Plate flexure

Physics of the Earth as a Planet, Lecture 6

(Fowler, The Solid Earth, CUP, Chapter 5, pages 218-224)

In lecture 4 we discussed topography and how we can use gravity measurements to determine if the topography is compensated by isostasy. We looked at several examples and found that most ridges and mountains are indeed compensated. However, we also found that the chain of ocean islands in Hawaii is not compensated, and there will be forces in the plate to make Hawaii stable. The lithosphere at Hawaii is bent, and we call this flexure. We can use flexure to determine the elastic thickness of the thickness of the lithosphere, again using topography and gravity measurements.

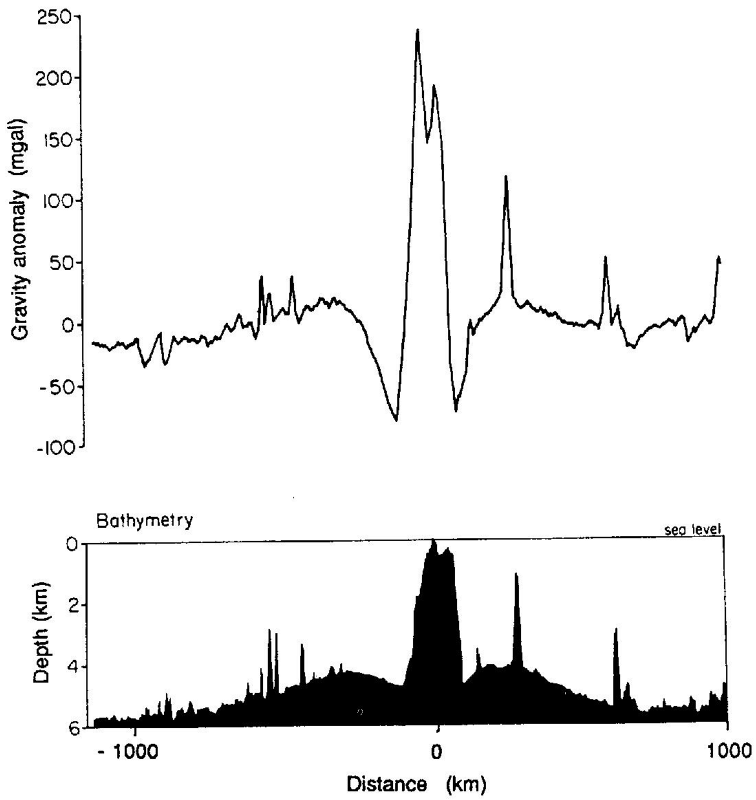

Figure 1: Gravity and bathymetry measurements in Hawaii. The significant gravity measurements suggest that Hawaii is not isostatically compensated. Forces in the plate must be supporting the load, which leads to flexure (i.e. bending) of the plate.

How can observations be used to find the thickness of the elastic part of the plate? The obvious way to do so is to look at how it bends when loads are applied to its surface, like volcanoes, sediments and topography produced by thrusting. The simplest problem is to look at the shape of the surface, like to ocean floor, where it is bent by, for instance, another plate thrusting on top at a trench. Either the topography or the gravity anomaly can be used. This handout does so first in the space domain. However, the results are better when the modelling is done in the frequency domain.

Both methods start from the equation that relates the deformation w of a floating plate to an applied load L whose elastic thickness is T_e, Young's modulus E, and Poisson's ratio σ, overlain by a fluid whose density is ρ_w (water), and overlying a fluid whose density is ρ_m (mantle),

$$D\nabla^4 w + g(\rho_m - \rho_w)w = L(x, y) \tag{1}$$

where

$$D = \frac{ET_e^3}{12(1-\sigma^2)}, \qquad \nabla^4 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2$$

Notice that the displacement and the load are positive in the $+z$ direction. D is called the *flexural rigidity* and describes the resistance of the plate to bending (formally it is the bending moment required to flex the plate to unit curvature). Equation (1) is derived in most engineering textbooks, such as Timoshenko, S., and D.H. Young, Elements of Strengths of Materials, Van Nostrand. The derivation assumes that the length over which w varies is large compared with T_e, so it used what is called the thin plate approximation.

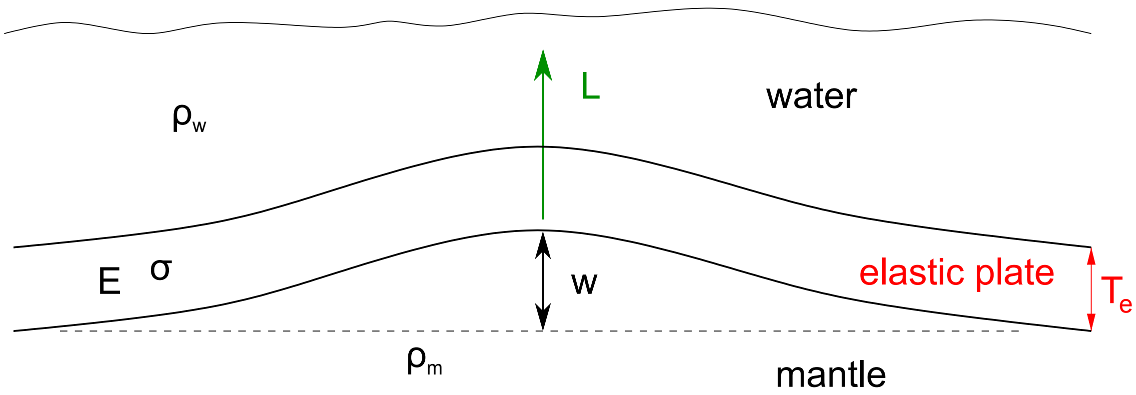

Figure 2: Schematic illustration of the parameters used in estimating the elastic thickness of the plates.

