The dynamics of the Universe

Relativistic Astrophysics and Cosmology: Lecture 17

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Pre-lecture question:

Why is radiation sub-dominant to matter today?

Last time

- The geometry of the Universe: The Friedmann-Robertson-Walker metric.
- The expansion of the Universe and redshift relation.

This lecture

- The dynamics of the universe.
- Physical derivation of the cosmological field equations.
- Types of cosmic fluid.

Next lecture

- The evolution of the universe: solutions to the cosmological equations.

The Friedmann-Robertson-Walker metric

- Last time we covered the FRW metric, which in its final (t, χ, θ, ϕ) form is:

$$\boxed{ds^2 = c^2 dt^2 - R^2(t)\left\{d\chi^2 + S^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)\right\},} \quad S^2(\chi) = \begin{cases} \sin^2\chi & k = +1, \\ \chi^2 & k = 0, \\ \sinh^2\chi & k = -1. \end{cases}$$

- t and (χ, θ, ϕ) are the cosmic time and comoving coordinates of fundamental observers.
- Working radially without loss of generality, proper distance is $\boxed{\int \sqrt{g_{\chi\chi}} d\chi = R(t)\chi}$.
- The spacetime tells matter how to move, e.g. the redshift relation $\boxed{1 + z = \frac{R(t_0)}{R(t_1)}}$.
- Now we cover the second half of Wheeler's epithet: Matter tells spacetime how to curve.
- but first let's calculate the curvature of FRW...

- Working radially again, for a 2d metric of form: $ds^2 = g_{tt}dt^2 + g_{\chi\chi}d\chi^2$ then Gauss' Curvature Theorem (as given e.g. in Lecture 3) is that the curvature is equal to:

$$K(t,\chi) = \frac{1}{2g_{tt}g_{\chi\chi}}\left\{-\partial_\chi^2 g_{tt} - \partial_t^2 g_{\chi\chi} + \frac{\partial_t g_{tt}\partial_t g_{\chi\chi} + {\partial_\chi g_{tt}}^2}{2g_{tt}} + \frac{\partial_\chi g_{tt}\partial_\chi g_{\chi\chi} + {\partial_t g_{\chi\chi}}^2}{2g_{\chi\chi}}\right\}$$

- We can see the curvature calculation is relatively straightforward since $g_{tt} = c^2$, and $g_{\chi\chi} = -R^2(t)$, the first and third term are zero

$$\begin{aligned}\partial_t g_{\chi\chi} &= -2\dot{R}R \\ \partial_t^2 g_{\chi\chi} &= -2\ddot{R}R - 2\dot{R}^2\end{aligned} \quad \Rightarrow \quad K(t,\chi) = \frac{1}{-2c^2R^2}\left\{2\ddot{R}R + 2\dot{R}^2 - \frac{\left(2R\dot{R}\right)^2}{2R^2}\right\} = -\frac{1}{c^2}\frac{\ddot{R}}{R}.$$

- The curvature of the Friedmann-Robertson-Walker spacetime is therefore

$$\boxed{c^2K(t) = -\frac{\ddot{R}}{R}.}$$

Cosmic matter

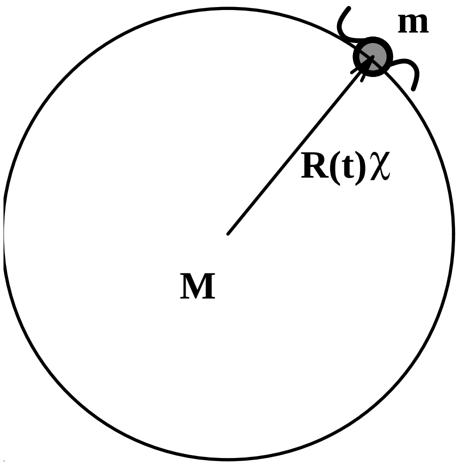
- By the cosmological principle, we assume that the universe is filled homogeneously and isotropically with material, which we model on the largest scales as a perfect fluid.
- A perfect fluid has stress-energy tensor

$$T^{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) u^\mu u^\nu - P g^{\mu\nu}.$$

- In general ρ, P & u^μ depend on space and time, but by the cosmological principle we note
  - Homogeneity implies $\rho(t)$, $P(t)$, $u^\mu(t)$ depend only on time.
  - Isotropy implies that spatial components of the four velocity $u^i = 0$.
- The universe is composed of a combination fluid components, which interact with each other to varying degrees, e.g. baryons, photons, neutrinos, dark matter, etc.
- More on this in Lecture 20.

