Accretion disks

Relativistic Astrophysics and Cosmology: Lecture 9

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Pre-lecture question:

Which events are the most energetic in the universe?

Last time

- Astronomy across the electromagnetic spectrum and beyond
- Radiation processes

This lecture

- Accretion disks
- Analysis of ISCO and radiation efficiency

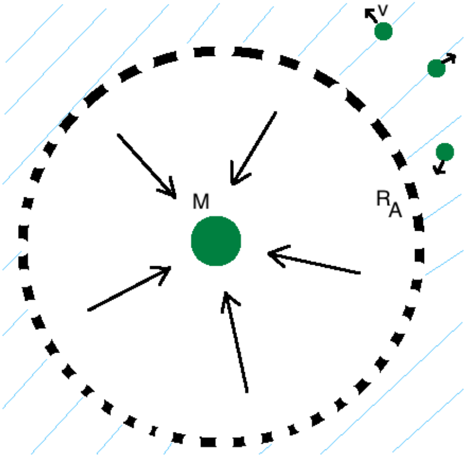
Next lecture

- Active galactic nuclei

Accretion

- Gravitational energy released by accretion onto supermassive black holes power the most luminous objects in the Universe - quasars.
- Accretion is also central to galaxy, star and planet formation.
- Angular momentum can be a bar to accretion.
- It can prevent or slow accretion and cause the formation of accretion discs, which are ubiquitous.

Bondi Accretion



- Here is a simple model of accretion/collapse.
- Uniform medium, density ρ, speed of sound c_s.
- Continuity: Matter in a region will accrete at a rate

$$\dot{M} \sim 4\pi R^2 \rho v.$$

- The relevant radius is the Bondi radius, found by equating the escape velocity $v_{\rm esc} = \sqrt{\frac{2GM}{r}}$ with c_s, $R_{\rm Bondi} = \frac{2GM}{c_s^2}$.
- Putting this together gives a relation for $\dot{M}$ and ρ

$$\dot{M} \sim \frac{4\pi\rho G^2 M^2}{c_s^3}.$$

Radiative Efficiency

- It is useful to consider radiative efficiency ϵ, defined by the fraction of rest mass energy converted i.e. $\boxed{L = \epsilon \dot{M} c^2}$.
- Take a simple model where gas free falls then decelerates at surface of central star.
- Energy liberated by infall of mass dm is $dE = GMdm/R_*$ so using $dm = \dot{M}dt$ and luminosity $L = dE/dt$:

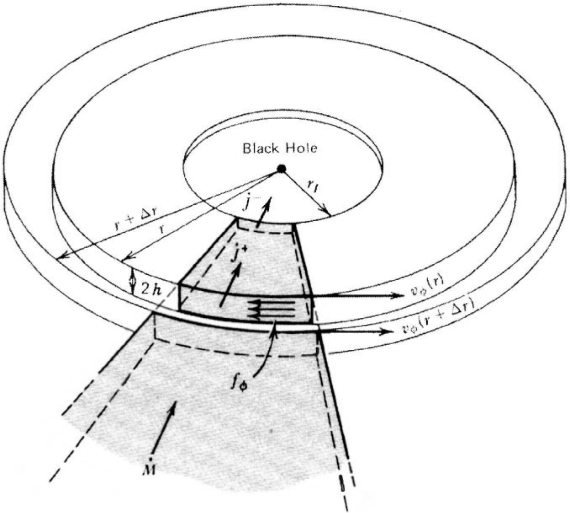
$$\boxed{L = \frac{GM\dot{M}}{R_*}} \Rightarrow \boxed{\epsilon = \frac{GM}{Rc^2}.}$$

- From this we can find estimates:

  White dwarf $\epsilon \sim 3 \times 10^{-4}$,
  Neutron star $\epsilon \sim 15\%$,
  Black hole $\epsilon = 0.5$,
  Nuclear burning $\epsilon \sim 0.7\%$,
  Chemistry $\epsilon < 10^{-10}$.

- For nuclear burning binding energy, $\sim$ MeV, but mass energy $\sim$ GeV.
- Angular momentum reduces these estimates (e.g. BH $\epsilon \sim \frac{1}{12} \sim 8\%$).
- Full GR calculation also reduces ($\epsilon \sim 6\%$).
- Black hole spin increases up to ($\epsilon \sim 32\%$).

