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Advanced Quantum Physics

Handout 2: Symmetries

1.1 Introduction

The results of an experiment would not, in general, be expected to depend on the particular location or orientation of the experimental apparatus, or on precisely when the experiment was carried out, i.e. the results are generally expected to be invariant under the operations of spatial translation, spatial rotation, and time translation. Symmetry transformations such as these, which result in no observable consequences, are of fundamental importance, and underpin the structure of (all?) physical theories. In particular, invariance under spatial translation, spatial rotation, and time translation have the consequence that the total linear momentum, the total angular momentum and the total energy of an isolated system are all conserved quantities.

In this handout[1], we begin by introducing the general formalism used to handle symmetry transformations in quantum mechanics, and demonstrate the connection between invariance under a symmetry transformation and the existence of a corresponding conservation law. This general formalism is then applied to the particular cases of invariance under spatial translation, and invariance under spatial rotation. Rotational symmetry places tight constraints on the structure of matrix elements of observables taken between angular momentum eigenstates. These constraints are embodied in the *Wigner-Eckart theorem*, which we obtain for the particular cases of scalar and vector operators. Finally, we use the Wigner-Eckart theorem to obtain the Landé projection formula for the matrix elements of vector operators, and apply this to the combination (addition) of two or more magnetic dipole moments.

1.2 Symmetry transformations

When a measurement of an observable $\hat{A}$ is carried out on a quantum system which is initially in a state $|\psi\rangle$, the probabilities of the various posssible measurement outcomes are given by

$$P(\psi \to \phi_i) = \left|\langle\phi_i|\psi\rangle\right|^2 ,$$

where the $|\phi_i\rangle$ are the eigenstates of $\hat{A}$. A *symmetry transformation* can be implemented by introducing an operator $\hat{U}$ which transforms all the states of the system such that

$$|\psi\rangle \to |\psi'\rangle = \hat{U}|\psi\rangle \ , \qquad |\phi_i\rangle \to |\phi_i'\rangle = \hat{U}|\phi_i\rangle \ .$$

[1] This handout is thanks to Dr J R Batley, previous lecturer of this course.

The probabilities of the various possible measurement outcomes are left unchanged (invariant) under the symmetry operation if, for all states $|\phi_i\rangle$, we have

$$\left|\langle\phi_i'|\psi'\rangle\right|^2 = \left|\langle\phi_i|\psi\rangle\right|^2 .$$

A theorem due to Wigner states that the condition above can be satisfied in two distinct ways. The first possibility is that the overlaps themselves remain invariant,

$$\langle\phi_i'|\psi'\rangle = \langle\phi_i|\psi\rangle .$$

In this case, we have

$$\langle\phi_i'|\psi'\rangle = \langle\hat{U}\phi_i|\hat{U}\psi|\rangle = \langle\phi_i|\hat{U}^\dagger\hat{U}|\psi|\rangle = \langle\phi_i|\psi\rangle \ ,$$

and the operator $\hat{U}$ must be linear and unitary:

$$\hat{U}|\lambda\phi_i\rangle = \lambda|\phi_i\rangle \ , \qquad \hat{U}^\dagger\hat{U} = I \ .$$

The second possibility is that the overlaps pick up a complex conjugation,

$$\langle\phi_i'|\psi'\rangle = \langle\phi_i|\psi\rangle^* \ ,$$

in which case the operator $\hat{U}$ is antilinear and antiunitary:

$$\hat{U}|\lambda\phi_i\rangle = \lambda^*|\phi_i\rangle \ , \qquad \hat{U}^*\hat{U} = I \ .$$

The symmetries we shall consider are all covered by the first possibility; they are described by linear, unitary operators $\hat{U}$. An important example of a symmetry requiring an antiunitary, rather than unitary, operator $\hat{U}$ is time-reversal invariance, $t \to -t$; this will be considered in the Particle Physics option courses, for example.

For *continuous* symmetries, such as spatial translations or rotations, the symmetry operation can be taken arbitrarily close to the identity operation (for example, a rotation through an infinitesimal angle). For such symmetries, the operator $\hat{U}$ can be taken arbitrarily close to the identity operator $\hat{I}$, in which case it must take the form

$$\hat{U}(\epsilon) = \hat{I} + i\epsilon\hat{T} + \mathcal{O}(\epsilon^2) \ , \tag{1}$$

where ϵ is an arbitrary, real number. Since the identity operator $\hat{I}$ is unitary, continuous symmetries must always be described by operators $\hat{U}$ which are unitary, rather than antiunitary. We then have

$$\hat{U}(\epsilon)^\dagger\hat{U}(\epsilon) = \big(\hat{I} - i\epsilon\hat{T}^\dagger + \mathcal{O}(\epsilon^2)\big)\big(\hat{I} + i\epsilon\hat{T} + \mathcal{O}(\epsilon^2)\big) = \hat{I} + i\epsilon(\hat{T} - \hat{T}^\dagger) + \mathcal{O}(\epsilon^2) \ ,$$

from which it follows that the operator $\hat{T}$ must be Hermitian:

$$\hat{T} = \hat{T}^\dagger .$$

(The factor of i was introduced in Equation (1) for convenience, to ensure that $\hat{T}$ is Hermitian rather than anti-Hermitian). The operator $\hat{T}$ is known as the *generator* of the symmetry. In general, a symmetry will be described by a set of parameters ϵ_k, and as a result, the symmetry will possess a set of generators $\hat{T}_k$.

Any finite transformation $\hat{U}$ can be built up from an infinite number of infinitesimal transformations. To see this, take $\epsilon = \theta/N$, where θ is a finite, N-independent parameter, and apply N successive infinitesimal transformations, ϵ. This corresponds to applying the operator

$$\left(\hat{I} + \frac{i\theta\hat{T}}{N}\right)^N = \hat{I} + N\left(\frac{i\theta\hat{T}}{N}\right) + \frac{1}{2}N(N-1)\left(\frac{i\theta\hat{T}}{N}\right)^2 + \frac{1}{6}N(N-1)(N-2)\left(\frac{i\theta\hat{T}}{N}\right)^3 + \ldots\ .$$

Taking the limit $N \to \infty$ (and hence $\epsilon \to 0$) then gives

$$\left(\hat{I} + i\theta\hat{T}/N\right)^N \to \exp\left(i\theta\hat{T}\right)\ . \tag{2}$$

Thus, for finite θ, the symmetry operator $\hat{U}$ can be written as

$$\hat{U}(\theta) = \exp\left(i\theta\hat{T}\right)\ .$$

Under a symmetry transformation, a matrix element of an observable $\hat{A}$ transforms as

$$\langle\phi|\hat{A}|\psi\rangle \to \langle\phi'|\hat{A}|\psi'\rangle = \langle\hat{U}\phi|\hat{A}|\hat{U}\psi\rangle = \langle\phi|\hat{U}^\dagger\hat{A}\hat{U}|\psi\rangle\ .$$

Hence the transformation properties of matrix elements can be found by subjecting operators to a *similarity transformation*

$$\hat{A} \to \hat{U}^{-1}\hat{A}\hat{U}\ ,$$

while leaving the states of the system unaltered.

For an infinitesimal transformation, the similarity transformation is

$$\hat{A} \to \left(\hat{I} - i\epsilon\hat{T}^\dagger + \mathcal{O}(\epsilon^2)\right)\hat{A}\left(\hat{I} + i\epsilon\hat{T} + \mathcal{O}(\epsilon^2)\right)\ ,$$

and hence

$$\hat{A} \to \hat{A} - i\epsilon[\hat{T},\hat{A}] + \mathcal{O}(\epsilon^2)\ .$$

In the particular case that the magnitudes of all matrix elements of an observable $\hat{A}$ are left unaltered by the symmetry transformation, we have

$$\left|\langle\phi|\hat{A}|\psi\rangle\right|^2 = \left|\langle\phi|\hat{U}^\dagger\hat{A}\hat{U}|\psi\rangle\right|^2\ .$$

For this to hold for all possible choices of the states $|\phi\rangle$ and $|\psi\rangle$, the operator $\hat{A}$ must satisfy the condition

$$\hat{A} = \hat{U}^\dagger\hat{A}\hat{U}\ .$$

Since $\hat{U}$ is unitary, this implies that $\hat{A}$ must commute with $\hat{U}$, and hence also with the generator $\hat{T}$:

$$[\hat{A},\hat{U}] = 0\ , \qquad [\hat{A},\hat{T}] = 0\ .$$

In particular, if all matrix elements of the Hamiltonian $\hat{H}$ of a system are unchanged by a symmetry transformation, then we must have

$$[\hat{H},\hat{U}] = 0\ , \qquad [\hat{H},\hat{T}] = 0\ .$$

Ehrenfest's theorem then immediately implies that, for any state of the system, the expectation value of the generator $\hat{T}$ must remain constant:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi|\hat{T}|\psi\rangle = 0\ .$$

Thus invariance of a system under a symmetry operation leads directly to a corresponding conservation law.

1.2.1 Spatial Translations

A spatial translation of a system through a vector displacement $\boldsymbol{a}$ changes the expectation value of the position $\hat{\boldsymbol{r}}_n$ of the n'th particle for any state $|\psi\rangle$ of the system such that

$$\langle\psi'|\hat{\boldsymbol{r}}_n|\psi'\rangle = \langle\psi|\hat{\boldsymbol{r}}_n|\psi\rangle + \boldsymbol{a} \ .$$

Introducing the symmetry operator $\hat{U}(\boldsymbol{a})$ defined such that

$$|\psi'\rangle = \hat{U}(\boldsymbol{a})|\psi\rangle$$

then gives

$$\langle\psi|\hat{U}^\dagger(\boldsymbol{a})\hat{\boldsymbol{r}}_n\hat{U}(\boldsymbol{a})|\psi\rangle = \langle\psi|\hat{\boldsymbol{r}}_n|\psi\rangle + \langle\psi|\boldsymbol{a}|\psi\rangle \ .$$

For this relation to hold for all possible choices of the state $|\psi\rangle$ then requires that

$$\hat{U}^\dagger(\boldsymbol{a})\hat{\boldsymbol{r}}_n\hat{U}(\boldsymbol{a}) = \hat{\boldsymbol{r}}_n + \boldsymbol{a} \ . \tag{3}$$

If the probabilities of all measurement outcomes are left unchanged by the spatial translation, then $\hat{U}(\boldsymbol{a})$ must be unitary:

$$\hat{U}^\dagger(\boldsymbol{a})\hat{U}(\boldsymbol{a}) = \hat{I} \ .$$

For an infinitesimal translation, the operator $\hat{U}(\boldsymbol{a})$ must take the form

$$\hat{U}(\boldsymbol{a}) = \hat{I} - \frac{i}{\hbar}(\hat{\boldsymbol{P}}\cdot\boldsymbol{a}) + \mathcal{O}(a^2) \ , \tag{4}$$

where $\hat{\boldsymbol{P}} = (\hat{P}_x, \hat{P}_y, \hat{P}_z)$ is a vector containing the generators $\hat{P}_k$ of spatial translations. The factor $1/\hbar$ has been introduced in order that the operator $\hat{\boldsymbol{P}}$ has dimensions of momentum. Substituting Equation (4) into Equation (3) gives

$$\left(\hat{I} + \frac{i}{\hbar}(\hat{\boldsymbol{P}}\cdot\boldsymbol{a}) + \mathcal{O}(a^2)\right)\hat{\boldsymbol{r}}_n\left(\hat{I} - \frac{i}{\hbar}(\hat{\boldsymbol{P}}\cdot\boldsymbol{a}) + \mathcal{O}(a^2)\right) = \hat{\boldsymbol{r}}_n + \boldsymbol{a} \ .$$

Multiplying out, and keeping only terms up to $\mathcal{O}(a)$, we obtain the commutation relation

$$\frac{i}{\hbar}\left[(\hat{\boldsymbol{P}}\cdot\boldsymbol{a}), \hat{\boldsymbol{r}}_n\right] = \boldsymbol{a} \ . \tag{5}$$

The k'th component of the above equation is

$$\frac{i}{\hbar}\left[\sum_{j=1}^{3}(\hat{P}_j a_j), \hat{r}_{nk}\right] = a_k \ ,$$

or, equivalently,

$$\frac{i}{\hbar}\sum_{j=1}^{3}\left[\hat{P}_j, \hat{r}_{nk}\right] a_j = \sum_{j=1}^{3}\delta_{jk}a_j \ .$$

For this relation to hold for all possible translations $\boldsymbol{a}$, the coefficients of a_j on each side of the equation must be equal:

$$\left[\hat{r}_{nk}, \hat{P}_j\right] = i\hbar\delta_{jk} \ .$$

The commutation relation of the operator $\hat{\boldsymbol{P}}$ with the position operator $\hat{\boldsymbol{r}}_n$ is the same for all particles in the system; we therefore interpret $\boldsymbol{P}$ as the *total* momentum operator for the system.

