VII. SPACETIME CURVATURE

The equivalence principle has led us to formulate physical laws as tensor equations that reduce to their usual special-relativistic form in local-inertial coordinates.

If we could construct such coordinates globally across spacetime, there would be no manifestations of gravity.

Gravity enters only via our inability to construct such coordinates, i.e., through the *curvature of spacetime.*

In general relativity, gravity is no longer regarded as a force in the conventional sense, but rather as a manifestation of spacetime curvature, where the curvature is itself due to the presence of matter.

In this topic we shall develop further the idea of the intrinsic curvature of a manifold, and in the next we shall learn how general relativity relates the curvature to the matter that is present.

1 Gravity as spacetime curvature

To capture gravitational effects, we need to model spacetime as a 4D manifold that is more complicated than Minkowski space.

However, we are restricted to pseudo-Euclidean manifolds by the equivalence principle: we must be able to find coordinates X^μ locally such that the line element reduces to the Minkowski form

$$ds^2 \approx \eta_{\mu\nu} dX^\mu dX^\nu \,, \tag{1}$$

and this is only the case when the interval is quadratic in the coordinate differentials.

In curved spacetime, the equivalence principle tells us that the equation of motion of a massive particle is

$$\frac{Du^\mu}{D\tau} = 0\,, \tag{2}$$

with the 4-velocity $u^\mu = dx^\mu/d\tau$, since this tensor equation reduces to the special-relativistic form $d^2X^\mu/d\tau^2 = 0$ in local-inertial coordinates.

In other words, the wordline of a particle freely-falling under gravity is a geodesic in curved spacetime.

1.1 Local-inertial coordinates

Close to any point P, we can always construct coordinates X^μ on a pseudo-Euclidean 4D manifold such that

$$g_{\mu\nu}(P) = \eta_{\mu\nu} \quad \text{and} \quad (\partial_\rho g_{\mu\nu})_P = 0\,; \tag{3}$$

these imply that the metric connection vanishes at P.

Physically, these local-inertial coordinates correspond to a free-falling, non-rotating, Cartesian reference frame over some limited region of spacetime.

In such coordinates, the coordinates basis vectors $\boldsymbol{e}_\mu \equiv \partial/\partial X^\mu$ are orthonormal at P:

$$\boldsymbol{g}(\boldsymbol{e}_\mu, \boldsymbol{e}_\nu) = \eta_{\mu\nu}\,. \tag{4}$$

The local-inertial coordinates are not unique at P; there are infinitely many such coordinate systems related by Lorentz transformations at P.

Close to the point P, in local-inertial coordinates

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{2}\left(\frac{\partial^2 g_{\mu\nu}}{\partial X^\rho \partial X^\sigma}\right)_P [X^\rho - X^\rho(P)][X^\sigma - X^\sigma(P)] + \cdots\,. \tag{5}$$

The size of the second derivative of the metric determines the extent of the region over which physics locally looks like special relativity in these coordinates.

Fermi-normal coordinates

It is possible to construct coordinates all along a timelike geodesic such that $g_{\mu\nu} = \eta_{\mu\nu}$ and $\Gamma^{\rho}_{\mu\nu} = 0$ on the geodesic.

These coordinates, called *Fermi-normal coordinates*, extend the idea of local-inertial coordinates from the vicinity of a point to the vicinity of a timelike geodesic.

Fermi-normal coordinates can be constructed physically by the following procedure:

- Take a free-falling observer carrying an orthonormal frame of vectors, $\hat{\boldsymbol{e}}_0(\tau)$ equal to their 4-velocity and three spacelike vectors $\{\hat{\boldsymbol{e}}_i(\tau)\}$, where τ is proper time for the observer. All four vectors are parallel transported along the observer's geodesic worldline (e.g., the $\hat{\boldsymbol{e}}_i(\tau)$ could be defined by the direction of gyroscopes supported at their centre of mass).
- At every proper time τ, the observer constructs a family of spacelike geodesics that have unit tangent vectors at the observer constructed as linear combinations of the $\hat{\boldsymbol{e}}_i(\tau)$.
- Any point close to the observer's worldline will lie on exactly one such spacelike geodesic $\mathcal{C}$; assign coordinates to this point with $T = \tau$ and the X^i equal to the products of the direction cosines of the unit tangent vector to $\mathcal{C}$ at the observer and the proper distance along $\mathcal{C}$.

