The Migdal-Kadanoff Method

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1. *The Migdal-Kadanoff Method*: The partition function for the one-dimensional ferromagnetic Ising model with nearest neighbour interaction is given by

$$\mathcal{Z} = \sum_{\{\sigma_i = \pm 1\}} e^{-\beta H[\sigma_i]}, \qquad \beta H = -\sum_{\langle ij \rangle} \left[J\sigma_i\sigma_j + \frac{h}{2}(\sigma_i + \sigma_j) + g \right],$$

where $\langle ij \rangle$ denotes the sum over neighbouring lattice sites. The Migdal-Kadanoff scheme involves an RG procedure which, by eliminating a certain fraction of the spins from the partition sum, reduces the number of degrees of freedom by a factor of b. Their removal induces an effective interaction of the remaining spins which renormalises the coefficients in the effective Hamiltonian. The precise choice of transformation is guided by the simplicity of the resulting RG. For $b = 2$ (known as *decimation*) a natural choice is to eliminate (say) the even numbered spins.

(a) By applying this procedure, show that the partition function is determined by a renormalised Hamiltonian involving spins at odd numbered sites σ'_i,

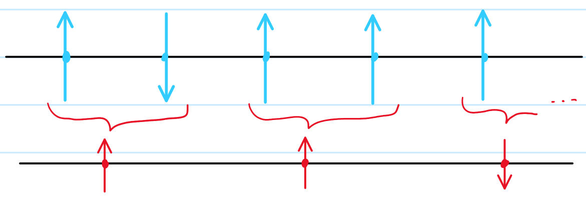
$$\mathcal{Z} = \sum_{\{\sigma'_i = \pm 1\}} e^{-\beta H'[\sigma'_i]},$$

where the Hamiltonian $\beta H'$ has the same form as the original but with renormalised interactions determined by the equation

$$\exp\left[J'\sigma'_1\sigma'_2 + \frac{h'}{2}(\sigma'_1 + \sigma'_2) + g' \right] = \sum_{s=\pm 1} \exp\left[Js(\sigma'_1 + \sigma'_2) + \frac{h}{2}(\sigma'_1 + \sigma'_2) + hs + 2g \right].$$

(b) Substituting different values for σ'_1 and σ'_2 obtain the relationship between the renormalised coefficients and the original. Show that the recursion relations take the general form

$$\begin{aligned} g' &= 2g + \delta g(J, h), \\ h' &= h + \delta h(J, h), \\ J' &= J'(J, h). \end{aligned}$$



PBC $\sigma_{N+1} \equiv \sigma_1$

$$\beta H = -\sum_{\langle ij \rangle} \left[J\sigma_i\sigma_j + h\sigma_j + g \right] = -\sum_{\langle ij \rangle} \left[J\sigma_i\sigma_j + \frac{h}{2}(\sigma_i + \sigma_j) + g \right] = -\sum_{i=1}^{N} \left[J\sigma_i\sigma_{i+1} + \frac{h}{2}(\sigma_i + \sigma_{i+1}) + g \right]$$

useful for the renormalisation procedure

$$\mathcal{Z}_N = \sum_{\sigma_i} e^{-\beta H(\{\sigma_i\})} = \sum_{\sigma_i} \exp\left\{ \sum_{i=1}^{N} \left[J\sigma_i\sigma_{i+1} + \frac{h}{2}(\sigma_i + \sigma_{i+1}) + g \right] \right\}$$

$$= \sum_{\sigma_i} \prod_{i=1}^{N} \exp\left\{ J\sigma_i\sigma_{i+1} + \frac{h}{2}(\sigma_i + \sigma_{i+1}) + g \right\}$$

$\rightarrow$ Define $T_{\sigma\sigma'} = \exp\left[J\sigma\sigma' + \frac{h}{2}(\sigma + \sigma') + g \right]$ transfer matrix

$$T = \begin{pmatrix} T_{++} & T_{+-} \\ T_{-+} & T_{--} \end{pmatrix} = \begin{pmatrix} e^{J+h+g} & e^{-J+g} \\ e^{-J+g} & e^{J-h+g} \end{pmatrix} = e^{g} \begin{pmatrix} e^{J+h} & e^{J} \\ e^{J} & e^{J-h} \end{pmatrix}$$

$$= \sum_{\sigma_i} \prod_{i=1}^{N} T_{\sigma_i\sigma_{i+1}} = \sum_{\sigma_i} T_{\sigma_1\sigma_2} T_{\sigma_2\sigma_3} \cdots T_{\sigma_N\sigma_1}$$

$$= \mathrm{tr}\left[T^N \right]$$

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To renormalise, group blocks of b spins together:

$$Z_N(J,h,g) = \mathrm{tr}\left[T^N\right] = \mathrm{tr}\left[(T^b)^{N/b}\right] = Z_{N/b}(J',h',g')$$

looks like an Ising partition function with $T' = T^b$

We will consider the case $b=2$.

$$T'(J',h',g') = T(J,h,g)^2 \qquad T'_{\sigma_1'\sigma_2'} = T_{\sigma_1'\tau}T_{\tau\sigma_2'}$$

$$e^{g'}\begin{pmatrix} e^{J'+h'} & e^{-J'} \\ e^{-J'} & e^{J'-h'} \end{pmatrix} = \left[e^{g}\begin{pmatrix} e^{J+h} & e^{-J} \\ e^{-J} & e^{J-h} \end{pmatrix}\right]^2$$

Rewrite: $u = e^{-J}$, $v = e^{-h}$, $c = e^{g}$

$$\rightsquigarrow c'\begin{pmatrix} \frac{1}{u'v'} & u' \\ u' & v'/u' \end{pmatrix} = \left[c\begin{pmatrix} 1/uv & u \\ u & v/u \end{pmatrix}\right]^2$$

$$\Rightarrow u' = \frac{\sqrt{v+v^{-1}}}{(u^4+u^{-4}+v^2+v^{-2})^{1/4}}, \qquad v' = \sqrt{\frac{u^4+v^2}{u^4+v^{-2}}}, \qquad c' = c^2\sqrt{v+v^{-1}}\,(u^4+u^{-4}+v^2+v^{-2})^{1/4}$$

$$\Rightarrow \begin{cases} J' = \frac{1}{4}\log\left[\frac{\cosh(2J+h)\cosh(2J-h)}{\cosh^2 h}\right], \qquad h' = h + \frac{1}{2}\log\left[\frac{\cosh(2J+h)}{\cosh(2J-h)}\right], \\ g' = 2g + \frac{1}{4}\log\left[16\cosh(2J+h)\cosh(2J-h)\cosh^2 h\right] \end{cases}$$

(c) For $h = 0$ check that no term h' is generated from the renormalisation (as is clear from symmetry). In this case, show that

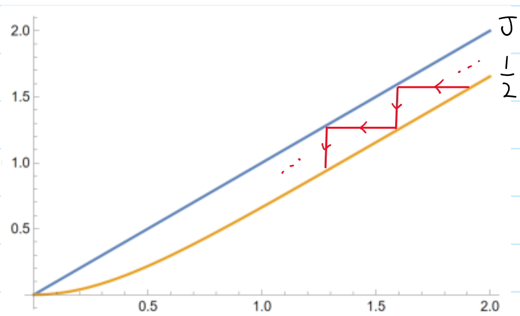
$$J' = \frac{1}{2}\ln\cosh(2J).$$

Show that this implies a stable (infinite T) fixed point at $J = 0$ and an unstable (zero T) fixed point at $J = \infty$. Any finite interaction renormalises to zero indicating that the one-dimensional chain is always disordered at sufficiently long length scales.

$h=0$

i) $h' = h + \frac{1}{2}\log\left[\frac{\cosh(2J+h)}{\cosh(2J-h)}\right] = 0 + \frac{1}{2}\log 1 = 0$ ✓

ii) $J' = \frac{1}{4}\log\left[\frac{\cosh(2J+h)\cosh(2J-h)}{\cosh^2 h}\right] = \frac{1}{4}\log\left[\cosh^2(2J)\right] = \frac{1}{2}\log\left[\cosh 2J\right]$ ✓



$J = \frac{1}{2}\log[\cosh 2J]$ solved only by $J = 0, \infty$.

To see stability look at the recursion relation plot.

(d) Linearising (in the exponentials) the recursion relations around the unstable fixed point, show that

$$e^{-J'} = \sqrt{2}e^{-J}, \qquad h' = 2h.$$

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d) For $h=0$, $J \to \infty$ we have $u = e^{-J} \ll 1$ and $v = e^{-h} = 1$.

$$u'\big|_{v=1} = \frac{\sqrt{2}}{(u^4 + u^{-4} + 2)^{1/4}} = \sqrt{2}\,u + \mathcal{O}(u^3) \quad \Rightarrow \quad e^{-J'} = \sqrt{2}\, e^{-J}$$

$$v'\big|_{u=0} = \sqrt{\frac{v^2}{v^{-2}}} = v^2 \quad \Rightarrow \quad h' = 2h$$

(e) Regarding e^{-J} and h as scaling fields, show that in the vicinity of the fixed point the correlation length satisfies the homogeneous form $(b=2)$

$$\begin{aligned} \xi(e^{-J}, h) &= 2\xi(\sqrt{2}e^{-J}, 2h) \\ &= 2^\ell \xi(2^{\ell/2}e^{-J}, 2^\ell h). \end{aligned}$$

$$\xi = \xi(u,v) = \underset{\uparrow}{b}\,\xi' = 2\xi'(u', v') = 2\xi(\sqrt{2}u, v^2)$$

all length scales change by a factor of $b=2$

Iterating ℓ times:

$$\xi(u,v) = 2^\ell(2^{\ell/2}u, v^{2^\ell}) = 2^\ell \xi(2^{\ell/2}e^{-J}, 2^\ell h)$$

The Lifshitz Point

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2. *The Lifshitz Point*: (see Chaikin and Lubensky, p. 184) A number of materials, such as liquid crystals, are highly anisotropic and behave differently along directions parallel and perpendicular to some axis. An example is provided below. The d spatial dimensions are grouped into one parallel direction, $x_\parallel$ and $d-1$ perpendicular directions, $\mathbf{x}_\perp$. Consider a one-component field m subject to the Hamiltonian

$$\begin{aligned}\beta H &= \beta H_0 + U,\\ \beta H_0 &= \int dx_\parallel \int d\mathbf{x}_\perp \left[\frac{K}{2}(\nabla_\parallel m)^2 + \frac{L}{2}(\nabla_\perp^2 m)^2 + \frac{t}{2}m^2 - hm\right],\\ U &= u\int dx_\parallel \int d\mathbf{x}_\perp m^4.\end{aligned}$$

A Hamiltonian of this kind is realised in the theory of fluctuations in stacked fluid membranes — the smectic liquid crystal. [Note that βH depends on the *first* gradient in the $x_\parallel$ direction, and on the *second* gradient in the $\mathbf{x}_\perp$ directions.]

(a) Write βH_0 in terms of the Fourier transforms $m(q_\parallel, \mathbf{q}_\perp)$.

$$m(q_\parallel, \underline{q}_\perp) = \int dx_\parallel \int d\underline{x}_\perp\, m(\underline{x})\, e^{-i\underline{q}\cdot\underline{x}} \quad \Leftrightarrow \quad m(x_\parallel, \underline{x}_\perp) = \int \frac{dq_\parallel}{2\pi}\int\frac{dq_\perp}{(2\pi)^{d-1}}\, m(\underline{q})\, e^{i\underline{q}\cdot\underline{x}}$$

$$\beta H_0 = \int dx_\parallel \int d\underline{x}_\perp \left[\frac{K}{2}(\nabla_\parallel m)^2 + \frac{L}{2}(\nabla_\perp^2 m)^2 + \frac{t}{2}m^2 - hm\right]$$

$$= \int\frac{dq_\parallel}{2\pi}\int\frac{dq_\perp}{(2\pi)^{d-1}}\left[\frac{1}{2}\underbrace{\left(t + K q_\parallel^2 + L q_\perp^4\right)}_{G_0^{-1}(\underline{q})}|m(\underline{q})|^2\right] - h\, m(\underline{q}=0)$$

comes from $m(\underline{q})\,m(-\underline{q}) = m(\underline{q})\,m(\underline{q})^*$

(b) Construct a renormalisation group transformation for βH_0 by rescaling distances such that $q_\parallel' = bq_\parallel$, $\mathbf{q}_\perp' = c\mathbf{q}_\perp$, and the field $m' = m/z$.

