Lecture 21: Equilibrium figures

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Outline and motivation

In the previous lecture we obtained the equations of motion for a self-gravitating and rotating earth model, and showed that these effects are relevant at sufficiently long periods. A particular conclusion was that the equilibrium state of such an earth model cannot be stress-free as has been assumed in earlier parts of the course. In this lecture we investigate the constraints imposed by equilibrium equations in further detail, and show that they are key to understanding the observed shapes of planets.

The equilibrium equations

In the previous lecture we obtained the following equilibrium equations

$$\rho\epsilon_{ijk}\epsilon_{klm}\Omega_j\Omega_l x_m - \frac{\partial T^0_{ij}}{\partial x_j} - \rho\gamma^0_i = 0, \tag{1}$$

for a self-gravitating and steadily rotating planet[1]. Here ρ is the density, Ω_i the angular velocity, T^0_{ij} the stress, and γ^0_i the gravitational acceleration. To simplify notations, let us write σ_{ij} for the equilibrium stress, and g_i for the gravitational acceleration. The equilibrium equations then become

$$\rho\epsilon_{ijk}\epsilon_{klm}\Omega_j\Omega_l x_m - \frac{\partial \sigma_{ij}}{\partial x_j} - \rho g_i = 0, \tag{2}$$

which are subject to the boundary conditions $\sigma_{ij}\hat{n}_j = 0$ on ∂M. Though these equations were derived from elasticity, the force balance expressed is valid in more general materials. This is important because over long time-scales the Earth is not elastic.

If we knew the shape, density distribution, and angular velocity of the planet, then all terms in the equilibrium equations are fixed except for the stress tensor. Recalling from Lecture 13 that equilibrium stress tensors are necessarily symmetric, we see that we have only three equations for six unknown functions. The problem is, therefore, under-determined, and we cannot expect to uniquely determine stress tensor[2].

Hydrostatic equilibrium

You have seen in the first part of this course that over thousands of years the interior of the Earth flows like a fluid, this being due to solid-state creep. Considering for simplicity

[1] Though our focus in this course is on the Earth, the material of this lecture applies so naturally to other planets it seems sensible to generalise the discussion somewhat.

[2] This counting argument is suggestive but not entirely correct. Rather, the relevant point is that there exist non-zero symmetric tensor fields having vanishing divergence. This can be compared to divergence-free vector fields with which you will be very familiar (i.e. divergence of a curl is zero). In fact, a method for generating divergence-free tensor fields exists using an extension of the curl operator.

a Newtonian fluid, we know that the stress tensor[3] takes the form

$$\sigma_{ij} = -p\,\delta_{ij} + \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\frac{\partial v_k}{\partial x_k}\delta_{ij}\right) \tag{3}$$

where p is the pressure, η the viscosity, and v_i the velocity. Note that the given expression is valid in a compressible fluid, but it reduces to that for an incompressible one by requiring the velocity be divergence-free. If the fluid is not flowing, then the stress tensor takes the simple **hydrostatic** form

$$\sigma_{ij} = -p\,\delta_{ij}, \tag{4}$$

and the same is true for non-Newtonian fluids. Such an expression must also be approximately valid if the fluid is flowing sufficiently slowly. To get a sense of this approximation for the Earth, we note that the size of the non-hydrostatic stress associated with mantle convection scales like $\eta V/L$ with $\eta \sim 10^{21}\,\mathrm{Pa\,s}$ a typical viscosity, $V \sim 10^{-8}\mathrm{m\,s^{-1}}$ a typical speed of a few centimetres per year, and $L \sim 10^6\,\mathrm{m}$ the length scale. From this we find that deviatoric stresses are around $\eta V/L \sim 10^7\,\mathrm{Pa}$. Pressures in the mantle are, however, known[4] to be of order $10^{10}\,\mathrm{Pa}$, and hence the stress is quite close to being hydrostatic. This motivates us to look for hydrostatic solutions of the equilibrium equations.

