Experimental and Theoretical Physics Part II
Soft Condensed Matter Lectures 1-11

Question Sheet 1

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Problems

Q 1 The equation for the drag force on a sphere can be obtained (within a numerical factor of order of unity) from dimensional analysis, by considering only the parameters that are physically relevant. Discuss the approximations one needs to make to derive the expressions for the viscous drag at low Reynolds numbers, and the inertial drag at high Reynolds numbers. The result at low Reynolds numbers was obtained in the notes.

Q 2 A long flat plate of width b, inclined at an angle α to the horizontal, is covered by a liquid film of uniform thickness a which is flowing steadily downhill. Discuss the boundary conditions at the free surface of the liquid, and show that the volumetric flow rate is $Q = (a^3b/3\eta)\rho g \sin\alpha$.

Q 3 In a Couette viscometer the fluid is contained between concentric cylinders, each of length L=150 mm. The inner cylinder, with radius r_1=95 mm, rotates at an angular velocity of ω_1=5 rad s^{-1}, while the outer one, with radius r_2=100 mm, is fixed. This is the geometry that is used in experiments. The viscometer is filled with an oil whose viscosity is 1 Pa s. What torque is measured on the outer cylinder?

Q 4 A pipeline transporting crude oil is a straight pipe of circular cross-section, of diameter 2m and length 100km. Estimate the minimal mean velocity of oil flow below which dissipation due to viscous friction will become significant. Take the density of crude oil to be $\rho = 800$kg/m^3 and the viscosity to be $\eta = 0.005$Pa s.

Q 5 A major river of the world can be approximated as a semi-circular trough of radius 20 m. Assuming the river uniformly drops 1 km over its 1000 km length, and assuming a laminar flow dominated by viscosity, calculate the terminal velocity of flow at the midpoint of its surface. What would be the river's maximum possible velocity, ignoring friction? Why are both these velocities unphysical?

Q 6 For a rectangular channel ($W \times H$), the dimensionless stokes equation is:

$$[\partial_{y'}^2 + \partial_{z'}^2]V(y', z') = 1 \tag{6.1}$$

with $y' = y/W$, $0 \leq y' \leq 1$, $z' = z/W$, $0 \leq z' \leq \beta$, and $\beta = H/W$. The boundary conditions are:

$$V(0, z') = V(1, z') = V(y', 0) = V(y', \beta) = 0 \tag{6.2}$$

(a) What is the expression of the Poiseuille flow as a function of $V(y', z')$?

(b) We can make the ansatz $V(y', z') = V_y(y')V_z(z')$. Show that this ansatz fails to easily separate the variables.

(c) We can modify the original equation to try to go further. Use this ansatz to solve, with the same boundary conditions,

$$[\partial_{y'}^2 + \partial_{z'}^2]V(y', z') = 0 \tag{6.3}$$

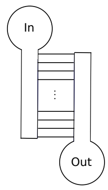
Do you run into any problems?

(d) From what we learned, we make the following ansatz:

$$V(y', z') = \sum_{nm} A_{nm} \sin(n\pi y') \sin(m\pi z'/\beta) \tag{6.4}$$

What expression for A_{nm} solves the original equation?

Q 7 A microfluidic chip is designed as two parallel channels linked with small grooves.



(a) If the hydraulic resistance of the channels is negligible and the resistance of the grooves is R_g, What is the resistance of the microfluidic chip between the inlet and the outlet for $N = 100$ grooves?

(b) What about if there is a resistance R_c in the channels between two grooves? You can consider a recursive argument with increasing number of grooves.

Q 8 A hollow sphere, with inner radius R_1 and outer radius R_2, is thin when compared to the radius. Knowing that the internal pressure p_1 is larger than the external pressure p_2, what is the sphere radius variation?

The fact that the sphere is thin means you can take mean values over the thickness.

Q 9 Consider a torsional pendulum consisting of a cylindrical rod (length l, radius R) with an inertial disk (moment of inertia I) attached at the bottom end. The upper end is clamped firmly, whilst the bottom disk can rotate by an angle θ.

(a) Show that for small angular rotations θ, the restoring torque is given by:

$$\text{torque} = \frac{G\pi R^4}{2l}\theta$$

where G is the shear modulus of the material of the cylinder.

(b) Now consider experiments in which the rod is set in oscillatory motion at a frequency ω. $G(\omega) = G'(\omega) + iG''(\omega)$ is the complex modulus of the cylinder. Show that the solution to the equation of motion can be written in the form

$$\theta = \theta_0 \exp(-\Lambda t) \exp(i\Omega t) \,.$$

Demonstrate that in the case of light damping Ω is independent of G''.

Q 10

(a) Calculate how long it would take a red blood cell to travel from the lungs to the tips of our fingers if it relied on diffusion. [Approximate the RBC as an 8 μm sphere, and take the viscosity of blood to be 0.01 Pa s. Ignore vessel wall effects.]

(b) Some animals, in particular insects, largely rely on gas diffusion for respiration. During the Carboniferous geologic period, the first trees began to develop and rapidly conquered the entire earth. The oxygen content of the atmosphere thus rose to 35% compared to the current 20%. This enabled insects to grow to sizes which were much larger than today. Assuming the following:

   (a) Insects are cylinders.

   (b) The wingspan is proportional to the weight.

