The evolution of the Universe

Relativistic Astrophysics and Cosmology: Lecture 18

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Wednesday 15th November 2023

Pre-lecture question:

How big a mistake can Einstein make?

Last time

- Dynamics of the Universe: the cosmological field equations.

This lecture

- Solving the cosmological field equations .
- Einstein-de-Sitter, Friedmann, Flat Λ, Einstein & de Sitter Universes.

Next lecture

- Measuring the Universe.

Feedback request

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- Feedback encouraged.
- Deadline Sunday 26^{th} November (after Lecture 22).
- 11 short, multiple choice questions.
- The free-form (optional) box is particularly useful for improving the course, e.g:
  - What would you have like to see more/less of?
  - What worked well/not-so-well?
  - How well does this fit with Part II/III Physics?
- Important for influencing future students, department and lecturer.

Recap: The equations of our Universe

- Universe described by a spatially homogeneous and isotropic spacetime with FRW metric

$$ds^2 = c^2 dt^2 - R(t)^2 dX^2, \qquad dX^2 = d\chi^2 + S_k^2(\chi) d\Omega, \qquad S_k \in \{\chi, \sin\chi, \sinh\chi\}.$$

- The scale factor $R(t)$ is related to the cosmic fluid density $\rho(t)$ and pressure $P(t)$ by any two of the cosmological field equations

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda c^2}{3}, \qquad \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3} + \frac{\Lambda c^2}{3} - \frac{kc^2}{R^2}, \qquad \dot{\rho} = -3\frac{\dot{R}}{R}\left(\rho + \frac{P}{c^2}\right).$$

- Many fluids can be described by an equation of state $P = w\rho c^2$:

$$w_r = \frac{1}{3}, \qquad w_m \approx 0, \qquad w_k = -\frac{1}{3}, \qquad w_\Lambda = -1.$$

The Friedmann equation

- As described last time, for a w-fluid we may solve the continuity equation immediately

$$\dot{\rho} = -3\frac{\dot{R}}{R}\rho(1+w) \quad \Rightarrow \quad \rho \propto R^{-3(1+w)}.$$

- If the fluids are non-interacting, then $\rho = \sum_i \rho_i$ and $P = \sum_i P_i = \sum_i \rho_i w_i$.
- Since the equations are linear in ρ and P, the acceleration and velocity equations modify

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\sum_i \rho_i\left(1+3w_i\right), \quad \frac{\dot{R}^2}{R^2} = \frac{8\pi G}{3}\sum_i \rho_i. \quad \dot{\rho}_i = -3\frac{\dot{R}}{R}\rho_i\left(1+w_i\right),$$

- Note that non-interaction means the continuity equation applies separately, so $\rho_r \propto R^{-4}$, $\rho_m \propto R^{-2}$, $\rho_k \propto R^{-2}$ and $\rho_\Lambda \propto$ const independent of the presence of other components.
- For non-interacting fluids the master equation for the scale factor R is therefore

$$\boxed{\frac{H^2}{H_0^2} = \Omega_{r,0}\left(\frac{R}{R_0}\right)^{-4} + \Omega_{m,0}\left(\frac{R}{R_0}\right)^{-3} + \Omega_{k,0}\left(\frac{R}{R_0}\right)^{-2} + \Omega_{\Lambda,0}.}$$

Classifying solutions

- On the face of it the master equation for our Universe is quite simple.
- However, unlike most of the equations in non-relativistic courses, it is non-linear.
- The range of consequences of the dynamical equations, in conjunction with the FRW metric, is therefore very large.
- It is straightforward to solve numerically (try it!).
- Analytic solutions difficult unless:

$$\begin{aligned} \Lambda = k = 0 &: \text{ Einstein-de-Sitter Universe,} \\ \Lambda = 0 &: \text{ Friedmann Universes,} \\ k = 0 &: \text{ Flat } \Lambda \text{ Universes.} \end{aligned}$$

- We also usually assume matter dominated or radiation dominated (not a bad approximation as in mixed Universes usually one is dominant depending on epoch).
- Other special cases are Einstein static universe and de Sitter accelerating universe.
- They cannot all be favoured by data, but were the object of intense study for many decades, and have nice mathematical forms, thus forming part of the knowledge one requires to understand a lot of (still existing) literature and even popular expositions.

