Relativistic Astrophysics and Cosmology — Answers 4 — 2023

1. In a de Sitter universe we take $k = 0$ and $\rho = 0$. The (B) equation thus becomes

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{\Lambda c^2}{3} = 0 \qquad (**)$$

telling us $H = \sqrt{\Lambda c^2/3}$. We therefore have

$$\chi(z) = \int_0^z \frac{dz'}{R_0\sqrt{\frac{\Lambda}{3}}} = \frac{z}{R_0}\sqrt{\frac{3}{\Lambda}}$$

which combined with the fact that for a flat universe $S(\chi) = \chi$, means that the result for d_θ yields:

$$d_\theta = \frac{z}{1+z}\sqrt{\frac{3}{\Lambda}}.$$

Since $d_\theta = D/\Delta\theta$ we see that $\Delta\theta \propto 1/z$ for small z, and tends to a constant for large z.

2. The relation (**) for $\dot{R}$ (see previous question) gives the solution

$$R(t) = R_0 \exp(\sqrt{(\Lambda/3)}c(t - t_0)).$$

Integrating c times the reciprocal of this from t_0 to $t_{\rm recep}$ must give the same χ as just found in the last question, since the χ coordinate of the object is constant. Carrying out the integration, this yields

$$\frac{z}{R_0}\sqrt{\frac{3}{\Lambda}} = -\frac{1}{R_0}\sqrt{\frac{3}{\Lambda}}\left\{\exp\left(-\sqrt{\frac{\Lambda}{3}}c(t_{\rm recep} - t_0)\right) - 1\right\}$$

i.e.

$$z = 1 - \exp\left(-\sqrt{\frac{\Lambda}{3}}c(t_{\rm recep} - t_0)\right)$$

Clearly this can only be solved for $z < 1$, in which case we have

$$ct_{\rm recep} = ct_0 - \sqrt{\frac{3}{\Lambda}}\ln(1-z),$$

as required.

3.

$$\chi_p = \int_{t_{min}}^{t_0} \frac{c\,dt}{R(t)} < \infty$$

implies a particle horizon exists.

In EdS, $R = R_0 \left(3tH_0/2\right)^{\frac{2}{3}}$ (and so $t_0 = 2/(3H_0)$) and thus performing integral $\chi_p = 2c/(R_0 H_0)$.

Present proper distance to particle horizon is $R(t_0)\chi_p$ and in EdS spatial geometry is Euclidean. Also, $\rho_0 = 3{H_0}^2/(8\pi G)$. Thus mass within present particle horizon is

$$\frac{4}{3}\pi {R_0}^3 {\chi_p}^3 \frac{3H_0^2}{8\pi G} = 6t_0 c^3 G^{-1}.$$

Note same analysis applies for any t. In particular, distance to the particle horizon at general time t is $R(t)\chi_p(t)$ and

$$\chi_p(t) = \frac{c}{R_0}\left(\frac{2}{3H_0}\right)^{\frac{2}{3}} \int_0^t \frac{dt'}{t'^{\frac{2}{3}}} = 3t^{\frac{1}{3}} \frac{c}{R_0}\left(\frac{2}{3H_0}\right)^{\frac{1}{3}} = \frac{3t^{\frac{1}{3}}}{R(t)} ct^{\frac{2}{3}}$$

(using $R = R_0 \left(\frac{3tH_0}{2}\right)^{\frac{2}{3}}$ again).

Thus, $R(t)\chi_p(t) = 3ct$. This equals 10^{-15}m ($\sim$ radius of proton), $\implies t = 1.1 \times 10^{-24}\,\mathrm{s} \implies$ mass within particle horizon is $6c^3tG^{-1} = 2.7 \times 10^{12}\,\mathrm{kg}$. So whatever matter was then, it wasn't protons!

