General Relativity and Matter

Relativistic Astrophysics and Cosmology: Lecture 4

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Pre-lecture question:

How can pressure act attractively?

Last time

- Built an intuitive understanding of curvature from two dimensions
- Understand the difference between intrinsic and embedded curvature
- Motivated and discussed general radial metrics
- Introduced and semi-derived the Schwarzschild metric

This lecture

Quite heavy-going (non-examinable at points)

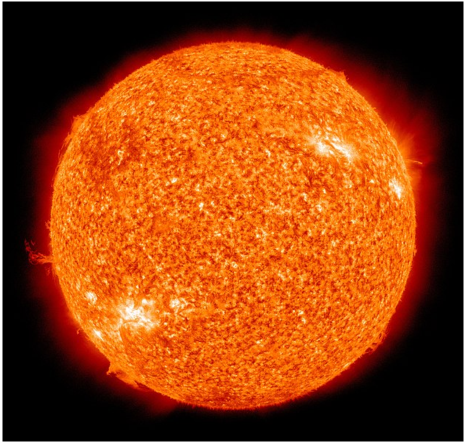
- Newtonian stars
- The stress-energy tensor
- The Oppenheimer–Volkoff equation
- Derivation of the Schwarzschild interior solution

Next lecture

- Particle motion in the Schwarzschild metric

Static solutions with matter

- So far (last year) you have only looked at the Einstein field equations in the presence of matter, in a cosmological context
- Have not looked at all, using GR, at the extremely important case of isolated objects of non-infinitesimal extent made from ordinary (baryonic) matter
- Otherwise known as stars!
- So here, want to look generally at how matter comes into the field equations, and then look at stars, and the differences GR makes
- Note a crucial quantity in discussing this is pressure, and will also discuss how this enters GR (some surprises!)



Newtonian Treatment

- We know the basic equation of hydrostatic equilibrium

$$\frac{1}{\rho}\frac{dP}{dr} = -\frac{GM(r)}{r^2}, \qquad \text{where} \qquad M(r) = \int_0^r 4\pi r^2 \rho(r) dr.$$

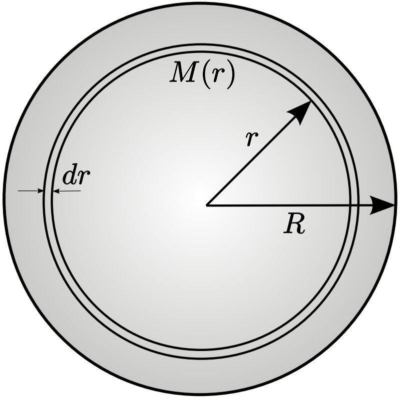
- Suppose ρ is uniform, $= \rho_0$ say, so that $M(r) = \frac{4}{3}\pi\rho_0 r^3$ and thus

$$\frac{dP}{dr} = -\frac{4}{3}\pi G\rho_0^2 r.$$

- We can immediately solve to get

$$P(r) = P_0 - \frac{2}{3}\pi G\rho_0^2 r^2$$

where P_0 is the central pressure.



- The other boundary condition on P is that it should vanish at the surface of the star. (Think about the force on an infinitesimal layer at the edge, if P didn't match the external pressure there.)
- Thus we get $\boxed{P_0 = \frac{2}{3}\pi G \rho_0^2 R^2}$ and $\boxed{P(r) = \frac{2}{3}\pi G \rho_0^2 R^2 \left(1 - \frac{r^2}{R^2}\right)}$.
- Note the following interesting point as regards later developments: if we henceforth let M (with no (r)) be the total mass of the star (so $M = M(R)$), then we see

$$P_0 = \rho_0 c^2 \frac{GM}{2c^2 R} = \rho_0 c^2 \frac{1}{4}\frac{R_S}{R},$$

  where R_S is the Schwarzschild radius.
- From this we see:
  - Pressure has the same units as energy density
  - The central pressure is tiny compared to energy density unless the R_S for the object starts to approach its actual radius.
  - (Compare the Sun, where $R_S \approx 3\,\mathrm{km}$, while $R = 695.000\,\mathrm{km}$.)

