Relativistic Astrophysics and Cosmology — Answers 2 — 2023

1. (i) From Handout 5 we know the expression for specific angular momentum h in this metric is

$$h^2 = \frac{c^2 r^3 A'}{2A - rA'}$$

and can note that $h^2 = r^4 \dot{\phi}^2$.

If we employ the rapidity α, we know this is given by $v = c\tanh\alpha$ where v is the ordinary velocity we would like to find.

$\frac{dt}{d\tau} = \cosh\alpha$, where τ is the particle proper time, implies

$$v = r\frac{d\phi}{dt} = r\frac{d\phi}{d\tau}\Big/\frac{dt}{d\tau} = \frac{r\dot{\phi}}{\cosh\alpha},$$

from which we deduce $r\dot{\phi} = c\sinh\alpha$. Now

$$\tanh^2\alpha = \frac{\sinh^2\alpha}{1+\sinh^2\alpha},$$

and $r^2\dot{\phi}^2 = \left(\frac{h}{r}\right)^2$, hence using the above expression for h^2

$$\frac{v^2}{c^2} = \tanh^2\alpha = \frac{h^2}{h^2+r^2} = \frac{rA'}{2A}.$$

(ii) Handout 4 tells us that

$$\frac{r}{A}\frac{dA}{dr} = \frac{8\pi G}{c^4}r^2 BP - 1 + B,$$

for a spherically symmetric perfect fluid with pressure P. Furthermore B works out to the simple form

$$B(r) = \left[1 - \frac{2Gm(r)}{c^2 r}\right]^{-1},$$

where $m(r)$ is the usual integral of density $4\pi\int_0^r \rho(\bar{r})\bar{r}^2\,d\bar{r}$. Inserting these into the expression for v^2 in terms of A, we indeed get

$$v^2 = \frac{G\left(m(r) + 4\pi r^3 P(r)/c^2\right)}{r - 2Gm(r)/c^2}.$$

For the next part of the question, we need the radial profile of pressure. We could use the exact profile given above, but the Newtonian pressure result

$$P(r) = \frac{2}{3}\pi G\rho_0^2 R^2\left(1 - \frac{r^2}{R^2}\right)$$

would be fine to the accuracy required (and in any case, the assumption we can treat the Earth as a perfect fluid is unlikely to be well justified). Suitable figures for M and R for the Earth are $R = 6371\,\mathrm{km}$ and $M = 5.972 \times 10^{24}\,\mathrm{kg}$, leading on the assumption of constant density to $\rho_0 = 5513\,\mathrm{kgm}^{-3}$. Working out the pressure at a depth of 100 km then gives a P of $5.371 \times 10^9\,\mathrm{Nm}^{-2}$ (so about 53,000 times atmospheric pressure!).

Inserting these numbers into the expression for v^2 yields a typical velocity of 7.785 km/s, with additions to this of $1.6 \times 10^{-9}\%$ corresponding to the P term, and $6.7 \times 10^{-8}\%$ for the effect of the $-2m(r)$ term in the denominator. Thus such effects are very small for the Earth (though can be very big for neutron stars).

2. Surface area of a Schwarzschild black hole is $\mathcal{A} = 4\pi R_S^2$, where $R_S = 2GM/c^2$ is the Schwarzschild radius.

   Inserting this in $S = k_{\mathrm{B}}\mathcal{A}/(4\ell_p^2)$, and using $\ell_p = \sqrt{\hbar G/c^3}$, we get

$$S = \frac{4\pi k_{\mathrm{B}} G M^2}{\hbar c}.$$

   Evaluating for $M = 2 \times 10^{30}\,\mathrm{kg}$ gives $S = 1.06 \times 10^{77} k_{\mathrm{B}} = 1.46 \times 10^{54}\,\mathrm{JK}^{-1}$.

   On the other hand, for the Sun a rough estimate of the number of baryons it contains is M/m_p, and so its entropy is about 20 times this in units of k_{B}, i.e. $\sim 2.4 \times 10^{58} k_{\mathrm{B}} = 3.3 \times 10^{35}\,\mathrm{JK}^{-1}$.

   The entropy of a solar mass black hole is therefore about 4.4×10^{18} larger than that of the Sun itself.

3. From the uncertainty principle, estimate p by $\hbar/a$, where a is the radius. The total energy then becomes

$$E = \frac{\hbar^2}{2m_e a^2} - \frac{e^2}{4\pi\epsilon_0 a}.$$

   This has a minimum at

$$a = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2} = a_0,$$

   which is the Bohr radius.

