Electrodynamics and Optics

Handout 3

Prof. Paul Alexander

Michaelmas Term 2021

§7 Special Relativity

§7.1 Background The wave equation for propagation of electromagnetic waves in vacuum is not invariant under a Galilean transformation between coordinate system S and S' moving with respect to each other with a velocity $\boldsymbol{v}$: $\boldsymbol{r'} = \boldsymbol{r} - \boldsymbol{v}t$ and $t' = t$.

$$(\sum_i \frac{\partial^2}{\partial x_i'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2})\Psi = 0 \tag{7.1}$$

$$(\sum_i \frac{\partial^2}{\partial x_i^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{2}{c^2}\boldsymbol{v}\cdot\nabla\frac{\partial}{\partial t} - \frac{1}{c^2}\boldsymbol{v}\cdot\nabla\boldsymbol{v}\cdot\nabla)\Psi = 0 \tag{7.2}$$

When this was realised there were three possible solutions: (i) Maxwell's equations were not correct. (ii) For electromagnetism there is a defined frame of reference with respect to which the ether is at rest. (iii) There must be a relativity principle different from the Galilean one. Einstein chose the third solution.

Experiments contradicting the existence of a preferred frame of reference led directly to Einstein's proposition of the:

Principle of Relativity, that the *Laws of Physics are identical in all inertial frames*

Law of Light Propagation, that *light in vacuum propagates rectilinearly with the same speed c in all inertial frames*, regardless of the motion of the source.

The foundations of Special Relativity (SR) and the formulation of electrodynamics as a classical field theory within Minkowski space-time have been covered in the Relativity course and will be only briefly reviewed here. Our focus in this section will be on discussing a number of important relativistic electromagnetic phenomena, such as the radiation emitted by charges moving at relativistic speeds.

§7.2 Revision from Part II Relativity course

You considered the Lorentz transformation in the four-dimensional Minkowski space-time $(x_0, \boldsymbol{x}) = (ct, \boldsymbol{x})$ as a special member of a more general class of coordinate transformations $x'^\alpha = x'^\alpha(x^0, x^1, x^2, x^3)$ of the Lorentz group that leaves $s^2 \equiv x_0^2 - x_1^2 - x_2^2 - x_3^2$ invariant when transforming between inertial frames S and S'. *Tensors of rank k* are defined by their transformation properties under coordinate transformations:

- Contravariant vectors: $A'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta} A^\beta$

- Covariant vectors: $B'_\alpha = \frac{\partial x^\beta}{\partial x'^\alpha} B_\beta$

- Contravariant tensor of rank two: $F'^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\gamma}\frac{\partial x'^\beta}{\partial x^\delta} F^{\gamma\delta}$

- Covariant tensor of rank two: $G'_{\alpha\beta} = \frac{\partial x^\gamma}{\partial x'^\alpha}\frac{\partial x^\delta}{\partial x'^\beta} G_{\gamma\delta}$

The scalar product of a covariant and a contravariant vector is an invariant scalar: $\boldsymbol{B}\cdot\boldsymbol{A} \equiv B_\alpha A^\alpha = \boldsymbol{B'}\cdot\boldsymbol{A'}$.

The *metric tensor* $g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ for the flat space-time of special relativity is defined through:

$$(ds)^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \equiv g_{\alpha\beta}dx^\alpha dx^\beta \qquad (7.3)$$

Contraction with $g_{\alpha\beta}$ or $g^{\alpha\beta}$ is the procedure for changing an index of a tensor from being contravariant to covariant, for example, $F^{\cdot\cdot\alpha\cdot\cdot}_{\cdot\cdot} = g^{\alpha\beta}F^{\cdot\cdot\cdot\cdot}_{\cdot\cdot\beta}$

For example: If a contravariant four vector has components A^0, A^1, A^2, A^3 then the corresponding covariant four vector has components $A_0 = A^0, A_1 = -A^1, A_2 = -A^2, A_3 = -A^3$.

$$A^\alpha = (A^0, \boldsymbol{A}) \qquad\qquad A_\alpha = (A^0, -\boldsymbol{A})$$

Differentiation with respect to a contravariant component transforms like the covariant component of a vector operator: $\frac{\partial}{\partial x'^\alpha} = \frac{\partial x^\beta}{\partial x'^\alpha}\frac{\partial}{\partial x^\beta}$

With $\partial^\alpha \equiv \frac{\partial}{\partial x_\alpha} = (\frac{\partial}{\partial x_0}, -\boldsymbol{\nabla})$ and $\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha} = (\frac{\partial}{\partial x_0}, \boldsymbol{\nabla})$ the four-divergence of a four-vector and the four-dimensional Laplacian are both invariant contractions:

$$\begin{aligned} \partial^\alpha A_\alpha &= \partial_\alpha A^\alpha = \frac{\partial A_0}{\partial x_0} + \boldsymbol{\nabla}\cdot\boldsymbol{A} \\ \Box^2 &\equiv \partial_\alpha \partial^\alpha = \frac{\partial^2}{\partial^2 x_0} - \boldsymbol{\nabla}^2 \end{aligned}$$

§7.3 Invariance of electric charge

There is very strong experimental evidence that the electric charge of an electron is a Lorentz invariant (LI).

(i) *Atoms* are very accurately known to be *neutral*, yet they contain slow-moving nuclei and fast-moving electrons and in different atoms the electrons move with different velocities. If charge were not a LI, this could not be so.

(ii) Atoms also *remain neutral* when heated or are viewed in a different inertial frame. Consider a 10 kg block of copper containing about 10^{26} atoms; the total charge of the conduction electrons is then about 10^7C. On heating, the *nuclear motion is increased* more significantly than the electron motion (phonons are induced). If the charge on the nuclear protons changed due to this motion then the copper would charge up. If the copper is in the form of a sphere (say 65mm radius) changes in potential of say 1 mV could be detected (easily done), the experiment would be sensitive to a net charge of about 10^{-14} C. No such effect is observed, hence the electronic and nuclear charges cannot differ by more than 1 part in 10^{21}.

(iii) In particle accelerators and mass spectrometers, the *dynamics* of electrons depend on e/m and their dynamics are *correctly predicted* by taking e as a LI. This has been verified to the highest energies attainable with modern accelerators.

The electric charge can be assumed to be a LI with high level of confidence.

From this it follows that the 4-current-density ("4-current") $\mathbf{J} = (c\rho, \boldsymbol{J})$ is a physical 4-vector. This can be seen in the following way. Consider a macroscopic charge Q occupying a volume V_0 in its IRF S_0. In S_0 the charge density is $\rho_0 = Q/V_0$.

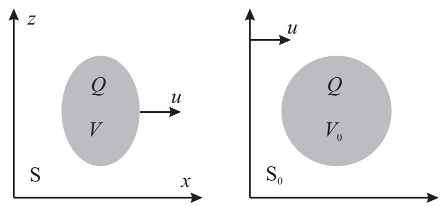

Figure 7.1: Charge invariance and the Lorentz contraction give $\rho = \gamma\rho_0$.

In S in which the charge is moving at speed u the volume occupied by the charge will be $V = V_0/\gamma$.

But Q is a Lorentz Invariant, so the charge density in S is therefore $\rho = \gamma\rho_0$.

Now consider two nearby points on the world line of Q in S, separated by the 4-vector $\mathrm{d}\mathbf{R} = (c\,\mathrm{d}t, \mathrm{d}\boldsymbol{r})$. In S_0 the current density $\boldsymbol{J}_0 = 0$ since Q is stationary. But in S

$$\boldsymbol{J} = \rho\boldsymbol{u} = \gamma\rho_0\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}$$

Therefore in S

$$\mathbf{J} = (c\rho, \boldsymbol{J}) = \left(\gamma c\rho_0, \gamma\rho_0\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right) = \rho_0\gamma\frac{\mathrm{d}}{\mathrm{d}t}(ct, \boldsymbol{r}) = \rho_0\frac{\mathrm{d}}{\mathrm{d}\tau}(ct, \boldsymbol{r})$$

where $\mathrm{d}\tau = \mathrm{d}t/\gamma$, with τ the proper time in the rest frame S′ of the charge.

ρ_0 (just a number defined in its instantaneous rest frame (IRF) of the charge) and the proper time τ are Lorentz Invariants.

So, since $\mathbf{R} = (ct, \boldsymbol{r})$ is a 4-vector, $\mathbf{J}$ *must be a 4-vector*.

From this follows the continuity equation and conservation of charge must hold in all frames:

$$\partial_\alpha J^\alpha = \frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{J} = 0$$

Applying the Lorentz Transformations to the 4-current-density gives the transformation properties for the 3-current and charge densities:

$$c\rho' = \gamma(c\rho - \beta J_x) \tag{7.4}$$
$$J'_x = \gamma(J_x - \beta c\rho) \tag{7.5}$$
$$J'_y = J_y \tag{7.6}$$
$$J'_z = J_z \tag{7.7}$$

Alternatively, with $\|$ and $\perp$ referring to the Ox axis:

$$c\rho' = \gamma(c\rho - \boldsymbol{\beta}\cdot\boldsymbol{J}) \qquad \boldsymbol{J}' = \gamma(\boldsymbol{J}_\| - \beta c\rho) + \boldsymbol{J}_\perp \tag{7.8}$$

§7.4 Maxwell's equations in covariant form

Maxwell's equations expressed in terms of the scalar and vector potentials are:

$$\nabla^2\phi - \frac{1}{c^2}\ddot{\phi} = -\frac{\rho}{\epsilon_0} \quad\longrightarrow\quad \Box^2\phi = \frac{\rho}{\epsilon_0}$$
$$\nabla^2\boldsymbol{A} - \frac{1}{c^2}\ddot{\boldsymbol{A}} = -\mu_0\boldsymbol{J} \quad\longrightarrow\quad \Box^2\boldsymbol{A} = \mu_0\boldsymbol{J}$$

substituting the d'Alembertian $\Box^2 = \partial_\alpha\partial^\alpha = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$. These equations can be written in terms of the 4-potential $\mathbf{A} = (\frac{\phi}{c}, \boldsymbol{A})$ by adding:

$$\Box^2\left(\frac{\phi}{c}, \boldsymbol{A}\right) = (\mu_0 c\rho, \mu_0\boldsymbol{J}) = \mu_0(c\rho, \boldsymbol{J})$$
$$\Box^2\mathbf{A} = \mu_0\mathbf{J} \tag{7.9}$$

Since $\Box^2$ is a Lorentz Invariant and $\mathbf{J}$ has been shown to be a 4-vector, *it must be that* $\mathbf{A}$ *is a 4-vector*. Maxwell's equations have now been expressed in an elegant form (Eqn 7.9) that makes no reference to any specific inertial frame – they have been "written in a *manifestly covariant* form".