General Relativity Treatment

- Now think about the treatment of matter in GR.
- We know that the way geometry and matter get linked together in GR is via the Einstein field equations

$$\boxed{G_{\mu\nu} = -\frac{8\pi}{c^4} G T_{\mu\nu}.}$$

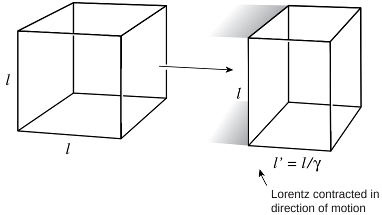
- L.h.s. is the Einstein tensor, which we discuss more later.
- R.h.s. is the stress-energy tensor (SET) which want to look at now (discussion useful also in cosmology).
- Note we are using the sign conventions in the Hobson, Efstathiou & Lasenby book.

Brief Revision of the SET

- Consider fluid as made up of electrically neutral, non-interacting particles, each of rest mass m_0 (commonly called dust).
- Fluid at a point is characterised by its matter density ρ and 3-velocity $\vec{u}$ in some inertial frame.
- For simplicity, consider the fluid in its instantaneous rest frame (IRF) S, in which $\vec{u} = \vec{0}$.
- In S the (proper) density is

$$\rho_0 = m_0 n_0,$$

  where n_0 is the number of particles in a unit volume.
- In some other frame S', moving with speed v relative to S, volume containing a fixed number of particles is Lorentz contracted along the direction of motion.



- In S' the number density of particles is $n' = \gamma_v n_0$ and effective mass of each particle is $m' = \gamma_v m_0 \Rightarrow$ matter density in S' is

$$\boxed{\rho' = \gamma_v^2 \rho_0.}$$

- $\Rightarrow$ matter density is not a scalar.
- But: it does transform as the tt-component of a rank-2 tensor.
- $\Rightarrow$ suggests source term for gravity is a rank-2 tensor.

- Since we have two lots of γ_v, and $c\gamma_v$ is what the time component of the 4-velocity vector in the IRF (which is just c) becomes under the transformation to the new frame, the obvious choice of tensor is

$$\boxed{T^{\mu\nu} = \rho_0 u^\mu u^\nu .} \tag{1}$$

  where $\rho_0(x)$ is proper density of fluid (i.e. that measured by observer comoving with local flow) and u^μ is its 4-velocity.
- The tensor $T^{\mu\nu}$ is the energy-momentum tensor or the stress-energy tensor of the matter distribution.
- Note: from now on, denote the proper density simply by ρ, i.e. without the zero subscript.

How does Pressure change things?

- Equivalence of mass and energy tells us pressure, as well as mass density, should be a source for gravity. Consider
$$T^{ij} = \rho u^i u^j = \gamma_u^2 \rho v^i v^j,$$
where the v^i are the components of ordinary velocity.
- Can see T^{ij} is rate of flow of the i-component of momentum per unit area in j-direction.
- Then since rate of change of momentum $\equiv$ force, we have T^{ij} = i-component of force per unit area perpendicular to j-direction. i.e. for a perfect fluid $T^{ij} = \delta^{ij} \times$ pressure.
- Thus, in the IRF, for a perfect fluid
$$[T^{\mu\nu}] = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}.$$
- Note the T^{tt} component here is the total energy density, including e.g. KE from random thermal motions.

- In the IRF this can be written as

$$T^{\mu\nu} = (\rho + P/c^2)u^\mu u^\nu - P\eta^{\mu\nu}. \tag{2}$$

  which must be valid in any local Cartesian inertial frame at the given point.
- Moreover, can obtain expression valid in arbitrary coordinate system by replacing $\eta^{\mu\nu}$ by $g^{\mu\nu}$ in the arbitrary system.
- $\Rightarrow$ fully covariant expression for a perfect fluid is

$$\boxed{T^{\mu\nu} = (\rho + P/c^2)u^\mu u^\nu - Pg^{\mu\nu}.} \tag{3}$$

- We see that $T^{\mu\nu}$ is symmetric, and made from two scalar fields ρ and P, and vector field $\vec{u}$ that characterise the perfect fluid. In the limit $P \to 0$ a perfect fluid becomes dust.
- Can also define energy-momentum tensors of imperfect fluids, charged fluids, and even the electromagnetic field.
- They are all symmetric.

The effects of pressure

- Can get a guide as to the relative contribution pressure is likely to make by considering its units, which are the same as those of energy density.
- In a relativistic theory, the energy density of matter with density ρ is of course ρc^2.
- For ordinary forms of matter and ordinary pressures, this hugely exceeds the likely contribution from pressure.
- For example, consider the Earth's atmosphere. We get

$$\frac{\text{pressure energy density}}{\text{atmospheric density} \times c^2} = \frac{10^5 \mathrm{Nm}^{-2}}{1\,\mathrm{kgm}^{-3} \times c^2} \approx 10^{-12}.$$

- The pressure contribution to gravity is usually very small.
- This is not true for radiation however. Particles with zero rest mass (i.e. completely relativistic) satisfy

$$\text{pressure} = \frac{\text{energy density}}{3},$$

  and thus their pressure contribution to gravity must be taken into account.
- So let's resume on the solution of Einstein's equations for STARS.

