

18 Poisson Process

A *counting process* is a random process $N(t)$, $t \geq 0$, such that

1. $N(t)$ is a nonnegative integer for each t ;
2. $N(t)$ is nondecreasing in t ; and
3. $N(t)$ is right-continuous.

The third condition is merely a convention: if the first two events happens at $t = 2$ and $t = 3$ we want to say $N(2) = 1$, $N(3) = 2$, $N(t) = 1$ for $t \in (2, 3)$, and $N(t) = 0$ for $t < 2$. Thus $N(t) - N(s)$ represents the *number of events in* $(s, t]$.

A *Poisson process* with *rate* (or *intensity*) $\lambda > 0$ is a counting process $N(t)$ such that

1. $N(0) = 0$;
2. it has independent increments: if $(s_1, t_1] \cap (s_2, t_2] = \emptyset$, then $N(t_1) - N(s_1)$ and $N(t_2) - N(s_2)$ are independent; and
3. number of events in any interval of length t is $\text{Poisson}(\lambda t)$.

In particular, then,

$$P(N(t+s) - N(s) = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}, \quad k = 0, 1, 2, \dots,$$

$$E(N(t+s) - N(s)) = \lambda t.$$

Moreover, as $h \rightarrow 0$,

$$P(N(h) = 1) = e^{-\lambda h} \lambda h \sim \lambda h,$$

$$P(N(h) \geq 2) = \mathcal{O}(h^2) \ll \lambda h.$$

Thus, in small time intervals, a single event happens with probability proportional to the length of the interval; this is why λ is called the rate.

A definition like this should be followed by the question whether the object in question exists — we may be wishing for contradictory properties. To demonstrate the existence, we will outline two constructions of *the* Poisson process. Yes, it is unique, but it would require some probabilistic sophistication to prove this, and so would the proof (or even the formulation) of convergence in the first construction we are about to give. Nevertheless, it is very useful, as it makes many properties of Poisson process almost instantly understandable.

Construction by tossing a low-probability coin very fast. Pick a large n , and assume that you have a coin with (low) Heads probability $\frac{\lambda}{n}$. Toss this coin at times which are positive integer multiples of $\frac{1}{n}$ (that is, very fast), and let $N_n(t)$ be the number of Heads in $[0, t]$. Clearly, as $n \rightarrow \infty$, the number of Heads in any interval $(s, t]$ is Binomial with number of trials $n(t-s) \pm 2$ and success probability $\frac{\lambda}{n}$, thus converges to $\text{Poisson}(t-s)$ as $n \rightarrow \infty$. Moreover,

N_n has independent increments for any n and so the same holds in the limit. We should note that the Heads probability does not need to be exactly $\frac{\lambda}{n}$, instead it suffices that this probability converges to λ when multiplied by n . Similarly, we do not need all integer multiples of $\frac{1}{n}$, it is enough that their number in $[0, t]$, divided by n , converges to t in probability for any fixed t .

An example of a property that follows immediately is the following. Let S_k be the time of the k 'th (say, 3rd) event (which is a *random* time), and let $N_k(t)$ be the number of additional events in time t after time S_k . Then $N_k(t)$ is another Poisson process, with the same rate λ , as starting to count your Heads afresh after the k 'th Heads gives you the same process as if you counted them from the beginning — we can *restart* a Poisson process at the time of k 'th event. In fact, we can do so at any *stopping time*, a random time T with the property that $T = t$ depends only on the behavior of the Poisson process up to time t (i.e., on the past but not on the future). That Poisson process, restarted at a stopping time, has the same properties as the original process started at time 0 is called the *strong Markov property*.

As each N_k is a Poisson process, $N_k(0) = 0$, so two events in the original Poisson $N(t)$ process do not happen at the same time.

Let T_1, T_2, \dots be the *interarrival times*, where T_n is the time elapsed between $(n - 1)$ 'st and n 'th event. A typical example would be the times between consecutive buses arriving at a station.

