Particle motion and energy in the Schwarzschild metric

Relativistic Astrophysics and Cosmology: Lecture 5

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Pre-lecture question:

How might dark energy disrupt galaxies?

Last time

- The stress-energy tensor for a perfect fluid
- The Oppenheimer–Volkoff equation and interior metric for constant density stars
- Buchdahl's theorem and the collapse point of a star

This lecture

- Conjugate momenta in general relativity
- Energy of circular orbits
- Stability of orbits in a general spherically symmetric metric

Next lecture

- Rotating black holes and the Kerr metric

Comments on the Schwarzschild internal solution

- Not trivial getting a matched internal and external solution.
- In Kerr case this has still not been done.
- Not even trivial getting an internal solution.
- In some 'modified gravity' theories getting a solution for a star is a problem which has already eliminated them.
- An area of great current interest.

- Return now to Schwarzschild exterior metric, and also more general spherically symmetric metrics.
- What's important for us astrophysically, and what do we want to be confident about?
- Geodesic equations and identification of particle energy.
- Capture of particles in non-circular orbits.
- Energies in circular orbits — key for energy release from accretion discs.
- Stability of orbits, and of orbits for light.
- We will discuss energies, angular momenta and stability for a general spherically symmetric metric of our A, B type (new), but to get started, and in particular to get clear what the conserved quantities mean, we will begin with ordinary Schwarzschild metric.

Solving the Schwarzschild geodesic equations

- Without loss of generality, we can work with θ fixed at $\pi/2$. (If motion not in that plane, we just re-align coordinates).
- Then $\mathcal{L}$ given by

$$\mathcal{L} = \left(1 - \frac{2GM}{c^2 r}\right) c^2 \dot{t}^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \dot{r}^2 - r^2 \dot{\phi}^2, \tag{1}$$

- Equations are

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0, \quad (\mu = 0, 1, 2, 3)$$

- But $\mathcal{L}$ is *independent of t and ϕ* (i.e. $g_{\mu\nu}$ is).
- So we have

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}}\right) = \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = 0,$$

- i.e.

$$
\begin{aligned}
u_t = \left(1 - \frac{2GM}{c^2 r}\right) c\dot{t} &= \text{constant} = kc \text{ say,} && (2) \\
\text{and} \quad u_\phi = r^2\dot{\phi} &= \text{constant} = h \text{ say.} && (3)
\end{aligned}
$$

- Though the connection with the former may not be obvious yet, equations (2) and (3) correspond to conservation of energy and angular momentum respectively.
- The radial equation is more complicated, but we can avoid the need for it by remembering that $\mathcal{L} = c^2$ or 0.
- Thus (1) itself gives us the third equation, i.e. it determines $\dot{r}$ once we know $\dot{t}$ and $\dot{\phi}$.
- Substituting in (1) from (2) and (3), we get

$$
\left(1 - \frac{2GM}{c^2 r}\right)^{-1} c^2 k^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \dot{r}^2 - \frac{r^2 h^2}{r^4} = c^2,
$$

- i.e.

$$\dot{r}^2 = c^2(k^2 - 1) - \frac{h^2}{r^2}\left(1 - \frac{2GM}{c^2 r}\right) + \frac{2GM}{r}. \tag{4}$$

- Rearranging this using (3) and multiplying by the mass of the test particle m, we get

$$\underbrace{\frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\left(r\dot{\phi}\right)^2\left(1 - \frac{2GM}{c^2 r}\right)}_{\text{K.E.-like}} \underbrace{-\frac{GMm}{r}}_{\text{P.E.-like}} = \underbrace{\frac{1}{2}mc^2(k^2 - 1)}_{\text{“Energy?”}}. \tag{5}$$

- This is suggestive as being the GR statement of energy conservation for orbital motion.
- The left hand side looks quite Newtonian, but it must be remembered that the dot refers to derivative with respect to *proper time* rather than coordinate time, and there is a GR factor of $\left(1 - \frac{2GM}{c^2 r}\right)$ modifying the transverse kinetic energy term.
- In fact, within GR, the most straightforward definition of the energy of a particle in its orbit, is not the right hand side of this equation, i.e. $\frac{1}{2}mc^2(k^2 - 1)$, though this is indeed constant, but a quantity directly proportional to k.
- Get to this via looking at alternative form of geodesic equation, best adapted to dealing with conserved quantities.