An even more elegant formulation can be obtained by realising that the relationships

$$E_x = -\frac{\partial A_x}{\partial t} - \frac{\partial\Phi}{\partial x} = -(\partial^0 A^1 - \partial^1 A^0)$$
$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = -(\partial^2 A^3 - \partial^3 A^2)$$

suggest that the electric and magnetic fields are the components of a *second-rank antisymmetric field-strength tensor*:

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha = \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix} \tag{7.10}$$

$$F_{\alpha\beta} = g_{\alpha\gamma}F^{\gamma\delta}g_{\delta\beta} = \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & -B_z & B_y \\ -\frac{E_y}{c} & B_z & 0 & -B_x \\ -\frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix} \tag{7.11}$$

In terms of $F_{\alpha\beta}$ the two inhomogeneous Maxwell equations (ME1 and ME4) take on the covariant form:

$$\partial_\alpha F^{\alpha\beta} = \mu_0 J^\beta \tag{7.12}$$

The two homogeneous Maxwell equations (ME2 and ME3) become:

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \tag{7.13}$$

where α, β and γ are any of the three integers 0,1,2,3 (four equations).

The Lorentz force equation and the equation for the rate of energy change for a particle with rest mass m and charge q when written in covariant form are:

$$\frac{dp^\alpha}{d\tau} = m\frac{dU^\alpha}{d\tau} = qF^{\alpha\beta}U_\beta \tag{7.14}$$

where $\mathbf{p}$ and $\mathbf{U}$ are the 4-momentum and 4-velocity and τ is the particle's proper time.

§7.5 The Lorenz Gauge

∂_α transforms as a 4-vector, and $\mathbf{A}$ is a 4-vector, so their inner product must be an *invariant*:

$$\partial_\alpha A^\alpha = \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\phi}{c}\right) + \boldsymbol{\nabla}\cdot\boldsymbol{A} = \text{invariant}$$

– the *Lorenz Gauge* condition

$$\frac{1}{c^2}\frac{\partial\phi}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$$

chosen earlier to simplify Maxwell's equations *is itself Lorentz invariant* and can be applied in all inertial frames.

§7.6 Transformations of $\boldsymbol{E}$ and $\boldsymbol{B}$

For practical calculations it is useful to know how the electric and magnetic fields transform between inertial frames. This can be obtained directly from the general transformation properties of the contravariant field-strength tensor:

$$F'^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\gamma}\frac{\partial x'^\beta}{\partial x^\delta}F^{\gamma\delta}$$

Using the Lorentz transformation we obtain:

$$\boxed{\begin{aligned} E'_x &= E_x & B'_x &= B_x \\ E'_y &= \gamma\left(E_y - vB_z\right) & B'_y &= \gamma\left(B_y + \frac{v}{c^2}E_z\right) \\ E'_z &= \gamma\left(E_z + vB_y\right) & B'_z &= \gamma\left(B_z - \frac{v}{c^2}E_y\right) \end{aligned}} \tag{7.15}$$

In terms of the components of $\boldsymbol{E}$ and $\boldsymbol{B}$ parallel and perpendicular to the velocity $\boldsymbol{u}$ indicating the transformation between the two frames:

$$\boldsymbol{E}'_\parallel = \boldsymbol{E}_\parallel \qquad \boldsymbol{E}'_\perp = \gamma\left[\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}\right]_\perp \tag{7.16}$$

$$\boldsymbol{B}'_\parallel = \boldsymbol{B}_\parallel \qquad \boldsymbol{B}'_\perp = \gamma\left[\boldsymbol{B} - \frac{1}{c^2}(\boldsymbol{v}\times\boldsymbol{E})\right]_\perp \tag{7.17}$$

The transformations between $\boldsymbol{B}$ and $\boldsymbol{E}$ are very revealing. They suggest that the *magnetic and electric fields* are *two aspects of the same phenomenon* which just depend on the frame in which the phenomenon is observed.

This is perhaps most apparent in the similarity between the transformations in vector form (Eqn 7.16) and the familiar form of the *Lorentz force* on a particle moving in $\boldsymbol{E}$ and $\boldsymbol{B}$ fields. See §7.8 ahead.

§7.7 Covariance of Maxwell's Equations II - FOR UNBELIEVERS ONLY !!

Maxwell's Equations have already been shown to be *covariant* (preserve their form) under relativistic transformations. The same physical result can be shown *explicitly*, using the transformation laws for the fields derived in §7.6.

Consider ME4 expressed in frame S:

$$\boldsymbol{\nabla} \times \boldsymbol{B} - \epsilon_0\mu_0 \frac{\partial \boldsymbol{E}}{\partial t} = \mu_0 \boldsymbol{J}$$

To be covariant, this must have exactly the same form in some other frame S′.

Using the transformation laws for the fields, currents and the 4-gradient operator, this is shown below to hold for the x' component of ME4. It also holds for all the other components, and for the other Maxwell Equations, which are therefore explicitly covariant. The tedious algebra emphasizes the power of the 4-vector approach, which demonstrated the same point more elegantly.

$$\begin{aligned}
\left(\boldsymbol{\nabla}' \times \boldsymbol{B}' - \epsilon_0\mu_0 \frac{\partial \boldsymbol{E}'}{\partial t'}\right)_{x'} &= \frac{\partial B'_z}{\partial y'} - \frac{\partial B'_y}{\partial z'} - \frac{1}{c^2}\frac{\partial E'_x}{\partial t'} \\
&= \frac{\partial}{\partial y}\left(\gamma B_z - \frac{\gamma v}{c^2}E_y\right) - \frac{\partial}{\partial z}\left(\gamma B_y + \frac{\gamma v}{c^2}E_z\right) \\
&\quad -\frac{\gamma}{c^2}\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)E_x \\
&= \gamma\left\{\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} - \frac{v}{c^2}\left(\frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} + \frac{\partial E_x}{\partial x}\right)\right. \\
&\qquad\left. - \frac{1}{c^2}\frac{\partial E_x}{\partial t}\right\} \\
&= \gamma\left\{\left(\boldsymbol{\nabla} \times \boldsymbol{B} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}\right)_x - \frac{v}{c^2}\boldsymbol{\nabla}\cdot\boldsymbol{E}\right\} \\
&= \gamma\left(\mu_0 J_x - \frac{v}{c^2}\frac{\rho}{\epsilon_0}\right) \\
&= \mu_0\gamma\left(J_x - v\rho\right) \\
&= \mu_0 J'_x \qquad \text{Q.E.D.}
\end{aligned}$$

§7.8 Magnetism as a Relativistic Effect

$\boldsymbol{E}$ and $\boldsymbol{B}$ are connected through Maxwell's Equations, but their transformation laws suggest they are very deeply linked indeed.

Consider a *neutral* current-carrying wire and a charge q moving parallel to it at a velocity u – a trivial problem to deal with in EM, but worth closer study using SR. Take S as the rest frame of the wire, and S′ as the rest frame of the carriers in the wire moving with velocity v w.r.t. S.

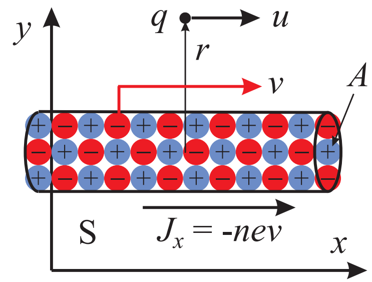

Figure 7.2: (a) A charge q moves parallel to a current-carrying wire with speed u.

Now consider *separately* the forces on q arising from the positive background charge and the negative carriers:

In S, the stationary positive background charge in the wire is $\rho_+ = +ne$, and using Gauss's Law produces a force on q given by

$$f_y^+ = \frac{qneA}{2\pi r\epsilon_0} \tag{7.18}$$

In S, the *charge neutrality* of the wire requires that the charge density of the moving negative carriers is $\rho_- = -ne$.

But in S′ the carriers are stationary, so there is *no Lorentz contraction* as in S, and the charge density is $\rho'_- = -ne/\gamma_v$.

In S′ this produces an electrostatic force on q given by

$$(f_y^-)' = -\frac{qneA}{\gamma_v 2\pi r\epsilon_0} \tag{7.19}$$

Using the Lorentz transformation for forces $f'_y = \frac{f_y}{\gamma_v(1-vu_x/c^2)}$ (with $u_x \to u$ for the charge q) to transform this to frame S:

$$f_y^- = -\frac{qneA}{\gamma_v 2\pi r\epsilon_0} \times \gamma_v \left(1 - \frac{vu}{c^2}\right) \tag{7.20}$$

So in S, the total force on the charge is

$$\begin{aligned} f_y = f_y^+ + f_y^- &= \frac{qneA}{2\pi r\epsilon_0} - \frac{qneA}{2\pi r\epsilon_0}\left(1 - \frac{vu}{c^2}\right) \\ &= \frac{qnevA}{2\pi r c^2\epsilon_0} u \\ &= \frac{\mu_0(-I)}{2\pi r} qu \end{aligned} \tag{7.21}$$

where I is the current in the wire – exactly the result expected from the expected $\boldsymbol{B}$ field (Ampère's Law) and the Lorentz force.

So, electrostatic forces, transformed between frames according to SR dynamics, give rise to the Lorentz force, more familiarly described in terms of the motion of a charge through a magnetic field.

B AND ***E*** CAN BE REGARDED AS DIFFERENT **MANIFESTATIONS** OF THE SAME UNDERLYING PHYSICS, **FUNDAMENTALLY LINKED** THROUGH SR.

§7.8.1 Spin-orbit coupling

An important real example is the magnetic field experienced by an *electron* as it *orbits an atomic nucleus.*

Eqn 7.17:

$$\boldsymbol{B}'_\perp = \gamma \left[\boldsymbol{B} - \frac{1}{c^2}(\boldsymbol{u} \times \boldsymbol{E})\right]_\perp$$

With S the frame of the nucleus, and S′ that of the electron, $\boldsymbol{B} = 0$ and $\boldsymbol{E}$ is the radial electric field of the nucleus.

