UNIVERSITY OF CAMBRIDGE

NST Part II Physics
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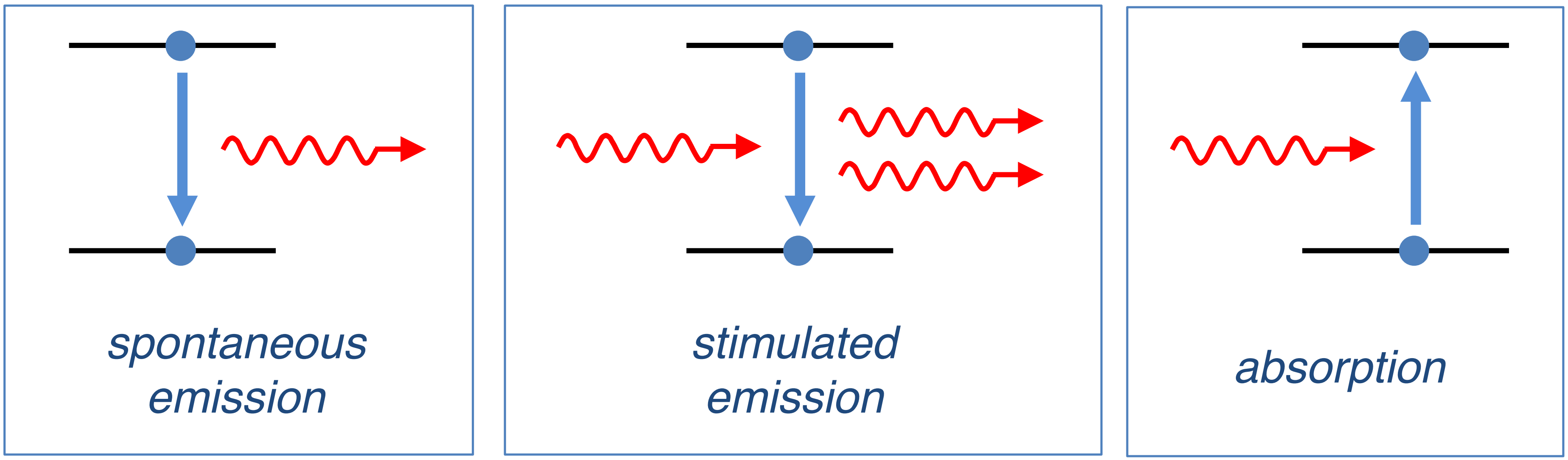
ADVANCED QUANTUM PHYSICS

Handout 10

- Quantising the EM field (QED)
- Photons

Quantum Electrodynamics

-- Atoms can undergo transitions to lower (higher) energy states by emitting (absorbing) a photon :



-- To compute transition rates for these (and other) processes, we need a *quantum* description of the electromagnetic field

$\rightarrow$ *Quantum Electrodynamics* (QED)

To obtain this, we quantise the mode (Fourier) expansion of the EM field

The Classical EM Field

-- Maxwell's equations in free space are

$$\boldsymbol{\nabla} \wedge \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \; ; \qquad \boldsymbol{\nabla} \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \; ; \qquad \boldsymbol{\nabla} \cdot \mathbf{E} = 0$$

-- The vector and scalar potentials are given by

$$\mathbf{B} = \boldsymbol{\nabla} \wedge \mathbf{A} \; ; \qquad \mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t}$$

-- In the Coulomb gauge, $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$, we obtain Laplace's equation $\nabla^2 \phi = 0$

The only solution of Laplace's equation with $\phi \to 0$ at infinity is $\phi(\mathbf{r}) = 0$; hence we have

$$\boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} \wedge \mathbf{A}) = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}$$

-- The identity $\boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} \wedge \mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ then gives, in the Coulomb gauge,

$$\boxed{\nabla^2 \mathbf{A} = \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2}} \qquad c^2 = \frac{1}{\mu_0 \epsilon_0}$$

The Classical EM Field (2)

-- This equation admits plane-wave solutions of the form (for any $\mathbf{k}$)

$$\mathbf{A}(\mathbf{r},t) = A_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\,\mathbf{e}(\mathbf{k})$$

where $A_{\mathbf{k}}$ is a complex overall constant, $\mathbf{e}(\mathbf{k})$ is a unit vector, and

$$\omega = \omega(\mathbf{k}) = c|\mathbf{k}|$$

-- Since

$$\boldsymbol{\nabla}\cdot\left[e^{i\mathbf{k}\cdot\mathbf{r}}\mathbf{e}(\mathbf{k})\right] = ik_x e^{i\mathbf{k}\cdot\mathbf{r}} e_x + ik_y e^{i\mathbf{k}\cdot\mathbf{r}} e_y + ik_z e^{i\mathbf{k}\cdot\mathbf{r}} e_z = i(\mathbf{k}\cdot\mathbf{e})e^{i\mathbf{k}\cdot\mathbf{r}}$$

the Coulomb gauge condition $\nabla\cdot\,\mathbf{A} = 0$ gives the constraint

$$\mathbf{k}\cdot\mathbf{e}(\mathbf{k}) = 0$$

-- For each vector $\mathbf{k}$, we therefore obtain two linearly independent plane-wave solutions, with

$$\mathbf{k}\cdot\mathbf{e}_1(\mathbf{k}) = 0\,; \qquad \mathbf{k}\cdot\mathbf{e}_2(\mathbf{k}) = 0\,; \qquad \mathbf{e}_1(\mathbf{k})\cdot\mathbf{e}_2(\mathbf{k}) = 0$$

-- The three-vector $\mathbf{k} = (k_x, k_y, k_z)$ defines the direction of propagation of the EM wave; $\mathbf{e}_1$ and $\mathbf{e}_2$ are unit vectors perpendicular to the wave direction

Polarisation States

-- For $\mathbf{k}$ directed along the z axis, $\mathbf{k} = (0,0,k)$, a possible choice of the vectors $\mathbf{e}_1(\mathbf{k})$, $\mathbf{e}_2(\mathbf{k})$ is

$$\mathbf{e}_1(\mathbf{k}) = (1,0,0) \; ; \qquad \mathbf{e}_2(\mathbf{k}) = (0,1,0)$$

The electric fields $\mathbf{E} = -\partial\mathbf{A}/\partial t$ corresponding to this choice are *plane polarisation* states of the form

$$\mathbf{E}_1 = (E_x,0,0) \; ; \qquad \mathbf{E}_2 = (0,E_y,0)$$

-- Alternatively, we can introduce the *circular polarisation* unit vectors

$$\mathbf{e}_{\mathrm{L}}(\mathbf{k}) = -\frac{1}{\sqrt{2}}\left[\mathbf{e}_1(\mathbf{k}) + i\mathbf{e}_2(\mathbf{k})\right] \; ; \qquad \mathbf{e}_{\mathrm{R}}(\mathbf{k}) = \frac{1}{\sqrt{2}}\left[\mathbf{e}_1(\mathbf{k}) - i\mathbf{e}_2(\mathbf{k})\right] \tag{10.5.1}$$

with normalisation

$$\mathbf{e}_{\mathrm{L}} \cdot \mathbf{e}_{\mathrm{L}}^* = 1 \; ; \qquad \mathbf{e}_{\mathrm{R}} \cdot \mathbf{e}_{\mathrm{R}}^* = 1 \; ; \qquad \mathbf{e}_{\mathrm{L}} \cdot \mathbf{e}_{\mathrm{R}}^* = 0$$

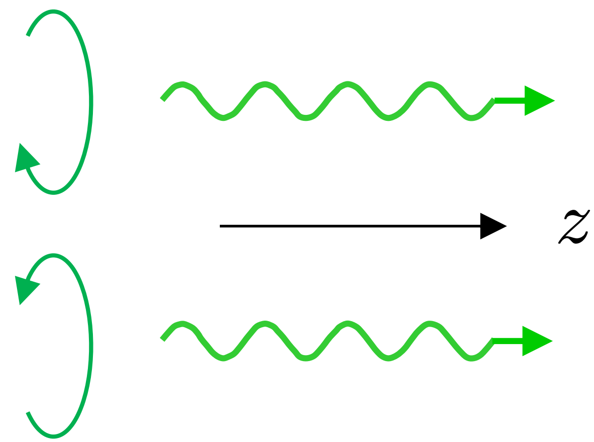
(in general, $\mathbf{e}(\mathbf{k})$ is a *complex* unit vector)