Proposition 18.1. *Distribution of interarrival times:*

$$T_1, T_2, \dots \text{ are independent and Exponential}(\lambda).$$

Proof. We have

$$P(T_1 > t) = P(N(t) = 0) = e^{-\lambda t},$$

which proves that T_1 is Exponential(λ). Moreover, for any $s > 0$ and $t > 0$,

$$P(T_2 > t | T_1 = s) = P(\text{no events in } (s, s + t] | T_1 = s) = P(N(t) = 0) = e^{-\lambda t},$$

as events in $(s, s + t]$ are not influenced by what happens in $[0, s]$. So T_2 is independent of T_1 and Exponential(λ). Similarly we can establish that T_3 is independent of T_1 and T_2 with the same distribution, and so on. \square

Construction by exponential interarrival times. We can use the above Proposition 18.1 for another construction of a Poisson process, which is very convenient for simulations. Let T_1, T_2, \dots be i. i. d. Exponential(λ) random variables and let $S_n = T_1 + \dots + T_n$ be the waiting time for the n 'th event. Then we define $N(t)$ to be the largest n so that $S_n \leq t$.

We know that $ES_n = \frac{n}{\lambda}$, but we can derive its density; the distribution is called Gamma(n, λ). We start with

$$P(S_n > t) = P(N(t) < n) = \sum_{j=0}^{n-1} e^{-\lambda t} \frac{(\lambda t)^j}{j!},$$

and then we differentiate to get

$$\begin{aligned}
 -f_{S_n}(t) &= \sum_{j=0}^{n-1} \frac{1}{j!} (-\lambda e^{-\lambda t} (\lambda t)^j + e^{-\lambda t} j (\lambda t)^{j-1} \lambda) \\
 &= \lambda e^{-\lambda t} \sum_{j=0}^{n-1} \left(-\frac{(\lambda t)^j}{j!} + \frac{(\lambda t)^{j-1}}{(j-1)!} \right) \\
 &= -\lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!},
 \end{aligned}$$

and so

$$f_{S_n}(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}.$$

Example 18.1. Consider a Poisson process with rate λ . Compute (a) $E(\text{time of the 10'th event})$, (b) $P(\text{the 10th event occurs 2 or more time units after the 9th event})$, (c) $P(\text{the 10th event occurs later than time 20})$, and (d) $P(2 \text{ events in } [1, 4] \text{ and } 3 \text{ events in } [3, 5])$.

The answer to (a) is $\frac{10}{\lambda}$ by Proposition 18.1. The answer to (b) is $e^{-2\lambda}$, as one can restart the Poisson process at any event. The answer to (c) is $P(S_{10} > 20) = P(N(20) < 10)$, so can either write the integral

$$P(S_{10} > 20) = \int_{20}^{\infty} \lambda e^{-\lambda t} \frac{(\lambda t)^9}{9!} dt,$$

or use

$$P(N(20) < 10) = \sum_{j=0}^9 e^{-20\lambda} \frac{(20\lambda)^j}{j!}.$$

To answer (d), we condition on the number of events in $[3, 4]$:

$$\begin{aligned}
 &\sum_{k=0}^2 P(2 \text{ events in } [1, 4] \text{ and } 3 \text{ events in } [3, 5] \mid k \text{ events in } [3, 4]) \cdot P(k \text{ events in } [3, 4]) \\
 &= \sum_{k=0}^2 P(2 - k \text{ events in } [1, 3] \text{ and } 3 - k \text{ events in } [4, 5]) \cdot P(k \text{ events in } [3, 4]) \\
 &= \sum_{k=0}^2 e^{-2\lambda} \frac{(2\lambda)^{2-k}}{(2-k)!} \cdot e^{-\lambda} \frac{\lambda^{3-k}}{(3-k)!} \cdot e^{-\lambda} \frac{\lambda^k}{k!} \\
 &= e^{-4\lambda} \left(\frac{1}{3} \lambda^5 + \lambda^4 + \frac{1}{2} \lambda^3 \right).
 \end{aligned}$$

Theorem 18.2. *Superposition of independent Poisson processes.*

Assume that $N_1(t)$ and $N_2(t)$ are independent Poisson processes with rates λ_1 and λ_2 . Combine them into a single process by taking the union of both sets of events, or equivalently $N(t) = N_1(t) + N_2(t)$. This is a Poisson process with rate $\lambda_1 + \lambda_2$.

