Topics you should be looking at:

1. Characteristic f^n and how we used it for C.L.T.

2. Evolution / Kolmogorov-Chapman eqs

3. Itoh Lemma — we used it only once in G.B.M. case

— practice on other $f(s)$ ...

— for multi-variable case $f(s,x)$

4. B-Sch. equation derived in Lecture 5 (via portfolio) and by Guest Lecturer in Lecture-6 (via average $\langle f(s) \rangle = 0$)

5. Ornstein-Uhlenbeck process, esp. with multi-variables:

$$\boxed{dx_i = -\theta_{ik} x_k \, dt + \sigma_{ik} \, dW_k}$$

6. How to derive the "Fokker-Planck" equation from

SDE $\rightarrow$ Kramers-Moyal $\rightarrow$ $\frac{\partial P}{\partial t} = \hat{L} P$

7. What is MFPT?

Q12 is an example of 2-variable O-U process.

$m\ddot{x}_1 = -k_1 x_1 + K(x_2 - x_1) - \gamma_1 \dot{x}_1 + W_1$

overdamped

$m\ddot{x}_2 = -k_2 x_2 + K(x_1 - x_2) - \gamma_2 \dot{x}_2 + W_2$

$\cancel{v_1 = \dot{x}_1}$

$\cancel{v_2 = \dot{x}_2}$

$\sqrt{2kT\gamma_1}$

$\gamma_1 d x_1 = (-(k_1+K)x_1 + K x_2)dt + \sigma_1 \xi_1 dW_1$

$\gamma_2 d x_2 = (-(k_2+K)x_2 + K x_1)dt + \sigma_2 \xi_2 dW_2$

$$\underline{\underline{\Theta}} = \begin{pmatrix} \frac{k_1+K}{\gamma_1} & -\frac{K}{\gamma_1} \\ -\frac{K}{\gamma_2} & \frac{k_2+K}{\gamma_2} \end{pmatrix} \qquad \underline{\underline{\sigma}} = \begin{pmatrix} \sqrt{\frac{2kT}{\gamma_1}} & 0 \\ 0 & \sqrt{\frac{2kT}{\gamma_2}} \end{pmatrix}$$

$$\Theta_{ik}\langle x_k x_l\rangle + \langle x_i x_k\rangle \Theta^T_{lk} = (GG^T)_{ik}$$

$$\begin{pmatrix} 2\frac{(k_1+k)}{\gamma_1}\langle x_1x_1\rangle - \frac{k}{\gamma_1}\langle x_2x_1\rangle + \langle x_1x_2\rangle & \frac{k_1+k}{\gamma_1}\langle x_1x_2\rangle - \frac{k}{\gamma_1}\langle x_2x_2\rangle \\ -\frac{k}{\gamma_2}\langle x_1x_1\rangle + \frac{k}{\gamma_1}\langle x_2x_2\rangle + \frac{k_2+k}{\gamma_2}\langle x_2x_1\rangle + \frac{k_2+k}{\gamma_1} & -\frac{k}{\gamma_2}\langle x_1x_2\rangle + \frac{k_2+k}{\gamma_2}\langle x_2x_2\rangle \end{pmatrix}$$

$$\begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix}\begin{pmatrix} x_1x_1 & x_1x_2 \\ x_2x_1 & x_2x_2 \end{pmatrix}\begin{pmatrix} \frac{k_1+k}{\gamma_1} & -\frac{k}{\gamma_2} \\ -\frac{k}{\gamma_1} & \frac{k_2+k}{\gamma_2} \end{pmatrix}$$

$$GG^T = \begin{pmatrix} \boxed{\frac{2kT}{\gamma_1}} & 0 \\ 0 & \frac{2kT}{\gamma_2} \end{pmatrix}$$

$$2\frac{k_1+k}{\gamma_1}\langle x_1x_1\rangle - 2\frac{k}{\gamma_1}\langle x_2x_1\rangle = \frac{2kT}{\gamma_1}$$

$$-\frac{k}{\gamma_2}\langle x_1x_1\rangle - \frac{k}{\gamma_1}\langle x_2x_2\rangle + (k_2+k)\left(\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right)\langle x_1x_2\rangle = 0$$

∴ Three linear equations need.

Solve these to find:

$$\langle x_1 x_1 \rangle = \frac{(k_2+K)\, k_B T}{2(k_1 k_2 + k_1 K + k_2 K)} \sim \frac{k_B T}{K}$$ e.g.

