VIII. THE GRAVITATIONAL FIELD EQUATIONS

We have seen how to formulate the laws of physics on a curved spacetime as tensor equations, thus ensuring consistency with the equivalence principle.

In this handout, we complete the development of general relativity by specifying how the curvature of spacetime is related to the matter that is present.

This will lead us to the Einstein equations, relating the energy–momentum tensor of the matter to the Einstein tensor.

In terms of components, these are non-linear, second-order partial differential equations for the metric functions, much as Maxwell's equations can be regarded as (linear) second-order equations relating the 4-vector potential to the current 4-vector.

1 The energy–momentum tensor

In Newtonian gravity, the Poisson equation relates the Laplacian of the gravitational potential to the mass density.

In general relativity, we need to find an appropriate tensor that generalises the mass density to describe the relativistic energy at each event in spacetime.

First consider the case of non-interacting particles, each of rest mass m, with no velocity dispersion.

Such matter is usually referred to as dust; it has the property that at each event P all particles present there have the same 4-velocity $u^\mu(x)$.

At P, the dust has *energy density* ρc^2 when measured in some local-inertial frame there, and the particles all have 3-velocity $\vec{u}$.

In particular, it is possible to find a local-inertial frame in which the particles at P are at rest; in this instantaneous rest frame, let the number density of particles be n_0 so that the energy density is $\rho_0 c^2 = m n_0 c^2$.

Transforming to the local-inertial frame S in which the 3-velocity of the dust is $\vec{u}$ at P, the number density will be $\gamma_u n_0$ (length contraction) and the energy of each particle $\gamma_u m c^2$.

It follows that in this frame, the energy density is

$$\rho c^2 = (\gamma_u n_0)\gamma_u m c^2 = \gamma_u^2 \rho_0 c^2 \,. \tag{1}$$

We see that energy density is not a Lorentz scalar; rather it transforms like the 00 component of the type-$(2,0)$ tensor

$$T^{\mu\nu}(x) = \rho_0(x) u^\mu(x) u^\nu(x) \,. \tag{2}$$

That this is a tensor follows from the fact that $\rho_0 c^2$ is a scalar field (it is *defined* to be the energy density in the instantaneous rest frame), and u^μ are the components of a 4-vector.

In the local-inertial frame S, in which the 3-velocity is $\vec{u}$, the 0-component of u^μ is $\gamma_u c$, so that

$$T^{00} = \gamma_u^2 \rho_0 c^2 \,, \tag{3}$$

as advertised.

What is the physical interpretation of the other components of $T^{\mu\nu}$?

Again, consider observing in the local-inertial frame S

in which the 3-velocity at P is $\vec{u}$; the 4-velocity of the dust has components $u^\mu = \gamma_u(c, \vec{u})$ and so

$$\begin{aligned} T^{i0} &= mn_0 \left(\gamma_u \vec{u}^i\right) (\gamma_u c) \\ &= c\,(\gamma_u n_0) \left(m\gamma_u \vec{u}^i\right) \,. \end{aligned} \tag{4}$$

This is the product of the number density of particles and the 3-momentum of each, and so is the *momentum density* (times c).

An alternative interpretation is as the energy flux in the ith direction, since

$$\text{energy flux} = \left(\gamma_u^2 n_0 mc^2\right) \vec{u}^i = cT^{i0} \,, \tag{5}$$

i.e., the product of the energy density and the particle 3-velocity.

Finally, consider the ij components:

$$\begin{aligned} T^{ij} &= mn_0 \left(\gamma_u \vec{u}^i\right) \left(\gamma_u \vec{u}^j\right) \\ &= \left(\gamma_u^2 mn_0 \vec{u}^i\right) \vec{u}^j \,. \end{aligned} \tag{6}$$

This is the ith component of the 3-momentum density multiplied by the jth component of the 3-velocity, i.e., the *flux* of the ith component of 3-momentum along the jth direction.

Collecting these results together, we have the following physical interpretation of the components of the tensor $T^{\mu\nu}$ in a local-inertial frame:

T^{00}	energy density
T^{i0}	ith component of 3-momentum density (times c)
T^{ij}	flux of i-component of 3-momentum in j-direction.

This association of a type-$(2,0)$ tensor $T^{\mu\nu}$ with the energy and momentum density and their fluxes generalises to other sources, such as the electromagnetic field.

