Rotating Black holes

Relativistic Astrophysics and Cosmology: Lecture 6

Sandro Tacchella

Wednesday 18th October 2023

Pre-lecture question:

Where does the entropy of the highly structured late-time Universe go?

Last time

- Conjugate momenta in general relativity
- Energy of circular orbits
- Stability of orbits in a general spherically symmetric metric

This lecture

- The Kerr Metric
- The structure of a spinning black hole
- The Penrose process
- Black hole thermodynamics

Next lecture

- Compact objects & the life cycles of stars

Angular momentum of collapsing objects

- The material from which a black hole is initially formed is very likely to have possessed angular momentum, and material swallowed over its lifetime will also carry angular momentum — therefore expect that nearly all the black holes in the universe will have spin as well as mass.
- Thus important astrophysically to get to grips with rotating black holes.
- Will see they allow much more efficient release of energy than non-rotating (Schwarzschild) black holes.
- Also rotation is probably vital for the formation of jets.
- Very important theoretically as well. The parameters of mass, angular momentum and black hole area form a thermodynamic statespace in which entropy can be identified with the area, and temperature is inversely proportional to mass
- Won't have time to discuss this in detail, but will discuss a key process which allows one to navigate around this space — the Penrose process

The Kerr Metric

- Schwarzschild derived metric applicable to a spherically symmetric body in 1915.
- Wait of 47 years had to occur before the first exact solution which could be applicable to an azimuthally symmetric body embedded in real space (i.e. tending to flat spacetime in all directions at infinity).
- This metric, applicable to a black hole with angular momentum, was found by Roy Kerr in 1962.
- We look at this in the form developed by Boyer & Lindquist in 1966, which uses coordinates closer to ordinary spherical coordinates.
- Note you won't be expected to remember mathematical details in the exam.

The Kerr Metric (in Boyer Lindquist form)

- A spacetime outside a mass M with angular momentum J has metric

$$ds^2 = \left(1 - \frac{2\mu r}{\rho^2}\right) c^2 dt^2 + \frac{4\mu r a \sin^2\theta}{\rho^2} c dt d\phi - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - \left(r^2 + a^2 + \frac{2\mu r a^2 \sin^2\theta}{\rho^2}\right) \sin^2\theta d\phi^2$$

where

$$\boxed{\mu = \frac{GM}{c^2}, \quad a = \frac{J}{Mc}, \quad \Delta = r^2 - 2\mu r + a^2, \quad \rho^2 = r^2 + a^2\cos^2\theta.}$$

- μ has the usual definition.
- a is Kerr parameter controlling spin, proportional to angular momentum per unit mass.
- Δ is known as the discriminant, due to mathematical origins from quadratic forms.
- ρ is length parameter defined for convenience.

A vacuum solution

- Calculating the Christoffel connection coefficients from this, then working out Riemann and Ricci tensors from it is a formidable job, and not how it was discovered!
- However, if you did do this, you would find $R_{\mu\nu} = 0$ i.e. it is indeed a vacuum solution of the Einstein field equations.
- We'll take this on trust. What we need for our purposes, comes mainly from understanding particle motion in this metric, i.e. being able to form the geodesic equations, and these are not too bad.
- With a little work this can be re-written as

$$ds^2 = \frac{\Delta}{\rho^2}(cdt - a\sin^2\theta d\phi)^2 - \frac{\sin^2\theta}{\rho^2}\left((r^2 + a^2)d\phi - acdt\right)^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2,$$

  which is closer to the form it was derived in, and more suggestive of a spinning system.
- NB: this is usually written in natural units, so more common to see μ replaced by M.

Initial remarks

- Can see that if $a \to 0$ it goes over into Schwarzschild metric.
- Other thing worth noting immediately is that none of the metric coefficients depend explicitly on t or ϕ.
- Therefore will expect moving particles to have a conserved energy and angular momentum.
- However, solution not static — the second term is sensitive to sign of dt, and therefore not time reverse symmetric/not irrotational.
- Such a solution called stationary (i.e. spinning does not change with time).
- However, if flip sign of a as well, then restores symmetry — clearly this corresponds to reversing direction of time and spin.