So, motion through the electrostatic field of the nucleus introduces a magnetic field in the electron frame, which acts upon the electron's intrinsic magnetic moment to produce *spin-orbit effects.*

This magnetic field experienced in the rest frame of the electron due to the electric field in the laboratory frame needs to be incorporated into the Hamiltonian of an electron in atoms or solids. If we write the electric field as $\boldsymbol{E} = -\frac{\boldsymbol{r}}{r}\frac{d\Phi}{dr}$ we obtain for this *spin-orbit interaction* H_{so} in the Hamiltonian:

$$H_{so} = -\frac{1}{2}\frac{ge}{2m}\boldsymbol{s} \cdot \boldsymbol{B'} = -\frac{ge}{4mc^2}\boldsymbol{s} \cdot (\boldsymbol{u} \times \boldsymbol{r})\frac{1}{r}\frac{d\Phi}{dr} = \frac{ge}{4m^2c^2}\boldsymbol{s} \cdot \boldsymbol{L}\frac{1}{r}\frac{d\Phi}{dr} \tag{7.22}$$

where g is the electron g-factor, $\boldsymbol{s}$ is the electron spin operator and $\boldsymbol{L}$ is its orbital angular momentum. Here we have neglected terms on the order of v^2/c^2, i.e. $\gamma \approx 1$. The correction factor of 1/2 included arises because of an effect called Thomas precession, which is caused by the rest frame of the electron rotating with respect to the laboratory frame (for a discussion see Jackson, Chapter 11).

The spin-orbit interaction has well known consequences for the electronic structure of atoms and solids, for example, Hund's rules in atomic physics.

§8 Radiation and Relativistic Electrodynamics

– the interaction between EM fields and *moving charged particles.*

§8.1 A Uniformly Moving Charge

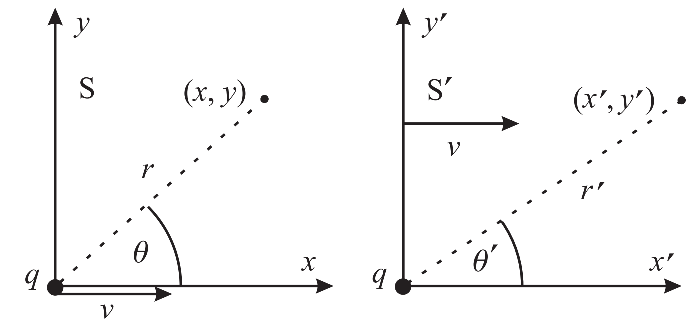

Figure 8.1: q is at the origin for both S and S′ at $t = t' = 0$, and is at rest in S′ and has velocity v in S.

At the point $(x', y', z' = 0)$ in S$'$ the fields are just the *static fields*:

$$\boldsymbol{E}' = \frac{q}{4\pi\epsilon_0}\left(\frac{x'}{r'^3}, \frac{y'}{r'^3}, 0\right) \qquad\qquad \boldsymbol{B}' = 0$$

Applying the Lorentz Transformations to these (p. 240) gives the fields in S:

$$E_x = E'_x = \frac{qx'}{4\pi\epsilon_0 r'^3} \qquad E_y = \gamma E'_y = \frac{\gamma q y'}{4\pi\epsilon_0 r'^3} \qquad E_z = 0$$

$$B_x = 0 \qquad B_y = 0 \qquad B_z = \frac{\gamma v}{c^2}E'_y$$

Taking the x-axis as the polar axis, $x = r\cos\theta$ and $y = r\sin\theta$, and with $x' = \gamma x$ and $y' = y$ at $t = t' = 0$:

$$\begin{aligned} r'^2 &= x'^2 + y'^2 = \gamma^2 r^2(1 - \sin^2\theta) + r^2\sin^2\theta \\ &= \gamma^2 r^2\left(1 - \frac{v^2}{c^2}\sin^2\theta\right) \end{aligned}$$

using $1 - \dfrac{1}{\gamma^2} = \dfrac{v^2}{c^2}$.

Therefore, with $\parallel$ and $\perp$ referring to the x axis, the direction of the charge's motion:

$$E_\parallel = \frac{q\cos\theta}{4\pi\epsilon_0\gamma^2 r^2\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{\frac{3}{2}}} \qquad E_\perp = \frac{q\sin\theta}{4\pi\epsilon_0\gamma^2 r^2\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{\frac{3}{2}}} \qquad (8.1)$$

$$B_r = B_\theta = 0 \qquad\qquad B_\phi = \frac{\gamma v}{c^2}E'_y = \frac{v}{c^2}E_\perp \qquad (8.2)$$

Note:

- There is now a *magnetic field* B_ϕ in the *azimuthal* direction (around Ox), arising from *motion* of charge.
- The electric field $\boldsymbol{E}$ has the usual $1/r^2$ dependence.
- Therefore the Poynting flux $\boldsymbol{N} \sim 1/r^4$: there is *no far-field radiation* from a *uniformly moving* charge (in vacuum).
- $\dfrac{E_\perp}{E_\parallel} = \dfrac{\sin\theta}{\cos\theta}$, so $\boldsymbol{E}$ is *purely radial*, although its magnitude has an angular dependence that depends on the Lorentz factor γ.

This last point is worth some further thought....

$\dfrac{E_\perp}{E_\parallel} = \dfrac{\sin\theta}{\cos\theta}$ so the field is directed away from the origin which is the *present* position of the charge in S.

This is a little surprising at first sight, since retardation means it should depend on where the charge was at *some earlier time*. But the field components are Lorentz transformed, and this produces the somewhat counter-intuitive result.

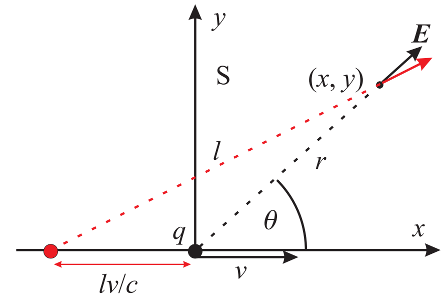

Figure 8.2: $\boldsymbol{E}$ at $\boldsymbol{r}$ is determined by the present ($t = 0$) position of the uniformly moving charge q.

How is the magnitude of the $\boldsymbol{E}$-field affected by the motion? From Eqn (8.1):

- For $\gamma \sim 1$: $\boldsymbol{E}$ has the usual isotropy, and $\boldsymbol{B}$ is azimuthal and small.
- For $\gamma \gg 1$: Near $\theta = 0$ or π: $E_\parallel \sim \dfrac{1}{\gamma^2}\dfrac{q}{4\pi\epsilon_0 r^2}$ i.e. very small
- For $\gamma \gg 1$: Near $\theta = \pi/2$:
  $E_\perp \sim \dfrac{1}{\gamma^2}\dfrac{q}{4\pi\epsilon_0 r^2(1 - v^2/c^2)^{\frac{3}{2}}} = \dfrac{\gamma q}{4\pi\epsilon_0 r^2}$ i.e. very large.

– the *motion flattens the* $\boldsymbol{E}$*-field into the* zy*-plane*:

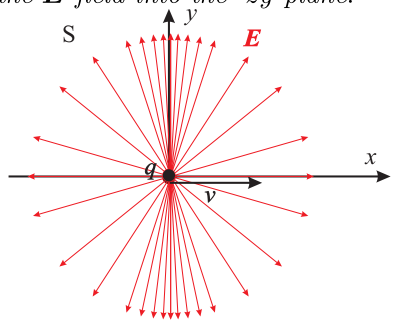

Figure 8.3: $\boldsymbol{E}$ is concentrated into the plane perpendicular to the charge's motion.

Also, $B_\phi = \dfrac{v}{c^2} E_\perp$, so overall:

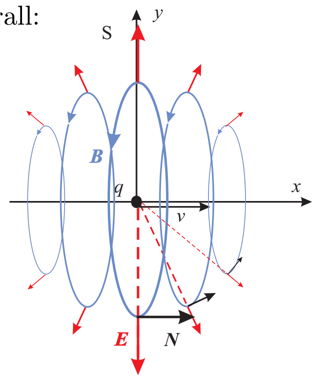

Figure 8.4: $\boldsymbol{E}$, $\boldsymbol{B}$ and $\boldsymbol{N}$ for a uniformly moving charge.

Note that the Poynting vector $\boldsymbol{N}$ is *tangentially oriented* as shown, and is maximum for $\theta = \pi/2$ and zero for $\theta = 0$.

From above, $\boldsymbol{N} \sim 1/r^4$, and does not give rise to radiation. A charge *moving uniformly* (in a vacuum) *does not radiate*. $\boldsymbol{N}$ simply moves the field energy along with the charge itself.

§8.2 Čerenkov Radiation

It was implicitly assumed in §8.1 that the charge is moving in vacuum. But what if it is moving through a *medium*?

Suppose $\epsilon = \epsilon(\omega)$ and $\mu = 1$, so $n = \sqrt{\epsilon(\omega)}$. Now the velocity of the charge can in principle *exceed the speed of light* in the medium: i.e. possibly:

$$\boxed{v > \frac{c}{n} \qquad \text{the } \textbf{\textit{Čerenkov}} \textit{ condition}} \tag{8.3}$$

This problem is not easily addressed as was for the case of a vacuum, since the *medium is moving in the rest frame S′ of the charge*, so the fields in S′ are not simple to obtain.

Some insights can be obtained from the Maxwell Equations expressed in terms of the potentials.

In the presence of a *medium*:

$$\frac{\epsilon\mu}{c^2}\ddot{\phi} - \nabla^2\phi = \frac{\rho}{\epsilon\epsilon_0} \qquad \text{MA1} \tag{8.4}$$

$$\frac{\epsilon\mu}{c^2}\ddot{\boldsymbol{A}} - \nabla^2\boldsymbol{A} = \mu\mu_0\boldsymbol{J} \qquad \text{MA2} \tag{8.5}$$

for which the solutions are:

$$\phi(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon\epsilon_0}\int_{\text{all space}} \frac{[\rho]\,\mathrm{d}V'}{|\boldsymbol{r} - \boldsymbol{r}'|} \tag{8.6}$$

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu\mu_0}{4\pi}\int_{\text{all space}} \frac{[\boldsymbol{J}]\,\mathrm{d}V'}{|\boldsymbol{r} - \boldsymbol{r}'|} \tag{8.7}$$

where [ ] denotes *retarded* time which is now $t - \dfrac{|\boldsymbol{r} - \boldsymbol{r}'|}{c/n}$.

The retarded time is determined by the *speed of light in the medium*, c/n.

