Relativistic Astrophysics and Cosmology — Answers 3 — 2023

1. Assuming a circular system:

   - $T = \frac{2\pi}{\Omega} = 3.892\,\text{days}$,
   - $a_1 = 1.6 \times 10^{10}\,\text{m}$, since the pulse delay time tells us that the radius of the orbit is $53.46\,\text{s} \times c$, since the clock signal shifts by $53.46\,\text{s}$ in and out of phase as the NS moves further and closer away by $53.46\,\text{lightseconds}$,
   - $M_2 = 17M_\odot$ from spectroscopy,
   - $v_2 = 19\,\text{kms}^{-1}$ from spectroscopy.

   We know $v_2 = \Omega a_2$, and from center of mass $M_1 a_1 = M_2 a_2$, Eliminating a_2, we find

   $$M_1 = \frac{M_2 v_2 T}{2\pi a_1} = 1.08 M_\odot$$

   which is plausible for a neutron star.

2. From Lecture 14 for circular orbits:

   $$-\frac{dE}{dt} = L_{\text{GW}} = \frac{2G}{5c^5} M^2 a^4 \Omega^6 = \frac{2G^4 M^5}{5c^5 a^5} \qquad \left(\text{since } \Omega^2 = \frac{GM}{a^3}\right)$$

   Using $M = 2M_\odot$

   $$\dot{E} = -\frac{64}{5} \frac{G^4 M_\odot^5}{c^5 a^5}.$$

   Now the total orbital energy is

   $$E = -\frac{GM_1 M_2}{2a} = -\frac{GM_\odot^2}{2a}$$

   and thus

   $$\dot{a} = -2a^2 \frac{L_{\text{GW}}}{GM_\odot^2} = -\frac{128}{5} \frac{G^3 M_\odot^3}{c^5 a^3}.$$

   Solving this we find

   $$a(t)^4 = a_0^4 - \frac{512}{5} \frac{G^3 M_\odot^3}{c^5} t,$$

   which tells us

   $$t = \frac{5}{512} \frac{a_0^4 c^5}{G^3 M_\odot^3}$$

   is the time for complete orbital decay. Finally, we can use Kepler's law evaluated for the initial period, $P = \frac{2\pi}{\Omega}$ and radius a_0, to eliminate a_0 and obtain

   $$t_{\text{decay}} = \frac{5}{2048} \frac{2^{2/3} P^{8/3} c^5}{\pi^{8/3} G^{5/3} M_\odot^{5/3}} = 12.4 \times 10^6\,\text{y} \left(\frac{P_{\text{orb}}}{1\,\text{h}}\right)^{8/3}.$$

3. (a) The key formula from lectures is

$$\dot{\Omega} = \frac{96}{5}\frac{G^{5/3}}{c^5}\mathcal{M}^{5/3}\Omega^{11/3}$$

We see the frequency rise from 35 Hz to 450 Hz in 1 s during the chirp. Now, this frequency needs to be changed (a) into radians ($\times 2\pi$), and (b) divided by 2 since a system orbiting at angular speed Ω has an intensity profile with frequency 2Ω (since squaring a sinusoid doubles the frequency). We thus should put in $\Omega_0 = 35\pi$ and $\Omega_1 = 450\pi$, $t = 1$ into the solution to the above

$$\frac{3}{8}(\Omega_0^{-8/3} - \Omega_1^{-8/3}) = \frac{96}{5}\frac{G^{5/3}}{c^5}\mathcal{M}^{5/3}t$$

to recover a chirp mass of $\mathcal{M} \approx 10M_\odot$, which compares well with the LIGO-derived GR value of $9M_\odot$ (see arXiv:1606.0485 for details).

(b) If we assume that the BHs coalesce when their separation is the sum of their Schwarzschild radii, i.e. $a_1 + a_2 = \frac{2GM_1}{c^2} + \frac{2GM_2}{c^2}$ i.e. $a = \left(\frac{GM}{\Omega_c^2}\right)^{1/3} = \frac{2GM}{c^2}$, this yields

$$\Omega_c = \frac{1}{\sqrt{8}}\frac{c^3}{GM}$$

where Ω_c is the angular frequency at coalescence. Using $\Omega_c = 450\pi$, this yields a total mass M of about $50M_\odot$, and then solving for M_1 and M_2 individually, using their sum and the chirp mass, gives a primary mass of about $46M_\odot$ and a secondary mass of about $4M_\odot$. (These numbers are not very realistic — see below.)