- We derive this in the Appendix, and show how it links with the Connection.
- It turns out that for discussing conserved quantities, it is the downstairs components of $\vec{p}$ which are the important ones.
- We can write

$$\vec{p} = p_a e^a, \quad \text{where} \quad p^a \equiv m\frac{dx^a}{d\tau}.$$

- We obtain the useful alternative form of geodesic equations,

$$\boxed{\dot{p}_a = \frac{1}{2m}(\partial_a g_{cd})p^c p^d.}$$

- This equation verifies earlier finding from the Lagrangian method that if the metric g_{cd} does not depend on the coordinate x^a then we have a conserved quantity, and now identifies it as p_a!
- I.e., it is p_a that is the correct 'conjugate momentum'.
- E.g. the 'momentum' conjugate to displacements in time is energy, so p_t should be the particle energy, and will be conserved if the metric is static.
- The 'momentum' conjugate to displacements in angle ϕ is angular momentum, so p_ϕ should be the particle angular momentum, and will be conserved if the metric is azimuthally symmetric.

Application to Schwarzschild metric

- Both these statements are completely general — can apply them to any metric with these properties.

$$p_t = m g_{t\mu} \dot{x}^\mu = m \left(1 - \frac{2GM}{rc^2}\right) c\dot{t} = kcm = E/c$$

- Therefore, putting in the c's,

$$\boxed{E_{\text{part}} = kmc^2}$$

  is the correct particle energy.
- Also

$$p_\phi = m g_{\phi\mu} \dot{x}^\mu = -mr^2 \dot{\phi} = -mh$$

  and so $L = -mh$ is the particle angular momentum.

Master equation for orbit shapes

- We would now like to proceed as in Newtonian theory, by finding the *shape* of the orbit, i.e. r as a function of ϕ rather than the proper time.
- To this end we use (3) to write $\frac{dr}{d\tau} = \frac{dr}{d\phi}\frac{d\phi}{d\tau} = \frac{h}{r^2}\frac{dr}{d\phi}$, and thus have from (4)

$$\left(\frac{h}{r^2}\frac{dr}{d\phi}\right)^2 + \frac{h^2}{r^2} = c^2(k^2 - 1) + \frac{2GM}{r} + \frac{2GMh^2}{c^2 r^3}.$$

- This suggests the usual Newtonian substitution $u \equiv 1/r$, which yields

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{c^2}{h^2}(k^2 - 1) + \frac{2GMu}{h^2} + \frac{2GMu^3}{c^2}.$$

- We now differentiate this equation through with respect to ϕ to obtain finally

$$\boxed{\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2} + \frac{3GM}{c^2}u^2.}$$

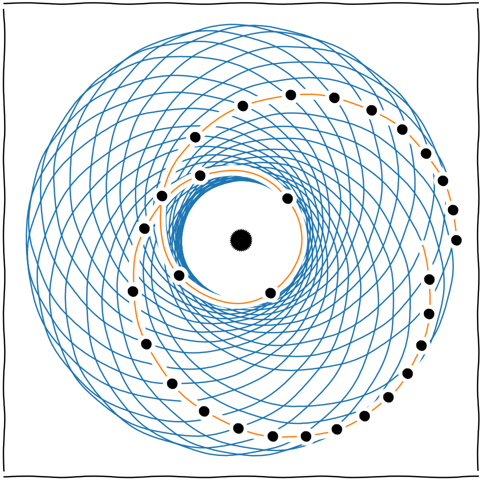
- In Newtonian gravity, the equation for planetary orbits, in the same notation, is