A charged particle moving at a speed $v > c/n$ is *moving faster than information about its presence propagates* in the medium.

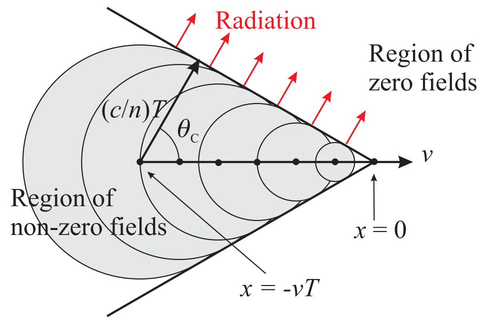

Figure 8.5: The spherical regions centred on earlier positions of the particle determine the region in which the potentials and fields are non-zero.

Suppose that at $t = 0$ the charge is at $x = 0$. At $t = -T$ it was at $x = -vT$, and at time $t = 0$ its presence at $x = -vT$ can have influenced only the region of space within a sphere of radius $(c/n)T$ centred on the point $x = -vT$.

So there is a region where the potentials (and therefore the fields) are *zero*.

The region of space where the potentials (and fields) are non-zero is obtained by considering spheres that expand at a speed c/n from points at which the particle has been. Fig. 8.5 shows the result of this construction for a particle moving at a uniform velocity $v > c/n$ – a *cone* around its current ($t = 0$) position.

Because the region of non-zero field is growing with time, there must be energy transfer into the region of zero-field – *radiation*. What is the direction of this radiation?

In simple physical terms, use the Huygens construction and draw the tangent to the wavelets at $t = 0$ emanating from points on the particle's track at earlier times.

The wavelet centred on $(x = -vT, t = -T)$ has radius cT/n, and the wavelet from $(x = 0, t = 0)$ has zero radius. So the tangent is as shown in Fig. 8.5, which is also tangent to all the wavelets from intermediate times.

Radiation is other directions is suppressed due to destructive interference between wavelets in these directions.

So clearly the radiation is propagating in the direction shown, with

$$\boxed{\cos\theta_{\mathrm{C}} = \frac{(cT/n)}{vT} = \frac{c}{nv} = \frac{1}{\beta n}} \tag{8.8}$$

where θ_{C} is the Čerenkov angle for the emitted radiation.

Note that it has been assumed that the rate of energy loss of the particle through radiation is small so that its *speed* can be taken to be *constant*.

Also, in principle $n = n(\omega)$, so that Čerenkov radiation of different frequencies will be emitted at *different angles*.

This very simplistic approach offers *no information on the spectral character* of the radiation and also does not make clear what the nature of the energy flow associated with the radiation is.

In the following section (non-examinable) we will attempt to understand how Čerenkov radiation is generated more quantitatively by calculating explicitly the electric and magnetic fields associated with a relativistic particle with charge q moving with velocity $\boldsymbol{v}$ through a medium with dielectric function $\epsilon(\omega)$. We will see that the Čerenkov radiation is related to the energy lost by the particle into regions of the dielectric medium that are far away from its path. We will solve MA1 and MA2 using Fourier transforms of the form:

$$F(\boldsymbol{r}, t) = \frac{1}{(2\pi)^4} \int\int \tilde{F}(\boldsymbol{k}, \omega)\, \mathrm{e}^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \mathrm{d}\boldsymbol{k}\, \mathrm{d}\omega$$

If we look for plane wave solutions of the form

$$\phi = \tilde{\phi}(\boldsymbol{k}, \omega)\mathrm{e}^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \qquad \boldsymbol{A} = \tilde{\boldsymbol{A}}(\boldsymbol{k}, \omega)\mathrm{e}^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}$$

arising from the source terms:

$$\rho(\boldsymbol{r}, t) = q\, \delta(\boldsymbol{r} - \boldsymbol{v}t) \quad \longrightarrow \quad \tilde{\rho}(\boldsymbol{k}, \omega) = q\, \delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v}) \tag{8.9}$$

$$\boldsymbol{J}(\boldsymbol{r}, t) = q\boldsymbol{v}\, \delta(\boldsymbol{r} - \boldsymbol{v}t) \quad \longrightarrow \quad \tilde{\boldsymbol{J}}(\boldsymbol{k}, \omega) = q\boldsymbol{v}\, \delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v}) \tag{8.10}$$

FT-ing MA1 and MA2 (with $\mu = 1$): $\ddot{\phi} \to -\omega^2\phi$, $\nabla^2\phi \to -k^2\phi$ etc.:

$$\left(k^2 - \epsilon(\omega)\frac{\omega^2}{c^2}\right)\tilde{\phi}(\boldsymbol{k}, \omega) = \frac{1}{\epsilon(\omega)\epsilon_0}\tilde{\rho}(\boldsymbol{k}, \omega)$$

$$\left(k^2 - \epsilon(\omega)\frac{\omega^2}{c^2}\right)\tilde{\boldsymbol{A}}(\boldsymbol{k}, \omega) = \mu_0\tilde{\boldsymbol{J}}(\boldsymbol{k}, \omega)$$

$$\tilde{\phi}(\boldsymbol{k}, \omega) = \frac{q}{\epsilon(\omega)\epsilon_0}\frac{\delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})}{\left(k^2 - \epsilon(\omega)\frac{\omega^2}{c^2}\right)} \tag{8.11}$$

$$\tilde{\boldsymbol{A}}(\boldsymbol{k}, \omega) = \mu_0 q\boldsymbol{v}\frac{\delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})}{\left(k^2 - \epsilon(\omega)\frac{\omega^2}{c^2}\right)} = \epsilon(\omega)\frac{\boldsymbol{v}}{c^2}\,\tilde{\phi}(\boldsymbol{k}, \omega) \tag{8.12}$$

The fields can then be obtained from $\boldsymbol{E} = -\nabla\phi - \dot{\boldsymbol{A}} \to \tilde{\boldsymbol{E}} = -i\boldsymbol{k}\tilde{\phi} + i\omega\tilde{\boldsymbol{A}}$ and $\boldsymbol{B} = \nabla\times\boldsymbol{A} \to \tilde{\boldsymbol{B}} = i\boldsymbol{k}\times\tilde{\boldsymbol{A}}$:

$$\tilde{\boldsymbol{E}}(\boldsymbol{k}, \omega) = i\left(\frac{\omega\epsilon(\omega)}{c}\frac{\boldsymbol{v}}{c} - \boldsymbol{k}\right)\,\tilde{\phi}(\boldsymbol{k}, \omega) \tag{8.13}$$

$$\tilde{\boldsymbol{B}}(\boldsymbol{k}, \omega) = i\left(\frac{\epsilon(\omega)}{c}\boldsymbol{k}\times\frac{\boldsymbol{v}}{c}\right)\,\tilde{\phi}(\boldsymbol{k}, \omega) \tag{8.14}$$

Let us now consider the flow of the electromagnetic energy through a cylinder of radius a around the path of the incident particle. By conservation of energy this is the energy lost per unit time by the incident particle:

$$(\frac{dE}{dx})_{b>a} = \frac{1}{v}\frac{dE}{dt} = -\frac{1}{\mu_0 v}\int_{-\infty}^{+\infty} 2\pi a B_3 E_1 dx \tag{8.15}$$

where E_1 is the electric field in the direction of motion of the particle and B_3 is the magnetic field in the azimuthal direction. The integral over dx at one instant in time must be equal to the integral over all time at one point on the cylinder (chosen to be at $x = 0$):

$$(\frac{dE}{dx})_{b>a} = -\frac{2\pi a}{\mu_0}\int_{-\infty}^{+\infty} B_3(t)E_1(t)dt \tag{8.16}$$

The fields are evaluated on the surface of a cylinder with radius $r = a$. This integral effectively captures the total energy lost by the incident particle in collisions with electrons of the dielectric medium that have an impact parameter $b > a$.

Using the expression for the Fourier transforms of the fields this can be converted into:

$$(\frac{dE}{dx})_{b>a} = -\frac{2\pi a}{\mu_0}\frac{1}{(2\pi)^4}\int_{-\infty}^{+\infty} dt \int d^3k \int d\omega B_3(\boldsymbol{k},\omega)e^{iak_2 - i\omega t}.$$

$$\int d^3k' \int d\omega' E_1(\boldsymbol{k'},\omega')e^{iak'_2 - i\omega' t} \tag{8.17}$$

Performing first the integration over t this can be expressed in the following form

$$(\frac{dE}{dx})_{b>a} = -\frac{8\pi a}{\mu_0}\Re\int_0^{+\infty} d\omega E_1(\omega)B_3^*(\omega) \tag{8.18}$$

where

$$E_1(\omega) = \frac{1}{(2\pi)^{3/2}}\int d^3k E_1(\boldsymbol{k},\omega)e^{iak_2} \tag{8.19}$$

$$B_3(\omega) = \frac{1}{(2\pi)^{3/2}}\int d^3k B_3(\boldsymbol{k},\omega)e^{iak_2} \tag{8.20}$$

We now need to calculate these integrals:

$$E_1(\omega) = \frac{iq}{\epsilon(\omega)\epsilon_0(2\pi)^{3/2}}\int d^3k e^{iak_2}[\frac{\omega\epsilon(\omega)v}{c^2} - k_1]\frac{\delta(\omega - vk_1)}{k^2 - \frac{\omega^2}{c^2}\epsilon(\omega)}$$

Performing first the integration over k_1 we obtain:

$$E_1(\omega) = -\frac{iq\omega}{(2\pi)^{3/2}\epsilon_0 v^2}[\frac{1}{\epsilon(\omega)} - \beta^2]\int_{-\infty}^{\infty} dk_2 e^{iak_2}\int_{-\infty}^{\infty}\frac{dk_3}{k_2^2 + k_3^2 + \kappa^2}$$

where $\kappa^2 = \frac{\omega^2}{v^2} - \frac{\omega^2}{c^2}\epsilon(\omega) = \frac{\omega^2}{v^2}[1 - \beta^2\epsilon(\omega)]$. The integral over dk_3 has the value $\pi/\sqrt{\kappa^2 + k_2^2}$.

$$\begin{aligned} E_1(\omega) &= -\frac{iq\omega}{2\sqrt{2\pi}\epsilon_0 v^2}[\frac{1}{\epsilon(\omega)} - \beta^2]\int_{-\infty}^{\infty} dk_2\frac{e^{iak_2}}{\sqrt{\kappa^2 + k_2^2}} \\ &= -\frac{iq\omega}{\epsilon_0 v^2}\sqrt{\frac{1}{2\pi}}[\frac{1}{\epsilon(\omega)} - \beta^2]K_0(\kappa a) \end{aligned} \tag{8.21}$$

where $K_0(\kappa a)$ is a modified Bessel function (of the second kind).

