Thermal structure of plates

Physics of the Earth as a Planet, Lecture 5

(Fowler, The Solid Earth, CUP, Chapter 7)

Conservation of Heat

Consider a volume V of material enclosed by a surface $\mathcal{S}$ (see Figure 1), where $\hat{\mathbf{n}}$ is an outward pointing vector whose magnitude is proportional to the area of the surface element shown, dV is a small volume element.

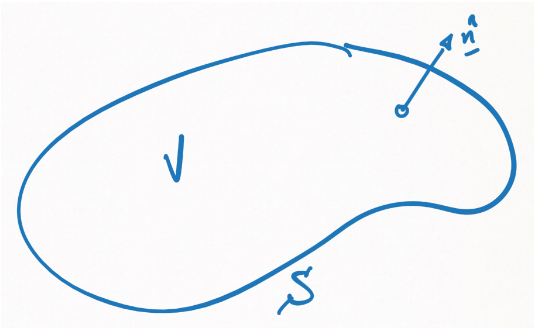


Figure 1: Definition of volume V, surface $\mathcal{S}$ and vector $\hat{\mathbf{n}}$.

Conservation of energy within the volume, at constant pressure, can therefore be expressed as

$$\frac{\partial}{\partial t}\int_V \rho H dV = -\int_{\mathcal{S}} \rho H \mathbf{v}\cdot\hat{\mathbf{n}} d\mathcal{S} - \int_{\mathcal{S}} \mathbf{q}\cdot\hat{\mathbf{n}} d\mathcal{S}, \tag{1}$$

where ρ is the density, H is the enthalpy (or internal energy), $\mathbf{v}$ is the velocity and $\mathbf{q}$ is the (diffusive) heat flux. Here the LHS of equation (1) is the total internal energy in the volume, and the RHS represents the advection and diffusion of thermal energy through the surface, respectively.

We may rewrite equation (1) using the divergence theorem,

$$\int_{\mathcal{S}} \mathbf{A}\cdot\hat{\mathbf{n}}\; d\mathcal{S} = \int_V \nabla\cdot\mathbf{A}\; dV, \tag{2}$$

so that energy conservation is now expressed as

$$\int_V \frac{\partial(\rho H)}{\partial t}\; dV = -\int_V \nabla\cdot(\rho H\mathbf{v})\; dV - \int_V \nabla\cdot\mathbf{q}\; dV. \tag{3}$$

Since the volume is arbitrary we may write the differential form of the equation

$$\frac{\partial(\rho H)}{\partial t} = -\nabla\cdot(\rho H\mathbf{v}) - \nabla\cdot\mathbf{q}. \tag{4}$$

Finally, we note that conservation of mass may be expressed as

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0, \tag{5}$$

that the specific heat capacity is defined as

$$c_p = \left.\frac{\partial H}{\partial T}\right|_P, \tag{6}$$

and that Fourier's law of heat conduction is

$$\mathbf{q} = -k\nabla T, \tag{7}$$

where k is the thermal conductivity (which in general is a tensor that can be a function of space, orientation, composition, temperature etc.). Using these relationships we may rewrite equation (4) as

$$\rho c_p \left[\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right] = \nabla \cdot (k \nabla T). \tag{8}$$

The thermal field is therefore affected by the processes of both thermal advection through motion of the medium, and through thermal diffusion which relies on temperature gradients. For constant material properties this simplifies to the familiar advection-diffusion equation for temperature

$$\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \kappa \nabla^2 T, \tag{9}$$

written in terms of a constant thermal diffusivity $\kappa = k/\rho c_p$.

Half-space cooling

When a plate is formed it is hot, and it cools as it moves away from the ridge. Since the upwelling velocity is large and the conductivity of rock is small, the upwelling is approximately isothermal. As a first approximation we consider a half-space cooling model of the ridge as shown in Figure 2