- Note that r, θ, ϕ are plane-polar only in the large $r \gg a, \mu$ limit.
- Consider spatial part of $\mu = 0$ line element:

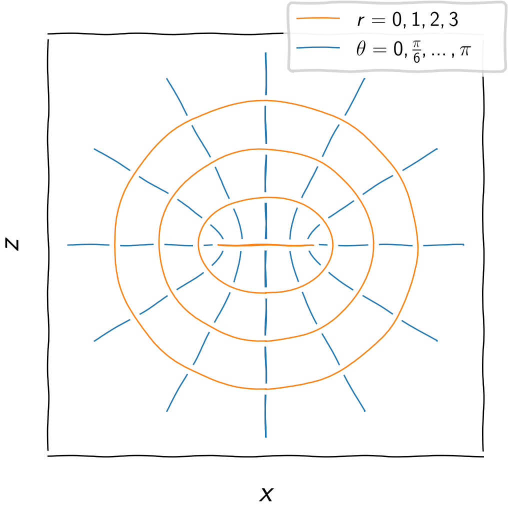
$$\frac{\rho^2}{r^2 + a^2} dr^2 + \rho^2 d\theta^2 + (r^2 + a^2) \sin^2\theta d\phi^2.$$

- Can get back to Cartesians with:

$$z = r\cos\theta, \qquad x = \sqrt{r^2 + a^2}\sin\theta\cos\phi$$

$$y = \sqrt{r^2 + a^2}\sin\theta\sin\phi,$$

- In empty space $r =$ constant are oblate ellipsoids: $\frac{x^2+y^2}{r^2+a^2} + \frac{z^2}{r^2} = 1$.
- This is complicated by matter $\mu \neq 0$.
- With time, the coordinates are also rotating.



Charged black holes

- If one replaces $\Delta \to \Delta + \frac{Q^2 G}{4\pi\epsilon_0 c^4}$ recover Kerr-Newman metric for charged spinning object.

	Non-rotating $J = 0$	Rotating $J \neq 0$
Uncharged $Q = 0$	Schwarzschild	Kerr
Charged $Q \neq 0$	Reissner-Nordström	Kerr–Newman

- Brandon Carter showed in 1971 that this metric (or coordinate transforms from it) is the unique azimuthally symmetric metric, tending to flat space at infinity, and possessing an event horizon (i.e. it's a rotating black hole).
- There's another family of solutions if the BH has charge, called the Kerr-Newman solutions.

Event horizon(s)!

- How do we identify where the event horizon is? This is defined as a null surface, corresponding to light cone being tipped over so that it lies in the surface.
- The Boyer-Lindquist coordinates are set up so that this surface is a function only of r — say it's defined by some function $F(r) = 0$, then can see that since a null vector is perpendicular to itself, the gradient of F should be null

$$g^{\mu\nu}\left(\partial_\mu F\right)\left(\partial_\nu F\right) = 0, \quad \text{i.e.} \quad g^{rr}\left(\partial_r F\right)^2 = 0,$$

- i.e. the horizon is where $\boxed{g^{rr} = 0}$. Thus need to look at contravariant components of the metric — easy to do for the (r, θ) sector since this is decoupled from (t, ϕ) sector and is diagonal.
- Hence

$$g^{rr} = -\frac{\Delta}{\rho^2} = 0 \quad \text{occurs where} \quad \boxed{\Delta = r^2 - 2\mu r + a^2 = 0.}$$

which has solutions $r = \mu \pm \sqrt{\mu^2 - a^2}$.

Stationary limit surfaces

- This is only one of the two important surfaces for the Kerr solution, however.
- Consider a photon initially sent off in one of the $\pm\phi$ directions – no r or θ motion initially.
- We know for this that (again initially)

$$ds^2 = 0 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2.$$

- Dividing by dt^2 we get

$$g_{\phi\phi}\left(\frac{d\phi}{dt}\right)^2 + 2g_{t\phi}\frac{d\phi}{dt} + g_{tt} = 0.$$

- Solving the quadratic we get

$$\frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}}.$$

- Now from the metric above, know

$$g_{t\phi} = \frac{2c\mu ra\sin^2\theta}{\rho^2}, \qquad g_{\phi\phi} = -\left(r^2 + a^2 + \frac{2\mu ra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta,$$

- $g_{t\phi}$ is positive (assuming a is) and $g_{\phi\phi}$ is always negative.
- We see there's a limiting case if $g_{tt} = 0$. Then the two solutions are

$$\frac{d\phi}{dt} = -2\frac{g_{t\phi}}{g_{\phi\phi}} \quad \text{and the second is} \quad \frac{d\phi}{dt} = 0.$$

- First corresponds to a photon emitted in same direction as black hole rotation, second to one trying to go against the black hole rotation, but failing.

$$\text{Can see that if } g_{tt} < 0, \text{ then} \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}} \quad \text{is always less than} \left|\frac{g_{t\phi}}{g_{\phi\phi}}\right|$$

  and in this case then both solutions for $d\phi/dt$ are > 0.
- So photon will get dragged along in same direction as black hole spin no matter what.
- This is called frame dragging, and we have shown that there is a *stationary limit surface* at $g_{tt} = 0$, i.e. where

$$\rho^2 = r^2 + a^2\cos^2\theta = 2\mu r, \quad \text{i.e. where} \quad r = \mu \pm \sqrt{\mu^2 - a^2\cos^2\theta}.$$