Deriving the dynamical equations for R

- From $Kc^2 = -\frac{\ddot{R}}{R}$ the spacetime is positively curved if the expansion of the universe is decelerating.
- This corresponds to the gravitating effect of the material in the universe.
- Another possibility, that empty space could be curved, is what happens when we have Λ as well (will discuss).
- We can find what this curvature should be, in terms of the effects of matter, by considering following simple setup.
- Consider a comoving volume of the cosmological fluid encompassed by a small sphere of proper radius $R\chi$ with mass M, and a test particle/galaxy with mass m.
- Note "small" here means small enough that we can just use Euclidean notions in calculating its volume.



The acceleration equation

- In Newtonian gravity, it is only the portion of the fluid interior to the sphere which has any effect on a galaxy, of mass m say, at its boundary.
- Taking the galaxy as comoving, so that it moves with R, we deduce from Gauss' law that:

$$\left[m\ddot{r} = -\frac{GMm}{r^2}\right]_{r=R(t)\chi} \Rightarrow m\ddot{R}\chi = -\frac{GMm}{(R\chi)^2}.$$

- The relevant mass M is the mass of the fluid in the sphere, so $M = (4/3)\pi R^3\chi^3\rho$ and so

$$M = \frac{4}{3}\pi R^3 \rho \chi^3 \quad \Rightarrow \quad \frac{\ddot{R}}{R} = -\frac{4\pi G\rho}{3}. \tag{1}$$

- We found just now that the Gaussian curvature of the 2d (t, χ) surface was

$$K(t,\chi) = K(t) = -\frac{1}{c^2}\frac{\ddot{R}}{R}, \quad \text{which means} \quad K(t) = \frac{4\pi G\rho}{3c^2}.$$

which is a 'continuous' version of our idea that curvature depends linearly on mass.

What about pressure?

- But the Newtonian-type argument ignores the effects of pressure.
- We can deal with this by going back to the Oppenheimer-Volkov equation in Lecture 4.
- For the constant density ρ case, we found $ds^2 = A(r)dt^2 - B(r)dr^2 - r^2 d\Omega$, with

$$A(r) = \frac{c^2}{4}\left[3\left(1-\frac{2\mu}{R}\right)^{1/2} - \left(1-\frac{2\mu r^2}{R^3}\right)^{1/2}\right]^2, \quad B(r) = \left(1-\frac{2\mu r^2}{R^3}\right)^{-1},$$

$$P(r) = \rho c^2 \frac{\left(1-\frac{2\mu r^2}{R^3}\right)^{1/2} - \left(1-\frac{2\mu}{R}\right)^{1/2}}{3\left(1-\frac{2\mu}{R}\right)^{1/2} - \left(1-\frac{2\mu r^2}{R^3}\right)^{1/2}}.$$

- From the Theorema Egregium again, this gives a spacetime curvature of

$$K = \frac{2\mu\left(1-\frac{2r^2\mu}{R^3}\right)^{1/2}}{R^3\left(3\left(1-\frac{2\mu}{R}\right)^{1/2} - \left(1-\frac{2\mu r^2}{R^3}\right)^{1/2}\right)} = \frac{\mu}{R^3\rho}\left(\rho + \frac{3P}{c^2}\right) = \boxed{\frac{4\pi G}{3c^2}\left(\rho + \frac{3P}{c^2}\right)}.$$

- So this tells us how to generalise (1) to include pressure, and we find

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\left(\rho + \boxed{\frac{3P}{c^2}}\right).$$

- In just the same way as we said empty space could have a constant spatial curvature earlier (when deriving the Schwarzschild de Sitter solution in Lecture 5), here we include a constant spacetime curvature and write

$$Kc^2 = -\frac{\ddot{R}}{R} = \frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) - \boxed{\frac{\Lambda c^2}{3}}.$$

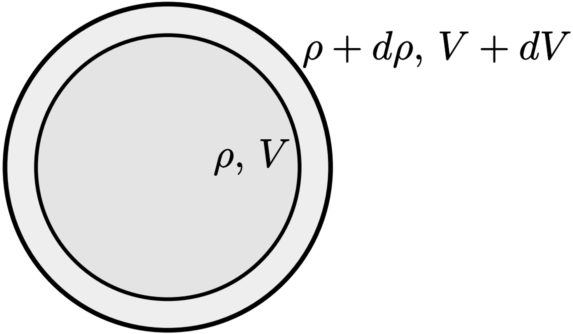
- The final equality of this represents our first dynamical field equation, linking the acceleration of the scale factor with the pressure and matter content, which we will call the "acceleration equation".

