XII. COSMOLOGY

In this topic, we shall apply General Relativity to model the Universe as a whole.

The late-time Universe is clearly very complicated and we cannot hope to find exact, analytic solutions of the Einstein field equations in this case.

However, if we smooth on sufficiently large spatial scales (of the order of $100\,\mathrm{Mpc} \sim 10^8$ light years), the Universe looks remarkably symmetric in space.

This symmetry allows us to find analytic solutions for the spacetime of a smoothed-out universe.

These solution are the starting point for all cosmological studies, with the structures that we observe on smaller scales treated as a (not necessarily small amplitude) perturbation to the highly symmetric background.

1 Homogeneity and isotropy

Cosmological observations, most notably of the cosmic microwave background (CMB) radiation, suggest that at any given time the Universe looks very nearly the same in all directions, i.e., is *isotropic* about us.

Of course, we can only make observations on our past lightcone and along one particular wordline through spacetime.

However, if we assume that we are not in a privileged location, then it should be possible to construct a whole class of observers, filling space, who all observe the Universe to be isotropic.

Moreover, with appropriate synchronisation of their clocks, the fundamental observers must agree on what they observe at any given proper time, i.e., the Universe is spatially *homogeneous*.

We shall call these preferred observers *fundamental observers*.

From these symmetries alone, it follows that the fundamental observers must have the following properties.

- They must comove with the matter in the Universe, since if they did not the 3-velocity of the matter they measure locally would break isotropy.
- They must be free-falling, since acceleration (i.e., departure from motion along a geodesic) would also break isotropy.
- If we consider the (smoothed out) matter in the Universe as a fluid, the hypersurfaces of constant proper density must be orthogonal to the worldlines of the fundamental observers. Local measurements of the density in the instantaneous rest-space of each fundamental observer must reveal no spatial gradients (or else these would break isotropy), so the local rest-spaces must be tangent to the hypersurfaces of constant density (see Fig. 1).

This last point also ensures that there are no pressure gradients in the instantaneous rest-space of the fluid, which would otherwise cause acceleration.

1.1 Synchronous coordinates

We adopt a coordinate system, called *synchronous coordinates*, adapted to the fundamental observers as follows.

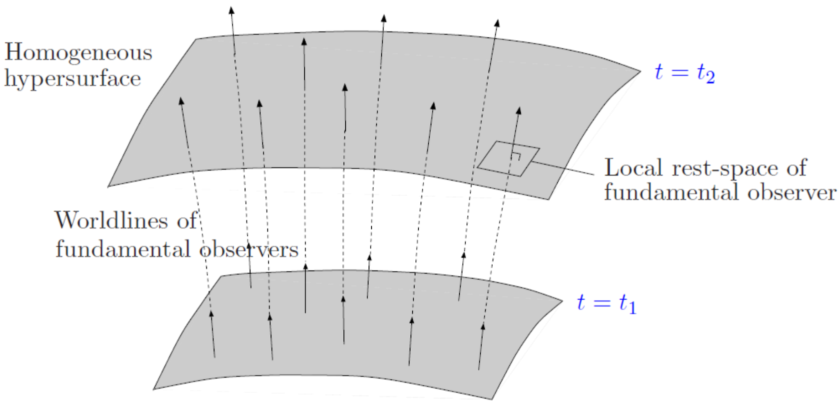


Figure 1: Worldlines of fundamental observers, who move with the matter in the Universe, are orthogonal to the spacelike hypersurfaces of homogeneity. This ensures that the instantaneous rest-spaces of each observer lie in the homogeneous hypersurfaces.

We assign fixed spatial coordinates x^i $(i = 1, 2, 3)$ to each fundamental observer.

We label the surfaces of homogeneity, i.e., the hypersurfaces of constant proper density, with proper time as measured by one of the fundamental observers.

We call this *synchronous* or *cosmic time* and denote it by $x^0 = t$.

By homogeneity, all fundamental observers will see cosmic time pass at the same rate as their proper time.

The line element in these synchronous coordinates must take the following form:

$$ds^2 = c^2dt^2 + g_{ij}(t, \vec{x})dx^i dx^j \ , \tag{1}$$

where there is an implicit summation over repeated spatial indices.

To see this, first note that the wordlines of fundamental observers are of the form $x^\mu = (t, x^i)$, where the x^i are constant for each observer.

It follows that $dx^\mu = \delta^\mu_0 dt$ along the wordline so that

$$ds^2 = g_{00}dt^2 = c^2 dt^2 \,. \tag{2}$$

This must equal $c^2 d\tau^2$, where τ is proper time for the observer, so that we can take $t = \tau$.

Second, note that any infinitesimal displacement in the hypersurface $t =$ const. is of the form $dx^\mu = (0, dx^i)$, and this must be orthogonal everywhere to the 4-velocity of the fundamental observers, $u^\mu = \delta^\mu_0$.

This requires $g_{0i} = 0$.

Finally, we must check that the worldlines $x^\mu = (t, x^i)$, for fixed x^i, are geodesics.

As t is an affine parameter, we require

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = 0 \qquad \Rightarrow \qquad \Gamma^\mu_{00} = 0 \,. \tag{3}$$

This is satisfied for the metric in Eq. (1) since

$$\Gamma^\mu_{00} = \frac{1}{2} g^{\mu\nu}\left(2\partial_0 g_{\nu 0} - \partial_\nu g_{00}\right) = 0 \,, \tag{4}$$

as $g_{00} = c^2$ and $g_{0i} = 0$.

