

ADVANCED QUANTUM PHYSICS

Handout 1

- ▶ Fundamentals of Quantum Mechanics
- ▶ Particles in potentials (Bound states)
- ▶ Angular momentum

Course overview

- The Part II AQP course follows on from, and is similar in spirit to, last year's Part IB Quantum Physics course
 - emphasis on basic principles and techniques
 - applications mainly in atomic physics
 - many more applications in next term's Part II courses and in Part III next year
- Begin with Quantum Mechanics (wave mechanics)
 - applies to systems with a fixed number of particles (mostly non-relativistic; will add some relativity "by hand")
- Later, will introduce some basic ideas of Quantum Field Theory (QFT), by quantising the electromagnetic field
 - allows particles (e.g. photons) to be created and annihilated

Synopsis

- **Review of Quantum Physics:** Review of Quantum Physics: Postulates of quantum mechanics, operator methods, time dependence. Solutions to the Schrödinger equation in one and three dimensions. Hydrogen atom. Angular momentum and spin; matrix representations. Addition of angular momentum, Clebsch-Gordan coefficients.
- **Approximate methods:** Time-independent perturbation theory, first and second order expansion; degenerate perturbation theory. Variational method: ground state energy and eigenfunctions.
- **Motion of charge particle in electromagnetic field:** Gauge invariance; Aharonov-Bohm effect. Particle magnetic moments; Stern-Gerlach experiment. Landau levels, two-dimensional electron gas.
- **Symmetries:** Translations and rotations, parity. Conservation laws. Wigner-Eckart theorem for scalar and vector operators. Selection rules. Landé projection formula.
- **Identical particles:** Particle indistinguishability and quantum statistics; exchange interactions. The helium atom.
- **Atomic and molecular structure:** The 'real' hydrogen atom; fine structure: relativistic corrections, spin-orbit coupling; hyperfine structure. Multi-electron atoms: LS coupling; Hund's rules; Stark effect; Zeeman effect. Born-Oppenheimer approximation; H₂⁺ ion; molecular orbitals; H₂ molecule; ionic and covalent bonding.
- **Time-dependent perturbations:** Transitions in two-level systems, Rabi oscillations; magnetic resonance. Spin transitions. Time-dependent perturbation theory, Fermi's Golden rule. Scattering, the Born approximation.
- **Elements of Quantum Field Theory:** Quantization of the electromagnetic field, photons; number states. Radiative transitions, electric dipole approximation, selection rules, spontaneous emission and absorption, stimulated emission, Einstein's A and B coefficients; Cavity rate equations and lasers, coherent states.

Practicalities

- 24 lectures: Mondays, Wednesday, and Fridays at 11.30am
- The **three substantial typeset handouts**, as well as most of the **presented material**, are thanks to Prof. Ben Simons and Dr Richard Batley, previous lecturers of the course
- All materials, including the **two Example Sheets**, will be posted on TIS
<https://www-teach.phy.cam.ac.uk/students/courses/advanced-quantum-physics/4>
- Recordings of the lectures will be uploaded to Moodle
<https://www.vle.cam.ac.uk/course/view.php?id=245692>
- If there are questions/problems with lectures or example sets, please feel free to contact me by e-mail (palvarez@hep.phy.cam.ac.uk) or in person (Rm 938, Rutherford building).

Quantum Physics, Gasiorowicz S (2nd edition, Wiley, 1996; 3rd edition, Wiley, 2003)

Introduction to Quantum Mechanics, Griffiths D J and Schroeter D F (3rd edition, CUP, 2018)

Quantum Mechanics, Bransden B H and Joachain C J (2nd edition, Pearson, 2000)

Physics of Atoms and Molecules, Bransden B H and Joachain C J (2nd edition, Pearson, 2003)

Principles of Quantum Mechanics Shankar R (2nd edition, Plenum, 1994)

Quantum Mechanics, Schwabl F (4th edition, Springer, 2007)

This Handout

Will review results from the Part IB Quantum Physics course on

- states, observables, measurement, Dirac notation
- the quantum harmonic oscillator
- Schrödinger's equation in 1D and 3D; the hydrogen atom
- angular momentum (orbital and spin)
- combining (adding) angular momenta

Results are presented without proof : the proofs are in Part IB QP

Notation will sometimes be different to Part IB :

esp. Clebsch-Gordan coefficients, $C_{l,m_l,s,m_s} \rightarrow \langle lm_l; sm_s | jm_j \rangle$

Fundamentals of QM

QM is based on a set of ingredients (postulates) incorporating wave-particle duality and covering

- $|\psi\rangle, \psi(\mathbf{r}, t)$ ▶ states vectors, wavefunctions
- $\hat{A} : (\hat{\mathbf{r}}, \hat{\mathbf{p}}, \hat{H}, \dots)$ ▶ observables (Hermitian operators)
- $\hat{A} |\psi\rangle = a |\psi\rangle$ ▶ measurements
- $\hat{H}\psi = i\hbar(\partial\psi/\partial t)$ ▶ Time evolution (Schrödinger's equation)

Constitutes the *Copenhagen interpretation*; inherently probabilistic, and justified ultimately by experiment

Nobel Prize in Physics 2022



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Alain Aspect

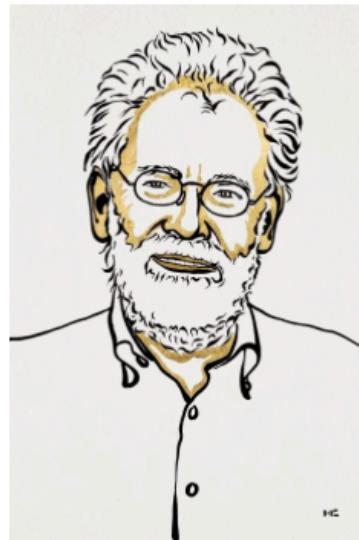
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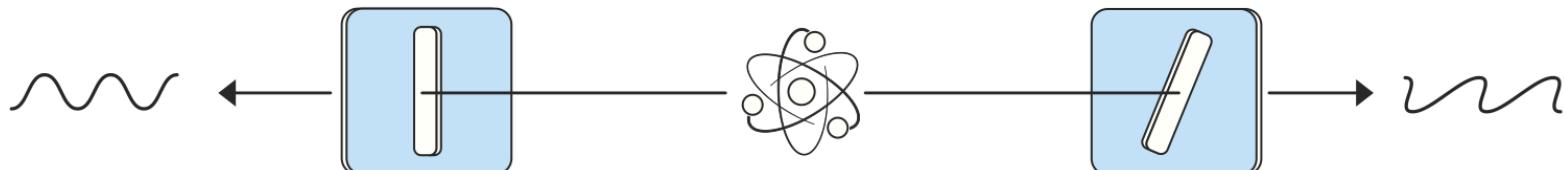
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“for experiments with entangled photons, establishing the violation of Bell inequalities and pioneering quantum information science”

<https://www.nobelprize.org/prizes/physics/2022/summary/>

Bell inequalities provide a way to experimentally test the compatibility of QM with a theory with local ‘hidden variables’

https://cds.cern.ch/record/111654/files/vol1p195-200_001.pdf



States / Wavefunctions

(1) *The state of a quantum mechanical system is completely specified by its state vector, $|\psi\rangle$*

- In ordinary space, $|\psi\rangle$ is a complex *wavefunction*, $\psi(\mathbf{r}, t)$, which must be single-valued, continuous, finite and (usually) square-integrable :

$$\int |\psi(\mathbf{r}, t)|^2 d^3\mathbf{r} = \int \psi^*(\mathbf{r}, t)\psi(\mathbf{r}, t) d^3\mathbf{r} = 1 \quad (1.7.1)$$

- For a single particle in a bound state :

$|\psi(\mathbf{r}, t)|^2 d^3\mathbf{r}$ { represents the probability that the particle lies in
the volume element $d^3\mathbf{r}$ at time t

- In general, ψ may depend additionally (or entirely) on other “coordinates”
e.g. ψ may be the spin state of a particle, in an abstract “spin space”

States / wavefunctions: Dirac notation

- *Dirac notation* is compact and flexible, and can cover all cases :

$$\text{“ket”} : \psi \rightarrow |\psi\rangle$$

$$\text{“bra”} : \psi^* \rightarrow \langle\psi|$$

[technically, $|\psi\rangle$ belongs to a type of vector space known as a *Hilbert space*]

- The *overlap (“braket”)* of two states $|\psi_1\rangle$ and $|\psi_2\rangle$ is defined as

$$\langle\psi_1|\psi_2\rangle = \int \psi_1^*(\mathbf{r}, t)\psi_2(\mathbf{r}, t) d^3\mathbf{r} = \langle\psi_2|\psi_1\rangle^*$$

Thus, for complex constants a_1, a_2 , we have

$$\langle a_1\psi_1|a_2\psi_2\rangle = a_1^*a_2\langle\psi_1|\psi_2\rangle$$

- The normalisation condition of eqn. (1.7.1) can be written compactly as

$$\boxed{\langle\psi|\psi\rangle = 1}$$

Observables

(2) *To every observable A in classical mechanics, there corresponds a linear, complete, Hermitian operator \hat{A} in quantum mechanics*

-- The *Hermitian conjugate* \hat{A}^\dagger of an operator \hat{A} is defined through

$$\int \phi^*(\mathbf{r}, t) \hat{A}^\dagger \psi(\mathbf{r}, t) d^3\mathbf{r} = \left[\int \psi^*(\mathbf{r}, t) \hat{A} \phi(\mathbf{r}, t) d^3\mathbf{r} \right]^*$$

More generally, in Dirac notation :

$$\langle \phi | \hat{A}^\dagger | \psi \rangle = \langle \psi | \hat{A} | \phi \rangle^* = \langle \hat{A} \phi | \psi \rangle$$

-- Equivalently, in terms of *matrix elements* :

$$\boxed{\langle \phi | \hat{A}^\dagger | \psi \rangle = \langle \psi | \hat{A} | \phi \rangle^*}$$

where

$$\hat{A}|\phi\rangle = |\hat{A}\phi\rangle ; \quad \langle \phi | \hat{A}^\dagger = \langle \hat{A}\phi |$$

Observables: Hermitian operators

- An operator A which is self-conjugate is said to be *Hermitian* :

$$\hat{A}^\dagger = \hat{A} ; \quad \langle \phi | \hat{A} | \psi \rangle = \langle \psi | \hat{A} | \phi \rangle^*$$

- The eigenvalues of a Hermitian operator A are real

$$\hat{A} | \psi_i \rangle = a_i | \psi_i \rangle ; \quad a_i^* = a_i$$

- If $g \geq 2$ distinct eigenstates share the same eigenvalue a ,

$$\hat{A} | \psi_1 \rangle = a | \psi_1 \rangle ; \quad \hat{A} | \psi_2 \rangle = a | \psi_2 \rangle ; \quad \dots ; \quad \hat{A} | \psi_g \rangle = a | \psi_g \rangle$$

then the eigenvalue a is said to have *degeneracy* g

("distinct" means not related by only a phase factor)

- The eigenstates of a Hermitian operator A can always be taken to be orthonormal :

$$\langle \psi_i | \psi_j \rangle = \delta_{ij}$$

(1.10.1)

(which may involve taking suitable linear combinations
of eigenstates in the case of degeneracy, $g \geq 2$)

Observables: completeness

- For a Hermitian operator A to qualify as an observable, the eigenstates of A must form a *complete* set of states :
 - any state $|\psi\rangle$ can be expanded in terms of the eigenstates $|\psi_i\rangle$ of A :

$$|\psi\rangle = \sum_i c_i |\psi_i\rangle ; \quad c_i = \langle\psi_i|\psi\rangle$$

where $c_i = \langle\psi_i|\psi\rangle$ follows from orthonormality, equation (1.10.1)

- For any state $|\psi\rangle$, we therefore have

$$|\psi\rangle = \sum_i |\psi_i\rangle\langle\psi_i|\psi\rangle = \left(\sum_i |\psi_i\rangle\langle\psi_i| \right) |\psi\rangle$$

The above equation is equivalent to the operator *completeness relation*

$$\sum_i |\psi_i\rangle\langle\psi_i| = \hat{I}$$

Observables: momentum and energy

- To fix the precise form of observable operators, we can initially work by analogy with classical physics, use intuition etc.
 - will see later how *symmetry principles* fix (some) operator forms
- For a free particle, wave-particle duality (de Broglie) gives

$$\lambda = \frac{h}{|\mathbf{p}|} ; \quad \mathbf{p} = \hbar \mathbf{k} ; \quad E = \hbar \omega$$

This suggests the travelling wave description

$$\psi(\mathbf{r}, t) \propto e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \propto e^{i(\mathbf{p} \cdot \mathbf{r} - Et)/\hbar}$$

- The eigenvalue equations

$$\left(-i\hbar \frac{\partial}{\partial x} \right) \psi(\mathbf{r}, t) = p_x \psi(\mathbf{r}, t) ; \quad \left(i\hbar \frac{\partial}{\partial t} \right) \psi(\mathbf{r}, t) = E \psi(\mathbf{r}, t)$$

then suggest taking the momentum and energy operators to be

$$\hat{\mathbf{p}} = -i\hbar \nabla$$

$$\hat{E} = i\hbar(\partial/\partial t)$$

Observables: commutation relations

- The commutator of two operators A and B is defined as

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$$

In general, operators (observables) need not commute :

e.g. unlike components of position and momentum commute,

$$[\hat{x}, \hat{y}] = [\hat{x}, \hat{p}_y] = [\hat{p}_x, \hat{p}_y] = \dots = 0$$

but like components do *not* :

$$[\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = [\hat{z}, \hat{p}_z] = i\hbar$$

- Overall, for the position and momentum operators, we have

$$\hat{\mathbf{r}} = (\hat{x}, \hat{y}, \hat{z}) = (\hat{r}_1, \hat{r}_2, \hat{r}_3) ;$$

$$\hat{\mathbf{p}} = (\hat{p}_x, \hat{p}_y, \hat{p}_z) = (\hat{p}_1, \hat{p}_2, \hat{p}_3)$$

$$(i, j = 1, 2, 3)$$

$$[\hat{r}_i, \hat{p}_j] = i\hbar \delta_{ij}$$

$$[\hat{r}_i, \hat{r}_j] = [\hat{p}_i, \hat{p}_j] = 0$$

Observables: commutation relations (2)

- For any function of position, $f(\mathbf{r})$, and any wavefunction $\psi(\mathbf{r})$, we have

$$\begin{aligned}[f(\mathbf{r}), \hat{\mathbf{p}}]\psi &= -i\hbar[f(\mathbf{r}), \nabla]\psi && (\hat{\mathbf{p}} = -i\hbar\nabla) \\ &= -i\hbar f(\mathbf{r})\nabla\psi + i\hbar\nabla(f(\mathbf{r})\psi) \\ &= i\hbar(\nabla f)\psi\end{aligned}$$