Proof. This is a consequence of the same property for Poisson random variables. \square

Theorem 18.3. *Thinning of a Poisson process.*

Toss an independent coin with probability p of Heads for every event in a Poisson process $N(t)$. Call Type I events those with Heads outcome and Type II events those with Tails outcome. Let $N_1(t)$ and $N_2(t)$ be the numbers of Type I and Type II events in $[0, t]$. These are independent Poisson processes with rates λp and $\lambda(1 - p)$.

The real substance of this theorem is independence, as the other claims follow from thinning properties of Poisson random variables (Example 11.4).

Proof. We argue by discrete approximation. Mark Type I Heads by flipping $\frac{p\lambda}{n}$ coins at integer multiples of $\frac{1}{n}$. Moreover, mark Type II Heads by flipping $\frac{\lambda(1-p)}{n}$ coins (a) at all integer multiples of $\frac{1}{n}$ or (b) at integer multiples of $\frac{1}{n}$ which are not Type I Heads. The option (b) gives in the limit as $n \rightarrow \infty$ exactly the construction in the statement of the theorem, as the probability of Type II Heads at any multiple of $\frac{1}{n}$ is $(1 - \frac{p\lambda}{n}) \cdot \frac{(1-p)\lambda}{n} \sim \frac{(1-p)\lambda}{n}$. On the other hand, the option (a) gives us *independent* Type I and Type II Heads. The two options, moreover, give the same limit, as the probability there is a difference at a site is the probability that option (a) produces two Heads (one of either type), and this happens with probability on the order $\frac{1}{n^2}$. For a fixed t , the expected number of integer multiples of $\frac{1}{n}$ in $[0, t]$ at which this happens is then on the order $\frac{1}{n}$, and so the probability that there is even one of these “double points” in $[0, t]$ goes to zero. \square

Example 18.2. Customers arrive at a store at the rate of 10 per hour. Each is either male or female with probability $\frac{1}{2}$. Assume that you know that exactly 10 women entered within some hour (say, 10 to 11am). (a) Compute the probability that exactly 10 men also entered. (b) Compute the probability that at least 20 customers have entered.

Male and female arrivals are independent Poisson processes, with parameter is $\frac{1}{2} \cdot 10 = 5$, so the answer to (a) is

$$e^{-5} \frac{5^{10}}{10!}.$$

The answer to (b) is

$$\sum_{k=10}^{\infty} P(k \text{ men entered}) = \sum_{k=10}^{\infty} e^{-5} \frac{5^k}{k!} = 1 - \sum_{k=0}^9 e^{-5} \frac{5^k}{k!}.$$

Example 18.3. Assume that cars arrive at rate 10 per hour. Assume each car will pick up a hitchhiker with probability $\frac{1}{10}$. You are second in line. What is the probability that you’ll have to wait for more than 2 hours?

Cars that pick up hitchhikers are a Poisson process with rate $10 \cdot \frac{1}{10} = 1$. For this process,

$$P(T_1 + T_2 > 2) = P(N(2) \leq 1) = e^{-2}(1 + 2) = 3e^{-2}.$$

Proposition 18.4. *Order of events in independent Poisson processes.*

Assume that you have two independent Poisson processes, $N_1(t)$ with rate λ_1 and $N_2(t)$ with rate λ_2 . The probability that n events occur in the first process before m events occur in the second process is

$$\sum_{k=n}^{n+m-1} \binom{n+m-1}{k} \left(\frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^k \left(\frac{\lambda_2}{\lambda_1 + \lambda_2} \right)^{n+m-1-k}.$$

We can easily extend this idea to more than two independent Poisson process; we will not make a formal statement, but instead illustrate by a few examples below.