A similar calculation yields:

$$B_3(\omega) = \frac{q}{2\epsilon_0 c^2}\sqrt{\frac{2}{\pi}}\kappa K_1(\kappa a) \tag{8.22}$$

We now consider the limit $|\kappa a| \gg 1$ which is relevant if one is interested only in the energy deposited in the dielectric medium far away from the particle's track. The modified Bessel functions can then be represented in their asymptotic form.

$$E_1(\omega) = \frac{iq\omega}{2\epsilon_0 c^2}[1 - \frac{1}{\beta^2\epsilon(\omega)}]\frac{e^{-\kappa a}}{\sqrt{\kappa a}} \tag{8.23}$$

$$B_3(\omega) = \frac{q}{2\epsilon_0 c^2}\sqrt{\frac{\kappa}{a}}e^{-\kappa a} \tag{8.24}$$

$$(\frac{dE}{dx})_{b>a} = \frac{2\pi q^2}{\mu_0\epsilon_0^2 c^4}\Re\int_0^{+\infty} d\omega\ i\sqrt{\frac{\kappa^*}{\kappa}}\omega[1 - \frac{1}{\beta^2\epsilon(\omega)}]e^{-(\kappa+\kappa^*)a} \tag{8.25}$$

If κ has a positive real part, as is generally true, the exponential factor will cause the energy deposited outside the cylinder to decrease exponentially, i.e. all energy is deposited near the particle's path. However, if κ is purely imaginary, the exponential is unity and some energy escapes to infinity as radiation. This can occur if $\epsilon(\omega)$ is real and $\beta^2\epsilon(\omega) > 1$, i.e. $v > \frac{c}{\sqrt{\epsilon(\omega)}}$. This provides the same criterion for the occurence of Čerenkov radiation as the Huygens wavelet construction above.

In this case $\sqrt{\kappa^*/\kappa} = i$ and

$$(\frac{dE}{dx})_{b>a} = \frac{2\pi q^2}{\mu_0\epsilon_0^2 c^4}\int_{\epsilon(\omega)>1/\beta^2} d\omega \;\; \omega[1 - \frac{1}{\beta^2\epsilon(\omega)}] \tag{8.26}$$

This expression was derived in 1937 by Frank and Tamm to explain the radiation observed by Čerenkov in 1934. The Čerenkov radiation is not emitted uniformly in frequency, but in frequency bands in which $\epsilon(\omega) > \beta^{-2}$ (see Fig 8.6).

The spectral character of the emitted radiation can be understood by considering the behaviour of the dielectric function in the vicinity of a resonance at frequency ω_0.

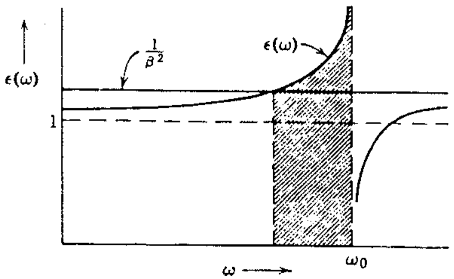

Figure 8.6: Frequency band of Čerenkov radiation: The radiation is only emitted in the shaded frequency region in which $\epsilon(\omega) > 1/\beta^2$

This explains qualitatively why Čerenkov radiation often appears as a blueish glow.

Čerenkov radiation is observed in:

(i) *Particle showers* as cosmic rays enter the atmosphere.

(ii) The bluish glow surrounding *water-cooled nuclear reactors*.

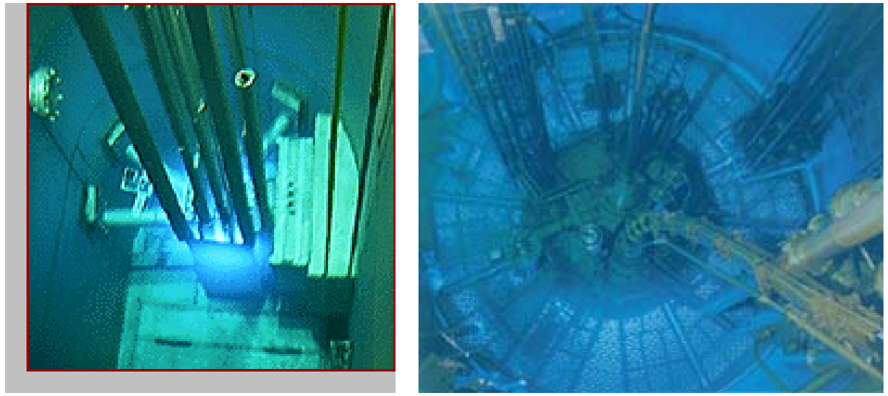

Figure 8.7: The blue glow – Čerenkov radiation – due to particles emitted from nuclear reactors passing through water at superluminal speeds.

(iii) Čerenkov *particle detectors* – tanks of liquid surrounded by photodetectors.

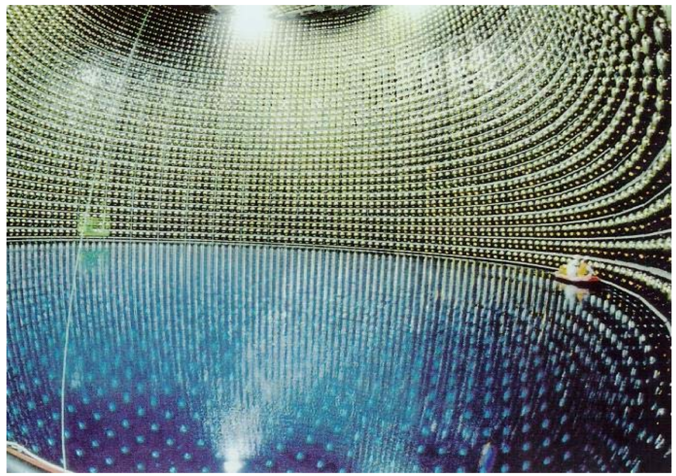

Figure 8.8: The Super-Kamiokande particle detector consists of a water bath surrounded by 11,000 photomultipliers.

§8.3 An Accelerated Charge

A uniformly moving charge does not radiate (except under "Čerenkov conditions").

But consider a *charged particle accelerating or decelerating* in vacuum. Fig. 8.9 shows how the field lines develop for a charge initially in steady motion with a speed $v = 0.8c$ to the right is *decelerated* and comes to rest.

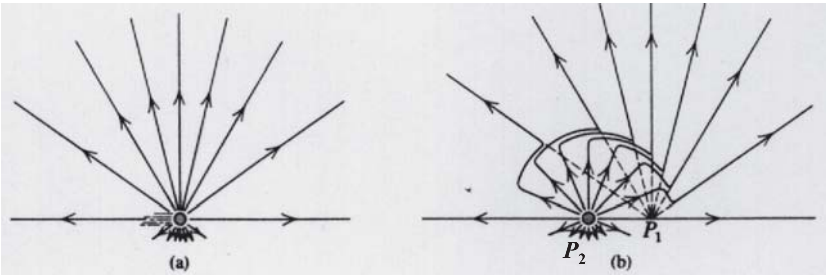

Figure 8.9: (a) The field lines of a uniformly moving charge emanate from its present position in S (§8.1). (b) The effect of a sudden deceleration of the charge; $\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0$ so the lines of force are *continuous*.

When the charge is suddenly decelerated, for a *distant* observer ("$r > ct$") the field still emanates from P_1, the point the charge *would have moved to* had it continued in uniform motion. But for a near observer ("$r < ct$") the field would emanate from the true position P_2.

So the field develops a *transverse kink* that propagates out at the speed of light, corresponding to a *radiative electromagnetic field.*

For a charge of *opposite sign* suddenly *accelerating*, a similar kink forms, with the field in the kink in the same direction as in Fig. 8.9.

So, unlike a charged particle moving uniformly in vacuum,

an acc(dec)elerating charged particle RADIATES

If the particle is oscillating, the period $T \sim |\boldsymbol{u}|/|\boldsymbol{a}|$ sets the typical *wavelength* of the radiation that is emitted, $\lambda \sim cT \sim c|\boldsymbol{u}|/|\boldsymbol{a}|$. For more general motion, $|\boldsymbol{u}|/|\boldsymbol{a}|$ determines some *characteristic time* which sets the typical wavelength of the radiation, $\lambda \sim c|\boldsymbol{u}|/|\boldsymbol{a}|$.

§8.4 Liénart-Wiechert Potentials

In §3.5 we derived expressions for the solutions of MA1 and MA2:

$$\phi(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon_0} \int_{\text{all space}} \frac{\rho\left(\boldsymbol{r}', t - \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right) \mathrm{d}V'}{|\boldsymbol{r} - \boldsymbol{r}'|}$$

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} \int_{\text{all space}} \frac{\boldsymbol{J}\left(\boldsymbol{r}', t - \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right) \mathrm{d}V'}{|\boldsymbol{r} - \boldsymbol{r}'|}$$

The condition for evaluating the charge/current density at the retarded time can also be expressed in the following covariant form with $\mathbf{x} = (ct, \boldsymbol{r})$:

$$A^\alpha(\mathbf{x}) = \frac{\mu_0}{4\pi} \int d^4x' \frac{\theta(x_0 - x_0')}{|\boldsymbol{r} - \boldsymbol{r}'|} \delta(x_0 - x_0' - |\boldsymbol{r} - \boldsymbol{r}'|) J^\alpha(\mathbf{x}') \tag{8.27}$$

The theta function is included to ensure that the source-point time x_0' is always earlier than the observation-point time x_0. This expression can be interpreted in terms of the Green function $\frac{1}{4\pi} \frac{\theta(x_0 - x_0')}{|\boldsymbol{r}-\boldsymbol{r}'|} \delta(x_0 - x_0' - |\boldsymbol{r} - \boldsymbol{r}'|)$ of the wave equation (see Jackson, chapter 12).