2 The Robertson–Walker metric

The intrinsic geometry of the hypersurfaces $t =$ const. is determined by the spatial components of the metric $g_{ij}(t, \vec{x})$.

This intrinsic geometry must be consistent with isotropy and homogeneity.

This will only be the case for all t if every component of g_{ij} evolves in the same way with t, so that we can write

$$g_{ij}(t, \vec{x}) = -a^2(t)\gamma_{ij}(\vec{x}) \,. \tag{5}$$

Here, $a(t)$ is the *scale factor*, which determines the overall scale of the intrinsic geometry of the $t =$ const. surfaces.

The $\gamma_{ij}(\vec{x})$ play the role (up to scaling) of the 3D metric in the surfaces $t =$ const.; under time-independent coordinate transformations $x^i \to x'^i(\vec{x})$ in spacetime, γ_{ij} transform as a 3D type-$(0,2)$ tensor.

We require γ_{ij} to describe a 3D space that is homogeneous and isotropic.

Isotropy implies that γ_{ij} must be spherically symmetric so we can always write the 3D line element in spherical-polar coordinates as

$$\begin{aligned} d\sigma^2 &= \gamma_{ij} dx^i dx^j \\ &= B(r) dr^2 + r^2 d\Omega^2 \,, \end{aligned} \tag{6}$$

where $d\Omega^2$ is the metric on the unit 2-sphere.

Here, we have used the residual freedom in our coordinates to adopt an area-like radial coordinate r.

It follows that the components of the metric and its inverse are

$$\begin{aligned} \gamma_{rr} &= B(r)\,, & \gamma^{rr} &= \frac{1}{B(r)}\,, \\ \gamma_{\theta\theta} &= r^2\,, & \gamma^{\theta\theta} &= \frac{1}{r^2}\,, \\ \gamma_{\phi\phi} &= r^2 \sin^2\theta\,, & \gamma^{\phi\phi} &= \frac{1}{r^2 \sin^2\theta}\,. \end{aligned} \tag{7}$$

We can construct a 3D metric connection, ${}^{(3)}\Gamma^i_{jk}$, from

γ_{ij} with (independent) non-zero components

$$
\begin{aligned}
{}^{(3)}\Gamma^r_{rr} &= \frac{1}{2B}\frac{dB}{dr}\,, & {}^{(3)}\Gamma^r_{\theta\theta} &= -\frac{r}{B}\,, \\
{}^{(3)}\Gamma^r_{\phi\phi} &= -\frac{r\sin^2\theta}{B}\,, & {}^{(3)}\Gamma^\theta_{r\theta} &= \frac{1}{r}\,, \\
{}^{(3)}\Gamma^\theta_{\phi\phi} &= -\sin\theta\cos\theta\,, & {}^{(3)}\Gamma^\phi_{r\phi} &= \frac{1}{r}\,, \\
{}^{(3)}\Gamma^\phi_{\theta\phi} &= \cot\theta\,.
\end{aligned}
\tag{8}
$$

The 3D Riemann curvature tensor is given by

$$
{}^{(3)}R_{ijk}{}^l = -\partial_i {}^{(3)}\Gamma^l_{jk} + \partial_j {}^{(3)}\Gamma^l_{ik} + {}^{(3)}\Gamma^m_{ik}{}^{(3)}\Gamma^l_{jm} - {}^{(3)}\Gamma^m_{jk}{}^{(3)}\Gamma^l_{im}\,. \tag{9}
$$

It has six independent components, but only three of these are non-zero for the spherically-symmetric metric:

$$
{}^{(3)}R_{r\theta r\theta} = \frac{r}{2B}\frac{dB}{dr}\,, \tag{10}
$$

$$
{}^{(3)}R_{r\phi r\phi} = \frac{r}{2B}\frac{dB}{dr}\sin^2\theta\,, \tag{11}
$$

$$
{}^{(3)}R_{\theta\phi\theta\phi} = \left(1-\frac{1}{B}\right) r^2\sin^2\theta\,. \tag{12}
$$

Contracting with γ_{ij}, we can form the 3D Ricci tensor:

$$
{}^{(3)}R_{ij} = \gamma^{kl}\,{}^{(3)}R_{kijl}\,, \tag{13}
$$

which has non-zero (independent) components

$$
{}^{(3)}R_{rr} = -\frac{1}{rB}\frac{dB}{dr}\,, \tag{14}
$$

$$
{}^{(3)}R_{\theta\theta} = -1 + \frac{1}{B} - \frac{r}{2B^2}\frac{dB}{dr}\,, \tag{15}
$$

$$
{}^{(3)}R_{\phi\phi} = \sin^2\theta\,{}^{(3)}R_{\theta\theta}\,. \tag{16}
$$

Finally, we can form the 3D Ricci scalar

$$
\begin{aligned}
{}^{(3)}R = \gamma^{ij}\,{}^{(3)}R_{ij} &= -\frac{2}{r^2}\left(1-\frac{1}{B}+\frac{r}{B^2}\frac{dB}{dr}\right) \\
&= -\frac{2}{r^2}\left[1-\frac{d}{dr}\left(\frac{r}{B}\right)\right]\,.
\end{aligned}
\tag{17}
$$

If the 3D space is homogeneous, the Ricci scalar cannot depend on position.

