

Astrophysical Fluid Dynamics

Example sheet 2

1. A planet is composed of a material that is incompressible, density ρ , at pressures $\leq p_0$. Show that the maximum mass of such a planet that is incompressible throughout its interior is given by

$$M_{\max} = \frac{2}{3\rho^2} \sqrt{\frac{1}{2\pi} \left(\frac{3p_0}{G}\right)^3}$$

2. An equilibrium ring of isothermal fluid orbits a star with mass M_* at radius R . In the plane of the ring, mechanical equilibrium results from a balance of centrifugal force and the gravitational force of the central object; normal to the ring (ie. vertically) equilibrium is between the vertical component of the gravitational force of the central object and vertical pressure gradients in the ring gas. Show that in the limit that the ring thickness $H \ll R$, the vertical density stratification in the ring is a Gaussian and determine its e-folding length in terms of the gas temperature and the angular velocity at the ring, Ω . Hence determine an upper limit to the temperature such that the ring is thin ($H \ll R$) and calculate this temperature if the ring's radius is that of the Earth's orbit around the Sun. [Hint: In the limit of $H \ll R$ you can find an approximate expression for gravitational acceleration.]

3. Show that if

$$\psi = -\frac{GM_s}{(r^2 + b^2)^{\frac{1}{2}}}$$

is the gravitational potential for a spherical distribution of matter then its density $\rho \propto \psi^5$.

Deduce the pressure and hence show that the equation of state is polytropic with $n = 5$.

Find the total internal energy U as a function of K, M_s and b where $K = \frac{P}{\rho^{6/5}}$ thus showing that it scales as $K M_s^{6/5} b^{-3/5}$.

4. Sketch the density distribution for an isothermal slab and discuss the asymptotic limits $z \rightarrow 0, z \rightarrow \infty$.

A galactic disc can be well approximated in its vertical structure by an isothermal slab of gas, temperature T , central density ρ_0 . If a star falls from rest from a height z_0 , show that its vertical velocity at height z is given by

$$z^2 = \frac{4R_*T}{\mu} \ln \left(\frac{\cosh(az_0)}{\cosh(az)} \right)$$

5. Explain (for the case of a polytrope, index n) why the internal energy per kg, ϵ , is equal to $\int_0^\rho \frac{P}{\rho^2} d\rho$, if and only if $\gamma = 1 + \frac{1}{n}$.

Calculate how the total internal energy of a polytropic star varies with stellar mass, assuming all stars share the same polytropic constant K . [Hint: Determine how the total mass $M = \int_0^{R_0} 4\pi r^2 \rho dr$ and the total internal energy $U = \int_0^{R_0} 4\pi r^2 \rho \epsilon dr$ scale with the central density ρ_c .]

6. Derive the mass-radius relation for polytropic stars [equation of state $P = K\rho^{1+\frac{1}{n}}$] on the assumption that K varies with stellar mass in such a way as to maintain a constant central temperature independent of mass.
7. Derive that for a linear sound wave (i.e. one in which $\Delta\rho/\rho$ is small) the velocity of fluid motion is $\ll c_s$. Estimate the maximum longitudinal fluid velocity in the case of a sound wave in air at s.t.p. in the case of a disturbance which sets up pressure fluctuations of order 0.1%.

8. Consider an oblique adiabatic shock wave where the gas approaching the shock front from the forward direction is inclined to the normal of the front at an angle θ . If the gas after the shock leaves at an angle χ with respect to the direction of motion of the gas before it reaches the shock, show that

$$\cot\chi = \cot\theta \left[\frac{(\gamma + 1)M^2}{2(M^2 \cos^2\theta - 1)} - 1 \right],$$

where M is the Mach number (i.e. the ratio of the *total* incident velocity in the frame of the shock to the sound speed) for the shock.

9. Two identical clouds, radius 3×10^{16} m, temperature 10K, collide with each other with relative velocity 4km/s. What is the time interval t_{coll} , over which each cloud falls into the shock? If the cooling rate in the shocked gas is $Q^- = 10^{-4}(Js^{-1}kg^{-1})$ decide whether the shock is approximately adiabatic or isothermal.

If the clouds colliding produces an isothermal shock, what is the thickness of the shocked layer, x , at the moment that the entirety of each cloud has been shocked? At later times the layer relaxes into a structure that can be approximated by a hydrostatic isothermal slab, column density 0.1 kg m^{-2} . What fraction of the cloud masses remains within thickness x in this hydrostatic structure? [Hint: Ignore edge effects and variations of column density in the plane of the slab.]