The geometry of the Universe

Relativistic Astrophysics and Cosmology: Lecture 16

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Friday 10^{th} November 2023

Pre-lecture question:

Can you measure how fast you are moving?

Last time

- Gravitational lensing

This lecture

- What is cosmology
- The geometry of the universe
- The Friedmann-Robertson-Walker metric

Next lecture

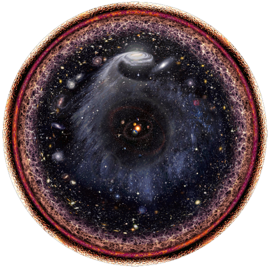
- The dynamics of the universe

Cosmology

- The remainder of the course covers cosmology
  - Week 6: The relativistic Universe
    - Lecture 16: The geometry of the Universe
    - Lecture 17: The dynamics of the Universe
    - Lecture 18: The evolution of the Universe
  - Week 7: Observational cosmology
    - Lecture 19: Measuring the Universe
    - Lecture 20: The constituents of the Universe
    - Lecture 21: Cosmological data and the Cosmic Microwave Background
  - Week 8: Beyond homogeneity and the quantum Universe
    - Lecture 22: The primordial universe
    - Lecture 23: The perturbed universe
    - Lecture 24: The quantum universe & wrap-up
- Aim here is to consolidate understanding of Part II material, and fill in many of the details.
- In particular, aim to give a physically- rather than mathematically-based approach, to complement what you did last year.
- Then in later part of course we can cover new material, such as observations, inflation & perturbation theory.

What is cosmology?

- Cosmology is the study of the universe as a whole, and on the largest scales.
- Modern precision cosmology involves using a combination of data, theory and inference to study.
  - Its geometry (spatial size, shape and age),
  - Its evolution (origin, dynamics & fate),
  - Its contents (matter, radiation, neutrinos, dark matter & dark energy),
  - . . . and the interplay between the above.
- Interestingly, there was never any Newtonian cosmology.
- This is completely possible – Galilean relativity allows for an infinite space filled with relatively expanding matter.
- However it took the innovation of having spacetime as a dynamical quantity for physicists to start thinking about cosmology physically.



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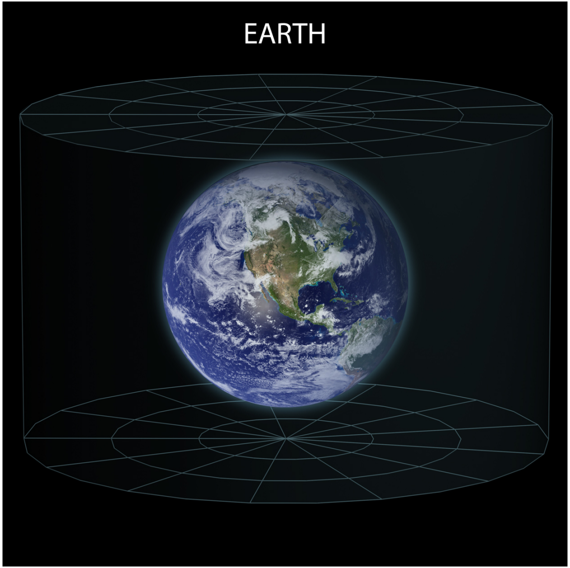

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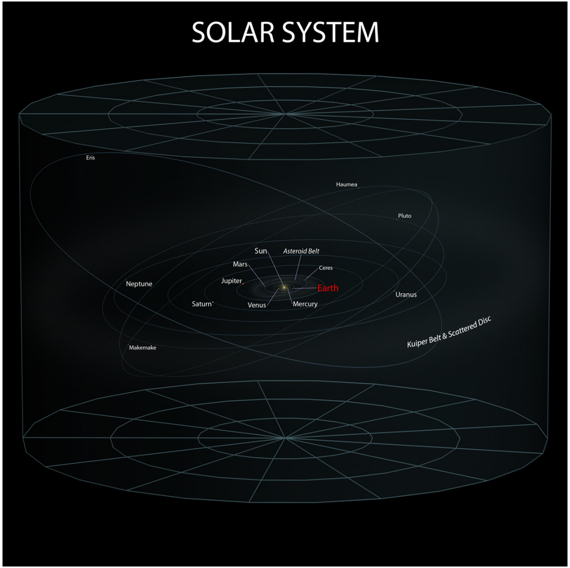
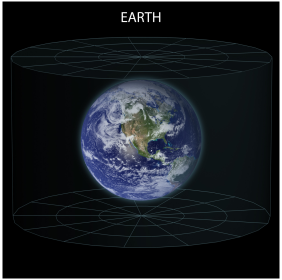

SOLAR SYSTEM
Eris
Haumea
Pluto
Sun
Asteroid Belt
Mars
Ceres
Jupiter
Earth
Neptune
Uranus
Saturn
Venus
Mercury
Kuiper Belt & Scattered Disc
Makemake


EARTH

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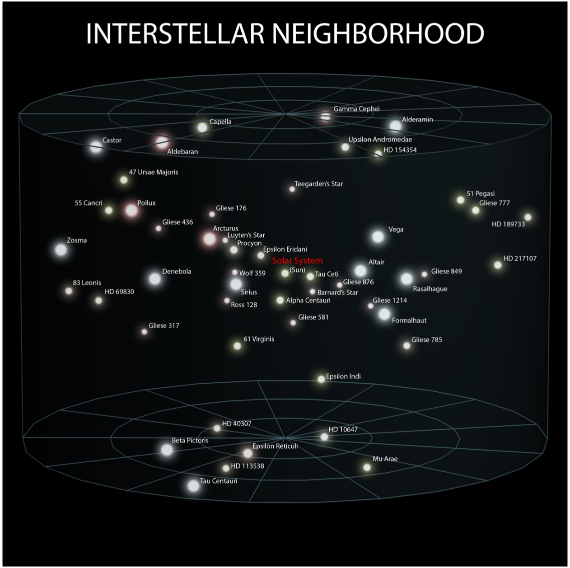
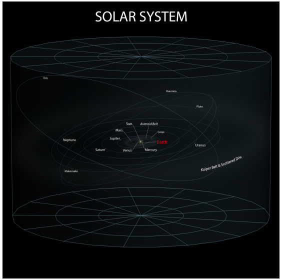


INTERSTELLAR NEIGHBORHOOD
Gamma Cephei
Capella
Alderamin
Castor
Aldebaran
Upsilon Andromedae
HD 154354
47 Ursae Majoris
Teegarden's Star
51 Pegasi
Gliese 777
55 Cancri
Pollux
Gliese 176
HD 189733
Gliese 436
Arcturus
Luyten's Star
Procyon
Vega
Zosma
Epsilon Eridani
Solar System
(Sun)
Altair
HD 217107
Denebola
Wolf 359
Tau Ceti
Gliese 849
83 Leonis
Gliese 876
Rasalhague
HD 69830
Sirius
Barnard's Star
Ross 128
Alpha Centauri
Gliese 1214
Gliese 581
Formalhaut
Gliese 317
61 Virginis
Gliese 785
Epsilon Indi
HD 40307
HD 10647
Beta Pictoris
Epsilon Reticuli
Mu Arae
HD 113538
Tau Centauri



SOLAR SYSTEM
Eris
Haumea
Pluto
Sun
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Ceres
Jupiter
Neptune
Earth
Uranus
Saturn
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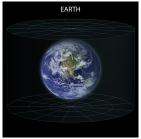
Mercury
Makemake
Kuiper Belt & Scattered Disc



