Electrodynamics and Optics

Handout 2

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§3 Electrodynamics

In this section we first want to achieve an understanding of the deeper physical significance of the magnetic vector potential $\boldsymbol{A}$ before using it practically to solve electromagnetic radiation problems due to time-varying sources.

§3.1 The Magnetic Vector Potential $\boldsymbol{A}$ (mostly revision from Part 1B)

A scalar potential can describe a conservative field such as an electro*static* field: $\boldsymbol{E} = -\boldsymbol{\nabla}\phi$.

But in general $\boldsymbol{E}$ is *time-dependent* $\oint \boldsymbol{E}\cdot\mathbf{d}\boldsymbol{l} = -\int \dot{\boldsymbol{B}}\cdot\mathbf{d}\boldsymbol{S}$, and $\boldsymbol{E}$ is not conservative and cannot be defined in terms of a scalar potential ϕ alone.

Likewise, $\oint \boldsymbol{H}\cdot\mathbf{d}\boldsymbol{l} = \int(\boldsymbol{J}+\dot{\boldsymbol{D}})\cdot\mathbf{d}\boldsymbol{S}$, so a *magnetic scalar potential* ϕ_m can be defined only in static circumstances and in regions where $\boldsymbol{J}=0$.

However $\boldsymbol{\nabla}\cdot\boldsymbol{B}=0$, and since "div curl" of any vector field is zero, $\boldsymbol{B}$ can always be written as:

$$\boxed{\boldsymbol{B}(\boldsymbol{r}) = \boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{r})} \tag{3.1}$$

This is the magnetic "*vector potential*" $\boldsymbol{A}$.

Some physical insight into its significance is suggested by considering a surface S bounded by the loop L:

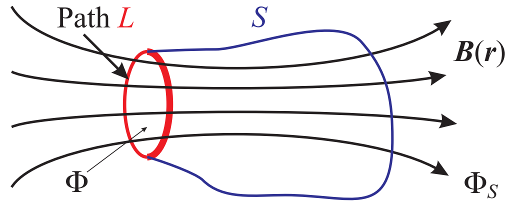

Figure 3.1: $\Phi_S = \Phi$ for *any* surface bounded by L.

$\boldsymbol{\nabla}\cdot\boldsymbol{B}=0$ so if Φ is the magnetic flux passing through the loop L, $\int_S \boldsymbol{B}\cdot\mathbf{d}\boldsymbol{S} = \Phi_S = \Phi$ is the same for *any* surface bounded by L.

Φ_S must therefore depend only on *the properties of* L – its path through space and physical quantities on that path.

Now:

$$\boxed{\Phi_S = \int \boldsymbol{B}\cdot\mathbf{d}\boldsymbol{S} = \int \boldsymbol{\nabla}\times\boldsymbol{A}\cdot\mathbf{d}\boldsymbol{S} = \oint_L \boldsymbol{A}\cdot\mathbf{d}\boldsymbol{L}} \tag{3.2}$$

– the flux through any S is equal to the line integral of $\boldsymbol{A}$ around L.

This is precisely the required condition: the flux through any surface bounded by the closed path L depends only on properties on the path L.

ME3: $\boldsymbol{\nabla} \times \boldsymbol{E} = -\dot{\boldsymbol{B}} = -\boldsymbol{\nabla} \times \dot{\boldsymbol{A}}$.

Integrating:

$$\boxed{\boldsymbol{E} = -\dot{\boldsymbol{A}} - \boldsymbol{\nabla}\phi} \tag{3.3}$$

where ϕ is in general *any* scalar field, and we have used the fact that $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\boldsymbol{\phi}$ is identically zero

In static situations $\boldsymbol{E} \longrightarrow -\boldsymbol{\nabla}\phi$, so ϕ must correspond to the familiar electrostatic potential.

So the vector potential $\boldsymbol{A}$ is required to describe both electric and magnetic fields in time-varying circumstances.

§3.1.1 Gauges in EM

$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ does not completely specify $\boldsymbol{A}$.

$\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\chi) \equiv 0$ for any scalar function χ, so for $\boldsymbol{A}' \rightarrow \boldsymbol{A} + \boldsymbol{\nabla}\chi$:

$$\boldsymbol{B}' \rightarrow \boldsymbol{\nabla} \times \boldsymbol{A}' = \boldsymbol{\nabla} \times \boldsymbol{A} + \boldsymbol{\nabla} \times (\boldsymbol{\nabla}\chi) = \boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{B}$$

So any number of $\boldsymbol{A}$s result in the same $\boldsymbol{B}$.

Correspondingly, $\boldsymbol{E} = -\dot{\boldsymbol{A}} - \boldsymbol{\nabla}\phi$ is unchanged for $\boldsymbol{A}' \rightarrow \boldsymbol{A} + \boldsymbol{\nabla}\chi$ provided also

$$\phi \rightarrow \phi' = \phi - \frac{\partial\chi}{\partial t}$$

χ can therefore be *chosen at will*, and the mathematically most convenient choice depends on the context in which $\boldsymbol{A}$ is used.

This choice is called a "***gauge***".

The physical fields $\boldsymbol{B}$ and $\boldsymbol{E}$ are unchanged under the "***gauge transformation***" (a mere mathematical manoeuvre):

$$\boxed{\boldsymbol{A} \rightarrow \boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla}\chi \qquad \phi \rightarrow \phi' = \phi - \frac{\partial\chi}{\partial t}} \tag{3.4}$$

Indeed, all physical results must be independent of the gauge – "***gauge invariant***".

Note that $\boldsymbol{\nabla} \cdot \boldsymbol{A}' = \boldsymbol{\nabla} \cdot \boldsymbol{A} + \nabla^2\chi$. Since χ is arbitrary so is $\nabla^2\chi$, so

$$\boxed{\boldsymbol{\nabla} \cdot \boldsymbol{A} \text{ can be chosen freely}}$$

– again, "choosing the gauge".

The following choices will prove useful:

$$\boxed{\textbf{Coulomb Gauge}: \qquad \boldsymbol{\nabla} \cdot \boldsymbol{A} = 0} \tag{3.5}$$

$$\boxed{\textbf{Lorenz Gauge}: \qquad \boldsymbol{\nabla} \cdot \boldsymbol{A} + \frac{1}{c^2}\frac{\partial\phi(\boldsymbol{r},t)}{\partial t} = 0} \tag{3.6}$$

[The Danish Lorenz $\not\equiv$ the Dutch Lorentz.]

§3.1.2 Steady Currents in vacuum

For steady currents in vacuum, $\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$ and since $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$:

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} = \mu_0 \boldsymbol{J}$$

Choosing the Coulomb gauge $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$:

$$\boxed{\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J}} \tag{3.7}$$

This is of exactly the same form as *Poisson's equation* for the scalar electrostatic potential:

$$\nabla^2 \phi = -\rho/\epsilon_0 \tag{3.8}$$

for which the solution is:

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \int_{\text{all space}} \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}^3\boldsymbol{r}' \tag{3.9}$$

Each component of $\boldsymbol{A}$ must obey a similar equation and therefore in general

$$\boxed{\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int_{\text{all space}} \frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}^3\boldsymbol{r}' \longrightarrow \frac{\mu_0}{4\pi} \int_{\text{all space}} \frac{I}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathbf{d}\boldsymbol{l}'} \tag{3.10}$$

where, for a line current I, $\boldsymbol{J}(\boldsymbol{r}')\mathrm{d}^3\boldsymbol{r}' \longrightarrow I\mathbf{d}\boldsymbol{l}'$, and then $\boxed{\mathbf{d}\boldsymbol{A} \parallel I\mathbf{d}\boldsymbol{l}'}$

Note:

$$\begin{aligned}
\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} &= \frac{\mu_0}{4\pi} \int I \boldsymbol{\nabla} \times \left(\frac{\mathbf{d}\boldsymbol{l}'}{|\boldsymbol{r} - \boldsymbol{r}'|} \right) = -\frac{\mu_0}{4\pi} \int I\mathbf{d}\boldsymbol{l}' \times \boldsymbol{\nabla} \left(\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \right) \\
&= -\frac{\mu_0}{4\pi} \int I\mathbf{d}\boldsymbol{l}' \times \frac{(-\boldsymbol{r} + \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \\
&= \frac{\mu_0}{4\pi} \int I \frac{\mathbf{d}\boldsymbol{l}' \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3}
\end{aligned} \tag{3.11}$$

– the *Biot-Savart Law.*

§3.2 The Vector Potential $\boldsymbol{A}$ for Simple Cases

$\boldsymbol{A}$ can be calculated by:

(i) Direct calculation from the current distribution – Eqn 3.10.

(ii) Integration of a known $\boldsymbol{B}$-field from $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$.

(iii) Equating the line integral of $\boldsymbol{A}$ to the flux through any surface bounded by the path of the line integral – Eqn 3.2.

In all cases, the *symmetry* properties of the vector potential reflect the symmetry of the problem.

In particular for steady currents, from Eqn 3.10: $\mathbf{d}\boldsymbol{A} \parallel \mathbf{d}\boldsymbol{J} \equiv I\mathbf{d}\boldsymbol{l}$.

§3.2.1 A Long Straight Wire

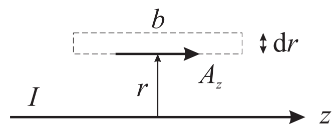

Figure 3.2: Geometry for a long straight wire carrying current I.

From Eqn 3.10, $\mathbf{d}\boldsymbol{A} \parallel \mathbf{d}\boldsymbol{J}$, so $\boldsymbol{A} \parallel I$: $\boldsymbol{A} = A_z \widehat{\boldsymbol{z}}$.

From the cylindrical symmetry, A_z is a function only of the distance r from the wire.

From Ampère's Law, $\boldsymbol{B} = (0, B_\phi = \mu_0 I/2\pi r, 0)$. From Eqn 3.2 and integrating $\boldsymbol{A}$ round the loop of length b and width $\mathrm{d}r$ as shown above:

$$\begin{aligned}
[A_z(r) - A_z(r + \mathrm{d}r)]b = -\frac{\partial A_z}{\partial r} \mathrm{d}r\, b = B_\phi b\, \mathrm{d}r &= \frac{\mu_0 I}{2\pi r} b\, \mathrm{d}r \\
A_z(r) = -\frac{\mu_0 I}{2\pi} \ln r + \ &\text{constant}
\end{aligned} \tag{3.12}$$

§3.2.2 A Distant Current Loop

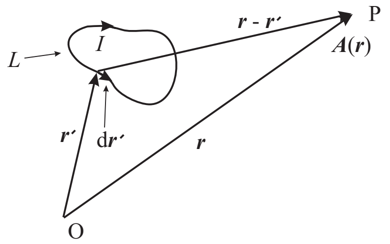

Figure 3.3: Geometry for a distant loop of wire L carrying current I.

At P:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int_{\text{all space}} \frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}^3\boldsymbol{r}' \longrightarrow \frac{\mu_0}{4\pi}\oint_L \frac{I}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}\boldsymbol{r}'$$

Taking O close to the loop and $\boldsymbol{r}$ distant: $|\boldsymbol{r}| \gg |\boldsymbol{r}'|$. So to first order in r'/r:

$$|\boldsymbol{r}-\boldsymbol{r}'|^{-1} = (r^2 + r'^2 - 2\boldsymbol{r}'\cdot\boldsymbol{r})^{-\frac{1}{2}} \approx \frac{1}{r}\left[1 + \frac{\boldsymbol{r}\cdot\boldsymbol{r}'}{r^2} + \ldots\right]$$

So

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi}\left[\frac{1}{r}\oint_L \mathrm{d}\boldsymbol{r}' + \frac{1}{r^3}\oint_L (\boldsymbol{r}'\cdot\boldsymbol{r})\mathrm{d}\boldsymbol{r}' + \ldots\right]$$

The first integral is clearly zero.

Now

$$(\boldsymbol{r}'\times\mathbf{d}\boldsymbol{r}')\times\boldsymbol{r} = -\boldsymbol{r}'(\boldsymbol{r}\cdot\mathbf{d}\boldsymbol{r}') + \mathbf{d}\boldsymbol{r}'(\boldsymbol{r}'\cdot\boldsymbol{r})$$

and, taking the differential of $\boldsymbol{r}'(\boldsymbol{r}\cdot\boldsymbol{r}')$ for a small change in $\boldsymbol{r}'$ ($\boldsymbol{r}$ is constant):

$$\mathrm{d}[\boldsymbol{r}'(\boldsymbol{r}\cdot\boldsymbol{r}')] = \boldsymbol{r}'(\boldsymbol{r}\cdot\mathbf{d}\boldsymbol{r}') + \mathbf{d}\boldsymbol{r}'(\boldsymbol{r}'\cdot\boldsymbol{r})$$

Adding:

$$\mathbf{d}\boldsymbol{r}'(\boldsymbol{r}'\cdot\boldsymbol{r}) = \tfrac{1}{2}\mathrm{d}[\boldsymbol{r}'(\boldsymbol{r}\cdot\boldsymbol{r}')] + \tfrac{1}{2}(\boldsymbol{r}'\times\mathbf{d}\boldsymbol{r}')\times\boldsymbol{r}$$

The first term, a perfect differential, integrated round the closed loop L gives zero.

So

$$\boxed{\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\left[\frac{I}{2}\oint_L \boldsymbol{r}'\times\mathbf{d}\boldsymbol{r}'\right]\times\frac{\boldsymbol{r}}{r^3} = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\boldsymbol{r}}{r^3}} \qquad (3.13)$$

where $\boldsymbol{m}$ is the *magnetic moment* of the loop. [Pt IB.]

