IX. THE SCHWARZSCHILD SOLUTION

The Einstein field equations are non-linear partial differential equations and are therefore hard to solve.

However, some exact solutions are known in situations where the spacetime possesses symmetries.

In this topic, we shall look at the first exact solution of Einstein's equations that was found, representing the spacetime in the vacuum region outside a spherically-symmetric mass distribution.

This solution was found by Karl Schwarzschild in 1915, the same year that Einstein published his General Theory of Relativity in its final form, while serving in the German army on the Russian front during World War I.

1 Spherically-symmetric spacetimes

In other areas of physics, we are used to thinking of symmetries of field configurations in an *active* sense: the fields and their sources can be actively transformed (e.g., rotated about some origin) and the resulting configuration is identical to the original if the transformation is a symmetry.

However, we can also adopt a *passive* viewpoint: we can change our coordinate system without changing the *functional form* of the fields on our coordinates.

In general relativity, we shall adopt such a passive viewpoint.

A spacetime possesses a symmetry if under some coordinate transformation $x^\mu \to x'^\mu$, the new components of the metric expressed as functions of the new coordinates, $g'_{\mu\nu}(x')$, have the same functional dependence as the original metric components on the original coordinates, $g_{\mu\nu}(x)$.

This means that the line element in the new coordinates has exactly the same dependence on x'^{μ} and dx'^{μ} as the original line element does on x^{μ} and dx^{μ}.

For the specific case of spherical symmetry, Cartesian-like coordinates x^i ($i = 1, 2, 3$) must exist such that under the *constant* coordinate transformation

$$x'^i = \sum_{j=1}^{3} (O^{-1})^i{}_j x^j \, , \tag{1}$$

where $O^i{}_j$ is an orthogonal matrix, the functional form of the line element is unchanged.

Writing $\vec{x} = (x^1, x^2, x^3)$, and the original line element as[1]

$$ds^2 = g_{00}(t, \vec{x})dt^2 + 2g_{0i}(t, \vec{x})dtdx^i + g_{ij}(t, \vec{x})dx^i dx^j \, , \tag{2}$$

the line element in the rotated coordinates is

$$\begin{aligned} ds^2 = g_{00}(t, \mathbf{O}\vec{x}')dt^2 &+ 2g_{0i}(t, \mathbf{O}\vec{x}')dtO^i{}_j dx'^j \\ &+ g_{ij}(t, \mathbf{O}\vec{x}')O^i{}_k O^j{}_l dx'^k dx'^l \, . \end{aligned} \tag{3}$$

If the spacetime is spherically-symmetric, this must have the same functional form as the original line element so that (dropping the primes)

$$g_{00}(t, \vec{x}) = g_{00}(t, \mathbf{O}\vec{x}) \tag{4}$$

$$g_{0j}(t, \vec{x})dx^j = g_{0i}(t, \mathbf{O}\vec{x})O^i{}_j dx^j \tag{5}$$

$$g_{kl}(t, \vec{x})dx^k dx^l = g_{ij}(t, \mathbf{O}\vec{x})O^i{}_k dx^k O^j{}_l dx^l \, . \tag{6}$$

To satisfy these constraints, the following must be true:

[1]Here, we extend the summation convention to include implicit summation over repeated "spatial" indices.

- $g_{00}(t, \vec{x})$ can only depend on $\vec{x}$ through the rotational invariant $r = \sqrt{\vec{x} \cdot \vec{x}}$ so we can write
$$g_{00}(t, \vec{x}) = A(t, r)\,; \tag{7}$$
- $g_{0i}(t, \vec{x})dx^i$ can only involve the invariant $\vec{x} \cdot d\vec{x}$ multiplying a function of t and r, so that
$$g_{0i}(t, \vec{x})dx^i = -B(t, r)\vec{x} \cdot d\vec{x}\,; \tag{8}$$
- $g_{ij}(t, \vec{x})dx^i dx^j$ must be of the form
$$g_{ij}(t, \vec{x})dx^i dx^j = -C(t, r)\left(\vec{x} \cdot d\vec{x}\right)^2 - D(t, r) d\vec{x} \cdot d\vec{x}\,. \tag{9}$$

We shall later discuss the following additional symmetries.

- *Stationary*: a spacetime is stationary if it has constant time shifts $t \to t + \text{const.}$ as a symmetry (for all constant shifts), which means that $g_{\mu\nu}$ cannot depend on t.
- *Static*: a spacetime is static if it additionally has time reversal, $t \to -t$, as a symmetry; this requires $g_{0i} = 0$.

To understand the distinction between a static spacetime and a stationary one, consider the analogous situation in electromagnetism.

There, a steady charge and current distribution is stationary (it looks the same for all time) and the fields and sources are independent of t.

However, under time reversal the current changes sign at a given point in space and this reverses the sign of the magnetic field everywhere.

For the configuration to be static, i.e., invariant under time reversal, would require that the current be zero (so the charges are at rest).