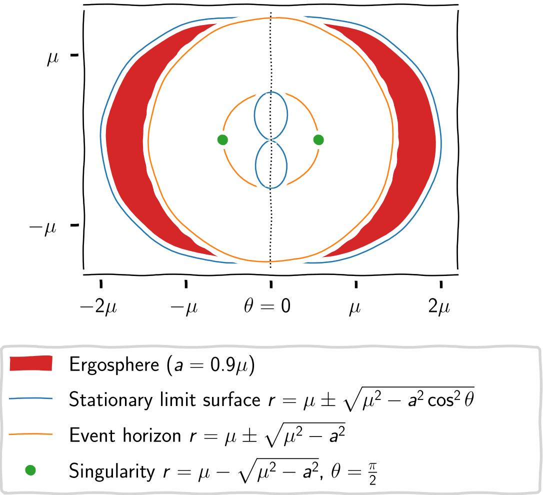
- Inside this surface, nothing can resist being swept round in same direction as the hole spin.

Singularities

- Kerr metric in Boyer-Lindquist form is singular when $\rho = 0$ or $\Delta = 0$.
- Physical singularities are defined by the divergence of the curvature i.e. the Ricci scalar
- Not shown here, but the $R \to \infty$ is equivalent to $\rho = 0$.
- Since
$$\rho^2 = r^2 + a^2 \cos^2\theta$$
- follows that $\rho = 0$ occurs when
$$r = 0, \qquad \theta = \frac{\pi}{2}.$$
- Remember that $r =$ const are oblate spheroids, not spheres.
- This is a ring singularity, radius a in equatorial plane $\theta = \frac{\pi}{2}$
- Note that like the Schwarzschild radius, most of the event horizon surfaces $\Delta = r^2 - 2\mu r + a^2 = 0$ are merely coordinate singularities.

Structure of a Kerr Black hole

- The region between the two (outer) surfaces is called the ergosphere.
- Possible to extract energy and angular momentum from the black hole by injecting particles into this region and using something called the Penrose process (will discuss shortly).
- Note diagram only shows the outer surface in each case — for $a = \mu$ the inner event horizon is the same as the outer one, hence $a = \mu$ is a limit, and beyond this we believe there is no horizon and we have a naked singularity.



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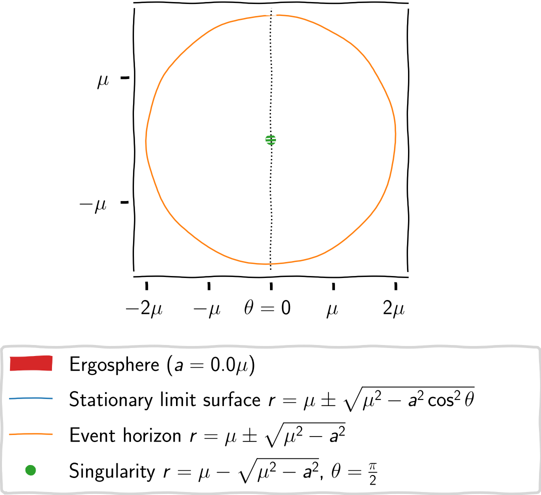

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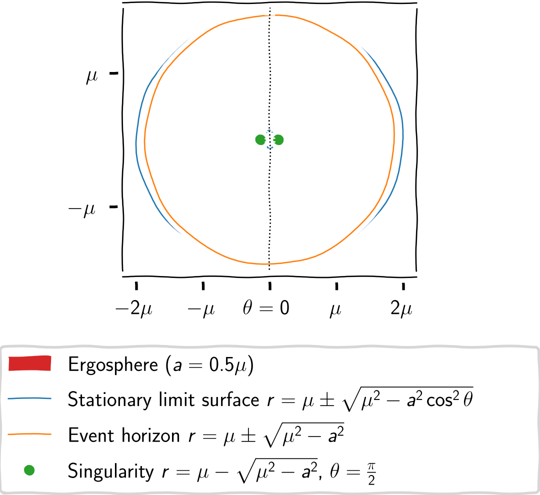