For a planar loop:

$$\frac{1}{2}\oint_L \boldsymbol{r}'\times\mathbf{d}\boldsymbol{r}' = A\widehat{\boldsymbol{n}}$$

where A is the area of the loop and $\widehat{\boldsymbol{n}}$ is the unit vector normal to plane of the loop. Then:

$$\boxed{\boldsymbol{m} = IA\widehat{\boldsymbol{n}}} \qquad (3.14)$$

§3.2.3 An Infinite Solenoid

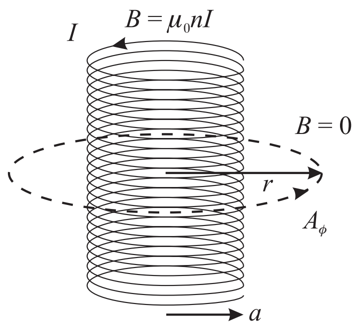

Figure 3.4: A solenoid of radius a, n turns per unit length, carrying current I.

Inside: $\boldsymbol{B} = (0, 0, \mu_0 nI)$ Outside: $\boldsymbol{B} = (0, 0, 0)$

From symmetry, $\boldsymbol{A}$ must follow the direction of the current elements, so must be in the $\widehat{\boldsymbol{\phi}}$-direction, and from Eqn 3.2:

$$\int \boldsymbol{B}\cdot\mathbf{d}\boldsymbol{S} = \oint_L \boldsymbol{A}\cdot\mathbf{d}\boldsymbol{L}$$

So integrating round circular loops co-axial with the solenoid:

(i) Inside: $\quad A_\phi 2\pi r = \mu_0 n I \pi r^2 \quad \longrightarrow \quad A_\phi = \dfrac{\mu_0 n I r}{2}$

(ii) Outside: $\quad A_\phi 2\pi r = \mu_0 n I \pi a^2 \quad \longrightarrow \quad A_\phi = \dfrac{\mu_0 n I a^2}{2r}$

Note that *outside* the solenoid, $\boldsymbol{A}$ ***is non-zero***, even though $\boldsymbol{B} = 0$.

So is $\boldsymbol{A}$ a real physical field, or just a mathematical construct?

If $\boldsymbol{A}$ is a real physical field, there should be *experimental effects* that sense $\boldsymbol{A}$ even where $\boldsymbol{B} = 0$.

Classically, there can be no such effect: the classical motion of a charged particle depends only on $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$.

We will now see how the *quantum* description of a charged particle *is* sensitive to $\boldsymbol{A}$, even when $\boldsymbol{B} = 0$.

§3.3 $\boldsymbol{A}$ in Quantum Mechanics

Forces are not part of the structure of QM. External influences on particles are included using *potentials* in the Hamiltonian. To understand how a quantum mechanical particle of mass m and charge q couples to the electromagnetic field we first construct a classical Hamiltonian that yields the Lorentz force equation of motion when applying the formalism of classical Hamiltonian dynamics.

$$\boxed{H = \frac{1}{2m}\left[\boldsymbol{p}_{\mathrm{k}}\right]^2 + q\phi(\boldsymbol{r},t) = \frac{1}{2m}\left[\boldsymbol{p}_{\mathrm{c}} - q\boldsymbol{A}(\boldsymbol{r},t)\right]^2 + q\phi(\boldsymbol{r},t)} \tag{3.15}$$

where the *kinetic* momentum $\boldsymbol{p}_{\mathrm{k}} = m\boldsymbol{v}$ is to be distinguished from the *canonical* or conjugate momentum $\boldsymbol{p}_{\mathrm{c}}$ which arises in the Hamiltonian and Lagrangian formulations of classical mechanics [TP1]. $H(.., p_i, ..., x_i, ...)$ is a function of the independent phase space co-ordinates p_i, x_i etc., where p_i is conjugate (in the Lagrangian sense) to x_i.

Eqn 3.15 can be validated using Hamilton's classical equations of motion:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i} \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial x_i} \tag{3.16}$$

whence:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{1}{m}[p_i - qA_i(\boldsymbol{r},t)] \tag{3.17}$$

and (remembering that the p_is and x_is are independent co-ordinates):

$$\begin{aligned}\frac{\mathrm{d}p_i}{\mathrm{d}t} &= -\frac{(-q)}{m}\sum_k [p_k - qA_k(\boldsymbol{r},t)]\frac{\partial A_k}{\partial x_i} - q\frac{\partial \phi}{\partial x_i} \\ &= \frac{q}{m}\sum_k \left(m\frac{\mathrm{d}x_k}{\mathrm{d}t}\right)\frac{\partial A_k}{\partial x_i} - q\frac{\partial \phi}{\partial x_i} \qquad \text{using Eqn 3.17}\end{aligned}$$

Differentiating Eqn 3.17 w.r.t. t:

$$\begin{aligned}m\frac{\mathrm{d}^2 x_i}{\mathrm{d}t^2} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(p_i - qA_i(\boldsymbol{r},t)\right) \\ &= \frac{\mathrm{d}p_i}{\mathrm{d}t} \quad -q\left(\frac{\partial A_i}{\partial t} + \sum_k \frac{\partial A_i}{\partial x_k}\frac{\mathrm{d}x_k}{\mathrm{d}t}\right) \\ &= \frac{q}{m}\sum_k\left(m\frac{\mathrm{d}x_k}{\mathrm{d}t}\right)\frac{\partial A_k}{\partial x_i} - q\frac{\partial \phi}{\partial x_i} \quad -q\frac{\partial A_i}{\partial t} - q\sum_k \frac{\partial A_i}{\partial x_k}\frac{\mathrm{d}x_k}{\mathrm{d}t} \\ &= -q\left(\frac{\partial \phi}{\partial x_i} + \frac{\partial A_i}{\partial t}\right) - q\sum_k \frac{\mathrm{d}x_k}{\mathrm{d}t}\left(\frac{\partial A_i}{\partial x_k} - \frac{\partial A_k}{\partial x_i}\right)\end{aligned}$$

Recalling $\boldsymbol{E} = -\dot{\boldsymbol{A}} - \boldsymbol{\nabla}\phi$:

$$\begin{aligned}m\frac{\mathrm{d}^2 x_i}{\mathrm{d}t^2} &= qE_i + q[\boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{A})]_i \\ &= qE_i + q(\boldsymbol{v} \times \boldsymbol{B})_i\end{aligned} \tag{3.18}$$

recovering the equation of motion for the *Lorentz force*.

So the classical Hamiltonian (Eqn 3.15) is consistent with the *experimentally validated* Lorentz force, and will be taken as correct. Writing it as a QM operator:

$$\widehat{H} = \frac{1}{2m}\left[\widehat{\boldsymbol{p}}_{\rm c} - q\boldsymbol{A}(\boldsymbol{r},t)\right]^2 + q\phi(\boldsymbol{r},t)$$

with $\widehat{\boldsymbol{p}}_{\rm c} = -i\hbar\boldsymbol{\nabla}$ as usual for the canonical momentum. Then

$$\boxed{\widehat{\boldsymbol{p}}_{\rm k}\ (= \text{“}\,\widehat{\boldsymbol{p}}\,\text{”}) = -i\hbar\boldsymbol{\nabla} - q\boldsymbol{A}(\boldsymbol{r},t)} \qquad (3.19)$$

The eigenstates of $\widehat{\boldsymbol{p}} = -i\hbar\boldsymbol{\nabla} - q\boldsymbol{A}(\boldsymbol{r},t)$ are

$$\psi \sim \exp\left[i\left(\boldsymbol{k}\cdot\boldsymbol{r} + \frac{q\boldsymbol{A}\cdot\boldsymbol{r}}{\hbar}\right)\right] \qquad (3.20)$$

with eigenvalues $\hbar\boldsymbol{k}$.

If $\phi = 0$, these are also obviously eigenstates of $\widehat{H}$ with eigenvalues $\hbar^2k^2/2m$.

Let us now consider the *gauge transformation*

$$\phi \to \phi' = \phi - \frac{\partial\chi(\boldsymbol{r},t)}{\partial t} \qquad\qquad \boldsymbol{A} \to \boldsymbol{A'} = \boldsymbol{A} + \boldsymbol{\nabla}\chi(\boldsymbol{r},t)$$

. The Schrödinger equation for the wavefunction $\Psi(\boldsymbol{r},t)$ in the first gauge is:

$$[\frac{1}{2m}\left[-i\hbar\boldsymbol{\nabla} - q\boldsymbol{A}(\boldsymbol{r},t)\right]^2 + q\phi(\boldsymbol{r},t)]\Psi(\boldsymbol{r},t) = i\hbar\frac{\partial}{\partial t}\Psi(\boldsymbol{r},t)$$

We will now show that the wavefunction $\Psi'(\boldsymbol{r},t)$ in the second gauge is:

$$\Psi'(\boldsymbol{r},t) = \exp(\frac{ie}{\hbar}\chi(\boldsymbol{r},t))\Psi(\boldsymbol{r},t)$$

By multiplying the Schrödinger equation by $\exp(\frac{ie}{\hbar}\chi(\boldsymbol{r},t))$ and using the identity $e^{f(y)}\frac{\partial}{\partial y} = (\frac{\partial}{\partial y} - \frac{\partial f}{\partial y})e^{f(y)}$ we obtain:

$$[\frac{1}{2m}\left[-i\hbar\boldsymbol{\nabla} - q\boldsymbol{A} + i\hbar\frac{ie}{\hbar}\boldsymbol{\nabla}\chi\right]^2 + q\phi]\exp(\frac{ie}{\hbar}\chi)\Psi = i\hbar(\frac{\partial}{\partial t} - \frac{ie}{\hbar}\frac{\partial\chi}{\partial t})\exp(\frac{ie}{\hbar}\chi)\Psi$$

This is obviously identical with

$$[\frac{1}{2m}\left[-i\hbar\boldsymbol{\nabla} - q\boldsymbol{A'}(\boldsymbol{r},t)\right]^2 + q\phi'(\boldsymbol{r},t)]\Psi'(\boldsymbol{r},t) = i\hbar\frac{\partial}{\partial t}\Psi'(\boldsymbol{r},t)$$

For the Schrödinger equation to be *gauge invariant* the gauge transformation requires an additional phasefactor of the wavefunction, which has, however, no directly observable consequences as it does not change $|\Psi|^2$.

We will now use this gauge transformation of the wavefunction to understand a particularly mysterious quantum mechanical phenomenon, the Aharonov-Bohm effect, a quantum mechanical electron interference effect that can be modulated by applying a magnetic flux through a localised region that the electrons never enter into.

§3.3.1 The Aharonov-Bohm Effect

Consider the motion of an electron in the presence of a time-independent magnetic field $\boldsymbol{B}(\boldsymbol{r})$ that vanishes in a certain region S, i.e., $\boldsymbol{B} = \nabla\times\boldsymbol{A} = 0$ in S. In this region the vector potential $\boldsymbol{A}$ can therefore be represented as the gradient of a scalar field χ, $\boldsymbol{A} = \boldsymbol{\nabla}\chi$ and $\chi(\boldsymbol{r}) = \int_{\boldsymbol{r}_0}^{\boldsymbol{r}} d\boldsymbol{s}\cdot\boldsymbol{A}(\boldsymbol{s})$, where $\boldsymbol{r}_0$ is an arbitrary reference point in the field-free region. We can find the wavefunction either by solving the Schrödinger equation in the presence of $\boldsymbol{A}$ or, more elegantly, by performing a gauge transformation $\boldsymbol{A'} = \boldsymbol{A} + \boldsymbol{\nabla}(-\chi) = 0$, and then solving simply the problem of the wavefunction Ψ' in the absence of a magnetic field (V represents whatever non-electrical potential is experienced by the electrons):

$$\frac{1}{2m}(-i\hbar\boldsymbol{\nabla})^2 + V]\Psi' = i\hbar\frac{\partial}{\partial t}\Psi'$$

For the wavefunction Ψ in the presence of the vector potential we then obtain:

$$\Psi = \Psi'_{A=0}\exp(\frac{ie}{\hbar}\chi) = \Psi'_{A=0}\exp(\frac{ie}{\hbar}\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} d\boldsymbol{s}\cdot\boldsymbol{A}(\boldsymbol{s})) \qquad (3.21)$$

Let's consider the situation below where the magnetic field is only non-zero in the red region, which is designed to be impenetrable to the electrons. An electron can travel from a to b along different paths, e.g. paths 1 and 2.

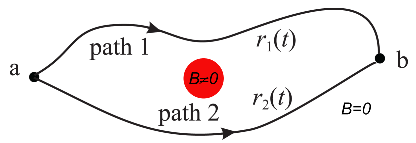

Figure 3.5: Paths between a and b for calculating the QM phases.

The wavefunctions of electrons traveling along Path 1 and Path 2 acquire different phase shifts:

$$\Psi_1 = \Psi'_{1,A=0}\exp\left(\frac{ie}{\hbar}\int_{path1} d\boldsymbol{s}\cdot\boldsymbol{A}(\boldsymbol{s})\right)$$

$$\Psi_2 = \Psi'_{2,A=0}\exp\left(\frac{ie}{\hbar}\int_{path2} d\boldsymbol{s}\cdot\boldsymbol{A}(\boldsymbol{s})\right)$$

The total wavefunction amplitude at b due to these two paths is obtained by linear superposition.

$$\Psi_{1+2} = \left(\Psi'_{1,A=0}\exp\left(\frac{ie}{\hbar}\Phi_B\right) + \Psi'_{2,A=0}\right)\exp\left(\frac{ie}{\hbar}\int_{path2} d\boldsymbol{s}\cdot\boldsymbol{A}(\boldsymbol{s})\right)$$

Φ_B is the magnetic flux enclosed in the red region:

$$\Phi_B = \int_{path1} d\boldsymbol{s}\cdot\boldsymbol{A}(\boldsymbol{s}) - \int_{path2} d\boldsymbol{s}\cdot\boldsymbol{A}(\boldsymbol{s}) = \oint d\boldsymbol{s}\cdot\boldsymbol{A}(\boldsymbol{s}) = \int dS\,\boldsymbol{\nabla}\times\boldsymbol{A}$$

The relative phase between $\Psi'_{1,A=0}$ and $\Psi'_{2,A=0}$ can be changed with the magnetic flux enclosed and affects the experimentally observable interference pattern that is created by the two electron paths at point b, although neither of the two electron paths ever penetrates the region where B is non-zero. (Aside: the relative phase is not influenced by any electrostatic potential that may also be present, because the electrostatic potential is conservative and the net phase shift along a closed path is zero.)