Classical accretion disc analysis



- For a rotating disk we assume circular motion with $v_r \ll v_\phi = \Omega(r)r$.
- Tangential viscous stress s_ϕ is proportional to viscosity η and velocity gradient

$$\frac{dv_\phi}{dr} = \frac{d}{dr}(\Omega r) = \Omega + r\frac{d\Omega}{dr}.$$

- Only the last term relevant here since the Ω term represents uniform rotation

$$s_\phi = \eta r \frac{d\Omega}{dr} = -\frac{3}{2}\eta\Omega, \qquad \left[\Omega = \frac{\sqrt{GM}}{r^{3/2}},\right]$$

where we have assumed Keplerian orbits.

- Now viscous torque f_ϕ is the rate of change of angular momentum so

$$\overbrace{\underbrace{s_\phi}_{\text{stress}} \; \underbrace{2\pi r 2h}_{\text{area}} \, r}^{\text{torque}} = \dot{j}^+ - \dot{j}^- = -\dot{M}(\sqrt{GMr} - \underbrace{\beta\sqrt{GMr_\star}}_{\text{ang.mom absorbed by star}}).$$

- If $\beta < 1$, then angular momentum is removed by jets/outflows.

- Now eliminate s_ϕ to find velocity gradient

$$\frac{3}{2}\Omega = \frac{\dot{M}(\sqrt{GMr} - \beta\sqrt{GMr_\star})}{4\pi r^2 h\eta}.$$

- Rate of working per unit volume is force/unit area $\times$ velocity gradient

$$\dot{Q} = \frac{3}{2}\eta\Omega\frac{\dot{M}(\sqrt{GMr} - \beta\sqrt{GMr_\star})}{4\pi r^2 h\eta}.$$

- Note this is independent of η.

- If energy radiated locally (disk thin) then the surface flux is

$$F(r) = \frac{1}{2}2h\dot{Q} = \frac{3\dot{M}}{8\pi r^2}\frac{GM}{r}\left(1 - \beta\left(\frac{r_\star}{r}\right)^{1/2}\right).$$

- Total luminosity

$$L = \int 2F \times 2\pi r dr = \left(\frac{3}{2} - \beta\right)\frac{GM\dot{M}}{r_\star}.$$

Radiation Spectrum

- Blackbody emission has temperature

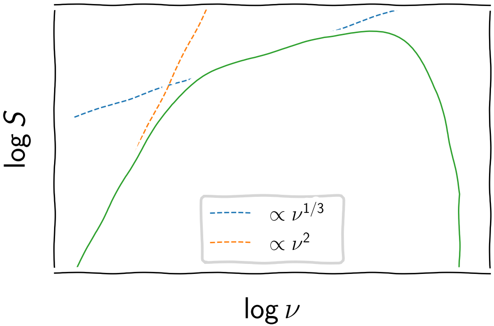
$$T = (F/\sigma_{SB})^{1/4}$$

(σ_{SB} is the Stefan-Boltzmann constant).

- Integrating the spectrum over the disk:

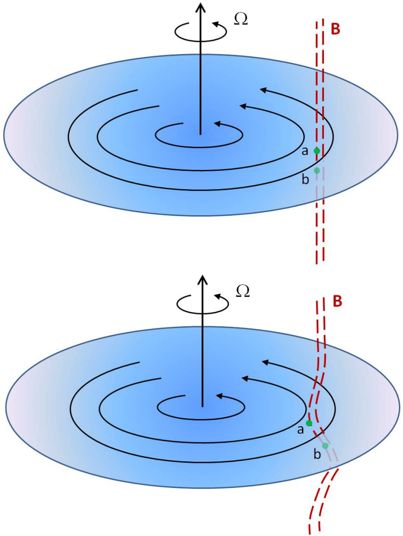
$$S(\nu) \propto \int_{r_\star}^{\infty} \frac{\nu^3}{e^{h\nu/kT(r)} - 1} 2\pi r dr$$

- Can approximate central region as $S \sim \nu^{1/3}$, outer edge with ν^2 and inner as $e^{-\nu}$.



- On a log-log scale this has a very clear spectral shape in the X-ray band.

Magnetic viscosity



- Viscosity due to MRI (Magneto Rotational Instability).
- Magnetic forces behave as though elements of fluid were connected with elastic bands.
- Magnetic field lines straddling different radii become stretched and amplified, linking gas across radii and acting as a shear viscosity.
- Attractive forces are usually restorative/stabilising, but in the context of accretion they are disruptive.
- Magnetic fields surrounding central objects can therefore increase the ISCO, changing the accretion power.
- Turbulent viscosity also separately important, which can be modelled in e.g. α-disks, where one assumes $\eta = \alpha c_s h$.
- For thin disks, much theory can be derived in terms of only the free parameter α.