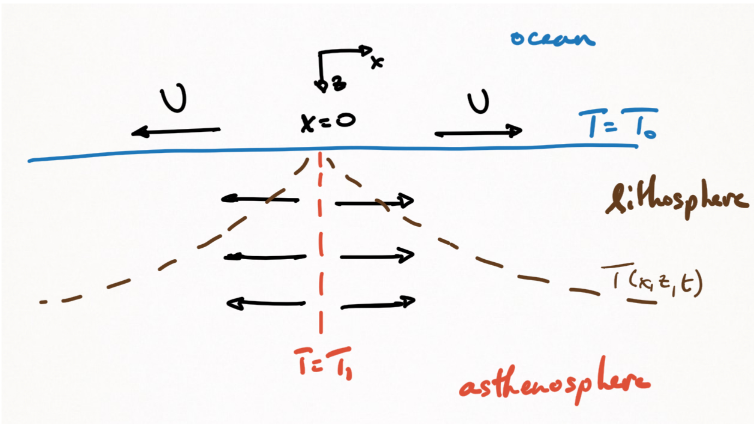


Figure 2: Schematic illustration of the half-space cooling model.

We look for steady-state solutions, $\partial_t = 0$, with uniform plate spreading velocity $\mathbf{v} = U\hat{\mathbf{x}}$. Since the aspect ratio of the plates is large, $x \gg z$ we may approximate equation (9) by

$$U \frac{\partial T}{\partial x} \simeq \kappa \frac{\partial^2 T}{\partial z^2}. \tag{10}$$

The characteristic temperature scale is the difference in temperature between the ocean and mantle, $\Delta T = T_1 - T_0$, but for half-space cooling there is no natural length scale and so we scale $z \sim \sqrt{\kappa x/U}$. Anticipating a self-similar solution we define the self-similar variable

$$\zeta = \frac{z}{2\sqrt{\kappa x/U}}, \tag{11}$$

along with a nondimensional temperature

$$\tilde{T}(\zeta) = \frac{T - T_0}{T_1 - T_0}. \tag{12}$$

Equation (10), written in terms of $\tilde{T}$ and η is therefore

$$-2\zeta \frac{\partial \tilde{T}}{\partial \zeta} = \frac{\partial^2 \tilde{T}}{\partial \zeta^2}, \tag{13}$$

subject to the boundary conditions

$$\tilde{T}(0) = 0, \ \tilde{T}(\infty) = 1. \tag{14}$$

Integrating twice, and applying the boundary conditions provides the half-space cooling model of plate temperature

$$T(x, z) = T_0 + (T_1 - T_0)\,\mathrm{erf}\left(\frac{z}{2\sqrt{\kappa x / U}}\right), \tag{15}$$

where the error function is defined as

$$\mathrm{erf}(\zeta) = \frac{2}{\sqrt{\pi}} \int_0^{\zeta} e^{-y^2}\, dy. \tag{16}$$

The half-space cooling model accurately captures the behaviour of the plate near the ridge, but fails to predict a finite plate thickness as $x \to \infty$.

Plate cooling

Far from the ridge axis, the heat flux associated with convection balances the conductive heat flux through the plate leading to a steady-state, far-field thickness a. Figure 3 shows a useful simple model of the process.

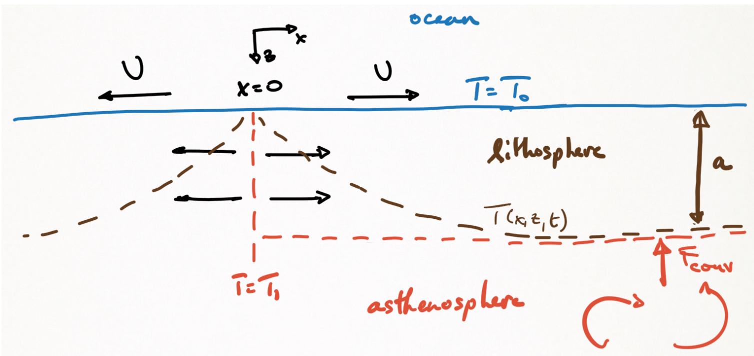


Figure 3: Schematic illustration of the plate model.