Returning to the most general spherically-symmetric spacetime, it is convenient to switch from the Cartesian-like spatial coordinates x^i to spherical-polar coordinates with

$$x^1 = r\sin\theta\cos\phi\,, \quad x^2 = r\sin\theta\sin\phi\,, \quad x^3 = r\cos\theta\,. \tag{10}$$

It follows that

$$\vec{x}\cdot d\vec{x} = r dr\,, \tag{11}$$
$$d\vec{x}\cdot d\vec{x} = dr^2 + r^2 d\Omega^2\,, \tag{12}$$

where $d\Omega^2$ is the spherical line element on the unit 2-sphere:

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2\,. \tag{13}$$

Using Eqs (7–9), after regrouping terms and absorbing factors of r into redefined functions A, B, C and D, the spherically-symmetric line element takes the form

$$ds^2 = A(t,r)dt^2 - 2B(t,r)dtdr - C(t,r)dr^2 - D(t,r)d\Omega^2\,. \tag{14}$$

We can simplify this further by applying various coordinate transformations.

First, we switch to a new radial coordinate $\bar{r}$ defined by

$$\bar{r}^2 = D(t,r)\,, \tag{15}$$

and so use coordinates $(t,\bar{r},\theta,\phi)$.

We can, in principle, express r as a function of $\bar{r}$ and t so that, for example, $A(t,r)$ becomes some (different) function of $\bar{r}$ and t.

Also,

$$\bar{r}^2 = D(t,r) \quad \Rightarrow \quad 2\bar{r}d\bar{r} = \frac{\partial D}{\partial t}dt + \frac{\partial D}{\partial r}dr\,, \tag{16}$$

so that

$$dr = \frac{1}{\partial D/\partial r}\left(2\bar{r}d\bar{r} - \frac{\partial D}{\partial t}dt\right) . \tag{17}$$

Substituting into the line element (14), and collecting terms and redefining functions A, B and C, we find

$$ds^2 = A(t,\bar{r})dt^2 - 2B(t,\bar{r})dtd\bar{r} - C(t,\bar{r})d\bar{r}^2 - \bar{r}^2 d\Omega^2 . \tag{18}$$

Note that $\bar{r}$ has a clear physical definition: it is an *area coordinate* in the sense that in the $t =$ const. hypersurface, the 2D surface defined by $\bar{r} =$ const. is a 2-sphere of area $4\pi\bar{r}^2$.

We can make a further coordinate transformation to remove the cross-term involving $dtd\bar{r}$ in the line element as follows.

Noting that

$$\begin{aligned} A(t,\bar{r})dt^2 - 2B(t,\bar{r})dtd\bar{r} = {} & \frac{1}{A(t,\bar{r})}\left[A(t,\bar{r})dt - B(t,\bar{r})d\bar{r}\right]^2 \\ & - \frac{B^2(t,\bar{r})}{A(t,\bar{r})}d\bar{r}^2 , \end{aligned} \tag{19}$$

we introduce a new time coordinate $\bar{t}$, which is a function of t and $\bar{r}$, such that

$$d\bar{t} = \Phi(t,\bar{r})\left[A(t,\bar{r})dt - B(t,\bar{r})d\bar{r}\right] . \tag{20}$$

The function $\Phi(t,\bar{r})$ is an integrating factor to ensure that the combination on the right-hand side is an exact differential.

In terms of $(\bar{t},\bar{r},\theta,\phi)$, the line element becomes

$$ds^2 = \frac{1}{A\Phi^2}d\bar{t}^2 - \left(\frac{B^2}{A} + C\right)d\bar{r}^2 - \bar{r}^2 d\Omega^2 , \tag{21}$$

where all functions depend implicitly on $\bar{t}$ and $\bar{r}$ through their arguments t and $\bar{r}$.

Redefining these functions, and dropping the bars, we obtain the diagonal form of the line element with spherical symmetry (the *isotropic* line element):

$$ds^2 = A(t,r)dt^2 - B(t,r)dr^2 - r^2 d\Omega^2 \,. \qquad (22)$$

Finally, we shall look for solutions that are static in these coordinates, so that the two functions A and B do not depend on time.

In this case, the *static, isotropic line element* takes the form

$$\boxed{ds^2 = A(r)dt^2 - B(r)dr^2 - r^2 d\Omega^2 \,.} \qquad (23)$$

2 Solution of the field equations in vacuum

Any static spherically-symmetric spacetime has a line element that can be written in the form (23).

The functions $A(r)$ and $B(r)$ are determined by solving the Einstein field equations given some (spherical and static) matter distribution.

We shall consider the important case of the vacuum region outside of such a spherical mass distribution.

In this case, the energy–momentum tensor vanishes and (ignoring the cosmological constant term) the Einstein field equations reduce to

$$R_{\mu\nu} = 0 \,, \qquad (24)$$

where the Ricci tensor can be written in terms of the metric connection as

$$R_{\mu\nu} = -\partial_\rho \Gamma^\rho_{\mu\nu} + \partial_\mu \Gamma^\rho_{\rho\nu} + \Gamma^\rho_{\sigma\nu}\Gamma^\sigma_{\mu\rho} - \Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\sigma\rho} \,. \qquad (25)$$