The properties of these densities and fluxes under Lorentz transformations ensure that they also form the components of a tensor for any source.

The tensor $T^{\mu\nu}$ is called the *energy–momentum tensor* (alternatively, the stress–energy tensor) and, as we shall see, acts as the source for spacetime curvature.

Note that the energy–momentum tensor is always symmetric, $T^{\mu\nu} = T^{\nu\mu}$; this can be shown to be necessary to ensure angular momentum conservation in all inertial frames.

1.1 Energy–momentum tensor of an ideal fluid

The energy density that appears in the energy–momentum tensor must include all sources of energy, for example, the kinetic energy of the particles in a gas due to their velocity dispersion and any interaction energies.

Similarly, the energy flux and momentum density must include contributions from heat conduction in the gas, as well as from bulk motions (the former giving non-zero components T^{i0} in the instantaneous rest frame).

For the flux of momentum, we must include effects from the velocity dispersion in the gas and any shear stresses that may be present.

In this course, we shall only consider *ideal fluids*.

These have the property that at any event one can find a local-inertial frame (the instantaneous rest frame) in which $T^{i0} = 0$ and in which the spatial components are isotropic: $T^{ij} \propto \delta^{ij}$.

In the case of a gas of particles, this requires the mean-free path in the gas to be short compared to the length scale of any temperature or velocity gradients (so that conduction and shear stresses are negligible).

In the instantaneous rest frame, the components of the energy–momentum tensor for an ideal fluid have to take the form

$$T^{\mu\nu} = \mathrm{diag}(\rho c^2, p, p, p)\,, \tag{7}$$

where ρc^2 is the rest-frame energy density (we now drop the subscript 0 on the rest-frame energy density that we included earlier) and p is the *isotropic pressure*.

We can write this in tensor form, and so valid in any coordinate system, as

$$\boxed{T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu - p g^{\mu\nu}\,,} \tag{8}$$

where u^μ is the 4-velocity of the fluid (defined by its instantaneous rest frame).

In the instantaneous rest frame, $u^\mu = (c, \vec{0})$ and $g^{\mu\nu} = \eta^{\mu\nu}$, and so we reover the correct diagonal components [Eq. (7)] from Eq. (8).

Note that the energy density ρc^2 and isotropic pressure p are scalar fields as they are defined in the instantaneous rest frame.

Note also that for $p \ll \rho c^2$, the energy–momentum tensor for an ideal fluid reduces to that for dust.

1.2 Conservation of energy and momentum

Recall that in electromagnetism, the conservation of charge is expressed through the (covariant) continuity equation $\nabla_\mu j^\mu = 0$.

The energy–momentum tensor satisfies a similar continuity equation:

$$\boxed{\nabla_\mu T^{\mu\nu} = 0\,.} \tag{9}$$

In local-inertial coordinates at some event P, this reduces to

$$\frac{\partial T^{00}}{\partial t} + c\sum_i \frac{\partial T^{i0}}{\partial x^i} = 0\,, \tag{10}$$

$$\frac{\partial (T^{i0}/c)}{\partial t} + \sum_j \frac{\partial T^{ij}}{\partial x^j} = 0\,. \tag{11}$$

The first expresses conservation of energy through a continuity equation of the form

$$\frac{\partial}{\partial t}(\text{energy density}) + \vec{\nabla}\cdot(\text{energy flux}) = 0\,. \tag{12}$$

The second, Eq. (11), expresses conservation of momentum in the form

$$\frac{\partial}{\partial t}(\text{momentum density}) + \vec{\nabla}\cdot(\text{momentum flux}) = 0\,. \tag{13}$$

The covariant continuity equation (9) thus encodes both conservation of energy and 3-momentum.

Example: energy and momentum conservation for the ideal fluid

For the ideal fluid, with energy–momentum tensor given by Eq. (8), conservation of the energy–momentum tensor gives

$$\nabla_\mu \left[\left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu - p g^{\mu\nu}\right] = 0\,, \tag{14}$$

so that

$$u^\nu u^\mu \nabla_\mu \left(\rho + \frac{p}{c^2}\right) + \left(\rho + \frac{p}{c^2}\right)(\nabla_\mu u^\mu)\, u^\nu + \left(\rho + \frac{p}{c^2}\right) u^\mu \nabla_\mu u^\nu - \nabla^\nu p = 0\,, \tag{15}$$

where we have used $\nabla_\mu g^{\mu\nu} = 0$.