Island chains

We want solutions to (1) that will allow us to estimate T_e from observations. One of the simplest solutions is a one dimensional solution for the shape where $L = 0$. This is a useful solution for the deflection caused by island chains (i.e. Hawaii), which only have a load L at $x = 0$, and have $L = 0$ everywhere else.

$$\frac{D}{g(\rho_m - \rho_w)} \frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + w = 0 \tag{2}$$

Writing

$$\alpha^4 = \frac{4D}{g(\rho_m - \rho_w)} \tag{3}$$

where α is called the *flexural parameter* (the natural lengthscale in the problem), gives

$$\frac{\alpha^4}{4} \frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + w = 0 \tag{4}$$

The general solution of (4) that decays as $x \to \infty$ is

$$w = A_1 \exp\left[-\left(\frac{x}{\alpha}\right)\right] \cos\left(\frac{x}{\alpha}\right) + A_2 \exp\left[-\left(\frac{x}{\alpha}\right)\right] \sin\left(\frac{x}{\alpha}\right) \tag{5}$$

where A_1 and A_2 are constants. The gravity anomaly Δg can then easily be calculated using

$$\Delta g = 2\pi G(\rho_m - \rho_w) w \tag{6}$$

so

$$\Delta g = B_1 \exp\left[-\left(\frac{x}{\alpha}\right)\right] \cos\left(\frac{x}{\alpha}\right) + B_2 \exp\left[-\left(\frac{x}{\alpha}\right)\right] \sin\left(\frac{x}{\alpha}\right) \tag{7}$$

where B_1 and B_2 are constants. T_e is estimated by stacking a number of profiles at right angles to some linear feature, like a trench or a foreland basin, and calculating the average value w_0 and its standard deviation σ_0 at N different points along the profile. The misfit H^s for a particular value of T_e is

$$H^s = \frac{1}{N}\sum_n \left(\frac{w_o - w_c}{\sigma_0}\right)^2 \tag{8}$$

or

$$H^s = \frac{1}{N}\sum_n \left(\frac{\Delta g_o - \Delta g_c}{\sigma_0}\right)^2 \tag{9}$$

where the subscripts o and c refer to the observed and calculated topography and gravity at the N points on the profile. The best fit is then the minimum of $H^s(T_e)$ with respect to A_1 and A_2, or B_1 and B_2, as T_e is varied.

Differentiating equation (8) with respect to A_1 and A_2 (or (9) with respect to B_1 and B_2) and setting

$$\frac{\partial H^s}{\partial A_1} = \frac{\partial H^s}{\partial A_2} = 0$$

gives two equations that can be written

$$\mathbf{M}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \tag{10}$$

where $\mathbf{M}$ is a 2×2 matrix and C_1 and C_2 are constants. Because w_c depends linearly on A_1 and A_2, $\mathbf{M}$, C_1 and C_2 are independent of A_1 and A_2. The best fitting profile can therefore be obtained directly by inverting $\mathbf{M}$ to give H^s_{min}. This process is repeated for a range of values of T_e to find the elastic thickness that gives the smallest value of H^s_{min}. Figure 4 shows $H^s_{min}(T_e)$ for two collections of profiles, from the foreland basin NE of Iran and from a region of India immediately S of the Himalaya. Though the agreement between the calculated and observed profiles is excellent in both cases, the value of T_e for N. India is poorly constrained. This problem is common. Another problem with this approach is that it uses the gravity and topography separately, and each only in one dimension. It would clearly be better to use the large amounts of gravity and topography that are available in two dimensions. The best way to do so is to work in the frequency domain, by Fourier transforming equation (1) and the observations.

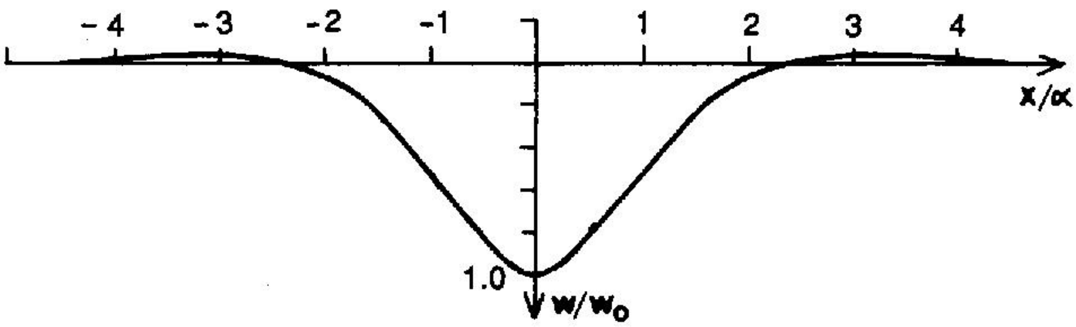

Figure 3: Schematic illustration of the deflection of an elastic plate for a line load at $x = 0$. The width of the buldge can be used to approximate α, leading to $x/\alpha \approx 2.4$. All other parameters are known, so T_e can in principle be computed.