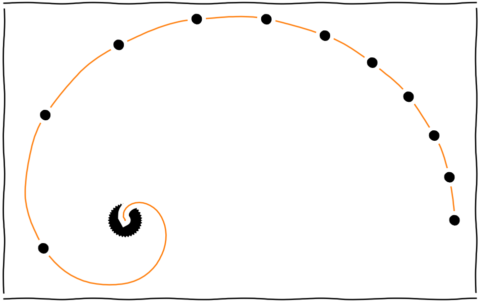
$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2},$$

  so we have *almost* found the Newtonian answer, except for the extra term $3GMu^2/c^2$.
- This is what gives all the relativistic effects, and we note it correctly goes to zero as $c \to \infty$.
- Thus, from our knowledge of Newtonian theory, we predict that the orbits of planets; motion around a black hole; motion of a projectile on Earth; will all be *modified ellipses*.
- An interesting point is that if one looks at the $\ddot{r}$ equation found by differentiating equation (4) w.r.t. τ

$$\frac{d^2r}{d\tau^2} = -\frac{GM}{r^2} + \frac{h^2}{r^3} - \frac{3h^2GM}{c^2r^4}.$$

  this shows that close to the hole, specifically within the radius $r = 3GM/c^2$, centrifugal force 'changes sign', and is directed inwards, thus hastening the demise of any particle that strays too close to the hole.
- This leads to typical spiral orbits (Fig. 2), whereas if we keep far enough out (Fig. 1), just get continued precession.





- Tangentially launch matter particle from $20R_S$.
- $h = 3.75$ on left (precession).
- $h = 3.5$ on right (capture).
- Dots are equally spaced in proper time.

Circular Orbits and Energy

- An important application of the orbit formula in the context of high energy astrophysics, is what it tells us about circular orbits in Schwarzschild geometry.
- These will be approximately the orbits of material accreting onto black holes, since infalling material nearly always has angular momentum; would not generally expect radial infall.
- If r is constant, then our equation for u

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2} + \frac{3GM}{c^2}u^2,$$

yields

$$h^2 = \frac{GMr^2}{r - 3GM/c^2}.$$

- Putting $\dot{r} = 0$ in equation (5) (earlier this handout) gives us

$$\frac{1}{2}m\left(r\dot{\phi}\right)^2\left(1 - \frac{2GM}{c^2r}\right) - \frac{GMm}{r} = \frac{1}{2}mc^2(k^2 - 1).$$

- Putting both these last two results together (and using $r^2\dot{\phi} = h$) yields an equation for k in terms of r alone:

$$k = \frac{1 - \frac{2GM}{rc^2}}{\sqrt{1 - \frac{3GM}{rc^2}}}.$$

- Now what is k? Know now we should identify $k = E/mc^2$, where E is the particle energy.
- Thus we have found that the energy of a particle in a circular orbit is

$$\boxed{E_{\text{circ}} = mc^2 \frac{1 - \frac{2GM}{rc^2}}{\sqrt{1 - \frac{3GM}{rc^2}}}.}$$

- Obvious check on this equation, is whether it can reproduce the Newtonian expression for the total energy of a circular orbit in the limit of large r.
- Using the binomial theorem we see that the first two terms in an asymptotic expansion in r

$$E_{\text{circ}} \sim mc^2 - \frac{GMm}{2r} + \ldots,$$

- Now usual Newtonian expression for a circular orbit is derived via

$$E_{\rm tot} = \text{K.E.} + \text{P.E.} = \frac{1}{2}mv^2 - \frac{GMm}{r} = \frac{-GMm}{2r} \text{ (where we used } \quad \frac{mv^2}{r} = \frac{GMm}{r^2}.\text{)}$$

- Thus get agreement provided we realise that Newtonian energy enters as a correction to the rest mass energy mc^2, which is the dominant term.
- The equation we have just found for the energy of a circular orbit, provides us with useful information about the nature of such orbits.
- First we see that in the limit $m \to 0$, the orbit $r \to 3GM/c^2$ is of interest, since the singularity in the denominator can cancel the zero at the top. In fact this is the circular photon orbit at $r = 3GM/c^2$, which we'll look at shortly.
- Secondly, we can see which orbits (for particles of non-zero rest mass) are *bound*. This will occur if $E_{\rm circ} < mc^2$, since then we have less energy than the value for a stationary particle at infinity. The condition for $E_{\rm circ} = mc^2$ is that

$$\left(1 - \frac{2GM}{rc^2}\right)^2 = 1 - \frac{3GM}{rc^2}$$

happens for $r = 4GM/c^2$ or $r = \infty \Rightarrow$ circular orbits bound over the range $4 < r < \infty$.

