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ADVANCED QUANTUM PHYSICS

Examples Sheet 2

17. **Atomic structure (Part IB Advanced Physics 1993)**

Explain the physical origin of the following terms in the Hamiltonian for one-electron atoms:

$$\begin{aligned}\hat{H}_1 &= -\frac{(\hat{\boldsymbol{p}}^2)^2}{8m_\mathrm{e}^3c^2}\\ \hat{H}_2 &= \frac{1}{2m_\mathrm{e}^2c^2}\frac{1}{r}\frac{\partial V}{\partial r}\,\hat{\boldsymbol{L}}\cdot\hat{\boldsymbol{S}}\\ \hat{H}_3 &= \frac{Ze^2}{4\pi\epsilon_0}\frac{\hbar^2}{8(m_\mathrm{e}c)^2}4\pi\delta^{(3)}(\boldsymbol{r})\,.\end{aligned}$$

Here m_e denotes the electron mass, V the electrostatic potential generated by the nucleus, and Z the atomic number.

The hydrogenic radial wavefunctions have the form

$$\psi_{n\ell}(r) = \left(\frac{Z}{a_0}\right)^{3/2} G_{n\ell}\left(\frac{Zr}{a_0}\right)\exp\left[-\frac{Zr}{na_0}\right]\,,$$

where a_0 denotes the Bohr radius and $G_{n\ell}$ is a polynomial function of its argument. Show that the expectation values of the energies associated with the three terms listed above all have the same dependence on Z.

(Explicit evaluation of numerical factors is **not** required.)

18. **Hyperfine structure**

The magnetic part of the Hamiltonian for a hydrogen atom in the 1s state, in the presence of a constant magnetic field B along the z axis, may be cast in the form

$$\hat{H} = B\left(\mu_\mathrm{e}\hat{\sigma}_z^{(\mathrm{e})} + \mu_\mathrm{p}\hat{\sigma}_z^{(\mathrm{p})}\right) + W\hat{\boldsymbol{\sigma}}^{(\mathrm{e})}\cdot\hat{\boldsymbol{\sigma}}^{(\mathrm{p})}\,,$$

where the superscripts (e) and (p) refer to the electron and proton, the vector components of $\hat{\boldsymbol{\sigma}}$ are the Pauli spin operators, $\mu_{\mathrm{e,p}}$ are the respective magnetic dipole moments, and W is a constant.

(a) Explain the physical origin of each term in the Hamiltonian.

(b) Using as a basis the states $|\uparrow\rangle_{\rm e}\otimes|\uparrow\rangle_{\rm p}$, $|\uparrow\rangle_{\rm e}\otimes|\downarrow\rangle_{\rm p}$, $|\downarrow\rangle_{\rm e}\otimes|\uparrow\rangle_{\rm p}$, $|\downarrow\rangle_{\rm e}\otimes|\downarrow\rangle_{\rm p}$, and neglecting the small term in $\mu_{\rm p}$, show that the operator $\hat{H}$ may be represented by the matrix

$$\begin{pmatrix} b+W & 0 & 0 & 0 \\ 0 & b-W & 2W & 0 \\ 0 & 2W & -b-W & 0 \\ 0 & 0 & 0 & -b+W \end{pmatrix},$$

where $b=\mu_{\rm B}B$.

(c) Determine the energy levels and sketch their evolution as a function of B, labelling them with as much information as possible about the total angular momenta of the states. Comment on the behaviour in the two limiting cases $B\to 0$ and $B\to\infty$.

19. **Rotational symmetry**

Write down the Wigner-Eckart theorem for matrix elements of the form $\langle\alpha_1 j_1 m_1|\hat{V}_m|\alpha_2 j_2 m_2\rangle$, where the operators $\hat{V}_m$ $(m=\pm1,0)$ are the spherical components of a vector operator $\hat{\boldsymbol{V}}$, and the α_i represent any other quantum numbers needed to uniquely identify the total angular momentum eigenstates $|\alpha_i j_i m_i\rangle$ of the system.

(a) For the case $j_1=1$, $j_2=0$, identify the matrix elements $\langle\alpha_1 j_1 m_1|\hat{V}_m|\alpha_2 j_2 m_2\rangle$ which can be non-zero, and show that they are all equal. Hence show that the Cartesian components $(\hat{V}_x,\hat{V}_y,\hat{V}_z)$ of $\hat{\boldsymbol{V}}$ have matrix elements of the form

$$\begin{aligned}\langle\alpha_1 10|(\hat{V}_x,\hat{V}_y,\hat{V}_z)|\alpha_2 00\rangle &= A(0,0,1)\\ \langle\alpha_1 1,\pm1|(\hat{V}_x,\hat{V}_y,\hat{V}_z)|\alpha_2 00\rangle &= \frac{A}{\sqrt{2}}(\mp1,i,0)\end{aligned}$$

where A is a constant.