Again, we seek a steady-state solution, $\partial_t = 0$ to equation (9) for constant spreading velocity $\mathbf{v} = U\hat{\mathbf{x}}$. The thermal equation is therefore

$$U\frac{\partial T}{\partial x} = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right), \tag{17}$$

now with boundary conditions

$$T = T_0 \text{ on } z = 0, \qquad T = T_1 \text{ on } x = 0, \tag{18}$$

$$T = T_1 \text{ on } z = a, \qquad \frac{\partial T}{\partial x} \to 0 \text{ as } x \to \infty. \tag{19}$$

We now make use of an idea that is of great importance, and is used throughout fluid mechanics, which is to scale all variables so that they are dimensionless. Putting $z = a\tilde{z}$, $x = a\tilde{x}$, $T = (T_1 - T_0)\tilde{T} + T_0$ then $\tilde{z}$ varies from 0 to 1 and $\tilde{T}$ from 1 to 0 across the layer, no matter what its actual thickness may be. Equation (17) now becomes

$$\mathrm{Pe}\frac{\partial \tilde{T}}{\partial \tilde{x}} = \frac{\partial^2 \tilde{T}}{\partial tx^2} + \frac{\partial^2 \tilde{T}}{\partial \tilde{z}^2} \tag{20}$$

where the behaviour is now characterised by a single nondimensional parameter, the Peclet number

$$\mathrm{Pe} = Ua/\kappa. \tag{21}$$

The thermal Peclet number is dimensionless, and determines the importance of the advection of heat by the motion in comparison to conduction. Fluid mechanics contains many such dimensionless numbers, which arise when the equations are reduced to dimensionless form. Their values in any situation control the physical behaviour of the system.

Equation (20) is easily solved by separation of variables. However, before we can begin with the separation of variables technique we must adjust the problem so that the boundary conditions on $z = 0, 1$ are homogeneous. This can be done by subtracting the far field ($x \to \infty$) solution $\tilde{T}_\infty(z)$. The boundary conditions on $z = 0, 1$ are that $\tilde{T} = 0, 1$, and in the far field we have

$$\frac{\mathrm{d}^2 \tilde{T}_\infty}{\mathrm{d}z^2} = 0 \implies \tilde{T}_\infty(z) = Cz + D, \tag{22}$$

for constants C, D. The boundary conditions require

$$\tilde{T}_\infty(z) = z. \tag{23}$$

Letting

$$\tilde{T}(x, z) = \tilde{T}_\infty(z) + \theta(x, z), \tag{24}$$

we see that θ satisfies the same partial differential equation as $\tilde{T}$,

$$\mathrm{Pe}\frac{\partial \theta}{\partial x} = \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial z^2}, \tag{25}$$

but now the boundary conditions on the top and bottom boundaries are homogeneous, $\theta = 0$ on $z = 0, 1$. We can now proceed with separation of variables

$$\theta(x, z) = X(x)Z(z) \tag{26}$$

which when substituted into (25) yields

$$\frac{1}{X}\left(\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} - \mathrm{Pe}\frac{\mathrm{d}X}{\mathrm{d}x}\right) = -\frac{1}{Z}\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} = \alpha^2 \tag{27}$$

where α is constant. The Z equation has solution

$$Z = A\sin\alpha z + B\cos\alpha z \tag{28}$$

and the boundary conditions on $z = 0, 1$ demand that $Z(0) = Z(1) = 0$. Hence

$$B = 0, \qquad \alpha = n\pi. \tag{29}$$

The X equation is then

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} - \mathrm{Pe}\frac{\mathrm{d}X}{\mathrm{d}x} - n^2\pi^2 X = 0. \tag{30}$$

Putting

$$X = \mathrm{e}^{\beta_n x} \tag{31}$$

gives

$$\beta_n^2 - \mathrm{Pe}\beta_n - n^2\pi^2 = 0, \tag{32}$$

$$\beta_n = (\mathrm{Pe}/2) \pm \sqrt{(\mathrm{Pe}/2)^2 + n^2\pi^2}. \tag{33}$$

There are two root for β_n, one positive, one negative. However, since $\theta \to 0$ as $x \to \infty$, only the negative root is acceptable, and

$$\beta_n = (\mathrm{Pe}/2) - \sqrt{(\mathrm{Pe}/2)^2 + n^2\pi^2}, \qquad \beta_n < 0. \tag{34}$$

Hence we can write the full solution for θ as

$$\theta(x, z) = \sum_n A_n e^{\beta_n x} \sin{(n\pi z)}. \tag{35}$$

To determine the coefficients A_n we must use the condition on $x = 0$, which in terms of θ is $\theta(0, z) = 1 - z$. The standard Fourier series expressions give that

$$A_n = 2\int_0^1 (1 - z)\sin n\pi z \,\mathrm{d}z = \frac{2}{n\pi}. \tag{36}$$

The full solution for $\tilde{T}$ is thus

$$\tilde{T}(x, z) = z + \sum_{n=1}^{\infty} \frac{2}{n\pi} \mathrm{e}^{\beta_n x} \sin{(n\pi z)}. \tag{37}$$