How about stability?

- Interesting to do this in more general case of a metric of the form

$$ds^2 = A(r)c^2dt^2 - B(r)dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\phi^2 \tag{6}$$

  i.e. of the type we used when discussing solutions with matter.
- Some quite interesting vacuum solutions fall into this class, in addition to the Schwarzschild solution.
- E.g., as a 'taster' for the later appearance of the 'cosmological constant', let us repeat our derivation of the space part of the Schwarzschild metric, but this time assuming that empty space could have a constant curvature, which we write as $\Lambda/3$ say, so now instead of the relation

$$K(r) = -\frac{GM}{c^2r^3} = \frac{f'}{2f^2r},$$

  we have

$$K(r) = -\frac{GM}{c^2r^3} + \frac{\Lambda}{3} = \frac{f'}{2f^2r},$$

- This has solution
$$f(r) = B(r) = \left(1 - \frac{2GM}{rc^2} - \frac{\Lambda r^2}{3}\right)^{-1}.$$
- Now, haven't shown this here, but it's a fact that $A(r)$ being different from $1/B(r)$ only has to happen if fluid pressure is involved, e.g. as with Schwarzschild interior solution. Will cover this when we come to cosmological (t, r) surface.
- Thus here $A = 1/B$ and complete metric for this case with a constant curvature for the empty space surrounding a mass M is

$$\boxed{ds^2 = \left(1 - \frac{2GM}{c^2 r} - \frac{\Lambda r^2}{3}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2.}$$

- This is known as the Schwarzschild-de Sitter metric.
- It's presumably the appropriate metric in the solar system, now that we know a cosmological constant exists.
- We will see how to deal with the time/space part properly when we get to cosmology.

- Note thinking about g_{00} in our usual way as $\left(1+\frac{2\Phi}{c^2}\right)$, we get

$$\Phi = -\frac{GM}{r} - \frac{\Lambda r^2 c^2}{6}.$$

- So the 'Newtonian force per unit mass' is

$$-\frac{d\Phi}{dr} = \frac{F_r}{m} = -\frac{GM}{r^2} + \frac{\Lambda c^2}{3} r,$$

  i.e. Λ leads to a repulsive force and increasing with distance! — this is a simple Newtonian way of thinking about its effects in the universe.
- OK, now repeat the question, but now for a general metric of form (6).

How about stability?

- Easy to see that our $\mathcal{L} \equiv c^2$ equation is now
$$Ac^2\dot{t}^2 - B\dot{r}^2 - r^2\dot{\phi}^2 = c^2,$$
and for our energy and angular momentum conserved quantities, we will get
$$k = A\dot{t}, \quad \text{and} \quad h = r^2\dot{\phi}.$$

- So can now rewrite $\dot{r}^2$ as
$$\dot{r}^2 = \frac{1}{B}\left(\frac{c^2k^2}{A} - \frac{h^2}{r^2} - c^2\right).$$

- Meanwhile, can get $\ddot{r}$ from
$$\frac{d}{dr}\left(\dot{r}^2\right) = \frac{d}{d\tau}\left(\dot{r}^2\right)\frac{d\tau}{dr} = \frac{2\ddot{r}\dot{r}}{\dot{r}} = 2\ddot{r}.$$

- So plan is, for a circular orbit, both $\dot{r}$ and $\ddot{r}$ have to vanish.