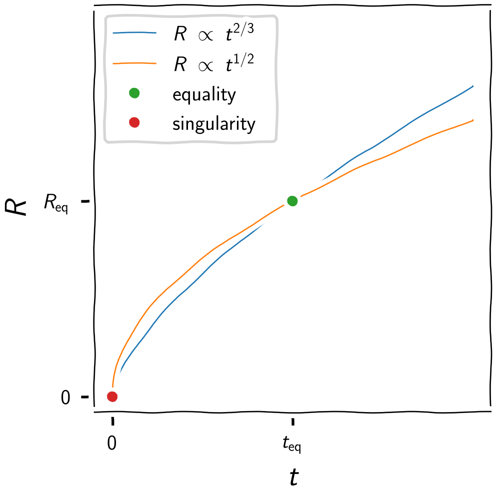
Einstein-de-Sitter ($\Lambda = k = 0$)

- Consider a flat Universe without dark energy.
- If $\Lambda = k = 0$, $w \neq 0$ the equations are

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3} \quad \text{and} \quad \rho \propto R^{-3(1+w)}$$

$$\Rightarrow \quad \dot{R} \propto R^{-(1+3w)/2} \quad \Rightarrow \quad \boxed{R \propto t^{\frac{2}{3(1+w)}}}.$$

- For a (matter dominated) EdS $\boxed{R \propto t^{2/3}}$.
- For a radiation dominated EdS $\boxed{R \propto t^{1/2}}$.
- NB, if content not specified, EdS means matter dominated ($\Lambda = k = w = 0$).
- Solutions useful for quick calculations.

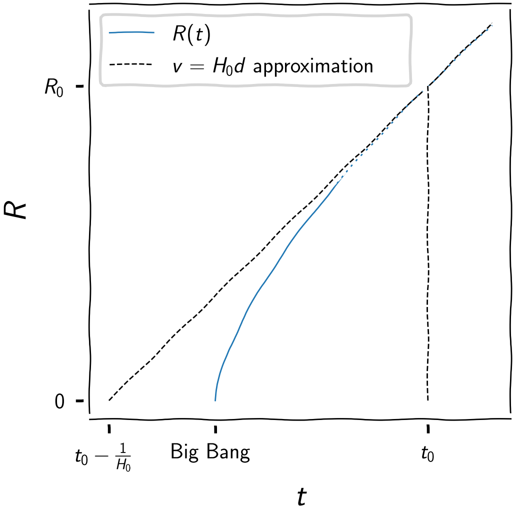


Friedmann models ($\Lambda = 0$): general properties

- Universes with no cosmological constant $\Lambda = 0$ are known as Friedmann solutions. (with EdS a special case if $k = 0$)
- The most obvious immediate thing for these, is that the acceleration equation tells us that

$$\frac{\ddot{R}}{R} = -\frac{4\pi G\rho}{3}(1+3w) \Rightarrow \ddot{R} < 0 \quad \forall\, t$$

- So R against t is convex towards the t axis
- This means that all $\Lambda = 0$ models have a Big-Bang origin! (i.e. $R = 0$ in finite past).
- This time is less than the Hubble time $\frac{1}{H_0}$ when "all the galaxies on top of each other" extrapolating linearly backward from today.