① Integrate out high-energy modes

② renormalise / rescale to match with original theory

Partition function:

$D[m(\underline{x})] = D[m(\underline{q})]$ as FT unitary

$$\mathcal{Z} = \int D[m(\underline{q})]\, e^{-\beta H_0[m(\underline{q})]} = \int D[m(\underline{q})]\exp\left\{-\int^{\Lambda}\frac{dq_\parallel}{2\pi}\int^{\lambda}\frac{dq_\perp}{(2\pi)^{d-1}}\left[\frac{1}{2}G_0^{-1}(\underline{q})|m(\underline{q})|^2\right] + h\, m(\underline{q}=0)\right\}$$

① Divide up field:

$$m(\underline{q}) = \begin{cases} m_<(\underline{q}) & |q_\parallel| < \Lambda/b \quad \textbf{and} \quad |q_\perp| < \lambda/c \\ m_>(\underline{q}) & |q_\parallel| > \Lambda/b \quad \textbf{or} \quad |q_\perp| > \lambda/c\end{cases}$$

Since βH_0 is quadratic in $m(\underline{q})$, the integration splits up neatly,

$$\mathcal{Z} = \int D[m(\underline{q})]\exp\left\{-\left(\int^{\Lambda/b}\frac{dq_\parallel}{2\pi} + \int_{\Lambda/b}^{\Lambda}\frac{dq_\parallel}{2\pi}\right)\left(\int^{\lambda/c}\frac{dq_\perp}{(2\pi)^{d-1}} + \int_{\lambda/c}^{\lambda}\frac{dq_\perp}{(2\pi)^{d-1}}\right)\left[\frac{1}{2}G_0^{-1}(\underline{q})|m(\underline{q})|^2\right] + h\, m(\underline{q}=0)\right\}$$

$$= \mathcal{Z}_>\int D[m_<(\underline{q})]\exp\left\{-\frac{1}{2}\int^{\Lambda/b}\frac{dq_\parallel}{2\pi}\int^{\lambda/c}\frac{dq_\perp}{(2\pi)^{d-1}}G_0^{-1}(\underline{q})|m_<(\underline{q})|^2 + h\, m_<(0)\right\}$$

explicitly: $\mathcal{Z}_> = \int D[m_>(\underline{q})]\exp\left\{-\frac{1}{2}\left(\int^{\Lambda/b}\frac{dq_\parallel}{2\pi}\int_{\lambda/c}\frac{dq_\perp}{(2\pi)^{d-1}} + \int_{\Lambda/b}\frac{dq_\parallel}{2\pi}\int^{\lambda/c}\frac{dq_\perp}{(2\pi)^{d-1}} + \int_{\Lambda/b}\frac{dq_\parallel}{2\pi}\int_{\lambda/c}\frac{dq_\perp}{(2\pi)^{d-1}}\right)G_0^{-1}(\underline{q})|m_>(\underline{q})|^2\right\}$

This is simply a multiplicative constant, so we can ignore it.

② Rescale / renormalise: $q_\parallel' = bq_\parallel$, $q_\perp' = cq_\perp$, $m_>'(\underline{q}') = m_>(\underline{q})/z$

$$\mathcal{Z} = \mathcal{Z}_>\int D[m_<(\underline{q})]\exp\left\{-\frac{1}{2}\int^{\Lambda/b}\frac{dq_\parallel}{2\pi}\int^{\lambda/c}\frac{dq_\perp}{(2\pi)^{d-1}}G_0^{-1}(\underline{q})|m_<(\underline{q})|^2 + h\, m_<(0)\right\}$$

$$= \mathcal{Z}_>\,\mathcal{N}\int D[m_<'(\underline{q}')]\exp\left\{-\frac{1}{2}\int^{\Lambda}\frac{dq_\parallel'}{2\pi b}\int^{\lambda}\frac{dq_\perp'}{(2\pi c)^{d-1}}\left[t + K(q_\parallel')^2 b^{-2} + L(q_\perp')^4 c^{-4}\right]z^2|m'(\underline{q}')|^2 + h\, z\, m'(0)\right\}$$

$\mathcal{N}$: $\mathrm{Det}[z] = \exp(\mathrm{Tr}\log z) = \exp\left(\int\frac{dq}{(2\pi)^d}\log z\right)$

We can thus read off

$$t' = \frac{z^2}{b\,c^{d-1}}\,t, \qquad K' = \frac{z^2}{b^3 c^{d-1}}\,K, \qquad L' = \frac{z^2}{b\,c^{d+3}}\,L, \qquad h' = zh.$$

(c) Choose c and z such that $K' = K$ and $L' = L$. At the resulting fixed point calculate the eigenvalues y_t and y_h.

(c) Choose c and z such that $K' = K$ and $L' = L$. At the resulting fixed point calculate the eigenvalues y_t and y_h.

$$\left.\begin{aligned} K' &\overset{!}{=} K \quad \Rightarrow \quad \frac{z^2}{b^3 c^{d-1}} = 1 \\ L' &\overset{!}{=} L \quad \Rightarrow \quad \frac{z^2}{b\, c^{d+3}} = 1 \end{aligned}\right\} \quad c = b^{1/2}, \qquad z = b^{(d+5)/4}$$

$$\Rightarrow \begin{cases} t' = b^2\, t & y_t = 2 \\ h' = b^{(d+5)/4}\, h & y_h = \dfrac{d+5}{4} \end{cases}$$

(d) Write down the relationship between the free energies $f(t,h)$ and $f(t',h')$ in the original and rescaled problems. Hence write the unperturbed free energy in the homogeneous form

$$f(t,h) = t^{2-\alpha} g_f(h/t^{\Delta}),$$

and identify the exponents α and Δ.

$$f(t,h) = \frac{1}{b c^{d-1}} f(t', h') = b^{-\frac{d+1}{2}} f\left(b^2 t,\, b^{(d+5)/4}\right)$$

$\rightsquigarrow$ f is a scaling function and we can find its form by choosing $b^2 t = 1$

$$\begin{aligned} f(t,h) &= t^{\frac{d+1}{4}} f\left(1,\, t^{-\frac{d+5}{8}} h\right) \\ &= t^{\frac{d+1}{4}}\, g\left(t^{-\frac{d+5}{8}} h\right) \end{aligned}$$

$$2 - \alpha = \frac{d+1}{4} \quad \rightsquigarrow \quad \alpha = \frac{8-d-1}{4} = \frac{7-d}{4}$$

$$\delta = \frac{d+5}{8}$$

(e) How does the unperturbed zero-field susceptibility $\chi(t,0)$ diverge as $t \to 0$?

$$\begin{aligned} \chi = \frac{\partial m}{\partial h} = \frac{\partial^2 f}{\partial h^2} &= t^{\frac{d+1}{4} - 2\cdot\frac{d+5}{8}}\, g''\left(t^{-\frac{d+5}{8}} h\right) \\ &= t^{-1}\, \tilde{g}\left(t^{-\frac{d+5}{8}} h\right) \end{aligned}$$

At $h = 0$, $\chi \sim t^{-1}$ i.e. $\gamma = 1$

In the remainder of this problem set $h = 0$, and treat U as a perturbation.

(f) In the unperturbed Hamiltonian calculate the expectation value $\langle m(\mathbf{q}) m(\mathbf{q}') \rangle_0$, and the corresponding susceptibility $\chi(\mathbf{q})$, where $\mathbf{q} = (q_\parallel, \mathbf{q}_\perp)$.

Couple the magnetisation to an external magnetic field $h(\underline{x})$ $\rightsquigarrow$ source term

$$\begin{aligned} \underset{U=0}{\mathcal{Z}_0[h]} &= \int \mathcal{D}[m(\underline{x})] \exp\left\{ -\int dx_\parallel \int d\underline{x}_\perp \left[\frac{K}{2} (\nabla_\parallel m)^2 + \frac{L}{2} (\nabla_\perp^2 m)^2 + \frac{t}{2} m^2 - h(\underline{x}) m(\underline{x}) \right] \right\} \\ &= \int \mathcal{D}[m(\underline{q})] \exp\left\{ -\int \frac{d^d q}{(2\pi)^d} \left[\frac{1}{2} G_0^{-1}(\underline{q}) |m(\underline{q})|^2 - h(-\underline{q}) m(\underline{q}) \right] \right\} \\ &= \int \mathcal{D}[m(\underline{q})] \exp\left\{ -\frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \left[m(\underline{q}) G_0^{-1}(\underline{q}) m(-\underline{q}) - h(-\underline{q}) m(\underline{q}) - h(\underline{q}) m(-\underline{q}) \right] \right\} \\ &= \int \mathcal{D}[m(\underline{q})] \exp\left\{ -\frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \left[\left(m(\underline{q}) - h(\underline{q}) G_0(\underline{q})\right) G_0^{-1}(\underline{q}) \left(m(-\underline{q}) - G_0(\underline{q}) h(-\underline{q})\right) - h(\underline{q}) G_0(\underline{q}) h(-\underline{q}) \right] \right\} \\ &\overset{m(\underline{q}) - G_0(\underline{q}) h(\underline{q}) \mapsto m(\underline{q})}{=} \mathcal{Z}_0[h=0] \exp\left\{ \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} h(\underline{q}) G_0(\underline{q}) h(-\underline{q}) \right\} \end{aligned}$$

$\mathcal{Z}_0[h]$ is the **generating functional** from which all correlators etc. can be derived.

$Z[h]$ is the **generating functional** from which all correlators etc. can be derived.

$$\langle m(q)\, m(q')\rangle_0 = \frac{1}{Z_0}\int D[m(q)]\; m(q)\, m(q')\, \exp\left\{-\int\frac{d^dq}{(2\pi)^d}\left[\frac{1}{2}G_0^{-1}(q)\,|m(q)|^2 - h(q)\,m(q)\right]\right\}\Bigg|_{h=0}$$

$$= \frac{1}{Z_0}\int D[m(q)]\,(2\pi)^{2d}\frac{\delta}{\delta h(-q)}\frac{\delta}{\delta h(-q')}\exp\left\{-\int\frac{d^dq}{(2\pi)^d}\left[\frac{1}{2}G_0^{-1}(q)\,|m(q)|^2 - h(q)\,m(q)\right]\right\}\Bigg|_{h=0}$$

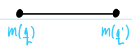
$$= \frac{1}{Z_0}(2\pi)^{2d}\frac{\delta}{\delta h(-q)}\frac{\delta}{\delta h(-q')}Z_0[h]\Bigg|_{h=0}$$

$$= (2\pi)^{2d}\frac{\delta}{\delta h(-q)}\frac{\delta}{\delta h(-q')}\exp\left\{\frac{1}{2}\int\frac{d^dk}{(2\pi)^d}h(k)\,G_0(k)\,h(-k)\right\}\Bigg|_{h=0}$$

$$= (2\pi)^{2d}\frac{\delta}{\delta h(-q)}\left(\frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\left[\delta(k+q')\,G_0(k)\,h(-k) + h(k)\,G_0(k)\,\delta(k-q')\right]\right)\exp\left\{\frac{1}{2}\int\frac{d^dk}{(2\pi)^d}h(k)\,G_0(k)\,h(-k)\right\}\Bigg|_{h=0}$$

$$= (2\pi)^{d}\frac{\delta}{\delta h(-q)}\underbrace{\frac{1}{2}\left[G_0(-q')+G_0(q')\right]}_{G_0(q')}h(q')\Bigg|_{h=0}$$

$$= (2\pi)^d\, G_0(q)\,\delta(q+q') = \frac{(2\pi)^d\,\delta(q+q')}{t + Kq_\parallel^2 + Lq_\perp^4} \equiv$$



This is exactly what we expect from Wick's theorem.

We can also compute the susceptibility from here:

$$\chi_0(q) = \frac{\delta\langle m(q)\rangle_0}{\delta h(q)} = \frac{\delta}{\delta h(q)}\left(\frac{(2\pi)^d}{Z_0[h]}\frac{\delta}{\delta h(-q)}Z_0[h]\right)$$

n.b. **not** setting $h=0$

n.b. field h now included in expectation value

$$= (2\pi)^d\frac{\delta^2}{\delta h(q)\,\delta h(-q)}\log Z_0[h]$$

$$= (2\pi)^d\frac{\delta^2}{\delta h(q)\,\delta h(-q)}\frac{1}{2}\int\frac{d^dq}{(2\pi)^d}h(q)\,G_0(q)\,h(-q)$$

can ignore $Z[0]$ since it is constant

$$= G_0(q).$$

essentially, we find $\chi_0(q)$ by removing a factor of $(2\pi)^d\,\delta(q+q')$ from $\langle m(q)\,m(q')\rangle_0$

(g) Write the perturbation U in terms of the Fourier modes $m(\mathbf{q})$.

$$U = u\int dx_\parallel\, d\underline{x}_\perp\, m^4 = u\int\prod_{i=1}^{4}\left[\frac{dq_i}{(2\pi)^d}\right]\delta(q_1+\dots+q_4)\,m(q_1)\cdots m(q_4)$$

(h) Obtain the expansion for $\langle m(\mathbf{q})m(\mathbf{q}')\rangle$ to first order in U, and reduce the correction term to a product of two-point expectation values.