- Relation just found for $\ddot{r}$ therefore says that

$$-\frac{k^2c^2}{A^2}A' + \frac{2h^2}{r^3} = 0,$$

where $A' \equiv dA/dr$, and combining this with expression for $\dot{r} = 0$ we can solve for k^2 and h^2, obtaining

$$k^2 = \frac{2A^2}{2A - rA'}, \quad \text{and} \quad h^2 = \frac{c^2 r^3 A'}{2A - rA'}.$$

- So we now have the energy and angular momentum for a particle in circular orbit for quite a wide range of metrics!
- E.g. metric for a charged black hole fits into this scheme as well — known as Reissner-Nordström metric — just replace $-(1/3)\Lambda r^2$ in SdS with $+q^2G/(4\pi\epsilon_0 r^2)$.
- Exercise: show that these results yield what we already know for the standard Schwarzschild form.
- Interesting at this point to get a general formula for the velocity in a circular orbit.
- We start with the expression just now for h^2 and note this is $r^4\dot{\phi}^2$.

- If we employ the rapidity α, we know this is given by

$$\frac{v}{c} = \tanh \alpha,$$

where v is the ordinary velocity we would like to find.

- Now introduce a new time coordinate t', which is the time of a stationary observer at the radius r (and in particular is *not* the Schwarzschild coordinate time t, which is the time of a stationary observer at infinity).
- Then $\frac{dt'}{d\tau} = \cosh \alpha$, where τ is the particle proper time, implies

$$v = r\frac{d\phi}{dt'} = r\frac{d\phi}{d\tau} \bigg/ \frac{dt'}{d\tau} = \frac{r\dot{\phi}}{\cosh \alpha},$$

from which we deduce

$$r\dot{\phi} = c \sinh \alpha.$$

- Finally we use

$$\tanh^2 \alpha = \frac{\sinh^2 \alpha}{1 + \sinh^2 \alpha} \qquad \text{and} \qquad r^2\dot{\phi}^2 = \left(\frac{h}{r}\right)^2 = \frac{c^2 rA'}{2A - rA'}.$$

- to deduce

$$\boxed{\frac{v^2}{c^2} = \frac{rA'}{2A},}$$

which is the result we need (and quoted in previous lecture).
- It is interesting, that like almost everything else connected to the orbits of particles, it is a function of just the A coefficient of the metric!

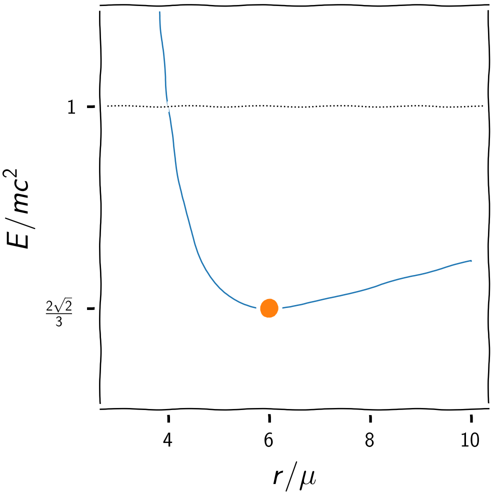
- To get requirement for stability, could now rewrite $\dot{r}^2$ expression in the standard form

$$\frac{1}{2}\dot{r}^2 + V_{\rm eff}(r) = \text{const.}, \qquad V_{\rm eff}(r) = \frac{c^2}{2B(r)} + \frac{h^2}{2r^2B(r)} - \frac{c^2k^2}{2A(r)B(r)}$$

and then require $d^2V_{\rm eff}/dr^2 > 0$ for stability. $\left(V_{\rm eff}^{\rm Schw}(r) = -\frac{GM}{r} + \frac{h^2}{2r^2} - \frac{GMh^2}{c^2r^3}\right)$
- However, turns out that this yields exactly the same boundary point (i.e. stable one side and unstable the other) as where k, treated as a function of the radius of circular orbit, has a minimum, so easier to work with latter.
- (Another exercise: show that these two conditions are indeed the same, and obtain the explicit expression for general A — note only depends on this — not B!).