Polarisation States (2)

-- For $\mathbf{k} = (0,0,k)$, the circular polarisation vectors are

$$\mathbf{e}_{\mathrm{R}}(\mathbf{k}) = \frac{1}{\sqrt{2}}(1, -i, 0)$$

$$\mathbf{e}_{\mathrm{L}}(\mathbf{k}) = -\frac{1}{\sqrt{2}}(1, i, 0)$$



right-handed

left-handed

-- The vectors $\mathbf{e}_{\mathrm{L}}$ and $\mathbf{e}_{\mathrm{R}}$ correspond to *left-circular* and *right-circular* polarisation according to the standard optical convention :

→ looking into an oncoming wave, the electric field vector rotates clockwise (anti-clockwise) for left- (right-) circular polarisation

-- Each wavevector $\mathbf{k}$ gives rise to two independent *modes* $(\mathbf{k},\lambda)$ of the EM field,

$$\mathbf{A}(\mathbf{r}, t) = A_{\mathbf{k},\lambda} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \mathbf{e}_{\lambda}(\mathbf{k})$$

where $\lambda = 1, 2$ or $\lambda = \mathrm{L}, \mathrm{R}$ denotes the *polarisation state* (λ is *not* $2\pi/k$)

Polarisation States (3)

-- We will need explicit expressions for the polarisation vectors $\mathbf{e}_\lambda(\mathbf{k})$ for a wave travelling in a *general* direction $\mathbf{k}$ (not just along the z axis) :

$$\mathbf{k} = |\mathbf{k}|(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

-- These can be obtained by applying two successive rotations to all vectors (leaving the coordinate axes fixed) :

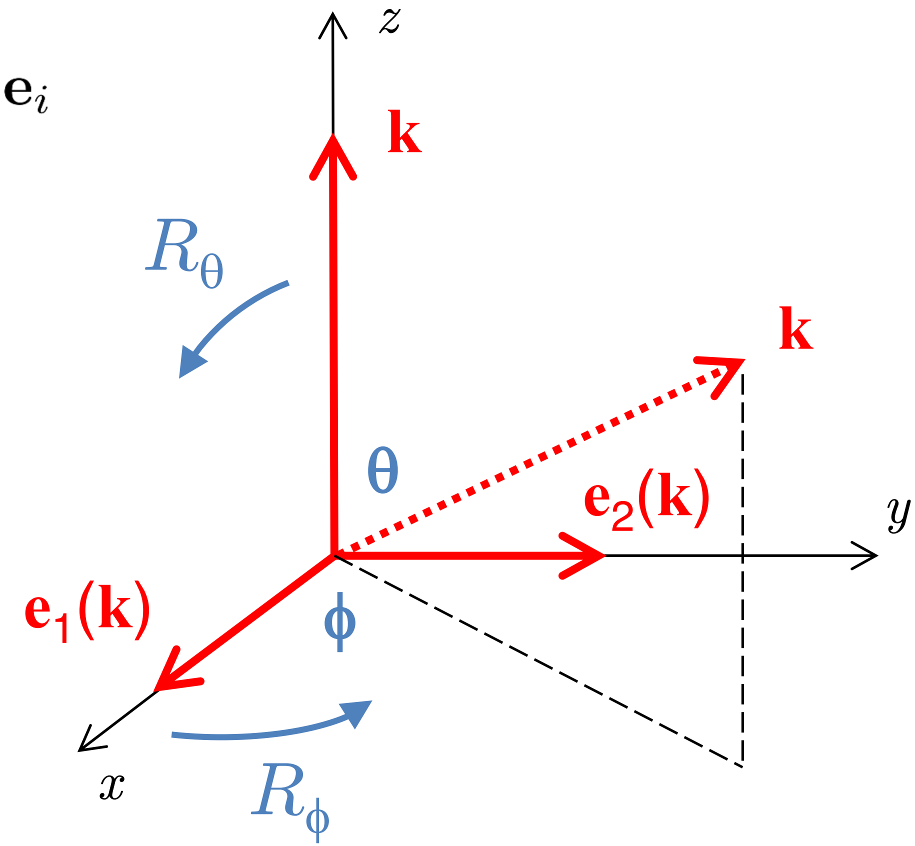
$$\mathbf{k} \to (R_\phi R_\theta)\mathbf{k} \; ; \qquad \mathbf{e}_i \to (R_\phi R_\theta)\mathbf{e}_i$$

$$R_\theta = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

(rotate through θ about the y axis)

$$R_\phi = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(rotate through ϕ about the z axis)



Polarisation States (4)

-- The combined effect of R_θ followed by R_ϕ on unit vectors along x,y,z is

$$(1,0,0) \to (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$$
$$(0,1,0) \to (-\sin\phi, \cos\phi, 0)$$
$$(0,0,1) \to (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

-- The vector $\mathbf{k} = (0,0,k) = |\mathbf{k}|(0,0,1)$ rotates to the direction (θ, ϕ) :

$$\mathbf{k} = |\mathbf{k}|(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

The plane polarisation vectors $\mathbf{e}_1 = (1,0,0)$ and $\mathbf{e}_2 = (0,1,0)$ rotate to

$$\mathbf{e}_1(\mathbf{k}) = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$$
$$\mathbf{e}_2(\mathbf{k}) = (-\sin\phi, \cos\phi, 0)$$

-- From equation (10.5.1), the circular polarisation vectors are then given by

$$\mathbf{e}_\mathrm{L}(\mathbf{k}) = -\frac{1}{\sqrt{2}}(\cos\theta\cos\phi - i\sin\phi,\ \cos\theta\sin\phi + i\cos\phi,\ -\sin\theta)$$
$$\mathbf{e}_\mathrm{R}(\mathbf{k}) = +\frac{1}{\sqrt{2}}(\cos\theta\cos\phi + i\sin\phi,\ \cos\theta\sin\phi - i\cos\phi,\ -\sin\theta)$$

Polarisation States (5)

-- In fact, it is the *spherical* *components* of the polarisation vectors which will be seen to have the most direct physical interpretation :

$$e_{+1} \equiv -\frac{1}{\sqrt{2}}(e_x + ie_y) \ , \qquad e_{-1} \equiv \frac{1}{\sqrt{2}}(e_x - ie_y) \ , \qquad e_0 \equiv e_z$$

e.g. the "$+1$" spherical component of the plane polarisation vector $\mathbf{e}_1$ is

$$(\mathbf{e}_1)_{+1} = -\frac{1}{\sqrt{2}}\left[(\mathbf{e}_1)_x + i(\mathbf{e}_1)_y\right]$$

$$= -\frac{1}{\sqrt{2}}(\cos\theta\cos\phi + i\cos\theta\sin\phi) = -\frac{1}{\sqrt{2}}\cos\theta\, e^{i\phi}$$

-- Overall, in spherical component form, the plane polarisation vectors are

$$\mathbf{e} = (e_{+1}, e_{-1}, e_0)$$

$$\boxed{\begin{aligned} \mathbf{e}_1(\mathbf{k}) &= -\frac{1}{\sqrt{2}}(\cos\theta e^{i\phi}, -\cos\theta e^{-i\phi}, -\sin\theta) \\ \mathbf{e}_2(\mathbf{k}) &= -\frac{i}{\sqrt{2}}(e^{i\phi}, e^{-i\phi}, 0) \end{aligned}}$$

Polarisation States (6)