Black hole astrophysics

A little history

- Idea of black holes first proposed in 1783 by John Michell (Fellow of Queens') (Philosophical Transactions of the Royal Society of London, vol **74**, p35 (1784)).
- Also had the idea that existence of 'dark stars' could be inferred from motions of nearby objects.
- Said would need an object 500 times larger in diameter than Sun if same density (ratio with modern values is about 484, so pretty close!).

42 *Mr.* Michell *on the Means of diſcovering the*

16. Hence, according to article 10, if the ſemi-diameter of a ſphære of the ſame denſity with the ſun were to exceed that of the ſun in the proportion of 500 to 1, a body falling from an infinite height towards it, would have acquired at its ſurface a greater velocity than that of light, and conſequently, ſuppoſing light to be attracted by the ſame force in proportion to its vis inertiæ, with other bodies, all light emitted from ſuch a body would be made to return towards it, by its own proper gravity.

29. If there ſhould really exiſt in nature any bodies, whoſe denſity is not leſs than that of the ſun, and whoſe diameters are more than 500 times the diameter of the ſun, ſince their light could not arrive at us; or if there ſhould exiſt any other bodies of a ſomewhat ſmaller ſize, which are not naturally luminous; of the exiſtence of bodies under either of theſe circumſtances, we could have no information from ſight; yet, if any other luminous bodies ſhould happen to revolve about them we might ſtill perhaps from the motions of theſe revolving bodies infer the exiſtence of the central ones with ſome degree of probability, as this might afford a clue to ſome of the apparent irregularities of the revolving bodies, which would not be eaſily explicable on any other hypotheſis; but as the con-

Densities

- Radius of the event horizon, R_S, corresponds to 3 km per Solar mass.
- So if the density of a proton is $\rho_p = 7 \times 10^{17}\,\mathrm{kgm}^{-3}$, the mean density of a black hole is:

$$\rho = \frac{M}{\frac{4\pi}{3} {R_S}^3} = \frac{M}{\frac{4\pi}{3}\left(\frac{2GM}{c^2}\right)^3} = 27\rho_p \left(\frac{M}{M_\odot}\right)^{-2}.$$

- Solar mass black holes are at super-proton densities.
- Black holes with masses above $10^8 M_\odot$, have average densities below that of water.
- Those above a few billion $M_\odot$ have densities below that of air.
- So from a mean density point of view, supermassive black holes are not particularly dense.
- The light-crossing time of the event horizon (i.e. a length equivalent to its diameter) is 0.2 ms for a $10 M_\odot$ black hole. 20 s for $10^6 M_\odot$ and about one day for $5 \times 10^9 M_\odot$.

Circular orbits

- Schwarzschild metric is

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \frac{1}{c^2}\left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - \frac{r^2}{c^2} d\theta^2 - \frac{r^2 \sin^2\theta}{c^2} d\phi^2.$$

- Energy and angular momentum in a circular orbit (Handout 5) are

$$E_{\mathrm{circ}} = mc^2 \frac{1 - \frac{2GM}{rc^2}}{\sqrt{1 - \frac{3GM}{rc^2}}}, \qquad h = \frac{\sqrt{GMr}}{\sqrt{1 - \frac{3GM}{rc^2}}}. \tag{1}$$

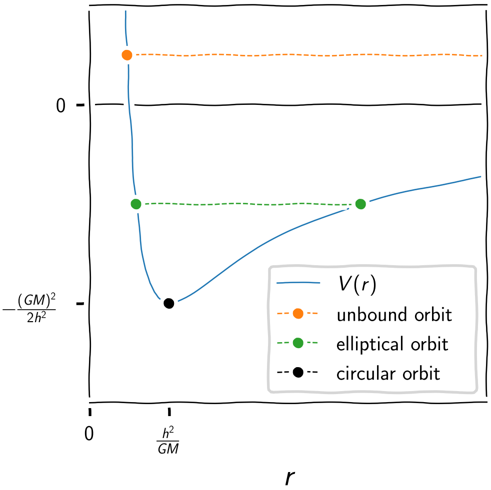
- Want to briefly revise how to use 'effective potential' to find innermost stable circular orbit (ISCO), and will then pass on to equivalent rotating black hole case.