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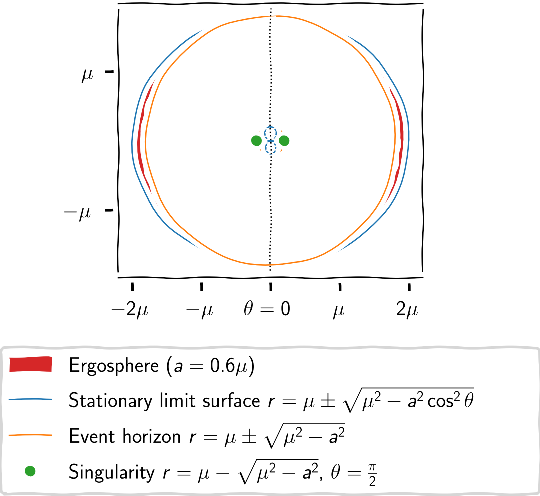

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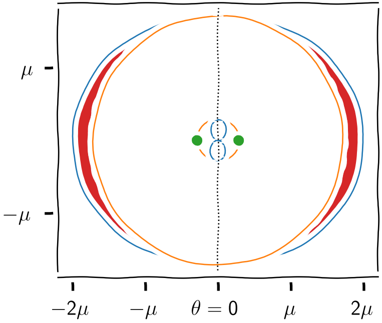

Ergosphere ($a = 0.7\mu$)
Stationary limit surface $r = \mu \pm \sqrt{\mu^2 - a^2 \cos^2\theta}$
Event horizon $r = \mu \pm \sqrt{\mu^2 - a^2}$
Singularity $r = \mu - \sqrt{\mu^2 - a^2}$, $\theta = \frac{\pi}{2}$

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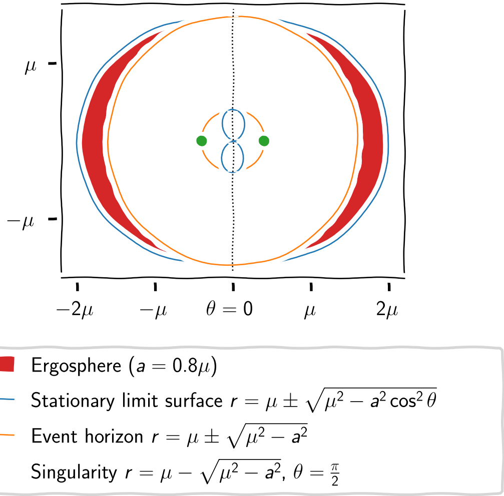

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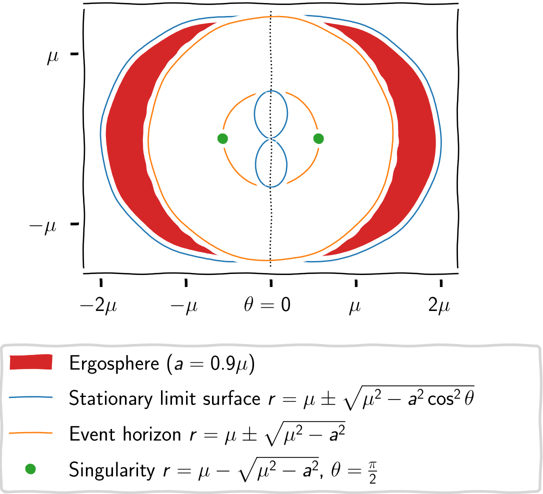

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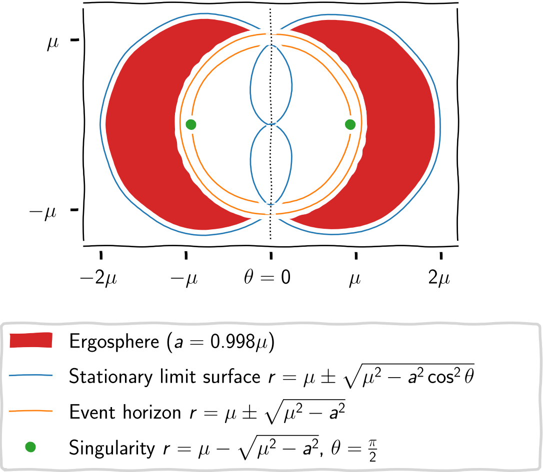