We can easily calculate the connection from the metric and its inverse; instead of using numerical coordinate labels $0, 1, 2, 3$, let us use the more transparent labels t, r, θ, ϕ, in which case

$$\begin{aligned}
g_{tt} &= A(r)\,, & g^{tt} &= 1/A(r)\,, \\
g_{rr} &= -B(r)\,, & g^{rr} &= -1/B(r)\,, \\
g_{\theta\theta} &= -r^2\,, & g^{\theta\theta} &= -1/r^2\,, \\
g_{\phi\phi} &= -r^2\sin^2\theta\,, & g^{\phi\phi} &= -1/(r^2\sin^2\theta)\,.
\end{aligned} \tag{26}$$

The non-zero, independent connection coefficients are

$$\begin{aligned}
\Gamma^t_{tr} &= A'/(2A)\,, & \Gamma^r_{tt} &= A'/(2B)\,, \\
\Gamma^r_{rr} &= B'/(2B)\,, & \Gamma^r_{\theta\theta} &= -r/B\,, \\
\Gamma^r_{\phi\phi} &= -r\sin^2\theta/B\,, & \Gamma^\theta_{r\theta} &= 1/r\,, \\
\Gamma^\theta_{\phi\phi} &= -\sin\theta\cos\theta\,, & \Gamma^\phi_{r\phi} &= 1/r\,, \\
\Gamma^\phi_{\theta\phi} &= \cot\theta\,,
\end{aligned} \tag{27}$$

where primes denote derivatives with respect to r.

It is now a matter of tedious, but routine, algebra to calculate the components of the Ricci tensor.

A useful intermediate result is that

$$\Gamma^\rho_{\rho\sigma} = \left(\frac{A'}{2A} + \frac{B'}{2B} + \frac{2}{r}\right)\delta^r_\sigma + \cot\theta\,\delta^\theta_\sigma\,. \tag{28}$$

The off-diagonal components of the Ricci tensor vanish, while the diagonal components are

$$R_{tt} = -\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB}\,, \tag{29}$$

$$R_{rr} = \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB}\,, \tag{30}$$

$$R_{\theta\theta} = \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right)\,, \tag{31}$$

$$R_{\phi\phi} = \sin^2\theta R_{\theta\theta}\,. \tag{32}$$

In vacuum, all these components must vanish.

Forming $AR_{rr}/B + R_{tt} = 0$ gives

$$\frac{A'}{A} + \frac{B'}{B} = 0 \quad \Rightarrow \quad AB = \alpha\,, \tag{33}$$

where α is an integration constant.

Substituting in $R_{\theta\theta} = 0$ then gives

$$rA' + A = \alpha\,, \tag{34}$$

so that

$$rA = \alpha(r + k)\,, \tag{35}$$

where k is a further integration constant.

It follows that

$$A(r) = \alpha\left(1 + \frac{k}{r}\right)\,, \qquad B(r) = \left(1 + \frac{k}{r}\right)^{-1}\,. \tag{36}$$

Note that if we substitute $B = \alpha/A$ into $R_{tt} = 0$ or $R_{rr} = 0$, we have

$$rA'' + 2A' = 0 \quad \Rightarrow \quad \left(r^2 A'\right)' = 0\,, \tag{37}$$

so that $r^2 A' = \text{const.}$; this is automatically satisfied by the solution in Eq. (36).

We can determine the constants α and k by considering the metric at large r, where the field becomes weak.

Recall that in the weak-field limit, consistency with Newtonian theory demands that the line element take the form

$$ds^2 \approx \left(1 + \frac{2\Phi}{c^2}\right) d(ct)^2 + \cdots\,, \tag{38}$$

where Φ is the Newtonian potential.

Outside a spherical mass of total mass M, we have $\Phi = -GM/r$ and so

$$A(r)dt^2 \to c^2\left(1 - \frac{2GM}{c^2 r}\right) dt^2\,, \tag{39}$$

which requires $\alpha = c^2$ and $k = -2GM/c^2$.

We this obtain the *Schwarzschild solution* for the vacuum region outside a static, spherically-symmetric body of mass M:

$$\boxed{ds^2 = c^2\left(1-\frac{2\mu}{r}\right)dt^2 - \left(1-\frac{2\mu}{r}\right)^{-1}dr^2 - r^2 d\Omega^2\,,} \tag{40}$$

where $\mu \equiv GM/c^2$.

Note the following properties of this solution.

- The solution is valid only down to the surface of the spherical body; in the interior region, $R_{\mu\nu} \neq 0$ and the solution is not valid there.
- The metric is singular at $r = 2\mu$. This turns out to be only a coordinate singularity, although the hypersurface $r = 2\mu$ does have interesting properties (if the radius of the body is smaller than 2μ, in which case we have a blackhole; see Handout XI). For the moment we restrict attention to $r > 2\mu$.
- As $r \to \infty$, the metric tends to the Minkowski metric and the spacetime is said to be *asymptotically flat*.

2.1 Birkhoff's theorem

Suppose we dropped the requirement that the metric be static; then

$$ds^2 = A(t,r)dt^2 - B(t,r)dr^2 - r^2 d\Omega^2\,. \tag{41}$$

If we repeated the calculation above, we would find additional terms in the connection and the components of the Ricci tensor, but solving $R_{\mu\nu} = 0$ would still lead to the same Schwarzschild solution.

This leads to *Birkhoff's theorem*:

$$\boxed{\text{Any spherically-symmetric solution of the Einstein field equations in vacuum is given by the Schwarzschild solution; it is static and asymptotically flat.}}$$

Birkhoff's theorem implies that a spherical star undergoing radial pulsations has a static external metric and so, for example, cannot emit gravitational waves.