Results for Schwarzschild and SdS cases

- Can consider the 'fractional binding energy' $E/(mc^2) - 1 = k - 1$ versus r. This is a plot of $E/(mc^2)$ versus r (the latter measured in units of $\mu = GM/c^2$) where E is the energy of a particle of mass m in a circular orbit at radius r about a Schwarzschild black hole.
- You can see that (considering smaller and smaller radii, starting from a point a long way out) $r = 6GM/c^2$ is the point where dE_{circ}/dr fails to have a positive gradient.



- Indeed makes sense we get a restoring force, and hence this should coincide with stability in the orbit, provided

$$\frac{dE_{\text{circ}}}{dr} > 0 \quad \text{(condition for stability)}.$$

- Can of course in the Schwarzchild case get this as well from direct differentiation of

$$k = \frac{1 - \frac{2GM}{rc^2}}{\sqrt{1 - \frac{3GM}{rc^2}}}.$$

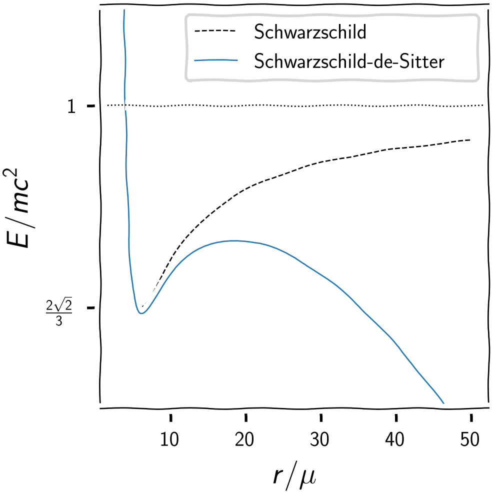
- So this happens at 3 Schwarzschild radii ($6GM/c^2$), and not 2 (which is the limit for where bound orbits exist). This is therefore most likely to be the inner edge of the accretion disc for a Schwarzschild black hole.
- Note the possible energy we could liberate from material gradually drifting into this lowest stable orbit from infinity is about 6% of the rest mass — this is a pretty efficient process, of the same order as the few percent which is the maximum obtainable from nuclear burning, and shows accretion around black holes is going to be a powerful energy source in astrophysics.

Stability for Schwarzschild-de Sitter?

- For Schwarzschild-de Sitter, we find

$$k = \left(1 - \frac{2GM}{c^2 r} - \frac{1}{3}\Lambda c^2 r^2\right) \bigg/ \sqrt{1 - \frac{3GM}{c^2 r}}.$$

- Unrealistic values for plot $\Lambda = 10^{-4}\mu/c^2$.
- For reasonable values, can show that to good accuracy stability breaks down where $r_{\text{stab}} \approx (3GM/(4\Lambda c^2))^{1/3}$.
- Can show that e.g. for a cluster of galaxies, for measured value of the cosmological constant, and $M = 10^{15} M_{\odot}$, $r_{\text{stab}} \approx 7.2\,\text{Mpc}$.
- Close to maximum cluster sizes observed (Nandra, Lasenby & Hobson, MNRAS, **422**, 2945 (2012)).



Photon orbits

- The only change (though it has profound consequences) we need for photons, is that instead of putting $\mathcal{L} = c^2$ in deriving the equations of motion from the Euler-Lagrange equations, we should put $\mathcal{L} = 0$ – which in general simplifies things.
- If look back to equation (4) and the equation before it, this means we now get:

$$\dot{r}^2 - k^2c^2 = -\frac{h^2}{r^2}\left(1 - \frac{2GM}{rc^2}\right)$$

- Also, tracking through to the 'shape' equation, (10), one finds that it is now the 'Newtonian' term that disappears on the rhs, and we just get

$$\frac{d^2u}{d\phi^2} + u = \frac{3GM}{c^2}u^2.$$

- Can immediately confirm from this that there is a photon orbit at $r = 3GM/c^2$. But is it stable?