The resulting coordinates can be shown to have the properties advertised above (see Example Sheet 3 and also *Gravitation* by Misner, Thorne and Wheeler, Section 13.6).

The $\hat{\boldsymbol{e}}_0$ and $\hat{\boldsymbol{e}}_i$ are the (orthonormal) coordinate basis vectors of the Fermi-normal coordinates along the observer's geodesic.

Local measurements made by the observer correspond to projections of tensors onto this orthonormal frame.

1.2 Newtonian limit for a free-falling particle

Let us verify that we can recover the correct Newtonian limit for free-falling particles from the geodesic equation in the limit of low speeds and weak gravitational fields.

In the absence of gravity, spacetime is Minkowski space; for weak gravitational fields we expect to be able to find global coordinates where the metric is close to Minkowski (spacetime is only "weakly curved"):

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{where} \quad |h_{\mu\nu}| \ll 1 \,. \tag{6}$$

We shall also assume that the metric is stationary in these coordinates, $\partial h_{\mu\nu}/\partial x^0 = 0$, which corresponds to our usual intuition of a static gravitational field.

For slow-moving particles relative to this coordinate system, $|dx^i/dt| \ll c$, where $ct = x^0$, and so

$$\left|\frac{dx^i}{d\tau}\right| \ll \frac{dx^0}{d\tau} \,. \tag{7}$$

In the geodesic equation,

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0 \,, \tag{8}$$

we can therefore ignore $dx^i/d\tau$ terms relative to $dx^0/d\tau$ in the connection part, so that

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00} c^2 \left(\frac{dt}{d\tau}\right)^2 \approx 0\,. \tag{9}$$

The relevant connection coefficients to first-order in $h_{\mu\nu}$ are

$$\begin{aligned}\Gamma^\mu_{00} &= \frac{1}{2} g^{\mu\nu}\left(2\frac{\partial g_{\nu 0}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^\nu}\right) \\ &\approx -\frac{1}{2}\eta^{\mu\nu}\frac{\partial g_{00}}{\partial x^\nu} \\ &\approx -\frac{1}{2}\sum_i \eta^{\mu i}\frac{\partial h_{00}}{\partial x^i}\,,\end{aligned} \tag{10}$$

where we used that derivatives of the metric are first order in $h_{\mu\nu}$ in passing to the second line on the right.

It follows that

$$\Gamma^0_{00} \approx 0 \quad \text{and} \quad \Gamma^i_{00} \approx \frac{1}{2}\frac{\partial h_{00}}{\partial x^i}\,. \tag{11}$$

The 0-component of the geodesic equation gives $d^2t/d\tau^2 \approx 0$, so that $dt/d\tau = \text{const}$.

The ith component of the geodesic equation then becomes

$$\begin{aligned} & \frac{d^2x^i}{d\tau^2} \approx -\frac{c^2}{2}\frac{\partial h_{00}}{\partial x^i}\left(\frac{dt}{d\tau}\right)^2 \\ \Rightarrow \qquad & \frac{d^2x^i}{dt^2} \approx -\frac{c^2}{2}\frac{\partial h_{00}}{\partial x^i}\,.\end{aligned} \tag{12}$$

This has the form of the Newtonian equation of motion in Cartesian coordinates,

$$\frac{d^2x^i}{dt^2} = -\frac{\partial \Phi}{\partial x^i}\,, \tag{13}$$

where Φ is the Newtonian gravitational potential, provided that we make the identification $h_{00} \approx 2\Phi/c^2$ or

$$g_{00} \approx \left(1 + \frac{2\Phi}{c^2}\right)\,. \tag{14}$$

The assumption of a small perturbation to the metric holds provided that $\Phi/c^2 \ll 1$.

This is an excellent approximation even for dense objects; for example, $\Phi/c^2 \sim 10^{-4}$ at the surface of a white dwarf.

