General Relativity Recap

Relativistic Astrophysics and Cosmology: Lecture 2

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Monday 9th October 2023

Pre-lecture question:

Why/how does gravity cause clocks to speed up and slow down?

Housekeeping

- Recommended book for the course:
  - "General Relativity for Physicists" (Hobson, Efstathiou and Lasenby)
  - Available in online form or hard copy from the university libraries
    https://ebookcentral.proquest.com/lib/cam/detail.action?docID=244399
  - Covers the mathematical material at an appropriate level and in the spirit of the course
- First examples sheet now available online and in printed form
  - There will be for example sheets in total
  - Please sign up to supervision via sign-up sheet on Moodle
- Lecture finishes before 11:00 – feel free to pick up tea & a biscuit at IoA coffee!

Overview

Last time

- Covered the broad outline of the course.
- Emphasised that we will build intuition and understanding of GR, leaving tensor calculus and advanced differential geometry to PhD/self study.
- Discussed the essence of the equivalence principles, i.e. the equivalence of:
  - local gravity and acceleration,
  - special relativity and free fall.

This lecture

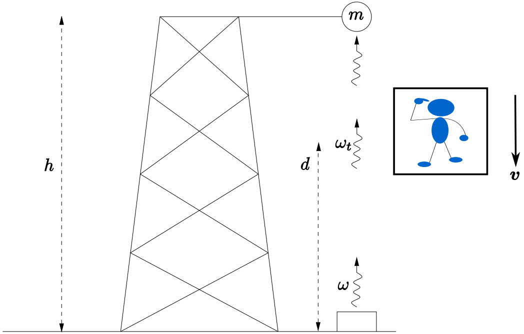
- Revise mathematics of GR.
- Calculating geodesics through curved space: "Spacetime tells matter how to move".

Next lecture

- Computing curvature for the Schwarzschild metric: "Matter tells spacetime how to curve".

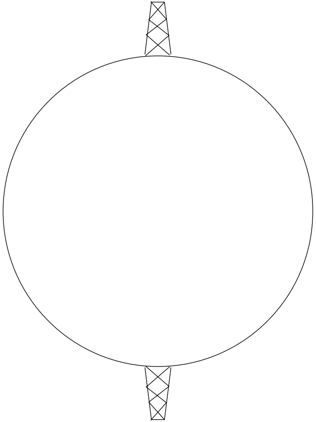
Einstein tower revisited

- In a freely falling frame we should get no redshift, since SR implies no gravitational redshift, via the Schild argument.
- Suppose the photon starts at the bottom of a tower with frequency ω in the earth frame. It climbs a height d, taking time d/c, and has a new frequency $\omega_t = \omega(1 - gd/c^2)$ in the earth frame.
- In the meantime an observer starts accelerating from the top of the tower. After time d/c the observer has a velocity $v = gd/c$ downwards. The photon frequency in the observer's frame is



$$
\begin{aligned}
\omega' &= \omega_t \left(\frac{1 + v/c}{1 - v/c} \right)^{1/2} \\
&= \omega(1 + gd/c^2)^{1/2}(1 - gd/c^2)^{1/2} \\
&\approx \omega
\end{aligned}
$$

- Thus one finds that in the freely falling frame the photon frequency does not change during the ascent, verifying (via the Pound-Rebka experiment) that the SEP works for electromagnetic waves (light) to an accuracy of $\sim 1\%$.
- Thus all looks well, and as though we have succeeded in getting rid of gravity.
- *But*, this only works over regions where the field can be taken as uniform.
- E.g. consider the tower experiment again, but for two towers, on opposite sides of the Earth.



- Here need two separate freely falling frames, one for each tower, in order to make the physics look special relativistic in each.
- Even over a limited region, if we make very precise measurements over an extended period of time, then we will see neighbouring test particles, initially on parallel courses, start deviating. This indicates real gravity effects, manifested by curvature.

As an example of particles on initially parallel courses deviating, consider an (x, y, z) frame inside the International Space Station (ISS), with the z axis pointing away from the Earth.