We firstly define the **interacting** generating functional

$$Z[h] = \int D[m(q)]\exp\left\{-\int\frac{d^dq}{(2\pi)^d}\left[\frac{1}{2}G_0^{-1}(q)\,|m(q)|^2 - h(q)\,m(q)\right] - u\int\prod_{i=1}^{4}\left[\frac{dq_i}{(2\pi)^d}\right]\delta(q_1+\dots+q_4)\,m(q_1)\cdots m(q_4)\right\}$$

$$= \int D[m(q)]\exp\left\{-u\int\prod_{i=1}^{4}\left[\frac{dq_i}{(2\pi)^d}\right]\delta(q_1+\dots+q_4)(2\pi)^{4d}\frac{\delta}{\delta h(-q_1)}\cdots\frac{\delta}{\delta h(-q_4)}\right\}\exp\left\{-\int\frac{d^dq}{(2\pi)^d}\left[\frac{1}{2}G_0^{-1}(q)\,|m(q)|^2 - h(q)\,m(q)\right]\right\}$$

$$= \exp\left\{-u\int\prod_{i=1}^{4}[dq_i]\,\delta(q_1+\dots+q_4)\frac{\delta}{\delta h(-q_1)}\cdots\frac{\delta}{\delta h(-q_4)}\right\}Z_0[h=0]\exp\left\{\frac{1}{2}\int\frac{d^dq}{(2\pi)^d}h(q)\,G_0(q)\,h(-q)\right\}.$$

From here we can compute correlators etc. in the interacting theory.

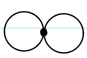
$$\langle m(q)\,m(q')\rangle = \frac{1}{Z}\int D[m(q)]\,m(q)\,m(q')\exp\left\{-\int\frac{d^dq}{(2\pi)^d}\left[\frac{1}{2}G_0^{-1}(q)\,|m(q)|^2 - h(q)\,m(q)\right] - u\int\prod_{i=1}^{4}\left[\frac{dq_i}{(2\pi)^d}\right]\delta(q_1+\dots+q_4)\,m(q_1)\cdots m(q_4)\right\}\Bigg|_{h=0}$$

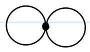
$$= \frac{1}{Z}(2\pi)^{2d}\frac{\delta}{\delta h(-q)}\frac{\delta}{\delta h(-q')}Z[h]\Bigg|_{h=0}$$

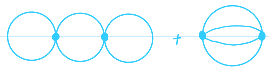
Note that the interacting partition function appears in the denominator; we will have to be careful when expanding in u.

i) $$Z = Z[h]\Big|_{h=0} = \exp\left\{-u\int\prod_{i=1}^{4}[dq_i]\,\delta(q_1+\dots+q_4)\frac{\delta}{\delta h(-q_1)}\cdots\frac{\delta}{\delta h(-q_4)}\right\}Z_0[h=0]\exp\left\{\frac{1}{2}\int\frac{d^dq}{(2\pi)^d}h(q)\,G_0(q)\,h(-q)\right\}\Bigg|_{h=0}$$

$$= Z_0[0]\left(1 - u\int\prod_{i=1}^{4}[dq_i]\,\delta(q_1+\dots+q_4)\frac{\delta}{\delta h(-q_1)}\cdots\frac{\delta}{\delta h(-q_4)} + O(u^2)\right)\exp\left\{\frac{1}{2}\int\frac{d^dq}{(2\pi)^d}h(q)\,G_0(q)\,h(-q)\right\}\Bigg|_{h=0}$$

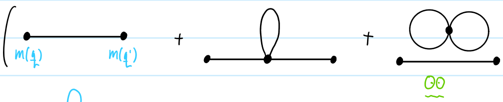
$$= Z_0[0]\Big(1 + $$  $$+ O(u^2)\Big)$$

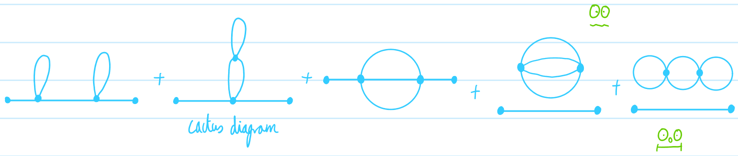
$$= \mathcal{Z}_0[0]\Big(1 + $$  $$+ \mathcal{O}(u^2) \Big)$$

↳ At $\mathcal{O}(u^2)$: 

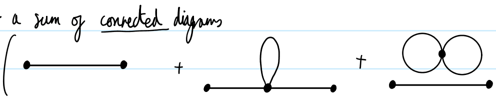
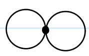
ii) $$(2\pi)^{2d} \frac{\delta}{\delta h(-q)} \frac{\delta}{\delta h(-q')} \mathcal{Z}[h]\Big|_{h=0} = (2\pi)^{2d} \frac{\delta}{\delta h(-q)} \frac{\delta}{\delta h(-q')} \exp\left\{ -u \int \prod_{i=1}^{4}[dq_i]\, \delta(q_1+\dots+q_4) \frac{\delta}{\delta h(-q_1)} \cdots \frac{\delta}{\delta h(-q_4)} \right\} \mathcal{Z}[h=0] \exp\left\{ \frac{1}{2}\int \frac{d^dq}{(2\pi)^d} h(q)\, G_0(q)\, h(-q)\right\}\Big|_{h=0}$$

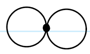
$$= (2\pi)^{2d} \mathcal{Z}_0[0] \frac{\delta}{\delta h(-q)} \frac{\delta}{\delta h(-q')} \left(1 - u\int \prod_{i=1}^{4}[dq_i]\, \delta(q_1+\dots+q_4) \frac{\delta}{\delta h(-q_1)} \cdots \frac{\delta}{\delta h(-q_4)} + \mathcal{O}(u^2) \right) \exp\left\{ \frac{1}{2}\int \frac{d^dq}{(2\pi)^d} h(q)\, G_0(q)\, h(-q)\right\}\Big|_{h=0}$$

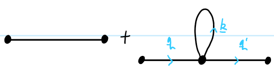
$$= \mathcal{Z}_0[0]\Big($$  $$+ \mathcal{O}(u^2)\Big)$$

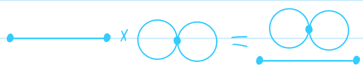
↳ at $\mathcal{O}(u^2)$: 

The two-point correlator reduces to a sum of connected diagrams

$$\langle m(q)\, m(q')\rangle = \frac{\mathcal{Z}_0[0]\Big(\text{diagrams} + \mathcal{O}(u^2)\Big)}{\mathcal{Z}_0[0]\Big(1 + \text{diagram} + \mathcal{O}(u^2)\Big)}$$  

$$= \Big(\ \cdots + \mathcal{O}(u^2)\Big)\Big(1 - \ \cdots + \mathcal{O}(u^2)\Big)$$  

$$= \ \cdots + \mathcal{O}(u^2)$$ 



$$= \langle m(q)\, m(q')\rangle_0 + 12\,(-u)\int dk\ G_0(q)\, G_0(k)\, G_0(q')\ \delta(q+q') + \mathcal{O}(u^2)$$

Symmetry factor: very annoying to get right in general!
See P & S for discussion

(i) Write down the expression for $\chi(\mathbf{q})$ in the first order of perturbation theory, and identify the transition point t_c at first order in u. [Do not evaluate the integral explicitly.]

The calculation of the susceptibility is very similar to before:

$$\chi(q) = \frac{\delta\langle m(q)\rangle}{\delta h(q)} = \frac{\delta}{\delta h(q)}\left(\frac{(2\pi)^d}{\mathcal{Z}[h]} \frac{\delta}{\delta h(-q)} \mathcal{Z}[h]\right)$$

$$= (2\pi)^d \frac{\delta^2}{\delta h(q)\, \delta h(-q)} \log \mathcal{Z}[h]$$

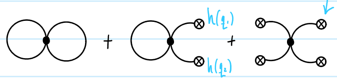
interacting generating functional

We are ultimately interested in the zero-field ($h=0$) susceptibility – but it is interesting/instructive to keep $h \neq 0$ for the time being

We again expand to 1st order in u:

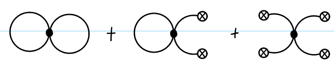
$$\mathcal{Z}[h] = \left(1 - u\int \prod_{i=1}^{4}[dq_i]\, \delta(q_1+\dots+q_4) \frac{\delta}{\delta h(-q_1)} \cdots \frac{\delta}{\delta h(-q_4)} + \mathcal{O}(u^2) \right) \exp\left\{ \frac{1}{2}\int \frac{d^dq}{(2\pi)^d} h(q)\, G_0(q)\, h(-q)\right\}$$

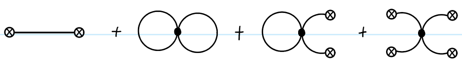
$h\,m$ generates a term $-\otimes$

$$= \Big(1 + \ \cdots + \mathcal{O}(u^2)\Big) \exp\left\{ \frac{1}{2}\int \frac{d^dq}{(2\pi)^d} h(q)\, G_0(q)\, h(-q)\right\}$$ 

$q_1 + q_2 = 0$

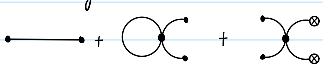
so the logarithm reads

$$\log \mathcal{Z}[h] = \frac{1}{2}\int \frac{d^dq}{(2\pi)^d} h(q)\, G_0(q)\, h(-q) + \log\Big(1 + \ \cdots + \mathcal{O}(u^2)\Big)$$ 

$$= \ \cdots + \mathcal{O}(u^2).$$ 

Differentiation w.r.t. h acts on the external legs $-\otimes$

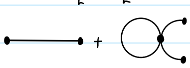
(possible missing factors of 2 etc. here)

$$(2\pi)^d \frac{\delta^2}{\delta h(q)\, \delta h(-q)} \log \mathcal{Z}[h] = \ \cdots + \mathcal{O}(u^2)$$ 

Thus, the zero-field susceptibility is

↳ if $h \neq 0$ then this diagram gives the nonlinear terms in the susceptibility

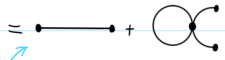
$$\chi(q)\Big|_{h=0} = (2\pi)^d \frac{\delta^2}{\delta h(q)\, \delta h(-q)} \log \mathcal{Z}[h]\Big|_{h=0}$$

$$= \ \cdots + \mathcal{O}(u^2)$$ 

This is the key result – don't worry too much about the above diagrammatic manipulations

$$= \langle \cdots \rangle + \dots \int dk \ \cdots + \mathcal{O}(u^2)$$

as before, we find $\chi(q)$ by removing a factor of

 $+\ \mathcal{O}(u^2)$

This is the key result - don't worry too much about the above diagrammatic manipulations

$$= G_0(\underline{q}) + 12(-u)\int\frac{d\underline{k}}{(2\pi)^d}\, G_0(\underline{q})\, G_0(\underline{k})\, G_0(\underline{q}) + \mathcal{O}(u^2).$$

as before, we find $\chi(\underline{q})$ by removing a factor of $(2\pi)^d\, \delta(\underline{q}+\underline{q}')$ from $\langle m(\underline{q})\, m(\underline{q}')\rangle$

This notation is slightly sloppy - really we ought to indicate that the external momenta sum to $\underline{0}$

At the transition point, the long wavelength susceptibility $\chi(\underline{q}=\underline{0})$ diverges, so $\chi(\underline{q}=\underline{0})^{-1} \to 0$.

$$\chi^{-1}(\underline{0}) = G_0(\underline{0})^{-1}\left[1 - 12u\, G_0(\underline{0})\int\frac{d^d\underline{k}}{(2\pi)^d}\, G_0(\underline{k}) + \mathcal{O}(u^2)\right]^{-1}$$

$$= G_0(\underline{0})^{-1}\left[1 + 12u\, G_0(\underline{0})\int\frac{d^d\underline{k}}{(2\pi)^d}\, G_0(\underline{k}) + \mathcal{O}(u^2)\right]$$

t

$$= t + 12u\int\frac{d^d\underline{k}}{(2\pi)^d}\, G_0(\underline{k}) \overset{!}{=} 0$$

$$\Rightarrow\quad t_c = -12u\int\frac{d^d\underline{k}}{(2\pi)^d}\, G_0(\underline{k}).$$

This is equivalent to the RPA - see official solutions for discussion. In general, truncating the power series at $\mathcal{O}(u)$ is not justified, but it turns out that inverting the power series as we have done here is equivalent to resumming an infinite series of diagrams - hence the expansion is more valid than it may appear. See also ch.3 of Kardar.

(j) Using RG, or any other method, find the upper critical dimension d_u for validity of the Gaussian exponents.