Karl Schwarzschild (1873-1916)

- Wrote two pivotal papers very soon after Einstein's 1915 paper on GR.
- The latter had looked at the advance of perihelion of Mercury in the gravitational field of the Sun, but worked with only an approximate solution for the GR field around a spherically symmetric body.
- Einstein was very impressed when Schwarzschild found the exact vacuum solution.
- Moreover, Schwarzschild then went on to derive the interior solutions appropriate to a body of constant density.
- Died on the front in the First World War (and moreover worked out his solutions there!).
- What we do today rests upon a later generalisation:

The Oppenheimer–Volkoff equation

- Will derive the Schwarzschild interior metric and OV equation by using as much as possible of what you did last year in Schwarzschild vacuum solution
- (Note you will not be asked a question on derivation of this in exam.)
- So last year you derived the Ricci Tensor components for a metric of the form

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2 \tag{4}$$

  where A and B are general functions of r.
- Here we want to set $G_{\mu\nu} = -\frac{8\pi}{c^4} G T_{\mu\nu}$ where $G_{\mu\nu}$ is the Einstein tensor defined by being the trace reversed version of the Ricci tensor, i.e. we have

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R.$$

  (Exercise: show that G^{μ}_{μ} is indeed minus $R \equiv R^{\mu}_{\mu}$.)
- So since the $R_{\mu\nu}$ are available to us from last year, the equations we want are

$$\boxed{R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu}.} \tag{5}$$

Einstein Tensor

- The Einstein tensor $G_{\mu\nu}$ is the tensor that describes the curvature of spacetime in the field equations of GR

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R,$$

i.e., the trace reversed version of the Ricci tensor.

- Ricci tensor $R_{\mu\nu}$ is directly related to the Riemann curvature tensor $R^{\lambda}{}_{\mu\nu\sigma}$:

$$R_{\mu\nu} = R^{\sigma}{}_{\mu\nu\sigma}.$$

- Relation to Gaussian curvature K

$$K = \frac{R_{1212}}{g},$$

where $g = \det[g_{\mu\nu}]$ is the Ricci tensor.

- Actually, since $R_{\mu\nu}$ is quite complicated, and the SET is relatively simple, then sensible to manipulate the trace a bit more.
- In particular, alternative form of Einstein's equations obtained by writing (5) in terms of mixed components

$$R^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu} R = -\frac{8\pi G}{c^4} T^{\mu}_{\nu},$$

  and contracting by setting $\mu = \nu$.
- We thus find that $R = \frac{8\pi G}{c^4} T$, where $T \equiv T^{\mu}_{\mu}$ $\Rightarrow$ can write Einstein's equations (5) as

$$\boxed{R_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right),} \tag{6}$$

  so this is the form we will use.

Derivation of the Oppenheimer–Volkoff equation [non-examinable]

- Next part NON-EXAMINABLE until point we indicate — will summarise important physical results after this.
- Here is what was given last year (on page 7 of 'Topic 9') for the Ricci entries for our chosen form of metric:
- The off-diagonal components of $R_{\mu\nu}$ vanish and diagonal components are

$$R_{tt} = -\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB}, \tag{7}$$

$$R_{rr} = \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB}, \tag{8}$$

$$R_{\theta\theta} = \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right), \tag{9}$$

$$R_{\phi\phi} = R_{\theta\theta}\sin^2\theta. \tag{10}$$

- Meanwhile, the r.h.s. of (6) is got from assuming the matter is is a perfect fluid, so

$$T_{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) u_\mu u_\nu - P g_{\mu\nu}, \tag{11}$$

where $\rho(r)$ is proper mass density and $P(r)$ is isotropic pressure in IRF of fluid (only functions of r if static).
- Using $u_\mu u^\mu = c^2$ gives

$$T = \left(\rho + \frac{P}{c^2}\right) c^2 - P\delta^\mu_\mu = \rho c^2 - 3P,$$

$\Rightarrow$ field equations (6) read

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}\left[\left(\rho + \frac{P}{c^2}\right) u_\mu u_\nu - \tfrac{1}{2}(\rho c^2 - P) g_{\mu\nu}\right]. \tag{12}$$