Writing

$$^{(3)}R = -6K\,, \tag{18}$$

where K is a constant (and the prefactor is for later convenience), we require

$$\begin{aligned} 1 - \frac{d}{dr}\left(\frac{r}{B}\right) &= 3Kr^2 \\ \Rightarrow \quad \frac{r}{B} &= A + r - Kr^3 \\ \Rightarrow \quad B &= \frac{1}{A/r + (1 - Kr^2)}\,. \end{aligned} \tag{19}$$

Here, A is an integration constant, but this must vanish for the space to be homogeneous.

To see this, note that we require all curvature invariants to be constant; for example, considering

$$R_{ij}R^{ij} = 12K^2 + \frac{3A^2}{2r^6}\,, \tag{20}$$

we find that this is only homogeneous for $A = 0$.

The final form of the 3D line element for isotropic and homogeneous spaces is thus

$$\boxed{d\sigma^2 = \frac{dr^2}{(1 - Kr^2)} + r^2 d\Omega^2\,.} \tag{21}$$

The spacetime line element is then of *Robertson–Walker* form

$$\boxed{ds^2 = c^2dt^2 - a^2(t)\left[\frac{dr^2}{(1 - Kr^2)} + r^2 d\Omega^2\right]\,.} \tag{22}$$

Maximally-symmetric spaces

Substituting for $B(r) = \left(1 - Kr^2\right)^{-1}$ in the components of the 3D Ricci tensor, we have

$$\begin{aligned} {}^{(3)}R_{rr} &= \frac{-2K}{(1-Kr^2)} = -2K\gamma_{rr} \\ {}^{(3)}R_{\theta\theta} &= -2Kr^2 = -2K\gamma_{\theta\theta} \\ {}^{(3)}R_{\phi\phi} &= -2Kr^2\sin^2\theta = -2K\gamma_{\phi\phi}\,. \end{aligned} \tag{23}$$

It follows that

$$ {}^{(3)}R_{ij} = -2K\gamma_{ij}\,, \tag{24}$$

which is a tensor equation valid in any coordinates x^i.

Moreover, evaluating the components of the 3D Riemann tensor, we find

$$ {}^{(3)}R_{ijkl} = K\left(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}\right)\,, \tag{25}$$

It is easy to check that the tensor on the right has all the symmetries required of the Riemann tensor.

Generally, manifolds that have a Riemann tensor of the form

$$R_{abcd} = K\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right) \tag{26}$$

are known as *maximally-symmetric spaces* and they have the same number of symmetries as Euclidean space of the same dimension.

The quantity K has to be constant by virtue of the Bianchi identity.

For example, in ND the Ricci tensor and scalar in a maximally-symmetric space are

$$R_{ab} = K(N-1)g_{ab}\,, \tag{27}$$

$$R = KN(N-1)\,, \tag{28}$$

so the contracted Bianchi identity gives

$$\nabla^a \left(R_{ab} - \frac{1}{2} g_{ab} R \right) = 0$$
$$\Rightarrow \quad \frac{1}{2}(N-1)(N-2)\nabla^a \left(K g_{ab}\right) = 0\,, \qquad (29)$$

which implies $\nabla_b K = 0$ (so K is constant) for $N > 2$.

The space with the line element in Eq. (21) is therefore a 3D maximally-symmetric space.

Isotropy is manifest, but homogeneity is rather hidden by having made an explicit choice of origin.

2.1 Geometry of the 3D spaces

The properties of the 3D maximally-symmetric spaces with line element

$$d\sigma^2 = \frac{dr^2}{(1-Kr^2)} + r^2 d\Omega^2 \qquad (30)$$

depend on whether K is positive, negative or zero.

$K = 0$

With $K = 0$, we have Euclidean space in spherical-polar coordinates:

$$d\sigma^2 = dr^2 + r^2 d\Omega^2\,. \qquad (31)$$

$K > 0$

For $K > 0$, it is convenient to switch from the dimensionfull r coordinate to another dimensionfull coordinate χ, where

$$r = \frac{1}{\sqrt{K}} \sin(\sqrt{K}\chi) \equiv S_K(\chi)\,. \qquad (32)$$

Using $dr = \cos(\sqrt{K}\chi)d\chi$, the line element becomes

$$d\sigma^2 = d\chi^2 + S_K^2(\chi)d\Omega^2 \,. \tag{33}$$

This is the same as the line element on the surface of a 3-sphere of radius $1/\sqrt{K}$ embedded in 4D Euclidean space, which we can see as follows.

Let (w, x, y, z) be Cartesian coordinates in $\mathbb{R}^4$, so that

$$ds^2 = dw^2 + dx^2 + dy^2 + dz^2 \,. \tag{34}$$

The 3-sphere has

$$w^2 + x^2 + y^2 + z^2 = 1/K \,, \tag{35}$$

and can be parameterised by

$$w = \frac{1}{\sqrt{K}} \cos(\sqrt{K}\chi) \,, \tag{36}$$

$$x = S_K(\chi) \sin\theta \cos\phi \,, \tag{37}$$

$$y = S_K(\chi) \sin\theta \sin\phi \,, \tag{38}$$

$$z = S_K(\chi) \cos\theta \,. \tag{39}$$

Here, θ and ϕ are the usual angular coordinates and $0 \leq \sqrt{K}\chi \leq \pi$.

