# Lecture 8 ... still on O-U process

If is hard to solve (or even comprehend) an SDE ...

$$\boxed{dX = -\theta x\, dt + \sigma\, dW}$$

Instead, it would be better if we had $\underbrace{P(x,t)}$

then normal calculus rules

How to systematically derive $P(x,t)$ from the SDE.

Start with Evolution Eq.

$$P(x, t+dt) = \int \underbrace{G(x, t+dt \,|\, y, t)}\, P(y,t)\, dy$$

Normally we think of a propagator $G(x|y)$ with fixed initial $y$, variable target $x$.

Here, the variable is $y$ ...

To address this, take $\Delta x = x - y$,

so $y = x - \Delta x$

$$P(x,t+\Delta t) = \int G(\overbrace{x - \Delta x + \Delta x}^{x}, t+\Delta t \mid x-\Delta x, t)\, P(x-\Delta x, t)\; d(-\Delta x)$$

uniform shift:

- G "steps" from $(x-\Delta x)$ into $(x-\Delta x)+\Delta x$:
- We have a uniform function of $(x-\Delta x)$

Taylor expand it:

step $\Delta x$

$$P(x,t+\Delta t) = \int \sum_{n=0}^{\infty} \frac{(-\Delta x)^n}{n!} \frac{\partial^n}{\partial x^n} G(x+\Delta x \mid x)\, P(x)\; \underbrace{d(\Delta x)}_{\text{whole range!}}$$

Define moments:

$\langle \Delta x^0 \rangle = 1$ : G normalised

for O-U

$\langle \Delta x^1 \rangle = \int \Delta x\, G(x+\Delta x \mid x)\, d\Delta x \quad \rightarrow \; -\theta x \cdot dt$

$\langle \Delta x^2 \rangle = \int \Delta x^2\, G(x+\Delta x \mid x)\, d\Delta x \quad \rightarrow \; \sigma^2 dt$

. . . . .

$$P(x,t+\Delta t) = P(x,t) - \frac{\partial}{\partial x}\left(\langle \Delta x \rangle P\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\langle \Delta x^2 \rangle P\right) + \ldots$$

terms $\sim dt$

(no more)

Now substitute moments:

$$\boxed{\frac{\partial P(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(\theta x \cdot P(x,t)\right) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}P(x,t)}$$

for O-U process. No more terms $\sim dt$ in r.h.s.

(limit $dt \to 0$)

external force

diffusion constant

$$\boxed{D = \frac{1}{2}\sigma^2}$$

This was an example of using "Kramers-Moyal expansion"

Evolution → Expand kernel in powers of "step"

identify moments

form the PDE for $P(x,t)$

# Kramers-Moyal Expansion (general)

## Evolution

$$P(x,t+\Delta t) = \int G(x,t+\Delta t \,|\, y,t)\, P(y,t)\, dy$$

or it could be Kolmogorov-Chapman if we care about the initial condition:

$$P(x,t+\Delta t \,|\, x_0 t_0) = \int G(x,t+\Delta t \,|\, y,t)\, P(y,t \,|\, x_0 t_0)\, dy$$

Again: $\Delta x = x-y$, so $y = x - \Delta x$

$$P(x,t+\Delta t) = \int G(x-\Delta x+\Delta x \,|\, x-\Delta x)\, P(x-\Delta x,t)\, d(-\Delta x)$$

shifted argument

step

Taylor expand it

$$= \int \sum_{n=0}^{\infty} \frac{(-\Delta x)^n}{n!} \frac{\partial^n}{\partial x^n} G(x+\Delta x \,|\, x)\, P(x,t)\, d(\Delta x)$$

Define "Kramers-Moyal coefficients"

$$\boxed{D^{(n)}(x) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int \frac{(\Delta x)^n}{n!} G(x+\Delta x \,|\, x)\, d\Delta x}$$

Note that this is general (whatever $G$ is), for any SDE. (For Wiener SDE, only two $D^{(1)}$ and $D^{(2)}$ are non-zero)

Then we have:

$$\boxed{\frac{\partial P(x,t)}{\partial t} = \sum_{n=1}^{\infty} (-1)^n \frac{\partial^n}{\partial x^n} \left[ D^{(n)}(x)\, P(x,t) \right]}$$

for $\forall$ SDE, not necessarily Wiener, or ......

$$\frac{\partial P}{\partial t} = \hat{L}_{KM}\, P(x,t)$$

K.-M. operator

$D^{(n)}(x)$ carry all information about the nature of stochastic proc.