- First determine consequences of vanishing off-diagonal components $R_{ti} = 0$ for $i = r, \theta, \phi$.
- From field equations (12) and using $g_{ti} = 0 \Rightarrow u_i u_t = 0$.
- Combine with $u_\mu u^\mu = c^2 \Rightarrow$ fluid 4-velocity is

$$[u_\mu] = c\sqrt{A}(1, 0, 0, 0). \tag{13}$$

- $\Rightarrow$ spatial 3-velocity of fluid must vanish everywhere.
- Thus the metric choice (4) being independent of t automatically ensures matter distribution is static.
- Now use diagonal $(\mu = \nu)$ components field equations (12) to get

$$\begin{aligned} R_{tt} &= -\frac{4\pi G}{c^4}(\rho c^2 + 3P)A, && (14)\\ R_{rr} &= -\frac{4\pi G}{c^4}(\rho c^2 - P)B, && (15)\\ R_{\theta\theta} &= -\frac{4\pi G}{c^4}(\rho c^2 - P)r^2, && (16)\\ R_{\phi\phi} &= R_{\theta\theta}\sin^2\theta. && (17) \end{aligned}$$

- These equations give

$$\frac{R_{tt}}{A} + \frac{R_{rr}}{B} + \frac{2R_{\theta\theta}}{r^2} = -\frac{16\pi G}{c^2}.$$

- Substituting expressions (7–10) for Ricci tensor components:

$$\left(1 - \frac{1}{B}\right) + \frac{rB'}{B^2} = \frac{8\pi G}{c^2}r^2\rho, \tag{18}$$

- which can be rewritten as
$$\frac{d}{dr}\left[r\left(1-\frac{1}{B}\right)\right]=\frac{8\pi G}{c^4}r^2\rho.$$
- Integrate and note constant of integration must be zero for $B(r)$ to be non-zero at origin (as demanded by (18)):
$$\boxed{B(r)=\left[1-\frac{2Gm(r)}{c^2r}\right]^{-1},}\tag{19}$$
where we define the function
$$m(r)=\int_0^r\rho(r)4\pi r^2\,dr.\tag{20}$$
- Function $m(r)$ appears to be mass contained within coordinate radius r, but this interpretation isn't quite correct — will come back to this!

Getting A and P

- Now consider differential equations satisfied by $A(r)$.
- Can in fact get A by using our Gauss Theorema Egregium on the 2d (t, r) subspace instead of the (r, ϕ) subspace we have used so far.
- However, this requires knowledge of how pressure enters the equations — will come back to this when doing cosmology — so for now stay with full GR.
- Easiest here to get a first order differential equation for A by using R_{tt} and $R_{\theta\theta}$ equations, solving them for A'' and B', and then inserting this into the equation involving R_{rr}.
- Doing this implies
$$\frac{r}{A}\frac{dA}{dr} = \frac{8\pi G}{c^4} r^2 BP - 1 + B. \tag{21}$$
- If we then differentiate *this* equation w.r.t. r, and use our knowledge so far about A'', B and A', one can see that one must end up with an equation for the derivative of P, with other derivatives not appearing in it.
- Doing this one gets the famous Oppenheimer–Volkoff (OV) equation (J. R, Oppenheimer and G. M. Volkoff, Physical Review, **55**, 374 (1939)) and reads with all constants in:

$$\boxed{\frac{dP}{dr} = -\frac{1}{r^2}(\rho c^2 + P)\left[\frac{4\pi G}{c^4}Pr^3 + \frac{Gm(r)}{c^2}\right]\left[1 - \frac{2Gm(r)}{c^2 r}\right]^{-1}} \tag{22}$$

End of non-examinable portion

- Now, want to point out that the OV equation, which looks quite forbidding in way usually written, is actually quite simple, indeed we can more or less guess it!
- Here it is (using units in which $c = 1$ for the moment, since structure clearer)

$$\boxed{\frac{dP}{dr} = -\frac{v^2}{r}(\rho + P).} \tag{23}$$

- Here v is the ordinary velocity in a circular orbit at radius r within the fluid (we're assuming such an orbit is possible — might have to clear a channel through the fluid to enable this if wanted to do it for real!)
- Compare with Newtonian equivalent can see we've just replaced $GM(r)/r^2$ with v^2/r:

$$\frac{1}{\rho}\frac{dP}{dr} = -\frac{GM(r)}{r^2}.$$

- i.e. equated inward gravitational force with centripetal force, and changed ρ to $\rho + P$
- That's it! Let's demonstrate the equivalence.
- Go back to the full equation (22), and let's rewrite it without all the constants (using Planck units, with $G = c = 1$) to see the structure of it better

$$\boxed{\frac{dP}{dr} = -\frac{(\rho + P)(m(r) + 4\pi r^3 P)}{r(r - 2m(r))}.} \tag{24}$$