- Since this holds for *all* wavefunctions $\psi(\mathbf{r})$, we obtain the *operator* relation

$$[f(\mathbf{r}), \hat{\mathbf{p}}] = i\hbar(\nabla f)$$

N.B. the brackets on the right hand side are essential, in order to distinguish $(\nabla f)\psi$ from $\nabla(f\psi)$

- In passing, we obtain a commutation relation which will be needed later :

$$[\hat{r}_j, \hat{p}_k^2] = [\hat{r}_j, \hat{p}_k]\hat{p}_k + \hat{p}_k[\hat{r}_j, \hat{p}_k] = 2i\hbar\hat{p}_k\delta_{jk}$$

$$[\hat{r}_j, \hat{\mathbf{p}}^2] = [\hat{r}_j, \hat{p}_1^2] + [\hat{r}_j, \hat{p}_2^2] + [\hat{r}_j, \hat{p}_3^2] = 2i\hbar\hat{p}_j$$

$$[\hat{r}, \hat{\mathbf{p}}^2] = 2i\hbar\hat{p}$$

Observables: commutation relations (3)

- If observables A and B commute,

$$\hat{A}\hat{B} = \hat{B}\hat{A} ; \quad [\hat{A}, \hat{B}] = 0$$

then all eigenstates of A are also eigenstates of B (and *v.v.*), and we can always find a complete set of *simultaneous eigenstates* of A and B :

$$\hat{A}|\phi_j\rangle = a_j|\phi_j\rangle ; \quad \hat{B}|\phi_j\rangle = b_j|\phi_j\rangle$$

- If operators A and B commute, then so also do all powers of A and B :

$$[\hat{A}, \hat{B}] = 0 \Rightarrow [\hat{A}^m, \hat{B}^n] = 0 \quad (m, n = 1, 2, 3, \dots)$$

- If an operator C commutes with operators A and B , then C also commutes with the product AB :

$$[\hat{C}, \hat{A}] = [\hat{C}, \hat{B}] = 0 \Rightarrow [\hat{C}, \hat{A}\hat{B}] = 0$$

(even if A and B do not themselves commute)

Measurement

- (3) *If the result of a measurement of A is the number a_i , then a_i must be one of the eigenvalues of A :*

$$\hat{A}|\psi_i\rangle = a_i|\psi_i\rangle$$

The probability that a measurement of A gives the result a_i when the system is in the state $|\psi\rangle$ is

$$P(a_i) = |\langle\psi_i|\psi\rangle|^2$$

A measurement of $|\psi\rangle$ that gives the eigenvalue a_i changes the state of the system to the corresponding eigenstate $|\psi_i\rangle$

(“collapse of the wavefunction”)

Measurement (2)

- The state $|\psi\rangle$ can be expanded in terms of the (complete set of) eigenstates $|\psi_i\rangle$ of A , as on slide 1.11 :

$$|\psi\rangle = \sum_i c_i |\psi_i\rangle ; \quad c_i = \langle \psi_i | \psi \rangle$$

The normalisation of $|\psi\rangle$ is then obtained as

$$\langle \psi | \psi \rangle = \left\langle \sum_i c_i \psi_i \right| \sum_j c_j \psi_j \rangle = \sum_{i,j} c_i^* c_j \langle \psi_i | \psi_j \rangle = \sum_{i,j} c_i^* c_j \delta_{ij} = \sum_i |c_i|^2$$

- Hence, with $|\psi\rangle$ normalised to unity, the probabilities of all possible measurement outcomes sum to one :

$$\langle \psi | \psi \rangle = 1 \quad \Rightarrow \quad \sum_i |c_i|^2 = 1$$

$$P(a_i) = |\langle \psi_i | \psi \rangle|^2 = |c_i|^2 ; \quad \sum_i P(a_i) = 1$$

Measurement : expectation values

- The *expectation value* of an observable A for a system in a state $|\psi\rangle$ is defined as

$$\langle \hat{A} \rangle \equiv \langle \psi | \hat{A} | \psi \rangle$$

$$\langle \hat{A} \rangle \equiv \int \psi^*(\mathbf{r}, t) \hat{A} \psi(\mathbf{r}, t) d^3\mathbf{r}$$

- $\langle A \rangle$ is the average value of many repeated measurements of A on $|\psi\rangle$
To see this, again expand $|\psi\rangle$ in terms of the eigenstates of A :

$$\begin{aligned}\langle \hat{A} \rangle &= \left\langle \sum_i c_i \psi_i \middle| \hat{A} \middle| \sum_j c_j \psi_j \right\rangle \\ &= \sum_{i,j} c_i^* c_j \langle \psi_i | \hat{A} | \psi_j \rangle \\ &= \sum_j |c_j|^2 a_j = \sum_j a_j P(a_j)\end{aligned}\quad \left\{ \begin{array}{l} \hat{A} |\psi_j\rangle = a_j |\psi_j\rangle \\ \langle \psi_i | \hat{A} | \psi_j \rangle = a_j \delta_{ij} \end{array} \right.$$

(the probability weighted sum of all possible measurement outcomes)

Measurement : uncertainties

- The *uncertainty* ΔA on a measurement of A is given by

$$(\Delta A)^2 \equiv \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2 = \langle \psi | \hat{A}^2 | \psi \rangle - \langle \psi | \hat{A} | \psi \rangle^2$$

ΔA is the r.m.s. spread of a large number of repeat measurements of A ;

→ it is an unavoidable uncertainty which is inevitably present, even for a “perfect” experiment

- If the system is in a state $|\psi\rangle$ which is already an eigenstate of A ,

$$\hat{A}|\psi\rangle = a|\psi\rangle$$

then the uncertainty ΔA vanishes :

$$\begin{aligned} \langle \hat{A} \rangle &= \langle \psi | \hat{A} | \psi \rangle = a ; & \langle \hat{A}^2 \rangle &= \langle \psi | \hat{A}^2 | \psi \rangle = a^2 \\ &&\Rightarrow \Delta A &= 0 \end{aligned}$$

Thus, for a system which is in an eigenstate of A with eigenvalue a , a measurement of A *always* results in the value a (with zero uncertainty)

Time Evolution

- (4) *The state of a system evolves with time according to the time-dependent Schrödinger equation*

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$$

- The Hamiltonian operator H corresponds to the total energy of the system, i.e. the eigenstates of H are states of definite total energy :

$$\hat{H}|\psi_i\rangle = E_i|\psi_i\rangle$$

e.g. for a free-particle, the Hamiltonian is $\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} = -\frac{\hbar^2}{2m}\nabla^2$

and Schrödinger's equation has the solutions

$$\psi(\mathbf{r}, t) \propto e^{i(\mathbf{p} \cdot \mathbf{r} - Et)/\hbar}; \quad E = \mathbf{p}^2/2m$$

Time evolution : Ehrenfest's Theorem

- From Schrödinger's equation, energy eigenstates evolve as

$$|\psi_i(t)\rangle = |\psi_i\rangle e^{-iE_i t/\hbar}$$

- The energy eigenstates form a complete set of states, so a general initial wavefunction can always be expanded as

$$|\psi(0)\rangle = \sum_i c_i |\psi_i\rangle \quad \Rightarrow \quad |\psi(t)\rangle = \sum_i c_i |\psi_i\rangle e^{-iE_i t/\hbar}$$

- From Schrödinger's equation and its conjugate,

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle ; \quad -i\hbar \frac{\partial}{\partial t} \langle\psi| = \langle\psi| \hat{H}$$

we obtain *Ehrenfest's theorem* for the expectation value of a general, time-independent operator A :

$$\frac{d}{dt} \langle\psi(t)|\hat{A}|\psi(t)\rangle = \frac{i}{\hbar} \langle\psi(t)|[\hat{H}, \hat{A}]|\psi(t)\rangle$$

Good Quantum Numbers

- If a time-independent observable A commutes with the Hamiltonian H , then so does A^2 ; for any state $|\psi\rangle$, Ehrenfest's theorem then gives

$$[\hat{H}, \hat{A}] = 0 \quad \Rightarrow \quad \frac{d}{dt} \langle \psi(t) | \hat{A} | \psi(t) \rangle = 0$$

$$[\hat{H}, \hat{A}^2] = 0 \quad \Rightarrow \quad \frac{d}{dt} (\Delta A)^2 = 0$$

i.e. the expectation $\langle A \rangle$ and uncertainty ΔA remain constant :

$$\langle A \rangle = \text{const.} ; \quad \Delta A = \text{const.}$$

- In particular, if a system is initially in an eigenstate of A ,

$$|\psi(0)\rangle = |a\rangle ; \quad \hat{A}|a\rangle = a|a\rangle$$

then the value of A is known precisely at all times :

$$\langle A \rangle = a ; \quad \Delta A = 0$$

- The eigenvalues of observables which commute with the Hamiltonian are said to be *good quantum numbers*

Stationary States

- If a system is in a state of well-defined energy :

$$\hat{H}|\psi\rangle = E|\psi\rangle ; \quad \langle\psi|\hat{H} = \langle\psi|E$$

then the expectation value of *any* time-independent operator A is time-independent, even if A does not commute with H :

$$\begin{aligned}\frac{d}{dt}\langle\psi(t)|\hat{A}|\psi(t)\rangle &= \frac{i}{\hbar}\langle\psi(t)|(\hat{H}\hat{A} - \hat{A}\hat{H})|\psi(t)\rangle \\ &= \frac{i}{\hbar}\left(E\langle\psi(t)|\hat{A}|\psi(t)\rangle - E\langle\psi(t)|\hat{A}|\psi(t)\rangle\right) = 0\end{aligned}$$

$$\langle A \rangle(t) = \text{const.} ; \quad \Delta A(t) = \text{const.}$$

- States of well-defined energy are known as *stationary states*

For an energy eigenstate with a wavefunction of the form

$$\psi_i(\mathbf{r}, t) = \psi_i(\mathbf{r})e^{-iE_i t/\hbar}$$

the spatial probability distribution remains constant (“stationary”) :

$$P(\mathbf{r}) d^3\mathbf{r} = |\psi_i(\mathbf{r}, t)|^2 d^3\mathbf{r} = |\psi_i(\mathbf{r})|^2 d^3\mathbf{r}$$

Matrix Representations

- Consider a given (complete) basis set of states $|\psi_i\rangle$

Then, *for this basis*, the matrix representation of an operator A is defined as

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \dots \\ A_{21} & A_{22} & A_{23} & \dots \\ A_{31} & A_{32} & A_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \begin{aligned} A_{jk} &\equiv \langle \psi_j | \hat{A} | \psi_k \rangle \\ &\text{("matrix elements")} \end{aligned}$$

- The basis states $|\psi_i\rangle$ are represented by the column matrices

$$|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \quad |\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \quad |\psi_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{pmatrix}, \dots$$

A general state $|\psi\rangle = \sum_j c_j |\psi_j\rangle$ is represented by

$$|\psi\rangle = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix}$$

Matrix representations : eigenvalues and eigenvectors

- The eigenvalues and eigenvectors of the matrix A ,

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & \dots \\ A_{21} & A_{22} & A_{23} & \dots \\ A_{31} & A_{32} & A_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix} = \lambda \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix}$$

can be found by solving the *secular equation*

$$\det(A - \lambda I) = 0$$

- The matrix elements A_{jk} *depend on the choice of basis*;
But the eigenvalues λ (i.e. the solutions of the secular equation, i.e. the possible results a_i of a measurement of A) are *basis-independent*
- If the basis used is the set of eigenstates of A , then the matrix A is diagonal, with the eigenvalues of A as the diagonal elements :

$$A = \text{diag}(a_1, a_2, a_3, \dots)$$

Matrix representations : observables

- The Hermitian conjugate operator A^\dagger is represented by the Hermitian conjugate matrix : $[A^\dagger]_{jk} = A_{kj}^* ; \quad A^\dagger = (A^*)^T$
- If the operator A is an observable, then the matrix representation of A (w.r.t. any basis) is Hermitian

$$A_{jk} = A_{kj}^* ; \quad (A^*)^T = A$$

- For a Hermitian matrix A :
 - the eigenvalues of A are always *real*
 - the eigenvectors corresponding to distinct eigenvalues of A are always *orthogonal*
 - the eigenvectors corresponding to repeated (degenerate) eigenvalues of A (if any) can always be *chosen* to be orthogonal

Thus we can always arrange that the eigenvectors ψ_i of a Hermitian matrix are mutually orthonormal :

$$\psi_i^{*T} \psi_j = \delta_{ij} ; \quad \langle \psi_i | \psi_j \rangle = \delta_{ij}$$

Orbital Angular Momentum

- The classical vector relation $\mathbf{L} = \mathbf{r} \wedge \mathbf{p}$ suggests that the quantum operators for orbital angular momentum should be defined as

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y ; \quad \hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z ; \quad \hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$$

- The position-momentum commutation relations of slide 1.13 lead to the angular momentum commutation relations

$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z ; \quad [\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x ; \quad [\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y$$

- Unlike components of \mathbf{L} , \mathbf{r} , \mathbf{p} are non-commuting :

$$[\hat{L}_x, \hat{y}] = i\hbar\hat{z} \quad [\hat{L}_x, \hat{p}_y] = i\hbar\hat{p}_z$$

$$[\hat{L}_y, \hat{x}] = -i\hbar\hat{z} \quad [\hat{L}_y, \hat{p}_x] = -i\hbar\hat{p}_z$$

(together with their cyclic permutations)

Orbital angular momentum (2)

- The totally antisymmetric *Levi-Civita tensor*, ϵ_{ijk} , is defined as

$$\epsilon_{ijk} = \begin{cases} +1 & (ijk) = (123), (231), (312) \\ -1 & (ijk) = (132), (213), (321) \\ 0 & \text{otherwise} \end{cases}$$

$$\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}$$

- Using ϵ_{ijk} , the commutation relations on the previous slide can be expressed more concisely as

$$[\hat{L}_i, \hat{L}_j] = i\hbar \sum_k \epsilon_{ijk} \hat{L}_k$$

$$\begin{aligned}\hat{\mathbf{L}} &= (\hat{L}_x, \hat{L}_y, \hat{L}_z) \\ &\equiv (\hat{L}_1, \hat{L}_2, \hat{L}_3)\end{aligned}$$

$$[\hat{L}_i, \hat{r}_j] = i\hbar \sum_k \epsilon_{ijk} \hat{r}_k ; \quad [\hat{L}_i, \hat{p}_j] = i\hbar \sum_k \epsilon_{ijk} \hat{p}_k$$

Orbital angular momentum (3)

- Consider a vector operator $\hat{\mathbf{V}} = (\hat{V}_x, \hat{V}_y, \hat{V}_z) = (\hat{V}_1, \hat{V}_2, \hat{V}_3)$ satisfying the commutation relations

$$[\hat{L}_i, \hat{V}_j] = i\hbar \sum_k \epsilon_{ijk} \hat{V}_k \quad (1.29.1)$$