-- The spherical components of the *circular* polarisation vectors can then be obtained using equation (10.5.1) again :

$$(\mathbf{e}_\mathrm{L})_m = -\frac{1}{\sqrt{2}}\left[(\mathbf{e}_1)_m + i(\mathbf{e}_2)_m\right] \; ; \qquad (\mathbf{e}_\mathrm{R})_m = \frac{1}{\sqrt{2}}\left[(\mathbf{e}_1)_m - i(\mathbf{e}_2)_m\right]$$

$$(m = +1, -1, 0)$$

-- This gives the spherical components of $\mathbf{e}_\mathrm{L}$ and e_R as

$$\mathbf{e} = (e_{+1}, e_{-1}, e_0)$$

$$\boxed{\begin{aligned} \mathbf{e}_\mathrm{L}(\mathbf{k}) &= -\left(\tfrac{1}{2}(1-\cos\theta)e^{i\phi},\ \tfrac{1}{2}(1+\cos\theta)e^{-i\phi},\ -\tfrac{1}{\sqrt{2}}\sin\theta\right) \\ \mathbf{e}_\mathrm{R}(\mathbf{k}) &= -\left(\tfrac{1}{2}(1+\cos\theta)e^{i\phi},\ \tfrac{1}{2}(1-\cos\theta)e^{-i\phi},\ \tfrac{1}{\sqrt{2}}\sin\theta\right) \end{aligned}}$$

-- Thus, when expressed in terms of spherical (rather than Cartesian) components :

→ all components of the polarisation vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_\mathrm{L}$, $\mathbf{e}_\mathrm{R}$, factorise into separate functions of θ and ϕ

Mode Expansion of the EM Field

-- Any real solution $\mathbf{A}(\mathbf{r},t)$ to the classical wave equation must be expressible as an expansion over all possible modes $(\mathbf{k},\lambda)$ of the EM field :

$$\mathbf{A}(\mathbf{r},t) = \sum_{\lambda=1}^{2} \int \mathrm{d}^3\mathbf{k} \left[A_{\mathbf{k},\lambda} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \mathbf{e}_\lambda(\mathbf{k}) + A^*_{\mathbf{k},\lambda} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \mathbf{e}^*_\lambda(\mathbf{k}) \right]$$

-- To integrate over $\mathbf{k}$, we employ the standard trick of imposing periodic boundary conditions on the surface of an arbitrary volume $V = L^3$

$\rightarrow$ the allowed vectors $\mathbf{k}$ occupy a regular 3D lattice of points with spacing $2\pi/L$ along k_x, k_y, k_z

-- The continuous integral over $\mathbf{k}$ space then becomes a summation over an infinite cubic lattice of discrete $\mathbf{k}$ points

The vector potential thus has the mode (plane wave) (Fourier) expansion

$$\boxed{\mathbf{A}(\mathbf{r},t) = \sum_{\mathbf{k},\lambda} \left[A_{\mathbf{k},\lambda} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \mathbf{e}_\lambda(\mathbf{k}) + A^*_{\mathbf{k},\lambda} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \mathbf{e}^*_\lambda(\mathbf{k}) \right]} \quad \textbf{(10.11.1)}$$

The EM Field : mode expansion (2)

-- The electric field, $\mathbf{E} = -\partial\mathbf{A}/\partial t$, has the mode expansion

$$\boxed{\mathbf{E}(\mathbf{r},t) = \sum_{\mathbf{k},\lambda} i\omega(\mathbf{k})\left[A_{\mathbf{k},\lambda}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\mathbf{e}_\lambda(\mathbf{k}) - A^*_{\mathbf{k},\lambda}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\mathbf{e}^*_\lambda(\mathbf{k})\right]}$$

-- To obtain the magnetic field $\mathbf{B} = \boldsymbol{\nabla}\wedge\mathbf{A}$, combine the vector identity

$$\boldsymbol{\nabla}\wedge(\alpha\mathbf{a}) = (\boldsymbol{\nabla}\alpha)\wedge\mathbf{a} + \alpha(\boldsymbol{\nabla}\wedge\mathbf{a})$$

with

$$\boldsymbol{\nabla}e^{i\mathbf{k}\cdot\mathbf{r}} = i(k_x,k_y,k_z)e^{i\mathbf{k}\cdot\mathbf{r}} = i\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{r}}$$

giving

$$\boldsymbol{\nabla}\wedge\left[A_{\mathbf{k},\lambda}e^{i\mathbf{k}\cdot\mathbf{r}}\mathbf{e}_\lambda(\mathbf{k})\right] = iA_{\mathbf{k},\lambda}e^{i\mathbf{k}\cdot\mathbf{r}}\left[\mathbf{k}\wedge\mathbf{e}_\lambda(\mathbf{k})\right]$$

-- This gives the mode expansion of the magnetic field as

$$\boxed{\mathbf{B}(\mathbf{r},t) = \sum_{\mathbf{k},\lambda} i\mathbf{k}\wedge\left[A_{\mathbf{k},\lambda}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\mathbf{e}_\lambda(\mathbf{k}) - A^*_{\mathbf{k},\lambda}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\mathbf{e}^*_\lambda(\mathbf{k})\right]}$$

Electromagnetic Field Energy

-- The total electromagnetic energy U contained within the normalisation volume V is

$$U = \frac{1}{2} \int_V \left(\epsilon_0 \mathbf{E} \cdot \mathbf{E} + \frac{1}{\mu_0} \mathbf{B} \cdot \mathbf{B} \right) \mathrm{d}V$$

-- Substituting the mode expansions of $\mathbf{E}$ and $\mathbf{B}$ from the previous slide into the above equation then gives the mode expansion of the energy U

A straightforward, but lengthy, calculation (see the Appendix if interested) gives the total energy as

$$\boxed{U = V \sum_{\mathbf{k},\lambda} \epsilon_0 \omega(\mathbf{k})^2 \left[A_{\mathbf{k},\lambda} A^*_{\mathbf{k},\lambda} + A^*_{\mathbf{k},\lambda} A_{\mathbf{k},\lambda} \right]}$$

-- Thus the total EM field energy U is in fact *constant* (independent of time and position) and depends only on the coefficients $A_{\mathbf{k},\lambda}$

EM Field Energy (2)

-- Thus far, everything has been completely classical, and, in that context, the above expression for U can be simplified as

$$U = 2V \sum_{\mathbf{k},\lambda} \epsilon_0 \omega(\mathbf{k})^2 |A_{\mathbf{k},\lambda}|^2$$

-- Alternatively, introduce dimensionless coefficients $a_{\mathbf{k},\lambda}$ by writing $A_{\mathbf{k},\lambda}$ as

$$A_{\mathbf{k},\lambda} = \sqrt{\frac{\hbar}{2\epsilon_0 \omega(\mathbf{k}) V}}\, a_{\mathbf{k},\lambda}\ , \qquad A^*_{\mathbf{k},\lambda} = \sqrt{\frac{\hbar}{2\epsilon_0 \omega(\mathbf{k}) V}}\, a^*_{\mathbf{k},\lambda}$$

The classical EM field energy U can then be written as

$$\boxed{U = \sum_{\mathbf{k},\lambda} \frac{1}{2} \hbar \omega(\mathbf{k}) \left[a_{\mathbf{k},\lambda} a^*_{\mathbf{k},\lambda} + a^*_{\mathbf{k},\lambda} a_{\mathbf{k},\lambda} \right]} \qquad \textbf{(10.14.1)}$$

-- In this form, U is reminiscent of the Hamiltonian for a quantum harmonic oscillator, which, expressed in terms of ladder operators, is

$$\hat{H} = \tfrac{1}{2} \hbar \omega \left[\hat{a} \hat{a}^\dagger + \hat{a}^\dagger \hat{a} \right]$$

This suggests a way of obtaining a *quantum* description of the EM field ...

