Fluid dynamcis:
modelling the dynamics of Earth's evolution
Physics of the Earth as a Planet, Lecture 3

The dynamics of the Earth as a planet often relies on understanding the deformation on very large length scales (100s or 1000s of kilometres), but where the response relies on the behaviour of materials whose composition may vary on the length scales of microns. However, simulating the dynamics of the planet while starting at the scale of individual crystals is not only onerous but unlikely to be fruitful. Instead, the physical description must be constructed at a scale appropriate to the phenomena in question. For example, when considering the large-scale convection of the Earth's mantle, or the flow of glacial ice, we consider deformation that occurs on length scales greater than many grains or crystals.

Conservation of mass

Consider the flow of mantle material whose density $\rho(\mathbf{x}, t)$ may depend on space $\mathbf{x}$ and time t and which deforms with a velocity $\mathbf{v}(\mathbf{x}, t)$. Within an arbitrary, fixed volume V mass can neither be created nor destroyed and so any change in mass of this volume must be due to the flux of mass across the boundary (denoted by surface $\mathcal{S}$). This may be written, in integral form as

$$\frac{d}{dt}\int_V \rho \, dV = \int_{\mathcal{S}} \rho \mathbf{v} \cdot \hat{\mathbf{n}} \, d\mathcal{S}, \tag{1}$$

where $\hat{\mathbf{n}}$ is the outward unit normal to the surface $\mathcal{S}$ bounding the volume V. Hence, $-\rho\mathbf{v}\cdot\hat{\mathbf{n}}$ is the mass flux across the surface. We may simplify this expression by using the *divergence theorem* and noting that V is fixed in time, so that

$$\int_V \left\{ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right\} dV = 0. \tag{2}$$

Since the volume V is arbitrary, this may therefore be written as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{3}$$

In many, if not all, contexts within the mantle (as well as a host of other geophysical contexts) the material is incompressible to good approximation, and therefore

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = 0. \tag{4}$$

Hence, conservation of mass may be written more simply as a divergence-free velocity field,

$$\nabla \cdot \mathbf{v} = 0. \tag{5}$$

This is sometimes referred to as the *continuity equation.*

Eulerian (referential) / Lagrangian (spatial) coordinates

In continuum mechanics we can either choose to describe the dynamics in a stationary framework, in which parcels of material move through the coordinate system as a function of time. In this Eulerian coordinate system $(\mathbf{x}, t)$ the spatial coordinate $\mathbf{x}$ is fixed in space. In contrast, we might describe our coordinates based on the deformation of the material, so that in the Lagrangian coordinates $(\mathbf{X}, t)$ the spatial coordinate is fixed instead to the material. It is common to choose $\mathbf{X} = \mathbf{x}$ at time $t = 0$ in many settings. However, we might notice that the time derivative in the Eulerian perspective, the static reference frame, is given by $\partial/\partial t$, while from the Lagrangian perspective, the moving reference frame the derivative following a parcel of fluid is given by the *material derivative*

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla, \tag{6}$$

which is sometime also called the *advective derivative* or the *total derivative.* Again, for an incompressible fluid, we may write that the density is unchanged,

$$\frac{D\rho}{Dt} = 0 \qquad \Longrightarrow \qquad \nabla \cdot \mathbf{v} = 0, \tag{7}$$

from which we find that the velocity is divergence free.

Conservation of momentum

We now consider the momentum of material in the arbitrary volume V, which is given by

$$\int_V \rho \mathbf{v} \, dV. \tag{8}$$

Changes in the momentum are driven by the advection of momentum in and out of the volume, and by any body and surface forces acting on the mass within the volume. Here we include the force of gravity acting on the mass and the action of stresses σ acting on the surface, and therefore we may write

$$\frac{d}{dt} \int_V \rho \mathbf{v} \, dV = - \int_{\mathcal{S}} \rho \mathbf{v}\mathbf{v} \cdot \hat{\mathbf{n}} \, d\mathcal{S} + \int_V \rho \mathbf{g} \, dV + \int_{\mathcal{S}} \sigma \cdot \hat{\mathbf{n}} \, d\mathcal{S}. \tag{9}$$

