaking we'd be better at hearing if we had three).



- Six years ago announced discovery.
- Discussed already in Lecture 7 [arxiv:1710.05832]
- Note how much longer this event lasted.
- Virgo detector joined the team.
- Component masses much smaller — about 1.17—1.60 $M_\odot$ each.

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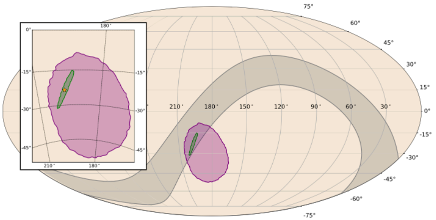

Figure 1. Final localizations. The 90% contour for the final sky-localization map from LIGO–Virgo is shown in green (LIGO Scientific Collaboration & Virgo Collaboration 2017a, 2017b, 2017c). The 90% GBM targeted search localization is overlaid in purple (Goldstein et al. 2017). The 90% annulus determined with *Fermi* and *INTEGRAL* timing information is shaded in gray (Svinkin et al. 2017). The zoomed inset also shows the position of the optical transient marked as a yellow star (Abbott et al. 2017f; Coulter et al. 2017a, 2017b). The axes are R.A. and decl. in the Equatorial coordinate system.

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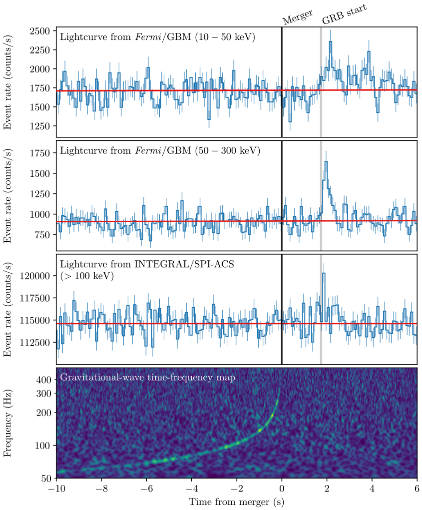
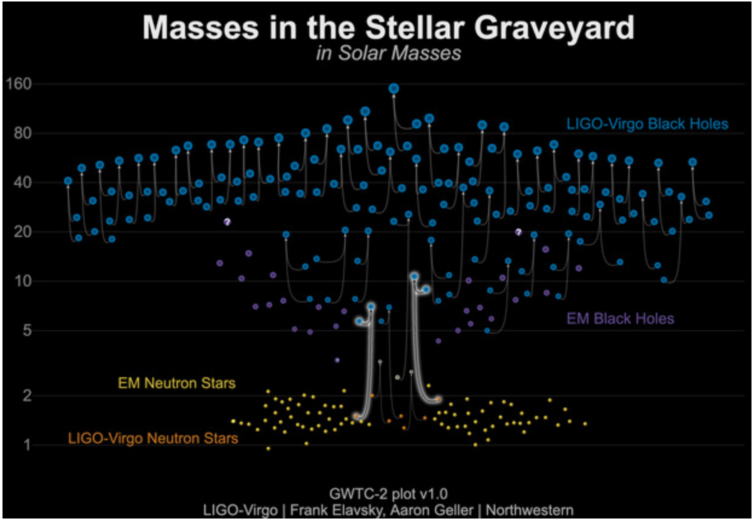


The story so far

- We've come a long way since 2016.
- Getting to the stage where we now have statistical power.
- e.g. beginning to constrain the Hubble constant using mergers as standard sirens (particularly with EM counterparts).
- Note we have a couple of highlighted NS-BH mergers.
- Also a mysterious mass gap (question mark).



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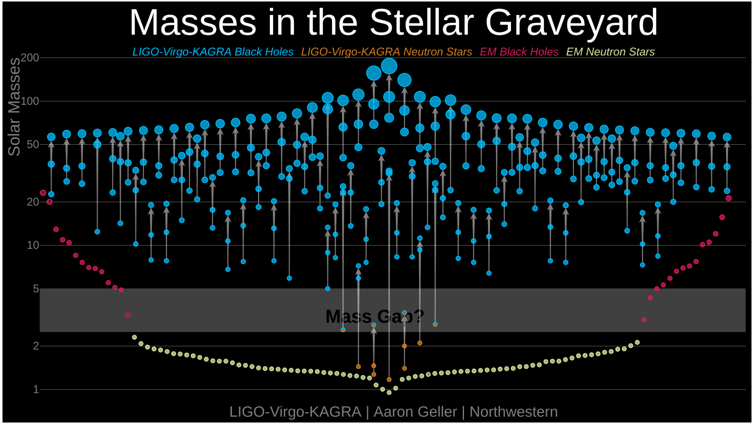
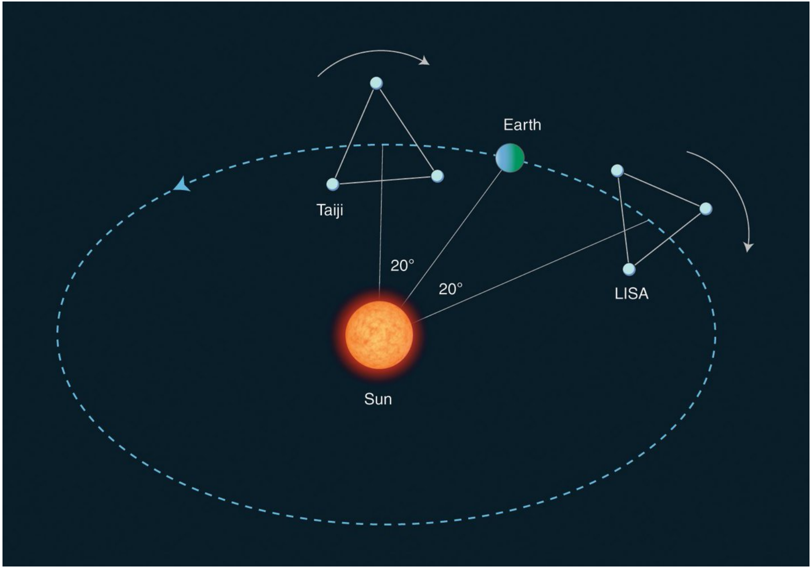


LISA: Laser interferometer space antenna

- With the success of LISA pathfinder, it is likely that we will see LISA this side of 2050
- Due to its size, LISA will be sensitive to different portions of the GW spectrum
- Before then, the Einstein telescope (LIGO II) and many other earth-bound systems will come online
- + Other approaches (atomic interferometry)



Summary

- Compact source approximation

$$\boxed{\bar{h}^{\mu\nu}(ct,\vec{x}) = -\frac{4G}{c^4 r}\int T^{\mu\nu}(ct-r,\vec{y})d^3\vec{y}.}$$

- Gravitational wave equations

$$\boxed{\bar{h}^{00} = -\frac{4GM}{c^2 r}, \qquad \bar{h}^{i0} = \bar{h}^{0i} = -\frac{4GP^i}{c^3 r} = 0, \qquad \bar{h}^{ij} = -\frac{2G}{c^6 r}\ddot{\mathcal{I}}^{ij}.}$$

- Gravitational wave luminosity

$$\boxed{\frac{dE}{dt} = -L_{\mathrm{GW}} = -\frac{G}{5c^9}\left\langle \dddot{\mathcal{I}}_{ij}\dddot{\mathcal{I}}^{ij}\right\rangle.}$$

- LIGO as a strain detector

$$\boxed{\frac{\delta l}{l_0} = -\frac{1}{2}h_{ij}n^i n^j.}$$

Next time

Gravitational lensing & the weak field limit