Problem Set 2

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1. Consider the regularised least squares solution to a linear(ised) inverse problem obtained by minimising the function

$$J(\mathbf{m}) = \frac{1}{2}(\mathbf{A}\mathbf{m} - \mathbf{d})^T \mathbf{C}^{-1}(\mathbf{A}\mathbf{m} - \mathbf{d}) + \frac{\lambda}{2}\mathbf{m}^T \mathbf{B}\mathbf{m}.$$

   Suppose that the data takes the form $\mathbf{d} = \mathbf{A}\mathbf{m}_{\mathrm{in}}$, with $\mathbf{m}_{\mathrm{in}}$ a specified input model. Show that the resulting least squares solution, $\mathbf{m}_{\mathrm{out}}$, takes the form

$$\mathbf{m}_{\mathrm{out}} = \mathbf{R}\mathbf{m}_{\mathrm{in}},$$

   where $\mathbf{R}$ is a matrix to be determined. Discuss the significance of this so-called *resolution matrix*, explaining why, in particular, it would be desirable for its value to be close to the identity matrix.

2. Consider the deformation of an isotropic elastic body due to an applied surface load, σ. The relevant linearised equations of motion are

$$-\frac{\partial T_{ij}}{\partial x_j} = 0, \quad T_{ij} = \lambda \frac{\partial u_k}{\partial x_k}\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

   within the reference body M, and subject to boundary conditions

$$T_{ij}\hat{n}_j = \sigma g_i,$$

   on its boundary, ∂M. Here g_i is the acceleration due to gravity prior to the deformation. Note that self-gravitation is being ignored within these equations.

   Suppose that observations, $\mathbf{u}_i^{\mathrm{obs}}$, of the displacement vector have been made at surface locations $\mathbf{x}_i$ for $i = 1, \ldots, m$. An inverse problem could then be formulated to estimate the Lamé parameters, λ and μ, as well as the load, σ. To this end, we can seek to minimise the misfit

$$J = \frac{1}{2}\sum_{i=1}^{m} \frac{1}{\sigma_i}\|\mathbf{u}(\mathbf{x}_i) - \mathbf{u}_i^{\mathrm{obs}}\|^2,$$

   where σ_i is the standard deviation for the ith observation. Adapt the approach within Lecture 19 to derive sensitivity kernels for J with respect to λ, μ, and σ, showing how they can be calculated through one solution of the forward problem and one of an appropriate adjoint problem.

3. Within Lecture 19 the idea of a cross correlation travel time was introduced. This is based on finding the shift, $\bar{\tau}$, that maximises the cross correlation

$$C(\tau) = \int_{-\infty}^{\infty} s^{\mathrm{obs}}(t - \tau)s(\mathbf{x}_s, t)\,\mathrm{d}t,$$

between an observed waveform, s^{obs}, at a surface location $\mathbf{x}_s$ and a synthetic one, s, both having been suitably windowed. By perturbing the optimality condition

$$\frac{\mathrm{d}C}{\mathrm{d}\tau}(\bar{\tau}) = 0,$$

with respect to s, show that to first-order accuracy

$$\delta\bar{\tau} = \frac{\int_{-\infty}^{\infty} \dot{s}^{\text{obs}}(t-\bar{\tau})\,\delta s(\mathbf{x}_s,t)\,\mathrm{d}t}{\int_{-\infty}^{\infty} \ddot{s}^{\text{obs}}(t-\bar{\tau}) s(\mathbf{x}_s,t)\,\mathrm{d}t},$$

where dots denote time derivatives. From this result, it is a simple matter to obtain adjoint tractions required for the calculation of banana doughnut kernels.

4. Recall that the Lagrangian for a self-gravitating elastic body is given by

$$\mathcal{L}(\mathbf{x},t,\boldsymbol{\varphi},\mathbf{v},\mathbf{F}) = \frac{1}{2}\rho(\mathbf{x})\|\mathbf{v}\|^2 - W(\mathbf{x},t,\mathbf{F}) - \frac{1}{2}\rho(\mathbf{x})\zeta(\mathbf{x},t),$$

where ζ is the referential gravitational potential

$$\zeta(\mathbf{x},t) = -G\int_M \frac{\rho(\mathbf{x}')}{\{[\varphi_i(\mathbf{x},t)-\varphi_i(\mathbf{x}',t)][\varphi_i(\mathbf{x},t)-\varphi_i(\mathbf{x}',t)]\}^{\frac{1}{2}}}\,\mathrm{d}^3\mathbf{x}'.$$

By calculating the appropriate functional derivative, show that

$$\frac{\delta\mathcal{L}}{\delta\varphi_i} = \rho\gamma_i,$$

where γ_i is the referential gravitational acceleration.