Then, for the squared operator $\hat{\mathbf{V}}^2 = \hat{V}_x^2 + \hat{V}_y^2 + \hat{V}_z^2$, we have

$$[\hat{L}_i, \hat{\mathbf{V}}^2] = \sum_j [\hat{L}_i, \hat{V}_j] \hat{V}_j + \sum_j \hat{V}_j [\hat{L}_i, \hat{V}_j] = i\hbar \sum_{j,k} \epsilon_{ijk} (\hat{V}_k \hat{V}_j + \hat{V}_j \hat{V}_k)$$

- Since ϵ_{ijk} is antisymmetric in j and k , while $\hat{V}_k \hat{V}_j + \hat{V}_j \hat{V}_k$ is symmetric, we immediately obtain

$$[\hat{L}_i, \hat{\mathbf{V}}^2] = 0 ; \quad [\hat{\mathbf{L}}, \hat{\mathbf{V}}^2] = 0$$

and hence also

$$[\hat{L}_i^2, \hat{\mathbf{V}}^2] = 0 ; \quad [\hat{\mathbf{L}}^2, \hat{\mathbf{V}}^2] = 0$$

- The operators $\mathbf{V} = \mathbf{L}, \mathbf{r}, \mathbf{p}$ all satisfy equation (1.29.1), giving immediately

$$[\hat{L}_i, \hat{\mathbf{L}}^2] = [\hat{L}_i, \hat{\mathbf{r}}^2] = [\hat{L}_i, \hat{\mathbf{p}}^2] = [\hat{\mathbf{L}}^2, \hat{\mathbf{r}}^2] = [\hat{\mathbf{L}}^2, \hat{\mathbf{p}}^2] = 0$$

Orbital angular momentum (4)

- The orbital angular momentum operator

$$\hat{\mathbf{L}} = \hat{\mathbf{r}} \wedge \hat{\mathbf{p}} = -i\hbar(\hat{\mathbf{r}} \wedge \nabla)$$

has components given, in Cartesian and spherical polar coordinates, by

$$\begin{aligned}\hat{L}_x &= -i\hbar \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) ; & \hat{L}_x &= -i\hbar \left(-\sin \phi \frac{\partial}{\partial \theta} - \frac{\cos \phi}{\tan \theta} \frac{\partial}{\partial \phi} \right) \\ \hat{L}_y &= -i\hbar \left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right) ; & \hat{L}_y &= -i\hbar \left(\cos \phi \frac{\partial}{\partial \theta} - \frac{\sin \phi}{\tan \theta} \frac{\partial}{\partial \phi} \right) \\ \hat{L}_z &= -i\hbar \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) ; & \hat{L}_z &= -i\hbar \frac{\partial}{\partial \phi}\end{aligned}$$

Thus the operator $\hat{\mathbf{L}}$ depends only on the angular variables θ and ϕ

- The square of the angular momentum operator, $\hat{\mathbf{L}}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$, is given in spherical polar coordinates as

$$\hat{\mathbf{L}}^2 = -\hbar^2 \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right)$$

Orbital angular momentum (5)

- Since the operator $\hat{\mathbf{L}}$ depends only on the angles θ and ϕ , we immediately obtain, for any function $f(r)$ of radius alone,

$$[f(r), \hat{L}_x] = [f(r), \hat{L}_y] = [f(r), \hat{L}_z] = 0$$

Thus, for any function $f(r)$, we have

$$[f(r), \hat{\mathbf{L}}] = [f(r), \hat{\mathbf{L}}^2] = 0$$

Spherical Harmonics

- The commutation relations for the operator $L = \mathbf{r} \wedge \mathbf{p}$ have the structure

$$[\hat{L}_x, \hat{\mathbf{L}}^2] = [\hat{L}_y, \hat{\mathbf{L}}^2] = [\hat{L}_z, \hat{\mathbf{L}}^2] = 0$$

$$[\hat{L}_x, \hat{L}_y] \neq 0 ; \quad [\hat{L}_y, \hat{L}_z] \neq 0 ; \quad [\hat{L}_z, \hat{L}_x] \neq 0$$

Hence we can find a set of simultaneous eigenstates of L^2 and one of the Cartesian components of L , the conventional choice being L_z

- In spherical polar coordinates, the simultaneous eigenstates of L^2 and L_z are the *spherical harmonics* $Y_{\ell m}(\theta, \phi)$:

$$\boxed{\begin{aligned}\hat{L}_z Y_{\ell m}(\theta, \phi) &= m\hbar Y_{\ell m}(\theta, \phi) \\ \hat{\mathbf{L}}^2 Y_{\ell m}(\theta, \phi) &= \ell(\ell + 1)\hbar^2 Y_{\ell m}(\theta, \phi)\end{aligned}}$$

where ℓ and m are integer quantum numbers :

$$\left. \begin{array}{l} \ell = 0, 1, 2, 3, \dots \\ m = -\ell, -\ell + 1, -\ell + 2, \dots, \ell - 1, \ell \end{array} \right\} \quad (2\ell + 1) \text{ values of } m$$

Spherical Harmonics (2)

- The $Y_{\ell m}$ constitute a complete, orthonormal set of functions :

$$\int_0^{2\pi} \int_{-1}^{+1} |Y_{\ell m}(\theta, \phi)|^2 d\cos\theta d\phi = 1$$
$$\int_0^{2\pi} \int_{-1}^{+1} Y_{\ell' m'}^*(\theta, \phi) Y_{\ell m}(\theta, \phi) d\cos\theta d\phi = \delta_{\ell' \ell} \delta_{m' m} \quad (1.33.1)$$

- The first few spherical harmonics are :

$$\ell = 0$$

$$Y_{00}(\theta, \phi) = \sqrt{\frac{1}{4\pi}}$$

$$\ell = 1$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$Y_{1,\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi}$$

$$\ell = 2$$

$$Y_{20} = \sqrt{\frac{5}{16\pi}} (3 \cos^2\theta - 1)$$

$$Y_{2,\pm 1} = \mp \sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{\pm i\phi}$$

$$Y_{2,\pm 2} = \sqrt{\frac{15}{32\pi}} \sin^2\theta e^{\pm 2i\phi}$$

Spherical Harmonics (3)

$\ell = 0$



$$|Y_{\ell m}(\theta, \phi)|^2$$

z

$\ell = 1$



The angular probability distributions,
 $|Y_{\ell m}|^2$, depend on θ only :

$$|Y_{\ell m}(\theta, \phi)|^2 \propto |P_{\ell}^{|m|}(\cos \theta)|^2$$

$\ell = 2$



The sum of each row is
a sphere (isotropic)

$\ell = 3$



$m = 0$

$m = \pm 1$

$m = \pm 2$

$m = \pm 3$

Spherical Harmonics (4)

- The spherical harmonics $Y_{\ell m}$ factorise as

$$Y_{\ell m}(\theta, \phi) \propto P_\ell^{|m|}(\cos \theta) e^{im\phi}$$

where the functions $P_\ell^m(\cos \theta)$ are Legendre polynomials

- An equal mix of all m 's is isotropic (a uniform sphere) :

$$\sum_{m=-\ell}^{\ell} |Y_{\ell m}(\theta, \phi)|^2 = \text{const.}$$

- Under a *parity operation* $\mathbf{r} \rightarrow -\mathbf{r}$, the $Y_{\ell m}$ have the property

$$Y_{\ell m}(\pi - \theta, \pi + \phi) = (-1)^\ell Y_{\ell m}(\theta, \phi)$$

i.e. the spherical harmonics are eigenstates of parity with eigenvalue

$$P = (-1)^\ell$$

Bound States

- Consider a non-relativistic particle of mass m moving in a potential $V(\mathbf{r})$:

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + V(\mathbf{r}) = -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r})$$

- The energy eigenstates are given by the solutions of

$$\hat{H}\psi(\mathbf{r}) = E\psi(\mathbf{r})$$

$$-\frac{\hbar^2}{2m} \nabla^2 \psi(\mathbf{r}) + V(\mathbf{r})\psi(\mathbf{r}) = E\psi(\mathbf{r})$$

- Imposing appropriate boundary conditions,

$$\int |\psi(\mathbf{r})|^2 d^3\mathbf{r} < \infty , \quad \psi(\mathbf{r}) \text{ continuous, single-valued , ...}$$

then results in *quantised* energy levels

- e.g. the 1D square well, the 1D harmonic oscillator, the H atom ...

1D infinite square well

-- Schrödinger's equation for $0 < x < a$:

$$-\frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} = E\psi(x)$$

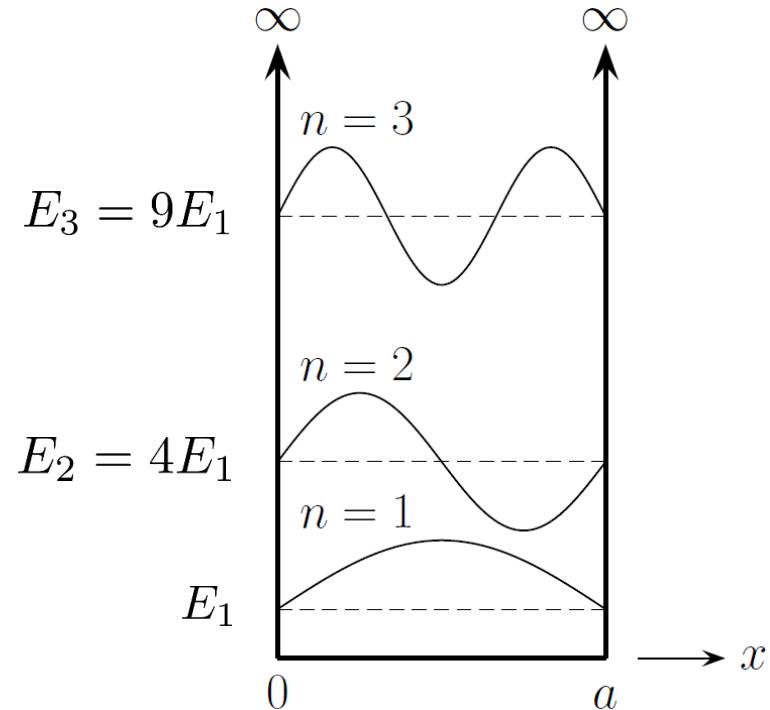
-- Boundary conditions :

$$\psi(0) = \psi(a) = 0$$

-- Energy eigenstates ($n = 1, 2, 3, \dots$) :

$$E_n = \frac{\hbar^2}{2m} \frac{n^2\pi^2}{a^2} \propto n^2 ; \quad \psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

- the bound state energies are *quantised*
- the ground state ($n = 1$) has *zero-point energy* $E_1 > 0$
- the eigenstates are *symmetric* (n odd) or *antisymmetric* (n even) about the centre of the well



Harmonic Oscillator

- For the 1D harmonic oscillator potential $V(x) = \frac{1}{2}m\omega^2x^2$, the Hamiltonian operator is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

- The eigenstates of H are quantised, with equally spaced energy levels :

$$\begin{aligned}\hat{H}\psi_n(x) &= E_n\psi_n(x) \\ \hat{H}|n\rangle &= E_n|n\rangle\end{aligned}$$

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

- The ground state ($n = 0$) is

$$|0\rangle = \psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar} \quad E_0 = \frac{1}{2}\hbar\omega$$

(zero-point energy)

- The first excited state ($n = 1$) is

$$|1\rangle = \psi_1(x) = \frac{\sqrt{2}}{\pi^{1/4}} \left(\frac{m\omega}{\hbar}\right)^{3/4} x e^{-m\omega x^2/2\hbar} \quad E_1 = \frac{3}{2}\hbar\omega$$

Harmonic oscillator energy levels

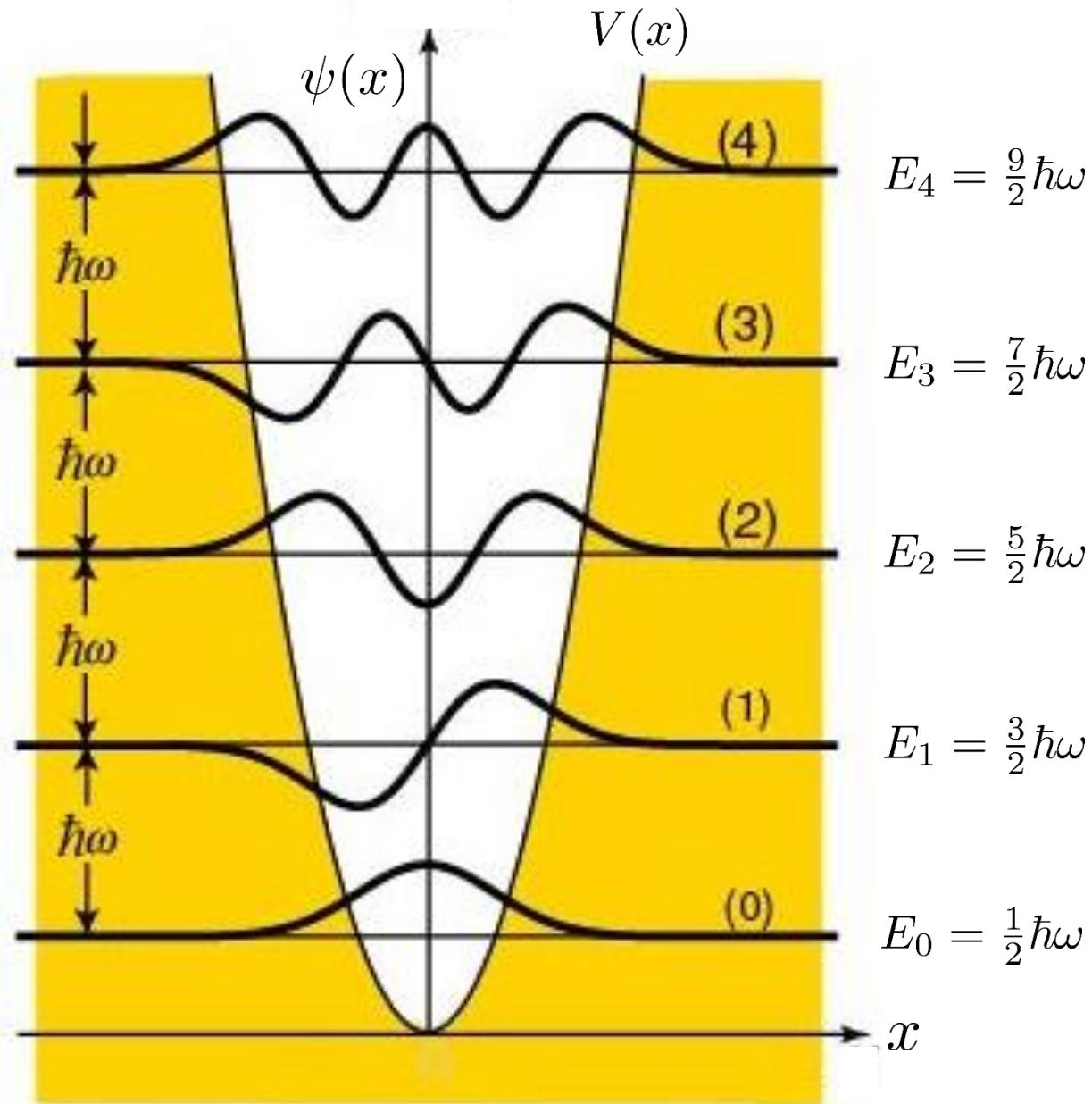
$$\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$$