The induced metric on the 3-sphere follows from

$$dx^2 + dy^2 + dz^2 = \cos^2(\sqrt{K}\chi)d\chi^2 + S_K^2(\chi)d\Omega^2 \,, \tag{40}$$

and

$$dw^2 = \sin^2(\sqrt{K}\chi)d\chi^2 \,; \tag{41}$$

adding these together gives Eq. (33).

It follows that 3D maximally-symmetric space with $K > 0$ is compact (or *closed*).

The area of a 2-sphere $\chi = \text{const.}$ is

$$A = 4\pi S_K^2(\chi) \,, \tag{42}$$

and grows from zero at $\chi = 0$ to a maximum of $4\pi/K$ at $\sqrt{K}\chi = \pi/2$, before shrinking back to zero as $\sqrt{K}\chi \to \pi$.

The space has finite volume

$$V = 4\pi \int_0^{\pi/\sqrt{K}} S_K^2(\chi)\, d\chi = \frac{2\pi^2}{K^{3/2}}\,. \tag{43}$$

$K < 0$

For $K < 0$, we write

$$r = \frac{1}{\sqrt{|K|}} \sinh(\sqrt{|K|}\chi) \equiv S_K(\chi)\,, \tag{44}$$

which defines $S_K(\chi)$ for $K < 0$.

Now, $dr = \cosh(\sqrt{|K|}\chi) d\chi$, and the line element becomes

$$d\sigma^2 = d\chi^2 + S_K^2(\chi) d\Omega^2\,. \tag{45}$$

Up to a sign, this is the same as the line element on the surface of spacelike hyperboloid,

$$w^2 - x^2 - y^2 - z^2 = \frac{1}{|K|}\,, \tag{46}$$

embedded in Minkowski space

$$ds^2 = dw^2 - dx^2 - dy^2 - dz^2\,. \tag{47}$$

The hyperboloid can be parameterised by

$$w = \frac{1}{\sqrt{|K|}} \cosh(\sqrt{|K|}\chi)\,, \tag{48}$$
$$x = S_K(\chi) \sin\theta \cos\phi\,, \tag{49}$$
$$y = S_K(\chi) \sin\theta \sin\phi\,, \tag{50}$$
$$z = S_K(\chi) \cos\theta\,, \tag{51}$$

where, now, $0 \leq \sqrt{|K|}\chi < \infty$.

The induced metric on the hyperboloid follows from

$$dx^2 + dy^2 + dz^2 = \cosh^2(\sqrt{|K|}\chi)d\chi^2 + S_K^2(\chi)d\Omega^2 \,, \quad (52)$$

and

$$dw^2 = \sinh^2(\sqrt{|K|}\chi)d\chi^2 \,, \quad (53)$$

which combine to give Eq. (45).

The 3D maximally-symmetric space with $K < 0$ is infinite (or *open*) and has infinite volume.

In terms of the radial coordinate χ, we can write the spacetime line element for all three cases as

$$ds^2 = c^2dt^2 - a^2(t)\left[d\chi^2 + S_K^2(\chi)d\Omega^2\right] \,, \quad (54)$$

where

$$S_K(\chi) \equiv \begin{cases} \sin(\sqrt{K}\chi)/\sqrt{K} & \text{for } K > 0 \,, \\ \chi & \text{for } K = 0 \,, \\ \sinh(\sqrt{|K|}\chi)/\sqrt{|K|} & \text{for } K < 0 \,. \end{cases} \quad (55)$$

Note that $S_K(\chi)$ is continuous in K at $K = 0$ as, e.g., $\lim_{K\to 0} S_K(\chi) = \chi$ from above and below.

Note further that (χ, θ, ϕ) are still comoving coordinates as χ is related to r through a time-independent transformation.

3 An expanding universe

Consider the fundamental observer at $\chi = 0$ and a neighbouring one at $\Delta\chi$.

The proper distance between these observers is $l(t) = a(t)\Delta\chi$.

If the scale factor depends on time, this distance will also change as

$$\frac{1}{l(t)}\frac{dl(t)}{dt} = \frac{1}{a}\frac{da}{dt} \equiv H(t)\,, \tag{56}$$

which defines the *Hubble parameter* $H(t)$.

For $H > 0$, the observer at the origin sees all fundamental observers (and hence the matter in the Universe) moving away isotropically with the same fractional rate.

The same is true for any other fundamental observer by isotropy.[1]

We know that our Universe is expanding as the light from distant galaxies is observed to be redshifted (see below).

The current value of the Hubble parameter is denoted by H_0; its value is around $68\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

3.1 Cosmological redshift

Consider a photon emitted by a fundamental observer at coordinates $(t_E, 0, 0, 0)$, which is received later by a fundamental observer at $(t_R, \chi_R, \theta_R, \phi_R)$.

The radial (symmetry!) path of the photon is

$$x^\mu(\lambda) = (t(\lambda), \chi(\lambda), \theta_R, \phi_R)\,, \tag{57}$$

where λ is an affine parameter.

The 4-momentum of the photon is

$$p^\mu = \frac{dx^\mu}{d\lambda} = (p^0, p^1, 0, 0)\,, \tag{58}$$

[1] An often-quoted analogy is the surface of a balloon covered in spots that is being inflated. Every spot sees all others moving away isotropically.

and lowering the index with the (diagonal) metric gives

$$p_\mu = (p_0, p_1, 0, 0)\,, \tag{59}$$

where, numerically, $p_0 = c^2 p^0$ and $p_1 = -a^2 p^1$.