Particle motion around a Kerr Black hole

- For simplicity, will work in equatorial plane ($\theta = \pi/2$), no longer without loss of generality.
- Use same method as for Schwarzschild, i.e. get equations for $\dot{t}$ and $\dot{\phi}$ from geodesic equation (which will be simple for them since no explicit dependence), and use $\mathcal{L} = u \cdot u = c^2$ (or $\mathcal{L} = 0$ for photons) to get $\dot{r}$ equation.
- Then in the latter we eliminate $\dot{t}$ and $\dot{\phi}$ via the corresponding conserved quantities k (particle energy) and h particle angular momentum (both defined per unit mass).
- Comparing with the Kerr metric above, with $\theta = \pi/2$, which also means $\rho = r$, get a relatively simple expression for the 'geodesic quantity':

$$\mathcal{L} = \left(1 - \frac{2\mu}{r}\right) c^2 \dot{t}^2 + \frac{4\mu a}{r} c \dot{t} \dot{\phi} - \left(1 - \frac{2\mu}{r} + \frac{a^2}{r^2}\right)^{-1} \dot{r}^2 - \left(r^2 + a^2 + \frac{2\mu a^2}{r}\right) \dot{\phi}^2.$$

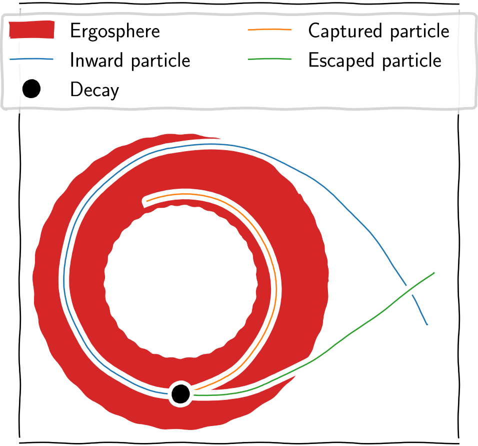
- Thus the differentiations to get $\partial\mathcal{L}/\partial\dot{t}$ and $\partial\mathcal{L}/\partial\dot{\phi}$ are (relatively) simple:

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\dot{t}} &= 2\left(1-\frac{2\mu}{r}\right)c^2\dot{t} + \frac{4\mu a}{r}c\dot{\phi} &&= 2c^2k, \\
\frac{\partial\mathcal{L}}{\partial\dot{\phi}} &= \frac{4\mu a}{r}c\dot{t} - 2\left(r^2+a^2+\frac{2\mu a^2}{r}\right)\dot{\phi} &&= -2h.
\end{aligned}
$$

- Can then invert these to get $\dot{t}$ and $\dot{\phi}$ in terms of k and h and then substitute in $\mathcal{L} = c^2$ (or $\mathcal{L} = 0$ for photons) to get $\dot{r}$ equation. Find

$$
\begin{aligned}
c\dot{t} &= \frac{kc\left((r+2\mu)a^2+r^3\right) - 2ha\mu}{r\Delta}, \\
\dot{\phi} &= \frac{h\left(r-2\mu\right) + 2kca\mu}{r\Delta}, \\
\dot{r}^2 &= c^2(k^2-1) + \frac{2\mu c^2}{r} - \frac{1}{r^2}\left(h^2 - a^2c^2(k^2-1)\right) + \frac{2\mu}{r^3}(h-ack)^2.
\end{aligned}
$$

- So now have first order equations which we can integrate for each coordinate, to find particle path.



- If working outside equatorial plane, turns out integration of simple first order equations is still possible, due to existence of another constant of the motion, known as the Carter constant.
- Not too much more complicated than Schwarzschild case, but can lead to really rich phenomena!
- A massive particle (blue) falls in from infinity and penetrates the ergosphere (red) where it decays (with conservation of energy and momentum) into two particles.
- One of these (orange) falls into the horizon, the other escapes to infinity (green).

- We see that particles can escape from the ergosphere — it is only from beyond the horizon they can never escape.
- However, $g_{tt} < 0$ inside ergosphere (its definition), means that particle and photon momenta can have the curious property of having negative energy.
- Basically, (in notation of Hobson, Efstathiou & Lasenby) the time basis vector e_t whose squared length is g_{tt} becomes spacelike inside the ergosphere, and this allows projection of 4-momentum onto this, $p_t = p \cdot e_t$ (which is conserved since metric is independent of t) to be negative as well as positive.
- Suppose the photon which is captured by the hole does indeed have negative energy $E_1 < 0$.
- Then since energy conserved in the disintegration, i.e. $E = E_1 + E_2$, the escaping photon must have energy E_2 greater than that of the incoming particle!
- If analyse angular momentum in same way (note again conserved during disintegration), find same thing — in circumstances just described, escaping photon will have greater angular momentum than incoming particle (see e.g. HEL, Section 13.9).
- Where has this and the extra energy come from? It's because the black hole has absorbed a photon with negative energy and angular momentum!
- Thus we have extracted rotational energy from the black hole!