- Consider now what generically happens for $R \mapsto 0$ in Friedmann ($\Lambda = 0$) models.
- We can write the velocity equation as

$$\frac{\dot{R}^2 + kc^2}{R^2} = \frac{8\pi G\rho}{3}.$$

- We also know $\rho \propto R^{-3(1+w)}$, from energy conservation, so

$$\dot{R}^2 + kc^2 \propto R^{-(1+3w)}.$$

- Since $w \geqslant 0$ and kc^2 is a constant, we deduce $\dot{R} \to \infty$ as $R \to 0$.
- Thus at sufficiently early epochs we have

$$\dot{R} \propto R^{-(1+3w)/2}$$

  regardless of the value of k.
- This means that space curvature can be ignored in the early universe.
- EdS is therefore a good approximation for the early universe.

Conformal time η

- We now want to consider the solution for full Friedmann $\Lambda = 0$, $k \neq 0$ models.
- In these cases we can't solve explicitly for R as a function of cosmic time t.
- There is however another important time coordinate: conformal time η defined by

$$ds^2 = c^2 dt^2 - R(t)^2 dX^2 = R(t)^2 (d\eta^2 - dX^2) \quad \Rightarrow \quad cdt = Rd\eta \quad \Rightarrow \quad \eta = \int \frac{cdt}{R}.$$

- Conformal time is preferred by theoreticians.
- Can be thought of as a dimensionless "comoving time".
- So-called because it renders manifest the conformal rescaling property of the FRW metric.
- Photons move in straight lines in the (η, χ) plane, so it a natural photonic time coordinate.
- It is also the time measured by a light clock expanding with the universe.

Friedmann ($\Lambda = 0$)

- In $\Lambda = 0$, $k \neq 0$ we can't solve explicitly for R as a function of cosmic time t, but can in conformal time η.
- Changing variable to η the $\dot{R}$ derivative becomes $\boxed{\dot{R} = \frac{dR}{dt} = \frac{dR}{d\eta}\frac{d\eta}{dt} = \frac{c}{R}\frac{dR}{d\eta}}$
- The energy conservation relation $\rho \propto R^{-3(1+w)}$ and velocity equation combine, and introducing $a = R/R_0$, we find.

$$\frac{1}{R^2}\left(\frac{c}{R}\frac{dR}{d\eta}\right)^2 - \frac{8\pi G\rho_0 R_0^{3(1+w)}}{3R^{3(1+w)}} = -\frac{kc^2}{R^2} \quad \Rightarrow \quad \frac{1}{a^2 R_0^2}\left(R_0\frac{da}{d\eta}\right)^2 - \frac{8\pi G\rho_0 R_0^2}{3c^2 a^{1+3w}} = -k.$$

- Introducing a single dimensionless parameter in place of the constant of integration ρ_0

$$\boxed{\left(\frac{1}{a}\frac{da}{d\eta}\right)^2 = \left(\frac{a_m}{a}\right)^{1+3w} - k} \qquad \text{where} \qquad \frac{8\pi G R_0^2\rho_0}{3c^2} = a_m^{1+3w}.$$

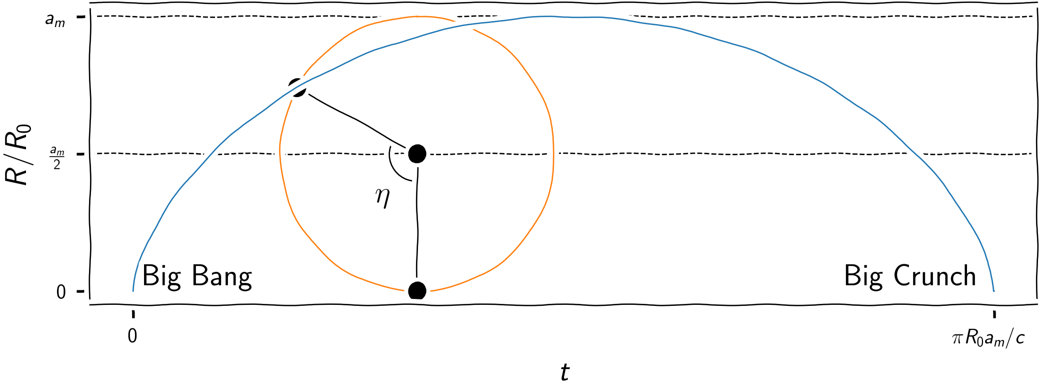
- The formal solution is then found by separation of variables, and $cdt = Rd\eta = R_0 a d\eta$

$$\eta = \int \frac{da}{a\sqrt{\left(\frac{a_m}{a}\right)^{1+3w} - k}} \qquad \text{and} \qquad t = \frac{R_0}{c}\int a d\eta.$$