The photon travels on a null geodesic, parallel transporting p^μ.

It is convenient to consider the geodesic equation in the form (see Handout IV)

$$\frac{dp_\mu}{d\lambda} = \frac{1}{2}\frac{\partial g_{\nu\rho}}{\partial x^\mu} p^\nu p^\rho\,. \tag{60}$$

We have

$$\begin{aligned} \frac{dp_1}{d\lambda} &= \frac{1}{2}\left(\frac{\partial g_{00}}{\partial \chi} p^0 p^0 + \frac{\partial g_{11}}{\partial \chi} p^1 p^1\right) \\ \Rightarrow \quad p_1 &= \text{const.}\,, \end{aligned} \tag{61}$$

where we have used $g_{00} = c^2$ and $g_{11} = -a^2$, both of which are independent of χ.

The energy of a photon of 4-momentum p^μ as measured by a fundamental observer with 4-velocity u^μ is

$$E = g_{\mu\nu} p^\mu u^\nu\,. \tag{62}$$

As discussed earlier, $u^\mu = \delta^\mu_0$, so that

$$E = p_\mu u^\mu = p_0\,. \tag{63}$$

We can relate p_0 to the conserved p_1 using the null condition

$$g^{\mu\nu} p_\mu p_\nu = 0 \quad \Rightarrow \quad c^{-2}(p_0)^2 - a^{-2}(p_1)^2 = 0\,. \tag{64}$$

It follows that $p_0 = cp_1/a$ so

$$Ea(t) = \text{const.} \tag{65}$$

Finally, we can compute the redshift of the photon:

$$1 + z \equiv \frac{\lambda_R}{\lambda_E} = \frac{E_E}{E_R} = \frac{a(t_R)}{a(t_E)}\,, \tag{66}$$

where λ_E and λ_R are the emitted and received wavelengths, respectively, and E_E and E_R are the associated energies.

We see that in an expanding universe, light from distant galaxies (assumed to be moving almost as fundamental observers) will be redshifted.

The first evidence for this effect was obtained by Edwin Hubble in 1929.

Note that we performed the calculation of the redshift for the simple case that one of the observers was at the origin.

However, by homogeneity, the same result will hold for any pair of fundamental observers separated by the same (shortest) proper distance χ_R.

4 Cosmological field equations

The Robertson–Walker metric contains a single function of time $a(t)$ whose evolution is determined by the Einstein field equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu}\,, \tag{67}$$

where, recall, $\kappa \equiv 8\pi G/c^4$.

We have included the cosmological constant term $\Lambda g_{\mu\nu}$ as this can be important on cosmological scales.

For the energy–momentum tensor, isotropy demands that we consider the ideal fluid form

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu - pg^{\mu\nu}\,, \tag{68}$$

since this is isotropic in the instantaneous rest-frame defined by u^μ.

We must take u^μ to coincide with the 4-velocity of the fundamental observers.

Homogeneity demands that the proper energy density ρc^2 and pressure p are functions of time only.

4.1 Friedmann equations

Our starting point is the spacetime line element, which we write in the form

$$ds^2 = c^2 dt^2 - a^2(t)\gamma_{ij} dx^i dx^j \,, \tag{69}$$

where γ_{ij} is the metric of a 3D maximally-symmetric space and depends only on the comoving coordinates x^i.

The components of the spacetime metric and its inverse are

$$g_{00} = c^2 \,, \qquad g^{00} = \frac{1}{c^2} \,, \tag{70}$$

$$g_{ij} = -a^2\gamma_{ij} \qquad g^{ij} = -\frac{1}{a^2}\gamma^{ij} \,, \tag{71}$$

where γ^{ij} is the inverse of γ_{ij}.

The metric connection has the following (independent) non-zero components:

$$\Gamma^0_{ij} = \frac{\dot{a}a}{c^2}\gamma_{ij} \,, \quad \Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^i_j \,, \quad \Gamma^i_{jk} = {}^{(3)}\Gamma^i_{jk} \,, \tag{72}$$

where overdots denote differentiation with respect to cosmic time t and ${}^{(3)}\Gamma^i_{jk}$ are the metric connection coefficients of the 3D metric γ_{ij}.

(The ${}^{(3)}\Gamma^i_{jk}$ were given in Eq. (8) in the (r, θ, ϕ) coordinates; however, we shall not require their specific form here.)

Connection coefficients for the Robertson–Walker metric

The calculation of the connection coefficients follows from

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\sigma\nu} - \partial_\sigma g_{\nu\rho} \right) . \tag{73}$$

For example,

$$\begin{aligned}
\Gamma^{0}_{ij} &= \frac{1}{2} g^{0\sigma} \left(\partial_i g_{\sigma j} + \partial_j g_{\sigma i} - \partial_\sigma g_{ij} \right) \\
&= \frac{1}{2} g^{00} \left(\partial_i g_{0j} + \partial_j g_{0i} - \partial_0 g_{ij} \right) \\
&= \frac{1}{2c^2} \frac{da^2}{dt} \gamma_{ij} \\
&= \frac{\dot{a}a}{c^2} \gamma_{ij} .
\end{aligned} \tag{74}$$

