ASM : Examples Class #1

6. Poisson 2

What is $P(N, t)$ given rate k_0

$\left(N = \frac{L}{a}\right)$

this is $P(N,t) = \frac{(k_0 t)^N}{N!} e^{-k_0 t}$

What is average time to reach N?

MFPT : Survival $S(t)$ = Not to reach N

$$S(t) = \sum_{n=1}^{N} P(n, t)$$

$$= \sum_n \frac{(k_0 t)^n}{n!} e^{-k_0 t} \rightarrow \frac{\Gamma(N, k_0 t)}{N!}$$

$$\tau = \int_0^\infty S(t)\, dt = \sum_{n=1}^{N} \frac{1}{n!} \int_0^\infty dt\, (k_0 t)^n \cdot e^{-k_0 t}$$

$$\rightarrow \frac{1}{k_0} N \qquad \frac{n!}{k_0} = \int_0^\infty t^n e^{-k_0 t}\, dt$$

Alternative method.

$$f(t) = -\frac{\partial S}{\partial t} = -\sum_n \left(\frac{k_0^n t^{n-1}}{(n-1)!} e^{-k_0 t} - \frac{k_0^{n+1} t^n}{n!} e^{-k_0 t} \right)$$

$$= \sum_{n=0}^{N-1} \left(U_n - U_{n+1} \right)$$

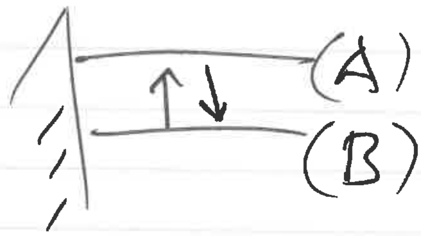
$$= \underline{U_{n=0} - U_N} \,||$$

$$\tau = \int_0^\infty t \, f(t) \, dt$$

7 Fluctuation-Dissipation...

$$\frac{\partial A}{\partial t} = -k_1 A + k_2 B$$

$$\frac{\partial B}{\partial t} = -k_2 B + k_1 A$$



$$\Delta = \frac{A - B}{2} \qquad A = A_{eq} + \Delta$$

$$B = B_{eq} - \Delta$$

$$\boxed{\frac{\partial \Delta}{\partial t} = -(k_1 + k_2)\Delta}$$

$$\frac{\partial (A-B)}{\partial t} = -2k_1 A + 2k_2 B = -\cancel{2}(k_1+k_2)\Delta$$

$$\boxed{(k_1 A - k_2 B)_{eq} = 0}$$

To have "fluctuations" we must add them!

Wiener...

$$\frac{\partial \Delta}{\partial t} = -(k_1+k_2)\Delta + \sigma \xi(t)$$

$$\Delta(t) = \int e^{-(k_1+k_2)(t-t')} \cdot \sigma \xi(t')\, dt'$$

Average

$$\langle \Delta(t_1)\Delta(t_2) \rangle = \text{produkt} \ldots$$

⑧ Simulation of GBM

⑨ 2-state model of Black-Scholes

Itoh Lemma !

$\odot$ $dx = \mu(x)\,dt + \sigma(x)\,dW_x$

and $f(x)$

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dx^2$$

$$\boxed{df = \left(\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}\right)dt + \sigma\frac{\partial f}{\partial x}dW}$$

$\odot\odot$ What if you have 2 variables?

$dy = \alpha\,dt + \xi\,dW_y$

and $f(x,y)$

$df = \ldots$ longer algebra cross term?

$$+ \frac{\partial^2 f}{\partial x \partial y}dx\,dy \ldots$$

is there a correlation?

$\sigma\xi\langle dW_x dW_y\rangle = \rho\,dt$

$= \text{Corr}$

"Black-Scholes" price of option

$$dS = \mu S\,dt + \sigma S\,dW_s$$

Also

Also stochastic.

an example

$$\boxed{d\sigma = -\theta\,\sigma\,dt + \gamma\,dW_\sigma}$$

why

$$V(S, \sigma) \Longrightarrow dV$$ Itoh

etc....