However, the weak-field approximation does break down in some extreme situations of astrophysical interest, such as close to the event horizon of a black hole.

In the next handout, we shall use the requirement (14) to fix the field equations in general relativity that relate the geometry of spacetime to the matter that is present.

2 Intrinsic curvature of a manifold

A manifold, or some extended region of one, is *flat* if it possible to find Cartesian coordinates X^a such that, throughout this region, the line element takes the Euclidean form

$$ds^2 = \epsilon_1(dX^1)^2 + \epsilon_2(dX^2)^2 + \cdots + \epsilon_N(dX^N)^2\,, \quad \epsilon_a = \pm 1\,. \tag{15}$$

Given points labelled with arbitrary coordinates in some manifold, how can we tell whether it is flat?

For example, we know that the 3D space with line element

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{16}$$

is just 3D Euclidean space written in spherical polar coordinates, and so is flat, but how could we tell this without spotting the coordinate transformation to Cartesian coordinates?

Fortunately, we can construct a tensor-valued measure of curvature, the *Riemann curvature tensor*, which allows us to test for curvature in an arbitrary coordinate system.

The physical relevance of curvature for general relativity is as follows: if a region of spacetime is flat, then we can construct global inertial coordinates, in which the metric is $\eta_{\mu\nu}$, and we recover special relativity *globally*, i.e., there is no gravitational field.

2.1 Riemann curvature tensor

We defined the covariant derivative such that its action was commutative on scalar fields, $\nabla_a \nabla_b \phi = \nabla_b \nabla_a \phi$.

This is not the case for vector fields.

Consider a dual-vector field v_a, and take two covariant derivatives:

$$\begin{aligned}
\nabla_a \nabla_b v_c &= \partial_a (\nabla_b v_c) - \Gamma^d_{ab} \nabla_d v_c - \Gamma^d_{ac} \nabla_b v_d \\
&= \partial_a \left(\partial_b v_c - \Gamma^d_{bc} v_d \right) - \Gamma^d_{ab} \left(\partial_d v_c - \Gamma^e_{dc} v_e \right) \\
&\quad - \Gamma^d_{ac} \left(\partial_b v_d - \Gamma^e_{bd} v_e \right) \, .
\end{aligned} \tag{17}$$

Now switch a and b and subtract; the term involving Γ^d_{ab} then cancels as do all terms involving derivatives of v_a (try it!), leaving

$$\begin{aligned}
\nabla_a \nabla_b v_c - \nabla_b \nabla_a v_c = -\partial_a \Gamma^d_{bc} v_d + \partial_b \Gamma^d_{ac} v_d + \Gamma^e_{ac} \Gamma^d_{be} v_d \\
- \Gamma^e_{bc} \Gamma^d_{ae} v_d \, .
\end{aligned} \tag{18}$$

We can therefore write

$$\boxed{\nabla_a \nabla_b v_c - \nabla_b \nabla_a v_c = R_{abc}{}^d v_d \, ,} \tag{19}$$

where

$$\boxed{R_{abc}{}^d = -\partial_a \Gamma^d_{bc} + \partial_b \Gamma^d_{ac} + \Gamma^e_{ac} \Gamma^d_{be} - \Gamma^e_{bc} \Gamma^d_{ae} \, .} \tag{20}$$

The quotient theorem tells us that this must be a type-$(1,3)$ tensor, which we call the *Riemann curvature tensor*.

Note how the curvature tensor is constructed from the metric and its first and second derivatives (via the connection).

If a manifold is flat, the Riemann curvature tensor vanishes since we can always choose Cartesian coordinates such that the connection vanishes; as the components of the Riemann curvature tensor vanish in such coordinates, the tensor itself is zero.

The converse is also true: if the Riemann tensor vanishes throughout some region of a manifold, the manifold is flat in that region (see *General Theory of Relativity* by Dirac, Chapter 12 for a proof).

2.2 Symmetries of the curvature tensor

It follows directly from the definition (19) that the Riemann tensor is antisymmetric in its first two indices:

$$R_{abc}{}^{d} = -R_{bac}{}^{d} \,. \tag{21}$$

The explicit construction (20) reveals the cyclic symmetry,

$$R_{abc}{}^{d} + R_{cab}{}^{d} + R_{bca}{}^{d} = 0 \,, \tag{22}$$

in which the first three indices are cyclically permuted.