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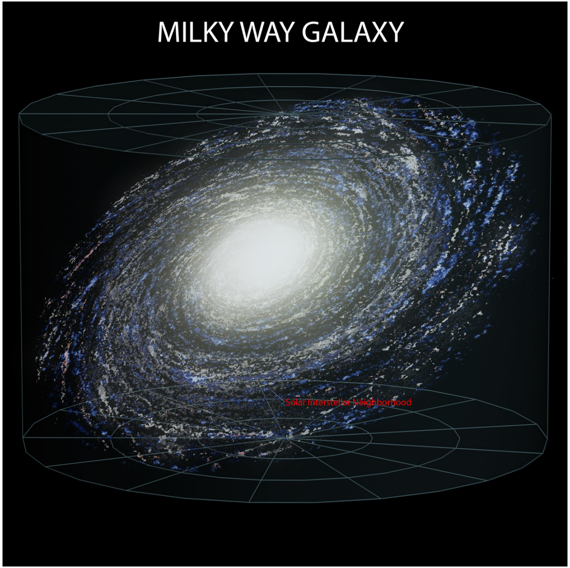
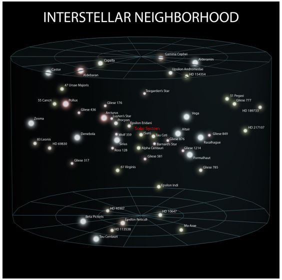
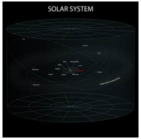
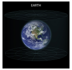

MILKY WAY GALAXY
Solar Interstellar Neighborhood


INTERSTELLAR NEIGHBORHOOD


SOLAR SYSTEM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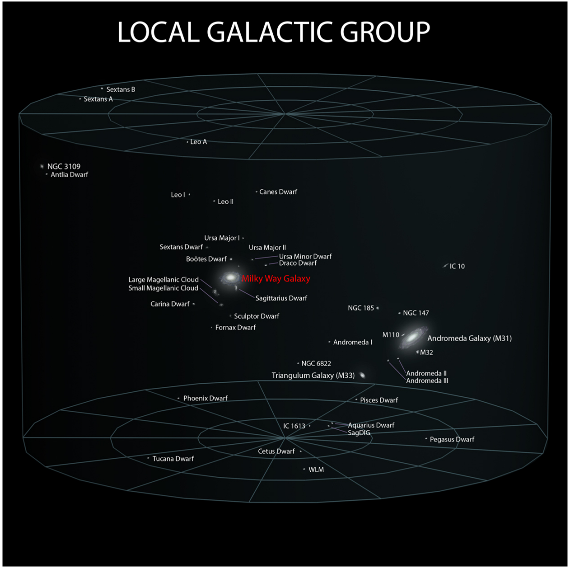
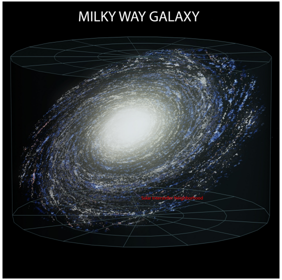
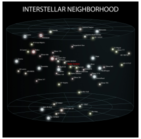

LOCAL GALACTIC GROUP
Sextans B
Sextans A
Leo A
NGC 3109
Antlia Dwarf
Leo I
Leo II
Canes Dwarf
Ursa Major I
Sextans Dwarf
Ursa Major II
Ursa Minor Dwarf
Draco Dwarf
Boötes Dwarf
IC 10
Large Magellanic Cloud
Small Magellanic Cloud
Milky Way Galaxy
Sagittarius Dwarf
Carina Dwarf
NGC 185
NGC 147
Sculptor Dwarf
Fornax Dwarf
M110
Andromeda Galaxy (M31)
Andromeda I
M32
NGC 6822
Triangulum Galaxy (M33)
Andromeda II
Andromeda III
Phoenix Dwarf
Pisces Dwarf
IC 1613
Aquarius Dwarf
SagDIG
Pegasus Dwarf
Cetus Dwarf
Tucana Dwarf
WLM


MILKY WAY GALAXY


INTERSTELLAR NEIGHBORHOOD

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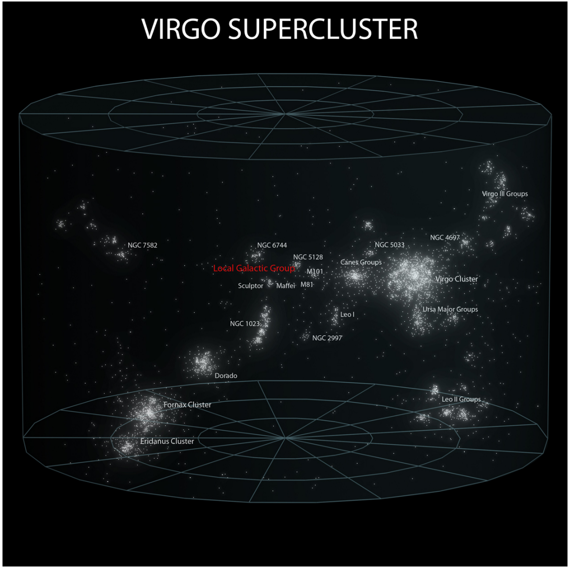
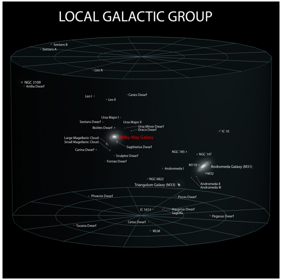
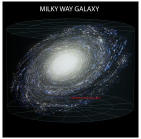

VIRGO SUPERCLUSTER
Virgo III Groups
NGC 7582
NGC 6744
NGC 5128
NGC 5033
NGC 4697
Local Galactic Group
M101
Canes Groups
Virgo Cluster
Sculptor
Maffei
M81
Ursa Major Groups
Leo I
NGC 1023
NGC 2997
Dorado
Fornax Cluster
Leo II Groups
Eridanus Cluster


LOCAL GALACTIC GROUP
Milky Way Galaxy
Andromeda Galaxy (M31)
Triangulum Galaxy (M33)


MILKY WAY GALAXY


INTERSTELLAR NEIGHBORHOOD

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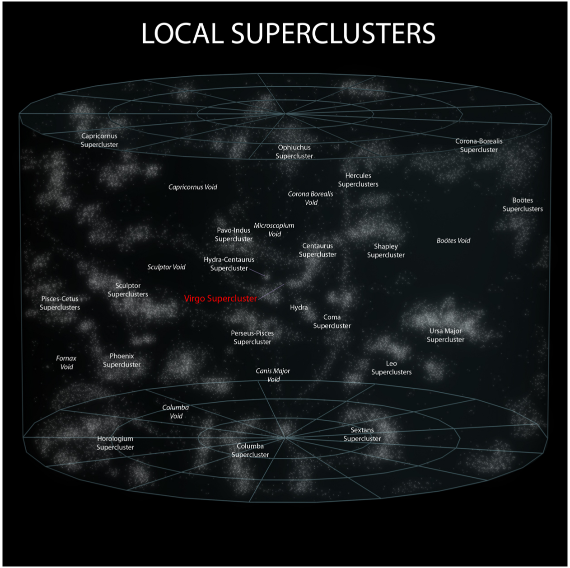
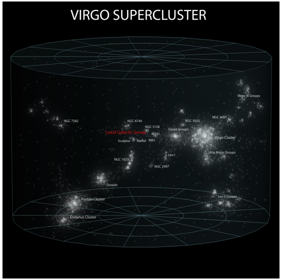
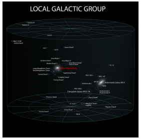
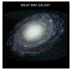