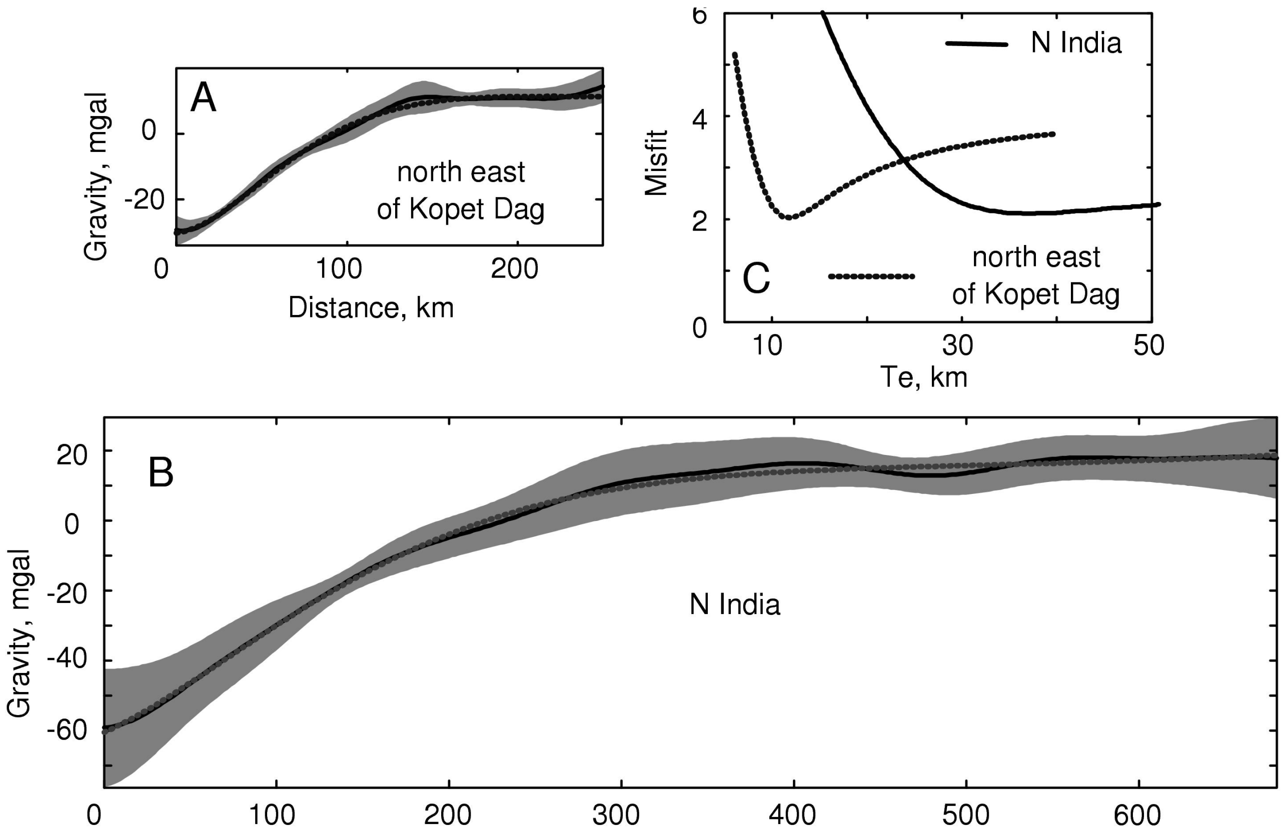

Figure 4: Fits to profiles in the space domain, from the Kopet Dag, on the northeastern edge of the Iranian Plateau, and from northern India, a foreland basin formed by the thrusting of Tibet over India.

Periodic load

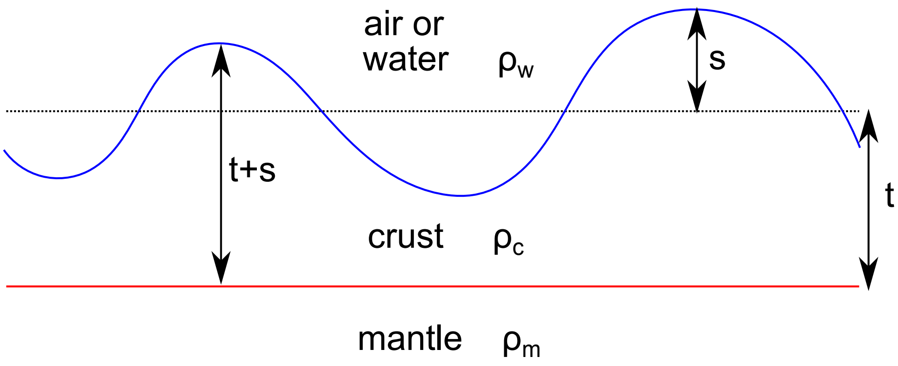

Figure 5: Schematic illustration of the loading before flexure.

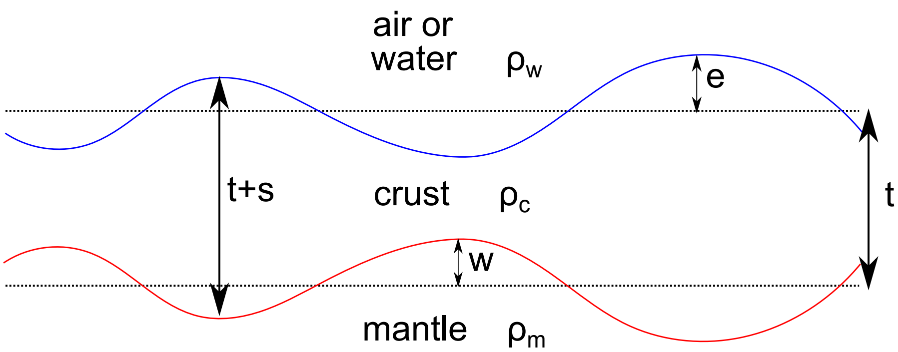

Figure 6: Schematic illustration after flexure.

