UNIVERSITY OF CAMBRIDGE

INSTITUTE OF ASTRONOMY

Astrophysical Fluid Dynamics

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NST PART II ASTROPHYSICS/PHYSICS LENT 2023

About

These notes accompany the 24 lectures in the Lent-2021 term on Astrophysical Fluid Dynamics, a course within the Part II Physics/Astrophysics of the Natural Sciences Tripos. They are not intended to be a complete or self-contained discussion of course material and must be used in conjunction with the lectures and the course textbook, Principles of Astrophysical Fluid Dynamics by C.J.Clarke and R.F.Carswell [**PAFD**].

I would like to express my sincere thanks to Zihan Yan who produced these LaTeXnotes based on the hand-written handouts from the Lent-2020 run of this course. Minor additions and modifications have been included for Lent-2021. If you find any typo or error, please contact csr12@cam.ac.uk for correction.

Chris Reynolds
(Cambridge, Jan 2021)

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CHAPTER A

Basic Principles

A.1 Introduction

Fluid Dynamics concerns itself with the dynamics of liquid, gases and (to some degree) plasmas. Phenomena considered in fluid dynamics are *macroscopic*. We describe a fluid as a *continuous medium* with well-defined macroscopic quantities (e.g. density ρ, pressure p...), even though, at a microscopic level, the fluid is composed of particles.

Most of the baryonic matter in the Universe can be treated as a fluid. Fluid dynamics is thus an extremely important topic within astrophysics. Astrophysical systems can display extremes of density (both low and high) and temperature beyond those accessible in terrestrial laboratories. In addition, gravity is often a crucial component of the dynamics in astrophysical systems. Thus the subject of *Astrophysical Fluid Dynamics* encompasses but significantly extends the study of fluids relevant to terrestrial systems and/or engineers.

In the astrophysical context, the liquid state is not very common (examples are high pressure environments of planetary surfaces and interiors), so our focus will be on the gas phase. Key difference is that gases are more compressible than liquids.

Examples (Fluids in the Universe)**.**

- Interiors of stars, white dwarfs, neutron stars;
- Interstellar medium (ISM), intergalactic medium (IGM), intracluster medium (ICM);
- Stellar winds, jets, accretion disks;
- Giant planets.

In our discussion, we shall use the concept of a *fluid element*. This is a region of fluid that is

(i) Small enough that there are no significant variations of any property q that interests us

$$l_{\text{region}} \ll l_{\text{scale}} \sim \frac{q}{|\boldsymbol{\nabla} q|}$$

(ii) Large enough to contain sufficient particles to be considered in the continuum limit

$$n l_{\text{region}}^3 \gg 1$$

where n is the number density of particles.

A.2 Collisional and Collisionless Fluids

In a *collisional fluid*, any relevant fluid element is large enough such that the constituent particles know about local conditions through interactions with each other, i.e.

$$l_{\text{region}} \gg \lambda$$

where λ is the mean free path. Particles will then attain a distribution of velocities that maximises the entropy of the system at a given temperature. Thus, a collisional fluid at a given density ρ and temperature T will have a well-defined distribution of particle speeds and hence a well-defined pressure, p. We can relate ρ, T and p as *equation of state*:

$$p = p(\rho, T)$$

In a *collisionless fluid*, particles do not interact frequently enough to satisfy $l_{\text{region}} \gg \lambda$. So, distribution of particle speeds locally does not correspond to maximum entropy solution, instead depending on initial conditions and non-local conditions.

Examples. Collisionless Fluids

- Stars in a galaxy;
- Grains in Saturn's rings;
- Dark matter;
- ICM (transitional from collisional to collisionless)

Example (Expand example of ICM)**.** Treat as fully ionised plasma of H(e^-, p). The mean free path is set by Coulomb collisions and an analysis gives

$$\lambda_e = \frac{3^{3/2}(k_B T_e)^2 \epsilon_0^2}{4\pi^{1/2} n_e e^4 \ln \Lambda}$$

where

$$n_e = \mathrm{e}^- \text{ number density}$$
$$\Lambda = \text{ratio of largest to smallest impact parameter}$$

and for $T \gtrsim 4 \times 10^5$ K we have $\ln \Lambda \sim 40$. So, if $T_i = T_e$, we have

$$\lambda_e = \lambda_i \approx 23 \text{ kpc} \left(\frac{T_e}{10^8 \text{ K}} \right)^2 \left(\frac{n_e}{10^{-3} \text{ cm}^{-3}} \right)^{-1}.$$

So we have

$$\overbrace{R_{\text{galaxy}} \sim \lambda_e}^{\text{collisionless}} \hspace{-1.2em} \underbrace{ \ll R_{\text{cluster}}}_{\text{collisional}} \sim 1 \text{ Mpc}.$$

CHAPTER B

Formulation of the Fluid Equations

B.1 Eulerian vs Lagrangian

Two main frameworks for understanding fluid flow:

(i) *Eulerian description* — one considers the properties of the fluid measured in a frame of reference that is fixed in space. So we consider quantities like

$$\rho(\mathbf{r},t), \quad p(\mathbf{r},t), \quad T(\mathbf{r},t), \quad \mathbf{v}(\mathbf{r},t)$$

(ii) *Lagrangian description* — one considers a particular fluid element and examines the change in the properties of that element. So, the spatial reference frame is co-moving with the fluid flow.

The Eulerian approach is more useful if the motion of particular fluid elements is not of interest.

The Lagranian approach is useful if we *do* care about the passage of given fluid elements (e.g. gas parcels that are enriched by metals).

These two different pictures lead to very different computational approaches to fluid dynamics which we will discuss later.

Mathematically, it is straightforward to relate these two pictures. Consider a quantity Q in a fluid element at position $\mathbf{r}$ and time t. At time $t+\delta t$ the element will be at position $\mathbf{r}+\delta\mathbf{r}$. The change in quantity Q of the fluid element is

$$\frac{\mathrm{D}Q}{\mathrm{D}t} = \lim_{\delta t\to 0}\left[\frac{Q(\mathbf{r}+\delta\mathbf{r},t+\delta t)-Q(\mathbf{r},t)}{\delta t}\right]$$

but

$$Q(\mathbf{r}+\delta\mathbf{r},t+\delta t) = Q(\mathbf{r},t)+\frac{\partial Q}{\partial t}\delta t+\delta\mathbf{r}\cdot\boldsymbol{\nabla}Q+\mathcal{O}(\delta t^2,|\delta\mathbf{r}|^2,\delta t|\delta\mathbf{r}|)$$

so

$$\frac{\mathrm{D}Q}{\mathrm{D}t} = \lim_{\delta t\to 0}\left[\frac{\partial Q}{\partial t}+\frac{\delta\mathbf{r}}{\delta t}\cdot\boldsymbol{\nabla}Q+\mathcal{O}(\delta t,|\delta\mathbf{r}|)\right]$$

which gives us

$$\boxed{\underbrace{\frac{\mathrm{D}Q}{\mathrm{D}t}}_{\substack{\text{Lagrangian}\\\text{time derivative}}} = \underbrace{\frac{\partial Q}{\partial t}}_{\substack{\text{Eulerian}\\\text{time derivative}}} + \underbrace{\mathbf{u}\cdot\boldsymbol{\nabla} Q}_{\substack{\text{“convective”}\\\text{derivative}}}}$$

B.2 Kinematics

Kinematics is the study of particle (and fluid element) trajectories.

Streamlines, streaklines and particle paths are field lines resulting from the velocity vector fields. If the flow is steady with time, they all coincide.

(i) *Streamline*: families of curves that are instantaneously tangent to the velocity vector of the flow $\mathbf{u}(\mathbf{r}, t)$. They show the direction of the fluid element.

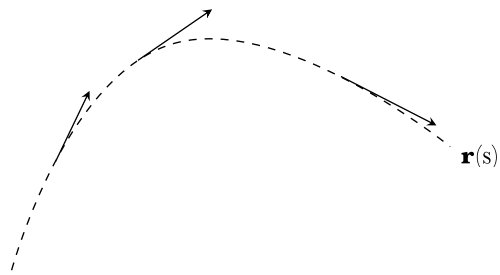

Figure B.1: Streamline

Parameterise streamline by label s such that

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \left(\frac{\mathrm{d}x}{\mathrm{d}s}, \frac{\mathrm{d}y}{\mathrm{d}s}, \frac{\mathrm{d}z}{\mathrm{d}s}\right)$$

and demand $\mathrm{d}\mathbf{r}/\mathrm{d}s \parallel \mathbf{u}$, we get

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} \times \mathbf{u} = 0 \qquad \Rightarrow \qquad \frac{\mathrm{d}x}{u_x} = \frac{\mathrm{d}y}{u_y} = \frac{\mathrm{d}z}{u_z}$$

(ii) *Particle paths*: trajectories of individual fluid elements given by

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{u}(\mathbf{r}, t)$$

For small time intervals, particle paths follow streamlines since $\mathbf{u}$ can be treated as approximately steady.

(iii) *Streaklines*: locus of points of all fluid that have passed through a given spatial point in the past.

$$\mathbf{r}(t) = \mathbf{r}_0$$

for some given t in the past.

We now proceed to discuss the equations that describe the dynamics of a fluid. These are essentially expressions of the conservation of mass, momentum and energy.

B.3 Conservation of Mass

Consider a fixed volume V bounded by a surface S. If there are no sources or sinks of mass within the volume, we can say

rate of change of mass in $V = -$ rate that mass is flowing out across S

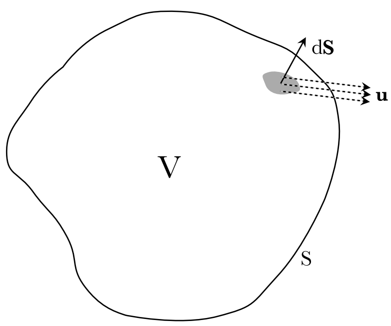

Figure B.2: Mass flow of a fluid element

this gives

$$\begin{aligned} \frac{\partial}{\partial t}\int_V \rho\,\mathrm{d}V &= -\int_S \rho\mathbf{u}\cdot \mathrm{d}\mathbf{S} \\ \Rightarrow \qquad \int_V \frac{\partial \rho}{\partial t}\,\mathrm{d}V &= -\int_V \boldsymbol{\nabla}\cdot(\rho\mathbf{u})\,\mathrm{d}V \\ \Rightarrow \qquad \int_V \left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u})\right)\mathrm{d}V &= 0. \end{aligned}$$

This is true for all volumes V. So we must have

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0} \qquad \text{Eulerian Continuity Equation}$$

The Lagrangian expression of mass conservation is easily found:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\partial\rho}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\rho = -\boldsymbol{\nabla}\cdot\rho\mathbf{u} + \mathbf{u}\cdot\boldsymbol{\nabla}\rho = -\rho\boldsymbol{\nabla}\cdot\mathbf{u}.$$

Thus we have

$$\boxed{\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\boldsymbol{\nabla}\cdot\mathbf{u} = 0} \qquad \text{Lagrangian Continuity Equation}$$

In an incompressible flow, fluid elements maintain a constant density, i.e.

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0.$$

We can now see that incompressible flows must be divergence free, $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$.

B.4 Conservation of Momentum

Pressure

Consider only collisional fluids where there are forces within the fluid due to particle-particle interactions. Thus there can be momentum flux across surfaces within the fluid even in the absence of bulk flows.

In a fluid with uniform properties, the momentum flux through a surface is balanced by an equal and opposite momentum flux through the other side of the surface. Therefore, there is no net acceleration even for non-zero pressure since pressure is defined as the momentum flux on *one* side of the surface.

If the particle motions within the fluid are isotropic, the momentum flux is locally independent of the orientation of the surface and the components parallel to the surface cancel out. Then, the force acting on one side of surface element is

$$\mathrm{d}\mathbf{F} = p\,\mathrm{d}\mathbf{S}$$

In the more general case, forces across surfaces are not perpendicular to the surface and we have

$$\mathrm{d}F_i = \sum_j \sigma_{ij}\mathrm{d}S_j.$$

where σ_{ij} is the stress tensor — the force in direction i acting on a surface with normal along j.

Isotropic pressure in a static fluid corresponds to

$$\sigma_{ij} = p\delta_{ij}.$$

Momentum Equation for a Fluid

Consider a fluid element that is subject to a gravitational field $\mathbf{g}$ and internal pressure forces. Let the fluid element have volume V and surface S.