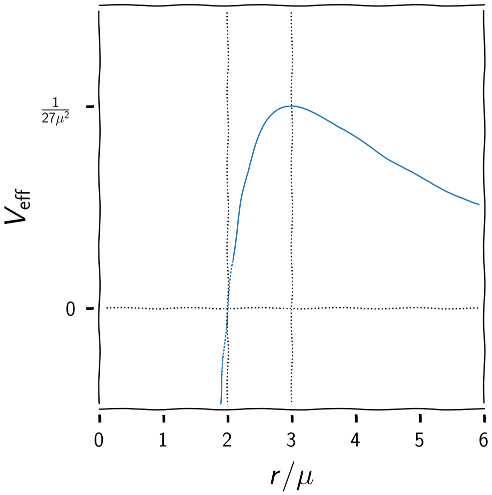
- To do stability analysis, rewrite energy equation as

$$\frac{\dot{r}^2}{h^2} + V_{\text{eff}}(r) = \frac{1}{b^2}, \tag{7}$$

where $b = h/ck$, $\mu = GM/c^2$ and the effective potential

$$V_{\text{eff}}(r) = \frac{1}{r^2}\left(1 - \frac{2\mu}{r}\right).$$

- So, $V_{\text{eff}}(r)$ has a single maximum at $r = 3\mu$ where the value of the potential is $1/(27\mu^2)$
- $\Rightarrow$ circular orbit at $r = 3\mu$ is unstable
- $\Rightarrow$ no stable circular photon orbits in Schwarzschild geometry



Summary

- Energy of a particle in a circular orbit

$$\boxed{E_{\text{circ}} = mc^2 \frac{1 - \frac{2GM}{rc^2}}{\sqrt{1 - \frac{3GM}{rc^2}}}.}$$

- Stability of circular orbits and ISCO at 6μ.
- The Schwarzschild de Sitter metric:

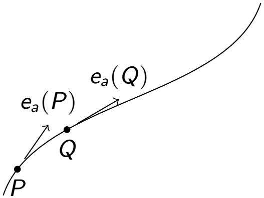
$$\boxed{ds^2 = \left(1 - \frac{2GM}{c^2 r} - \frac{\Lambda r^2}{3}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 - r^2 d\Omega.}$$

- Reissner-Nordström metric for charged black holes: $-\frac{\Lambda r^2}{3} \to \frac{q^2 G}{4\pi\epsilon_0 r^2}$.

Next time

The Kerr metric

- The connection arises where we want to differentiate frame vectors
- Consider basis vectors $\vec{e}_a$ of some coordinate system x^a at nearby points P and Q with coordinates x^a and $x^a + \delta x^a$:



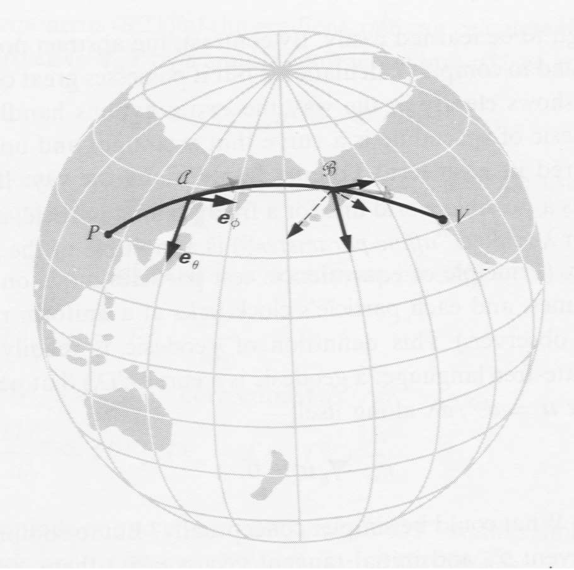
- In general, the basis vectors at Q will differ infinitesimally from those at P, and to first-order we can write

$$\vec{e}_a(Q) = \vec{e}_a(P) + \delta\vec{e}_a.$$

- The standard partial derivative $\lim_{\delta x^c \to 0} \delta\vec{e}_a/\delta x^c$ of the basis vector will, however, in general not lie in the tangent plane at P (denoted T_P), hence $\Rightarrow$ not a useful construction