LOCAL SUPERCLUSTERS
Capricornus Supercluster
Ophiuchus Supercluster
Corona-Borealis Supercluster
Hercules Superclusters
Capricornus Void
Corona Borealis Void
Boötes Superclusters
Microscopium Void
Pavo-Indus Supercluster
Centaurus Supercluster
Shapley Supercluster
Boötes Void
Sculptor Void
Hydra-Centaurus Supercluster
Sculptor Superclusters
Pisces-Cetus Superclusters
Virgo Supercluster
Hydra
Coma Supercluster
Perseus-Pisces Supercluster
Ursa Major Supercluster
Fornax Void
Phoenix Supercluster
Canis Major Void
Leo Superclusters
Columba Void
Sextans Supercluster
Horologium Supercluster
Columba Supercluster


VIRGO SUPERCLUSTER
Local Galactic Group
Virgo Cluster
Fornax Cluster
Eridanus Cluster


LOCAL GALACTIC GROUP


MILKY WAY GALAXY

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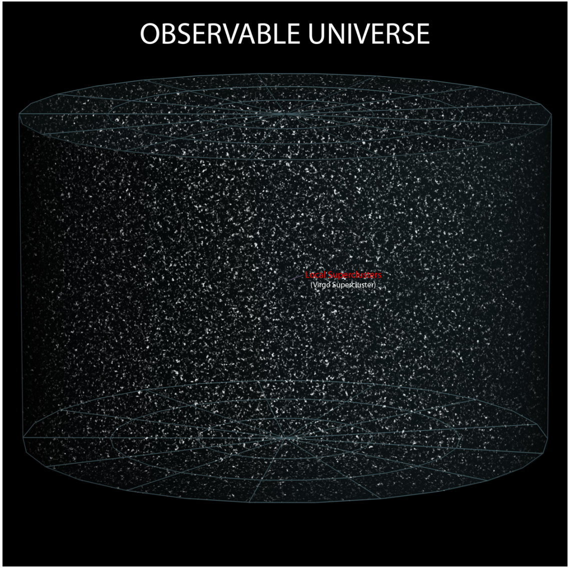
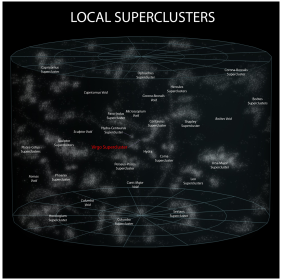
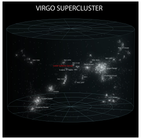
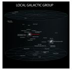
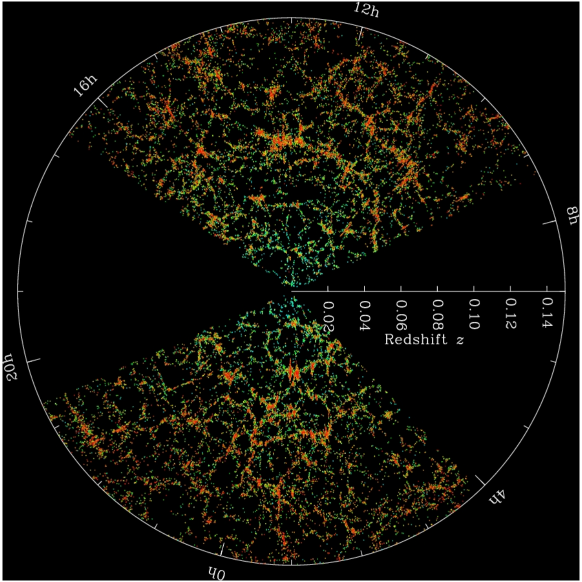

OBSERVABLE UNIVERSE
Local Superclusters
(Virgo Supercluster)


LOCAL SUPERCLUSTERS
Virgo Supercluster


VIRGO SUPERCLUSTER


LOCAL GALACTIC GROUP

- Start by looking again at the concepts underlying the Friedmann-Robertson-Walker metric, discussed last year in Part II.
- Given our Gauss Theorema Egregium approach of using the curvature of 2d subsurfaces, we can quickly find both the metric and dynamical equations.
- We will then move on to consider what we can infer from the metric itself — this includes the concept of redshift, Hubble's law, and the important concept of the measurement of distances in the universe.
- Right: Distribution of galaxies in space as observed by the SLOAN digital sky survey (SDSS).



- Our fundamental cosmological assumption is that on the largest scales, the universe satisfies the Extended Copernican Principle:
- All positions in the universe are equivalent for the purposes of cosmology.
- This implies *homogeneity*.
- Approximation to a metric describing the whole universe – spatial part completely isotropic and homogeneous.
- Work at a constant time – what is the most general 3-space metric with complete isotropy and homogeneity?
- To get isotropy we use spherical coordinates and require

$$ds^2_{r=\mathrm{const}} = r^2(d\theta^2 + \sin^2\theta d\phi^2) \equiv r^2 d\Omega.$$

- To get homogeneity, but allow for a true curvature of space, we write (Lecture 3)

$$ds^2 = f(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \equiv f(r)dr^2 + r^2 d\Omega, \tag{1}$$

- $\sqrt{f(r)}dr$ is the proper distance between neighbouring points (r, θ, ϕ) and $(r + dr, \theta, \phi)$. r is just a label which we use and is defined by how we write the metric.

- For spatial homogeneity $f(r)$ must give constant curvature $K(r)$.
- This was discussed in Lecture 3 in the context of trying to achieve a *2-surface* with constant curvature everywhere, using Gauss's formula for curvature. We may immediately use the result obtained before:

$$K = \frac{f'}{2f^2 r} \qquad \Rightarrow \qquad f(r) = \frac{1}{1 - Kr^2}. \tag{2}$$

- K is the Gaussian curvature of the 2-dimensional surface $\theta = \pi/2$.
- Isotropy implies this should be independent of the specific choice of coordinate system, and equations (1) and (2) provide a metric which is genuinely homogeneous over the whole of 3-d space.

What sort of spaces are these?

$K = 0$ (flat universe)

- This is just ordinary Euclidean space $ds^2 = dr^2 + r^2 d\Omega$.

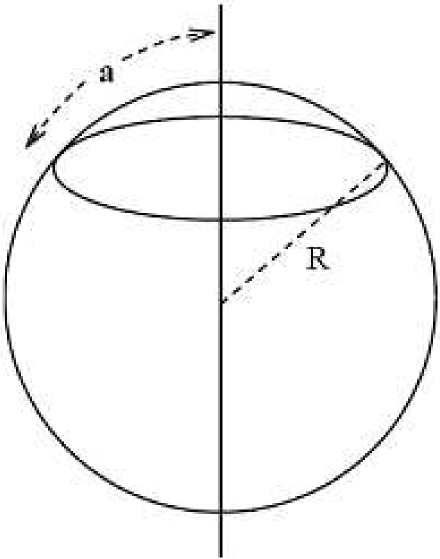
$K > 0$ positive (closed universe)

- The area of the sphere $r =$ const is just $A = 4\pi r^2$, but what *actual* distance from the origin does the coordinate distance r correspond to?
- This is given by the *proper* interval $a(r)$

$$a(r) = \int_0^{a(r)} ds = \int_0^r \frac{dr}{\sqrt{1 - Kr^2}} = \frac{1}{\sqrt{K}} \sin^{-1}(r\sqrt{K}) \quad \Rightarrow \quad r = \frac{1}{\sqrt{K}} \sin(a\sqrt{K}). \tag{3}$$

- The proper distance is what we would physically measure if we could simultaneously lay down metre sticks over the cosmic distances required.