(b) Verify this result explicitly for the matrix elements of the position operator $\hat{\boldsymbol{r}}=(\hat{x},\hat{y},\hat{z})$ for a system such as the hydrogen atom for which the $j=0$ and $j=1$ angular momentum eigenstates $|\alpha jm\rangle$ are spatial wavefunctions of the form

$$|\alpha 00\rangle = R_{\alpha 0}(r)Y_{00}(\theta,\phi)\ ; \qquad |\alpha 1m\rangle = R_{\alpha 1}(r)Y_{1m}(\theta,\phi)\ .$$

[The $\ell=0$ and $\ell=1$ spherical harmonics $Y_{\ell m}(\theta,\phi)$ are

$$Y_{00}=\sqrt{\frac{1}{4\pi}}\ , \qquad Y_{10}=\sqrt{\frac{3}{4\pi}}\cos\theta\ , \qquad Y_{1,\pm1}=\mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}\ .\ ]$$

(c) What is the equivalent result to part (a) for the case $j_1=j_2=0$?

20. **Identical particles**

Two non-interacting, indistinguishable particles of mass m move in the one-dimensional potential $V(x)$ given by

$$V(x) = \begin{cases} 0 & 0 < x < L \\ \infty & \text{otherwise} \end{cases} .$$

Show that the energy of the system is of the form $E = (n_1^2 + n_2^2)\varepsilon$, where n_1 and n_2 are integers, and find an expression for ε.

Consider the state with $E = 5\varepsilon$ for each of the following three cases:

(a) spin-zero particles;

(b) spin-1/2 particles in a spin-singlet state;

(c) spin-1/2 particles in a spin-triplet state.

In each case, state the symmetries of the spin and spatial components of the two-particle wavefunction. Write down the spatial wavefunction $\psi(x_1, x_2)$, and sketch the probability density $|\psi(x_1, x_2)|^2$ in the (x_1, x_2) plane.

Describe qualitatively how the energies of these states would change if the particles carried electric charge and hence interacted with each other.

21. **Hund's rules**

Determine the possible spectroscopic terms, ${}^{2S+1}\mathrm{L}_J$, for each of the following electron configurations: (2s)(3p), $(2\mathrm{p})^2$, $(3\mathrm{d})^2$, $(3\mathrm{d})^{10}$, and $(3\mathrm{d})^9$.

Using Hund's rules, determine the angular momentum quantum numbers of the ground state of Sm, which has electron configuration $(4\mathrm{f})^6$.

22. **Atomic spectra (Tripos 1997)**

What are the selection rules for electric dipole transitions in atomic spectra, and what additional rules apply if the atom is accurately described by LS coupling?

The absorption spectrum of helium shows one series of single lines, the two of longest wavelength occurring at 58.4 nm and 53.7 nm. If the sample is excited by an electrical discharge, two new prominent single absorption lines are observed, at 2058 nm and 501.6 nm, and two prominent multiplet absorptions at 1083 nm and 389 nm. In emission, in addition to all the above wavelengths, single emission lines are seen at 728 and 668 nm, and multiplets at 707 and 588 nm.

Write the term symbol for the ground state $1\mathrm{s}^2$ of helium, and for all the excited states of the form $(1\mathrm{s})^1(n\ell)^1$ with $n = 2$, 3. Draw a diagram of the energy levels (see the lecture notes), and mark on it the spectral transitions described above, giving your reasoning.

Describe qualitatively how the spectra would be modified in the series of increasing atomic number Be, Mg, Ca, each of which has electron configuration $(ns)^2$. Why does Ca display two absorption transitions, at 647 nm and 423 nm, from the ground state to the first excited state: $(4\mathrm{s})^2 \to (4\mathrm{s})^1(4\mathrm{p})^1$, and why is each a single line?

23. **Linear Stark effect**

A hydrogen atom is placed in an external electric field of strength $\mathcal{E}$, resulting in shifts in the atomic energy levels which are large relative to atomic fine structure. The effect of the electric field on the level with principal quantum number $n = 3$ is to be analysed using first-order degenerate perturbation theory, with a perturbation $\hat{H}' = e\mathcal{E}z$, and working in the basis of states $|n\ell m_\ell\rangle$ ordered as

$$|300\rangle, \quad |310\rangle, \quad |320\rangle, \quad |311\rangle, \quad |321\rangle, \quad |31,-1\rangle, \quad |32,-1\rangle, \quad |322\rangle, \quad |32,-2\rangle \; .$$

The reduced matrix elements for the electron position operator $\hat{\boldsymbol{r}}$ for $n = 3$ are $\langle 3\mathrm{s}\,\|\,\hat{\boldsymbol{r}}\,\|\,3\mathrm{p}\rangle = 9\sqrt{2}a_0$, and $\langle 3\mathrm{d}\,\|\,\hat{\boldsymbol{r}}\,\|\,3\mathrm{p}\rangle = -(9/\sqrt{2})a_0$, where a_0 is the Bohr radius.