Birkhoff's theorem is similar to Gauss's theorem in Newtonian theory; in particular, any *redistribution* of the mass in some spherical system that preserves the total mass has a static external gravitational field.

3 Geodesics in Schwarzschild spacetime

We now consider the motion of free-falling particles in the Schwarzschild solution.

It is simplest to follow the alternative "Lagrangian" approach of Handout IV, so we consider

$$L = c^2\left(1-\frac{2\mu}{r}\right)\dot{t}^2 - \left(1-\frac{2\mu}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2\,, \tag{42}$$

where overdots denote derivatives with respect to the affine parameter (which we shall denote here by λ).

The Euler–Lagrange equations are

$$\frac{\partial L}{\partial x^\mu} = \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right)\,, \tag{43}$$

where $x^0 = t$, $x^1 = r$, $x^2 = \theta$ and $x^3 = \phi$.

Consider first the equation for θ:

$$-2r^2\sin\theta\cos\theta\dot{\phi}^2 = -2\frac{d(r^2\dot{\theta})}{d\lambda}$$

$$\Rightarrow \qquad \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 = 0\,. \tag{44}$$

A possible solution of this is $\theta = \pi/2$, i.e., planar motion in the equatorial plane; given the spherical symmetry we can always consider motion in this plane without loss of generality.

The Euler–Lagrange equation for t reduces to $\partial L/\partial \dot{t} =$ const. since L has no explicit dependence on t, a consequence of the solution being stationary.

We can therefore write

$$\left(1 - \frac{2\mu}{r}\right) \dot{t} = k\,, \tag{45}$$

where k is a constant.

Note that this conservation law is equivalent to $t_0 =$ const., where $t^\mu = \dot{x}^\mu$ is the tangent vector to the world-line.

The Lagrangian has no explicit dependence on ϕ and so $\partial L/\partial \dot{\phi} =$ const.

In the plane $\theta = \pi/2$, this reduces to

$$r^2 \dot{\phi} = h \tag{46}$$

for constant h, which is equivalent to $t_3 =$ const.

Finally, for r the Euler–Lagrange equation gives

$$\left(1 - \frac{2\mu}{r}\right)^{-1} \ddot{r} + \frac{\mu c^2}{r^2}\dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-2} \frac{\mu}{r^2}\dot{r}^2 - r\dot{\phi}^2 = 0\,. \tag{47}$$

However, it is often more convenient to use a further first integral of the motion, which follows directly from $L = c^2$ for a massive particle, and $L = 0$ for a massless one:

$$\left(1 - \frac{2\mu}{r}\right) c^2\dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} \dot{r}^2 - r^2\dot{\phi}^2 = \begin{cases} c^2 & \text{massive}\,, \\ 0 & \text{massless}\,. \end{cases} \tag{48}$$

3.1 Interpretation of the integration constants k and h

The constant k in $(1-2\mu/r)\dot{t} = k$ is related to the energy of the particle as measured by a stationary observer.

Consider an observer at rest in the (r, θ, ϕ) coordinates.

Their 4-velocity is of the form $u^\mu = \mathcal{A}\delta^\mu_0$, with $\mathcal{A}$ determined from

$$c^2 = g_{\mu\nu}u^\mu u^\nu \quad \Rightarrow \quad \mathcal{A} = \left(1 - \frac{2\mu}{r}\right)^{-1/2} . \tag{49}$$

The energy of the particle with 4-momentum $\boldsymbol{p}$ as measured by this observer is[2]

$$E = \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{p}) = g_{00}\mathcal{A}p^0 , \tag{50}$$

which evaluates to

$$E = mc^2\dot{t}\left(1 - \frac{2\mu}{r}\right)^{1/2} = kmc^2\left(1 - \frac{2\mu}{r}\right)^{-1/2} \tag{51}$$

for the massive case.

We see that kmc^2 is the energy of the particle as measured by a stationary observer as $r \to \infty$.

For the massive particle to reach spatial infinity we require $k \geq 1$ (since the measured energy cannot be less than mc^2).

In the case of a massless particle, $p^0 = \dot{t}$ and we have

$$E = c^2\dot{t}\left(1 - \frac{2\mu}{r}\right)^{1/2} = kc^2\left(1 - \frac{2\mu}{r}\right)^{-1/2} . \tag{52}$$

We require $k \geq 0$ for the massless particle to reach infinity.

[2] In the rest-frame of the observer, in local inertial coordinates, the observer's 4-velocity has components $u'^\mu = (c, 0, 0, 0)$ and $p'^0 = E/c$. It follows that $E = \eta_{\mu\nu}u'^\mu p'^\nu = g_{\mu\nu}u^\mu p^\nu$.

The constant h in $r^2\dot{\phi} = h$ arises from the symmetry of the spacetime under rotations about the z-axis, and can be interpreted as the *specific angular momentum.*

3.2 The energy equation and effective potential

In Newtonian theory, we can understand a lot about orbital motion by considering the effective potential.

Let us begin by recalling how this works in Newtonian theory, before extending to general relativity.