The order in which two distinct spatial translations defined by vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are applied to a system is immaterial. The corresponding symmetry operators $\hat{U}(\boldsymbol{a})$ and $\hat{U}(\boldsymbol{b})$ therefore commute with each other:

$$\hat{U}(\boldsymbol{b})\hat{U}(\boldsymbol{a}) = \hat{U}(\boldsymbol{a})\hat{U}(\boldsymbol{b}) \ .$$

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are both infinitesimal translations, this becomes

$$\left(\hat{I} - \frac{i}{\hbar}(\hat{\boldsymbol{P}} \cdot \boldsymbol{b}) + \ldots\right)\left(\hat{I} - \frac{i}{\hbar}(\hat{\boldsymbol{P}} \cdot \boldsymbol{a}) + \ldots\right) = \left(\hat{I} - \frac{i}{\hbar}(\hat{\boldsymbol{P}} \cdot \boldsymbol{a}) + \ldots\right)\left(\hat{I} - \frac{i}{\hbar}(\hat{\boldsymbol{P}} \cdot \boldsymbol{b}) + \ldots\right) .$$

Multiplying out, we obtain

$$(\hat{\boldsymbol{P}} \cdot \boldsymbol{b})(\hat{\boldsymbol{P}} \cdot \boldsymbol{a}) = (\hat{\boldsymbol{P}} \cdot \boldsymbol{a})(\hat{\boldsymbol{P}} \cdot \boldsymbol{b}) \ .$$

Writing the above equation in component form, and equating the coefficients of $a_i b_j$ on both sides then shows that the components $\hat{P}_x$, $\hat{P}_y$, $\hat{P}_z$ of the total momentum operator $\hat{\boldsymbol{P}}$ are mutually commuting:

$$[\hat{P}_i, \hat{P}_j] = 0 \, .$$

If the expectation values of an operator $\hat{A}$ are invariant under translations for all possible states of the system,

$$\langle \psi' | \hat{A} | \psi' \rangle = \langle \psi | \hat{A} | \psi \rangle \ ,$$

then we must have

$$\hat{U}^\dagger(\boldsymbol{a}) \hat{A} \hat{U}(\boldsymbol{a}) = \hat{A} \ .$$

Since $\hat{U}(\boldsymbol{a})$ is unitary, this implies that $\hat{A}$ commutes with $\hat{U}(\boldsymbol{a})$:

$$[\hat{A}, \hat{U}(\boldsymbol{a})] = 0 \ .$$

Taking $\hat{U}(\boldsymbol{a})$ to correspond to an infinitesimal translation, as in Equation (4), then gives

$$[\hat{A}, \hat{\boldsymbol{P}} \cdot \boldsymbol{a}] = 0 \ .$$

For this to hold for all possible translations $\boldsymbol{a}$, the operator $\hat{A}$ must commute with the generators of the translation:

$$[\hat{A}, \hat{P}_i] = 0 \ , \qquad [\hat{A}, \hat{\boldsymbol{P}}] = 0 \ .$$

In particular, if the expectation values of the Hamiltonian $\hat{H}$ are invariant under spatial translation for all states of the system (as they must be for an isolated system, for example), then we have

$$[\hat{H}, \hat{P}_i] = 0 \ .$$

From Ehrenfest's theorem, the expectation values, $\langle \hat{P}_x \rangle$, $\langle \hat{P}_y \rangle$, $\langle \hat{P}_z \rangle$, of each component of the total linear momentum are therefore conserved:

$$\frac{\mathrm{d}\langle \hat{P}_i \rangle}{\mathrm{d}t} = 0 \ .$$

We thus establish that conservation of total linear momentum for an isolated system is a consequence of invariance under spatial translations.

1.2.2 Spatial Rotations

A spatial rotation is a real, linear transformation of spatial three-vectors $\boldsymbol{r} = (r_x, r_y, r_z) = (r_1, r_2, r_3)$ of the form

$$r_i \to r'_i = \sum_{j=1}^{3} R_{ij} r_j$$

that leaves all three-vector scalar products $\boldsymbol{x} \cdot \boldsymbol{y}$ invariant:

$$\sum_i \left(\sum_j R_{ij} x_j \right) \left(\sum_k R_{ik} y_k \right) = \sum_i x_i y_i \ .$$

Writing the right-hand side of the above equation as

$$\sum_i x_i y_i = \sum_i \left(\sum_j \delta_{ij} x_j \right) \left(\sum_k \delta_{ik} y_k \right) \ ,$$

and equating the coefficients of $x_j y_k$ on both sides then gives

$$\sum_i R_{ij} R_{ik} = \delta_{jk} \ .$$

In matrix notation, this is

$$R^T R = I \ , \tag{6}$$

where R and R^T are matrices which contain the elements

$$[R]_{ij} = [R^T]_{ji} = R_{ij} \ .$$

Using the identities $\det(R^T R) = \det(R^T) \det(R)$ and $\det(R^T) = \det(R)$, Equation (6) then implies that the determinant of R is

$$\det(R) = \pm 1 \ .$$

Transformations with $\det(R) = +1$ are spatial rotations; transformations with $\det(R) = -1$ are spatial inversions.

For an infinitesimal rotation, we can write the matrix R in the form

$$R = I + \omega + \mathcal{O}(\omega^2) \ ; \qquad R_{ij} = \delta_{ij} + \omega_{ij} + \mathcal{O}(\omega^2) \ ,$$

where the ω_{ij} are infinitesimal parameters. The condition in Equation (6) then gives

$$I = \left[I + \omega^T + \mathcal{O}(\omega^2) \right] \left[I + \omega + \mathcal{O}(\omega^2) \right] = I + \omega^T + \omega + \mathcal{O}(\omega^2) \ .$$

Hence the matrix ω must satisfy the constraint

$$\omega^T = -\omega \ , \qquad \omega_{ji} = -\omega_{ij} \ ,$$

and so is of the form

$$\omega = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{pmatrix} \ .$$

To find the explicit form of the matrix ω for a general infinitesimal rotation, we begin by considering a rotation through a finite angle ϕ about the z-axis. The rotation matrix R_z corresponding to such a rotation is

$$R_z = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{7}$$

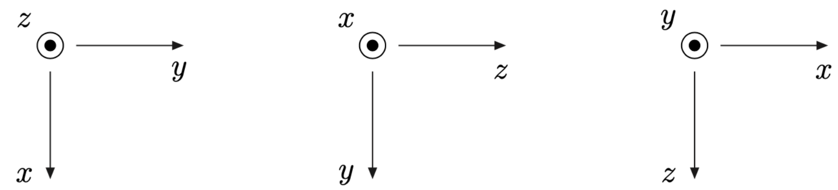
This can be considered to be either an *active* rotation of the position vector $\boldsymbol{r}$, keeping the coordinate axes fixed, or, equivalently, a *passive* rotation of the coordinate axes, keeping all positions (the apparatus) fixed.

The sign convention in Equation (7) corresponds to the left-hand rule (right-hand rule) for active (passive) rotations: when the thumb is oriented along the rotation axis (the $+z$ direction), the direction of the fingers gives the sense of the rotation for positive $\phi > 0$.

When the angle ϕ is infinitesimal, the matrix R_z becomes, to leading order,

$$R_z = \begin{pmatrix} 1 & \phi & 0 \\ -\phi & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \mathcal{O}(\phi^2) , \qquad \omega = \begin{pmatrix} 0 & \phi & 0 \\ -\phi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{8}$$

To find the corresponding matrices for rotations about the x and y axes, cycle the coordinates as $(x, y, z) \to (y, z, x) \to (z, x, y)$:



For example, the matrix element $[R_z]_{xy}$ (equal to $\sin\phi$) for a rotation about the z-axis becomes the matrix element $[R_y]_{zx}$ for a rotation about the y-axis. This gives

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} , \qquad R_y = \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix} . \tag{9}$$

For a general infinitesimal rotation, combining three matrices of the forms given in Equations (8) and (9) then gives

$$\omega = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} .$$

This corresponds to a rotation through an infinitesimal angle $|\boldsymbol{\omega}|$ about an axis oriented in the direction of the vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$.

For two successive rotations R_1 and R_2, satisfying $R_1^T R_1 = I$ and $R_2^T R_2 = I$, we have

$$(R_2 R_1)^T (R_2 R_1) = (R_1)^T (R_2)^T (R_2 R_1) = R_1^T R_1 = I .$$

Thus the matrix product R_2R_1 satisfies the condition in Equation (6), showing that two successive rotations correspond to a single, overall spatial rotation.

Since, in general, matrices do not commute, neither do rotations; applying R_1 followed by R_2 will in general give a different overall rotation to applying R_2 followed by R_1.

1.3 Rotational symmetry in quantum mechanics

Under a rotation R, the states of a quantum system transform as

$$|\psi\rangle \to |\psi'\rangle = \hat{U}(R)|\psi\rangle \, .$$

If all measurement outcomes remain invariant under spatial rotations, then the operator $\hat{U}(R)$ is a unitary operator :

$$\hat{U}(R)^\dagger \hat{U}(R) = \hat{I} \ .$$

For an infinitesimal rotation described by a vector $\boldsymbol{\omega}$, it must be possible to write the unitary operator $\hat{U}(R)$ in the form

$$\hat{U}(\omega) = \hat{I} + \frac{i}{\hbar}(\boldsymbol{\omega} \cdot \hat{\boldsymbol{J}}) + \mathcal{O}(\omega^2) \ , \tag{10}$$

where the generators $\hat{\boldsymbol{J}} = (\hat{J}_1, \hat{J}_2, \hat{J}_3)$ are a vector of Hermitian operators. The factor of $1/\hbar$ is introduced in order that the generators have dimension of angular momentum. Since rotations do not in general commute with each other, the generator components $\hat{J}_i$ will not be mutually commuting (unlike the generators $\hat{P}_i$ of spatial translations).

Since two successive rotations, R_1 followed by R_2, are equivalent to a single rotation described by the matrix product R_2R_1, we must have

$$\hat{U}(R_2)\hat{U}(R_1) = \hat{U}(R_2R_1) \ .$$

To obtain the commutation relations satisfied by the components $\hat{J}_i$, we consider a sequence of three infinitesimal rotations:

$$R_1 = I + \omega' + \ldots \, ; \qquad R_2 = I + \omega + \ldots \, ; \qquad R_3 = (R_1)^{-1} = I - \omega' + \ldots \ .$$

where we have used $(R_1)^{-1} = (R_1)^T$ and $(\omega')^T = -\omega'$. This corresponds to an overall rotation given by

$$R_3R_2R_1 = (I - \omega' + \ldots)(I + \omega + \ldots)(I + \omega' + \ldots) = I + \omega + \omega\omega' - \omega'\omega + \ldots \ ,$$

where we have kept the quadratic terms of order $\omega\omega'$ and $\omega'\omega$ as these will be needed in what follows. We thus have

$$R_3R_2R_1 = I + \omega + \Omega + \ldots \ ,$$

where

$$\Omega \equiv \omega\omega' - \omega'\omega \ .$$

Explicit multiplication gives the matrix product $\omega\omega'$ as

$$\begin{aligned}\omega\omega' &= \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \omega_3' & -\omega_2' \\ -\omega_3' & 0 & \omega_1' \\ \omega_2' & -\omega_1' & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\omega_3\omega_3' - \omega_2\omega_2' & \omega_2\omega_1' & \omega_3\omega_1' \\ \omega_1\omega_2' & -\omega_1\omega_1' - \omega_3\omega_3' & \omega_3\omega_2' \\ \omega_1\omega_3' & \omega_2\omega_3' & -\omega_2\omega_2' - \omega_1\omega_1' \end{pmatrix} .\end{aligned}$$

The combination $\Omega = \omega\omega' - \omega'\omega$ is therefore

$$\omega\omega' - \omega'\omega = \begin{pmatrix} 0 & \omega_2\omega_1' - \omega_1\omega_2' & \omega_3\omega_1' - \omega_1\omega_3' \\ \omega_1\omega_2' - \omega_2\omega_1' & 0 & \omega_3\omega_2' - \omega_2\omega_3' \\ \omega_1\omega_3' - \omega_3\omega_1' & \omega_2\omega_3' - \omega_3\omega_2' & 0 \end{pmatrix} .$$

Thus the matrix Ω corresponds to an infinitesimal rotation,

$$\Omega = \begin{pmatrix} 0 & \Omega_3 & -\Omega_2 \\ -\Omega_3 & 0 & \Omega_1 \\ \Omega_2 & -\Omega_1 & 0 \end{pmatrix} ,$$

with components

$$\Omega_1 = \omega_3\omega_2' - \omega_2\omega_3' , \qquad \Omega_2 = \omega_1\omega_3' - \omega_3\omega_1' , \qquad \Omega_3 = \omega_2\omega_1' - \omega_1\omega_2' .$$

The equation $\hat{U}(R_3)\hat{U}(R_2)\hat{U}(R_1) = \hat{U}(R_3R_2R_1)$ then gives

$$\left(\hat{I} - \frac{i}{\hbar}(\boldsymbol{\omega}' \cdot \hat{\boldsymbol{J}}) + ..\right)\left(\hat{I} + \frac{i}{\hbar}(\boldsymbol{\omega} \cdot \hat{\boldsymbol{J}}) + ..\right)\left(\hat{I} + \frac{i}{\hbar}(\boldsymbol{\omega}' \cdot \hat{\boldsymbol{J}}) + ..\right) = \hat{I} + \frac{i}{\hbar}(\boldsymbol{\omega} \cdot \hat{\boldsymbol{J}}) + \frac{i}{\hbar}(\boldsymbol{\Omega} \cdot \hat{\boldsymbol{J}}) + .. ,$$

where $\boldsymbol{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$. Multiplying out the left-hand side above then gives

$$(\boldsymbol{\omega}' \cdot \hat{\boldsymbol{J}})(\boldsymbol{\omega} \cdot \hat{\boldsymbol{J}}) - (\boldsymbol{\omega} \cdot \hat{\boldsymbol{J}})(\boldsymbol{\omega}' \cdot \hat{\boldsymbol{J}}) = i\hbar(\boldsymbol{\Omega} \cdot \hat{\boldsymbol{J}}) . \tag{11}$$

Writing the scalar products in Equation (11) explicitly in terms of the vector components, and equating the coefficients of $\omega_1\omega_2'$ on both sides, gives

$$\hat{J}_1\hat{J}_2 - \hat{J}_2\hat{J}_1 = i\hbar\hat{J}_3 .$$

Similarly, equating the coefficients of $\omega_2\omega_3'$ gives

$$\hat{J}_2\hat{J}_3 - \hat{J}_3\hat{J}_2 = i\hbar\hat{J}_1 ,$$

and equating the coefficients of $\omega_3\omega_1'$ gives

$$\hat{J}_3\hat{J}_1 - \hat{J}_1\hat{J}_3 = i\hbar\hat{J}_2 .$$

Overall, we obtain the commutation relations

$$\boxed{[\hat{J}_i, \hat{J}_j] = i\hbar\sum_k \epsilon_{ijk}\hat{J}_k} . \tag{12}$$

The vector operator $\hat{\boldsymbol{J}}$ therefore satisfies the standard commutation relations for angular momentum. Since $\hat{\boldsymbol{J}}$ is a property of the system as a whole, rather than of a particular particle, we identify $\hat{\boldsymbol{J}}$ as the *total* angular momentum operator for the system.