(a) Show that the matrix representation of $\hat{H}'$ in the basis above is block diagonal, with sub-matrices of the form

$$H'_0 = \begin{pmatrix} 0 & a & 0 \\ a & 0 & b \\ 0 & b & 0 \end{pmatrix} , \qquad H'_{+1} = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix} , \qquad H'_{-1} = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix} ,$$

where a, b, c are constants such that $a = \sqrt{2}b = \sqrt{8/3}c$, and $c = -(9/2)ea_0\mathcal{E}$.

[You may find the $1 \otimes 1$ table of Clebsch-Gordan coefficients useful.]

(b) Show that the electric field splits the $n = 3$ level into five equally spaced levels with energy separation $(9/2)ea_0\mathcal{E}$. State the values of the quantum number m_ℓ associated with each of these five levels.

(c) Show that, in the electric field, the $n = 3$ level of highest energy corresponds to the zeroth-order eigenstate

$$|\psi\rangle = \sqrt{\frac{1}{3}}|300\rangle - \sqrt{\frac{1}{2}}|310\rangle + \sqrt{\frac{1}{6}}|320\rangle \; ,$$

and (*optionally*) determine the zeroth-order eigenstates for the other four levels.

24. **Landé g-factors (Tripos 2004)**

Explain why, in the L-S (Russell-Saunders) coupling scheme, the good quantum numbers for the electronic configuration of an atom are L, S, J and m_J.

A magnetic field B in the z-direction is applied to an atom described by the L-S coupling scheme so that, to first order in B, the energies of the states change by

$$\Delta E_{L,S,J,m_J} = \big\langle \phi_{L,S,J,m_J} \big| \frac{eB}{2m_e}(\hat{L}_z + 2\hat{S}_z) \big| \phi_{L,S,J,m_J} \big\rangle \, .$$

Explain why the state $|\phi_{L=2,S=1,J=3,m_J=3}\rangle$ can be written as

$$|\phi_{L=2,S=1,J=3,m_J=3}\rangle = |\psi_{L=2,m_L=2}\rangle \otimes |\chi_{S=1,m_S=1}\rangle \, ,$$

where $|\psi_{L,m_L}\rangle$ are orbital angular momentum eigenstates and $|\chi_{S,m_S}\rangle$ are spin angular momentum eigenstates.

Calculate the change in energy of the state with quantum numbers $L = 2$, $S = 1$, $J = 3$, $m_J = 3$ on application of the magnetic field.

Show that your result is consistent with the following formula for the Landé g-factor:

$$g = \frac{3}{2} - \frac{L(L+1) - S(S+1)}{2J(J+1)} \, .$$

Calculate the change in energy of the state with quantum numbers $L = 2$, $S = 1$, $J = 3$, $m_J = 2$ and of the state with quantum numbers $L = 2$, $S = 1$, $J = 2$, $m_J = 2$. Show that these are also consistent with the formula for the Landé g-factor.

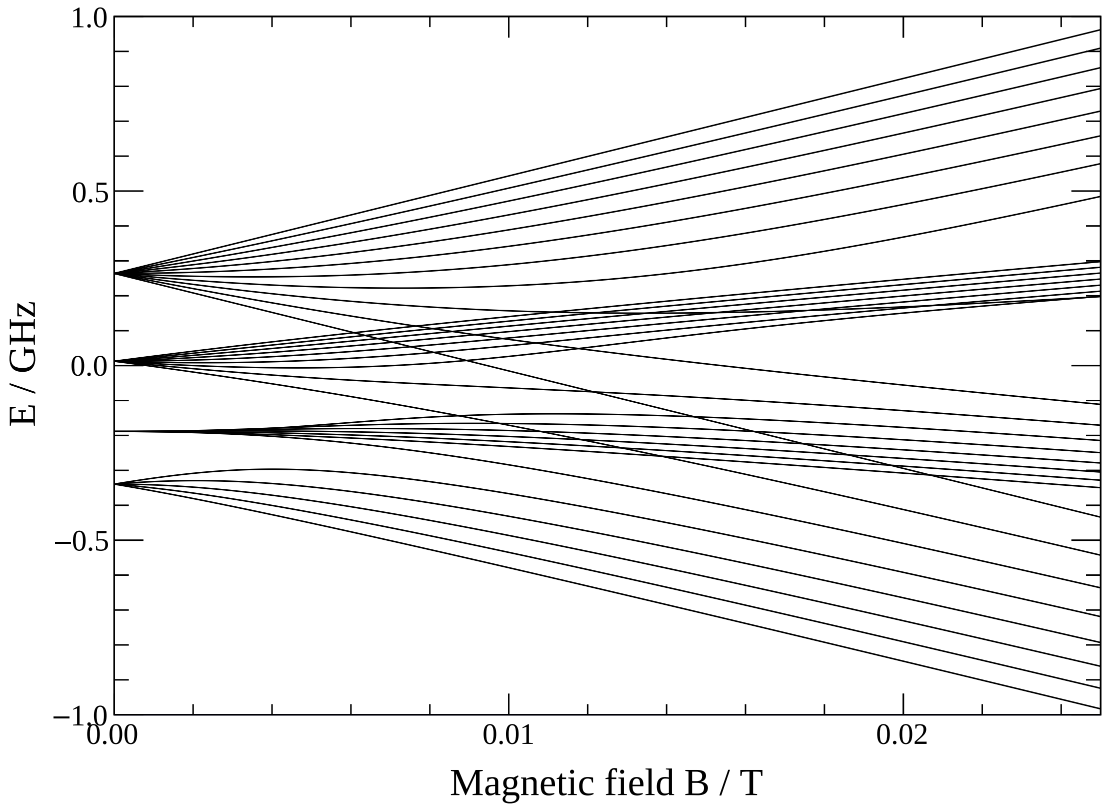
[The raising and lowering operators for orbital angular momentum are defined as follows:

$$\hat{L}_{\pm}|\psi_{L,m_L}\rangle = \hbar\sqrt{L(L+1) - m_L(m_L \pm 1)}\,|\psi_{L,m_L\pm1}\rangle \, ,$$

and similarly for $\hat{J}_{\pm}$ and $\hat{S}_{\pm}$.]

25. **Zeeman effect for hyperfine levels**

The figure shows the splitting (in frequency units, with an arbitrary offset) of the hyperfine levels associated with a term ${}^{2S+1}\mathrm{L}_J$ of a neutral atom due to an applied magnetic field.



(a) What is the total angular momentum quantum number F of each of the four hyperfine levels seen at zero field? What combinations of nuclear spin I and total electronic angular momentum J can produce these values of F?

(b) What is the value of the Landé g-factor g_F for the hyperfine level with a frequency in zero field of about -0.2 GHz? Use this observation to determine unambiguously the quantum numbers I and J.

(c) Explain why these values of I and J are consistent with the pattern of splitting seen at high magnetic field.

(d) From the plot, estimate g_F for one of the three hyperfine levels showing a distinct splitting at low field, and hence estimate the Landé g-factor g_J for the term. [In principle, any of the three hyperfine levels can be used, but one of them is a better choice than the others.]

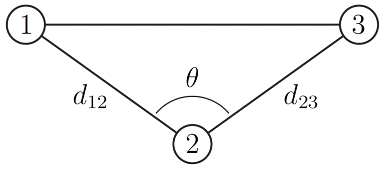
(e) Use your estimate of g_J to determine the quantum numbers L and S.

(f) Is the observed pattern of energy separations at zero field between the four hyperfine levels consistent with an interaction proportional to $\hat{\boldsymbol{I}} \cdot \hat{\boldsymbol{J}}$?

26. **Molecular bonding (Tripos 1998)**

Explain what is meant by the Born-Oppenheimer approximation and discuss how molecular wavefunctions can be formed within this approximation by using the Linear Combination of Atomic Orbitals (LCAO).

The H_3^+ ion exists as an isosceles triangle (distance $d_{12} = d_{23}$) as shown, with $60^o \leq \theta \leq 180^o$.



Treat this ion in the LCAO approximation by introducing the 1s wavefunctions $|\psi_i\rangle$ for the i'th atom. Show that, in the basis $\{(|\psi_1\rangle - |\psi_3\rangle)/\sqrt{2},\ (|\psi_1\rangle + |\psi_3\rangle)/\sqrt{2},\ |\psi_2\rangle\}$, the Hamiltonian $\hat{H}$ for the ion has the matrix representation

$$H = \begin{pmatrix} \alpha - \gamma\beta & 0 & 0 \\ 0 & \alpha + \gamma\beta & \beta\sqrt{2} \\ 0 & \beta\sqrt{2} & \alpha \end{pmatrix} ,$$

where we assume $\alpha = \langle\psi_1|\hat{H}|\psi_1\rangle = \langle\psi_2|\hat{H}|\psi_2\rangle = \langle\psi_3|\hat{H}|\psi_3\rangle$, $\beta = \langle\psi_1|\hat{H}|\psi_2\rangle = \langle\psi_2|\hat{H}|\psi_3\rangle < 0$, $\beta\gamma = \langle\psi_1|\hat{H}|\psi_3\rangle$, and the overlap integrals are neglected: $\langle\psi_i|\psi_j\rangle = \delta_{ij}$.

By solving the secular equation, obtain expressions for the energy levels for the ion. Sketch the energy levels in the range $0 \leq \gamma \leq 1$ and explain why the ground state must be a spin singlet.

27. **Time-dependent perturbation theory**

As shown in lectures, the probability that a system prepared in an energy eigenstate ψ_0 at time $t = 0$ is subsequently found in a state ψ_n when a weak perturbation $V(t)$ is applied is given approximately by $|c_n(t)|^2$ where

$$c_n(t) = \frac{1}{i\hbar} \int_0^t e^{i(E_n - E_0)t'/\hbar} \langle\psi_n|V(t')|\psi_0\rangle \, dt' \, .$$

At times $t > 0$, an electric field $\mathcal{E}_z = \mathcal{E}_0 \exp(-t/\tau)$ is applied to a hydrogen atom, initially prepared in its ground state. Working to first order in the electric field, find the probability that, after a long time, $t \gg \tau$, the atom is in (i) the 2s state, and (ii) one of the 2p states (state which one).