Given the antisymmetry on the first two indices, this can also be written as $R_{[abc]}{}^{d} = 0$.

There are remaining symmetries that are most easily seen by lowering the final index to form the type-$(0,4)$ tensor R_{abcd}.

On a general curved manifold, let us work in local Cartesian coordinates at an arbitrary point P; then the connection vanishes at P and we have

$$(R_{abcd})_P = -\left(g_{de}\partial_a\Gamma^e_{bc} - g_{de}\partial_b\Gamma^e_{ac}\right)_P \,. \tag{23}$$

The (metric) connection is, generally,

$$\Gamma^e_{bc} = \frac{1}{2} g^{ef} \left(\partial_b g_{cf} + \partial_c g_{bf} - \partial_f g_{bc} \right) , \tag{24}$$

so that, in local Cartesian coordinates,

$$(g_{de} \partial_a \Gamma^e_{bc})_P = \frac{1}{2} \left(\partial_a \partial_b g_{cd} + \partial_a \partial_c g_{bd} - \partial_a \partial_d g_{bc} \right)_P . \tag{25}$$

Using this in Eq. (23), we find

$$(R_{abcd})_P = \frac{1}{2} \left(\partial_a \partial_d g_{bc} + \partial_b \partial_c g_{ad} - \partial_a \partial_c g_{bd} - \partial_b \partial_d g_{ac} \right)_P . \tag{26}$$

Since symmetries are preserved under general coordinate transformations, and given that the point P is arbitrary, any symmetry of the components R_{abcd} in local Cartesian coordinates at P will imply a general symmetry of the Riemann tensor.

Inspection of Eq. (26) reveals two further symmetries:

$$R_{abcd} = -R_{abdc} \tag{27}$$

$$R_{abcd} = R_{cdab} , \tag{28}$$

the first implying antisymmetry on the third and fourth indices (as for the first and second), and the second implying symmetry under swapping the first pair of indices with the second pair.

Let us consider what these symmetries mean for the number of independent components of the curvature tensor.

In 1D, the curvature tensor necessarily vanishes since there is only one possible component R_{1111}, but this vanishes by antisymmetry.[1]

[1] You might think a line can be curved, and indeed it can, but this curvature reflects the embedding in the plane and is not *intrinsic curvature*. We can always use the length along the curve as a coordinate, in which case the metric is trivially $g_{11} = 1$ everywhere implying intrinsic flatness.

In 2D, antisymmetry implies there is only one independent component, say R_{1212}.

In 3D, there are six independent components, as the following argument shows:

- There are three distinct combinations of the first pair of indices that give non-zero components of the curvature tensor: 12, 13 and 23. The same is true for the second pair.
- Given the symmetry under swapping the first and second pair of indices, there are three independent curvature components where the pairs are the same and three where they are different (like the diagonal and off-diagonal components of a 3×3 matrix, respectively). This gives six independent components.
- Finally, we should check that the cyclic symmetry (22) does not imply any further dependencies amongst these six components. Generally, the cyclic symmetry $R_{[abc]d} = 0$ is trivially satisfied if any of abc are equal. Moreover, it is also trivial if the fourth index is equal to any of the first three, by virtue of the other symmetries (antisymmetry in the first pair and second pair of indices and symmetry under swapping these pairs). So in 3D the cyclic symmetry implies no new constraints and there are six independent components of the curvature tensor.

In 4D, there are 20 independent components.

The argument is similar to that in 3D, but now there are six distinct combinations for either the first or second pair of indices, and so 21 independent components before we consider the cyclic symmetry.

However, in 4D the cyclic symmetry is not trivial since it is possible for all four indices to be different.

This implies one further constraint, of the form (letting indices run from 0–3)

$$R_{0123} + R_{1203} + R_{2013} = 0 \,, \tag{29}$$

which reduces the number of independent components of the curvature tensor from 21 to 20.