(for Wiener process, there are only two non-zero K-M coeffs)

**Practice**:

1

$$\dot{v} = -\frac{\gamma}{m} v + \frac{\sqrt{2kT\gamma}}{m} \xi(t)$$

$$dv = -\frac{\gamma}{m} v\, dt + \frac{\sqrt{2kT\gamma}}{m} dW$$

standard O-U format

$$\langle dv \rangle = -\frac{\gamma}{m} v\, dt$$

$$\langle dv^2 \rangle = \frac{2kT\gamma}{m^2} \underbrace{\langle dW^2 \rangle}_{= dt}$$

then:

$$\boxed{\frac{\partial P(v,t)}{\partial t} = \frac{\gamma}{m} \frac{\partial}{\partial v} (v \cdot P) + \frac{\cancel{2}kT\gamma}{\cancel{2}m^2} \frac{\partial^2 P}{\partial v^2}}$$

This equation describes relaxation of P(v,t) towards the equilibrium Maxwell distribution test the steady state (eq.):

$$0 = \frac{\partial}{\partial v}\Big[ \underbrace{\frac{\gamma}{m} v \cdot p + \frac{kT\gamma}{m^2}\frac{\partial p}{\partial v}} \Big]$$

$$\frac{\partial p}{\partial v} = \frac{m^2}{kT\gamma}\left(-\frac{\gamma}{m} v \cdot p\right)$$

$$\int \frac{dp}{p} = -\int \frac{m}{kT} v \, dv$$

thermostat equilib. $\boxed{p_{eq.} = \text{norm}\, e^{-\frac{mv^2}{2kT}}}$

## (2) Relaxation to a fixed equilibrium ( in a harmonic potential V(x) )

$$\gamma \dot{x} = -\theta(x - x_0) + \sqrt{2kT\gamma} \cdot \xi(t)$$

overdamped limit

$$dx = -\frac{\theta}{\gamma}(x - x_0) + \sqrt{\frac{2kT}{\gamma}}\, dw$$

standard O-U format

will test later

$$\boxed{\frac{\partial P(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\theta}{\gamma}(x - x_0) P\right) + \frac{kT}{\gamma}\frac{\partial^2 P}{\partial x^2}}$$

Smoluchowski equation:
(diffusion with external force)

Test: steady state: $\frac{dP}{dx} = -\frac{\theta}{kT}(x - x_0) P$ : $\boxed{P = P_0 e^{-\frac{\frac{\theta}{2}(x-x_0)^2}{kT}}}$

i.e. Boltzmann distr.

③ GBM $\quad dS = \mu S\,dt + \sigma S\,dW$

$$\langle dS \rangle = \mu S\,dt$$

$$\langle dS^2 \rangle = \sigma^2 S^2 \langle dW^2 \rangle = dt$$

$$\boxed{\frac{\partial P(S,t)}{\partial t} = -\frac{\partial}{\partial S}\left(\mu S\,P(S,t)\right) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial S^2}\left(S^2 P\right)}$$

④ Multi-variable process.

(see the last lecture for 'matrix notation')

Brownian motion with force

$v(t)$
$x(t)$

$$\begin{cases} m\dot{v} = -\gamma v - \theta(x - x_0) + \sqrt{2kT\gamma}\,\xi(t) \\ \dot{x} = v \end{cases}$$

$\boxed{\text{O-U process}}$ — linear drift terms.

Matrices: $\underline{\underline{\theta}} = \begin{pmatrix} \frac{\gamma}{m} & \frac{\theta}{m} \\ -1 & 0 \end{pmatrix}$ ; $\underline{\underline{\sigma}} = \begin{pmatrix} \frac{\sqrt{2kT\gamma}}{m} & 0 \\ 0 & 0 \end{pmatrix}$

Let us work out the PDE for $P(v, x, t)$ ...

1st K-M coefficient 2nd K-M coefficient

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x_i}\left(\theta_{ik}\, x_k \cdot P\right) + \frac{1}{2}\left(\sigma\sigma^T\right)_{ik} \frac{\partial^2 P}{\partial x_i \partial x_k}$$

$$= \frac{\partial}{\partial v}\left(\frac{\gamma}{m} v \cdot P\right) + \frac{\partial}{\partial v}\left(\frac{\theta}{m}(x - x_0)\, P\right)$$

$$- \frac{\partial}{\partial x}\left(v P\right) + \frac{\cancel{2} kT\gamma}{\cancel{2} m^2} \frac{\partial^2 P}{\partial v^2}$$

$$\boxed{\underbrace{\frac{\partial P(v,x,t)}{\partial t} + \frac{\partial}{\partial x}\left(v \cdot P\right)}_{\text{convective derivative}} = \frac{\partial}{\partial v}\left(\frac{\overbrace{\gamma v + \theta(x - x_0)}^{\text{full force}}}{m} \cdot P\right) + \frac{kT\gamma}{m^2}\frac{\partial^2 P}{\partial v^2}}$$

General Fokker-Planck equation for Brownian motion in  pot.