1.3.1 Scalar operators

A *scalar operator* is an operator whose matrix elements are invariant under rotations,

$$\langle\psi'|\hat{K}|\phi'\rangle = \langle\psi|\hat{K}|\phi\rangle \ , \tag{13}$$

for all possible states of the system, $|\psi\rangle$ and $|\phi\rangle$. Since $|\psi'\rangle = U(R)|\psi\rangle$ and $|\phi'\rangle = U(R)|\phi\rangle$, the left-hand side is

$$\langle\psi'|\hat{K}|\phi'\rangle = \langle\psi|\hat{U}(R)^\dagger\hat{K}\hat{U}(R)|\phi\rangle \ .$$

For Equation (13) to hold for all possible states $|\psi\rangle$ and $|\phi\rangle$ then requires that $\hat{K}$ satisfy the relation

$$U(R)^\dagger\hat{K}U(R) = \hat{K}$$

for all rotations R. Since $U(R)$ is unitary, this is equivalent to the condition that $\hat{K}$ commutes with $U(R)$ for all rotations R:

$$[U(R), \hat{K}] = 0 \ .$$

Applying this result to the case of infinitesimal rotations, with $U(R)$ of the form given in Equation (10), then shows that $\hat{K}$ must commute with all components of the total angular momentum operator, $\hat{\boldsymbol{J}}$:

$$[\hat{J}_i, \hat{K}] = 0 \ ; \qquad [\hat{\boldsymbol{J}}, \hat{K}] = 0 \ . \tag{14}$$

From this, it follows immediately that $\hat{K}$ also commutes with $\hat{\boldsymbol{J}}^2$ and with the ladder operators $\hat{J}_\pm = \hat{J}_1 \pm i\hat{J}_2$:

$$[\hat{\boldsymbol{J}}^2, \hat{K}] = 0 \ , \qquad [\hat{J}_\pm, \hat{K}] = 0 \ .$$

Equation (14) can also serve as the *definition* of a scalar operator $\hat{K}$ with respect to an angular momentum operator $\hat{\boldsymbol{J}}$.

1.3.2 Vector operators

An operator

$$\hat{\boldsymbol{V}} = (\hat{V}_x, \hat{V}_y, \hat{V}_z) = (\hat{V}_1, \hat{V}_2, \hat{V}_3)$$

whose expectation values transform under rotations like an ordinary vector,

$$\langle\psi|\hat{V}_i|\psi\rangle \to \langle\psi'|\hat{V}_i|\psi'\rangle = \sum_j R_{ij}\langle\psi|\hat{V}_j|\psi\rangle \ , \tag{15}$$

is said to be a *vector operator*. The left-hand side of Equation (15) is

$$\langle\psi'|\hat{V}_i|\psi'\rangle = \langle\psi|\hat{U}(R)^\dagger\hat{V}_j\hat{U}(R)|\psi\rangle \ .$$

For Equation (15) to hold for all possible states $|\psi\rangle$, the operator $\hat{\boldsymbol{V}}$ must satisfy the relation

$$\hat{U}^\dagger(R)\hat{V}_i\hat{U}(R) = \sum_j R_{ij}\hat{V}_j \tag{16}$$

for all possible rotations R.

For the case that R is an infinitesimal rotation, $R = I + \omega$, Equation (16) gives

$$\left(I - \frac{i}{\hbar}(\boldsymbol{\omega}\cdot\hat{\boldsymbol{J}}) + \ldots\right)\hat{V}_i\left(I + \frac{i}{\hbar}(\boldsymbol{\omega}\cdot\hat{\boldsymbol{J}}) + \ldots\right) = \hat{V}_i + \sum_j \omega_{ij}\hat{V}_j + \ldots\ .$$

Expanding the left-hand side up to first-order in ω then gives

$$\frac{i}{\hbar}\hat{V}_i(\boldsymbol{\omega}\cdot\hat{\boldsymbol{J}}) - \frac{i}{\hbar}(\boldsymbol{\omega}\cdot\hat{\boldsymbol{J}})\hat{V}_i = \sum_j \omega_{ij}\hat{V}_j \qquad (i = 1, 2, 3)\ . \tag{17}$$

Written out explicitly, the right-hand side of the above equation is

$$\sum_j \omega_{ij}\hat{V}_j = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}\begin{pmatrix} \hat{V}_1 \\ \hat{V}_2 \\ \hat{V}_3 \end{pmatrix} = \begin{pmatrix} \omega_3\hat{V}_2 - \omega_2\hat{V}_3 \\ -\omega_3\hat{V}_1 + \omega_1\hat{V}_3 \\ \omega_2\hat{V}_1 - \omega_1\hat{V}_2 \end{pmatrix}\ .$$

Hence, for $i = 1$, Equation (17) gives

$$\hat{V}_1(\boldsymbol{\omega}\cdot\hat{\boldsymbol{J}}) - (\boldsymbol{\omega}\cdot\hat{\boldsymbol{J}})\hat{V}_1 = -i\hbar(\omega_3\hat{V}_2 - \omega_2\hat{V}_3)\ .$$

Expanding $\boldsymbol{\omega}\cdot\hat{\boldsymbol{J}} = \omega_1\hat{J}_1 + \omega_2\hat{J}_2 + \omega_3\hat{J}_3$ on the left-hand side, and rearranging, we obtain

$$\omega_1(\hat{V}_1\hat{J}_1 - \hat{J}_1\hat{V}_1) + \omega_2(\hat{V}_1\hat{J}_2 - \hat{J}_2\hat{V}_1) + \omega_3(\hat{V}_1\hat{J}_3 - \hat{J}_3\hat{V}_1) = -i\hbar(\omega_3\hat{V}_2 - \omega_2\hat{V}_3)\ .$$

This relation must hold for all possible choices of $\boldsymbol{\omega}$. Equating the coefficients of ω_1, ω_2 and ω_3 on both sides gives

$$\hat{V}_1\hat{J}_1 - \hat{J}_1\hat{V}_1 = 0\ , \qquad \hat{V}_1\hat{J}_2 - \hat{J}_2\hat{V}_1 = i\hbar\hat{V}_3\ , \qquad \hat{V}_1\hat{J}_3 - \hat{J}_3\hat{V}_1 = -i\hbar\hat{V}_2\ .$$

Repeating this process for $i = 2$ and $i = 3$ gives, similarly,

$$\begin{aligned} &\hat{V}_2\hat{J}_2 - \hat{J}_2\hat{V}_2 = 0\ , \qquad \hat{V}_2\hat{J}_3 - \hat{J}_3\hat{V}_2 = i\hbar\hat{V}_1\ , \qquad \hat{V}_2\hat{J}_1 - \hat{J}_1\hat{V}_2 = -i\hbar\hat{V}_3\ , \\ &\hat{V}_3\hat{J}_3 - \hat{J}_3\hat{V}_3 = 0\ , \qquad \hat{V}_3\hat{J}_1 - \hat{J}_1\hat{V}_3 = i\hbar\hat{V}_2\ , \qquad \hat{V}_3\hat{J}_2 - \hat{J}_2\hat{V}_3 = -i\hbar\hat{V}_1\ . \end{aligned}$$

Altogether, for *any* vector operator $\hat{V}$, the equations above can be written compactly as

$$\boxed{[\hat{J}_i, \hat{V}_j] = i\hbar\sum_k \epsilon_{ijk}\hat{V}_k}\ , \tag{18}$$

Equivalently, using the antisymmetry properties of ϵ_{ijk}, this is

$$[\hat{V}_i, \hat{J}_j] = i\hbar\sum_k \epsilon_{ijk}\hat{V}_k\ .$$

Explicitly, the non-zero commutation relations between the components of the operators $\hat{\boldsymbol{J}}$ and $\hat{\boldsymbol{V}}$ can be summarised as

$$\begin{aligned} &[\hat{J}_1, \hat{V}_2] = i\hbar\hat{V}_3\ , \qquad [\hat{J}_2, \hat{V}_3] = i\hbar\hat{V}_1\ , \qquad [\hat{J}_3, \hat{V}_1] = i\hbar\hat{V}_2\ , \\ &[\hat{J}_1, \hat{V}_3] = -i\hbar\hat{V}_2\ , \qquad [\hat{J}_2, \hat{V}_1] = -i\hbar\hat{V}_3\ , \qquad [\hat{J}_3, \hat{V}_2] = -i\hbar\hat{V}_1\ . \end{aligned}$$

Equation (18) can also serve as the *definition* of a vector operator $\hat{\boldsymbol{V}}$ with respect to an angular momentum operator $\hat{\boldsymbol{J}}$.

1.3.2.1 Spherical components of a vector operator

The *spherical components* $\hat{V}_{+1}$, $\hat{V}_{-1}$ and $\hat{V}_0$ of a vector operator are defined as

$$\boxed{\hat{V}_{+1} \equiv -\frac{1}{\sqrt{2}}(\hat{V}_1 + i\hat{V}_2), \qquad \hat{V}_{-1} \equiv \frac{1}{\sqrt{2}}(\hat{V}_1 - i\hat{V}_2), \qquad \hat{V}_0 \equiv \hat{V}_3} \ . \tag{19}$$

Note that the spherical component $\hat{V}_{+1}$ is distinct from the Cartesian component $\hat{V}_1 = \hat{V}_x$ and from the ladder operator $\hat{V}_+ = \hat{V}_1 + i\hat{V}_2$.