$$\begin{array}{c} \hat{a}^\dagger \\ \hline \hline \hat{a} \end{array}$$

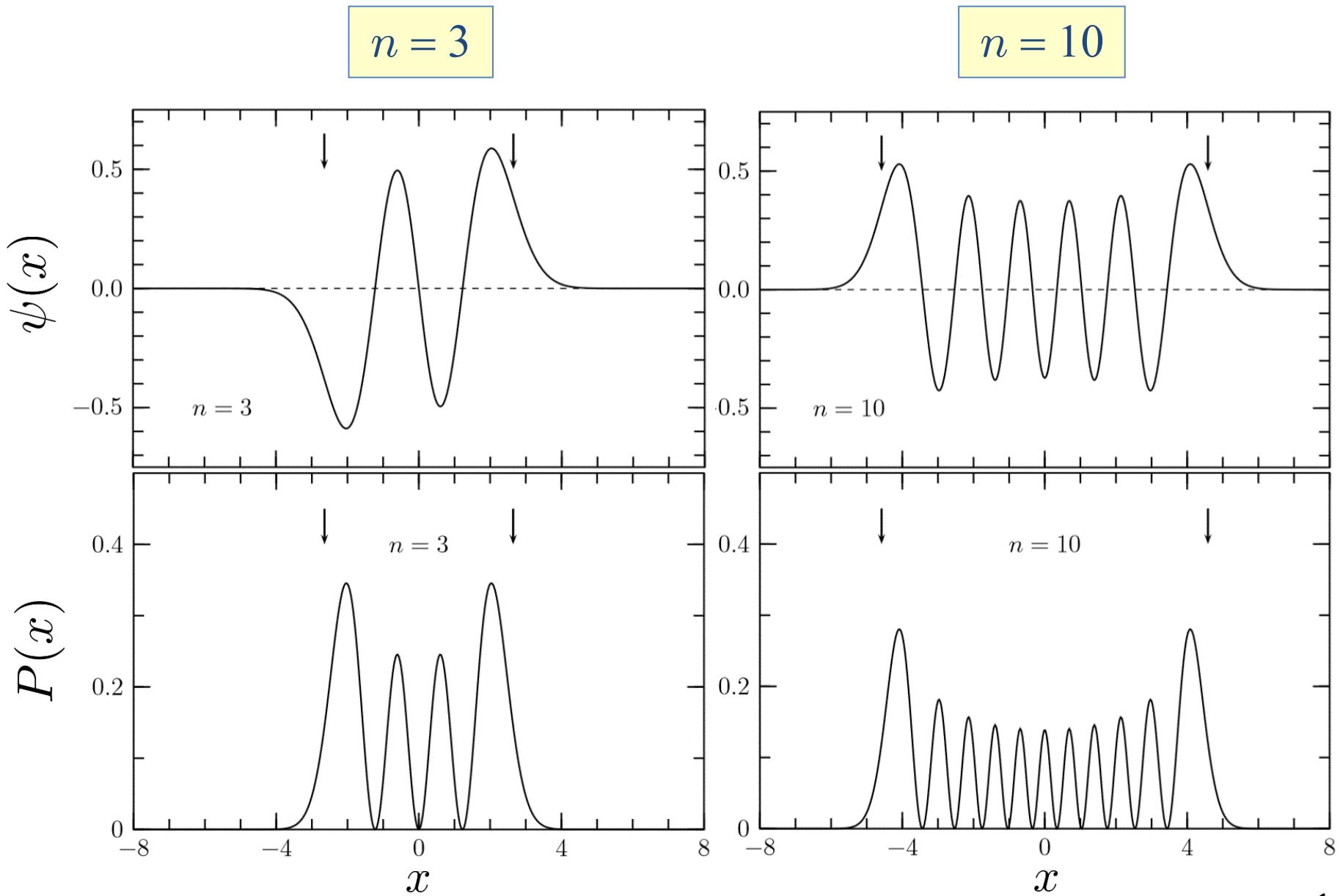
$$\hat{a}|n\rangle = \sqrt{n} |n-1\rangle$$

$$\hat{a}|0\rangle = 0$$

$$\begin{array}{c} \vdash \\ \hline \hline \vdash \end{array}$$



Harmonic oscillator wavefunctions



Harmonic oscillator : ladder operators

- The (non-Hermitian) ladder operators a and a^\dagger are defined as

$$\begin{aligned}\hat{a} &= \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + \frac{i}{\sqrt{2m\hbar\omega}}\hat{p} \\ \hat{a}^\dagger &= \sqrt{\frac{m\omega}{2\hbar}}\hat{x} - \frac{i}{\sqrt{2m\hbar\omega}}\hat{p}\end{aligned}\quad (\hat{a}^\dagger \neq \hat{a})$$

- They satisfy the commutation relation

$$[\hat{x}, \hat{p}] = i\hbar \quad \Rightarrow \quad [\hat{a}, \hat{a}^\dagger] = 1$$

- They provide easy movement between neighbouring states :

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle ; \quad \hat{a} |n\rangle = \sqrt{n} |n-1\rangle \quad \hat{a} |0\rangle = 0$$

- A given state $|n\rangle$ can be obtained by repeatedly working up from the ground state $|0\rangle$:

$$|n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n |0\rangle$$

Harmonic oscillator : the number operator

- In terms of ladder operators, the oscillator Hamiltonian is

$$\hat{H} = \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \hbar\omega$$

The operator product $\hat{a}^\dagger \hat{a}$ is known as the *number operator* :

$$(\hat{a}^\dagger \hat{a})|n\rangle = n|n\rangle$$

- The equations defining a and a^\dagger can be inverted to give

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^\dagger) ; \quad \hat{p} = -i\sqrt{\frac{m\hbar\omega}{2}} (\hat{a} - \hat{a}^\dagger)$$

Using the ladder operators allows many results to be obtained very simply

e.g. the property $\langle x \rangle = \langle p \rangle = 0$ follows immediately from

$$\langle n | \hat{a} | n \rangle = \langle n | \hat{a}^\dagger | n \rangle = 0$$

while expressions for $\langle x^2 \rangle$ and $\langle p^2 \rangle$ can be obtained using

$$\langle n | \hat{a} \hat{a} | n \rangle = \langle n | \hat{a}^\dagger \hat{a}^\dagger | n \rangle = 0 ; \quad \langle n | \hat{a}^\dagger \hat{a} | n \rangle = n$$

The “Zeroth-Order” Hydrogen Atom

- At “zeroth-order” of approximation (non-relativistic electron, infinitely heavy nucleus, neglect spin), the Hamiltonian for a hydrogen-like atom is

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(r) = -\frac{\hbar^2}{2m}\nabla^2 + V(r) ; \quad V(r) = -\frac{Ze^2}{4\pi\epsilon_0 r}$$

- In spherical polar coordinates (r, θ, ϕ) , the operator ∇^2 is given by

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{1}{\hbar^2 r^2} \hat{\mathbf{L}}^2$$

$$\hat{\mathbf{L}}^2 = -\hbar^2 \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right)$$

- The time-independent Schrödinger equation, $\hat{H}\psi(\mathbf{r}) = E\psi(\mathbf{r})$, can be solved by looking for separated-variable solutions of the form

$$\psi(r, \theta, \phi) = R(r) \Theta(\theta) \Phi(\phi) = R(r)Y(\theta, \phi)$$

The “zeroth-order” hydrogen atom (2)

- Setting $u(r) \equiv rR(r)$, the radial function $R(r)$ is given by the solution of

$$-\frac{\hbar^2}{2m} \frac{d^2 u(r)}{dr^2} + \left[V(r) + \frac{\ell(\ell+1)\hbar^2}{2mr^2} \right] u(r) = E u(r) \quad (1.44.1)$$

- Requiring the wavefunction ψ to be single-valued, square-integrable, and to satisfy appropriate boundary conditions at the origin and at infinity, results in *quantised* energy eigenstates

$$\psi_{n\ell m}(r, \theta, \phi) = R_{n\ell}(r)Y_{\ell m_\ell}(\theta, \phi)$$

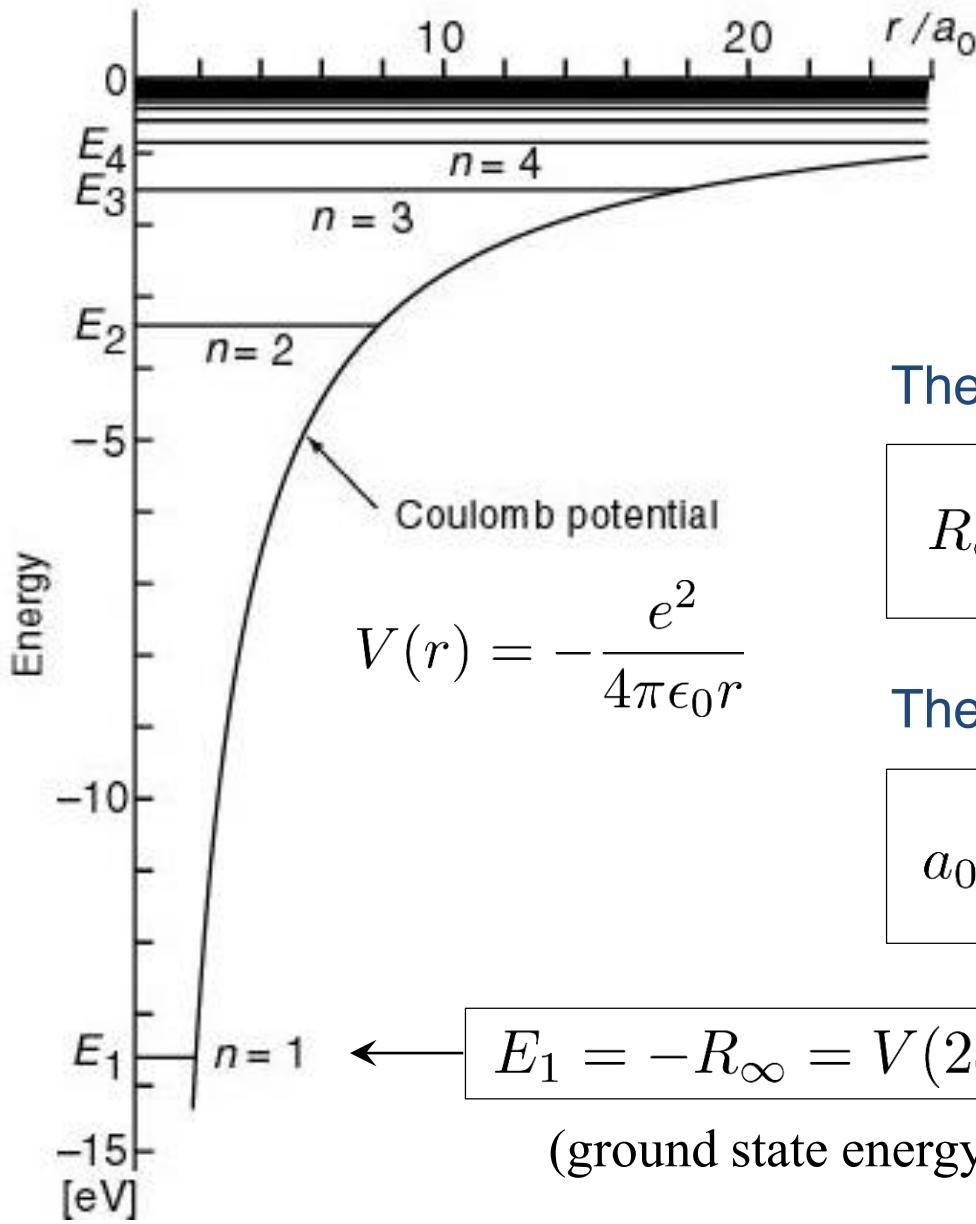
where n is the *principal quantum number*, and $\ell < n$:

$$n = 1, 2, 3, \dots \quad \begin{cases} \ell = 0, 1, 2, \dots, (n-1) \\ m_\ell = -\ell, -\ell+1, -\ell+2, \dots, +\ell \end{cases}$$

- The energy eigenvalues depend only on n :

$$\hat{H}|n\ell m_\ell\rangle = E_n |n\ell m_\ell\rangle$$

The “zeroth-order” hydrogen atom (3)



$$E_n = -\frac{Z^2}{n^2} R_\infty$$

The *Rydberg constant* R_∞ is

$$R_\infty = \frac{\hbar^2}{2m_e a_0^2} = 13.6056981(40) \text{ eV}$$

The *Bohr radius* a_0 is

$$a_0 = \frac{4\pi\epsilon_0}{e^2} \frac{\hbar^2}{m_e} = 0.5297 \times 10^{-10} \text{ m}$$

$$E_1 = -R_\infty = V(2a_0)$$

The “zeroth-order” hydrogen atom (4)

- The possible quantum numbers of the states $|n\ell m_\ell\rangle$ are

n	ℓ	subshells
1	0	1s
2	0, 1	2s, 2p
3	0, 1, 2	3s, 3p, 3d
4	0, 1, 2, 3	4s, 4p, 4d, 4f
...
n	$0, 1, 2, 3, \dots, (n - 1)$	ns, np, ...