The interaction U renormalizes as

$$u\int\prod_{i=1}^{4}\left[\frac{d\underline{q}_i}{(2\pi)^d}\right]\delta(\underline{q}_1+\dots+\underline{q}_4)\, m(\underline{q}_1)\cdots m(\underline{q}_4) = u\int\prod_{i=1}^{4}\left[\frac{d\underline{q}_i^{\perp}}{(2\pi)^{d-1}}\,\frac{dq_i^{\parallel}}{2\pi}\right]\delta(\underline{q}_1+\dots+\underline{q}_4)\, m(\underline{q}_1)\cdots m(\underline{q}_4)$$

$$\mapsto\quad u\int\prod_{i=1}^{4}\left[\frac{d\underline{q}_i^{\perp\prime}}{(2\pi c)^{d-1}}\,\frac{dq_i^{\parallel\prime}}{2\pi b}\right]\delta(\underline{q}_1'+\dots+\underline{q}_4')\, m'(\underline{q}_1')\cdots m'(\underline{q}_4')\, z^4$$

$$\text{i.e.}\quad u' = \frac{z^4}{(b\, c^{d-1})^3}\, u \underbrace{=}_{c\,=\,b^{1/2},\quad z\,=\,b^{(d+5)/4}} b^{(7-d)/2}\, u$$

If $d > 7$ then u becomes smaller under the RG flow $\Rightarrow d_c = 7$.

Perturbative RG

15 February 2024 13:29

1. *Perturbative RG (adpated from 2018 Part III NatSci Tripos):* Consider the long-wavelength expansion of the Hamiltonian of the 2-dimensional XY model:

$$\beta\mathcal{H}[\phi(\mathbf{r})] = \int d^2\mathbf{r}\left(\frac{K}{2}(\nabla\phi)^2 + u(\nabla\phi)^4\right),$$

where $\phi(\mathbf{r})$ is the azimuthal angle that describes the transverse fluctuations of the magnetisation. Longitudinal fluctuations can be assumed to be frozen out. $\mathbf{r}$ spans 2-dimensional Euclidean space.

(a) By integrating out the Fourier modes of $\phi(\mathbf{r})$ with wavevectors $\Lambda e^{-l} < |\mathbf{q}| < \Lambda$, implement the momentum-shell renormalisation group procedure to first order in u and derive the following flow equations

$$\frac{dK}{dl} = \frac{4u\Lambda^2}{K\pi}, \qquad \frac{du}{dl} = -2u.$$

$$\beta H[\phi] = \int d^2\underline{r}\left[\frac{K}{2}(\nabla\phi)^2 + u(\nabla\phi)^4\right]$$

Strictly speaking, the path integral is not well-defined until we introduce the uv cutoff

$$= \underbrace{\int^{\Lambda}\frac{d^2\underline{q}}{(2\pi)^2}\left[\frac{K}{2}\,\underline{q}^2\,|\phi(\underline{q})|^2\right]}_{\beta H_0[\phi]} + \underbrace{u\int^{\Lambda}\prod_i\left[\frac{d^2\underline{q}_i}{(2\pi)^2}\right](2\pi)^2\delta(\underline{q}_1+\dots+\underline{q}_4)(\underline{q}_1\cdot\underline{q}_2)(\underline{q}_3\cdot\underline{q}_4)\,\phi(\underline{q}_1)\cdots\phi(\underline{q}_4)}_{\beta U[\phi]}$$

Split up the field into fast and slow modes:

$$\phi(\underline{q}) = \begin{cases}\phi_<(\underline{q}) & |\underline{q}| < \Lambda e^{-l} \equiv \Lambda/b\\ \phi_>(\underline{q}) & \Lambda e^{-l} < |\underline{q}| < \Lambda.\end{cases}$$

note that $\phi(\underline{q}) = 0$ for $|\underline{q}| > \Lambda$ already

Then

$$\mathcal{Z} = \int D[\phi(\underline{q})]\, e^{-\beta H_0[\phi] - \beta U[\phi]}$$

$$= \int D[\phi_<(\underline{q})]\,D[\phi_>(\underline{q})]\, e^{-\beta H_0[\phi_< + \phi_>] - \beta U[\phi_< + \phi_>]}$$

$$= \int D[\phi_<(\underline{q})]\, e^{-\beta H_0[\phi_<]}\int D[\phi_>(\underline{q})]\, e^{-\beta H_0[\phi_>] - \beta U[\phi_< + \phi_>]}$$

where we have used

$$\beta H_0[\phi_< + \phi_>] = \frac{K}{2}\int^{\Lambda}\frac{d^2\underline{q}}{(2\pi)^2}\,\underline{q}^2\,|\phi_<(\underline{q}) + \phi_>(\underline{q})|^2 = \frac{K}{2}\int^{\Lambda}\frac{d^2\underline{q}}{(2\pi)^2}\,\underline{q}^2\left(|\phi_<(\underline{q})|^2 + |\phi_>(\underline{q})|^2 + 2\,\mathrm{Re}\left[\phi_<(\underline{q})\,\phi_>(\underline{q})\right]\right)$$

$$= \underbrace{\frac{K}{2}\int^{\Lambda/b}\frac{d^2\underline{q}}{(2\pi)^2}\,\underline{q}^2\,|\phi_<(\underline{q})|^2}_{\beta H_0[\phi_<]} + \underbrace{\frac{K}{2}\int_{\Lambda/b}^{\Lambda}\frac{d^2\underline{q}}{(2\pi)^2}\,\underline{q}^2\,|\phi_>(\underline{q})|^2}_{\beta H_0[\phi_>]} + K\,\mathrm{Re}\Big\{\underbrace{\int^{\Lambda}\frac{d^2\underline{q}}{(2\pi)^2}\,\underline{q}^2\,\phi_<(\underline{q})\,\phi_>(\underline{q})}_{\substack{0 \text{ since if } \phi_<(\underline{q})\neq 0 \text{ then}\\ \phi_>(\underline{q}) = 0 \text{ and vice versa}}}\Big\} = \beta H_0[\phi_<] + \beta H_0[\phi_>]$$

We must therefore compute the partition function

$$\mathcal{Z}[\phi_<] = \int D[\phi_>(\underline{q})]\, e^{-\beta H_0[\phi_>] - \beta U[\phi_< + \phi_>]}$$

or, equivalently, the effective action / Hamiltonian

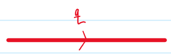
$$-\beta H_{eff}[\phi_<] = \log\mathcal{Z}[\phi_<]$$

which contains only the connected diagrams from $\mathcal{Z}$.

The next step is to work out the Feynman rules. Firstly, the quadratic Hamiltonian $\beta H_0[\phi_>]$ gives the propagator for the fast fields:

$$\beta H_0[\phi_>] = \frac{K}{2}\int_{\Lambda/b}^{\Lambda}\frac{d^2\underline{q}}{(2\pi)^2}\,\underline{q}^2\,|\phi_>(\underline{q})|^2 \equiv \frac{1}{2}\int_{\Lambda/b}^{\Lambda}\frac{d^2\underline{q}}{(2\pi)^2}\,|\phi_>(\underline{q})|^2\,G_0^{-1}(\underline{q})$$

so



$$= G_0^>(\underline{q}) = \frac{1}{K\underline{q}^2}\,\Theta^>(\underline{q}) \equiv \begin{cases}1/K\underline{q}^2 & \Lambda/b < |\underline{q}| < \Lambda\\ 0 & \text{otherwise}\end{cases}$$

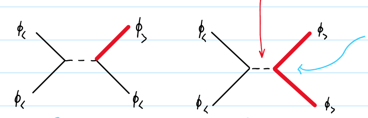
Additional factor of $(2\pi)^d\delta(\underline{q}+\underline{q}')$ required for computing expectation values

Interaction vertices between $\phi_<$ and $\phi_>$ arise from $U[\phi_< + \phi_>]$:

$$\beta U[\phi_< + \phi_>] = \int^{\Lambda}\prod_i\left[\frac{d^2\underline{q}_i}{(2\pi)^2}\right](2\pi)^2\delta(\underline{q}_1+\dots+\underline{q}_4)(\underline{q}_1\cdot\underline{q}_2)(\underline{q}_3\cdot\underline{q}_4)\left[\phi_<(\underline{q}_1)+\phi_>(\underline{q}_1)\right]\cdots\left[\phi_<(\underline{q}_4)+\phi_>(\underline{q}_4)\right]$$

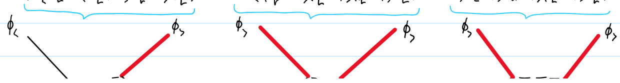
Caution: the dashed line here is not a propagator

Convention: dot together the momenta that meet at the dashed line

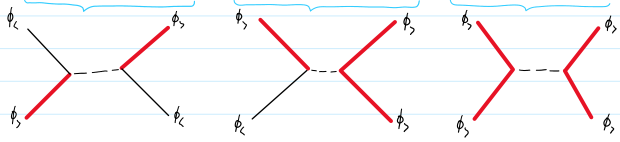


$$= \int^{\Lambda}\prod_i\left[\frac{d^2\underline{q}_i}{(2\pi)^2}\right](2\pi)^2\delta(\underline{q}_1+\dots+\underline{q}_4)(\underline{q}_1\cdot\underline{q}_2)(\underline{q}_3\cdot\underline{q}_4)\Big[\underbrace{\phi_<(\underline{q}_1)\phi_<(\underline{q}_2)\phi_<(\underline{q}_3)\phi_<(\underline{q}_4)}_{\beta U[\phi_<]\ \rightsquigarrow\ \text{drops out of } \phi_> \text{ integration}} + \underbrace{4\,\phi_<(\underline{q}_1)\phi_<(\underline{q}_2)\phi_<(\underline{q}_3)\phi_>(\underline{q}_4)} + \underbrace{2\,\phi_<(\underline{q}_1)\phi_<(\underline{q}_2)\phi_>(\underline{q}_3)\phi_>(\underline{q}_4)}$$

$$+ \underbrace{4\,\phi_<(\underline{q}_1)\phi_<(\underline{q}_3)\phi_>(\underline{q}_2)\phi_>(\underline{q}_4)} + \underbrace{4\,\phi_<(\underline{q}_1)\phi_>(\underline{q}_2)\phi_>(\underline{q}_3)\phi_>(\underline{q}_4)} + \underbrace{\phi_>(\underline{q}_1)\phi_>(\underline{q}_2)\phi_>(\underline{q}_3)\phi_>(\underline{q}_4)}\Big]$$

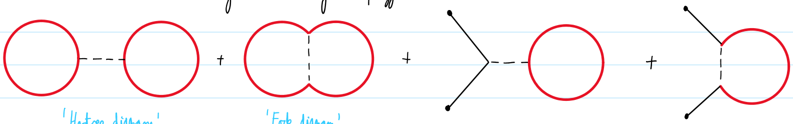


$+ \, 4\,\underbrace{\phi_<(q_1)\,\phi_<(q_3)\,\phi_>(q_2)\,\phi_>(q_4)} \; + \; 4\,\underbrace{\phi_<(q_1)\,\phi_>(q_2)\,\phi_>(q_3)\,\phi_>(q_4)} \; + \; \underbrace{\phi_>(q_1)\,\phi_>(q_2)\,\phi_>(q_3)\,\phi_>(q_4)} \Big]$



From here we can write down the (connected) diagrams contributing to βH_{eff} at $O(u)$:

$-\beta H_{eff} =$



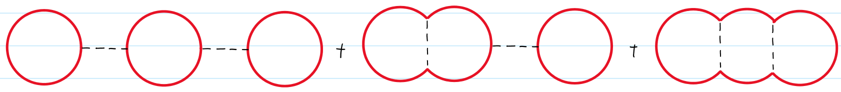
'Hartree diagram' 'Fock diagram'

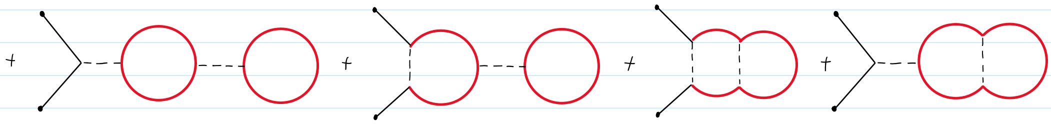
These diagrams contain no external legs so only contribute an overall shift in vacuum energy. We will ignore them.

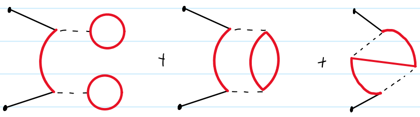
Important: as far as $\phi_>$ is concerned, $\phi_<$ is an external field with no dynamics. In other words, $\phi_<$ can only have external legs, and there are no $\phi_<$ propagators in the diagrams we are concerned with. This means that diagrams such as

do not appear as they contain a $\phi_<$ propagator.