We may use the divergence theorem and mass conservation to rewrite this as

$$\int_V \left\{ \rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} - \rho \mathbf{g} - \nabla \cdot \sigma \right\} = 0, \tag{10}$$

which, because the volume is arbitrary, may be rewritten as

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = \nabla \cdot \sigma + \rho \mathbf{g}. \tag{11}$$

At this point we have not yet stated anything about how the material may respond to the applied forces. That information is encoded in the stress tensor, σ, whose components σ_{ij} represent the force per unit area acting in direction i on the face with normal in direction j. For example, in cartesian coordinates we might write the stress tensor as

$$\sigma = \sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}. \tag{12}$$

Since the torques on the infinitesimal volume must balance, or equivalently the infinitesimal volume must have zero moment of inertia, the stress tensor must be symmetric and hence there are 6 independent components of σ_{ij}. We can further simplify the description by decomposing the stress tensor into an isotropic pressure and a deviatoric stress tensor. The pressure is therefore defined to be

$$p = -\frac{1}{3}\sigma_{ii} = -\frac{1}{3}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right), \tag{13}$$

and hence define the deviatoric stress, τ_{ij}, by the relationship

$$\sigma = \sigma_{ij} = -p\mathbf{I} + \tau = -p\delta_{ij} + \tau_{ij}, \tag{14}$$

where $\delta_{ij} = \mathbf{I}$ is the identity matrix.

Material rheology

We have said little about the material properties, and to make further headway we need to specify the manner in which the material deforms in response to the various forces. In the second half of the course, stresses within the material will be related to strains, as is the case for elastic materials. Here, we will instead describe the rate of strain in response to the applied stress. Hence, we will be describing the strain rate as characterised by the spatial distribution of the velocity field in terms of the deviatoric stress. It is worth noting that a uniform velocity field would equate to uniform translation, so it is the gradients and higher moments which are important in describing the deformation of the material. At it's simplest, we therefore describe the rheology as a function of the strain-rate tensor,

$$\dot{\epsilon} = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right), \quad \text{or} \quad \dot{\epsilon}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{15}$$

so that $\tau = \tau(\dot{\epsilon})$, or equivalently $\dot{\epsilon} = \dot{\epsilon}(\tau)$. Note that the strain-rate tensor is symmetric by definition, and that for an incompressible material the diagonal components sum to zero. These diagonal elements represent stretching deformations, while the off-diagonal elements represent shearing.

The simplest of rheologies is one in which the relationship between the stress and deformation rate is local, linear, instantaneous and isotropic. If this is the case, the fluid is Newtonian and the stress may be written as

$$\sigma = -p\mathbf{I} + 2\eta\dot{\epsilon}, \tag{16}$$

where η is the dynamic viscosity. Thus, the equations describing fluid motion for incompressible, Newtonian fluids with constant viscosity are

$$\nabla\cdot\mathbf{u} = 0, \tag{17}$$

$$\rho\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \eta\nabla^2\mathbf{u} + \rho\mathbf{g}, \tag{18}$$

and together are called the Navier-Stokes equations.

The rheology may be a complex function of the temperature, composition, shear-rate and time,

$$\eta = \eta(T, C, \dot{\epsilon}, t, \ldots), \tag{19}$$

and indeed many modern questions of interest in Earth and planetary sciences involve a non-trivial effect of complex rheology. We'll explore some of these questions in later lectures where we look at the rheology of mantle materials, but for the bulk of problems in this course you should assume a simple constant rheology (η = constant) unless explicitly told otherwise.

Conservation of energy and composition

The same approach we've applied here may be effectively utilised to construct similar statements of conservation. Perhaps the most useful (at least in this course) is the statement of conservation of thermal energy. That derivation will be given in full in the lecture on the thermal structure of plates, but it may be summarised here as

$$\rho c_p\left(\frac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T\right) = \nabla\cdot(k\nabla T) + Q, \tag{20}$$

where c_p is the specific heat capacity, T is the temperature, k is the thermal conductivity and Q accounts for the internal generation of heat (for example by radioactive decay).

Similarly, we might write a statement conserving the concentration C of some tracer (salt in the ocean, water in the mantle) as described by

$$\frac{DC}{Dt} = \frac{\partial C}{\partial t} + \mathbf{v} \cdot \nabla C = \nabla \cdot (D \nabla C), \tag{21}$$

where here D is the diffusivity of the concentration C.