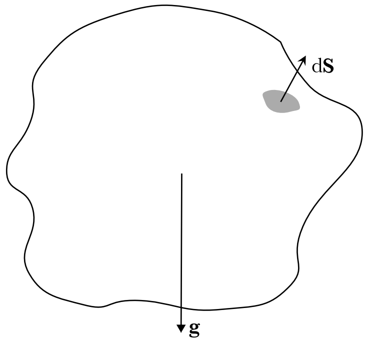

Figure B.3: A fluid element subject to gravity

Pressure acting on the surface element gives force $-p\,\mathrm{d}\mathbf{S}$. Pressure force on element projected in direction $\hat{\mathbf{n}}$ is $-p\hat{\mathbf{n}}\cdot\mathrm{d}\mathbf{S}$. So, net pressure force in direction $\hat{\mathbf{n}}$ is

$$\mathbf{F}\cdot\hat{\mathbf{n}} = -\int_S p\hat{\mathbf{n}}\cdot\mathrm{d}\mathbf{S} = -\int_V \boldsymbol{\nabla}\cdot(p\hat{\mathbf{n}})\,\mathrm{d}V = -\int_V \hat{\mathbf{n}}\cdot\boldsymbol{\nabla}p\,\mathrm{d}V\,.$$

Rate of change of momentum of fluid element in direction $\hat{\mathbf{n}}$ is the total force in that direction:

$$\left(\frac{\mathrm{D}}{\mathrm{D}t}\int_V \rho\mathbf{u}\,\mathrm{d}V\right)\cdot\hat{\mathbf{n}} = -\int_V \hat{\mathbf{n}}\cdot\boldsymbol{\nabla}p\,\mathrm{d}V + \int_V \rho\mathbf{g}\cdot\hat{\mathbf{n}}\,\mathrm{d}V\,.$$

In limit that $\int \mathrm{d}V \to \delta V$ we have

$$\frac{\mathrm{D}}{\mathrm{D}t}(\rho\mathbf{u}\;\delta V)\cdot\hat{\mathbf{n}} = -\delta V\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}p + \delta V\rho\mathbf{g}\cdot\hat{\mathbf{n}}$$

$$\Rightarrow \qquad \hat{\mathbf{n}}\cdot\mathbf{u}\;\underbrace{\frac{\mathrm{D}}{\mathrm{D}t}(\rho\delta V)}_{\substack{=0\text{ by mass}\\ \text{conservation}}} + \rho\delta V\hat{\mathbf{n}}\cdot\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\delta V\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}p + \delta V\rho\mathbf{g}\cdot\hat{\mathbf{n}}$$

$$\Rightarrow \qquad \delta V\hat{\mathbf{n}}\cdot\left(\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \boldsymbol{\nabla}p - \rho\mathbf{g}\right) = 0.$$

This must be true for all $\hat{\mathbf{n}}$ and all δV. So,

$$\boxed{\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\boldsymbol{\nabla}p + \rho\mathbf{g}} \qquad \textsc{Lagrangian Momentum Equation}$$

or

$$\boxed{\rho\frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p + \rho\mathbf{g}} \qquad \textsc{Eulerian Momentum Equation}$$

Now consider the Eulerian rate of change of momentum density $\rho\mathbf{u}$ and introduce more compact notation

$$\begin{aligned}\frac{\partial}{\partial t}(\rho u_i) &\equiv \partial_t(\rho u_i)\\ &= \rho\partial_t u_i + u_i\partial_t\rho\\ &= -\rho u_j\partial_j u_i - \partial_j p\delta_{ij} + \rho g_i - u_i\partial_j(\rho u_j)\end{aligned}$$

where we have used notation

$$\partial_j \equiv \frac{\partial}{\partial x_j}$$

and employed summation convention (summation over the repeated indices). This gives

$$\partial_t(\rho u_i) = -\partial_j(\underbrace{\rho u_i u_j}_{\substack{\text{stress tensor}\\ \text{due to bulk flow}\\ \text{`Ram Pressure'}}} + \underbrace{p\delta_{ij}}_{\substack{\text{stress tensor}\\ \text{due to random}\\ \text{thermal motions}}}) + \rho g_i = -\partial_j\sigma_{ij} + \rho g_i$$

where we have generalised the stress tensor to include the momentum flux from the bulk flow,

$$\sigma_{ij} = p\delta_{ij} + \rho u_i u_j.$$

In component free language we write

$$\partial_t(\rho\mathbf{u}) = -\boldsymbol{\nabla}\cdot\underbrace{(\rho\mathbf{u}\otimes\mathbf{u} + p\underline{\underline{\mathbf{I}}})}_{\substack{\text{flux of}\\ \text{momentum}\\ \text{density}}} + \rho\mathbf{g}$$

Example (Flow in a pipe in the y-direction).

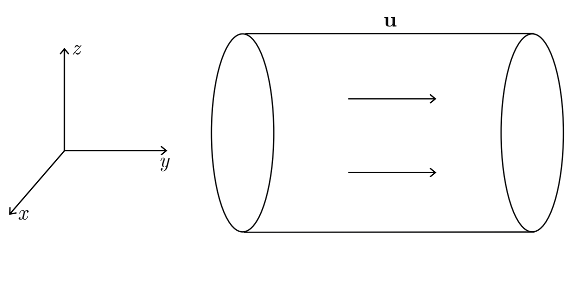

Figure B.4: Flow in a pipe

Any surface will experience a momentum flux p due to pressure. Only surfaces with a normal that has a component parallel to flow will experience ram pressure.

$$\sigma_{ij} = \begin{pmatrix} p & 0 & 0 \\ 0 & p + \rho u^2 & 0 \\ 0 & 0 & p \end{pmatrix}$$

The remaining equation of fluid dynamics is based on the conservation of energy. We will defer a discussion of that until later.

CHAPTER C

GRAVITATION

C.1 Basics

Define the gravitational potential Ψ s.t. the gravitational acceleration $\mathbf{g}$ is

$$\boxed{\mathbf{g} = -\boldsymbol{\nabla}\Psi}$$

If l is some closed loop, we have

$$\oint_l \mathbf{g} \cdot \mathrm{d}\mathbf{l} = \int_S (\boldsymbol{\nabla} \times \mathbf{g}) \cdot \mathrm{d}\mathbf{S} = -\int_S [\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\Psi)] \cdot \mathrm{d}\mathbf{S} = 0$$

as curl of any gradient is zero. So gravity is a conservative force — the work done around a closed loop is zero.

As a consequence, the work needed to take a mass from point $\mathbf{r}$ to ∞ is

$$-\int_{\mathbf{r}}^{\infty} \mathbf{g} \cdot \mathrm{d}\mathbf{l} = \int_{\mathbf{r}}^{\infty} \boldsymbol{\nabla}\Psi \cdot \mathrm{d}\mathbf{l} = \Psi(\infty) - \Psi(\mathbf{r})$$

which is independent of path.

A particular important case is the gravity of a point mass, which has

$$\begin{aligned} \Psi &= -\frac{GM}{r} && \text{if mass at origin} \\ \Psi &= -\frac{GM}{|\mathbf{r} - \mathbf{r}'|} && \text{if mass at location } \mathbf{r}' \end{aligned}$$

For system of point masses we have

$$\Psi = -\sum_i \frac{GM_i}{|\mathbf{r} - \mathbf{r}_i'|}$$

$$\Rightarrow \qquad \mathbf{g} = -\boldsymbol{\nabla}\Psi = -\sum_i \frac{GM_i(\mathbf{r} - \mathbf{r}_i')}{|\mathbf{r} - \mathbf{r}_i'|^3}$$

Replacing $M_i \to \rho_i \delta V_i$ and going to the continuum limit we have

$$\mathbf{g}(\mathbf{r}) = -G \int \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \, \mathrm{d}V'$$

Take divergence of both sides

$$\begin{aligned}\boldsymbol{\nabla}_{\mathbf{r}} \cdot \mathbf{g} &= -G \int \rho(\mathbf{r}') \underbrace{\boldsymbol{\nabla}_{\mathbf{r}} \cdot \left[\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}\right]}_{4\pi\delta(\mathbf{r}-\mathbf{r}')} \mathrm{d}V' \\ &= -4\pi G \int \rho(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}') \, \mathrm{d}V' \\ &= -4\pi G\rho(\mathbf{r})\end{aligned}$$

So,

$$\boxed{\boldsymbol{\nabla} \cdot \mathbf{g} = -\nabla^2\Psi = -4\pi G\rho} \qquad \text{Poisson's Equation}$$

We can also express Poisson's equation in integral form: for some volume V bounded by surface S we have

$$\begin{aligned} & \int_V \boldsymbol{\nabla} \cdot \mathbf{g} \, \mathrm{d}V = -4\pi G \int_V \rho \, \mathrm{d}V \\ \Rightarrow \quad & \int_S \mathbf{g} \cdot \mathrm{d}\mathbf{S} = -4\pi GM \end{aligned}$$

This is useful for calculating $\mathbf{g}$ when the mass distribution obeys some symmetry.

Example (Spherical distribution of mass)**.**

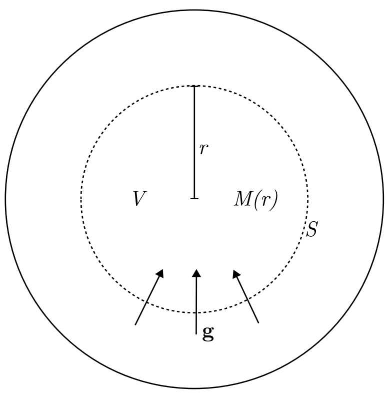

Figure C.1: Spherical distribution of mass

By symmetry $\mathbf{g}$ is radial and $|\mathbf{g}|$ is constant over a $r=$ const. shell. So

$$
\begin{aligned}
& \int \mathbf{g} \cdot \mathrm{d}\mathbf{S} = -4\pi G \underbrace{M(r)}_{\substack{\text{mass}\\\text{enclosed}}} \\
\Rightarrow \quad & -4\pi r^2 |\mathbf{g}| = -4\pi G M(r) \\
\Rightarrow \quad & |\mathbf{g}| = \frac{GM(r)}{r^2} \\
\therefore \quad & \mathbf{g} = -\frac{GM(r)}{r^2}\hat{\mathbf{r}}
\end{aligned}
$$

Example (Infinite cylindrically symmetric mass)**.**

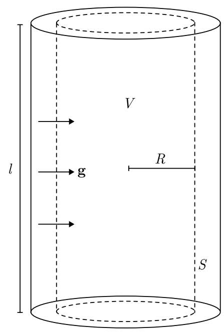

Figure C.2: Cylindrical distribution of mass

By symmetry, $\mathbf{g}$ is uniform and radial on the curved sides of the cylindrical surface, and is zero on the flat side, then

$$
\begin{aligned}
& \int \mathbf{g} \cdot \mathrm{d}\mathbf{S} = -4\pi G \int_V \rho \, \mathrm{d}V \\
\Rightarrow \quad & -2\pi R l |\mathbf{g}| = -4\pi G l \cdot \underbrace{M(r)}_{\substack{\text{enclosed mass}\\\text{per unit length}}} \\
\Rightarrow \quad & \mathbf{g} = -\frac{2GM(R)}{R}\hat{\mathbf{R}}
\end{aligned}
$$

Example (Infinite planar distribution of mass)**.** Assume infinite and homogeneous in x and y, $\rho = \rho(z)$.

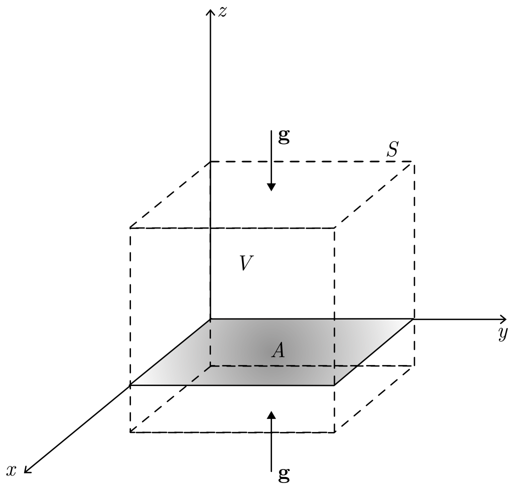

Figure C.3: Planar distribution of mass

By symmetry, $\mathbf{g}$ is in $-\hat{\mathbf{z}}$ direction and is constant on a $z =$ const. surface. So, if we also have reflection symmetry about $z = 0$,

$$\begin{aligned} \int_S \mathbf{g} \cdot \mathrm{d}\mathbf{S} &= -4\pi G \int_V \rho \, \mathrm{d}V \\ \Rightarrow \qquad -2|\mathbf{g}|A &= -4\pi G A \int_{-z}^{z} \rho(z) \, \mathrm{d}z \\ \Rightarrow \qquad \mathbf{g} &= -4\pi G \hat{\mathbf{z}} \int_0^z \rho(z) \, \mathrm{d}z \end{aligned}$$

(For planar distribution of finite height $z_{\max}$, $\mathbf{g}$ is constant for $z \geq z_{\max}$.)

Example (Finite axisymmetric disk)**.**

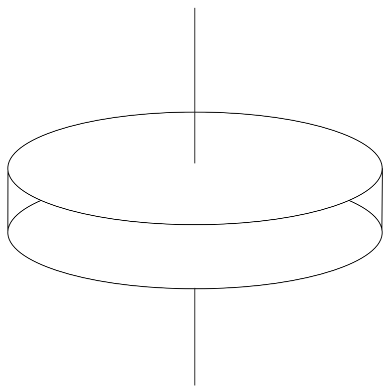

Figure C.4: Finite axisymmetric disk

$$\oint_S \mathbf{g} \cdot \mathrm{d}\mathbf{S} = ?$$

No surfaces where $\mathbf{g}$ vanishes by symmetry, and no easily determined surfaces where $|\mathbf{g}|$ is a constant. We would need to solve Poisson's equation directly, e.g. using separation of variables.

C.2 Potential of a Spherical Mass Distribution

We found that, for a spherical distribution,

$$\mathbf{g} = -|\mathbf{g}|\hat{\mathbf{r}}, \qquad |\mathbf{g}| = \frac{G}{r^2}\int_0^r 4\pi\rho(r')r'^2\,\mathrm{d}r' = \frac{\mathrm{d}\Psi}{\mathrm{d}r}$$

so,

$$\Psi = \int_\infty^{r_0} \frac{G}{r^2}\left\{\int_0^r 4\pi\rho(r')r'^2\,\mathrm{d}r'\right\}\mathrm{d}r$$

Taking $\Psi(\infty) = 0$ by convention, integrate this by parts:

$$\Psi = -\left\{\frac{G}{r}\int_0^r 4\pi\rho(r')r'^2\,\mathrm{d}r'\right\}\Bigg|_{r=\infty}^{r_0} + \int_\infty^{r_0}\frac{G}{r}4\pi\rho(r)r^2\,\mathrm{d}r$$

$$\Rightarrow \qquad \Psi = -\frac{GM(r_0)}{r_0} + \int_\infty^{r_0} 4\pi G\rho(r)r\,\mathrm{d}r$$

where we have made an assumption that $M(r)/r \to 0$ as $r \to \infty$.

