Compact objects and the life cycles of stars

Relativistic Astrophysics and Cosmology: Lecture 7

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Friday 20th October 2023

Pre-lecture question:

Where does gold come from?

Last time

- The Kerr metric
- Black hole thermodynamics

This lecture

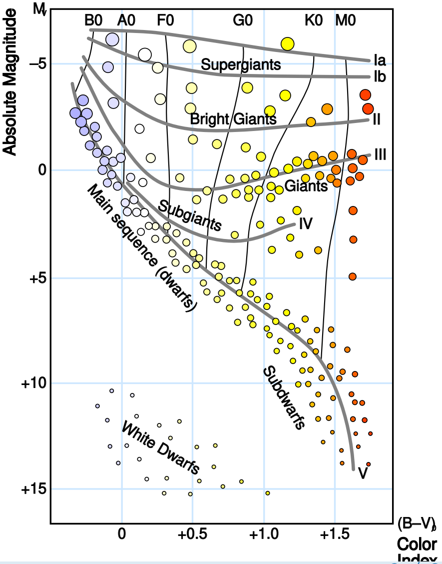
- Stellar evolution
- Limits on stability of stars (Eddington, collapse & degeneracy pressure)
- Chandrasekhar limit and beyond
- Supernovae

Next lecture

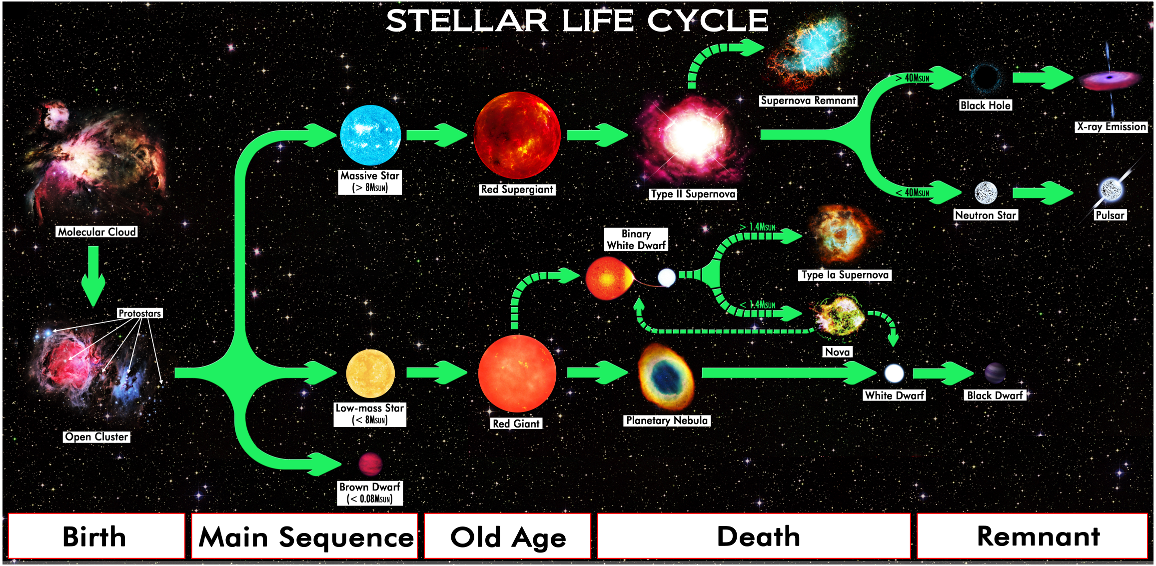
- Principles & physics of electromagnetic astronomy

Stars

- Stars come in a great variety of shapes and sizes [xkcd:2360].
- A unifying definition of "normal stars" is that they are pressure-supported.
- This means that stars are a stable balance between gravity acting inward and a thermal pressure gradient acting outward.
- The Hertzsprung-Russell diagram is a key diagnostic tool for understanding stellar evolution.



