Binary systems and gravitational waves

Relativistic Astrophysics and Cosmology: Lecture 13

Sandro Tacchella

Friday 3rd November 2023

Pre-lecture question:

When is a gravitational wave not a wave?

Last time

- Neutron stars
- Pulsars
- Elliptical orbits

This lecture

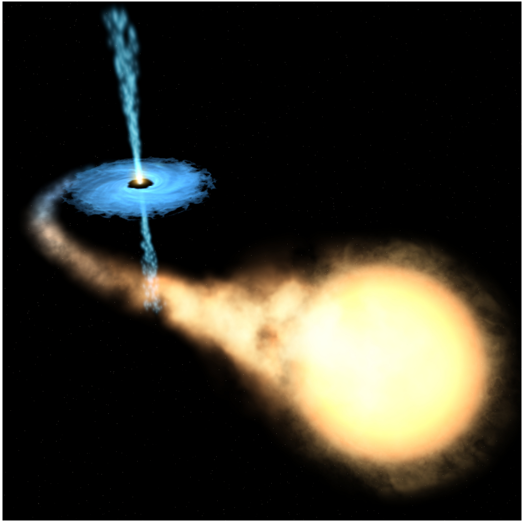
- Binary systems & the binary pulsar
- Linearised gravity
- Geometry of gravitational waves

Next lecture

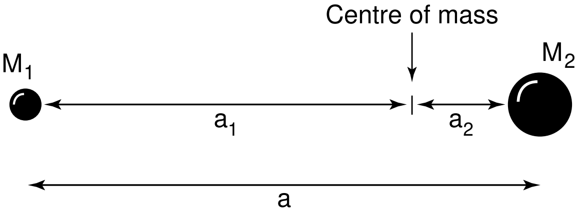
- Emission and detection of gravitational waves

Binary stars

- Most (85%) of stars exist in binary systems.
- Reason is not analytically clear (as it is a three body problem)
- In simulations, initially randomly distributed particles with inverse square-law forces either are "thrown out" or occupy binary systems.
- Very important for astronomers, as we can measure period and velocity with spectral redshift variation.
- This then gives us access to mass, distance and other critical properties.
- Transits between stars give us even more information.



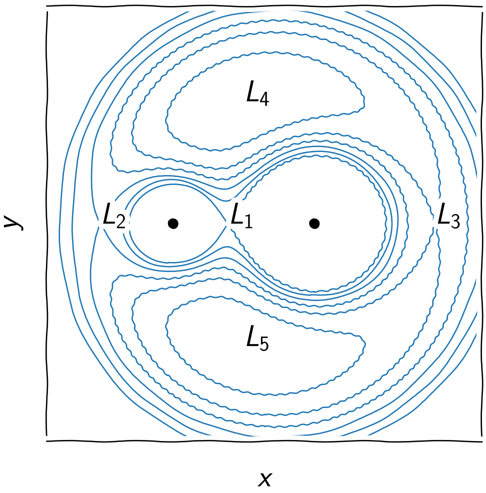
Binary Stars: two-body basics



- Consider two masses M_1, M_2 in circular orbits at distances a_1, a_2 about their centre of mass.
- Total mass $M = M_1 + M_2$.
- Separation $a = a_1 + a_2$.
- Centre of mass implies $M_1 a_1 = M_2 a_2$.
- Orbital angular speed and period $\Omega = \frac{2\pi}{P}$.

- Kepler's law $\boxed{\Omega^2 = \frac{GM}{a^3}.}$
- Moment of inertia $\boxed{I = \frac{M_1 M_2}{M} a^2.}$
- Total energy $\boxed{E = -\frac{GM_1 M_2}{2a}.}$ (bound)
- Angular momentum $\boxed{J = \frac{M_1 M_2}{M} a^2 \Omega.}$
- Reduced mass $\mu = \frac{M_1 M_2}{M}$.
- This picture is complicated by inclination of system i, so we only measure projected velocities/masses.