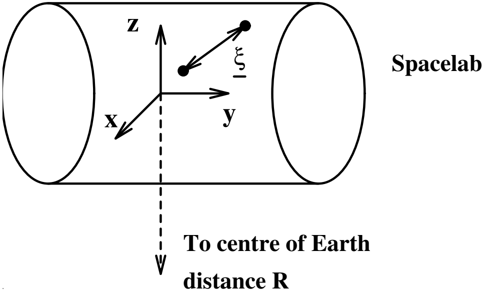


Figure: *An experiment to show that a freely falling frame is only SR over a limited region of space and time.*

Two test particles are released from rest, with an initial separation vector $\boldsymbol{\xi}$. In an example you are asked to show that this separation vector varies in time as

$$\frac{d^2}{dt^2}\begin{pmatrix}\xi^x\\ \xi^y\\ \xi^z\end{pmatrix} = \frac{GM_E}{R^3}\begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & +2\end{pmatrix}\begin{pmatrix}\xi^x\\ \xi^y\\ \xi^z\end{pmatrix}$$

Note particularly the R^{-3} dependence of these 'forces'. For a spherical system this is characteristic of 'true' gravity, that part not transformable away by an acceleration, and has the dependence on distance of what we commonly call 'tidal forces'. This raises several points about the true nature of gravity:

True Gravity

- 'True' gravity manifests itself via the relative separation, or coming together, of test particles initially on parallel tracks.
- To cope with this, as is immediately apparent in the example with two towers on opposite sides of the Earth, we may need different inertial frames at different places and times.
- Physics is SR over a sufficiently localized region and over a sufficiently small time (this is the content of the SEP), but in time, or further away from the origin, discrepancies will be noticed.
- Thus there is no such thing as a global Lorentz frame in the presence of a non-uniform gravitational field.
- To try to patch these different frames together, is what **curvature** is about.

Curvature as the description of gravity

- We have seen that the equivalence principle is experimentally verified.
- This means, that we can always locally remove a gravitational force by going to the freely falling frame at that point.
- This is because all objects accelerate the same way at the point.
- Since behaviour is identical for all objects, then maybe it is a property of the structure of spacetime at this point.
- These observations led Einstein to make a profound proposal that simultaneously provides for a relativistic description of gravity and naturally incorporates the equivalence principle.
- **Einstein's proposal: gravity should no longer be regarded as a force in the conventional sense, but rather as a manifestation of the curvature of the spacetime, where this curvature is induced by the presence of matter.**
- This is the central idea underpinning general relativity.
- **Gravity doesn't cause time to speed up and slow down, gravity is the bending and stretching of time (and space) by matter.**

- To understand how to apply this, one needs some mathematical apparatus.
- Can get a long way in GR via using and understanding
  - The metric — i.e. how distances relate to coordinates
  - Geodesics — i.e. how spacetime tells particles to move
  - Curvature — i.e. how spacetime responds to matter
- We will be able to get a long way with this by using a simplified approach to curvature that avoids too many long calculations!
- But we will also make sure that we get things right by using results from full field equations of GR where appropriate
- Don't worry if haven't covered this before — won't be asked about full tensor calculus derivations in the exam, even if used here

Special Relativity in one slide

- Unification of spacetime into four vectors in Minkowski geometry

$$x = (ct, \vec{x}), \quad u = \dot{x} = \gamma(c, \vec{v}),$$
$$p = mu = (E/c, \vec{p}), \quad \gamma \equiv \tfrac{dt}{d\tau} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where $\dot{x} = \frac{dx}{d\tau}$ is four-velocity wrt proper time τ, and $\vec{v} = \frac{d\vec{x}}{dt}$ is coordinate velocity.

