Problem Set 1

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1. Within Lecture 12, we derived the following equation

$$\rho \frac{\partial v_i}{\partial t} - \frac{\partial T_{ij}}{\partial x_j} = 0,$$

for an elastic body. Here ρ is the referential density, φ_i the motion, and T_{ij} the first Piola-Kirchhoff stress tensor. For an arbitrary sub-body, U, of the reference body, M, show that the equality

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_U \rho v_i \, \mathrm{d}^3\mathbf{x} = \int_{\partial U} T_{ij} \hat{n}_j \, \mathrm{d}S,$$

holds, where ∂U is the boundary of U. What is the physical significance of this result? In particular, what does the vector $T_{ij}\hat{n}_j$ represent?

The angular momentum of the sub-body U is defined by

$$\int_U \rho \epsilon_{ijk} \varphi_j v_k \, \mathrm{d}^3\mathbf{x}.$$

Show that a necessary and sufficient condition for this quantity to be conserved for any such U is that

$$\mathbf{F}\mathbf{T}^T$$

is a symmetric tensor. Defining the second Piola-Kirchhoff stress, $\mathbf{S}$, through

$$\mathbf{T} = \mathbf{F}\mathbf{S},$$

conclude that $\mathbf{S}$ is symmetric.

2. The Lagrangian density for the *linearised* motion for an elastic body is given by

$$\mathcal{L} = \frac{1}{2}\rho \frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t} - \frac{1}{2}A_{ijkl}\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l} + \bar{S}_{ij}\frac{\partial u_i}{\partial x_j},$$

where u_i is the displacement vector, A_{ijkl} the elastic tensor, and $\bar{S}_{ij}$ the stress glut. Using this expression, write down the equations of motion and natural boundary conditions; here you may assume the standard Euler Lagrange equations.

The energy density for this system is defined by

$$E = \frac{1}{2}\rho \frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t} + \frac{1}{2}A_{ijkl}\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l}.$$

In the absence of a stress glut, show the following equality holds

$$\frac{\partial E}{\partial t} + \frac{\partial s_i}{\partial x_i} = 0.$$

Here s_i is known as the *elastic Poynting vector*; the form and physical significance of this vector should be determined as part of your solution.

3. The principle of material frame indifference shows that the strain energy function, W, of an elastic body depends on the deformation gradient, $\mathbf{F}$, only through the right Cauchy-Green deformation tensor, $\mathbf{C} = \mathbf{F}^T\mathbf{F}$, and hence we can write

$$W(\mathbf{x}, \mathbf{F}) = U(\mathbf{x}, \mathbf{C}),$$

for some auxiliary function, U. The elastic tensor, A_{ijkl}, is defined by

$$A_{ijkl} = \frac{\partial^2 W}{\partial F_{ij} \partial F_{kl}},$$

where the partial derivative on the right hand side is evaluated at the equilibrium value $\mathbf{F} = \mathbf{1}$. Assuming that the equilibrium is stress-free, show that the elastic tensor possesses the symmetries

$$A_{ijkl} = A_{jikl} = A_{ijlk} = A_{klij}.$$

How many independent components does such an elastic tensor have?

Do these symmetries remain if there is a non-zero equilibrium stress?

4. Consider plane wave propagation in a homogeneous and transversely isotropic whole space. For such a material the elastic tensor takes the form

$$\begin{aligned} A_{ijkl} =& \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj}) + 8\gamma\hat{\nu}_i\hat{\nu}_j\hat{\nu}_k\hat{\nu}_l + 4\xi(\hat{\nu}_i\hat{\nu}_j\delta_{kl} + \delta_{ij}\hat{\nu}_k\hat{\nu}_l) \\ &- \zeta(\hat{\nu}_i\hat{\nu}_k\delta_{jl} + \hat{\nu}_j\hat{\nu}_k\delta_{il} + \hat{\nu}_j\hat{\nu}_l\delta_{ik} + \hat{\nu}_i\hat{\nu}_l\delta_{jk}), \end{aligned}$$

where λ, μ, γ, ξ, and ζ are elastic constants, and $\hat{\nu}_i$ is a unit vector along the symmetry axis. Using first-order perturbation theory, obtain an expression for the phase speed of a quasi P-wave as a function of the angle, θ, between the propagation direction, $\hat{p}_i$ and the symmetry axis.