Lagrange points



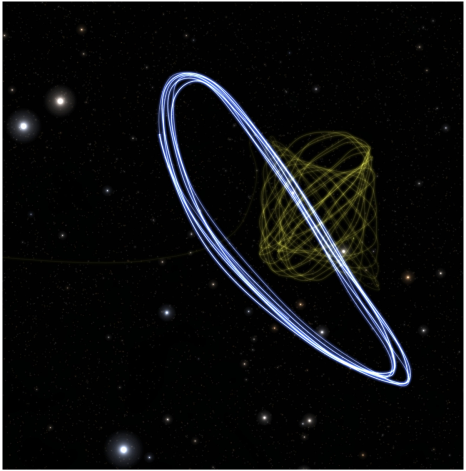
- A useful construction is to consider the potential energy ϕ in the rotating frame, written with the origin at the centre of mass.
- The potential is composed of:
  1. gravitational from star 1 at $(-a_1, 0, 0)$ $-\frac{GM_1}{\sqrt{(x+a_1)^2+y^2+z^2}}$,
  2. gravitational from star 2 at $(+a_2, 0, 0)$ $-\frac{GM_2}{\sqrt{(x-a_2)^2+y^2+z^2}}$,
  3. centrifugal $-\frac{1}{2}\Omega^2(x^2+y^2)$.

$$\boxed{\phi = -\frac{\Omega^2(x^2+y^2)}{2} - \frac{GM_1}{\sqrt{(x+a_1)^2+y^2+z^2}} - \frac{GM_2}{\sqrt{(x-a_2)^2+y^2+z^2}}.}$$

- Five Lagrange points where gravity and centrifugal force cancel. $L_{4,5}$ stable if $\frac{M_1}{M_2} > 24.96$. $L_{1,2,3}$ pseudo-stable.

Lagrange points in astronomy

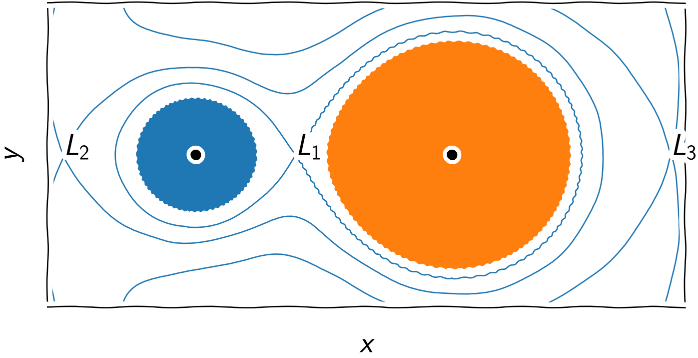
- Sun-Earth Lagrange points are useful for making observations
  - L_1 SOHO, ACE, WIND, DSCOVR (earth-sun observatories)
  - L_2 Gaia, Spektr-RG, JWST
  - L_3 None (that we know of)
  - $L_{4,5}$ Trojan asteroids (observed in passing by STEREO A,B & others)
- As are Earth-moon, Sun-*planet* and *Planet-moon* are other regions of interest for stores of matter



The pseudo-stable orbits at L_2 of Gaia (yellow) and JWST (blue)

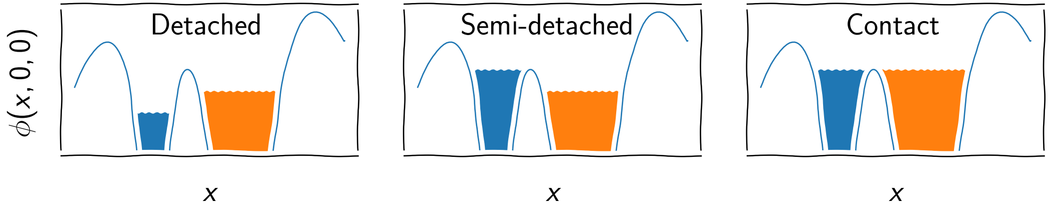
Roche Lobe overflow

- The surface of a star will follow an equipotential.
- When stars are in close binary systems they may exchange matter.



- The first equipotential common to both stars is called the Roche lobe.
- The common point between the stars is called the inner Lagrangian or L_1 point.
- If a star swells up enough over it's evolution, it spills mass onto the other through inner Lagrange point the Roche lobe overflow process.