Stellar evolution

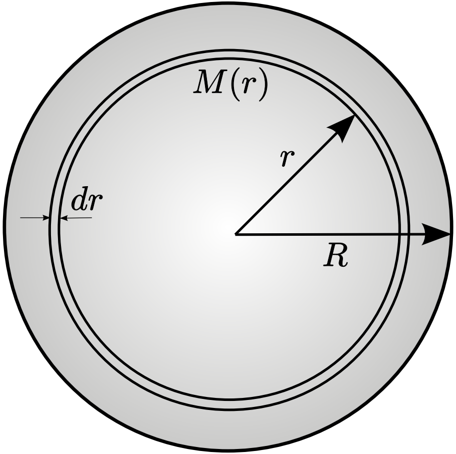


Hydrostatic equilibrium

- We discussed this in the context of the Oppenheimer Volkov equation in Lecture 4 (which gives both the relativistic and Newtonian equations).
- Modelling can be very sophisticated, but in essence it amounts to the equation of hydrostatic support:

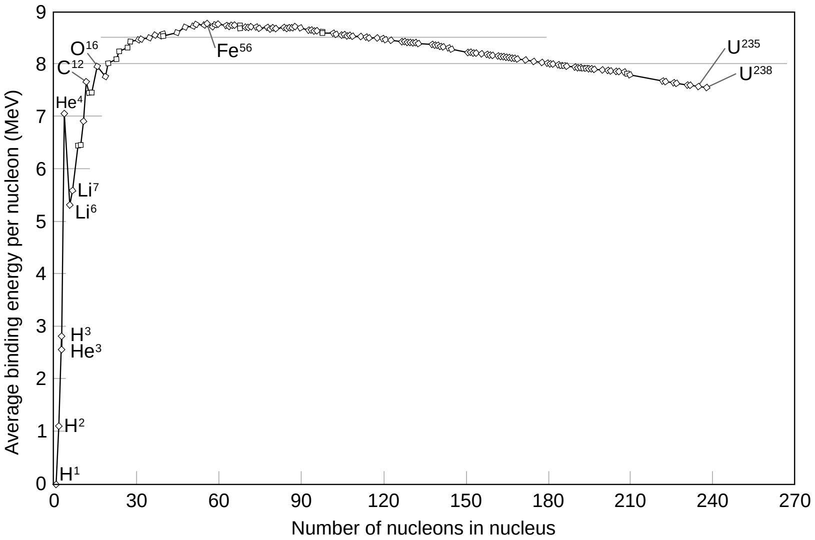
$$\frac{dP}{dr} = -\rho(r)\frac{GM(r)}{r^2}(= -\rho g).$$

- Radiation pressure is provide by nuclear fusion.
- The left-hand side encodes all of the stellar nuclear physics, which provides the outward pressure, and depends on the chemical composition of the star (which changes over its lifetime due to nuclear fusion).



The nuclear stellar forge

- Stars act to fuse lower mass elements like H & He to heavier elements.
- Nuclear binding energy released provides thermal pressure for hydrostatic equilibrium.
- Stellar composition steadily changes.
- Fails when it reaches the peak at Fe, and stars transition from "main sequence" into compact objects or supernovae.



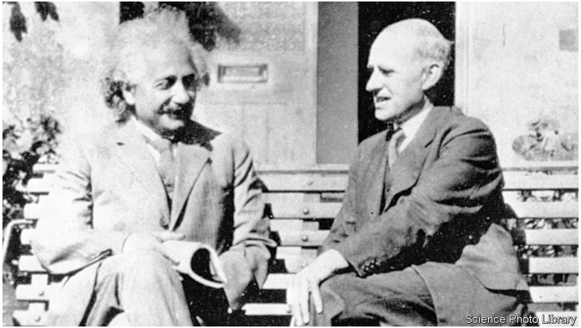
The Eddington limit

- Fundamental limit to the brightness of an object.
- Occurs when the brightness of a star is high enough to separate electrons from protons.
- NB: This applies to any radiating body (e.g. accretion disks).
- Equating the gravitational force on the electron-proton system (i.e. on the proton) with the radiative force (i.e. on the electron).