Generally, in ND the number of independent components of the curvature tensor is $N^2(N^2-1)/12$, which is exactly the number of physical degrees of freedom in the second derivative of the metric (i.e., after accounting for the freedom to perform coordinate transformations; see Handout I).

2.3 The Bianchi identity

The curvature tensor satisfies the differential *Bianchi identity*, which is very important for the development of general relativity:

$$\boxed{\nabla_a R_{bcd}{}^e + \nabla_b R_{cad}{}^e + \nabla_c R_{abd}{}^e = 0 \,.} \tag{30}$$

Note that this is a tensor identity since it involves the covariant derivative.

The Bianchi identity can be written in the equivalent form

$$\nabla_{[a} R_{bc]d}{}^e = 0 \tag{31}$$

using the antisymmetry of the curvature tensor in its first pair of indices.

It is simplest to prove the Bianchi identity by working in local Cartesian coordinates at some arbitrary point P; as it is a tensor identity, if we can show that it holds in one coordinate system it will necessarily hold in all.

Since the covariant derivative reduces to a simple covariant derivative in local Cartesian coordinates, we have

$$\begin{aligned}(\nabla_a R_{bcd}{}^e)_P &= \left(\partial_a \left[-\partial_b \Gamma^e_{cd} + \partial_c \Gamma^e_{bd} + \Gamma^f_{bd}\Gamma^e_{cf} - \Gamma^f_{cd}\Gamma^e_{bf}\right]\right)_P \\ &= (-\partial_a\partial_b\Gamma^e_{cd} + \partial_a\partial_c\Gamma^e_{bd})_P \,. \end{aligned} \tag{32}$$

Adding in the cyclic permutations of a, b and c, the right-hand side vanishes thus proving the Bianchi identity.

2.4 Ricci tensor and Ricci scalar

Lower-rank tensors can be formed from the curvature tensor by contraction.

Given the antisymmetry $R_{abcd} = R_{[ab]cd} = R_{ab[cd]}$, the only option is to contract across the first and second pair; the contraction on the first and last indices[2] defines the *Ricci tensor*

$$R_{ab} \equiv R_{cab}{}^c \,. \tag{33}$$

The Ricci tensor is symmetric, $R_{ab} = R_{ba}$, which follows from contracting the cyclic identity:

$$\begin{aligned} 0 &= \delta^c_d \left(R_{abc}{}^d + R_{cab}{}^d + R_{bca}{}^d\right) \\ &= R_{ab} - R_{ba} \,, \end{aligned} \tag{34}$$

where we used $R_{abcd} = R_{[ab]cd} = R_{ab[cd]}$.

We can contract the Ricci tensor to obtain the *Ricci scalar* (or curvature scalar):

$$R \equiv g^{ab} R_{ab} \,. \tag{35}$$

If a manifold is flat in some region, the curvature tensor will vanish and hence so will the Ricci tensor and scalar.

[2]This choice is not universal in the literature; often the first and third are chosen instead, which reverses the sign of the Ricci tensor.

However, it is possible for the Ricci tensor to vanish but for the full curvature tensor to be non-zero and the manifold to be curved – we shall see that this situation generally arises in vacuum regions of spacetime, where only tidal gravitational effects arise.

Contracting the Bianchi identity gives

$$\begin{aligned} 0 &= \delta^b_e \left(\nabla_a R_{bcd}{}^e + \nabla_c R_{abd}{}^e + \nabla_b R_{cad}{}^e\right) \\ &= \nabla_a R_{cd} - \nabla_c R_{ad} + \nabla^b R_{cadb} \, . \end{aligned} \tag{36}$$

Taking a further contraction over a and d gives

$$\begin{aligned} 0 &= g^{ad} \left(\nabla_a R_{cd} - \nabla_c R_{ad} + \nabla^b R_{cadb}\right) \\ &= \nabla^d R_{cd} - \nabla_c R + \nabla^b R_{cb} \\ &= 2\nabla^d R_{cd} - \nabla_c R \, . \end{aligned} \tag{37}$$

This gives the *contracted Bianchi identity*:

$$\boxed{\nabla^a \left(R_{ab} - \frac{1}{2} g_{ab} R\right) = 0 \, ,} \tag{38}$$

which involves the divergence of the *Einstein tensor*

$$\boxed{G_{ab} \equiv R_{ab} - \frac{1}{2} g_{ab} R \, .} \tag{39}$$

The Einstein tensor is symmetric and divergence-free.