Again, it's worth stressing that these are continuum descriptions which average over a large number of constituents (molecules, crystals, etc) and hence material quantities c_p, k, D represent appropriate averages over the representative volume. For simple fluids, averaged over many molecular free paths, these are the molecular specific heat, conductivity and diffusivity respectively, but for complex materials these quantities may depend on the internal structure of the medium, and k, D may in general be tensorial, for example reflecting an underlying material fabric.

Boundary conditions

No problem is specified completely until a sufficient number of boundary and/or initial conditions are specified. That may involve estimates of the initial distribution of temperature and composition, the geometry of the boundaries, and the forces acting on those boundaries. Here, we make an important distinction between fixed boundaries on which the quantities are prescribed and those which are free, and hence which must be solved as part of the problem. For fixed boundaries, we typically impose constraints on the velocity components and on tractions at the boundaries of the material. In contrast, for free boundaries we impose a kinematic condition which relates the fluid velocities along the boundary to the evolution of the boundary itself.

Kinematic boundary conditions

A the interface between two fluids, we must prescribe conditions ensuring the continuity of mass and momentum. For example, at a fluid-fluid interface between fluid 1 and 2 we might impose mass conservation. This is perhaps easiest to express in a coordinate frame attached to the moving interface, the normal velocity of which we can describe by a vector $\mathbf{v}_n = v_n \hat{\mathbf{n}}$, where $\mathbf{v}_n$ is the velocity of the interface moving at speed v_n in direction $\hat{\mathbf{n}}$, which is the unit normal to the interface (here defined as pointing from fluid 1 into fluid 2). Mass conservation at the evolving interface can therefore be expressed generally as

$$\rho_1 \left(v_n - \mathbf{v}_1 \cdot \hat{\mathbf{n}} \right) = \rho_2 \left(v_n - \mathbf{v}_2 \cdot \hat{\mathbf{n}} \right). \tag{22}$$

It is perhaps helpful to consider two applications of this general expression. If there is no change of phase and $\rho_1 = \rho_2$ then the problem reduces to a purely kinematic description of the advection of the interface,

$$v_n = \mathbf{v}_1 \cdot \hat{\mathbf{n}} = \mathbf{v}_2 \cdot \hat{\mathbf{n}}, \tag{23}$$

or equivalently

$$\mathbf{v}_1 \cdot \hat{\mathbf{n}} - \mathbf{v}_2 \cdot \hat{\mathbf{n}} = 0. \tag{24}$$

A compact means of indicating this jump in quantities across an interface is

$$[\mathbf{v} \cdot \hat{\mathbf{n}}] = \mathbf{v}_1 \cdot \hat{\mathbf{n}} - \mathbf{v}_2 \cdot \hat{\mathbf{n}} = 0, \tag{25}$$

which should be interpreted as the jump in quantities across the interface, where all expressions are evaluated at the (possibly moving) interface. Note that in cases where the second fluid or material is inviscid (or indeed is not present, as in a vacuum) this may instead be written as

$$v_n = \mathbf{v} \cdot \hat{\mathbf{n}}, \tag{26}$$

where $v_n(t)$ is again the normal velocity of the boundary.

A second example might be that of the evolution of a boundary involving phase change, for example the solidification of fluid 2 into solid 1. In this case we might have $\mathbf{v}_1 = 0$ and for arbitrary densities $\rho_1 \neq \rho_2$ the velocity within the fluid is therefore given by

$$\mathbf{v}_2 \cdot \hat{\mathbf{n}} = \frac{\rho_2 - \rho_1}{\rho_1} v_n. \tag{27}$$

That is the change of density on solidification induces a flow within the liquid due to the change in density on change of phase.

Dynamic boundary conditions

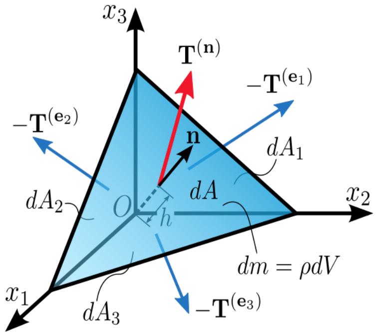

Figure 1: Cauchy tetrahedron.