In Newtonian theory, conservation of energy (kinetic plus gravitational potential) for motion in the equatorial plane gives

$$\frac{m}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - \frac{GMm}{r} = E_{\rm N}\,, \tag{53}$$

where $E_{\rm N}$ is the Newtonian energy.

Here r is radial distance from the mass M and the overdots are with respect to time t.

Angular momentum conservation takes the form $r^2\dot{\phi} = h$, and using this in Eq. (53) gives

$$\frac{1}{2}\dot{r}^2 - \frac{GM}{r} + \frac{h^2}{2r^2} = \frac{E_{\rm N}}{m}\,. \tag{54}$$

We write this as

$$\frac{1}{2}\dot{r}^2 + V_{\rm eff,N}(r) = \frac{E_{\rm N}}{m}\,, \tag{55}$$

where the Newtonian effective potential is

$$V_{\rm eff,N}(r) = -\frac{GM}{r} + \frac{h^2}{2r^2}\,. \tag{56}$$

This is plotted in Fig. 1 for non-zero h.

The main features of the Newtonian effective potential are as follows.

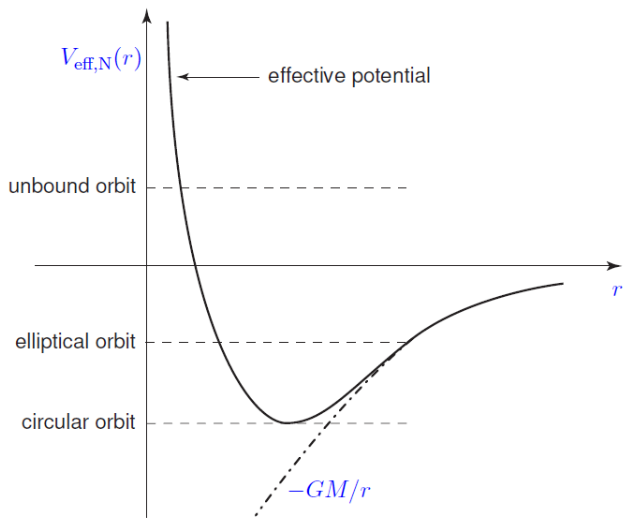


Figure 1: Effective potential in Newtonian theory (for non-zero angular orbital momentum). Note how there is an angular momentum barrier that prevents particles from reaching $r = 0$.

- There is a "centrifugal barrier" at small r, which prevents the particle reaching $r = 0$ (for non-zero angular momentum h) for any value of the energy $E_{\rm N}$.
- Bound orbits have $E_{\rm N} < 0$ so the particle cannot reach spatial infinity. In this case, there are two turning points in r during the orbit (corresponding to the extrema of an elliptical orbit) given by the solutions of $V_{\rm eff,N}(r) = E_{\rm N}$.
- The effective potential has a single turning point at $r = h^2/(GM)$. It is a minimum and corresponds to a *stable circular orbit* at this radius.

3.2.1 Massive particles in general relativity

In general relativity, an analogous energy equation can be derived by eliminating $\dot{t}$ and $\dot{\phi}$ (with their conservation equations 45 and 46) from

$$\left(1 - \frac{2\mu}{r}\right) c^2 \dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} \dot{r}^2 - r^2 \dot{\phi}^2 = c^2 \,. \quad (57)$$

which, recall, follows from $L = c^2$.

This results in

$$\frac{1}{2}\dot{r}^2 - \frac{GM}{r} + \frac{h^2}{2r^2}\left(1 - \frac{2\mu}{r}\right) = \frac{1}{2}c^2\left(k^2 - 1\right) , \qquad (58)$$

which we write as

$$\frac{1}{2}\dot{r}^2 + V_{\rm eff}(r) = \frac{1}{2}c^2\left(k^2 - 1\right) , \qquad (59)$$

where the relativistic effective potential is

$$V_{\rm eff}(r) = -\frac{GM}{r} + \frac{h^2}{2r^2}\left(1 - \frac{2\mu}{r}\right) . \qquad (60)$$

Equation (59) has the same structure as in Newtonian theory, but it must be remembered that the time derivative is with respect to the proper time of the particle, and the coordinate r does not simply measure radial distance from the origin (it is an area coordinate).

The effective potential (60) is also very similar to the Newtonian case, but the centrifugal term is modified by the factor $(1 - 2\mu/r)$.

This modification has a significant effect on the effective potential at small r for non-zero h, reversing the sign of the centrifugal barrier as shown in Fig. 2.

The figure shows that the form of the effective potential depends strongly on the angular momentum h; to understand these dependencies, consider the stationary points of $V_{\rm eff}(r)$.

These can be found from

$$\frac{dV_{\rm eff}}{dr} = \frac{\mu c^2}{r^2} + \frac{h^2}{r^3}\left(\frac{3\mu}{r} - 1\right) , \qquad (61)$$

where we have used $\mu \equiv GM/c^2$.