We shall see that the contracted Bianchi identity is related to the conservation of energy and momentum in general relativity.

3 Physical manifestations of curvature

3.1 Curvature and parallel transport

We noted in Handout IV that parallel transport is generally path dependent, the exception being when the manifold is flat.

We can relate the curvature of the manifold, as expressed by the curvature tensor, to the path dependence of parallel transport by considering an infinitesimal loop defined by a curve $\mathcal{C}$.

Let $\mathcal{C}$ be parameterised by $x^a(u)$, and consider parallel transporting a vector $\boldsymbol{v}$ around this loop from the point P back to itself.

The equation of parallel transport is

$$\frac{dv^a}{du} = -\Gamma^a_{bc}\frac{dx^b}{du}v^c\,, \tag{40}$$

and so, at the point with coordinates $x^a(u)$, the result of parallel transporting $\boldsymbol{v}$ from P is the vector with components

$$v^a(u) = v^a_P - \int_{u_P}^{u} \Gamma^a_{bc}\frac{dx^b}{du'}v^c\,du'\,. \tag{41}$$

Since the closed loop is small, we can expand the connection and components v^a in the right-hand side to first order in the coordinate differences $x^a(u) - x^a_P$ as

$$\Gamma^a_{bc}(u) = (\Gamma^a_{bc})_P + (\partial_d\Gamma^a_{bc})_P\,[x^d(u) - x^d_P] + \cdots\,, \tag{42}$$

$$v^c(u) = v^c_P - \left(\Gamma^c_{ef}\right)_P\, v^f_P[x^e(u) - x^e_P] + \cdots\,. \tag{43}$$

Substituting these into Eq. (41), for the closed path

$$\Delta v^a = -\left(\partial_d\Gamma^a_{bc} - \Gamma^a_{be}\Gamma^e_{dc}\right)_P v^c_P \oint x^d\,dx^b\,, \tag{44}$$

where we have retained terms to first order in $x^a(u)-x^a_P$ and used $\oint dx^b = 0$.

We can simplify this further by noting that

$$\oint d(x^b x^d) = 0 \quad \Rightarrow \quad \oint x^b\,dx^d = -\oint x^d\,dx^b \tag{45}$$

so that

$$\Delta v^a = (\partial_d \Gamma^a_{bc} - \Gamma^a_{be}\Gamma^e_{dc})_P \, v^c_P \oint x^{[b} \, dx^{d]} \, . \tag{46}$$

We can now antisymmetrise over the indices b and d in the first-term on the right to find

$$\Delta v^a = \frac{1}{2} \underbrace{(\partial_d \Gamma^a_{bc} - \partial_b \Gamma^a_{dc} - \Gamma^a_{be}\Gamma^e_{dc} + \Gamma^a_{de}\Gamma^e_{bc})_P}_{-(R_{dbc}{}^a)_P} \, v^c_P \oint x^{[b} \, dx^{d]} \, . \tag{47}$$

Finally, relabelling indices, we have

$$\Delta v^a = \frac{1}{2} \, (R_{bcd}{}^a)_P \, v^d_P \oint x^{[b} dx^{c]} \, . \tag{48}$$

For an infinitesimal loop, the integral $\oint x^{[b} dx^{c]}$ is a type-$(2,0)$ tensor at P that encodes the planar area of the loop, so the right-hand side of Eq. (48) is a vector there, as required.

We see that the vector $\boldsymbol{v}$ does not change on parallel transport around a closed loop near P if the curvature tensor vanishes there.

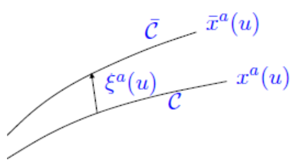
These ideas are illustrated in Fig. 1 for the case of a curved manifold (the 2-sphere) and a manifold with no intrinsic curvature (the surface of a cyclinder embedded in $\mathbb{R}^3$).