Putting eq.(4) into eq.(5) we obtain

$$\rho\epsilon_{ijk}\epsilon_{klm}\Omega_j\Omega_l x_m + \frac{\partial p}{\partial x_i} - \rho g_i = 0, \tag{5}$$

along with the boundary condition $p = 0$ on ∂M. Note that in assuming the stress is hydrostatic we have moved from an under-determined problem to an over-determined one, with three equations for a single unknown, p. The form of this **hydrostatic equilibrium equation** can be usefully simplified by noting first that

$$g_i = -\frac{\partial \phi}{\partial x_i}, \tag{6}$$

with ϕ the equilibrium gravitational potential. Similarly, the centrifugal term can also be written as the gradient of a potential

$$\epsilon_{ijk}\epsilon_{klm}\Omega_j\Omega_l x_m = \frac{\partial \psi}{\partial x_i}, \tag{7}$$

where we have defined

$$\psi = \frac{1}{2}\left(\Omega_i\Omega_j - \Omega_k\Omega_k\delta_{ij}\right)x_i x_j. \tag{8}$$

Moving to vector notation, the hydrostatic equilibrium equation takes the simple form

$$\nabla p + \rho\nabla\gamma = \mathbf{0}, \tag{9}$$

where we have introduced the **gravity potential** $\gamma = \phi + \psi$. Note that "gravity" is used here for the combined effect of gravitational and centrifugal forces.

An immediate consequence of eq.(9) is that ∇p and $\nabla\gamma$ are parallel, and hence the level surfaces of these fields are coincident. In particular, pressure vanishes on the surface

[3] This is the Cauchy stress that occurs within spatial formulations of continuum mechanics. But note that at equilibrium the referential and spatial pictures coincide due to our choice of reference body.

[4] Values for the pressure can be calculated once the density profile is known.

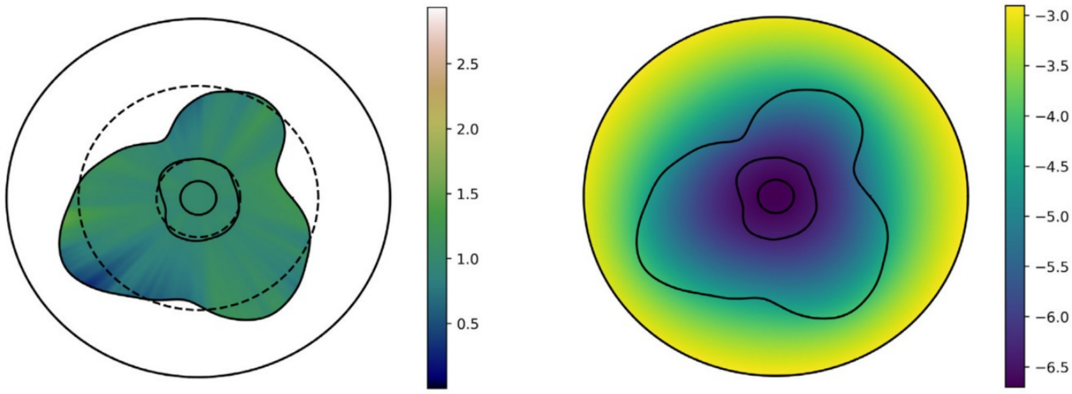


Fig. 1: On the left is shown a slice through the density structure of a non-rotating aspherical planet, with the values suitably non-dimensionalised. The corresponding gravitational potential is then shown on the right. It can be observed that the planet's surface is not an equipotential, nor are the level surfaces of the density and gravitational potential related in any obvious manner. This illustrates that very strong constraints on a planet's shape and density are imposed by requiring it to be in hydrostatic equilibrium. Figure taken from Maitra & Al-Attar (2021).

of the planet, and so this must also be an equipotential of gravity. If we take the curl of eq.(9), recalling $\nabla \times \nabla f = \mathbf{0}$ for any scalar field, we find

$$\nabla \rho \times \nabla \gamma = \mathbf{0}, \tag{10}$$

and so note that ρ and γ also have coincident level surfaces. In conclusion we have seen that for an planet to be in hydrostatic equilibrium the level surfaces of p, ρ, and γ must all be the same.

Suppose we arbitrarily specified the shape, density variations, and angular velocity of an planet. We could then, in principle, determine its gravitational potential by solving Poisson's equation, and hence find the gravity potential. There would be no reason, however, that this field would have level surfaces that agree with those for the chosen density, nor that γ would be constant on the free-surface. This suggests that in requiring a planet to be in hydrostatic equilibrium we are placing rather strong restriction on its possible shape, density distribution, and angular velocity.