The commutator of the spherical component $\hat{V}_{+1}$ of $\hat{\boldsymbol{V}}$ with the component $\hat{J}_3 = \hat{J}_z$ of $\hat{\boldsymbol{J}}$ can be evaluated as

$$[\hat{J}_3, \hat{V}_{+1}] = -\frac{1}{\sqrt{2}}[\hat{J}_3, \hat{V}_1] - \frac{i}{\sqrt{2}}[\hat{J}_3, \hat{V}_2] = -\frac{1}{\sqrt{2}}(i\hbar\hat{V}_2) - \frac{i}{\sqrt{2}}(-i\hbar\hat{V}_1) = \hbar\,\hat{V}_{+1} \ .$$

For $\hat{V}_{-1}$ we similarly obtain

$$[\hat{J}_3, \hat{V}_{-1}] = -\hbar\,\hat{V}_{-1} \ ,$$

while for $\hat{V}_0$ we have simply

$$[\hat{J}_3, \hat{V}_0] = [\hat{J}_3, \hat{V}_3] = 0 \ .$$

The above relations can be summarised as

$$\boxed{[\hat{J}_3, \hat{V}_m] = \hbar m\,\hat{V}_m} \qquad (m = 0, \pm 1) \ .$$

The spherical component $\hat{V}_{+1}$ commutes with the ladder operator $\hat{J}_+ = \hat{J}_1 + i\hat{J}_2$,

$$[\hat{J}_+, \hat{V}_{+1}] = -\frac{1}{\sqrt{2}}[\hat{J}_1 + i\hat{J}_2, \hat{V}_1 + i\hat{V}_2] = -\frac{1}{\sqrt{2}}\Big(i(i\hbar\hat{V}_3) + i(-i\hbar\hat{V}_3)\Big) = 0 \ ,$$

while for $\hat{V}_{-1}$ and $\hat{V}_0$, the commutators with $\hat{J}_+$ are

$$[\hat{J}_+, \hat{V}_{-1}] = \frac{1}{\sqrt{2}}[\hat{J}_1 + i\hat{J}_2, \hat{V}_1 - i\hat{V}_2] = \frac{1}{\sqrt{2}}\Big(-i(i\hbar\hat{V}_3) + i(-i\hbar\hat{V}_3)\Big) = \sqrt{2}\hbar\,\hat{V}_3 = \sqrt{2}\hbar\,\hat{V}_0 \ ,$$

and

$$[\hat{J}_+, \hat{V}_0] = [\hat{J}_1 + i\hat{J}_2, \hat{V}_3] = -i\hbar\hat{V}_2 + i(i\hbar\hat{V}_1) = \sqrt{2}\hbar\,\hat{V}_{+1} \ .$$

For the ladder operator $\hat{J}_- = \hat{J}_1 - i\hat{J}_2$, the equivalent expressions are

$$[\hat{J}_-, \hat{V}_{+1}] = \sqrt{2}\hbar\,\hat{V}_0 \ , \qquad [\hat{J}_-, \hat{V}_{-1}] = 0 \ , \qquad [\hat{J}_-, \hat{V}_0] = \sqrt{2}\hbar\,\hat{V}_{-1} \ .$$

All the above commutation relations involving the ladder operators $\hat{J}_\pm$ can be summarised as

$$[\hat{J}_\pm, \hat{V}_m] = \hbar\sqrt{2 - m(m \pm 1)}\,\hat{V}_{m\pm 1} \qquad (m = 0, \pm 1) \ ,$$

or, equivalently, as

$$\boxed{[\hat{J}_\pm, \hat{V}_m] = \hbar\sqrt{j(j+1) - m(m \pm 1)}\,\hat{V}_{m\pm 1}} \qquad (j \equiv 1,\ m = 0, \pm 1) \ . \tag{20}$$

1.3.2.2 Properties of vector operators

If the vector operator $\hat{\boldsymbol{V}}$ corresponds to an observable quantity, then its Cartesian components $\hat{V}_i$ must all be Hermitian. It then follows directly from Equation (19) that the spherical components $\hat{V}_m$ of $\hat{\boldsymbol{V}}$ have Hermitian conjugates given by

$$\hat{V}_m^\dagger = (-1)^m \hat{V}_{-m} \ .$$

Inverting Equation (19), the Cartesian components $\hat{V}_i$ are given in terms of the spherical components $\hat{V}_m$ as

$$\hat{V}_1 = \frac{1}{\sqrt{2}}(\hat{V}_{-1} - \hat{V}_{+1}), \qquad \hat{V}_2 = \frac{i}{\sqrt{2}}(\hat{V}_{-1} + \hat{V}_{+1}), \qquad \hat{V}_3 = \hat{V}_0 \ .$$

The commutation relations between the scalar product $\hat{\boldsymbol{V}} \cdot \hat{\boldsymbol{W}} = \hat{V}_1\hat{W}_1 + \hat{V}_2\hat{W}_2 + \hat{V}_3\hat{W}_3$ of two vector operators $\hat{\boldsymbol{V}}$ and $\hat{\boldsymbol{W}}$ and the component $\hat{J}_1$ of $\hat{\boldsymbol{J}}$ can then be evaluated as

$$\begin{aligned}
[\hat{J}_1, \hat{V}_1\hat{W}_1] &= [\hat{J}_1, \hat{V}_1]\hat{W}_1 + \hat{V}_1[\hat{J}_1, \hat{W}_1] = 0 \\
[\hat{J}_1, \hat{V}_2\hat{W}_2] &= [\hat{J}_1, \hat{V}_2]\hat{W}_2 + \hat{V}_2[\hat{J}_1, \hat{W}_2] = i\hbar(\hat{V}_3\hat{W}_2 + \hat{V}_2\hat{W}_3) \\
[\hat{J}_1, \hat{V}_3\hat{W}_3] &= [\hat{J}_1, \hat{V}_3]\hat{W}_3 + \hat{V}_3[\hat{J}_1, \hat{W}_3] = -i\hbar(\hat{V}_2\hat{W}_3 + \hat{V}_3\hat{W}_2)
\end{aligned}$$

Summing the above equations shows that $\hat{J}_1$ commutes with the scalar product $\hat{\boldsymbol{V}} \cdot \hat{\boldsymbol{W}}$:

$$[\hat{J}_1, \hat{\boldsymbol{V}} \cdot \hat{\boldsymbol{W}}] = 0 \ .$$

Since the choice of the component $\hat{J}_1$ was arbitrary, this relation must be true for all components of $\hat{\boldsymbol{J}}$, not just $\hat{J}_1$:

$$[\hat{\boldsymbol{J}}, \hat{\boldsymbol{V}} \cdot \hat{\boldsymbol{W}}] = 0 \ .$$

Hence the scalar product $\hat{\boldsymbol{V}} \cdot \hat{\boldsymbol{W}}$ of two vector operators (for example, the spin-orbit coupling product $\hat{\boldsymbol{L}} \cdot \hat{\boldsymbol{S}}$) is a scalar operator. In particular, taking $\hat{\boldsymbol{W}} = \hat{\boldsymbol{V}}$, we also obtain

$$[\hat{\boldsymbol{J}}, \hat{\boldsymbol{V}}^2] = 0 \ .$$

The scalar product $\hat{\boldsymbol{V}} \cdot \hat{\boldsymbol{W}}$ can be expressed in terms of spherical components as

$$\begin{aligned}
\hat{\boldsymbol{V}} \cdot \hat{\boldsymbol{W}} &= \hat{V}_1\hat{W}_1 + \hat{V}_2\hat{W}_2 + \hat{V}_3\hat{W}_3 \\
&= \frac{1}{2}(\hat{V}_{-1} - \hat{V}_{+1})(\hat{W}_{-1} - \hat{W}_{+1}) - \frac{1}{2}(\hat{V}_{-1} + \hat{V}_{+1})(\hat{W}_{-1} + \hat{W}_{+1}) + \hat{V}_0\hat{W}_0 \\
&= -\hat{V}_{+1}\hat{W}_{-1} - \hat{V}_{-1}\hat{W}_{+1} + \hat{V}_0\hat{W}_0 \ .
\end{aligned}$$

Similar arguments show that the vector product $\hat{\boldsymbol{V}} \wedge \hat{\boldsymbol{W}}$ of two vector operators $\hat{\boldsymbol{V}}$ and $\hat{\boldsymbol{W}}$ is also a vector operator.

1.4 The Wigner-Eckart theorem for scalar operators

Let $\hat{K}$ be a scalar operator with respect to an angular momentum operator $\hat{\boldsymbol{J}}$:

$$[\hat{\boldsymbol{J}}, \hat{K}] = 0 \ .$$

Then the Wigner-Eckart theorem states that matrix elements of $\hat{K}$ taken between total angular momentum eigenstates $|\alpha''j''m''\rangle$ and $|\alpha'j'm'\rangle$ must be of the form

$$\boxed{\langle\alpha''j''m''|\hat{K}|\alpha'j'm'\rangle = C(\alpha''\alpha';j')\,\delta_{j''j'}\,\delta_{m''m'}} \ , \tag{21}$$

where $C(\alpha''\alpha';j')$ is a complex constant known as the *reduced matrix element* which is independent of the quantum numbers m'' and m'. The quantities α'' and α' collectively label all other quantum numbers needed to uniquely identify the angular momentum eigenstates involved.

For completeness, the proof of Equation (21) is provided in the Appendix.

The Wigner-Eckart theorem shows that matrix elements between total angular momentum eigenstates for rotationally invariant (scalar) operators do not depend on the quantum number m. This is intuitively reasonable; the angular momentum eigenstates $|jm\rangle$ are defined with respect to a particular choice of quantisation axis, conventionally taken to be the z axis. A different choice of axis, the x axis for example, would result in a different set of eigenstates, $|jm_x\rangle$, which are linear combinations of the conventional z-axis states $|jm\rangle$. For a rotationally invariant operator, all spatial directions are equivalent. Hence the matrix elements of $\hat{K}$ cannot depend on a particular choice of quantisation axis; hence they cannot depend on the quantum number m (or m_x or ...) whose definition depends on the choice of axis.

The reduced matrix element $C(\alpha''\alpha';j')$ is usually written in Dirac-style notation as

$$C(\alpha''\alpha';j') \equiv \langle\alpha''j''||\hat{K}||\alpha'j'\rangle \ . \tag{22}$$

Despite its name, the reduced matrix element, $\langle\alpha''j''||\hat{K}||\alpha'j'\rangle$, is *not* a matrix element; it is a common, constant component of a family of $(2j+1)(2j'+1)$ matrix elements.

The product of Kronecker-δ factors in Equation (21) can be expressed as a single Clebsch-Gordan coefficient. To see this, consider the (trivial) angular momentum combination $0 \otimes j'' = j''$. The connection between the total (coupled) and uncoupled eigenstates is simply

$$|j''m''\rangle = |00\rangle \otimes |j''m''\rangle \equiv |00\rangle|j''m''\rangle \ .$$

Comparing with the expansion

$$|j''m''\rangle = \sum_{j'm'} |00\rangle|j'm'\rangle\langle 00;j'm'|j''m''\rangle$$

then shows that the relevant Clebsch-Gordan coefficients are either zero or unity :

$$\langle 00;j'm'|j''m''\rangle = \delta_{j''j'}\,\delta_{m''m'} \ . \tag{23}$$

Equations (22) and (23) allow the Wigner-Eckart theorem for scalar operators, Equation (21), to be written in the form

$$\boxed{\langle\alpha''j''m''|\hat{K}|\alpha'j'm'\rangle = \langle\alpha''j''||\hat{K}||\alpha'j'\rangle\langle 00;j'm'|j''m''\rangle}\ .$$

The reason for writing the Wigner-Eckart theorem in this way will become clear once we have also considered the equivalent result for *vector* operators.

1.4.1 Consequences of the Wigner-Eckart theorem (for scalars)

Consider the Hamiltonian operator $\hat{H}$ for an isolated system. Since there can be no preferred spatial direction for an isolated system, the expectation values of $\hat{H}$ (the energy eigenvalues) cannot depend on the spatial orientation of the system. Hence $\hat{H}$ must be a scalar operator, commuting with (all components of) the total angular momentum operator $\hat{\boldsymbol{J}}$:

$$[\hat{J}_i, \hat{H}] = 0\ , \qquad [\hat{\boldsymbol{J}}, \hat{H}] = 0\ , \qquad [\hat{\boldsymbol{J}}^2, \hat{H}] = 0\ .$$

Since $\hat{H}$ and $\hat{\boldsymbol{J}}$ commute, it follows from Ehrenfest's theorem that the expectation values $\langle\hat{J}_i\rangle$ of each component of $\hat{\boldsymbol{J}}$ are constant :

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi(t)|\hat{\boldsymbol{J}}|\psi(t)\rangle = 0\ .$$

Thus, for an isolated system, total angular momentum is conserved; this is a direct consequence of rotational symmetry.

Since $\hat{H}$ and $\hat{\boldsymbol{J}}$ commute, they possess a simultaneous set of eigenstates $|\alpha jm\rangle$, where "α" represents all other quantum numbers needed to uniquely identify a particular energy eigenstate of $\hat{H}$. From the Wigner-Eckart theorem, the matrix elements of $\hat{H}$ between these eigenstates must be of the form

$$\langle\alpha''j''m''|\hat{H}|\alpha'j'm'\rangle = \langle\alpha''j''||\hat{H}||\alpha'j'\rangle\,\delta_{j''j'}\,\delta_{m''m'}\ ,$$

where the reduced matrix element $\langle\alpha''j''||\hat{H}||\alpha'j'\rangle$ is a constant, independent of the quantum number m. In particular, the expectation values of $\hat{H}$ are

$$\langle\alpha jm|\hat{H}|\alpha jm\rangle = \langle\alpha j||\hat{H}||\alpha j\rangle\ .$$

Hence the energy eigenvalues for an isolated system can depend only on α and j, and must be independent of m. The energy levels therefore have a degeneracy $2j+1$ with respect to angular momentum. Further degeneracy is possible if the Hamiltonian $\hat{H}$ possesses additional symmetry properties involving the quantum numbers α.

This illustrates a general principle, that degeneracies in quantum mechanics are hardly ever accidental, but indicate the existence of an underlying symmetry (in this case, rotational invariance).

1.5 The Wigner-Eckart theorem for vector operators

Consider an operator $\hat{\boldsymbol{V}}$ which is a vector operator with respect to an angular momentum operator $\hat{\boldsymbol{J}}$, and thus satisfies Equation (18) :

$$[\hat{J}_i, \hat{V}_j] = i\hbar \sum_k \epsilon_{ijk} \hat{V}_k \ .$$

Then the Wigner-Eckart theorem states that the matrix elements of $\hat{\boldsymbol{V}}$ between eigenstates $|\alpha' j' m'\rangle$ and $|\alpha'' j'' m''\rangle$ of $\hat{\boldsymbol{J}}$ must be of the form

$$\boxed{\langle \alpha'' j'' m'' | \hat{V}_m | \alpha' j' m' \rangle = \langle \alpha'' j'' || \hat{\boldsymbol{V}} || \alpha' j' \rangle \langle 1m; j'm' | j''m'' \rangle} \ , \tag{24}$$

where the quantum number m takes the values $m = \pm 1, 0$, and $\hat{V}_m$ is a spherical component of the operator $\hat{\boldsymbol{V}}$. The reduced matrix element $\langle \alpha'' j'' || \hat{\boldsymbol{V}} || \alpha' j' \rangle$ is a complex constant which is independent of the quantum numbers m, m' and m''. The final factor, $\langle 1m; j'm' | j''m'' \rangle$, is a Clebsch-Gordan coefficient which can be obtained by considering the angular momentum combination $j'' = 1 \otimes j' = j', j' \pm 1$.

