Gravitational lensing

Relativistic Astrophysics and Cosmology: Lecture 15

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Wednesday 8$^{\text{th}}$ November 2023

Pre-lecture question:

What's the biggest telescope we can use?

Last time

- Gravitational waves

This lecture

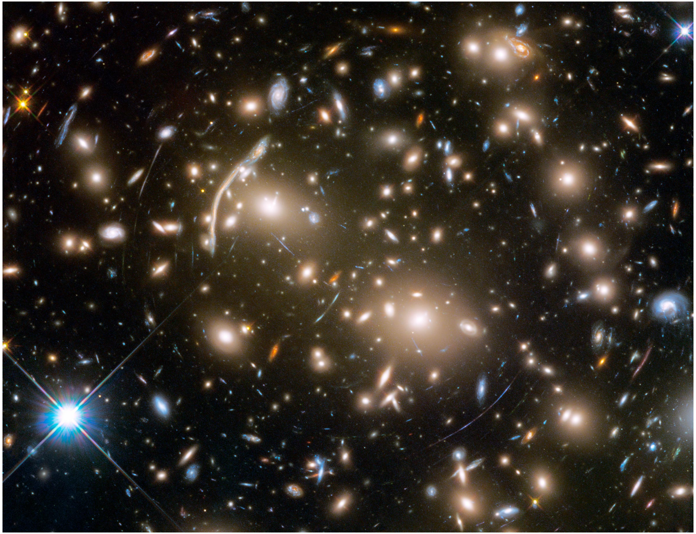
- Gravitational lensing
- The weak field metric
- The event horizon telescope

Next lecture

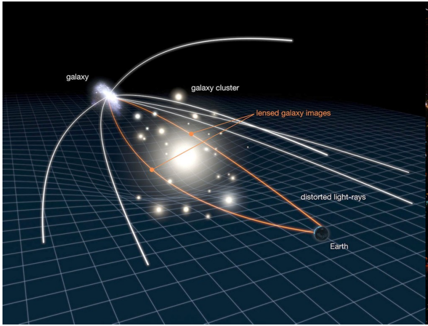
- Cosmology: the geometry of the Universe

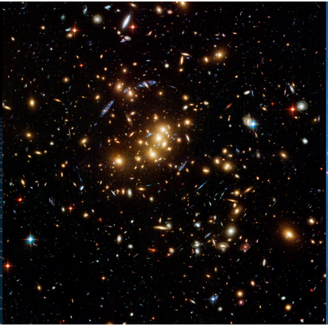
Gravitational lensing

- Gravity bends light.
- This is technically true in the Newtonian case (if you model photons as $m \to 0$ with speed c).
- An important prediction of Einstein's gravity is that it bends light by twice the angle.
- Can see in image "arcs" of lensed galaxies whose light is being bent around the central object.



Natural cosmic telescopes

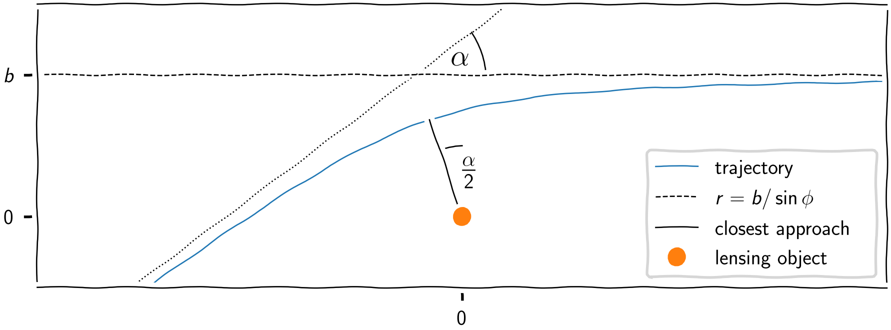





- Massive objects are capable of bending light from objects behind them.
- These act as "lenses" the same as glass.
- This means they act magnify images from behind them (as cosmic scale telescopes).
- Can also reconstruct the matter distribution (right hand picture) from lensing.
- "Pure geometry" (i.e. gravity) – this is a powerful probe of matter distribution.
- Note much of the lensing matter is invisible.