We find that Ψ *is* affected by matter outside of r through our choice of setting $\Psi = 0$ at infinity. So $\Psi \neq -GM(r)/r$ unless there is no mass outside of r.

C.3 Gravitational Potential Energy

For a given system of point masses,

$$\Psi = -\sum_i \frac{GM_i}{|\mathbf{r} - \mathbf{r}_i|}$$

and the energy required to take a unit mass to ∞ is $-\Psi$. Energy required to take a system of point masses to ∞ is

$$\Omega = -\frac{1}{2}\sum_{j\neq i}\sum_i \frac{GM_iM_j}{|\mathbf{r}_j - \mathbf{r}_i|} = \frac{1}{2}\sum_j M_j\Psi_j$$

where the half is present to avoid double counting pairs.

For a continuum matter distribution,

$$\Omega = \frac{1}{2}\int \rho(\mathbf{r})\Psi(\mathbf{r})\,\mathrm{d}V$$

Specialising to the spherically symmetric case gives

$$\Omega = \frac{1}{2}\int_0^\infty 4\pi\rho(r) r^2 \Psi(r)\,\mathrm{d}r$$

Integrate by parts, choosing parts $u \equiv \Psi, \mathrm{d}v \equiv 4\pi\rho r^2$ so that $v = \int_0^r 4\pi\rho r'^2\,\mathrm{d}r' = M(r)$, then

$$\Omega = \frac{1}{2}\left[M(r)\Psi(r)\Big|_0^\infty - \int_0^\infty M(r)\frac{\mathrm{d}\Psi}{\mathrm{d}r}\,\mathrm{d}r\right].$$

Assuming that we have a finite distribution of mass with a non-singular behaviour at $r = 0$, the first term on the RHS (i.e. the boundary term) is zero. Noting further that

$$\frac{\mathrm{d}\Psi}{\mathrm{d}r} = \frac{GM(r)}{r^2}$$

we conclude

$$\Omega = -\frac{1}{2}\int_0^\infty \frac{GM(r)^2}{r^2}\,\mathrm{d}r\,.$$

Integrate again by parts, choosing $u \equiv GM(r)^2, \mathrm{d}v \equiv 1/r^2$,

$$\Omega = \underbrace{\frac{1}{2}\,GM(r)^2\frac{1}{r}\Big|_0^\infty}_{=0} - \frac{1}{2}\int_0^\infty \frac{1}{r}2GM\frac{\mathrm{d}M}{\mathrm{d}r}\,\mathrm{d}r$$

$$\Rightarrow \qquad \Omega = -G\int_0^\infty \frac{M(r)}{r}\,\mathrm{d}M$$

This is equivalent to the assembly of spherical shells of mass, each brought from ∞ with potential energy

$$\frac{GM(r)}{r}\,\mathrm{d}M(r)\,.$$

C.4 The Virial Theorem

We now come to a powerful result that greatly helps in the understanding of isolated gravitating systems.

Consider the motion of a cloud of particles (atoms, stars, galaxies...). Particle with mass m_i at $\mathbf{r}_i$ is acted upon by a force

$$\mathbf{F} = m_i\frac{\mathrm{d}^2\mathbf{r}_i}{\mathrm{d}t^2}$$

Consider the 2nd derivative of the scalar moment of inertia, $I_i = m_i r_i^2$

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}(m_i r_i^2) &= m_i \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{r}_i \cdot \frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t}\right) \\ &= m_i \mathbf{r}_i \cdot \frac{\mathrm{d}^2\mathbf{r}_i}{\mathrm{d}t^2} + m_i \left(\frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t}\right)^2 \\ &= \mathbf{r}_i \cdot \mathbf{F}_i + \underbrace{m_i \left(\frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t}\right)^2}_{2\times\text{Kinetic Energy } T_i}\end{aligned}$$

If $I \equiv \sum_i m_i r_i^2$ then we can sum the previous equation over all particles to give

$$\frac{1}{2}\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = \underbrace{\sum_i (\mathbf{r}_i \cdot \mathbf{F}_i)}_{\substack{V\text{, the virial} \\ \text{(R. Clausius)}}} + 2T$$

In the absence of external forces (i.e. an isolated system), we have that $\mathbf{F}_i = \sum_j \mathbf{F}_{ij}$ where $\mathbf{F}_{ij}$ is the force exerted on the ith particle by the jth particle. Consider any two particles with m_i and m_j at $\mathbf{r}_i$ and $\mathbf{r}_j$, Newton's 3rd Law says

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji}$$

and so their contribution to the virial is $\mathbf{F}_{ij} \cdot (\mathbf{r}_i - \mathbf{r}_j)$. We then have

$$V = \sum_i \sum_{j>i} \mathbf{F}_{ij} \cdot (\mathbf{r}_i - \mathbf{r}_j)$$

If there are no non-gravitational interactions except for possibly when $\mathbf{r}_i = \mathbf{r}_j$, all forces other than gravitational can be neglected and

$$\mathbf{F}_{ij} = -\frac{G m_i m_j}{r_{ij}^3}\mathbf{r}_{ij} \qquad \text{where} \qquad \mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$$

Thus we have

$$V = -\sum_i \sum_{j>i} \frac{G m_i m_j}{r_{ij}}$$

where each term is the work done to separate each pair of particles to infinity against gravity.

And so, $V = \Omega$ and we can use above to write

$$\frac{1}{2}\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 2T + \Omega$$

If the system is in a steady state ("relaxed"), then $I =$ const. and we can say

$$\boxed{2T + \Omega = 0} \qquad \text{THE VIRIAL THEOREM}$$

Here, the kinetic energy T has contributions from local flows and random/thermal motions.

$$\text{Virial theorem} \quad \Rightarrow \quad \begin{array}{l}\text{gravitational potential sets the “temperature”} \\ \text{or velocity dispersion of the system}\end{array}$$

CHAPTER D

Equations of State and the Energy Equation

D.1 The Equation of State

In 3-dimensions, the (scalar) equation of mass conservation and the (vector) equation of momentum conservation can be written as 4 independent scalar equations. Given appropriate boundary conditions, these must be solved in order to find the density (scalar field), pressure (scalar field), gravitational potential (scalar field), and velocity components (3-d vector field); a total of six degrees of freedom.

To close the system of equations, we need additional information. Specifically, we need to find relations between Ψ, p and the other fluid variables such as ρ and $\mathbf{u}$.

$\Psi(\mathbf{r})$ and ρ are related via Poisson's equation (and/or we sometimes consider an externally imposed gravitational potential).

p and the other thermodynamic properties of the system are related by the *equation of state* (EoS). This is only valid for collisional fluids.

Most astrophysical fluids are quite dilute (particle separation much larger than effective particle size) and can be well approximated as ideal gases. The corresponding EoS is

$$p = nk_B T = \frac{k_B}{\mu m_p}\rho T$$

where μ is the mean particle mass in units of the proton mass m_p. (Exceptions, where significant deviation from ideal gas behaviour occurs, can be found in high density environments of planets, neutron stars and white dwarfs.)

The ideal gas EoS introduces another scalar field into the description of the fluid, the temperature $T(\mathbf{r}, t)$. In general, we need to solve another PDE that describes heating and cooling processes in order to close the set of equations. We shall move on to this soon — see "Section D.2 : The Energy Equation".

However, for special cases, we can relate T and ρ without the need to solve a separate energy equation. Fluids for which p is *only* a function of ρ are known as barotropic fluids.

Example (Isothermal case)**.** T is constant so that $p \propto \rho$. Valid when the fluid is locally in thermal equilibrium with strong heating and cooling processes that are in balance.

Example (Adiabatic case)**.** Ideal gas undergoes reversible thermodynamic changes such that

$$p = K\rho^\gamma$$

where K, γ are constants.

Derivation. First law of thermodynamics is

$$\underbrace{đQ}_{\substack{\text{heat absorbed by}\\ \text{unit mass of fluid}\\ \text{from surronding}}} = \underbrace{\mathrm{d}\mathcal{E}}_{\substack{\text{change in internal}\\ \text{energy of unit}\\ \text{mass of fluid}}} + \underbrace{p\,\mathrm{d}V}_{\substack{\text{work done by}\\ \text{unit mass of fluid}}}$$

Here đ is a Pfaffian operator — change in quantity depends on the path taken through the thermodynamic phase space. For an ideal gas, we can write

$$p = \frac{\mathcal{R}_*}{\mu}\rho T, \qquad \mathcal{E} = \mathcal{E}(T)$$

where $\mathcal{R}_*$ is a modified gas constant.

So, first law reads

$$\begin{aligned} đQ &= \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}T}\,\mathrm{d}T + p\,\mathrm{d}V \\ &= C_V\,\mathrm{d}T + \frac{\mathcal{R}_* T}{\mu V}\,\mathrm{d}V \end{aligned}$$

where we define specific heat capacity at constant volume as $C_V \equiv \mathrm{d}\mathcal{E}/\mathrm{d}T$ and have noted that for unit mass we have $\rho = 1/V$.

For a reversible change we have $đQ = 0$, so

$$\begin{aligned} & C_V\,\mathrm{d}T + \frac{\mathcal{R}_* T}{\mu V}\,\mathrm{d}V = 0 \\ \Rightarrow\qquad & C_V\,\mathrm{d}(\ln T) + \frac{\mathcal{R}_*}{\mu}\,\mathrm{d}(\ln V) = 0 \\ \Rightarrow\qquad & V \propto T^{-C_V\mu/\mathcal{R}_*} \\ \Rightarrow\qquad & p \propto T^{1+C_V\mu/\mathcal{R}_*} \end{aligned}$$

C_V depends on the number of degrees of freedom with which the gas can store kinetic energy, f such that

$$C_V = f\frac{\mathcal{R}_*}{2\mu}$$

Monatomic gas has $f = 3 \Rightarrow C_V = 3\mathcal{R}_*/2\mu$; diatomic gas at a few$\times$100 K (two rotational modes excited) has $f = 5 \Rightarrow C_V = 5\mathcal{R}_*/2\mu$.

Returning to the ideal gas law,

$$\begin{aligned} & p = \frac{\mathcal{R}_*}{\mu}\rho T \qquad \text{with } \rho = 1/V \text{ for a unit mass of fluid} \\ \Rightarrow \qquad & pV = \frac{\mathcal{R}_* T}{\mu} \\ \Rightarrow \qquad & p\,\mathrm{d}V + V\,\mathrm{d}p = \frac{\mathcal{R}_*}{\mu}\,\mathrm{d}T \end{aligned}$$

but

$$\begin{aligned} đQ &= \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}T}\,\mathrm{d}T + p\,\mathrm{d}V \\ &= \underbrace{\left(\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}T} + \frac{\mathcal{R}_*}{\mu}\right)}_{\substack{\text{specific heat capacity} \\ \text{at constant pressure, } C_p}} \mathrm{d}T - V\,\mathrm{d}p \end{aligned}$$

so,

$$C_p - C_V = \frac{\mathcal{R}_*}{\mu}$$

Let us define

$$\gamma \equiv \frac{C_p}{C_V}$$

so that, for the reversible/adiabatic processes discussed above, we have

$$\begin{aligned} p \propto T^{1+C_V\mu/\mathcal{R}_*} \qquad &\Rightarrow \qquad p \propto T^{\gamma/(\gamma-1)} \\ V \propto T^{-C_V\mu/\mathcal{R}_*} \qquad &\Rightarrow \qquad V \propto T^{-1/(\gamma-1)} \end{aligned}$$

which we can combine to give

$$p \propto \rho^\gamma$$

$\square$

We say that a fluid element behaves *adiabatically* if $p = K\rho^\gamma$ with $K =$ constant. A fluid is *isentropic* if all fluid elements behave adiabatically with the same value of K. K is related to the entropy per unit mass.

D.2 The Energy Equation

In general, the equation of state will not be barotropic and we will need to solve a separate differential equation which follows the heating and cooling processes in the gas, the *energy equation*.