For completeness, the proof of Equation (24) is provided in the Appendix.

The definition of the quantum numbers m, m' and m'' depends on the arbitrary choice of quantisation axis. The Wigner-Eckart theorem shows that the dependence of vector operator matrix elements on this arbitrary choice of axis is carried entirely by Clebsch-Gordan coefficients; these are independent of the particular vector operator, $\hat{\boldsymbol{V}}$, being considered.

The reduced matrix element $\langle \alpha'' j'' || \hat{\boldsymbol{V}} || \alpha' j' \rangle$ is a common, constant component of a set of $3(2j' + 1)(2j'' + 1)$ matrix elements, $\langle \alpha'' j'' m'' | \hat{V}_m | \alpha' j' m' \rangle$. To determine the reduced matrix element, only *one* of the matrix elements belonging to this set needs to be evaluated explicitly. Once this has been done, all other matrix elements of $\hat{\boldsymbol{V}}$ are determined straightforwardly by looking up the appropriate Clebsch-Gordan coefficients.

The Wigner-Eckart theorem allows various general properties of the matrix elements of vector operators (positions, linear momenta, angular momenta, ...) to be obtained. For example, it leads directly to selection rules in atomic transitions, to general expressions for atomic energy level shifts in electric and magnetic fields (the Stark and Zeeman effects), as well as to many applications involving the scattering or decay of nuclei and elementary particles.

1.5.1 Selection rules for vector operator matrix elements

For the case $j' = j'' = 0$, and hence also $m' = m'' = 0$, the Clebsch-Gordan coefficient $\langle 1m; j'm' | j''m'' \rangle$ vanishes for all m :

$$\langle 1m; 00 | 00 \rangle = 0 \qquad (m = 0, \pm 1) \ . \tag{25}$$

The Wigner-Eckart theorem, Equation (24), then shows that the matrix elements of any vector operator $\hat{\boldsymbol{V}}$ taken between eigenstates of zero angular momentum must vanish :

$$\langle \alpha'' 00 | \hat{V}_m | \alpha' 00 \rangle = 0 \ .$$

Equivalently, this can be written as

$$\boxed{\langle\alpha''00|\hat{\boldsymbol{V}}|\alpha'00\rangle = 0}\ . \tag{26}$$

For all other cases, with $j' > 0$ or $j'' > 0$, the Clebsch-Gordan coefficient $\langle 1m; j'm'|j''m''\rangle$ vanishes unless

$$j'' = j',\ j' \pm 1\ , \qquad m'' = m + m' \qquad (m = 0, \pm 1)\ . \tag{27}$$

The conditions in Equations (25) and (27) for a non-zero Clebsch-Gordan coefficient can be summarised as the *selection rules*

$$\boxed{|\Delta j| = 0, 1\ ; \qquad j' + j'' \geq 1\ ; \qquad |\Delta m| = 0, 1}\ .$$

The Wigner-Eckart theorem shows that the matrix element $\langle\alpha''j''m''|\hat{V}_m|\alpha'j'm'\rangle$ will vanish if these selection rules are not satisfied.

Satisfying the above selection rules does not, on its own, guarantee that the matrix element will be non-zero. The Clebsch-Gordan coefficient may anyway still vanish, for example $\langle 10; 10|10\rangle = 0$, or the operator $\hat{\boldsymbol{V}}$ may possess additional symmetry properties which lead to additional selection rules.

1.5.2 The Landé projection formula

The angular momentum operator $\hat{\boldsymbol{J}}$ is itself a vector operator, and so must satisfy the Wigner-Eckart theorem. Therefore, matrix elements of the spherical components $\hat{J}_{+1}$, $\hat{J}_{-1}$, $\hat{J}_0$ of $\hat{\boldsymbol{J}}$ taken between angular momentum eigenstates must be of the form

$$\langle\alpha''j''m''|\hat{J}_m|\alpha'j'm'\rangle = \langle\alpha''j''||\hat{\boldsymbol{J}}||\alpha'j'\rangle\langle 1m; j'm'|j''m''\rangle\ ,$$

where the reduced matrix element $\langle\alpha''j''||\hat{\boldsymbol{J}}||\alpha'j'\rangle$ is an overall constant which is independent of m, m' and m''. In particular, taking $\alpha'' = \alpha'$ and $j'' = j'$, we have

$$\langle\alpha'j'm''|\hat{J}_m|\alpha'j'm'\rangle = \langle\alpha'j'||\hat{\boldsymbol{J}}||\alpha'j'\rangle\langle 1m; j'm'|j'm''\rangle\ . \tag{28}$$

Similarly, for any vector operator $\hat{\boldsymbol{V}}$, the Wigner-Eckart theorem gives

$$\langle\alpha'j'm''|\hat{V}_m|\alpha'j'm'\rangle = \langle\alpha'j'||\hat{\boldsymbol{V}}||\alpha'j'\rangle\langle 1m; j'm'|j'm''\rangle\ . \tag{29}$$

Eliminating the Clebsch-Gordan coefficient between Equations (28) and (29), we obtain

$$\langle\alpha'j'm''|\hat{V}_m|\alpha'j'm'\rangle = \frac{\langle\alpha'j'||\hat{\boldsymbol{V}}||\alpha'j'\rangle}{\langle\alpha'j'||\hat{\boldsymbol{J}}||\alpha'j'\rangle}\langle\alpha'j'm''|\hat{J}_m|\alpha'j'm'\rangle\ .$$

This can equivalently be written as

$$\langle\alpha'j'm''|\hat{\boldsymbol{V}}|\alpha'j'm'\rangle = \frac{\langle\alpha'j'||\hat{\boldsymbol{V}}||\alpha'j'\rangle}{\langle\alpha'j'||\hat{\boldsymbol{J}}||\alpha'j'\rangle}\langle\alpha'j'm''|\hat{\boldsymbol{J}}|\alpha'j'm'\rangle\ .$$

This expression no longer explicitly involves the quantum number $m = \pm 1, 0$. Thus we can simplify the notation by dropping one prime throughout, and write

$$\boxed{\langle \alpha j m' | \hat{\boldsymbol{V}} | \alpha j m \rangle = g_{\alpha j} \langle \alpha j m' | \hat{\boldsymbol{J}} | \alpha j m \rangle} \ , \tag{30}$$

where $g_{\alpha j}$, the ratio of reduced matrix elements, is defined as

$$g_{\alpha j} \equiv \frac{\langle \alpha j || \hat{\boldsymbol{V}} || \alpha j \rangle}{\langle \alpha j || \hat{\boldsymbol{J}} || \alpha j \rangle} \ , \tag{31}$$

and is a constant which is independent of m and m'. Equation (30) can be considered as holding for each of the *spherical* components of $\hat{\boldsymbol{V}}$ and $\hat{\boldsymbol{J}}$, or equivalently, for each of the *Cartesian* components of $\hat{\boldsymbol{V}}$ and $\hat{\boldsymbol{J}}$, with a common value of the constant $g_{\alpha j}$.

The expectation value $\langle \alpha j m | \hat{\boldsymbol{V}} \cdot \hat{\boldsymbol{J}} | \alpha j m \rangle$ of the scalar product operator $\hat{\boldsymbol{V}} \cdot \hat{\boldsymbol{J}}$ is

$$\langle \alpha j m | \hat{\boldsymbol{V}} \cdot \hat{\boldsymbol{J}} | \alpha j m \rangle = \sum_k \langle \alpha j m | \hat{V}_k \hat{J}_k | \alpha j m \rangle \ . \tag{32}$$

The eigenstates $|\alpha j m\rangle$ of the total angular momentum operator $\hat{\boldsymbol{J}}$ satisfy the completeness relation

$$\sum_{\alpha', j', m'} |\alpha' j' m' \rangle \langle \alpha j' m' | = \hat{I} \ .$$

Using this completeness relation to insert a factor of identity into the matrix element of Equation (32) then gives

$$\langle \alpha j m | \hat{\boldsymbol{V}} \cdot \hat{\boldsymbol{J}} | \alpha j m \rangle = \sum_{\alpha', j', m'} \langle \alpha j m | \hat{\boldsymbol{V}} | \alpha' j' m' \rangle \cdot \langle \alpha' j' m' | \hat{\boldsymbol{J}} | \alpha j m \rangle \ .$$

The matrix elements $\langle \alpha' j' m' | \hat{\boldsymbol{J}} | \alpha j m \rangle$ vanish unless $\alpha' = \alpha$ and $j' = j$:

$$\langle \alpha' j' m' | \hat{\boldsymbol{J}} | \alpha j m \rangle = \langle \alpha j m' | \hat{\boldsymbol{J}} | \alpha j m \rangle \delta_{\alpha' \alpha} \delta_{j' j} \ .$$

Hence the summations over α' and j' can be trivially carried out, leaving

$$\langle \alpha j m | \hat{\boldsymbol{V}} \cdot \hat{\boldsymbol{J}} | \alpha j m \rangle = \sum_{m'} \langle \alpha j m | \hat{\boldsymbol{V}} | \alpha j m' \rangle \cdot \langle \alpha j m' | \hat{\boldsymbol{J}} | \alpha j m \rangle \ .$$

Substituting from Equation (30) then gives

$$\langle \alpha j m | \hat{\boldsymbol{V}} \cdot \hat{\boldsymbol{J}} | \alpha j m \rangle = g_{\alpha j} \sum_{m'} \langle \alpha j m | \hat{\boldsymbol{J}} | \alpha j m' \rangle \cdot \langle \alpha j m' | \hat{\boldsymbol{J}} | \alpha j m \rangle \ .$$

The sum over j' can now be reinstated, giving

$$\langle \alpha j m | \hat{\boldsymbol{V}} \cdot \hat{\boldsymbol{J}} | \alpha j m \rangle = g_{\alpha j} \sum_{j', m'} \langle \alpha j m | \hat{\boldsymbol{J}} | \alpha j' m' \rangle \cdot \langle \alpha j' m' | \hat{\boldsymbol{J}} | \alpha j m \rangle \ .$$

Invoking completeness again, the summation over j' and m' just gives a factor of identity. Hence we have

$$\begin{aligned} \langle \alpha j m | \hat{\boldsymbol{V}} \cdot \hat{\boldsymbol{J}} | \alpha j m \rangle &= g_{\alpha j} \langle \alpha j m | \hat{\boldsymbol{J}}^2 | \alpha j m \rangle \\ &= g_{\alpha j} \, j(j+1)\hbar^2 \ . \end{aligned}$$

Provided $j > 0$, the constant $g_{\alpha j}$ is therefore given by

$$g_{\alpha j} = \frac{\langle \alpha j m | \hat{\boldsymbol{V}} \cdot \hat{\boldsymbol{J}} | \alpha j m \rangle}{j(j+1)\hbar^2} \ . \tag{33}$$

Although the quantum number m appears inside the matrix element on the right-hand side of Equation (33), the constant $g_{\alpha j}$ is in fact independent of m, as is evident from its original definition as a ratio of reduced matrix elements in Equation (31).

Substituting for $g_{\alpha j}$ in Equation (30) gives (for $j > 0$) the *Landé projection formula*

$$\boxed{\langle \alpha j m' | \hat{\boldsymbol{V}} | \alpha j m \rangle = \frac{\langle \alpha j m | \hat{\boldsymbol{V}} \cdot \hat{\boldsymbol{J}} | \alpha j m \rangle}{j(j+1)\hbar^2} \langle \alpha j m' | \hat{\boldsymbol{J}} | \alpha j m \rangle} \ . \tag{34}$$

The z-component of the projection formula is

$$\langle \alpha j m' | \hat{V}_z | \alpha j m \rangle = g_{\alpha j} \langle \alpha j m' | \hat{J}_z | \alpha j m \rangle = g_{\alpha j}\, m\hbar\, \delta_{m'm} \ .$$

Taking $m' = m$ then gives the expectation values of $\hat{V}_z$ as

$$\boxed{\langle \alpha j m | \hat{V}_z | \alpha j m \rangle = g_{\alpha j}\, m\hbar} \ . \tag{35}$$

Finally, for the case $j = 0$, Equation (26) gives simply

$$\langle \alpha 00 | \hat{\boldsymbol{V}} | \alpha 00 \rangle = 0 \ .$$

Thus in Equation (30), for $j = 0$, the Landé g-factor vanishes:

$$\boxed{g_{\alpha 0} = 0} \ .$$

1.6 Magnetic dipole moments

The Hamiltonian operator describing the motion of a particle of charge q and mass m in a magnetic field $\boldsymbol{B}$ includes an interaction which can be described in terms of an orbital magnetic dipole moment operator $\hat{\boldsymbol{\mu}}_L$ as

$$\hat{H}_B = -\hat{\boldsymbol{\mu}}_L \cdot \boldsymbol{B} \ ; \qquad \hat{\boldsymbol{\mu}}_L = \frac{q}{2m}\hat{\boldsymbol{L}} \ ,$$

where $\hat{\boldsymbol{L}} = \hat{\boldsymbol{r}} \wedge \hat{\boldsymbol{p}} = -i\hbar \hat{\boldsymbol{r}} \wedge \nabla$ is the orbital angular momentum operator for the particle. Similarly, a spin-half particle is predicted by the Dirac equation to possess an intrinsic magnetic dipole moment $\hat{\boldsymbol{\mu}}_S$ which interacts with a magnetic field as

$$\hat{H}_B = -\hat{\boldsymbol{\mu}}_S \cdot \boldsymbol{B} \ ; \qquad \hat{\boldsymbol{\mu}}_S = \frac{q}{m}\hat{\boldsymbol{S}} \ ,$$

where $\hat{\boldsymbol{S}}$ is the spin operator for the particle. Thus, both orbital and spin magnetic moments have interactions with a magnetic field which are generically of the form

$$\hat{H} = -\hat{\boldsymbol{\mu}} \cdot \boldsymbol{B} \ , \qquad \hat{\boldsymbol{\mu}} = \gamma \hat{\boldsymbol{J}} \ ,$$

where $\hat{\boldsymbol{J}}$ is an angular momentum operator, and γ is a constant known as the *gyromagnetic ratio*.