For most spreading ridges $\mathrm{Pe} > 10$ and therefore $\mathrm{Pe}/2 > n\pi$ when n is small. For large Peclet numbers (fast spreading)

$$\beta_n \approx -\frac{n^2\pi^2}{\mathrm{Pe}} \tag{38}$$

and away from the ridge the behaviour is dominated by the first term in the expansion

$$\tilde{T}(x, z) \approx z + \frac{2}{\pi}\exp\left(-\frac{\pi^2 x}{\mathrm{Pe}}\right)\sin{(\pi z)}. \tag{39}$$

Converting back to dimensional variables gives

$$T(x, z) \approx T_0 + (T_1 - T_0)\left[\frac{z}{a} + \frac{2}{\pi}\exp{(-x/l)}\sin{(\pi z/a)}\right] \tag{40}$$

where

$$l = \frac{Ua^2}{\pi^2\kappa} = \upsilon\tau. \tag{41}$$

In this expression τ is the thermal time constant of a slab of thickness a. If this equation is written in terms of the age t of the sea floor, rather than its distance from the ridge x, then

$$x = Ut \tag{42}$$

and

$$T(x,z) \approx T_0 + (T_1 - T_0)\left[\frac{z}{a} + \frac{2}{\pi}\exp(-t/\tau)\sin(\pi z/a)\right], \qquad \tau = \frac{a^2}{\pi^2\kappa}. \tag{43}$$

The temperature structure is then independent of the spreading velocity U. Though the exponential factor could have been written down using the thermal constant τ of a slab of thickness a, the constants can only be found using the correct boundary conditions. This expression is the basis of the depth-age and heatflow-age relationships. The heat flow F is easily obtained from (9)

$$F = -k\left.\frac{\partial T}{\partial z}\right|_{z=a} \approx -\frac{k(T_1 - T_0)}{a}[1 + 2\exp(-t/\tau)]. \tag{44}$$

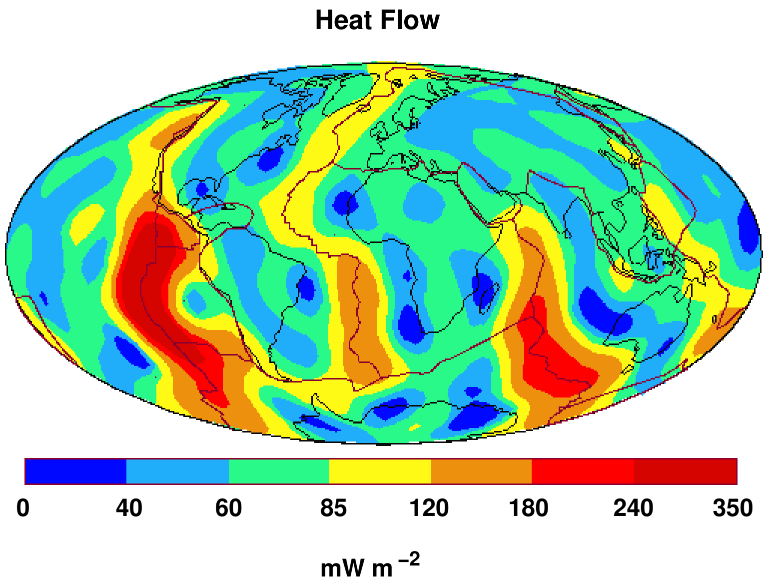


Figure 4: Global observations of heatflow. The largest flow is observed at the mid oceanic ridges, with values decaying away from the ridges, in agreement with equation (44).

Bathymetry and heatflow

The calculation of the bathymetry $w(x)$ of a spreading ridge requires more information. The important observation is that the gravity anomaly over ridges is almost zero and therefore that they are isostatically compensated. Figure 5 shows the method of calculating the elevation of the sea floor.