In 1959 Aharonov and Bohm suggested a conceptual setup to detect this effect:

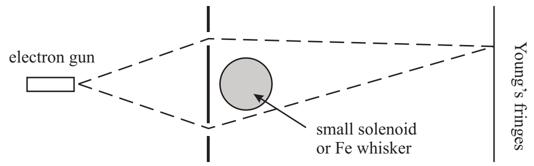

Figure 3.6: Schematic of a Young's slit experiment for electrons.

The electrons pass only through regions where $\boldsymbol{B} = 0$, but the flux Φ enclosed by the two paths (produced by $\boldsymbol{B}$ in the solenoid or ferromagnet) introduces a phase difference $\Delta = e\Phi/\hbar$ – the Aharonov-Bohm effect.

Δ should cause the interference pattern on the screen to shift as Φ is varied.

In 1988 Tonomura achieved this experimentally by bending the ferromagnetic whisker in Fig. 3.6 into a *ring* – in fact a Permalloy ring encased in Nb:

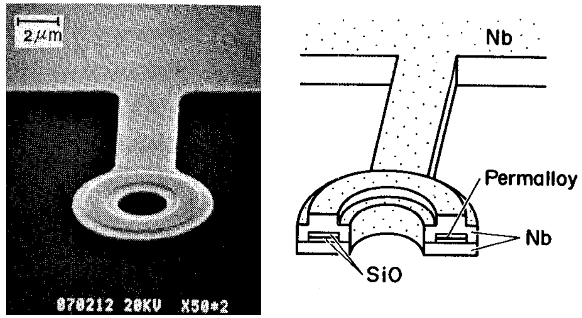

Figure 3.7: Electron micrograph and schematic of Tonomura's ring. (Physica **B 151** (1988) 206-213)

Permalloy is *ferromagnetic*, so a magnetic field runs around the ring, but none outside it. So electrons passing through the ring experience a different phase shift Δ from those passing outside, and this is manifest in the shift of the fringe patterns in Fig. 3.8

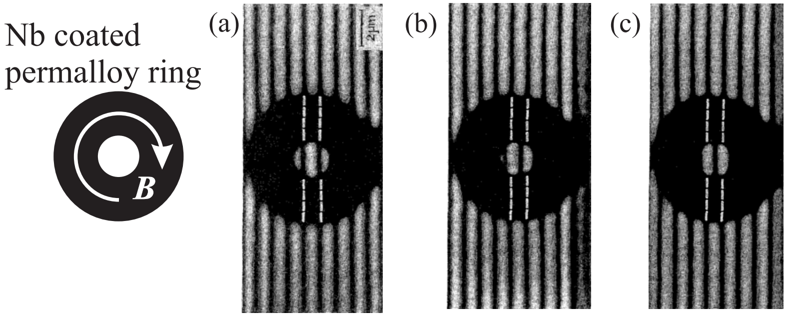

Figure 3.8: Interference of electron waves through Tonomura's ring. (a) $T = 300$ K, $\Delta = 0.3\pi$; (b) $T = 15$ K, $\Delta = 0.8\pi$; (c) $T = 4.5$ K, $\Delta = \pi$.

The pattern shifts by a one fringe-spacing each time $\Delta = e\Phi/\hbar$ increases by 2π: i.e. when Φ increases by $\Phi_0 = h/e$.

As T falls from 300 K in (a), the permalloy becomes more magnetized as thermal disorder of the spins falls, so B, Φ and therefore Δ increase in (b).

At 4.5 K, the Nb coating is superconducting, and expels the B-field entirely (the Meissner effect).

The result is that the flux Φ in the permalloy ring must be an integer number n of *flux quanta* (or *fluxoids*) $h/2e$.

Then

$$\Delta = \frac{e}{\hbar}\,\Phi = \frac{e}{\hbar}\,\frac{nh}{2e} = n\pi \tag{3.22}$$

n is odd in (c).

§3.4 Maxwell's Equations in Terms of $\boldsymbol{A}$ and ϕ

Maxwell's four equations include some redundancy in that they do not all give independent information. For example, taking the divergence of:

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\dot{\boldsymbol{B}}$$

gives:

$$\begin{aligned} \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{E}) &= -\boldsymbol{\nabla} \cdot \dot{\boldsymbol{B}} \\ \Rightarrow 0 &= -\frac{\partial}{\partial t} \boldsymbol{\nabla} \cdot \boldsymbol{B} \end{aligned}$$

But this is already known from $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$; the divergent part of $\boldsymbol{\nabla} \times \boldsymbol{E} = -\dot{\boldsymbol{B}}$ tells us nothing further.

Maxwell's equations include *8 independent unknowns* – 4 independent components of $\boldsymbol{E}$ and $\boldsymbol{B}$, 3 components of $\boldsymbol{J}$, and ρ.

A more compact, simpler, way of expressing the same physical concepts involves the use of the potentials ϕ and $\boldsymbol{A}$.

Maxwell's equations in the presence of a medium (ϵ, μ) and arbitrary free charges and currents (ρ, $\boldsymbol{J}$) are (from p. 9):

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{D} &= \rho && \text{ME1} \\ \boldsymbol{\nabla} \cdot \boldsymbol{B} &= 0 && \text{ME2} \\ \boldsymbol{\nabla} \times \boldsymbol{E} &= -\dot{\boldsymbol{B}} && \text{ME3} \\ \boldsymbol{\nabla} \times \boldsymbol{H} &= \boldsymbol{J} + \dot{\boldsymbol{D}} && \text{ME4} \end{aligned}$$

Writing the fields in terms of the potentials:

$$\mu\mu_0 \boldsymbol{H} = \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \qquad \frac{\boldsymbol{D}}{\epsilon\epsilon_0} = \boldsymbol{E} = -\dot{\boldsymbol{A}} - \boldsymbol{\nabla}\phi$$

effectively replaces ME2 and ME3.

Putting $\boldsymbol{E} = -\dot{\boldsymbol{A}} - \boldsymbol{\nabla}\phi$ into ME1:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = -\frac{\partial}{\partial t} \boldsymbol{\nabla} \cdot \boldsymbol{A} - \nabla^2 \phi = \frac{\rho}{\epsilon\epsilon_0}$$

gives

$$-\frac{\partial}{\partial t}\boldsymbol{\nabla}\cdot\boldsymbol{A} - \nabla^2\phi = \frac{\rho}{\epsilon\epsilon_0} \tag{3.23}$$

Substituting for $\boldsymbol{D}$ and $\boldsymbol{H}$ in ME4:

$$\mu\mu_0\boldsymbol{\nabla}\times\boldsymbol{H} = \boldsymbol{\nabla}\times\boldsymbol{B} = \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A}) - \nabla^2\boldsymbol{A} = \mu\mu_0\boldsymbol{J} - \epsilon\epsilon_0\mu\mu_0(\ddot{\boldsymbol{A}} + \boldsymbol{\nabla}\dot{\phi})$$

$$\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\boldsymbol{A} + \epsilon\epsilon_0\mu\mu_0\dot{\phi}\right) - \nabla^2\boldsymbol{A} = \mu\mu_0\boldsymbol{J} - \epsilon\epsilon_0\mu\mu_0\ddot{\boldsymbol{A}} \tag{3.24}$$

Eqns 3.23 and 3.24 are coupled, since $\boldsymbol{A}$ and ϕ appear in both, and *complicated*, but can be simplified by choosing a suitable *gauge* (choice of χ in §3.1.1) such that $\boldsymbol{A}$ and ϕ satisfy the "***Lorenz condition***"

$$\boxed{\boldsymbol{\nabla}\cdot\boldsymbol{A} + \epsilon\epsilon_0\mu\mu_0\dot{\phi} = \boldsymbol{\nabla}\cdot\boldsymbol{A} + \epsilon\mu\frac{\dot{\phi}}{c^2} = 0} \tag{3.25}$$

Applying the Lorenz condition to Eqns 3.23 and 3.24 gives

$$\frac{\epsilon\mu}{c^2}\ddot{\phi} - \nabla^2\phi = \frac{\rho}{\epsilon\epsilon_0} \qquad \text{MA1} \tag{3.26}$$

$$\frac{\epsilon\mu}{c^2}\ddot{\boldsymbol{A}} - \nabla^2\boldsymbol{A} = \mu\mu_0\boldsymbol{J} \qquad \text{MA2} \tag{3.27}$$

Eqns MA1 and MA2 express Maxwell's Equations in terms of the *potentials* rather than the fields, and *replace 8* component equations implicit in Maxwell Equations with *4 component equations.*

Eqns MA1 and MA2 are *wave equations* with *source terms* given by the free charges and currents of the system.

For static fields they reduce to Poisson's equation, the Lorenz condition is then equivalent to the Coulomb gauge $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$.

With the earlier equations:

$$\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A} \qquad \text{C1} \tag{3.28}$$

$$\boldsymbol{E} = -\dot{\boldsymbol{A}} - \boldsymbol{\nabla}\phi \qquad \text{C2} \tag{3.29}$$

Eqns MA1 and MA2 allow a *complete description of Electromagnetism* in terms of *the vector and scalar potentials.*

§3.5 General Solution for $\boldsymbol{A}$ and ϕ

Using physical (rather than mathematically rigorous) arguments, the general solution to these equations can be found as follows. For simplicity, the medium will be taken to be the *vacuum*: $\epsilon = 1$ and $\mu = 1$, so Eqns MA1 and MA2 become:

$$\frac{1}{c^2}\ddot{\phi} - \nabla^2\phi = \frac{\rho}{\epsilon_0} \tag{3.30}$$

$$\frac{1}{c^2}\ddot{\boldsymbol{A}} - \nabla^2\boldsymbol{A} = \mu_0\boldsymbol{J} \tag{3.31}$$

– wave equations with *sources* ρ and $\boldsymbol{J}$, with general solutions of the form $f(t \pm r/c)$.

Consider a time varying charge at the origin, $q(r = 0, t)$. The solution for ϕ must be spherically symmetric ($\boldsymbol{r} \to r$) and probably follow an inverse square law:

$$\phi(r,t) \sim \frac{1}{r}g\left(t - \frac{r}{c}\right)$$

where the "$-$" sign corresponds to an *outgoing* wave. So the solution at r at any time depends on the charge at a time *earlier* by $-r/c$, the *retarded time.*

If the characteristic time over which the charge at O changes is τ, then for $r \ll c\tau$, the solution must be much like the static case, which (from Poisson's equation) is:

$$\phi(r,t) = \frac{1}{4\pi\epsilon_0}\frac{q(t)}{r}$$

Continuity between these means that the complete solution must be:

$$\phi(r,t) = \frac{1}{4\pi\epsilon_0}\frac{q\left(t - \frac{r}{c}\right)}{r}$$

For a source term that is a charge *distribution* $\rho(\boldsymbol{r}', t)$ the solution follows from *superposition* since the wave equations are *linear*. So:

$$\boxed{\phi(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon_0} \int_{\text{all space}} \frac{\rho\left(\boldsymbol{r}', t - \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right) \mathrm{d}V'}{|\boldsymbol{r}-\boldsymbol{r}'|}} \tag{3.32}$$

– the *retarded scalar potential* which allows for the propagation of information at a *finite speed* c. ϕ at $\boldsymbol{r}$ at time t depends on the charge density at $\boldsymbol{r}'$ at a time $|\boldsymbol{r}-\boldsymbol{r}'|/c$ *earlier*.

Each component of $\boldsymbol{A}$ satisfies a similar wave equation plus current source term. So in the same way, the *retarded vector potential* is:

$$\boxed{\boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} \int_{\text{all space}} \frac{\boldsymbol{J}\left(\boldsymbol{r}', t - \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right) \mathrm{d}V'}{|\boldsymbol{r}-\boldsymbol{r}'|}} \tag{3.33}$$

§3.5.1 Notation

Denote

$$\rho\left(\boldsymbol{r}', t - \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right) \quad \text{by} \quad [\rho\,(\boldsymbol{r}', \boldsymbol{r}, t)] \quad \text{or} \quad [\rho]$$

where the square brackets mean "evaluate at the *retarded time*" $t - \dfrac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}$.

Then Eqns 3.32 and 3.33 take the form:

$$\phi(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon_0} \int_{\text{all space}} \frac{[\rho]\,\mathrm{d}V'}{|\boldsymbol{r}-\boldsymbol{r}'|} \tag{3.34}$$

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} \int_{\text{all space}} \frac{[\boldsymbol{J}]\,\mathrm{d}V'}{|\boldsymbol{r}-\boldsymbol{r}'|} \tag{3.35}$$

Differentials of retarded quantities F such as

$$\frac{\partial}{\partial r}[F] = \frac{\partial}{\partial r} F(t - r/c)$$

are often required.

Straightforwardly using the chain rule:

$$\boxed{\frac{\partial}{\partial r}[F] = -\frac{1}{c}[F'] \qquad \text{and} \qquad \frac{\partial}{\partial t}[F] = [F']} \tag{3.36}$$

....where F' is the derivative of F wrt its argument.