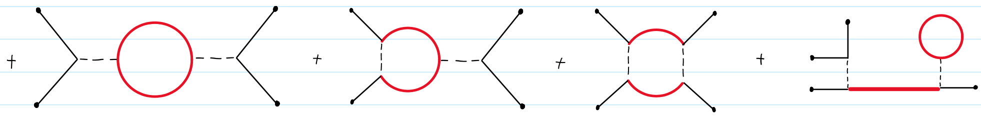
Aside: at $O(u^2)$ we have

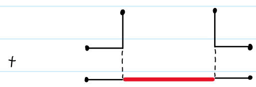






Note that corrections to the 4-point vertex contribute only at $O(u^2)$

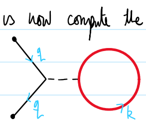




The RG generates a 6-point vertex. However, this interaction is irrelevant.
Note that no diagrams with an odd number of $\phi_<$ legs can appear as the RG preserves the parity symmetry $\phi \mapsto -\phi$.

Let us now compute the required diagrams.

i)



$$= 2\int \frac{d^2q}{(2\pi)^2}\, q^2\, \phi_<(q)\,\phi_<(-q) \int \frac{d^2k}{(2\pi)^2}(-u)\, G_0^>(k)\, k^2$$

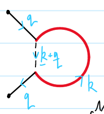
$$= -\frac{2u}{(2\pi)^2}\int^{\Lambda/b}\frac{d^2q}{(2\pi)^2}\, q^2\, |\phi_<(q)|^2 \int_{\Lambda/b}^{\Lambda} d^2k \cdot \frac{1}{K k^2}\, k^2 \qquad \int_q \phi_<^\alpha\, \phi_<^\alpha \int_k G_>^{\beta\beta}(k)$$

$$= -\frac{u}{2\pi^2 K}\int^{\Lambda e^{-l}}\frac{d^2q}{(2\pi)^2}\, q^2\, |\phi_<(q)|^2 \int_{\Lambda e^{-l}}^{\Lambda} 2\pi k\, dk$$

l infinitesimal

$$= -\frac{u\Lambda^2}{2\pi K}\left(1-e^{-2l}\right)\int^{\Lambda e^{-l}}\frac{d^2q}{(2\pi)^2}\, q^2\, |\phi_<(q)|^2 = -\frac{u\Lambda l}{\pi K}\int^{\Lambda e^{-l}}\frac{d^2q}{(2\pi)^2}\, q^2\, |\phi_<(q)|^2$$

ii)



$$= 4\int \frac{d^2q}{(2\pi)^2}\frac{d^2k}{(2\pi)^2}(q\cdot k)^2\, \phi_<(q)\,\phi_<(-q)\,(-u)\, G_0^>(k) \qquad \int_{kq} \phi_{<(q)}^{\alpha}\, G_>^{\alpha\beta}(k)\, \phi_<^{\beta}(q)$$

$\cos^2\theta_{qk} \rightsquigarrow \langle \cos^2\theta \rangle = 1/2$ indep. of q

$$= -\frac{4u}{(2\pi)^2}\int^{\Lambda/b}\frac{d^2q}{(2\pi)^2}\, |\phi_<(q)|^2 q^2 \int_{\Lambda/b}^{\Lambda} d^2k\, \frac{1}{K}\, \frac{\overbrace{(q\cdot k)^2}}{k^2}$$

$$= -\frac{u}{\pi^2 K}\cdot\frac{1}{2}\cdot 2\pi\cdot\frac{\Lambda^2}{2}\left(1-e^{-2l}\right)\int^{\Lambda/b}\frac{d^2q}{(2\pi)^2}\, |\phi_<(q)|^2 q^2$$

$$= -\frac{u\Lambda^2 l}{\pi K}\int^{\Lambda/b}\frac{d^2q}{(2\pi)^2}\, |\phi_<(q)|^2 q^2 \quad + O(l^2)$$

Overall, we find

$$-\beta H_{eff} = -\beta U[\phi_<] + (\text{const.}) - \frac{2u\Lambda^2 l}{\pi K}\int^{\Lambda/b}\frac{d^2q}{(2\pi)^2}\, |\phi_<(q)|^2 q^2 + O(u^2)$$

Overall, we find

$$-\beta H_{\text{eff}} = -\beta U[\phi_<] + (\text{const.}) - \frac{2u\Lambda^2 l}{\pi K}\int^{\Lambda/b}\frac{d^2 q}{(2\pi)^2}|\phi_<(\mathbf{q})|^2 q^2 + \mathcal{O}(u^2)$$

so that

$$\mathcal{Z} = \int D[\phi_<(\mathbf{k})]\, e^{-\beta H_0[\phi_<] - \beta H_{\text{eff}}[\phi_<]}$$

$$= \mathcal{N}\int D[\phi_<(\mathbf{k})]\exp\left\{-\int^{\Lambda/b}\frac{d^2 q}{(2\pi)^2}\left[\frac{K}{2}q^2|\phi_<(\mathbf{q})|^2\right] - u\int^{\Lambda/b}\prod_i\left[\frac{d^2 q_i}{(2\pi)^2}\right](2\pi)^2\delta(\mathbf{q}_1+\dots+\mathbf{q}_4)(\mathbf{q}_1\cdot\mathbf{q}_2)(\mathbf{q}_3\cdot\mathbf{q}_4)\phi_<(\mathbf{q}_1)\cdots\phi_<(\mathbf{q}_4) - \frac{2u\Lambda^2 l}{\pi K}\int^{\Lambda/b}\frac{d^2 q}{(2\pi)^2}|\phi_<(\mathbf{q})|^2 q^2 + \mathcal{O}(u^2)\right\}$$

unimportant multiplicative constant from exponential of vacuum diagrams

$$= \mathcal{N}'\int D[\phi'(\mathbf{k})]\exp\left\{-\int^{\Lambda}\frac{d^2 q'}{(2\pi b)^2}\left(\frac{K}{2} + \frac{2u\Lambda^2 l}{\pi K}\right)\frac{q'^2}{b^2}|\phi'(\mathbf{q}')|^2 z^2 - u\int^{\Lambda}\prod_i\left[\frac{d^2 q_i'}{(2\pi b)^2}\right](2\pi)^2\delta(\mathbf{q}_1'+\dots+\mathbf{q}_4')\, b^2(\mathbf{q}_1'\cdot\mathbf{q}_2')(\mathbf{q}_3'\cdot\mathbf{q}_4')\frac{1}{b^4}\phi'(\mathbf{q}_1')\cdots\phi'(\mathbf{q}_4')\, z^4 + \dots\right\}$$

$\mathcal{N}' = \mathcal{N}\,\mathrm{Det}[z]$

note that the δ-function gives a factor of b^2

rescaling $\mathbf{q}' = b\mathbf{q}$, $\phi'(\mathbf{q}') = \phi_<(\mathbf{q})/z$

We can now read off that

$$\frac{K'}{2} = \frac{1}{b^4}\left(\frac{K}{2} + \frac{2u\Lambda^2 l}{\pi K}\right)z^2, \qquad u' = \frac{1}{b^8}\cdot b^2\cdot\frac{1}{b^4}z^4 u$$

i.e.

$$K' = e^{-4l}\left(K + \frac{4u\Lambda^2 l}{\pi K}\right)z^2 \qquad u' = e^{-10l}z^4 u$$

Let

$$G(r, K, u) \equiv \langle e^{i\phi(\mathbf{r}) - i\phi(\mathbf{0})}\rangle_{\beta\mathcal{H}},$$

where the expectation value with respect to the above Hamiltonian depends on the parameters (K, u).

(b) Show that

$$G(r, K, u = 0) = \frac{1}{(r/a)^{\frac{1}{2\pi K}}},$$

where a is the lattice constant.

$$G(r, K, u=0) = \left\langle e^{i\phi(\mathbf{r}) - i\phi(\mathbf{0})}\right\rangle_{\beta H_0} = \frac{1}{\mathcal{Z}_0}\int D[\phi(\mathbf{r})]\, e^{i\phi(\mathbf{r}) - i\phi(\mathbf{0}) - \beta H_0}$$

$$\phi(\mathbf{r}) = \int^{\Lambda}\frac{d^2 q}{(2\pi)^2}e^{i\mathbf{q}\cdot\mathbf{r}}\phi(\mathbf{q})$$

$$= \frac{1}{\mathcal{Z}_0}\int D[\phi(\mathbf{k})]\exp\left\{-\int^{\Lambda}\frac{d^2 q}{(2\pi)^2}\left[\frac{K}{2}q^2|\phi(\mathbf{q})|^2 - \frac{i}{2}(e^{i\mathbf{q}\cdot\mathbf{r}} - 1)\phi(\mathbf{q}) - \frac{i}{2}(e^{-i\mathbf{q}\cdot\mathbf{r}} - 1)\phi(-\mathbf{q})\right]\right\}$$

$$\phi(\mathbf{q}) \mapsto \phi(\mathbf{q}) + i\frac{e^{-i\mathbf{q}\cdot\mathbf{r}} - 1}{Kq^2}$$

$$= \frac{1}{\mathcal{Z}_0}\underbrace{\int D[\phi(\mathbf{k})]\exp\left\{-\int^{\Lambda}\frac{d^2 q}{(2\pi)^2}\frac{K}{2}q^2|\phi(\mathbf{q})|^2\right\}}_{\mathcal{Z}_0}\exp\left\{\underbrace{-\frac{1}{2}\int^{\Lambda}\frac{d^2 q}{(2\pi)}\frac{(e^{i\mathbf{q}\cdot\mathbf{r}} - 1)(e^{-i\mathbf{q}\cdot\mathbf{r}} - 1)}{Kq^2}}_{\langle[\phi(\mathbf{r}) - \phi(0)]^2\rangle}\right\}$$

2D XY correlator

$$= \exp\left\{\left\langle[\phi(\mathbf{r}) - \phi(0)]^2\right\rangle\right\}.$$

Now

$$\left\langle[\phi(\mathbf{r}) - \phi(0)]^2\right\rangle = \frac{-1}{2}\int^{\Lambda}\frac{d^2 q}{(2\pi)^2}\frac{(e^{i\mathbf{q}\cdot\mathbf{r}} - 1)(e^{-i\mathbf{q}\cdot\mathbf{r}} - 1)}{Kq^2} = -\frac{1}{2}\int^{\Lambda}\frac{d^2 q}{(2\pi)^2}\frac{2 - 2\cos(\mathbf{q}\cdot\mathbf{r})}{Kq^2} = \frac{1}{(2\pi)^2}\int_0^{2\pi}d\theta\int_0^{\Lambda}dq\, q\cdot\frac{\cos(qr\cos\theta) - 1}{Kq^2}.$$

We may use the expansion

$$I(x, \Lambda) = \int_0^{\Lambda}dq\,\frac{1 - \cos(qx)}{q} \overset{\Lambda\to\infty}{\sim} \gamma + \log|x\Lambda| + \mathcal{O}\left(\frac{1}{x\Lambda}\right) \quad \text{Eq. 1}$$

note: $I(x, \infty)$ is divergent; the lattice regularisation is necessary

'asymptotic to'

Euler-Mascheroni constant $\gamma \simeq 0.577$

the next term is $\mathrm{sinc}(x\Lambda)$

to write

$$\left\langle[\phi(\mathbf{r}) - \phi(0)]^2\right\rangle \overset{\Lambda\to\infty}{\sim} -\frac{1}{(4\pi)^2 K}\int_0^{2\pi}d\theta\left[\gamma + \log|\Lambda r\cos\theta|\right] = -\frac{\gamma}{2\pi K} - \frac{1}{2\pi K}\log(\Lambda r) - \frac{1}{8\pi^2 K}\underbrace{\int_0^{2\pi}\log(\cos^2\theta)\,d\theta}_{-2\pi\log 4}$$

$$= -\frac{\gamma - \log 2}{2\pi} - \frac{1}{2\pi K}\log(\Lambda r)$$

This is an asymptotic series in $r/a \to \infty$

Therefore

$$G(r, K, u=0) = \exp\left\{-\frac{\gamma - \log 2}{2\pi} - \frac{1}{2\pi K}\log(\Lambda r) + \mathcal{O}((\Lambda r)^{-1})\right\}$$

$$\propto (\Lambda r)^{-\frac{1}{2\pi K}} \simeq \frac{1}{(r/a)^{\frac{1}{2\pi K}}}$$

$\Lambda \sim 1/a$

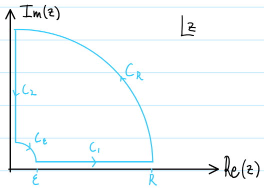
$(r/a)^{-2\pi K}$

$\Lambda \sim 1/a$

Aside: derivation of Eq. 1

Define $I(R) = \int_0^R \frac{1-e^{it}}{t}\,dt$ so that $\mathrm{Re}\left\{I\left(\frac{\Lambda}{2}\right)\right\} = \int_0^{\Lambda/2} \frac{1-\cos q}{q}\,dq = \int_0^{\Lambda} \frac{1-\cos(\Lambda q)}{q}\,dq.$

Integrate $f(z) = \frac{1-e^{iz}}{z}$ along the contour:



