Lecture 16: Ray theory

David Al-Attar, Michaelmas term 2023

Outline and Motivation

In the previous lecture we examined the propagation of plane elastic waves along with their interaction with boundaries. Here we consider what happens when the material parameters are not constant, developing a ray theory that provides a good approximation in situations when the length-scale of the heterogeneity is large relative to that of the waves. Essentially, the waves always look locally planar, but as they propagate the wavefronts gradually deform, and the amplitudes can be changed by focusing and defocusing effects. This ray theory is related to elastodynamics in the same manner that geometric optics is related to Maxwell's equations, with both being "high frequency" approximations in a suitable sense. There are also close links with what is known as **semi-classical analysis**, this being the subject in which the behaviour of quantum mechanical systems are studied in the "classical limit" $\hbar \to 0$.

Relative length scales of heterogeneity

We consider the linearised motion of an elastic body occupying a whole space whose material parameters are spatially constant everywhere except a bounded region in which both ρ and A_{ijkl} vary smoothly. For definiteness, let p_i^0 be the slowness vector of elastic waves within the homogeneous part of the body, and assume that the heterogeneous region is contained within the half-plane $p_i^0 x_i > 0$. Within the homogeneous half plane $p_i^0 x_i < 0$ we know that plane wave solutions are possible of the form

$$u_i(\mathbf{x}, t) = f(t - p_i^0 x_i)\, a_i^0, \tag{1}$$

and our intention is to model what happens to such a wave as it enters the heterogeneous region. In general there exists no exact solution to this problem, and so it must be studied either through numerical calculations or using approximate methods. While numerical calculations are of huge practical value, they provide relatively little physical insight, and so we shall focus on physically motivated approximate methods.

In broad terms there are three possible approaches for studying this problem, with the appropriate choice being determined by the relative length scales of the incident wave and the heterogeneous structure. Let λ denote the dominant wavelength for the plane wave, and denote by L the smallest length scale over which the material parameters vary spatially to an appreciable extent. If the ratio of these two length scales satisfies

$$\frac{L}{\lambda} \approx 1, \tag{2}$$

then we expect the incident wave to be **scattered** by the heterogeneity. A numerical example of such scattering is shown in fig.1. There is not sufficient time in this course to discuss scattering of seismic waves further, though it is an important physical process

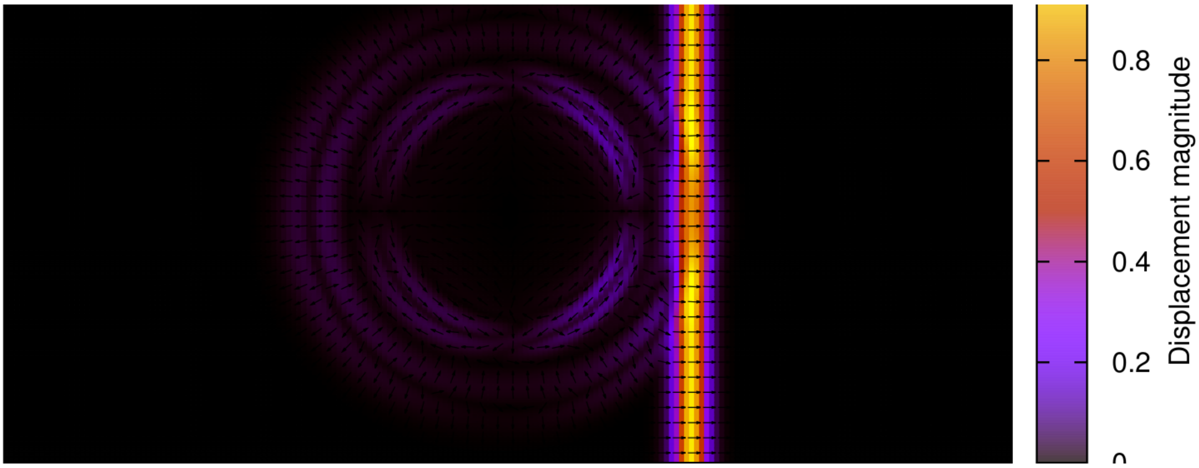

Fig. 1: Scattering of a plane p-wave in a homogeneous background model due to its interaction with a heterogeneity having a typical length-scale approximately equal to the length-scale of the incident wave. The arrows show the particle motion, and we see that the scattered wave field is comprised of both p-waves and s-waves.