- Invariant lengths of four vectors

$$w \cdot w = w_t^2 - \vec{w}^2, \quad u \cdot u = c^2 \text{ or } 0$$
$$p \cdot p = m^2 c^2$$

- Lorentz transformation (active or passive)

$$\begin{pmatrix} w_t' \\ w_x' \end{pmatrix} = \begin{pmatrix} \gamma & \gamma v/c \\ \gamma v/c & \gamma \end{pmatrix} \begin{pmatrix} w_t \\ w_x \end{pmatrix}$$

- Rapidity defined by $v = c \tanh \psi$ gives $\gamma = \cosh \psi$ and $\gamma v = c \sinh \psi$
- Length contraction and time dilation derived by putting $(0, 1)$ and $(c, 0)$ into LT.
- Loss of simultaneity resolves most paradoxes

General relativity: coordinates and notation

- Given a spacetime, we lay down coordinates x^μ, which label physical events.
- Coordinates are labels, nothing more. They are not vectors in the general case.
- Good choices may have physical interpretations, e.g. something measured by a ruler, or a clock, but this is neither essential, nor always desirable and sometimes even impossible
- We use Einstein summation convention:
  - Single "free" indexes range from 0 to 3, with 0 indicating the time-like component,
  - Doubly repeated indexes (one upstairs, one downstairs) are assumed summed:

$$\sum_{a=0}^{3} v^a w_a \equiv v^a w_a$$

- Index notation & summation convention, while in my view one of Einstein's greatest contributions to Physics & Mathematics, is terrible for doing specific calculations.
- As a rule of thumb, you should never explicitly set an index to a value e.g. x^0.
- Instead of using x^0, x^1, x^2, x^3 or u^0, u^1, u^2, u^3 as your variables, use e.g. t, r, θ, ϕ, and u^t, u^r, u^θ, u^ϕ when referring to specific coordinates.

The dot product

- The dot product is generalised to a form involving a metric

$$u \cdot u = c^2 = \left(\frac{cdt}{d\tau}\right)^2 - \left(\frac{d\vec{x}}{d\tau}\right)^2 \qquad \Rightarrow \qquad ds^2 = c^2 d\tau^2 = c^2 dt^2 - d\vec{x}^2 = g_{\mu\nu} dx^\mu dx^\nu .$$

- The equivalence principle states that SR applies locally, which means that there are always "Locally inertial coordinates (LIC)" $(t, \vec{x})$ where the metric takes Minkowski form
- In general however it may/need not take this form globally, either due to
  - coordinate choice (e.g. in 3D cylindrical polars $ds^2 = dr^2 + r^2 d\theta^2 + dz^2$),
  - curvature (e.g. in 2D a sphere $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$).
- For notational convenience we define this Minkowski form as $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.
- What is the metric?

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- For notational convenience we define this Minkowski form as $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$.
- What is the metric? **The metric is the dot product**
  - i.e. the geometrical function encoding angles, lengths, proper times & rapidities

$$a\cdot b = a^t b^t - |\vec{a}||\vec{b}|\cos\theta = (a\cdot a)^{1/2}(b\cdot b)^{1/2}\cosh\psi$$

  expressed in an arbitrary coordinate system $a\cdot b = a^\mu g_{\mu\nu} b^\nu$.

Summation convention mnemonics

- The metric raises and lowers indices $w^\mu g_{\mu\nu} \equiv w_\nu$, $A^{\mu\nu} g_{\nu\lambda} = A^\mu_\lambda$
- This is useful as a shorthand for removing the number of metric terms

$$a \cdot b = a^\mu g_{\mu\nu} b^\nu = a^\mu b_\mu = a_\nu b^\nu = a_\nu g^{\nu\mu} b_\mu$$

- From this we can observe that $g^{\mu\nu}$ is the inverse of the metric, or equivalently $g^\mu_\nu = \delta^\mu_\nu$.
- Mathematicians will then enjoy going into great detail about covectors, and identification of vectors with directional derivatives, but this is not necessary to do any calculations.
- General point with tensor geometry: Don't memorise equations with indices in.
- Memorise the general shape, and fill in indices
- Usually only need to remember at most one or two tricks to break the symmetry and let summation convention do the rest for you.