We also have

$$\begin{aligned}
\Gamma^{i}_{0j} &= \frac{1}{2} g^{i\sigma} \left(\partial_0 g_{\sigma j} + \partial_j g_{\sigma 0} - \partial_\sigma g_{0j} \right) \\
&= \frac{1}{2} g^{ik} \left(\partial_0 g_{kj} + \partial_j g_{k0} - \partial_k g_{0j} \right) \\
&= \frac{1}{2a^2} \gamma^{ik} \frac{da^2}{dt} \gamma_{kj} \\
&= \frac{\dot{a}}{a} \delta^i_j .
\end{aligned} \tag{75}$$

The only other non-zero connection coefficient is

$$\begin{aligned}
\Gamma^{i}_{jk} &= \frac{1}{2} g^{i\sigma} \left(\partial_j g_{\sigma k} + \partial_k g_{\sigma j} - \partial_\sigma g_{jk} \right) \\
&= \frac{1}{2} g^{il} \left(\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk} \right) \\
&= \frac{1}{2} \gamma^{il} \left(\partial_j \gamma_{lk} + \partial_k \gamma_{lj} - \partial_l \gamma_{jk} \right) \\
&= {}^{(3)}\Gamma^{i}_{jk} .
\end{aligned} \tag{76}$$

We also require the spacetime Ricci tensor, given by

$$R_{\mu\nu} = -\partial_\rho \Gamma^\rho_{\mu\nu} + \partial_\mu \Gamma^\rho_{\rho\nu} + \Gamma^\sigma_{\rho\nu}\Gamma^\rho_{\mu\sigma} - \Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\rho\sigma} \,. \qquad (77)$$

A useful intermediate result, which follows from $\Gamma^\rho_{\rho\mu} = \Gamma^0_{0\mu} + \Gamma^i_{i\mu}$, is

$$\Gamma^\rho_{\rho 0} = 3\dot{a}/a \quad \text{and} \quad \Gamma^\rho_{\rho i} = {}^{(3)}\Gamma^j_{ji} \,. \qquad (78)$$

By isotropy, $R_{0i} = 0$; the remaining components of the Ricci tensor are

$$R_{00} = 3\frac{\ddot{a}}{a} \,, \qquad (79)$$

$$R_{ij} = -\frac{1}{c^2}\left(\ddot{a}a + 2\dot{a}^2 + 2Kc^2\right)\gamma_{ij} \,. \qquad (80)$$

Ricci tensor for the Robertson–Walker metric

For R_{00} we have

$$\begin{aligned} R_{00} &= -\partial_\rho \underbrace{\Gamma^\rho_{00}}_{0} + \partial_0 \underbrace{\Gamma^\rho_{\rho 0}}_{3\dot{a}/a} + \Gamma^\sigma_{\rho 0}\Gamma^\rho_{0\sigma} - \underbrace{\Gamma^\sigma_{00}}_{0}\Gamma^\rho_{\rho\sigma} \\ &= 3\frac{d}{dt}\left(\frac{\dot{a}}{a}\right) + \Gamma^i_{j0}\Gamma^j_{0i} \\ &= 3\frac{d}{dt}\left(\frac{\dot{a}}{a}\right) + \left(\frac{\dot{a}}{a}\right)^2 \delta^i_j \delta^j_i \\ &= 3\left[\frac{d}{dt}\left(\frac{\dot{a}}{a}\right) + \left(\frac{\dot{a}}{a}\right)^2\right] \\ &= 3\frac{\ddot{a}}{a} \,. \end{aligned} \qquad (81)$$

For R_{ij}, we have

$$R_{ij} = -\partial_\rho \Gamma^\rho_{ij} + \partial_i \Gamma^\rho_{\rho j} + \Gamma^\sigma_{\rho j}\Gamma^\rho_{i\sigma} - \Gamma^\sigma_{ij}\Gamma^\rho_{\rho\sigma} \,.$$

The first term is

$$\begin{aligned} -\partial_\rho \Gamma^\rho_{ij} &= -\partial_0 \Gamma^0_{ij} - \partial_k \Gamma^k_{ij} \\ &= -\frac{1}{c^2}\frac{d(\dot{a}a)}{dt}\gamma_{ij} - \partial_k {}^{(3)}\Gamma^k_{ij} \,. \end{aligned} \qquad (82)$$

The second term is

$$\partial_i \Gamma^\rho_{\rho j} = \partial_i {}^{(3)}\Gamma^k_{kj} \,. \tag{83}$$

For the third, we have

$$\begin{aligned}
\Gamma^\sigma_{\rho j}\Gamma^\rho_{i\sigma} &= \Gamma^0_{\rho j}\Gamma^\rho_{i0} + \Gamma^k_{\rho j}\Gamma^\rho_{ik} \\
&= \Gamma^0_{kj}\Gamma^k_{i0} + \Gamma^k_{0j}\Gamma^0_{ik} + \Gamma^k_{lj}\Gamma^l_{ik} \\
&= \frac{\dot{a}a}{c^2}\gamma_{kj}\frac{\dot{a}}{a}\delta^k_i + \frac{\dot{a}}{a}\delta^k_j\frac{\dot{a}a}{c^2}\gamma_{ik} + {}^{(3)}\Gamma^k_{lj}\,{}^{(3)}\Gamma^l_{ik} \\
&= 2\frac{\dot{a}^2}{c^2}\gamma_{ij} + {}^{(3)}\Gamma^k_{lj}\,{}^{(3)}\Gamma^l_{ik} \,.
\end{aligned} \tag{84}$$