- Each energy level n , of energy E_n , is degenerate with respect to the quantum numbers ℓ and m_ℓ :

The degeneracy g_n of each energy level (neglecting spin) is

$$g_n = \sum_{\ell=0}^{n-1} (2\ell + 1) = n^2$$

The “zeroth-order” hydrogen atom (5)

- The first few hydrogen atom wavefunctions $|n\ell m_\ell\rangle$ are :

$n = 1, \ell = 0$	$ 100\rangle = \left(\frac{Z^3}{\pi a_0^3}\right)^{1/2} e^{-Zr/a_0}$	$g_1 = 1$
$n = 2, \ell = 0$	$ 200\rangle = \left(\frac{Z^3}{32\pi a_0^3}\right)^{1/2} e^{-Zr/2a_0} \left(2 - \frac{Zr}{a_0}\right)$	$g_2 = 4$
$m_\ell = 0$ $n = 2, \ell = 1$ $m_\ell = \pm 1$	$ 210\rangle = \left(\frac{Z^3}{32\pi a_0^3}\right)^{1/2} e^{-Zr/2a_0} \frac{Zr}{a_0} \cos \theta$ $ 21, \pm 1\rangle = \mp \left(\frac{Z^3}{64\pi a_0^3}\right)^{1/2} e^{-Zr/2a_0} \frac{Zr}{a_0} \sin \theta e^{\pm i\phi}$	$g_2 = 4$

- Only s-state ($\ell = 0$) wavefunctions $|n00\rangle$ are finite at the origin :

$$\psi_{n00}(r = 0) = \sqrt{\frac{Z^3}{\pi n^3 a_0^3}}$$

The “zeroth-order” hydrogen atom (6)

- Normalisation of the hydrogen atom wavefunctions gives

$$\begin{aligned} 1 &= \int_0^\infty \int_{-1}^{+1} \int_0^{2\pi} |\psi_{n\ell m}(r, \theta, \phi)|^2 r^2 dr d\cos\theta d\phi \\ &= \int_0^\infty r^2 |R_{n\ell}(r)|^2 dr \times \int_{-1}^{+1} \int_0^{2\pi} d\cos\theta d\phi |Y_{\ell m}(\theta, \phi)|^2 \\ &= \int_0^\infty r^2 |R_{n\ell}(r)|^2 dr \end{aligned}$$

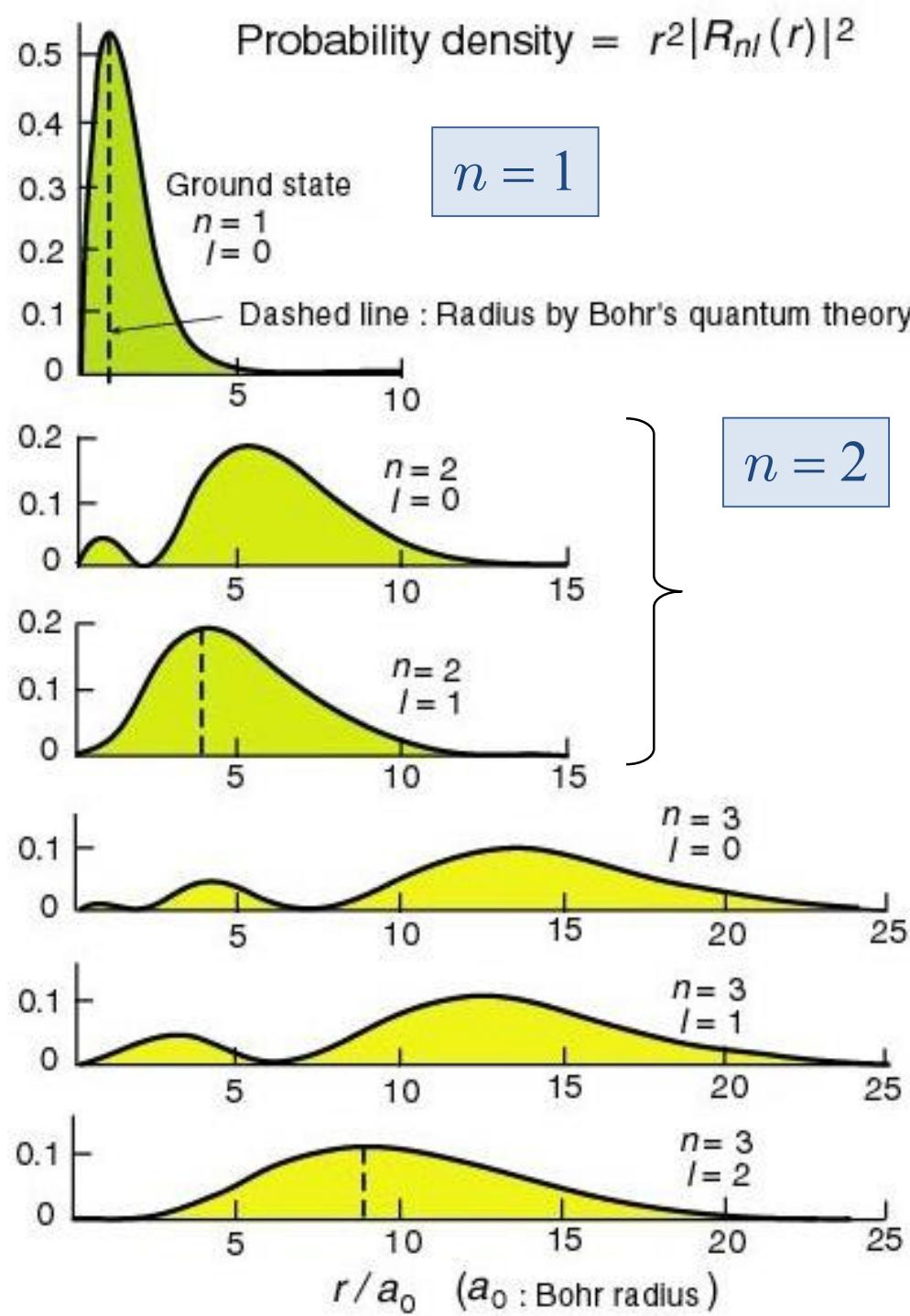
Hence the radial probability distribution $P_{n\ell}(r)$ is given by

$$P_{n\ell}(r) = r^2 |R_{n\ell}(r)|^2 ; \quad \int_0^\infty P_{n\ell}(r) dr = 1$$

- The average radial position $\langle r \rangle$ of the electron is

$$\begin{aligned} \langle r \rangle &= \int_0^\infty \int_{-1}^{+1} \int_0^{2\pi} r |\psi_{n\ell m}(r, \theta, \phi)|^2 r^2 dr d\cos\theta d\phi \\ &= \int_0^\infty r^3 |R_{n\ell}(r)|^2 dr = \int_0^\infty r P_{n\ell}(r) dr \end{aligned}$$

Probability density = $r^2|R_{nl}(r)|^2$



$$R_{10}(r) = \left(\frac{Z}{a_0}\right)^{3/2} 2e^{-Zr/a_0}$$

$$R_{20}(r) = \left(\frac{Z}{a_0}\right)^{3/2} \frac{1}{\sqrt{2}} \left(1 - \frac{Zr}{2a_0}\right) e^{-Zr/2a_0}$$

$$R_{21}(r) = \left(\frac{Z}{a_0}\right)^{3/2} \frac{1}{2\sqrt{6}} \frac{Zr}{a_0} e^{-Zr/2a_0}$$

$n = 3$

The “zeroth-order” hydrogen atom (8)

- The average radius $\langle r \rangle$ can be shown to be

$$\langle r \rangle_{n\ell} = \frac{3a_0}{2Z} \left(n^2 - \frac{\ell(\ell+1)}{3} \right)$$

- The states $|n\ell m_\ell\rangle$ form a complete, orthonormal set :

$$\begin{aligned} \langle n'\ell'm'_\ell | n\ell m_\ell \rangle &= \int_0^\infty \int_{-1}^{+1} \int_0^{2\pi} R_{n'\ell'}(r) Y_{\ell'm'_\ell}^*(\theta, \phi) \\ &\quad \times R_{n\ell}(r) Y_{\ell m_\ell}(\theta, \phi) r^2 dr d\cos\theta d\phi \\ &= \delta_{n'n} \delta_{\ell'\ell} \delta_{m'_\ell m_\ell} \end{aligned}$$

The orthogonality with respect to n is due to the radial component,

$$\int_0^\infty r^2 R_{n'\ell}(r) R_{n\ell}(r) dr = \delta_{n'n}$$

while orthogonality with respect to ℓ and m_ℓ arises from orthogonality of the spherical harmonics, equation (1.33.1)

The “zeroth-order” hydrogen atom (9)

- The energy eigenstates $|n\ell m_\ell\rangle$ form a complete set of simultaneous eigenstates of the operators H , \hat{L}^2 and L_z (see slide 1.32) :

$$\hat{H}|n\ell m_\ell\rangle = E_n|n\ell m_\ell\rangle ; \quad \hat{L}_z|n\ell m_\ell\rangle = m_\ell \hbar |n\ell m_\ell\rangle$$
$$\hat{\mathbf{L}}^2|n\ell m_\ell\rangle = \ell(\ell + 1)\hbar^2 |n\ell m_\ell\rangle$$

- From slides 1.29 and 1.31, the angular momentum operator $\hat{\mathbf{L}}$ commutes with $\hat{\mathbf{p}}^2$ and with the potential $V(r)$:

$$[\hat{\mathbf{L}}, \hat{\mathbf{p}}^2] = [\hat{\mathbf{L}}^2, \hat{\mathbf{p}}^2] = 0 ; \quad [V(r), \hat{\mathbf{L}}] = [V(r), \hat{\mathbf{L}}^2] = 0$$

Therefore $\hat{\mathbf{L}}$ also commutes with the Hamiltonian $\hat{H} = \hat{\mathbf{p}}^2/2m + V(r)$:

$$[\hat{H}, \hat{L}_z] = [\hat{H}, \hat{\mathbf{L}}] = [\hat{H}, \hat{\mathbf{L}}^2] = 0$$

- Thus :
$$\{\hat{H}, \hat{\mathbf{L}}^2, \hat{L}_z\}$$
 form a mutually commuting set of operators, consistent with the existence of a set of simultaneous eigenstates $|n\ell m_\ell\rangle$

The “zeroth-order” hydrogen atom (10)

- As already noted on slide 1.46, each energy level n is degenerate with respect to the angular momentum quantum numbers ℓ and m_ℓ :
 - The degeneracy with respect to ℓ is a particular feature of the $1/r$ Coulomb potential ;
 - for a more general central potential $V(r)$, as used to describe multi-electron atoms for example, the energy eigenvalues depend on both n and ℓ :
 - The degeneracy with respect to m_ℓ is a consequence of *rotational symmetry* (see later)

$$E_n \longrightarrow E_{n\ell}$$

Angular Momentum

- The *orbital* angular momentum operator $L = \mathbf{r} \wedge \mathbf{p}$ is characterised by the commutation relations

$$[\hat{L}_i, \hat{L}_j] = i\hbar \sum_k \epsilon_{ijk} \hat{L}_k$$

- All elementary particles (except the Higgs boson) possess an *internal* angular momentum S (*spin*) satisfying the same commutation relations as L :

$$[\hat{S}_i, \hat{S}_j] = i\hbar \sum_k \epsilon_{ijk} \hat{S}_k$$

- This leads us to consider a *generic* angular momentum operator

$$\hat{\mathbf{J}} = (\hat{J}_x, \hat{J}_y, \hat{J}_z) = (\hat{J}_1, \hat{J}_2, \hat{J}_3)$$

satisfying the commutation relations

$$[\hat{J}_i, \hat{J}_j] = i\hbar \sum_k \epsilon_{ijk} \hat{J}_k$$

$$\left\{ \begin{array}{l} [\hat{J}_x, \hat{J}_y] = i\hbar \hat{J}_z \\ [\hat{J}_y, \hat{J}_z] = i\hbar \hat{J}_x \\ [\hat{J}_z, \hat{J}_x] = i\hbar \hat{J}_y \end{array} \right.$$

Angular momentum (2)

- The eigenstates of the operator J are characterised by quantum numbers j and m which can be integer or half-integer :

$$\hat{J}_z |jm\rangle = m\hbar |jm\rangle$$

$$\hat{\mathbf{J}}^2 |jm\rangle = j(j+1)\hbar^2 |jm\rangle$$

$$\langle j'm' | jm \rangle = \delta_{j'j} \delta_{m'm}$$

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \dots$$

with $(2j + 1)$ possible values of m for each possible value of j :

$$m = j, j-1, j-2, \dots, -j+1, -j$$

- For *orbital* angular momentum, $J = L$, the quantum number j is restricted to *integer* values : $j = \ell = 0, 1, 2, \dots$
(this follows from the additional requirement that the eigenstates of L be single-valued after a spatial rotation through 2π)

Angular momentum : ladder operators

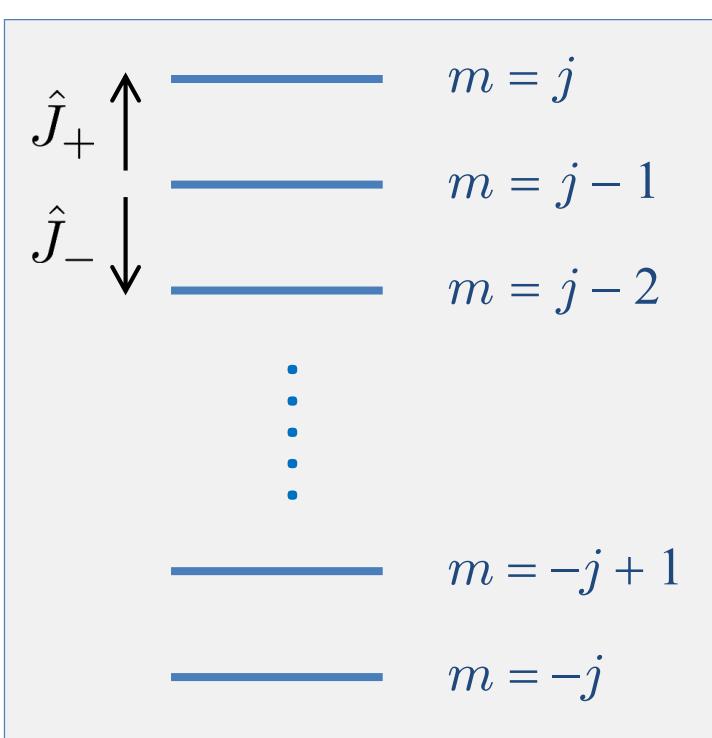
- The *ladder operators* J_{\pm} are defined as

$$\hat{J}_{\pm} \equiv \hat{J}_x \pm i\hat{J}_y \quad (\hat{J}_{\pm}^{\dagger} = \hat{J}_{\mp})$$

$$[\hat{J}_+, \hat{J}_-] = 2\hbar\hat{J}_z$$

$$\hat{\mathbf{J}}^2 = \hat{J}_+ \hat{J}_- + \hat{J}_z^2 - \hbar \hat{J}_z$$

- The ladder operators J_{\pm} change the quantum number m by $\Delta m = \pm 1$:



$$\hat{J}_{\pm}|jm\rangle = \hbar\sqrt{j(j+1) - m(m \pm 1)} |j, m \pm 1\rangle$$

or, equivalently : $\hat{J}_{\pm}|jm\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)} |j, m \pm 1\rangle$