   The relativistic expression for the kinetic energy is

$$\mathrm{K.E.} = m_e c^2\left(1 + \frac{p^2}{m_e^2 c^2}\right)^{1/2} - m_e c^2,$$

   where p is the relativistic momentum, and again we replace p by $\hbar/a$. The total energy is now

$$E = m_e c^2\left(1 + \frac{\hbar^2}{m_e^2 c^2 a^2}\right)^{1/2} - m_e c^2 - \frac{e^2}{4\pi\epsilon_0 a},$$

   which now has a minimum at a given by

$$a^2 = a_0^2 - \frac{\hbar^2}{m_e^2 c^2}.$$

The correction term $\hbar/m_e c$ is the Compton wavelength of the electron. As the nuclear charge is increased, get

$$a^2 = \left(\frac{a_0}{Z}\right)^2 - \left(\frac{\hbar}{m_e c}\right)^2$$

which has no solution if

$$Z > \frac{4\pi\epsilon_0 \hbar c}{e^2} = \frac{1}{\alpha_f}.$$

This limit corresponds to attempting to localise an electron within its Compton wavelength. The same limit is found in the complete quantum analysis (needs the Dirac equation). Smearing out the nuclear charge can increase the limit on Z.

4. The Eddington limit is where the force due to radiation pushing matter out balances the gravitational attraction:

$$\frac{GMm_p}{R^2} = \underbrace{\frac{L}{4\pi R^2 h\nu}}_{\text{number flux}} \underbrace{\sigma_T}_{\text{cross-section}} \underbrace{\frac{h\nu}{c}}_{\text{momentum}}$$

Gravitational force acts on the protons (or ions). Radiation pressure acts on the free electrons in the plasma. So for ionised H we get

$$L_{\text{Edd}} = \frac{4\pi GMm_p c}{\sigma_T}$$

in the usual way. For ionised Helium (mass $4m_p$, with 2 electrons), we get

$$L_{\text{Edd}} = \frac{4\pi GM(4m_p)\,c}{2\sigma_T} = 2 \times \text{that for H}$$

For partially ionised iron (K and L shells retained), the ion mass is $56m_p$, the K shell has 2 electrons and the L 8, therefore 10 are retained and $26 - 10 = 16$ are free. Thus

$$L_{\text{Edd}} = \frac{4\pi GM(56m_p)\,c}{16\sigma_T} = \frac{7}{2}\times \text{that for H}$$

The idea of the next part of the question is to pretend that human beings are bound not by electrostatic forces, but by gravity. Also, that we are made of ionised material, so that radiation pressure can be exerted on free electrons.

The surface area of a human is in the range $1\,\text{m}^2$ to $2\,\text{m}^2$, so treated as a black body radiator at 300 K we get a luminosity of about

$$L = A\sigma_{SB}T^4 \approx 690\,\text{W}$$

(This looks impressive, but of course human beings absorb heat as well as radiating, and the *net* emission is much less.)

Assuming humans are made of ionised H(!), with a mass of 75 kg say, then their Eddington luminosity is

$$L_{\mathrm{Edd}} = \frac{4\pi G M m_p c}{\sigma_T} \approx 470\,\mathrm{W}$$

but we could raise this to higher values for higher ion mass. Thus human luminosity is at or around the Eddington limit, depending on the assumptions made.

For the final part, we have that the ULX has $L = 10^{34}$W and we know $L_{\mathrm{Edd}} = 10^{31}\mathrm{W}M_\odot^{-1}$ for H, and only 7/2 times this for ionised iron. Thus to be sub-Eddington it must be a black hole with a mass of at least $10^3 M_\odot$. (Note this ignores possible beaming effects.)

5. We are considering energies where the Inverse Compton effect is just Thomson scattering in the electron's rest frame. Specifically, consider a collision between a photon and a relativistic electron as seen in the laboratory frame of reference S and in the rest frame of the electron S'. Since $h\nu << m_{\mathrm{e}}c^2$ in S', the centre of momentum frame is very closely that of the relativistic electron. If the energy of the photon is $h\nu$ and the angle of incidence θ in S, its energy in the frame S' is

$$h\nu' = h\nu\gamma[1 + \beta\cos\theta]$$

according to the standard relativistic Doppler shift formula with β being the electron velocity. Similarly, the angle of incidence θ' in the frame S' is related to θ by the formulae

$$\sin\theta' = \frac{\sin\theta}{\gamma[1 + \beta\cos\theta]}$$

$$\cos\theta' = \frac{\cos\theta + \beta}{1 + \beta\cos\theta}$$

Now, provided $h\nu' << m_{\mathrm{e}}c^2$, the Compton interaction in the rest frame of the electron is simply Thomson scattering and hence the energy loss rate of the electron in S' is just the rate at which energy is reradiated by the electron. According to the analysis of Thomson scattering, the loss rate is

$$-(\mathrm{d}E/\mathrm{d}t)' = \sigma_T\, c\, U'$$

where U' is the energy density of radiation in the rest frame of the electron. It is of no importance whether or not the radiation is isotropic. The free electron oscillates in response to any incident radiation field. Our strategy is therefore to work out U' in the frame of the electron S' and then to use this to work out $(\mathrm{d}E/\mathrm{d}t)'$. Because $\mathrm{d}E/\mathrm{d}t$ is an invariant between inertial frames, this is also the loss rate $(\mathrm{d}E/\mathrm{d}t)$ in the observer's frame S.