We can project this parallel and perpendicular to u^ν.

Contracting with u_ν extracts the parallel component:

$$c^2 u^\mu \nabla_\mu \left(\rho + \frac{p}{c^2}\right) + c^2 \left(\rho + \frac{p}{c^2}\right) (\nabla_\mu u^\mu) - u^\mu \nabla_\mu p = 0$$
$$\Rightarrow \quad \nabla_\mu (\rho u^\mu) + \frac{p}{c^2} \nabla_\mu u^\mu = 0\,. \tag{16}$$

Here, we have used that $u^\mu \nabla_\mu u^\nu$ is orthogonal to u^ν because of $g_{\nu\rho} u^\nu u^\rho = c^2$:

$$\begin{aligned} u^\mu \nabla_\mu (g_{\nu\rho} u^\nu u^\rho) &= 0 \\ \Rightarrow \quad g_{\nu\rho} (u^\mu \nabla_\mu u^\nu) u^\rho + g_{\nu\rho} u^\nu (u^\mu \nabla_\mu u^\rho) &= 0 \\ \Rightarrow \quad 2u_\nu (u^\mu \nabla_\mu u^\nu) &= 0\,. \end{aligned} \tag{17}$$

The part of Eq. (15) that is perpendicular to u^ν can now be extracted by subtracting u^ν times the parallel component; we find

$$\left(\rho + \frac{p}{c^2}\right) u^\mu \nabla_\mu u^\nu = \left(g^{\mu\nu} - \frac{u^\mu u^\nu}{c^2}\right) \nabla_\mu p\,. \tag{18}$$

These parallel and perpendicular projections describe energy and momentum conservation, respectively.

We can gain physical insight into these equations by adopting local-inertial coordinates in the vicinity of some event P.

The metric is $\eta_{\mu\nu}$ at P, and is close to this in the immediate vicinity of P (with the extent of this region determined by the spacetime curvature).

For the 4-velocity of the fluid, we have

$$u^\mu = \frac{dx^\mu}{d\tau} = \frac{dt}{d\tau} \left(c, \frac{dx^i}{dt}\right) = \frac{dt}{d\tau} \left(c, \vec{u}^i\right)\,. \tag{19}$$

We can determine $dt/d\tau$ in terms of the 3-velocity $\vec{u}^i$ using the normalisation $g_{\mu\nu} u^\mu u^\nu = c^2$.

Consider the Newtonian limit $|\vec{u}^i| \ll c$; then we have $dt/d\tau \approx 1$ and

$$u^\mu \approx \left(c, \vec{u}^i\right) . \tag{20}$$

We further assume that $p \ll \rho c^2$ (i.e., the speed of all particles in the rest-frame of the fluid are much less than c).

Since the metric connection vanishes at P, Eq. (16) reduces there to

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^i}\left(\rho \vec{u}^i\right) \approx 0 . \tag{21}$$

Given that ρ is approximately the mass density in this Newtonian limit, we see that we recover the *continuity equation* of Newtonian fluid mechanics.

For the perpendicular projection, Eq. (18), we have at P

$$\rho u^\mu \frac{\partial u^\nu}{\partial x^\mu} = \left(\eta^{\mu\nu} - \frac{u^\mu u^\nu}{c^2}\right) \frac{\partial p}{\partial x^\mu} . \tag{22}$$

The $\nu = 0$ component vanishes identically at leading order in $p/(\rho c^2)$ and $|\vec{u}^i|/c$.

For $\nu = i$, we have

$$\rho \left(\frac{\partial \vec{u}^i}{\partial t} + \sum_j \vec{u}^j \frac{\partial \vec{u}^i}{\partial x^j} \right) \approx -\sum_j \delta^{ij} \frac{\partial p}{\partial x^j} - \frac{\vec{u}^i}{c^2} \frac{\partial p}{\partial t} , \tag{23}$$

which, on retaining the leading-order terms, gives

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} \approx -\frac{\vec{\nabla} p}{\rho} . \tag{24}$$

This is the *Euler equation* (for an ideal fluid) of Newtonian fluid mechanics.