- At the top ($m = j$) or bottom ($m = -j$) of the ladder, there are no states left :

$$\hat{J}_+|jj\rangle = 0 ; \quad \hat{J}_-|j, -j\rangle = 0$$

$$j = 0$$

$$\hat{\mathbf{J}}^2|00\rangle = 0$$

$$m = 0$$

$$\begin{aligned}\hat{J}_z|00\rangle &= 0 \\ \hat{J}_{\pm}|00\rangle &= 0\end{aligned}$$

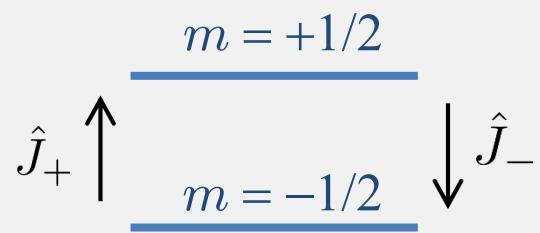
$$j = 1/2$$

$$\hat{\mathbf{J}}^2|\frac{1}{2}, \pm \frac{1}{2}\rangle = (3/4)\hbar^2|\frac{1}{2}, \pm \frac{1}{2}\rangle$$

$$\hat{J}_z|\frac{1}{2}, \pm \frac{1}{2}\rangle = \pm \frac{1}{2}\hbar|\frac{1}{2}, \pm \frac{1}{2}\rangle$$

$$\hat{J}_+|\frac{1}{2}, +\frac{1}{2}\rangle = 0$$

$$\hat{J}_+|\frac{1}{2}, -\frac{1}{2}\rangle = \hbar|\frac{1}{2}, +\frac{1}{2}\rangle$$



$$\hat{J}_-|\frac{1}{2}, +\frac{1}{2}\rangle = \hbar|\frac{1}{2}, -\frac{1}{2}\rangle$$

$$\hat{J}_-|\frac{1}{2}, -\frac{1}{2}\rangle = 0$$

$$j = 1$$

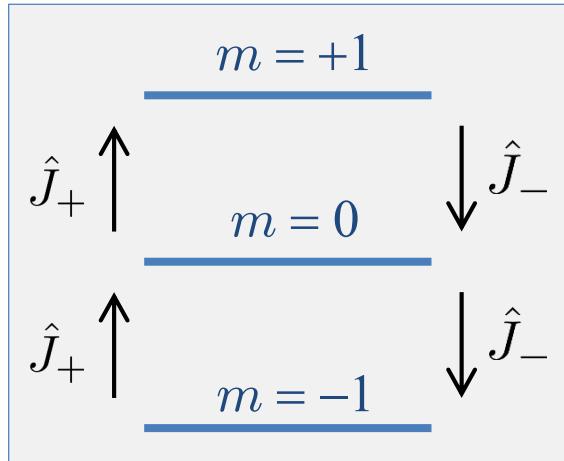
$$\hat{\mathbf{J}}^2|1, m\rangle = 2\hbar^2|1, m\rangle$$

$$\begin{aligned}\hat{J}_z|1, \pm 1\rangle &= \pm \hbar|1, \pm 1\rangle \\ \hat{J}_z|1, 0\rangle &= 0\end{aligned}$$

$$\hat{J}_+|1, +1\rangle = 0$$

$$\hat{J}_+|1, 0\rangle = \sqrt{2}\hbar|1, 1\rangle$$

$$\hat{J}_+|1, -1\rangle = \sqrt{2}\hbar|1, 0\rangle$$



$$\hat{J}_-|1, +1\rangle = \sqrt{2}\hbar|1, 0\rangle$$

$$\hat{J}_-|1, 0\rangle = \sqrt{2}\hbar|1, -1\rangle$$

$$\hat{J}_-|1, -1\rangle = 0$$

Angular momentum : matrix representations

- Angular momentum eigenstates are orthonormal :

$$\langle j_2 m_2 | j_1 m_1 \rangle = \delta_{j_1 j_2} \delta_{m_1 m_2}$$

Matrix elements of J_+ , J_- , J_z therefore vanish unless $j_1 = j_2$:

$$\langle j_2 m_2 | \hat{J}_+ | j_1 m_1 \rangle = \hbar \sqrt{j_1(j_1 + 1) - m_1(m_1 + 1)} \delta_{j_2 j_1} \delta_{m_2, m_1 + 1}$$

$$\langle j_2 m_2 | \hat{J}_- | j_1 m_1 \rangle = \hbar \sqrt{j_1(j_1 + 1) - m_1(m_1 - 1)} \delta_{j_2 j_1} \delta_{m_2, m_1 - 1}$$

$$\langle j_2 m_2 | \hat{J}_z | j_1 m_1 \rangle = \hbar m_1 \delta_{j_2 j_1} \delta_{m_2 m_1}$$

The same is true also for matrix elements of J_x , J_y , since

$$\hat{J}_x = \frac{1}{2}(\hat{J}_+ + \hat{J}_-) ; \quad \hat{J}_y = -\frac{i}{2}(\hat{J}_+ - \hat{J}_-)$$

- Thus matrix elements $\langle j_2 m_2 | \hat{\mathbf{J}} | j_1 m_1 \rangle$ vanish unless $j_1 = j_2$

We can therefore consider the matrix representation of \mathbf{J} one j at a time
(the overall matrix representation is block-diagonal)

Angular momentum $j = 1/2$: the Pauli matrices

- For $j = 1/2$, in the basis of states $|\uparrow\rangle, |\downarrow\rangle$, the operator J_z has matrix elements

$$J_z = \begin{pmatrix} \langle \uparrow | \hat{J}_z | \uparrow \rangle & \langle \uparrow | \hat{J}_z | \downarrow \rangle \\ \langle \downarrow | \hat{J}_z | \uparrow \rangle & \langle \downarrow | \hat{J}_z | \downarrow \rangle \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- For J_{\pm} , we have

$$\hat{J}_+ |\downarrow\rangle = \hbar |\uparrow\rangle ; \quad \hat{J}_- |\uparrow\rangle = \hbar |\downarrow\rangle$$

$$J_+ = \begin{pmatrix} \langle \uparrow | \hat{J}_+ | \uparrow \rangle & \langle \uparrow | \hat{J}_+ | \downarrow \rangle \\ \langle \downarrow | \hat{J}_+ | \uparrow \rangle & \langle \downarrow | \hat{J}_+ | \downarrow \rangle \end{pmatrix} = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ; \quad J_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$J_x = \frac{1}{2}(J_+ + J_-) = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \quad J_y = -\frac{i}{2}(J_+ - J_-) = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

- Overall, for $j = 1/2$, we obtain the matrix representation $\mathbf{J} = \frac{\hbar}{2} \boldsymbol{\sigma}$:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where $(\sigma_x, \sigma_y, \sigma_z)$ are the (Hermitian) *Pauli matrices*

Angular momentum : matrices for higher j

-- For $j = 1$, in the basis $|j,m\rangle = |1,+1\rangle, |1,0\rangle, |1,-1\rangle$: **[EXAMPLES SHEET]**

$$J_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad J_y = \frac{i\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}; \quad J_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

-- For $j = 3/2$, in the basis $|3/2,+3/2\rangle, |3/2,+1/2\rangle, |3/2,-1/2\rangle, |3/2,-3/2\rangle$:

$$J_x = \frac{\hbar}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}; \quad J_y = -i\frac{\hbar}{2} \begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$$J_z = \hbar \begin{pmatrix} 3/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & -3/2 \end{pmatrix}$$

e.g.

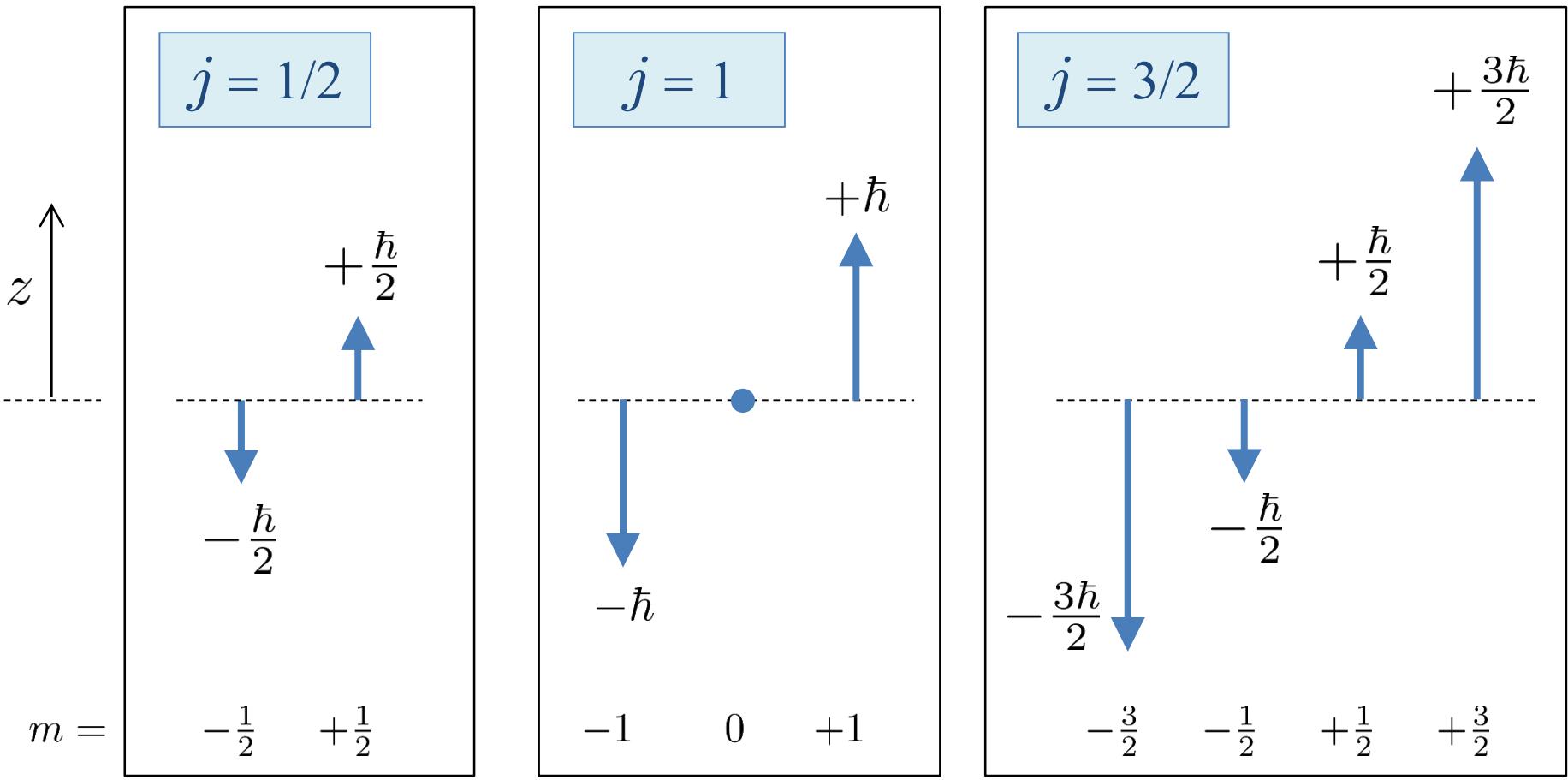
$$\langle \frac{3}{2}, -\frac{3}{2} | \hat{J}_y | \frac{3}{2}, -\frac{1}{2} \rangle = -i\hbar \frac{\sqrt{3}}{2}$$

Angular momentum : expectation and uncertainty

- For an eigenstate $|j, m\rangle$, the expectation values of the operator \mathbf{J} are

$$\langle \hat{\mathbf{J}} \rangle = (\langle \hat{J}_x \rangle, \langle \hat{J}_y \rangle, \langle \hat{J}_z \rangle) = (0, 0, m\hbar) ; \quad \langle \hat{\mathbf{J}}^2 \rangle = j(j+1)\hbar^2$$

Thus the vector $\langle \mathbf{J} \rangle$ points “up” or “down” along the z axis :



Angular momentum : expectation and uncertainty (2)

- Each arrow represents the *average result of many repeated measurements* of the angular momentum components of a given state $|j,m\rangle$

In general, such repeat measurements result in a *spread* of measured values, with a non-zero uncertainty (except for the case $j = 0$)

- e.g. for the $j = 1/2$ “spin-up” eigenstate $|\uparrow\rangle$ (defined w.r.t. the z -axis), a measurement of J_z will *always* give the result $+\hbar/2$, with no uncertainty :

$$\langle \hat{J}_z \rangle = \hbar/2 ; \quad \Delta \hat{J}_z = 0$$

However a measurement of J_x or J_y for this state will randomly give one of the two results $+\hbar/2$ or $-\hbar/2$, with equal probability :

$$\langle \hat{J}_x \rangle = \langle \hat{J}_y \rangle = 0 ; \quad \Delta \hat{J}_x = \Delta \hat{J}_y = \frac{\hbar}{2}$$

- In general, for repeat measurements on a state $|j,m\rangle$, we obtain

$$\langle \hat{J}_z \rangle = m\hbar ; \quad \langle \hat{J}_x \rangle = \langle \hat{J}_y \rangle = 0 ; \quad \Delta \hat{J}_z = 0 ; \quad \Delta \hat{J}_{x,y} > 0$$

Addition of Angular Momentum

- It is often necessary to combine (add) two or more independent angular momentum operators to obtain the total angular momentum
 - e.g. the total angular momentum of an atomic electron : $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$
 - e.g. the total spin of a two-electron system : $\hat{\mathbf{S}} = \hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2$
- How are the eigenvalues and eigenstates of J (or S) related to those of L and S (or S_1 and S_2) ?
 - e.g. what is the total angular momentum j of an electron in a 3d eigenstate ($\ell = 2$, $s = 1/2$) of the hydrogen atom ?
- This was answered in Part IB for the example of $J = L + S$, and the cases ($\ell = 1$, $s = 1/2$) and ($s_1 = 1/2$, $s_2 = 1/2$) were treated in detail
The arguments developed there are completely general, and to emphasise this we consider the addition of *generic* angular momentum operators :

$$\hat{\mathbf{J}} = \hat{\mathbf{J}}' + \hat{\mathbf{J}}''$$

Addition of angular momentum (2)

- Consider two independent angular momentum operators \hat{J}' and \hat{J}''

$$[\hat{J}'_i, \hat{J}'_j] = i\hbar \sum_k \epsilon_{ijk} \hat{J}'_k ; \quad [\hat{J}''_i, \hat{J}''_j] = i\hbar \sum_k \epsilon_{ijk} \hat{J}''_k ; \quad [\hat{J}'_i, \hat{J}''_j] = 0$$

Then, for the operator $\hat{J} = \hat{J}' + \hat{J}''$, we obtain the commutation relations

$$[\hat{J}_i, \hat{J}_j] = i\hbar \sum_k \epsilon_{ijk} \hat{J}_k$$

Thus (not surprisingly) the sum, J , is also an angular momentum operator

- The operators \hat{J}' and \hat{J}'' possess eigenstates $|j'm'\rangle$ and $|j''m''\rangle$:

$$\left. \begin{array}{l} \hat{J}'_z |j'm'\rangle = m' \hbar |j'm'\rangle \\ \hat{J}'^2 |j'm'\rangle = j'(j'+1) \hbar^2 |j'm'\rangle \end{array} \right\} \begin{array}{l} j' = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \dots \\ m' = j', j'-1, \dots, -j' \end{array}$$

$$\left. \begin{array}{l} \hat{J}''_z |j''m''\rangle = m'' \hbar |j''m''\rangle \\ \hat{J}''^2 |j''m''\rangle = j''(j''+1) \hbar^2 |j''m''\rangle \end{array} \right\} \begin{array}{l} j'' = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \dots \\ m'' = j'', j''-1, \dots, -j'' \end{array}$$

Addition of angular momentum (3)

- The operator J must itself possess a complete set of eigenstates $|jm\rangle$:

$$\left. \begin{array}{l} \hat{J}_z|jm\rangle = m\hbar|jm\rangle \\ \hat{J}^2|jm\rangle = j(j+1)\hbar^2|jm\rangle \end{array} \right\} \quad \begin{array}{l} j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \dots \\ m = j, j-1, \dots, -j \end{array}$$

- how are the quantum numbers j, m related to the quantum numbers j', m', j'', m'' ?
- how are the eigenstates $|jm\rangle$ related to the eigenstates $|j'm'\rangle$ and $|j''m''\rangle$?
- For m the answer is straightforward; a product state $|j'm'\rangle \otimes |j''m''\rangle$ is easily seen to be an eigenstate of J_z :

$$\hat{J}_z|j'm'\rangle|j''m''\rangle = (\hat{J}'_z + \hat{J}''_z)|j'm'\rangle|j''m''\rangle = (m' + m'')\hbar|j'm'\rangle|j''m''\rangle$$

Hence we must have

$$m\hbar = (m' + m'')\hbar ;$$

$$m = m' + m''$$

Addition of angular momentum (4)

- However a *single* product state $|j'm'\rangle \otimes |j''m''\rangle$ is not in general also an eigenstate of \mathbf{J}^2 ; we need to take suitable *linear combinations* ...