5. Consider a planet at equilibrium occupying the volume M, and with density ρ. Its gravitational potential, ϕ, is a solution of Poisson's equation

$$\nabla^2\phi = 4\pi G\rho,$$

where it is understood that ρ vanishes outside of M. Across the surface ∂M, we then have continuity conditions on ϕ and its normal derivative, and finally require that ϕ tend to zero at infinity.

Suppose that the planet is only slightly aspherical, with boundary

$$r = b + s\,h_1(\theta,\varphi) + \cdots,$$

in spherical polar co-ordinates, and its density decomposed as

$$\rho = \rho_0 + s\,\rho_1 + \cdots,$$

with ρ_0 spherically symmetric. Here s is a perturbation parameter, and the dots denote higher-order terms that are to be neglected. Similarly writing the potential in the form $\phi = \phi_0 + s\,\phi_1 + \cdots$, show that across the reference surface $r = b$ we require continuity of both ϕ_1 and

$$\frac{\partial\phi_1}{\partial r} + 4\pi G\rho_0 h_1.$$

6. Using results from Lecture 22 along with residue calculus (which you don't need to know for this course), it can be shown that the time-domain solution of the elastodynamic equations in a non-rotating earth model can be expressed as the following eigenfunction expansion

$$\mathbf{u}(\mathbf{x},t) = \sum_{km} \left\{ \int_0^t \frac{\sin[\omega_k(t-t')]}{\omega_k} \int_M \bar{S}_{ij}(\mathbf{x}',t') \frac{\partial \,|\, km \,\rangle^*}{\partial x_j}(\mathbf{x}')\,\mathrm{d}^3\mathbf{x}'\,\mathrm{d}t' \right\} |\, km \,\rangle\,(\mathbf{x}).$$

   Simplify this expression in the case of a *point source* for which

$$\bar{S}_{ij}(\mathbf{x},t) = M_{ij}\delta(\mathbf{x}-\mathbf{x_s})H(t-t_s),$$

   with M_{ij} known as the moment tensor, $\mathbf{x}_s$ the source location, and t_s the source time. Describe briefly and qualitatively how an inverse problem might be formulated to estimate the ten source parameters $(M_{ij}, \mathbf{x}_s, t_s)$ from recorded seismograms.

7. Consider the eigenvalue problem for a rotating earth model

$$-\omega^2 \,\langle\, \mathbf{w} \,|\, P \,|\, \mathbf{s} \,\rangle + \mathrm{i}\omega \,\langle\, \mathbf{w} \,|\, W \,|\, \mathbf{s} \,\rangle + \langle\, \mathbf{w} \,|\, H \,|\, \mathbf{s} \,\rangle = 0,$$

   where $\mathbf{s}$ is the eigenfunction, ω the eigenvalue, and $\mathbf{w}$ an arbitrary test function. Assuming the eigenfunction normalisation condition

$$\langle\, \mathbf{s} \,|\, P \,|\, \mathbf{s} \,\rangle = 1,$$

   show that the eigenfrequency can be written

$$\omega = \frac{1}{2}\mathrm{i}\,\langle\, \mathbf{s} \,|\, W \,|\, \mathbf{s} \,\rangle \pm \sqrt{\langle\, \mathbf{s} \,|\, H \,|\, \mathbf{s} \,\rangle - \frac{1}{4}\,\langle\, \mathbf{s} \,|\, W \,|\, \mathbf{s} \,\rangle^2}.$$

   Use this result to discuss qualitatively the stability of a rotating earth model, including, in particular, the role of the Coriolis and centrifugal forces.

Comments on past exam questions: The seismology half of this course was modernised in 2016. As a result, exam questions from earlier years are not applicable. The following is a guide so you know what is relevant.

- All seismology questions prior to the 2015 exam should be ignored.
- On the 2015 exam, question 1, part (a) asks for definitions of some terms related to earthquake sources. We did not cover "dip", "strike", nor "rake", so these can be ignored. In the rest of the question, the notations differ somewhat from what was done in lectures, but the content is comparable.
- On the 2016 exam question 3, part (c) asks for a qualitative description of waves in a layer above a half space. Though this material was discussed briefly, it was not done in enough detail to answer such a question, and it can be ignored.
- All later seismology questions are based on the current course, though notations used do vary somewhat.