5. Show that the time-dependent Schrödinger equation for a free particle

$$\mathrm{i}\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi,$$

admits plane wave solutions of the form

$$\psi(\mathbf{x}, t) = a \exp\left[\frac{-\mathrm{i}(E\,t - p_i x_i)}{\hbar}\right],$$

and obtain the relationship between the energy E and momentum $\mathbf{p}$.

Consider the corresponding equation for a particle in a non-constant but smoothly varying potential field V:

$$\mathrm{i}\hbar\frac{\partial \psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi.$$

Starting from the ansatz

$$\psi(\mathbf{x}, t) = a(\mathbf{x}) \exp\left\{\frac{-\mathrm{i}[E\,t - \varphi(\mathbf{x})]}{\hbar}\right\},$$

where the amplitude a and phase function φ are to be determined, and retaining only the leading-order terms in powers of $1/\hbar$, obtain the eikonal equation

$$H(\mathbf{x}, \nabla\varphi) = E,$$

where H is the classical Hamiltonian. Applying the method of characteristics, show that the particle's phase function can be determined through solution of the classical equations of motion.

6. Consider P-wave rays within an isotropic body. The travel-time of a ray between two points $\mathbf{x}_1$ and $\mathbf{x}_2$ can be written

$$T = \int_0^1 \frac{1}{\alpha(\mathbf{x})}\sqrt{\frac{\mathrm{d}x_i}{\mathrm{d}\gamma}\frac{\mathrm{d}x_i}{\mathrm{d}\gamma}}\,\mathrm{d}\gamma,$$

where the curve $\gamma \mapsto \mathbf{x}(\gamma)$ is the ray path between $\mathbf{x}_1$ and $\mathbf{x}_2$ parameterised by a generating parameter $\gamma \in [0,1]$. Regarding this travel time as a functional of the given ray path, Fermat's principle states that the true ray path is a stationary point of this functional. Obtain the corresponding Euler-Lagrange equations, and show that they are equivalent to the Hamiltonian ray tracing equations discussed within lectures. [*Hint: it will be use to express both sets of ordinary differential equations in terms of the arc length along the ray.*]

7. Consider an isotropic elastic elastic half-space in which the material parameters vary only with depth $z \geq 0$. Suppose that the P-wave speed α increases monotonically, and consider a ray starting at the surface whose initial tangent vector makes an angle $0 < \theta_0 < \pi/2$ with the z-axis and lies in the x-z plane. Assume that the ray travels monotonically down into the half-space until a depth $z_t > 0$ at which point it is travelling horizontally, and subsequently it turns upwards, returning to the surface in a symmetric manner.

   By parameterising this ray in terms of the depth co-ordinate z, show that its travel time can be written

$$T = \int_0^{z_t} \frac{2}{\alpha}\sqrt{1 + \frac{\mathrm{d}x_i}{\mathrm{d}z}\frac{\mathrm{d}x_i}{\mathrm{d}z}}\,\mathrm{d}z,$$

   where $\mathbf{x}$ denotes the *horizontal* position vector of the ray given as a function of z.

   Apply Fermat's principle to show that the ray remains in its initial plane, and that along the ray path

$$\frac{\sin\theta}{\alpha} = q,$$

   where θ denotes the local angle between the vertical direction and the tangent to the ray, and q is a constant known as the ray parameter. Obtain, in particular, an implicit equation for the turning depth z_t in terms of q.

   Show that the travel time T and total horizontal distance X of the ray can be obtained through the following integrals

$$T = \int_0^{z_t} \frac{2}{\alpha}\frac{\mathrm{d}z}{\sqrt{1-q^2\alpha^2}}, \quad X = \int_0^{z_t} \frac{2q\alpha\,\mathrm{d}z}{\sqrt{1-q^2\alpha^2}}.$$