[To identify which of the matrix elements $\langle 2\mathrm{s}|\hat{z}|1\mathrm{s}\rangle$, $\langle 2\mathrm{p}_m|\hat{z}|1\mathrm{s}\rangle$ might be non-zero, you may find it helpful to refer back to question 19.]

28. **Time-dependent perturbation theory (Tripos 2001)**

A one-dimensional harmonic oscillator with Hamiltonian $\hat{H}_0 = (\hat{p}_x^2/2m) + \frac{1}{2}m\omega^2\hat{x}^2$, initially in the ground state, is subjected to a perturbation:

$$\hat{H}'(t) = \begin{cases} 0 & t < 0 \text{ and } t > T \\ \lambda\hat{x}(1 - t/T) & 0 \le t \le T \end{cases} .$$

Find, to first order in λ, the probability that the oscillator is in the first excited state at time $t > T$.

Verify that for $\omega T \gg 1$ this probability approaches the value of $|\langle\psi_1|\psi_0'\rangle|^2$ where $|\psi_0'\rangle$ is the ground state for the Hamiltonian $\hat{H}_0 + \lambda\hat{x}$.

[The ground state and first excited state of a harmonic oscillator have wavefunctions:

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right), \qquad \psi_1(x) = \left(\frac{2m\omega}{\hbar}\right)^{1/2} x\psi_0(x)\,. \,]$$

29. **Scattering (Born Approximation)**

Neglecting atomic recoil, the elastic scattering of electrons from stationary hydrogen atoms can be modelled as scattering from a fixed potential $V(\boldsymbol{r})$. Show that a suitable form for $V(\boldsymbol{r})$ is

$$V(\boldsymbol{r}) = \frac{e^2}{4\pi\epsilon_0}\left(-\frac{1}{r} + \int \frac{|\psi(\boldsymbol{r}')|^2}{|\boldsymbol{r} - \boldsymbol{r}'|}\,\mathrm{d}^3\boldsymbol{r}'\right) ,$$

where $\psi(\boldsymbol{r})$ is the wavefunction for the atomic electron.

For incoming electrons of momentum $\hbar k$, and for hydrogen atoms in their ground state, $\psi_{1\mathrm{s}}(\boldsymbol{r}) = (\pi a_0^3)^{-1/2}e^{-r/a_0}$, show that the scattering amplitude $f(\theta,\phi)$ predicted in the Born approximation is

$$f(\theta,\phi) = -2a_0\frac{8 + (qa_0)^2}{(4 + (qa_0)^2)^2} ,$$

where $q = 2k\sin(\theta/2)$, and a_0 is the Bohr radius: $a_0 = 4\pi\epsilon_0\hbar^2/(e^2 m_\mathrm{e}) = 0.53 \times 10^{-10}$ m.

[Hint: the integral in the expression above for $V(\boldsymbol{r})$ is a convolution. The Fourier transforms of the radial functions $1/r$ and $e^{-\lambda r}$ are

$$\int \frac{1}{r}e^{i\boldsymbol{q}\cdot\boldsymbol{r}}\,\mathrm{d}^3\boldsymbol{r} = \frac{4\pi}{q^2} ; \qquad \int e^{-\lambda r}e^{i\boldsymbol{q}\cdot\boldsymbol{r}}\,\mathrm{d}^3\boldsymbol{r} = \frac{8\pi\lambda}{(\lambda^2 + q^2)^2} ,$$

where $q = |\boldsymbol{q}|$.]

Figure 1 shows measurements of the differential cross section for elastic electron-hydrogen scattering, for incoming electron energies of 20 eV and 200 eV. For each electron energy, obtain the predicted differential cross section in the Born approximation (in units of a_0^2) for a scattering angle $\theta = 100°$, and compare with the measured values.

What is the predicted cross section for small scattering angles, $\theta \to 0°$? Comment on the agreement, or otherwise, of the Born approximation prediction with data in this limit.