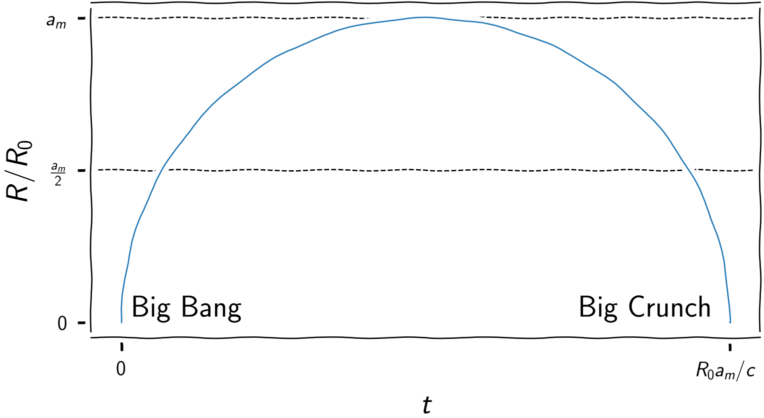
- Let's do this explicitly for matter $w = 0$, in closed universes $k = +1$, integrating from $(0,0,0)$ to (η, a, t), making the substitution $a = a_m \sin^2 x$.

$$\eta = \int_0^a \frac{da}{\sqrt{a(a_m - a)}} = \int_0^{\sin^{-1}\sqrt{a/a_m}} \frac{2a_m \sin x \cos x}{\sqrt{a_m \sin^2 x \, a_m \cos^2 x}} dx = 2\sin^{-1}\sqrt{a/a_m}.$$

- Applying half angle formulae gives the solution $\boxed{R = R_0 \frac{a_m}{2}(1 - \cos\eta)}$,
- Integrating gives $\boxed{t = R_0 \frac{a_m}{2c}(\eta - \sin\eta)}$.



- For matter-dominated closed universes we get a cycloid in the $t - R$ plane!
- a_m is well-named – when multiplied by R_0 it gives the scale factor of the universe at maximum expansion, before the recollapse begins.
- Image of the universe's evolution being given by a circle rolling along a line is suggestive
- Could there be further revolutions after the 'big crunch', or before the 'big bang'?



- For radiation dominated $w = \frac{1}{3}$ closed universes $k = +1$ the solution is a semicircle

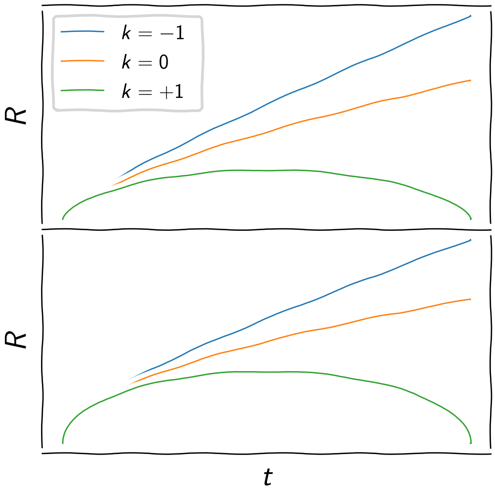
$$\boxed{R = R_0 a_m \sin\eta}, \qquad t = \frac{R_0}{c} a_m (1 - \cos\eta)$$

- Could we "cycle" this below the axis and go full circle?