- From (3) the proper area of a sphere with proper radius a is
$$A = (4\pi/K)\sin^2(a\sqrt{K}).$$
- For small spheres, with $a \ll 1/\sqrt{K}$, $A \approx 4\pi a^2$ – the Euclidean value.
- As a increases we get a maximum $A_{\max} = 4\pi/K$.
- A decreases as we continue to increase a.
- Compare: circumference of a circle of latitude on a sphere $C = 2\pi R\sin(a/R)$.
- These universes are spheres with three-dimensional surfaces, and hence termed closed.



$K < 0$ negative (open universe)

- The same integration here gives

$$r = \frac{1}{\sqrt{-K}} \sinh(a\sqrt{-K}), \qquad A = \frac{4\pi}{|K|} \sinh^2(a\sqrt{|K|}).$$

- The area thus now increases *faster* than in flat space, and tends to ∞ as $a \to \infty$.
- This case is called an **open** universe, or *hyperbolic* space.

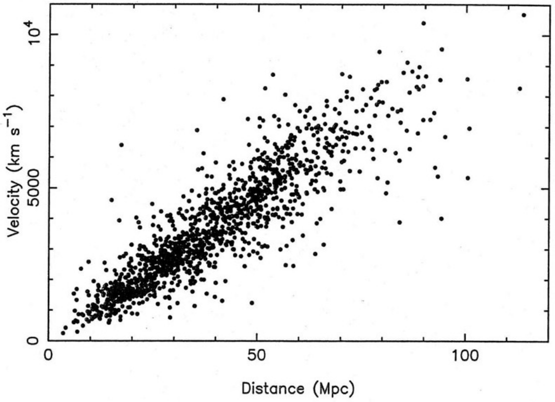
- So we know the spatial part of the metric. What do we need to get a **time** part?

Derivation of time part of FRW metric

- The fundamental observation is that the universe is expanding, and therefore the metric is changing in time.
- This expansion is captured by *Hubble's Law* (Hubble, 1929 — *predicted* by Weyl, 1923).
- The law is:

$$\boxed{v = H_0 d}$$

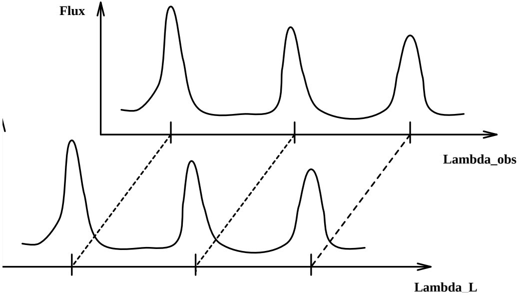
- i.e. galaxies are systematically moving away from us with speed proportional to distance.
- Will discuss how one could actually *define* the quantities involved in this later.
- This is consistent with a uniform expansion

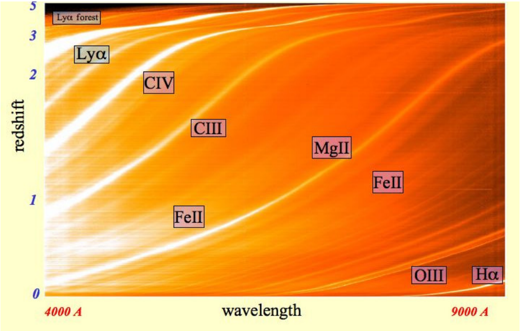


- Infer recession velocity from looking at spectra (recall quasars, Lecture 10)
- Identifiable pattern of emission (or absorption) lines (H, Mg, Ca, Na, etc. all give optical lines).
- Express this quantitatively via the redshift z defined by

$$z = \frac{\lambda_{\rm obs} - \lambda_{\rm em}}{\lambda_{\rm em}}.$$

- Note that other definitions of redshift would be possible (e.g. dividing by. $\lambda_{\rm obs}$) — the one given is the one used in cosmology.
- If we choose to interpret this as a (non-relativistic) Doppler shift, then $v \approx zc$ (since $\nu_{\rm obs} = \nu_{\rm em}(1 - v/c)$).



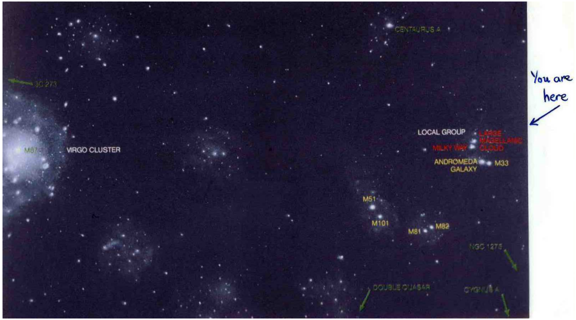


- So the Hubble effect, is that for cases where we know the distance d to the object (e.g. as provided by Cepheids) then we find

$$v \approx zc = H_0 d.$$

- H_0 is Hubble's Constant.
- Exact value has continued to be controversial, and over the last 30 years or so has spanned the range $H_0 = 50 - 80 \mathrm{kms}^{-1}\mathrm{Mpc}^{-1}$.
- (Will discuss up-to-date determinations later.)
- $1\,\mathrm{Mpc} = 3.0857 \times 10^{22}\,\mathrm{m}$, so translated into SI units this is $H_0 = 1.6 - 2.6 \times 10^{-18}\,\mathrm{s}^{-1}$.
- This has big implications for the time evolution of the universe.

- Consider a galaxy a distance d from us, e.g. M101 which is $\approx 4\,\mathrm{Mpc}$ away.

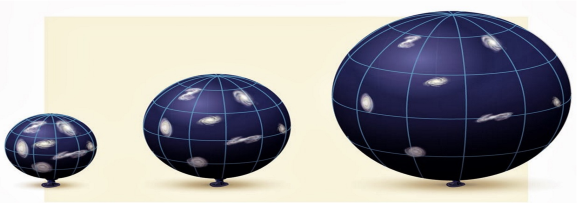


- Projecting back in time, when would it have been on top of us?

$$v = 4 \times H_0 \mathrm{kms}^{-1} \Longrightarrow \qquad t = \frac{d}{v} = \frac{4\,\mathrm{Mpc}}{4\,\mathrm{Mpc} \times H_0} = H_0^{-1}\mathrm{s}$$

- I.e. t_{collide} is approximately:
  - 15 billion years (byr) for H_0 in the middle of the above range;
  - 12.5 byr if $H_0 = 80\,\mathrm{kms}^{-1}\mathrm{Mpc}^{-1}$;
  - and 20 byr if $H_0 = 50\,\mathrm{kms}^{-1}\mathrm{Mpc}^{-1}$.