The continuity equation

- We have three dynamical quantities in our acceleration equation: R, ρ and P, but so far only one equation, so our system is undetermined.
- What physics can give us a further equation?
- An additional equation comes from considering energy conservation in a comoving sphere.

- Consider a small sphere of comoving radius χ and approximate proper volume $V = (4/3)\pi R^3\chi^3$.
- (This is exact for non-small χ in spatially flat cases) .
- As it expands, conservation of energy yields

$$\rho c^2 V = (\rho c^2 + d\rho c^2)(V + dV) + PdV.$$



- Equating terms at first order yields

$$d\rho c^2 V + \rho c^2 dV + PdV = 0,$$

which using the expression for V plus the fact that χ is constant for a comoving sphere, yields

$$d\rho c^2 R^3 + (\rho c^2 + P) 3R^2 dR = 0.$$

- Dividing through by $R^3 c^2 dt$

$$\dot{\rho} = -3\left(\rho + \frac{P}{c^2}\right)\frac{\dot{R}}{R}.$$

- This expression is very useful (particularly when we come to scalar fields in Lecture 22).
- However, it is in fact more useful to synthesise this with the acceleration equation to form an additional equation without time derivatives of ρ.

The energy equation (or velocity equation)

- It is essentially the 'first integral' of the acceleration equation, and we can quickly derive it this way, with help from the continuity equation.
- Start with the acceleration equation, and multiply by $2R\dot{R}$

$$2\dot{R}\ddot{R} + \frac{8\pi G}{3} R\dot{R}\left(\rho + \frac{3P}{c^2}\right) - \frac{2\Lambda c^2 R\dot{R}}{3} = 0.$$

- We can integrate the first and third terms using $x\dot{x} = \frac{1}{2}\frac{d}{dt}(x^2)$ for $x = R$ and $x = \dot{R}$.
- For the middle term we integrate the ρ portion by parts and use the continuity equation to eliminate $\dot{\rho}$ before rearranging the recursive expression.

$$\int R\dot{R}\rho dt = \rho\frac{R^2}{2} - \int \frac{R^2}{2}\dot{\rho} dt = \rho\frac{R^2}{2} + \int \frac{3}{2}\left(\rho + \frac{P}{c^2}\right) R\dot{R} dt \quad \Rightarrow \quad \int R\dot{R}\left(\rho + \frac{3P}{c^2}\right) dt = -\rho R^2.$$

- Integrating the acceleration equation therefore gives $\dot{R}^2 - \dfrac{8\pi G R^2 \rho}{3} - \dfrac{\Lambda c^2 R^2}{3} = C$, where C is a constant of integration.

- Let us rewrite this as
$$\frac{\dot{R}^2}{R^2} - \frac{8\pi G\rho}{3} - \frac{\Lambda c^2}{3} = \frac{C}{R^2}.$$
- What value should C have?
- We can see that the absolute scale of R drops out of both the acceleration and the continuity equations, as well as the l.h.s. of this equation.
- Thus we are in fact free to scale R in any way so as to make C simple.
- Let's choose $C = -kc^2$, with k being the spatial curvature index appearing in
$$K(t) = \frac{k}{R^2(t)} \quad \begin{cases} k = +1 & \text{if curvature positive,} \\ k = 0 & \text{if space is flat,} \\ k = -1 & \text{is curvature is negative.} \end{cases}$$
- This means the r.h.s. of our equation will be just minus the Gaussian spatial curvature expressed in units of $1/\text{time}^2$. Note this is not the same as the spacetime curvature.
- Note we can justify the sign of C in terms of as in previous parts of the course wanting positive curvature to go with positive ρ in the homogeneous case.