Newtonian gravitational lensing

- Note that Newtonian gravity does in fact predict gravitational lensing (Söldner in 1801).
- GR predicts that light should be bent twice as much as the Newtonian value.
- Inertial mass and gravitational mass cancel, so around a large object all fast deflected masses move in hyperbolae. We model photons as objects with speed c.



- Taking the shape equation with initial conditions $u = 1/r = 0$ at $\phi = 0$, and $r \sin\phi \to b$

$$\frac{d^2 u}{d\phi^2} + u = \frac{GM}{h^2} \quad \Rightarrow \quad u = \frac{1}{r} = \frac{GM}{h^2}(1 - \cos\phi) + \frac{1}{b}\sin\phi.$$

- The second initial condition $r \sin\phi \to b$ as $\phi \to 0$ defines the impact parameter via the equation of a straight line in plane polar coordinates.
- This also allows us to identify $h = \vec{r} \times \vec{v} = |\vec{r}||\vec{v}| \sin\phi = bc$.
- Maximising u gives closest approach location $-\cot\phi_* = \tan(\phi_* - \pi/2) = \frac{GMb}{h^2} = \frac{GM}{bc^2}$.
- Symmetry about the line between origin and closest approach gives tells us that the deflection angle is $\alpha = 2(\phi_* - \pi/2) = 2\arctan\frac{GM}{bc^2}$.
- For small α this gives the Newtonian expression

$$\alpha_{\text{Newton}} = \frac{2GM}{bc^2} = \frac{R_S}{b}.$$

Return to the weak field metric: Stationary sources

- Recall from last lecture the general solution to the linearised field equations:

$$\bar{h}^{\mu\nu}(ct,\vec{x}) = -\frac{4G}{c^4}\int \frac{T^{\mu\nu}(ct-|\vec{x}-\vec{y}|,\vec{y})}{|\vec{x}-\vec{y}|}d^3\vec{y}.$$

- Previous this was specialised to the compact source approximation.
- This time we will make the specialisation that the source is stationary, i.e. does not depend on time so $\partial_t T^{\mu\nu} = 0$. Retardation is now irrelevant and $T^{\mu\nu} = T^{\mu\nu}(\vec{y})$ so

$$\boxed{\bar{h}^{\mu\nu}(\vec{x}) = \frac{4G}{c^4}\int \frac{T^{\mu\nu}(\vec{y})}{|\vec{x}-\vec{y}|}d^3\vec{y}.}$$

- We next assume a pressureless stress-energy tensor $T^{\mu\nu} = \rho u^\mu u^\nu$ which is non-relativistic $u \ll c$, i.e. $\left|\frac{T^{ij}}{T^{00}}\right| \sim \frac{u^2}{c^2} \ll 1$, so our metric has components:

$$\bar{h}^{00} = \frac{4\Phi}{c^2}, \quad \bar{h}^{0i} = \frac{A^i}{c}, \quad \bar{h}^{ij} = 0, \quad \boxed{\Phi(\vec{x}) = -G\int\frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|}d^3\vec{y}, \quad \vec{A}(\vec{x}) = -\frac{4G}{c^2}\int\frac{\rho(\vec{y})\vec{u}(\vec{y})}{|\vec{x}-\vec{y}|}d^3\vec{y}.}$$

The stationary weak field metric

- Finally noting that
  - $h^{\mu\nu} = \bar{h}^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\bar{h}$,
  - since $\bar{h}^{ij} = 0$, the trace $\bar{h} = h^{00}$,
  - we lower indices with $\eta_{\mu\nu}$,
  - we find $h_{00} = h_{11} = h_{22} = h_{33} = \frac{2\Phi}{c^2}$ and $h_{0i} = \frac{A_i}{c}$, so

$$\boxed{ds^2 = \left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 + 2\vec{A} \cdot d\vec{x} cdt - \left(1 - \frac{2\Phi}{c^2}\right) d\vec{x}^2}.$$

- This metric is valid for relativistic motion (i.e. photons) around non-relativistic sources.
- Note the similarity in structure with the Schwarzschild and Kerr metrics.
- Note we may make this static so that it is invariant under $t \to -t$ by setting $\vec{A} = 0$.