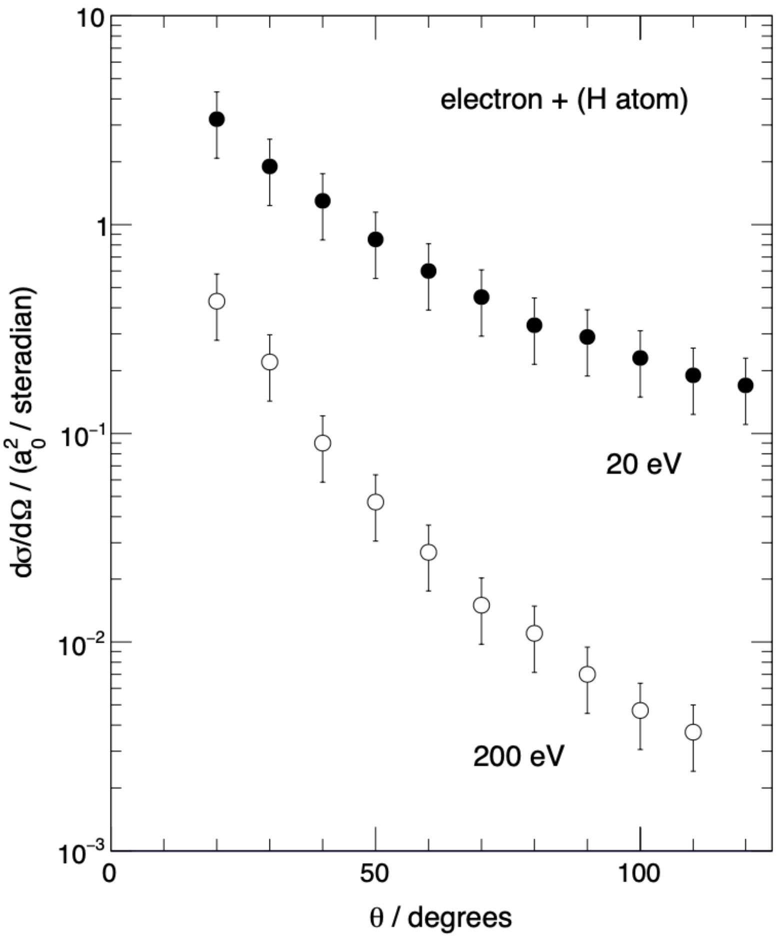


Figure 1: Measurements of the elastic scattering of electrons from hydrogen atoms, for incoming electron energies of 20 eV (solid points) and 200 eV (open points). Data from: P. J. O. Teubner, C. R. Lloyd and E. Weigold, Phys. Rev. A **9** (1974) 2552 and C. R. Lloyd, P. J. O. Teubner, E. Weigold and B. R. Lewis, Phys. Rev. A **10** (1974) 174. (The error bars represent a common overall uncertainty of $\pm 35\%$, as alluded to in the references given.)

30. **Photons**

The total linear momentum operator $\hat{\boldsymbol{P}}$ for an electromagnetic field was obtained in lectures as

$$\hat{\boldsymbol{P}} = \sum_{\boldsymbol{k},\lambda} \hbar\boldsymbol{k}\, \hat{a}^{\dagger}_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda} \ .$$

Show that a photon of wave vector $\boldsymbol{k}$ (in any polarisation state λ) has linear momentum $\hbar\boldsymbol{k}$.

Similarly, the intrinsic (spin) angular momentum operator $\hat{\boldsymbol{J}}_{\mathrm{s}}$ was given as

$$\hat{\boldsymbol{J}}_{\mathrm{s}} = \hbar \sum_{\boldsymbol{k}} \frac{\boldsymbol{k}}{|\boldsymbol{k}|} \left[\hat{a}^{\dagger}_{\boldsymbol{k},L}\hat{a}_{\boldsymbol{k},L} - \hat{a}^{\dagger}_{\boldsymbol{k},R}\hat{a}_{\boldsymbol{k},R} \right] \ .$$

Show that for left-handed (right-handed) photons, the spin is oriented parallel to the photon direction of motion, with spin projection $+\hbar$ ($-\hbar$).

31. **Spontaneous emission lifetimes**

Explain why (neglecting fine structure) the 2p states $|n\ell m_\ell\rangle = |211\rangle$, $|210\rangle$ and $|21,-1\rangle$ of atomic hydrogen must all have the same lifetime. [You may find it helpful to refer back to question 19(a).]

Calculate the lifetime of the 2p state of atomic hydrogen. [A suitable choice of m_ℓ will allow the matrix elements already computed in question 27 to be reused.]

Without detailed calculation, explain why:

(a) the 3s level of hydrogen has a lifetime roughly 100 times longer than the 2p lifetime;

(b) the 2s level of hydrogen has a lifetime very much longer than the 2p lifetime (in fact, by a factor of about 10^8);

(c) ("switching on" fine structure) the fine structure levels $2^2\mathrm{P}_{1/2}$ and $2^2\mathrm{P}_{3/2}$ of hydrogen both have lifetimes closely equal to the 2p lifetime computed above (in fact, to within about 1 part in 10^5).

32. **Stark effect: spontaneous decays**

Spontaneous decays from the $n = 3$ to the $n = 2$ level of the hydrogen atom (the Balmer line H_α) are examined in the presence of a strong electric field. Using the results for the linear Stark effect obtained in lectures (for $n = 2$) and in question 23 (for $n = 3$):

(a) What is the multiplicity of the H_α emission line when viewed in a general direction in the presence of the electric field?

(b) How many components of the H_α line are seen when the transition is viewed along, or perpendicular to, the electric field direction? What is the polarisation state of the emitted radiation in each case?