- Now we can start to see it is pretty much like what we started with, equation (23). They would be the same if the circular orbit velocity satisfied

$$v^2 = \frac{m(r) + 4\pi r^3 P}{r - 2m(r)}. \tag{25}$$

- But next lecture we will show generally that for a metric of our general A, B form, then

$$v^2 = \frac{rA'}{2A}.$$

- Evaluating this using the A' from (21) we get back precisely (25), so everything ties together! (and have an interesting formula for circular velocity inside a fluid with pressure).

Understanding the B result

- We can apply our 2d Gauss Theorema Egregium to the (r, ϕ) subspace of the A, B metric.
- Only B will figure in this, and we can just use our result for the Schwarzschild metric g_{rr} function appropriate to the exterior of a mass $m(r)$, to get

$$\boxed{B(r) = \left[1 - \frac{2Gm(r)}{c^2 r}\right]^{-1},} \tag{26}$$

- Note

$$m(r) = \int_0^r \rho(r) 4\pi r^2 \, dr. \tag{27}$$

  and we use a small $m(r)$ since then M can be reserved for the total mass (at the edge of the star or object).
- Note we're assuming a version of Gauss' Theorem (i.e. that can ignore the effects of mass outside the spherical surface one is working with) but this works here!
- This is fine, but there are some subtleties, which we can quickly see:

- We're saying the function $m(r)$ appears to be mass contained within coordinate radius r.
- But: this interpretation not quite correct, since proper spatial volume element for metric (4) is
$$d^3V = \sqrt{B(r)}\, r^2 \sin\theta\, dr\, d\theta\, d\phi.$$
- Thus proper integrated 'mass' (i.e. energy/c^2) within coordinate radius r is
$$\begin{aligned}\tilde{m}(r) &= \int_0^r \rho(r)\sqrt{B(r)}\, 4\pi r^2\, dr \\ &= \int_0^r \rho(r)\left[1 - \frac{2Gm(r)}{c^2 r}\right]^{-1/2} 4\pi r^2\, dr.\end{aligned}$$
- But: $m(r)$, not $\tilde{m}(r)$, in metric coefficient $B(r)$ in (26).
- If object extends to $r = R \Rightarrow$ spacetime geometry outside is Schwarzschild metric with mass parameter $M = m(R)$, rather than $\widetilde{M} = \tilde{m}(R)$.
- Difference $E = (\widetilde{M} - M)c^2$ corresponds gravitational binding energy of object = energy required to disperse material comprising object to infinite spatial separation.

Physical implications of the OV equation

- Going back to the OV equation in the form (24), one can see that the nice thing about this equation, is that since $m(r)$ is just the spherical integral of $\rho(r)$, then given a density distribution $\rho(r)$ we can integrate this equation to find $P(r)$, without having to know anything about A! (or indeed B, though we already have an equation for this we used en route).
- Alternatively, suppose we know a relation between ρ and P of the form

$$P = P(\rho).$$

- This is known as an equation of state. If we have this, plus the definition of $m(r)$, plus the OV equation, then can determine all the physics.
- Note: for many astrophysical systems, matter obeys polytropic equation of state $P = K\rho^{\gamma}$, where K and γ are constants.
- Usual notation: $\gamma = 1 + 1/n$, where n is polytropic index (NB: a polytropic process obeys the relation $PV^n =$ constant).

- So now have a closed system of three equations (turn $m(r)$ integral expression into derivative relation, since easier to work with)

$$\boxed{\frac{dm(r)}{dr} = 4\pi r^2 \rho(r),} \tag{28}$$

$$\boxed{\frac{dP}{dr} = -\frac{1}{r^2}(\rho c^2 + P)\left[\frac{4\pi G}{c^4} P r^3 + \frac{Gm(r)}{c^2}\right]\left[1 - \frac{2Gm(r)}{c^2 r}\right]^{-1},} \tag{29}$$

$$\boxed{P = P(\rho).} \tag{30}$$

- Have two coupled first-order differential equations.
- $\Rightarrow$ need two boundary conditions to obtain unique solution.
- First BC straightforward: must have $m(0) = 0$.
- One further BC to be specified: most common to choose central pressure $P(0)$, or equivalently central density $\rho(0)$.
- Very few exact solutions known for realistic equations of state $\Rightarrow$ in practice system of equations integrated numerically.