Friedmann models

		matter $w \approx 0$	radiation $w = \frac{1}{3}$
closed $k = +1$	R ct	$\frac{1}{2} R_0 a_m (1 - \cos\eta)$ $\frac{1}{2} R_0 a_m (\eta - \sin\eta)$	$R_0 a_m \sin\eta$ $R_0 a_m (1 - \cos\eta)$
flat $k = 0$	R ct	$\frac{1}{4} R_0 a_m \eta^2$ $\frac{1}{12} R_0 a_m \eta^3$	$R_0 a_m \eta$ $\frac{1}{2} R_0 a_m \eta^2$
open $k = -1$	R ct	$\frac{1}{2} R_0 a_m (\cosh\eta - 1)$ $\frac{1}{2} R_0 a_m (\sinh\eta - \eta)$	$R_0 a_m \sinh\eta$ $R_0 a_m (\cosh\eta - 1)$

- Completing the set we find that in Friedmann models, geometry is generically linked to fate.
- Note that eliminating η in the flat case recovers $R \sim t^{2/3}$ for matter and $R \sim t^{1/2}$ for radiation.



Historical context

- Einstein publishes his final theory of general relativity in 1915 (convinced his non-linear equations are insoluble).
- Schwarzschild solves interior and exterior of a spherical body in 1916 from the trenches.
- Friedmann, Robertson, Walker and Lemaître then independently develop their solutions for the Universe in the 1920s and 30s.
- The fact that $R = 0$ at some finite time in the past caused a lot of difficulty.
- This was partly due to philosophical bias, but mostly because their observed universe was much smaller than ours!
- Many astronomers at that time believed our Milky way galaxy to be the entire universe.
- It was believed that the universe was infinite in time (as well as possibly in space), uniformly filled with stars, which were observed to be fixed distances from us.
- Einstein came to the rescue to explain how a universe could be static, using his cosmological constant which he had previously discarded in his derivation of the Einstein Field equations.

The Einstein static universe

- There is a static solution to the cosmological equations:

$$\dot{R} = \ddot{R} = 0 \quad \Rightarrow \quad \frac{4\pi G\rho}{3}(1+3w) = \frac{\Lambda c^2}{3}, \qquad \frac{8\pi G\rho}{3} = -\frac{\Lambda c^2}{3} + \frac{kc^2}{R^2}.$$

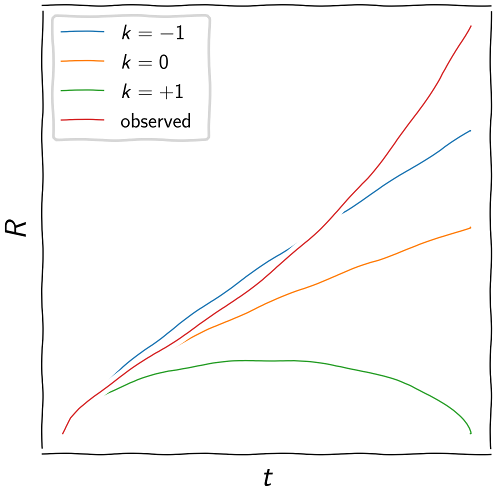
- Solving these for R^2 and ρ we find

$$R^2 = \frac{kc^2}{4\pi G\rho(1+w)}, \quad \Lambda = \frac{4\pi G}{c^2}\rho(1+3w).$$

- We therefore see that if the universe is closed $k = +1$ then for a very specific value of $\Lambda \neq 0$ the universe is static with a constant (specific) radius which depends on the density.
- We can think of this as a balance of repulsive cosmological constant with attractive gravity.
- For typical $\rho_0 \sim 1 m_p \mathrm{m}^{-3}$ (one proton per cubic meter) then $R = 8.2\,\mathrm{Gpc}$ – big enough to encompass the known universe.
- Not immediately and obviously wrong! However:
  1. It disagrees with the observed Hubble expansion.
  2. It is unstable to small perturbations $R(t) = R_{\mathrm{Einstein}} + \delta R(t)$.