Relativistic lensing

- We can view the Newtonian case as proceeding from the specific Lagrangian density

$$\mathcal{L} \sim T - V \sim \frac{1}{2}\left(\frac{d\vec{x}}{dt}\right)^2 - \Phi(\vec{x}).$$

- In the static weak field metric, the correct Lagrangian is

$$\mathcal{L} = \left(1+\frac{2\Phi}{c^2}\right)c^2dt^2 - \left(1-\frac{2\Phi}{c^2}\right)d\vec{x}^2 \quad \propto \quad \frac{1}{2}\left(\frac{d\vec{x}}{dt}\right)^2 - \frac{c^2}{2}\frac{1+\frac{2\Phi}{c^2}}{1-\frac{2\Phi}{c^2}} \quad \approx \quad \frac{1}{2}\left(\frac{d\vec{x}}{dt}\right)^2 - 2\Phi(\vec{x})$$

- This is very heuristic – real gravitational lensing calculations are very involved.
- However, in all cases the essence of where the factor 2 comes from is from a correct treatment of photons relativistically taking into account the curvature of space(time).
- This adjustment comes from the extra $\left(1-\frac{2\Phi}{c^2}\right)$ in the space term, in addition to the $\left(1+\frac{2\Phi}{c^2}\right)$ in the time term which recovers Newtonian gravity.
- In the end, we find that instead of $\alpha_{\text{Newton}} = \frac{2GM}{bc^2} = \frac{R_S}{b}$, we find $\boxed{\alpha_{\text{Einstein}} = \frac{4GM}{bc^2}} = \frac{2R_S}{b}$.

Historical Proof of GR

- The prediction of twice as much lensing is what crowned general relativity as the successor to Newton's theory.
- Eddington's 1919 image shows eclipse, and in the background pairs of thin horizontal lines.
- These lines mark where stars were seen during the eclipse (usually invisible in daylight).
- Their locations were different from their usual ones, consistent with Einsteinian rather than Newtonian lensing.

LIGHTS ALL ASKEW IN THE HEAVENS

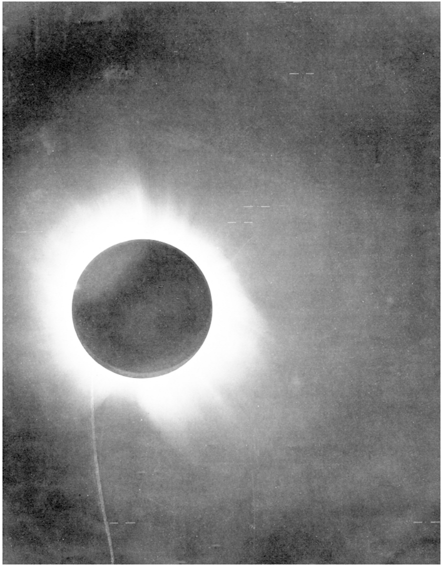
Men of Science More or Less Agog Over Results of Eclipse Observations.

EINSTEIN THEORY TRIUMPHS

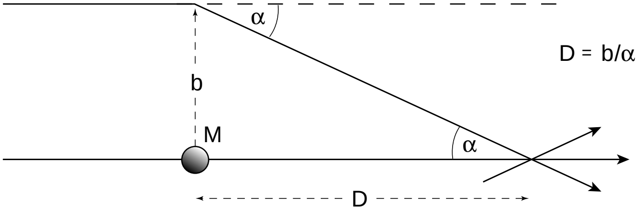
Stars Not Where They Seemed or Were Calculated to be, but Nobody Need Worry.

A BOOK FOR 12 WISE MEN

No More in All the World Could Comprehend It, Said Einstein When His Daring Publishers Accepted It.