One of the main sources of loading of the plate is crustal thickness variations. To model this we consider a three layer system of a half space of mantle of density ρ_m overlain by crust of density ρ_c and average thickness t overlain by water of density ρ_w (Figures 5 and 6). If the variations in crustal thickness about its average value is given by $s(x, y)$, then equation (1) becomes

$$D\nabla^4 w + g(\rho_m - \rho_w)w = -g(\rho_c - \rho_w)s \tag{11}$$

The resulting topography e is given by the sum of the thickness added and the deflection w

$$e = s + w \tag{12}$$

Assuming that the load s is periodic with wavenumber k_x, k_y, periodic topography e and deflection w are produced, i.e.

$$w = W \exp[i(k_x x + k_y y)], s = S \exp[i(k_x x + k_y y)]$$

$$e = E \exp[i(k_x x + k_y y)]$$

Substitution into (11) gives

$$W = -\left[\frac{g(\rho_c - \rho_w)}{Dk^4 + g(\rho_m - \rho_w)}\right] S \tag{13}$$

where

$$k^2 = k_x^2 + k_y^2$$

Hence

$$E = \left[1 - \frac{g(\rho_c - \rho_w)}{Dk^4 + g(\rho_m - \rho_w)}\right] S \tag{14}$$

As $k \to 0$ (long wavelength)

$$E = \left(\frac{\rho_m - \rho_c}{\rho_m - \rho_w} \right) S \tag{15}$$

and the layer is isostatically compensated.

To calculate the gravity field it is necessary to solve Poisson's equation

$$\nabla^2 \phi = -4\pi G \rho \tag{16}$$

The solution for the Fourier-transformed gravity anomaly $\overline{\Delta g}$ is the total contribution from both the topography e and the layer with thickness t deflected by an amount w, i.e.

$$\overline{\Delta g} = 2\pi G[(\rho_c - \rho_w)E + (\rho_m - \rho_c)\exp(-kt)W] \tag{17}$$

Equations (13) and (14) give

$$W = \frac{-g(\rho_c - \rho_w)}{Dk^4 + g(\rho_m - \rho_c)} E \tag{18}$$

and equation (17) then becomes

$$\overline{\Delta g} = 2\pi G(\rho_c - \rho_w) \left[1 - \frac{\exp(-kt)}{1 + \frac{Dk^4}{g(\rho_m - \rho_c)}} \right] E \tag{19}$$

The equation can also be written

$$\overline{\Delta g} = Z(k; T_e)E \tag{20}$$

where Z is called the admittance. It is generally measured in mGal/km.

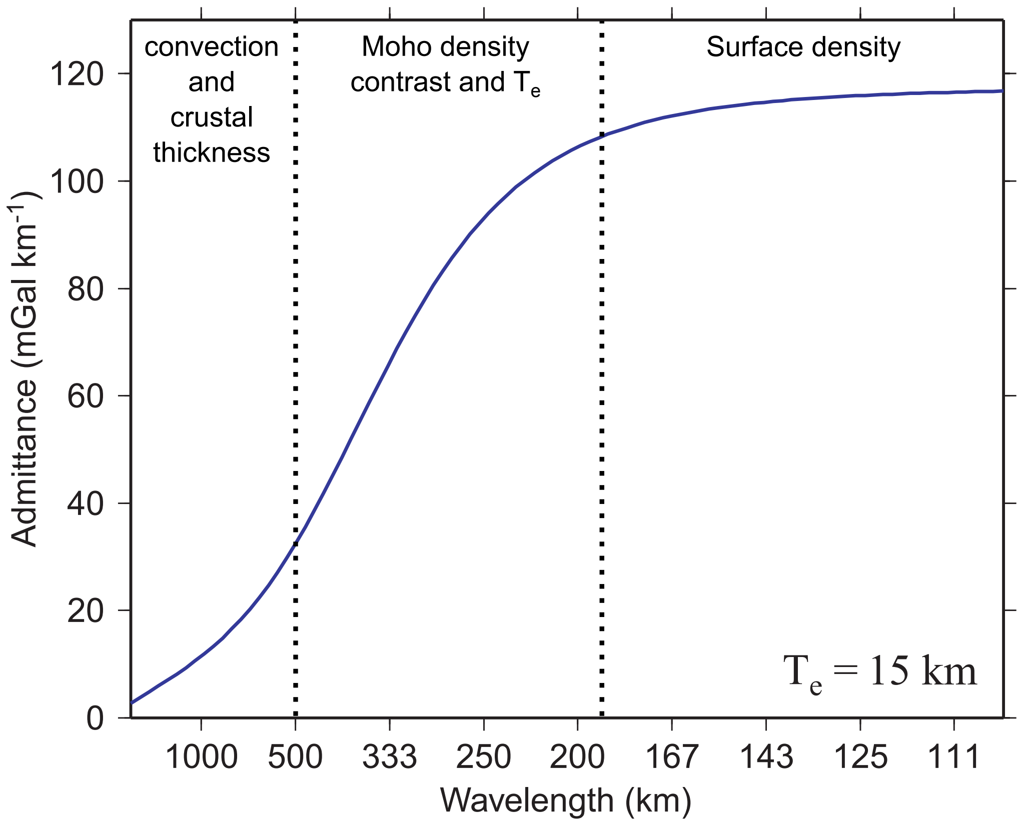

Figure 7: Plot showing the controls on the admittance in different wavelength bands.