- There are 3 classes of close (interacting) binaries:



- In Roche lobe overflow, mass and angular momentum conservation of $J = \frac{M_1 M_2}{M} a^2 \Omega$ causes the separation of the stars a to change.
- If the mass transfer is conservative ($\dot{J} = 0,\ \dot{M} = 0$):

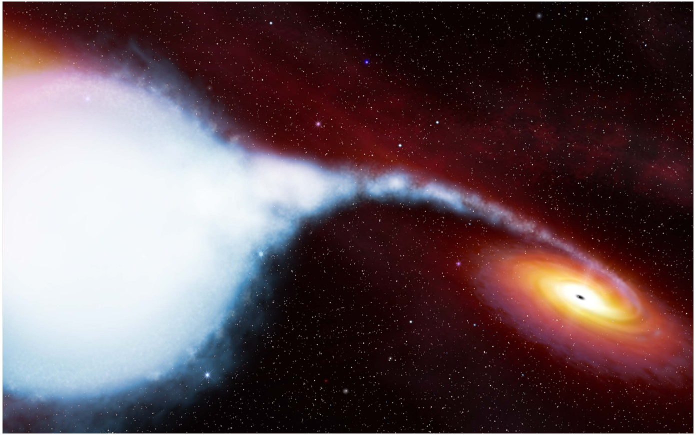
$$(\dot{JM}) = \dot{M}_1 M_2 a^2 \Omega + M_1 \dot{M}_2 a^2 \Omega + 2 M_1 M_2 a \dot{a} \Omega + M_1 M_2 a^2 \dot{\Omega} = 0, \qquad \dot{M}_1 + \dot{M}_2 = 0.$$

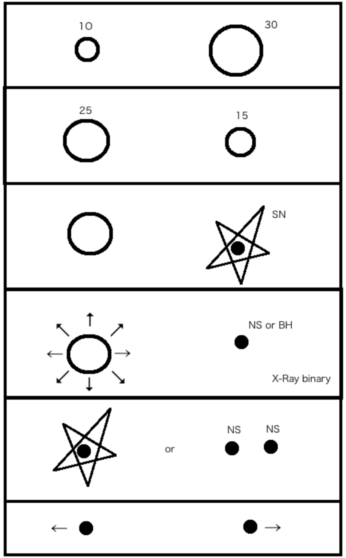
- Kepler's law $\frac{\dot{a}}{a} = -\frac{2\dot{\Omega}}{3\Omega}$ then gives the relation for magnitude and direction of spin-up:

$$\boxed{\frac{3\dot{M}_1 (M_1 - M_2)}{M_1 M_2} = -\frac{\dot{\Omega}}{\Omega} = \frac{\dot{P}}{P} = \frac{3\dot{a}}{2a}.}$$

X-ray binaries

- These are binary stars whose mass transfer results in an accretion disk luminous in X-rays.