4. For this question, we have to assume an efficiency for the conversion (in the quasars) of mass-energy to radiation energy. Let this factor be ϵ. ($\epsilon = 0.1$ would be a typical assumption.) Radiation energy density scales as $(1+z)^4$ and so with E's representing energy density, we have

$$E_{z=2}(\text{quasar light}) = (1+z)^4 E_{\text{now}}(\text{quasar light}) = 8.1 \times 10^{-15}\,\mathrm{Jm}^{-3}.$$

This implies a mass energy density at $z = 2$ of $1/\epsilon c^2$ times this number, i.e. $9 \times 10^{-31}\,\mathrm{kgm}^{-3}$. The mass density scales as $(1+z)^3$, implying a mass density today in dead black holes of $3.3 \times 10^{-32}\,\mathrm{kgm}^{-3} = 5 \times 10^5 M_\odot \mathrm{Mpc}^{-3}$.

A number density of 10^{-2} galaxies per cubic megaparsec means the mean mass per galaxy of dead black holes is $4.8 \times 10^7 M_\odot$. (Our own galaxy, which while it may have been active, was presumably never a proper quasar, is thought to have a black hole of mass about $3 \times 10^6 M_\odot$ in its centre.)

5. Space density then $= (1+z)^3 \times$ now $= 0.128\,\mathrm{Mpc}^{-3}$. Imagine a box on the sky 1° wide on each side and with depth extending from $z = 3 - \Delta z$ (front face) to $z = 3$ (back face). Here $z = 3 - \Delta z$ is the redshift corresponding to a time of 10^8y after $z = 3$.

Distance corresponding to $\Delta\theta = 1^\circ$ is

$$D = \frac{2c[1 - (1+z)^{-1/2}]}{H_0(1+z)} \Delta\theta$$

at redshift z, assuming EdS. Now, will show in a moment that $\Delta z \ll 3$. We can thus approximate the entire volume and faces of the box as being at $z = 3$. Thus find $D = c\Delta\theta/(4H_0) = 26.2\,\mathrm{Mpc}$ for $H_0 = 50\,\mathrm{km/sMpc}$.

The depth of the box is 10^8 light years $= 30.7\,\mathrm{Mpc}$. Thus will see $0.128 \times (26.2)^2 \times 30.7 = 2690$ galaxies within the box, i.e. an areal density on the sky of 2690 galaxies per square degree.

Finally, we have to show $\Delta z \ll 3$. Let $z_1 = 3$, $z_2 = 3 - \Delta z$, then we know

$$\begin{aligned} t_2 - t_1 = 10^8\mathrm{y} &= -\int_{z_1}^{z_2} \frac{dz}{H(z)(1+z)} \\ &= \frac{2}{3H_0}\left\{\frac{1}{(1+z_2)^{3/2}} - \frac{1}{(1+z_1)^{3/2}}\right\} \quad \text{for EdS.} \end{aligned}$$

Solving this we find $z_2 = 2.844$ and so indeed $\Delta z \ll 3$ and we can treat all of the box as effectively at $z = 3$ in terms of number densities etc.

6. This question can be done using the same principle as mentioned in Lectures for calculating the optical depth experienced by the CMB due to reionization of the universe at late times (which provides a slight screen of free electrons between ourselves and the epoch of recombination). There we imagined a tube of constant proper area σ_T stretching between the observer and coordinate radius χ. The key part was to multiply this proper area σ_T by the element of proper distance at radius χ, i.e. $Rd\chi$, and then multiply this by the number density of scatterers at epoch χ and finally integrate to get the total number of scatterers for a given line of sight. This was then converted to an integral in z using the relation

$$\frac{d\chi}{dz} = \frac{c}{R_0 H(z)}$$

derived in Lecture 19

Here we do the same thing, but the cross-sectional area we use is now that of the galactic halo, σ, and our argument is that if a galaxy has its centre anywhere within this tube, its halo will intersect the line of sight running along the centre of the tube between the quasar and the observer. Thus the expected number of galaxies in the tube is

$$\int_0^z n(z')\sigma R(z')\frac{d\chi}{dz'}\,dz' = \int_0^z N_0(1+z')^3\sigma R\frac{c\,dz'}{R_0 H(z')} = \int_0^z N_0(1+z')^2\frac{\sigma c\,dz'}{H(z')},$$

We now just need to specify a form for $H(z)$. Since the answer mentions Ω_{m0} and but not $\Omega_{\Lambda 0}$, and we are not told to assume flatness (which is covered in the next question), we are working with a universe with general matter density and zero cosmological constant. We prove in a moment that in this case $H(z) = H_0(1+z)(1+\Omega_{m0}z)^{1/2}$ and so the number of galaxies whose haloes intersect the line of sight, is

$$\int_0^z \left(\frac{N_0\sigma c}{H_0}\right)(1+z')(1+\Omega_{m0}z')^{-1/2}\,dz',$$

as required.