Einstein's greatest mistake

- The observations of Hubble overturned the case for the static universe:
  1. We live in a galactic 'continent' of 10^{11} stars, bound together by gravity,
  2. There are many other galaxies of similar size,
  3. These galaxies all appear to be moving away, at a speed proportional to their distance.
- Einstein discarded the cosmological constant (again), calling it his "greatest mistake".
- Astronomers reverted to Friedmann models.
- However, in the 1990s, when measuring the expansion of the Universe using Type Ia supernovae, Adam Riess found the universe was accelerating – Λ was back!



Flat Λ universes ($k = 0$)

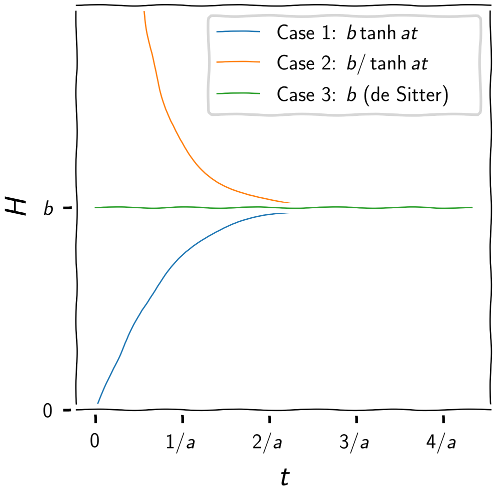
- Cosmological models which are close to spatial flatness, but have a non-zero cosmological constant, are currently favoured by the data.
- Although generally difficult to solve, one special case, that of exact spatial flatness, everything becomes reasonably simple again, and we can give explicit formulae for quantities of interest.
- Combine the dynamical equations for $w = 0$ (matter-dominated) via $(C) = 2 \times (A) + (B)$:

$$\frac{\ddot{R}}{R} + \frac{4\pi G\rho}{3} - \frac{\Lambda c^2}{3} = 0, (A) \quad \left(\frac{\dot{R}}{R}\right)^2 - \frac{8\pi G\rho}{3} - \frac{\Lambda c^2}{3} = 0, (B) \quad \Rightarrow \quad 2\frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2 - \Lambda c^2 = 0. (C)$$

- Defining the Hubble parameter as usual, we can write

$$H = \frac{\dot{R}}{R} \quad \Rightarrow \quad \dot{H} = \frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} \quad \Rightarrow \quad \dot{H} + H^2 = \frac{\ddot{R}}{R}.$$

- Our reduced cosmological equation (C) therefore becomes $\boxed{2\dot{H} + 3H^2 = \Lambda c^2}$.



- Assuming $\Lambda > 0$, there are three basic solutions, the first two of which are in some sense *dual* to each other:

$$H(t) = \begin{cases} b \tanh(at) & \text{Case 1,} \\ b/\tanh(at) & \text{Case 2,} \\ b & \text{Case 3.} \end{cases}$$

- Case 1: increases with time, starting from 0.
- Case 2: decreases, starting from ∞ at $t = 0$.
- Case 3: remains constant at this value for all time. This is called the *de Sitter universe*.
- In all H tends asymptotically to b.
- Will discuss the nature of these in a moment, but let's try to fix the constants a and b.

- Try out in Case 2 (which it turns out is a very good approximation to current data)

$$H = \frac{b}{\tanh at}, \quad \text{hence} \quad \dot{H} = \frac{-ba}{\sinh^2 at}.$$

- So need

$$\frac{-2ba}{\sinh^2 at} + \frac{3b^2}{\tanh^2 at} = \Lambda c^2, \quad \text{i.e.} \quad 3b^2 \cosh^2 at - 2ba = \Lambda c^2 \sinh^2 at.$$