- In Newtonian dynamics the equation of motion of a particle in a central potential is

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 + V(r) = E, \qquad (2)$$
$$V(r) = \frac{h^2}{2r^2} - \frac{GM}{r},$$

where $V(r)$ is an "effective potential", and h is specific angular momentum.

- In the effective potential, bound orbits have two turning points and a circular orbit corresponds to the special case where the particle sits at the minimum of the potential.



General relativistic case

- In Newtonian dynamics, a finite angular momentum provides an *angular momentum barrier* preventing a particle reaching $r = 0$. This is not true in General Relativity.
- Starting with the Lagrangian in the equatorial plane and the conservation equations

$$\mathcal{L} = Ac^2\dot{t}^2 - B\dot{r}^2 - r^2\dot{\phi}^2 = c^2, \qquad A\dot{t} = k, \qquad r^2\dot{\phi} = h.$$

  we can derive the analogous equation for $\dot{r}$: $A^{-1}k^2c^2 - B\dot{r}^2 - h^2/r^2 = c^2$.
- Substituting in the Schwarzschild A and B, we can write this as eq. (2) with

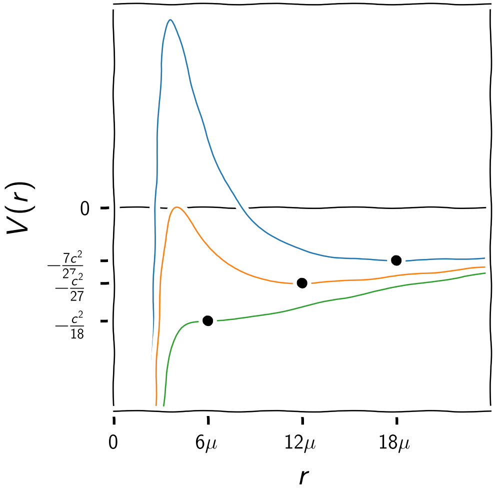
$$V(r) = \frac{h^2}{2r^2}\left(1 - \frac{2GM}{c^2r}\right) - \frac{GM}{r}.$$

- Note that the relativistic term $(1 - 2GM/c^2r)$ weakens the centrifugal effect of angular momentum at small r, and that $c \to \infty$ recovers the Newtonian result.

- Differentiating this expression, the extrema of the effective potential are located at the solutions of a quadratic equation

$$r = \frac{h^2}{2GM}\left\{1 \pm \sqrt{1 - 12\left(\frac{GM}{hc}\right)^2}\right\}.$$

- If $h = \sqrt{12}\frac{GM}{c}$ *there is only one extremum*, and there are no turning points in the orbit for lower values of h.
- At this point $r = 6GM/c^2 = 3R_S$. Next figure shows the effective potential for several values of h.
- The dots show the locations of stable circular orbits. The maxima in the potential are the locations of *unstable* circular orbits.



The innermost stable circular orbit (ISCO)

- The smallest stable circular orbit has $r_{\rm ISCO} = 6\frac{GM}{c^2} = 3R_S$.
- Gas in an accretion disc settles into circular orbits around the compact object.
- However, the gas slowly loses angular momentum because of turbulent viscosity (the turbulence is thought to be generated by magnetohydrodynamic instabilities). As the gas loses angular momentum it moves slowly towards the black hole, gaining gravitational potential energy and heating up.
- Eventually it loses enough angular momentum that it can no longer follow a stable circular orbit and so it falls into the black hole.
- The maximum efficiency is thus of order the gravitational binding energy at the smallest stable circular orbit divided by the rest mass energy of the gas. Non-relativistic approx. is

$$\epsilon_{\rm acc} \approx \frac{1}{2}\frac{GMm}{r_{\rm min}}\frac{1}{mc^2} \simeq \frac{1}{12} \sim 8\%.$$

- The correct value from $(1 - E_{\rm circ}/mc^2)$ using equation (1) is 5.7%.

Adding spin

- We can collect together the potentials we calculated in previous lectures

$$
\begin{aligned}
V_{\text{Newton}}(r) &= -\frac{GM}{r} + \frac{h^2}{2r^2}, \\
V_{\text{Schwarzschild}}(r) &= -\frac{GM}{r} + \frac{h^2}{2r^2}\left(1 - \frac{2GM}{c^2 r}\right), \\
V_{\text{Kerr}}(r) &= -\frac{GM}{r} + \frac{(h-ack)^2}{2r^2}\left(1 - \frac{2GM}{c^2 r}\right) + \frac{hack}{r^2} - \frac{a^2c^2}{2r^2}.
\end{aligned}
$$

- Note V_{Kerr} has been rearranged to show that the effect of spin is to adjust the effect of h, and add additional centrifugal terms.
- Note that this is complicated by the change in the interpretation of r.