Proof. Start with a Poisson process with $\lambda_1 + \lambda_2$, then independently decide for each event whether it belongs to the first process, with probability $\frac{\lambda_1}{\lambda_1 + \lambda_2}$, or the second process, with probability $\frac{\lambda_2}{\lambda_1 + \lambda_2}$. The obtained processes are independent and have the correct rates. The probability we are interested in is the probability that among the first $m + n - 1$ events in the combined process, n or more events belong to the 1st process, which is the binomial probability in the statement. \square

Example 18.4. Assume that $\lambda_1 = 5$, $\lambda_2 = 1$. Then

$$P(5 \text{ events in the first process before 1 in the second}) = \left(\frac{5}{6} \right)^5.$$

and

$$P(5 \text{ events in the first process before 2 in the second}) = \sum_{k=5}^6 \binom{6}{k} \left(\frac{5}{6} \right)^k \left(\frac{1}{6} \right)^{6-k} = \frac{11 \cdot 5^5}{6^6}.$$

Example 18.5. You have three friends, A , B , and C . Each will call you after an Exponential amount of time with expectation 30 minutes, 1 hour, and 2.5 hours respectively. You will go out with the first friend that calls. What is the probability that you go out with A ?

We could evaluate the triple integral, but we will avoid that. Interpret each call as the first event in the appropriate one of three Poisson processes with rates 2, 1, and $\frac{2}{5}$, assuming the time unit to be one hour. (Recall that the rates are inverses of the expectations.)

We will solve the general problem with rates λ_1 , λ_2 and λ_3 . Start with rate $\lambda_1 + \lambda_2 + \lambda_3$ Poisson process, distribute the events with probability $\frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3}$, $\frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3}$, and $\frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}$, respectively. Probability of A calling first is then clearly $\frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3}$ which in our case works out to be

$$\frac{2}{2 + 1 + \frac{2}{5}} = \frac{10}{17}.$$

Our next theorem illustrates what we can say about previous event times if we either know their number by time t is k , or we know that the k 'th one happens exactly at time t .

Theorem 18.5. *Uniformity of previous event times.*

1. Given that $N(t) = k$, the conditional distribution of the interarrival times, S_1, \dots, S_k , is distributed as order statistics of k independent uniform variables: the set $\{S_1, \dots, S_k\} = \{U_1, \dots, U_i\}$, where U_i are independent and uniform on $[0, t]$.
2. Given that $S_k = t$, S_1, \dots, S_{k-1} are distributed as order statistics of $k - 1$ independent uniform random variables on $[0, t]$.

Proof. Again, we discretize, and the discrete counterpart is as follows. Assume you toss a coin N times in succession, and you know the number of Heads in these N tosses is k . Then these Heads occur on any of the $\binom{N}{k}$ subsets (of k tosses out of a total of N) with equal probability, simply by symmetry. This is exactly the statement of the theorem, in the appropriate limit. \square

Example 18.6. Assume that passengers arrive at a bus station as a Poisson process with rate λ .

(a) The only bus departs after a deterministic time T . Let W be the combined waiting time for all passengers. Compute EW .

If S_1, S_2, \dots , are the arrival times in $[0, T]$, then the combined waiting time is $W = T - S_1 + T - S_2 + \dots$. Let $N(t)$ be the number of arrivals in $[0, t]$. We obtain the answer by conditioning on the value of $N(T)$:

$$\begin{aligned} E(W) &= \sum_{k=0}^{\infty} E[W | N(T) = k] P(N(T) = k) \\ &= \sum_{k=0}^{\infty} k \cdot \frac{T}{2} P(N(T) = k) \\ &= \frac{T}{2} EN(T) = \frac{\lambda T^2}{2}. \end{aligned}$$

(b) Now two buses depart, one at T and one at $S < T$. What is now EW ?

We have two independent Poisson processes in time intervals $[0, S]$ and $[S, T]$, so that the answer is

$$\lambda \frac{S^2}{2} + \lambda \frac{(T - S)^2}{2}.$$

(c) Now assume T , the only bus arrival time, is $\text{Exponential}(\mu)$, independent of the passengers' arrivals.

This time,

$$\begin{aligned} EW &= \int_0^\infty E(W|T=t) f_T(t) dt = \int_0^\infty \lambda \frac{t^2}{2} f_T(t) dt = \frac{\lambda}{2} E(T^2) \\ &= \frac{\lambda}{2} (\text{Var}(T) + (ET)^2) = \frac{\lambda}{2} \frac{2}{\mu^2} = \frac{\lambda}{\mu^2}. \end{aligned}$$

(d) Finally, two buses now arrive as first two events in a rate μ Poisson process.