The final term is

$$\begin{aligned}
-\Gamma^\sigma_{ij}\Gamma^\rho_{\rho\sigma} &= -\Gamma^0_{ij}\Gamma^\rho_{\rho 0} - \Gamma^k_{ij}\Gamma^\rho_{\rho k} \\
&= -3\frac{\dot{a}^2}{c^2}\gamma_{ij} - {}^{(3)}\Gamma^k_{ij}\,{}^{(3)}\Gamma^l_{lk} \,.
\end{aligned} \tag{85}$$

Adding these four terms, the parts involving ${}^{(3)}\Gamma^i_{jk}$ assemble to give the 3D Ricci tensor, leaving

$$\begin{aligned}
R_{ij} &= -\frac{1}{c^2}\left(\ddot{a}a + 2\dot{a}^2\right)\gamma_{ij} + {}^{(3)}R_{ij} \\
&= -\frac{1}{c^2}\left(\ddot{a}a + 2\dot{a}^2 + 2Kc^2\right)\gamma_{ij} \,,
\end{aligned} \tag{86}$$

where we used the result ${}^{(3)}R_{ij} = -2K\gamma_{ij}$ for the maximally-symmetric 3D space.

We now write the Einstein field equations as

$$R_{\mu\nu} = -\kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) + \Lambda g_{\mu\nu} \,, \tag{87}$$

where $T \equiv g_{\mu\nu}T^{\mu\nu}$ is the trace of the energy–momentum tensor.

For the ideal fluid, we have

$$\begin{aligned}
T &= g_{\mu\nu}\left(\rho + \frac{p}{c^2}\right)u^\mu u^\nu - p g_{\mu\nu}g^{\mu\nu} \\
&= \rho c^2 - 3p \,,
\end{aligned} \tag{88}$$

so that

$$T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T = \left(\rho + \frac{p}{c^2}\right) u_\mu u_\nu - \frac{1}{2}\left(\rho c^2 - p\right) g_{\mu\nu} \,. \quad (89)$$

The 4-velocity has components $u^\mu = \delta^\mu_0$, so that

$$u_\mu = g_{\mu 0} = c^2 \delta_{\mu 0} \,. \quad (90)$$

It follows that the non-zero Einstein equations reduce to

$$R_{00} = -4\pi G \left(\rho + \frac{3p}{c^2}\right) + \Lambda c^2 \,, \quad (91)$$

$$R_{ij} = -\frac{4\pi G}{c^2}\left(\rho - \frac{p}{c^2}\right) a^2 \gamma_{ij} - \Lambda a^2 \gamma_{ij} \,. \quad (92)$$

Using the explicit form for the components of the Ricci tensor that we worked out above, the first equation reduces to

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{1}{3}\Lambda c^2 \,,} \quad (93)$$

and the second to

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{2Kc^2}{a^2} = 4\pi G\left(\rho - \frac{p}{c^2}\right) + \Lambda c^2 \,. \quad (94)$$

Eliminating $\ddot{a}/a$, this reduces to

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 + \frac{Kc^2}{a^2} = \frac{8\pi G}{3}\rho + \frac{1}{3}\Lambda c^2 \,.} \quad (95)$$

Equations (93) and (95) are known as the *Friedmann equations.*

4.2 Conservation of the energy–momentum tensor

Recall that conservation of energy and momentum is described by

$$\nabla_\mu T^{\mu\nu} = 0 \,. \quad (96)$$

For the energy–momentum tensor

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu - p g^{\mu\nu}\,, \tag{97}$$

we have

$$\begin{aligned} 0 = \nabla_\mu T^{\mu\nu} &= u^\nu u^\mu \nabla_\mu \left(\rho + \frac{p}{c^2}\right) \\ &\quad + \left(\rho + \frac{p}{c^2}\right) \left(u^\nu \nabla_\mu u^\mu + u^\mu \nabla_\mu u^\nu\right) - \nabla^\nu p\,. \end{aligned} \tag{98}$$

Since $u^\mu = \delta^\mu_0$, we have $u^\mu \nabla_\mu \rho = \dot{\rho}$.

Furthermore, with $u^\mu = dx^\mu/d\tau$, we have

$$u^\mu \nabla_\mu u^\nu = \frac{Du^\nu}{D\tau}\,, \tag{99}$$

i.e., the 4-acceleration of the fluid.

However, as the fluid moves along geodesics, this vanishes and $u^\mu \nabla_\mu u^\nu = 0$.

Finally, we have

$$\begin{aligned} \nabla_\mu u^\mu &= \partial_\mu u^\mu + \Gamma^\mu_{\mu\nu} u^\nu \\ &= \Gamma^\mu_{\mu 0} = 3\dot{a}/a\,. \end{aligned} \tag{100}$$

It follows that Eq. (98) becomes

$$u^\nu \left[\dot{\rho} + \frac{\dot{p}}{c^2} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right)\right] - \nabla^\nu p = 0\,. \tag{101}$$

The component along u^ν is obtained by contracting with u_ν, which gives

$$\boxed{\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0\,;} \tag{102}$$

the projection perpendicular to u^ν vanishes identically since p is homogeneous.

Equation (102) expresses conservation of energy.

For example, for dust ($p = 0$), we have

$$\frac{\dot{\rho}}{\rho} = -3\frac{\dot{a}}{a} \quad \Rightarrow \quad \rho a^3 = \text{const.} \tag{103}$$

Since the proper volume of a given set of fluid particles scales as a^3, the proper number density of particles and hence energy density goes as a^{-3}.