(c) The matrix elements $\langle n'\ell' m'_\ell|z|n\ell m_\ell\rangle$ for $n' = 3$, $n = 2$ and $m'_\ell = m_\ell = 0$ have the values

$$\langle 310|z|200\rangle = 1.769\, a_0\,, \qquad \langle 300|z|210\rangle = 0.542\, a_0\,, \qquad \langle 320|z|210\rangle = 2.452\, a_0\ ,$$

where a_0 is the Bohr radius. Show that, in the presence of an electric field, and integrated over all directions, the emission intensity from the highest energy $n = 3$ level to the highest energy $n = 2$ level is about 1700 times greater than the intensity of the corresponding transition to the lowest energy $n = 2$ level.

(d) Explain why the information given in part (c) is sufficient to determine the intensities of *all* H_α transitions in the presence of the electric field.

33. **Lasers**

A laser operates between the upper two levels of a three-level system with level energies $E_2 > E_1 > E_0$. Levels 2 and 1 have degeneracies g_2 and g_1, respectively. The system is pumped from level 0 to level 2 at a rate R per atom. The level populations N_2, N_1, N_0, and the cavity photon population in the lasing mode, N_γ, satisfy the rate equations

$$\dot{N}_2 = N_0 R - N_2 A_{21} - N_2 A_{20} + N_1 B'_{12} N_\gamma - N_2 B'_{21} N_\gamma \quad (1)$$
$$\dot{N}_1 = N_2 A_{21} - N_1 A_{10} + N_2 B'_{21} N_\gamma - N_1 B'_{12} N_\gamma \quad (2)$$
$$\dot{N}_0 = -N_0 R + N_2 A_{20} + N_1 A_{10} \quad (3)$$
$$\dot{N}_\gamma = N_2 B'_{21}(N_\gamma + 1) - N_1 B'_{12} N_\gamma - \Gamma_{\mathrm{cav}} N_\gamma \ , \quad (4)$$

where the total number of atoms $N_0 + N_1 + N_2 = N$ is a constant, and $g_1 B'_{12} = g_2 B'_{21}$.

(a) Justify the form of Equation (4), and explain why the coefficients governing stimulated and spontaneous $2 \to 1$ transitions must satisfy the condition $B'_{21} \ll A_{21}$.

(b) The pumping rate is limited to $R \ll A_{21}, A_{20}, A_{10}$. Show that steady-state lasing action ($N_\gamma \gg 1$) can be achieved provided $A_{10} > (g_2/g_1)A_{21}$, and obtain an estimate of the minimum (threshold) pumping rate required.

[Hint: you may find it helpful to eliminate the bilinear terms $N_1 B'_{12} N_\gamma$ and $N_2 B'_{21} N_\gamma$, leaving only terms which are linear in the various populations.]

(c) For a helium-neon laser operating at $\lambda = 632.8\,\mathrm{nm}$ (red), the level degeneracies are $g_2 = 3$ and $g_1 = 5$, and the spontaneous decay constants are $A_{21} = 3.39 \times 10^6\,\mathrm{s}^{-1}$, $A_{10} = 5.27 \times 10^7\,\mathrm{s}^{-1}$ and $A_{20} = 3.95 \times 10^6\,\mathrm{s}^{-1}$. Typical laser parameters are: $N = 2.2 \times 10^{15}$, $\Gamma_{\mathrm{cav}} = 5 \times 10^6\,\mathrm{s}^{-1}$ and $B'_{21} = 0.064\,\mathrm{s}^{-1}$. Using the approximate steady-state solution obtained in part (b), estimate the laser beam power produced for a pumping rate $R = 1.6\,\mathrm{s}^{-1}$, and obtain the ratio of the stimulated and spontaneous $2 \to 1$ transition rates for this case. Estimate also the threshold pumping rate required for lasing action to occur.

(d) (*optional*) Show that Equations (1)-(4) admit an exact steady-state solution, valid for any value of the pumping rate R and imposing no restrictions on the values of any quantities involved.

[For example, show that the steady-state photon population N_γ satisfies a quadratic equation, and that N_1 and N_2 can be obtained from N_γ.]

Use MatLab or similar to plot the exact steady-state populations N_1, N_2 and N_γ as a function of R for the numbers given in part (c), and hence assess the validity of the approximate solution obtained in part (b).

34. **Coherent states**

(a) By using the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$, show that

$$e^{-\beta\hat{a}^\dagger}\hat{a}e^{\beta\hat{a}^\dagger} = \beta + \hat{a}\,.$$

[Hint: Consider the β derivative of this expression.]
Using this result, show that $|\beta\rangle = Ne^{\beta\hat{a}^\dagger}|0\rangle$ is a coherent state, i.e. $\hat{a}|\beta\rangle = \beta|\beta\rangle$. Finally, show that the normalization, $N = e^{-|\beta|^2/2}$.