The appropriate linear combinations were determined in Part IB, in the context of $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$:

$$\Psi_{j,m_j} = \sum_{m_\ell, m_s} C_{\ell, m_\ell, s, m_s} Y_{\ell, m_\ell}(\theta, \phi) |s, m_s\rangle$$

The constants C_{ℓ, m_ℓ, s, m_s} are known as *Clebsch-Gordan coefficients*

- Below, we translate the Part IB results from $\mathbf{J} = \mathbf{L} + \mathbf{S}$ to $\mathbf{J} = \mathbf{J}' + \mathbf{J}''$, and in the process change to a Dirac style notation :

$$(\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}) \rightarrow (\hat{\mathbf{J}} = \hat{\mathbf{J}}' + \hat{\mathbf{J}}'')$$
$$\begin{aligned}\Psi_{j,m_j} &\rightarrow |jm\rangle \\ Y_{\ell, m_\ell}(\theta, \phi) &\rightarrow |j'm'\rangle \\ |s, m_s\rangle &\rightarrow |j''m''\rangle \\ C_{\ell, m_\ell, s, m_s} &\rightarrow \langle j'm'; j''m''|jm\rangle\end{aligned}$$

Addition of angular momentum (5)

- Thus, for given values of j' and j'' , the eigenstates $|jm\rangle$ of J can be expressed as linear combinations of the product states $|j'm'\rangle \otimes |j''m''\rangle$:

$$|jm\rangle = \sum_{m', m''} |j'm'\rangle |j''m''\rangle \langle j'm'; j''m''| jm\rangle$$

The sum runs over all values of m' and m'' such that

$$m = m' + m''$$

(or, equivalently, the C-G coefficient vanishes if $m \neq m' + m''$)

- The Dirac notation emphasises that the Clebsch-Gordan coefficients are simply the overlap between the total and product eigenstates:

$$\langle j'm'; j''m''| jm\rangle = [\langle j'm' | \otimes \langle j''m'' |] | jm\rangle$$

Addition of angular momentum (6)

- For given values of j' and j'' , it was shown in Part IB that the quantum number j could take on a range of allowed values :

$$j = j' + j'', \quad j' + j'' - 1, \quad \dots, \quad |j' - j''| + 1, \quad |j' - j''| \quad (1.67.1)$$

- Using a notation $j' \otimes j'' = j$, this can be written as

$$j' \otimes j'' = j' + j'', \quad j' + j'' - 1, \quad \dots, \quad |j' - j''| + 1, \quad |j' - j''|$$

e.g. $\frac{1}{2} \otimes \frac{1}{2} = 1, \quad 0 \quad 2 \otimes 2 = 4, \quad 3, \quad 2, \quad 1, \quad 0$

$$3 \otimes \frac{3}{2} = \frac{9}{2}, \quad \frac{7}{2}, \quad \frac{5}{2}, \quad \frac{3}{2} \quad 3 \otimes 2 = 5, \quad 4, \quad 3, \quad 2, \quad 1$$

- Note that a total angular momentum $j = 0$ can only be obtained if $j' = j''$:

$$\frac{1}{2} \otimes \frac{1}{2} = 0, \dots ; \quad 1 \otimes 1 = 0, \dots ; \quad \frac{3}{2} \otimes \frac{3}{2} = 0, \dots$$

Addition of angular momentum (7)

- Note also that a given total angular momentum j can be obtained from multiple different combinations of j' and j'' :

e.g. for $j = 3$: $3 \otimes 0 = 3$; $2 \otimes 1 = 3, \dots$; $2 \otimes 2 = \dots, 3, \dots$

If necessary, the different possibilities can be distinguished by explicitly including the quantum numbers j' and j'' in the total angular momentum eigenstate :

$$|jm\rangle \longrightarrow |jmj'j''\rangle$$

- Note also that, for given values of j' and j'' :

- the maximum possible value of m' is $m' = j'$
- the maximum possible value of m'' is $m'' = j''$
- the maximum possible value of m is therefore $m = j' + j''$

Consulting equation (1.67.1), the only possible value of j which can accommodate this maximum value $m = j' + j''$ is $j = j' + j''$

Finding Clebsch-Gordan coefficients

- The numerical values of the Clebsch-Gordan coefficients can be determined by following a standard procedure :
 - 1) Start with the total angular momentum state with the maximum possible values of j and m (namely $j = m = j' + j''$) :
This maximal state can be written down by inspection
$$\text{e.g. for } j' = 3/2, j'' = 2 : \quad \left| \frac{7}{2}, +\frac{7}{2} \right\rangle = \left| \frac{3}{2}, +\frac{3}{2} \right\rangle \otimes \left| 2, +2 \right\rangle$$
 - 2) Repeatedly use ladder operators and orthogonality to obtain all other states :
Applying $\hat{J}_- = \hat{J}'_- + \hat{J}''_-$ to a product state $|j'm'\rangle|j''m''\rangle$ gives
$$(\hat{J}'_- + \hat{J}''_-) \left[|j'm'\rangle \otimes |j''m''\rangle \right] = \left[\hat{J}'_- |j'm'\rangle \right] \otimes |j''m''\rangle + |j'm'\rangle \otimes \left[\hat{J}''_- |j''m''\rangle \right]$$
where
$$\begin{cases} \hat{J}'_- |j'm'\rangle = \hbar \sqrt{j'(j'+1) - m'(m'-1)} |j', m'-1\rangle \\ \hat{J}''_- |j''m''\rangle = \hbar \sqrt{j''(j''+1) - m''(m''-1)} |j'', m''-1\rangle \end{cases}$$

Best illustrated by example

Finding C-G coefficients : $\frac{1}{2} \otimes \frac{1}{2}$

- Consider, for example, the case $j' = 1/2, j'' = 1/2 :$

$$\frac{1}{2} \otimes \frac{1}{2} = 0, 1$$

Starting at the top, with $m = +1$, the maximal state must be

$$|1, +1\rangle = |\frac{1}{2}, +\frac{1}{2}\rangle |\frac{1}{2}, +\frac{1}{2}\rangle = |\uparrow\rangle|\uparrow\rangle \quad (1.70.1)$$

$$|j, m\rangle \quad |j', m'\rangle \quad |j'', m''\rangle$$

- Applying J_- to the left-hand side of equation (1.70.1) gives

$$\hat{J}_- |1, +1\rangle = \sqrt{2}\hbar |1, 0\rangle \quad (1.70.2)$$

Applying $(J'_- + J''_-)$ to the right-hand side of (1.70.1) gives the sum of

$$\hat{J}'_- |\frac{1}{2}, +\frac{1}{2}\rangle |\frac{1}{2}, +\frac{1}{2}\rangle = \hbar |\frac{1}{2}, -\frac{1}{2}\rangle |\frac{1}{2}, +\frac{1}{2}\rangle \quad (1.70.3)$$

$$\hat{J}''_- |\frac{1}{2}, +\frac{1}{2}\rangle |\frac{1}{2}, +\frac{1}{2}\rangle = \hbar |\frac{1}{2}, +\frac{1}{2}\rangle |\frac{1}{2}, -\frac{1}{2}\rangle \quad (1.70.4)$$

- Equating (1.70.2) to the sum of (1.70.3) and (1.70.4) then fixes $|jm\rangle = |10\rangle$ as

$$|1, 0\rangle = \frac{1}{\sqrt{2}} \left(|\frac{1}{2}, +\frac{1}{2}\rangle |\frac{1}{2}, -\frac{1}{2}\rangle + |\frac{1}{2}, -\frac{1}{2}\rangle |\frac{1}{2}, +\frac{1}{2}\rangle \right) \quad (1.70.5)$$

Finding C-G coefficients : $\frac{1}{2} \otimes \frac{1}{2}$ (2)

- The state $|jm\rangle = |00\rangle$ must be orthogonal to the state $|jm\rangle = |10\rangle$ of equation (1.70.5) :

$$|0,0\rangle = \frac{1}{\sqrt{2}} \left(|\frac{1}{2}, +\frac{1}{2}\rangle |\frac{1}{2}, -\frac{1}{2}\rangle - |\frac{1}{2}, -\frac{1}{2}\rangle |\frac{1}{2}, +\frac{1}{2}\rangle \right)$$

- It is easily verified that the state above really does have total angular momentum $j = 0$, $m = 0$, by checking that $J_{\pm} |0,0\rangle = 0$:

$$\left(\hat{J}'_{\pm} + \hat{J}''_{\pm} \right) \left[\frac{1}{\sqrt{2}} \left(|\frac{1}{2}, +\frac{1}{2}\rangle |\frac{1}{2}, -\frac{1}{2}\rangle - |\frac{1}{2}, -\frac{1}{2}\rangle |\frac{1}{2}, +\frac{1}{2}\rangle \right) \right] = 0$$

- Finally, applying J_- and $(J'_- + J''_-)$ to either side of equation (1.70.5) fixes the state $|jm\rangle = |1,-1\rangle$ as

$$|1, -1\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle |\frac{1}{2}, -\frac{1}{2}\rangle$$

This minimal state, with $m = -1$, could also have been written down by inspection, using similar arguments as used for equation (1.70.1)

Finding C-G coefficients : $\frac{1}{2} \otimes \frac{1}{2}$ (3)

-- In summary, for $j' = 1/2, j'' = 1/2$, we obtain a *triplet* plus a *singlet* of states :

$j = 1$	$(\frac{1}{2} \otimes \frac{1}{2} = 1)$
$ 1, +1\rangle = \uparrow\rangle \uparrow\rangle$	
$ 1, -0\rangle = \frac{1}{\sqrt{2}}(\uparrow\rangle \downarrow\rangle + \downarrow\rangle \uparrow\rangle)$	
$ 1, -1\rangle = \downarrow\rangle \downarrow\rangle$	<i>(symmetric)</i>

$j = 0$	$(\frac{1}{2} \otimes \frac{1}{2} = 0)$
	$ 0, 0\rangle = \frac{1}{\sqrt{2}}(\uparrow\rangle \downarrow\rangle - \downarrow\rangle \uparrow\rangle)$

Clebsch-Gordan
coefficients

$\langle \frac{1}{2}, +\frac{1}{2}; \frac{1}{2}, +\frac{1}{2} 1, +1 \rangle = 1$
$\langle \frac{1}{2}, +\frac{1}{2}; \frac{1}{2}, -\frac{1}{2} 1, 0 \rangle = \frac{1}{\sqrt{2}}$
$\langle \frac{1}{2}, -\frac{1}{2}; \frac{1}{2}, +\frac{1}{2} 1, 0 \rangle = \frac{1}{\sqrt{2}}$
$\langle \frac{1}{2}, -\frac{1}{2}; \frac{1}{2}, -\frac{1}{2} 1, -1 \rangle = 1$

$\langle \frac{1}{2}, +\frac{1}{2}; \frac{1}{2}, -\frac{1}{2} 0, 0 \rangle = \frac{1}{\sqrt{2}}$
$\langle \frac{1}{2}, -\frac{1}{2}; \frac{1}{2}, +\frac{1}{2} 0, 0 \rangle = -\frac{1}{\sqrt{2}}$

(see below for a more digestible way of presenting the C-G coeffs.)