$f(z)$ has no poles inside the contour, so

$$\oint_C \frac{1-e^{iz}}{z} \overset{\text{limit } \varepsilon\to 0 \text{ understood}}{=} 0 = I(R) + \underbrace{\int_0^{\pi/2} \frac{1-e^{iRe^{i\phi}}}{Re^{i\phi}}\, iRe^{i\phi}\,d\phi}_{\text{write } e^{i\phi}\to z \text{ here}} + \int_R^0 \frac{1-e^{-t}}{t}\,dt + \underbrace{\int_{\pi/2}^0 \frac{1-e^{i\varepsilon e^{i\phi}}}{\varepsilon e^{i\phi}}\, i\varepsilon e^{i\phi}\,d\phi}_{\propto\varepsilon\to 0}$$

$$\Rightarrow I(R) = \int_0^R \frac{1-e^{-t}}{t}\,dt - \int_1^{e^{i\pi/2}} \frac{1-e^{iRz}}{z}\,dz$$

$\downarrow$ IBP $u = 1-e^{-t}$, $v' = 1/t$ IBP $u = 1/z$, $v' = e^{iRz}$

$$= \left(\left[\ln(t)(1-e^{-t})\right]_0^R - \int_0^R e^{-t}\ln t\,dt\right) - \left(\left[\ln z\right]_1^i - \left[\frac{1}{iR}e^{iRz}\cdot\frac{1}{z}\right]_1^i - \int_1^i \frac{e^{iRz}}{z^2}\,dz\right)$$

$$= \left(\ln R - \underbrace{\int_0^\infty e^{-t}\ln t\,dt}_{-\gamma} + \text{exp. small in } R\right) - \left(\frac{i\pi}{2} + \frac{e^{iR}}{iR} + \text{exp. small in } R + \mathcal{O}(1/R^2)\right)$$

Thus

$$\mathrm{Re}\{I(R)\} = \gamma + \ln R - \mathrm{sinc}\,R + \mathcal{O}\left(\frac{1}{R^2}\right).$$

(c) Show that an RG trajectory starting at the point (K_0, u_0) in the (K, u) plane flows towards the point $(\sqrt{K_0^2 + 4u_0\Lambda^2/\pi}, 0)$.

$$\frac{du}{dl} = -2u \quad \rightsquigarrow \quad \int_{u_0}^{u} \frac{du'}{u'} = -2\int_0^l dl' \quad ; \quad u = u_0 e^{-2l}$$

$$\Rightarrow \frac{dK}{dl} = \frac{4u\Lambda^2}{K\pi} = \frac{4u_0\Lambda^2}{K\pi}e^{-2l} \quad \rightsquigarrow \quad \int_{K_0}^{K} K'\,dK' = \frac{4u_0\Lambda^2}{\pi}\int_0^l e^{-2l'}\,dl' \quad ; \quad \frac{1}{2}(K^2 - K_0^2) = \frac{2u_0\Lambda^2}{\pi}(1-e^{-2l})$$

As $l\to\infty$, $u(l) \to 0$, $K(l) \to \sqrt{K_0^2 + \frac{4u_0\Lambda^2}{\pi}}$ ✓

$$\frac{dK}{du} = \frac{4u\Lambda^2}{K\pi}\cdot\frac{1}{(-2u)} = -\frac{2\Lambda^2}{K\pi}$$

$$\int K\,dK = -\frac{2\Lambda^2}{\pi}\int du$$

(d) Considering an infinitesimal RG flow from l to $l+\delta l$ starting at the point $(K \gg 1, u)$, show that

$$G(r,K,u) = e^{-\frac{\delta l}{2\pi K}}G\left(r(1-\delta l), K + \frac{dK}{dl}\delta l, u + \frac{du}{dl}\delta l\right). \quad (1)$$

$$G(r,K,u) = \left\langle e^{i\phi(\mathbf{r}) - i\phi(0)}\right\rangle_{\beta H} = \frac{1}{Z}\int D[\phi(\mathbf{r})]\,e^{i\phi(\mathbf{r}) - i\phi(0) - \beta H}$$

$$= \frac{1}{Z}\int D[\phi(\mathbf{q})]\exp\left\{-\beta H[\phi(\mathbf{q})] + i\int\frac{d^2q}{(2\pi)^2}\left(e^{i\mathbf{q}\cdot\mathbf{r}} - 1\right)\phi(\mathbf{q})\right\}$$

$$= \frac{1}{Z(K,u)}\int D[\phi_>(\mathbf{q})]\,D[\phi_<(\mathbf{q})]\exp\left\{-\beta H_0[\phi_>] - \beta H[\phi_<] - \beta U[\phi_< + \phi_>] + i\int_{\Lambda/b}^{\Lambda}\frac{d^2q}{(2\pi)^2}\left(e^{i\mathbf{q}\cdot\mathbf{r}} - 1\right)\phi_>(\mathbf{q}) + i\int^{\Lambda/b}\frac{d^2q}{(2\pi)^2}\left(e^{i\mathbf{q}\cdot\mathbf{r}} - 1\right)\phi_<(\mathbf{q})\right\}$$

$$= \frac{1}{Z(K,u)}\int D[\phi_<(\mathbf{q})]\exp\left\{-\beta H_0[\phi_<](K,u) + i\int^{\Lambda/b}\frac{d^2q}{(2\pi)^2}\left(e^{i\mathbf{q}\cdot\mathbf{r}} - 1\right)\phi_<(\mathbf{q})\right\}\int D[\phi_>(\mathbf{q})]\exp\left\{-\beta H_0[\phi_>] - \beta U[\phi_<,\phi_>] + i\int_{\Lambda/b}^{\Lambda}\frac{d^2q}{(2\pi)^2}\left(e^{i\mathbf{q}\cdot\mathbf{r}} - 1\right)\phi_>(\mathbf{q})\right\}$$

Consider the integral over the fast modes:

$$\int D[\phi_>(\mathbf{k})]\exp\left\{-\int_{\Lambda/b}^{\Lambda}\frac{d^2q}{(2\pi)^2}\left[\frac{K}{2}q^2|\phi_>(\mathbf{q})|^2 - \frac{i}{2}\left(e^{i\mathbf{q}\cdot\mathbf{r}} - 1\right)\phi_>(\mathbf{q}) - \frac{i}{2}\left(e^{-i\mathbf{q}\cdot\mathbf{r}} - 1\right)\phi_>^*(\mathbf{q})\right] - \beta U[\phi_< + \phi_>]\right\}$$

$$= \exp\left\{-\frac{1}{2}\int_{\Lambda/b}^{\Lambda}\frac{d^2q}{(2\pi)^2}\frac{\left(e^{i\mathbf{q}\cdot\mathbf{r}} - 1\right)\left(e^{-i\mathbf{q}\cdot\mathbf{r}} - 1\right)}{Kq^2}\right\}\int D[\phi_>(\mathbf{k})]\exp\left\{-\int_{\Lambda/b}^{\Lambda}\frac{d^2q}{(2\pi)^2}\left[\frac{K}{2}q^2|\phi_>(\mathbf{q})|^2\right] - \beta U\left[\phi_< + \phi_> + i\underbrace{\frac{e^{i\mathbf{q}\cdot\mathbf{r}} - 1}{Kq^2}}_{K\gg u\,\to\,\text{neglect}}\right]\right\}$$

$\phi_>(\mathbf{q}) \mapsto \phi_>(\mathbf{q}) + i\frac{e^{-i\mathbf{q}\cdot\mathbf{r}} - 1}{Kq^2}$

$$\simeq \exp\left\{\frac{1}{(2\pi)^2}\int_0^{2\pi}d\theta\int_{\Lambda/b}^{\Lambda}dq\,q\cdot\frac{\cos(qr\cos\theta) - 1}{Kq^2}\right\}\underbrace{\int D[\phi_<(\mathbf{k})]\,e^{-\beta H_0[\phi_<] - \beta U[\phi_< + \phi_>]}}_{e^{-\beta H_{\text{eff}}[\phi_<]}}$$

To compute the integral in the exponential we use

$$\int_{\Lambda/b}^{\Lambda}\frac{1-\cos(qx)}{q}\,dq = \left(\int_0^{\Lambda} - \int_0^{\Lambda/b}\right)\frac{1-\cos(qx)}{q}\,dq$$

$$= I(x,\Lambda) - I(x,\Lambda/b) \sim \left(\gamma + \log|x\Lambda| + \mathcal{O}\left(\frac{1}{x\Lambda}\right)\right) - \left(\gamma + \log|x\Lambda/b| + \mathcal{O}\left(\frac{b}{x\Lambda}\right)\right)$$

$$\int_{\Lambda/b} \frac{\cdots}{q}\, dq = \left(\int_0 - \int_0\right) \frac{\cdots}{q}\, dq$$

$$= I(x,\Lambda) - I(x, \Lambda/b) \sim \left(\gamma + \log|x\Lambda| + \mathcal{O}\left(\frac{1}{x\Lambda}\right)\right) - \left(\gamma + \log|x\Lambda/b| + \mathcal{O}\left(\frac{b}{x\Lambda}\right)\right)$$

$$= \log b + \mathcal{O}\left(\frac{1}{x\Lambda}\right)$$

which implies that

$$\frac{1}{(2\pi)^2}\int_0^{2\pi} d\theta \int_{\Lambda/b}^{\Lambda} dq\, \frac{\cos(qr\cos\theta)-1}{Kq} \sim -\frac{1}{2\pi K}\log b = -\frac{l}{2\pi K} + \mathcal{O}(l^2)$$

Putting everything back together, and using $\mathcal{Z}(K,u) = \underset{\text{Det}[z]}{\mathcal{N}}\, \mathcal{Z}(K',u')$, we have

$$G(r,K,u) \underset{K \gg u}{\simeq} \frac{1}{\mathcal{N}\mathcal{Z}(K',u')}\int D[\phi_<(q)] \exp\left\{-\beta H_0[\phi_<](K,u) + i\int^{\Lambda/b} \frac{d^2q}{(2\pi)^2}\left(e^{iq\cdot r}-1\right)\phi_<(q)\right\} e^{-l/2\pi K - \beta H_{eff}[\phi_<]}$$

$$= e^{-l/2\pi K}\frac{1}{\mathcal{N}\mathcal{Z}(K',u')}\int D[\phi_<(q)] \exp\left\{-\beta H_0[\phi_<](\tilde{K},\tilde{u}) + i\int^{\Lambda/b} \frac{d^2q}{(2\pi)^2}\left(e^{iq\cdot r}-1\right)\phi_<(q)\right\}$$

rescaling $q' = bq$, $\phi'(q') = \phi_<(q)/z$

$$= e^{-l/2\pi K}\frac{1}{\mathcal{N}\mathcal{Z}(K',u')}\underbrace{\text{Det}[z]}_{\mathcal{N}}\int D[\phi(q)] \exp\left\{-\beta H_0[\phi](K',u') + i\int^{\Lambda} \frac{d^2q'}{(2\pi)^2}\left(e^{iq'\cdot r}-1\right)\phi'(q')\, \underset{z=b^2}{z}\right\}$$

$$= e^{-l/2\pi K}\frac{1}{\mathcal{Z}(K',u')}\int D[\phi(q)] \exp\left\{-\beta H_0[\phi](K',u') + i\int^{\Lambda} \frac{d^2q}{(2\pi)^2}\left(e^{iq\cdot \underset{r'=r/b}{r'}}-1\right)\phi(q)\right\}$$

$$= e^{-l/2\pi K}\, G(r'(l), K'(l), u'(l)) \quad \text{as required.}$$

for l infinitesimal

(e) Now, consider a series of infinitesimal RG flows from $l = 0$ to $l = \ln\frac{r_0}{r}$, starting at the point (K_0, u_0) and ending at the point $\left(K\left(\ln\frac{r_0}{r}\right), u\left(\ln\frac{r_0}{r}\right)\right)$, to show that

$$G(r_0, K_0, u_0) = \exp\left(-\int_0^{\ln(r_0/r)} \frac{dl}{2\pi K(l)}\right) G\left(r, K\left(\ln\frac{r_0}{r}\right), u\left(\ln\frac{r_0}{r}\right)\right). \qquad (2)$$

Iterating the infinitesimal RG step:

$$G(r_0, K_0, u_0) = e^{-\frac{\delta l}{2\pi K(0)}}\, G(r(\delta l), K(\delta l), u(\delta l))$$

$$= e^{-\frac{\delta l}{2\pi K(0)}} e^{-\frac{\delta l}{2\pi K(\delta l)}}\, G(r(2\delta l), K(2\delta l), u(2\delta l))$$

$$= \cdots = e^{-\frac{\delta l}{2\pi K(0)}} e^{-\frac{\delta l}{2\pi K(\delta l)}} \cdots e^{-\frac{\delta l}{2\pi K(l_*)}}\, G(r(l_*), K(l_*), u(l_*))$$

$$\underset{\delta l \to 0}{=} \exp\left\{-\int_0^{l_*}\frac{dl}{2\pi K(l)}\right\} G(r(l_*), K(l_*), u(l_*))$$

$$r = e^{-l_*} r_0 \quad \Rightarrow \quad l_* = \ln(r/r_0) \quad ; \quad r(l_*) = r.$$

(f) Hence, show that the asymptotic limit of the correlator is given by

$$G(r_0, K_0, u_0) \overset{r_0/r \to \infty}{=} (1 + \mathcal{O}(u_0))\frac{1}{(r_0/a)^{\frac{1}{2\pi K_*}}},$$

i.e. the long-distance physics is given by the quadratic theory, but with a renormalised coupling constant $K_* = \sqrt{K_0^2 + 4u_0\Lambda^2/\pi}$.