This makes

$$EW = 2 \frac{\lambda}{\mu^2}.$$

Example 18.7. You have two machines. Machine 1 has lifetime T_1 , which is $\text{Exponential}(\lambda_1)$, and Machine 2 has lifetime T_2 , which is $\text{Exponential}(\lambda_2)$. Machine 1 starts at time 0 and Machine 2 starts at a time T .

(a) Assume that T is deterministic. Compute the probability that M_1 is the first to fail.

We could compute this via a double integral (which is a good exercise!), but instead we proceed like this:

$$\begin{aligned} P(T_1 < T_2 + T) &= P(T_1 < T) + P(T_1 \geq T, T_1 < T_2 + T) \\ &= P(T_1 < T) + P(T_1 < T_2 + T | T_1 \geq T) P(T_1 \geq T) \\ &= 1 - e^{-\lambda_1 T} + P(T_1 - T < T_2 | T_1 \geq T) e^{-\lambda_1 T} \\ &= 1 - e^{-\lambda_1 T} + P(T_1 < T_2) e^{-\lambda_1 T} \\ &= 1 - e^{-\lambda_1 T} + \frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-\lambda_1 T}. \end{aligned}$$

The key observation above is that $P(T_1 - T < T_2 | T_1 \geq T) = P(T_1 < T_2)$. Why does it hold? We can simply quote the memoryless property of Exponential distribution, but it is instructive to make a short argument using Poisson processes. Embed the failure times into appropriate Poisson processes. Then $T_1 \geq T$ means that no events in the first process occur during time $[0, T]$. Under this condition, $T_1 - T$ is time of the first event of the same process restarted at T , but this restarted process is not influenced by what happened before T , so the condition (which in addition does not influence T_2) drops out.

(b) Answer the same question when T is $\text{Exponential}(\mu)$ (and of course independent of the machines. Now, by the same logic,

$$\begin{aligned} P(T_1 < T_2 + T) &= P(T_1 < T) + P(T_1 \geq T, T_1 < T_2 + T) \\ &= \frac{\lambda_1}{\lambda_1 + \mu} + \frac{\lambda_1}{\lambda_1 + \lambda_2} \frac{\mu}{\lambda_1 + \lambda_2}. \end{aligned}$$

Example 18.8. *Impatient hitchhikers.* Two people, Alice and Bob, are hitchhiking. Cars that would pick up a hitchhiker arrive as a Poisson process with rate λ_C . Alice is first in line for a ride. Moreover, after $\text{Exponential}(\lambda_A)$ time, Alice quits, and after $\text{Exponential}(\lambda_B)$ time, Bob quits. Compute the probability that Alice is picked up before she quits, and the same for Bob.

Embed each quitting time into an appropriate Poisson process, call these A and B processes, and the car arrivals the C process. Clearly, Alice gets picked if the first event in the combined A and C process is a C event:

$$P(\text{Alice gets picked}) = \frac{\lambda_C}{\lambda_A + \lambda_C}.$$

Moreover,

$$\begin{aligned} &P(\text{Bob gets picked}) \\ &= P(\{\text{at least 2 } C \text{ events before a } B \text{ event}\} \\ &\quad \cup \{\text{at least one } A \text{ event before either a } B \text{ or a } C \text{ event,} \\ &\quad \text{then at least one } C \text{ event before a } B \text{ event}\}) \\ &= P(\text{at least 2 } C \text{ events before a } B \text{ event}) \\ &\quad + P(\text{at least one } A \text{ event before either a } B \text{ or a } C \text{ event,} \\ &\quad \text{then at least one } C \text{ event before a } B \text{ event}) \\ &\quad - P(\text{at least one } A \text{ event before either a } B \text{ or a } C \text{ event,} \\ &\quad \text{then at least two } C \text{ events before a } B \text{ event}) \\ &= \left(\frac{\lambda_C}{\lambda_B + \lambda_C}\right)^2 \\ &\quad + \left(\frac{\lambda_A}{\lambda_A + \lambda_B + \lambda_C}\right) \left(\frac{\lambda_C}{\lambda_B + \lambda_C}\right) \\ &\quad - \left(\frac{\lambda_A}{\lambda_A + \lambda_B + \lambda_C}\right) \left(\frac{\lambda_C}{\lambda_B + \lambda_C}\right)^2 \\ &= \frac{\lambda_A + \lambda_C}{\lambda_A + \lambda_B + \lambda_C} \cdot \frac{\lambda_C}{\lambda_B + \lambda_C}. \end{aligned}$$