3.2 Curvature and geodesic deviation

A further important consequence of curvature is that two nearby geodesics that are initially parallel will either converge or diverge depending on the local curvature.

Consider two nearby affinely-parameterised geodesics, $\mathcal{C}$ given by $x^a(u)$, and $\bar{\mathcal{C}}$ given by $\bar{x}^a(u)$.

Let the initial values of the affine parameters be chosen so that the coordinate difference $\xi^a(u) = \bar{x}^a(u) - x^a(u)$ is infinitesimal (see figure to the right).



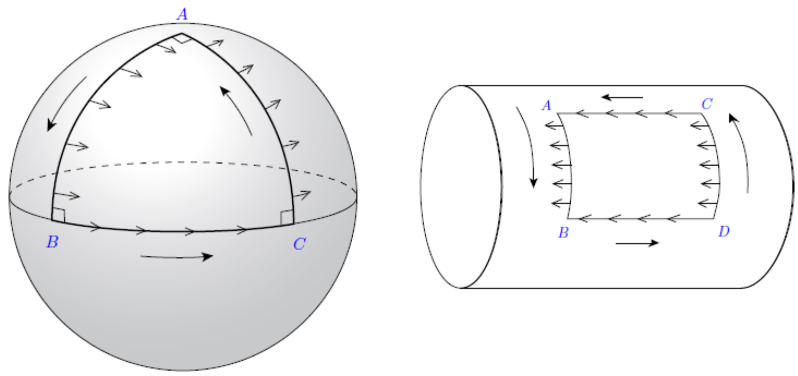


Figure 1: Parallel transport of a vector around closed loops on the 2-sphere (left) and the surface of a cylinder embedded in $\mathbb{R}^3$ (right). The 2-sphere has (constant) intrinsic curvature and as a result a vector is rotated after undergoing parallel transport around a closed loop. In contrast, the cylinder has vanishing intrinsic curvature and a vector is unchanged by parallel transport around a closed loop.

For infinitesimal separations, the ξ^a form the components of a vector.

Let us consider how $\xi^a(u)$ changes with u.

Since $\mathcal{C}$ and $\bar{\mathcal{C}}$ are geodesics, we have

$$\frac{d^2x^a}{du^2} + \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^c}{du} = 0 \quad \text{and} \quad \frac{d^2\bar{x}^a}{du^2} + \bar{\Gamma}^a_{bc}\frac{d\bar{x}^b}{du}\frac{d\bar{x}^c}{du} = 0\,, \tag{49}$$

where the bar on the metric connection denotes that it is evaluated at the point with coordinates $\bar{x}^a(u)$.

Taking the difference of the barred and unbarred geodesic equations, we have

$$\frac{d^2\xi^a}{du^2} + \bar{\Gamma}^a_{bc}\dot{\bar{x}}^b\dot{\bar{x}}^c - \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0\,, \tag{50}$$

where overdots denote derivatives with respect to u.

Expanding to first order in ξ^a, we have

$$\bar{\Gamma}^a_{bc}(u) = \Gamma^a_{bc}(u) + \partial_d\Gamma^a_{bc}\xi^d\,, \tag{51}$$

and so

$$\frac{d^2\xi^a}{du^2} + 2\Gamma^a_{bc}\dot{x}^b\dot{\xi}^c + \partial_d\Gamma^a_{bc}\dot{x}^b\dot{x}^c\xi^d = 0\,. \tag{52}$$

We now look to write this in terms of objects that are manifestly tensor-valued; we use

$$\begin{aligned}\frac{D}{Du}\left(\frac{D\xi^a}{Du}\right) &= \frac{d}{du}\left(\dot{\xi}^a + \Gamma^a_{bc}\dot{x}^b\xi^c\right) + \Gamma^a_{bc}\dot{x}^b\left(\dot{\xi}^c + \Gamma^c_{de}\dot{x}^d\xi^e\right) \\ &= \frac{d^2\xi^a}{du^2} + \partial_d\Gamma^a_{bc}\dot{x}^b\dot{x}^d\xi^c + \Gamma^a_{bc}\ddot{x}^b\xi^c \\ &\quad + 2\Gamma^a_{bc}\dot{x}^b\dot{\xi}^c + \Gamma^a_{bc}\Gamma^c_{de}\dot{x}^b\dot{x}^d\xi^e \,. \end{aligned} \tag{53}$$