$$F_{\text{grav}} = \frac{GMm_p}{R^2} \sim \underbrace{\frac{L}{4\pi R^2 h\nu}}_{\text{photon flux}} \underbrace{\sigma_T}_{\substack{\text{Thomson}\\\text{cross}\\\text{section}}} \underbrace{\frac{h\nu}{c}}_{\substack{\text{photon}\\\text{momentum}}} = F_{\text{rad}} \qquad \Rightarrow \qquad \boxed{L_{\text{edd}} \sim \frac{4\pi GMm_p c}{\sigma_T}.}$$

- NB: the relevant mass is the larger proton mass m_p, and the relevant flux absorber is the cross section for electron-photon scattering σ_T.

Einstein & Eddington



- Eddington features heavily in the history of relativity alongside Einstein.
- Photo taken 150m from lecture theatre (90ly/c in time).
- Journalist: “Sir Arthur, it is said that only three people in the world understand relativity!”
  Eddington: “Yes I’ve heard that. I am trying to work out who the third person is...”

The virial theorem (recap)

- This is a critical mathematical theorem of Newtonian mechanics relating to stable systems bound by potential forces.
$$\langle 2T\rangle = -\sum_{i\in\text{system}}\langle F_i\cdot r_i\rangle$$
- Relates the time-averaged kinetic energy T to the time-averaged projected forces
- If the forces result from a potential energy $V \propto r^n$, then this (non-trivially) becomes
$$\langle 2T\rangle = n\langle V\rangle,$$
where V is the total potential energy of the system (defined as zero at infinite separation).
- Importantly therefore for an inverse-square law force with $n=-1$, we have for a stable system:
$$\boxed{2T\sim -V.}$$
- NB: Time-averaged total energy $E = T+V\sim \frac{V}{2}\sim -T$ indicates system is bound (<0).

The virial theorem (check for stars)

- For a star, relevant potential energy is all gravitational,
$$E_{\mathrm{grav}} = -\int_0^R \frac{GM(r)}{r}\rho 4\pi r^2 dr.$$
- from the equation of hydrostatic support ($\frac{dP}{dr} = -\rho\frac{GM(r)}{r^2}$) this gives
$$E_{\mathrm{grav}} = \int_0^R \frac{dP}{dr} 4\pi r^3 dr.$$
- Integration by parts (noting that $P = 0$ when $r = R$) then gives
$$E_{\mathrm{grav}} = -3\int_0^R PdV,$$
where V is volume.
- For a perfect gas, $P = nkT$, and $E_{\mathrm{kin}} = \frac{3}{2}nkT$, so
$$E_{\mathrm{grav}} = -2E_{kin}.$$

Stellar Collapse limit

- In general, stars are self regulating in that increased collapse increases the thermal energy of the system via nuclear burning which acts to decreases collapse.
- We can use the Virial theorem to place a limit on at what point we expect this balance to break down, causing full collapse or supernova instability.
- We should include all forms of energy: i.e. not just thermal $E_T = \frac{3}{2}NkT$, but the total internal $U = C_V T$ where C_V is the heat capacity at constant volume.
- We also have the standard relations $C_P - C_V = Nk$ and $C_P/C_V = \gamma$ for the heat capacity at constant pressure C_P and the heat capacity ratio γ characterising the substance.
- Eliminating these we find the relationship between thermal energy and internal energy

$$E_T = \frac{3}{2}(\gamma - 1)U,$$

as well as the useful relations

$$C_V = \frac{Nk}{\gamma - 1}, \qquad C_P = \gamma\frac{Nk}{\gamma - 1}.$$

- By the virial theorem we can relate the thermal (kinetic) energy with gravitational energy

$$E_T = -E_{\rm grav}/2 \qquad \left(= \frac{3}{2}(\gamma - 1)U \text{ from previous slide}\right).$$