   (c) All the cells have the same oxygen need (They are sinks of oxygen).

   (d) Steady state.

   Give an estimate of how much wider the wingspan of a dragonfly was during this period.

Q 11 For a particle moving in a harmonic potential $U(x) = \frac{1}{2}\kappa x^2$, and whose motion is described by a Langevin equation

$$\frac{dx}{dt} = -\frac{1}{\xi}\frac{\partial U}{\partial x} + g(t)$$

where $g(t)$ is a stochastic term representing thermal fluctuations, evaluate the time autocorrelation function of $x(t)$ and the equilibrium mean square displacement of the particle.

Q 12 A Gaussian chain with fluorescent tag on one end and its quencher on the other end is dissolved in a non-quenching solvent. If there is no specific interaction between the two ends other than the fluorescence quenching, how does the fluorescence intensity change with the chain length? The fluorescence is quenched when the quencher is in close proximity.

Q 13

(a) Go through the calculation in the lectures to make sure you can derive the end to end distance of a wormlike chain. Specifically, for a wormlike chain, the magnitude of the orientation function between two segments a distance s apart depends only on this distance. This means that we can write:

$$\langle \cos(\theta(s) - \theta(0)) \rangle = \exp(-s/L_p)$$

where L_p is the persistence length. Show that the mean end-to-end length of the chain is given by

$$\langle R^2 \rangle = 2L_pL \left[1 - \frac{L_p}{L} \left(1 - \exp\left(-\frac{L}{L_p} \right) \right) \right]$$

where L is the total contour length of the chain. Examine and discuss the limits of very stiff and very flexible chains.

(b) The size of a typical cell is 10 µm. The bending rigidity is $C_B = E\pi r^4/2$. A cytoskeletal filament has to be straight on the lengthscale of the cell. Compute the minimal diameter of a cytoskeletal filament that can achieve this. You can assume a Young's modulus of 10 GPa.

Q 14 In a simple model for the entropic elasticity, the polymer molecule is treated as a one-dimensional freely jointed chain of N segments. Each segment can be described by a two state variable, which has a value of +1 if the segment points forward in the direction of the constant applied force f, and -1 if it points backward to oppose the force. Within this framework the total extension z can be written as: $z = a \sum_{i=1}^{N} \sigma_i$, where a is the length of a segment. Show that the relationship between $\langle z \rangle$, the total length of the chain L and the applied force f can be expressed by

$$\langle z \rangle = L \tanh\left(\frac{fa}{k_B T} \right).$$

Sketch the force-extension relation and examine the limits of low and high force.

Q 15 Obtain an expression for the stress along the x-direction for a rubber sheet when it is stretched biaxially by an amount $\lambda_x = \alpha_1$ along the x-direction, and $\lambda_y = \alpha_2$ along the y-direction.

Q 16 Compare the end-to-end distances of a single polymer chain with monomer size a and degree of polymerisation N, in the cases that the chain is in bulk solution, or when it is confined to a planar surface. In both cases assume good solvent conditions.

Q 17 Consider a solution of a polymer A, consisting of N monomers, in a monomolecular solvent B. The effective interaction of A and B is characterised by the Flory interaction parameter χ. The volume fraction of the A component is $\phi = \phi_A = 1 - \phi_B$.

Write down an expression for the free energy of mixing within the Flory-Huggins theory and obtain the condition for stability against phase separation. What is the critical point χ_c and how does it vary with temperature and N?

Now write down the free energy in the form of a series expansion for low polymer concentrations ($\phi \ll 1$) and discuss the chain conformation for good and poor solvent conditions. Define the "θ-point" and describe the conformation of the chains at this point.

Q 18 Sketch the phase diagram for the polymer solution and discuss the different solubility regimes.

Calculate the overlap volume fraction, ϕ^*, for polystyrene with statistical segment length of $a = 6.7\,\mathring{A}$ and a degree of polymerisation of $N = 1000$ in cyclohexane at 34.5^{o}C (a θ-solvent).

Q 19 A non-physical feature of the Gaussian chain is that it has a non-zero probability of having an end-to-end distance which is larger than the chain length. This probability, however, is very small. Evaluate it for $N = 100$ and $b_0 = 1nm$.

Q 20 Solute molecules self-assemble in solution to form clusters of aggregation number N per cluster. The equation between monomers (A) and aggregates (B) in solution can be expressed as

$$A + A + A + ... \rightleftharpoons B.$$

X_A and X_B are concentrations of A and B in mol fraction units. K is the equilibrium constant for the reaction and c is the total concentration of solute molecules. Obtain a relationship between K, N, c and X_A and then show that for $K \gg 1$ and $N \gg 1$ the concentration of monomers X_A can never exceed $(NK)^{-\frac{1}{N}}$.

Q 21 Actin is a cytoskeletal protein that can polymerize in solution to form linear filaments, by the addition of successive monomers. Show that if $\alpha k_B T$ is the free energy gain of binding a monomer to the aggregate, and X_N is the concentration of molecules in aggregates of number N, then

$$X_N = N\left[X_1 \exp\left(\alpha\left(1-\frac{1}{N}\right)\right)\right]^N.$$

Hence show that for a total concentration C of molecules, the peak in the distribution of N (for concentrations above the critical micelle concentration) is given by $N \simeq \sqrt{C \exp \alpha}$.