Suppose the number density of photons in a beam of radiation incident at angle θ to the x-axis is N. Then, the energy density of these photons in S is $Nh\nu$. The flux density of photons incident upon an electron stationary in S is $Uc = Nh\nu c$. Now let us work out the flux density of this beam in the frame of reference of the electron S'. We need two things, the energy of each photon in S' and the rate of arrival of these photons at the electron in S'. The first of these is given $h\nu' = h\nu\gamma[1+\beta\cos\theta]$. For estimating the second one, we consider two photons which arrive at the origin of S' at times t_1' and t_2'. The coordinates of these events in S are

$$[x_1, 0, 0, t_1] = [\gamma v t_1', 0, 0, \gamma t_1'] \quad \text{and} \quad [x_2, 0, 0, t_2] = [\gamma v t_2', 0, 0, \gamma t_2'].$$

This calculation makes the important point that the photons in the beam are propagated along parallel but separate trajectories in S. It is apparent that the time difference when the photons arrive at a plane perpendicular to their direction of propagation in S is

$$\Delta t = t_2 + \frac{x_2 - x_1}{c}\cos\theta - t_1 = (t_2' - t_1')\gamma[1+\beta\cos\theta],$$

that is, the time interval between the arrival of photons from the direction θ is shorter by a factor $\gamma[1+\beta\cos\theta]$ in S' than it is in S. Hence the contribution to the energy density in the electron rest frame of an incoming photon moving at angle θ (according to the observer in the frame in which the radiation is isotropic) will be enhanced by a factor $\gamma^2(1+\beta\cos\theta)^2$. On reflection, we should not be surprised by this result because these are two different aspects of the same relativistic transformation between the frames S and S', in one case the frequency interval and, in the other, the time interval. We now need to average this over all angles to get U'.

This gives

$$\begin{aligned} U' &= \frac{U\gamma^2}{4\pi} 2\pi \int_0^{\pi} (1+\beta\cos\theta)^2 \sin\theta d\theta \\ &= \frac{U\gamma^2}{2} 2(1+\beta^2/3) \end{aligned}$$

(for example use $dx = -\sin\theta d\theta$ as a change of variable).

Now $\beta^2 = 1 - 1/\gamma^2$, therefore

$$U' = \frac{U\gamma^2}{3}(3 + (1 - 1/\gamma^2)) = \left(\frac{4}{3}\gamma^2 - \frac{1}{3}\right) U \quad (*)$$

The second way we can get this result, is by using the stress-energy tensor for radiation. We know generally for a fluid this is given by

$$T^{\mu\nu} = (\rho + P/c^2)u^\mu u^\nu - P\eta^{\mu\nu}$$

and for radiation, where $P = \frac{1}{3}\rho c^2$, this gives

$$T^{\mu\nu} = \left(\frac{4}{3}u^\mu u^\nu - \frac{c^2}{3}\eta^{\mu\nu}\right)\rho$$

Now U' is just T^{00} evaluated in the electron rest frame and the 0 component of u in this frame is just $c\gamma$, so with $U = \rho c^2$ we recover $(*)$.

Using the result that the rate at which the radiation field has energy put into it is $\sigma_T c U'$, we see $(*)$ means that a photon which has been scattered has on average an energy of

$$\bar{\epsilon}_f = \left(\frac{4}{3}\gamma^2 - \frac{1}{3}\right)\epsilon$$

(For a further method of obtaining this result, see Rybicki and Lightmann, 'Radiative Processes in Astrophysics', Section 7.2.)

6. The gravitational redshift on a photon emitted from $r = 6GM/c^2$ with frequency ω results in a frequency at ∞ of

$$\omega\left(1 - \frac{2GM}{c^2 r}\right)^{1/2} = \sqrt{2/3}\,\omega.$$

Now $r^2\dot{\phi} = h$ and h is determined by (from p14, Handout 5)

$$h^2 = \frac{GMr^2}{r - 3GM/c^2}, \quad \text{implies} \quad h = \frac{cr}{\sqrt{3}}.$$

The tangential velocity is given by $r\dot{\phi} = \gamma v = c/\sqrt{3}$, which gives $v = c/2$ and $\gamma = 2/\sqrt{3}$. In the transverse case get an overall factor of $\sqrt{2/3} \times \sqrt{3}/2 = 1/\sqrt{2}$, which defines a redshift of $z = \sqrt{2} - 1$.