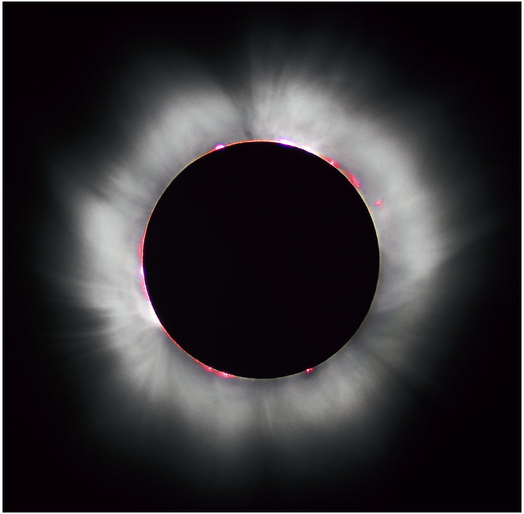
Gravitational Lensing



- In the solar system, this bending is tiny

$$\alpha = \frac{4GM}{bc^2} = \frac{2R_S}{b} = 1.75\left(\frac{M}{M_\odot}\right)\left(\frac{b}{R_\odot}\right)^{-1} \text{ arcsec.}$$

- NB: Degrees, minutes and seconds: arcsecond = measure of 'arc'/angle on the sky in seconds (1/3600 of a degree).
- This is the right at the limit of what can be resolved from the ground ($\sim 1\,$arcsec).

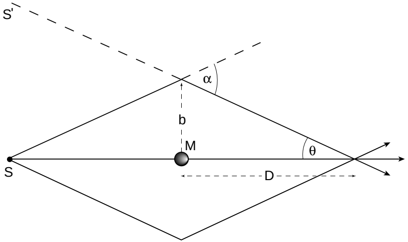


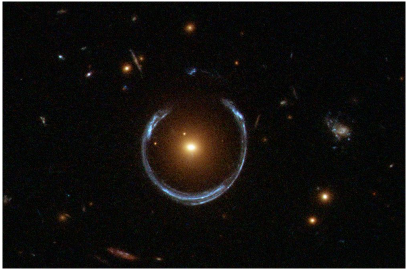
Lensing at cosmological distances

- Consider a source and lens at *cosmological distance* ($D \sim 10^{10}$ ly $= 10^{26}$ m). For simplicity consider the source and observer to be equidistant from the lens.

$$\theta = \frac{\alpha}{2} = \frac{R_S}{b} = \frac{b}{D} \Rightarrow \boxed{b = \sqrt{R_S D}} \sim 10^{-2}\left(\frac{M}{M_\odot}\right)^{1/2} \text{ pc},$$

and $\boxed{\theta \sim 1(M/M_\odot)^{1/2} \text{ mas.}}$

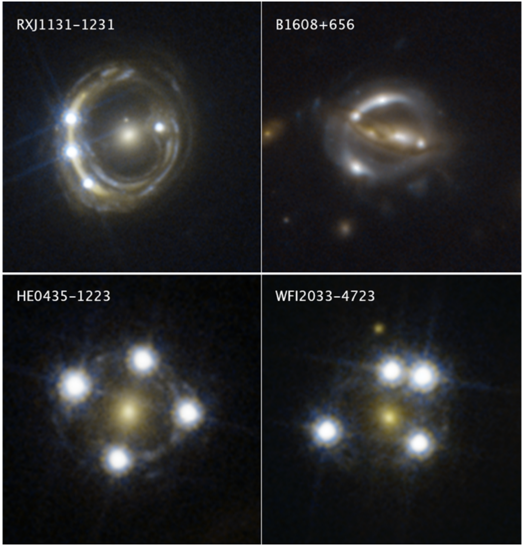




- The more general case is $\boxed{b = \sqrt{2R_S(D_s - D_l)D_l/D_s}}$ for a source at D_s and lens at D_l (Examples sheet 3).
- The lens can (if the geometry is right) then produce an Einstein ring of angular radius $\sim \theta$.
- To be resolved from the ground need $\theta \sim 1$ arcsec, therefore $M \gtrsim 10^{12} M_\odot$, and $b < 10$ kpc at the lens (a massive compact galaxy).