To understand the force balance on a plane it is useful to consider forces acting on an arbitrary plane - the so-called *Cauchy* stress tetrahedron. We may define the state of stress at a point, or on a plane, by the stress vector $\mathbf{T}^{(\hat{\mathbf{n}})}$, which may be defined in terms of the stress tensor as

$$\mathbf{T}^{(\hat{\mathbf{n}})} = \hat{\mathbf{n}} \cdot \sigma \tag{28}$$

A balance of forces on the tetrahedron may be written as

$$\rho \mathbf{g} \left(\frac{h}{3} dA \right) + \mathbf{T}^{(\hat{\mathbf{n}})} dA - \mathbf{T}^{(\mathbf{e}_1)} dA_1 - \mathbf{T}^{(\mathbf{e}_2)} dA_2 - \mathbf{T}^{(\mathbf{e}_3)} dA_3 = \rho \left(\frac{h}{3} dA \right) \mathbf{a}. \tag{29}$$

However, we may replace the areas dA_i by projecting the infinitesimal area element in each of the three directions with the projection onto the area element dA, such that

$$\begin{aligned} dA_1 &= (\hat{\mathbf{n}} \cdot \mathbf{e}_1) dA, \\ dA_2 &= (\hat{\mathbf{n}} \cdot \mathbf{e}_2) dA, \\ dA_3 &= (\hat{\mathbf{n}} \cdot \mathbf{e}_3) dA. \end{aligned}$$

Substituting this back into our force balance equation therefore gives

$$\rho \left(\frac{h}{3} \right) \mathbf{g} + \mathbf{T}^{(\hat{\mathbf{n}})} - \mathbf{T}^{(\mathbf{e}_1)} (\hat{\mathbf{n}} \cdot \mathbf{e}_1) - \mathbf{T}^{(\mathbf{e}_2)} (\hat{\mathbf{n}} \cdot \mathbf{e}_2) - \mathbf{T}^{(\mathbf{e}_3)} (\hat{\mathbf{n}} \cdot \mathbf{e}_3) = \rho \left(\frac{h}{3} \right) \mathbf{a}. \tag{30}$$

Taking the limit as $h \to 0$, the body force and acceleration vanish, and we therefore find that

$$\mathbf{T}^{(\hat{\mathbf{n}})} = \mathbf{T}^{(\mathbf{e}_1)}(\hat{\mathbf{n}} \cdot \mathbf{e}_1) + \mathbf{T}^{(\mathbf{e}_2)}(\hat{\mathbf{n}} \cdot \mathbf{e}_2) + \mathbf{T}^{(\mathbf{e}_3)}(\hat{\mathbf{n}} \cdot \mathbf{e}_3). \tag{31}$$

This allows us to calculate the traction $\mathbf{T}^{(\hat{\mathbf{n}})}$ at a point on an arbitrary plane, which is what is required of a force balance at the boundary.

For a fluid, taking as an example a Newtonian, viscous rheology, we may relate the continuity of forces (tractions) across a boundary. In this case, it is often convenient to separately express continuity of the normal forces (the pressure) across the boundary, along with continuity of the tangential forces (for example the shear stress).

In general, we may express continuity of traction at the boundary by

$$[\sigma \cdot \hat{\mathbf{n}}] = 0, \tag{32}$$

and may subsequently decompose the stress tensor into isotropic and deviatoric components $\sigma = -p\mathbf{I} + \tau$ where for a Newtonian, viscous fluid the deviatoric stress is related to the local strain rate,

$$\tau = \frac{\eta}{2}\left(\nabla \mathbf{v} + \nabla \mathbf{v}^T\right), \tag{33}$$

where η is the viscosity.

Hence, we may write for the normal component of the traction

$$[\hat{\mathbf{n}} \cdot \sigma \cdot \hat{\mathbf{n}}] = [-p + \hat{\mathbf{n}} \cdot \tau \cdot \hat{\mathbf{n}}] = 0. \tag{34}$$

In many settings, this largely constrains the pressure at the free surface.