- The total energy of the star is written as

$$E_{\rm Total} = U + E_{\rm grav} = -\frac{E_{\rm grav}}{3(\gamma - 1)} + E_{\rm grav},$$

so

$$\boxed{E_{\rm Total} = E_{\rm grav}\frac{3\gamma - 4}{3\gamma - 3}.}$$

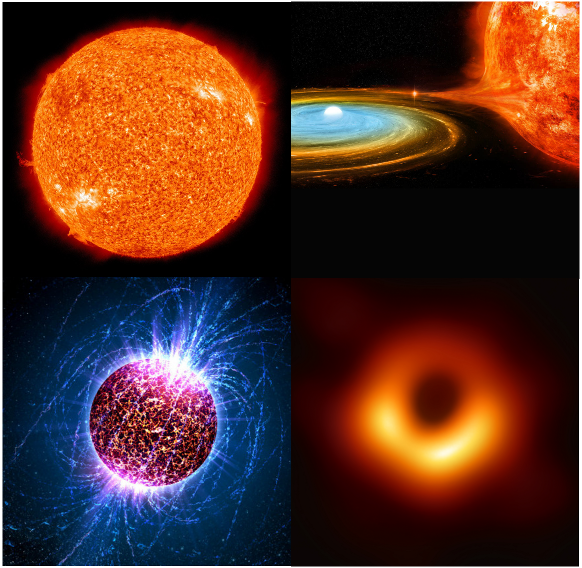
- If $E_{\rm Total}$ is positive, then the star is "unbound" i.e. unstable to collapse or infinite expansion. This happens if

$$\boxed{\gamma < 4/3.}$$

- Since $\gamma = 1 + \frac{2}{f}$ with f the number of degrees of freedom, this requires $f < 6$ (which for a plasma is fine).

Beyond thermal stars: Degeneracy pressure

- The above analysis applies well for stars supported by thermal pressure.
- Stars which have exhausted their nuclear fuel however can support themselves by degeneracy pressure.
- This should be familiar, since it is what stops the earth from collapsing.
- Also prevents collapse of compact objects such as White Dwarfs (WD), Neutron Stars (NS).
- It is the failure of degeneracy pressure support which allows the formation of a black hole (BH).



Non-relativistic degeneracy pressure

- Start from the uncertainty principle $\Delta p \Delta x \sim \hbar$.
- For nonrelativistic particles, we have a Fermi energy per particle of

$$E_{\mathrm{deg}} = \frac{p^2}{2m} \sim \frac{\hbar^2}{2m\Delta x^2},$$

  which shows that compression (decreasing Δx) yields higher energy.
- For a Fermi gas we can relate degeneracy pressure P_{deg} to E_{deg} and the system number density $n = N/V$. If we also take $\Delta x \sim (V/N)^{1/3}$, we find

$$P_{\mathrm{deg}} \sim E_{\mathrm{deg}} \frac{N}{V} \sim \frac{\hbar^2}{2m} n^{5/3}.$$

- We should use the mass of an electron m_e rather than protons or neutrons since $m_p/m_e \sim 2000$.
- This means that the system behaves like an ideal gas with $\gamma = 5/3$ and is therefore stable.

Relativistic degeneracy pressure

- This however changes as matter becomes more relativistic. Recall from special relativity:

$$E = \gamma_v mc^2, \qquad p = \gamma_v mv, \qquad \gamma_v = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \quad \Rightarrow \quad E = mc^2\sqrt{1+\frac{p^2}{m^2c^2}}.$$

  (note the relativistic factor γ_v is different from the heat capacity ratio γ).
- In either limit have

$$E_{\text{non relativistic}} = mc^2 + \frac{p^2}{2m} + \mathcal{O}\left(\frac{p}{mc}\right)^4, \qquad E_{\text{relativistic}} = pc.$$

- For the relativistic limit therefore have

$$P_{\text{deg}} \sim E_{\text{deg}}\frac{N}{V} \sim pc\frac{N}{V} \sim \frac{\hbar c}{\Delta x}n \sim \hbar c n^{4/3}.$$

- This means that the system behaves like an ideal gas with $\gamma = 4/3$, and so is unstable to gravitational collapse. Relativity acts to break hydrostatic balance.