From the first law of thermodynamics we have

$$đQ = \mathrm{d}\mathcal{E} + \underbrace{p\,\mathrm{d}V}_{\mathrm{d}W=-p\mathrm{d}V} \qquad \text{in absence of dissipative processes}$$

so,

$$\frac{\mathrm{D}\mathcal{E}}{\mathrm{D}t} = \frac{\mathrm{D}W}{\mathrm{D}t} + \frac{đQ}{\mathrm{d}t}$$

with

$$\frac{\mathrm{D}W}{\mathrm{D}t} = -p\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{\rho}\right) = \frac{p}{\rho^2}\frac{\mathrm{D}\rho}{\mathrm{D}t}$$

and

$$\frac{đQ}{\mathrm{d}t} \equiv -\dot{Q}_{cool} \qquad \text{rate of cooling per unit mass}$$

therefore,

$$\frac{\mathrm{D}\mathcal{E}}{\mathrm{D}t} = \frac{p}{\rho^2}\frac{\mathrm{D}\rho}{\mathrm{D}t} - \dot{Q}_{\mathrm{cool}}$$

The total energy per unit volume is

$$E = \rho(\underbrace{\frac{1}{2}u^2}_{\text{kinetic}} + \underbrace{\Psi}_{\text{potential}} + \underbrace{\mathcal{E}}_{\text{internal}})$$

so,

$$\frac{\mathrm{D}E}{\mathrm{D}t} = \frac{\mathrm{D}\rho}{\mathrm{D}t}\frac{E}{\rho} + \rho\left(\mathbf{u}\cdot\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \frac{\mathrm{D}\Psi}{\mathrm{D}t} + \frac{p}{\rho^2}\frac{\mathrm{D}\rho}{\mathrm{D}t} - \dot{Q}_{\mathrm{cool}}\right)$$

where

$$\frac{\mathrm{D}E}{\mathrm{D}t} \equiv \frac{\partial E}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}E$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho\boldsymbol{\nabla}\cdot\mathbf{u}$$

$$\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\boldsymbol{\nabla}p + \rho\mathbf{g} = -\boldsymbol{\nabla}p - \rho\boldsymbol{\nabla}\Psi$$

$$\frac{\mathrm{D}\Psi}{\mathrm{D}t} \equiv \frac{\partial\Psi}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\Psi$$

Putting all together

$$\begin{aligned}\frac{\mathrm{D}E}{\mathrm{D}t} =& -\frac{E}{\rho}\rho\boldsymbol{\nabla}\cdot\mathbf{u} - \mathbf{u}\cdot\boldsymbol{\nabla}p - \rho\mathbf{u}\cdot\boldsymbol{\nabla}\Psi + \rho\frac{\partial\Psi}{\partial t}\\ &+ \rho\mathbf{u}\cdot\boldsymbol{\nabla}\Psi - \frac{p}{\rho}\rho\boldsymbol{\nabla}\cdot\mathbf{u} - \rho\dot{Q}_{\mathrm{cool}}\end{aligned}$$

$$\Rightarrow \qquad \frac{\partial E}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}E = -(E+p)\boldsymbol{\nabla}\cdot\mathbf{u} - \mathbf{u}\cdot\boldsymbol{\nabla}p + \rho\frac{\partial\Psi}{\partial t} - \rho\dot{Q}_{\mathrm{cool}}$$

which gives

$$\boxed{\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot [(E+p)\mathbf{u}] = \rho \frac{\partial \Psi}{\partial t} - \rho \dot{Q}_{\text{cool}}} \qquad \text{Energy Equation}$$

In many settings, $\partial \Psi / \partial t = 0$, i.e. Ψ depends on position only. If, further, we have no cooling ($\dot{Q}_{\text{cool}} = 0$), then this equation expresses the conservation of energy in which the Eulerian change in total energy density E is driven by the divergence of the enthalpy flux $(E+p)\mathbf{u}$.

D.3 Heating and Cooling Processes

The $\dot{Q}_{\text{cool}}$ term in the energy equation describes processes that locally cool ($\dot{Q}_{\text{cool}} > 0$) or locally heat ($\dot{Q}_{\text{cool}} < 0$) the fluid. There are many such processes and a full discussion of them would be lengthy. Here, we discuss just a small number of important cases.

(1) Cooling by radiation : energy carried away from fluid by photons.

- (i) Energy loss by recombination of an ionized gas (line emission as electrons cascade down energy levels);
- (ii) Energy loss by free-free emission (free electrons accelerated in electric fields of ions)

$$L_{\text{ff}} \propto n_{\text{e}} n_{\text{p}} T^{1/2}$$

- (iii) Collisionally-excited atomic line radiation (electron collides with atom in ground state $\rightarrow$ produces excited atomic state which returns to ground state by emitting a photon with energy χ)

$$L_{\text{c}} \propto n_{\text{e}} n_{\text{ion}} e^{-\chi/kT} \chi / \sqrt{T}$$

In cold gas clouds with $T \sim 10^4$ K, H cannot be excited so cooling occurs through trace species ($\text{O}^+, \text{O}^{++}, \text{N}^+$).

These are all two-body interactions $\Rightarrow$ cooling rate per unit volume proportional to ρ^2. Recalling that $\dot{Q}_{\text{cool}}$ is defined per unit mass, such processes give $\dot{Q}_{\text{cool}} = \rho f(T)$.

(2) Heating by cosmic rays : heating and energy transport via high-energy (often relativistic) particles that are diffusing/streaming through the fluid.

- High energy particles ionise atoms in fluid, excess energy put into freed e^- which ends up as heat in fluid.

$$\text{ionisation rate per unit volume} \;\propto\; \text{CR flux} \times \rho$$

$$\Rightarrow \qquad \dot{Q}_{\text{cool}} \propto \text{CR flux. (independent of } \rho)$$

Combining these cases, we can parametrise $\dot{Q}_{\text{cool}}$ as:

$$\dot{Q}_{\text{cool}} = \underbrace{A\rho T^{\alpha}}_{\substack{\text{radiative}\\\text{cooling}}} - \underbrace{H}_{\text{CR heating}}$$

where α depends upon the physics of the dominant radiative cooling process.

D.4 Energy Transport Processes

Transport processes move energy through the fluid. Important examples are:

(1) Thermal conduction — transport of thermal energy by diffusion of the hot e^- into cooler regions. Relevant in, for example

- Interiors of white dwarfs;
- Supernova shock fronts;
- ICM plasma.

There is also thermal conduction associated with ions, but it is smaller than the electron thermal conduction by a factor of $\sqrt{m_{\text{ion}}/m_e} \sim 43$.

The energy flux per unit area is

$$\mathbf{F}_{\text{cond}} = -\kappa \boldsymbol{\nabla} T$$

where κ is thermal conductivity (computed from kinetic theory).

The rate of change of E per unit volume is

$$-\boldsymbol{\nabla} \cdot \mathbf{F}_{\text{cond}} = \kappa \nabla^2 T$$

(2) Convection — transport of energy due to fluctuating or circulating fluid flows in presence of entropy gradient. Important in cores of massive stars, or interiors of some planets, or envelopes of low-mass stars.

(3) Radiation transport — relevant in optically-thick systems (mean free path of photon much shorter than size of system).

If scattering opacity dominates, then we have radiative diffusion. If ϵ_{rad} is the energy density of the radiation field, the radiative flux through the fluid is

$$\mathbf{F}_{\text{rad}} \propto -\boldsymbol{\nabla} \epsilon_{\text{rad}}$$

The general topic of radiation transport through a fluid flow is very complex and beyond the scope of this course.

CHAPTER E

Hydrostatic Equilibrium, Atmospheres and Stars

We now have the full set of equations describing the dynamics of an ideal (inviscid, dilute, unmagnetized) non-relativistic fluid:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0 \qquad \text{Continuity Equation}$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p + \rho \mathbf{g} \qquad \text{Momentum Equation}$$

$$\nabla^2 \Psi = 4\pi G \rho \qquad \text{Poisson's Equation}$$

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot [(E + p)\mathbf{u}] = \rho \frac{\partial \Psi}{\partial t} - \rho \dot{Q}_{\text{cool}} \qquad \text{Energy Equation}$$

$$E = \rho \left(\frac{1}{2} u^2 + \Psi + \mathcal{E} \right) \qquad \text{Defn of total energy}$$

$$p = \frac{k_B}{\mu m_p} \rho T \qquad \text{EoS for ideal gas}$$

$$\mathcal{E} = \frac{3p}{2\rho} \qquad \text{Internal energy (monoatomic)}$$

We proceed to use those equations to explore astrophysically relevant situations.

This chapter starts with the simplest, but important, case — fluid systems that are a static equilibrium with pressure forces balancing gravity.

E.1 Hydrostatic Equilibrium

A fluid system is in a state of *hydrostatic equilibrium* if

$$\mathbf{u} = 0, \qquad \frac{\partial}{\partial t} = 0$$

Then, continuity equation is trivially satisfied

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0$$

Momentum equation gives

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p - \rho \boldsymbol{\nabla} \Psi = 0$$

resulting in

$$\boxed{\frac{1}{\rho}\boldsymbol{\nabla} p = -\boldsymbol{\nabla}\Psi} \qquad \text{EQUATION OF HYDROSTATIC EQM.}$$

Assuming a barotropic equation of state $p = p(\rho)$, this system of equations can be solved.

Example (Isothermal atmosphere with constant (externally imposed) $\mathbf{g}$)**.** Suppose $\mathbf{g} = -g\hat{\mathbf{z}}$. Then equation of hydrostatic equilibrium with isothermal equation of state reads

$$\begin{aligned} & A \cdot \frac{1}{\rho}\boldsymbol{\nabla}\rho = -\boldsymbol{\nabla}\Psi = -g\hat{\mathbf{z}} \\ \Rightarrow \quad & \ln\rho = -\frac{gz}{A} + \text{const.} \\ \Rightarrow \quad & \rho = \rho_0 \exp\left(-\frac{\mu g}{\mathcal{R}_* T} z\right) \end{aligned}$$

i.e. exponential atmosphere.

Examples of this is the Earth's atmosphere: $T \sim 300$ K and $\mu \sim 28 \Rightarrow$ e-folding ~ 9 km. The highest astronomical observatories are at $z \sim 4$ km, so have ρ and $p \sim 60\%$ of sea level.

If system is self-gravitating (rather than having an externally imposed gravitational field), we also have

$$\nabla^2 \Psi = 4\pi G \rho$$

This must be solved together with the equation of hydrostatic equilibrium.

Example (Isothermal self-gravitating slab)**.** Consider static, isothermal slab in x and y which is symmetric about $z = 0$ (e.g. two clouds collide and generate a shocked slab of gas between them).

$$\text{Isothermal} \quad \Rightarrow \quad p = \frac{\mathcal{R}_*}{\mu}\rho T \quad \Rightarrow \quad p = A\rho, \quad A \text{ const.}$$

also, $\boldsymbol{\nabla} = \frac{\partial}{\partial z}$ due to symmetry, $p = p(z)$, $\Psi = \Psi(z)$.

Then the equation of hydrostatic equilibrium becomes

$$\begin{aligned}
& A\frac{1}{\rho}\boldsymbol{\nabla}\rho = -\boldsymbol{\nabla}\Psi \\
\Rightarrow \quad & A\frac{\mathrm{d}}{\mathrm{d}z}(\ln\rho) = -\frac{\mathrm{d}\Psi}{\mathrm{d}z} \\
\Rightarrow \quad & \Psi = -A\ln(\rho/\rho_0) + \Psi_0 \qquad (\rho_0 = \rho(z=0)) \\
\Rightarrow \quad & \rho = \rho_0 e^{-(\Psi-\Psi_0)/A}.
\end{aligned}$$

Since $A \propto T$, we note that this last equation has the form of a Boltzmann distribution.

Poisson's equation is

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}z^2} = 4\pi G\rho_0 e^{-(\Psi-\Psi_0)/A}.$$

Let's change variables to $\chi = -(\Psi-\Psi_0)/A$, $Z = z\sqrt{2\pi G\rho_0/A}$ so that Poisson's equation becomes

$$\begin{aligned}
& \frac{\mathrm{d}^2\chi}{\mathrm{d}Z^2} = -2e^{\chi} \qquad \chi = \frac{\mathrm{d}\chi}{\mathrm{d}Z} = 0 \text{ at } Z = 0 \\
\Rightarrow \quad & \frac{\mathrm{d}\chi}{\mathrm{d}Z}\frac{\mathrm{d}^2\chi}{\mathrm{d}Z^2} = -2\frac{\mathrm{d}\chi}{\mathrm{d}Z}e^{\chi} \\
\Rightarrow \quad & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}Z}\left[\left(\frac{\mathrm{d}\chi}{\mathrm{d}Z}\right)^2\right] = -2\frac{\mathrm{d}}{\mathrm{d}Z}(e^{\chi}) \\
\Rightarrow \quad & \left(\frac{\mathrm{d}\chi}{\mathrm{d}Z}\right)^2 = C_1 - 4e^{\chi}.
\end{aligned}$$

But we have boundary condition $\mathrm{d}\chi/\mathrm{d}Z = 0$ when $\chi = 0 \Rightarrow C_1 = 4$.

$$\therefore \qquad \frac{\mathrm{d}\chi}{\mathrm{d}Z} = 2\sqrt{1-e^{\chi}} \qquad \Rightarrow \qquad \int\frac{\mathrm{d}\chi}{\sqrt{1-e^{\chi}}} = 2\int\mathrm{d}Z\,.$$

Change variables $e^{\chi} = \sin^2\theta$

$$\Rightarrow \qquad e^{\chi}\,\mathrm{d}\chi = 2\sin\theta\cos\theta\,\mathrm{d}\theta \qquad \text{or} \qquad \mathrm{d}\chi = \frac{2\cos\theta}{\sin\theta}\,\mathrm{d}\theta\,.$$

So, we can evaluate χ integral

$$\begin{aligned}
\int\frac{\mathrm{d}\chi}{\sqrt{1-e^{\chi}}} &= \int\frac{2\cos\theta\,\mathrm{d}\theta}{\sin\theta\sqrt{1-\sin^2\theta}} \\
&= \int\frac{2\,\mathrm{d}\theta}{\sin\theta} \\
&= \int 2\cdot\frac{1}{2}\frac{1+t^2}{t}\,\mathrm{d}\theta \\
&= 2\int\frac{\mathrm{d}t}{t} \\
&= 2\ln t + C_2
\end{aligned}$$

by setting

$$t = \tan\frac{\theta}{2} \qquad \Rightarrow \qquad \mathrm{d}t = \frac{1}{2}(1+t^2)\,\mathrm{d}\theta$$

and by noting

$$\sin\theta \equiv 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{2t}{1+t^2} = e^{\chi/2}$$

So, Poisson's equation becomes

$$2\ln t = 2Z + C_2$$

Now, $\chi = 0$ at $Z = 0 \Rightarrow \theta = \pi/2, t = 1 \Rightarrow C_2 = 0$, so $t = e^Z$

$$\Rightarrow \qquad \sin\theta = e^{\chi/2} = \frac{2e^Z}{1+e^{2Z}} = \frac{1}{\cosh Z}$$

This gives

$$\boxed{\begin{aligned} \Psi - \Psi_0 &= 2A\ln\cosh\left(\sqrt{\frac{2\pi G\rho_0}{A}}z\right) \\ \rho &= \frac{\rho_0}{\cosh^2\left(\sqrt{\frac{2\pi G\rho_0}{A}}z\right)} \end{aligned}}$$

.