Similarly, examining the tangential component of the traction we find

$$[\hat{\mathbf{n}} \times \sigma \cdot \hat{\mathbf{n}}] = [\hat{\mathbf{n}} \times \tau \cdot \hat{\mathbf{n}}] = 0, \tag{35}$$

which equates the shear stress on either side of the boundary.

Finally, we imposed that the tangential velocity $\mathbf{v}_b$ be continuous along the boundary such that

$$\mathbf{v} \times \hat{\mathbf{n}} = \mathbf{v} - (\mathbf{v} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} = \mathbf{v}_b, \tag{36}$$

where a force balance involving the shear stress would typically be used to determine $\mathbf{v}_b$.

A sometimes helpful way of considering this boundary condition is to consider a discontinuity in the tangential velocity across a small length scale δ at the boundary. This of course implies a shear stress whose magnitude depends on the jump in viscosity divided by this small length scale δ. As we take the limit $\delta \to 0$ the shear stress becomes unbounded, and hence we require continuity of the tangential velocity.

These boundary conditions provide a useful approximation when considering the behaviour of the mantle in contact with much less viscous oceans and atmosphere.

Rigid boundary conditions

An important subset, or limiting set, of boundary conditions is that of a rigid boundary. In this case conservation of mass may be written as a no flux or no penetration boundary condition,

$$\mathbf{v} \cdot \hat{\mathbf{n}} = 0, \tag{37}$$

applied along the boundary. Similarly, a force balance along the interface can be used to infer the tangential velocity along the boundary. If the boundary is shear stress free, consider for example the stress the atmosphere exerts on the mantle, then the shear stress along the boundary may be set to zero,

$$\eta \frac{\partial \mathbf{v}}{\partial n} = 0. \tag{38}$$

Alternatively, if the boundary is rigid (for example its apparent viscosity $\eta \to \infty$) then the tangential velocity must match that of the boundary

$$\mathbf{v} \times \hat{\mathbf{n}} = \mathbf{u} - (\mathbf{u} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} = 0, \tag{39}$$

also commonly referred to as 'no-slip' boundary conditions.

Dynamic boundary conditions

At a free surface, the traction (normal component of the stress tensor) is continuous across the boundary,

$$[\sigma \cdot \hat{\mathbf{n}}] = 0, \tag{40}$$

unless there are surface forces acting. At very small scales, those surface forces are most commonly surface tension (or similar). In contrast, at planetary scales this is most commonly satisfied by a pressure matching condition. For example, at the Earth's surface we might set

$$\sigma \cdot \hat{\mathbf{n}} = -p_a \hat{\mathbf{n}}, \tag{41}$$

where p_a is the atmospheric pressure. Correspondingly, this suggests that the normal stress is simply equal to the atmospheric pressure, and that the tangential stress is zero. As an example, consider the outward normal in the $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ direction. The normal component of the dynamic boundary condition therefore gives

$$\hat{\mathbf{z}} \cdot \sigma \cdot \hat{\mathbf{z}} = -p + \eta \frac{\partial w}{\partial z} = -p - \frac{\partial u}{\partial x} = p_a, \tag{42}$$

where we have used mass conservation to rewrite the gradient in the velocity. The tangential component of the dynamic boundary condition is

$$\hat{\mathbf{z}} \times \sigma \cdot \hat{\mathbf{z}} = \eta \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = 0. \tag{43}$$

Hence, from the single dynamical boundary condition we find that, to leading order the pressure is continuous, as is the shear stress.

Extension to partially molten flows

As a simple example, motivated by the partial melting of the mantle due to depressurisation during upwelling beneath mid-ocean ridges, let's consider a medium composed of a solid of density ρ_s occupying a volume fraction ϕ, and a liquid melt of density ρ_l occupying a volume fraction $1 - \phi$. Our representative volume now encompasses a large number of solid grains, and a small, interstitial volume of melt. However, the behaviour of the aggregate must still obey statements of mass and momentum conservation. Hence, we may write statements of conservation for the solid and liquid mass,

$$\begin{aligned} \frac{\partial}{\partial t}(\rho_s \phi) + \nabla \cdot (\rho_s \phi \mathbf{v}_s) &= \rho_s \Gamma, & (44) \\ \frac{\partial}{\partial t}[\rho_l (1-\phi)] + \nabla \cdot [\rho_l (1-\phi) \mathbf{v}_l] &= -\rho_l \Gamma, & (45) \end{aligned}$$