The cosmological equations

- We therefore have arrived at our Friedmann equations

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda c^2}{3}, \qquad \text{(acceleration)}$$

$$\dot{\rho} = -3\frac{\dot{R}}{R}\left(\rho + \frac{P}{c^2}\right), \qquad \text{(continuity)}$$

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi G\rho}{3} + \frac{\Lambda c^2}{3} - \frac{kc^2}{R^2}. \qquad \text{(velocity/energy)}$$

- Alternatively, get this link via the full GR derivation given in the Appendix to this handout.
- Note that there are only two independent equations here (any two can derive the third)
- We are still not quite in a position to solve these, since we have two independent equations in three dynamical variables (R, ρ, P)
- As with the Oppenheimer-Volkov equation, we can use an equation of state linking P and ρ to close the system.

The equation of state parameter for w-fluids

- Equation of states can take many forms (e.g. general barotropic $P = P(\rho)$).
- For cosmology, in many cases it suffices to consider a simple fluid parameterised by w

$$P = w\rho c^2 \qquad \Rightarrow \qquad \boxed{w = \frac{P}{\rho c^2}}.$$

- Energy density and pressure have the same units, so w is dimensionless.
- Matter can be modelled as pressureless dust $\boxed{w_m \approx 0}$.
- Radiation can be modelled as a gas of photons $\boxed{w_r = \frac{1}{3}}$.
- Mnemonically we can interpret the cosmological constant as a fluid with $\boxed{w_\Lambda = -1}$, and curvature as a fluid with $\boxed{w_k = -\frac{1}{3}}$.
- There are a couple of alternative parameterisations you may see in the literature:

  $\epsilon = 3w$ gives $\epsilon = 0, 1$ for matter and radiation respectively – previous versions of the course used this – watch out when practising past tripos questions.

  $c_s^2 = wc^2$ gives a sound speed from direct ratio P/ρ.

- For a universe dominated by a fluid with equation of state w, we therefore have

$$\frac{\ddot{R}}{R} = -\frac{4\pi G\rho}{3}(1+3w) + \frac{\Lambda c^2}{3}, \qquad \text{(acceleration)}$$

$$\dot{\rho} = -3\frac{\dot{R}}{R}\rho(1+w), \qquad \text{(continuity)}$$

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi G\rho}{3} + \frac{\Lambda c^2}{3} - \frac{kc^2}{R^2}. \qquad \text{(velocity/energy)}$$

- In many cases we can model the density instead as a set of noninteracting fluids $\rho = \sum_i \rho_i$ with differing equations of state w_i and $i \in \{r, m, k, \Lambda\}$.
- Taking the mnemonic definitions for curvature and dark energy we therefore have

$$\boxed{\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\sum_i \rho_i(1+3w_i) \quad \dot{\rho}_i = -3\frac{\dot{R}}{R}\rho_i(1+w_i) \quad \frac{\dot{R}^2}{R^2} = \frac{8\pi G}{3}\sum_i \rho_i.}$$

Continuity for w-fluids

- Note that for w-fluids we can immediately solve the continuity equation

$$\dot{\rho} = -3\frac{\dot{R}}{R}\rho\,(1+w) \quad \Rightarrow \quad \frac{d\rho}{\rho} = -3(1+w)\frac{dR}{R} \quad \Rightarrow \quad \boxed{\rho \propto R^{-3(1+w)}.}$$

- This immediately recovers the expected result for matter $\boxed{\rho_m = \rho_{m,0}\left(\frac{R}{R_0}\right)^{-3}}$.
- For matter, density decaying with the "volume" of the universe as the scale factor expands, i.e. number conservation in a comoving volume.
- For photons we have $\boxed{\rho_r = \rho_{r,0}\left(\frac{R}{R_0}\right)^{-4}}$.
- This photonic decay in energy density can be interpreted as number conservation (as for matter) combined with the energy of the photons dropping with frequency due to redshift.
- This explains why matter dominates over radiation today, but at early times radiation was much more important (as illustrated by the CMB).