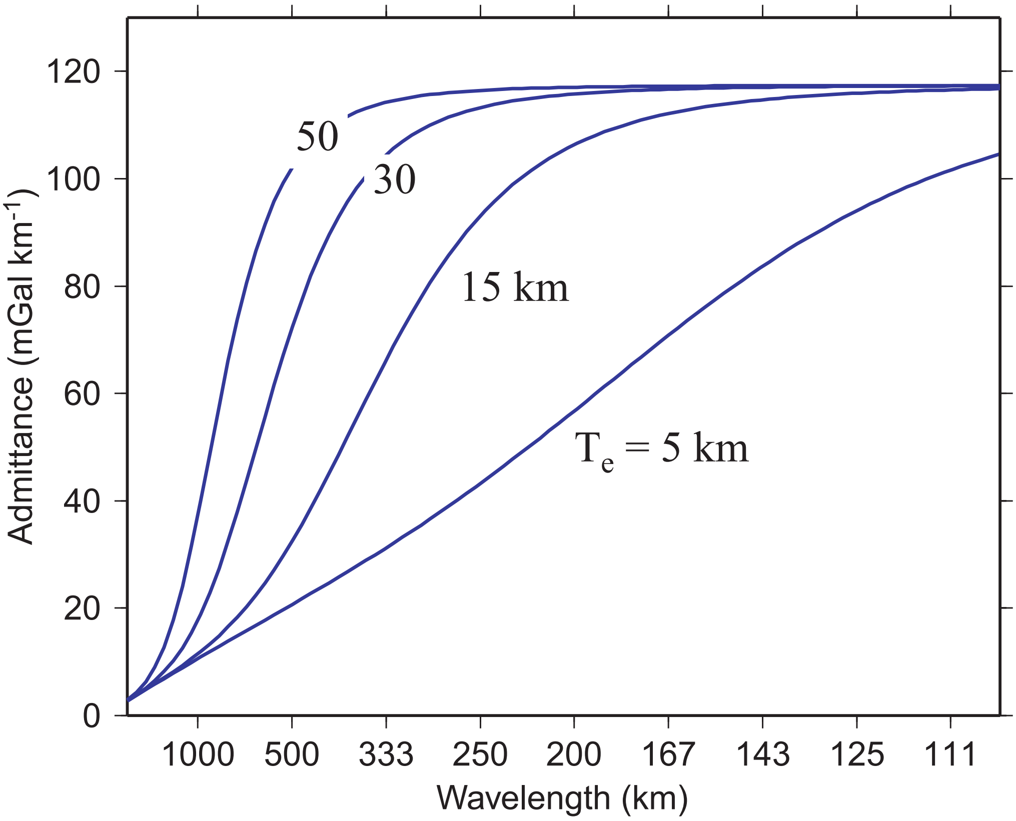

Figure 8: Plot showing the variation in admittance as a function of wavelength for different values of T_e.

Figure 7 shows the behaviour of $Z(k)$ and the parameters that control its behaviour in various wavelength bands. When $k << 1$ the load has a long wavelength and is compensated and $\overline{\Delta g} \to 0$. The admittance therefore tends to zero. However, at short wavelengths

$$k >> 1, \quad \overline{\Delta g} \to 2\pi G(\rho_c - \rho_w)E \tag{21}$$

and the topographic load is uncompensated. If $\rho_c = 2.7\ \mathrm{Mg\,m^{-3}}$, a typical crustal density, then

$$Z(k) = \frac{\overline{\Delta g}}{E} \to 116\ \mathrm{mGal/km} \tag{22}$$

at short wavelengths on continents, where $\rho_w = 0$. Whereas under water $\rho_w = 1.03\ \mathrm{Mg\,m^{-3}}$ and

$$Z(k) = \frac{\overline{\Delta g}}{E} \to 72\ \mathrm{mGal/km} \tag{23}$$

Modern workstations can easily Fourier transform grids of gravity and topography consisting of 10^6 grid points, and so find $Z(k)$. The best fitting value of T_e is calculated by minimising

$$H^f = \left[\frac{1}{N} \sum_n \left(\frac{Z_o - Z_c}{\sigma_0} \right)^2 \right]^{1/2} \tag{24}$$

where σ_0 is the standard deviation of the admittance in a particular wavenember band.

Figure 9 shows the calculated and observed curves for Hawaii and Siberia. The minima are deeper and better determined than are those in Figure 4.
Watts has collected a large number of estimates of T_e for oceanic regions, and Figure 10 shows these plotted as a function of age. He argues that the elastic thickness is controlled by the temperature, and that mantle material whose temperature exceeds 450°C, or a homologous temperature greater than about 0.5, cannot maintain elastic stresses for geological periods of time. Watts' arguments have been generally accepted.