Proof of $H(z) = H_0(1+z)(1+\Omega_{m0}z)^{1/2}$: We use the (B) field equation without cosmological constant:

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8\pi G\rho}{3} = -\frac{kc^2}{R^2}.$$

For matter, $\rho = \rho_0(1+z)^3$. Also, we know that (writing Ω_0 in place of Ω_{m0} for brevity) $\Omega_0 = \rho_o/(3H_0^2/8\pi G)$ and thus (B) is equivalent to

$$H^2 - H_0^2\Omega_0(1+z)^3 = -\frac{kc^2}{R_0^2}(1+z)^2$$

This equation, evaluated at the present time, gives $H_0^2(1-\Omega_0) = -kc^2/R_0^2$.

Thus, in general,

$$\begin{aligned} H^2 &= H_0^2(1-\Omega_0)(1+z)^2 + H_0^2\Omega_0(1+z)^3 \\ &= H_0^2(1+z)^2\{1-\Omega_0+\Omega_0+\Omega_0 z\} \end{aligned}$$

i.e. $H(z) = H_0(1+z)(1+\Omega_0 z)^{\frac{1}{2}}$.

7. The equation we work with in this question is the velocity equation without Λ:

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8\pi G\rho}{3} = -\frac{kc^2}{R^2}$$

If at some time the r.h.s. is negligible, then it follows that the two terms on the left must be of the same order. To assess whether kc^2/R^2 is negligible or not, we can therefore just take its ratio to the first term. The modulus of this (if $k \neq 0$) is $(c/RH)^2$, so we have simply to show that this is small up to the end of the radiation dominated phase. Since in this phase we have $R \propto t^{1/2}$, it follows that $H = 1/(2t)$ and $RH \propto t^{-1/2}$, thus the quantity we need to show is small is growing throughout this phase, and we only need to show it is small at the end of the era.

Working back to the epoch of equality from today assuming matter dominance yields

$$H(z_{\text{eq}}) = H_0(1+\Omega_{m0}z_{\text{eq}})^{1/2}(1+z_{\text{eq}})$$

(see end of question 6 answer). Also we know $R_0/R_{\text{eq}} = 1+z_{\text{eq}}$ and that

$$R_0 = \frac{c}{H_0}\,|\Omega_{m0}-1|^{-1/2}$$

(from Lecture 19). Putting these together yields

$$\left.\left(\frac{c}{RH}\right)^2\right|_{\text{eq}} = \frac{|\Omega_{m0}-1|}{1+\Omega_{m0}z_{\text{eq}}}$$

We know $\Omega_{m0} \gtrsim 0.2$, and if we assume z_{eq} is before recombination ($z \sim 1400$), then it's clear that the ratio is indeed very small.

Now at the epoch of equality (and using R to stand for $R_{\rm eq}$ from now on) we have

$$\rho_0^{\rm rad}\left(\frac{R_0}{R}\right)^4 = \Omega_{m0}\frac{3H_0^2}{8\pi G}\left(\frac{R_0}{R}\right)^3$$

where $\rho_0^{\rm rad}$ is the present radiation energy density. Thus

$$\rho_0^{\rm rad} = \Omega_{m0}\frac{3H_0^2}{8\pi G}\frac{R_0}{R}$$

But we know $9600\Omega_{m0}h^2 = R_0/R$ (strictly $1 + 9600\Omega_{m0}h^2 = R_0/R$, but we ignore the 1). Thus

$$\rho_0^{\rm rad} = \frac{3H_0^2}{8\pi G.9600.h^2}$$

Meanwhile, the velocity equation tells us that during the radiation dominated phase,

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho_0^{\rm rad}}{3}\left(\frac{R_0}{R}\right)^4$$