In contrast, for radiation ($p = \rho c^2/3$), we have

$$\frac{\dot{\rho}}{\rho} = -4\frac{\dot{a}}{a} \quad \Rightarrow \quad \rho a^4 = \text{const.} \tag{104}$$

In this case, the energy density falls more quickly as the Universe expands due to the pV work done by the fluid.

Note that Eq. (102) is implied by the two Friedmann equations (because of the contracted Bianchi identity), and so is not independent.

5 Cosmological models

We now briefly consider some properties of the evolution of the scale factor $a(t)$ that is implied by the Friedmann equations.

We have

$$H^2 + \frac{Kc^2}{a^2} = \frac{8\pi G}{3}\rho + \frac{1}{3}\Lambda c^2 \,, \tag{105}$$

where, recall, $H \equiv \dot{a}/a$ is the Hubble parameter with value H_0 today.

If we define a *critical density*

$$\rho_{\text{crit}} \equiv \frac{3H^2}{8\pi G} \,, \tag{106}$$

then, for $\Lambda = 0$,

$$\begin{aligned} \rho > \rho_{\text{crit}} \quad &\Rightarrow \quad K > 0 \quad \text{(closed)} \\ \rho = \rho_{\text{crit}} \quad &\Rightarrow \quad K = 0 \quad \text{(flat)} \\ \rho < \rho_{\text{crit}} \quad &\Rightarrow \quad K < 0 \quad \text{(open)}\,. \end{aligned} \tag{107}$$

$\Lambda = 0$

Consider $\Lambda = 0$, and "ordinary" matter with $\rho > 0$ and $p \geq 0$.

Then, since

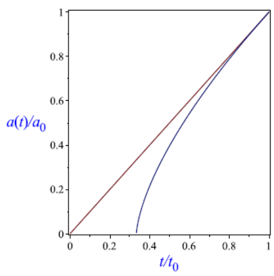
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{1}{3}\Lambda c^2\,, \tag{108}$$

we have $\ddot{a} < 0$.

This implies $a = 0$ at some time in the past, i.e., the Universe emerged from a singularity in the past (the *big bang*).

If we take $t = t_0$ today, in an expanding universe $\ddot{a} < 0$ implies

$$\text{age of universe} < \frac{a(t_0)}{\dot{a}(t_0)} = \frac{1}{H_0}\,. \tag{109}$$



We see that the age of the Universe is less than the *Hubble time* $1/H_0$ (see figure to the right).

For a current value of $H_0 = 68\,\mathrm{km\,s^{-1}\,Mpc^{-1}} = (14\,\mathrm{Gyr})^{-1}$, the age of the Universe would be less than 14 Gyr if Λ were zero.[2]

In a flat or open universe ($K \leq 0$), we have

$$H^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2} > 0\,, \tag{110}$$

so that the expansion never stops.

In contrast, in a closed universe there exists a maximum scale factor a_{max} where

$$\frac{Kc^2}{a_{\mathrm{max}}^2} = \frac{8\pi G}{3}\rho \tag{111}$$

[2]The age of the Universe is known actually to be very close to 14 Gyr; this is due to the late-time acceleration of our Universe (which increases the age for a given Hubble parameter today) compensating for $\ddot{a} < 0$ at earlier times.

and $\dot{a} = 0$.

(Here, we have used that ρ falls at least as fast as a^{-3} in an expanding universe with $p \geq 0$.)

Since $\ddot{a} < 0$, the universe subsequently contracts back to $a = 0$ (a future singularity).

Important special cases for $\Lambda = 0$ include models with $K = 0$ and $p = 0$ (the *Einstein–de Sitter model*), which is a good model of our Universe for most of its history.

In this case, $\rho \propto a^{-3}$ and so

$$\left(\frac{\dot{a}}{a}\right)^2 \propto \frac{1}{a^3} \quad \Rightarrow \quad a \propto t^{2/3} \quad \text{and} \quad H = \frac{2}{3t}\,, \qquad (112)$$

where we have taken $t = 0$ at $a = 0$.

As $\rho \propto a^{-4}$ for radiation, at sufficiently early times the energy density of our Universe is dominated by radiation rather than pressure-free (dark) matter.

For radiation domination,

$$\left(\frac{\dot{a}}{a}\right)^2 \propto \frac{1}{a^4} \quad \Rightarrow \quad a \propto t^{1/2} \quad \text{and} \quad H = \frac{1}{2t}\,. \qquad (113)$$

$\Lambda > 0$

Models with sufficiently large cosmological constant can undergo accelerated expansion, $\ddot{a} > 0$ (see Eq. 108).

A range of cosmological observations show that our Universe is accelerating at the current time, presumably under the action of something like a cosmological constant.

Consider a flat universe ($K = 0$) with $\Lambda > 0$.

This will expand forever, and at late times

$$H^2 = \frac{8\pi G}{3}\rho + \frac{1}{3}\Lambda c^2 \to \frac{1}{3}\Lambda c^2 \,. \tag{114}$$

The Hubble parameter therefore tends to a constant value and, asymptotically,

$$a \propto \exp\left(\sqrt{\frac{\Lambda c^2}{3}} t\right) \,. \tag{115}$$

The resulting solution is known as the *de Sitter* universe.