(b) Calculate the expectation values, $x_0 = \langle\hat{x}\rangle$ and $p_0 = \langle\hat{p}\rangle$, with respect to $|\beta\rangle$ and, by considering $\langle\hat{x}^2\rangle$ and $\langle\hat{p}^2\rangle$, show that

$$(\Delta p)^2(\Delta x)^2 = \frac{\hbar^2}{4}\,,$$

where $(\Delta p)^2 = \langle(\hat{p} - \langle\hat{p}\rangle)^2\rangle$ (and similarly $(\Delta x)^2$).
[Hint: Remember how the creation and annihilation operators are related to the phase space operators $\hat{x}$ and $\hat{p}$. Also, note that taking the Hermitian conjugate of the eigenvalue equation $\hat{a}|\beta\rangle = \beta|\beta\rangle$ leads to the relation $\langle\beta|\hat{a}^\dagger = \langle\beta|\beta^*$.]

(c) Show that the eigenvalue equation $\hat{a}|\beta\rangle = \beta|\beta\rangle$ translates to the equation

$$\sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)\psi(x) = \beta\psi(x)\,.$$

for the coordinate representation, $\psi(x)$, of the coherent state. Show that this equation has the solution

$$\psi(x) = N\exp\left[-\frac{(x-x_0)^2}{4(\Delta x)^2} + i\frac{p_0 x}{\hbar}\right]\,,$$

where x_0 and p_0 are defined in part (b) above.

(d) By expressing $|\beta\rangle$ in the number basis, show that

$$|\beta(t)\rangle = e^{-i\omega t/2}|\beta e^{-i\omega t}\rangle\,.$$

As a result, deduce expressions for $x_0(t)$ and $p_0(t)$ and show they represent solutions to the classical equations of motion. How does the width of the coherent state wavepacket evolve with time?

18. (c) $E = W \pm b$, $E = -W \pm \sqrt{4W^2 + b^2}$.

21. $(4f)^6 \to {}^7F_0$.

23. (b) $m_\ell = 0$ for $\Delta E = \pm 9ea_0\mathcal{E}$,
$m_\ell = \pm 1$ for $\Delta E = \pm 4.5\, ea_0\mathcal{E}$,
$m_\ell = 0, \pm 2$ for $\Delta E = 0$;

(c) summary of the linear Stark effect for the $n = 3$ level:

$\Delta E/ea_0\mathcal{E}$	$m_\ell = 0$	$m_\ell = \pm 1$	$m_\ell = \pm 2$	g
+9.0	$\sqrt{\frac{1}{3}}\,\|300\rangle - \sqrt{\frac{1}{2}}\,\|310\rangle + \sqrt{\frac{1}{6}}\,\|320\rangle$			1
+4.5		$\sqrt{\frac{1}{2}}\,(\|31,\pm1\rangle - \|32,\pm1\rangle)$		2
0	$\sqrt{\frac{1}{3}}\|300\rangle - \sqrt{\frac{2}{3}}\|320\rangle$		$\|32,\pm2\rangle$	3
−4.5		$\sqrt{\frac{1}{2}}\,(\|31,\pm1\rangle + \|32,\pm1\rangle)$		2
−9.0	$\sqrt{\frac{1}{3}}\,\|300\rangle + \sqrt{\frac{1}{2}}\,\|310\rangle + \sqrt{\frac{1}{6}}\,\|320\rangle$			1

24. $4\mu_B B$, $(8/3)\mu_B B$, $(7/3)\mu_B B$.

25. (a) $F = 2, 3, 4, 5$; $\{\, I = 3/2,\ J = 7/2 \,\}$ or $\{\, I = 7/2,\ J = 3/2 \,\}$;
(b) $g_F = 0$; $I = 7/2$, $J = 3/2$;
(d) $g_F = 0.40$, $g_J \approx 4/3$;
(e) $L = 1$, $S = 1/2$ $({}^2P_{3/2})$;

26. $E = \alpha - \gamma\beta$, $E = \alpha + \frac{1}{2}\beta\left(\gamma \pm \sqrt{\gamma^2 + 8}\right)$.

27. (i) $P(2s) = 0$; (ii) $P(2p_{\pm 1}) = 0$, $P(2p_0) = (e^2\mathcal{E}_0^2 a_0^2 2^{15}/3^{10})/(\Delta E^2 + \hbar^2/\tau^2)$.

32. (a) 15; (b) 7, 15.

29. $(0.17)a_0^2$ $(E_e = 20\,\mathrm{eV})$, $(3.3 \times 10^{-3})a_0^2$ $(E_e = 200\,\mathrm{eV})$.

31. $\tau(2p) = 1.56 \times 10^{-9}\,\mathrm{s}$.

33. (c) $P = 0.8\,\mathrm{mW}$, $R_{\mathrm{stim}}(2 \to 1)/R_{\mathrm{spon}}(2 \to 1) = 10.0$, $R_{\mathrm{th}} = 0.3\,\mathrm{s}^{-1}$.