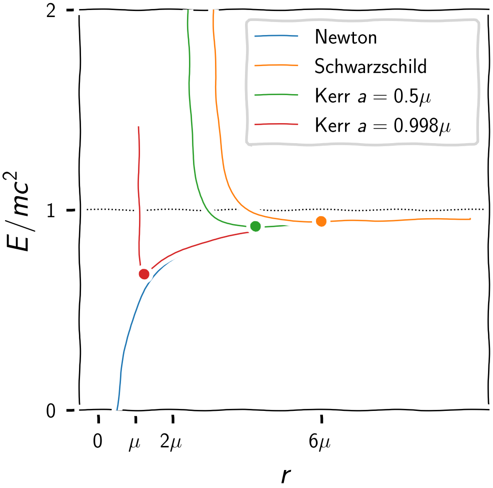
Energy in circular orbits

- Demanding that the orbits are circular means that we know k (and h) as functions of (constant) r.
- Compare again the structure emerging as we turn on (GR $c \ll \infty$) and then spin $(a \neq 0)$.

$$
\begin{aligned}
E &= kmc^2, \\
E_{\text{Newton}} &= \left(1 - \frac{GM}{2c^2 r}\right) mc^2, \\
E_{\text{Schwarzschild}} &= \frac{1 - \frac{2GM}{c^2 r}}{\sqrt{1 - \frac{3GM}{c^2 r}}} mc^2, \\
E_{\text{Kerr}} &= \frac{1 - \frac{2GM}{c^2 r} + a\sqrt{\frac{GM}{c^2 r^3}}}{\sqrt{1 - \frac{3GM}{c^2 r} + 2a\sqrt{\frac{GM}{c^2 r^3}}}} mc^2.
\end{aligned}
$$

Energy in circular orbits

- The effective potential graphs are a useful tool for interpreting elliptical orbits as oscillations about a potential, where r is a dynamical variable.
- Plotting the binding energy of *circular* orbits as a function of (constant) r allows one to identify the ISCO more directly as the "smallest r with a non-negative gradient".
- $-\frac{dE}{dr}$ can be interpreted as a restoring force, which if negative indicates stability, and if positive indicates instability



- An accretion disc can convert 5-20 percent of the rest mass energy of the gas into radiation, depending on spin.
- For spinning black holes, recall we found $\frac{a}{\mu} = \frac{2\left(2\sqrt{2}\sqrt{1-k^2}-k\right)}{3\sqrt{3}(1-k^2)}$:

Spin	Efficiency $(1-k)$	Interpretation
$a=0$	$1-\frac{2\sqrt{2}}{3}=5.7\%$	Schwarzschild case
$a=\mu$	$1-\frac{1}{\sqrt{3}}=42\%$	prograde orbit extremal black hole
$a=-\mu$	$1-\frac{5\sqrt{3}}{9}=3.8\%$	retrograde orbit extremal black hole
$a=0.998\mu$	32%	Thorne limit

- Thorne limit comes from the counteracting torque felt by the hole in absorbing radiation from the accretion disc.
- Accretion discs are capable of converting rest mass energy into radiation with an efficiency that is about 10 times greater than the efficiency of nuclear burning of hydrogen to helium (26 MeV per He nucleus), $\epsilon_{\text{nuclear}} \sim 0.7\%$.
- The 'accretion power' of black holes causes the most energetic phenomena known in the Universe. (N.B. this is different from most Luminous).

Summary

- Accretion luminosity & Radiative efficiency

$$\boxed{L \sim \frac{GM\dot{M}}{R_*}} \qquad \boxed{L = \epsilon \dot{M} c^2}$$

- Classical models of accretion disks & viscosity (laminar, MRI, turbulent)
- The effective potential

$$\boxed{V = \frac{h^2}{2r^2}\left(1 - \frac{2GM}{c^2 r}\right) - \frac{GM}{r}}$$

- Relativistic models of accretion and the ISCO $\boxed{r_{\rm ISCO} = \frac{6GM}{c^2} = 3R_S}$

Next time

Active galactic nuclei