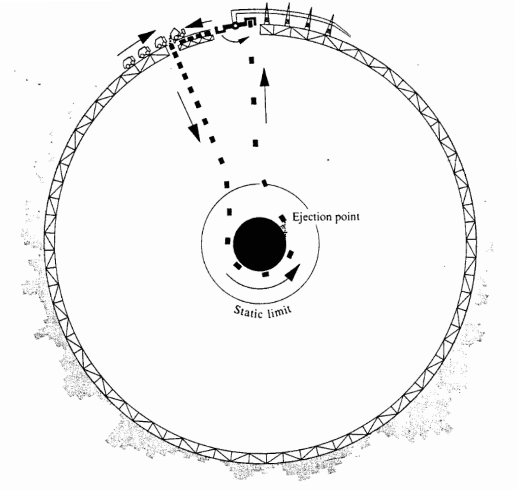


Figure 33.2.
An advanced civilization has constructed a rigid framework around a black hole, and has built a huge city on that framework. Each day trucks carry one million tons of garbage out of the city to the garbage dump. At the dump the garbage is shoveled into shuttle vehicles which are then, one after another, dropped toward the center of the black hole. Dragging of inertial frames whips each shuttle vehicle into a circling, inward-spiraling orbit near the horizon. When it reaches a certain "ejection point," the vehicle ejects its load of garbage into an orbit of negative energy-at-infinity, $E_{\text{garbage}} < 0$. As the garbage flies down the hole, changing the hole's total mass-energy by $\Delta M = E_{\text{garbage ejected}} < 0$, the shuttle vehicle recoils from the ejection and goes flying back out with more energy-at-infinity than it took down

- Figure from Misner, Thorne & Wheeler (Chap 33).
- Shows a fanciful way of making use of this process! (called the Penrose process).
- More realistically, an electromagnetic version of this process is possible when black hole is sitting in an ambient magnetic field.
- Called the Blandford-Znajek process and can harness the rotational energy of a Kerr back hole to drive a 'wind' of material up the rotation axis.
- May be important in powering jets (see later).

Circular orbits

- Let's return to properties of particle motion and consider circular orbits (as for Schwarzschild black holes, these are likely to be approximately the orbits of particles in the accretion disc).
- This time demanding $\dot{r} = \ddot{r} = 0$ leads to the following relations (for prograde orbits).

$$k = \frac{1 - \frac{2\mu}{r} + a\sqrt{\frac{\mu}{r^3}}}{\sqrt{1 - \frac{3\mu}{r} + 2a\sqrt{\frac{\mu}{r^3}}}}, \quad h = c\frac{\sqrt{\mu r} - \frac{2a\mu}{r} + a^2\sqrt{\frac{\mu}{r^3}}}{\sqrt{1 - \frac{3\mu}{r} + 2a\sqrt{\frac{\mu}{r^3}}}}. \tag{1}$$

- We can see how these generalise the corresponding relations in the Schwarzschild case (remember $E = kmc^2$).
- A key question here, as for the Schwarzschild case, is the one of stability. To what radius are the circular orbits of particles still stable?

- We can answer this is the same way as mentioned before from Handout 5, by defining an 'effective potential' via

$$\frac{1}{2}\dot{r}^2 + V_{\mathrm{Kerr}}(r) = \mathrm{const.}$$

and then using (1) this results in

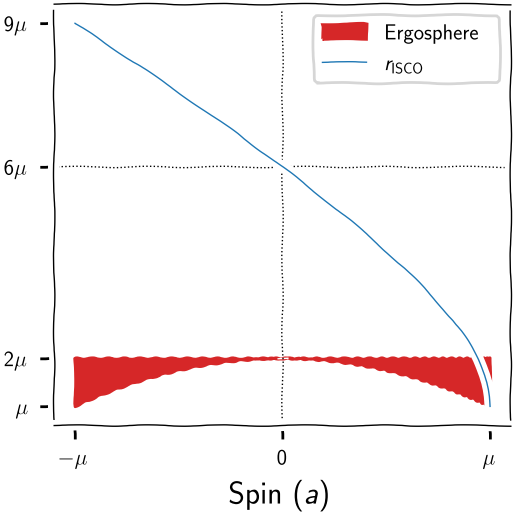
$$V_{\mathrm{Kerr}}(r) = -\frac{\mu c^2}{r} + \frac{1}{2r^2}\left(h^2 - a^2c^2(k^2-1)\right) - \frac{\mu}{r^3}(h - ack)^2.$$

- This is not too much more complicated than in the Schwarzschild case.
- Can proceed by asking that its second derivative is positive where the first derivative vanishes.
- Leads to the relatively simple criterion for stability

$$r^2 - 6\mu r + 8a\sqrt{\mu r} - 3a^2 > 0. \tag{2}$$

- Can use this to generate the following plot of r_{ISCO} (the r corresponding to the innermost stable circular orbit) versus spin.

