Lecture 4 : Wiener process

random variable X : $p(x)$

Stochastic process $y_x \to y(t)$

when independent steps each have Gaussian probability

$W(t)$

- Increment $\Delta W = W(t+\Delta t) - W(t)$

has
$$\boxed{p(\Delta W) = \frac{1}{\sqrt{2\pi\Delta t}}\, e^{-\frac{\Delta W^2}{2\Delta t}}}$$

(Normalise if: Variance $= \Delta t$)

- Characteristic function of ΔW

$$\phi(k,t) = \left\langle e^{ik\Delta W} \right\rangle_{p(\Delta W)}$$

$$\boxed{= e^{ikW_0} \cdot e^{-\frac{1}{2}k^2(\Delta t - t_0)}}$$

if we had a condition W_0 at $t = t_0$

- Scaling: $V(t) = \frac{1}{\sqrt{c}} W(ct)$ is also a Wiener
- time-inversion: $V(t) = t \cdot W(1/t)$ — " —
- time-reversal: $V(t) = W(A) - W(A-t)$ — " —

Could we find the full probability of the process $W(t)$: $\boxed{P(w,t)}$?

Let's start from Evolution eq:

$$P(w+\Delta w, t+\Delta t) = \int \underbrace{G(\overset{(w+\Delta w)-w_i}{\Delta w}/\Delta t)}_{\text{propagator : step } \Delta w} P(w_i, t)\, dw$$

or it could be the Kolmogorov-Chapman:

$$P(w+\Delta w, t+\Delta t \,/\, w_0, t_0) =$$
$$= \int \underline{G(\Delta w/\Delta t)}\, P(w, t \,/\, w_0, t_0)$$

Change variable: $\Delta W = W(t+\Delta t) - W(t)$

$dw \to (-) d_\Delta w$ from $dw \to -d(\Delta w)$

$$P(w+\Delta w / t_1 + \Delta t) = (-) \int G(\Delta w/\Delta t)\, P\big(W(t+\Delta t) - \Delta W(t), t\big)$$

$(-) d_\Delta W$

change the limits of integral back

Small ΔW : Taylor expansion

0-order:

$$P(w+\Delta w, t+\Delta t) = \int \underbrace{G(\Delta w/\Delta t)}_{1=\int G\,dw} \cdot P(w(t+\Delta t), t) \, d_\Delta w$$

1^{st} order:

$$-\int G(\Delta w/\Delta t) \cdot \frac{\partial P}{\partial W} \Delta W \, d_\Delta W$$

Gaussian: $\int w G \, dw = 0$

2^{nd} order:

$$+\int G(\Delta w/\Delta t) \frac{1}{2} \frac{\partial^2 P(w,t)}{\partial W^2} \Delta W^2 \, d_\Delta w$$

! variance of G

(+ ... stop)

All together this makes:

$$\frac{P(.. t+\Delta t) - P(.. t)}{\Delta t} = \frac{1}{2} \frac{(\text{variance})}{\Delta t} \cdot \frac{\partial^2 P}{\partial W^2}$$

$$\boxed{\frac{\partial P}{\partial t} = D \frac{\partial^2 P}{\partial W^2}}$$

Equivalently

$$D = \frac{(\text{Variance})}{2\Delta t}$$

$$\frac{\partial P}{\partial t} = -\operatorname{div} J$$

flux: prob. current $J = -D \frac{\partial P}{\partial W}$

Equations of this nature, for full $P(w,t)$, or $\phi_k(w)$, are called "kinetic" or macroscopic.

On the microscopic level of $w(t)$ we have "stochastic differential equation" (SDE)

1905 Einstein, Smoluchowski
→ 1911 Langevin equation

We saw this in simple Brownian motion:

$$m\frac{dv}{dt} = \sum \text{forces} = \underbrace{-\gamma v}_{} + \underbrace{\xi(t)}_{\text{stochastic force}}$$

More generally:

$$\text{SDE} \quad \frac{dx}{dt} = \underbrace{F(x)}_{\text{drift term}} + \underbrace{\sigma(x)}_{\text{diffusion term}} \boxed{\xi(t)}$$

$\boxed{\text{Normalised Wiener process}}$

$$D = \frac{1}{2}\sigma^2$$

Mathematical format of SDE:

$$\boxed{dX_t = \underbrace{\mu(x,t)\,dt}_{\text{drift}} + \underbrace{\sigma(x,t)\,dW_t}_{\text{diffusion}}}$$

Gaussian $P(dW)$

Wiener

Example of SDE:

Geometric Brownian Motion

Multiplicative

Wiener

SDE: $\boxed{dS_t = (\mu \cdot S_t)\,dt + (\sigma \cdot S_t)\,dW_t}$

constant.

$$\underbrace{\frac{dS}{S}}_{\substack{\text{looks like} \\ d(\ln S)}} = \underbrace{\mu\,dt + \sigma\,dW}_{\substack{\text{looks like simple} \\ \text{Brownian motion...}}}$$

But we discover that a lot is hidden in how one deals with calculus, e.g. $\int dW_t$?

$\Downarrow$

"Itoh Calculus"

$\boxed{\text{Itoh Lemma}}$

if we have a stochastic process $X(t)$, with SDE

$$\boxed{dX_t = \mu\, dt + \sigma\, dW_t}$$

and there is a function $f(x)$, then what is its SDE?

$$df = f(x+dx) - f(x)$$

chain rule

$$= \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} dx^2 + \dots$$

none there

We only want terms linear in (dt), not higher-order.

$$= \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x}(\mu\, dt + \sigma\, dW)$$

0 small

$$+ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\left(\mu^2 \cancel{dt^2} + 2\mu\sigma\, dt\, dW + \sigma^2 dW^2\right)$$

$$\approx \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x}(\mu\, dt + \sigma\, dW)$$

$$+ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2 dW^2 \dots \sim \langle dW^2 \rangle = dt$$

assume fast steps

So we have:

$$df = \underbrace{\left(\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \boxed{\frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2}} \right)}_{\text{drift term}} dt$$

$$+ \sigma \underbrace{\frac{\partial f}{\partial x}}_{\text{diffusion term}} dW$$

Itoh! Lemma

Now: GBM

$$dS = \mu S\, dt + \sigma S\, dW$$

and $f(S) = \ln(S)$

$$d[\ln(S)] = \frac{\partial f}{\partial S} dS + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} dS^2$$

$$= \frac{dS}{S} - \frac{1}{2S^2} dS^2$$

$$= \frac{dS}{S} - \frac{1}{2S^2} \left(\cancel{\mu^2 S^2 dt^2} + \cancel{2\mu\sigma S^2 dt\, dW} + \sigma^2 S^2 \boxed{dW^2} \right) \quad dW^2 \to dt$$

$$\boxed{d[\ln(S)] = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma\, dW}$$

gives

$$S = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W}$$

Look here!

Still stochastic

Stochasticity (σ) can reduce, or even revert exponential growth $e^{\mu t}$...