- Clearly this time is independent of which galaxy we are considering, and thus the whole of the universe must have been in a very small volume this time ago.
- But the age of the Earth is known to be $\sim 4.5\,\mathrm{byr}$ and the oldest stars in the Galaxy are $\sim 11\,\mathrm{byr}$ old.
- This poses great problems for higher values of H_0, since the universe is then in danger of being younger than some of its constituents!
- (If include dynamics, find t_{collide} is modified somewhat, and is most likely even shorter than we have just deduced!)
- In any case, it is clear that the universe must have come from an earlier stage at which it was much smaller and denser.
- Roughly, conservation of matter tells us that $\rho \propto R^{-3}$, where ρ is the matter density and R some characteristic size — the scale factor — associated with the universe — we will now look at how to define this physically.



- Must preserve isotropy and homogeneity at each instant of time for the 3-space geometry corresponding to spatial sections.
- Only possible change with time is for the curvature to vary $K = K(t)$.
- Since $K \sim 1/(\text{radius of space})^2$, we define a *scale factor* $R(t)$ via

$$K(t) = \frac{k}{R^2(t)} \begin{cases} k = +1 & \text{if curvature positive,} \\ k = 0 & \text{if space is flat,} \\ k = -1 & \text{is curvature is negative.} \end{cases}$$

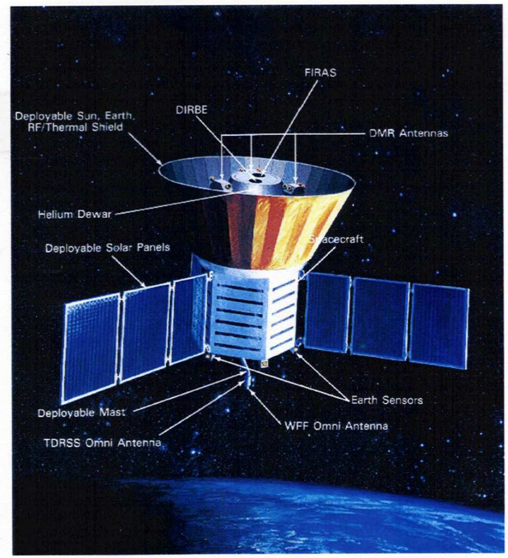
- Define a dimensionless coordinate $\sigma = r/R(t)$, then

$$\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

- becomes
$$R^2(t)\left\{\frac{d\sigma^2}{1-k\sigma^2}+\sigma^2(d\theta^2+\sin^2\theta d\phi^2)\right\}.$$
- Corresponds in time to a metric which is just a simple scaling of a fixed 3-metric.
- Objects with a fixed σ strictly obey the overall uniform expansion:
  - they have no *peculiar motion* of their own,
  - on the scales we are thinking of would correspond to galaxies with no peculiar motion,
  - called 'fundamental', or 'comoving' observers – f.o.'s for short.
- How do we distinguish fundamental observers? – And what did we mean by 'time' when we referred to wanting the same curvature over the whole of a given spacelike slice at the same instant?
- Not obvious how we can define an absolute time covering the whole universe, which observers can agree upon, and at successive instants get the same answers for curvature, density etc.
- There *is* an overall rest-frame, and therefore choice of time axis.
- It is provided by the cosmic microwave background radiation (CMBR)!
- We can detect our motion relative to it via the **dipole anisotropy**

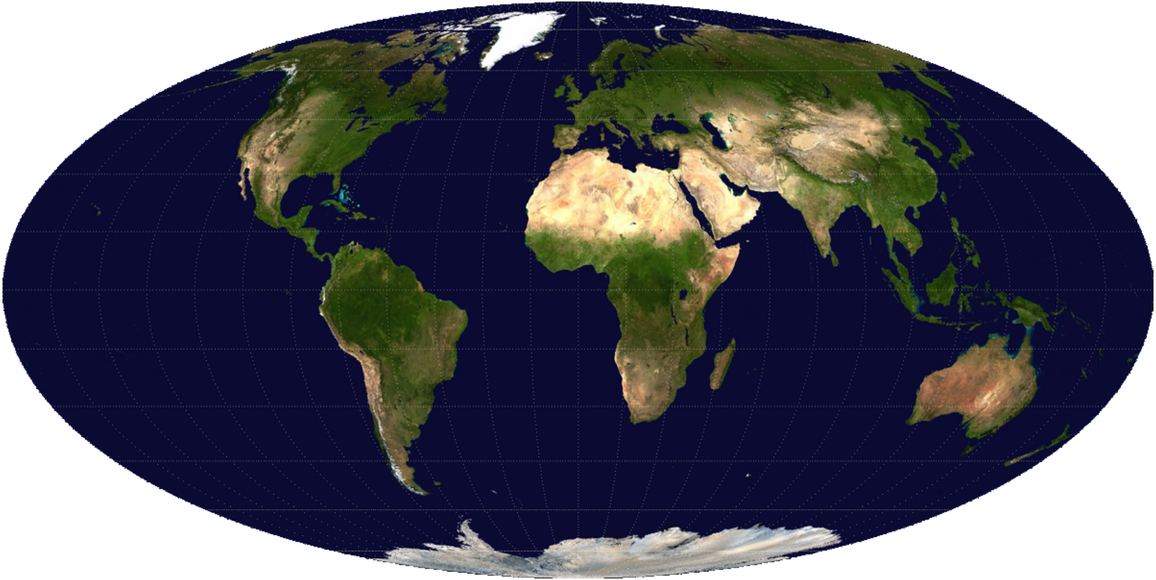
Fundamental Observers

Fundamental observers are those with zero velocity with respect to the frame defined by the microwave background radiation