where $\mathbf{v}_s$ and $\mathbf{v}_l$ are the velocities of the solid and liquid respectively and Γ is the (volumetric) melt rate. Here we can see that, when $\rho_s \simeq \rho_l =$ constant, we may define a divergence-free bulk velocity $\overline{\mathbf{v}} = \phi \mathbf{v}_s + (1-\phi)\mathbf{u}_l$ such that

$$\nabla \cdot \overline{\mathbf{v}} = 0, \tag{46}$$

from the two statements of solid and liquid mass conservation. Nevertheless, the individual solid and liquid velocities may not be divergence free due to melting/solidification or change in solid fraction,

$$\nabla \cdot (\phi \mathbf{v}_s) = \Gamma - \frac{\partial \phi}{\partial t}, \tag{47}$$

$$\nabla \cdot [(1-\phi)\mathbf{v}_l] = -\Gamma + \frac{\partial \phi}{\partial t}. \tag{48}$$

Scaling of motion in the mantle - Stokes flow

The Navier-Stokes equations, let alone any equations for partial molten systems or indeed for magnetohydrodynamics, are difficult to solve in general. This is not only due to their mathematical complexity (the nonlinear advection of momentum for example) but also because of the potential range of length and time scales involved (which often present difficulties for numerical schemes). For these reasons, many approaches commonly start by determining the dominant balances between terms in the governing equations. In this way, mathematical and computational effort can be focused on the key physical processes, which is often crucial in detecting the physical process particularly when comparing to naturally noisy geophysical data.

As an example, consider the viscous motion of the Earth's mantle as described by the Navier-Stokes equations,

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = -\nabla p + \eta\nabla^2\mathbf{v} + \rho\mathbf{g}, \quad \nabla\cdot\mathbf{v} = 0. \tag{49}$$

The mantle has a depth of roughly $L \sim 3000$ km $= 3 \times 10^6$ m, and has characteristic plate spreading rates of $U \sim 3$ cm/yr $\sim 10^{-10}$ m/s from which we might infer the characteristic horizontal velocities, at least at the top of the mantle. A scaling of the statement of conservation of mass suggests that horizontal and vertical velocities are comparable,

$$\nabla\cdot\mathbf{u} = 0 \quad \Longrightarrow \quad \frac{U}{L} \sim \frac{W}{L} \Longrightarrow U \sim W. \tag{50}$$

Examining the statement of conservation of momentum, we see that inertia, pressure, viscous dissipation and buoyancy scale as

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = -\nabla p + \eta\nabla^2\mathbf{v} + \rho\mathbf{g} \quad \Longrightarrow \frac{\rho U^2}{L} \sim \frac{P}{L} + \frac{\eta U}{L^2} + \rho g, \tag{51}$$

respectively, where P is the pressure scale. Here it's useful to know that the pressure field effectively acts to enforce mass conservation, and the buoyancy field drives the flow. The ratio of inertia to viscous dissipation is given by the Reynolds number

$$Re = \frac{\rho U L}{\eta} \simeq 10^{-22} \ll 1, \tag{52}$$

which is very small for a representative mantle viscosity of $\eta = 10^{21}$ Pa s. This strongly suggests that for an effective model of the mantle we can neglect (completely) the inertia of fluid parcels, so that the motion may instead be determined by Stokes equations

$$-\nabla p + \eta\nabla^2\mathbf{v} + \rho\mathbf{g} = 0, \quad \nabla\cdot\mathbf{v} = 0, \tag{53}$$

together with statements of how $\rho = \rho(T)$ and conservation of energy (or heat), both of which will be reviewed in later lectures.

Stokes equations have several helpful properties, which have important implications for our understanding of the mantle. They are instantaneous (no time derivative) and hence there is not inherent inertia or memory in the flow. The resultant behaviour is solely determined by the boundary conditions and applied forces, including the temperature and hence buoyancy distribution in the case of mantle convection. They are also linear (no nonlinear advection) and hence the flow is linearly proportional to the forcing. Finally, they are reversible (though this statement must be handled with care, since the thermal field is not reversible as we will see in later lectures!).