Mass-radius relations for white dwarfs

- Can use degeneracy pressure arguments to derive how mass M might depend on radius R.
- The Newtonian gravitational energy of the star of mass M, radius R is found via:

$$E_{\text{grav}} = \int_0^R -\frac{Gm(r)}{r}\rho(r)4\pi r^2 dr, \quad m(r) = \int_0^r \rho(r)4\pi r^2 dr, \quad M = m(R),$$

and for a constant density, example sheets show $\boxed{E_{\text{grav}} \sim -\frac{GM^2}{R}.}$

- As before, using $V \sim R^3$, $\Delta x^3 \sim V/N$ and $N \sim M/m_p$

$$NE_{\text{deg}} = N\frac{p^2}{2m} \sim N\frac{\hbar^2}{2m_e\Delta x^2} \sim N^{5/3}\frac{\hbar^2}{2m_eR^2} \sim \left(\frac{M}{m_p}\right)^{5/3}\frac{\hbar^2}{2m_eR^2}.$$

- The total energy of the star is composed of $E = NE_{\rm deg} + E_{\rm grav}$, so minimising

$$E = \left(\frac{M}{m_p}\right)^{5/3} \frac{\hbar^2}{2m_e R^2} - \frac{GM^2}{R} \Rightarrow \text{minimise wrt } R \text{ (or } M) \Rightarrow \boxed{M \sim \frac{\hbar^6}{G^3 m_e^3 m_p^5} R^{-3}.}$$

- Note the radius therefore decreases with increasing mass.
- The equivalent calculation for the relativistic case has

$$E_{\rm rel} = \left(\frac{M}{m_p}\right)^{4/3} \frac{\hbar c}{R} - \frac{GM^2}{R}.$$

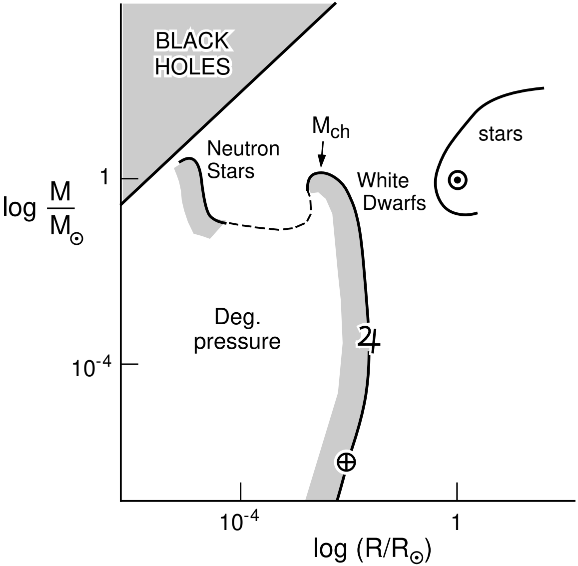
  which as expected has no minimum, and equating the two terms ($E_{\rm rel} = 0$) yields the Chandrasekhar mass – the maximum mass of a white dwarf star:

$$M_{\rm Ch} = \left(\frac{\hbar c}{G}\right)^{3/2} \frac{1}{m_p^2} = \frac{M_{\rm planck}^3}{m_p^2}$$

- A more sophisticated calculation for a reasonable Carbon-Oxygen star yields $M_{\rm Ch} \approx 1.4 M_\odot$ (the above gives $1.85 M_\odot$).