Recalling that $\delta(\alpha x) = \delta(x)/|\alpha|$ we have

$$\begin{aligned} \delta[(\mathbf{x} - \mathbf{x}')^2] &= \delta[(x_0 - x_0')^2 - |\boldsymbol{r} - \boldsymbol{r}'|^2] \\ &= \delta[(x_0 - x_0' - |\boldsymbol{r} - \boldsymbol{r}'|)(x_0 - x_0' + |\boldsymbol{r} - \boldsymbol{r}'|)] \\ &= \frac{1}{2|\boldsymbol{r} - \boldsymbol{r}'|} [\delta(x_0 - x_0' - |\boldsymbol{r} - \boldsymbol{r}'|) + \delta(x_0 - x_0' + |\boldsymbol{r} - \boldsymbol{r}'|)] \end{aligned}$$

therefore we can write

$$A^\alpha(\mathbf{x}) = \frac{\mu_0}{2\pi} \int d^4x' \theta(x_0 - x_0') \delta[(\mathbf{x} - \mathbf{x}')^2] J^\alpha(\mathbf{x}') \tag{8.28}$$

We now consider the motion of a point particle carrying a charge e:

$$J^\alpha(\mathbf{x}') = ec \int d\tau \quad V^\alpha(\tau) \delta^4[\mathbf{x}' - \mathbf{r}(\tau)] \tag{8.29}$$

where $V^\alpha(\tau) = (\gamma c, \gamma \boldsymbol{v})$ is the 4-velocity of the charge and $\mathbf{r}(\tau)$ is its position as a function of the proper time τ.

By performing the integration over d^4x' in the expression for A^α we obtain:

$$A^\alpha(\mathbf{x}) = \frac{\mu_0 e c}{2\pi}\int d\tau \quad V^\alpha(\tau)\theta[x_0 - r_0(\tau)]\delta[(\mathbf{x}-\mathbf{r}(\tau))^2] \tag{8.30}$$

The remaining integral only gives a contribution at $\tau = \tau_0$ where τ_0 is defined by the light-cone condition $[\mathbf{x}-\mathbf{r}(\tau_0)]^2 = 0$.

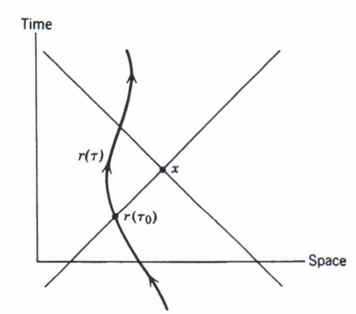

Figure 8.10: Wordline of the charged particle as a function of τ drawn together with the lightcone around the observation point $\mathbf{x}$.

Using $\delta[f(x)] = \sum_i \frac{\delta(x-x_i)}{|(df/dx_i)_{x=x_i}|}$ with $d/d\tau[\mathbf{x}-\mathbf{r}(\tau)]^2 = -2[\mathbf{x}-\mathbf{r}(\tau)]_\beta V^\beta(\tau)$ we obtain for the 4-potential:

$$A^\alpha(\mathbf{x}) = \frac{\mu_0 e c}{4\pi}\left[\frac{V^\alpha(\tau)}{\mathbf{V}\cdot[\mathbf{x}-\mathbf{r}(\tau)]}\right]_{\tau=\tau_0} \tag{8.31}$$

These solutions are called the *Liénard-Wiechert potentials.* They can also be expressed in a non-covariant, but maybe practically more useful form. The light-cone condition implies $x_0 - r_0(\tau_0) = |\boldsymbol{x} - \boldsymbol{r}(\tau_0)| \equiv R$.

$$\begin{aligned}
\mathbf{V}\cdot(\mathbf{x}-\mathbf{r}) &= V_0[x_0 - r_0(\tau_0)] - \boldsymbol{V}\cdot[\boldsymbol{x}-\boldsymbol{r}(\tau_0)] \\
&= \gamma c R - \gamma \boldsymbol{v}\cdot\boldsymbol{n} R \\
&= \gamma c R(1-\boldsymbol{\beta}\cdot\boldsymbol{n})
\end{aligned}$$

where $\boldsymbol{n}$ is a unit vector in the direction of $\boldsymbol{x}-\boldsymbol{r}(\tau_0)$ and $\boldsymbol{\beta} = \boldsymbol{v}/c$. We obtain:

$$\Phi(\boldsymbol{x},t) = \frac{1}{4\pi\epsilon_0}\left[\frac{e}{(1-\boldsymbol{\beta}\cdot\boldsymbol{n})R}\right]_{ret} \qquad \boldsymbol{A}(\boldsymbol{x},t) = \frac{\mu_0}{4\pi}\left[\frac{e\boldsymbol{v}}{(1-\boldsymbol{\beta}\cdot\boldsymbol{n})R}\right]_{ret} \tag{8.32}$$

The electric and magnetic fields can now be calculated from these non-covariant expressions or by calculating the components of the field-strength tensor. The detailed calculation can be found in Jackson, chapter 14 (note the use of Gaussian, not SI units in Jackson); the resulting fields are:

$$\begin{aligned}
\boldsymbol{E}(\boldsymbol{x},t) &= \frac{e}{4\pi\epsilon_0}\left\{\left[\frac{\boldsymbol{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\boldsymbol{n})^3R^2}\right]_{ret} + \left[\frac{\boldsymbol{n}\times((\boldsymbol{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}})}{c(1-\boldsymbol{\beta}\cdot\boldsymbol{n})^3R}\right]_{ret}\right\} \\
\boldsymbol{B}(\boldsymbol{x},t) &= \frac{1}{c}[\boldsymbol{n}\times\boldsymbol{E}]_{ret}
\end{aligned} \tag{8.33}$$

The first term describes *velocity fields* that are independent of the acceleration $\dot{\boldsymbol{\beta}} = d\boldsymbol{\beta}/dt$. They are essentially static fields and drop off as R^{-2}. For a charge moving with uniform velocity this term can be shown to reproduce the electric field calculated in §8.1 using the Lorentz transformation from the rest frame of the charge.

The second term describes *acceleration fields* which are typical radiation fields and drop off as R^{-1}. This term will now be investigated in more detail.

§8.5 Larmor and Liénart Formulae

As a first application of the Lienart-Wiechert potentials let's consider the power radiated by an accelerated, but non-relativistic charge, that is moving with a velocity small compared to that of light. In this case the acceleration field in Eq. 8.33 can be simplified to:

$$\boldsymbol{E}_a = \frac{e}{4\pi\epsilon_0 c}\left[\frac{\boldsymbol{n}\times(\boldsymbol{n}\times\dot{\boldsymbol{\beta}})}{R}\right]_{ret} \tag{8.34}$$

From the Poynting vector $\boldsymbol{S} = \frac{1}{\mu_0 c}|\boldsymbol{E}_a|^2\boldsymbol{n}$ we obtain the power radiated per unit solid angle:

$$\frac{dP}{d\Omega} = \frac{1}{\mu_0 c}|R\boldsymbol{E}_a|^2 = \frac{e^2}{16\pi^2\epsilon_0^2\mu_0 c^3}|\boldsymbol{n}\times(\boldsymbol{n}\times\dot{\boldsymbol{\beta}})|^2 = \frac{\mu_0 e^2}{16\pi^2 c}|\dot{\boldsymbol{v}}|^2\sin^2(\theta) \tag{8.35}$$

where θ is the angle between the acceleration $\dot{\boldsymbol{v}}$ and $\boldsymbol{n}$. The total radiated power is given by the *Larmor formula*:

$$P = \frac{\mu_0 e^2}{6\pi c}|\dot{\boldsymbol{v}}|^2 \tag{8.36}$$

This result can be interpreted as the radiation from a non-relativistic charge that experiences an acceleration $|\boldsymbol{a}| = |\dot{\boldsymbol{v}}| = \ddot{z}$ being equivalent to that of an electric dipole with dipole moment $p = ez$.

The corresponding result in a *general frame* for which the motion is relativistic can be obtained either from Eq. 8.33 (see below) or by noting that:

(i) the *total power radiated is a Lorentz Invariant*, as shown below:

Consider a small amount of energy $\mathrm{d}E'$ emitted in a time $\mathrm{d}t'$ in the IRF (S′) of the charge. ($\mathrm{d}t'$ is therefore the proper time.) The power loss in the IRF is then $P' \equiv \mathrm{d}E'/\mathrm{d}t'$. In the IRF, the radiation is emitted with a $\sin^2\theta$ distribution and carries no nett 3-momentum. But *4-momentum is lost*: $\mathrm{d}\mathbf{P'} = (\mathrm{d}E'/c, 0, 0, 0)$. Lorentz transforming $\mathbf{P'}$ to the frame S then gives $\mathrm{d}E = \gamma \mathrm{d}E'$. $\mathrm{d}t = \gamma \mathrm{d}t'$ as usual, so the power loss in S is

$$P \equiv \frac{\mathrm{d}E}{\mathrm{d}t} = \frac{(\gamma \mathrm{d}E')}{(\gamma \mathrm{d}t')} = P'$$

The *power loss is the same in all inertial frames.*

(ii) In SR the acceleration 4-vector is:

$$\mathbf{A_{cc}} = (\gamma_u c\dot{\gamma_u}, \gamma_u^2 \boldsymbol{a} + \gamma_u \dot{\gamma_u} \boldsymbol{u})$$

where $\boldsymbol{a} = \dfrac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}$ is the 3-acceleration in frame S.