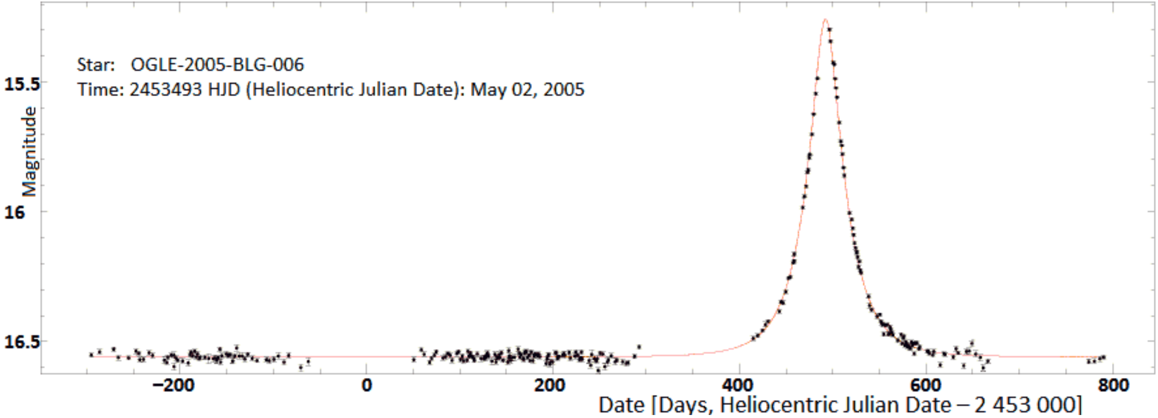
Multiply lensed objects

- If the source is offset however then the ring becomes a pair or more of images separated by $\sim 2\theta$.
- Many gravitational-lensed quasars and supernovae are now known.
- Recognisable by having identical spectra, often with time delays of seconds to days due to different light travel times.
- Very powerful geometrical probes for measuring the Hubble constant H_0.
- Time-delay supernovae cosmology is an emerging field being pioneered by people in Cambridge.



Microlensing

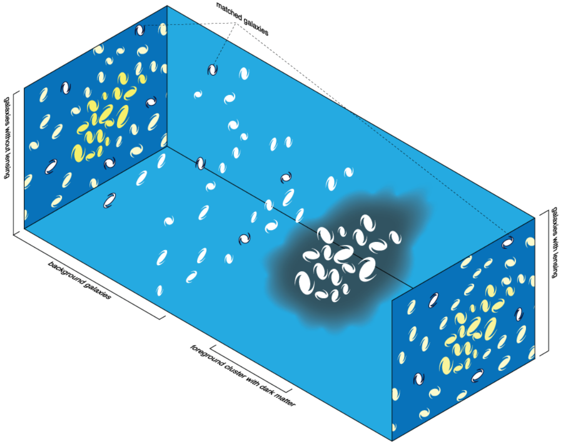
- Microlensing is used to detect non-luminous objects ranging from planets to stars.
- When an object with transverse velocity passes between us and a distant luminous object (e.g. a quasar or galaxy) lensing causes an abrupt spike in the magnitude.
- A lack of microlensing was used to rule out MACHO dark matter (e.g. brown dwarfs).



- As the microlens will have a very compact geometry, the amplification $A = \frac{u^2+2}{u\sqrt{u^2+4}}$ ($u = \frac{\delta\theta}{b}$ – see Examples sheet 3) and velocity are straightforward to fit to such curves.
- Individual microlenses are rather improbable, but the universe is a big place.
- Take for example the Large Magellanic cloud (30 kpc away – one of our satellite galaxies).
- For an object at this distance the lensing is $\theta \sim 400(M/M_\odot)^{1/2}$µas.
- This means that the fraction of one square degree of the LMC being lensed is $\sim \theta^2 \sim 10^{-14}(M/M_\odot)$.
- Now if the halo of our Galaxy is composed of $10^{12} M_\odot$ in objects each of mass M then in one square degree ($\sim 1/42,000$ of sky) there will be roughly $10^7(M/M_\odot)^{-1}$ of them so the probability of any particular star in the LMC being lensed is the product of these quantities or 10^{-7}.
- Since there are about 10^7 stars per square degree in the LMC there should be one star being lensed at any given time. This is detected because the halo object should be moving and thus the LMC star will brighten then dim symmetrically (and achromatically) on a timescale $\sim b/v \sim 10^7(M/M_\odot)^{1/2}$s (i.e. months).
- Several such events have been seen by the MACHO and EROS collaborations which have monitored the LMC. Many more lensing events are seen towards the Galactic Centre.