§4 Dipole Radiation

§4.1 Time-varying Fields: a Physical Approach

The retarded potentials derived in §3.5 can now be exploited to discuss time-varying fields and *radiation*:

$$\phi(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon_0} \int_{\text{all space}} \frac{\rho\left(\boldsymbol{r}', t - \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right) \mathrm{d}V'}{|\boldsymbol{r}-\boldsymbol{r}'|} \tag{4.1}$$

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} \int_{\text{all space}} \frac{\boldsymbol{J}\left(\boldsymbol{r}', t - \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right) \mathrm{d}V'}{|\boldsymbol{r}-\boldsymbol{r}'|} \tag{4.2}$$

Moving charges cause time-varying fields. But, for a charge moving at *constant* velocity in a vacuum a frame can be found in which it is *stationary* and therefore *does not lose energy*. So *a charge moving at constant velocity in vacuum cannot radiate.*

So *accelerating* charges are of interest for radiation.

Consider (in free space) an electric dipole $\boldsymbol{p}$ at O, and its field $\boldsymbol{E}$ at X:

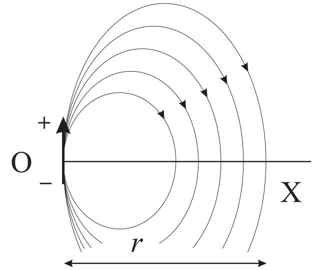

Figure 4.1: Static field of an electric dipole.

Statically, $\boldsymbol{E}$ at X is vertically downward. If the dipole is flipped over, $\boldsymbol{E}$ reverses direction – but at time r/c later because of the *retardation effects* in the propagation of ϕ.

So, if $\boldsymbol{p}(t) = \boldsymbol{p}_0 \mathrm{e}^{-i\omega t}$, $\boldsymbol{E}$ also oscillates but *shifted in phase* because of retardation. $\boldsymbol{E}(t) \sim \mathrm{e}^{-i\omega(t-r/c)}$, reflecting the dipole's situation at an earlier time – the *retarded time*.

The oscillating charges also involve *currents*, producing a *magnetic* field at X.

§4.2 The Hertzian Dipole

To calculate the fields in detail, a simple model dipole is required – the *Hertzian dipole*:

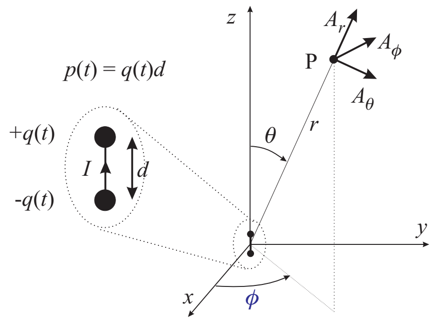

Figure 4.2: A Hertzian dipole aligned along Oz.

Context should prevent confusion between ϕ the potential and ϕ the angle!

The two charges q at O are separated by d and *vary harmonically in time* – so there must also be a *current* $I = \dot{q} = -i\omega q_0 \mathrm{e}^{-i\omega t}$ flowing between them.

The dipole moment and the current are then:

$$\boldsymbol{p} = (0,\ 0,\ p_0 \mathrm{e}^{-i\omega t}) \qquad\qquad I = \frac{\dot{p}}{d} = -i\omega \frac{p_0}{d} \mathrm{e}^{-i\omega t}$$

with $p_0 = q_0 d$.

ω is the frequency of oscillation of the dipole, and therefore that of the fields and any radiation it produces.

The *dipole's physical size* is taken to be *small* compared to the wavelength of the expected radiation, i.e.:

$$d \ll \lambda = \frac{2\pi c}{\omega}$$

... the ***dipole approximation***.

[A system in which this constraint does not hold will be discussed later.]

The time-varying fields at some distant point P at $\boldsymbol{r} = (r,\ \theta,\ \phi)$ are now obtained from the known *retarded potentials* with $r \gg d$.

Step 1: Vector Potential

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} \int \frac{[\boldsymbol{J}]\mathrm{d}V'}{|\boldsymbol{r} - \boldsymbol{r}'|}$$

The dipole is at O and $d \ll r$, so $|\boldsymbol{r} - \boldsymbol{r}'| \to r$ – *variations in* r within the integral can be *neglected*.

$$\boldsymbol{A}(r, t) = \frac{\mu_0}{4\pi r} \int [\boldsymbol{J}]\mathrm{d}V' \tag{4.3}$$

The integrated (retarded) current density is

$$\int \boldsymbol{J}(\boldsymbol{r}', t - r/c)\mathrm{d}V' \equiv Id\ \widehat{\boldsymbol{z}} = \dot{q}(t - r/c)d\ \widehat{\boldsymbol{z}} = \dot{\boldsymbol{p}}(t - r/c)\ \widehat{\boldsymbol{z}} = [\dot{\boldsymbol{p}}]$$

So

$$\boxed{\boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi r}[\dot{\boldsymbol{p}}]} \tag{4.4}$$

So $\boldsymbol{A}$ is parallel to $\dot{\boldsymbol{p}}$:

$$\boldsymbol{A} = A\hat{\boldsymbol{z}}$$

and its components in polar co-ordinates are

$$A_r = A\cos\theta \qquad A_\theta = -A\sin\theta \qquad A_\phi = 0$$

from which the magnetic field $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ can be found.

Step 2: Magnetic Field $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$, so:

$$B_r = \frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}(A_\phi \sin\theta) - \frac{\partial}{\partial\phi}A_\theta\right\} = 0 \tag{4.5}$$

$$B_\theta = \frac{1}{r}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}A_r - \frac{\partial}{\partial r}(rA_\phi)\right\} = 0 \tag{4.6}$$

$$\begin{aligned} B_\phi &= \frac{1}{r}\left\{\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial\theta}A_r\right\} = \frac{1}{r}\left\{-A\sin\theta - r\sin\theta\frac{\partial A}{\partial r} + A\sin\theta\right\} \\ &= -\sin\theta\frac{\partial A}{\partial r} \end{aligned} \tag{4.7}$$

Therefore

$$B_\phi = -\frac{\mu_0}{4\pi}\sin\theta\frac{\partial}{\partial r}\left(\frac{[\dot{p}]}{r}\right)$$

Using Eqn 3.36:

$$B_\phi = \frac{\mu_0}{4\pi}\sin\theta\left(\frac{[\dot{p}]}{r^2} + \frac{[\ddot{p}]}{rc}\right) \tag{4.8}$$

Step 3: Electric Field

$\boldsymbol{E} = -\dot{\boldsymbol{A}} - \boldsymbol{\nabla}\phi$ so the scalar potential ϕ is needed as well as $\boldsymbol{A}$.

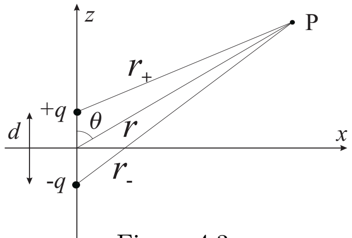

Figure 4.3:

The potential at P due to a charge $q(t - r_+/c)$ at a distance r_+ and a charge $-q(t - r_-/c)$ at a distance r_- is, including retardation:

$$\phi(r,\theta,t) = \frac{1}{4\pi\epsilon_0}\left\{\frac{+q(t - r_+/c)}{r_+} + \frac{-q(t - r_-/c)}{r_-}\right\} \tag{4.9}$$

For a Hertzian dipole, $d \ll r$ and $r_\pm \simeq r \mp (d/2)\cos\theta$, and this difference can be calculated from the derivative:

$$\begin{aligned} \phi(r,\theta,t) &= \frac{1}{4\pi\epsilon_0}(r_+ - r_-)\frac{\partial}{\partial r}\left\{\frac{q(t - r/c)}{r}\right\} \\ &= \frac{1}{4\pi\epsilon_0}(-d\cos\theta)\left\{-\frac{q(t - r/c)}{r^2} - \frac{\dot{q}(t - r/c)}{rc}\right\} \\ &= \frac{\cos\theta}{4\pi\epsilon_0}\left\{\frac{q(t - r/c)d}{r^2} + \frac{\dot{q}(t - r/c)d}{rc}\right\} \end{aligned}$$

$$\boxed{\phi(r,\theta,t) = \frac{\cos\theta}{4\pi\epsilon_0}\left\{\frac{[p]}{r^2} + \frac{[\dot{p}]}{rc}\right\}} \tag{4.10}$$

Now $\boldsymbol{E} = -\dot{\boldsymbol{A}} - \boldsymbol{\nabla}\phi$ and $\boldsymbol{A}(\boldsymbol{r},t) = \dfrac{\mu_0}{4\pi r}[\dot{\boldsymbol{p}}]$ (Eqn 4.4) and $\dfrac{\partial}{\partial r}[F] = -\dfrac{1}{c}[F']$ and $\dfrac{\partial}{\partial t}[F] = [F']$ (Eqn 3.36) so:

$$\begin{aligned} E_r &= -\frac{\mu_0[\ddot{p}]}{4\pi r}\cos\theta - \frac{\cos\theta}{4\pi\epsilon_0}\left\{-\frac{2[p]}{r^3} - \frac{[\dot{p}]}{r^2c}\right\} - \frac{\cos\theta}{4\pi\epsilon_0}\left(\frac{-1}{c}\right)\left\{\frac{[\dot{p}]}{r^2} + \frac{[\ddot{p}]}{rc}\right\} \\ &= \frac{2\cos\theta}{4\pi\epsilon_0}\left\{\frac{[p]}{r^3} + \frac{[\dot{p}]}{r^2c}\right\} \end{aligned} \tag{4.11}$$

$$\begin{aligned} E_\theta &= -\frac{\mu_0[\ddot{p}]}{4\pi r}(-\sin\theta) - \frac{1}{4\pi\epsilon_0 r}(-\sin\theta)\left\{\frac{[p]}{r^2} + \frac{[\dot{p}]}{rc}\right\} \\ &= \frac{\sin\theta}{4\pi\epsilon_0}\left\{\frac{[p]}{r^3} + \frac{[\dot{p}]}{r^2c} + \frac{[\ddot{p}]}{rc^2}\right\} \end{aligned} \tag{4.12}$$

$$E_\phi = -0 - \frac{1}{r\sin\theta} \times 0 = 0 \tag{4.13}$$

Step 4: Interpretation

Collecting together the non-zero components of the fields

$$\boxed{\begin{aligned} B_\phi &= \frac{\mu_0 \sin\theta}{4\pi}\left\{\frac{[\dot p]}{r^2} + \frac{[\ddot p]}{rc}\right\} && (4.14)\\ E_\theta &= \frac{\sin\theta}{4\pi\epsilon_0}\left\{\frac{[p]}{r^3} + \frac{[\dot p]}{r^2 c} + \frac{[\ddot p]}{rc^2}\right\} && (4.15)\\ E_r &= \frac{2\cos\theta}{4\pi\epsilon_0}\left\{\frac{[p]}{r^3} + \frac{[\dot p]}{r^2 c}\right\} && (4.16)\end{aligned}}$$

The terms in E_θ and E_r varying as $1/r^3$ are just the fields expected from a *static* dipole with a dipole moment that changes in time according to the retarded dipole moment $[p]$.

Consider the *magnitudes* of various terms with $p = p_0 \exp(-i\omega t)$:

$$\begin{aligned} \frac{[p]}{r^3} &\sim && && \sim \frac{p_0}{r^3} && \propto \frac{1}{r^3}\\ \frac{[\dot p]}{r^2 c} &\sim \frac{p_0\omega}{r^2 c} = \frac{p_0 k}{r^2} = \frac{p_0}{r^3}kr && && = \frac{p_0}{r^3}\left(\frac{2\pi r}{\lambda}\right) && \propto \frac{1}{r^2}\\ \frac{[\ddot p]}{rc^2} &\sim \frac{p_0\omega^2}{rc^2} = \frac{p_0 k^2}{r} = \frac{p_0}{r^3}(kr)^2 && && = \frac{p_0}{r^3}\left(\frac{2\pi r}{\lambda}\right)^2 && \propto \frac{1}{r}\end{aligned}$$

"*Dipole* term" $\propto \dfrac{[p]}{r^3} \sim \dfrac{p_0}{r^3}$ dominates "near" to the dipole for $(d \ll)\ r \ll \lambda$

"*Induction* term" $\propto \dfrac{[\dot p]}{r^2 c} \sim \dfrac{p_0}{r^3}\left(\dfrac{2\pi r}{\lambda}\right)$ never dominates

"*Radiation* term" $\propto \dfrac{[\ddot p]}{rc^2} \sim \dfrac{p_0}{r^3}\left(\dfrac{2\pi r}{\lambda}\right)^2$ dominates at large $r \gg \lambda\ (\gg d)$

The $1/r$-dependence suggests that the last term represents *radiation*.

§4.2.1 Field Pattern

The following figure shows lines of $\boldsymbol{E}$ for a Hertzian dipole at different times. The dipole is vertical at the centre. Lines of $\boldsymbol{B}$ are circles perpendicular to the plane of the paper and centred on the dipole.

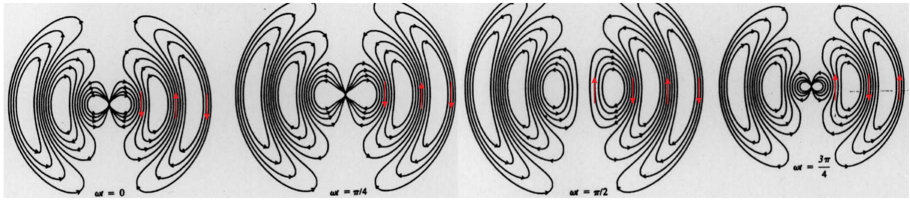

Figure 4.4: The electric field pattern of a Hertzian dipole aligned vertically.