Solving for the locations of the extrema gives

$$r_\pm = \frac{h}{2\mu c^2}\left(h \pm \sqrt{h^2 - 12\mu^2 c^2}\right) , \qquad (62)$$

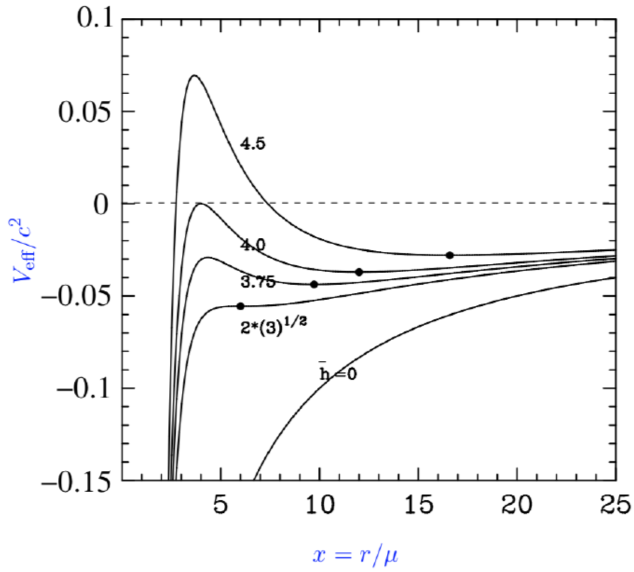


Figure 2: Effective potential in general relativity for several values of the dimensionless angular momentum $\bar{h} = h/(c\mu)$. The dots show the locations of stable circular orbits.

so that for $h > \sqrt{12}\mu c$ there are two stationary points, but none for smaller values of h.

As $h/(\mu c) \to \infty$, the locations of the stationary points tend to

$$r_+ \to \infty \quad \text{and} \quad r_- \to 3\mu \text{ (from above)}\,, \tag{63}$$

while as $h \to \sqrt{12}\mu c$ they merge at $r_\pm = 6\mu$.

The nature of the stationary points follows from

$$\frac{d^2V_{\rm eff}}{dr^2} = -\frac{2\mu c^2}{r^3} + \frac{3h^2}{r^4}\left(1 - \frac{4\mu}{r}\right)\,. \tag{64}$$

Evaluating at $r_\pm$ gives

$$\left.\frac{d^2V_{\rm eff}}{dr^2}\right|_{r_\pm} = \frac{h^2}{r_\pm^5}\left(r_\pm - 6\mu\right)\,, \tag{65}$$

so that r_- corresponds to a local maximum and r_+ to a minimum.

The effective potential at the stationary points evaluates to

$$V_{\text{eff}}(r_\pm) = \frac{h^2}{2r_\pm^3}\left(4\mu - r_\pm\right) , \tag{66}$$

which is always negative at the minimum (r_+), but is positive at the maximum for $r_- < 4\mu$ (which occurs for $h > 4\mu c$).

For $h = \sqrt{12}\mu c$ we have only an inflection point at $r = 6\mu$ and the effective potential there takes the value $V_{\text{eff}} = -c^2/18$.

The implications of the shape of $V_{\text{eff}}(r)$ for orbital motion are as follows.

- The reversal in sign of the centrifugal barrier at small r means that a particle with sufficient energy (i.e., large enough k) can always reach $r = 0$ for any h, unlike the case in Newtonian gravity. Such an in-falling particle spirals into $r = 0$.
- For $h > \sqrt{12}\mu c$, circular orbits are possible at two radii (the locations of the stationary points, $r_\pm$), with the innermost orbit being unstable and the outermost being stable.
  - The radius of the unstable circular orbit is always in the range $3\mu < r_- \leq 6\mu$, while the radius of the stable circular orbit has $r_+ > 6\mu$.
  - The *innermost stable circular orbit* is at $r = 6\mu$ and the particle has angular moment $h = \sqrt{12}\mu c$ in this orbit.
  - The circular orbits with $r > 4\mu$ are bound ($k < 1$) – the effective potential is negative there, and if the particle's motion is perturbed at constant h it cannot reach infinity.

Example: Accretion power.

Gas in an *accretion disc* around a compact object moves in quasi-circular orbits.

Due to viscosity, a packet of gas in the disc loses angular momentum causing it to move slowly inwards until it can no longer follow a stable circular orbit, at which point it falls into the compact object.

Vast amounts of energy can be radiated as gas moves though the disc, which we can estimate as follows.

First consider the Newtonian calculation: the total energy (kinetic plus gravitational potential energy) of a particle of mass m in a circular orbit at radius r is

$$E_{\mathrm{N}} = -\frac{GMm}{2r} = -\frac{\mu}{2r}mc^2 \,. \tag{67}$$

The difference in E_{N} between an orbit at $r = \infty$ and one at r can be radiated as the particle moves through the accretion disc in a series of quasi-static orbits.

It follows that

$$\frac{\Delta E_{\mathrm{N}}}{mc^2} = \frac{\mu}{2r} \,. \tag{68}$$

This can be a significant fraction of the rest-mass energy mc^2 if the particle descends to an orbit with r comparable to μ.

However, we cannot use Newtonian arguments for such compact orbits, so let us perform the analogous calculation relativistically.

We imagine a particle in a circular orbit at large r with parameter k, being disturbed in such a way that it moves to a radius r preserving the constant k (this would be like the Newtonian particle conserving its total energy).

At r, it will be moving too fast to enter a circular orbit there, and this excess energy can be shed as radiation.

We can figure out the parameter k required for a circular orbit at r as follows.