Direct differentiation of $\gamma_u = 1/\sqrt{(1 - u^2/c^2)}$ gives $\dot{\gamma}_u = \gamma_u^3 \dfrac{\dot{u}u}{c^2}$, so

$$\mathbf{A_{cc}} = \left(\gamma_u^4 \frac{\dot{u}u}{c}, \gamma_u^4 \frac{\dot{u}u}{c^2} \boldsymbol{u} + \gamma_u^2 \boldsymbol{a} \right)$$

In the particle's IRF S′, $\boldsymbol{u} = 0$, $\gamma_u \to 1$, and $\boldsymbol{a} \to \boldsymbol{\alpha}$ say, so

$$\mathbf{A'_{cc}} \to (0, \boldsymbol{\alpha})$$

The inner product of $\mathbf{A_{cc}}$ with itself is a *Lorentz Invariant*, so:

$$\mathbf{A_{cc}} \cdot \mathbf{A_{cc}} = -\alpha^2 \quad \text{in } \textit{all} \text{ frames, including S} \tag{8.37}$$

Evaluating $\mathbf{A_{cc}} \cdot \mathbf{A_{cc}}$ in the general frame S gives

$$\begin{aligned} -\mathbf{A_{cc}} \cdot \mathbf{A_{cc}} = \alpha^2 &= -\left(\gamma_u^4 \frac{\dot{u}u}{c} \right)^2 + \left(\gamma_u^4 \frac{\dot{u}u}{c^2} \boldsymbol{u} + \gamma_u^2 \boldsymbol{a} \right)^2 \\ &= -\gamma_u^8 \frac{(\dot{u}u)^2}{c^2} + \gamma_u^8 \frac{(\dot{u}u)^2 u^2}{c^4} + \gamma_u^4 a^2 + 2\gamma_u^6 \frac{\dot{u}u}{c^2} \boldsymbol{a} \cdot \boldsymbol{u} \end{aligned} \tag{8.38}$$

Now using

$$\boldsymbol{a} \cdot \boldsymbol{u} = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{u} \cdot \boldsymbol{u}) = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} u^2 = \dot{u}u$$

gives

$$\begin{aligned} \alpha^2 &= -\gamma_u^8 \frac{(\dot{u}u)^2}{c^2} + \gamma_u^8 \frac{(\dot{u}u)^2 u^2}{c^4} + \gamma_u^4 a^2 + 2\gamma_u^6 \frac{(\dot{u}u)^2}{c^2} \\ &= -\frac{(\dot{u}u)^2}{c^2} \left[\gamma_u^8 - \gamma_u^8 \beta^2 - 2\gamma_u^6 \right] + \gamma_u^4 a^2 \\ &= \gamma_u^6 \frac{(\dot{u}u)^2}{c^2} + \gamma_u^4 a^2 \end{aligned}$$

Now writing $\boldsymbol{a} = \boldsymbol{a}_{\|} + \boldsymbol{a}_{\perp}$, the accelerations $\|$ and $\perp$ to the velocity $\boldsymbol{u}$:

$$\dot{u} = a_{\|} \qquad a^2 = a_{\|}^2 + a_{\perp}^2$$

so

$$\alpha^2 = \gamma^6 a_{\|}^2 + \gamma^4 a_{\perp}^2 \tag{8.39}$$

The *emitted power is a Lorentz Invariant*, so in frame S is the same as that given by the *Larmor Formula* (Eqn 8.36):

$$P = \frac{\mu_0 e^2 |\boldsymbol{\alpha}|^2}{6\pi c}$$

which, when expressing $\boldsymbol{\alpha}$ in terms of the acceleration $\boldsymbol{a}$ in frame S (Eqn 8.39) gives:

$$\boxed{P = \frac{\mu_0 e^2}{6\pi c} (\gamma^6 a_{\|}^2 + \gamma^4 a_{\perp}^2)} \tag{8.40}$$

– the relativistic Larmor Formula, or *Liénard Formula.*

§8.6 Circular Motion

§8.6.1 Motion in a $\boldsymbol{B}$-field

Now consider a particle with charge e and rest mass m moving with velocity $\boldsymbol{v}$ in a uniform magnetic field $\boldsymbol{B} = B\widehat{\boldsymbol{z}}$.

Its equation of motion is:

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} \equiv \boldsymbol{f} = e\boldsymbol{v} \times \boldsymbol{B}$$

Here $\boldsymbol{f} \perp \boldsymbol{v}$ so the particle's energy is "constant".

BUT it will emerge that the particle radiates energy, so it is effectively being assumed that the energy loss is small compared with its kinetic energy, or that its kinetic energy is artificially maintained, as in a particle accelerator.

Writing $\gamma_v \to \gamma$ since v is constant, and using the transverse mass γm since $\boldsymbol{f} \perp \boldsymbol{v}$ (as opposed to the longitudinal mass $\gamma^3 m$):

$$\boldsymbol{f} = \gamma m \boldsymbol{a}$$

and since $\boldsymbol{a} = \dot{\boldsymbol{v}}$ the equation of motion becomes:

$$\gamma m \dot{\boldsymbol{v}} = eB\boldsymbol{v} \times \widehat{\boldsymbol{z}} \tag{8.41}$$

Taking $v_z = 0$ this corresponds to *circular motion* in the xy-plane with angular frequency

$$\boxed{\omega_B = \frac{eB}{\gamma m}} \tag{8.42}$$

– the *gyrofrequency*. The orbital radius $R = u/\omega_B$.

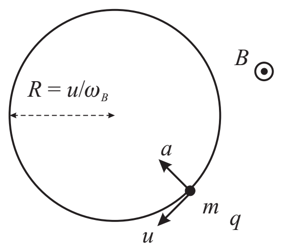

Figure 8.11: Relativistic circular motion in a magnetic field B.

The radial acceleration is:

$$a_\perp = \frac{evB}{\gamma m}$$

and substituting this into the Liénard formula (Eqn 8.40) gives, for the *instantaneous total power* radiated:

$$\boxed{P = \frac{\mu_0 e^4 \gamma^2 B^2 v^2}{6\pi c m^2}} \tag{8.43}$$

Note: In the frame S, the acceleration is circular and therefore oscillatory. But in the particle's IRF the acceleration is *continuous*.

The emitted EM radiation may *not necessarily* be thought of as *harmonic*. This will depend on the wavelength as compared with the orbital radius R, as shown below.

The general results above have *two important limits*: $v \ll c$ and $v \to c$.

§8.6.2 Cyclotron Radiation

For $v \ll c$, $\gamma \to 1$, and the motion is essentially classical, non-relativistic circular motion.

The wavelength of the radiation will be

$$\lambda = \frac{2\pi c}{\omega_B} \gg \frac{v}{\omega_B} = R$$

so the *dipole approximation* applies, and the circular motion can be described by the *superposition of two perpendicular Hertzian dipoles* oscillating in *quadrature*:

$$\boldsymbol{p}(t) = \widehat{\boldsymbol{x}} p_0 \cos\omega_B t + \widehat{\boldsymbol{y}} p_0 \sin\omega_B t$$

with $p_0 = eR$.

The radiation is therefore *monochromatic* at angular frequency ω_B, and the results derived earlier (§**??**) for the fields of the individual dipoles can be *superposed* for the present case.

Since the two dipoles here are $\pi/2$ out of phase, the *time-averaged Poynting fluxes add*, and the average power loss is (with $ev = |\dot{p}| = |\omega_B p_0|$):

$$\langle P \rangle = 2 \times \frac{\omega_B^4 p_0^2}{12\pi\epsilon_0 c^3} = \frac{\omega_B^2 e^2 v^2}{6\pi\epsilon_0 c^3} \tag{8.44}$$

The *polarization state* of the radiation depends on the *direction of emission*:

(i) in the *equatorial plane* (the plane of the circular orbit) only one component of the circular dipole is effective, so the radiation is *plane-polarized* in that plane.

(ii) in the *polar directions* both components contribute, equally in amplitude but $\pi/2$ out of phase, so the radiation is *circularly polarized.*

(iii) in other directions the radiation is *elliptically polarized.*

Note, again, the radiation is *monochromatic* at angular frequency ω_B. The situation is quite different for accelerated charges moving at *relativistic speeds* ....

§8.6.3 Synchrotron Radiation

For $v \longrightarrow c$, $\gamma \gg 1$, and the motion is highly-relativistic circular motion such as that occurring in high energy electron and proton *synchrotrons.*

As before, the acceleration $\boldsymbol{a}$ is perpendicular to the velocity $\boldsymbol{v}$, and in the IRF it is $\boldsymbol{\alpha}$, a *constant*, not oscillatory.

But here it turns out (see Eqn 8.50 below) that, in contrast with the cyclotron case, the characteristic wavelength emitted

$$\lambda_s \ll R = \frac{v \to c}{\omega_B}$$

so the rotating/oscillating dipole picture in S does not apply.

So the radiated fields are *not harmonically time-varying* as for the cyclotron case. Recall Fig. 8.9: there is a continuous shear of the lines of force. There is no "natural" frequency, but (see below) a *continuum* of frequencies centred on $\nu_s = \dfrac{c}{\lambda_s}$.

But *in the particle's IRF nothing has changed – the earlier results still apply.*

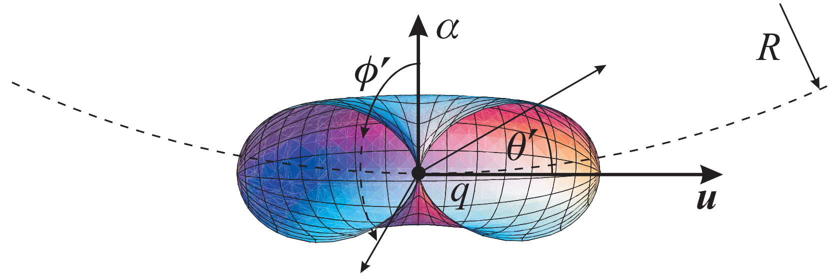

Figure 8.12: $G(\theta', \phi') \propto 1 - \sin^2\theta' \cos^2\phi'$ for radiation from a relativistic particle of charge e in its IRF S′.

The instantaneous radiated power is still given by the Larmor formula Eqn 8.36:

$$P = \frac{\mu_0 e^2 |\boldsymbol{\alpha}|^2}{6\pi c}$$

The angular distribution of the radiated power in the IRF S′ is

$$G(\theta', \phi') \propto 1 - \sin^2\theta' \cos^2\phi' \qquad \text{(formerly } \sin^2\theta) \tag{8.45}$$

where Oz is now taken to be the instantaneous direction of the velocity $\boldsymbol{u}$ of the particle (rather than, as earlier, its acceleration $\boldsymbol{\alpha}$) as shown above.

In the *laboratory frame S* the angular distribution of the radiated power for a relativistic particle can be calculated from Eqn 8.33. The radial component of the Poynting vector is:

$$[\boldsymbol{S} \cdot \boldsymbol{n}]_{ret} = \frac{e^2 \mu_0 c}{16\pi^2} \frac{1}{R^2} \left| \frac{\boldsymbol{n} \times ((\boldsymbol{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}})}{(1 - \boldsymbol{\beta} \cdot \boldsymbol{n})^3} \right|^2_{ret} \tag{8.46}$$

When calculating the energy radiated over a finite period of time it is convenient to calculate this with respect to the particle's own time:

$$E = \int_{t=T_1+[R(T_1)/c]}^{t=T_2+[R(T_2)/c]} dt\, [\boldsymbol{S} \cdot \boldsymbol{n}]_{ret} = \int_{t'=T_1}^{t'=T_2} dt' (\boldsymbol{S} \cdot \boldsymbol{n}) \frac{dt}{dt'}$$

We obtain for the power radiated per unit angle in terms of the charge's own time:

$$\begin{aligned}
\frac{dP(t')}{d\Omega} &= R^2(\boldsymbol{S}\cdot\boldsymbol{n})\frac{dt}{dt'} = R^2(\boldsymbol{S}\cdot\boldsymbol{n})(1-\boldsymbol{\beta}\cdot\boldsymbol{n}) \\
&= \frac{e^2\mu_0 c}{16\pi^2}\left[\frac{\left|\boldsymbol{n}\times((\boldsymbol{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}})\right|^2}{(1-\boldsymbol{\beta}\cdot\boldsymbol{n})^5}\right]_{ret} \qquad (8.47)
\end{aligned}$$

From this expression there are two main relativistic effects on the angular distribution of the emitted radiation:

- The angular distribution depends on the relative orientation between $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$.
- The factor $1-\boldsymbol{\beta}\cdot\boldsymbol{n}$ in the denominator results in a strong forward focussing of the emitted radiation in a direction $\boldsymbol{n}\|\boldsymbol{\beta}$. This effect dominates the angular distribution for ultrarelativistic particles.