Near the dipole ($r \lesssim \lambda$) the field has complex form, but for $r \gg \lambda$ considerable simplification occurs.

In the far field region $r \gg \lambda$, the *non-zero* fields are the *radiation fields* E_θ and B_ϕ which vary as $1/r$ (Eqns 4.15 and 4.14):

$$E_\theta = \frac{\sin\theta}{4\pi\epsilon_0}\frac{[\ddot p]}{rc^2} \qquad B_\phi = \frac{\mu_0 \sin\theta}{4\pi}\frac{[\ddot p]}{rc}$$

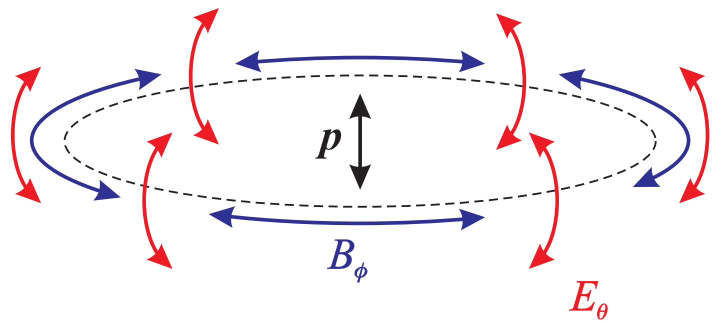

Figure 4.5: The fields of a vertical Hertzian dipole.

Note:

- The field components are *in phase* with one another and (allowing for retardation) with the second time-derivative of the dipole moment.
- If the dipole consisted of two fixed charges $\pm q$ with oscillating separation, this would correspond to their *acceleration*.
- The field components are
  - *proportional* to each other: $E_\theta = cB_\phi$, as expected for EM waves.
  - *orthogonal* in space, making $\boldsymbol{N}$ purely outwardly *radial* (at these large distances).
- The radiation is
  - *linearly polarized* in the direction of E_θ.
  - *vertically* polarized in the equatorial plane $\theta = \pi/2$.

§4.2.2 Power Radiated from a Hertzian Dipole

The energy flow arising from the fields of the dipole is given by $\boldsymbol{N} = \boldsymbol{E} \times \boldsymbol{H}$.

Clearly there is a Poynting flux associated with, say, the induction fields, but it is not purely radial and since it falls of like $1/r^4$ cannot represent a nett outward energy flux associated with EM waves. The flux merely *redistributes energy* within the fields close to the dipole as the fields there change with time.

In the *far field* the instantaneous magnitude of the Poynting flux is

$$N(r,\theta,\phi) = \frac{1}{\mu_0} E_\theta B_\phi = \frac{\mu_0}{16\pi^2 c} \sin^2\theta \frac{[\ddot{p}]^2}{r^2} \tag{4.17}$$

$|\boldsymbol{N}| \propto \dfrac{1}{r^2}$ as expected for radiation from a point source.

The *angular distribution* of the radiated power is

$$G(\theta,\phi) \propto \sin^2\theta \tag{4.18}$$

... obviously *cylindrically symmetric*, independent of ϕ. *Maximum* power is radiated in the *equatorial plane* of the dipole, and *zero* along its *polar axes*. (See §5.1 for further discussion of angular distributions.)

The *instantaneous* ***total*** *radiated power* P is obtained by integrating over the surface of a sphere at radius r to give

$$P = \int \boldsymbol{N}\cdot\mathbf{d}\boldsymbol{S} = \frac{\mu_0}{16\pi^2 c}\int_0^\pi \sin^2\theta \frac{[\ddot{p}]^2}{r^2} 2\pi r^2 \sin\theta \mathrm{d}\theta$$

$$\boxed{P = \frac{\mu_0}{6\pi c}[\ddot{p}]^2} \tag{4.19}$$

P is independent of r, so there is *no accumulation of energy* in space. All the energy from the radiation fields propagates outwards.

For $p = p_0 \mathrm{e}^{-i\omega t}$, $\ddot{p} = -\omega^2 p_0 \mathrm{e}^{-i\omega t}$, so

$$\langle [\ddot{p}]^2 \rangle = \langle \ddot{p}^2 \rangle = \tfrac{1}{2}\omega^4 p_0^2$$

since retardation is irrelevant when time-averaging over a number of cycles.

Then Eqn 4.17 becomes

$$\boxed{\langle N(r,\theta,\phi)\rangle = \frac{\mu_0 \omega^4 p_0^2}{32\pi^2 c}\frac{\sin^2\theta}{r^2}} \tag{4.20}$$

and the time-averaged total power radiated is therefore

$$\boxed{\langle P \rangle = \frac{\mu_0\omega^4 p_0^2}{12\pi c} = \frac{\omega^4 p_0^2}{12\pi\epsilon_0 c^3}} \tag{4.21}$$

Here is a plot of the angular distribution of the radiated power for a Hertzian dipole.

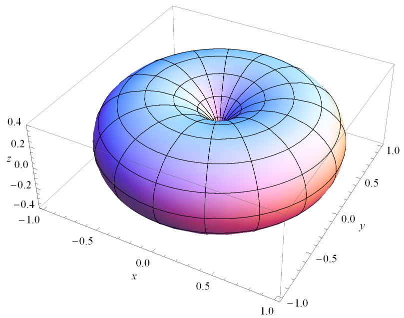

Figure 4.6: Angular distribution of radiation emitted by a Hertzian dipole.

The radiated power of an electric dipole is isotropic in the plane perpendicular to the dipole and this is a desirable characteristic for using such electric dipole transmitters in cellular mobile networks operating in a typical frequency range of 800 MHz - 2.1 GHz.

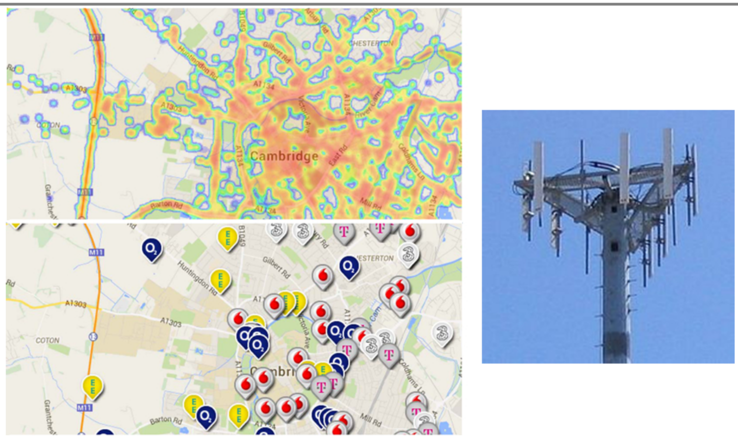

Figure 4.7: Map of 2G/3G/4G network coverage and base station locations around Cambridge (from www.opensignal.com); Photograph of typical base station tower

Another application of electric dipole radiation we will encounter below in the context of Rayleigh light scattering.

§4.3 Multipole expansion in the far field

We now investigate the more general case of a system of charges and currents that vary sinusoidally in time, i.e., $\rho(\boldsymbol{r},t) = \rho(\boldsymbol{r})e^{-i\omega t}$ and $\boldsymbol{J}(\boldsymbol{r},t) = \boldsymbol{J}(\boldsymbol{r})e^{-i\omega t}$, but we will restrict our consideration to the far field. From Eqn 3.33 we obtain for the vector potential:

$$\boldsymbol{A}(\boldsymbol{r},t) = e^{-i\omega t}\frac{\mu_0}{4\pi}\int \boldsymbol{J}(\boldsymbol{r'})\frac{e^{ik|\boldsymbol{r}-\boldsymbol{r'}|}}{|\boldsymbol{r}-\boldsymbol{r'}|}dV' \equiv \boldsymbol{A}(\boldsymbol{r})e^{-i\omega t} \tag{4.22}$$

In the far field $kr \gg 1$ we can use the approximation $|\boldsymbol{r}-\boldsymbol{r'}| \simeq r - \boldsymbol{n}\cdot\boldsymbol{r'}$ where $\boldsymbol{n}$ is a unit vector in the direction of $\boldsymbol{r}$:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \boldsymbol{J}(\boldsymbol{r'})e^{-ik\boldsymbol{n}\cdot\boldsymbol{r'}}dV' \tag{4.23}$$

It is easy to show that the electric and magnetic fields calculated from Eqn 4.23 are transverse to $\boldsymbol{r}$ and drop off as r^{-1}, i.e. they are radiation fields.

If the source dimension d is small compared to the wavelength it is appropriate to expand Eqn 4.23 in powers of k.

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\sum_n \frac{(-ik)^n}{n!}\int \boldsymbol{J}(\boldsymbol{r'})(\boldsymbol{n}\cdot\boldsymbol{r'})^n dV' \tag{4.24}$$

Since the magnitude of $\boldsymbol{r'}$ is on the order of d and $kd \ll 1$ terms with n > 1 fall off rapidly and the dominant terms in the expansion are the terms of n=0,1.

n=0 term: Using integration by parts as well as the continuity equation $\nabla\cdot\boldsymbol{J} = i\omega\rho$ this can be transformed into

$$\begin{aligned}\boldsymbol{A}(\boldsymbol{r}) &= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \boldsymbol{J}(\boldsymbol{r'})dV' = -\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \boldsymbol{r'}(\nabla'\cdot\boldsymbol{J})dV' \\ &= \frac{i\omega\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \boldsymbol{r'}\rho(\boldsymbol{r'})dV' = -\frac{i\omega\mu_0}{4\pi}\boldsymbol{p}\frac{e^{ikr}}{r}\end{aligned} \tag{4.25}$$

This terms corresponds to the *electrical dipole radiation* discussed above.

Let's now consider the *n=1 term*.

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-ik)\int \boldsymbol{J}(\boldsymbol{r'})(\boldsymbol{n}\cdot\boldsymbol{r'})dV' \tag{4.26}$$

This can be split into two contributions, one that is symmetric in $\boldsymbol{J}$ and $\boldsymbol{r'}$ and a second one that is antisymmetric:

$$(\boldsymbol{n}\cdot\boldsymbol{r'})\boldsymbol{J} = \frac{1}{2}[(\boldsymbol{n}\cdot\boldsymbol{r'})\boldsymbol{J} + (\boldsymbol{n}\cdot\boldsymbol{J})\boldsymbol{r'}] + \frac{1}{2}(\boldsymbol{r'}\times\boldsymbol{J})\times\boldsymbol{n} \tag{4.27}$$

The *antisymmetric term* is dependent on the magnetic dipole moment $\boldsymbol{m}$ of the current distribution $\boldsymbol{m} = \frac{1}{2}\int(\boldsymbol{r}\times\boldsymbol{J})dV$:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{ik\mu_0}{4\pi}(\boldsymbol{n}\times\boldsymbol{m})\frac{e^{ikr}}{r} \tag{4.28}$$

This term is the *magnetic dipole radiation* emitted by the time-varying current distribution.

The fields can now be determined simply by noting that the vector potential of a magnetic dipole is proportional to the magnetic field of an electric dipole (Eqn 4.14). Because $\boldsymbol{B} = \nabla\times\boldsymbol{A}$ and $\boldsymbol{E} = i\frac{Z_0}{k\mu_0}\nabla\times\boldsymbol{B}$ the magnetic induction of the magnetic dipole is equal to $\frac{\mu_0}{Z_0}$ times the electric field of the electric dipole. Likewise the electric field of the magnetic dipole is $-\frac{Z_0}{\mu_0}$ times the magnetic induction of an electric dipole provided we sustitute $c\boldsymbol{p}\to\boldsymbol{m}$.

$$\begin{aligned}\boldsymbol{B}(\boldsymbol{r}) &= \frac{\mu_0 k^2}{4\pi}(\boldsymbol{n}\times\boldsymbol{m})\times\boldsymbol{n})\frac{e^{ikr}}{r} \\ \boldsymbol{E}(\boldsymbol{r}) &= -\frac{Z_0 k^2}{4\pi}(\boldsymbol{n}\times\boldsymbol{m})\frac{e^{ikr}}{r}\end{aligned}$$

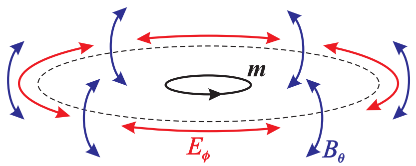

Figure 4.8: The fields of a magnetic dipole aligned vertically.

The time averaged total power radiated (Eqn 4.21 for the ED):

$$\langle P^{\mathrm{ED}}\rangle = \frac{\omega^4 p_0^2}{12\pi\epsilon_0 c^3} \qquad \to \qquad \langle P^{\mathrm{MD}}\rangle = \frac{\mu_0\omega^4 m_0^2}{12\pi c^3} \tag{4.29}$$

If the ED and MD carry similar currents of amplitude I_0 and have similar geometric sizes a:

$$p_0 \sim \frac{I_0}{\omega}a \qquad\qquad m_0 \sim I_0 a^2$$

then

$$\frac{\langle P^{\mathrm{MD}}\rangle}{\langle P^{\mathrm{ED}}\rangle} = \frac{m_0^2}{c^2 p_0^2} \sim \frac{1}{c^2}\frac{I_0^2 a^4\omega^2}{I_0^2 a^2} \sim \left(\frac{2\pi a}{\lambda}\right)^2 \ll 1 \tag{4.30}$$

Small ($a\ll\lambda$) magnetic dipoles are *much less efficient* radiation sources than Hertzian electric dipoles.