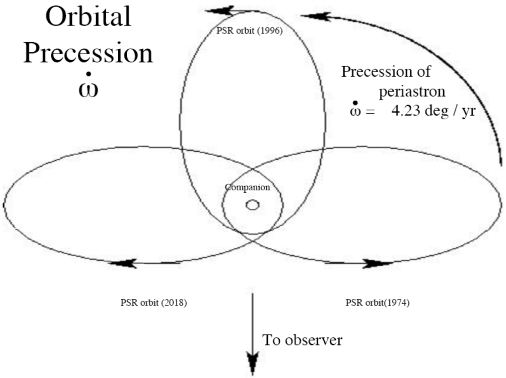


The Original Binary Pulsar

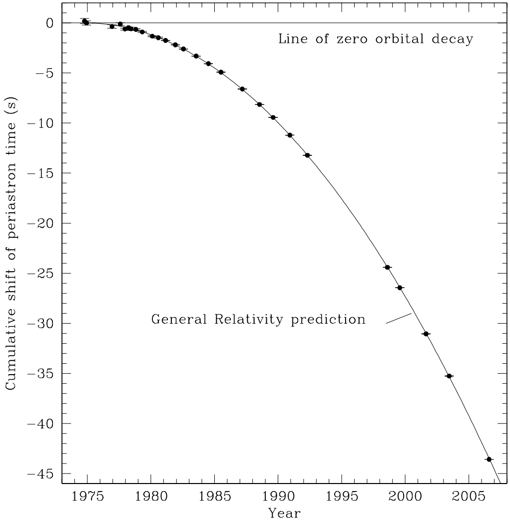
- PSR 1913+16 is the first binary pulsar, discovered in 1974 by Hulse & Taylor.
- It is a radio pulsar which shows large Doppler shifts on an 8 h period.
- Essentially a high precision clock orbiting with a velocity of $v \sim 300\,\mathrm{km s^{-1}}$ in an eccentric orbit close to a $\sim M_\odot$ companion (thought to be an old neutron star).
- Rewriting formula from last time

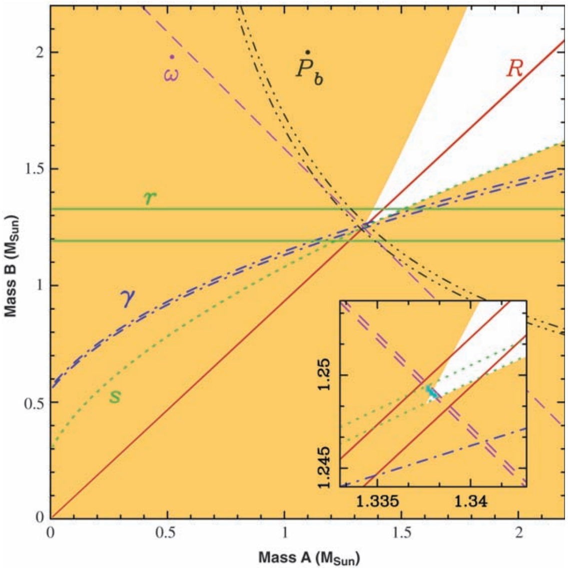
$$\dot{\omega} = \frac{6\pi G M_2}{a_1(1-e^2)Pc^2} \approx 4.23^\circ/\text{year}.$$

- About 35,000 times value for Mercury!



- Even more exciting is decay of the orbital period.
- The orbit is steadily tightening at a rate consistent with emission of gravitational waves.
- Plot on next slide shows how we can combine constraints to severely test GR and alternative theories.
- It shows mass constraints obtained from the double pulsar PSR J0737.
- Relativistic periastron advance is $\dot{\omega}$, gravitational redshift is γ, curvature of spacetime causes.
- r and s, orbital decay due to gravitational waves is dP_b/dt and the ratio of orbit sizes is R.
- 'General Relativity Prediction' shows how orbit should change if gravitational radiation exists and carries away energy (from Weisberg *et al.* 2010).