- we can read off $3b^2 = \Lambda c^2 = 2ba$, meaning $b = \sqrt{\Lambda/3}c$, $a = 3/2\sqrt{\Lambda/3}c$.
- In fact find $\boxed{b = \sqrt{\Lambda/3}c}$, $\boxed{a = 3/2\sqrt{\Lambda/3}c}$ in all cases!
- It may seem odd that we can have such drastically different behaviour from the same equation, but cases distinguished by behaviour of the density.
- From acceleration equation:

$$\rho = \frac{1}{8\pi G}(3H^2 - \Lambda c^2) \Rightarrow \frac{8\pi G}{c^2}\rho(t) = \begin{cases} -\Lambda \,\mathrm{sech}^2(at) & \text{Case 1: negative density (unphysical),} \\ \Lambda \,\mathrm{cosech}^2(at) & \text{Case 2: positive density,} \\ 0 & \text{Case 3: de Sitter, empty of matter.} \end{cases}$$

- For the physical case 2

$$H(t) = \frac{\sqrt{\Lambda/3}c}{\tanh\left(3/2\sqrt{\Lambda/3}ct\right)}, \quad R(t) \propto \left(\sinh \frac{\sqrt{3\Lambda}ct}{2}\right)^{2/3}, \quad \rho(t) = \frac{\Lambda c^2}{8\pi G} \frac{1}{\sinh^2(3/2\sqrt{\Lambda/3}ct)}.$$

- Inverting the first of these, and remembering $\Omega_\Lambda = \Lambda c^2/(3H^2)$ we can obtain a useful relation for the age of the universe

$$\boxed{t = \frac{2}{3H} \frac{\tanh^{-1}\sqrt{\Omega_\Lambda}}{\sqrt{\Omega_\Lambda}}.}$$

- This goes over to the Einstein de Sitter relation $t = 2/(3H)$ as Ω_Λ tends to 0.
- Also, as $t \to \infty$, then $H \to H_\infty = \sqrt{\Lambda/3}c$ and $\Omega_\Lambda \to 1$.
- For this reason, our current universe is described as *asymptotically de Sitter* — it currently has a non-negligible matter content but as time progresses this gets diluted, and it tends towards pure de Sitter.
- It is not just in a flat model that the universe is asymptotically de Sitter.
- For virtually all the ways in which one could come out of the big bang, then Λ acts as a 'focussing' term, driving the universe towards an asymptotic end state of pure de Sitter.

de Sitter Universes $\Lambda \neq 0$

- de Sitter Universes are defined by $H \equiv \frac{\dot{R}}{R} = \text{const.}$
- This solves immediately to give $R \propto e^{Ht}$, i.e. exponentially expanding.
- These are a favourite of theoreticians as they obey the Maximal Copernican principle.
- In moving us from a geocentric to a heliocentric model, Copernicus forced us to realise the Earth was not the centre of creation.
- Cosmology applies the extended Copernican principle, to say nowhere is special.
- Equivalently: there is a Universal set of fundamental observers who observe the Universe to be homogeneous and isotropic.
- The maximal Copernican principle extends this to time, i.e. nowhen is special.
- $R \propto e^{Ht}$ with $H = 0$ obviously satisfies this, but the scaling symmetry of exponentials makes this true for $H \neq 0$.
- More mathematically, de Sitter spacetime is a Lorentzian space of constant curvature, i.e. the analogue of a sphere/pseudosphere, but in Minkowski geometry.

Summary

- Einstein-de Sitter solutions $\Lambda = k = 0$:

$$w = 0 \quad \text{matter:} \quad \boxed{R \propto t^{2/3},}$$
$$w = \tfrac{1}{3} \quad \text{radiation:} \quad \boxed{R \propto t^{1/2}.}$$

- Friedmann solutions $\Lambda = 0$, fate fixed by geometry.
- The Einstein static universe, $\boxed{R = \dfrac{c}{4\pi G\rho(1+w)} = \text{const}}$ uses Λ to balance gravity.
- Flat Λ solutions $k = 0$: late-time acceleration powered by Λ.
- The de Sitter maximally symmetric universe.
- Conformal time $\boxed{cdt = Rd\eta \Rightarrow \eta = \int \dfrac{cdt}{R}}$

Next time

Measuring the Universe