Let us finally consider the *symmetric, n=1 term*. We can transform the integral of this term in Eqn 4.26 by integration by parts:

$$\frac{1}{2}\int[(\boldsymbol{n}\cdot\boldsymbol{r'})\boldsymbol{J} + (\boldsymbol{n}\cdot\boldsymbol{J})\boldsymbol{r'}]dV' = -\frac{i\omega}{2}\int\boldsymbol{r'}(\boldsymbol{n}\cdot\boldsymbol{r'})\rho(\boldsymbol{r'})dV' \tag{4.31}$$

$$\boldsymbol{A}(\boldsymbol{r}) = -\frac{\mu_0 ck^2}{8\pi}\frac{e^{ikr}}{r}\int\boldsymbol{r'}(\boldsymbol{n}\cdot\boldsymbol{r'})\rho(\boldsymbol{r'})dV' \tag{4.32}$$

The fields can be obtained from:

$$\boldsymbol{B} = ik\boldsymbol{n}\times\boldsymbol{A} = -\frac{ick^3\mu_0}{24\pi}\frac{e^{ikr}}{r}\boldsymbol{n}\times\boldsymbol{Q}(\boldsymbol{n}) \tag{4.33}$$

$$\boldsymbol{E} = \frac{ikZ_0}{\mu_0}(\boldsymbol{n}\times\boldsymbol{A})\times\boldsymbol{n} \tag{4.34}$$

Here $\boldsymbol{Q}(\boldsymbol{n}) = 3\int\boldsymbol{r'}(\boldsymbol{n}\cdot\boldsymbol{r'})\rho(\boldsymbol{r'})dV'$ with $Q_\alpha = \sum_{\alpha,\beta}Q_{\alpha\beta}n_\beta$ and $Q_{\alpha\beta}$ being the components of the *quadrupole moment tensor*:

$$Q_{\alpha\beta} = \int(3x_\alpha x_\beta - r^2\delta_{\alpha\beta})\rho(\boldsymbol{r})dV \tag{4.35}$$

This term therefore corresponds to *electric quadrupole radiation*.

With some algebra (see, for example, Jackson, chapter 9.3) it is possible to derive the angular distribution of the quadrupole radiation

$$N(r,\theta,\phi) = \frac{\mu_0\omega^6}{1152\pi^2 c^3}\frac{1}{r^2}|(\boldsymbol{n}\times\boldsymbol{Q}(\boldsymbol{n}))\times\boldsymbol{n}|^2 \tag{4.36}$$

and the total instantaneous power radiated

$$\langle P\rangle^{\mathrm{EQ}} = \frac{\mu_0\omega^6}{1440\pi c^3}\sum_{\alpha,\beta}|Q_{\alpha,\beta}|^2. \tag{4.37}$$

For the same q and a the quadrupole is much less effective as a radiator than the corresponding electric dipole if $ka \ll 1$. Also note that $\langle P\rangle^{\mathrm{EQ}} \propto \omega^6$ rather than ω^4.

Let's discuss a specific example of a so-called lateral quadrupole defined by four charges in the y-z plane, two charges +q at (0,a,a) and (0,-a,-a) and two charges -q at (0,a,-a) and (0,-a,a). Like in our discussion of the Hertzian dipole we assume the magnitude of the charge to be oscillating.

This corresponds to $Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 3qa^2 \\ 0 & 3qa^2 & 0 \end{pmatrix}$.

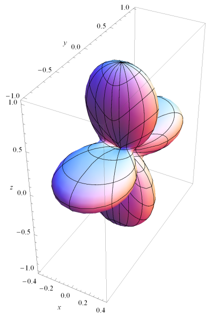

Figure 4.9: Power gain for a lateral quadrupole.

In contrast to the ED the radiated power of the quadrupole is no longer cylindrically symmetric, but has *lobes* along the $\pm y$ and $\pm z$-axes. *No power* is radiated along the $\pm x$-axes ($\phi = 0, \pi$), since the geometric phases of the two dipoles are anti-phase leading to *complete cancelation.*

§5 Antennas

The Hertzian ED, the MD and EQ are simple examples of antennas – devices designed to *emit EM waves.*

Any antenna will lose energy from whatever circuit is driving the charge/current oscillation, and in circuit theory this appears as a resistor – the *radiation resistance* R_{r}.

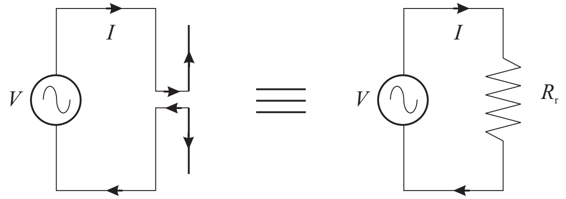

Figure 5.1: An antenna is effectively a resistance R_{r} in an electrical circuit.

R_{r} is defined as:

$$R_{\mathrm{r}} \equiv \langle P\rangle/\langle I^2\rangle = \langle V^2\rangle/\langle P\rangle \tag{5.1}$$

with P the total radiated power, and I the current at the antenna contact.

For a Hertzian dipole (Eqn 4.21):

$$\langle P\rangle = \frac{\mu_0\omega^4 p_0^2}{12\pi c} = \frac{\mu_0\omega^4\langle p^2\rangle}{6\pi c} = \frac{\mu_0\omega^2\langle \dot{p}^2\rangle}{6\pi c}$$

so with $\dot{p} = Id$:

$$R_{\mathrm{r}} \equiv \frac{\langle P\rangle}{\langle I^2\rangle} = \frac{1}{\langle I^2\rangle}\frac{\mu_0\omega^2}{6\pi c}\langle I^2\rangle d^2$$

$$\boxed{R_{\mathrm{r}} = \frac{\mu_0\omega^2 d^2}{6\pi c} = \frac{\omega^2 d^2}{6\pi\epsilon_0 c^3} = \frac{2\pi}{3}Z_0\left(\frac{d}{\lambda}\right)^2 = 789\left(\frac{d}{\lambda}\right)^2 \;\; \Omega} \tag{5.2}$$

where the impedance of free space is $Z_0 = (\mu_0/\epsilon_0)^{\frac{1}{2}} = 377\ \Omega$, $\omega = 2\pi c/\lambda$ and $c^2 = 1/\epsilon_0\mu_0$.

This is *valid provided* $d \ll \lambda$, the initial approximation made when considering the Hertzian dipole.

§5.1 Power Gain

An important property of an emitting antenna is the *directionality* of the emitted radiation, described by the *Power Gain* $G(\theta, \phi)$.

For a (time-averaged) *radial Poynting flux* $N(\theta, \phi)$ describing the angular distribution, the power gain is defined as:

$$\boxed{G(\theta, \phi) = \frac{N(\theta, \phi)}{\frac{1}{4\pi} \int N(\theta, \phi)\, d\Omega}} \tag{5.3}$$

where $d\Omega = \sin\theta\, d\theta\, d\phi$ and the integral is over the total solid angle 4π.

For a Hertzian dipole using results derived in §4.2:

$$G^{\text{ED}}(\theta, \phi) = \frac{3}{2} \sin^2\theta \tag{5.4}$$

§5.2 Antennas as Receivers

Incident EM waves induce a voltage in the antenna, so it appears as a *generator* with *open circuit voltage* V and *internal resistance* R_r. The equivalent circuit is:

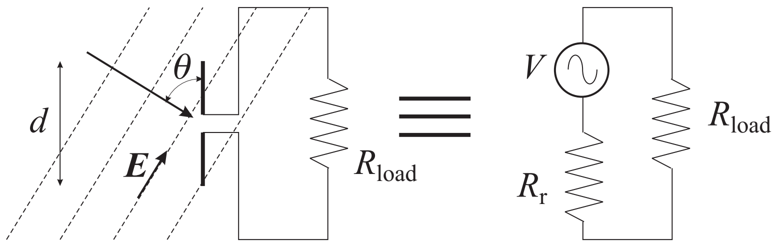

Figure 5.2: A receiving antenna as a voltage source in a circuit.

Its performance is quantified by the *power delivered to a load* resistance for a given incident flux of radiation.

The total power absorbed from the EM wave P_tot and the power delivered to the load P_load are:

$$\langle P_\text{tot} \rangle = \langle I^2 \rangle (R_\text{r} + R_\text{load}) = \langle V^2 \rangle \frac{1}{R_r + R_\text{load}}$$
$$\langle P_\text{load} \rangle = \langle I^2 \rangle R_\text{load} \qquad = \langle V^2 \rangle \frac{R_\text{load}}{(R_r + R_\text{load})^2}$$

$\langle P_\text{load} \rangle$ is maximized for a *matched load*, $R_\text{load} = R_\text{r}$. Then:

$$\langle P_\text{tot} \rangle = \frac{\langle V^2 \rangle}{2R_\text{r}} \qquad \langle P_\text{load}^\text{max} \rangle = \frac{\langle V^2 \rangle}{4R_\text{r}} \tag{5.5}$$

The *Effective Area* of an antenna (its "absorption cross section") is:

$$A_\text{eff}(\theta, \phi) \equiv \frac{\text{Power delivered to a matched load}}{\text{Energy flux density of polarized radiation}} = \frac{\langle P_\text{load} \rangle}{\langle N_\text{incident} \rangle} \tag{5.6}$$

when the polarization is oriented for best reception.

Example: The Hertzian dipole. EM radiation incident at angle θ on a Hertzian dipole oriented in the plane of polarization to maximize $\boldsymbol{E}$ along the dipole axis.

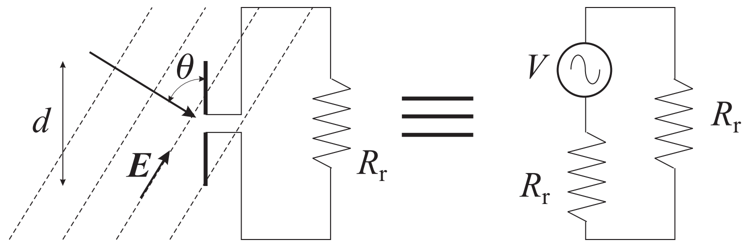

Figure 5.3: EM radiation incident at angle θ on a Hertzian dipole of length d connected to a matched load R_r.

The open circuit voltage across the antenna is $V = Ed\sin\theta$, so the mean power absorbed in a matched load R_r is:

$$\langle P_\text{load} \rangle = \frac{\langle V^2 \rangle}{4R_\text{r}} = \frac{\langle E^2 \rangle d^2 \sin^2\theta}{4R_\text{r}} = \frac{\langle E^2 \rangle 3\lambda^2 \sin^2\theta}{8\pi Z_0} = \frac{3\lambda^2 \sin^2\theta}{8\pi} \langle N_\text{incident} \rangle$$

So:

$$\boxed{A_{\text{eff}}(\theta,\phi) = \frac{3\lambda^2 \sin^2\theta}{8\pi}} \tag{5.7}$$

Since $d \ll \lambda$ for a Hertzian dipole, $A_{\text{eff}} \gg d^2$, the actual geometrical area of the antenna. How can this be?

Since current flows in the circuit, the antenna also *radiates* – some of the incident power is *re-radiated*, or *scattered*.

The re-radiated power $\langle P_{\text{rad}}\rangle$ is the difference between the total power absorbed and the power delivered to the load. For a matched load:

$$\langle P_{\text{rad}}\rangle = \langle P_{\text{tot}}\rangle - \langle P_{\text{load}}\rangle = \frac{\langle V^2\rangle}{2R_{\text{r}}} - \frac{\langle V^2\rangle}{4R_{\text{r}}} = \frac{\langle V^2\rangle}{4R_{\text{r}}} = \langle P_{\text{load}}\rangle$$

As much power is *scattered* from the antenna as is absorbed by the load.

The incident and scattered fields *interfere*, and the *total* Poynting flux must be computed from the resulting total fields. In some regions of space near the dipole, the *total Poynting vector can turn in* towards the dipole, increasing the power incident on the dipole above that expected on purely geometrical grounds. In detail the fields are complicated to compute, but illustratively:

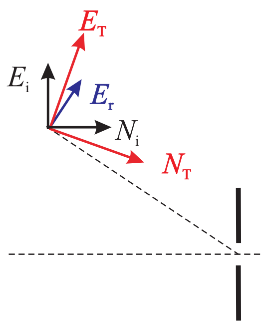

Figure 5.4: EM radiation normally incident on a Hertzian dipole; the re-radiated field turns in the Poynting vector towards the dipole.

§5.3 Effective Area and Power Gain

For a Hertzian dipole the reception and emission efficiencies have the same polar form and are related by Eqn 5.7:

$$A_{\text{eff}}(\theta,\phi) = \frac{\lambda^2}{4\pi} G^{\text{ED}}(\theta,\phi) \tag{5.8}$$

In fact this relation is true for *any* antenna, independent of any details of the its construction, and *even of its size*.

A remarkable result, proved *thermodynamically*.

Consider *any antenna* of effective area $A_{\text{eff}}(\theta,\phi)$ and power gain $G(\theta,\phi)$ connected to a matched load R_{r}, all in *thermal equilibrium* at temperature T in a *black-body* environment with radiation *energy density* $U(\nu)$.

It is reasonable to assume that $h\nu \ll k_{\text{B}}T$, so $U(\nu)$ is given by the classical *Rayleigh-Jeans formula*: $U(\nu) = 8\pi \frac{\nu^2}{c^3} k_{\text{B}}T$.

The antenna responds to only one polarization, so effectively $U(\nu) \to \frac{1}{2}U(\nu)$.