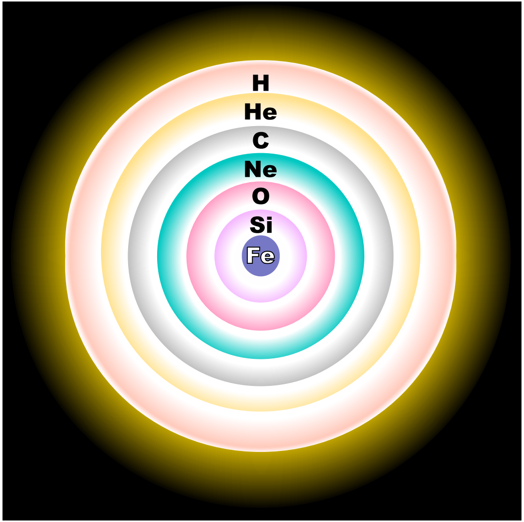
Beyond white dwarfs

- Mass-radius phase diagram (M, R) in units of solar mass $M_\odot$ and solar radius $R_\odot$.
- Sun $\odot$, Earth $\oplus$ & Jupiter ♃ are marked.
- The degeneracy pressure line shows increasing mass with a tip around white dwarfs with $M \sim R^{-3}$, followed by a break to Neutron stars and black holes.
- Neutron stars are planet-sized atomic nuclei.
- From the mass-radius relation $\rightarrow R_{NS} \approx R_{WD} \frac{m_e}{m_p} \approx 10^4 m$.
- More on these in later lectures!



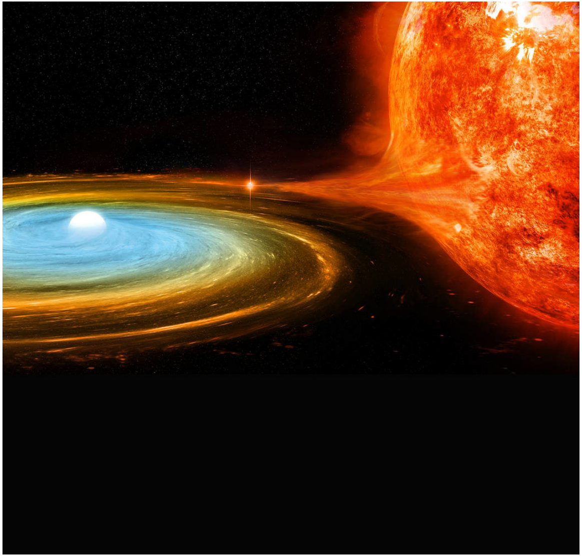
Supernovae (Type II)

- As massive stars $M > M_\odot$ age, cores turn to iron "ash" which cannot burn to provide pressure.
- The star collapses, and a large fraction of energy is released as neutrinos from inverse β-decay ($e^- + p \to n + \nu$).
- When the core reaches nuclear density (i.e. a neutron star), the collapse bounces and the rest of the matter is ejected at high velocity.
- Luminous outburst initially very bright (up to $10^9 L_\odot$) and outshines host galaxy.
- The onslaught of material creates an r-process where neutrons pile into nuclei more rapidly than weak decay, generating heavy elements.
- Final core is a Neutron star or Black Hole.