within the Earth, and can be analysed using the Born approximation that you will know from quantum mechanics. In this lecture we shall focus on situations when

$$\frac{L}{\lambda} \gg 1, \tag{3}$$

which is to say the length scale of the heterogeneity is much greater than the length scale of the incident wave. Here we do not expect to see significant scattering of the incident wave, but instead a smooth evolution of the wavefront as it propagates through the heterogeneity, along with associated amplitude variations due to focusing effects (see fig.2). Finally, we might ask what happens in the case that

$$\frac{L}{\lambda} \ll 1. \tag{4}$$

This is actually very applicable to the Earth, as rocks are made up of small grains of different minerals that can have quite distinct physical properties, and yet observed seismic waves have wavelengths on the order of kilometres to thousands of kilometres. We also lack the time to enter into this topic in any detail, but it can be shown that such waves "see" a suitably averaged version of the body that appears smooth on the length scale of the incident waves.

The ray series ansatz

As noted above, we shall focus on situations where the length scale of the heterogeneity is long relative to the length scale of the incident plane wave. As a first guess, we might expect that the continuation of the plane wave into the heterogeneous region can be approximated in the following manner

$$u_i(\mathbf{x}, t) = f[t - T(\mathbf{x})]\, a_i(\mathbf{x}) \tag{5}$$

with this expression taking account of the following physical ideas:

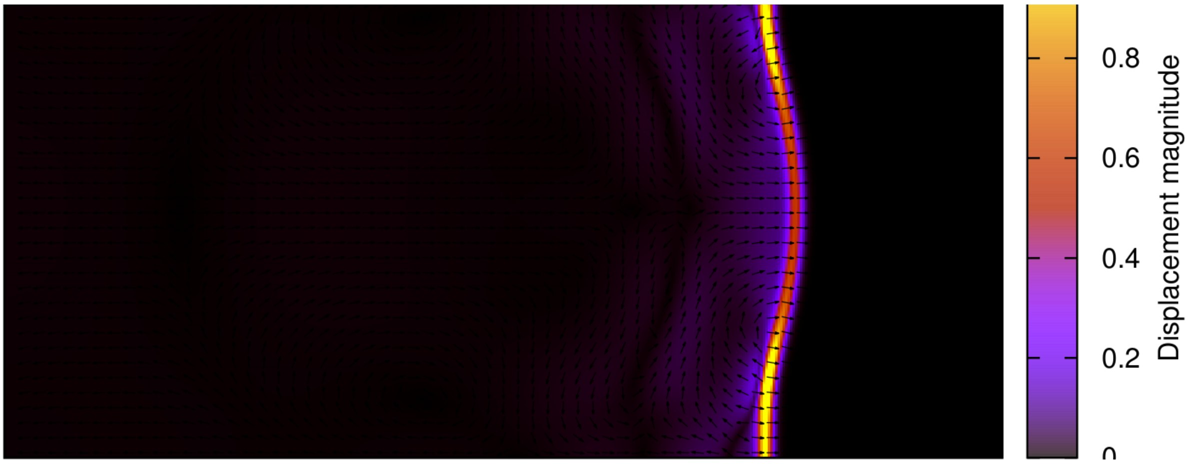

Fig. 2: Passage of a plane p-wave through a heterogeneous region having length-scale much larger than that of the incident wave. Here we see minimal scattering, along with the smooth distortion of the wave front and associated amplitude and polarisation variations. This is a situation to which ray theory can be usefully applied.

1. The linear phase term $t - p_i^0 x_i$ has been replaced with the *non-linear* term $t - T(\mathbf{x})$. This allows the wavefronts in the heterogeneous region to curve as it propagates through regions with different wave speeds;
2. The amplitude and polarisation term a_i^0 has been replaced with a function of position $a_i(\mathbf{x})$ which allows for (i) **focusing** and **defocusing** effects during propagation and (ii) changes in the particle motion associated with both wavefront curvature and variations in anisotropic properties.

We will call the function T the **travel time**, and note that the wavefronts of eq.(5) are, by definition level surfaces

$$T(\mathbf{x}) = \text{constant}. \tag{6}$$

It will be useful to define the **local slowness vector** in terms of this travel time by setting

$$p_i = \frac{\partial T}{\partial x_i} \tag{7}$$

which is everywhere orthogonal to the wavefronts.