Quantising the Electromagnetic Field

-- To quantise the EM field, we convert the coefficients $a_{\mathbf{k},\lambda}$ for each mode ($\mathbf{k}$,λ) into harmonic oscillator ladder operators for that mode :

$$a_{\mathbf{k},\lambda} \longrightarrow \hat{a}_{\mathbf{k},\lambda} \ , \qquad a^*_{\mathbf{k},\lambda} \longrightarrow \hat{a}^\dagger_{\mathbf{k},\lambda}$$

-- The classical energy U of equation (10.14.1) then becomes the quantum Hamiltonian operator for the EM field :

$$\boxed{\hat{H} = \sum_{\mathbf{k},\lambda} \frac{1}{2}\hbar\omega(\mathbf{k}) \left[\hat{a}_{\mathbf{k},\lambda}\hat{a}^\dagger_{\mathbf{k},\lambda} + \hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda} \right]}$$

-- Each mode ($\mathbf{k}$,λ) of the EM field becomes a separate quantum harmonic oscillator

i.e. the EM field consists of an infinite number of independent quantum harmonic oscillators

[two oscillators for each possible wavevector $\mathbf{k}$, corresponding to the two available polarisation states ($\lambda = 1, 2$ or $\lambda = \mathrm{L, R}$)]

The Quantum EM Field (2)

-- A single quantum harmonic oscillator has eigenstates $|n\rangle$ characterised by an integer quantum number $n = 0, 1, 2, 3, \ldots$

The ladder operators $a^\dagger$ and a increase or decrease n by one :

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$$
$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$\hat{a}|0\rangle = 0$$

-- Therefore the EM field has eigenstates characterised by an infinite number of integer quantum numbers $n_{\mathbf{k},\lambda} = 0, 1, 2, 3, \ldots$ (one for each mode)

-- The ladder operators for a given mode $(\mathbf{k},\lambda)$ change the quantum number (the mode occupancy) $n_{\mathbf{k},\lambda}$ for that mode by ± 1 :

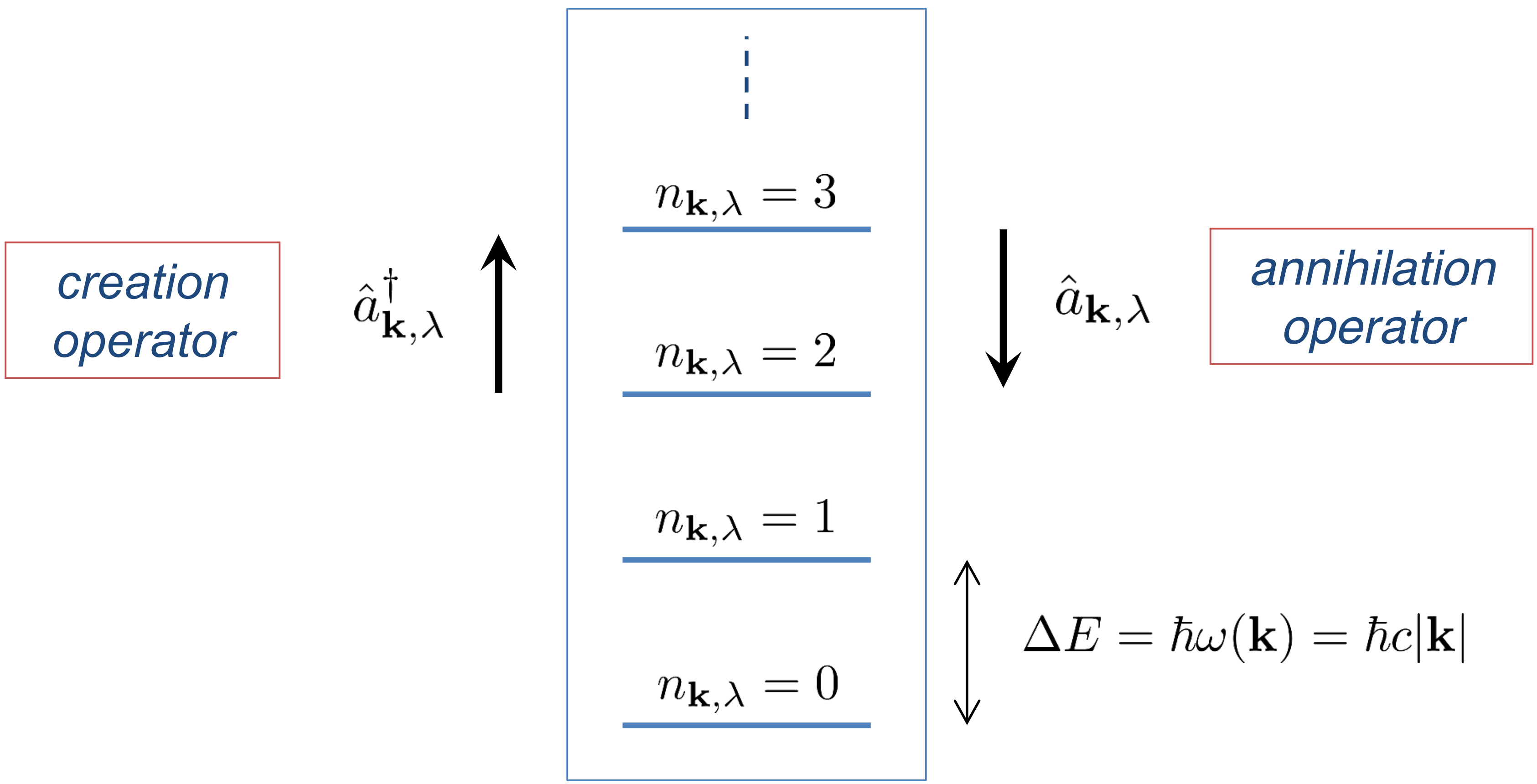
$$\hat{a}^\dagger_{\mathbf{k},\lambda}|n_{\mathbf{k},\lambda}\rangle = \sqrt{n_{\mathbf{k},\lambda}+1}\,|n_{\mathbf{k},\lambda}+1\rangle$$
$$\hat{a}_{\mathbf{k},\lambda}|n_{\mathbf{k},\lambda}\rangle = \sqrt{n_{\mathbf{k},\lambda}}\,|n_{\mathbf{k},\lambda}-1\rangle$$

$$\hat{a}_{\mathbf{k},\lambda}|n_{\mathbf{k},\lambda}=0\rangle = 0$$

All other mode occupancies, $n_{\mathbf{k}',\lambda'}$, are left unchanged

The Quantum EM Field (3)

-- The ladder operators for a given mode create and annihilate *photons* for that mode :



-- We interpret $|\ldots, n_{\mathbf{k},\lambda}, \ldots\rangle$ as a state which contains $n_{\mathbf{k},\lambda}$ photons in the mode with wave vector $\mathbf{k}$ and polarisation state λ

The Quantum EM Field (4)

-- The vacuum state $|0\rangle$ is the state containing no photons in *any* mode :

$$\hat{a}_{\mathbf{k},\lambda}|0\rangle = 0 \qquad \text{for } \textit{all } (\mathbf{k},\lambda)$$

-- States containing 1,2,3,... photons can then be built by operating on the vacuum state repeatedly with the creation operators $a^\dagger$:

e.g. a state containing a single photon in a particular mode $(\mathbf{k},\lambda)$:

$$\hat{a}^\dagger_{\mathbf{k},\lambda}|0\rangle = |\mathbf{k},\lambda\rangle = |1_{\mathbf{k},\lambda}\rangle$$

e.g. a state containing n photons in a single mode $(\mathbf{k},\lambda)$:

$$\frac{1}{\sqrt{n_{\mathbf{k},\lambda}!}}\left(\hat{a}^\dagger_{\mathbf{k},\lambda}\right)^{n_{\mathbf{k},\lambda}}|0\rangle = |n_{\mathbf{k},\lambda}\rangle$$

e.g. a two-photon state with the photons occupying two *different* modes :

$$|\mathbf{k}_1,\lambda_1;\mathbf{k}_2,\lambda_2\rangle = \hat{a}^\dagger_{\mathbf{k}_1,\lambda_1}\hat{a}^\dagger_{\mathbf{k}_2,\lambda_2}|0\rangle = |1_{\mathbf{k}_1,\lambda_1},1_{\mathbf{k}_2,\lambda_2}\rangle$$

The Quantum EM Field (5)

-- The quantum oscillators for distinct modes are independent of each other, so the creation and annihilation operators for distinct modes commute :

cf. slide 1.41 :

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1$$

$$\boxed{\begin{gathered}\left[\hat{a}_{\mathbf{k},\lambda}, \hat{a}^\dagger_{\mathbf{k}',\lambda'}\right] = \delta_{\mathbf{k}'\mathbf{k}}\, \delta_{\lambda'\lambda} \\ \left[\hat{a}_{\mathbf{k},\lambda}, \hat{a}_{\mathbf{k}',\lambda'}\right] = \left[\hat{a}^\dagger_{\mathbf{k},\lambda}, \hat{a}^\dagger_{\mathbf{k}',\lambda'}\right] = 0\end{gathered}}$$