Consider two flows:

i) $$G(r_0, K_0, u_0) = \exp\left\{-\int_0^{l_*}\frac{dl}{2\pi K(l)}\right\} G(r(l_*), K(l_*), u(l_*))$$

ii) $$G(r_0, K_*, 0) = \frac{1}{(r_0/a)^{1/2\pi K_*}} = \exp\left\{-\int_0^{l_*}\frac{dl}{2\pi K_*}\right\} G(r(l_*), K_*, 0)$$

$dK/dl \propto u = 0$

$$\Rightarrow G(r_0, K_0, u_0)\cdot (r_0/a)^{\frac{1}{2\pi K_*}} = \exp\left\{-\int_0^{l_*}\left[\frac{1}{2\pi K(l)} - \frac{1}{2\pi K_*}\right]dl\right\} \underbrace{\frac{G(r(l_*), K(l_*), u(l_*))}{G(r(l_*), K_*, 0)}}_{\to 1 \text{ as } l \to \infty}$$

Thus, taking $l_* \to \infty$ (equivalent to $r_0/r \to \infty$)

$$G(r_0, K_0, u_0) = \exp\left\{-\int_0^{l_*}\left[\frac{1}{2\pi K(l)} - \frac{1}{2\pi K_*}\right]dl\right\}\frac{1}{(r_0/a)^{1/2\pi K_*}}$$

The statement of the problem now follows from the fact that $K(\ell)$ varies between K_0 and $K_* = K_0 + \mathcal{O}(u_0)$, therefore, the whole integral to be exponentiated is $\mathcal{O}(u_0)$. That is, the quadratic theory predicting power-law decay of spin correlations is also valid to the next order of the low-temperature expansion. On the other hand, the original XY model has a Kosterlitz–Thouless type phase transition which destroys these correlations: this is, however, a nonperturbative effect due to quantised vortices which would not be captured by a coarse-grained theory like this.

Note. Given the RG flow (3.12) of K, it is relatively easy to evaluate the integral in (3.20). Defining $w = 4\Lambda^2 u/\pi$ and noting that $\mathrm{d}\ell = -\mathrm{d}u/2u = -\mathrm{d}w/2w$, we have

$$\int_0^\infty \mathrm{d}\ell \left(\frac{1}{K(\ell)} - \frac{1}{K_*}\right) = \int_0^{w_0} \frac{\mathrm{d}w}{2w} \left(\frac{1}{\sqrt{K_*^2 - w}} - \frac{1}{K_*}\right) = -\frac{1}{K_*} \log\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{w_0}{K_*^2}}\right), \tag{3.21}$$

and so (3.20) becomes (note that $K_*^2 = K_0^2 + w_0$)

$$G(r_0, K_0, u_0) = \left(\frac{K_0 + K_*}{2K_*} \frac{1}{r_0/a}\right)^{1/2\pi K_*} \tag{3.22}$$

RG of the O(n) model

16 February 2024 15:21

2. Using Wilson's perturbative renormalisation group, the aim of this problem is to obtain the second-order $\epsilon = 4-d$ expansion of the Ginzburg-Landau functional

$$\beta H = \int d\mathbf{x}\left[\frac{t}{2}\mathbf{m}^2 + \frac{K}{2}(\nabla\mathbf{m})^2 + u(\mathbf{m}^2)^2\right],$$

where $\mathbf{m}$ denotes an n-component field.

(a) Treating the quartic interaction as a perturbation, show that an application of the momentum shell RG generates a Hamiltonian of the form

$$\beta H[\mathbf{m}_<] = \int_0^{\Lambda/b}(d\mathbf{q})\frac{G^{-1}(\mathbf{q})}{2}|\mathbf{m}_<(\mathbf{q})|^2 - \ln\langle e^{-U}\rangle_{\mathbf{m}_>}, \qquad G^{-1}(\mathbf{q}) = t + K\mathbf{q}^2,$$

where we have used the shorthand $(d\mathbf{q}) \equiv d\mathbf{q}/(2\pi)^d$.

This is exactly the same as part (a) of Perturbative RG.

(b) Expressing the interaction in terms of the Fourier modes of the Gaussian Hamiltonian, represent *diagrammatically* those contributions from the second order of the cummulant expansion. [Remember that the cummulant expansion involves only those diagrams which are *connected*.]

This is again very similar to part (a) of Perturbative RG.

To find all possible diagrams we must first determine the Feynman rules by expanding $U(\mathbf{m}_< + \mathbf{m}_>)$. Firstly we have

$$\beta U[\mathbf{m}] = \int d^d\mathbf{x}\; u(\mathbf{m}^2)^2 = u\int d^d\mathbf{x}\; \underbrace{m_\alpha m_\alpha m_\beta m_\beta}_{\alpha,\beta=1,\dots,N} = u\int\prod_{i=1}^{4}\left[\frac{d^d\mathbf{q}_i}{(2\pi)^d}\right](2\pi)^d\delta(\mathbf{q}_1+\dots+\mathbf{q}_4)\, m_\alpha(\mathbf{q}_1)m_\alpha(\mathbf{q}_2)m_\beta(\mathbf{q}_3)m_\beta(\mathbf{q}_4)$$

so that

$$\beta U[\mathbf{m}_< + \mathbf{m}_>] = u\int\prod_{i=1}^{4}\left[\frac{d^d\mathbf{q}_i}{(2\pi)^d}\right](2\pi)^d\delta(\mathbf{q}_1+\dots+\mathbf{q}_4)\left[m_<^\alpha(\mathbf{q}_1)+m_>^\alpha(\mathbf{q}_1)\right]\left[m_<^\alpha(\mathbf{q}_2)+m_>^\alpha(\mathbf{q}_2)\right]\left[m_<^\beta(\mathbf{q}_3)+m_>^\beta(\mathbf{q}_3)\right]\left[m_<^\beta(\mathbf{q}_4)+m_>^\beta(\mathbf{q}_4)\right]$$

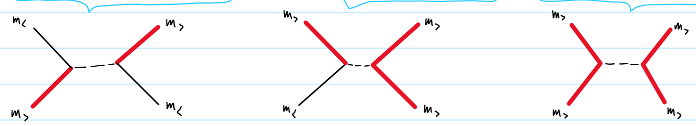
Where we previously had to be careful about pairing momenta, we now need to be careful about pairing indices.

Convention: contract the indices of neighbouring legs.

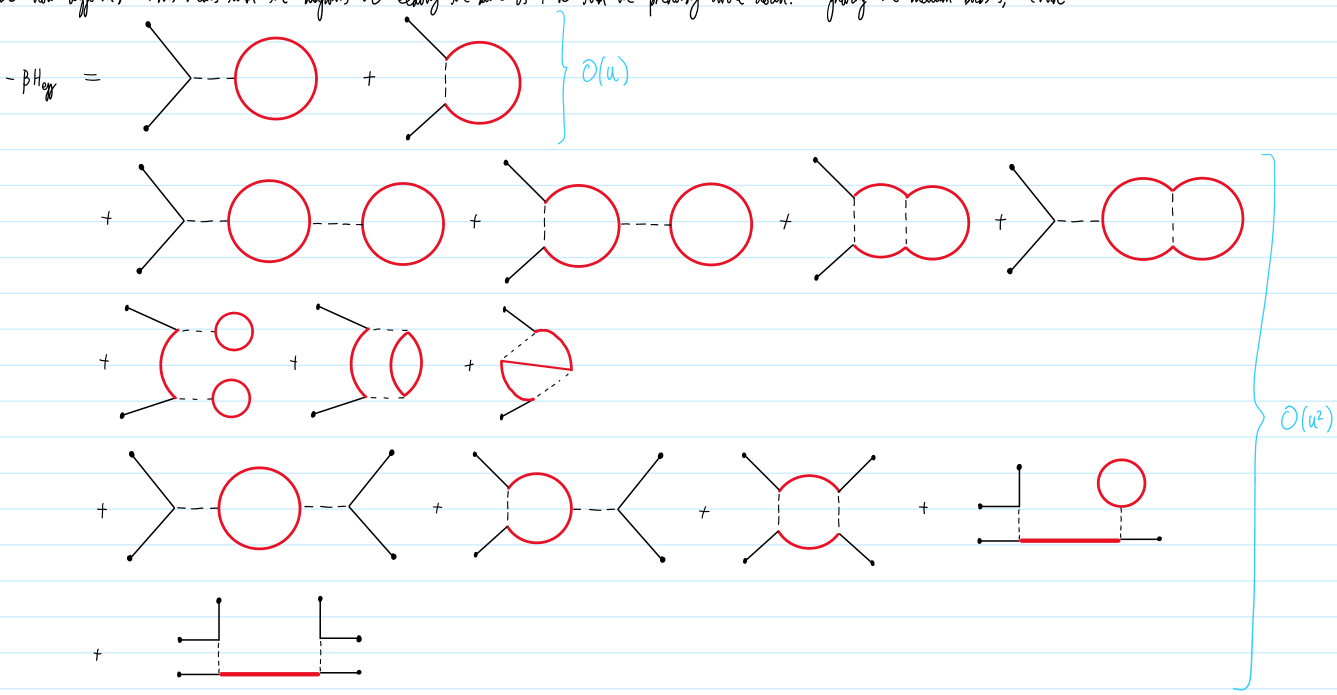
$\beta U[\mathbf{m}_<]$ ⇝ drops out of $\mathbf{m}_>$ integration

$$= u\int\prod_{i=1}^{4}\left[\frac{d^d\mathbf{q}_i}{(2\pi)^d}\right](2\pi)^d\delta(\mathbf{q}_1+\dots+\mathbf{q}_4)\Big[m_<^\alpha(\mathbf{q}_1)m_<^\alpha(\mathbf{q}_2)m_<^\beta(\mathbf{q}_3)m_<^\beta(\mathbf{q}_4) + 4\,m_<^\alpha(\mathbf{q}_1)m_<^\alpha(\mathbf{q}_2)m_<^\beta(\mathbf{q}_3)m_>^\beta(\mathbf{q}_4) + 2\,m_<^\alpha(\mathbf{q}_1)m_<^\alpha(\mathbf{q}_2)m_>^\beta(\mathbf{q}_3)m_>^\beta(\mathbf{q}_4)$$

$$+4\,m_<^\alpha(\mathbf{q}_1)m_<^\beta(\mathbf{q}_2)m_>^\alpha(\mathbf{q}_3)m_>^\beta(\mathbf{q}_4) + 4\,m_<^\alpha(\mathbf{q}_1)m_>^\alpha(\mathbf{q}_2)m_>^\beta(\mathbf{q}_3)m_>^\beta(\mathbf{q}_4) + m_>^\alpha(\mathbf{q}_1)m_>^\alpha(\mathbf{q}_2)m_>^\beta(\mathbf{q}_3)m_>^\beta(\mathbf{q}_4)\Big]$$



The structure of the vertices is identical to that from problem Perturbative RG (with the caveat that the Feynman rules corresponding to each diagram are now different). This means that the diagrams are exactly the same as those that we previously wrote down. Ignoring the vacuum bubbles, we have



+ 

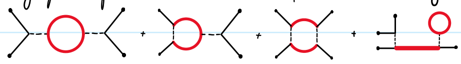
$+\ \mathcal{O}(u^3)$

(c) *Focusing only on those second order contributions that renormalise the quartic interaction,* show that the renormalised coefficient u takes the form

$$\tilde{u} \;=\; u - 4u^2(n+8)\int_{\Lambda/b}^{\Lambda}(d\mathbf{q})G(\mathbf{q})^2.$$

Comment on the nature of those additional terms generated at second-order.

At $\mathcal{O}(u^2)$, precisely four diagrams contribute to the 4-point vertex, namely

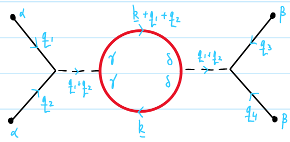
 .