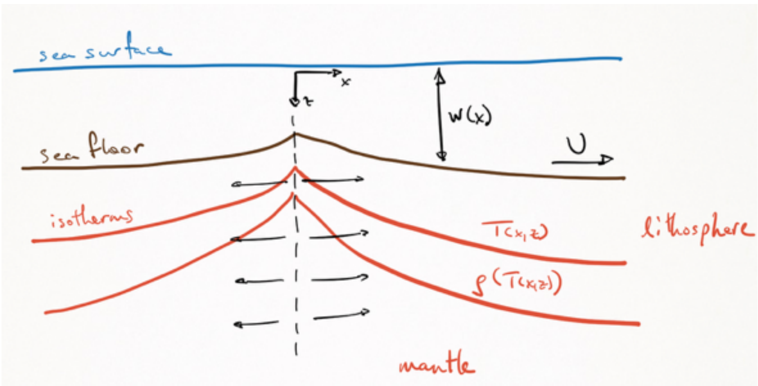


Figure 5: Schematic illustration of the bathymetry.

We begin by noting that the density of the plate is a function of temperature and may be given by

$$\rho = \rho_m \left[1 - \alpha(T - T_1)\right], \tag{45}$$

where ρ_m is the density of the mantle at temperature T_1, and α is the thermal expansion coefficient. For a plate which is isostatically compensated to the depth of the uniform mantle temperature (at $z = a$) we can therefore write

$$\rho_w w_0 + \rho_m(a - w_0) = \rho_w w(x) + \int_w^a \rho[T(x,z)]\, dz, \tag{46}$$

where ρ_w is the density of water, and w_0 is the depth of the ocean over the ridge.

Taking the limit where the ocean depth is small compared to the depth of the plate, $w \ll a$, we may use the approximate thermal model (43) to evaluate the isostatic balance (46) to find the depth of the ocean as a function of distance (or age) from the ridge axis,

$$w - w_0 \simeq -\frac{\rho_m a \alpha \Delta T}{\rho_m - \rho_w}\left\{-\frac{1}{2} + \frac{4}{\pi^2} e^{-x/l}\right\}. \tag{47}$$

A more robust data analysis is for the height of the ridge above the old sea floor

$$w_\infty - w(x) \simeq \frac{\rho_m a \alpha \Delta T}{\rho_m - \rho_w}\frac{4}{\pi^2} e^{-x/l}. \tag{48}$$

The values of three quantities are well known from oceanic measurements: the elevation of ridges above old sea floor $w_\infty - w(0)$, the mean heat flux through old lithosphere $F(\infty)$, and the thermal time constant of the oceanic lithosphere τ. The thermal time constant is determined from the variation of depth with age. The expressions for these quantities are given by the plate model as

$$w_\infty - w(0) = \frac{a\rho_m\alpha(T_1 - T_0)}{2(\rho_m - \rho_w)}, \qquad F(\infty) = \frac{k(T_1 - T_0)}{a}, \qquad \tau = \frac{a^2}{\pi^2\kappa}. \tag{49}$$

Laboratory measurements give values of the material constants

$$\begin{gathered} k = 3.8\,\mathrm{W\,{}^\circ C^{-1}\,m^{-1}},\ \kappa = 8.8\times10^{-7}\,\mathrm{m^2 s^{-1}}, \alpha = 4.0\times10^{-5}\,\mathrm{{}^\circ C^{-1}}, \\ \rho_0 = 3.3\times10^3\,\mathrm{kg\,m^{-3}},\ \rho_w = 1\times10^3\,\mathrm{kg\,m^{-3}}, \end{gathered} \tag{50}$$

and geophysical observations give $w_\infty - w(0) = 3.7$ km, $F = 40$ mW m^{-2} and $\tau_0 = 60$ Ma. τ then gives $a = 128$ km, $F(\infty)$ then gives $T_1 - T_0 = 1350$°C, and substitution into the expression for $w_\infty - w(0)$ gives a value of 4 km, so the heat flow and bathymetric observations are consistent. Importantly, the observations have allowed us to estimate the plate thickness.