-- Consider the two-photon state equivalent to that on the previous slide, but built by applying the creation operators in reverse order :

$$\left|\mathbf{k}_2, \lambda_2; \mathbf{k}_1, \lambda_1\right\rangle = \hat{a}^\dagger_{\mathbf{k}_2,\lambda_2} \hat{a}^\dagger_{\mathbf{k}_1,\lambda_1} \left|0\right\rangle$$

-- Since the creation operators commute, the two states are identical :

$$\left|\mathbf{k}_1, \lambda_1; \mathbf{k}_2, \lambda_2\right\rangle = \left|\mathbf{k}_2, \lambda_2; \mathbf{k}_1, \lambda_1\right\rangle$$

A two-photon state is therefore *symmetric* under particle interchange

i.e. photons are *bosons*

The Quantum EM Field (6)

-- To construct a relativistic quantum field theory of *spin-half* particles, it turns out that *anti-commutation* relations must be used instead of commutation relations; for example

$$\left[\hat{a}^\dagger_{\mathbf{k},\lambda}, \hat{a}^\dagger_{\mathbf{k}',\lambda'}\right]_+ \equiv \hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}^\dagger_{\mathbf{k}',\lambda'} + \hat{a}^\dagger_{\mathbf{k}',\lambda'}\hat{a}^\dagger_{\mathbf{k},\lambda} = 0$$

where the anti-commutator of two operators is defined as

$$\left[\hat{A}, \hat{B}\right]_+ \equiv \hat{A}\hat{B} + \hat{B}\hat{A}$$

-- Thus the creation operators for spin-half particles in different modes *anti-commute*, and states are *antisymmetric* under particle interchange :

$$|\mathbf{k}_1, \lambda_1; \mathbf{k}_2, \lambda_2\rangle = -|\mathbf{k}_2, \lambda_2; \mathbf{k}_1, \lambda_1\rangle$$

i.e. spin-half particles are *fermions*

-- This result (from quantum field theory) is known as the *spin-statistics theorem*

The Quantum EM Field (7)

-- An n-photon state $|n_{\mathbf{k},\lambda}\rangle$ is an eigenstate of the number operator

cf. slide 1.42 :

$$\hat{n} = \hat{a}^\dagger \hat{a}$$

$$\hat{n}|n\rangle = n|n\rangle$$

$$\boxed{\hat{n}_{\mathbf{k},\lambda} = \hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda}}$$

$$\hat{n}_{\mathbf{k},\lambda}|n_{\mathbf{k},\lambda}\rangle = n_{\mathbf{k},\lambda}|n_{\mathbf{k},\lambda}\rangle$$

-- The uncertainty Δn in the number of photons in a state $|n_{\mathbf{k},\lambda}\rangle$ is given by

$$(\Delta n_{\mathbf{k},\lambda})^2 \equiv \langle(\hat{n}_{\mathbf{k},\lambda})^2\rangle - \langle\hat{n}_{\mathbf{k},\lambda}\rangle^2$$

We have

$$\langle\hat{n}_{\mathbf{k},\lambda}\rangle = \langle n_{\mathbf{k},\lambda}|\hat{n}_{\mathbf{k},\lambda}|n_{\mathbf{k},\lambda}\rangle = n_{\mathbf{k},\lambda}$$

$$\langle(\hat{n}_{\mathbf{k},\lambda})^2\rangle = \langle n_{\mathbf{k},\lambda}|\hat{n}_{\mathbf{k},\lambda}\hat{n}_{\mathbf{k},\lambda}|n_{\mathbf{k},\lambda}\rangle = (n_{\mathbf{k},\lambda})^2$$

and hence

$$\boxed{\Delta n_{\mathbf{k},\lambda} = 0}$$

-- The n-photon states (also known as *Fock states*) therefore contain a well-defined number of photons (with no uncertainty in photon number)

The Fock states $|n_{\mathbf{k},\lambda}\rangle$ form a complete, orthonormal set of eigenstates

Zero Point Energy

-- The Hamiltonian for the EM field (slide 10.15) is

$$\hat{H} = \sum_{\mathbf{k},\lambda} \frac{1}{2}\hbar\omega(\mathbf{k}) \left(\hat{a}_{\mathbf{k},\lambda}\hat{a}^{\dagger}_{\mathbf{k},\lambda} + \hat{a}^{\dagger}_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda} \right)$$

-- The commutation relation (slide 10.19)

$$\hat{a}_{\mathbf{k}',\lambda'}\hat{a}^{\dagger}_{\mathbf{k},\lambda} - \hat{a}^{\dagger}_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k}',\lambda'} = \delta_{\mathbf{k}'\mathbf{k}}\,\delta_{\lambda'\lambda}$$

allows the operators in the first term of H to be reordered as

$$\hat{a}_{\mathbf{k},\lambda}\hat{a}^{\dagger}_{\mathbf{k},\lambda} = \hat{I} + \hat{a}^{\dagger}_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda}$$

-- The Hamiltonian H then becomes

$$\hat{H} = \sum_{\mathbf{k},\lambda} \frac{1}{2}\hbar\omega(\mathbf{k}) \left(\hat{I} + 2\hat{a}^{\dagger}_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda} \right)$$

or equivalently

$$\hat{H} = \sum_{\mathbf{k},\lambda} \hbar\omega(\mathbf{k}) \left(\hat{n}_{\mathbf{k},\lambda} + \frac{1}{2} \right)$$

Zero point energy (2)

-- For the vacuum state $|0\rangle$, the expectation value of the EM field energy is

$$U_0 = \langle 0|\hat{H}|0\rangle$$

Since, by definition, the vacuum state contains no photons in any mode, we have $a_{\mathbf{k},\lambda}|0\rangle = 0$ for all $\mathbf{k}$, λ ; hence the vacuum energy is

$$U_0 = \sum_{\mathbf{k},\lambda} \frac{1}{2}\hbar\omega(\mathbf{k}) = \infty \quad !!!$$

-- For our purposes (leading-order perturbation theory only), infinity is not a problem – we can just throw it away;

the justification being that only *relative* energies are observable

Defining all energies relative to the (infinite) energy U_0 of the vacuum state $|0\rangle$, the Hamiltonian of the EM field can then be taken to be

$$\boxed{\hat{H} = \sum_{\mathbf{k},\lambda} \hbar\omega(\mathbf{k})\hat{a}^{\dagger}_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda}} \quad \Rightarrow \quad \langle 0|\hat{H}|0\rangle = 0$$

Zero point energy (3)

-- The (infinite) zero-point energy of the EM field does in fact have genuine observable consequences :

e.g. it produces a small, but measurable, attractive force (the *Casimir force*) between two conductors at extremely small separations

-- The zero-point energy is an “easy” infinity to deal with :

Beyond leading-order, perturbation theory for QED and other quantum field theories is plagued by infinities and divergences of various type

A complex procedure known as *renormalisation* allows finite predictions to be obtained which can be compared with experiment

e.g. magnetic dipole moments (slide 3.44) and the Lamb shift (slide 4.48)

Photons

-- Operating with the revised Hamiltonian H of slide 10.23 on a single-photon state gives

$$\hat{H}|\mathbf{k},\lambda\rangle = \hat{H}\hat{a}^{\dagger}_{\mathbf{k},\lambda}|0\rangle = \left(\sum_{\mathbf{k}',\lambda'} \hbar\omega(\mathbf{k}')\hat{a}^{\dagger}_{\mathbf{k}',\lambda'}\hat{a}_{\mathbf{k}',\lambda'}\right)\hat{a}^{\dagger}_{\mathbf{k},\lambda}|0\rangle \qquad \text{(10.25.1)}$$

-- The commutation relation (slide 10.19)

$$\hat{a}_{\mathbf{k}',\lambda'}\hat{a}^{\dagger}_{\mathbf{k},\lambda} - \hat{a}^{\dagger}_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k}',\lambda'} = \delta_{\mathbf{k}'\mathbf{k}}\,\delta_{\lambda'\lambda}$$

allows the order of the rightmost two operators in equation (10.25.1) to be reversed :

$$\hat{a}_{\mathbf{k}',\lambda'}\hat{a}^{\dagger}_{\mathbf{k},\lambda} = \delta_{\mathbf{k}'\mathbf{k}}\,\delta_{\lambda'\lambda} + \hat{a}^{\dagger}_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k}',\lambda'}$$