Hedging options

$$\Pi = V - \alpha S$$

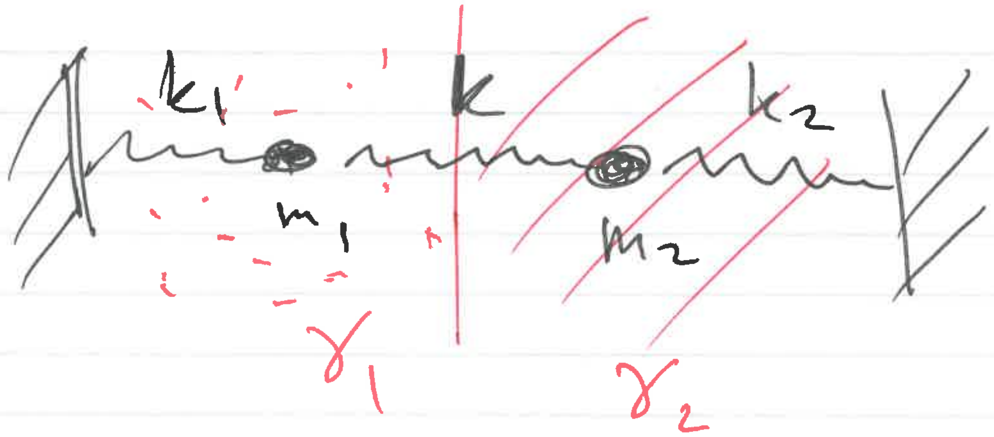
find $\alpha = -\frac{\partial V}{\partial S}$

Q12 2-variable O-U system

Derive Fokker-Planck

$$dx_i = -\theta_{ik} x_k + \sigma_{ik} \, dW_k$$

Q13 2-variable O-U system



$$\gamma_1 \dot{x}_1 = -k_1 x_1 - K(x_1 - x_2) + \sigma_1 \xi_1$$

$$\gamma_2 \dot{x}_2 = -k_2 x_2 - K(x_2 - x_1) + \sigma_2 \xi_2$$

$$\sqrt{2kT \cdot \gamma_{..}}$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = -\begin{pmatrix} \frac{k_1+K}{\gamma_1} & -\frac{K}{\gamma_1} \\ -\frac{K}{\gamma_1} & \frac{k_2+K}{\gamma_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{\sigma_1}{\gamma_1} & 0 \\ 0 & \frac{\sigma_2}{\gamma_2} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

$$\theta_{ik} \underbrace{\langle x_k x_\ell \rangle}_{M_{k\ell}} + \langle x_i x_k \rangle \theta_{\ell k} = (\sigma^T \sigma)$$

$$\underbrace{\langle x_1 x_1\rangle \cdot \frac{k_1+k}{\gamma_1} + \langle x_2 x_1\rangle\left(-\frac{k}{\gamma_1}\right)}_{11 \text{ element}} = \left(\frac{\sigma_1}{\gamma_1}\right)^2$$

$$\dots\dots\dots = \frac{2kT}{\gamma_1}$$

$$\underbrace{\langle x_1 x_2\rangle\left(-\frac{k}{\gamma_1}\right) + \langle x_2 x_1\rangle \frac{k_2+k}{\gamma_2}}_{21 \text{ element}} = 0$$

e.g.

find $\langle x_1 x_1\rangle = \dfrac{(k_2+k)\, k_B T}{2(k_1k_2+k_1k+k_2k)}$

$\langle x_1 x_2\rangle = \dots$

also $\langle x_2 x_2\rangle = \dots$

in Q12 $\quad \dfrac{dx}{dt} = \dfrac{1}{m} p$

$m\dot{v} = \dfrac{dp}{dt} = -m\omega^2 x - \gamma \dfrac{p}{m} + \sigma \xi(t) \qquad \sqrt{2kT\gamma}$

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = -\begin{pmatrix} 0 & -\frac{1}{m} \\ m\omega^2 & \frac{\gamma}{m} \end{pmatrix}\begin{pmatrix} x \\ p \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

Expect $\langle x^2\rangle = \dfrac{k_BT}{m\omega^2}$; $\langle p^2\rangle = 2k_BT\cdot m$;