- E.g., consider standard spherical polar coordinates on the surface of a sphere, and the basis frame $\vec{e_\theta}$ and $\vec{e_\phi}$ they give rise to (picture from Misner, Thorne & Wheeler).
- Can see that $\delta\vec{e_\theta}/\delta\theta$ has no component in the tangent plane at all — points wholly in the (negative) radial direction!
- Instead we define a derivative of $\vec{e}_a$ that does lie in the right space (T_P), namely

$$\frac{\partial\vec{e}_a}{\partial x^c} \equiv \left(\lim_{\delta x^c\to 0}\frac{\delta\vec{e}_a}{\delta x^c}\right)_{\parallel T_P}$$



- That is, we forcibly project our previous derivative into the tangent space, and then it's in the right space by construction.
- Notice, further, we have notated this derivative the same way as the partial derivative — what we mean is that this is *the partial derivative in the manifold* — not the same as the partial derivative in the full space. (Note acting on scalars defined on the manifold, rather than vectors, there is no projecting we need to do, and $\partial/\partial x^c$ is just the ordinary partial derivative.)
- Since the result lies in the tangent space, we can expand it in basis vectors $\vec{e}_a(P)$ at P as

$$\boxed{\frac{\partial \vec{e}_a}{\partial x^c} \equiv \Gamma^b{}_{ac}\vec{e}_b.} \tag{8}$$

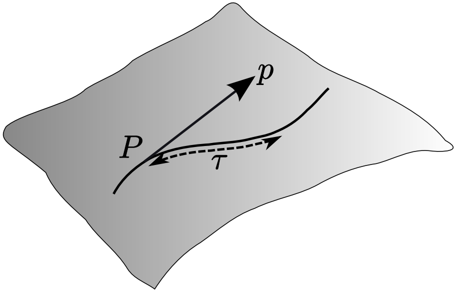
  The N^3 coefficients $\Gamma^b{}_{ac}$ are called the connection.
- Use this to get Alternative form of geodesic equations

- It turns out that for discussing conserved quantities, it is the downstairs components of $\vec{p}$ which are the important ones.
- We can write

$$\vec{p} = p_a e^a, \quad \text{where} \quad p^a \equiv m\frac{dx^a}{d\tau}$$

- $\vec{p}$ being a tangent vector to a curve, already lives in the tangent space at point P, and thus the equivalence principle tells us that it should have its Special Relativistic behaviour of *no change of direction* at each point, i.e. we want

$$\boxed{\left(\frac{d\vec{p}}{d\tau}\right)_{\| T_P} = 0}$$



- Let's expand this out.

$$\begin{aligned}\left(\frac{d\vec{p}}{d\tau}\right)_{\| T_P} &= \frac{d}{d\tau}\left(p_a\right) e^a + p_a \left(\frac{d\vec{e^a}}{d\tau}\right)_{\| T_P} \\ &= \frac{dp_a}{d\tau} e^a + p_a \frac{\partial e^a}{\partial x^c}\frac{dx^c}{d\tau} \\ &= \frac{dp_a}{d\tau} e^a - p_a {\Gamma^a}_{bc} e^b \frac{dx^c}{d\tau} \\ &= \frac{dp_a}{d\tau} e^a - p_b {\Gamma^b}_{ac} e^a \frac{dx^c}{d\tau} \\ &= \left(\frac{dp_a}{d\tau} - p_b {\Gamma^b}_{ac}\frac{dx^c}{d\tau}\right) e^a = \mathbf{0}\end{aligned}$$

- This tells us

$$\begin{aligned}\frac{dp_a}{d\tau} &= p_b {\Gamma^b}_{ac}\left(\frac{1}{m}p^c\right) \\ &= \frac{1}{m}{\Gamma^b}_{ac}\, g_{bd} p^d p^c \\ &= \frac{1}{2m}(\partial_a g_{dc} + \partial_c g_{ad} - \partial_d g_{ac}) p^d p^c\end{aligned}$$

- Symmetry of metric tensor $\Rightarrow$ last two terms in the summation on d and c cancel.
- Thus, obtain a useful alternative form of geodesic equations,

$$\boxed{\dot{p}_a = \tfrac{1}{2m}(\partial_a g_{cd}) p^c p^d} \tag{9}$$