This gives

$$\hat{H}|\mathbf{k},\lambda\rangle = \sum_{\mathbf{k}',\lambda'} \hbar\omega(\mathbf{k}')\hat{a}^{\dagger}_{\mathbf{k}',\lambda'}\left(\delta_{\mathbf{k}'\mathbf{k}}\,\delta_{\lambda'\lambda} + \hat{a}^{\dagger}_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k}',\lambda'}\right)|0\rangle$$

Photons : energy

-- Since $a_{\mathbf{k},\lambda}|0\rangle = 0$ for all modes, the final term above gives no contribution, leaving

$$\hat{H}|\mathbf{k},\lambda\rangle = \sum_{\mathbf{k}',\lambda'} \hbar\omega(\mathbf{k}')\hat{a}^{\dagger}_{\mathbf{k}',\lambda'}\delta_{\mathbf{k}'\mathbf{k}}\,\delta_{\lambda'\lambda}|0\rangle$$

$$= \hbar\omega(\mathbf{k})\hat{a}^{\dagger}_{\mathbf{k},\lambda}|0\rangle$$

Hence

$$\boxed{\hat{H}|\mathbf{k},\lambda\rangle = \hbar\omega(\mathbf{k})|\mathbf{k},\lambda\rangle}$$

-- Thus a single-photon state $|\mathbf{k},\lambda\rangle$ is an eigenstate of H with an energy (relative to the vacuum) given by

$$E_{\mathbf{k},\lambda} = \hbar\omega(\mathbf{k}) \qquad (\omega = c|\mathbf{k}|)$$

Photons : linear momentum

-- Classically, the Poynting vector $\mathbf{N} = \mathbf{E} \wedge \mathbf{H}$ represents the EM energy flux (energy flow per unit time per unit area)

Since EM energy propagates at speed c, the total linear momentum associated with the EM field is then (classically)

$$\mathbf{P} = \int \frac{1}{c^2}(\mathbf{E} \wedge \mathbf{H})\, \mathrm{d}^3\mathbf{r} = -\epsilon_0 \int \frac{\partial \mathbf{A}}{\partial t}(\mathbf{r}, t) \wedge \left[\boldsymbol{\nabla} \wedge \mathbf{A}(\mathbf{r}, t) \right] \mathrm{d}^3\mathbf{r}$$

-- In QED, the classical vector potential of equation (10.11.1) is converted to a quantum *field operator* (i.e. an operator which is a function of position $\mathbf{r}$)

$$\boxed{\hat{\mathbf{A}}(\mathbf{r}, t) = \sum_{\mathbf{k},\lambda} N(\mathbf{k}) \left[\hat{a}_{\mathbf{k},\lambda} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \mathbf{e}_\lambda(\mathbf{k}) + \hat{a}^\dagger_{\mathbf{k},\lambda} e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \mathbf{e}^*_\lambda(\mathbf{k}) \right]}$$

where (see slide 10.14) we have introduced, for convenience,

$$N(\mathbf{k}) \equiv \sqrt{\frac{\hbar}{2\epsilon_0 \omega(\mathbf{k}) V}}$$

Photons : linear momentum

-- The classical linear momentum vector $\mathbf{P}$ becomes the vector operator

$$\hat{\boldsymbol{P}} = -\epsilon_0 \int \frac{\partial \hat{\boldsymbol{A}}}{\partial t}(\mathbf{r},t) \wedge \left[\boldsymbol{\nabla} \wedge \hat{\boldsymbol{A}}(\mathbf{r},t)\right] \mathrm{d}^3\mathbf{r}$$

-- After substituting for $\boldsymbol{A}$ from the previous slide (and a straightforward but lengthy calculation), the total linear momentum operator is found to be

$$\hat{\boldsymbol{P}} = \sum_{\mathbf{k},\lambda} \hbar\mathbf{k}\, \hat{a}^{\dagger}_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda}$$

-- Applying this operator to a single-photon state, and using a similar argument to that used for the photon energy on slides 10.25 - 10.26, gives

$$\boxed{\hat{\boldsymbol{P}}|\mathbf{k},\lambda\rangle = \hbar\mathbf{k}|\mathbf{k},\lambda\rangle}$$

[EXAMPLES SHEET]

Thus a single photon in a mode $|\mathbf{k},\lambda\rangle$ has linear momentum $\hbar\mathbf{k}$

Photons : angular momentum (spin)

-- Classically, the total angular momentum of the EM field about a given point $\mathbf{r}_0$ has the form

$$\mathbf{J}(\mathbf{r}_0) = \mathbf{J}_{\mathrm{o}}(\mathbf{r}_0) + \mathbf{J}_{\mathrm{s}}$$

where the component $\mathbf{J}_{\mathrm{s}}$ is independent of $\mathbf{r}_0$:

$$\mathbf{J}_{\mathrm{s}} = \int \mathbf{E}(\mathbf{r},t) \wedge \mathbf{A}(\mathbf{r},t)\, \mathrm{d}^3\mathbf{r}$$

-- After quantisation (and another lengthy calculation) the angular momentum operator $\boldsymbol{J}_{\mathrm{s}}$ is found to be

$$\hat{\boldsymbol{J}}_{\mathrm{s}} = \hbar \sum_{\mathbf{k}} \frac{\mathbf{k}}{|\mathbf{k}|} \left[\hat{a}^{\dagger}_{\mathbf{k},L} \hat{a}_{\mathbf{k},L} - \hat{a}^{\dagger}_{\mathbf{k},R} \hat{a}_{\mathbf{k},R} \right]$$

where the creation operators for the circular polarisation states are defined by the operator equivalent of equation (10.5.1) :

$$\hat{a}^{\dagger}_{\mathbf{k},L} \equiv -\frac{1}{\sqrt{2}} \left[\hat{a}^{\dagger}_{\mathbf{k},1} + i \hat{a}^{\dagger}_{\mathbf{k},2} \right] \; ; \qquad \hat{a}^{\dagger}_{\mathbf{k},R} \equiv \frac{1}{\sqrt{2}} \left[\hat{a}^{\dagger}_{\mathbf{k},1} - i \hat{a}^{\dagger}_{\mathbf{k},2} \right]$$

Photons : spin (2)

-- Operating on circularly polarised single photon states then gives

$$\hat{\boldsymbol{J}}_{\mathrm{s}} \, |\mathbf{k}, L\rangle = +\hbar \frac{\mathbf{k}}{|\mathbf{k}|} \, |\mathbf{k}, L\rangle \, ; \qquad \hat{\boldsymbol{J}}_{\mathrm{s}} \, |\mathbf{k}, R\rangle = -\hbar \frac{\mathbf{k}}{|\mathbf{k}|} \, |\mathbf{k}, R\rangle$$

[EXAMPLES SHEET]

