X. CLASSIC TESTS OF GENERAL RELATIVITY

General Relativity was conceived by pure thought.

Over the 100 years since its formulation, General Relativity has been tested experimentally and observationally over a variety of length scales (from Solar System and smaller to the entire observable universe).

No deviation from the predictions of the theory has ever been found!

Many experimental tests are based on the Schwarzschild geometry for $r > 2\mu$, and involve the trajectories of massive particles or light.

Here, we shall discuss two such classic tests: the perihelion advance of the planet Mercury; and the bending of light by the Sun.

These are classic predictions, dating back to Einstein's seminal papers laying down the theory of General Relativity, of effects that are either absent or differ in value in Newtonian gravity.

Many more recent tests of General Relativity are targeting the strong-field region, $\Phi \sim c^2$.

A very significant recent breakthrough for the field was the detection of gravitational waves by LIGO (see Handout 0), which is now allowing tests of General Relativity in truly extreme environments.

For an extensive review of experimental tests of General Relativity, see the review article *The Confrontation between General Relativity and Experiment* by Clifford Will, available at `https://arxiv.org/abs/1403.7377`.

1 Shapes of orbits for massive and massless particles

Recall from Handout IX that for free-falling massive and massless particles in Schwarzschild spacetime

$$\left(1-\frac{2\mu}{r}\right)c^2\dot{t}^2-\left(1-\frac{2\mu}{r}\right)^{-1}\dot{r}^2-r^2\dot{\phi}^2=\begin{cases}c^2 & \text{massive}\,,\\ 0 & \text{massless}\,.\end{cases} \tag{1}$$

These combine with the conserved quantities,

$$\left(1-\frac{2\mu}{r}\right)\dot{t}=k\,, \tag{2}$$

$$r^2\dot{\phi}=h\,, \tag{3}$$

to determine the orbits.

Using the conserved quantities k and h in Eq. (1), we have

$$\frac{1}{2}\dot{r}^2-\frac{GM}{r}+\frac{h^2}{2r^2}\left(1-\frac{2\mu}{r}\right)=\frac{1}{2}c^2\left(k^2-1\right) \tag{4}$$

for massive particles and

$$\frac{1}{2}\dot{r}^2+\frac{h^2}{2r^2}\left(1-\frac{2\mu}{r}\right)=\frac{1}{2}c^2k^2 \tag{5}$$

for massless particles.

Here, we shall be interested in the *shape* of the orbits, i.e., r as a function of ϕ, so we use

$$\dot{r}=\dot{\phi}\frac{dr}{d\phi}=\frac{h}{r^2}\frac{dr}{d\phi}=-h\frac{du}{d\phi}\,, \tag{6}$$

where $u\equiv 1/r$.

Substituting in Eq. (4), we have

$$\frac{1}{2}\left(\frac{du}{d\phi}\right)^2-\frac{GM}{h^2}u+\frac{1}{2}\left(u^2-2\mu u^3\right)=\frac{c^2}{2h^2}\left(k^2-1\right)\,. \tag{7}$$

Differentiating with respect to ϕ gives

$$\frac{d^2u}{d\phi^2} + u - 3\mu u^2 = \frac{GM}{h^2} \qquad \text{(massive)}\,, \tag{8}$$

which determines the shape of the orbit.

Repeating for the massless case, we have

$$\frac{d^2u}{d\phi^2} + u - 3\mu u^2 = 0 \qquad \text{(massless)}\,. \tag{9}$$

1.1 Newtonian orbits of massive particles

Let us recall some of the properties of the orbits of massive particles in Newtonian theory.

If we repeat the analysis above for Newtonian dynamics, the orbit equation (for massive particles) is

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2} \qquad \text{(Newtonian)}\,. \tag{10}$$

This differs from the relativistic equation (8), which has the additional term $-3\mu u^2$ on the left-hand side.

We shall see that this has important consequences for the shape of the orbits in General Relativity.

The solution of Eq. (10) is

$$u = \frac{GM}{h^2}\left(1 + e\cos\phi\right)\,, \tag{11}$$

where the constant e is the eccentricity of the orbit and we have chosen the orbit to have $du/d\phi = 0$ at $\phi = 0$ without loss of generality.