Non-rotating hydrostatic planets

As a starting point, suppose that the planet does not rotate, and hence $\psi = 0$. Assume also that the planet is spherical in shape, and that density varies only with radius. We then know that ϕ is also spherically symmetric, and hence eq.(9) reduces to

$$\nabla p + \rho \frac{\mathrm{d}\phi}{\mathrm{d}r}\,\hat{\mathbf{r}} = \mathbf{0}, \tag{11}$$

with $\hat{\mathbf{r}}$ a radial unit vector. To solve this equation we can clearly take p to be a function of r alone, and hence need only integrate

$$\frac{\mathrm{d}p}{\mathrm{d}r} + \rho \frac{\mathrm{d}\phi}{\mathrm{d}r} = 0, \tag{12}$$

using the boundary condition $p = 0$ on the surface $r = b$. This argument shows that for a non-rotating planets it is possible to find at least one class of hydrostatic equilibrium states, namely those that are **spherically symmetric**. From a physical perspective, it seems reasonable that these are the *only* hydrostatic equilibria possible. This is indeed true, but a mathematical proof of this result is not easy[5].

Slowly rotating hydrostatic planets

Given that a non-rotating and hydrostatic planet must be spherically symmetric, we would expect that a slowly rotating planet should have an equilibrium figure that is close to spherical. In this section, we show that this is indeed the case. First, however, we should clarify what is meant by **slowly rotating**. The essential idea is that the ratio of the centrifugal to gravitational potentials should be small. Looking at these quantities on the planet's surface we have approximately

$$\psi \sim \frac{1}{2}\Omega^2 b^2, \tag{13}$$

where Ω is the angular frequency and b the mean radius, while also

$$\phi \sim \frac{4}{3}\pi G\bar{\rho} b^2, \tag{14}$$

with $\bar{\rho}$ the mean density. The ratio of these terms is given by

$$\frac{\text{centrifugal potential}}{\text{gravitational potential}} \sim \frac{3\Omega^2}{8\pi G\bar{\rho}}. \tag{15}$$

For the Earth, we have $\bar{\rho} \sim 5000\,\mathrm{kg\,m^{-3}}$ and $\Omega \sim 7\times 10^{-5}\,\mathrm{s}$, and hence we find

$$\frac{\text{centrifugal potential}}{\text{gravitational potential}} \sim 2\times 10^{-3}, \tag{16}$$

which is indeed a fairly small quantity.

We model the rotating planet through a first-order perturbation of a spherical reference model. The density in the reference model will be written ρ_0, and similarly for other quantities. Within the rotating planet we then have

$$\rho = \rho_0 + s\,\rho_1 + \cdots\,, \tag{17}$$
$$p = p_0 + s\,p_1 + \cdots\,, \tag{18}$$
$$\phi = \phi_0 + s\,\phi_1 + \cdots \tag{19}$$
$$\psi = s\,\psi_1, \tag{20}$$

with s a perturbation parameter introduced for book-keeping purposes. We must also consider the aspherical shape of the planet. For simplicity we neglect internal boundaries[6], and so need only parameterise the outer surface in the form

$$r(\theta,\varphi) = b + s\,h_1(\theta,\varphi) + \cdots\,, \tag{21}$$

[5] In fact, I am aware of a proof only in the case of a homogeneous planet, with the argument summarised in Chandrasekhar's book "Ellipsoidal figures of equilibrium".

[6] But these can be easily included into the theory.

with (θ, φ) spherical polar co-ordinates.

Putting the above expansions into the equilibrium equations we find at first-order

$$\nabla p_1 + \rho_0 \nabla \gamma_1 + \rho_1 g_0 \hat{\mathbf{r}} = \mathbf{0}, \tag{22}$$

where $\gamma_1 = \phi_1 + \psi_1$, and we have written $g_0 = \frac{\mathrm{d}\phi_0}{\mathrm{d}r}$ for convenience. The boundary condition on pressure is applied on the perturbed surface, and hence

$$p[b + s\, h_1(\theta, \varphi) + \cdots, \theta, \varphi] = 0. \tag{23}$$

Expanding this out to first-order in s, we arrive at

$$p_1 + \frac{\mathrm{d}p_0}{\mathrm{d}r}\, h_1 = 0, \tag{24}$$

on the reference surface $r = b$. The zeroth-order equilibrium equation tells us that

$$\frac{\mathrm{d}p_0}{\mathrm{d}r} + \rho_0 g_0 = 0, \tag{25}$$

and hence the boundary conditions simplify to

$$p_1 - \rho_0 g_0 h_1 = 0. \tag{26}$$

We will show that eq.(22) and (26) allow the density, pressure, and boundary perturbations to be expressed in terms of that for the gravity potential. First, we take the cross product of eq.(22) with the unit vector $\hat{\mathbf{r}}$ to obtain

$$\hat{\mathbf{r}} \times \nabla p_1 + \hat{\mathbf{r}} \times \rho_0 \nabla \gamma_1 = \mathbf{0}. \tag{27}$$