The effective incident energy flux in the frequency range $\nu \to \nu + \mathrm{d}\nu$ from a solid angle $\mathrm{d}\Omega = \sin\theta \mathrm{d}\theta \mathrm{d}\phi$ is therefore:

$$\mathrm{d}W = \frac{1}{2}U(\nu)\mathrm{d}\nu \, c \, \frac{\mathrm{d}\Omega}{4\pi} = \frac{\nu^2}{c^2} k_{\text{B}}T \mathrm{d}\Omega \mathrm{d}\nu$$

The antenna absorbs some of this radiation, and the power dissipated in the matched load R_{r} is:

$$P_{\text{abs}} = A_{\text{eff}}(\theta,\phi)\mathrm{d}W = A_{\text{eff}}(\theta,\phi)\frac{\nu^2}{c^2} k_{\text{B}}T \mathrm{d}\Omega \mathrm{d}\nu$$

In thermal equilibrium, *detailed balance* requires there to be a *compensating back-flow* of power from R_{r} into space.

This arises from the *Johnson noise* voltage fluctuations of R_{r}. The (classical) *Nyquist formula* for the mean-square voltage fluctuation in the frequency range $\nu \to \nu + \mathrm{d}\nu$ is:

$$\langle V^2\rangle = 4k_{\text{B}}TR_{\text{r}}\mathrm{d}\nu$$

For the equivalent circuit of the load-matched antenna, $I = V/(2R_\mathrm{r})$. So:

$$\langle I^2 \rangle = \frac{k_\mathrm{B}T}{R_\mathrm{r}} \mathrm{d}\nu$$

So, using the definition of the antenna power gain $G(\theta, \phi)$ (Eqn 5.3), the power radiated into $\mathrm{d}\Omega$ is:

$$P_\mathrm{rad} = \langle I^2 \rangle R_\mathrm{r} \frac{G(\theta,\phi)\mathrm{d}\Omega}{4\pi} = k_\mathrm{B}T\frac{G}{4\pi}\mathrm{d}\Omega\mathrm{d}\nu$$

Equating $P_\mathrm{rad} = P_\mathrm{abs}$ in equilibrium:

$$A_\mathrm{eff}(\theta,\phi)\frac{\nu^2}{c^2}k_\mathrm{B}T\mathrm{d}\Omega\mathrm{d}\nu = k_\mathrm{B}T\frac{G(\theta,\phi)}{4\pi}\mathrm{d}\Omega\mathrm{d}\nu$$

and using $\lambda = c/\nu$:

$$\boxed{A_\mathrm{eff}(\theta,\phi) = \frac{\lambda^2}{4\pi}G(\theta,\phi)} \qquad (5.9)$$

for *any antenna*. High radiative efficiency $\Leftrightarrow$ high detection sensitivity.

§5.4 Centre-fed Linear Antennas

The Hertzian dipole consists of two point charges oscillating out of phase, and a uniform oscillating current joining them. This is not realistic. Practical antennas are made from wires, and simple examples can be imagined by opening up the ends of a long parallel-wire air-spaced transmission line. For the line driven by an oscillating voltage source, standing sinusoidal current waves are set up in both wires, with *current nodes* (and voltage anti-nodes) at the end of the line. Now suppose the last $L/2$ length of the line is opened out:

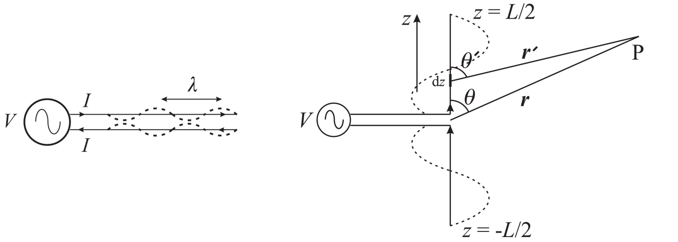

Figure 5.5: Standing current waves on a transmission line.

For such an antenna the time-averaged Poynting flux at radius $r \gg L$ is:

$$\langle N(r,\theta)\rangle = \frac{\mu_0 c I_0^2}{8\pi^2 r^2 \sin^2\theta}\left[\cos\left(\frac{kL}{2}\right) - \cos\left(\frac{kL\cos\theta}{2}\right)\right]^2 \qquad (5.10)$$

This has – *resonant* – behaviour with L, and a special case is $L = \lambda/2$ ($kL/2 = \pi/2$), the so-called half-wave antenna, which has power gain:

$$G_{\lambda/2}(\theta) = \frac{4\pi\cos^2(\frac{\pi}{2}\cos\theta)}{2\pi \times 1.219\sin^2\theta} \qquad (5.11)$$

This is maximal in the equatorial plane $\theta = \pi/2$, but is slightly more confined to the equatorial plane than the Hertzian dipole. The radiation resistance is found to be:

$$R_\mathrm{r} = \frac{1.219\mu_0 c}{2\pi} = \frac{1.219}{2\pi}Z_0 = 73.1\ \Omega \qquad (5.12)$$

The impedance of a half-wave antenna can be matched more easily to that of a signal feed cable than for a Hertzian dipole ($R_\mathrm{r} \leq 1\ \Omega$).

Antennas closely related to the half-wave dipole have become the workhorse antennas in cell phone communication systems at frequencies of 800 MHz - 2.1 GHz (3G/4G networks) ($\lambda = 0.37 - 0.14m$).

Figure 5.6: Photograph of a typical base station of a cellular network.

§5.5 Antenna arrays

Consider an array of dipole antennas oriented parallel to Oz arranged regularly along the Oy axis as shown below:

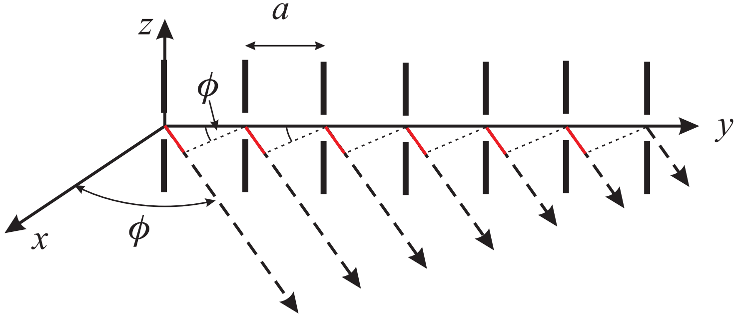

Figure 5.7: A regular dipole array of spacing a. The red line corresponds to a phase shift of $ka\sin\phi$.

If the dipoles are driven *in phase* at frequency $\omega = ck$ then the total $\boldsymbol{E}$-field at some distant point in the xy-plane ($\theta = \pi/2$) in the ϕ-direction will be:

$$\boldsymbol{E} \sim \sum_{j}^{N} \mathrm{e}^{-ika\sin\phi(j-1)} = \frac{1-\mathrm{e}^{-iNka\sin\phi}}{1-\mathrm{e}^{-ika\sin\phi}} \tag{5.13}$$

since when the fields are combined at the observation point those from adjacent dipoles differ in phase by $-ka\sin\phi$. So the Poynting flux is modified from the power gain function of the individual dipoles by the *ϕ-dependent factor* of Eqn 5.13.

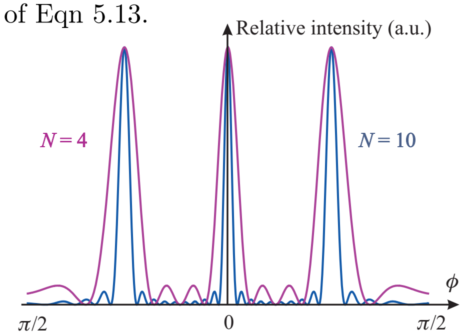

Figure 5.8: Intensity distribution from a regular dipole array of spacing a, for $N = 4$ and $N = 10$, and with $\lambda = 0.7a$.

The problem parallels to the *N-slit diffraction* problem, with predictable results:

The array produces *directed beams* at angles depending on the ratio of the wavelength of the radiation to the dipole spacing.

The widths of the beams narrows as the number of dipoles increases.

If an *extra phase difference* is introduced between each successive dipole by *electronic* means, the beam directions can be *controlled.* This is the basis of flat, *electronically-steerable radars.*

Recall Eqn 5.9: $A_{\mathrm{eff}}(\theta,\phi) \sim G(\theta,\phi)$. The effective area is proportional to the power gain, so these *phased arrays* of dipoles also act as *directionally sensitive receivers*, again useful in radar applications.

Figure 5.9: Phased array radar at RAF Fylingdales, part of the Ballistic Missile Early Warning System (BMEWS)

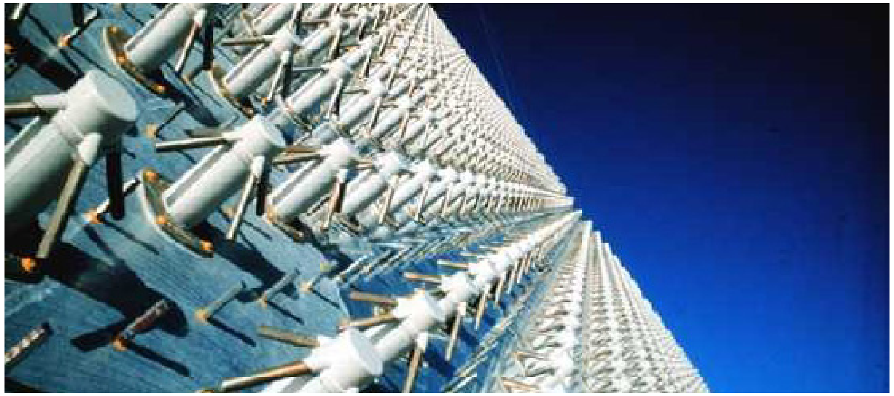

Figure 5.10: Individual elements of the phased array radar at RAF Fylingdales.

§5.6 Directional Antennas

Directionality can also be achieved for a single dipole antenna by using conducting reflectors and directional elements, as in a *Yagi-Uda antenna*:

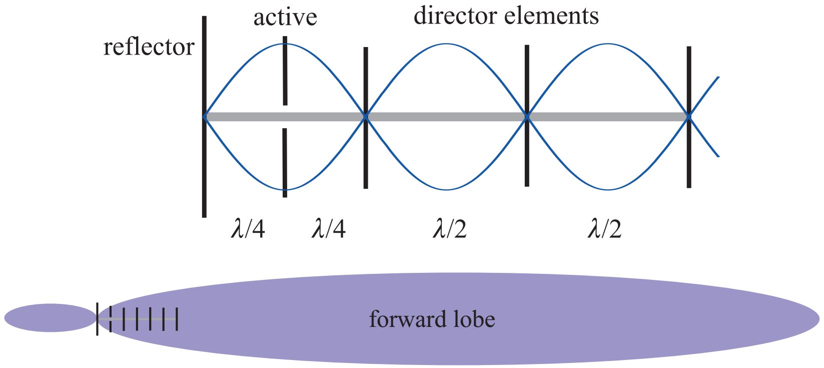

Figure 5.11: *Highly schematic* representation of a Yagi-Uda antenna and its radiation pattern. In practice the spacing of the directors is not as simple as shown.

The active element, the dipole itself, is mounted in line with a reflector and several "directors". The exact dimensions and spacing are subtle to design, but the overall effect is to direct the radiated waves strongly in the forward direction – a *highly directional transmitter* – and therefore a highly directional and efficient receiver in widespread use.

Figure 5.12: Photographs of Yagi antennas in a TV aerial (a) and in a cellular base station for mobile communications (b).

§6 Light scattering

Radiation incident on antennas induces oscillating dipole moments which cause further, outward, radiation, *scattered* in different directions.

Scattering also occurs if EM radiation is incident upon a *particle*.

Consider a *small* particle with size $a \ll \lambda$ so that phase variations across the particle are negligible. If the particle is *polarizable*, a dipole moment is induced and the particle radiates energy taken from the incident field at a rate (Eqn 4.19):

$$\langle P \rangle = \frac{\mu_0 \langle \ddot{p}^2 \rangle}{6\pi c}$$

The *cross section* σ is defined as:

$$\sigma = \frac{\langle P \rangle}{\text{Incident electromagnetic flux}} = \frac{\langle P \rangle}{E_0^2/2\mu_0 c} = \frac{\mu_0^2 \langle \ddot{p}^2 \rangle}{3\pi E_0^2} \qquad (6.1)$$

where E_0 is the amplitude of the incident field.

Usually $p_0 \propto E_0$ and σ does not depend on E_0.

§6.1 Polarization of Scattered Waves

Consider radiation scattered through angle α for two orthogonal planes of polarization of the incident field.

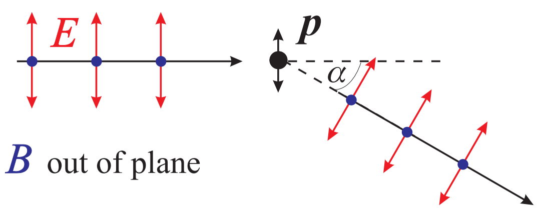

Figure 6.1: Scattering with $\boldsymbol{E}$ in the scattering plane.

(i) For $\boldsymbol{E}$ in the scattering plane, the polar angle is $\pi/2-\alpha$ and the re-radiated flux is given by (Eqn 4.17):

$$N_1=\frac{\mu_0\langle\ddot{p}_z^2\rangle}{16\pi^2r^2c}\sin^2(\pi/2-\alpha)=\frac{\mu_0\langle\ddot{p}_z^2\rangle}{16\pi^2r^2c}\cos^2\alpha$$

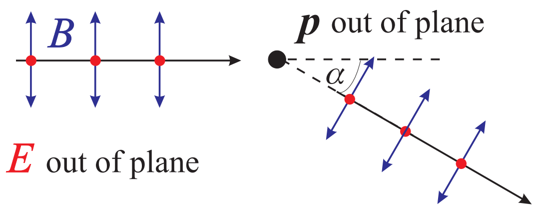

Figure 6.2: Scattering with $\boldsymbol{E}$ perpendicular to the scattering plane.