- Procedure: 'integrate outwards' from $r = 0$ until the pressure drops to zero $\Rightarrow$ condition defines surface $(r = R)$ of star.
- Before looking for particular solutions to equations, first consider their Newtonian limit.
- In fact, (28) and (30) remain unchanged in this limit.
- Only equation (29) for pressure gradient is simplified.
- In Newtonian limit: $P \ll \rho \Rightarrow 4\pi r^3 P \ll mc^2$. Also, metric close to Minkowski $\Rightarrow$ $2Gm/(c^2 r) \ll 1$ (equivalently can just take $c \to \infty$).
- Thus, Oppenheimer–Volkoff equation reduces to

$$\frac{dP}{dr} = -\frac{Gm(r)\rho(r)}{r^2} \tag{31}$$

which is Newtonian equation for hydrostatic equilibrium.
- Comparing (29) and (31) $\Rightarrow$ relativistic effects steepen pressure gradient relative to Newtonian case.
- Thus, for object to remain in hydrostatic equilibrium, fluid experiences stronger internal forces when general-relativistic effects taken into account

Schwarzschild constant-density solution

- Simplest analytic interior solution for relativistic star obtained by assuming that throughout the star
$$\boxed{\rho = \text{constant},}$$
which constitutes an equation of state.
- No physical justification, but on borderline of realistic.
- Corresponds to ultra-stiff equation of state for incompressible fluid $\Rightarrow$ speed of sound fluid $(dP/d\rho)^{1/2}$ is infinite.
- But: believed interiors of dense neutron stars of nearly uniform density $\Rightarrow$ simple case of some practical interest
- We find from the OV equation in this case (see Appendix) the following result:

$$\boxed{P(r) = \rho c^2 \frac{\left(1 - \frac{2\mu r^2}{R^3}\right)^{1/2} - \left(1 - \frac{2\mu}{R}\right)^{1/2}}{3\left(1 - \frac{2\mu}{R}\right)^{1/2} - \left(1 - \frac{2\mu r^2}{R^3}\right)^{1/2}}} \qquad \text{where } \mu = GM/c^2 = R_S/2.$$

- This looks not very pleasant, but note the actual r dependence is quite simple — just an r^2 in the first term at the top, and second term at the bottom.
- Setting $r = 0$, we get

$$P_0 = \rho c^2 \frac{1 - \left(1 - \frac{2\mu}{R}\right)^{1/2}}{3\left(1 - \frac{2\mu}{R}\right)^{1/2} - 1}, \tag{32}$$

- Here $\mu = GM/c^2$, where M is $m(r)$ evaluated at the surface $r = R$.
- Note that a series expansion of P in μ, yields

$$P(r) \approx \frac{\rho_0 c^2}{4} \frac{2\mu}{R} \left(1 - \frac{r^2}{R^2}\right)$$

- i.e. exactly the Newtonian expression, to first order — very sensible.
- So what happens when we plot the functions and investigate domains outside Newtonian experience?

- First, how does the total solution look (interior plus exterior)?
- From Appendix $ds^2 = Adt^2 + Bdr^2 + r^2 d\Omega$ so inside the star:

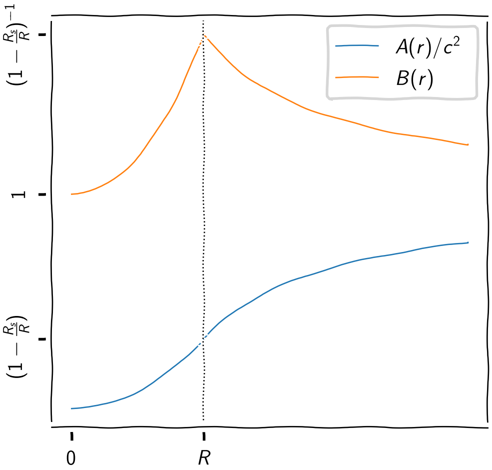
$$A = \frac{c^2}{4}\left[3\left(1-\frac{2\mu}{R}\right)^{1/2} - \left(1-\frac{2\mu r^2}{R^3}\right)^{1/2}\right]^2$$

$$B = \left(1-\frac{2\mu r^2}{R^3}\right)^{-1}.$$

- Outside we have the usual Schwarzschild solution

$$A = 1/B = \left(1-\frac{2\mu}{r}\right).$$

- Exercise: show from the that A, unlike B, has continuous first derivatives.