Putting in the solution $R = bt^{1/2}$ determines the constant b as

$$b = \left(\frac{32}{3}\pi G\rho_0^{\rm rad}\right)^{1/4} R_0$$

Thus the square root of the time spent in the radiation dominated era is given by

$$t_{\rm eq}^{1/2} = \frac{R}{b} = \frac{R}{R_0}\left(\frac{32}{3}\pi G\rho_0^{\rm rad}\right)^{-1/4}$$

and substituting for $\rho_0^{\rm rad}$ from above yields (after some cancelations)

$$t_{\rm eq} = \frac{9600^{-3/2}}{(\Omega_{m0}h^2)^2}\left(\frac{4H_0^2}{h^2}\right)^{-1/2}$$

Evaluating this numerically for the definition of h used here yields

$$t_{\rm eq} \sim 10,400(\Omega_{m0}h^2)^{-2}{\rm y}$$

Finally, for the last part, we just use that in an Einstein de Sitter universe, we have the relations

$$t = \frac{2}{3H} \qquad \text{and} \qquad H = H_0(1+z)^{3/2}$$

Inserting $9600\,h^2$ for z (note Ω_{m0} is assumed 1 for this case), yields $t_{\rm eq} = 13\,870h^{-4}{\rm y}$ for this way of doing the calculation.

8. We know that $TV^{\gamma-1} = \text{const}$ where
$$\gamma = \begin{cases} \frac{4}{3} & \text{before recombination} \\ \frac{5}{3} & \text{after} \end{cases}$$
($\frac{4}{3}$ is the standard value of γ for radiation).
Now since $V \propto R^3$ and $(1+z) \propto R^{-1}$ we find
$$T \propto (1+z)^2 \qquad \text{for matter after recombination}$$
Taking $T_{\rm rec} = 4000\,\mathrm{K}$, $z_{\rm rec} = 1400$, and a CMB temperature today of 2.76 K, thus yields a matter temperature today of 2.0 mK. This is lower than the CMB temperature, so that if there is any of this matter we might hope to detect it via absorption, e.g. of neutral hydrogen, against the CMB.

9. Proper distance across particle horizon at recombination would be $D = R(t_{\rm rec})\chi_p$ where
$$\chi_p = \int_0^{t_{\rm rec}} \frac{c}{R(t)}\, dt.$$
Now
$$R(t) = R(t_{\rm rec})\left(\frac{t}{t_{\rm rec}}\right)^{1/2} \implies D = \int_0^{t_{\rm rec}} c\left(\frac{t}{t_{\rm rec}}\right)^{-1/2} dt = 2ct_{\rm rec}.$$
Angular size subtended today given by
$$\Delta\theta = \frac{D(1+z)}{S(\chi)R_0} \approx \frac{D(1+z_{\rm rec})H_0}{2c} = (1+z_{\rm rec})(t_{\rm rec}H_0).$$
where in the second equality we have used the approximation that at high redshift in a matter-dominated EdS model $\chi(z) \sim 2c/(R_0H_0)$. Now in such a model,
$$R_{\rm rec} = R_0\left(\frac{3t_{\rm rec}H_0}{2}\right)^{2/3} \implies t_{\rm rec}H_0 = \left(\frac{R_{\rm rec}}{R_0}\right)^{3/2} \times \frac{2}{3} = \frac{2}{3}(1+z_{\rm rec})^{-3/2}.$$
Thus
$$\Delta\theta = \frac{2}{3}\frac{1}{(1+z_{\rm rec})^{1/2}}.$$
This means that points on the sky separated by greater than about 1° were causally disconnected at recombination. However the CMB is isotropic to about 1 part in 10^5 on these scales — this is one of the reasons for invoking the theory of inflation.

10. We start with the (B) equation, for a scalar field, which reads
$$H^2 = \frac{8\pi G}{3}\left(\tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}m^2\phi^2\right)$$
Differentiating this w.r.t. time, we can note that $\dot{H} = \ddot{R}/R - H^2$, and that from the continuity equation for ϕ, then $\ddot{\phi} + m^2\phi = -3H\dot{\phi}$. Putting these together, we obtain
$$\frac{\ddot{R}}{R} = -\frac{8\pi G}{3}\dot{\phi}^2 + \frac{4\pi G}{3}m^2\phi^2$$
as used in the Lecture.