We can get an estimate of the energy release by looking at the total energy in the final orbit. With the above assumption that the final $a = 2G(M_1 + M_2)/c^2 = 2GM/c^2$, then using

$$E_{\rm tot} = -1/2\frac{GM_1M_2}{a} \quad \text{yields} \quad E_{\rm rad} \approx \frac{1}{4}\frac{M_1M_2c^2}{M}$$

which with the masses just given is about $0.9M_\odot c^2$, comparing well with the LIGO GR value of $\approx 1.0M_\odot c^2$.

What is not realistic about the numbers got so far in this case, are the total and individual masses. LIGO GR finds $M \approx 22M_\odot$ and $M_1 \approx 14M_\odot$, $M_2 \approx 7.5M_\odot$. We can get close to these numbers without affecting agreement elsewhere by increasing the BH separation corresponding to the final (maximum) frequency. If this distance is $\beta 2GM/c^2$ instead of $2GM/c^2$, then (as an example) for $\beta = 1.6$ we get

$$M = 25M_\odot, \quad M_1 = 16M_\odot, \quad M_2 = 9M_\odot \quad \text{and} \quad E_{\rm rad} = 0.9M_\odot c^2$$

which is in much better agreement for the masses.

(See arXiv:1609.09349 and arXiv:1608.01940 (mentioned in the lectures) for further details on the Newtonian approximations and comparisons with the LIGO GR values.)

Bonus mark: GW151226 is termed the “Boxing day” event – can you see why?

4. In Handout 13 the following is given for the $h^{\mu\nu}$ corresponding to a plane polarised gravitational wave in 'TT' gauge

$$h^{\mu\nu} = A^{\mu\nu} \exp(ik_\rho x^\rho), \quad \text{where } k^\mu = (k, 0, 0, k) \text{ and } \quad A^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a^+ & a^\times & 0 \\ 0 & a^\times & -a^+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The total metric is $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and hence for the a^+ part of the wave, this leads to an interval

$$ds^2 = c^2 dt^2 - \left(1 - a^+ e^{ik(ct-z)}\right) dx^2 - \left(1 + a^+ e^{ik(ct-z)}\right) dy^2 - dz^2$$

(Note it's easy to show (from $g^{\mu\nu} g_{\nu\sigma} = \delta^\mu_\sigma$) that to first order the upstairs metric perturbation is $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$. However, for the metric we need the downstairs $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and since the downstairs h_{ij} are equal to the upstairs h^{ij} for spatial indices i and j (due to two flips of sign in transferring from one to another), and we are given the explicit expression for $h^{\mu\nu}$, the signs work out as shown in terms of a^+.)

From the metric we can read off that the Lagrangian $\mathcal{L}(x^\mu, \dot{x}^\mu)$ is given by

$$\mathcal{L} = c^2 \dot{t}^2 - \left(1 - a^+ e^{ik(ct-z)}\right) \dot{x}^2 - \left(1 + a^+ e^{ik(ct-z)}\right) \dot{y}^2 - \dot{z}^2.$$

The Euler-Lagrange geodesic equations then read

$$\begin{aligned}
\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}}\right) - \frac{\partial \mathcal{L}}{\partial t} &= 2c^2 \ddot{t} - ikca^+ e^{ik(ct-z)} \left(\dot{x}^2 - \dot{y}^2\right) = 0 \\
\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} &= -\frac{d}{d\tau}\left(\left(1 - a^+ e^{ik(ct-z)}\right) \dot{x}\right) = 0 \\
\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) - \frac{\partial \mathcal{L}}{\partial y} &= -\frac{d}{d\tau}\left(\left(1 + a^+ e^{ik(ct-z)}\right) \dot{y}\right) = 0 \\
\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{z}}\right) - \frac{\partial \mathcal{L}}{\partial z} &= -2\ddot{z} + ika^+ e^{ik(ct-z)} \left(\dot{x}^2 - \dot{y}^2\right) = 0
\end{aligned}$$