E.2 Stars as Self-Gravitating Polytropes

Consider a spherically-symmetric self-gravitating system in hydrostatic equilibrium; from now on we will refer to this as a "star". We have

$$\begin{aligned} &\boldsymbol{\nabla} p = -\rho\boldsymbol{\nabla}\Psi \\ \Rightarrow \qquad &\frac{\mathrm{d}p}{\mathrm{d}r} = -\rho\frac{\mathrm{d}\Psi}{\mathrm{d}r} \qquad \text{(spherical polar)} \end{aligned}$$

Now, $\rho > 0$ within star $\Rightarrow p$ is monotonic function of Ψ. Also,

$$\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{\mathrm{d}p}{\mathrm{d}\Psi}\frac{\mathrm{d}\Psi}{\mathrm{d}r} = -\rho\frac{\mathrm{d}\Psi}{\mathrm{d}r} \qquad \Rightarrow \qquad \rho = -\frac{\mathrm{d}p}{\mathrm{d}\Psi}$$

So ρ is monotonic function of Ψ.

$$\therefore \qquad p = p(\Psi), \rho = \rho(\Psi) \qquad \Rightarrow \qquad p = p(\rho)$$

i.e. non-rotating stars are barotropes!

A barotropic EoS can be written as

$$p = K\rho^{1+1/n}$$

where in general $n = n(\rho)$. When $n =$ constant, we say that we have a *polytropic EoS* and the structure is called a *polytrope*. Real stars are in fact well approximated as polytropes.

It is important to note that in general we will have

$$1 + \frac{1}{n} \neq \gamma.$$

We only have $1 + 1/n = \gamma$ (i.e. $p \propto \rho^\gamma$) if the star is isentropic (constant entropy throughout) due to, for example, mixing by convective motions.

Assuming a polytropic EoS, the equation of hydrostatic equilibrium gives

$$\begin{aligned} & -\boldsymbol{\nabla}\Psi = \frac{1}{\rho}\boldsymbol{\nabla}\left(K\rho^{1+1/n}\right) = (n+1)\boldsymbol{\nabla}\left(K\rho^{1/n}\right) \\ \Rightarrow \quad & \rho = \left(\frac{\Psi_T - \Psi}{[n+1]K}\right)^n \qquad \Psi_T \equiv \Psi \text{ where } \rho = 0 \text{, the surface.} \end{aligned}$$

If the central density if ρ_c and central potential is Ψ_c, we have

$$\rho_c = \left(\frac{\Psi_T - \Psi_c}{[n+1]K}\right)^n$$

so we can write,

$$\rho = \rho_c\left(\frac{\Psi_T - \Psi}{\Psi_T - \Psi_c}\right)^n$$

Feeding this into Poisson's equation gives

$$\nabla^2\Psi = 4\pi G\rho_c\left(\frac{\Psi_T - \Psi}{\Psi_T - \Psi_c}\right)^n$$

Define $\theta = (\Psi_T - \Psi)/(\Psi_T - \Psi_c)$, we then get

$$\nabla^2\theta = -\frac{4\pi G\rho_c}{\Psi_T - \Psi_c}\theta^n$$

In spherical polars, this becomes

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\theta}{\mathrm{d}r}\right) = -\frac{4\pi G\rho_c}{\Psi_T - \Psi_c}\theta^n$$

Defining a scaled radial coordinate $\xi = r\sqrt{(4\pi G\rho_c)/(\Psi_T - \Psi_c)}$, we finally get

$$\boxed{\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n} \qquad \text{LANE-EMDEN EQN. OF INDEX } n$$

The appropriate boundary conditions for the Lane-Emden equation are $\theta = 1$, $\mathrm{d}\theta/\mathrm{d}\xi = 0$ at $\xi = 0$. (Zero force at $\xi = 0$, enclosed mass $\to 0$ as $\xi \to 0$.)

The Lane-Emden equation can be solved analytically for $n = 0, 1$ and 5; otherwise solve numerically.

Solution for $n = 0$

This is a somewhat singular case, physically corresponding to a fluid that is constant density and incompressible.

$$
\begin{aligned}
& \frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n = -1 \\
\Rightarrow \quad & \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\xi^2 \\
\Rightarrow \quad & \xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi} = -\frac{1}{3}\xi^3 - C \\
\Rightarrow \quad & \theta = -\frac{\xi^2}{6} + \frac{C}{\xi} + D
\end{aligned}
$$

We need $\theta = 1$ at $\xi = 0 \Rightarrow C = 0, D = 1$.

$$
\therefore \qquad \theta = 1 - \frac{\xi^2}{6}
$$

For solutions for $n = 1$ or $n = 5$ cases, see the book [**PAFD**] (section 5.5.2 & 5.5.3).

E.3 Isothermal Spheres (Case $n \to \infty$)

The isothermal case $p = K\rho$ corresponds to $n \to \infty$. Let's combine

$$
\begin{aligned}
& \frac{\mathrm{d}p}{\mathrm{d}r} = -\rho\frac{\mathrm{d}\Psi}{\mathrm{d}r} \qquad \text{and} \qquad p = K\rho \\
\Rightarrow \quad & \frac{\mathrm{d}\Psi}{\mathrm{d}r} = -\frac{K}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} \\
\Rightarrow \quad & \Psi - \Psi_c = -K\ln(\rho/\rho_c)
\end{aligned}
$$

From Poisson's equation

$$
\begin{aligned}
& \nabla^2\Psi = 4\pi G\rho \\
\Rightarrow \quad & \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Psi}{\mathrm{d}r}\right) = 4\pi G\rho \\
\Rightarrow \quad & \frac{K}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) = -4\pi G\rho
\end{aligned}
$$

Let $\rho = \rho_c e^{-\Psi}$ (defining $\Psi_c = 0$), and set

$$
r = a\xi, \qquad a = \sqrt{\frac{K}{4\pi G\rho_c}}
$$

then

$$
\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\Psi}{\mathrm{d}\xi}\right) = e^{-\Psi}
$$

with $\Psi = \mathrm{d}\Psi/\mathrm{d}\xi = 0$ at $\xi = 0$.

This replaces the Lane-Emden equation in the case where the system is isothermal.

At large radii, this has solutions of the form $\rho \propto r^{-2}$, so the enclosed mass $\propto r$. Thus, the mass of an isothermal sphere of self-gravitating gas tends to ∞ as the radius tends to ∞. This is why we cannot adopt our usual convention of defining $\Psi = 0$ at ∞.

So, to be physical, isothermal spheres need to be truncated at some finite radius. There needs to be some continuing pressure by an external medium. These are called Bonnor-Ebert spheres, whose density profile depends on $\xi_{\rm cut}$.

E.g. dense gas core in a molecular cloud is well-described by a Bonnor-Ebert sphere.

E.4 Scaling Relations

In many circumstances, stars behave as polytropes, e.g. fully convective stars with $p-\rho$ close to the adiabatic relation. In such a star, assuming monatomic gas with $\gamma = 5/3$, we have $p = K\rho^{5/3} \Rightarrow n = 3/2$.

Consider a set of stars which share a given polytropic index n and a given adiabat K. They will then form a one-parameter family characterised by their central density ρ_c.

Thus one can find how mass and radius vary as a function of ρ_c and, eliminating ρ_c, obtain *scaling relations* relating the mass and radius.

All stars with given n have the same $\theta(\xi)$ since the Lane-Emden equation does not depend on ρ_c. Recall

$$
\begin{aligned}
\rho &= \left[\frac{\Psi_T - \Psi}{(n+1)K}\right]^n \qquad &&\Rightarrow \qquad \Psi_T - \Psi_c = K(n+1)\rho_c^{1/n} \\
\xi &= \sqrt{\frac{4\pi G\rho_c}{\Psi_T - \Psi_c}}\,r \qquad &&\Rightarrow \qquad \xi = \sqrt{\frac{4\pi G\rho_c^{1-1/n}}{K(1+n)}}\,r \\
\rho &= \rho_c\left[\frac{\Psi_T - \Psi}{\Psi_T - \Psi_c}\right]^n = \rho_c\theta^n
\end{aligned}
$$

The surface of the polytrope is at $\xi = \xi_{\max}$ defined as location where we have $\theta(\xi) = 0$. Let $r_{\max}$ be the corresponding physical radius. Then the total mass of the polytrope is

$$
\begin{aligned}
M &= \int_0^{r_{\max}} 4\pi r^2 \rho \,\mathrm{d}r \\
&= 4\pi\rho_c \left[\frac{4\pi G \rho_c^{1-1/n}}{K(1+n)}\right]^{-3/2} \underbrace{\int_0^{\xi_{\max}} \theta^n \xi^2 \,\mathrm{d}\xi}_{\substack{\text{same for all} \\ \text{polytrope of index } n}} \\
\Rightarrow \qquad M &\propto \rho_c^{\frac{1}{2}\left(\frac{3}{n}-1\right)}
\end{aligned}
$$

From definition of ξ above, we also know that

$$
r_{\max} \propto \rho_c^{\frac{1}{2}\left(\frac{1}{n}-1\right)}
$$

Eliminating ρ_c gives

$$
\boxed{M \propto R^{\frac{3-n}{1-n}}} \qquad \text{Mass-Radius Relation for Polytropic Stars}
$$

For $\gamma = 5/3, n = 3/2$ this gives $M \propto R^{-3}$ or $R \propto M^{-1/3}$. This suggests more massive stars have smaller radii.

This relation actually works well for white dwarfs (where the polytropic EoS is due to e^- degeneracy pressure rather than gas pressure). But for most main-sequence stars we observe $M \propto R$.

Reason is that stars do *not* share the same polytropic constant K. Let's write the temperature at the core in terms of the central density and K

$$
\left.\begin{aligned}
p &= K\rho^{1+1/n} \\
p &= \frac{\mathcal{R}_*}{\mu}\rho T
\end{aligned}\right\} \qquad \Rightarrow \qquad T_c = \frac{\mu K}{\mathcal{R}_*}\rho_c^{1/n}
$$

Nuclear reactions in the core tend to keep T_c similar in the cores of stars of different masses. So we can say that

$$
K \propto \rho_c^{-1/n}
$$

Substitute this into above expression for mass when $n = 3/2$ gives

$$
M \propto \rho_c^{-1/2}, \quad R \propto \rho_c^{-1/2} \qquad \Rightarrow \qquad M \propto R
$$

When can the $K = $ const. relation be applied? Answer: when new mass is added to a star adiabatically and the nuclear processes have not had time to adjust. For Sun we have

– Time to adjust to new hydrostatic equilibrium is

$$
t_{\mathrm{h}} \sim R/C_s \sim 1 \text{ day}
$$

– Time to lose significant energy

$$t_{\text{th}} \sim E_{\text{tot}}/L \sim \frac{GM^2}{RL} \sim 30 \text{ Myr}$$

So, mass loss/gain is followed by rapid re-adjustment of hydrostatic equilibrium but true thermal equilibrium is reached after a much longer time.

Example (Spherical rotating star)**.** Spherical rotating star with angular velocity Ω gains non-rotating mass. How does Ω evolve?

Conservation of angular momentum gives $MR^2\Omega = \text{const}$. So, if $\Omega \to \Omega + \Delta\Omega$ then

$$\begin{aligned} & MR^2\Delta\Omega + \Omega\Delta(MR^2) = 0 \\ \Rightarrow \quad & \frac{\Delta\Omega}{\Omega} = -\frac{\Delta(MR^2)}{MR^2} \end{aligned}$$

But we can use

$$R \propto M^{(1-n)/(3-n)}$$

to say

$$\begin{aligned} & \frac{\Delta\Omega}{\Omega} \propto -\Delta\left(M^{(5-3n)/(3-n)}\right) \\ \Rightarrow \quad & \frac{\Delta\Omega}{\Omega} \propto -\left(\frac{5-3n}{3-n}\right)\Delta M \end{aligned}$$

so,

$$\Delta M > 0 \quad \Rightarrow \quad \begin{cases} \Delta\Omega < 0 & \text{if} \quad \frac{5-3n}{3-n} > 0 \qquad (\text{e.g. } n = \frac{3}{2}) \text{ Spin Down} \\ \Delta\Omega > 0 & \text{if} \quad \frac{5-3n}{3-n} < 0 \qquad (\text{e.g. } n = 2) \text{ Spin Up} \end{cases}$$

Example (Star in a binary system)**.** Star in a binary system loses mass to its companion.

Donor star loses mass, $\Delta M < 0$. So since $R \propto M^{(1-n)/(3-n)}$, the radius will increase if $1 < n < 3$.

So there is the potential for unstable (runaway) mass transfer (need to look at evolution of the size of the Roche lobe to conclusively decide whether process is unstable).