The (scalar) magnetic dipole moment asociated with a (vector) magnetic dipole operator $\hat{\boldsymbol{\mu}}$ is defined as

$$\mu = \langle jj|\hat{\mu}_z|jj\rangle \ .$$

This is the expectation value of the operator $\hat{\mu}_z$ when the particle is in its "most spin-up" state, $|jj\rangle \equiv |j, m_j = j\rangle$, for which the angular momentum quantum number m_j (defined with respect to the z axis) takes its maximum possible value.

The state $|jj\rangle$ is an eigenstate of $\hat{J}_z$ with eigenvalue $m_j\hbar = j\hbar$:

$$\hat{J}_z|jj\rangle = j\hbar|jj\rangle \ .$$

The magnetic moment μ is therefore related to the gyromagnetic ratio γ as

$$\mu = \gamma\langle jj|\hat{J}_z|jj\rangle = \gamma j\hbar \ . \tag{36}$$

1.6.1 g-factors

It is often convenient to express magnetic moments in terms of a common overall scale, typically either the Bohr magneton $\mu_\mathrm{B} \equiv e\hbar/2m_\mathrm{e}$ or the nuclear magneton $\mu_\mathrm{N} \equiv e\hbar/2m_\mathrm{p}$. In atomic physics, for example, since the magnetic moment of the electron is $\mu_\mathrm{e} \approx -\mu_\mathrm{B}$, the Bohr magneton μ_B is a convenient scale and it is common to set

$$\mu = -gj\mu_\mathrm{B} \ ,$$

where the constant g is referred to simply as the *g-factor*. In nuclear physics, the nuclear magneton μ_N is a more convenient scale, and the g-factor is defined using

$$\mu = gj\mu_\mathrm{N} \ .$$

The signs in the expressions above reflect the most common sign conventions used in each area. Using Equation (36), the connections between the gyromagnetic ratio and the g-factor for each choice of scale are seen to be

$$\gamma = -g\frac{\mu_\mathrm{B}}{\hbar} \ ; \qquad \gamma = g\frac{\mu_\mathrm{N}}{\hbar} \ . \tag{37}$$

The magnetic moment of the electron, for example, with spin $j = 1/2$, can be expressed in units of the Bohr magneton as

$$\mu_\mathrm{e} = -\frac{1}{2}g_\mathrm{e}\mu_\mathrm{B} \ ,$$

where the g-factor for the electron is predicted by the Dirac equation to be $g_\mathrm{e} = 2$, while the measured value is $g_\mathrm{e} \approx 2.0023$. The equivalent expression for the magnetic moment of the proton is

$$\mu_\mathrm{p} = -\frac{1}{2}g_\mathrm{p}\mu_\mathrm{B} \ ,$$

where the measured g-factor is $g_\mathrm{p} \approx -0.0031$. In nuclear and particle physics applications, the proton magnetic moment is more conveniently expressed in terms of μ_N rather than μ_B, as

$$\mu_\mathrm{p} = \frac{1}{2}g_\mathrm{p}\mu_\mathrm{N} \ .$$

In this case, the measured g-factor becomes $g_\mathrm{p} \approx 5.586$. Similarly, the magnetic moment of the neutron is expressed as

$$\mu_\mathrm{n} = \frac{1}{2} g_\mathrm{n} \mu_\mathrm{N} \ ,$$

with $g_\mathrm{n} \approx -3.826$.

The table below gives some examples of the magnetic moments and g-factors of various particles and nuclei, expressed in terms of both μ_B and μ_N .

	spin J, I	"nuclear" μ/μ_N	g_J	"atomic" μ/μ_B	g_I
e^-	1/2			−1.0011	−2.0023
e^+	1/2			+1.0011	+2.0023
p	1/2	+2.793	+5.586	+0.00152	−0.00304
n	1/2	−1.913	−3.826	−0.00104	+0.00208
^{4}He	0	0	0	0	0
^{6}Li	1	+0.822	+0.822	+0.00045	−0.00045
^{7}Li	3/2	+3.256	+2.171	+0.00177	−0.00118
^{12}C	0	0	0	0	0
^{85}Rb	5/2	+1.353	+0.541	+0.00074	−0.00029
^{87}Rb	3/2	+2.750	+1.833	+0.00150	−0.00100
^{107}Ag	1/2	−0.114	−0.228	−0.00006	+0.00012
^{109}Ag	1/2	−0.131	−0.262	−0.00007	+0.00014

1.6.2 Combining magnetic moments

It is often the case that we need to determine the combined effects of two or more magnetic dipole moment interactions, for example the combined effect of the orbital and spin magnetic dipole moments of a particle, or the combined effect of the spin magnetic dipole moments of two spin-half particles.

Thus, in general, we are led to consider the combined effect of two magnetic dipole moment operators, $\hat{\boldsymbol{\mu}}_1 = \gamma_1 \hat{\boldsymbol{J}}_1$ and $\hat{\boldsymbol{\mu}}_2 = \gamma_2 \hat{\boldsymbol{J}}_2$. In a magnetic field $\boldsymbol{B}$, the Hamiltonian contains the sum of the two interaction terms

$$\hat{H}_B = -\hat{\boldsymbol{\mu}}_1 \cdot \boldsymbol{B} - \hat{\boldsymbol{\mu}}_2 \cdot \boldsymbol{B} \ . \tag{38}$$

The overall interaction with the magnetic field is therefore

$$\hat{H}_B = -\hat{\boldsymbol{\mu}} \cdot \boldsymbol{B} \ , \tag{39}$$

where $\hat{\boldsymbol{\mu}}$ is the total magnetic moment operator:

$$\hat{\boldsymbol{\mu}} = \hat{\boldsymbol{\mu}}_1 + \hat{\boldsymbol{\mu}}_2 = \gamma_1 \hat{\boldsymbol{J}}_1 + \gamma_2 \hat{\boldsymbol{J}}_2 \ . \tag{40}$$

If we choose to orient the z-axis along the magnetic field direction,

$$\boldsymbol{B} = (0, 0, B_z) \ ,$$

then the overall interaction becomes

$$\hat{H}_B = -\hat{\mu}_z B_z \ , \qquad \hat{\mu}_z = \gamma_1 \hat{J}_{1z} + \gamma_2 \hat{J}_{2z} \ .$$

In the uncoupled basis of states,

$$|\alpha j_1 m_1 j_2 m_2\rangle = |\alpha j_1 m_1\rangle \otimes |\alpha j_2 m_2\rangle \ ,$$

finding the matrix elements of $\hat{\mu}_z$ is straightforward. We have

$$\begin{aligned} \hat{J}_{1z}|\alpha j_1 m_1 j_2 m_2\rangle &= m_1\hbar|\alpha j_1 m_1 j_2 m_2\rangle \\ \hat{J}_{2z}|\alpha j_1 m_1 j_2 m_2\rangle &= m_2\hbar|\alpha j_1 m_1 j_2 m_2\rangle \ , \end{aligned}$$

and hence

$$\boxed{\langle\alpha j_1 m_1' j_2 m_2'|\hat{\mu}_z|\alpha j_1 m_1 j_2 m_2\rangle = (\gamma_1 m_1 + \gamma_2 m_2)\hbar\, \delta_{m_1' m_1}\delta_{m_2' m_2}} \ .$$

Thus, in the uncoupled basis, for given j_1 and j_2, the matrix representation of $\hat{\mu}_z$ is diagonal with respect to the quantum numbers m_1 and m_2.

This is the case that applies, for example, to the Zeeman effect at high magnetic field.

However, in other cases, for example the Zeeman effect at low magnetic field, or the determination of the magnetic moment of the proton in terms of its constituent quark magnetic moments, the matrix elements of $\hat{\mu}_z$ must be determined in the *coupled* basis $|\alpha jm\rangle$ of total angular momentum eigenstates, rather than in the uncoupled basis $|\alpha j_1 m_1 j_2 m_2\rangle$.

In the coupled basis, if the two gyromagnetic ratios involved are the same, $\gamma_1 = \gamma_2$, finding the matrix elements is trivial. We have

$$\hat{\boldsymbol{\mu}} = \gamma \hat{\boldsymbol{J}} \ ,$$

where $\hat{\boldsymbol{J}}$ is the total angular momentum of the combined system,

$$\hat{\boldsymbol{J}} = \hat{\boldsymbol{J}}_1 + \hat{\boldsymbol{J}}_2 \ ,$$

and where

$$\gamma = \gamma_1 = \gamma_2 \ .$$

The matrix elements of $\hat{\mu}_z$ in the coupled basis are then given simply by

$$\langle\alpha jm'|\hat{\mu}_z|\alpha jm\rangle = \gamma m\hbar\, \delta_{m'm} \ .$$

However, in the more general case that the gyromagnetic ratios are different, $\gamma_1 \neq \gamma_2$, the operator $\hat{\boldsymbol{\mu}}$ is *not* directly proportional to the total angular momentum $\hat{\boldsymbol{J}}$:

$$\hat{\boldsymbol{\mu}} \neq \gamma \hat{\boldsymbol{J}} \ .$$

In this case, finding the matrix elements of $\hat{\mu}_z$ between total angular momentum eigenstates is no longer straightforward. However, as we now show, the Landé projection formula allows proportionality between the operators $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{J}}$ to be *effectively* restored for states of given total angular momentum j, with a constant of proportionality γ_j which depends on j (and j_1 and j_2).

1.6.2.1 The Landé g-factor

Since $\hat{\boldsymbol{\mu}}$ is a vector operator, the projection formula of Equation (34) (obtained from the Wigner-Eckart theorem) is applicable. This gives the matrix elements of the operator $\hat{\boldsymbol{\mu}}$ between eigenstates of total angular momentum $\hat{\boldsymbol{J}}$ as

$$\langle \alpha j m' | \hat{\boldsymbol{\mu}} | \alpha j m \rangle = \gamma_{\alpha j} \langle \alpha j m' | \hat{\boldsymbol{J}} | \alpha j m \rangle \ , \tag{41}$$

where the constant $\gamma_{\alpha j}$ is independent of m and m'. The z component of this equation gives

$$\langle \alpha j m' | \hat{\mu}_z | \alpha j m \rangle = \gamma_{\alpha j} \langle \alpha j m' | \hat{J}_z | \alpha j m \rangle = \gamma_{\alpha j} m \hbar \, \delta_{m' m} \ .$$

Thus, for a given total angular momentum j, the matrix representation is diagonal with respect to m. However, we still need to find $\gamma_{\alpha j}$ in terms of γ_1 and γ_2.