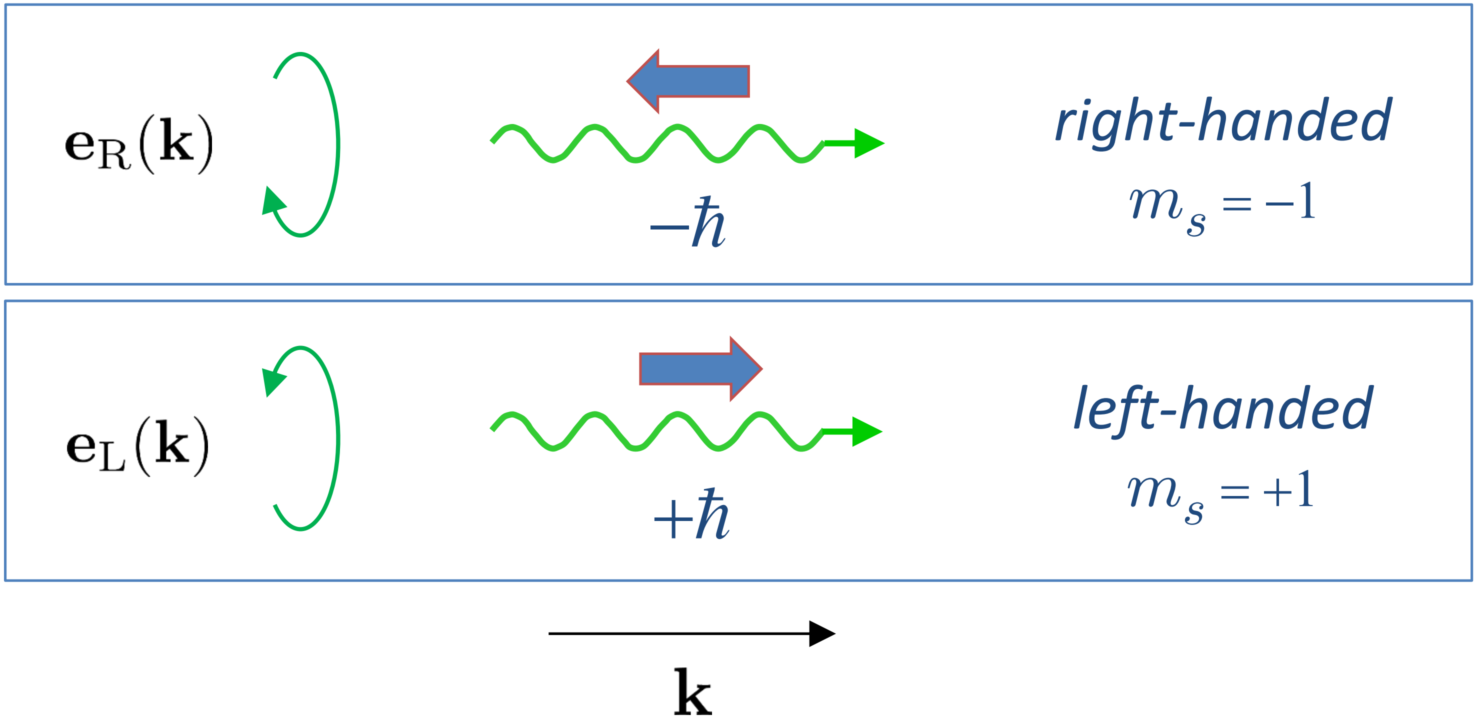
-- Since the operator $\boldsymbol{J}_{\mathrm{s}}$ is independent of position, we interpret it as an *intrinsic* angular momentum (spin) associated with the photon itself

-- The result above shows that, for circularly polarised photons, the photon spin is oriented parallel (L) or antiparallel (R) to the photon direction

The spin component projected along the photon direction is

$$\boxed{(J_{\mathrm{s}})_{\parallel} = \begin{cases} -\hbar & \text{for right-handed photons} \\ +\hbar & \text{for left-handed photons} \end{cases}}$$

Photons : spin (3)



-- Photons are therefore spin-one particles, but possess only *two* (not three) spin degrees of freedom :

- the longitudinal spin (polarisation) state $m_s = 0$ is “missing” ;

This missing state would correspond to photons with a longitudinal $\mathbf{E}$ field; thus photons are always *transversely polarised*

Photons : spin (4)

-- Beware that the convention used for L, R is not universal :

e.g. in particle physics : $\begin{cases} m_s = -1 & \Rightarrow \text{ "left-handed"} \\ m_s = +1 & \Rightarrow \text{ "right-handed"} \end{cases}$

(and similarly for spin-half particles and antiparticles)

-- In electromagnetism, the interactions of left-handed and right-handed particles have equal strength, as a result of which parity is conserved

In the weak interactions however, left-handed and right-handed particles interact with *different* strengths; this is the origin of *parity violation*

Appendix : Mode Expansion of the EM Energy

-- Here we provide the derivation of the result stated on slide 10.13, that the total classical EM field energy contained within a given volume V,

$$U = \frac{1}{2}\int_V \left(\epsilon_0 \mathbf{E}\cdot\mathbf{E} + \frac{1}{\mu_0}\mathbf{B}\cdot\mathbf{B}\right)\,\mathrm{d}V \qquad \textbf{(10.33.1)}$$

has the mode expansion

$$\boxed{U = V\sum_{\mathbf{k},\lambda} \epsilon_0 \omega(\mathbf{k})^2 \left[A_{\mathbf{k},\lambda}A^*_{\mathbf{k},\lambda} + A^*_{\mathbf{k},\lambda}A_{\mathbf{k},\lambda}\right]}$$

-- This result is obtained by using the mode expansions of $\mathbf{E}$ and $\mathbf{B}$ from slide 10.12 :

$$\mathbf{E}(\mathbf{r},t) = \sum_{\mathbf{k},\lambda} i\omega(\mathbf{k})\left[A_{\mathbf{k},\lambda}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\mathbf{e}_\lambda(\mathbf{k}) - A^*_{\mathbf{k},\lambda}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\mathbf{e}^*_\lambda(\mathbf{k})\right]$$

$$\mathbf{B}(\mathbf{r},t) = \sum_{\mathbf{k},\lambda} i\mathbf{k}\wedge\left[A_{\mathbf{k},\lambda}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\mathbf{e}_\lambda(\mathbf{k}) - A^*_{\mathbf{k},\lambda}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\mathbf{e}^*_\lambda(\mathbf{k})\right]$$

Mode expansion of EM field (2)

-- Substituting the mode expansions of $\mathbf{E}$ and $\mathbf{B}$ into equation (10.33.1) gives

$$U = \frac{\epsilon_0}{2}\int_V \left(\sum_{\mathbf{k},\lambda} i\omega(\mathbf{k})[...]\right)\cdot\left(\sum_{\mathbf{k}',\lambda'} i\omega(\mathbf{k}')[...]'\right)\mathrm{d}V$$
$$+\frac{1}{2\mu_0}\int_V \left(\sum_{\mathbf{k},\lambda} i\mathbf{k}\wedge[...]\right)\cdot\left(\sum_{\mathbf{k}',\lambda'} i\mathbf{k}'\wedge[...]'\right)\mathrm{d}V$$

where

$$[...] = \left[A_{\mathbf{k},\lambda}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\mathbf{e}_\lambda(\mathbf{k}) - A^*_{\mathbf{k},\lambda}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\mathbf{e}^*_\lambda(\mathbf{k})\right]$$
$$[...]' = \left[A_{\mathbf{k}',\lambda'}e^{i(\mathbf{k}'\cdot\mathbf{r}-\omega' t)}\mathbf{e}_{\lambda'}(\mathbf{k}') - A^*_{\mathbf{k}',\lambda'}e^{-i(\mathbf{k}'\cdot\mathbf{r}-\omega' t)}\mathbf{e}^*_{\lambda'}(\mathbf{k}')\right]$$

-- Multiplying out gives an expansion of the form

$$U = \frac{\epsilon_0}{2}\left[U_E(AA) + U_E(AA^*) + ...\right] + \frac{1}{2\mu_0}\left[U_B(AA) + U_B(AA^*) + ...\right]$$

where, for example, the term $U_E(AA^*)$ is

$$\int_V \left(\sum_{\mathbf{k},\lambda} i\omega(\mathbf{k})A_{\mathbf{k},\lambda}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\mathbf{e}_\lambda(\mathbf{k})\right)\cdot\left(\sum_{\mathbf{k}',\lambda'} -i\omega(\mathbf{k}')A^*_{\mathbf{k}',\lambda'}e^{-i(\mathbf{k}'\cdot\mathbf{r}-\omega' t)}\mathbf{e}^*_{\lambda'}(\mathbf{k}')\right)\mathrm{d}V$$

Mode expansion of EM field (3)

-- The integral over $\mathrm{d}V$, can be carried out using

$$\int_V e^{\pm i\mathbf{k}\cdot\mathbf{r}}\,\mathrm{d}V = \begin{cases} 0 & (\mathbf{k}\neq 0) \\ V & (\mathbf{k}=0) \end{cases} \quad \Rightarrow \quad \int_V e^{\pm i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}\,\mathrm{d}V = \delta^{(3)}(\mathbf{k}-\mathbf{k}')\,V$$