Derivatives

- Taking the derivative of a scalar (field or quantity) is easy. To differentiate vectors:

$$\nabla v = \partial v + \Gamma v, \qquad \Rightarrow \qquad .$$

- Mnemonics: (a) Upstairs is positive (b) Derivative is the last term on Γ
- The Γ connection coefficients are mostly coordinate compensations adjusting components of a changing vector on a changing grid. Very hard to intepret (i.e. don't try).
- Generalisation to tensors is straightforward:

$$\nabla T = \partial T + \Gamma T + \ldots \Gamma T \qquad \Rightarrow \qquad .$$

- Parallel transport along a curve $x^\mu(\sigma)$ of scalar/vector/tensor v:

$$\frac{D}{D\sigma} v = \dot{x}^\mu \nabla_\mu v.$$

- NB, useful notation $\frac{\partial}{\partial x^\mu} \equiv \partial_\mu$, or $\frac{\partial}{\partial x^r} \equiv \frac{\partial}{\partial r} \equiv \partial_r$
- Remember, coordinates are always upstairs, which disambiguates $r \equiv x^1 \equiv x^r$ (never x_1).

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The principle of curvature

- General relativity uses spacetime curvature to explain why objects gravitate toward one another independent of their properties, by travelling on straight lines (geodesics) through curved spacetime.
- Two planes setting out from one location on the surface of the earth in different directions
- They would appear to be inexorably drawn toward one another by a force proportional to their mass, so that the plane, and everyone in them, would be moved at the same rate toward the other plane
- The force is of course ficticious, and a manifestation of moving on a straight path in a curved spacetime.

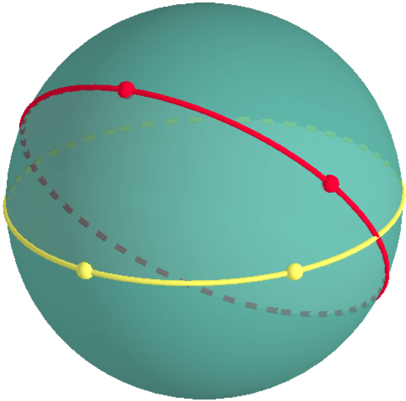

Figure: Two gravitating/converging geodesics on a sphere

Geodesic geometry

- We need then to be able to calculate geodesics $x^\mu(s)$ given coordinates and metric $g_{\mu\nu}$.
- Two definitions of geodesic:

  Shortest path Global definition, intuitive, technically "extremal" rather than shortest.

  Straightest line Direction of path doesn't change/tangent vector is parallel transported
- In torsionless manifolds, these are the same.
  - Einstein's general relativity assumes zero torsion, but this is not essential
  - Indeed, extensions to GR naturally incorporate the torsion tensor as being sourced by spin, in the same way that the curvature tensor is sourced by mass.
- Taking the first, we seek to minimise the length

$$L = \int_A^B ds = \int_A^B (g_{\mu\nu} dx^\mu dx^\nu)^{1/2} = \int_A^B (g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu)^{1/2} ds/c.$$

- We can therefore identify this as a calculus of variations problem, with a corresponding Euler-Lagrange equation applied to $G = (g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu)^{1/2}$.

The Euler-Lagrange equations

- In its most unrefined form, to get geodesics we just solve:

$$\frac{d}{ds}\left(\frac{\partial G}{\partial \dot{x}^\mu}\right) - \frac{\partial G}{\partial x^\mu} = 0, \text{ where } G = (g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu)^{1/2} \tag{1}$$

- However, we should also note that $G^2 = u \cdot u = c^2$ from the equivalence principle.
- Given that $\dot{G} = 0$ therefore, it is easier to define a Lagrangian $\mathcal{L} = G^2$ and solve

$$\boxed{\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0, \text{ where } \mathcal{L} = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \quad (= c^2 \text{ or } 0),}$$

which follows because

$$\frac{d}{ds}\left(\frac{\partial G^2}{\partial \dot{x}^\mu}\right) - \frac{\partial G^2}{\partial x^\mu} = 2\dot{G}\frac{\partial G}{\partial \dot{x}^\mu} + 2G\frac{d}{ds}\left(\frac{\partial G}{\partial \dot{x}^\mu}\right) - 2G\frac{\partial G}{\partial x^\mu} = 2G\left(\frac{d}{ds}\left(\frac{\partial G}{\partial \dot{x}^\mu}\right) - \frac{\partial G}{\partial x^\mu}\right) = 0$$