2 The Einstein equations

Recall that in Newtonian gravity, the gravitational potential Φ and mass density ρ are related by Poisson's equation

$$\nabla^2\Phi = 4\pi G\rho\,. \tag{25}$$

We also know that in the weak-field limit, where $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$, we recover the correct Newtonian equation of motion for a test particle from the geodesic equation if

$$g_{00} \approx \left(1 + \frac{2\Phi}{c^2}\right)\,. \tag{26}$$

We have seen that the proper relativistic treatment of the distribution of matter is through the energy–momentum tensor, and that for a slow-moving fluid $T_{00} \approx \rho c^2$ in local-inertial coordinates.

Putting these results together, the Poisson equation in the limit of weak fields and non-relativistic speeds is equivalent to

$$\vec{\nabla}^2 g_{00} \approx \frac{8\pi G}{c^4} T_{00}\,. \tag{27}$$

The second derivative of the metric is a measure of the curvature of spacetime, so this suggests we look for a relativistic field equation of the form

$$K_{\mu\nu} = \kappa T_{\mu\nu} \quad \text{with} \quad \kappa \equiv \frac{8\pi G}{c^4}\,, \tag{28}$$

where $K_{\mu\nu}$ is a symmetric type-$(0,2)$ tensor related to the curvature of spacetime (with dimensions of inverse-squared length).

The energy–momentum tensor is conserved, $\nabla_\mu T^{\mu\nu}{=}0$, and if this is to be consistent with the field equation (28) we must have

$$\nabla^\mu K_{\mu\nu} = 0\,. \tag{29}$$

We know a good candidate for $K_{\mu\nu}$: the contracted Bianchi identity (see Handout VII) in spacetime is

$$\nabla^\mu \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 0 \,, \tag{30}$$

so the (symmetric) Einstein tensor, $G_{\mu\nu} \equiv R_{\mu\nu} - g_{\mu\nu} R/2$, identically satisfies

$$\nabla^\mu G_{\mu\nu} = 0 \,. \tag{31}$$

This suggests the field equation

$$\boxed{G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\kappa T_{\mu\nu} \,.} \tag{32}$$

The constant of proportionality (and the minus sign) on the right is required for consistency with the weak-field limit, as we shall prove shortly.

Equation (32) is the *Einstein field equation* of general relativity.

In spacetime, the symmetric tensors $G_{\mu\nu}$ and $T_{\mu\nu}$ have 10 independent components so the Einstein field equations are really 10 non-linear partial differential equations for the metric functions $g_{\mu\nu}$ (and so are hard to solve in general!).

In contrast, Newtonian gravity has the single Poisson equation, which is linear in the potential Φ.

We can obtain an alternative (but equivalent) form of the Einstein equations by contracting with $g^{\mu\nu}$:

$$\begin{aligned} g^{\mu\nu} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) &= -\kappa g^{\mu\nu} T_{\mu\nu} \\ \Rightarrow \quad R - 2R &= -\kappa T \,, \end{aligned} \tag{33}$$

where

$$T \equiv g^{\mu\nu} T_{\mu\nu} = T^\nu{}_\nu \tag{34}$$

is the trace of the energy–momentum tensor.

It follows that

$$R_{\mu\nu} = -\kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) . \tag{35}$$

2.1 The Einstein equations in empty space

The energy–momentum tensor must include all sources of energy and momentum.

In some empty region of spacetime (i.e., vacuum), the energy–momentum tensor vanishes and we have

$$R_{\mu\nu} = 0 \qquad \text{(vacuum)} . \tag{36}$$

Generally, the vanishing of the Ricci tensor does not imply that spacetime is flat in that region; it is still possible for the Riemann tensor to be non-zero and so there to be gravitational tidal effects in the region.

Interestingly, four spacetime dimensions is the minimum in which gravitational effects do not necessarily vanish in empty space.

To see this, note that general relativity in ND in empty space would imply $N(N+1)/2$ field equations (the independent components of $R_{ab} = 0$), while the Riemann curvature tensor would have $N^2(N^2-1)/12$ independent components.

Only for $N \geq 4$ is the number of independent components in the curvature tensor larger than the number of field equations, which is necessary to have a non-zero Riemann tensor.

3 Weak-field limit of Einstein's equations

We aim to show that we recover Poisson's equation (25) from the Einstein equations in the weak-field limit.