Within the previous lecture we noted that it is useful to decompose displacement vector fields into an integral over harmonic plane waves, and here we similarly write eq.(5) as

$$u_i(\mathbf{x}, t) = \frac{1}{2\pi} \int_{\mathbb{R}} \tilde{f}(\omega) \mathrm{e}^{\mathrm{i}\omega[t - T(\mathbf{x})]} \, a_i(\mathbf{x}) \, \mathrm{d}\omega, \tag{8}$$

It is clear that eq.(5) is the temporal convolution of the singular plane wave

$$u_i(\mathbf{x}, t) = \delta[t - T(\mathbf{x})] \, a_i(\mathbf{x}) \tag{9}$$

with the waveform f. In what follows, it will be convenient to focus on such singular waves, and from now on we set $\tilde{f}(\omega) = 1$ or equivalently $f(t) = \delta(t)$. We could try to use the ansatz in eq.(9) as a starting point for the development of ray theory, but experience has shown that it is necessary to consider instead the following generalisation

$$u_i(\mathbf{x}, t) = \frac{1}{2\pi} \int_{\mathbb{R}} \sum_{n=0}^{\infty} \frac{a_i^n(\mathbf{x})}{(\mathrm{i}\omega)^n} \mathrm{e}^{\mathrm{i}\omega[t - T(\mathbf{x})]} \, \mathrm{d}\omega, \tag{10}$$

where we note that the n in a_i^n is a superscript and not an exponent. This expression is known as the **elastodynamic ray series**. The $n = 0$ term within this expression is equivalent to the eq.(9). The need for the additional terms within the expansion is less clear, and indeed, these terms are typically ignored within practical applications. We note, however, that each successive term within the series is divided by a factor of $1/(\mathrm{i}\omega)$, and within the Fourier domain this is equivalent to a temporal integration. Thus the terms within this series are getting progressively smoother and smoother.

Substitution into the equations of motion

We recall that the linearised equations of motion for an elastic body can be written

$$\rho \frac{\partial^2 u_i}{\partial t^2} - \frac{\partial}{\partial x_j}\left(A_{ijkl} \frac{\partial u_k}{\partial x_l} \right) = 0, \tag{11}$$

where we are not here allowing a body force associated with a seismic source. Within the temporal Fourier domain this equation becomes

$$-\omega^2 \rho u_i - \frac{\partial}{\partial x_j}\left(A_{ijkl} \frac{\partial u_k}{\partial x_l} \right) = 0, \tag{12}$$

where for simplicity we use the same symbol for the displacement vector and its Fourier transform. From eq.(10) we see that the ray series in the Fourier domain is simply

$$u_i(\mathbf{x}, \omega) = \sum_{n=0}^{\infty} \frac{a_i^n(\mathbf{x})}{(\mathrm{i}\omega)^n} \mathrm{e}^{-\mathrm{i}\omega T(\mathbf{x})}. \tag{13}$$

From eq.(7) and eq.(13) we find that

$$\frac{\partial u_k}{\partial x_l} = \sum_{n=0}^{\infty} \frac{1}{(\mathrm{i}\omega)^n} \left(\frac{\partial a_k^n}{\partial x_l} - \mathrm{i}\omega p_l a_k^n \right) \mathrm{e}^{-\mathrm{i}\omega T}, \tag{14}$$

and so the linearised stress tensor can then be written

$$A_{ijkl} \frac{\partial u_k}{\partial x_l} = \sum_{n=0}^{\infty} \frac{1}{(\mathrm{i}\omega)^n} A_{ijkl} \left(\frac{\partial a_k^n}{\partial x_l} - \mathrm{i}\omega p_l a_k^n \right) \mathrm{e}^{-\mathrm{i}\omega T}. \tag{15}$$

Taking the divergence of this expression, we similarly obtain

$$\begin{aligned} \frac{\partial}{\partial x_j}\left(A_{ijkl} \frac{\partial u_k}{\partial x_l} \right) = \sum_{n=0}^{\infty} \frac{1}{(\mathrm{i}\omega)^n} \Bigg[& \frac{\partial}{\partial x_j}\left(A_{ijkl} \frac{\partial a_k^n}{\partial x_l} \right) - \mathrm{i}\omega \frac{\partial}{\partial x_j}\left(A_{ijkl} p_l a_k^n \right) \\ & - \mathrm{i}\omega A_{ijkl} p_j \frac{\partial a_k^n}{\partial x_l} + (\mathrm{i}\omega)^2 A_{ijkl} p_j p_l a_k^n \Bigg] \mathrm{e}^{-\mathrm{i}\omega T}. \end{aligned} \tag{16}$$

and substituting into the frequency domain equations of motion in eq.(12) we arrive at