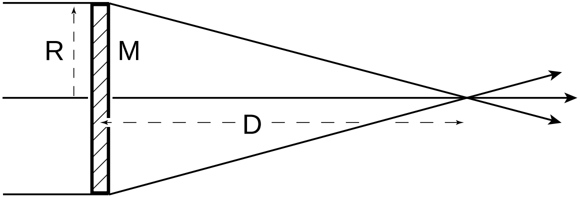
Weak lensing

- Mass structures along the line-of-sight to a distant galaxy can also cause weak gravitational lensing effects.
- Here the distortion is too weak to create multiple images or even appreciably magnify the galaxy, but the lensing leads to a slight twist in the image.
- The twist is correlated between neighbouring images and its measurement over a field of many thousands of background galaxies can reveal the projected mass distribution along that line of sight.
- Technique underlies the Dark Energy Survey (DES) and Kilo-degree survey (KiDS).



Extended gravitational lenses

- Consider a self-gravitating object of size R causing a deflection $\alpha = 4GM/Rc^2$.
- From the Virial Theorem $v^2 \sim GM/R$, so $\alpha \approx 4v^2/c^2$.
- Large angular deflections thus require objects with large internal velocities ($> 300\,\mathrm{kms}^{-1}$ for 1 arcsec). These are relatively rare objects.
- Rich clusters of galaxies have $v \sim 1000\,\mathrm{kms}^{-1}$ and $R \sim 3 \times 10^6$ ly, so can we expect $\alpha > 10$ arcsec? What is needed is that $R < b$, so for a cluster to act as a lens, it must have a compact core.



Lensing by a distributed mass. Only that mass M within the focussed beam causes the focussing.

- We require (from slide 11)

$$R^2 < b^2 = \frac{4GM}{c^2} D,$$

so surface mass density in a lens has to exceed

$$\frac{M}{\pi R^2} = \Sigma_{lens} \gtrsim \frac{c^2}{4\pi G D}.$$

- For cosmological distances (say $D \sim c/H_0$) this is $\sim 1\,\mathrm{g\,cm^{-2}} \sim 10\,\mathrm{kg\,m^{-2}}$.
- It is similar to the surface density of your hand or of a typical paperback book.
- Note also it is similar to the mean surface density in a universe of closure density, $\rho_{\rm crit} c/H_0$.
- An Einstein-de-Sitter Universe is essentially the ultimate gravitational lens.
- The probability that a class of objects gives rise to lensing along a random line of sight is proportional to their fractional contribution to the closure density, $\rho_\ell/\rho_{\rm crit} = \Omega_\ell$ (See Lecture 17).
- Thus since galaxies have $\Omega_{gal} \sim 0.01$, there is somewhat less than a 1% chance that lensing occurs due to a galaxies along a random line of site.

- Clusters which have such high central surface densities produce 'arcs' of light.
- These are images of distant galaxies that have been magnified by large amounts and so become visible.
- Spectra of the arcs show much higher redshifts than the cluster.
- The cluster cores are acting as 'giant telescopes' with apertures of 300.000 ly!
- The most distant galaxies ($z \sim 10$) currently known have been lensed by intervening clusters.
- As noted previously, gravitational lenses are highly non-linear and do not give simple images like a convex spectacle lens does.

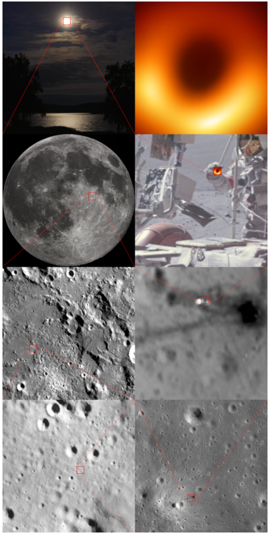
An optical image of the core of a distant cluster RCS032727 from the Hubble Space Telescope. Large arcs due to gravitational lensing of distant galaxies are seen.