Our starting point is

$$R_{00} = -\kappa \left(T_{00} - \frac{1}{2} g_{00} T \right) . \tag{37}$$

For the matter source, we consider a non-relativistic fluid ($p \ll \rho c^2$) that is slowly moving in coordinates in which the metric takes the weak-field form, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.

Moreover, we assume that the mass distribution and metric are independent of time in these coordinates (a *stationary* situation).

Neglecting p compared to ρc^2, the energy-momentum tensor is

$$T_{\mu\nu} \approx \rho u_\mu u_\nu \, , \tag{38}$$

and so

$$T \approx g^{\mu\nu} \rho u_\mu u_\nu = \rho c^2 \, . \tag{39}$$

Also, using $u^\mu \approx (c, \vec{u}^i)$, we have

$$u_0 = g_{0\mu} u^\mu \approx g_{00} c \approx c \, , \tag{40}$$

so that

$$T_{00} \approx \rho u_0 u_0 \approx \rho c^2 \, . \tag{41}$$

Putting these pieces together, Eq. (37) reduces to

$$R_{00} \approx -\frac{1}{2} \kappa \rho c^2 \, . \tag{42}$$

For the approximate Ricci tensor, we use

$$R_{00} = -\partial_\mu \Gamma^\mu_{00} + \partial_0 \Gamma^\mu_{\mu 0} + \Gamma^\nu_{\mu 0} \Gamma^\mu_{0\nu} - \Gamma^\nu_{00} \Gamma^\mu_{\mu\nu} \, . \tag{43}$$

The connection coefficients are first-order in $h_{\mu\nu}$ since they vanish for the Minkowski metric, and so we can neglect the last two terms involving products of the connection.

It follows that for a stationary metric, we have

$$R_{00} \approx -\sum_i \frac{\partial \Gamma^i_{00}}{\partial x^i} \,. \tag{44}$$

We showed in Handout VII that

$$\Gamma^i_{00} \approx \frac{1}{2}\frac{\partial h_{00}}{\partial x^i} \,, \tag{45}$$

and so

$$R_{00} \approx -\frac{1}{2}\vec{\nabla}^2 h_{00} \,. \tag{46}$$

Finally, the 00 component of the Einstein equation becomes

$$\vec{\nabla}^2 h_{00} \approx \frac{8\pi G}{c^2}\rho \,, \tag{47}$$

which, on recalling the identification $h_{00} \approx 2\Phi/c^2$ that is required from the geodesic equation, gives the usual Poisson equation

$$\vec{\nabla}^2 \Phi = 4\pi G \rho \,. \tag{48}$$

This justfies our choice of proportionality constant on the right-hand side of the Einstein field equation (32).

4 The cosmological constant

The Einstein equation (32) is the simplest field equation that we can write down if we demand that the tensor on the left has the following properties:

1. divergence free;
2. constructed from the metric and its first two derivatives; and

3. linear in second derivatives of the metric.

However, the Einstein tensor is not the only such tensor that satisfies these conditions.

It can be shown (*Lovelock's theorem*) that the only other possibility is to add a constant multiple of the metric tensor, giving a new field equation of the form

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu}\,. \tag{49}$$

The tensor on the left is still divergence free since $\nabla_\rho g_{\mu\nu} = 0$.

The quantity Λ is called the *cosmological constant*; it would be a new universal constant of nature with dimensions of inverse length squared.

Einstein originally included this term in the field equations in order to construct static cosmological solutions (see Handout XII).

However, with the discovery of the expanding universe he dismissed the new term $\Lambda g_{\mu\nu}$, describing its introduction as his "biggest blunder".

We now know from several different cosmological observations (including the observed fluxes of distant supernova, the clustering of galaxies, and the fluctuations in the cosmic microwave background) that Λ is non-zero, with

$$\begin{aligned}\Lambda &\approx 1.1 \times 10^{-52}\,\mathrm{m}^{-2} \\ &= (3.04\,\mathrm{Gpc})^{-2}\,,\end{aligned} \tag{50}$$

where 1 Gpc (Gigaparsec) equals 3.1×10^{25} m.

This is a very large length scale – comparable to the size of the observable universe – so the cosmological constant is a small correction in a system much smaller than this (i.e., where the curvature scale is small compared to $\Lambda^{-1/2}$).