The Newtonian energy equation

$$\begin{aligned} \frac{E_{\mathrm{N}}}{m} &= \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\phi}^2 - \frac{GM}{r} \\ \Rightarrow \quad \frac{E_{\mathrm{N}}}{mh^2} &= \frac{1}{2}\left(\frac{du}{d\phi}\right)^2 + \frac{1}{2}u^2 - \left(\frac{GM}{h^2}\right)u\,, \end{aligned} \tag{12}$$

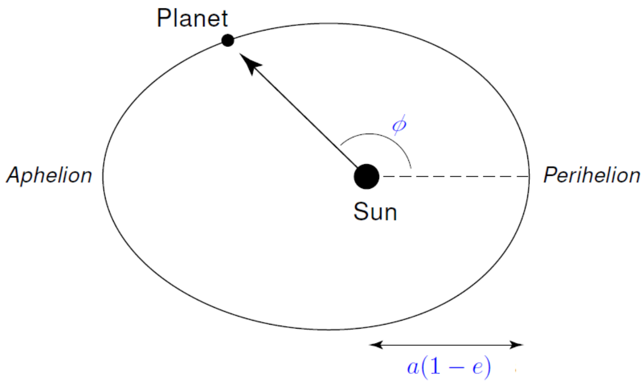


Figure 1: Elliptical orbit of a planet around the Sun in Newtonian gravity. The Sun is at the focus of the ellipse. The semi-major axis length is a and the ellipticity is e.

relates the eccentricity to the energy:

$$\frac{E_{\mathrm{N}}}{mh^2} = \frac{1}{2}\left(\frac{GM}{h^2}\right)^2 \left(e^2 - 1\right) . \tag{13}$$

Bound orbits have $E_{\mathrm{N}} < 0$ and so eccentricity $e < 1$.

For $e = 0$, we have circular orbits with radius r_0, where

$$r_0 \equiv \frac{h^2}{GM} . \tag{14}$$

For $0 < e < 1$, we have elliptical orbits with the mass M at a focus (see Fig. 1).

The radial distance at the point of closest approach (*perihelion*) is $r_0/(1+e)$ and the greatest distance (*aphelion*) is $r_0/(1-e)$.

If the ellipse has semi-major axis length a (so the distance between perihelion and aphelion is $2a$), we have $a = r_0/(1-e^2)$ or

$$a = \frac{h^2}{GM\,(1-e^2)} . \tag{15}$$

2 Precession of planetary orbits

We shall solve the relativistic equation of motion, which we now write as

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2} + \frac{3GM}{c^2}u^2\,, \tag{16}$$

in the limit where the relativistic correction is weak ($GM \ll rc^2$).

In this case, we can solve the equation by perturbing around a Newtonian orbit.

Let us introduce a dimensionless inverse radius, U, so that

$$u = \frac{GM}{h^2}U\,. \tag{17}$$

Writing Eq. (16) in terms of U, we have

$$\frac{d^2U}{d\phi^2} + U = 1 + \underbrace{\frac{3(GM)^2}{c^2h^2}}_{\alpha}U^2\,. \tag{18}$$

The dimensionless quantity $\alpha = 3\mu/r_0$ is assumed small (for Mercury, $\alpha = 8 \times 10^{-8}$, for example).

We therefore look for solutions as an expansion in α, i.e.,

$$U = U_0 + \alpha U_1 + \alpha^2 U_2 + \cdots\,, \tag{19}$$

where $U_0 = 1 + e\cos\phi$ is the Newtonian solution.

Substituting in Eq. (18), we have

$$\alpha\frac{d^2U_1}{d\phi^2} + \alpha U_1 - \alpha\left(1 + e\cos\phi\right)^2 + O(\alpha^2) = 0\,, \tag{20}$$

so that at first-order in α,

$$\begin{aligned}\frac{d^2U_1}{d\phi^2} + U_1 &= \left(1 + e\cos\phi\right)^2 \\ &= \left(1 + \frac{1}{2}e^2\right) + 2e\cos\phi + \frac{1}{2}e^2\cos 2\phi\,. \end{aligned}\tag{21}$$

The particular integral of this equation is

$$U_1(\phi) = \left(1 + \frac{1}{2}e^2\right) + e\phi\sin\phi - \frac{1}{6}e^2\cos 2\phi\,. \qquad (22)$$

The corrections to the Newtonian orbit are very small from the first and third terms on the right since they get multiplied by the small quantity α; however, the amplitude of the second term can grow over many orbits so we retain this.