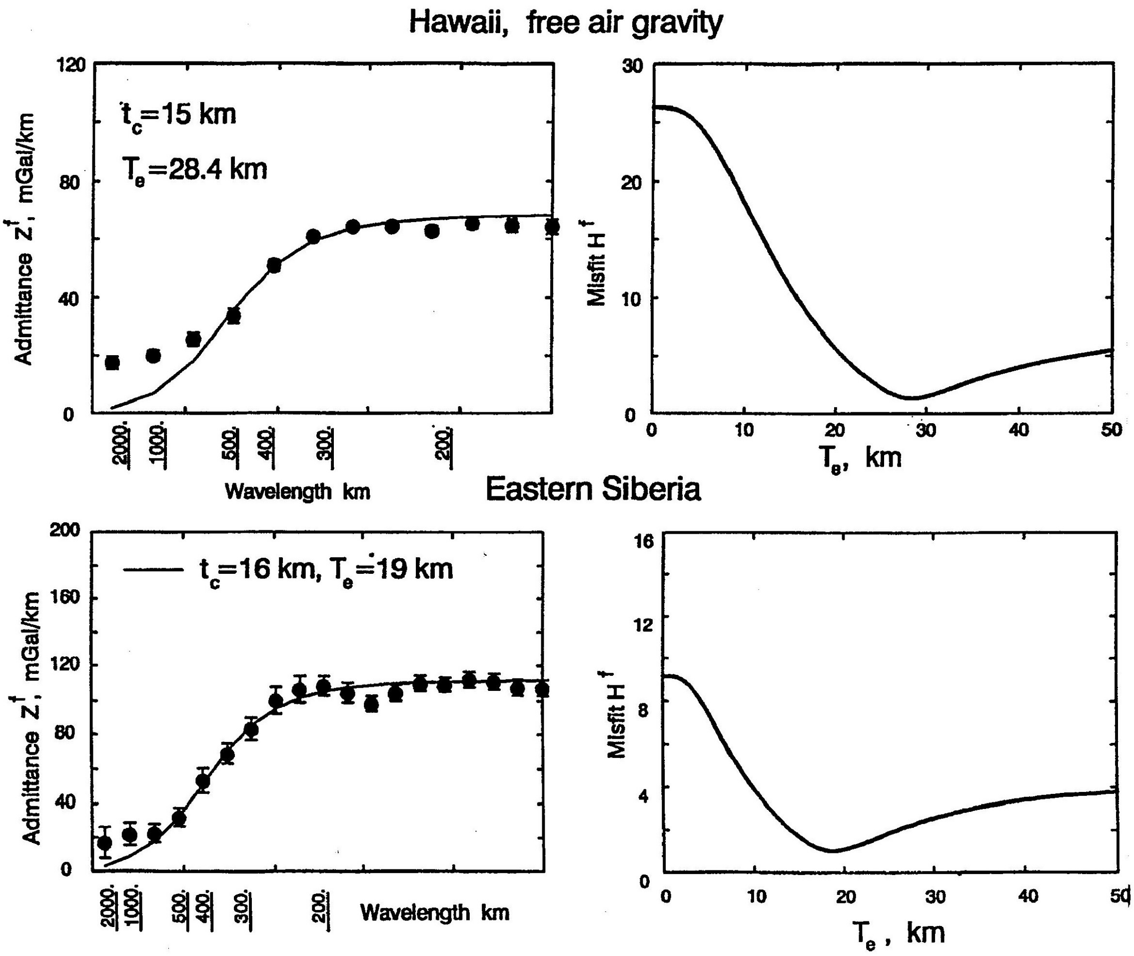

Figure 9: Elastic thickness for Hawaii and Eastern Siberia determined using a better technique.

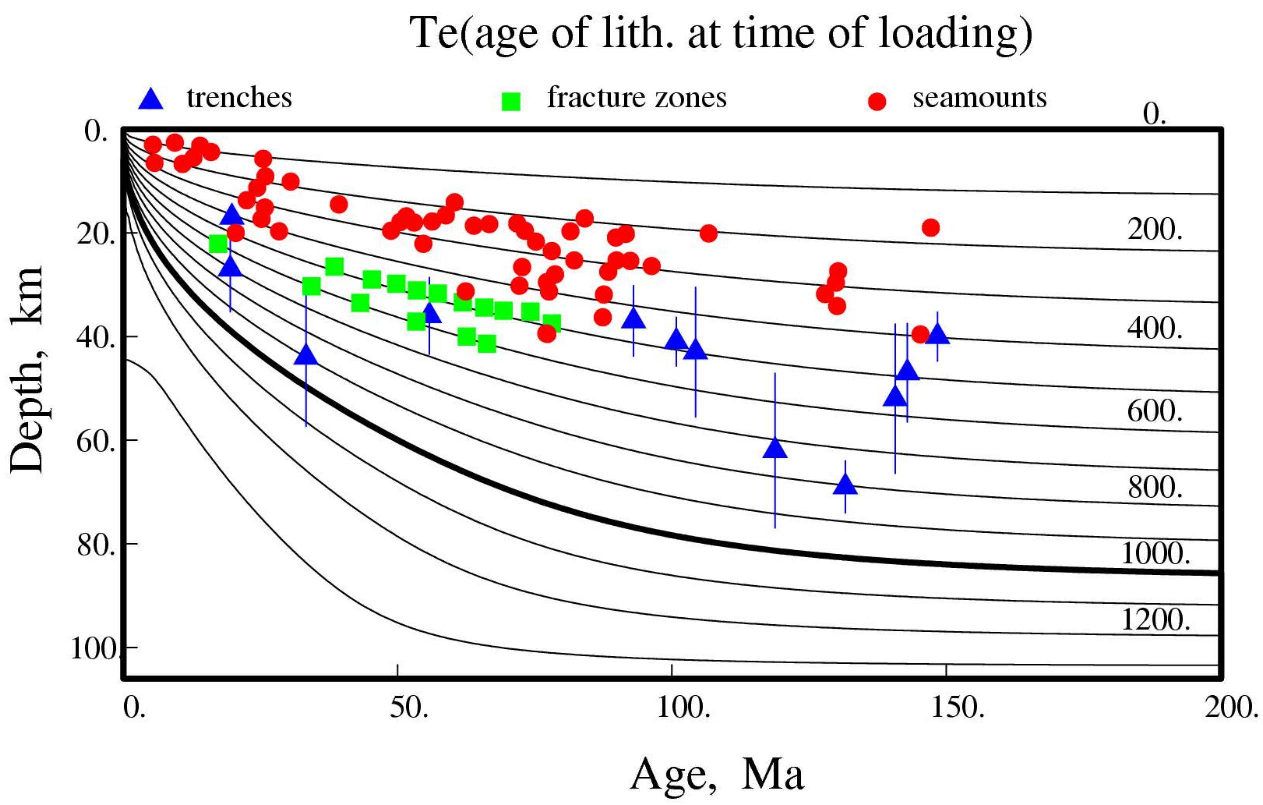

Figure 10: Estimates of T_e from oceanic regions, plotted on the thermal structure of oceanic plates

The situation for continents is much less satisfactory, and is still controversial.