In the plane case get a Doppler factor of

$$\left(\frac{1 + v/c}{1 - v/c}\right)^{1/2} = \begin{cases} \sqrt{3} & \text{towards observer} \\ 1/\sqrt{3} & \text{away from observer} \end{cases}$$

which is multiplied by a further factor of $\sqrt{2/3}$ from the gravitational redshift. This produces a redshift of -0.29 for motion towards the observer (*i.e.* a blueshift) and 1.12 away from the observer.

7. Radiation is produced as energy is dissipated in achieving the innermost stable orbit. The total energy of a particle in this orbit for Newtonian gravity is

$$E_{\text{circ}} = -\frac{GMm}{2r} = -\frac{1}{12}mc^2$$

for $r = 6GM/c^2$.

The efficiency ϵ is the energy dissipated divided by the total energy (mc^2), hence we get

$$\epsilon_{\text{Newt}} = \frac{1}{12} = 0.083.$$

In the GR case, the energy in a circular orbit is given by (p15, Handout 5),

$$E_{\rm circ} = mc^2 \frac{1 - 2GM/(c^2 r)}{[1 - 3GM/(c^2 r)]^{1/2}} = \frac{2\sqrt{2}}{3} mc^2.$$

leading to

$$\epsilon_{\rm GR} = 1 - \frac{2\sqrt{2}}{3} = 0.057.$$

slightly less than the Newtonian value.

8. We use results in the slides for Lecture 9.

   The temperature is

$$T(r) = \left(\frac{F(r)}{\sigma}\right)^{1/4}$$

   so we just need the maximum in

$$F(r) = \frac{3\dot{M}GM}{8\pi}\frac{1}{r^3}\left(1 - \beta\left(\frac{r_*}{r}\right)^{1/2}\right)$$

   Differentiating, this will be where

$$\frac{3}{r^4}\left(1 - \beta\left(\frac{r_*}{r}\right)^{1/2}\right) = \frac{r_*}{r^5}\left(\frac{r_*}{r}\right)^{-1/2}\tfrac{1}{2}\beta$$

   which unwraps to $r^{1/2} = \frac{7}{6}\beta r_*^{1/2}$, i.e.

$$r = \frac{49}{36}\beta^2 r_*$$

   To get the radial inflow velocity, we can use

$$\dot{M} = 2\pi r \rho 2h v_r, \quad s_\phi(2\pi r 2h)r = \dot{M}\left(\sqrt{GMr} - \beta\sqrt{GMr_*}\right)$$

   the first being the mass flow and the second the angular momentum flow controlled by viscous torque. Also, for an "α-disk" we know that $s_\phi = \alpha P$, where P is the pressure. We thus get

$$v_r = \frac{\alpha(P/\rho)r}{\sqrt{GMr} - \beta\sqrt{GMr_*}}$$

   Now $c_s^2 = P/\rho$ and since $r\Omega = \sqrt{GM/r}$ we have $\sqrt{GMr} = r^2\Omega$.

   We now need to use hydrostatic equilibrium $\rho\nabla\Phi = -\nabla P$ to derive the "standard result" $h \sim c_s/\Omega$. Resolving the disk vertically, the restoring force downward from the central mass is $-\nabla\Phi = \frac{GM}{r^2}\sin\theta \approx \frac{GM}{r^2}\frac{h}{r} = \Omega^2 h$. Writing $\nabla P \sim \frac{P}{h}$ recovers $h \sim c_s/\Omega$.

   Putting this all together yields

$$v_r = \frac{\alpha c_s^2 r}{r^2\Omega} = \frac{\alpha c_s h}{r}$$

   where we have ignored the $\beta\sqrt{GMr_*}$ term in comparison to $\sqrt{GMr}$, i.e. we are assuming we are far out in the disc.

9. The expression is
$$v_{\rm app} = \frac{v\sin\theta}{1-\frac{v}{c}\cos\theta}.$$
Taking $dv_{\rm app}/d\theta$ and setting it equal to 0, we find
$$\cos\theta = \frac{v}{c}, \quad \text{which implies} \quad \sin\theta = \frac{1}{\gamma},$$
where γ is the Lorentz boost factor.

10. The intensity scales as $D^{3+\alpha}$ where
$$D = \frac{1}{\gamma\left(1-\frac{v}{c}\cos\theta\right)}.$$
Solving $v_{\rm app} = 2.5\,c$ for v, we obtain $v = 0.977c$ and so $\gamma = 4.72$. Thus, using $\alpha = 1$, we see that D^4 is ≈ 0.5 for the jet approaching us, and ≈ 0.0002 for the receding side. This explains why the counterjet is not detected.