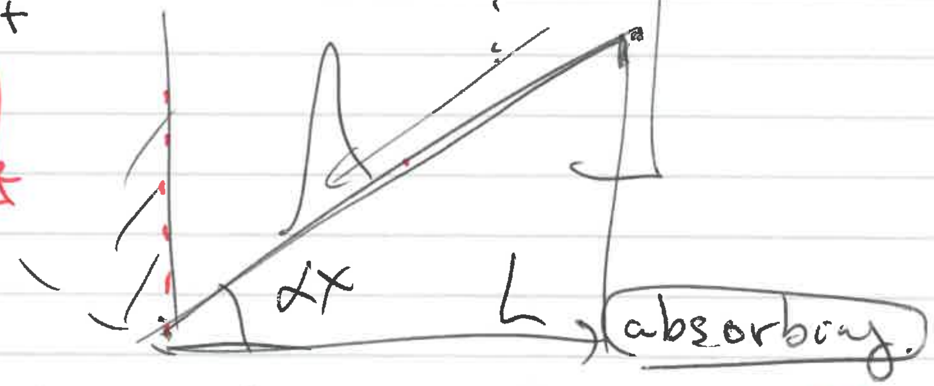
and so all others ....

Q13 MFPT in Linear potential.

1) We do have a general solution for linear potential (drift)

$$\frac{1}{\sqrt{4\pi D t}} e^{-\frac{x^2}{4Dt}} \qquad \frac{1}{\sqrt{4\pi D t}} e^{-\frac{[x - x_0 - (f/\gamma) t]^2}{4Dt}}$$

+ image



2) Use the ∫-formula

$$\tau = \frac{1}{D}\int_{x_0}^{L} dx\, e^{+\beta V(x)} \int_0^x dy\, e^{-\beta V(y)}$$

$$\int_0^x e^{-\alpha y}\, dy = \frac{1}{\alpha}\left(1 - e^{-\alpha x}\right)$$

$$\frac{1}{\alpha D}\int dx \left(e^{\alpha x} - 1\right)$$

3) Using Kramers escape (via flux)

const

$$J = -D e^{-\beta V(x)} \frac{\partial}{\partial x}\left(e^{\beta V(x)} p(x,t)\right)$$

$$\boxed{\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{f}{\gamma} p\right) + D \frac{\partial^2}{\partial x^2} p}$$

$$\int_0^\infty J e^{\beta V(x)}\, dx = -D\left(e^{\beta V(x)} p(x,t)\right)\Big|_0^{\infty}$$

zero

$$J \cancel{\beta V_0} \int_0^L e^{+\alpha x}\, dx = D\, p(x=0)$$

$$\frac{1}{\alpha} J \left(e^{\beta V_0} - \underbrace{\dots}_{\text{small}}\right) = D\, p(x=0)$$

$$\boxed{J = \alpha D\, p(x=0) \cdot e^{-\beta V_0}}$$

$$\langle N \rangle = \int_0^\infty p(x=0)\, e^{-\beta V(x)}\, dx$$

$$= p(x=0)\, \frac{1}{\alpha}$$

$$\text{rate} = \frac{J}{\langle N \rangle_{\text{at } x=0}} = \alpha^2 D e^{-\beta V_0}$$

$$\parallel$$

$$\frac{1}{2\,\cancel{\text{MFPT}}}$$

$$\boxed{\text{MFPT} = \frac{1}{2}\alpha^2 D e^{-\beta V_0}}$$

Black-Scholes

Basic underlying SDE

GBM — $dS = \mu S\,dt + \sigma S\,dW$

$\downarrow$

Lecture 4-5 — $S = S_0 e^{(\mu - \frac{\sigma^2}{2})t}$

We derived B-S via Hedging.

$\Pi = V(S) - \alpha S$

adjustible parameter

$d\Pi = (\ldots)dt + \underbrace{(\ldots)}_{=0}dW = r(V - \cancel{\alpha} S)\cancel{dt}$

(via Itoh)

at fixed α : risk free

Separately claim $d\Pi = r\Pi\,dt$

prescribed rate of growth

B-S equation

$$\boxed{(\ldots) = rV - \alpha S}$$

Alternatively: Guest lecture 6

Fluctuating Volatility!

$$dS = \mu S\,dt + \sigma S\,dW^S$$

$$\boxed{d\sigma = -\theta\sigma^2\,dt + (\;)\,dW^\sigma}$$

To derive B-S alternatively:

$V(S) \rightarrow$ Itoh will give

$$dV(S) = (\ldots)\,dt + (\ldots)\,dW$$

$\boxed{\text{Arbitrage assumption}}$ there is a standard rate of growth r

All values grow exponentially e^{rt} : inflation.

$f(S) = e^{-rt}V(S)$

$\langle d[e^{-rt}V(S)]\rangle$

$\langle d[\,e^{-rt}V(S)\,]\rangle$ corrected for future legislative inflation

on average this $\neq 0$

all terms $\langle (\ldots)\,dW\rangle = 0$

$$\boxed{\langle e^{-rt}\,dV(S)\rangle = 0} \Rightarrow \text{this is B-S eq}$$

with all $\langle dW\rangle = 0$