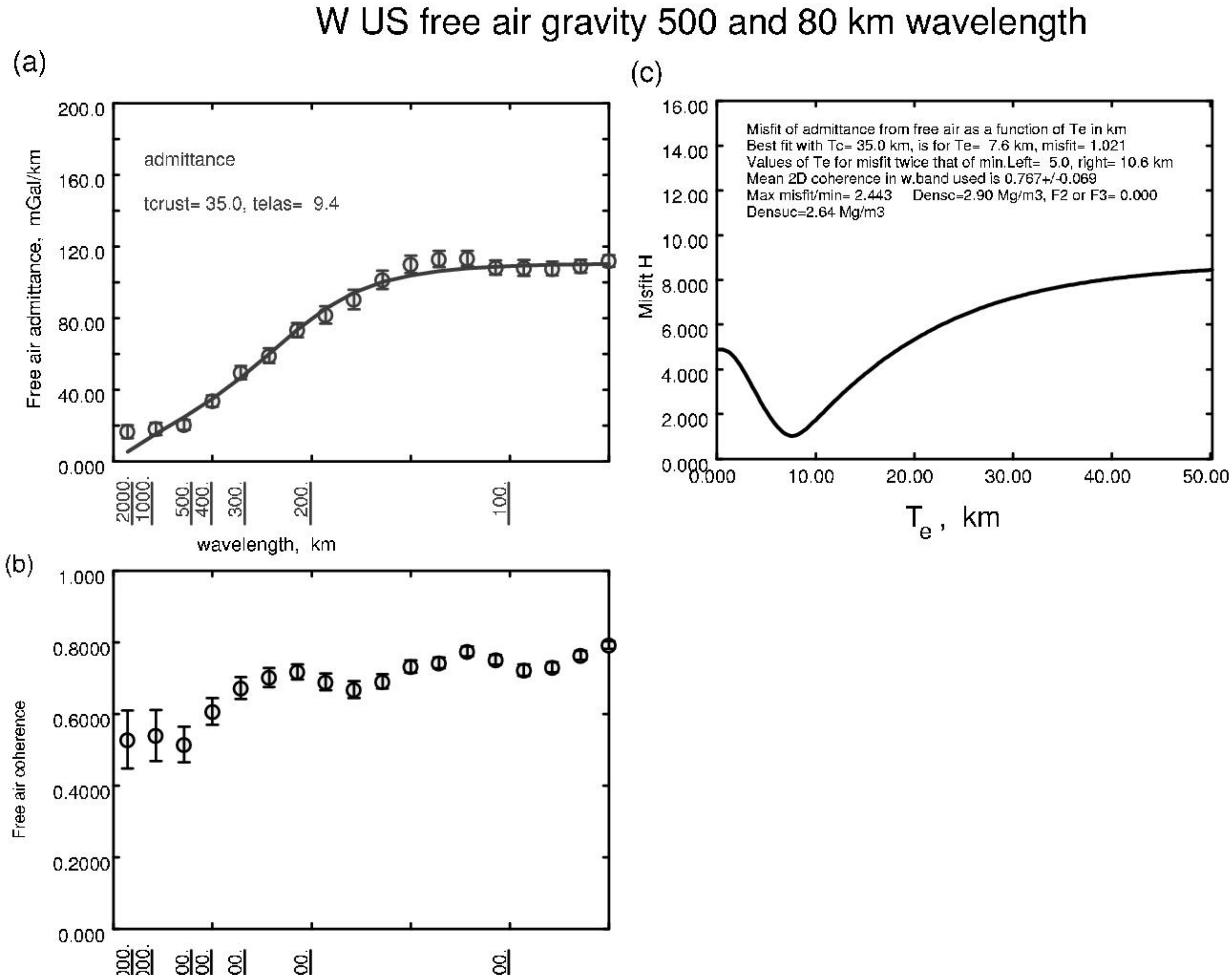

Figure 11: Admittance from the W USA, mostly the Basin and Range

Figure 11 shows the admittance for the western USA, and includes the Rockies and the Basin and Range. The misfit has a minimum with $T_e = 7.6$ km (Figure 11c) and the gravity and topography are coherent (Figure 11b). Regions of active tectonics and rough topography generally give good estimates of T_e which are generally less than 10 km (Figure 13). Where thrust faults bend down surrounding regions to form foreland basins the values of T_e are generally larger (see Figure 13), and are as large as 30–40 km for the oldest parts of the continents.

The problem arises when attempts are made to estimate T_e for regions with little topography in the frequency domain, and it is this that has produced the controversy.