Further Development of the Dynamical Equations

- It is useful to transform these equations into dimensionless form – not least because there are then fewer constants to remember!
- We define the Hubble and deceleration parameters

$$\boxed{H = \frac{\dot{R}}{R},} \qquad \boxed{q = -\frac{\ddot{R}R}{\dot{R}^2},} \qquad \boxed{\Omega = \frac{8\pi G\rho}{3H^2},} \qquad \boxed{\Omega_\Lambda = \frac{\Lambda c^2}{3H^2},} \qquad \boxed{\Omega_k = \frac{-kc^2}{R^2 H^2}.}$$

- Then the equations become

$$\begin{aligned}\frac{\dot{R}^2}{R^2} &= \frac{8\pi G\rho}{3} + \frac{\Lambda c^2}{3} - \frac{kc^2}{R^2},\\ \frac{\ddot{R}}{R} &= -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda c^2}{3}.\end{aligned} \qquad \Rightarrow \qquad \begin{aligned}1 &= \Omega + \Omega_\Lambda + \Omega_k,\\ q &= \frac{1}{2}(1+3w)\Omega - \Omega_\Lambda.\end{aligned}$$

- Note that in the same way we may write the density as a sum of noninteracting components $\rho = \rho_m + \rho_r + \ldots$, we may similarly write $\Omega = \Omega_m + \Omega_r + \ldots$ i.e.

$$1 = \Omega_r + \Omega_m + \Omega_\Lambda + \Omega_k, \qquad q = \Omega_r + \tfrac{1}{2}\Omega_m - \Omega_\Lambda.$$

- Ω is the ratio of the density to what's called the critical density $\rho_{\text{crit}} = 3H^2/8\pi G$. This is the solution when $k = \Lambda = 0$.
- Note $\Omega_\Lambda \equiv \Lambda c^2/(3H^2)$ may not look like it, but just like Ω it's a ratio of energy density to the critical density.
- This makes sense if we think of the cosmological constant as a dark energy fluid with $w = -1$ since as a fluid we will discuss later (when we get to considering scalar fields) $\rho_\Lambda = \Lambda c^2/8\pi G$.
- The universe is thus

$$\begin{array}{lll} \text{closed} & (+\text{ve curvature}) & \text{if } \Omega_m + \Omega_\Lambda > 1, \\ \text{flat} & (\text{zero curvature}) & \text{if } \Omega_m + \Omega_\Lambda = 1, \\ \text{open} & (\text{-ve curvature}) & \text{if } \Omega_m + \Omega_\Lambda < 1. \end{array}$$

- The q equation is interesting in telling us that the universe is accelerating if

$$\frac{\Omega_m}{2}(1 + 3w) < \Omega_\Lambda.$$

- Yet another way of writing the equations is to fix constants of proportionality by defining values today at t_0, so

$$\boxed{\left(\frac{H}{H_0}\right)^2 = \Omega_{r,0}\left(\frac{R}{R_0}\right)^{-4} + \Omega_{m,0}\left(\frac{R}{R_0}\right)^{-3} + \Omega_{k,0}\left(\frac{R}{R_0}\right)^{-2} + \Omega_{\Lambda,0}.}$$

- Somewhat confusingly, if it is "obvious" from context, cosmologists drop the subscript 0's indicating quantities evaluated today.
- e.g. if $\boxed{a = \frac{R}{R_0}}$ then under this convention, the system takes the compact form[1]

$$\left(\frac{H}{H_0}\right)^2 = \Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda.$$

- Note that nowadays some people use the notation a for the scale factor in place of R.
- Even older texts (e.g. the first papers in inflation) use S for "scale factor" in place of R!

[1]You may note that there is a a^{-1} missing – this can be interpreted as "double dark energy", but there is no evidence for this in present data [arxiv:1208.2542].

Summary

- Three (dependent) cosmological field equations

$$\boxed{\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda c^2}{3},} \quad \boxed{\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3} + \frac{\Lambda c^2}{3} - \frac{kc^2}{R^2},} \quad \boxed{\dot{\rho} = -3\frac{\dot{R}}{R}\left(\rho + \frac{P}{c^2}\right).}$$

- Equation of state $\boxed{P = w\rho c^2}$, with $\boxed{w_m \approx 0}$, $\boxed{w_r = \frac{1}{3}}$, $\boxed{w_\Lambda = -1}$, $\boxed{w_k = -\frac{1}{3}}$ gives:

$$\boxed{\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\sum_i \rho_i\,(1+3w_i)\,, \quad \dot{\rho}_i = -3\frac{\dot{R}}{R}\rho_i\,(1+w_i)\,, \quad \frac{\dot{R}^2}{R^2} = \frac{8\pi G}{3}\sum_i \rho_i, \quad i \in \{r, m, k, \Lambda\}.}$$

- $\boxed{\rho_w \propto R^{-3(1+w)}}$ so $\boxed{\rho_m \propto R^{-3}}$ and $\boxed{\rho_r \propto R^{-4}}$.
- Hubble $\boxed{H = \dot{R}/R}$ and deceleration $\boxed{q = -\ddot{R}R/\dot{R}^2}$ and $\boxed{\Omega = \frac{8\pi G\rho}{3H^2}}$ parameters.

