Relativistic Astrophysics and Cosmology — Answers 1 — 2023

1. Gravitational redshift z is defined by $z = (\lambda_r - \lambda_e)/\lambda_e$ where λ_r is the wavelength measured at the receiver, and λ_e is the wavelength in the rest frame of the emitter. The approximate Newtonian result is $z = \Delta\phi/c^2$, which gives

$$z = \frac{GM}{Rc^2}$$

   - Earth — $z \approx 7 \times 10^{-10}$. Has been observed with atomic clocks and using the Mössbauer effect.
   - Sun — $z \approx 2 \times 10^{-6}$. Hard to observe due to thermal Doppler broadening, but has been done.
   - White Dwarf — $z \approx 3 \times 10^{-4}$. Much easier to observe. Has been done for Sirius B (though need to correct for orbital and proper motion).
   - Neutron Star — $z \approx 0.5$. A huge effect. Hope to see this effect in positronium emission lines (e^+e^- decay).

2. Take centre of earth as origin and consider two particles with position vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. These satisfy

$$\ddot{\boldsymbol{x}}_i = -\frac{GM\boldsymbol{x}_i}{|\boldsymbol{x}_i|^3}.$$

Define $\boldsymbol{\epsilon} = \boldsymbol{x}_1 - \boldsymbol{x}_2$ and $\boldsymbol{d} = \frac{1}{2}(\boldsymbol{x}_1 + \boldsymbol{x}_2)$. It follows that

$$|\boldsymbol{x}_1|^2 = (\boldsymbol{d} + \tfrac{1}{2}\boldsymbol{\epsilon})^2 = d^2 + \boldsymbol{\epsilon}\cdot\boldsymbol{d} + \tfrac{1}{4}\boldsymbol{\epsilon}^2,$$

where $d = |\boldsymbol{d}|$. To first order in $\boldsymbol{\epsilon}/d$ we have

$$|\boldsymbol{x}_1| = d(1 + \tfrac{1}{2}\boldsymbol{\epsilon}\cdot\boldsymbol{d}/d^2), \qquad |\boldsymbol{x}_2| = d(1 - \tfrac{1}{2}\boldsymbol{\epsilon}\cdot\boldsymbol{d}/d^2).$$

We now have

$$\begin{aligned}\ddot{\boldsymbol{\epsilon}} &= -\frac{GM}{d^3}[(\boldsymbol{d} + \tfrac{1}{2}\boldsymbol{\epsilon})(1 - \tfrac{3}{2}\boldsymbol{\epsilon}\cdot\boldsymbol{d}/d^2) - (\boldsymbol{d} - \tfrac{1}{2}\boldsymbol{\epsilon})(1 + \tfrac{3}{2}\boldsymbol{\epsilon}\cdot\boldsymbol{d}/d^2)] \\ &= -\frac{GM}{d^3}(\boldsymbol{\epsilon} - (3\boldsymbol{\epsilon}\cdot\boldsymbol{d}/d^2)\boldsymbol{d}),\end{aligned}$$

valid to first order, and ignoring self attraction. Going to the instantaneous rest frame (IRF), we can align $\boldsymbol{d}$ with the z-axis, leading to the equation on p9 of Handout 2.

For motion in the z direction alone (ignoring the variation with time of $\boldsymbol{d}$), find

$$\epsilon_z(t) = \epsilon_z(0)\cosh(\alpha t),$$

where $\alpha^2 = 2GM_E/d^3$. Get a displacement of 1 mm when $(\alpha t)^2 = 2 \times 10^{-3}$. With $d \approx 6.4 \times 10^6$ m, get a time of 26 secs.

Note if we bring in the rotation of the IRF that occurs over the period of the measurement (or equivalently include the z variation of centrifugal force in the calculation), we obtain a slightly shorter time equal to $\sqrt{(2/3)}$ times the previous answer (i.e. about 20 seconds).

For a sphere of particles we work in the IRF and again ignore the time dependence of $\boldsymbol{d}$. The particles initially move inwards in the x–y plane and outwards in the z-direction. After a small time δt the cylindrical radius changes from r to $r(1-\Delta)$, where

$$\Delta = \frac{GM}{2d^3}\,\delta t^2.$$

Similarly, the z displacement gives a new ϵ_z of $\epsilon_z(1+2\Delta)$. The resultant shape is an ellipsoid governed by the equation

$$\frac{r^2}{(1-\Delta)^2} + \frac{z^2}{(1+2\Delta)^2} = R^2,$$

with R the initial radius. The volume of the ellipsoid is

$$\frac{4}{3}\pi R^3(1-\Delta)^2(1+2\Delta) \approx \frac{4}{3}\pi R^3.$$

3. We first go through the argument in the case where the observer is dropped from rest.

Suppose the photon starts at the bottom of a tower with frequency ω in the earth frame. It climbs a height d, taking time d/c, and has a new frequency $\omega_t = \omega(1-gd/c^2)$ in the earth frame. In the meantime the observer starts accelerating from the top of the tower. After time d/c the observer has a velocity $v = gd/c$ downwards. The photon frequency in the observer's frame is

$$\omega' = \omega_t\left(\frac{1+v/c}{1-v/c}\right)^{1/2} = \omega(1+gd/c^2)^{1/2}(1-gd/c^2)^{1/2} \approx \omega.$$

i.e. to first order in gd/c^2, unchanged.