C-G coefficients : $1 \otimes \frac{1}{2}$

$$1 \otimes \frac{1}{2} = \frac{3}{2}, \quad \frac{1}{2}$$

-- Similarly, for the case $j' = 1, j'' = 1/2$ (see Part IB) :

$$\underbrace{|jm\rangle}_{\text{}} \quad \underbrace{|m'\rangle|m''\rangle}_{\text{}} \equiv |1, m'\rangle|\frac{1}{2}, m''\rangle$$

$$|\frac{3}{2}, +\frac{3}{2}\rangle = |+1\rangle|+\frac{1}{2}\rangle$$

$$j = 3/2$$

$$|\frac{3}{2}, +\frac{1}{2}\rangle = \sqrt{\frac{1}{3}}|+1\rangle|-\frac{1}{2}\rangle + \sqrt{\frac{2}{3}}|0\rangle|+\frac{1}{2}\rangle$$

$$(1 \otimes \frac{1}{2} = \frac{3}{2})$$

$$|\frac{3}{2}, -\frac{1}{2}\rangle = \sqrt{\frac{2}{3}}|0\rangle|-\frac{1}{2}\rangle + \sqrt{\frac{1}{3}}|-1\rangle|+\frac{1}{2}\rangle$$

$$|\frac{3}{2}, -\frac{3}{2}\rangle = |-1\rangle|-\frac{1}{2}\rangle$$

$$j = 1/2$$

$$|\frac{1}{2}, +\frac{1}{2}\rangle = \sqrt{\frac{2}{3}}|+1\rangle|-\frac{1}{2}\rangle - \sqrt{\frac{1}{3}}|0\rangle|+\frac{1}{2}\rangle$$

$$(1 \otimes \frac{1}{2} = \frac{1}{2})$$

$$|\frac{1}{2}, -\frac{1}{2}\rangle = \sqrt{\frac{1}{3}}|0\rangle|-\frac{1}{2}\rangle - \sqrt{\frac{2}{3}}|-1\rangle|+\frac{1}{2}\rangle$$

C-G coefficients : $1 \otimes 1$

$1 \otimes 1 = 0, 1, 2$

$$|2, +2\rangle = |+1\rangle |+1\rangle$$

$j = 2$

$$|2, +1\rangle = \sqrt{\frac{1}{2}}|0\rangle |+1\rangle + \sqrt{\frac{1}{2}}|+1\rangle |0\rangle \quad (1 \otimes 1 = 2)$$

$$|2, 0\rangle = \sqrt{\frac{1}{6}}|-1\rangle |+1\rangle + \sqrt{\frac{2}{3}}|0\rangle |0\rangle + \sqrt{\frac{1}{6}}|+1\rangle |-1\rangle$$

$$|2, -1\rangle = \sqrt{\frac{1}{2}}|0\rangle |-1\rangle + \sqrt{\frac{1}{2}}|-1\rangle |0\rangle$$

$$|2, -2\rangle = |-1\rangle |-1\rangle \quad (\text{symmetric})$$

$j = 1$

$$|1, +1\rangle = \sqrt{\frac{1}{2}}|0\rangle |+1\rangle - \sqrt{\frac{1}{2}}|+1\rangle |0\rangle$$

$$(1 \otimes 1 = 1)$$

$$|1, 0\rangle = \sqrt{\frac{1}{2}}|-1\rangle |+1\rangle - \sqrt{\frac{1}{2}}|+1\rangle |-1\rangle$$

$$(\text{antisymmetric})$$

$$|1, -1\rangle = -\sqrt{\frac{1}{2}}|0\rangle |-1\rangle + \sqrt{\frac{1}{2}}|-1\rangle |0\rangle$$

$$|0, 0\rangle = \sqrt{\frac{1}{3}}|-1\rangle |+1\rangle - \sqrt{\frac{1}{3}}|0\rangle |0\rangle + \sqrt{\frac{1}{3}}|+1\rangle |-1\rangle$$

$$(\text{symmetric})$$

$j = 0$

$$(1 \otimes 1 = 0)$$

Clebsch-Gordan coefficient matrices

- For given values of j' and j'' , the C-G coefficients form a unitary (in fact orthogonal) matrix, U , of dimension $(2j' + 1)(2j'' + 1)$:

$$U^\dagger U = (U^*)^T U = U^T U = I$$

e.g. for $j' = 1, j'' = 1/2$: slide 1.73 can be written in matrix form as

$$\begin{pmatrix} | \frac{3}{2}, +\frac{3}{2} \rangle \\ | \frac{3}{2}, +\frac{1}{2} \rangle \\ | \frac{1}{2}, +\frac{1}{2} \rangle \\ | \frac{3}{2}, -\frac{1}{2} \rangle \\ | \frac{1}{2}, -\frac{1}{2} \rangle \\ | \frac{3}{2}, -\frac{3}{2} \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{1/3} & \sqrt{2/3} & 0 & 0 & 0 \\ 0 & \sqrt{2/3} & -\sqrt{1/3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2/3} & \sqrt{1/3} & 0 \\ 0 & 0 & 0 & \sqrt{1/3} & -\sqrt{2/3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} | 1, +1 \rangle | \frac{1}{2}, +\frac{1}{2} \rangle \\ | 1, +1 \rangle | \frac{1}{2}, -\frac{1}{2} \rangle \\ | 1, 0 \rangle | \frac{1}{2}, +\frac{1}{2} \rangle \\ | 1, 0 \rangle | \frac{1}{2}, -\frac{1}{2} \rangle \\ | 1, -1 \rangle | \frac{1}{2}, +\frac{1}{2} \rangle \\ | 1, -1 \rangle | \frac{1}{2}, -\frac{1}{2} \rangle \end{pmatrix}$$

$$|j, m\rangle = \sum_{m', m''} \langle 1m'; \frac{1}{2}m'' | jm \rangle \times |1, m'\rangle | \frac{1}{2}.m'' \rangle$$

(all rows are mutually orthonormal, as are all columns)

CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Read the instructions

Note: A square-root sign is to be understood over *every* coefficient, e.g., for $-8/15$ read $-\sqrt{8/15}$.

$1/2 \times 1/2$	$\begin{matrix} 1 \\ +1 \end{matrix}$	$\begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix}$
$+1/2 + 1/2$	$\begin{matrix} 1 \\ 1 \end{matrix}$	$\begin{matrix} 0 & 0 \\ -1/2 & -1/2 \end{matrix}$
$+1/2 - 1/2$	$\begin{matrix} 1/2 & 1/2 \\ -1/2 & +1/2 \end{matrix}$	$\begin{matrix} 1 \\ -1/2 - 1/2 \end{matrix}$
$-1/2 + 1/2$	$\begin{matrix} 1/2 & -1/2 \\ +1/2 & 1/2 \end{matrix}$	$\begin{matrix} 1 \\ -1/2 - 1/2 \end{matrix}$
$-1/2 - 1/2$	$\begin{matrix} -1/2 & -1/2 \\ -1/2 & -1/2 \end{matrix}$	$\begin{matrix} 1 \\ 1 \end{matrix}$

slide 1.72

$$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos \theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi}$$

$$Y_0^0 = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\phi}$$

$$Y_2^2 = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{2i\phi}$$

$1 \times 1/2$	$\begin{matrix} 3/2 \\ +3/2 \end{matrix}$	$\begin{matrix} 3/2 & 1/2 \\ 1 & +1/2 + 1/2 \end{matrix}$
$+1 + 1/2$	$\begin{matrix} 1 \\ 1 \end{matrix}$	$\begin{matrix} 1/3 & 2/3 \\ 2/3 & -1/3 \end{matrix}$
$+1 - 1/2$	$\begin{matrix} 1/3 & 2/3 \\ 0 + 1/2 \end{matrix}$	$\begin{matrix} 3/2 & 1/2 \\ -1/2 - 1/2 \end{matrix}$
$0 + 1/2$	$\begin{matrix} 2/3 & -1/3 \\ 0 - 1/2 \end{matrix}$	$\begin{matrix} 3/2 \\ -3/2 \end{matrix}$
$0 - 1/2$	$\begin{matrix} -1/2 & 1/2 \\ -1 + 1/2 \end{matrix}$	$\begin{matrix} 3/2 \\ -3/2 \end{matrix}$

slide 1.73

2×1	$\begin{matrix} 3 \\ +3 \end{matrix}$	$\begin{matrix} 3 & 2 & 1 \\ +2 + 1 & 1 & +2 + 2 \end{matrix}$
$+2 + 1$	$\begin{matrix} 1 \\ 1 \end{matrix}$	$\begin{matrix} 1/3 & 2/3 \\ 2/3 - 1/3 \end{matrix}$
$+2$	$\begin{matrix} 2 \\ 2 \end{matrix}$	$\begin{matrix} 3 & 2 & 1 \\ +1 & +1 & +1 \end{matrix}$
$+1 + 1$	$\begin{matrix} 0 \\ 2/3 \end{matrix}$	$\begin{matrix} 1/3 & 2/3 \\ 2/3 - 1/3 \end{matrix}$
$+1 - 1$	$\begin{matrix} 2/3 \\ -1/3 \end{matrix}$	$\begin{matrix} 3 & 2 & 1 \\ -1 & -1 & -1 \end{matrix}$

1×1	$\begin{matrix} 2 \\ +2 \end{matrix}$	$\begin{matrix} 2 & 1 \\ +1 + 1 & 1 \end{matrix}$
$+1 + 1$	$\begin{matrix} 1 \\ 1 \end{matrix}$	$\begin{matrix} 1/15 & 1/3 & 3/5 \\ 8/15 & 1/6 - 3/10 \\ 0 + 1 & 2/5 - 1/2 & 1/10 \end{matrix}$
$+1 - 1$	$\begin{matrix} 0 \\ 0 \end{matrix}$	$\begin{matrix} 3 & 2 & 1 \\ 0 & 0 & 0 \end{matrix}$
$0 + 1$	$\begin{matrix} 8/15 \\ 2/5 \end{matrix}$	$\begin{matrix} 1/5 & 1/2 & 3/10 \\ 3/5 & 0 & -2/5 \\ -1/2 + 1 & 1/5 - 1/2 & 3/10 \end{matrix}$
$0 - 1$	$\begin{matrix} 1/6 \\ -1/2 \end{matrix}$	$\begin{matrix} 3 & 2 & 1 \\ -1 & -1 & -1 \end{matrix}$

$$Y_{\ell}^{-m} = (-1)^{\ell} Y_{\ell}^m$$

$$Y_{\ell}^m = \begin{pmatrix} 0 & -1 & 1/2 & 1/2 & 2 \\ -1 & 0 & 1/2 & -1/2 & -2 \\ -1 & -1 & 1 \end{pmatrix}$$

slide 1.74

$$d_{\ell m,0}^{\ell} = \sqrt{\frac{4\pi}{2\ell + 1}} Y_{\ell}^m e^{-im\phi}$$

J	J	...
M	M	...
m_1	m_2	
m_1	m_2	
\vdots	\vdots	
		Coefficients

$2 \times 1/2$	$\begin{matrix} 5/2 \\ +5/2 \end{matrix}$	$\begin{matrix} 5/2 & 3/2 \\ +3/2 + 3/2 \end{matrix}$
$+2 + 1/2$	$\begin{matrix} 1 \\ 1 \end{matrix}$	$\begin{matrix} 1/5 & 4/5 \\ 4/5 - 1/5 \end{matrix}$
$+2 - 1/2$	$\begin{matrix} 1/5 \\ +1/2 \end{matrix}$	$\begin{matrix} 5/2 & 3/2 \\ +1/2 + 1/2 \end{matrix}$
$+1 + 1/2$	$\begin{matrix} 4/5 \\ -1/2 \end{matrix}$	$\begin{matrix} 2/5 & 3/5 \\ 3/5 - 2/5 \end{matrix}$
$+1 - 1/2$	$\begin{matrix} 3/5 \\ 0 + 1/2 \end{matrix}$	$\begin{matrix} -1/2 & -1/2 \\ -1 + 1/2 \end{matrix}$

$3/2 \times 1/2$	$\begin{matrix} 2 \\ +2 \end{matrix}$	$\begin{matrix} 2 & 1 \\ +1 + 1 \end{matrix}$
$+3/2 + 1/2$	$\begin{matrix} 1 \\ 1 \end{matrix}$	$\begin{matrix} 1/4 & 3/4 \\ 3/4 - 1/4 \end{matrix}$
$+3/2 - 1/2$	$\begin{matrix} 1/4 \\ +1/2 \end{matrix}$	$\begin{matrix} 2 & 1 \\ 0 & 0 \end{matrix}$
$+1/2 + 1/2$	$\begin{matrix} 3/4 \\ -1/2 \end{matrix}$	$\begin{matrix} 1/2 & 1/2 \\ 1/2 - 1/2 \end{matrix}$
$+1/2 - 1/2$	$\begin{matrix} 1/2 \\ -1/2 \end{matrix}$	$\begin{matrix} 2 & 1 \\ -1 & -1 \end{matrix}$

$3/2 \times 1$	$\begin{matrix} 5/2 \\ +5/2 \end{matrix}$	$\begin{matrix} 5/2 & 3/2 \\ +3/2 + 3/2 \end{matrix}$
$+3/2 + 1$	$\begin{matrix} 1 \\ 1 \end{matrix}$	$\begin{matrix} 2/5 & 3/5 \\ 3/5 - 2/5 \end{matrix}$
$+3/2 - 1$	$\begin{matrix} 2/5 \\ +1/2 \end{matrix}$	$\begin{matrix} 5/2 & 3/2 \\ +1/2 + 1/2 \end{matrix}$
$+1/2 + 1$	$\begin{matrix} 3/5 \\ -1/2 + 1 \end{matrix}$	$\begin{matrix} 1/10 & 2/5 & 1/2 \\ 3/10 & -8/15 & 1/6 \end{matrix}$
$+1/2 - 1$	$\begin{matrix} 1/10 \\ -1/2 + 1 \end{matrix}$	$\begin{matrix} 5/2 & 3/2 & 1/2 \\ -1/2 & -1/2 & -1/2 \end{matrix}$

$1/2 \times 1$	$\begin{matrix} 3 \\ +3 \end{matrix}$	$\begin{matrix} 3 & 2 & 1 \\ +2 + 1 & 1 & +2 + 2 \end{matrix}$
$+1/2 + 1$	$\begin{matrix} 1 \\ 1 \end{matrix}$	$\begin{matrix} 1/15 & 1/3 & 3/5 \\ 8/15 & 1/6 - 3/10 \\ 0 + 1 & 2/5 - 1/2 & 1/10 \end{matrix}$
$+1/2 - 1$	$\begin{matrix} 0 \\ 0 \end{matrix}$	$\begin{matrix} 1/5 & 1/2 & 3/10 \\ 3/5 & 0 & -2/5 \\ -1/2 + 1 & 1/5 - 1/2 & 3/10 \end{matrix}$
$+1/2 + 1$	$\begin{matrix} 1/5 \\ -1/2 + 1 \end{matrix}$	$\begin{matrix} 3 & 2 & 1 \\ -1 & -1 & -1 \end{matrix}$
$+1/2 - 1$	$\begin{matrix} 3/5 \\ -1/2 + 1 \end{matrix}$	$\begin{matrix} 1/10 & 2/5 & 1/2 \\ 3/10 & -1/15 & -1/3 \\ -3/2 + 1 & 1/10 & -2/5 \end{matrix}$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 JM \rangle = (-1)^{J-j_1-j_2} \langle j_2 j_1 m_2 m_1 | j_2 j_1 JM \rangle$$

C-G coefficients : $j \otimes 0$

$$j \otimes 0 = j$$

- Finally, we consider the “trivial” case, $0 \otimes j = j$, corresponding to the addition of zero angular momentum :

$$0 \otimes 0 = 0 ; \quad 1 \otimes 0 = 1 ; \quad 2 \otimes 0 = 2 ; \quad \dots$$

- For any j and m , the total angular momentum states can be written down directly as

$$|jm\rangle = |00\rangle \otimes |jm\rangle$$

This corresponds to a Clebsch-Gordan coefficient of unity :

$$\langle 00; jm | jm \rangle = 1$$

- All other Clebsch-Gordan coefficients vanish, so overall we have

$$\langle 00; jm | j'm' \rangle = \delta_{j'j} \delta_{m'm}$$