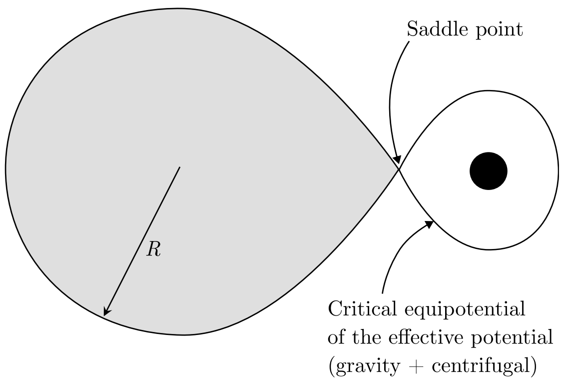

Figure E.1: Roche lobe overflow

CHAPTER F

Sound Waves, Supersonic Flows and Shock Waves

F.1 Sound Waves

We now start discussion of how disturbances can propagate in a fluid. We begin by talking about sound waves in a uniform medium (no gravity). We proceed by conducting a first-order perturbation analysis of the fluid equations:

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) &= 0 \\ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} &= -\frac{1}{\rho}\boldsymbol{\nabla} p\end{aligned}$$

The equilibrium around which we will perturb is

$$\begin{aligned}\rho &= \rho_0 && \text{(uniform \& constant)} \\ p &= p_0 && \text{(uniform \& constant)} \\ \mathbf{u} &= \mathbf{0}\end{aligned}$$

We consider small perturbations and write in Lagrangian terms (Lagrangian meaning the change of quantities are for *a given fluid element*)

$$\begin{aligned}p &= p_0 + \Delta p \\ \rho &= \rho_0 + \Delta \rho \\ \mathbf{u} &= \Delta \mathbf{u}\end{aligned}$$

The relation between Lagrangian and Eulerian perturbations is:

$$\underbrace{\delta \rho}_{\substack{\text{Eulerian} \\ \text{pert.}}} = \underbrace{\Delta \rho}_{\substack{\text{Lagrangian} \\ \text{pert.}}} - \underbrace{\boldsymbol{\xi} \cdot \boldsymbol{\nabla} \rho_0}_{\substack{\text{Element displacement dot} \\ \text{Gradient of unpert. state}}}$$

In present example, $\boldsymbol{\nabla} \rho_0 = 0$ and so $\delta \rho = \Delta \rho$. But the distinction between Lagrangian and Eulerian perturbations will be important for other situations that we'll address later.

Substitute the perturbations into fluid equations and ignore terms that are 2nd order (or higher) in the perturbed quantities:

Start with continuity equation:

$$\frac{\partial}{\partial t}(\rho_0 + \Delta\rho) + \boldsymbol{\nabla} \cdot [(\rho_0 + \Delta\rho)\Delta\mathbf{u}] = 0$$

$$\Rightarrow \quad \cancelto{0}{\frac{\partial \rho_0}{\partial t}} + \frac{\partial \Delta\rho}{\partial t} + \cancelto{0}{\boldsymbol{\nabla}\rho_0 \cdot \Delta\mathbf{u}} + \cancelto{\text{2nd order}}{\boldsymbol{\nabla}(\Delta\rho) \cdot \Delta\mathbf{u}} + \rho_0 \boldsymbol{\nabla} \cdot (\Delta\mathbf{u}) + \cancelto{\text{2nd order}}{\Delta\rho\boldsymbol{\nabla} \cdot (\Delta\mathbf{u})} = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial t}(\Delta\rho) + \rho_0 \boldsymbol{\nabla} \cdot (\Delta\mathbf{u}) = 0 \qquad ① \qquad \boxed{\texttt{eq.f.1.1}}$$

And similarly, the momentum equation:

$$\frac{\partial}{\partial t}(\Delta\mathbf{u}) = -\frac{1}{\rho_0}\boldsymbol{\nabla}(\Delta p)$$

$$\Rightarrow \quad \frac{\partial}{\partial t}(\Delta\mathbf{u}) = -\left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0} \frac{\boldsymbol{\nabla}(\Delta\rho)}{\rho_0}. \qquad \text{assuming barotropic EoS} \qquad ② \qquad \boxed{\texttt{eq.f.1.2}}$$

Now, take $\partial/\partial t$ of ①:

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}(\Delta\rho) &= -\rho_0 \frac{\partial}{\partial t}[\boldsymbol{\nabla} \cdot (\Delta\mathbf{u})] \\
&= -\rho_0 \boldsymbol{\nabla} \cdot \left[\frac{\partial}{\partial t}(\Delta\mathbf{u})\right] \\
&= \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0} \nabla^2(\Delta\rho).
\end{aligned}$$

We get

$$\boxed{\frac{\partial^2(\Delta\rho)}{\partial t^2} = \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0} \nabla^2(\Delta\rho).} \qquad \text{Wave Equation}$$

This admits solutions of the form $\Delta\rho = \Delta\rho_0 e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)}$. Substituting into the wave equation we get

$$\begin{aligned}
(-\mathrm{i}\omega)^2 \Delta\rho_0 &= \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0} (\mathrm{i}k)^2 \\
\Rightarrow \qquad \omega^2 &= \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0} k^2
\end{aligned}$$

The (phase) speed of the wave is $v_p = \omega/k$, so the sound wave travels at speed

$$c_s = \sqrt{\left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0}} \qquad \text{Sound Speed as the Derivative of } p(\rho)$$

Consider a 1-D wave and substitute

$$\Delta\rho = \Delta\rho_0 e^{\mathrm{i}(kx-\omega t)}$$
$$\Delta u = \Delta u_0 e^{\mathrm{i}(kx-\omega t)}$$

into ①. We get

$$-\mathrm{i}\omega\Delta\rho + \rho_0 \mathrm{i}k\Delta u = 0$$
$$\Rightarrow \qquad \Delta u = \frac{\omega}{k}\frac{\Delta\rho}{\rho_0} = c_s\frac{\Delta\rho}{\rho_0}$$

So we learn that

- Fluid velocity and density perturbations are in phase (since $\Delta u/\Delta\rho \in \mathbb{R}$);
- Disturbance propagates at a much higher speed than that of the individual fluid elements, provided density perturbations are small, since

$$\Delta u_0 = c_s\frac{\Delta\rho_0}{\rho_0} \ll c_s$$

Sound waves propagate because density perturbations give rise to a pressure gradient which then causes acceleration of the fluid elements, this induces further density perturbations, making disturbances propagate.

Sound speed depends on how the pressure forces react to density changes. If the EoS is "stiff" (i.e. high $\mathrm{d}p/\mathrm{d}\rho$), then restoring force is large and propagation is rapid.

Examples of $\mathrm{d}p/\mathrm{d}\rho$:

Example (Isothermal case)**.**

$$c_s^2 = \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_T$$

In this case, compressions and rarefactions are effective at passing heat to each other to maintain constant T. Then

$$p = \frac{\mathcal{R}_*}{\mu}\rho T$$
$$\Rightarrow \qquad c_{s,\mathrm{I}} = \sqrt{\frac{\mathcal{R}_* T}{\mu}}$$

Example (Adiabatic case)**.**

$$c_s^2 = \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_S$$

No heat exchange between fluid elements; compressions heat up and rarefactions cool down from $p\,\mathrm{d}V$ work. So

$$
\begin{aligned}
& p = K\rho^{\gamma} \\
\Rightarrow \quad & \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_S = \gamma K \rho^{\gamma-1} = \frac{\gamma p}{\rho} \\
\Rightarrow \quad & c_{s,\mathrm{A}} = \sqrt{\frac{\gamma \mathcal{R}_* T}{\mu}}
\end{aligned}
$$

Notes about these two examples:

- We see that $c_{s,\mathrm{I}}$ and $c_{s,\mathrm{A}}$ differ by only $\sqrt{\gamma}$;
- Thermal behaviour of the perturbations does *not* have to be the same as that of the unperturbed structure!

$$
\text{E.g. Earth's atmosphere} \begin{cases} \text{Background is approximately isothermal} \\ \text{Sound waves are adiabatic} \end{cases}
$$

- Waves for which c_s is not a function of ω are called non-dispersive. The shape of a wave packet is preserved.

F.2 Sound Waves in a Stratified Atmosphere

We now move to the more subtle problem of sound waves propagating in a fluid with background structure. For concreteness, let's consider an isothermal atmosphere with constant $\mathbf{g} = -g\hat{\mathbf{z}}$.

Horizontally travelling sound waves are unaffected by the (vertical) structure. So let's just focus on z-dependent terms, taking $\mathbf{u} = u\hat{\mathbf{z}}$. Continuity and momentum equations are:

$$
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}(\rho u) = 0 \qquad ① \qquad \text{eq.f.2.1}
$$

$$
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g, \qquad ② \qquad \text{eq.f.2.2}
$$

and the equilibrium is

$$
\begin{aligned}
& u_0 = 0 \\
& \rho_0(z) = \tilde{\rho} e^{-z/H}, \quad H \equiv \frac{\mathcal{R}_* T}{g\mu} \\
& p_0(z) = \frac{\mathcal{R}_* T}{\mu}\rho_0(z) = \tilde{p} e^{-z/H}.
\end{aligned}
$$

Consider a Lagrangian perturbation:

$$\begin{aligned} u &\to \Delta u \\ \rho_0 &\to \rho_0 + \Delta\rho \\ p_0 &\to p_0 + \Delta p. \end{aligned}$$

Remember that $\delta\rho = \Delta\rho - \boldsymbol{\xi} \cdot \boldsymbol{\nabla}\rho$. So we have

$$\left.\begin{aligned} \delta\rho &= \Delta\rho - \xi_z \frac{\partial \rho_0}{\partial z} \\ \delta p &= \Delta p - \xi_z \frac{\partial p_0}{\partial z} \\ \delta u &= \Delta u, \end{aligned}\right\} \text{Eulerian to Lagrangian perturbation relation}$$

and

$$\Delta\mathbf{u} = \frac{D\boldsymbol{\xi}}{Dt} = \frac{\partial\boldsymbol{\xi}}{\partial t} + \cancelto{\text{2nd order}}{\mathbf{u}\cdot\boldsymbol{\nabla}\boldsymbol{\xi}} = \frac{\partial\boldsymbol{\xi}}{\partial t}$$

Substituting perturbed quantities into the Eulerian continuity equation,

$$\begin{aligned} & \frac{\partial}{\partial t}(\rho_0 + \delta\rho) + \frac{\partial}{\partial z}[(\rho_0 + \delta\rho)\delta u_z] = 0 \\ \Rightarrow \quad & \frac{\partial}{\partial t}\left(\rho_0 + \Delta\rho - \xi_z \frac{\partial \rho_0}{\partial z}\right) + \frac{\partial}{\partial z}(\rho_0 \Delta u_z) = 0 \quad \text{(ignoring 2nd order terms)} \\ \Rightarrow \quad & \cancelto{0}{\frac{\partial \rho_0}{\partial t}} + \frac{\partial \Delta\rho}{\partial t} - \frac{\partial \xi_z}{\partial t}\frac{\partial \rho_0}{\partial z} - \xi_z \cancelto{0}{\frac{\partial}{\partial t}\frac{\partial \rho_0}{\partial z}} + \frac{\partial \rho_0}{\partial z}\Delta u_z + \rho_0 \frac{\partial \Delta u_z}{\partial z} = 0 \\ \Rightarrow \quad & \frac{\partial \Delta\rho}{\partial t} - \underbrace{\Delta u_z}_{\partial\xi_z/\partial t}\frac{\partial \rho_0}{\partial z} + \frac{\partial \rho_0}{\partial z}\Delta u_z + \rho_0 \frac{\partial \Delta u_z}{\partial z} = 0 \\ \Rightarrow \quad & \frac{\partial \Delta\rho}{\partial t} + \rho_0 \frac{\partial \Delta u_z}{\partial z} = 0. \end{aligned} \qquad ③ \quad \boxed{\texttt{eq.f.2.3}}$$

A similar calculation for the momentum equation gives

$$\begin{aligned} & \frac{\partial \Delta u_z}{\partial t} = -\frac{1}{\rho_0}\frac{\partial \Delta p}{\partial z} \\ \Rightarrow \quad & \frac{\partial \Delta u_z}{\partial t} = -\frac{c_u^2}{\rho_0}\frac{\partial \Delta\rho}{\partial z}, \quad c_u^2 \equiv \left.\frac{\partial p}{\partial \rho}\right|_{\rho_0}. \end{aligned} \qquad ④ \quad \boxed{\texttt{eq.f.2.4}}$$

To perform this calculation (which we leave as an exercise!), you need a relation that is obtained from the Lagrangian continuity equation:

$$\begin{aligned} & \frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\boldsymbol{\nabla}\cdot\mathbf{u} = 0 \\ \Rightarrow \quad & \Delta\rho + \left(\rho_0 \boldsymbol{\nabla}\cdot\frac{\partial\boldsymbol{\xi}}{\partial t}\right)\Delta t = 0 \quad \text{(integrating over a short time } \Delta t\text{)} \\ \Rightarrow \quad & \Delta\rho + \rho_0 \boldsymbol{\nabla}\cdot\boldsymbol{\xi} = 0. \end{aligned}$$

Let's now derive the wave equation and dispersion relation. Take $\partial/\partial t$ of ③:

$$\begin{aligned} &\frac{\partial^2 \Delta\rho}{\partial t^2} + \rho_0 \frac{\partial}{\partial z}\left(\frac{\partial \Delta u_z}{\partial t}\right) = 0 \\ \Rightarrow \qquad &\frac{\partial^2 \Delta\rho}{\partial t^2} - \rho_0 \frac{\partial}{\partial z}\left(\frac{c_u^2}{\rho_0}\frac{\partial \Delta\rho}{\partial z}\right) = 0, \end{aligned}$$

where the last step involved substitution from ④. If the medium is isothermal, then c_u is independent of z. So,

$$\begin{aligned} &\frac{\partial^2 \Delta\rho}{\partial t^2} - \cancel{\rho_0}\frac{c_u^2}{\cancel{\rho_0}}\frac{\partial^2 \Delta\rho}{\partial z^2} + \cancel{\rho_0}\frac{c_u^2}{\rho_0^{\cancel{2}}}\frac{\partial \rho_0}{\partial z}\frac{\partial \Delta\rho}{\partial z} = 0 \\ \Rightarrow \qquad &\underbrace{\frac{\partial^2 \Delta\rho}{\partial t^2} - c_u^2\frac{\partial^2 \Delta\rho}{\partial z^2}}_{\text{normal sound wave equation}} + \underbrace{\frac{c_u^2}{\rho_0}\frac{\partial \rho_0}{\partial z}\frac{\partial \Delta\rho}{\partial z}}_{\substack{\text{extra piece associated} \\ \text{with stratification}}} = 0 \end{aligned}$$

Now,

$$\begin{aligned} \frac{\partial \rho_0}{\partial z} &= \frac{\partial}{\partial z}\left(\tilde{\rho} e^{-z/H}\right) \\ &= -\frac{1}{H}\tilde{\rho} e^{-z/H} \\ &= -\frac{\rho_0}{H} \end{aligned}$$

So,

$$\frac{\partial^2 \Delta\rho}{\partial t^2} - c_u^2\frac{\partial^2 \Delta\rho}{\partial z^2} - \frac{c_u^2}{H}\frac{\partial \Delta\rho}{\partial z} = 0$$

Look for solutions of the form $\Delta\rho \propto e^{\mathrm{i}(kz-\omega t)}$

$$\begin{aligned} \Rightarrow \qquad & -\omega^2 = -c_u^2 k^2 + c_u^2 \frac{\mathrm{i}k}{H} \\ \Rightarrow \qquad & \boxed{\omega^2 = c_u^2\left(k^2 - \frac{\mathrm{i}k}{H}\right)} \qquad \textsc{Dispersion Relation} \end{aligned}$$

We can also write this as

$$k^2 - \frac{\mathrm{i}k}{H} - \frac{\omega^2}{c_u^2} = 0$$

and solve the quadratic for $k(\omega)$:

$$k = \frac{\mathrm{i}}{2H} \pm \sqrt{\frac{\omega^2}{c_u^2} - \frac{1}{4H^2}}$$

Let's take $\omega \in \mathbb{R}$. We have two cases to examine if we wish to understand the implications of this dispersion relation.