The general relativistic orbit is then

$$\begin{aligned} u(\phi) &\approx \frac{GM}{h^2}\left(1 + e\cos\phi + e\alpha\phi\sin\phi\right) \\ &\approx \frac{GM}{h^2}\left\{1 + e\left[\cos\phi\cos(\alpha\phi) + \sin\phi\sin(\alpha\phi)\right]\right\}\,, \end{aligned} \qquad (23)$$

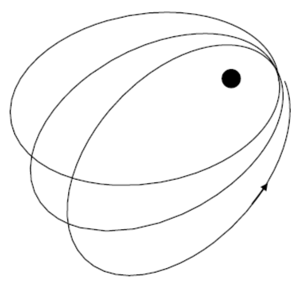
where the second line is correct to first-order in $\alpha\phi$.

It follows that

$$u(\phi) \approx \frac{GM}{h^2}\left\{1 + e\cos\left[\phi(1-\alpha)\right]\right\}\,. \qquad (24)$$

We see from this expression that the orbit is not closed since r is periodic in ϕ with period $2\pi/(1-\alpha)$.

This means that the ellipse *precesses*, with the angle ϕ at perihelion increasing by



$$\Delta\phi = 2\pi\left(\frac{1}{1-\alpha} - 1\right) \approx 2\pi\alpha \qquad (25)$$

per revolution (see figure to the right).

Substituting for α, and expressing h^2 in terms of the semi-major axis using Eq. (15), we have

$$\Delta\phi = \frac{6\pi GM}{a(1-e^2)c^2}\,. \qquad (26)$$

This is largest when the orbit is small and highly eccentric.

In the Solar System, the largest effect is for Mercury, which has $a = 5.8 \times 10^{10}\,\mathrm{m}$ and $e = 0.2$.

Combining with the Solar mass, $M_\odot = 2 \times 10^{30}\,\mathrm{kg}$ (so $\mu = 1.5\,\mathrm{km}$), we predict $\Delta\phi = 0.10\,\mathrm{arcsec}$ or

$$\Delta\phi = 43\,\mathrm{arcsec\ per\ century} \tag{27}$$

using the period of 88 days.

The measured precession is

$$\Delta\phi = (574.1 \pm 0.1)\ \mathrm{arcsec\ per\ century}\,, \tag{28}$$

but most of this is due to the perturbing effect of the other planets.

Once these are corrected for, the residual is

$$\Delta\phi = (43.1 \pm 0.5)\ \mathrm{arcsec\ per\ century}\,, \tag{29}$$

in beautiful agreement with the prediction of General Relativity.

3 The bending of light

To describe the gravitational deflection of light, we use Eq. (9), which we write in the form

$$\frac{d^2u}{d\phi^2} + u = \frac{3GM}{c^2}u^2\,. \tag{30}$$

For $M = 0$, this is solved by the straight line

$$u = \frac{\sin\phi}{b}\,, \tag{31}$$

where b is the impact parameter (see Fig. 2).

For $\beta \ll 1$, where

$$\beta \equiv \frac{3GM}{c^2 b}\,, \tag{32}$$

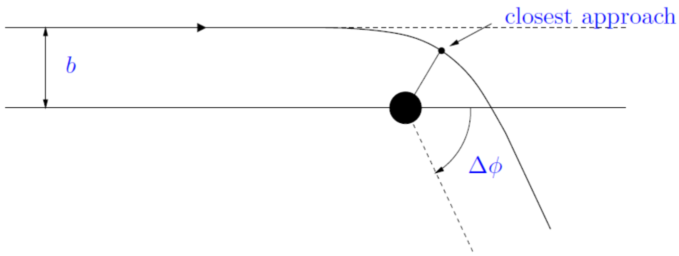


Figure 2: Bending of light with impact parameter b by a spherical mass. The total deflection angle is $\Delta\phi$.

we can proceed perturbatively writing

$$bu(\phi) = \sin\phi + \beta U_1(\phi) + \beta^2 U_2(\phi) + \cdots\,, \tag{33}$$

where the U_n are dimensionless.