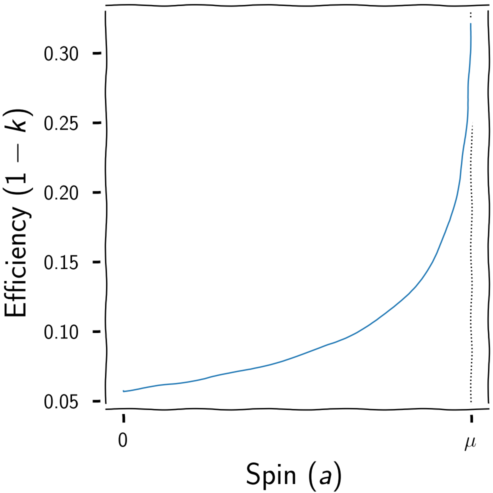
ISCO radius for spinning black holes



- If $a = 0$, we get the single solution $r = 6\mu$ corresponding to the Schwarzschild case.
- If $a = \mu$, an 'extremal' black hole, then the solutions are $r = \mu$ and $r = 9\mu$ (for prograde and retrograde orbits respectively — note have displayed in terms of a and a negative is same as a retrograde orbit about positive a!).
- This shows that stable circular orbits (in the prograde direction) persist right up to the event horizon for a black hole spinning at the maximum mathematically allowed rate, i.e. $a = \mu$.

- However, various effects are likely to intervene before this maximum spin rate can be achieved (mainly the counteracting torque felt by the hole in absorbing radiation from the accretion disc — first discussed by Kip Thorne in 1974).
- In fact $a = 0.998\mu$ is considered the most likely maximum attainable value, leading to a minimum attainable r_{ISCO} of 1.24μ — note this is well inside the ergosphere, which is at 2μ (for any a) in the equatorial plane.
- By eliminating r between (2) and the expression for k in (1) can get a useful relation between the black hole spin and the efficiency of energy release for a particle which has reached the ISCO.

Efficiency of spinning black holes



- For prograde orbits find

$$\frac{a}{\mu} = \frac{2\left(2\sqrt{2}\sqrt{1-k^2}-k\right)}{3\sqrt{3}(1-k^2)}$$

- Again taking the maximum attainable value of a as 0.998μ, this leads to a maximum attainable efficiency of 32%.
- Note also the non-spinning case of $a = 0$ with efficiency $1 - \frac{2\sqrt{2}}{3} \approx 6\%$.

Theoretical importance

- So can see from all this that the properties of rotating black holes are very important astrophysically.
- They are also very important theoretically. Just want to state a few things on this — no details!
- Hawking radiation from the horizon occurs for Kerr black holes just as for Schwarzschild ones.
- The temperature of the emitted black body radiation, for all black holes, in fact for all cases where there is a horizon, is given by Unruh's formula

$$T = \frac{\hbar g}{2\pi k_{\mathrm{B}} c},$$

where g is the 'surface gravity', i.e. 'force per unit mass', at the horizon.

- Need some technicalities for how to define this (basically is what an observer at infinity would say is force required) but for Schwarzschild case g works out to

$$\frac{GM}{r^2} \quad \text{(not surprising!) evaluated at} \quad r = \frac{2GM}{c^2} \quad \text{i.e.} \quad g = \frac{c^4}{4GM},$$

which immediately gives

$$T_{\mathrm{Hawking}} = \frac{\hbar c^3}{8\pi k_{\mathrm{B}} GM}.$$

- The black hole radiates black body radiation at this temperature, though all effects are very tiny unless one gets to very small mass black holes $\sim 10^{12}$kg — might these exist?

Black hole entropy

- The fact that black holes radiate like black bodies, suggest that they should have an entropy. If we identify Mc^2 as the hole's energy, then the thermodynamic relation

$$dU = TdS.$$

along with our identification of the BH temperature, yields

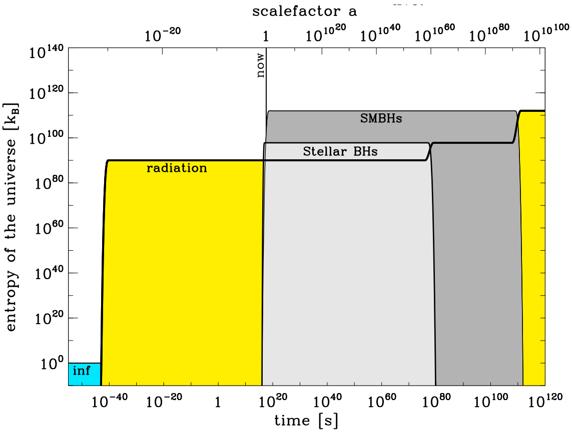
$$dM = \frac{\hbar c^3}{Gk_\mathrm{B}} \frac{1}{8\pi M} dS, \quad \text{so that} \quad S = \frac{Gk_\mathrm{B}}{\hbar c^3} 4\pi M^2 = \frac{1}{4} k_\mathrm{B} \frac{\mathcal{A}}{\ell_p^2},$$