Suppose instead the observer is projected downwards with initial speed u. Now the frequency seen initially is

$$\omega_i = \omega\left(\frac{1+u/c}{1-u/c}\right)^{1/2}$$

After time t the observer's downward velocity is $u+gt$, and the photon will have climbed a distance $d = ct$. Therefore the frequency seen is

$$\omega' = \omega_t\left(\frac{1+\frac{u+gt}{c}}{1-\frac{u+gt}{c}}\right)^{1/2}$$

where $\omega_t = (1-gd/c^2)\omega$ from the gravitational redshift effect. If we let $\epsilon = u/c$ and $\delta = gd/c^2$ then carrying out a first order binomial expansion of each term individually, we are left with expanding out

$$(1+\tfrac{1}{2}\epsilon+\tfrac{1}{2}\delta)(1+\tfrac{1}{2}\epsilon+\tfrac{1}{2}\delta)(1-\delta) \approx (1+\epsilon)$$

to first order in ϵ and δ. This can be identified with the doppler-expanded initial frequency $\omega_i = \omega(1+\epsilon)$ i.e. we get i.e. there is still no change compared to the initially doppler shifted frequency seen by the projected observer.

These results support the SEP, since they show that in a freely falling frame, there is no gravitational redshift, i.e. the results are those one would expect from special relativity alone.

4. The metric on a sphere is

$$ds^2 = a^2\, d\theta^2 + a^2 \sin^2\theta\, d\phi^2$$

so we have

$$\mathcal{L} = a(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2), \quad \text{where} \quad \dot{\theta} = \frac{d\theta}{ds}, \quad \dot{\phi} = \frac{d\phi}{ds}.$$

$\mathcal{L}$ is now a function of θ as well as $\dot{\theta}$ and $\dot{\phi}$. Since we are free to set $\mathcal{L} = a^2$, once the functional dependencies of $\mathcal{L}$ have been used, we arrive at the pair of equations

$$\sin^2\theta\,\dot{\phi} = L, \qquad \ddot{\theta} - \sin\theta\,\cos\theta\,\dot{\phi}^2 = 0,$$

where L is a constant. One immediate solution is $\theta = \pi/2$, giving the equator spanned at a constant rate. The general solution is given by

$$\cot\theta = \tan\psi\,\sin\phi$$

(where ψ is a constant), which is the equation of a great circle inclined at ψ to the equator.

On a cylinder the geodesics are simple helices.

5. The tension τ due to the tidal force at a distance r from a source of mass M, acting on a wire of length l with masses m at either end is given by half the force difference, i.e.

$$\tau = \frac{GmMl}{r^3}.$$

(We can get the same result from full GR, by using the 'equation of geodesic deviation' discussed in the Part II Relativity course, for example in Section 3.2 of Topic 7 (2019 notes). This shows we get a deviation by $+2GM/r^3$ per unit length, which over the length l will be felt by the masses as a force difference of $2GMml/r^3$.)

At the horizon, $r = 2GM/c^2$, so the tension becomes

$$\tau = \frac{mc^6 l}{8G^2M^2}.$$

For (a) this is $\approx 5\times 10^{11}\,\mathrm{N}$, and for (b) $\approx 5\times 10^{-5}\,\mathrm{N}$. Slightly surprisingly the force drops with increasing mass. This is reassuring, because if the universe is closed, we are in some sense already inside a horizon!