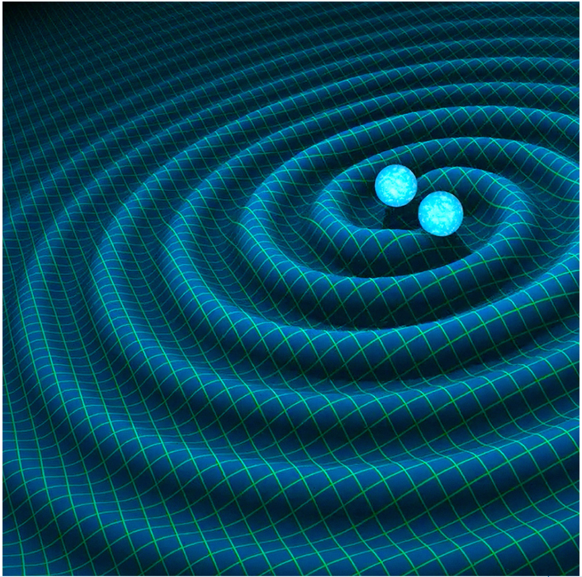




Timing parameter	PSR J0737-3039A	PSR J0737-3039B
Right ascension α	$07^h37^m51^s.24927(3)$	—
Declination δ	$-30°39'40''.7195(5)$	—
Proper motion in the RA direction (mas year^{-1})	$-3.3(4)$	—
Proper motion in declination (mas year^{-1})	2.6(5)	—
Parallax π (mas)	3(2)	—
Spin frequency ν (Hz)	44.054069392744(2)	0.36056035506(1)
Spin frequency derivative $\dot{\nu}$ (s^{-2})	$-3.4156(1) \times 10^{-15}$	$-0.116(1) \times 10^{-15}$
Timing epoch (MJD)	53,156.0	53,156.0
Dispersion measure DM (cm^{-3} pc)	48.920(5)	—
Orbital period P_b (day)	0.10225156248(5)	—
Eccentricity e	0.0877775(9)	—
Projected semimajor axis $x = (a/c)\sin i$ (s)	1.415032(1)	1.5161(16)
Longitude of periastron ω (°)	87.0331(8)	87.0331 + 180.0
Epoch of periastron T_0 (MJD)	53,155.9074280(2)	—
Advance of periastron $\dot{\omega}$ (°/year)	16.89947(68)	[16.96(5)]
Gravitational redshift parameter γ (ms)	0.3856(26)	—
Shapiro delay parameter s	0.99974(−39,+16)	—
Shapiro delay parameter r (μs)	6.21(33)	—
Orbital period derivative $\dot{P}_b$	$-1.252(17) \times 10^{-12}$	—
Timing data span (MJD)	52,760 to 53,736	52,760 to 53,736
Number of time offsets fitted	10	12
RMS timing residual σ (μs)	54	2169
Total proper motion (mas year^{-1})	4.2(4)	
Distance d(DM) (pc)	~500	
Distance $d(\pi)$ (pc)	200 to 1,000	
Transverse velocity (d = 500 pc) (km s^{-1})	10(1)	
Orbital inclination angle (°)	88.69(−76,+50)	
Mass function ($M_\odot$)	0.29096571(87)	0.3579(11)
Mass ratio R	1.0714(11)	
Total system mass ($M_\odot$)	2.58708(16)	
Neutron star mass ($m_\odot$)	1.3381(7)	1.2489(7)

Gravitational waves

- Gravitational waves are one of the major predictions of General Relativity (alongside gravitational lensing & in contrast to the mercurial perihelion shift postdiction).
- Perturbations in spacetime propagate at the speed of light.
- Such ripples are caused and are caused by quadrupolar distortions (in contrast to dipolar ones for electromagnetism).
- This half of the lecture is quite mathematically heavy, but boxed equations are highlighted as the take-home points.
- Highly recommend chapters 17 & 18 of Hobson, Efstathiou & Lasenby.



Gravity in the weak field limit

- If we assume that gravity is weak, we may expand the metric about the Minkowski form

$$\boxed{g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad \text{where } |h^{\nu}_{\mu}| \ll 1.}$$

- Upstairs downstairs formulation in "smallness" is only for dimensional correctness.
- $h_{\mu\nu}$ is symmetric.
- Note that the inverse metric $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ from which it can be seen $g_{\mu\sigma}g^{\sigma\nu} = \delta^{\nu}_{\mu}$.
- We can also raise small quantities using $\eta_{\mu\nu}$ rather than $g_{\mu\nu}$.
- After some effort (non-examinable exercise for the reader), the Einstein equations $G_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}$ become

$$\partial_\mu \partial_\nu h + \partial^\sigma \partial_\sigma h_{\mu\nu} - \partial_\mu \partial_\rho h^{\rho}_{\nu} - \partial_\nu \partial_\rho h^{\rho}_{\mu} - \eta_{\mu\nu}(\partial^\sigma \partial_\sigma h - \partial_\rho \partial_\sigma h^{\sigma\rho}) = -\frac{16\pi G}{c^4} T_{\mu\nu}.$$

- It is convenient to notate $\Box^2 = \partial_\sigma \partial^\sigma$ and to define a trace-reversed perturbation $\bar{h}_{\mu\nu}$

$$\boxed{\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h} \quad \Rightarrow \quad \Box^2 \bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial_\rho \partial_\sigma \bar{h}^{\rho\sigma} - \partial_\mu \partial_\rho \bar{h}^{\rho}_{\nu} - \partial_\nu \partial_\rho \bar{h}^{\rho}_{\mu} = -\frac{16\pi G}{c^4} T_{\mu\nu}.$$