- The effect from a complex gravitational field is more similar to that produced by the surface of a swimming pool on sunlight illuminating the bottom of the pool, with complex bands of light and dark (caustics) seen.
- We, the observer on the bottom, see a bright image if we lie on a bright band. (To see caustics, look at reflections on the bottom of the inside of a cylindrical mug illuminated by a strong light, or the underside of a bridge illuminated by sunlight reflected off the water.)
- In fact, things are not quite that bad – clusters do not have constant surface density. A better approximation is an isothermal sphere.
- The mass distribution has density $\rho = \sigma^2/2\pi G r^2$.
- The line-of-sight velocity dispersion σ is then constant with radius (as is the energy per bound particle hence the term isothermal).
- The surface density $\Sigma = \sigma^2/2Gr$ and the mass projected within radius r, $M(<r) = \pi\sigma^2 r/G$.
- This results in the deflection angle $\alpha = 4\pi\sigma^2/c^2$ being constant with radius (where the isothermal approximation is relevant).

How do we take a picture of a black hole?

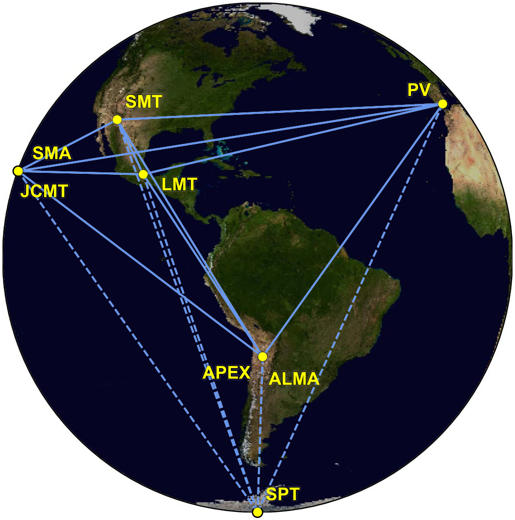
- Principal problem is that they are quite far away, and pretty small (astronomical speaking).
- Recall, diffraction limited resolution of a tele/microscope $1.22\frac{\lambda}{D}$.
- The largest dish size D we can get is $2R_{\mathrm{Earth}} = 12\,000\,\mathrm{km}$.
- We can achieve an effective dish size using interferometry, so we need radio waves, the largest for which it is practicable is $1.3\,\mathrm{mm}$.
- Diffraction limited resolution of $\boxed{\alpha = 25\,\mathrm{\mu as}}$.
- Black hole needs to be large or close enough such that $\frac{R_S}{d} > \alpha \Rightarrow \frac{M}{d} > \frac{1.22\lambda c^2}{4R_{\mathrm{Earth}}G}$.
- There are two options:

M87*	Mass $6.5 \times 10^9 M_\odot$, distance $16.4\,\mathrm{Mpc}$,
Sgr A*	Mass $4.1 \times 10^6 M_\odot$, distance $7.9\,\mathrm{kpc}$.



The event horizon telescope

- The nucleus of M87 was imaged at 1.3mm using the Event Horizon Telescope (EHT), a global interferometer composed of telescopes in Hawaii, North and South America and Europe.
- Calibration was enhanced with the addition of the South Pole Telescope in Antarctica. The data were gathered in April 2017 and after two years of processing the result published in April 2019.
- Eight stations of the EHT 2017 campaign over six geographic locations as viewed from the equatorial plane.
- Solid baselines represent mutual visibility on M87*. The dashed baselines were used for the calibration source [arxiv:1906.11238].