For $j > 0$, the effective gyromagnetic ratio $\gamma_{\alpha j}$ is given by Equation (33) as

$$\gamma_{\alpha j} = \frac{\langle \alpha j m | \hat{\boldsymbol{\mu}} \cdot \hat{\boldsymbol{J}} | \alpha j m \rangle}{j(j+1)\hbar^2} \ .$$

The numerator in the above expression is

$$\langle \alpha j m | \hat{\boldsymbol{\mu}} \cdot \hat{\boldsymbol{J}} | \alpha j m \rangle = \gamma_1 \langle \alpha j m | \hat{\boldsymbol{J}}_1 \cdot \hat{\boldsymbol{J}} | \alpha j m \rangle + \gamma_2 \langle \alpha j m | \hat{\boldsymbol{J}}_2 \cdot \hat{\boldsymbol{J}} | \alpha j m \rangle \ .$$

Squaring the relation $\hat{\boldsymbol{J}}_2 = \hat{\boldsymbol{J}} - \hat{\boldsymbol{J}}_1$ gives the scalar product $\hat{\boldsymbol{J}}_1 \cdot \hat{\boldsymbol{J}}$ as

$$\hat{\boldsymbol{J}}_1 \cdot \hat{\boldsymbol{J}} = \frac{1}{2}\big(\hat{\boldsymbol{J}}^2 + \hat{\boldsymbol{J}}_1^2 - \hat{\boldsymbol{J}}_2^2\big) \ .$$

For given j_1 and j_2, the state $|\alpha j m\rangle$ is an eigenstate of $\hat{\boldsymbol{J}}_1^2$ and $\hat{\boldsymbol{J}}_2^2$ (as well as of $\hat{\boldsymbol{J}}^2$) with

$$\hat{\boldsymbol{J}}_1^2 |\alpha j m\rangle = j_1(j_1+1)\hbar^2 |\alpha j m\rangle \ , \qquad \hat{\boldsymbol{J}}_2^2 |\alpha j m\rangle = j_2(j_2+1)\hbar^2 |\alpha j m\rangle \ .$$

Hence the expectation value of $\hat{\boldsymbol{J}}_1 \cdot \hat{\boldsymbol{J}}$ is

$$\langle \alpha j m | \hat{\boldsymbol{J}}_1 \cdot \hat{\boldsymbol{J}} | \alpha j m \rangle = \frac{\hbar^2}{2}\big[j(j+1) + j_1(j_1+1) - j_2(j_2+1)\big] \ .$$

The equivalent result for $\hat{\boldsymbol{J}}_2 \cdot \hat{\boldsymbol{J}}$ is obtained by interchanging j_1 and j_2:

$$\langle \alpha j m | \hat{\boldsymbol{J}}_2 \cdot \hat{\boldsymbol{J}} | \alpha j m \rangle = \frac{\hbar^2}{2}\big[j(j+1) + j_2(j_2+1) - j_1(j_1+1)\big] \ .$$

Combining the various equations above, we obtain $\gamma_{\alpha j}$ as

$$\boxed{\gamma_j = \gamma_1 \frac{j(j+1) + j_1(j_1+1) - j_2(j_2+1)}{2j(j+1)} + \gamma_2 \frac{j(j+1) + j_2(j_2+1) - j_1(j_1+1)}{2j(j+1)}} \ . \tag{42}$$

As anticipated, the constant $\gamma_{\alpha j}$ is independent of the quantum number m. Equation (42) shows that it is also independent of the quantum numbers α, so we can rename $\gamma_{\alpha j} \to \gamma_j$.

Equation (41) can now be written as

$$\langle \alpha j m' | \hat{\boldsymbol{\mu}} | \alpha j m \rangle = \gamma_j \langle \alpha j m' | \hat{\boldsymbol{J}} | \alpha j m \rangle \ .$$

From Equation (40), we have therefore shown that, for given values of j, j_1 and j_2, matrix elements of magnetic dipole moment operators combine as

$$\langle \alpha j m' | \gamma_1 \hat{\boldsymbol{J}}_1 + \gamma_2 \hat{\boldsymbol{J}}_2 | \alpha j m \rangle = \gamma_j \langle \alpha j m' | \hat{\boldsymbol{J}} | \alpha j m \rangle \, . \tag{43}$$

Thus, when a composite particle or system built from j_1 and j_2 is known to be in an eigenstate with total angular momentum quantum number j, the interaction of the composite system with a magnetic field $\boldsymbol{B}$, Equation (38), is *effectively* described by the Hamiltonian

$$\hat{H}_B = -\gamma_j \hat{\boldsymbol{J}} \cdot \boldsymbol{B} \, .$$

Equivalently, in an angular momentum state j, the total magnetic moment operator of Equation (40) is *effectively* given by

$$\hat{\boldsymbol{\mu}}_j = \gamma_j \hat{\boldsymbol{J}} \ .$$

The magnetic moments can be expressed in terms of g-factors rather than gyromagnetic ratios. For example, using the Bohr magneton as a common overall scale, Equation (37), we can introduce g-factors for each magnetic dipole as

$$\gamma_j = -g_j \frac{\mu_\mathrm{B}}{\hbar} \ , \qquad \gamma_1 = -g_1 \frac{\mu_\mathrm{B}}{\hbar} \ , \qquad \gamma_2 = -g_2 \frac{\mu_\mathrm{B}}{\hbar} \, .$$

Equation (42) retains the same form; we obtain

$$\boxed{g_j = g_1 \frac{j(j+1) + j_1(j_1+1) - j_2(j_2+1)}{2j(j+1)} + g_2 \frac{j(j+1) + j_2(j_2+1) - j_1(j_1+1)}{2j(j+1)}} \, . \tag{44}$$

Equations (42) and (44) apply to the case $j > 0$. For the case $j = 0$, Equation (26) gives the matrix elements of $\hat{\boldsymbol{\mu}}$ as simply

$$\boxed{\langle \alpha 00 | \hat{\boldsymbol{\mu}} | \alpha 00 \rangle = 0} \ .$$

Thus a composite system in a state with zero total angular momentum has no net dipole interaction with a magnetic field $\boldsymbol{B}$; we *effectively* have $\gamma = 0$.

1.6.2.2 Particular cases

For the case that the two g-factors are equal, $g_1 = g_2$, Equation (44) reduces simply to

$$g_j = g_1 = g_2 \ ,$$

as expected.

For the case $g_1 = 1$ and $g_2 = 2$, Equation (44) simplifies as

$$g_j = \frac{3}{2} - \frac{1}{2} \frac{j_1(j_1+1) - j_2(j_2+1)}{j(j+1)} \ .$$

This case is relevant to combining the orbital ($g_\ell = 1$) and spin ($g_e = 2$) magnetic dipole moments of atomic electrons, for example. In this context, the equation above would typically be written as

$$g_J = \frac{3}{2} - \frac{1}{2}\frac{L(L+1) - S(S+1)}{J(J+1)} \ .$$

This applies both to a single electron, in which case $S = 1/2$, or to an N-electron system, in which case S could take any integer or half-integer value up to a maximum of $S = N/2$.

For the case $j_2 = 1/2$, and hence $j = j_1 \pm (1/2)$, Equation (44) simplifies as

$$g_j = g_1 \pm \frac{g_2 - g_1}{2j_1 + 1} \qquad (j = j_1 \pm 1/2) \ . \tag{45}$$

This case is relevant to combining the orbital and spin magnetic moments of a single spin-half particle. For an electron, with $j = \ell \otimes s = \ell \pm (1/2)$, and with $g_1 = g_\ell = 1$, $g_2 = g_\mathrm{e} \approx 2$, Equation (45) further simplifies as

$$g_j = 1 \pm \frac{1}{2\ell + 1} \qquad (j = \ell \pm 1/2) \ .$$

Similarly, for a proton, we have $g_\ell = 1$ and $g_2 = g_\mathrm{p} = 5.586$, and hence

$$g_j = 1 \pm \frac{4.586}{2\ell + 1} \qquad (j = \ell \pm 1/2) \ . \tag{46}$$

For a neutron, we have $g_\ell = 0$ and $g_2 = g_\mathrm{n} = -3.826$, and hence

$$g_j = \mp\frac{3.826}{2\ell + 1} \qquad (j = \ell \pm 1/2) \ . \tag{47}$$

Equations (46) and (47) have application in the shell model of nuclear structure.

1.7 Summary

Conservation laws and degeneracies in quantum systems are generally (always?) the consequence of an underlying symmetry. For example, invariance of a system under spatial rotations implies that the total angular momentum of the system is conserved. Rotational invariance also has important consequences for the structure of matrix elements taken between eigenstates of total angular momentum.

For an operator $\hat{K}$ which commutes with all components $\hat{J}_i$ of an angular momentum operator $\hat{\boldsymbol{J}}$ (i.e. for a *scalar operator*), the Wigner-Eckart theorem states that matrix elements of $\hat{K}$ must be of the form

$$\langle \alpha'' j'' m'' | \hat{K} | \alpha' j' m' \rangle = \langle \alpha'' j'' || \hat{K} || \alpha' j' \rangle \langle 00; j' m' | j'' m'' \rangle \ .$$

The Clebsch-Gordan coefficient above is given by $\langle 00; j'm' | j''m'' \rangle = \delta_{j''j'} \delta_{m''m'}$.

For an operator $\hat{\boldsymbol{V}}$ whose commutation relations with $\hat{\boldsymbol{J}}$ are given by Equation (18) (i.e. for a *vector operator*), the equivalent result is

$$\langle \alpha'' j'' m'' | \hat{V}_m | \alpha' j' m' \rangle = \langle \alpha'' j'' || \hat{\boldsymbol{V}} || \alpha' j' \rangle \langle 1m; j' m' | j'' m'' \rangle \ ,$$

where the operators $\hat{V}_m$ $(m = \pm 1, 0)$ are the spherical components of $\hat{\boldsymbol{V}}$.

The reduced matrix elements $\langle \alpha'' j'' || \hat{K} || \alpha' j' \rangle$ and $\langle \alpha'' j'' || \hat{\boldsymbol{V}} || \alpha' j' \rangle$ in the expressions above are complex constants which are independent of the quantum numbers m, m' and m''. This independence reflects the lack of any preferred spatial direction for a rotationally symmetric system. The m-dependence is carried entirely by Clebsch-Gordan coefficients; these are independent of any particular observable, $\hat{K}$ or $\hat{\boldsymbol{V}}$, and incorporate the common "trivial" geometrical consequences of rotational transformations.

An important consequence of the scalar operator version of the Wigner-Eckart theorem is that the energy eigenstates of isolated systems are degenerate with respect to the quantum number m. For vector operators, the Wigner-Eckart theorem implies the existence of *selection rules* for matrix elements taken between angular momentum eigenstates, such as those governing electric dipole transitions in atoms:

$$|\Delta j| = 0, 1 \ ; \qquad j' + j'' \geq 1 \ ; \qquad |\Delta m_j| = 0, 1 \ .$$

The Wigner-Eckart theorem also leads to the Landé *projection formula*,

$$\langle \alpha j m' | \hat{\boldsymbol{V}} | \alpha j m \rangle = \frac{\langle \alpha j m | \hat{\boldsymbol{V}} \cdot \hat{\boldsymbol{J}} | \alpha j m \rangle}{j(j+1)\hbar^2} \langle \alpha j m' | \hat{\boldsymbol{J}} | \alpha j m \rangle \ .$$

The projection formula can be used to show that the effective gyromagnetic ratio (the Landé g-factor) of a combination of magnetic dipole moments for states of total angular momentum $j = j_1 \otimes j_2$ is given by

$$g_j = g_1 \frac{j(j+1) + j_1(j_1+1) - j_2(j_2+1)}{2j(j+1)} + g_2 \frac{j(j+1) + j_2(j_2+1) - j_1(j_1+1)}{2j(j+1)} \ .$$

This result has application to the shell model of nuclear structure, to the quark model of hadrons, and to the Zeeman effect, for example.

APPENDIX A

A.1 The Wigner-Eckart theorem

Proofs of the Wigner-Eckart theorem for the particular cases of a scalar operator, $\hat{K}$, and a vector operator, $\hat{\boldsymbol{V}}$, are presented below. These proofs are *non-examinable*.

A.1.1 Scalar and vector operators

The Wigner-Eckart theorem dictates the form of matrix elements such as $\langle\alpha''j''m''|\hat{K}|\alpha'j'm'\rangle$ and $\langle\alpha''j''m''|\hat{\boldsymbol{V}}|\alpha'j'm'\rangle$ between eigenstates $|\alpha jm\rangle$ of an angular momentum operator $\hat{\boldsymbol{J}}$. In this context, the operator $\hat{\boldsymbol{J}}$ is any operator whose components $(\hat{J}_1, \hat{J}_2, \hat{J}_3)$ satisfy the commutation relations

$$[\hat{J}_i, \hat{J}_j] = i\hbar \sum_k \epsilon_{ijk} \hat{J}_k \ . \tag{48}$$

An operator $\hat{K}$ is a scalar operator with respect to $\hat{\boldsymbol{J}}$ if it commutes with all components of $\hat{\boldsymbol{J}}$:

$$[\hat{J}_i, \hat{K}] = 0 \ ; \qquad [\hat{\boldsymbol{J}}, \hat{K}] = 0 \ ; \qquad [\hat{\boldsymbol{J}}^2, \hat{K}] = 0 \ . \tag{49}$$

An operator $\hat{\boldsymbol{V}} = (\hat{V}_1, \hat{V}_2, \hat{V}_3)$ is a vector operator with respect to $\hat{\boldsymbol{J}}$ is its components satisfy the commutation relations

$$[\hat{J}_i, \hat{V}_j] = i\hbar \sum_k \epsilon_{ijk} \hat{V}_k \ . \tag{50}$$

Equations (48)-(50) are the only requirements which the operators $\hat{\boldsymbol{J}}$, $\hat{K}$ and $\hat{\boldsymbol{V}}$ must satisfy in order for the Wigner-Eckart theorem to be applicable.