Gauge freedoms

- This procedure is only valid up to equally small perturbations in the coordinates

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x) \quad \Rightarrow \quad h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu,$$

  which is found by demanding that $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g'_{\mu\nu} dx'^\mu dx'^\nu$ to first order in h.
- This very similar to in electromagnetism, where we write expressions in terms of a vector potential A_μ, and note that $F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and all the equations are invariant under the transformation $A_\mu \to A'_\mu = A_\mu + \partial_\mu \psi$.
- As you may see from other courses this year, Gauge theories are how much of 20th and 21st century physics is written.
- This redundancy in the dynamical variables in which we write down our theories are a blessing and a curse:

  Positive We are free to choose the gauge to simplify the problem (c.f. coordinates).

  Negative Disentangling physics from gauge considerations adds complexity.

The Lorenz gauge

- A little algebra shows derivatives of $\bar{h}_{\mu\nu}$ transform as $\partial_\rho \bar{h}^{\mu\rho} \to \partial_\rho \bar{h}'^{\mu\rho} = \partial_\rho \bar{h}^{\mu\rho} - \Box^2 \xi^\mu$.
- Since we are free to choose $\xi^\mu(x)$ to be any (small) function we like, we can choose them to make the left hand side zero: $\Box^2 \xi^\mu = \partial_\rho \bar{h}^{\mu\rho}$.
- This means that we can choose a gauge such that $\partial_\rho \bar{h}'^{\mu\rho} = 0$ (independent of primes $'$).
- Our Einstein equations therefore greatly simplify

$$\boxed{\Box^2 \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}, \quad \partial_\mu \bar{h}^{\mu\nu} = 0.}$$

- These are linear wave equations, sourced by the stress energy tensor, which remain true under the gauge transformation $x^\mu \to x^\mu + \xi^\mu$ providing $\Box^2 \xi^\mu = 0$.
- Compare with electromagnetism: $\Box^2 A^\mu = \mu_0 j^\mu$ if $\partial_\mu A^\mu = 0$ coming from the gauge freedom $A^\mu \to A^\mu + \partial^\mu \psi$ with requirement $\Box^2 \psi = 0$.
- We therefore name this in both cases the Lorenz gauge.
- Contrast with electromagnetism in that in gravity we are gauging spacetime x itself, rather than fields on spacetime $A(x)$.

Vacuum solutions

- In the vacuum $T^{\mu\nu} = 0$, our equations become $\Box^2 h^{\mu\nu} = 0$ and it should hopefully not surprise you too much that these have (complex) plane wave solutions:

$$\boxed{\bar{h}^{\mu\nu} = A^{\mu\nu} \exp\left(ik_\rho x^\rho\right), \quad k_\rho k^\rho = 0, \quad A^{\mu\nu} k_\nu = 0.}$$

- Note the subtlety that in this weak field limit we are working in Minkowski space where we treat coordinates x^ρ like vectors again, and gravity $\bar{h}^{\mu\nu}(x)$ as a flat-space field.
- Taking the first of the above expressions as an *anzatz* and substituting into the field equation yields the second expression:

$$\Box^2 e^{ik\cdot x} = \partial_\rho \partial^\rho e^{ik\cdot x} = \partial_\rho i k^\rho e^{ik\cdot x} = -k_\rho k^\rho e^{ik\cdot x} \Rightarrow k_\rho k^\rho = 0.$$

- This indicates that the wavevector is null (like photons) with dispersion relation $\omega^2 = c^2 |\vec{k}|^2$, and therefore group and phase velocity c.
- Third expression derived similarly from Lorenz gauge condition $\partial_\rho \bar{h}^{\rho\mu} = 0 \Rightarrow A^{\mu\nu} k_\nu = 0$.
- Note wave equation and gauge condition are linear, so general solution is superposition of plane waves.