Let us now each of these diagrams one by one. In all that follows we will use the shorthand notations

$$\int_{4<} \equiv \int \prod_{i=1}^{4}\left[\frac{d^d q_i}{(2\pi)^d}\right](2\pi)^d\delta(q_1+\dots+q_4), \qquad \int_{k} \equiv \int\frac{d^d k}{(2\pi)^d}$$

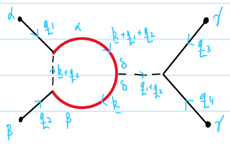
and make use of the Einstein summation convention throughout.

$$G_>^{\gamma\delta}(q) = \delta^{\gamma\delta}G_>(q)$$

i) 

$$= 4\,(-u)^2\int_{4<}\int_{k} m_<^{\alpha}(q_1)\,m_<^{\alpha}(q_2)\,G_>^{\gamma\delta}(k+q_1+q_2)\,G_>^{\delta\gamma}(k)\,m_<^{\beta}(q_3)\,m_<^{\beta}(q_4)$$

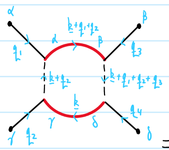
$$= 4u^2\int_{4<} m_<^{\alpha}(q_1)\,m_<^{\alpha}(q_2)\,m_<^{\beta}(q_3)\,m_<^{\beta}(q_4)\;\underbrace{\delta^{\gamma\delta}\delta^{\delta\gamma}}_{\delta^{\gamma\gamma}=n}\int_{k} G_>(k+q_1+q_2)\,G_>(k)$$

ii) 

$$= 16\,(-u)^2\int_{4<}\int_{k} m_<^{\alpha}(q_1)\,m_<^{\beta}(q_2)\,G_>^{\alpha\delta}(k+q_1+q_2)\,G_>^{\delta\beta}(k)\,m_<^{\gamma}(q_3)\,m_<^{\gamma}(q_4)$$

$$= 16u^2\int_{4<} m_<^{\alpha}(q_1)\,m_<^{\beta}(q_2)\,m_<^{\gamma}(q_3)\,m_<^{\gamma}(q_4)\;\delta^{\alpha\delta}\delta^{\delta\beta}\int_{k} G_>(k+q_1+q_2)\,G_>(k)$$

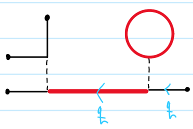
$$= 16u^2\int_{4<} m_<^{\alpha}(q_1)\,m_<^{\alpha}(q_2)\,m_<^{\beta}(q_3)\,m_<^{\beta}(q_4)\int_{k} G_>(k+q_1+q_2)\,G_>(k)$$

iii) 

$$= 16\,(-u)^2\int_{4<}\int_{k} m_<^{\alpha}(q_1)\,m_<^{\gamma}(q_2)\,G_>^{\alpha\beta}(k+q_1+q_2)\,G_>^{\gamma\delta}(k)\,m_<^{\beta}(q_3)\,m_<^{\delta}(q_4)$$

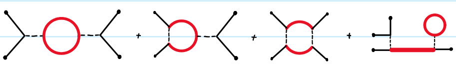
$$= 16u^2\int_{4<} m_<^{\alpha}(q_1)\,m_<^{\gamma}(q_2)\,m_<^{\beta}(q_3)\,m_<^{\delta}(q_4)\;\delta^{\alpha\beta}\delta^{\gamma\delta}\int_{k} G_>(k+q_1+q_2)\,G_>(k)$$

$$= 16u^2\int_{4<} m_<^{\alpha}(q_1)\,m_<^{\alpha}(q_2)\,m_<^{\beta}(q_3)\,m_<^{\beta}(q_4)\int_{k} G_>(k+q_1+q_2)\,G_>(k)$$

iv) 

$$\propto\ G_>(q_<) \;=\; 0$$

Thus, we see that



$$= 4u^2(n+8)\int_{4<} m_<^{\alpha}(q_1)\,m_<^{\alpha}(q_2)\,m_<^{\beta}(q_3)\,m_<^{\beta}(q_4)\int_{k} G_>(k+q_1+q_2)\,G_>(k)$$

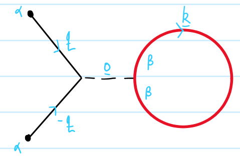
which implies that

$$\tilde{u} = u - 4u^2(n+8)\int_{k} G_>(k+q_1+q_2)\,G_>(k).$$

(d) Applying the rescaling $\mathbf{q} = \mathbf{q}'/b$, performing the renormalisation $\mathbf{m}_< = z\mathbf{m}$, and arranging that $K' = K$, show that the differential recursion relations take the form ($b = e^\ell$)

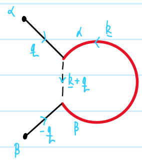
$$\begin{aligned} \frac{dt}{d\ell} &= 2t + 4u(n+2)G(\Lambda)K_d\Lambda^d - u^2 A(\mathbf{q}=0), \\ \frac{du}{d\ell} &= (4-d)u - 4(n+8)u^2 G(\Lambda)^2 K_d \Lambda^d. \end{aligned}$$

We must now evaluate the Feynman diagrams at $\mathcal{O}(u)$ and $\mathcal{O}(u^2)$ for the two-point vertex.

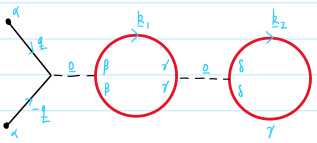
i) 

$$= 2(-u)\int_{q,k} m_<^\alpha(q)\, m_<^\alpha(-q)\, G_>^{\beta\beta}(k)$$

$$= -2nu \int_q m^\alpha(q)\, m^\alpha(-q) \int_k G_>(k)$$

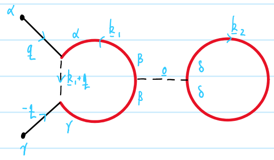
ii) 

$$= 4(-u)\int_{q,k} m_<^\alpha(q)\, m_<^\beta(-q)\, G_>^{\alpha\beta}(k) = -4u\int_q m^\alpha(q)\, m^\alpha(-q)\int_k G_>(k)$$

iii) 

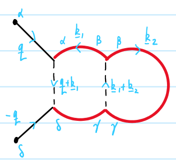
$$= 16(-u)^2\int_{q,k_1,k_2} m_<^\alpha(q)\, m_<^\alpha(-q)\, G_>^{\beta\gamma}(k_1)\, G_>^{\gamma\beta}(k_1)\, G_>^{\delta\delta}(k_2)$$

$$= 16nu^2\int_q m_<^\alpha(q)\, m_<^\alpha(-q)\int_{k_1} G_>(k_1)^2\int_{k_2} G_>(k_2)$$

iv) 

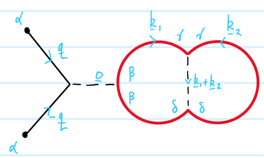
$$= 32(-u)^2\int_{q,k_1,k_2} m_<^\alpha(q)\, m_<^\gamma(-q)\, G_>^{\alpha\beta}(k_1)\, G_>^{\beta\gamma}(k_1)\, G_>^{\delta\delta}(k_2)$$

$$= 32nu^2\int_q m_<^\alpha(q)\, m_<^\alpha(-q)\int_{k_1} G_>(k_1)^2\int_{k_2} G_>(k_2)$$

v) 

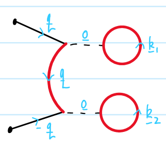
$$= 64(-u)^2\int_{q,k_1,k_2} m_<^\alpha(q)\, m_<^\delta(-q)\, G_>^{\alpha\beta}(k_1)\, G_>^{\gamma\delta}(k_1)\, G_>^{\beta\gamma}(k_2)$$

$$= 64u^2\int_q m_<^\alpha(q_1)\, m_<^\alpha(q_2)\int_{k_1} G_>(k_1)^2\int_{k_2} G_>(k_2)$$

vi) 

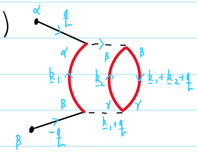
$$= 32(-u)^2\int_{q,k_1,k_2} m_<^\alpha(q)\, m_<^\alpha(-q)\, G_>^{\beta\gamma}(k_1)\, G_>^{\delta\beta}(k_1)\, G_>^{\gamma\delta}(k_2)$$

$$= 32u^2 n\int_q m_<^\alpha(q)\, m_<^\alpha(-q)\int_{k_1} G_>(k_1)^2\int_{k_2} G_>(k_2)$$

vii) 

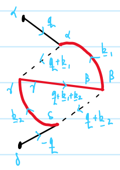
$$\propto G_>(q_<) = 0$$

viii) 

viii) 

$$= 32(-u)^2 \int_{q,k_1,k_2} m^\alpha_<(q)\, m^\beta_<(-q)\, G^{\alpha\beta}_>(k_1)\, G^{\beta\gamma}_>(k_1+k_2+q)\, G^{\gamma\beta}_>(k_2)$$

$$= 32\, n\, u^2 \int_q m^\alpha_<(q)\, m^\alpha_<(-q) \int_{k_1,k_2} G_>(k_1)\, G_>(q+k_1+k_2)\, G_>(k_2)$$

ix) 

$$= 64(-u)^2 \int_{q,k_1,k_2} m^\alpha(q)\, m^\delta(-q)\, G^{\beta\alpha}_>(k_1)\, G^{\gamma\beta}_>(q+k_1+k_2)\, G^{\delta\gamma}_>(k_2)$$

$$= 64u^2 \int_q m^\alpha(q)\, m^\alpha(-q) \int_{k_1,k_2} G_>(k_1)\, G_>(q+k_1+k_2)\, G_>(k_2)$$

Overall, we see that the sum of all diagrams is given by

$$-\beta H_{\text{eff}}[m_<]\Big|_{m^2} = \int_q m^\alpha_<(q)\, m^\alpha_<(-q) \Big[-(2n+4)u\, I_1 + (16n^2+32n+64+32n)u^2 I_1 I_2 + (32n+64)u^2 I_3(q) \Big] + \mathcal{O}(u^3)$$

where

$$I_1 = \int_k G_>(k), \qquad I_2 = \int_k G_>(k)^2, \qquad I_3(q) = \int_{k_1,k_2} G_>(k_1)\, G_>(k_1+k_2+q)\, G_>(k_2).$$

Thus

$$\frac{\tilde{t}}{2} + \frac{\tilde{K}}{2} q^2 = \frac{t}{2} + \frac{K}{2} q^2 + 2u(n+2)\, I_1 - 16u^2(n+2)^2 I_1 I_2 - 32u^2(n+2)\, I_3(q) + \mathcal{O}(u^3)$$

$$\equiv \frac{t}{2} + \frac{K}{2} q^2 + 2u(n+2)\, I_1 - \frac{1}{2}u^2 A(q)$$

$$= \frac{1}{2}\Big[t + 4u(n+2)\, I_1 - u^2 A(0)\Big] + \frac{1}{2}\Big[K - \frac{u^2}{2} \nabla^2_k A(k)\Big|_{k=0}\Big] q^2 + \mathcal{O}(q^4)$$

Note that $A(q)$ contains only even powers of q in its Taylor expansion since $G_>(k) \sim k^2$

where

$$2A(q) = 16(n+2)^2 I_1 I_2 + 32(n+2)\, I_3(q).$$

Thus

$$\tilde{t} = t + 4u(n+2)\, I_1 - u^2 A(0),$$

$$\tilde{K} = K - \frac{u^2}{2} \nabla^2_k A(k)\Big|_{k=0} \equiv K - \frac{u^2}{2} B$$

We can now perform the usual RG transformations (analogous to the earlier part (a) in Perturbative RG).

$$K' = \frac{1}{b^d} \cdot \underbrace{\frac{1}{b^2}}_{\text{from } q^2} \underbrace{z^2}_{\text{from } |m|^2} \tilde{K} = \frac{z^2}{b^{d+2}}\Big(K - \frac{u^2}{2}B\Big) \overset{!}{=} K \quad\Rightarrow\quad z^2 = \frac{b^{d+2}}{1 - \frac{u^2}{2K}B} = b^{d+2} + \mathcal{O}(\epsilon^2)$$

We shall later see that $u = \mathcal{O}(\epsilon)$ in the region of interest

$$t' = \underbrace{\frac{1}{b^d}}_{\text{from } d^d m} \underbrace{z^2}_{\text{from } |m|^2} \tilde{t} = b^2\Big[t + 4u(n+2)\, I_1 - u^2 A(0)\Big] + \mathcal{O}(\epsilon^2, u^3),$$

$$u' = \frac{1}{b^{3d}} z^4 \tilde{u} = b^{4-d}\Big[u - 4u^2(n+8)\, I_2\Big] + \mathcal{O}(\epsilon^2, u^3).$$

$$I_1 = \int_k G_>(k) = \int \frac{d^d k}{(2\pi)^d}\, \Theta_>(k)\, G(k) = \frac{S_d}{(2\pi)^d} \int_{\Lambda/b}^{\Lambda} k^{d-1}\, dk\, G(k) = \frac{S_d}{(2\pi)^d} \Lambda^{d-1} G(\Lambda) + \mathcal{O}(\ell^2)$$

See official solutions for the rest of the problem