where

- $\mathcal{A} = 4\pi R_S^2 = 4\pi(2GM/c^2)^2$ is the black hole area, and
- $\ell_p = \sqrt{\hbar G/c^3}$ is the Planck length.

The entropy budget of the Universe

- For an astrophysical black hole, this entropy is large, and as we have seen scales like M^2.
- The result of this is that it is thought that supermassive black holes (SMBH) in the centres of galaxies dominate the entropy budget of the observable universe.
- In fact SMBH may contribute up to 19 orders of magnitude more entropy than stellar mass BHs, and approximately 26 orders of magnitude more than the next largest component, due to photons [arxiv:0909.3983].



Extension to rotating black holes

- The "first law of black hole thermodynamics" now becomes (in natural units)

$$dU = TdS + \Omega_H dJ,$$
$$\text{i.e.} \quad dM = \frac{g}{8\pi} d\mathcal{A} + \Omega_H dJ.$$

  where Ω_H is the black hole angular velocity at the horizon, and we have again made the identifications

$$\text{temperature} \leftrightarrow \frac{\text{surface gravity}}{2\pi} \qquad \text{and} \qquad \text{entropy} \leftrightarrow \frac{\text{surface area}}{4}.$$

- The new feature here is that work can be done to change the angular momentum J of the black hole. (Recall that this is related to the mass and spin parameters via $J = aM$.)

- We can now change both the mass and angular momentum *down* (as well as up) via a classical process using the Penrose process (which only exists for rotating BHs, since needs the ergosphere).
- Navigating in the space of M and J using the Penrose process, find the remarkable result that even if both M and J decrease, the horizon area still either increases, or remains the same (the latter a *reversible process*).
- In fact one can prove the second law of black hole thermodynamics:

  In any classical process, the area of a black hole horizon, which measures the black hole's entropy, does not decrease.
- The restriction to classical processes is necessary, since as we have seen, Hawking radiation succeeds in reducing both mass and surface area. However, if include the entropy of the emitted radiation in the totals, then again one can show total entropy does not decrease.
- So shows us that there is a very profound connection between gravity and thermodynamics, though precisely where this comes from is not yet clear.

Final comment

- We said, the Kerr solution has been proved to be the unique solution for an azimuthally symmetric spacetime with a horizon present.
- Mirrors the case for the Schwarzschild solution, where this is unique if spherical symmetry imposed.
- However, in Schwarzschild case, know use of metric extends well beyond black holes — it's the correct vacuum metric around a s.s. body, and we demonstrated explicitly how it joins onto an interior solution for the body.
- No equivalent in the Kerr case — no material source known for it, and not proven as the spacetime surrounding any material body.
- Still an outstanding problem of research!
- For further reading on rotating black holes, see Chap. 13 of HEL, Chap. 8 of Introduction to Black Hole Physics by Frolov & Zelnikov and Chap. 7 of Einstein Gravity in a Nutshell by Zee.

Summary

- Kerr metric for spacetime outside a mass M with angular momentum J

$$ds^2 = \left(1 - \frac{2\mu r}{\rho^2}\right) c^2 dt^2 + \frac{4\mu r a \sin^2\theta}{\rho^2} c dt d\phi - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - \left(r^2 + a^2 + \frac{2\mu r a^2 \sin^2\theta}{\rho^2}\right) \sin^2\theta d\phi^2$$

$$\mu = \frac{GM}{c^2}, \quad a = \frac{J}{Mc}, \quad \Delta = r^2 - 2\mu r + a^2, \quad \rho^2 = r^2 + a^2 \cos^2\theta$$

- Event horizon $\boxed{\Delta = 0}$, stationary limit surface $\boxed{g_{tt} = 0}$, and singularity $\boxed{\rho = 0}$.
- Black hole thermodynamics

$$T = \frac{g}{2\pi} \qquad S = \frac{\mathcal{A}}{4} \qquad dM = \frac{g}{8\pi} d\mathcal{A} + \Omega_H dJ$$

- Gravity as a mechanism for transferring disorder into black holes to create cosmic structure

Next time

Compact objects & the life cycles of stars