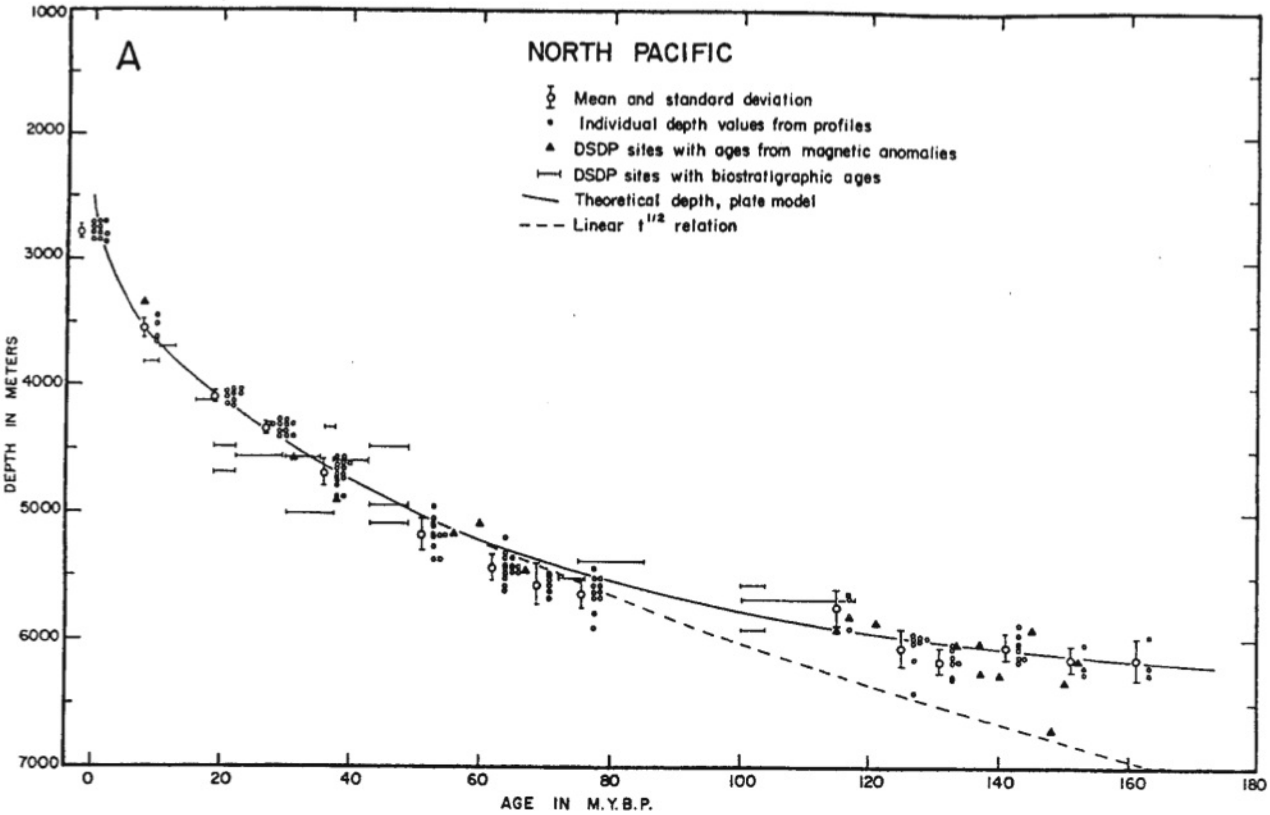


Figure 6: Comparison between data measurements for the depth (or bathymetry) of the North Pacific and the depth predicted by the plate model.

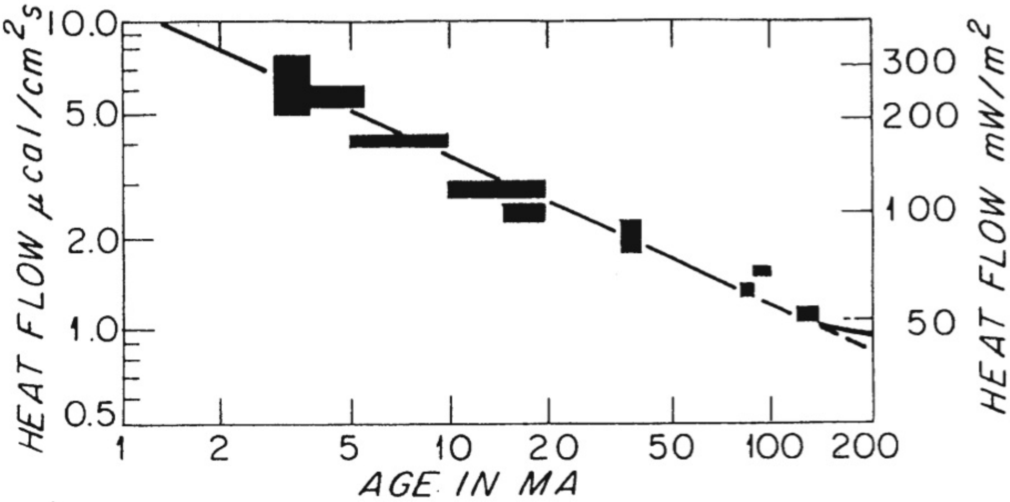


Figure 7: Mean heat flux as a function of age for the oceans. The width of the boxes represents the age range, and the height the scatter of the data.