Eliminating $\ddot{x}^b$ with the geodesic equation, and substituting into Eq. (52), after some index relabelling we find

$$\frac{D}{Du}\left(\frac{D\xi^a}{Du}\right) + \underbrace{\left(\partial_d\Gamma^a_{bc} - \partial_b\Gamma^a_{dc} - \Gamma^e_{dc}\Gamma^a_{be} + \Gamma^e_{bc}\Gamma^a_{de}\right)}_{-R_{dbc}{}^a}\dot{x}^b\dot{x}^c\xi^d = 0\,. \tag{54}$$

We see that the evolution of the connecting vector is given by the equation of *geodesic deviation*:

$$\frac{D}{Du}\left(\frac{D\xi^a}{Du}\right) - R_{dbc}{}^a\dot{x}^b\dot{x}^c\xi^d = 0\,. \tag{55}$$

If a manifold is flat over some region, the curvature tensor vanishes there and, in Cartesian coordinates, the intrinsic derivative reduces to the ordinary derivative d/du and Eq. (55) reduces to $d^2\xi^a/du^2 = 0$ – the Cartesian components of ξ^a grow linearly with u, as expected.

However, if there is curvature, two geodesics that are initially parallel (i.e., $D\xi^a/Du = 0$) will converge or diverge due to the curvature.

This latter behaviour is familiar on the 2-sphere, where neighbouring lines of longitude are geodesics that intersect at the poles, but are parallel at the equator.

In spacetime, the geodesic deviation equation describes the *relative* acceleration of neighbouring free-falling particles due to *tidal* gravitational effects.

Consider free-falling particles with geodesics $x^\mu(\tau)$ and $\bar{x}^\mu(\tau)$, where τ is proper time for each particle.

The connecting vector $\xi^\mu(\tau) \equiv \bar{x}^\mu(\tau) - x^\mu(\tau)$ evolves according to

$$\frac{D}{D\tau}\left(\frac{D\xi^\mu}{D\tau}\right) = \underbrace{R_{\nu\alpha\beta}{}^{\mu} u^\alpha u^\beta}_{S_\nu{}^\mu} \xi^\nu \,, \tag{56}$$

where $u^\mu = dx^\mu/d\tau$ is the 4-velocity and we have introduced the tidal tensor $S_\nu{}^\mu$.

Note that $S_{\mu\nu}$ is symmetric.

Equation (56) is analogous to the result for the tidal acceleration in Newtonian gravity, as we now show.

Take two free-falling particles with neighbouring trajectories $x^i(t)$ and $\bar{x}^i(t)$ in Cartesian coordinates, and define their connecting vector as $\xi^i(t) \equiv \bar{x}^i(t) - x^i(t)$.

The particles free-fall in a gravitational potential Φ as

$$\frac{d^2x^i}{dt^2} = -\left(\frac{\partial\Phi}{\partial x^i}\right)_{x(t)} \quad \text{and} \quad \frac{d^2\bar{x}^i}{dt^2} = -\left(\frac{\partial\Phi}{\partial x^i}\right)_{\bar{x}(t)} . \tag{57}$$

Taking the difference, and expanding to first-order in ξ^i, gives

$$\frac{d^2\xi^i}{dt^2} \approx -\left(\frac{\partial^2\Phi}{\partial x^i \partial x^j}\right)\xi^j \,. \tag{58}$$

In free space, where $\vec{\nabla}^2\Phi = 0$, the tidal tensor $-\partial_i\partial_j\Phi$ is symmetric and trace-free and generates a volume-preserving distortion of a set of free-falling particles.

It can be shown (see Examples Sheet 3) in the weak-field limit, and for slow speeds, the Newtonian tidal equation (58) and the appropriate components of the geodesic deviation equation (55) are equivalent.