Euler Lagrange equations: mathematical consequences

- We thus find ourselves with a set of non-linear coupled ordinary differential equations in the spacetime paths $x^a(s)$
$$\ddot{x}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{x}^c = 0.$$
- With a little tensor geometry (see appendix) these $\Gamma^a{}_{bc}$ come out as
$$\Gamma = \frac{1}{2}g\left(\partial g + \partial g - \partial g\right) \qquad \Rightarrow$$
- We link to the second definition of geodesic "straightest line", via the definition of parallel transporting a vector v: $\frac{D}{Ds}v^b \equiv \dot{x}^a\nabla_a v^b \equiv \dot{x}^a\partial_a v^b + \dot{x}^a\Gamma^b{}_{ca}v^c \equiv \dot{v}^b + \Gamma^b{}_{ca}v^c\dot{x}^a$ and then substituting $v^b = \dot{x}^b$ to find that eq. (1) states $\frac{D}{Ds}\dot{x}^a = 0$.
- If you are worried that the thing we are varying ($\mathcal{L}$ or G) is a constant, remember that it is that it is its *functional* dependence upon x^μ and $\dot{x}^\mu$ that matters
- For those who have studied Hamiltonian dynamics, the same thing is true there, where use of the Hamiltonian function is the basis of the method, but numerically it is a constant, the total energy.

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$$\Gamma = \frac{1}{2}g\left(\partial g + \partial g - \partial g\right) \qquad \Rightarrow \qquad \Gamma^a{}_{bc} = \frac{1}{2}g^{ad}\left(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}\right)$$
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Euler Lagrange equations: practical consequences

$$\boxed{\frac{d}{ds}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\right)-\frac{\partial\mathcal{L}}{\partial x^\mu}=0,\ \text{where } \mathcal{L}=g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \quad (=c^2 \text{ or } 0).}$$

- We have five equations in four unknowns $x^\mu(s)$
  - Four differential equations (EL $\mu = 0, 1, 2, 3$)
  - One constraint equation ($\mathcal{L} = u \cdot u = c^2$ or 0)
- It is prudent to use the four easiest of these (and drop the nastiest differential equation).
- Equally important:

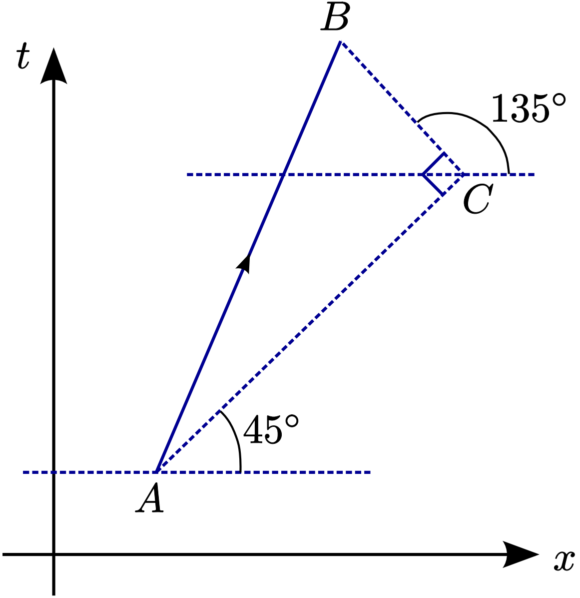
$$\boxed{\text{if } \frac{\partial\mathcal{L}}{\partial x^\mu}=0,\ \text{i.e. } \partial_\mu g_{ab}=0,\ \text{then } \frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\propto g_{\mu\nu}\dot{x}^\nu=u_\mu=\text{const.}}$$

- In words: if the metric does not depend on a coordinate, then the downstairs four velocity in that coordinate is conserved.
- This means that in most cases you do not need any differential equations at all!