6. For a photon the proper path length ds^2 must vanish. Travelling radially in a Schwarzschild metric this implies

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2\, dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 = 0,$$

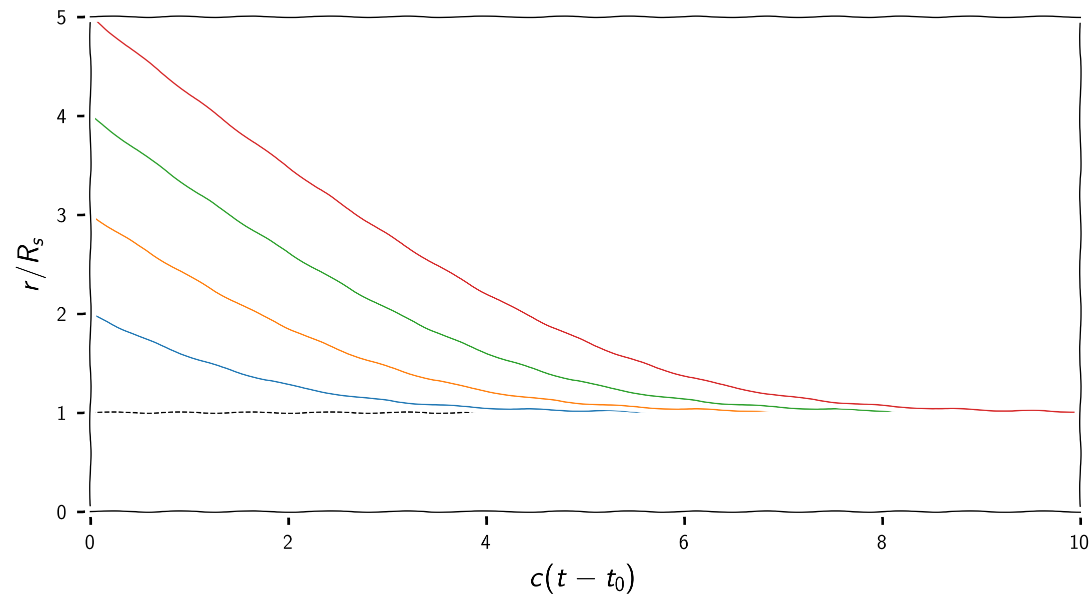
hence

$$\frac{dr}{dt} = \pm c \left(1 - \frac{2GM}{c^2 r}\right).$$

Integrating this for an inward-travelling photon we find that

$$ct = ct_0 - r + r_0 - R_S \ln\left(\frac{r - R_S}{r_0 - R_S}\right),$$

where $R_S = 2GM/c^2$ is the Schwarzschild radius. According to external observers a photon (or indeed any matter) appears to take an infinite coordinate time to cross the horizon:



7. The geodesic energy equation for radial motion gives

$$\tfrac{1}{2} m \dot{r}^2 - \frac{GMm}{r} = \tfrac{1}{2} mc^2 (k^2 - 1).$$

If the particle is dropped from rest at infinity have $\dot{r}(\infty) = 0$, so $k = 1$ and

$$\frac{dr}{d\tau} = -\left(\frac{2GM}{r}\right)^{1/2}.$$

Integrating from the horizon to the origin find that the proper time elapsed, τ, is given by

$$c\tau = -\int_{R_S}^{0} \left(\frac{r}{R_S}\right)^{1/2} dr = \frac{2}{3} R_S.$$

For a solar mass black hole we find a time of $\approx 7 \times 10^{-6}$ s.

8. For a photon falling towards a source, starting from r_0, get (from Question 6)

$$c(t-t_0) = r_0 - r + \frac{2GM}{c^2}\ln\left(\frac{r_0 - 2GM/c^2}{r - 2GM/c^2}\right)$$

with a similar formula holding for outgoing photons. This produces an overall *delay* in the round trip time of

$$\frac{4GM}{c^3}\ln\left(\frac{(D_\oplus - 2GM/c^2)(D_♀ - 2GM/c^2)}{(R_\odot - 2GM/c^2)^2}\right) \approx \frac{4GM}{c^3}\ln\left(\frac{D_\oplus D_♀}{{R_\odot}^2}\right),$$

where $D_\oplus$, $D_♀$ are the Earth–Sun and Venus–Sun distances respectively. This evaluates to $\approx 200\,\mu\mathrm{s}$.

9. For photons, $\mathcal{L} = 0$ replaces the usual $\mathcal{L} = c^2$ for massive particles, so the 'energy equation' rearranges to

$$\frac{\dot{r}^2}{h^2} + \frac{1}{r^2}\left(1 - \frac{2\mu}{r}\right) = \frac{1}{b^2}$$

where $\mu = GM/c^2$ and b has been defined as h/kc. Note the dot here is not derivative w.r.t. proper time, which is zero for a photon, but w.r.t. an affine parameter along the path. We don't need to identify this parameter explicitly, since we are going to work with the *shape* of the orbit. Specifically, noting $r^2\dot{\phi} = h$, we have

$$\left(\frac{\dot{r}^2}{r^4\dot{\phi}^2}\right)^{1/2} = \frac{1}{r^2}\frac{dr}{d\phi} = \left(\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2\mu}{r}\right)\right)^{1/2} \qquad (*)$$

Now why is b the impact parameter? This follows from considering the equation in the limit where the only role of the central mass is to define the origin of the (r,ϕ) system, but it has no gravitational effects. This corresponds to μ vanishing, and this will be a good approximation at large r.