Thermal structure of slabs beneath island arcs

We can use exactly the same method to find the temperature structure in the slabs beneath island arcs. If the plate is old when it starts to descend the temperature within it is the steady state temperature distribution

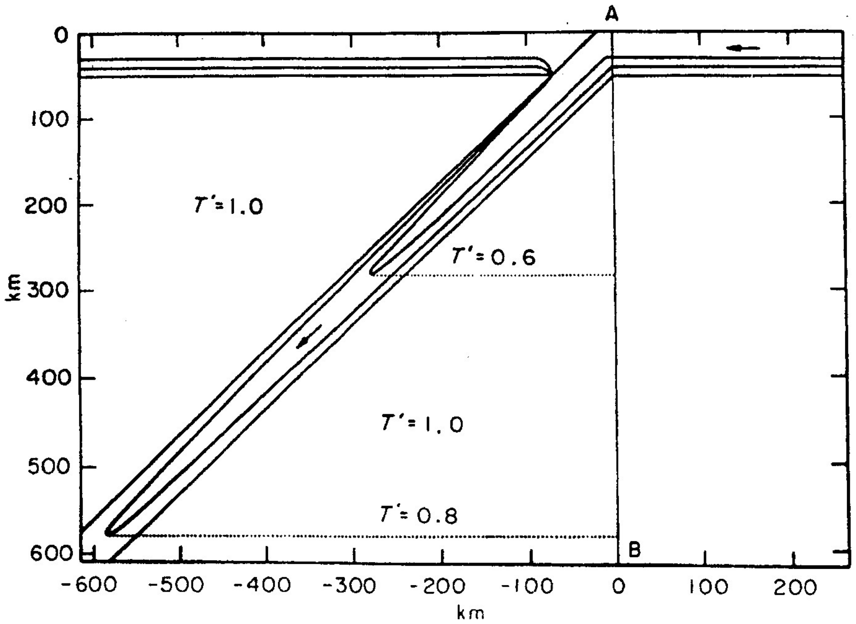
$$T = z. \tag{51}$$

The boundary conditions are $T = 1$ on $z = 0, 1$ when the slab is in the mantle, and the dimensionless temperature within it is

$$T = 1 + \sum_{n=1}^{\infty} \frac{2}{n\pi} \exp\left[\left((\mathrm{Pe}/2) - \sqrt{(\mathrm{Pe}/2)^2 + n^2\pi^2}\right) x\right] \sin(n\pi z). \tag{52}$$

The dimensionless temperature contours and the earthquake distribution beneath the Tonga-Fiji-Kermadec Island Arc are shown in Figure 8.

(a)



(b)