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{(\mathrm{i}\omega)^n} \Bigg[& \frac{\partial}{\partial x_j}\left(A_{ijkl} \frac{\partial a_k^n}{\partial x_l} \right) - \mathrm{i}\omega \frac{\partial}{\partial x_j}\left(A_{ijkl} p_l a_k^n \right) \\ & - \mathrm{i}\omega A_{ijkl} p_j \frac{\partial a_k^n}{\partial x_l} + (\mathrm{i}\omega)^2 A_{ijkl} p_j p_l a_k^n - (\mathrm{i}\omega)^2 \rho a_i^n \Bigg] \mathrm{e}^{-\mathrm{i}\omega T} = 0. \end{aligned} \tag{17}$$

This equation must hold for all frequencies, and this can only be true the coefficient of each power of iω is *separately* equal to zero. From the coefficients of the highest-order term $(\mathrm{i}\omega)^2$ we obtain

$$[A_{ijkl}p_jp_l - \rho\delta_{ik}]\, a_k^0 = 0, \tag{18}$$

while those for the iω term take the form

$$[A_{ijkl}p_jp_l - \rho\delta_{ik}]\, a_k^1 = \frac{\partial}{\partial x_j}\left(A_{ijkl}p_l a_k^0\right) + A_{ijkl}p_j\frac{\partial a_k^0}{\partial x_l}, \tag{19}$$

and then for a general term with $n \geq 2$

$$[A_{ijkl}p_jp_l - \rho\delta_{ik}]\, a_k^n = \frac{\partial}{\partial x_j}\left(A_{ijkl}p_l a_k^{n-1}\right) + A_{ijkl}p_j\frac{\partial a_k^{n-1}}{\partial x_l} - \frac{\partial}{\partial x_j}\left(A_{ijkl}\frac{\partial a_k^{n-2}}{\partial x_l}\right). \tag{20}$$

The first of these equations depends on only the local slowness vector and the zeroth-order polarisation vector. We will see shortly that this equation can be solved for the travel time T and also gives the orientation of $\mathbf{a}^0(\mathbf{x})$ at each point. Having done this, solution of eq.(19) gives the amplitude of the zeroth-order term, while eq.(20) can be used to successively determine the higher-order polarisation vectors. We will not, however, discuss the solution of these higher-order equations within this course.

At this stage it is reasonable to ask where exactly we have made an approximation? Assuming that the above set of equations can be solved for T and all the $\mathbf{a}^n$ then it would seem that we have obtained an exact solution to the equations of motion. For this to be a valid conclusions, however, we must then think carefully about the sense in which the ray series converges. Though we will not enter into this topic in detail, the key point is the ray series simply is *not convergent* in any usual sense. Indeed, the problem is at low frequencies where it is seen that this series is very badly behaved at $\omega = 0$. To force this series to converge, we could zero-out a certain region around $\omega = 0$, but by doing this we are left with only an *approximate* solution to the equations of motion. However, because the error lies only within a bounded range of frequencies it must be a smooth function. Building on such ideas, it can be shown for small enough times that the ray theoretic solution *exactly* predicts the initial singular part of the wavefield, but that there is a *smooth* error term whose size need not be small.

The eikonal equation

Within this section, we show that eq.(18) can be solved for the travel time T using the so-called **method of characteristics**. This is a useful method that can be applied to both linear and non-linear first-order partial differential equations, and reduces their solution to the integration of systems of ordinary differential equations. In fact, it is through solution of these ordinary differential equations that **rays** will enter into the theory in a natural manner. Before applying this technique, we must first reduce eq.(18) into a single partial differential equation for the travel time. To do this, it will be useful to define the **local Christoffel operator** through

$$\Gamma_{ik}(\mathbf{x}, \mathbf{p}) = \frac{1}{\rho(\mathbf{x})}\Lambda_{ijkl}(\mathbf{x})p_jp_l, \tag{21}$$