Special relativistic check

- Let us check first that it makes sense in flat SR space, using Cartesian coordinates.
- Here $\mathcal{L}$ does not depend explicitly on the x^μ, since $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is constant. We thus have
$$\mathcal{L} = c^2 \dot{t}^2 - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$
- and the geodesic equation reduces to
$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) = 0. \qquad \Rightarrow \qquad \ddot{t} = \ddot{x} = \ddot{y} = \ddot{z} = 0,$$
which are the equations of an arbitrary straight line in spacetime.
- Thus we predict that test particles move along straight lines, in a region without gravity.
- We have also proved that in Minkowski space the extremal proper time between two points is got by joining them by a straight line.

- Note this cannot be the *minimum* proper time — a photon taking a zig-zag path between them would give this — the route AC followed by CB is composed of null intervals and therefore takes zero proper time
- Instead the route AB corresponds to the *maximum* proper time.
- This is the resolution of the twin paradox — the unaccelerated observer is the one who ages most.
- Thus acceleration keeps you young, and applying the equivalence principle we can predict that gravity keeps you young as well, which we know to be correct from the existence of gravitational redshift.



Summary

- Understanding of coordinates and good notation habits
- Intuition of gravity as curvature: "spacetime tells matter how to move"
- Coordinates are always upstairs, metric raises and lowers indices
- The Lagrangian $\mathcal{L} = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = u \cdot u = c^2$ or 0

$$\boxed{\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0, \text{ where } \mathcal{L} = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \quad (= c^2 \text{ or } 0).}$$

- Downstairs components of the four-velocity $\dot{x}_\mu = u_\mu$ are conserved if $g_{\mu\nu}$ independent of x^μ

$$\boxed{\text{if } \frac{\partial \mathcal{L}}{\partial x^\mu} = 0, \text{ i.e. } \partial_\mu g_{ab} = 0, \text{ then } \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \propto g_{\mu\nu}\dot{x}^\nu = u_\mu = \text{const.}}$$

- Armed with these, you often do not need geodesic equations for systems we will consider

Next time

Computing curvature & the Schwarzschild metric ("matter tells spacetime how to curve")

Appendix (non-examinable)

Demonstrating equivalence of the two forms of geodesic equation

Taking our Euler-Lagrange equations (new separated and distinguishable indices):

$$\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^a}\right) - \frac{\partial \mathcal{L}}{\partial x^a} = 0, \text{ where } \mathcal{L} = g_{bc}\dot{x}^b\dot{x}^c,$$

and substituting the second into the first

$$\frac{d}{ds}\left(\frac{\partial}{\partial \dot{x}^a} g_{bc}\dot{x}^b\dot{x}^c\right) - \partial_a g_{bc}\dot{x}^b\dot{x}^c = 0.$$

Expanding the first term's derivatives yields

$$\begin{aligned}\frac{d}{ds}\left(\frac{\partial}{\partial \dot{x}^a} g_{bc}\dot{x}^b\dot{x}^c\right) &= \frac{d}{ds}\left(g_{ba}\dot{x}^b + g_{ac}\dot{x}^c\right)\\ &= \dot{g}_{ba}\dot{x}^b + \dot{g}_{ac}\dot{x}^c + g_{ba}\ddot{x}^b + g_{ac}\ddot{x}^c\end{aligned}$$

We can chain rule the first two terms $\dot{g} = \dot{x}^d\partial_d g$ and relabel summed indices (noting metric is also symmetric)

$$2g_{ba}\ddot{x}^b + \dot{x}^d\partial_d g_{ba}\dot{x}^b + \dot{x}^d\partial_d g_{ac}\dot{x}^c - \partial_a g_{bc}\dot{x}^b\dot{x}^c = 0.$$

Contracting with g^{ad}, dividing by 2 and relabelling indices yields

$$\ddot{x}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{x}^c = 0,$$

where

$$\Gamma^a{}_{bc} = \frac{1}{2}g^{ad}\left(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}\right).$$