The x and y equations here follow immediately from the fact that none of the metric coefficients depend explicitly on x or y. Hence just from these two equations we can deduce what the question asks. E.g. since $\left(1 - a^+ e^{ik(t-z)}\right) \dot{x}$ is constant then $\dot{x} = 0$ initially means that it remains so throughout the motion. Similarly for y, and hence if the particle starts at rest in the (x, y) plane it remains so thereafter, despite the passage of the wave. The other two equations show us that in this case $\ddot{t}$ and $\ddot{z}$ are zero, and hence its motion in the (t, z) plane is uniform, in a straight line.

To understand what is measurable physically, however, we need to think about proper distance between neighbouring points. E.g. for two points (t, x, y, z) and $(t, x + \Delta x, y + \Delta y, z)$, the proper distance squared between them at time t and position z is

$$\left(1 - a^+ e^{ik(t-z)}\right) \Delta x^2 + \left(1 + a^+ e^{ik(t-z)}\right) \Delta y^2$$

which has an oscillating component $2a^+ \cos(ik(t-z))(\Delta y^2 - \Delta x^2)$ once we take the real part (which is implicitly assumed in the complex exponential notation). It is this physical proper distance change which can be detected e.g. in a two-armed interferometer experiment, and which is plotted for a ring of neighbouring particles on Handout 14.

5. The main equation we use for lensing is:

$$\alpha = 2\frac{R_S}{b}$$

where α is the lensed angle, R_S in Schwarzschild radius of the lens and b is the impact parameter, mnemonically this can be remembered by Einstein lensing having twice the Newtonian result.

In this case we therefore have:

$$2\frac{R_S}{b} = \frac{b}{D_s - D_l} + \frac{b}{D_l} \qquad (*)$$

where the l.h.s. is our Einstein lens equation, and the r.h.s. is the two angles added together (with a small angle approximation). In the small angle approximation we can also identify $b = R_E$ in the question, and solving this for b yields

$$b = R_E = \sqrt{2R_S(D_s - D_l)D_l/D_s} = \sqrt{2R_S D} \qquad (**)$$

as required.

For the second part we note that adding an offset r affects b on only one side of the equation $(*)$ – which side depends on whether you put the misalingment of r on the lens or the source, and the sign on which way you move it, but fundamentally you end up changing $\sqrt{b^2}$ in $(**)$ to $\sqrt{R(R+r)}$ to recover the required equation

$$R^2 + rR - R_E^2 = 0 \Rightarrow R_\pm = \frac{-r \pm \sqrt{r^2 + 4R_E^2}}{2}$$

Using the amplitude formula given for these values of R

$$A_\pm = \left|\frac{dR_\pm}{dr}\frac{R_\pm}{r}\right| = \left|\left(\frac{-1}{2} \pm \frac{r}{2\sqrt{r^2 + 4R_E^2}}\right)\left(\frac{-r \pm \sqrt{r^2 + 4R_E^2}}{2r}\right)\right|.$$

To get the result we need to add the total amplification $A = A_+ + A_-$, and taking care with positive signs one finds one can get this by considering $A_+ - A_-$ in the above with out absolutes to recover the result

$$A = \frac{r + 2R_E^2}{r\sqrt{r^2 + 4R_E^2}} = \frac{u^2 + 2}{u\sqrt{u^2 + 4}}.$$

(if you do $A_+ + A_-$ directly without absolutes you get 1, i.e. no magnification).

To get the numerical answers for an einstein ring produced by a solar mass star, the angular size of the ring's diameter is $2\frac{R_E}{D_l} = \sqrt{\frac{8R_S(D_s-D_l)}{D_sD_l}}$, which is largest if $D_s \gg D_l$ in which case it is $\sqrt{\frac{8R_E}{D_l}}$. Putting in $R_S = 3\,\mathrm{km}$ with (a) $D_l = 8\,\mathrm{kpc}$ for our galaxy to find 2 mas, or (b) 16 Mpc for the distance to M87 gives 45 µas, which would be only just detectable with event horizon telescope resolution, though the EHT only detects radio waves. This should be contrasted with the direct measurement of a self-lensed black hole, rather than the measurement of a lens of a distant source.