The angular momentum for a circular orbit at radius r follows from $dV_{\rm eff}/dr = 0$, which gives

$$\mu c^2 r^2 = h^2(r - 3\mu)\,. \tag{69}$$

The value of k in the orbit follows from

$$\begin{aligned}\frac{1}{2}c^2\left(k^2-1\right) &= V_{\rm eff}(r) \\ &= -\frac{\mu c^2}{r} + \frac{h^2}{2r^2}\left(1-\frac{2\mu}{r}\right) \\ &= -\frac{h^2}{2r^2}\left(1-\frac{4\mu}{r}\right)\,,\end{aligned} \tag{70}$$

where we have used Eq. (69) to substitute for μc^2 in passing to the final line.

It follows that

$$k^2 - 1 = -\frac{\mu}{r}\frac{(1-4\mu/r)}{(1-3\mu/r)}\,, \tag{71}$$

which, on solving for k, gives

$$k = \frac{(1-2\mu/r)}{(1-3\mu/r)^{1/2}}\,. \tag{72}$$

The particle starts in a circular orbit at $r \gg \mu$, so that $k \approx 1$.

When it reaches r, the energy of the particle as measured locally by a stationary observer is, from Eq. (51) with $k = 1$,

$$E = mc^2\,(1-2\mu/r)^{-1/2}\,. \tag{73}$$

In contrast, the energy for a particle in a circular orbit there is

$$E = mc^2\left(\frac{1-2\mu/r}{1-3\mu/r}\right)^{1/2}\,, \tag{74}$$

where we have used k from Eq. (72).

The difference between these two energies must be lost if the particle is to enter a circular orbit at r, which gives

$$\frac{\Delta E}{mc^2} = \left(1 - \frac{2\mu}{r}\right)^{-1/2} - \left(\frac{1 - 2\mu/r}{1 - 3\mu/r}\right)^{1/2} . \tag{75}$$

In the limit $\mu \ll r$, we have

$$\frac{\Delta E}{mc^2} \approx \frac{\mu}{2r} , \tag{76}$$

which recovers the Newtonian result (68).

Evaluating ΔE for $r = 6\mu$, corresponding to the innermost stable circular orbit, we find

$$\frac{\Delta E}{mc^2} = \frac{\sqrt{3}}{6}\left(3\sqrt{2} - 4\right) = 0.07 . \tag{77}$$

It follows that around 7 % of the rest-mass energy must be radiated as a packet of gas moves through the accretion disc to the inner-most edge at $r = 6\mu$.

This is around 10 times as efficient as fusing hydrogen into helium (which releases 0.7 % of the initial rest mass).

Such accretion powers some of the most extreme phenomena known in the Universe.

3.2.2 Massless particles in general relativity

For massless particles in general relativity, our starting point is

$$\left(1 - \frac{2\mu}{r}\right) c^2 \dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} \dot{r}^2 - r^2 \dot{\phi}^2 = 0 , \tag{78}$$

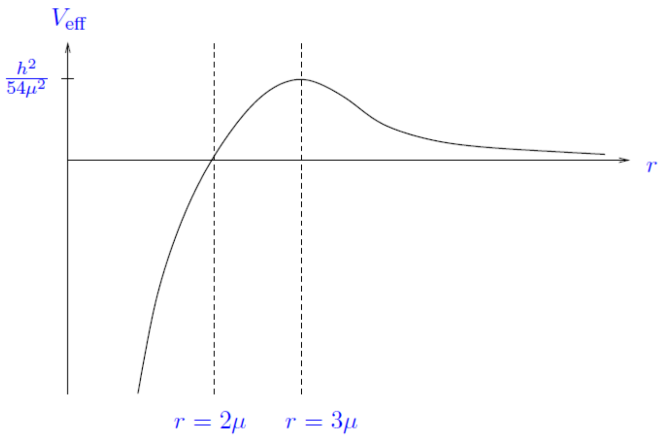


Figure 3: Relativistic effective potential for massless particles.

which follows from $L = 0$.

Eliminating $\dot{t}$ and h, as in the massive case, gives

$$\frac{1}{2}\dot{r}^2 + \frac{h^2}{2r^2}\left(1 - \frac{2\mu}{r}\right) = \frac{1}{2}c^2k^2\,, \tag{79}$$

which we write as

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}}(r) = \frac{1}{2}c^2k^2\,, \tag{80}$$

where the relativistic effective potential is

$$V_{\text{eff}}(r) = \frac{h^2}{2r^2}\left(1 - \frac{2\mu}{r}\right)\,. \tag{81}$$

The effective potential has the same shape for all non-zero h; it is plotted in Fig. 3.

We have

$$\frac{dV_{\text{eff}}}{dr} = -\frac{h^2}{r^3}\left(1 - \frac{3\mu}{r}\right)\,, \tag{82}$$

so there is a single stationary point at $r = 3\mu$, which is a maximum.

The value of V_{eff} there is

$$V_{\text{eff}}(3\mu) = \frac{h^2}{54\mu^2}\,, \tag{83}$$

and is the global maximum of the function.

We see that massless particles have a single circular orbit at $r = 3\mu$ and it is unstable.