Eqn 8.47 is a general result for the angular distribution of the radiation emitted by an accelerated charge in vacuum. We now return to the case of circular motion for which the acceleration $\dot{\boldsymbol{\beta}}$ is perpendicular to $\boldsymbol{\beta}$. Using polar angles as defined below the angular distribution becomes:

$$\frac{dP(t')}{d\Omega} = \frac{e^2\mu_0}{16\pi^2 c}\frac{|\dot{v}|^2}{(1-\beta\cos\theta)^3}\left(1-\frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right) \qquad (8.48)$$

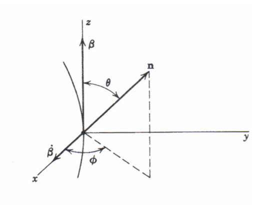

Figure 8.13: Polar angles ϕ and θ as used in Eqn 8.48

In the non-relativistic limit $\beta \to 0$ and $\gamma \to 1$ Eqn 8.48 reduces to Eqn 8.45.

The angular profile Eqn 8.48 is shown below for increasing values of $\beta = v/c$:

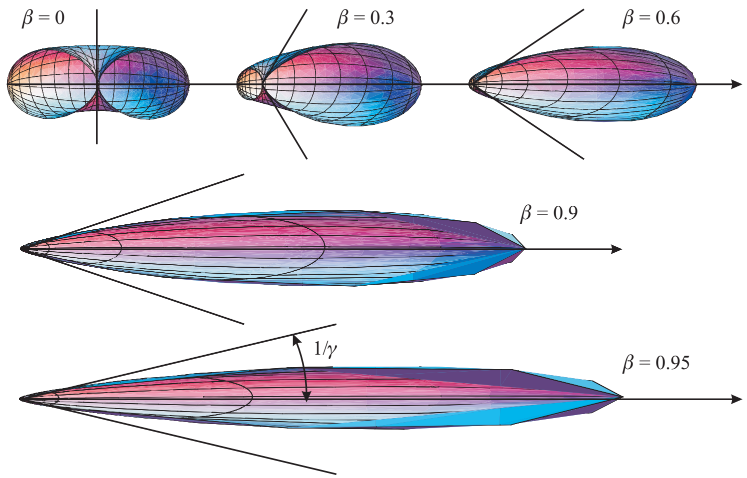

Figure 8.14: The angular profile (not to scale) of radiation emitted by a relativistic charged particle for various values of $\beta = u/c$.

As β increases, the radiated *power becomes more closely confined to the forward direction*, the direction of motion, and the rearward lobe becomes relatively weaker.

At highly relativistic speeds $\gamma \gg 1$, the angle separating the rearward and forward lobes $\to 1/\gamma$.

So a stationary observer O receives radiation only when the angle between the observer's line of sight to the particle and the particle's velocity is $< 1/\gamma$. i.e. when the observer's line of sight is approximately *tangential* to the particle's circular path.

The field at O therefore is therefore *pulsed*. The pulse width can be estimated as follows...

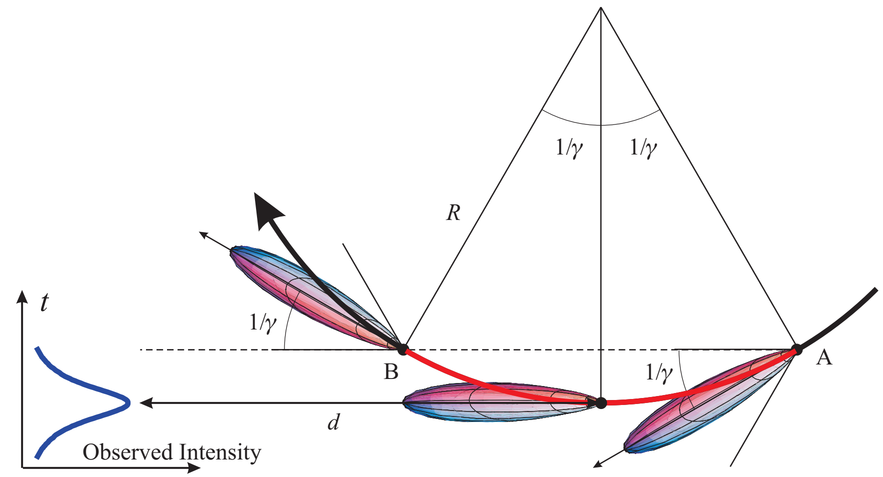

Figure 8.15: Radiation emitted by a relativistic charged particle reaches a stationary observer O only from the sector AB (of length $2R/\gamma$) of the particle's orbit shown in red.

O first receives radiation from the point A (emitted at time t_A) at a time

$$t_\mathrm{A} + \frac{d + R/\gamma}{c}$$

The end of the pulse (emitted at time $t_\mathrm{B} = t_\mathrm{A} + 2R/\gamma v$) is detected at time

$$t_\mathrm{B} + \frac{d - R/\gamma}{c}$$

The *duration* Δt *of the pulse* is therefore:

$$\begin{aligned}\Delta t &\sim \frac{1}{c}\left(d - \frac{R}{\gamma} - d - \frac{R}{\gamma}\right) + \frac{2R}{\gamma v} \\ &= \frac{2R}{\gamma}\left(\frac{1}{v} - \frac{1}{c}\right) \\ &= \frac{2}{\gamma\omega_B}\left(1 - \frac{v}{c}\right) \\ &\sim \frac{1}{\gamma^3\omega_B}\end{aligned} \tag{8.49}$$

using $(1 - v/c) \approx 1/2\gamma^2$.

The radiation is *pulsed in time*, so is *spectrally broad* with some characteristic frequency and wavelength:

$$\nu_\mathrm{s} \sim \frac{1}{\Delta t} \sim \gamma^3\omega_B \qquad\qquad \lambda_\mathrm{s} \sim \frac{c}{\gamma^3\omega_B} \sim \frac{R}{\gamma^3} \ll R \tag{8.50}$$

More detailed calculations give:

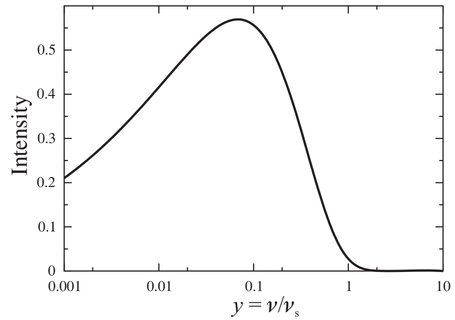

Figure 8.16: Spectral emission intensity curve for synchrotron radiation.

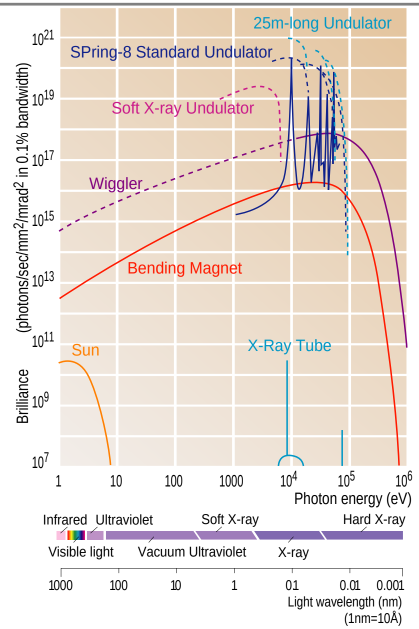

Figure 8.17: Synchrotron spectrum comparison.

This process produces significant energy loss in electron synchrotrons, and *limits the energy* that can be reached in circular *accelerators*. Bad for particle physicists.

For high γ, $\nu_{\rm s} \gg \omega_B = \dfrac{v(\approx c)}{R}$ producing *hard X-rays and a broad continuum at lower frequencies.*

Figure 8.18: Aerial view of the European Synchrotron Research Facility.

E.S.R.F.: $\gamma \sim 12000$ $2\pi R = 844$ m $\lambda_{\rm s} \sim 5$ Å

So synchrotrons now used as a powerful *source of highly collimated X-rays* for spectroscopy and X-ray diffraction studies. Good for everyone except particle physicists.

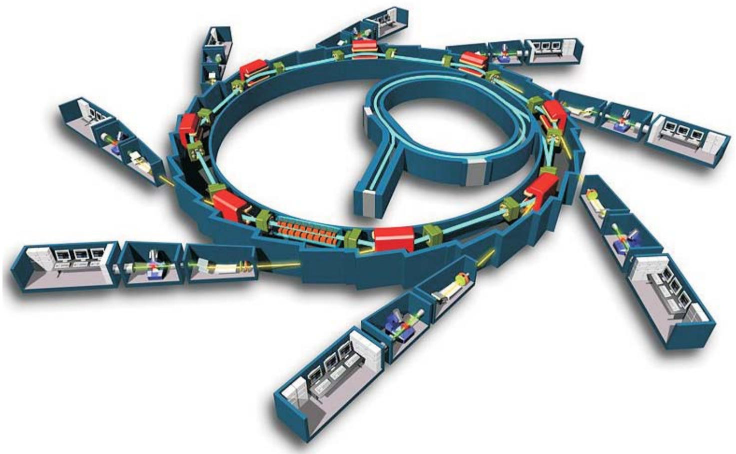

Figure 8.19: Schematic of a synchrotron with tangential experimental stations.

Synchrotron radiation is also very important astronomically, as it also occurs naturally in astronomical objects such as supernovae and radio galaxies.

Figure 8.20: Naturally occurring synchrotron radiation in the Crab Nebula.