Buchdahl's theorem

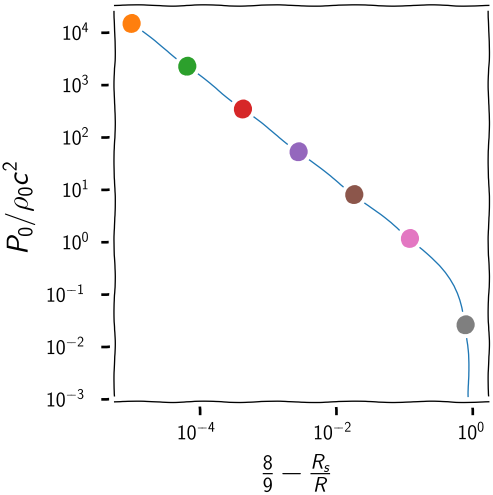
- Most important feature of Schwarzschild constant-density solution is it imposes constraint connecting star 'mass' M and its (coordinate) radius R.
- Equation (32) for central pressure is

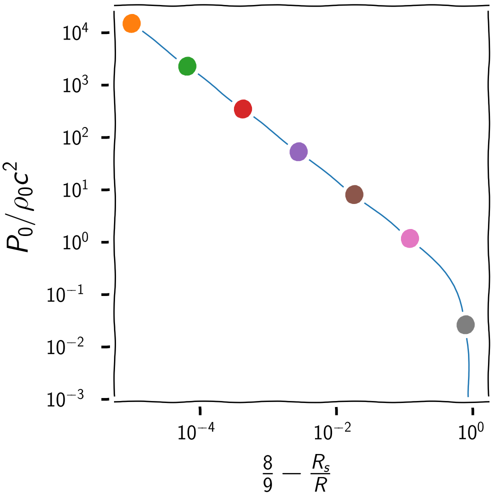
$$P_0 = \rho c^2 \frac{1 - (1 - 2\mu/R)^{1/2}}{3(1 - 2\mu/R)^{1/2} - 1},$$

- We can see the denominator vanishes if

$$3\left(1 - \frac{2\mu}{R}\right)^{1/2} = 1.$$

- Unwrapping this gives $2\mu/R = 8/9$ i.e. $\boxed{R_S/R = 8/9}$.
- Thus any (constant density) star for which the Schwarzschild radius starts to approach 8/9 of the actual radius, will have central pressure $\to \infty$!





- Central pressure as $R_S \to \frac{8}{9}R$.
- Note the log scale on both axes.

- Pressure profile as $R_S \to \frac{8}{9}R$.
- Colours correspond between diagrams.

- This is a physical thing: pressure is a scalar $\Rightarrow$ infinity persists in any coordinate system.
- We deduce the following limit:

$$\boxed{\frac{GM}{c^2 R} < \frac{4}{9}.} \tag{33}$$

- Constraint as proved here holds for object of constant density, but Buchdahl's theorem states it is valid for any equation of state.
- Makes a certain intuitive sense — nothing can be 'stiffer' than infinitely stiff.
- Equation (33) $\Rightarrow$ upper limit on star mass for a fixed radius.
- If attempt to pack more mass inside R than is allowed by (33)
- $\Rightarrow$ GR admits no static solution.
- $\Rightarrow$ hydrostatic equilibrium broken by increased attraction.
- $\Rightarrow$ star must therefore collapse inwards without stopping.

- Throughout collapse, exterior geometry described by Schwarzschild metric $\Rightarrow$ obtain Schwarzschild black hole.
- Limit (33) quite easily reached. For density appropriate to neutron star, $\sim 5 \times 10^{17}\,\mathrm{kg m^{-3}}$ $\Rightarrow M < 10^{31}\mathrm{kg} \sim 5M_{\odot}$.
- So we would predict $5M_{\odot}$ as an upper limit on any neutron star's mass.
- Actual limit depends on detailed calculations involving a more realistic equation of state (though what this is isn't clear yet!), and also have to consider stability, not just static calculations we have done.
- About $3M_{\odot}$ is reasonable, so we weren't far off!