Solving (*) for $\mu = 0$, we get $r = b/\sin\phi$. which is the equation of a straight line, which if it continued all the way past the central object, would pass it with perpendicular distance b, so this is indeed the impact parameter.

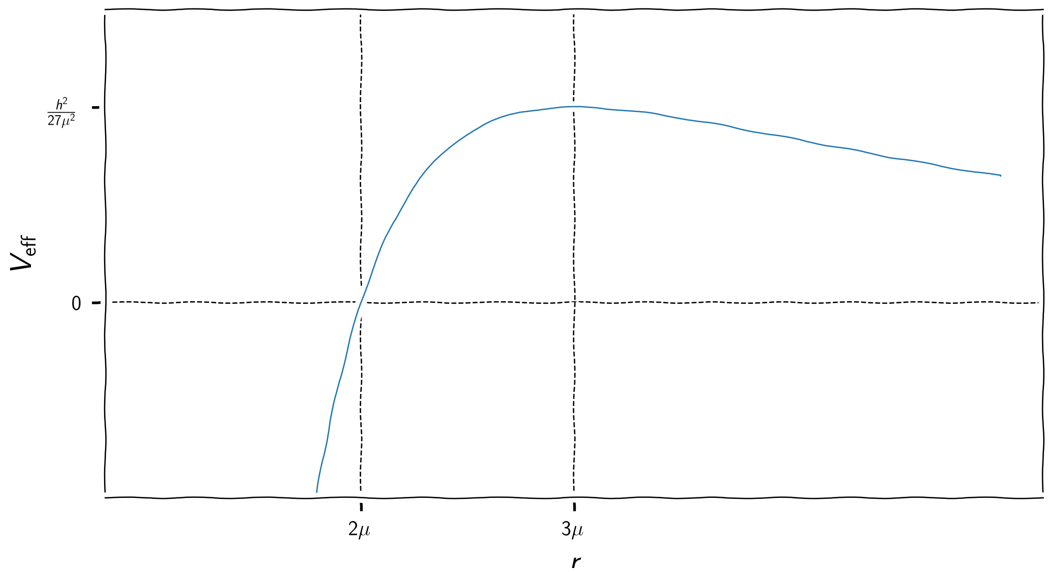
Now we consider the exact equation again. If the photon path just grazes the object at its coordinate radius, which we will call R rather than r to avoid confusion, then we have

$$\left.\frac{dr}{d\phi}\right|_{r=R} = 0$$

This tells us that

$$\frac{1}{b^2} = \frac{1}{R^2}\left(1 - \frac{2\mu}{R}\right) \quad \text{i.e.} \quad b = R\left(\frac{R}{R-2\mu}\right)^{1/2} \quad \text{as stated.}$$

The restriction to $R > 3GM/c^2$ arises from examination of the effective potential plot for photons



$V_{\text{eff}}(r)$ has a single maximum at $r = 3\mu$ where the value of the potential is $1/(27\mu^2)$. This implies that the circular photon orbit, which we know exists at $r = 3\mu$, is *unstable* and in fact we can see there are *no* stable circular photon orbits in Schwarzschild geometry. Thus straying inside $r = 3GM/c^2$ leads to capture. Conversely, if $R > 3GM/c^2$ then we see the photon can escape to infinity, where the b we have found will be its impact parameter. Thus the apparent diameter of the object will be

$$2b = 2R\left(\frac{R}{R-2\mu}\right)^{1/2} \approx 2R\left(1+\frac{\mu}{R}\right)$$

leading to an increase by approximately the Schwarzschild radius (approximately 3 km for the Sun).

10. We have

$$\widetilde{m}(r) = 4\pi\int_0^r \rho(r)\left[1-\frac{2Gm(r)}{c^2 r}\right]^{-1/2} r^2\,dr.$$

In the weak field case, one can expand the bracket$^{-1/2}$ part using the binomial theorem, to obtain $1+\frac{Gm(r)}{c^2 r}$. The '1' part in the integral will reproduce $m(r)$ so the difference $\widetilde{m}(r) - m(r)$, evaluated at $r = R$, is

$$\int_0^R \frac{Gm(r)}{c^2 r} 4\pi r^2\,dr.$$

But this is just $1/c^2$ times what one would calculate to obtain the Newtonian binding energy (i.e. the energy required to disperse the material comprising the object to infinite spatial separation), hence result. (Note in terms of total mass for a constant density star the result is $\frac{3GM^2}{5R}$.)