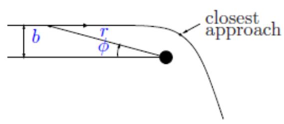
Now consider more general orbits, in particular, a photon moving inwards from large radii with angular momentum h.

If $c^2k^2/2 < h^2/(54\mu^2)$, the photon cannot cross the barrier in $V_{\rm eff}(r)$ at $r = 3\mu$; there will be a single turning point of closest approach, where $c^2k^2/2 = V_{\rm eff}(r)$, and the photon will escape to infinity again.

However, if $c^2k^2/2 > h^2/(54\mu^2)$, the photon will be captured by the massive body and will spiral inwards to $r = 0$.

Note that the behaviour of the photon orbit depends on the parameters k and h only through the ratio h/k.

The physical interpretation of this parameter is that it is the *impact parameter* of the orbit (see figure to the right):



$$b = \frac{h}{ck}\,. \tag{84}$$

To see this, we combine the energy equation (80) with $r^2\dot{\phi} = h$ to find the shape equation

$$\begin{aligned}\left(\frac{dr}{d\phi}\right)^2 &= \frac{\dot{r}^2}{\dot{\phi}^2} = \frac{r^4\dot{r}^2}{h^2}\\ &= r^2\left[\frac{c^2k^2}{h^2}r^2 - \left(1 - \frac{2\mu}{r}\right)\right]\,,\end{aligned} \tag{85}$$

so that

$$\frac{dr}{d\phi} = \pm r\left[\frac{c^2k^2}{h^2}r^2 - \left(1 - \frac{2\mu}{r}\right)\right]^{1/2}\,. \tag{86}$$

At large radii, the spacetime metric tends to the Minkowski metric and we expect photon orbits to approach straight lines there.

If the impact parameter is b and, without loss of generality, we assume that $\phi \to 0$ as $r \to \infty$, the straight line path is

$$r \sin\phi = b\,. \tag{87}$$

Differentiating gives

$$\frac{dr}{d\phi}\sin\phi + r\cos\phi = 0 \quad \Rightarrow \quad \frac{dr}{d\phi} = \pm r\left(\frac{r^2}{b^2} - 1\right)^{1/2}. \tag{88}$$

Comparing with Eq. (86) for $r \gg 2\mu$, we see that, indeed, $b = h/(ck)$.

It follows that if the impact parameter $b < \sqrt{27}\mu$, the photon will be captured by the massive body.

The reason the orbit depends only on the ratio h/k is because the effects of gravitational fields on massless particles are achromatic (i.e., independent of energy).

Since we have taken $p^\mu = dx^\mu/d\lambda$, considering a particle of different energy amounts to a constant scaling of the affine parameter (preserving its affine character).

Under such a change, the constants k and h both scale in the same way, preserving their ratio, but if the orbit depended on k and h separately the orbits would be different for massless particles with the same intitial conditions (i.e., spacetime position and spatial direction of propagation) but different frequencies.

4 Gravitational redshift

Finally, we consider the change in the frequency of a photon, as measured by observers at constant (r, θ, ϕ), as it propagates through Schwarzschild spacetime.

We have already seen that the energy of the photon relative to a stationary observer at r is

$$E = c^2 \dot{t} \left(1 - \frac{2\mu}{r}\right)^{1/2} = kc^2 \left(1 - \frac{2\mu}{r}\right)^{-1/2} , \tag{89}$$

and the energy is related to the observed frequency by $E = h\nu$.

As k is constant, if a photon is emitted at r_E, with frequency ν_E as measured by a stationary observer there, and is received by a stationary observer at r_R who measures the frequency to be r_E, we have

$$\frac{\nu_R}{\nu_E} = \left(\frac{1 - 2\mu/r_E}{1 - 2\mu/r_R}\right)^{1/2} . \tag{90}$$

This is usually expressed in terms of the *redshift* z, which is the ratio of received to emitted wavelengths:

$$1 + z = (\nu_R/\nu_E)^{-1} . \tag{91}$$

For observations at infinity, the redshift becomes

$$1 + z_\infty = \left(1 - \frac{2\mu}{r_E}\right)^{-1/2} . \tag{92}$$

Note how this tends to infinity as the point of emission approaches $r = 2\mu$, something we shall return to in Handout XI.

The calculation of the gravitational redshift generalises to any *static* metric.

We then have the "Lagrangian"

$$L = g_{00}\dot{t}^2 + \sum_{ij} g_{ij}\dot{x}^i\dot{x}^j \,, \tag{93}$$

where the metric functions are independent of t.

The Euler–Lagrange equation for t now gives

$$g_{00}\dot{t} = kc^2 \,, \tag{94}$$

with k a constant.

Furthermore, the 4-velocity of a stationary observer has

$$u^\mu = \frac{c}{\sqrt{g_{00}}}\delta^\mu_0 \,, \tag{95}$$

and so the energy of the photon as measured by such an observer is

$$E = g_{00}\frac{c}{\sqrt{g_{00}}}\dot{t} = \frac{kc^2}{\sqrt{g_{00}/c^2}} \,. \tag{96}$$

The formula for the gravitational redshift in this case generalises Eq. (90) to

$$\frac{\nu_R}{\nu_E} = \sqrt{\frac{g_{00}(E)}{g_{00}(R)}} \,. \tag{97}$$