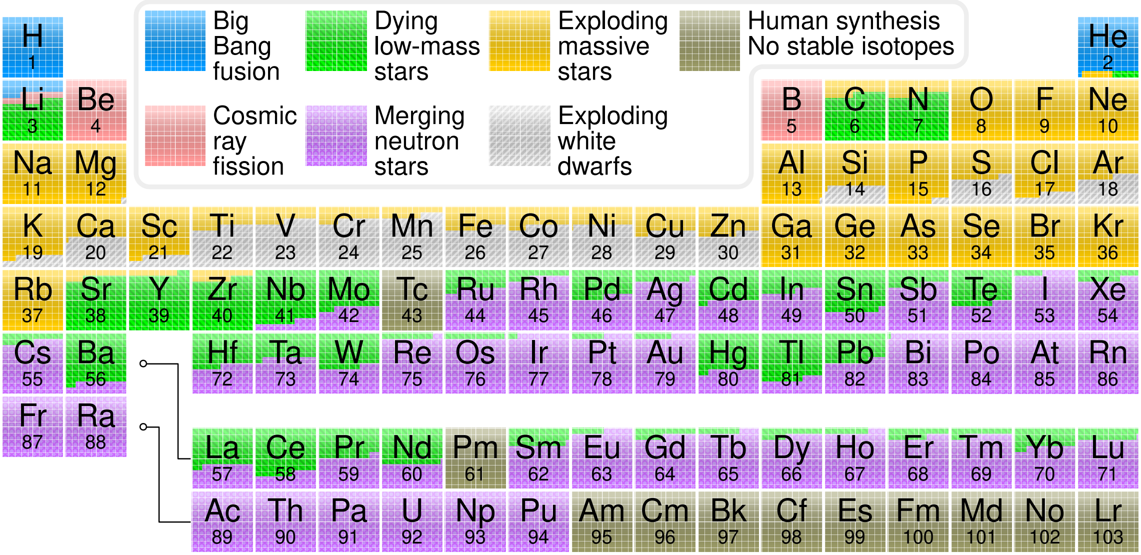


Supernovae (Type Ia)

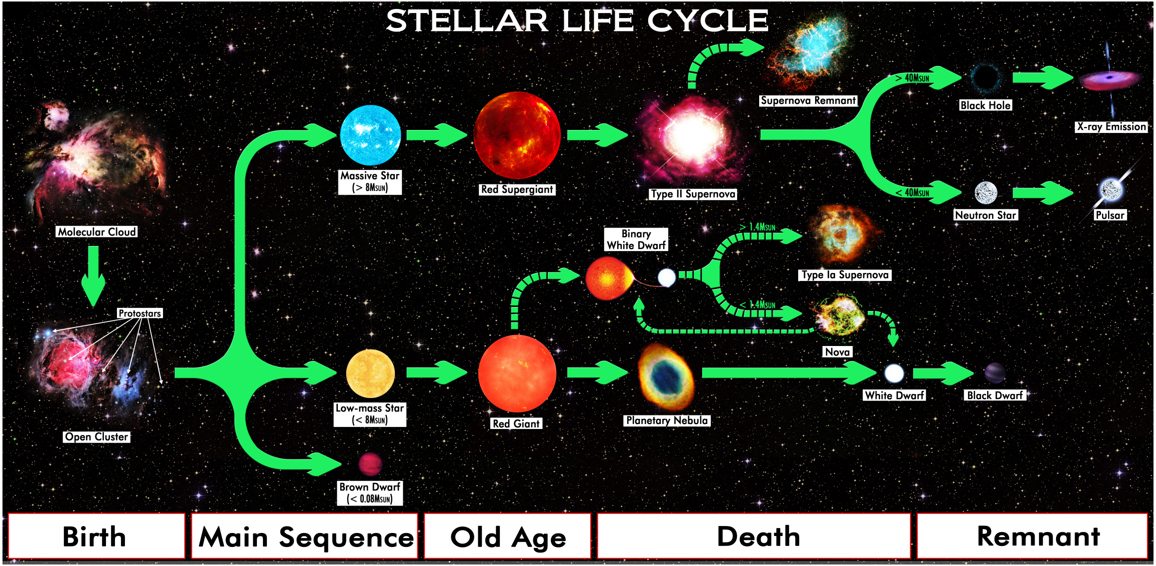
- Occur when a WD accretes so much mass that it exceeds the Chandraskhar mass (e.g. in binary system).
- This process is reliable, meaning these Supernovae form cosmological "Standard candles".
- No stellar remnant is expected after the explosion and the iron formed is expelled.
- Distinguishable from Type II by a lack of hydrogen features in spectrum.
- Usually more optically luminous than core collapse SN (Type II) at $5 \times 10^9\, L_\odot$.
- Types IB & IC are rarer core collapse events.



The origin of the elements



Stellar evolution



Summary

- The Eddington limit

$$\boxed{L_{\rm edd} \sim \frac{4\pi G M m_p c}{\sigma_T}.}$$

- The virial theorem

$$\boxed{2T \sim -V.}$$

- The collapse limit of stars at $\gamma < 4/3$, with consequences for degeneracy pressure with $P \sim n^\gamma$: non-relativistic ($\gamma = 5/3$) and relativistic ($\gamma = 4/3$).
- Compact objects (white dwarfs and neutron stars).
- Type II & Type Ia supernovae.
- Many elements do not come from nuclear fusion or supernovae!

Next time

Principles & physics of electromagnetic astronomy