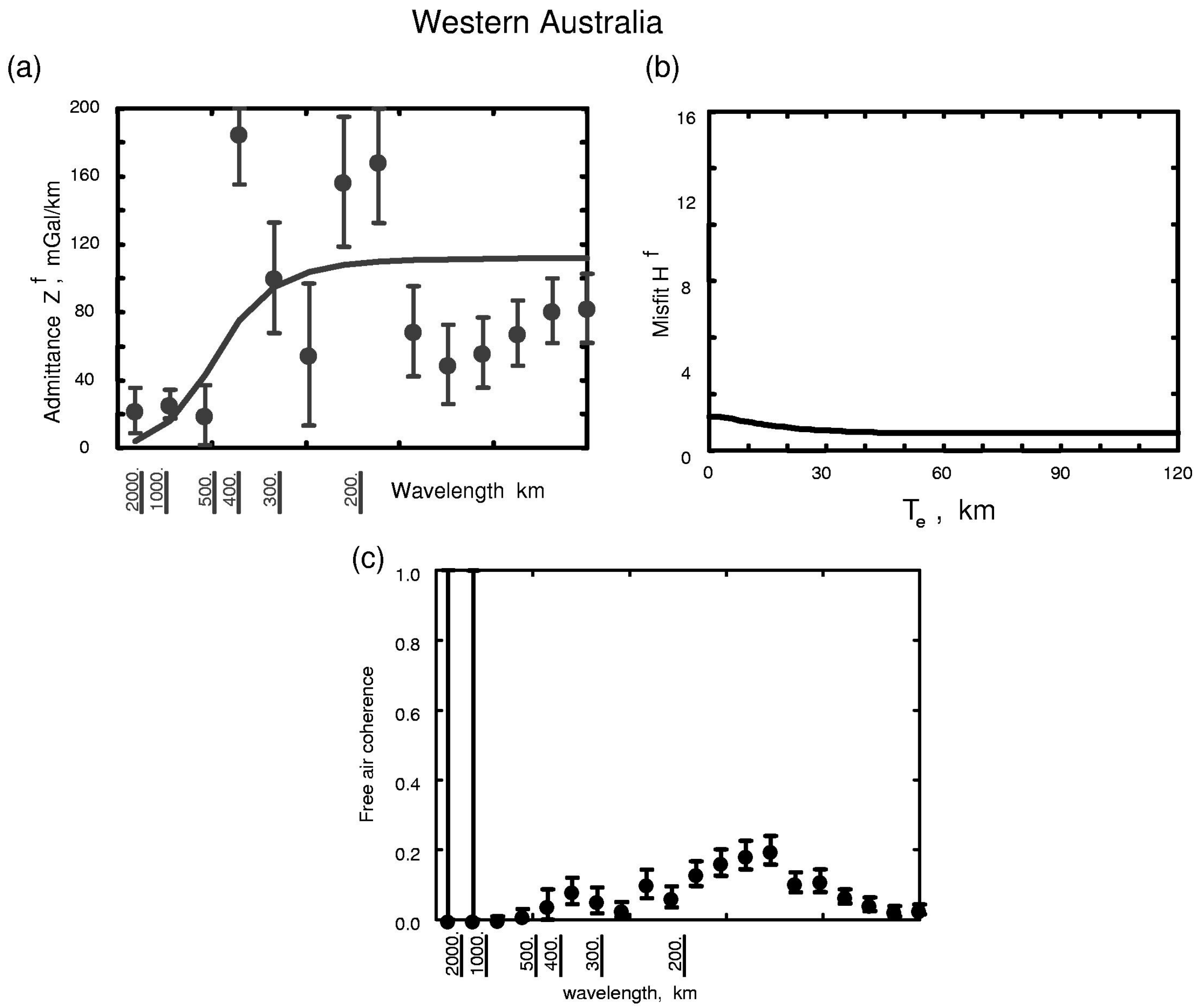

Figure 12: Admittance from western Australia. Because the coherence between the topography and gravity is so small no estimate of T_e can be obtained

Figure 12 shows Z, the coherence and H^f for western Australia. Because the region has not been affected by tectonics for several hundred million years, it has been deeply eroded and is almost flat. The topography therefore produces scarcely any gravity anomalies. Those that are present (and some of them are very large) result from density contrasts within the crust, and have no topographic expression. Since there are no gravity anomalies associated with the topography, which is the load we know about, we cannot use the relationship between the load and its gravity anomaly to estimate T_e. Many similar regions, such as Canada and Fennoscandia, suffer from the same problem. No reliable estimates of T_e can be obtained from the relationship between gravity and topography where there is essentially no topography.

Figure 13 shows estimates of T_e for a number of parts of Eurasia which are reasonably reliable. Most of the values are smaller than that for Hawaii. The isotherm that corresponds to T_e is about 350°C, corresponding to a homologous temperature of about 0.45.

The thickness of the lithosphere determined from seismology and thermal studies is considerably greater than the elastic thickness obtained from flexure studies.

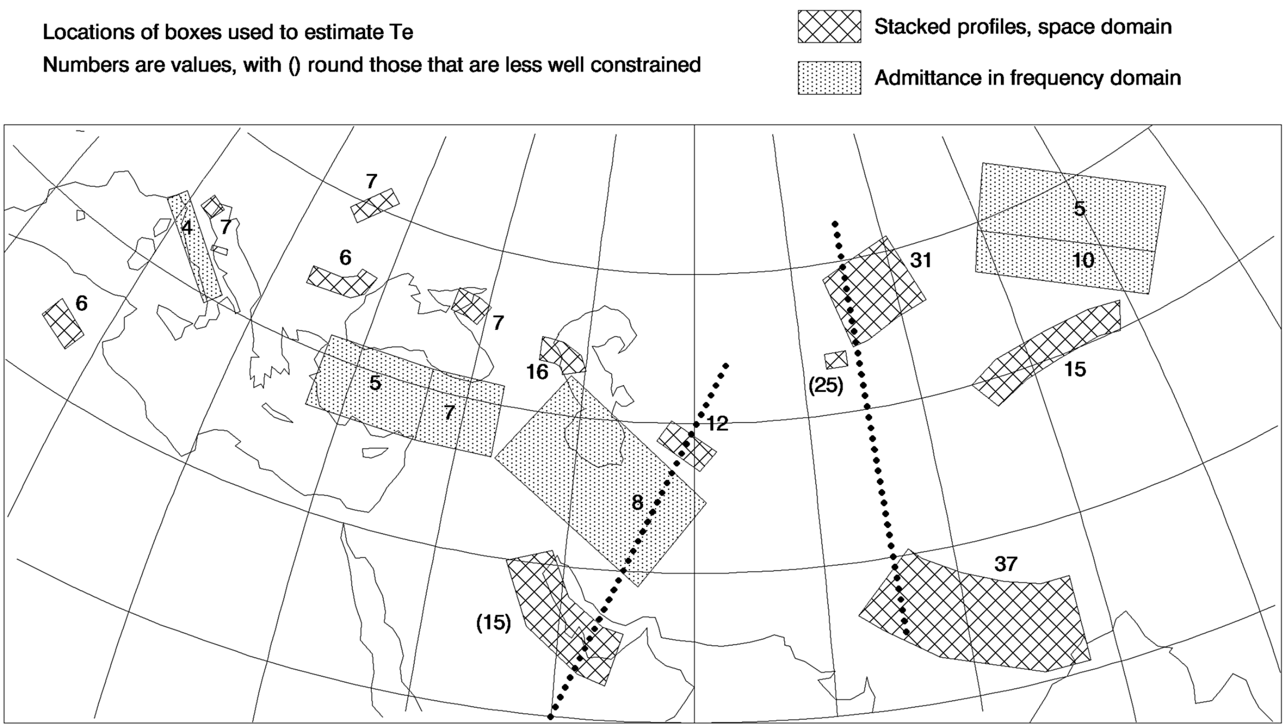

Figure 13: Estimates of T_e from various parts of Eurasia, obtained from stacking profiles or from Fourier transforming gravity and topography