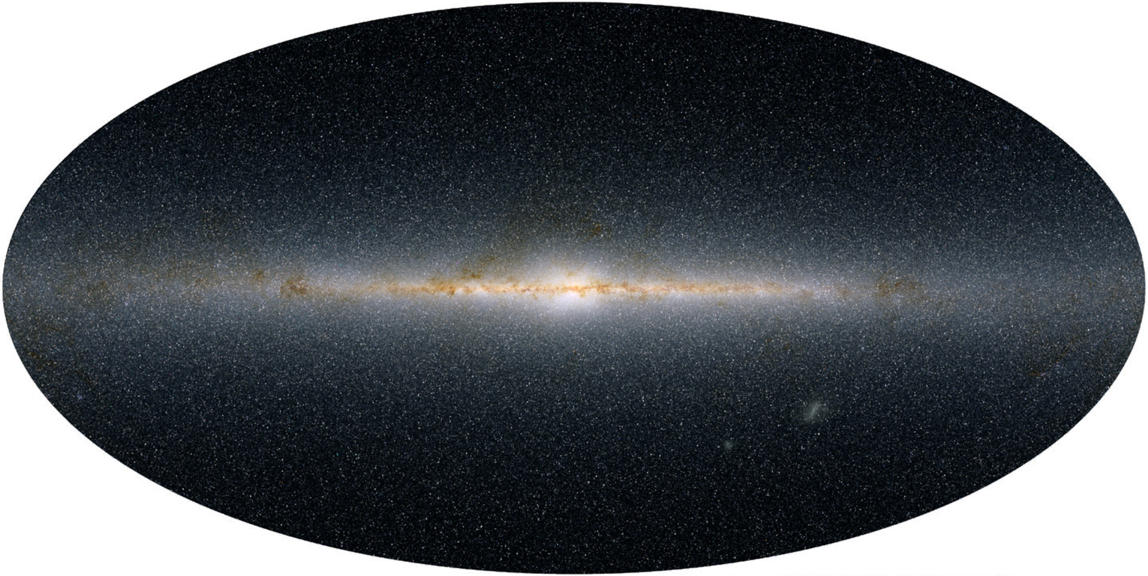


The COBE Satellite, showing the main instruments

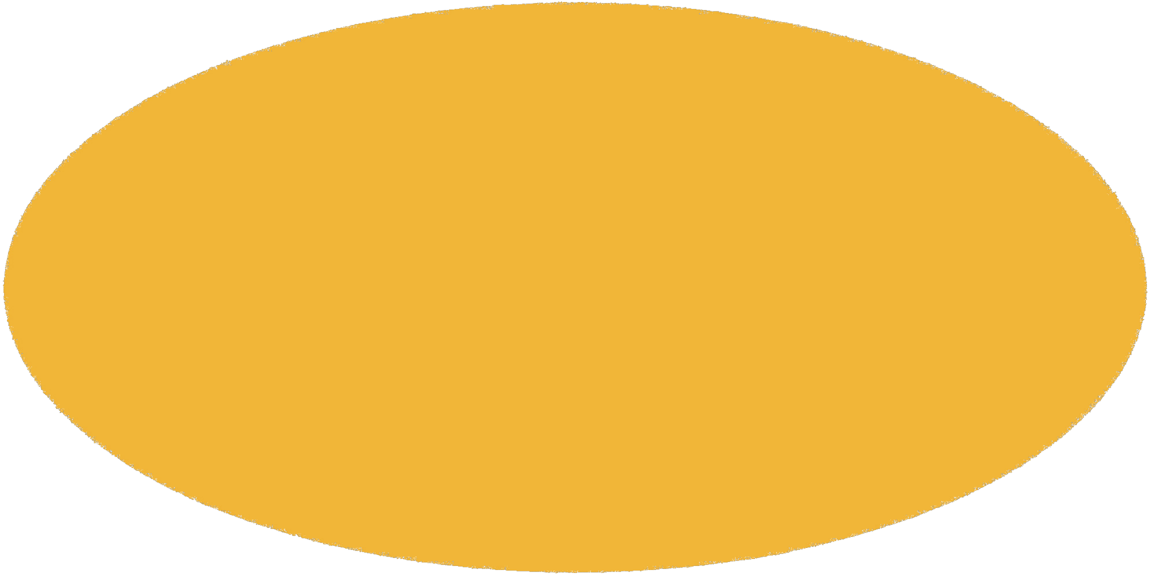
- Mollweide projection of the Earth's surface. Astro plots the same principle, but looking up.



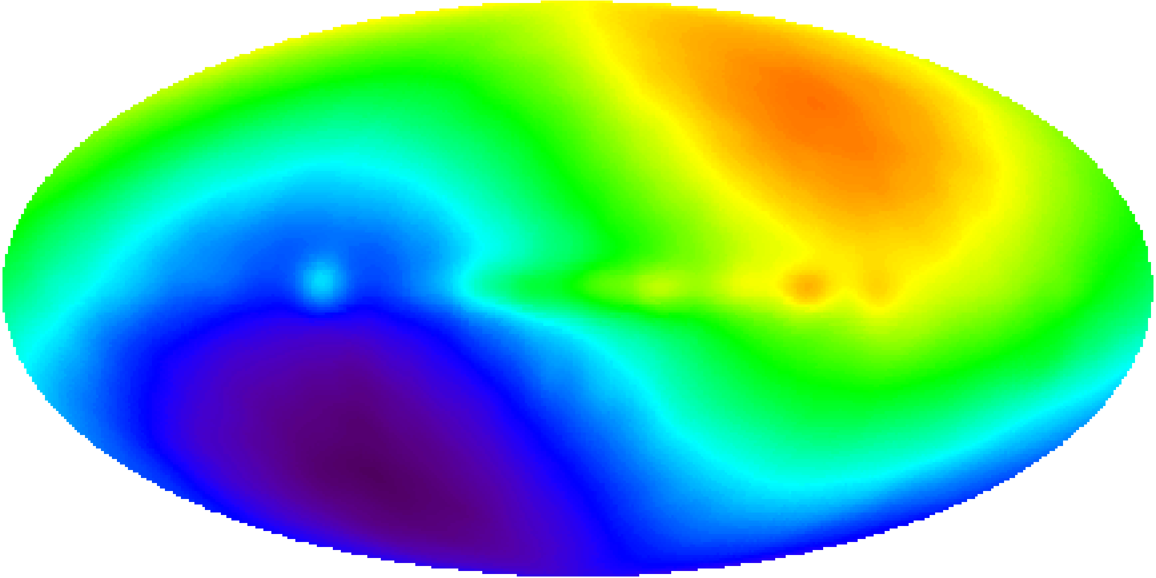
- Image of the (galactic aligned) milky way sky in the infrared



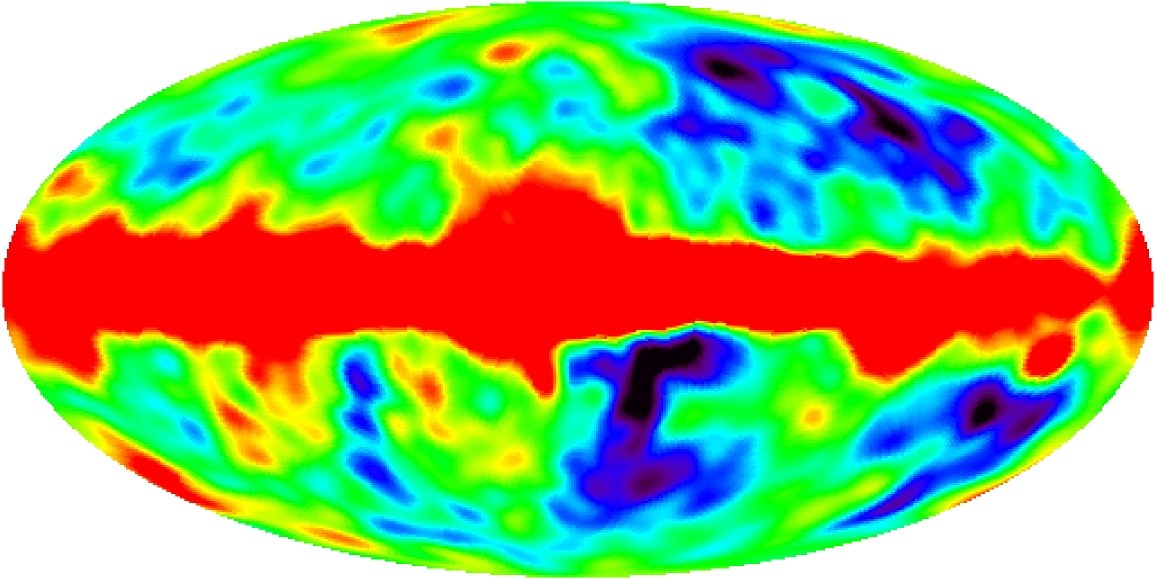
- Image of the sky in microwaves. Blackbody spectrum $T = 2.725$ K



