Relativistic Astrophysics and Cosmology — Examples 1 — 2023

1. Calculate the gravitational redshift between a point at infinity and the surface of (a) the Earth, (b) the Sun, (c) a white dwarf and (d) a neutron star. Could any of these redshifts be measured? If so, how? (You will need to consider whether any features exist which could be used to measure the redshift, and if so, whether or not the redshift would be visible in the presence of other broadening effects.)

2. Consider an experiment on a sealed space vehicle, with no windows, in circular near-Earth orbit, in which two test particles are released inside the spacecraft and their relative separations are monitored. The experiment is being carried out in order to determine if there is a gravitating body nearby. The two test particles are released from rest, with an initial separation vector $\boldsymbol{\xi}$. If the z axis points away from the Earth, find the differential equation satisfied by the separation vector $\boldsymbol{\xi}$ between the particles. (This was given in Handout 2.) If a measurement accuracy of about 1 mm is available, and the particles are started with a separation in the z direction of 1 m, how long will it be before the presence of the Earth can be inferred from this experiment?

   If, instead of two particles, a spherical shell of particles was released from rest, show that, for small displacements $\boldsymbol{\xi}$, the motion of the shell is volume-preserving.

3. In the lectures an 'Einstein Tower' experiment was discussed and it was shown that a freely-falling observer who drops from rest from above the top of the tower (height h) at the moment when the photon begins climbing up from the bottom, finds that to first order in gh/c^2 there is no change in the photon energy between top and bottom of the tower. Generalise this analysis to the case where instead of falling from rest, the (unfortunate) observer is projected downwards with an initial speed u. Explain why these results support the Strong Equivalence Principle.

4. Find the geodesic equations on the surface of a sphere, radius a, using the standard (θ, ϕ) coordinate system. Show that these equations are satisfied by great circles. What are the geodesics on the surface of a cylinder?

5. It is sometimes said that an astronaut falling into a black hole would feel nothing peculiar as he or she crossed over the event horizon. Discuss whether this is true, and how the statement depends upon the black hole mass, using the following approach. First, use the Newtonian force law to estimate the tidal force affecting two compact objects each of mass m, joined by a taut wire of length l, falling radially into a distant black hole of mass M.

   The tidal force derived using this Newtonian approach is in fact equal to the proper result derived using full general relativity *(Optional exercise: demonstrate this using a suitable notion of force in GR)*, and can therefore be applied all the way up to the hole. Thus show that at the horizon the wire must support a tension of

$$\frac{mc^6 l}{8G^2M^2}.$$

Using this, estimate numerically the tidal force on a human being at the event horizon for (a) a $1M_\odot$ mass black hole, and (b) a black hole of mass $10^8 M_\odot$.

6. Starting from the Schwarzschild metric, show that photons moving radially travel at a coordinate speed
$$\frac{dr}{dt} = \pm c\left(1 - \frac{2GM}{c^2 r}\right).$$
Suppose a photon is emitted from a position $r_0 > R_S$ (where R_S is the Schwarzschild horizon) at time t_0, and moves radially inwards towards the horizon. Find an equation for the t coordinate of the photon as a function of r, and sketch this in the (r, t) plane. What coordinate time does it take the photon to reach the horizon?

7. Use the geodesic energy equation, to show that a particle of non-zero rest mass, falling from rest at infinity radially towards a black hole, satisfies
$$\frac{dr}{d\tau} = -\left(\frac{2GM}{r}\right)^{\frac{1}{2}},$$
where τ is the particle's proper time. Use this to find an expression for the elapse of proper time between the event that the particle crosses the Schwarzschild horizon, and the event that it reaches $r = 0$. What is this time for a solar mass black hole?

8. A radar signal is sent from Earth towards Venus, when Venus is on the far side of the Sun. The signal is reflected off Venus and travels back to Earth, where the total round trip time is recorded. Ignoring the effects of gravitational bending, i.e. assuming a *straight line* path between Venus and the Earth, just passing the Sun's disc, and approximating all the motion as purely radial, use the results of Question 6 to get a numerical estimate of the effect of the Sun's gravity on the total round trip travel time. Does the Sun's presence speed up or slow down the arrival of the signal back at Earth? ($R_\odot = 7 \times 10^8$ m; Earth–Sun distance $= 1.5 \times 10^{11}$ m; Venus–Sun distance $= 1.1 \times 10^{11}$ m.)

9. All massive objects look larger than they really are. Show that a light ray grazing the surface of a massive sphere of coordinate radius $r > 3GM/c^2$ will arrive at infinity with impact parameter
$$b = r\left(\frac{r}{r - 2GM/c^2}\right)^{1/2}.$$
Hence show that the apparent diameter of the Sun ($M_\odot = 2 \times 10^{30}$ kg, $R_\odot = 7 \times 10^8$ m) exceeds the coordinate diameter by nearly 3 km.

10. In Handout 4, two different expressions for 'mass' were discussed, $m(r)$ and $\tilde{m}(r)$. Show that, in the limit of weak fields, and for a uniform density object, their difference (times c^2) at the edge of a star of radius R coincides with the Newtonian gravitational binding energy of the star.