Noting that

$$\hat{\mathbf{r}} \times \rho_0 \nabla \gamma_1 = \hat{\mathbf{r}} \times \left[\nabla(\rho_0 \gamma_1) - \gamma_1 \frac{\mathrm{d}\rho_0}{\mathrm{d}r} \hat{\mathbf{r}} \right] = \hat{\mathbf{r}} \times \nabla(\rho_0 \gamma_1), \tag{28}$$

eq.(27) can be written

$$\hat{\mathbf{r}} \times \nabla(p_1 + \rho_0 \gamma_1) = \mathbf{0}. \tag{29}$$

This relation tells us that the gradient of $p_1 + \rho_0 \gamma_1$ is parallel to $\hat{\mathbf{r}}$, and hence this function depends on radius alone. We are free to assume that the perturbations all average to zero over spherical surfaces[7], and hence we arrive at

$$p_1 = -\rho_0 \gamma_1, \tag{30}$$

which expresses the pressure perturbation in terms of that to the gravity potential. From the perturbed boundary condition we then immediately obtain

$$h_1 = -\frac{\gamma_1}{g_0}. \tag{31}$$

For the second stage of the argument, we take the curl of eq.(22) to obtain

$$\frac{\mathrm{d}\rho_0}{\mathrm{d}r} \hat{\mathbf{r}} \times \nabla \gamma_1 + g_0 \nabla \rho_1 \times \hat{\mathbf{r}} = \mathbf{0}. \tag{32}$$

[7] If not, these terms could be combined into the spherical reference model.

Using the same trick as before, this is equivalent to

$$\hat{\mathbf{r}} \times \nabla \left(\frac{\mathrm{d}\rho_0}{\mathrm{d}r}\gamma_1 - g_0\rho_1 \right) = \mathbf{0}, \tag{33}$$

from which we see that the density perturbation is given by

$$\rho_1 = \frac{1}{g_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}r}\gamma_1. \tag{34}$$

To summarise, we started from the hydrostatic equilibrium equations in a slowly rotating planet, and have obtained the following identities

$$p_1 = -\rho_0\gamma_1, \quad h_1 = -\frac{\gamma_1}{g_0}, \quad \rho_1 = \frac{1}{g_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}r}\gamma_1, \tag{35}$$

which express the other perturbed quantities in terms of that for the gravity potential, γ_1. This latter term is comprised of the centrifugal potential perturbation ψ_1, determined by the planet's angular velocity, and the gravitational potential perturbation, ϕ_1. To complete the discussion, we recall that ϕ_1 is also related to ρ_1 and h_1 through solution of Poisson's equation. Indeed, we have

$$\nabla^2\phi_1 = \begin{cases} 4\pi G\rho_1 & r < b \\ 0 & r > b \end{cases}, \tag{36}$$

while the appropriate continuity conditions at $r = b$ are

$$[\phi_1]_-^+ = 0, \quad [\hat{\mathbf{r}} \cdot \nabla\phi_1 + 4\pi G\rho_0 h_1]_-^+ = 0, \tag{37}$$

whose derivation is left as an exercise in the second problem set. Using eq.(35), we see that Poisson's equation is transformed into

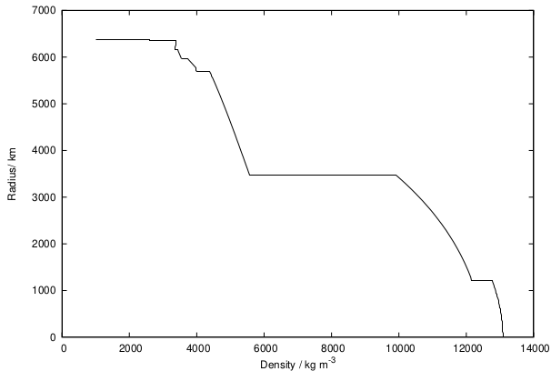
$$\nabla^2\phi_1 = \begin{cases} 4\pi G\frac{1}{g_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}r}(\phi_1 + \psi_1) & r < b \\ 0 & r > b \end{cases}, \tag{38}$$