Summary

- The stress-energy tensor for a perfect fluid

$$\boxed{T^{\mu\nu} = (\rho + P/c^2)u^\mu u^\nu - Pg^{\mu\nu}.}$$

- The Oppenheimer–Volkoff equation for pressure and gravity.
- Gravitational binding energy, m and $\tilde{m}$.
- The Schwarzschild interior solution for a constant density star:

$$\boxed{ds^2 = \frac{1}{4}\left[3\left(1 - \frac{2\mu}{R}\right)^{1/2} - \left(1 - \frac{2\mu r^2}{R^3}\right)^{1/2}\right]^2 c^2 dt^2 - \left(1 - \frac{2\mu r^2}{R^3}\right)^{-1} dr^2 - r^2 d\Omega.}$$

- Buchdahl's theorem for the collapse point of a star $\frac{R_S}{R} = \frac{8}{9}$.

Next time

Particle motion and energy in the Schwarzschild metric.

Appendix:

Schwarzschild constant-density solution

- Simplest analytic interior solution for relativistic star obtained by assuming that throughout the star

$$\boxed{\rho = \text{constant},}$$

  which constitutes an equation of state.
- Believed interiors of dense neutron stars of nearly uniform density $\Rightarrow$ simple case of some practical interest.
- Equation (20) immediately integrates to give

$$\boxed{m(r) = \begin{cases} \frac{4}{3}\pi\rho r^3 & \text{for } r \leqslant R \\ \frac{4}{3}\pi\rho R^3 \equiv M & \text{for } r > R, \end{cases}} \tag{34}$$

  where R is the radius of the star, as yet undetermined, and M is mass parameter for Schwarzschild metric describing exterior spacetime geometry.

- Oppenheimer–Volkoff equation (22) becomes

$$\frac{dP}{dr} = -\frac{4\pi G}{3c^4} r(\rho c^2 + P)(\rho c^2 + 3P)\left(1 - \frac{8\pi G}{3c^2}\rho r^2\right)^{-1}.$$

which is separable and so

$$\int_{P_0}^{P(r)} \frac{d\bar{P}}{(\rho c^2 + \bar{P})(\rho c^2 + 3\bar{P})} = -\frac{4\pi G}{3c^4}\int_0^r \frac{\bar{r}\,d\bar{r}}{1 - 8\pi G\rho\bar{r}^2/(3c^2)},$$

where $P_0 = P(0)$ is central pressure of the star.

- Performing these standard integrals, one finds

$$\frac{\rho c^2 + 3P}{\rho c^2 + P} = \frac{\rho c^2 + 3P_0}{\rho c^2 + P_0}\left(1 - \frac{8\pi G}{3c^2}\rho r^2\right)^{1/2}. \tag{35}$$

- At surface $r = R$ of star, $P = 0 \Rightarrow$ LHS equals unity, so

$$R^2 = \frac{3c^2}{8\pi G\rho}\left[1 - \left(\frac{\rho c^2 + P_0}{\rho c^2 + 3P_0}\right)^2\right],$$

$\Rightarrow$ radius of star of uniform density ρ with central pressure P_0.

- Alternatively, rearrange this result and use (34) $\Rightarrow$ expression for central pressure:

$$P_0 = \rho c^2 \frac{1-(1-2\mu/R)^{1/2}}{3(1-2\mu/R)^{1/2}-1}, \tag{36}$$

  where $\mu = GM/c^2$.
- Using this expression to replace P_0 in (35) gives (for $r \leqslant R$)

$$\boxed{P(r) = \rho c^2 \frac{(1-2\mu r^2/R^3)^{1/2}-(1-2\mu/R)^{1/2}}{3(1-2\mu/R)^{1/2}-(1-2\mu r^2/R^3)^{1/2}}.} \tag{37}$$

- To complete solution, remains to determine $A(r)$ and $B(r)$.
- From (19) and (34), we immediately find that

$$B(r) = \left(1-\frac{2\mu r^2}{R^3}\right)^{-1}. \tag{38}$$

- Note: at star surface $r = R$, solution matches Schwarzschild metric for exterior solution
- Function $A(r)$ obtained from (21), (34) and (38).

- Fix integration constant arising from (21) by imposing boundary condition that $A(r)$ matches corresponding expression in Schwarzschild metric at $r = R$:

$$A(r) = \frac{c^2}{4}\left[3\left(1 - \frac{2\mu}{R}\right)^{1/2} - \left(1 - \frac{2\mu r^2}{R^3}\right)^{1/2}\right]^2. \tag{39}$$

- Expressions (38) and (39) constitute Schwarzschild's interior solution for a constant-density object.