Next time

Evolution of the universe – the solutions to these equations

Appendix: Full GR derivation of the cosmological field equations (non-examinable)

- Dynamics of the spacetime geometry characterised entirely through the scale factor $R(t)$.
- To determine the function $R(t)$, we must solve the gravitational field equations in the presence of matter and with a cosmological constant present:

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu},$$

- More convenient to express the field equations in the trace-reversed form

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \tfrac{1}{2} T g_{\mu\nu}\right) + \Lambda g_{\mu\nu}. \tag{2}$$

- To solve these equations, we clearly need a model for the energy-momentum tensor of the matter that fills the Universe.

- For simplicity assume a perfect fluid (Handout 4) with proper density ρ and the pressure P in the instantaneous rest frame.

$$T^{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) u^\mu u^\nu - P g^{\mu\nu}. \tag{3}$$

- Since we are seeking solutions for a homogeneous and isotropic Universe, the density ρ and pressure p must be functions of cosmic time t alone.
- We shall perform the calculation in the comoving coordinates $[x^\mu] = (t, r, \theta, \phi)$, in which the FRW metric takes the particular form

$$ds^2 = c^2\, dt^2 - R^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \right].$$

- **Note:** This is actually the 'σ' form of the metric, as given in Handout 15.
- We don't want to write σ everywhere though, so have written r in place of it.
- So note this r is the r of Handout 7 divided by the scale factor R.
- The covariant components $g_{\mu\nu}$ of the metric are

$$g_{tt} = c^2, \quad g_{rr} = -R^2(t)/(1 - kr^2), \quad g_{\theta\theta} = -R^2(t) r^2, \quad g_{\phi\phi} = -R^2(t) r^2 \sin^2\theta.$$

- Since the metric is diagonal, the contravariant components $g^{\mu\nu}$ are simply reciprocals of the covariant components.
- The connection is given in terms of the metric by

$$\Gamma^{\sigma}{}_{\mu\nu} = \tfrac{1}{2} g^{\sigma\rho}(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu}),$$

- from which it is straightforward to show that the only non-zero coefficients are

$$\begin{array}{lll}
\Gamma^{t}{}_{rr} = R\dot{R}/c^2(1-kr^2), & \Gamma^{t}{}_{\theta\theta} = R\dot{R}r^2/c^2, & \Gamma^{t}{}_{\phi\phi} = (R\dot{R}r^2\sin^2\theta)/c^2, \\
\Gamma^{r}{}_{tr} = \dot{R}/R, & \Gamma^{r}{}_{rr} = kr/(1-kr^2), & \\
\Gamma^{r}{}_{\theta\theta} = -r(1-kr^2), & \Gamma^{r}{}_{\phi\phi} = -r(1-kr^2)\sin^2\theta, & \\
\Gamma^{\theta}{}_{t\theta} = \dot{R}/R, & \Gamma^{\theta}{}_{r\theta} = 1/r, & \Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta, \\
\Gamma^{\phi}{}_{t\phi} = \dot{R}/R, & \Gamma^{\phi}{}_{r\phi} = 1/r, & \Gamma^{\phi}{}_{\theta\phi} = \cot\theta,
\end{array}$$

  where the dots denote differentiation with respect to cosmic time t.
- (Note there are some misprints in HEL at the point where these relations are given (p377).)

- We next substitute these expressions for the connection coefficients into the expression for the Ricci tensor,

$$R_{\mu\nu} = \partial_\nu \Gamma^\sigma{}_{\mu\sigma} - \partial_\sigma \Gamma^\sigma{}_{\mu\nu} + \Gamma^\rho{}_{\mu\sigma}\Gamma^\sigma{}_{\rho\nu} - \Gamma^\rho{}_{\mu\nu}\Gamma^\sigma{}_{\rho\sigma}.$$