To first order in β, we have

$$\begin{aligned}\frac{d^2U_1}{d\phi^2} + U_1 &= \sin^2\phi \\ &= \frac{1}{2}(1-\cos 2\phi)\,. \end{aligned} \tag{34}$$

This is solved by

$$U_1(\phi) = C_1\sin\phi + C_2\cos\phi + \frac{1}{2}\left(1+\frac{1}{3}\cos 2\phi\right)\,, \tag{35}$$

where C_1 and C_2 are integration constants.

For impact parameter b, we require that $bu \to \sin\phi$ as $\phi \to \pi$, so that $C_1 = 0$ and $C_2 = 2/3$.

It follows that

$$u(\phi) = \frac{\sin\phi}{b} + \frac{3GM}{c^2b^2}\left[\frac{2}{3}\cos\phi + \frac{1}{2}\left(1+\frac{1}{3}\cos 2\phi\right)\right]\,. \tag{36}$$

At the far end of the light path, $u \to 0$ as $\phi \to -\Delta\phi$ (with $|\Delta\phi| \ll 1$) where

$$-\frac{\Delta\phi}{b} + \frac{3GM}{c^2b^2}\times\frac{4}{3} = 0 \tag{37}$$

to first order in β.

It follows that the total deflection of the light ray is[1]

$$\Delta\phi = \frac{4GM}{c^2 b}\,. \tag{38}$$

For light grazing the Sun, $b = R_\odot = 6.96 \times 10^5\,\mathrm{km}$, and the total deflection predicted is

$$\Delta\phi = 1.75\,\mathrm{arcsec}\,. \tag{39}$$

This *prediction* of General Relativity was first verified in the famous 1919 eclipse expeditions led by Arthur Eddington.

High-precision tests of light bending have subsequently been made using extragalactic radio sources rather than stars in our Galaxy (as used by Eddington).

The radio sources (quasars) can be measured close to the Sun even when there is no lunar eclipse and are not affected by the atmosphere; the angular shift in the position of quasars when they are eclipsed by the Sun has been used to test General Relativity with a relative accuracy approaching 10^{-4}.

More extreme manifestations of light bending have now been observed in astrophysical systems.

For example, if a distant source is sufficiently aligned with a foreground galaxy, it is possible for multiple light paths to connect the source to the observer.

This phenomenon, known as strong gravitational lensing, leads to strong distortion of the image of the source and even multiple images.

[1] This is twice the value obtained with a Newtonian calculation for a particle travelling at c. This reflects the fact that for a massless particle both the g_{tt} and g_{rr} metric perturbations contribute to the particle dynamics, while for a slowly-moving massive particle only g_{tt} is relevant.

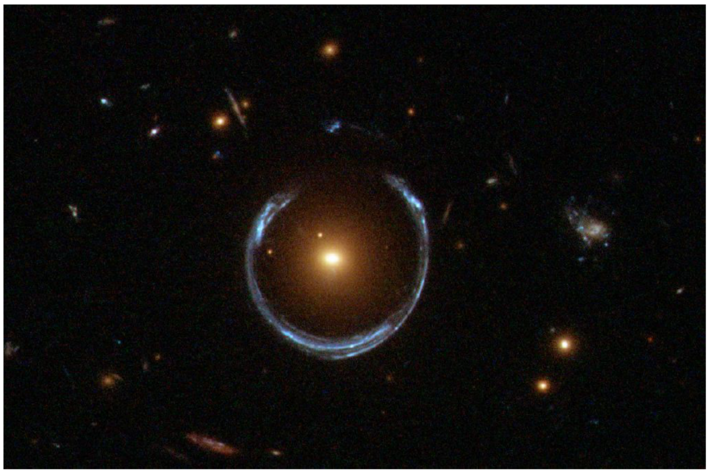


Figure 3: The *Cosmic Horseshoe* is a beautiful example of strong gravitational lensing. A distant galaxy (blue) lies directly behind a foreground luminous red galaxy on our line of sight. The light from the former is bent by the massive foreground galaxy. Due to the close alignment, multiple light paths from the background galaxy can reach us on Earth, giving rise to the extreme ring-like distortion (called an *Einstein ring*) in the image of the background galaxy.

An example of strong lensing is the *Cosmic Horseshoe*; see `https://en.wikipedia.org/wiki/Cosmic_Horseshoe` and Fig. 3.