Case I: $\omega > c_u/2H$

Examine the real and imaginary parts of k:

$$\operatorname{Im} k = \frac{1}{2H}$$
$$\operatorname{Re} k = \pm\sqrt{\left(\frac{\omega}{c_u}\right)^2 - \left(\frac{1}{2H}\right)^2}$$

So the density perturbation is

$$\Delta\rho \propto \underbrace{e^{-z/2H}}_{(\dagger)} \underbrace{e^{\mathrm{i}\left(\pm\sqrt{(\omega/c_u)^2-(1/2H)^2}z-\omega t\right)}}_{(\ddagger)}$$

corresponding to

(†) Exponentially decaying amplitude with increasing height;

(‡) Wave with phase velocity

$$v_{\mathrm{ph}} = \frac{\omega}{\mathbb{K}}, \quad \mathbb{K} \equiv \pm\sqrt{\left(\frac{\omega}{c_u}\right)^2 - \left(\frac{1}{2H}\right)^2}$$

where v_{ph} is function of ω, meaning that the wave is dispersive. Wave packet consisting of different ω's will change shape as it propagates.

As before, we can relate Δu to $\Delta\rho$:

$$\Delta u_z = \frac{\Delta\rho}{\rho_0}\frac{\omega}{k}$$

with

$$\Delta\rho \propto e^{-z/2H}$$
$$\rho_0 \propto e^{-z/H}$$

giving

$$\Delta u_z \propto e^{+z/2H}, \qquad \frac{\Delta\rho}{\rho_0} \propto e^{+z/2H}$$

Thus the perturbed velocity and the fractional density variation both *increase* with height. In the absence of dissipation (e.g. viscosity), the kinetic energy flux ($\propto \Delta\rho\Delta u$) is conserved and the amplitude of the wave increases until

$$\Delta u \sim c_s, \qquad \frac{\Delta\rho}{\rho_0} \sim 1$$

where the linear treatment breaks down and the sound wave "steepens" into a shock. So, in the absence of dissipation, an upward propagating sound wave from a hand clapping would generate shocks in the upper atmosphere!

Case II: $\omega < c_u/2H$

In this case, we find that k is purely imaginary. So,

$$\Delta\rho \propto e^{|k|z} e^{\mathrm{i}\omega t}$$

This is a non-propagating, evanescent wave. In essence the wave cannot propagate since the properties of the atmosphere change significantly over one wavelength, giving rise to reflections.

F.3 Transmission of Sound Waves at Interfaces

Consider two non-dispersive media with a boundary at $x = 0$. Suppose we have a sound wave travelling from $x < 0$ to $x > 0$. Let the incident wave have unity amplitude (in, say, the density perturbation), and denote by r and t the amplitude of the reflected and transmitted waves, respectively:

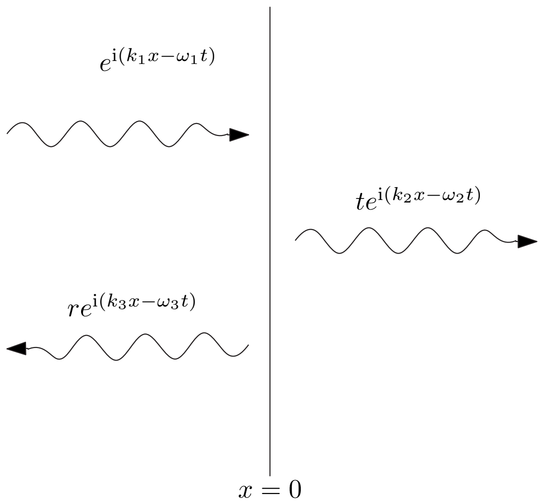

Figure F.1: Waves at boundary $x = 0$

At the boundary $x = 0$, variables must be single valued and the accelerations are finite, thus oscillates in the second medium must have the same frequency

$$\therefore \quad \omega_1 = \omega_2 = \omega_3 = \omega.$$

The reflected wave is in the same medium as the incident

$$\therefore \quad k_3 = -k_1. \qquad \text{(phase speed reversed)}$$

Amplitude of sound wave continuous at $x = 0$

$$\therefore \quad 1 + r = t,$$

and the derivative of the amplitude is continuous at $x = 0$

$$\therefore \quad k_1(1 - r) = k_2 t.$$

We can combine these relations to get

$$t = \frac{2k_1}{k_1 + k_2} \qquad r = \frac{k_1 - k_2}{k_1 + k_2}$$

From these relations we can see that the reflection/transmission of sound waves strongly depends on the relative sound speeds in the two media:

(i) If $c_{s,2} > c_{s,1} \Rightarrow k_2 < k_1 \Rightarrow r > 0$, i.e reflected wave in phase with incident;

(ii) If $c_{s,2} < c_{s,1} \Rightarrow r < 0 \Rightarrow$ reflected wave is π out of phase with incident wave;

(iii) If $c_{s,2} \ll c_{s,1} \Rightarrow k_2 \gg k_1 \Rightarrow t \ll 1$, i.e. wave almost completely reflected.

F.4 Supersonic Fluids and Shocks

Shocks occur when there are disturbances in the fluid caused by compression by a large factor, or acceleration to velocities comparable to or exceeding c_s The linear theory applied to sound waves breaks down.

When thinking about the sound speed, recall that the chemical composition of the fluid matters, $c_s \propto \mu^{-1/2}$

$$c_s \text{ in} \underbrace{\text{atomic Hydrogen}}_{\substack{\mu\approx 1 \\ \text{e.g. ISM}}} \gg \quad c_s \text{ in} \underbrace{\text{diatomic Nitrogen}}_{\substack{\mu=28 \\ \text{e.g. Earth} \\ \text{atmosphere}}} \qquad \text{for given } T$$

Disturbances in a fluid always propagate at the sound speed relative to the fluid itself. Consider an observer at the centre of a spherical disturbance, watching the fluid flow past is at speed v.

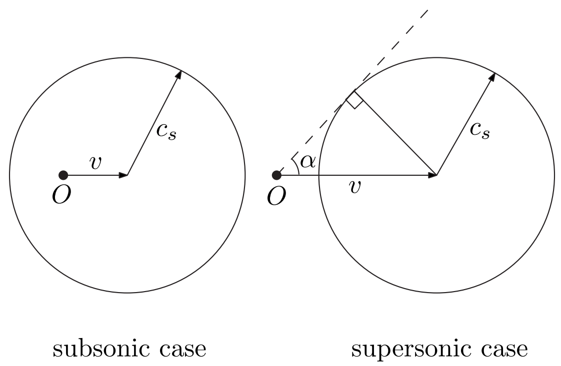

Figure F.2: Subsonic flow vs supersonic flow

The velocity of the disturbance relative to the observer, v', is the vector sum of the fluid velocity and the disturbance velocity relative to the fluid.

- Subsonic case: v' sweeps 4π steradians;
- Supersonic case: disturbance always to the right. If we continuously produce a disturbance, the envelope of the disturbances will define a cone with opening angle α given by

$$\sin\alpha = \frac{c_s}{v} \qquad \text{Mach Cone}$$

The ratio of the flow speed to the sound speed is called the Mach number

$$M \equiv \frac{v}{c_s}$$

$$\therefore \quad \sin\alpha = \frac{1}{M}$$

Imagine an obstacle in a supersonic flow — disturbances cannot propagate upstream from the obstacle so the flow cannot adjust to the presence of obstacle. The flow properties must change discontinuously once the obstacle is reached, giving shock!

F.5 The Rankine-Hugoniot Relations

We analyse a shock by applying conservation of mass, momentum and energy across the shock front.

In the frame of the shock, lets assume following geometry

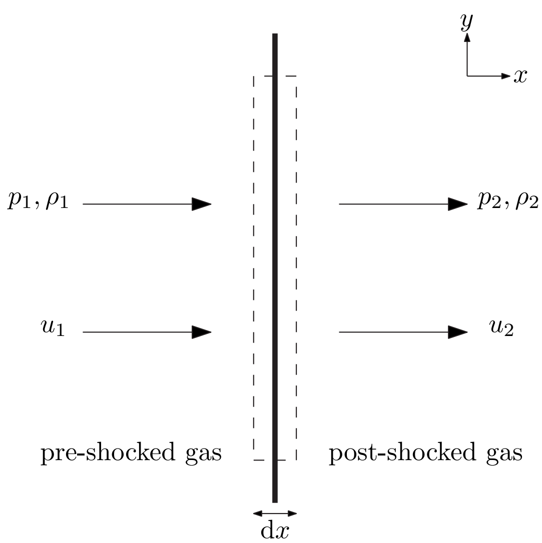

Figure F.3: Geometry of shock front

Continuity gives

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_x) = 0 \\
\Rightarrow\quad &\frac{\partial}{\partial t}\left(\int_{-\mathrm{d}x/2}^{\mathrm{d}x/2} \rho\,\mathrm{d}x\right) + \rho u_x\Big|_{x=\mathrm{d}x/2} - \rho u_x\Big|_{x=-\mathrm{d}x/2} = 0
\end{aligned}
$$

where we have integrated over a small region $\mathrm{d}x$ around the shock.

Let's take $\mathrm{d}x \to 0$ and assume that mass does not continually accumulate at $x = 0$. Then

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\int \rho\,\mathrm{d}x\right) = 0 \\
\Rightarrow\quad &\boxed{\rho_1 u_1 = \rho_2 u_2} \qquad \text{1st Rankine-Hugoniot Relation}
\end{aligned}
$$

Apply similar analysis to the momentum equation:

$$
\begin{aligned}
&\frac{\partial}{\partial t}(\rho u_x) = -\frac{\partial}{\partial x}(\rho u_x u_x + p) - \rho\frac{\partial \Psi}{\partial x} \\
\Rightarrow\quad &\frac{\partial}{\partial t}\left(\int \rho u_x\,\mathrm{d}x\right) = -\,(\rho u_x u_x + p)\Big|_{\mathrm{d}x/2} + (\rho u_x u_x + p)\Big|_{-\mathrm{d}x/2} \\
\Rightarrow\quad &\boxed{\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2} \qquad \text{2nd R-H Relation}
\end{aligned}
$$

We note that u_y and u_z do not change across the shock front (can be immediately seen by looking at the y- and z-components of the momentum equation).