Contracting Eq. (49) with $g^{\mu\nu}$, we have

$$R - 4\Lambda = \kappa T \,, \tag{51}$$

so that

$$R_{\mu\nu} = -\kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) + \Lambda g_{\mu\nu} \,. \tag{52}$$

Repeating the calculation of the weak-field limit in the presence of small Λ gives

$$\vec{\nabla}^2 \Phi = 4\pi G \rho - \Lambda c^2 \,. \tag{53}$$

The solution for a point source of mass M at rest at the origin gives

$$-\vec{\nabla}\Phi(\vec{x}) = -\frac{GM}{|\vec{x}|^3}\vec{x} + \frac{\Lambda c^2}{3}\vec{x} \,, \tag{54}$$

showing that Λ provides a *gravitational repulsion* that increases linearly with distance from the central mass.

Although the cosmological constant is irrelevant on many scales of interest, it is important at current times on the scale of the entire universe.

The repulsive nature of the cosmological constant may be responsible for the observed late-time acceleration in the expansion of our universe.[1]

4.1 The cosmological constant as vacuum energy

The energy–momentum tensor for an ideal fluid is

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2} \right) u^\mu u^\nu - p g^{\mu\nu} \,. \tag{55}$$

[1] Other alternatives are also being actively considered including scalar fields (*quintessence*) and modifications to the laws of gravitation on cosmological scales.

Consider a fluid with the strange property that $p = -\rho c^2$, implying that it has a large negative pressure (i.e., tension); then

$$T_{\mu\nu} = -p g_{\mu\nu} = \rho c^2 g_{\mu\nu} \,. \tag{56}$$

In *any* local-inertial frame, $T^{00} = \rho c^2$, so *all* observers measure an energy density ρc^2 (and pressure $p = -\rho c^2$).

Such an energy–momentum tensor would therefore have to be a fundamental property of the vacuum.

The term $\Lambda g_{\mu\nu}$ in the Einstein equation (49) has exactly the form of the energy–momentum tensor with $p = -\rho c^2$.

We can write

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R &= -\kappa \left(T_{\mu\nu} + \frac{\Lambda}{\kappa} g_{\mu\nu} \right) \\ &= -\kappa \left(T_{\mu\nu} + T_{\mu\nu}^{\mathrm{vac}} \right) \,, \end{aligned} \tag{57}$$

where we have introduced the energy–momentum tensor of the vacuum

$$T_{\mu\nu}^{\mathrm{vac}} = \rho_{\mathrm{vac}} c^2 g_{\mu\nu} \,, \quad \text{with} \quad \rho_{\mathrm{vac}} c^2 = \frac{\Lambda c^4}{8\pi G} \,. \tag{58}$$

Should we expect the vacuum to have a non-zero energy density?

Indeed, we should because of the quantum zero-point energies of all the fields in nature.

Recall that a simple harmonic oscillator with classical frequency ω has a quantum ground-state energy $\hbar\omega/2$.

In quantum field theory, the Fourier modes of free quantum fields behave like quantum harmonic oscillators with a frequency $\omega(k)$ that depends on the Fourier wavenumber k.

The ground-state configuration of the field (i.e., the vacuum) has a non-zero energy due to the zero-point energies of each of these harmonic oscillators.

For example, for the electromagnetic field, we expect a ground-state energy

$$\rho_{\mathrm{vac,EM}} c^2 = \frac{2}{(2\pi)^3} \int \frac{\hbar c k}{2} \, d^3\vec{k} \,, \tag{59}$$

where we have used $\omega = ck$.

The integral is clearly divergent, but we cannot expect quantum field theory to be valid on all scales, particularly as $1/k$ approaches the Planck length.[2]

If we cut off the integral in Eq. (59) at the Planck scale, and include all known fields, we predict a vacuum energy density some 120 orders of magnitude larger than current cosmological limits!

We do not (yet) understand why the energy density of the vacuum, as inferred from the observed cosmological constant, is so much smaller than the naive theoretical prediction.

[2]The Planck scale is where we expect quantum gravity effects to become important; it is given by

$$l_{\mathrm{P}l} = \sqrt{\frac{\hbar G}{c^3}} = 1.62 \times 10^{-35}\,\mathrm{m}\,. \tag{60}$$