along with the continuity conditions

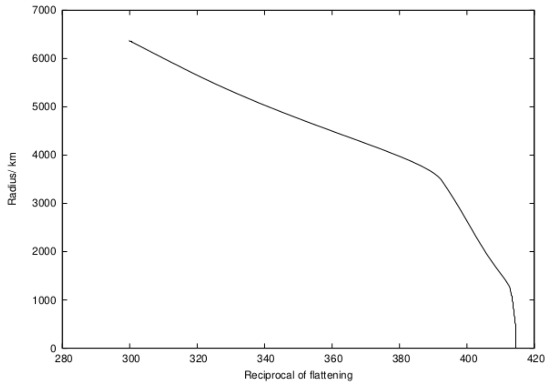
$$[\phi_1]_-^+ = 0, \quad \left[\hat{\mathbf{r}} \cdot \nabla\phi_1 - 4\pi G\rho_0\frac{(\phi_1 + \psi_1)}{g_0}\right]_-^+ = 0. \tag{39}$$

Eq.(38) and (39) constitute a boundary value problem that can be solved for ϕ_1 in terms of ψ_1. Having done this we can then use eq.(35) to determine the other parameters for the perturbed planet. These equations, albeit in a less general form, were first obtained by Clairaut (1713–1765), this work building on Newton's original discussion of equilibrium figures in the *Principia*. To implement these equations in practice we must know the variation of spherically averaged density with radius (see Fig.2), and in later lectures we will see how this was determined for the Earth. It is worth noting these arguments apply to a general applied potential ψ. The theory can, therefore, also be applied to model things like equilibrium tides within gaseous planets.

Another point to make is that the equations are linear in ϕ_1 and ψ_1, and hence symmetries of ψ_1 are directly mirrored in ϕ_1 and the other perturbed quantities. In terms of spherical harmonics, the aspherical part of the centrifugal potential perturbation can be



(a) Radial variation in density.



(b) The calculated values of the reciprocal flattening $1/\epsilon$.

Fig. 2: The upper panel shows variations in the Earth's spherically averaged density structure as determined from free oscillation data along with mass and moment of inertia constraints. The lower figure then plots the reciprocal of the ellipticity as determined through numerical solution of Clairaut's equation.

shown to be proportional to Y_{20}. It is for this reason that equilibrium figures of slowly rotating planets are found to **oblate spheroids**, and characterised in terms of a single radial function, ϵ, known as the **ellipticity** and defined such that

$$\phi_1 = \sqrt{\frac{4\pi}{5}}\frac{2}{3} r g_0 \epsilon Y_{20}. \tag{40}$$

At the mean surface $r = b$, it can be shown that the ellipticity is such that

$$\epsilon(b) = \frac{\text{equatorial radius} - \text{polar radius}}{\text{mean radius}}, \tag{41}$$

and for the Earth this value is around $1/300$.

Using properties of spherical harmonics, it can be shown[8] that eq.(38) reduces to

$$\frac{\mathrm{d}^2\epsilon}{\mathrm{d}r^2} + 8\pi G \rho_0 g_0^{-1}\left(\frac{\mathrm{d}\epsilon}{\mathrm{d}r} + \frac{\epsilon}{r}\right) - \frac{6\epsilon}{r^2} = 0, \tag{42}$$

which is known as **Clairaut's equation** and is subject to the boundary conditions

$$\left.\frac{\mathrm{d}\epsilon}{\mathrm{d}r}\right|_{r=0} = 0, \quad \left.\frac{\mathrm{d}\epsilon}{\mathrm{d}r}\right|_{r=b} = \frac{1}{b}\left[\frac{5\Omega^2 b^3}{2GM} - 2\epsilon(b)\right], \tag{43}$$

[8] But you do not need to know how.

with M the planet's mass. This problem can be solved numerically using a shooting method, whereby the initial conditions at $r = 0$ are progressively adjusted so that the required condition at $r = b$ is met.

What you need to know and be able to do

(i) Know the form of the equilibrium equations, and that they hold in a general planet and not just an elastic one.

(ii) That the equilibrium equations are under-determined, but they admit non-unique solutions subject to the two compatibility conditions.

(iii) What a hydrostatic equilibrium state is, and be able to derive the constraints imposed on the level surfaces of ρ, p, and γ.

(iv) Derive the hydrostatic equilibrium state in a non-rotating planet.

(v) Know in outline how solutions can be obtained within a slowly rotating planet. Here you will not need to reproduce the full derivation, but may be asked to perform related calculations with appropriate guidance.

(vi) How minimum stress fields can be defined and their relation to slow viscous flow. Again, you do not need to learn this derivation in full, but may be asked to perform related calculations given the necessary information.