for all $\mathbf{x} \in \mathbb{R}^3$ and vectors $\mathbf{a}$ and $\mathbf{p}$. We can then write eq.(18) in the form

$$\{\boldsymbol{\Gamma}[\mathbf{x}, \mathbf{p}(\mathbf{x})] - \mathbf{1}\}\mathbf{a}^0(\mathbf{x}) = \mathbf{0}, \tag{22}$$

which closely resembles the Christoffel equation we encountered in the study of elastic plane waves. Indeed, the whole point of ray theory is that *locally* the solutions behave like plane waves. At each point, $\mathbf{x}$, we know that eq.(22) may be satisfied by a non-trivial polarisation vector $\mathbf{a}^0(\mathbf{x})$ if and only if the local slowness vector $\mathbf{p}(\mathbf{x})$ is a solution of

$$\det\{\mathbf{\Gamma}[\mathbf{x}, \mathbf{p}(\mathbf{x})] - \mathbf{1}\} = 0, \tag{23}$$

which is known in the context of ray theory as an **eikonal equation**. Eq.(23) can be interpreted as requiring that at each $\mathbf{x}$, the the local slowness vector $\mathbf{p}(\mathbf{x})$ lies on the **local slowness surface**.

Having determined the appropriate local slowness vector $\mathbf{p}(\mathbf{x})$, eq.(22) then implies that $\mathbf{a}^0(\mathbf{x})$ lies within the eigenspace of $\mathbf{\Gamma}[\mathbf{x}, \mathbf{p}(\mathbf{x})]$ with eigenvalue equal to one. For simplicity, we shall only consider materials whose local slowness surfaces are everywhere non-degenerate, and so we can write

$$\mathbf{a}^0(\mathbf{x}) = A_0(\mathbf{x})\hat{\mathbf{a}}^0(\mathbf{x}), \tag{24}$$

where $A_0(\mathbf{x})$ a **scalar amplitude**, and $\hat{\mathbf{a}}^0(\mathbf{x})$ is the appropriate normalised eigenvector of $\mathbf{\Gamma}[\mathbf{x}, \mathbf{p}(\mathbf{x})]$ which is determined uniquely up to its sign. We have assumed that the local slowness surfaces within the body are everywhere non-degenerate, and can, therefore, always write the eikonal equation unambiguously in the form

$$\left\{\|\mathbf{p}(\mathbf{x})\|^2 c_1[\mathbf{x}, \hat{\mathbf{p}}(\mathbf{x})]^2 - 1\right\}\left\{\|\mathbf{p}(\mathbf{x})\|^2 c_2[\mathbf{x}, \hat{\mathbf{p}}(\mathbf{x})]^2 - 1\right\}\left\{\|\mathbf{p}(\mathbf{x})\|^2 c_3[\mathbf{x}, \hat{\mathbf{p}}(\mathbf{x})]^2 - 1\right\} = 0, \tag{25}$$

where $c_k(\mathbf{x}, \hat{\mathbf{p}})$ denotes the local phase speed at $\mathbf{x}$ corresponding to the kth sheet and with propagation direction $\hat{\mathbf{p}}$. At an initial point only one of these factors will vanish, say the kth one, and by continuity it is the same factor that vanishes elsewhere. As a result, the eikonal equation is reduced to the simpler form

$$\left\{\|\mathbf{p}(\mathbf{x})\|^2 c_k[\mathbf{x}, \hat{\mathbf{p}}(\mathbf{x})]^2 - 1\right\} = 0, \tag{26}$$

and this is the equation we will aim to solve.

The method of characteristics

To solve the reduced form of the eikonal equation we will proceed in a way that will likely seem rather unmotivated. We introduce the **ray Hamiltonian** for the kth sheet of the local slowness surface

$$H_k(\mathbf{x}, \mathbf{p}) = \frac{1}{2}\|\mathbf{p}(\mathbf{x})\|^2 c_k[\mathbf{x}, \hat{\mathbf{p}}(\mathbf{x})]^2 \tag{27}$$

which is defined on a six-dimensional **phase space** comprised of points $(\mathbf{x}, \mathbf{p})$, and allows us to write the eikonal equation eq.(26) in the alternate form

$$H_k[\mathbf{x}, \mathbf{p}(\mathbf{x})] = \frac{1}{2}, \quad \mathbf{p}(\mathbf{x}) = (\nabla T)(\mathbf{x}). \tag{28}$$

Eq.(28) states that the desired solution T of the eikonal equation corresponds to a **three-dimensional level surfaces** $H_k(\mathbf{x}, \mathbf{p}) = \frac{1}{2}$ of the ray Hamiltonian. This geometric viewpoint is central to the method of characteristics.