- After some tedious but straightforward algebra, we find that the off-diagonal components of the Ricci tensor are zero, and the diagonal components are given by

$$\begin{aligned} R_{tt} &= 3\ddot{R}/R, \\ R_{rr} &= -(R\ddot{R} + 2\dot{R}^2 + 2c^2k)c^{-2}/(1-kr^2), \\ R_{\theta\theta} &= -(R\ddot{R} + 2\dot{R}^2 + 2c^2k)c^{-2}r^2, \\ R_{\phi\phi} &= -(R\ddot{R} + 2\dot{R}^2 + 2c^2k)c^{-2}r^2\sin^2\theta \end{aligned}$$

- We must now consider the RHS of the field equations. In comoving coordinates (t, r, θ, ϕ), the 4-velocity of the fluid is

$$[u^\mu] = (1, 0, 0, 0),$$

i.e. $u^\mu = \delta^\mu_0 \Rightarrow u_\mu = g_{\mu\nu}\delta^\nu_0 = g_{\mu 0} = c^2\delta^0_\mu$.

- So can write the energy-momentum tensor as

$$T_{\mu\nu} = (\rho c^2 + P)c^2\delta^0_\mu\delta^0_\nu - Pg_{\mu\nu}.$$

- Also, using the normalisation condition $u^\mu u_\mu = c^2$, the contraction of the energy-momentum tensor is

$$T = T^\mu_\mu = \left(\rho + \frac{p}{c^2}\right)c^2 - p\delta^\mu_\mu = \rho c^2 - 3p.$$

- Hence, we can write the terms on the RHS of the field that depend on the energy-momentum tensor, as

$$T_{\mu\nu} - \tfrac{1}{2}Tg_{\mu\nu} = (\rho c^2 + p)c^2\delta^0_\mu\delta^0_\nu - \tfrac{1}{2}(\rho c^2 - p)g_{\mu\nu}.$$

- Including the cosmological constant term, we find that the RHS of the field equations vanishes for $\mu \neq \nu$, and

$$
\begin{aligned}
-\frac{8\pi G}{c^4}(T_{tt} - \frac{1}{2}Tg_{tt}) + \Lambda g_{tt} &= -\frac{4\pi G}{c^4}(\rho c^2 + 3p)c^2 + \Lambda c^2, \\
-\frac{8\pi G}{c^4}(T_{rr} - \frac{1}{2}Tg_{rr}) + \Lambda g_{rr} &= -[\frac{4\pi G}{c^4}(\rho c^2 - p) + \Lambda]R^2/(1 - kr^2), \\
-\frac{8\pi G}{c^4}(T_{\theta\theta} - \frac{1}{2}Tg_{\theta\theta}) + \Lambda g_{\theta\theta} &= -[\frac{4\pi G}{c^4}(\rho c^2 - p) + \Lambda]R^2 r^2, \\
-\frac{8\pi G}{c^4}(T_{\phi\phi} - \frac{1}{2}Tg_{\phi\phi}) + \Lambda g_{\phi\phi} &= -[\frac{4\pi G}{c^4}(\rho c^2 - p) + \Lambda]R^2 r^2 \sin^2\theta.
\end{aligned}
$$

- Comparing these expressions with the Ricci tensor, we see that the three spatial field equations are equivalent, which is essentially due to the homogeneity and isotropy of the FRW metric.

- Thus the gravitational field equations yield just the two independent equations

$$\begin{aligned} 3\ddot{R}/R &= -\frac{4\pi G}{c^4}(\rho c^2 + 3p)c^2 + \Lambda c^2, \\ R\ddot{R} + 2\dot{R}^2 + 2c^2 k &= [\tfrac{4\pi G}{c^4}(\rho c^2 - p) + \Lambda]c^2 R^2. \end{aligned}$$

- Eliminating $\ddot{R}$ from the second equation, we finally arrive at the cosmological field equations

$$\begin{aligned} \ddot{R} &= -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)R + \tfrac{1}{3}\Lambda c^2 R, \\ \dot{R}^2 &= \frac{8\pi G}{3}\rho R^2 + \tfrac{1}{3}\Lambda c^2 R^2 - c^2 k. \end{aligned}$$

- These equations are two differential equations describing the time evolution of the scale factor $R(t)$ and are known as the Friedmann-Lemaitre equations. In the case where $\Lambda = 0$, they are often called simply the Friedmann equations.
- Note that differentiating the second equation w.r.t. time, and using the first equation for $\ddot{R}$, we can **deduce** the continuity equation for ρ given in the main text.