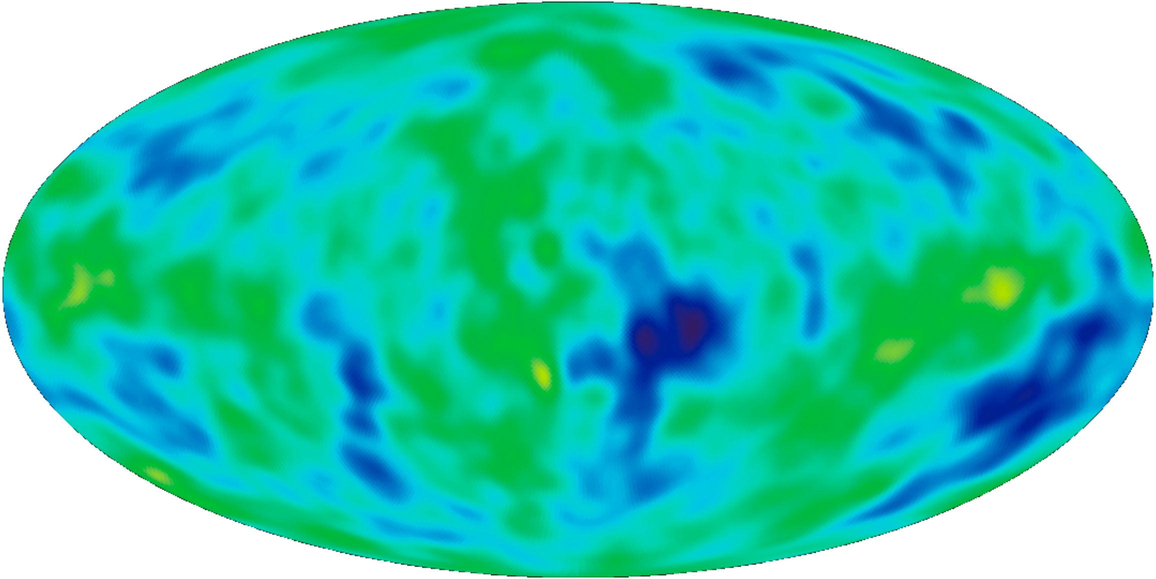
- Subtract off this constant term $T - 2.725$ to reveal dipole



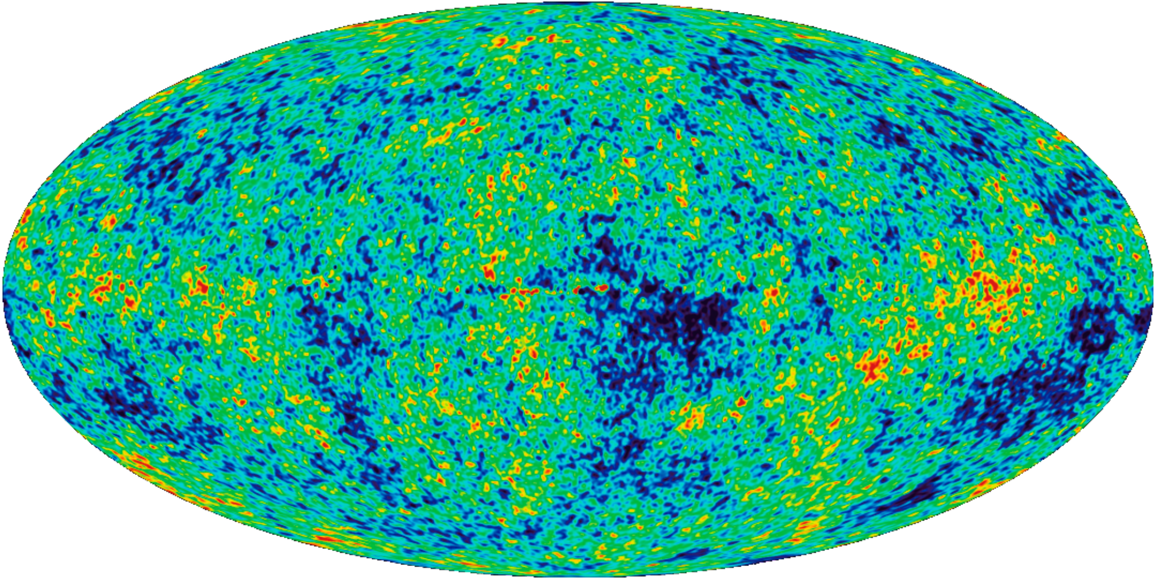
- Subtract off dipole to reveal anisotropy



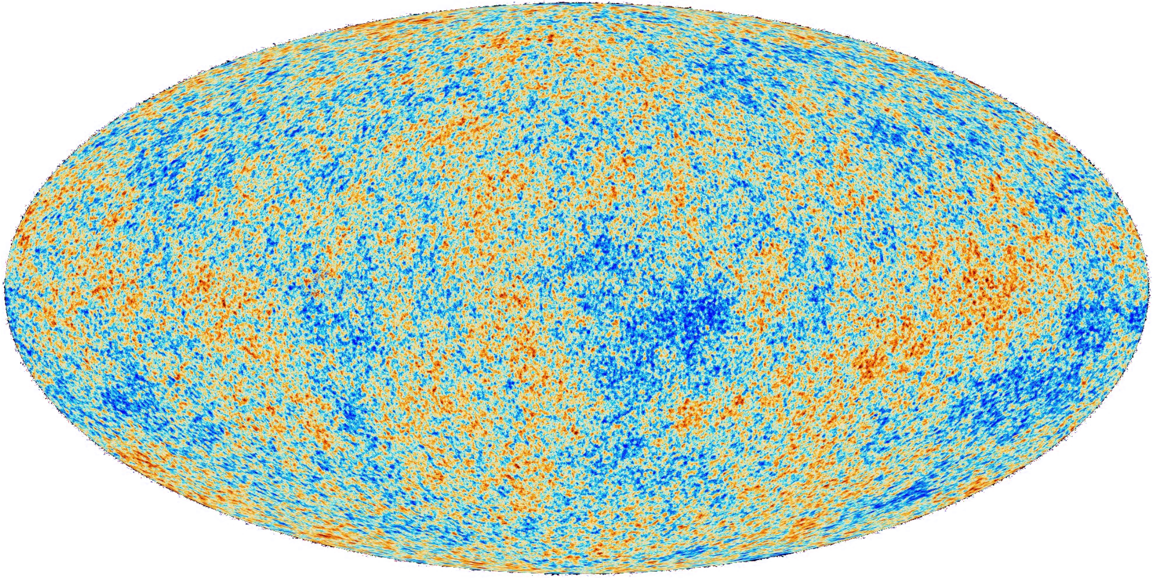
- Subtract off galaxy to reveal primordial anisotropy from the beginning of the Universe

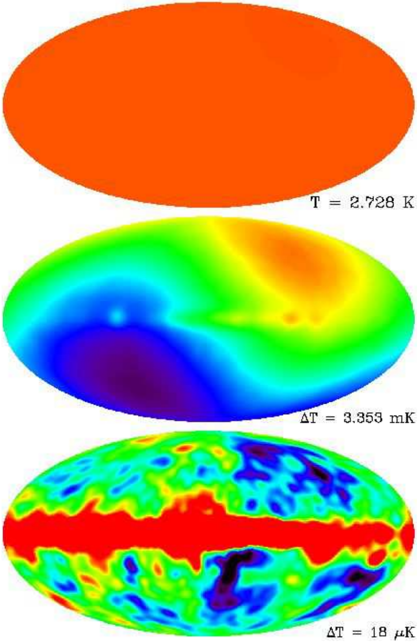


- Replace COBE with WMAP



- Replace WMAP with Planck





Maps made by COBE. Top: The sky 'monopole';
Middle: what's left after subtraction of the monopole;
Bottom: after subtraction of monopole and dipole

Cosmic time

- Cosmic time corresponds to the *proper time* (in the usual special relativistic sense) as measured by these fundamental observers.
- Having seen that for comoving observers the interval of cosmic time, dt say, that they will measure, is just the interval of proper time ds, we have that the total metric for the universe, i.e. with space and time parts joined together, is

$$\boxed{ds^2 = c^2 dt^2 - R^2(t) \left\{ \frac{d\sigma^2}{1 - k\sigma^2} + \sigma^2 (d\theta^2 + \sin^2\theta d\phi^2) \right\}.}$$

- In the exercises you will be asked to show that comoving observers follow *geodesics* of this metric, which is called the Friedmann-Robertson-Walker metric.
- This means that comoving observers are freely falling, and can apply all the usual SR laws in their local frame.