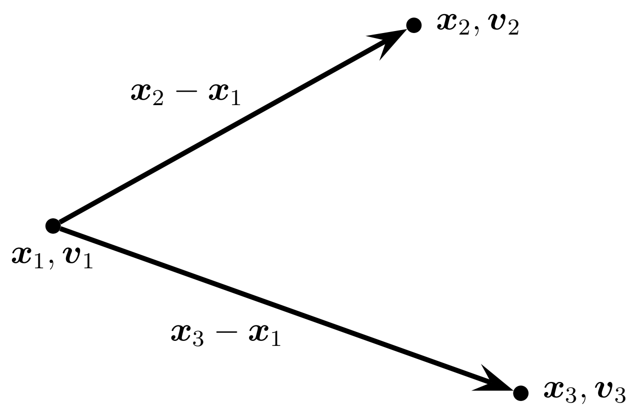
See `http://arxiv.org/abs/astro-ph/9604011` for more details.

6. Suppose each object has intrinsic luminosity L. The flux received from a spherical shell Δr at distance r is

$$\begin{aligned} \frac{L}{4\pi r^2} \times \text{no. of objects in shell} &= \frac{L}{4\pi r^2}\,\rho\, 4\pi r^2 \Delta r = \rho L \Delta r \\ \Longrightarrow \text{total flux} &= \int_0^\infty \rho L\, dr = \infty! \end{aligned}$$

However, objects in front block those behind, thus every line of sight will terminate on the surface of a star (generally in an external galaxy) and therefore the night sky should appear as bright as the surface of the Sun. This is not observed, therefore there is a paradox!

7. Suppose that every galaxy has a position vector $\boldsymbol{x}_i$ relative to some arbitrary origin, with velocity vector $\boldsymbol{v}_i$.



An observer at $\boldsymbol{x}_1$ sees a galaxy at $\boldsymbol{x}_2$ receeding at $\boldsymbol{v}_2 - \boldsymbol{v}_1$ along a radial path, so the Hubble law requires

$$\boldsymbol{v}_2 - \boldsymbol{v}_1 = H_0(\boldsymbol{x}_2 - \boldsymbol{x}_1).$$

The same formula holds for a galaxy at $\boldsymbol{x}_3$. An observer at $\boldsymbol{x}_2$ therefore sees galaxy $\boldsymbol{x}_3$ receding at

$$\boldsymbol{v}_3 - \boldsymbol{v}_2 = (\boldsymbol{v}_3 - \boldsymbol{v}_1) - (\boldsymbol{v}_2 - \boldsymbol{v}_1) = H_0(\boldsymbol{x}_3 - \boldsymbol{x}_1) - H_0(\boldsymbol{x}_2 - \boldsymbol{x}_1) = H_0(\boldsymbol{x}_3 - \boldsymbol{x}_2)$$

so also sees everything receding according to the Hubble law.

Similarly, if everything is rotating have $\boldsymbol{v}_i = \boldsymbol{\omega} \times \boldsymbol{x}_i$. The relative velocity between $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ is now

$$\boldsymbol{v}_j - \boldsymbol{v}_i = \boldsymbol{\omega} \times \boldsymbol{x}_j - \boldsymbol{\omega} \times \boldsymbol{x}_i = \boldsymbol{\omega} \times (\boldsymbol{x}_j - \boldsymbol{x}_i)$$

so all see the same law.