The initial data we have for the problem is that that on the plane

$$p_i^0 x_i = 0 \tag{29}$$

the travel time is constant, and we are free to set this value equal to zero. We then wish to continue this solution into the heterogeneous region $p_i^0 x_i \geq 0$. Each point $\mathbf{x}^0$ on this initial plane corresponds to a point $(\mathbf{x}^0, \mathbf{p}^0)$ within the phase space, and at such points eq.(28) is satisfied. In this manner, the initial data for the problem specifies a **two-dimensional level surface** S^0 of the ray Hamiltonian, and to solve the eikonal equation, we must construct a three-dimensional level surface S containing S^0.

Let $(\mathbf{x}^0, \mathbf{p}^0)$ be a point on the initial plane S^0, and consider the solution of **Hamilton's canonical equations** for the ray Hamiltonian H_k as defined by

$$\frac{\mathrm{d}x_i}{\mathrm{d}\sigma} = \frac{\partial H_k}{\partial p_i}, \quad \frac{\mathrm{d}p_i}{\mathrm{d}\sigma} = -\frac{\partial H_k}{\partial x_i}, \tag{30}$$

subject to the initial conditions

$$x_i(0) = x_i^0, \quad p_i(0) = p_i^0. \tag{31}$$

Within these equations, σ is a generating parameter whose physical interpretation will become clear as we proceed. A key property of Hamilton's equations is that the Hamiltonian is **conserved** along their solutions. Indeed, using the chain rule we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\sigma} H_k[\mathbf{x}(\sigma), \mathbf{p}(\sigma)] &= \frac{\partial H}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}\sigma} + \frac{\partial H}{\partial p_i}\frac{\mathrm{d}p_i}{\mathrm{d}\sigma} \\ &= -\frac{\mathrm{d}p_i}{\mathrm{d}\sigma}\frac{\mathrm{d}x_i}{\mathrm{d}\sigma} + \frac{\mathrm{d}x_i}{\mathrm{d}\sigma}\frac{\mathrm{d}p_i}{\mathrm{d}\sigma} = 0, \end{aligned} \tag{32}$$

where on the right hand side we have suppressed arguments for clarity. As we have $H_k(\mathbf{x}^0, \mathbf{p}^0) = \frac{1}{2}$ at this point on the initial plane, it follows that $H_k[\mathbf{x}(\sigma), \mathbf{p}(\sigma)] = \frac{1}{2}$ along the curve within phase space obtained by integration of eq.(30) subject to the given initial conditions. What we can then do is take in turn each point $(\mathbf{x}^0, \mathbf{p}^0)$ of the initial plane, and integrate Hamilton's equations to produce a curve on which the ray Hamiltonian is equal to one half. The union of all these curves defines a smooth three-dimensional surface within the phase space, and this gives us the desired solution of the eikonal equation in geometric form. In detail, this solution is usually defined only locally near to S^0, and we shall return to this point in a moment. Furthermore, it is not actually clear that such level surfaces need to be formed from the graph of ∇T for some travel time function T. To show that this is (locally) the case requires some ideas from **symplectic geometry** that we will not discuss.

Having constructed the level surface S, how do we recover the travel time T? We first make use of eq.(7) and (30) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}\sigma} T[\mathbf{x}(\sigma)] = \frac{\partial T}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}\sigma} = p_i \frac{\partial H}{\partial p_i}. \tag{33}$$

To simplify the result, we note that the ray Hamiltonian $H_k(\mathbf{x}, \mathbf{p})$ is a homogeneous function of $\mathbf{p}$ of degree two, meaning

$$H_k(\mathbf{x}, \gamma\mathbf{p}) = \gamma^2 H_k(\mathbf{x}, \mathbf{p}), \tag{34}$$

for all real γ. Euler's homogeneous function theorem[1] then implies

$$p_i \frac{\partial H}{\partial p_i} = 2H_k(\mathbf{x}, \mathbf{p}), \tag{35}$$

[1] You do not need to know this theorem. In isotropic media the necessary calculation can be performed explicitly, and this is the only case you might be asked to reproduce this derivation.

and from eq.(28) we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}T[\mathbf{x}(\sigma)] = 1. \tag{36}$$

On the initial surface S^0 the travel time is everywhere equal to zero, and so eq.(36) has the trivial solution

$$T[\mathbf{x}(\sigma)] = \sigma, \tag{37}$$

along the curve starting from each initial point $(\mathbf{x}^0, \mathbf{p}^0)$.