Final form of the FRW metric

- Most of our information in cosmology comes from radial propagation of photons.
- Therefore define a coordinate transformation which puts the FRW metric into a form which is especially convenient for this.
- Introduce a new comoving radial coordinate χ:

$$d\chi = \frac{d\sigma}{\sqrt{1-k\sigma^2}}, \quad \text{i.e.} \quad \chi = \begin{cases} \sin^{-1}\sigma & k=+1, \\ \sigma & k=0, \\ \sinh^{-1}\sigma & k=-1. \end{cases}$$

- The overall FRW metric then becomes

$$\boxed{ds^2 = c^2dt^2 - R^2(t)\left\{d\chi^2 + S^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)\right\},} \quad \text{where} \quad \boxed{S^2(\chi) = \begin{cases} \sin^2\chi & k=+1, \\ \chi^2 & k=0, \\ \sinh^2\chi & k=-1. \end{cases}}$$

- Have arrived at this after two radial coordinate transformations (first $r \to \sigma$, then $\sigma \to \chi$), and is essentially our final form.

Propagation of photons — the redshift

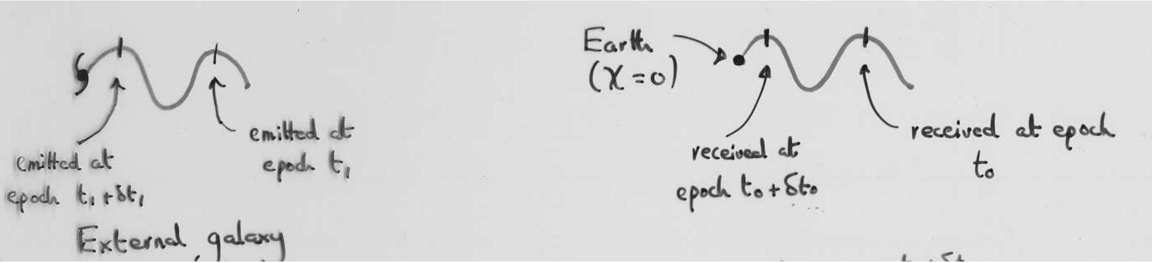
- We now apply the FRW metric to photon propagation to deduce the redshift formula.
- Suppose a photon is travelling to us from an external galaxy. Its motion is radial, so we can ignore the θ and ϕ coordinates.
- The condition for photon propagation is that the interval should vanish. Thus

$$ds^2 = c^2 dt^2 - R^2(t) d\chi^2 = 0 \quad \Rightarrow \quad \frac{d\chi}{dt} = -\frac{c}{R(t)},$$

  for an incoming photon. This is how its χ coordinate changes with time.
- Suppose the photon was emitted at cosmic time t_1 and is received by us at time t_0.
- (Conventionally t_0 denotes the time here now.) The χ coordinate of the emitter is given by

$$\chi_1 = \int_{t_1}^{t_0} \frac{c\, dt}{R(t)}.$$

- The crucial point is that if the galaxy can be treated as a fundamental observer (i.e. moving with the cosmic fluid without a peculiar velocity of its own), then its χ coordinate is fixed.
- This enables us to work out the redshift.



If the interval between emission of successive wavecrests is δt_1 and that between reception is δt_0, we have

$$\chi_1 = \int_{t_1}^{t_0} \frac{c\,dt}{R(t)} = \int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{c\,dt}{R(t)} = \left(\int_{t_1+\delta t_1}^{t_1} + \int_{t_1}^{t_0} + \int_{t_0}^{t_0+\delta t_0} \right) \frac{c\,dt}{R(t)}.$$

Thus

$$\int_{t_1}^{t_1+\delta t_1} \frac{c\,dt}{R(t)} = \int_{t_0}^{t_0+\delta t_0} \frac{c\,dt}{R(t)}.$$

- We now approximate by assuming that $R(t)$ hardly changes between emission of successive crests, i.e. that

$$\frac{R(t)}{\dot{R}(t)} \gg \delta t, \quad \text{i.e. “age of universe”} \gg \text{“period of oscillation of wave”}.$$

(This is one of the best approximations you are likely to make in physics!)

- In this case

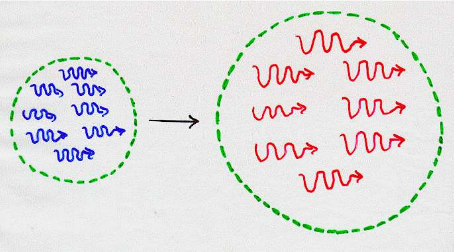
$$\frac{\delta t_1}{R(t_1)} = \frac{\delta t_0}{R(t_0)}.$$

- Now frequency $\nu = 1/\delta t = c/\lambda$, and hence $\lambda_0/\lambda_1 = R(t_0)/R(t_1)$.
- With redshift following our usual definition of $(\lambda_{\text{obs}} - \lambda_{\text{em}})/\lambda_{\text{em}}$, we derive

$$z = \frac{(\lambda_0 - \lambda_1)}{\lambda_1} = \frac{R(t_0)}{R(t_1)} - 1,$$

- i.e. $\boxed{1 + z = \frac{R(t_0)}{R(t_1)} = \frac{\text{scale factor of universe on reception}}{\text{scale factor of universe when emitted}}}$

- This coincides nicely with our conception of redshift as being due to *'stretching'* of the photon wavelength:



- Can also derive it from the fact that radially p_χ is conserved, so identifying $p^\mu = (\hbar\omega, \hbar\omega/R(t_1))$, and conserving $p_\chi = R(t_1)\hbar\omega_1$ at t_1, we find that at time t_0 that $p^\chi = \frac{R(t_1)}{R(t_0)^2}\hbar\omega_1 = \hbar\omega_0/R(t_0)$, so $1 + z = \frac{\lambda_0}{\lambda_1} = \frac{\omega_1}{\omega_0} = \frac{R(t_0)}{R(t_1)}$.
- To get much further with applying the FRW metric, we will need to know the time history of $R(t)$, i.e. we need to know something about the 'dynamical history' of the universe.
- Thus next in cosmology we will aim to get to grips with **the basic dynamical equations**

Summary

- Flat, closed and open universes.
- Fundamental observers, the CMB dipole, and cosmic time.
- The Friedmann-Robertson-Walker metric.

$$\boxed{ds^2 = c^2dt^2 - R(t)^2dx^2,} \quad \boxed{dx^2 = d\chi^2 + S^2(\chi)d\Omega,} \quad \boxed{S^2(\chi) = \begin{cases} \sin^2\chi & k = +1, \\ \chi^2 & k = 0, \\ \sinh^2\chi & k = -1. \end{cases}}$$

- The redshift formula

$$\boxed{1 + z = \frac{R(t_0)}{R(t_1)}.}$$

Next time

The dynamics of the universe from the Einstein Field equations.