This gives $U_E(AA^*)$ as

$$V\left(\sum_{\mathbf{k},\lambda} i\omega(\mathbf{k})A_{\mathbf{k},\lambda}e^{-i\omega t}\mathbf{e}_\lambda(\mathbf{k})\right)\cdot\left(\sum_{\mathbf{k}',\lambda'} -i\omega(\mathbf{k}')A^*_{\mathbf{k}',\lambda'}e^{i\omega' t}\mathbf{e}^*_{\lambda'}(\mathbf{k}')\right)\delta^{(3)}(\mathbf{k}-\mathbf{k}')$$

-- The δ-function imposes the constraint

$$\mathbf{k}=\mathbf{k}' \quad \Rightarrow \quad \omega' \equiv \omega(\mathbf{k}') = \omega(\mathbf{k}) \equiv \omega$$

Hence, after carrying out the summation (integral) over $\mathbf{k}'$, the time-dependent phase factors cancel, leaving

$$U_E(AA^*) = V\left(\sum_{\mathbf{k},\lambda} i\omega(\mathbf{k})A_{\mathbf{k},\lambda}\mathbf{e}_\lambda(\mathbf{k})\right)\cdot\left(\sum_{\lambda'} -i\omega(\mathbf{k})A^*_{\mathbf{k},\lambda'}\mathbf{e}^*_{\lambda'}(\mathbf{k})\right)$$

Mode expansion of EM field (4)

-- This tidies up as

$$U_E(AA^*) = V \sum_{\mathbf{k},\lambda,\lambda'} \omega(\mathbf{k})^2 A_{\mathbf{k},\lambda} A^*_{\mathbf{k},\lambda'} \left[\mathbf{e}_\lambda(\mathbf{k}) \cdot \mathbf{e}^*_{\lambda'}(\mathbf{k})\right]$$

$$= V \sum_{\mathbf{k},\lambda,\lambda'} \omega(\mathbf{k})^2 A_{\mathbf{k},\lambda} A^*_{\mathbf{k},\lambda'} \, \delta_{\lambda,\lambda'} = V \sum_{\mathbf{k},\lambda} \omega(\mathbf{k})^2 A_{\mathbf{k},\lambda} A^*_{\mathbf{k},\lambda}$$

The contribution to the stored energy from this term is thus

$$\frac{\epsilon_0}{2} U_E(AA^*) = \frac{\epsilon_0}{2} V \sum_{\mathbf{k},\lambda} \omega(\mathbf{k})^2 A_{\mathbf{k},\lambda} A^*_{\mathbf{k},\lambda}$$

-- The $U_E(A^*A)$ term gives a similar (equal) contribution

$$\frac{\epsilon_0}{2} U_E(A^*A) = \frac{\epsilon_0}{2} V \sum_{\mathbf{k},\lambda} \omega(\mathbf{k})^2 A^*_{\mathbf{k},\lambda} A_{\mathbf{k},\lambda}$$

-- The $U_E(AA)$ term involves the integral

$$\int_V \left(\sum_{\mathbf{k},\lambda} i\omega(\mathbf{k}) A_{\mathbf{k},\lambda} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \mathbf{e}_\lambda(\mathbf{k}) \right) \cdot \left(\sum_{\mathbf{k}',\lambda'} i\omega(\mathbf{k}') A_{\mathbf{k}',\lambda'} e^{i(\mathbf{k}'\cdot\mathbf{r}-\omega' t)} \mathbf{e}_{\lambda'}(\mathbf{k}') \right) \mathrm{d}V$$

Mode expansion of EM field (5)

-- We now have

$$\int_V e^{\pm i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}}\,\mathrm{d}V = \delta^{(3)}(\mathbf{k}+\mathbf{k}')\,V$$

$$\Rightarrow \quad \mathbf{k} = -\mathbf{k}' \quad \Rightarrow \quad \omega' \equiv \omega(\mathbf{k}') = \omega(-\mathbf{k}) = \omega(\mathbf{k}) \equiv \omega$$

The time-dependent phase factors no longer cancel :

$$U_E(AA) = -V \sum_{\mathbf{k},\lambda,\lambda'} e^{-2i\omega t}\omega(\mathbf{k})^2 A_{\mathbf{k},\lambda} A_{-\mathbf{k},\lambda'} \left[\mathbf{e}_\lambda(\mathbf{k})\cdot\mathbf{e}_{\lambda'}(-\mathbf{k})\right]$$

-- Similarly, for $U_E(A^*A^*)$,

$$U_E(A^*A^*) = -V \sum_{\mathbf{k},\lambda,\lambda'} e^{2i\omega t}\omega(\mathbf{k})^2 A^*_{\mathbf{k},\lambda} A^*_{-\mathbf{k},\lambda'} \left[\mathbf{e}^*_\lambda(\mathbf{k})\cdot\mathbf{e}^*_{\lambda'}(-\mathbf{k})\right]$$

-- The $\mathbf{B}$ field energy includes terms such as $U_B(AA^*)$:

$$\int_V \left(\sum_{\mathbf{k},\lambda} i\mathbf{k}\wedge A_{\mathbf{k},\lambda} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\mathbf{e}_\lambda(\mathbf{k})\right)\cdot\left(\sum_{\mathbf{k}',\lambda'} -i\mathbf{k}'\wedge A^*_{\mathbf{k}',\lambda'} e^{-i(\mathbf{k}'\cdot\mathbf{r}-\omega' t)}\mathbf{e}^*_{\lambda'}(\mathbf{k}')\right)\mathrm{d}V$$

Mode expansion of EM field (6)

-- The AA^* and A^*A integrals can be evaluated using

$$\frac{1}{|\mathbf{k}|^2}[\mathbf{k} \wedge \mathbf{e}_\lambda(\mathbf{k})] \cdot [\mathbf{k} \wedge \mathbf{e}^*_{\lambda'}(\mathbf{k})] = \delta_{\lambda\lambda'}$$

$$\Rightarrow \quad \frac{1}{2\mu_0} U_B(AA^*) = \frac{1}{2\mu_0} V \sum_{\mathbf{k},\lambda} |\mathbf{k}|^2 A_\lambda(\mathbf{k}) A^*_\lambda(\mathbf{k})$$

Since

$$c^2 = \frac{1}{\mu_0 \epsilon_0} = \frac{\omega(\mathbf{k})^2}{|\mathbf{k}|^2}$$

this is *equal* to the corresponding $\mathbf{E}$ field term :

$$\frac{1}{2\mu_0} U_B(AA^*) = \frac{\epsilon_0}{2} U_E(AA^*)$$

-- The $\mathbf{B}$ field AA and A^*A^* terms involve

$$\frac{1}{|\mathbf{k}|^2}[\mathbf{k} \wedge \mathbf{e}_\lambda(\mathbf{k})] \cdot [-\mathbf{k} \wedge \mathbf{e}_{\lambda'}(-\mathbf{k})] = -\mathbf{e}_\lambda(\mathbf{k}) \cdot \mathbf{e}_{\lambda'}(-\mathbf{k})$$

and *cancel* against the equivalent $\mathbf{E}$ field terms :

$$\frac{1}{2\mu_0} U_B(AA) = -\frac{\epsilon_0}{2} U_E(AA)$$

Mode expansion of EM field (7)

-- Combining the various contributions, the AA and A^*A^* terms are all time-dependent but cancel each other, leaving just the constant AA^* terms :

$$U = \frac{\epsilon_0}{2}\left[U_E(AA^*) + U_E(A^*A)\right] + \frac{1}{2\mu_0}\left[U_B(AA^*) + U_B(A^*A)\right]$$

-- The $\mathbf{E}$ field and $\mathbf{B}$ field contributions are equal :

$$U = \epsilon_0\left[U_E(AA^*) + U_E(A^*A)\right]$$

-- Thus the total, classical EM energy in a volume V is

$$\boxed{U = V\sum_{\mathbf{k},\lambda}\epsilon_0\omega(\mathbf{k})^2\left[A_{\mathbf{k},\lambda}A^*_{\mathbf{k},\lambda} + A^*_{\mathbf{k},\lambda}A_{\mathbf{k},\lambda}\right]}$$

This simplifies as

$$U = 2V\sum_{\mathbf{k},\lambda}\epsilon_0\omega(\mathbf{k})^2|A_{\mathbf{k},\lambda}|^2$$