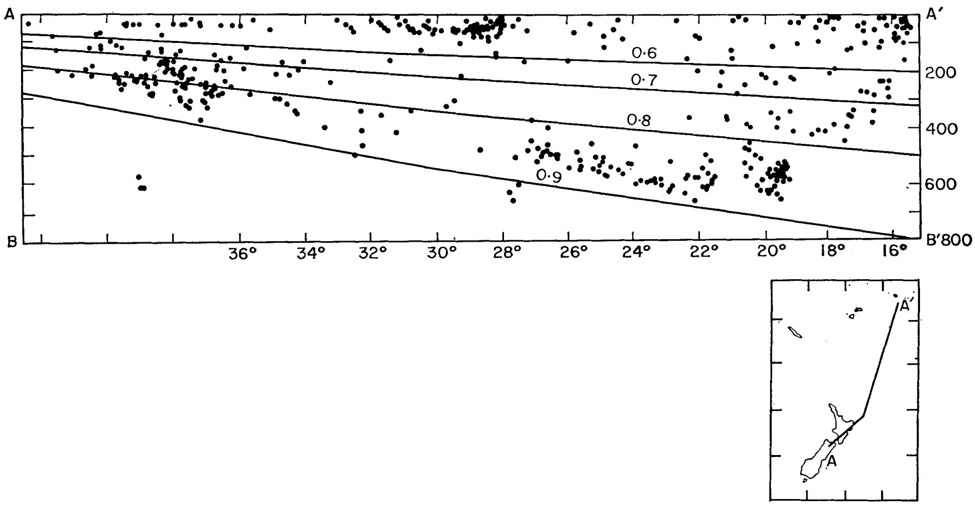


Figure 8: (a) Temperature structure beneath island arcs with no vertical or horizontal exaggeration. (b) Projections of foci of earthquakes and isotherms onto a vertical plane along the Tonga-Fiji-Kermadec New Zealand region (see inset). Except for six anomalous deep earthquakes, intermediate and deep focus activity ceases at about $T' = 0.85$ or $T = 680°C$.

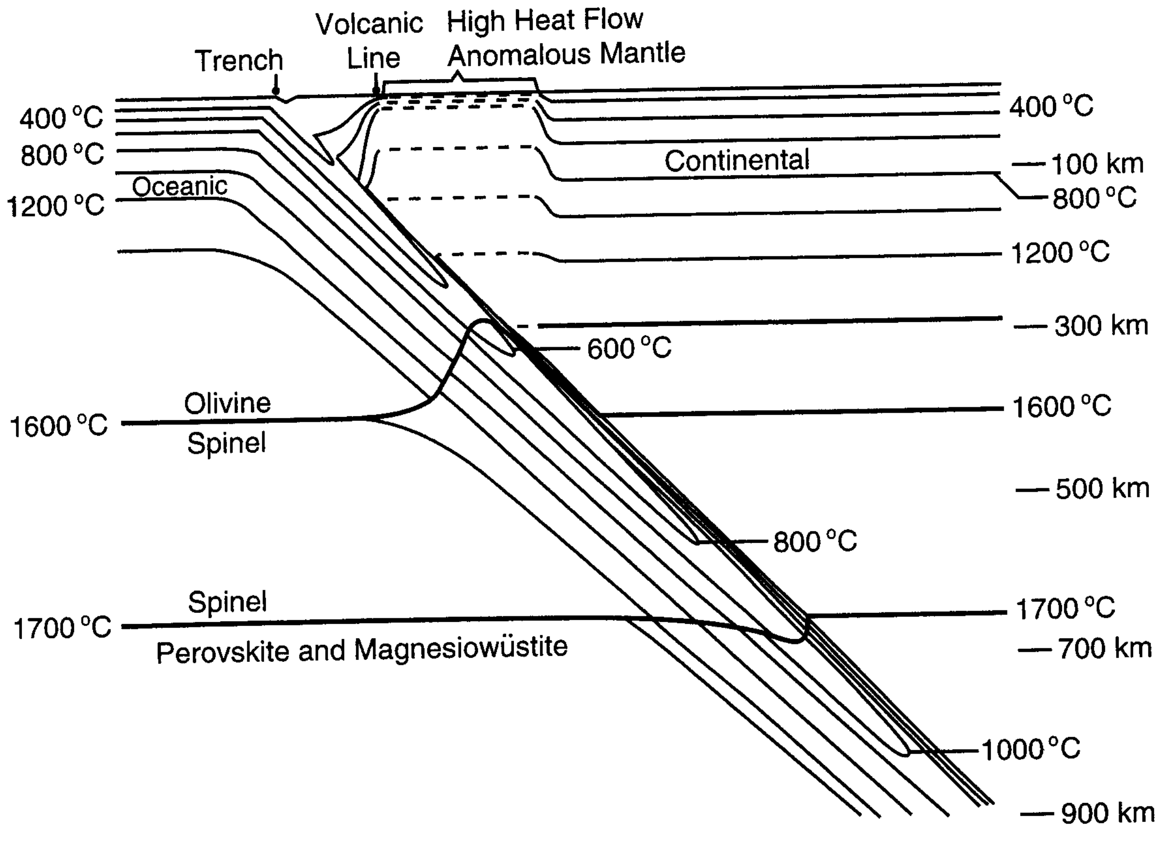


Figure 9: More detailed image showing the thermal structure of a subducting slab, incorporating the effect of the slab's thermal structure on the surrounding mantle.