A.1.2 Angular momentum relations

The operators $\hat{\boldsymbol{J}}^2$ and $\hat{J}_z = \hat{J}_3$ possess a set of simultaneous eigenstates, $|\alpha jm\rangle$, such that

$$\hat{\boldsymbol{J}}^2|\alpha jm\rangle = j(j+1)\hbar^2|\alpha jm\rangle \ , \qquad \hat{J}_z|\alpha jm\rangle = m\hbar|\alpha jm\rangle \ ,$$

where j and m are the usual angular momentum quantum numbers, and α collectively labels all other quantum numbers needed to uniquely identify the states. The ladder operators $\hat{J}_\pm \equiv \hat{J}_1 \pm i\hat{J}_2$ acting on the state $|\alpha jm\rangle$ raise or lower the index m by ± 1:

$$\hat{J}_\pm|\alpha jm\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}|\alpha j, m\pm 1\rangle \ . \tag{51}$$

Noting that $(\hat{J}_\pm)^\dagger = \hat{J}_\mp$, the conjugate of this relation is

$$\langle\alpha jm|\hat{J}_\mp = \hbar\sqrt{j(j+1) - m(m\pm 1)}\langle\alpha j, m\pm 1| \ .$$

Writing this equation in terms of $\hat{J}_\pm$ rather than $\hat{J}_\mp$ (i.e. interchanging all "$\pm$" and "$\mp$" signs), we obtain

$$\langle\alpha jm|\hat{J}_\pm = \hbar\sqrt{j(j+1) - m(m\mp 1)}\langle\alpha j, m\mp 1| \ . \tag{52}$$

A.1.3 The Wigner-Eckart theorem for scalar operators

Since the scalar operator $\hat{K}$ commutes with $\hat{J}_z$, we have, for any pair of total angular momentum eigenstates $|\alpha'' j'' m''\rangle$ and $|\alpha' j' m'\rangle$,

$$\begin{aligned} 0 &= \langle \alpha'' j'' m'' | [\hat{J}_z, \hat{K}] | \alpha' j' m' \rangle \\ &= \hbar (m'' - m') \langle \alpha'' j'' m'' | \hat{K} | \alpha' j' m' \rangle \ . \end{aligned}$$

Therefore, if $m'' \neq m'$, the matrix element $\langle \alpha'' j'' m'' | \hat{K} | \alpha' j' m' \rangle$ must vanish. Similarly, since $\hat{K}$ commutes with $\hat{\boldsymbol{J}}^2$, we have

$$0 = \langle \alpha'' j'' m'' | [\hat{\boldsymbol{J}}^2, \hat{K}] | \alpha' j' m' \rangle = \hbar^2 \left[j''(j''+1) - j'(j'+1) \right] \langle \alpha'' j'' m'' | \hat{K} | \alpha' j' m' \rangle \ .$$

Hence, if $j'' \neq j'$, the matrix element $\langle \alpha'' j'' m'' | \hat{K} | \alpha' j' m' \rangle$ must vanish. Overall therefore, we must have

$$\langle \alpha'' j'' m'' | \hat{K} | \alpha' j' m' \rangle = \delta_{j'' j'} \delta_{m'' m'} C(\alpha'' \alpha'; j' m') \ , \tag{53}$$

where $C(\alpha'' \alpha'; j' m')$ is a constant. For $j'' = j'$ and $m'' = m'$, the equation above becomes

$$\langle \alpha'' j' m' | \hat{K} | \alpha' j' m' \rangle = C(\alpha'' \alpha'; j' m') \ . \tag{54}$$

Since $\hat{K}$ commutes with the ladder operators $\hat{J}_\pm$, we have

$$\langle \alpha'' j'' m'' | \hat{J}_\pm \hat{K} | \alpha' j' m' \rangle = \langle \alpha'' j'' m'' | \hat{K} \hat{J}_\pm | \alpha' j' m' \rangle \ . \tag{55}$$

Using Equations (51) and (52) then gives

$$\begin{aligned} &\sqrt{j''(j''+1) - m''(m'' \mp 1)} \langle \alpha'' j'', m'' \mp 1 | \hat{K} | \alpha' j' m' \rangle \\ &\qquad = \sqrt{j'(j'+1) - m'(m' \pm 1)} \langle \alpha'' j'' m'' | \hat{K} | \alpha' j', m' \pm 1 \rangle \ . \end{aligned}$$

For $j'' = j'$ and $m'' = m' \pm 1$, the square-root factors on either side of the equation above are equal to each other; cancelling these factors then leaves

$$\langle \alpha'' j' m' | \hat{K} | \alpha' j' m' \rangle = \langle \alpha'' j', m' \pm 1 | \hat{K} | \alpha' j', m' \pm 1 \rangle \ .$$

Comparing with Equation (54) then gives

$$C(\alpha'' \alpha'; j' m') = C(\alpha'' \alpha'; j', m' \pm 1) \ .$$

Thus the constants $C(\alpha'' \alpha'; j' m')$ are, in fact, independent of the quantum number m', and can be written as $C(\alpha'' \alpha'; j')$. Equation (53) then gives the *Wigner-Eckart theorem* for scalar operators,

$$\boxed{\langle \alpha'' j'' m'' | \hat{K} | \alpha' j' m' \rangle = C(\alpha'' \alpha'; j') \, \delta_{j'' j'} \, \delta_{m'' m'}} \ ,$$

as in Equation (21).

A.1.4 The Wigner-Eckart theorem for vector operators

The commutation relations of the vector operator $\hat{\boldsymbol{V}}$ with the ladder operators $\hat{J}_\pm$ were obtained in Equation (20):

$$[\hat{J}_\pm, \hat{V}_m] = \hbar\sqrt{j(j+1) - m(m\pm 1)}\,\hat{V}_{m\pm 1} \qquad (j \equiv 1,\ m = 0, \pm 1)\ . \tag{56}$$

Taking matrix elements of Equation (20) between eigenstates $|\alpha' j' m'\rangle$ and $|\alpha'' j'' m''\rangle$ of the total angular momentum operator $\hat{\boldsymbol{J}}$ gives

$$\langle\alpha'' j'' m''|(\hat{J}_\pm \hat{V}_m - \hat{V}_m \hat{J}_\pm)|\alpha' j' m'\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}\langle\alpha'' j'' m''|\hat{V}_{m\pm 1}|\alpha' j' m'\rangle\,, \tag{57}$$

where

$$j \equiv 1\ ; \qquad m = 0, \pm 1\ .$$

From Equations (51) and (52), the ladder operators act on the states $\langle\alpha'' j'' m''|$ and $|\alpha' j' m'\rangle$ as

$$\begin{aligned}\langle\alpha'' j'' m''|\hat{J}_\pm &= \hbar\sqrt{j''(j''+1) - m''(m''\mp 1)}\langle\alpha'' j'', m''\mp 1|\\ \hat{J}_\pm|\alpha' j' m'\rangle &= \hbar\sqrt{j'(j'+1) - m'(m'\pm 1)}|\alpha' j', m'\pm 1\rangle\ .\end{aligned}$$

Substituting the above into the left-hand side of Equation (57) and rearranging, we obtain the following relation between spherical component matrix elements:

$$\begin{aligned}&\sqrt{j''(j''+1) - m''(m''\mp 1)}\,\langle\alpha'' j'', m''\mp 1|\hat{V}_m|\alpha' j' m'\rangle\\ &\qquad = \sqrt{j(j+1) - m(m\pm 1)}\,\langle\alpha'' j'' m''|\hat{V}_{m\pm 1}|\alpha' j' m'\rangle\\ &\qquad + \sqrt{j'(j'+1) - m'(m'\pm 1)}\,\langle\alpha'' j'' m''|\hat{V}_m|\alpha' j', m'\pm 1\rangle\ .\end{aligned} \tag{58}$$

This relation connects matrix elements with common values of j' and j'' throughout, but with varying m, m' and m''.

Now consider the addition of two angular momentum operators $\hat{\boldsymbol{J}}$ and $\hat{\boldsymbol{J}}'$ to form a combined angular momentum $\hat{\boldsymbol{J}}''$:

$$\hat{\boldsymbol{J}}'' = \hat{\boldsymbol{J}} + \hat{\boldsymbol{J}}'\ .$$

The operator sum $\hat{\boldsymbol{J}} + \hat{\boldsymbol{J}}'$ has eigenstates which are the direct product of the individual eigenstates of $\hat{\boldsymbol{J}}$ and $\hat{\boldsymbol{J}}'$:

$$|jm; j'm'\rangle \equiv |jm\rangle \otimes |j'm'\rangle\ .$$

The action of the ladder operator $\hat{J}''_\pm = \hat{J}_\pm + \hat{J}'_\pm$ on these product states is

$$\begin{aligned}\hat{J}''_\pm|jm; j'm'\rangle &= \hbar\sqrt{j(j+1) - m(m\pm 1)}|j, m\pm 1; j'm'\rangle\\ &+ \hbar\sqrt{j'(j'+1) - m'(m'\pm 1)}|jm; j', m'\pm 1\rangle\ .\end{aligned}$$

Premultiplying by an eigenstate $\langle j'' m''|$ of $\hat{\boldsymbol{J}}''$ then gives

$$\begin{aligned}\langle j'' m''|\hat{J}''_\pm|jm; j'm'\rangle &= \hbar\sqrt{j(j+1) - m(m\pm 1)}\langle j'' m''|j, m\pm 1; j'm'\rangle\\ &+ \hbar\sqrt{j'(j'+1) - m'(m'\pm 1)}\langle j'' m''|jm; j', m'\pm 1\rangle\,.\end{aligned}$$

Taking the complex conjugate of the above equation, and using $(\hat{J}''_{\pm})^{\dagger} = \hat{J}''_{\mp}$, we obtain

$$\langle jm; j'm'|\hat{J}''_{\mp}|j''m''\rangle = \hbar\sqrt{j(j+1) - m(m \pm 1)}\langle j, m \pm 1; j'm'|j''m''\rangle$$
$$+ \hbar\sqrt{j'(j'+1) - m'(m' \pm 1)}\langle jm; j', m' \pm 1|j''m''\rangle \, .$$

The matrix element on the left-hand side can be evaluated using

$$\hat{J}''_{\mp}|j''m''\rangle = \sqrt{j''(j''+1) - m''(m'' \mp 1)}|j'', m'' \mp 1\rangle \ .$$

Hence we obtain the following relation between Clebsch-Gordan coefficients of fixed j, j' and j'', but varying m, m' and m'' :

$$\sqrt{j''(j''+1) - m''(m'' \mp 1)}\langle jm; j'm'|j'', m'' \mp 1\rangle$$
$$= \sqrt{j(j+1) - m(m \pm 1)}\langle j, m \pm 1; j'm'|j''m''\rangle$$
$$+ \sqrt{j'(j'+1) - m'(m' \pm 1)}\langle jm; j', m' \pm 1|j''m''\rangle \ . \qquad (59)$$

This relation is *identical in form* to Equation (58), under the replacements

$$\langle \alpha'' j'' m''|\hat{V}_{m\pm 1}|\alpha' j' m'\rangle \longleftrightarrow \langle j, m \pm 1; j'm'|j''m''\rangle \ .$$

Thus Equations (58) and (59) represent two identical sets of linear, homogenous equations of the form

$$\sum_j A_{ij} x_j = 0 \ , \qquad \sum_j A_{ij} y_j = 0 \ . \qquad (60)$$

In the first set of equations, the quantities x_j are matrix elements $\langle \alpha'' j'' m''|\hat{V}_{m\pm 1}|\alpha' j' m'\rangle$ of $\hat{\boldsymbol{V}}$. In the second set, the y_j are Clebsch-Gordan coefficients $\langle j, m \pm 1; j'm'|j''m''\rangle$. The matrix A_{ij} contains all the factors of the form $\sqrt{j(j+1) - m(m \pm 1)}$, and is common to both sets of equations.

The solution to linear equations of the form given in Equation (60) is unique up to an overall constant. Hence the solutions x_j and y_j must be directly proportional to each other :

$$\langle \alpha'' j'' m''|\hat{V}_{m\pm 1}|\alpha' j' m'\rangle = C(\alpha''\alpha'; j''j')\langle j, m \pm 1; j'm'|j''m''\rangle \ . \qquad (61)$$

The (complex) constant of proportionality, $C(...)$, must be independent of m, m' and m'' because these parameters vary from element to element within the matrix A_{ij}. On the other hand, the parameters α'', α', j'' and j' are common to all the elements A_{ij}.

Replacing $m \pm 1$ by m throughout Equation (61), we obtain the Wigner-Eckart theorem for a vector operator $\hat{\boldsymbol{V}}$:

$$\langle \alpha'' j'' m''|\hat{V}_m|\alpha' j' m'\rangle = C(\alpha''\alpha'; j''j')\langle jm; j'm'|j''m''\rangle \ .$$

The constant $C(\alpha''\alpha'; j''j')$, the *reduced matrix element*, is usually written in Dirac-style notation as $\langle \alpha'' j''||\hat{\boldsymbol{V}}||\alpha' j'\rangle$, giving the Wigner-Eckart Theorem in standard form as

$$\boxed{\langle \alpha'' j'' m''|\hat{V}_m|\alpha' j' m'\rangle = \langle \alpha'' j''||\hat{\boldsymbol{V}}||\alpha' j'\rangle\langle jm; j'm'|j''m''\rangle} \ ,$$

as in Equation (24). In the equation above, the quantum number j is, by definition, $j \equiv 1$, and the quantum number m takes the values $m = \pm 1, 0$.

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