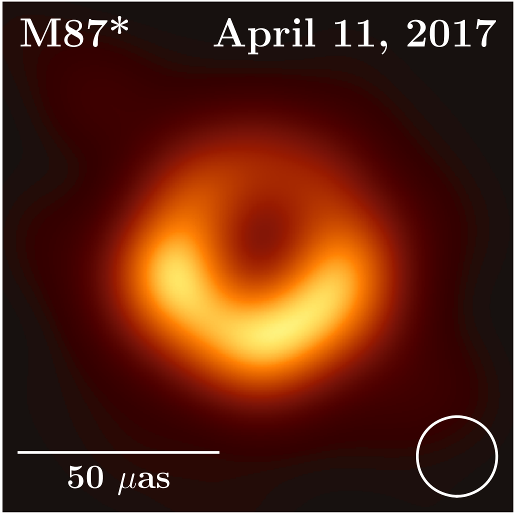


The shadow of the black hole

- Gravitational self-lensing occurs in the strong field around a black hole.
- The apparent angular size of the event horizon of a black hole, as seen by a distant observer, is magnified by the strong bending of light close to the hole.
- The minimum impact parameter to avoid capture by the black hole gives the apparent radius of the black hole
$$R_{\rm app} = b_{\rm min} = \frac{3\sqrt{3}}{2} R_S \approx 2.6 R_S.$$
- We can get this from the impact parameter calculation in Example Sheet 1 by setting $r = 3GM/c^2$, which is the smallest radius a photon coming from infinity can reach and still escape to infinity.
- The apparent radius for a maximally spinning black hole is $4.5GM/c^2$, which is only 15% smaller than for the non-spinning hole above.
- The shadow size is not therefore very sensitive to the spin of the black hole.
- A fairly complete discussion of Black Hole Shadows is in [arxiv:1906.00873].

The black hole M87*

- The ring in the image has a radius of $42 \pm 3\mu as$.
- This implies that the BH mass is 6.5 ± 0.7 billion Solar masses, in agreement with optical measurements of the dynamics of the stars in the nucleus.
- EHT image of M87* from observations on 2017 April 11 as a representative example of the images collected in the 2017 campaign.
- Sgr A* has been imaged by the EHT but is highly variable which makes the data much more challenging to analyse (image released 2022).
- The effective image smoothing kernel (20 mas FWHM) is shown lower right [arxiv:1906.11238].
- Can also image in polarised light



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The black hole M87* (polarised)

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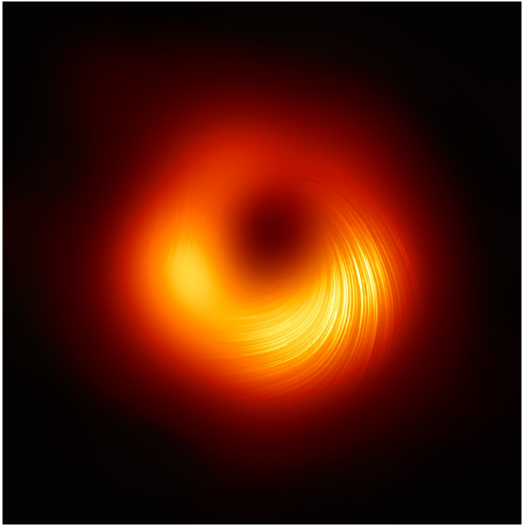

The black hole Sgr A*

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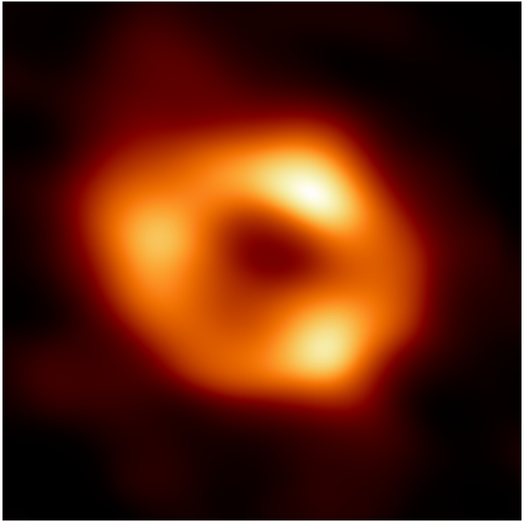

Summary

- The stationary weak field metric

$$\boxed{ds^2 = \left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 + 2\vec{A} \cdot d\vec{x} cdt - \left(1 - \frac{2\Phi}{c^2}\right) d\vec{x}^2.}$$

- The lensing angle

$$\boxed{\alpha = \frac{4GM}{bc^2} = \frac{2R_S}{b} = 2\alpha_{\mathrm{Newton}}.}$$

- The lensing scale

$$\boxed{b^2 = 2R_S(D_s - D_l)D_l/D_s.}$$

- Strong gravitational lensing & microlensing around compact objects.
- Weak lensing & lensing from extended objects (e.g. clusters).

Next time

Cosmology: the geometry of the Universe.