The transverse traceless gauge

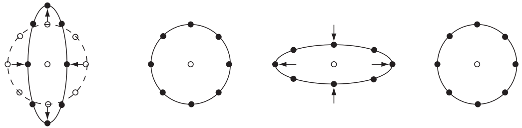
- $A^{\mu\nu}$ initially has 16 components. Symmetry $A^{\mu\nu} = A^{\nu\mu}$ reduces this to 10.
- The Lorenz condition $A^{\mu\nu} k_\nu = 0$ with four equations reduces this to 6.
- We still have residual gauge freedom: solutions valid up to $x^\mu \to x^\mu + \xi^\mu$ where $\Box^2 \xi^\mu = 0$.
- We can solve for our remaining gauge perturbations as $\xi^\mu = \epsilon^\mu e^{ik\cdot x}$ with $k^2 = 0$,[1] so our amplitudes transform as $A^{\mu\nu} \to A'^{\mu\nu} = A^{\mu\nu} - i\epsilon^\mu k^\nu - i\epsilon^\nu k^\mu + i\eta^{\mu\nu}\epsilon^\rho k_\rho$.
- Under this freedom, we can choose ϵ^μ to reduce $A^{\mu\nu}$ to only 2 components, usually chosen to be transverse and traceless, and if travelling in the z-direction takes the form.

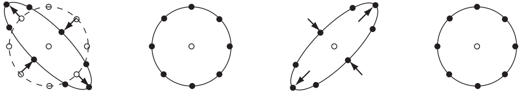
$$\boxed{\bar{h}^{\mu\nu} = h^{\mu\nu} = A^{\mu\nu} \exp\left(ik_\rho x^\rho\right), \qquad k^\mu = \begin{pmatrix} k \\ 0 \\ 0 \\ k \end{pmatrix}, \qquad A^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a^+ & a^\times & 0 \\ 0 & a^\times & -a^+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.}$$

[1] Note this should give us pause for thought, as it means our coordinates adjustments are also oscillatory – historically it took people a long time to convince themselves that anything physical was actually waving.

Gravitational waves

- We therefore have our $z-$travelling gravitational wave, parameterised by k, a^+, $a^\times$.
- These clearly satisfy all the constraints, and note h (the trace) is indeed zero in this case, so in fact $A^{\mu\nu}\exp(ik_\rho x^\rho)$ is the solution for $h^{\mu\nu}$ as well as $\bar{h}^{\mu\nu}$.
- The (generally complex) constants a^+ and $a^\times$ control the two polarisations of the wave:
  - a^+ controls terms which expand proper distances in the x direction at the same time as they squeeze them in the y direction, and vice versa:
  - $a^\times$ is just this pattern rotated by 45° (e.g. if transform via $x = \frac{1}{2}(u+v)$, $y = \frac{1}{2}(u-v)$, have $dx\,dy = \frac{1}{4}\left(du^2 - dv^2\right)$), so get:





- In both diagrams $k(t-z) = 2n\pi$, $(2n+\frac{1}{2})\pi$, $(2n+1)\pi$, $(2n+\frac{3}{2})\pi$.
- hence the two names "plus" and "cross", which give pictures of how a ring of particles would be distorted in terms of the proper (not coordinate) distance between them.

Summary

- Binary systems parameterised by $M = M_1 + M_2$, $a = a_1 + a_2$, $\Omega = \frac{2\pi}{P}$.

$$\boxed{M_1 a_1 = M_2 a_2, \quad \Omega^2 = \frac{GM}{a^3}, \quad I = \frac{M_1 M_2}{M} a^2, \quad E = -\frac{G M_1 M_2}{2a}, \quad J = I\Omega.}$$

- Linearised gravity

$$\boxed{\Box^2 \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}, \quad \partial_\mu \bar{h}^{\mu\nu} = 0, \quad \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h, \quad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad \text{where } |h^\nu_\mu| \ll 1.}$$

- Gravitational plane-wave solution in transverse traceless gauge

$$\boxed{\begin{aligned} &\bar{h}^{\mu\nu} = h^{\mu\nu} = A^{\mu\nu} \exp\left(i k_\rho x^\rho\right), \\ &k_\rho k^\rho = 0, \quad A^{\mu\nu} k_\nu = 0, \end{aligned} \qquad k^\mu = \begin{pmatrix} k \\ 0 \\ 0 \\ k \end{pmatrix}, \qquad A^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a^+ & a^\times & 0 \\ 0 & a^\times & -a^+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.}$$

Next time