This leaves us with an excellent hint that there may be a shorter way, and indeed there is:

$$P(\text{Bob gets picked}) = P(\text{first event is either } A \text{ or } C, \text{ then next event among } B \text{ and } C \text{ is } C).$$

Problems

1. An office has two clerks, and three people, A , B and C , enter simultaneously. A and B begin service at the two clerks, while C waits for the first available clerk. Assume that the service time is Exponential(λ). (a) Compute the probability that A is the last to finish the service. (b) Compute the expected time before C is done (i.e., C 's combined waiting and service time).
2. A car wash has two stations, 1 and 2, with Exponential(λ_1) and Exponential(λ_2) service times. A car enters at station 1. Upon completing the service at station 1, the car then proceeds to station 2, provided station 2 is free; otherwise, the car has to wait at station 1, blocking the entrance of other cars. The car exits the wash after service at station 2 is completed. When you arrive at the wash there is a single car at station 1. Compute your expected time before you exit.
3. A system has two server stations, 1 and 2, with Exponential(λ_1) and Exponential(λ_2) service times. Whenever a new customer arrives, any customer in the system immediately departs. Customer arrivals are a rate μ Poisson process. (a) What proportion of customers complete their service? (b) What proportion of customers stay in the system for more than 1 time unit, but do not complete the service?
4. A machine needs frequent maintenance to stay on. The maintenance times occur as a Poisson process with rate μ . Once the machine receives no maintenance for a time interval of length h , it breaks down. It then needs to be repaired, which takes an Exponential(λ) time, after which it goes back on. (a) After the machine is started, find the probability that the machine will break down before receiving its first maintenance. (b) Find the expected time for the first breakdown. (c) Find the proportion of time the machine is on.
5. Assume that certain events (say, power surges) occur as a Poisson process with rate 3 per hour. These events cause damage to certain system (say, a computer), thus a special protecting unit has been designed. That unit now has to be removed from the system for 10 minutes for service.
 - (a) Assume that a single event occurring in the service period will cause the system to crash. What is the probability that the system will crash?
 - (b) Assume that the system will survive a single event, but two events occurring in the service period will cause it to crash. What is now the probability that the system will crash?
 - (c) Assume that crash will not happen unless there are two events within 5 minutes of each other. Compute the probability that the system will crash.
 - (d) Solve (b) assuming that the protective unit will be out of the system for time which is exponentially distributed with expectation 10 minutes.

Solutions to problems

1. (a) This is the probability that two events happen in a rate λ Poisson process before a single one in a independent rate λ process, that is, $\frac{1}{4}$. (b) First C has to wait for the first event in two combined Poisson processes, which is a single process with rate 2λ , and then for the service time; the answer is $\frac{1}{2\lambda} + \frac{1}{\lambda} = \frac{3}{2\lambda}$.

2. Your total time is (time the other car spends at station 1) + (time you spend at station 2) + (maximum of the time the other car spends at station 2 and the time you spend at station 1). If T_1 and T_2 are $\text{Exponential}(\lambda_1)$ and $\text{Exponential}(\lambda_2)$, then you need to compute

$$E(T_1) + E(T_2) + E(\max\{T_1, T_2\}).$$

Now use that

$$\max\{T_1, T_2\} = T_1 + T_2 - \min\{T_1, T_2\}$$

and $\min\{T_1, T_2\}$ is $\text{Exponential}(\lambda_1 + \lambda_2)$, to get

$$\frac{2}{\lambda_1} + \frac{2}{\lambda_2} - \frac{1}{\lambda_1 + \lambda_2}.$$

3. (a) A customer need to complete the service at both stations before a new one arrives, thus the answer is

$$\frac{\lambda_1}{\lambda_1 + \mu} \cdot \frac{\lambda_2}{\lambda_2 + \mu}.$$

(b) Let T_1 and T_2 be the customer's times at stations 1 and 2. The event will happen if either

- $T_1 > 1$, no newcomers during time 1 and a newcomer during time $[1, T_1]$; or
- $T_1 < 1$, $T_1 + T_2 > 1$, no newcomers during time 1 and a newcomer during time $[1, T_1 + T_2]$.