The curves $\sigma \mapsto \mathbf{x}(\sigma)$ formed by projection of solutions of Hamilton's equations in phase space onto the position variable are known as **rays**. In order for the travel time T

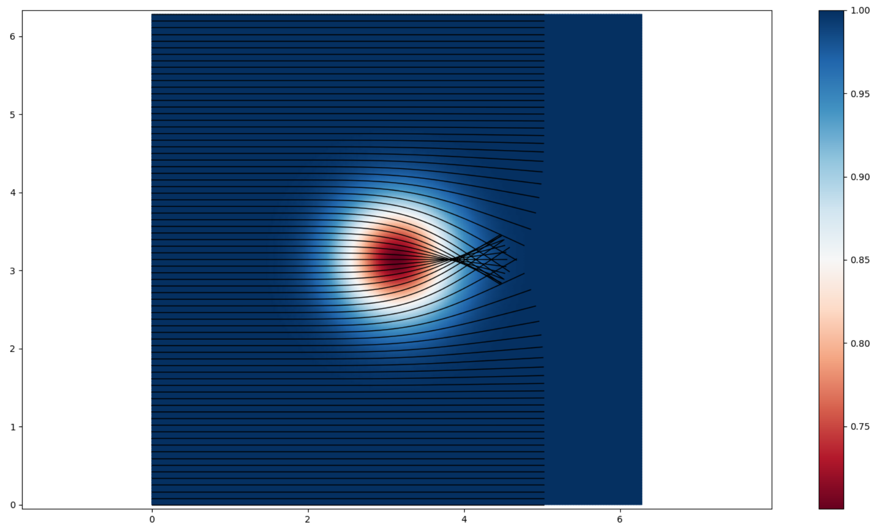

Fig. 3: An example of ray tracing within a heterogeneous body, and of the formation of a caustic due to an area of low velocity.

constructed by the above method to be a *single-valued* function of $\mathbf{x}$, then it is necessary that each point $\mathbf{x}$ within the heterogeneous region is joined to a point $\mathbf{x}^0$ on the initial plane by a *unique* ray. Indeed, if there were two such rays, then for each we could calculate a travel-time through the above method, and there is no reason why these two times should be equal. Points $\mathbf{x}$ within the heterogeneous region that can be reached by rays starting at two or more points on the initial plane are called **caustics**, and at such points ray theory breaks down (see fig.3 for an example). Though we do not have time to look quantitatively at the the behaviour of the scalar amplitude $A^0(\mathbf{x})$ introduced in eq.(24), it can be shown that this ray theoretical amplitude becomes *infinite* at caustics. In reality, caustics are characterised by large but finite amplitudes (the name comes from the Latin for "to burn"), and more sophisticated techniques based on Fourier integral operators allow them to be handled quantitatively.

Though within this lecture, we have focused on a problem of an initially plane wave entering into a heterogeneous region, it should be apparent that ray theory can be applied to problems having more general initial conditions. For example, the initial data for the travel time can be specified on a a curve or even at a point. Furthermore, it is possible to extend the method to allow for discontinuities in material parameters along internal boundaries so long as they are smooth relative to the wavelength of the waves considered. At such boundaries, things look locally just like the case of a plane elastic wave already considered, and there will be various reflected, transmitted, and converted phases produced.

What you need to know and be able to do

(i) The form of the ray-series expansion, and understand its underlying physical motivation.

(ii) How to obtain the eikonal equation and transport equations by substituting the ray expansion into the equations of motion. The full derivation given in the lecture is too involved to occur within an exam, but you may be asked to apply these methods within simpler situations.

(iii) Understand how the eikonal equation can be interpreted geometrically within a six-dimensional phase space, and be able to explain how solutions can be obtained through integration of Hamilton's equations.

(iv) Be able to solve the Hamiltonian ray equations in simple cases, and also to deduce some of their properties. Practice in this will be given in the next lecture and within the final example sheet.

(v) What a caustic is, and why ray theory breaks down here.