(ii) For $\boldsymbol{E}$ perpendicular to the scattering plane the polar angle is $\pi/2$, and the scattered flux is:

$$N_2=\frac{\mu_0\langle\ddot{p}_x^2\rangle}{16\pi^2r^2c}$$

If the incoming radiation is unpolarized (an incoherent equal superposition of cases (i) and (ii)):

$$\langle\ddot{p}_x^2\rangle=\langle\ddot{p}_z^2\rangle=\frac{\langle\ddot{\boldsymbol{p}}^2\rangle}{2}$$

$N_2>N_1$ so write $N_2=N_1+(N_2-N_1)$.

The first term combined with the mutually incoherent flux N_1 polarized in the perpendicular direction corresponds to an *unpolarized* beam of flux $2N_1$.

The second term is the only remaining polarized flux in this description.

The degree of polarization of the radiation scattered through angle α is then:

$$P=\frac{I_{\rm pol}}{I_{\rm tot}}=\frac{N_2-N_1}{2N_1+(N_2-N_1)}=\frac{N_2-N_1}{N_2+N_1}=\frac{1-\cos^2\alpha}{1+\cos^2\alpha} \tag{6.2}$$

– so the scattered radiation is partially polarized even if the incident radiation is unpolarized.

No scattered flux is produced in the direction *parallel to the induced dipole moment*, which accounts physically for the *Brewster condition* when the reflected and refracted beams at an interface are perpendicular.

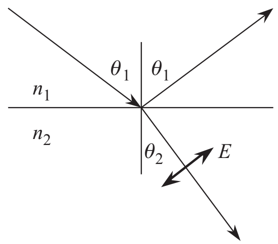

Figure 6.3: For p-polarised light and $\tan\theta_i=n_2/n_1$, the induced dipoles in medium 2 oscillate along the direction shown, so no reflected (scattered) beam is produced.

§6.2 Rayleigh Scattering

Rayleigh scattering concerns scattering from small *neutral* particles.

In general, for each of the particles a dipole moment with amplitude $p_0 = \alpha E_0$ will be induced, α is the *polarizability*.

The scattering cross section becomes:

$$\sigma = \frac{\mu_0 \omega^4 p_0^2 / 12\pi c}{c\epsilon_0 E_0^2/2} = \frac{\mu_0^2 \omega^4 \alpha^2}{6\pi} = \frac{8\pi^3}{3} \mu_0^2 c^4 \alpha^2 \frac{1}{\lambda^4} \tag{6.3}$$

Note in particular the λ^{-4} dependence.

The classic example is the blue of the sky, where the scattering particles are the molecules in the air.

The concomitant *polarization properties of light from the sky* can easily be observed using polaroid film and understood in terms of the scattering geometry. (Examples Sheet, and §6.1.)

§6.3 Thomson Scattering by Free Electrons

Consider z-polarized EM radiation incident on a free electron. The $\boldsymbol{E}$-field accelerates the electron in the z-direction. Lorentz forces due to the $\boldsymbol{B}$-field are smaller by a factor v/c and will be ignored (though ultra-intense lasers with power densities of 10^{18} Wm^{-2} allow relativistic speeds to be attained).

The equation of motion for the electron is

$$m_{\mathrm{e}} \ddot{z} = -eE_0 \mathrm{e}^{-i\omega t}$$

and it behaves like a dipole with $p = -ez = p_0 \mathrm{e}^{-i\omega t}$. So:

$$\ddot{p} = -e\ddot{z} = \frac{e^2}{m_{\mathrm{e}}} E_0 \mathrm{e}^{-i\omega t} = -\omega^2 p_0 \mathrm{e}^{-i\omega t}$$

and the total time-averaged scattered power (Eqn 4.21) is then:

$$\langle P \rangle = \frac{\mu_0 \omega^4 p_0^2}{12\pi c} = \frac{\mu_0 e^4 E_0^2}{12\pi c m_{\mathrm{e}}^2}$$

The time-averaged incident power is $\frac{1}{2} c\epsilon_0 E_0^2$ so the *Thomson cross section* is:

$$\boxed{\sigma_{\mathrm{T}} = \frac{\mu_0^2 e^4}{6\pi m_{\mathrm{e}}^2} = 6.65 \times 10^{-29} \mathrm{m}^2} \tag{6.4}$$

Note that σ_{T} is a constant, independent of ω.

(This classical analysis fails for very high frequencies when the photon momentum is significant – then *Compton Scattering* occurs.)

σ_{T} can be written in terms of the "classical electron radius" r_{e} defined such that the electrostatic self-energy of the electron equals its rest mass energy:

$$e^2/4\pi\epsilon_0 r_{\mathrm{e}} = m_{\mathrm{e}} c^2 \quad \rightarrow \quad r_{\mathrm{e}} = 2.8 \times 10^{-15} \text{ m}$$

so

$$\sigma_T = \frac{\mu_0^2 e^4}{6\pi m_{\mathrm{e}}^2} = \frac{e^4}{6\pi \epsilon_0^2 c^4 m_{\mathrm{e}}^2} = \frac{8\pi}{3} \left(\frac{e^2}{4\pi\epsilon_0 c^2 m_{\mathrm{e}}} \right)^2 = \frac{8\pi}{3} r_{\mathrm{e}}^2 \tag{6.5}$$

– roughly compatible.

Thomson scattering is important in plasma diagnosis, incoherent scattering from the atmosphere, and the scattering of light in astrophysical plasmas such as those surrounding a black hole and in the solar corona:

Figure 6.4: The Sun's corona visible during the eclipse of 11 Aug 1999. The halo is partly due to Thomson scattering of sunlight by free electrons emerging from the Sun. (What should its polarization properties be?)

§6.4 Collections of Scatterers

The cross sections derived above were for *single scatterers*. What if *many* scatterers are present – "*dense*" media?

Two new effects are relevant.

§6.4.1 Multiple scattering

Light scattered by one particle is subsequently (incoherently) scattered by another particle, and so on.

The degree to which this happens depends on the particles' scattering cross section and their density....

For particles with scattering cross-section σ and mean separation d occupying a volume of linear dimension L, the probability of a single scattering event is $\sim (\sigma/L^2) \times (N \sim L^3/d^3)$.

So it is valid to ignore multiple scattering when $\sigma L/d^3 \ll 1$.

§6.4.2 Interference

The resultant field depends on the interference of the *coherent* radiation scattered from individual particles. Consider N identical scatterers at positions $\boldsymbol{r}_j$ $(j = 1, N)$ coherently illuminated by a plane wave with wavevector $\boldsymbol{k}_\mathrm{i}$.

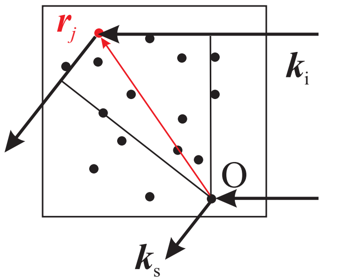

Figure 6.5: Scattering by a collection of particles.

At a distant observation point B, let the electric field of the radiation scattered from a particle at the origin O be $\tilde{\boldsymbol{E}}\mathrm{e}^{-i\omega t}$. The *additional phase* accumulated by a wave scattered from a particle at $\boldsymbol{r}_j$ is $(\boldsymbol{k}_\mathrm{i} - \boldsymbol{k}_\mathrm{s}) \cdot \boldsymbol{r}_j$, where $\boldsymbol{k}_\mathrm{s}$ is the wavevector of the scattered radiation.

With $\boldsymbol{q} \equiv \boldsymbol{k}_\mathrm{i} - \boldsymbol{k}_\mathrm{s}$ the *scattering wavevector*, the total field at B is then:

$$\tilde{\boldsymbol{E}}_\mathrm{tot} = \tilde{\boldsymbol{E}}\mathrm{e}^{-i\omega t} \sum_{j=1}^{N} \mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{r}_j} \tag{6.6}$$

The total (time averaged) radiated power is then

$$P_\mathrm{tot} = \frac{1}{2}\frac{|\tilde{\boldsymbol{E}}_\mathrm{tot}|^2}{Z_0} = P_1(\boldsymbol{q}) \left| \sum_{j=1}^{N} \mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{r}_j} \right|^2 \tag{6.7}$$

where $P_1(\boldsymbol{q}) \equiv |\tilde{\boldsymbol{E}}|^2/2Z_0$ is the "*form factor*", the power that would be scattered from a single scatterer in the direction $\boldsymbol{k}_\mathrm{s} = \boldsymbol{k}_\mathrm{i} - \boldsymbol{q}$.

$P_1(\boldsymbol{q})$ generally has a relatively weak dependence on $\boldsymbol{q}$, and the angular dependence (*via* $\boldsymbol{q}$) of the scattering is dominated by the "*structure factor*" $\mathcal{F}(\boldsymbol{q})$, caused by interference:

$$\mathcal{F}(\boldsymbol{q}) \equiv \left| \sum_{j=1}^{N} \mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{r}_j} \right|^2 \equiv \left| \mathrm{FT}\left[\sum_{j=1}^{N} \delta(\boldsymbol{r} - \boldsymbol{r}_j) \right] \right|^2 = |\mathrm{FT}[n(\boldsymbol{r})]|^2 = |\tilde{n}(\boldsymbol{q})|^2 \tag{6.8}$$

i.e. $\mathcal{F}$ is determined by the positions $\boldsymbol{r}_j$ of the scatterers *via* the (mod square) of the Fourier transform $\tilde{n}(\boldsymbol{q})$ of the *particle density* $n(\boldsymbol{r})$, and so depends on the *positional correlations* of the scattering centres.

For an ideal gas, the particle positions are uncorrelated. So each $\mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{r}_j}$ is a *phasor* with unit length and an independent random phase (i.e. direction).

The sum of these phasors has the form of a random walk in the complex plane, so the mean squared end-to-end distance is equal to the squared step length times the number of steps. So, on average:

$$\left| \sum_{j=1}^{N} \mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{r}_j} \right|^2 = N \qquad \text{so that} \qquad P_\mathrm{tot} = NP_1$$

For *random* scatterers, the scattered power is simply the sum of those scattered by the individual molecules – interference plays no role.

But when the particle positions are *correlated* (as in liquids and solids) interference plays a crucial role.

If $n(\boldsymbol{r})$ has peaks (at $\boldsymbol{r} = L$ say) corresponding to crystal structure, molecular correlations in liquids, or density variations of a certain size, so will $|\tilde{n}(\boldsymbol{q})|^2$ at $\boldsymbol{q} \sim 2\pi/L$.

Through $\mathcal{F}(\boldsymbol{q}) = |\tilde{n}(\boldsymbol{q})|^2$ these density variations produce strong scattering at $\boldsymbol{q}$.

Since $|\boldsymbol{q}| \sim |\boldsymbol{k}_\mathrm{i}|$, scattering will be strongest from *density variations* on *lengthscales* comparable to the *wavelength* of the incident radiation.

e.g. for rock salt crystals: the scatterers (Na, Cl atoms) have spacing $\sim$0.28 nm, and are very numerous.

$n(\boldsymbol{r}) \sim$ constant on lengthscales of the order the wavelength of visible light ($\sim$500 nm), so the $\mathcal{F}(\boldsymbol{q}) \to 0$. Perfect (non-absorbing) crystals scatter no light (except from their surfaces).

However, $n(\boldsymbol{r})$ is strongly modulated on the lengthscale of suitable X-rays (0.1 $\sim$ 1 nm) by the crystal lattice.

$\mathcal{F}(\boldsymbol{q})$ is then sharply peaked in particular directions given by $\boldsymbol{q} \to \boldsymbol{G}$, the reciprocal lattice vectors – *diffraction* maxima.

$\boldsymbol{k}_\mathrm{s} = \boldsymbol{k}_\mathrm{i} - \boldsymbol{G}$ is then *Bragg's Law*.

§6.4.3 Critical Opalescence

Even for transparent substances, scattering of visible light occurs from defects (including surfaces) or *density fluctuations*.

Density fluctuations on a lengthscale $\sim \lambda_\mathrm{vis}$ appear in *liquids* close to the liquid/gas *critical point*, or in liquid mixtures close to a *phase-separation*, leading to strong scattering of the visible light – ***critical opalescence***.

Methanol (CH_3OH) has polar molecules and is immiscible with non-polar hexane (C_6H_6) at room temperature. Both liquids are transparent.

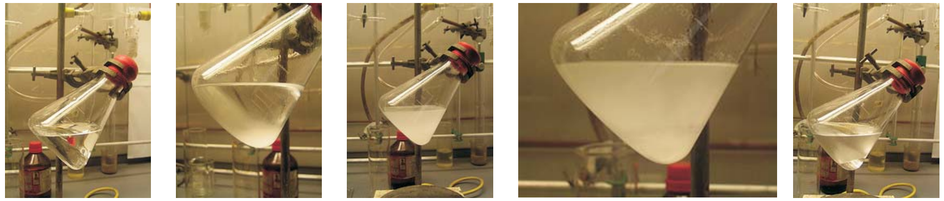

Figure 6.6: Critical opalescence from a 0.435 methanol : hexane 0.665 mixture.

Above $T = 42.4°$ C they mix to form a uniform fluid (a).

Cooling the mixture extremely slowly, two distinct phases form at some point below $T = 43°$ C, but a meniscus does not form immediately. As $T \to 42.4°$ C, large *fluctuations* in the fluid's density and therefore *local refractive index* arise, causing light scattering – the fluid appears cloudy, even though the component liquids are individually transparent. (b),(c).

Ultimately, the fluids separate (d),(e), and each appears clear again.