For the first case, nothing will happen by time 1, which has probability $e^{-(\mu+\lambda_1)}$. Then, after time 1, a newcomer has to appear before the service time at station 1, which has probability $\frac{\mu}{\lambda_1 + \mu}$.

For the second case, conditioned on $T_1 = t < 1$, the probability is

$$e^{-\mu} e^{-\lambda_2(1-t)} \frac{\mu}{\lambda_2 + \mu}.$$

Therefore, the probability is

$$\frac{\mu}{\lambda_2 + \mu} \int_0^1 e^{-\lambda_2(1-t)} \lambda_1 e^{-\lambda_1 t} dt = \frac{\lambda_1 \mu}{\lambda_2 + \mu} e^{-\lambda_2} \frac{e^{\lambda_2 - \lambda_1} - 1}{\lambda_2 - \lambda_1},$$

where the last factor is 1 when $\lambda_1 = \lambda_2$. The answer is

$$e^{-(\mu+\lambda_1)} \frac{\mu}{\lambda_1 + \mu} + \frac{\lambda_1 \mu}{\lambda_2 + \mu} e^{-\lambda_2} \frac{e^{\lambda_2 - \lambda_1} - 1}{\lambda_2 - \lambda_1}.$$

4. (a) The answer is $e^{-\mu h}$. (b) Let W be the waiting time for a maintenance such that the next maintenance is at least time h in the future, and T_1 the time of the first maintenance. Then, provided $t < h$,

$$E(W|T_1 = t) = t + EW$$

as the process is restarted at time t . Therefore

$$EW = \int_0^h (t + EW) \mu e^{-\mu t} dt = \int_0^h t \mu e^{-\mu t} dt + EW \int_0^h \mu e^{-\mu t} dt.$$

Computing the two integrals and solving for EW gives

$$EW = \frac{1 - \mu h e^{-\mu h} - e^{-\mu h}}{e^{-\mu h}}.$$

The answer to (b) is $EW + h$ (the machine waits for h more units before it breaks down). The answer to (c) is

$$\frac{EW + h}{EW + h + \frac{1}{\lambda}}.$$

5. Assume the time units is 10 minutes, $\frac{1}{6}$ of an hour. The answer to (a) is

$$P(N(\frac{1}{6}) \geq 1) = 1 - e^{-\frac{1}{2}},$$

and to (b)

$$P(N(\frac{1}{6}) \geq 2) = 1 - \frac{3}{2}e^{-\frac{1}{2}}.$$

For (c), if there are 0 or 1 events in the 10 minutes, there will be no crash, but 3 or more events in the 10 minutes will cause a crash. The final possibility is exactly two events, in which case the crash will happen with probability

$$P(|U_1 - U_2| < \frac{1}{2}),$$

where U_1 and U_2 are independent uniform variables on $[0, 1]$. By drawing a picture, this probability can be computed to be $\frac{3}{4}$. Therefore,

$$\begin{aligned} P(\text{crash}) &= P(X > 2) + P(\text{crash}|X = 2)P(X = 2) \\ &= 1 - e^{-\frac{1}{2}} - \frac{1}{2}e^{-\frac{1}{2}} - e^{-\frac{1}{2}} \frac{(\frac{1}{2})^2}{2} + \frac{3}{4} \cdot \frac{(\frac{1}{2})^2}{2} e^{-\frac{1}{2}} \\ &= 1 - \frac{49}{32}e^{-\frac{1}{2}} \end{aligned}$$

Finally, for (d), we need to calculate the probability that two events in a rate 3 Poisson(3) occur before an event occurs in a rate 6 Poisson process occurs. This probability is

$$\left(\frac{3}{3+6}\right)^2 = \frac{1}{9}.$$