8. The radial part of the FRW metric with $c = 1$ is

$$ds^2 = dt^2 - R^2(t)d\chi^2$$

This means that the Lagrangian we use in the geodesic Euler-Lagrange procedure is

$$\mathcal{L} = \dot{t}^2 - R^2\dot{\chi}^2$$

Now

$$\frac{\partial \mathcal{L}}{\partial \dot{\chi}} = -2R^2\dot{\chi}$$

and since $\mathcal{L}$ has no explicit χ dependence, the EL equation for χ is

$$\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \dot{\chi}}\right) = 0, \quad \text{i.e.} \quad R^2\dot{\chi} = \text{const.}$$

For t we have $\partial\mathcal{L}/\partial\dot{t} = 2\dot{t}$ and

$$\frac{\partial \mathcal{L}}{\partial t} = -2R\frac{dR}{dt}\dot{\chi}^2$$

Thus using $dR/dt = HR$, the EL equation for t is

$$\ddot{t} = -R^2H\dot{\chi}^2$$

(i) Fundamental observers are comoving, i.e. they have fixed χ, θ and ϕ. We can see that χ fixed, i.e. $\dot{\chi} = 0$, is indeed a solution of our equations provided $\ddot{t} = 0$, i.e. that $t = as + b$ for constants a and b. By choosing these appropriately ($a = 1$, $b = 0$), the proper time of a fundamental observer can be aligned with the cosmic time t.

(ii) We have

$$\frac{d\chi}{dt} = \frac{d\chi}{ds}\frac{ds}{dt} = \frac{\dot{\chi}}{\dot{t}}$$

Thus

$$\begin{aligned}
\frac{dv}{dt} &= \frac{d}{dt}\left(R\frac{\dot{\chi}}{\dot{t}}\right) = \frac{d}{dt}\left(\frac{R^2\dot{\chi}}{R\dot{t}}\right) \\
&= \frac{1}{R\dot{t}}\frac{d}{ds}\left(R^2\dot{\chi}\right)\frac{ds}{dt} - \frac{1}{R^2\dot{t}^2}\frac{d}{ds}\left(R\dot{t}\right)\frac{ds}{dt}R^2\dot{\chi} \\
&= -\frac{\dot{\chi}}{\dot{t}^3}\left(\frac{dR}{dt}\dot{t}^2 - R^3H\dot{\chi}^2\right) \\
&= -HR\frac{\dot{\chi}}{\dot{t}}\left(1 - R^2\frac{\dot{\chi}^2}{\dot{t}^2}\right) = -Hv(1 - v^2)
\end{aligned}$$

where the geodesic equations were used for the derivatives of $R^2\dot{\chi}$ and $\dot{t}$.

We see that the expansion of the universe, as represented by H, acts like a 'drag' term on peculiar velocities, i.e. proportional to velocity (at least for non-relativistic speeds) but of opposite sign. This is an example of what is sometimes called 'Hubble drag'.

9. We just need the second field equation for the flat, $k = 0$, case:

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8\pi G\rho}{3} - \frac{\Lambda c^2}{3} = 0$$

i.e. substituting for ρ,

$$H^2 - \frac{8\pi G\rho_0(1+z)^3}{3} - \frac{\Lambda c^2}{3} = 0 \qquad (*)$$

Using $\Omega_{m0} = \rho_0/(3H_0^2/8\pi G)$, and the fact that flatness implies $1 = \Omega_{m0} + \Omega_{\Lambda 0}$ (e.g. just evaluate (*) at the present day) we find

$$\begin{aligned} H^2 &= \frac{\Lambda c^2}{3} + \Omega_{m0} H_0^2 (1+z)^3 \\ &= H_0^2 \left(\Omega_{\Lambda 0} + (1-\Omega_{\Lambda 0})(1+z)^3\right) \end{aligned}$$

10. Using the (B) equation and the equation of continuity $\rho \propto R^{-(3+\epsilon)}$, it is easy to show that with $k = \Lambda = 0$ we have

$$\dot{R}^2 \propto = \begin{cases} \frac{1}{R} & \text{matter domination} \\ \frac{1}{R^2} & \text{radiation domination} \end{cases}$$

This has solutions of the type $R \to 0$ as $t \to 0$, of $R \propto t^{2/3}$ and $R \propto t^{1/2}$ respectively. Inserting these into the expressions for $H(t)$ and $q(t)$ we then get:

$$H = \frac{\dot{R}}{R} = \begin{cases} \frac{2}{3t} & \text{matter} \\ \frac{1}{2t} & \text{radiation} \end{cases}$$

and

$$q = -\frac{\ddot{R}R}{\dot{R}^2} = \begin{cases} \frac{1}{2} & \text{matter} \\ 1 & \text{radiation} \end{cases}$$