Now for the energy equation. Start with the *adiabatic case* so that the gas cannot cool and hence we have $\dot{Q}_{\mathrm{cool}} = 0$. Also take gravitational potential to have no time-dependence. Then

$$
\begin{aligned}
&\frac{\partial E}{\partial t} + \boldsymbol{\nabla}\cdot[(E+p)\mathbf{u}] = \cancelto{0}{-\rho\dot{Q}_{\mathrm{cool}}} + \rho\cancelto{0}{\frac{\partial \Psi}{\partial t}} \\
\Rightarrow\quad &\frac{\partial E}{\partial t} + \boldsymbol{\nabla}\cdot[(E+p)\mathbf{u}] = 0 \\
\Rightarrow\quad &\frac{\partial}{\partial t}\left(\int E\,\mathrm{d}x\right) + (E+p)u_x\Big|_{\mathrm{d}x/2} - (E+p)u_x\Big|_{-\mathrm{d}x/2} = 0 \\
\Rightarrow\quad &(E_1 + p_1)u_1 = (E_2 + p_2)u_2
\end{aligned}
$$

Since $E = \rho\left(\frac{1}{2}u^2 + \mathcal{E} + \Psi\right)$, this becomes

$$
\begin{aligned}
&\frac{1}{2}\rho_1 u_1^3 + \rho_1\mathcal{E}_1 u_1 + \rho_1\Psi_1 u_1 + p_1 u_1 \\
&\qquad\qquad = \frac{1}{2}\rho_2 u_2^3 + \rho_2\mathcal{E}_2 u_2 + \rho_2\Psi_2 u_2 + p_2 u_2
\end{aligned}
$$

But $\Psi_1 = \Psi_2$ and $\rho_1 u_1 = \rho_2 u_2$, so terms involving Ψ cancel out. We are left with

$$\boxed{\frac{1}{2}u_1^2 + \mathcal{E}_1 + \frac{p_1}{\rho_1} = \frac{1}{2}u_2^2 + \mathcal{E}_2 + \frac{p_2}{\rho_2}} \qquad \text{3rd R-H Relation}$$

For an ideal gas, we have

$$\left.\begin{aligned} \mathcal{E} &= C_V T \\ p &= \frac{\mathcal{R}_*}{\mu}\rho T \end{aligned}\right\} \quad \Rightarrow \quad \mathcal{E} = \frac{C_V \mu}{\mathcal{R}_*}\frac{p}{\rho}$$

$$\left.\begin{aligned} \gamma &= \frac{C_p}{C_V} \\ C_p - C_V &= \frac{\mathcal{R}_*}{\mu} \end{aligned}\right\} \quad \Rightarrow \quad C_V(\gamma - 1) = \frac{\mathcal{R}_*}{\mu}$$

which combine to give

$$\boxed{\mathcal{E} = \frac{1}{\gamma - 1}\frac{p}{\rho}} \qquad \text{(internal energy per unit mass)}$$

If we assume that γ does not change across the shock (e.g. there are no disassociation of molecules), the 3rd R-H relation becomes

$$\begin{aligned} & \frac{1}{2}u_1^2 + \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} = \frac{1}{2}u_2^2 + \frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2} \\ \Rightarrow \qquad & \frac{1}{2}u_1^2 + \frac{c_{s,1}^2}{\gamma - 1} = \frac{1}{2}u_2^2 + \frac{c_{s,2}^2}{\gamma - 1} \end{aligned}$$

since, for adiabatic case, the sound speed is

$$c_s^2 = \left.\frac{\partial p}{\partial \rho}\right|_S = \frac{\gamma p}{\rho}$$

Using all three R-H relations and after some algebra we get

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma + 1)p_2 + (\gamma - 1)p_1}{(\gamma + 1)p_1 + (\gamma - 1)p_2}$$

In the limit of strong shocks, $p_2 \gg p_1$, we get

$$\frac{\rho_2}{\rho_1} \to \frac{\gamma + 1}{\gamma - 1}$$

For $\gamma = 5/3$, this gives $\rho_2 = 4\rho_1$. So there is a maximum possible density contrast across an adiabatic shock — with stronger and stronger shocks, the thermal pressure of the shocked gas increases and prevents further compression.

Note that, since $p_2 \gg p_1$, and $\rho_2 \leq 4\rho_1$, we have

$$\frac{p_1}{\rho_1^\gamma} \neq \frac{p_2}{\rho_2^\gamma} \quad \text{i.e. } K_1 \neq K_2$$

The gas has jumped adiabats during its passage through the shock. Shocking the gas produces a non-reversible change, due to viscous processes operating within shock.

While the R-H conditions are symmetric in the up- and down-stream quantities, the thermodynamic requirement that entropy increases dictates the direction of the jump (i.e. a fast/cold upstream flow shocking to produce a slow/fast downstream flow).

It is interesting that we can derive R-H conditions using the inviscid equations that do not explicitly include dissipation/entropy-generating terms.

Not all shocks are adiabatic! To consider the other extreme, let's discuss *isothermal shocks*. Here we have $\dot{Q}_{\rm cool} \neq 0$ such that the shocked gas cools to produce $T_2 = T_1$. Whether a shock is isothermal or adiabatic depends on whether the "cooling length" is smaller or larger than the system size, respectively.

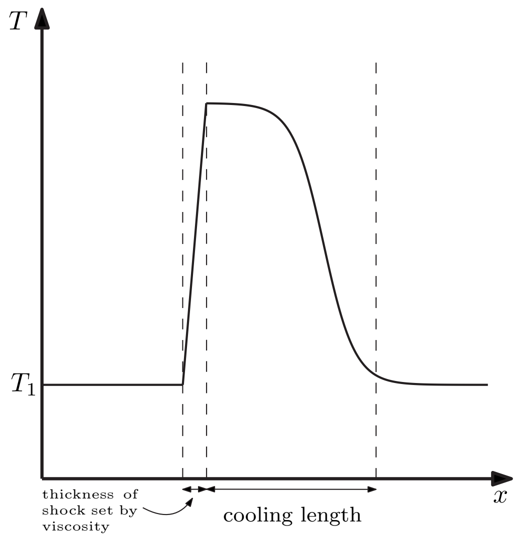

Figure F.4: Temperature profile through a shock

For isothermal shocks, the first two R-H equations are unchanged:

$$\rho_1 u_1 = \rho_2 u_2$$
$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$

but the 3rd R-H equation is replaced by

$$T_1 = T_2$$

Now,

$$
\begin{aligned}
c_{s,\mathrm{I}} &= \sqrt{\frac{\mathcal{R}_* T}{\mu}} \quad \Rightarrow \quad c_{s,1} = c_{s,2} \\
&= \sqrt{\frac{p}{\rho}} \quad \Rightarrow \quad p = c_{s,\mathrm{I}}^2 \rho
\end{aligned}
$$

So, 2nd R-H equation becomes

$$
\begin{aligned}
& \rho_1(u_1^2 + c_s^2) = \rho_2(u_2^2 + c_s^2) \\
\Rightarrow \quad & u_1 + \frac{c_s^2}{u_1} = u_2 + \frac{c_s^2}{u_2} \quad (\text{since } \rho_1 u_1 = \rho_2 u_2) \\
\Rightarrow \quad & c_s^2 \left(\frac{1}{u_1} - \frac{1}{u_2} \right) = u_2 - u_1 \\
\Rightarrow \quad & c_s^2 \frac{u_2 - u_1}{u_1 u_2} = u_2 - u_1 \\
\Rightarrow \quad & \boxed{c_s^2 = u_1 u_2}
\end{aligned}
$$

Thus we see that

$$
\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \left(\frac{u_1}{c_s} \right)^2 = M_1^2
$$

where M_1 is the Mach number of the upstream flow. So the density compression can be very large.

Note that since $c_s^2 = u_1 u_2$ and $u_1 > c_s$ (condition for a shock), we must have $u_2 < c_s$. So flow behind the shock is subsonic. In fact this is always true for any shock and is necessary to preserve causality (the post shock gas must know about the shock!).

F.6 Theory of Supernova Explosions

An important application of shock wave theory is to supernova explosions in the interstellar medium (ISM). A supernova (SN) deposits about 10^{51} erg ($= 10^{44}$ J) of energy into the surrounding medium, then shocked medium expands, sweeps up more gas, and creates large bubbles in the ISM.

Consider following system:

- Uniform density medium with density ρ_0;
- Point explosion with energy E;
- Ignore temperature of the ambient ISM ($T_0 = 0$), thus no confinement of explosion by an external pressure.

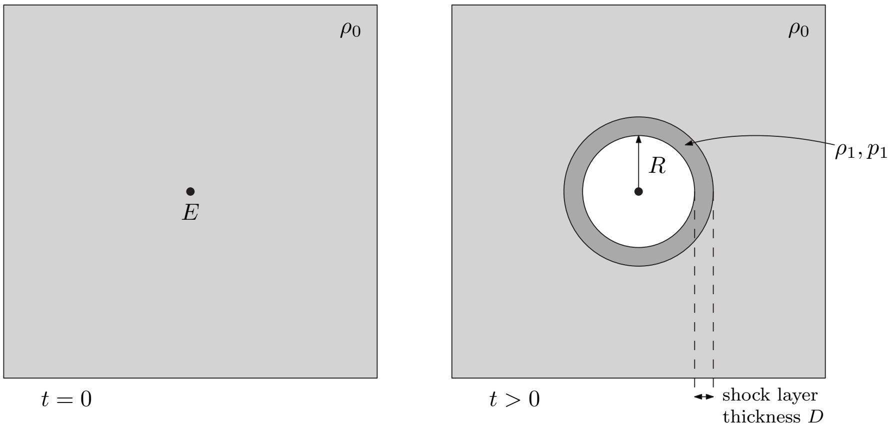

Figure F.5: Supernova explosion

Given that $T_0 = 0$, the shock has $M \to \infty$. Assuming an adiabatic shock, we sweep mass into a shell with density ρ, given by

$$\rho_1 = \rho_0 \frac{\gamma + 1}{\gamma - 1}$$

If all mass is swept up into shell then

$$\frac{4\pi}{3}\rho_0 R^3 = 4\pi\rho_1 R^2 D \qquad \text{(assuming } D \ll R\text{)}$$

$$\Rightarrow \qquad D = \frac{1}{3}\left(\frac{\gamma - 1}{\gamma + 1}\right) R$$

For $\gamma = 5/3$, we have $D \approx 0.08R$ which justifies the assumption $D \ll R$.

Assume that all gas in the shell moves with a common velocity. In the frame of a local patch of the shock we have

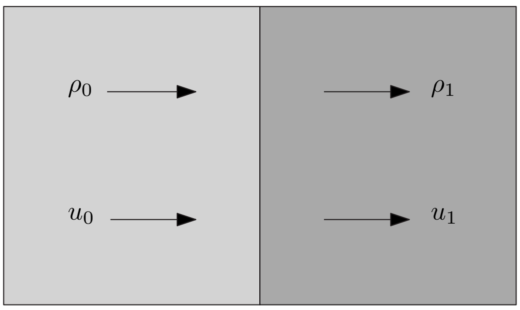

Figure F.6: Situation seen in shock frame and so

$$
\begin{aligned}
& \rho_0 u_0 = \rho_1 u_1 \\
\Rightarrow \qquad & u_1 = \frac{\rho_0}{\rho_1} u_0 = \frac{\gamma - 1}{\gamma + 1} u_0. \qquad ① \qquad \text{eq.f.6.1}
\end{aligned}
$$

Thus, relative to the unshocked gas, the velocity of the shocked gas U is

$$
U = u_0 - u_1 = \frac{2u_0}{\gamma + 1}.
$$

Then, the rate of change of momentum of the shocked shell is

$$
\frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{4\pi}{3} \rho_0 R^3 \frac{2u_0}{\gamma + 1} \right]
$$

This momentum gain is provided by pressure acting on the inside surface of the shell — call this p_{in}. Let's make the ansatz that this is related to the pressure within the shell by

$$
p_{\text{in}} = \alpha p_1,
$$

and we now relate p_1 and u_0 using the R-H jump condition: we have

$$
\begin{aligned}
& p_0 + \rho_0 u_0^2 = p_1 + \rho_1 u_1^2 \\
\Rightarrow \qquad & p_1 = \rho_0 u_0^2 \left[1 - \frac{\rho_1 u_1^2}{\rho_0 u_0^2} \right] \quad (\text{since } p_0 = 0 \text{ by assumption }) \\
& \quad = \rho_0 u_0^2 \left[1 - \frac{\gamma - 1}{\gamma + 1} \right] \quad (\text{assuming a strong shock}) \\
& \quad = \frac{2}{\gamma + 1} \rho_0 u_0^2 \qquad ② \qquad \text{eq.f.6.2}
\end{aligned}
$$

So, equating rate of change of momentum of the shocked shell to the pressure acting on the inside surface of the shell, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{4\pi}{3} \rho_0 R^3 \frac{2u_0}{\gamma + 1} \right] &= 4\pi R^2 p_{\text{in}} \\
&= 4\pi R^2 \alpha p_1 \\
&= 4\pi R^2 \frac{2}{\gamma + 1} \rho_0 u_0^2
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \qquad & \frac{\mathrm{d}}{\mathrm{d}t} \left[R^3 u_0 \right] = 3\alpha R^2 u_0^2 \\
\Rightarrow \qquad & \frac{\mathrm{d}}{\mathrm{d}t} \left[R^3 \dot{R} \right] = 3\alpha R^2 \dot{R}^2 \quad \text{since } u_0 \equiv \dot{R}
\end{aligned}
$$

This admits solutions of the form $R \propto t^b$:

$$\Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(t^{3b}bt^{b-1}\right) = 3\alpha t^{2b}\left(bt^{b-1}\right)^2$$

$$\Rightarrow \qquad b(4b-1)\cancel{t^{4b-2}} = 3\alpha b^2 \cancel{t^{4b-2}} \quad \text{(cancellation justifies assumed form of solution)}$$

$$\Rightarrow \qquad b = 0 \quad \text{(not physical)} \qquad \text{or} \qquad b = \frac{1}{4-3\alpha}$$

$$\Rightarrow \qquad R \propto t^{1/(4-3\alpha)}, \quad u_0 \propto t^{(3\alpha-3)/(4-3\alpha)} \propto R^{3\alpha-3}$$

To determine α, we need to consider energy conservation. For an adiabatic shock, the explosion energy is conserved and transformed into kinetic and internal energy:

- Kinetic energy of the shell is
$$\frac{1}{2}\cdot\frac{4\pi}{3}\rho_0 R^3 U^2$$
- Internal energy per unit mass is
$$\mathcal{E} = \frac{1}{\gamma-1}\frac{p}{\rho}$$
and so the internal energy per unit volume is
$$\rho\mathcal{E} = \frac{1}{\gamma-1}p$$
Since the shell is very thin, it has small volume and so most of the internal energy is in the central cavity which contains little mass
$$\text{Int. Energy of cavity} \approx \frac{4\pi}{3}R^3\frac{p_{\text{in}}}{\gamma-1} = \frac{4\pi}{3}R^3\alpha\frac{p_1}{\gamma-1}$$

So, energy conservation says that

$$\begin{aligned} E &= \frac{1}{2}\cdot\frac{4\pi}{3}\rho_0 R^3 U^2 + \frac{4\pi}{3}R^3\alpha\frac{p_1}{\gamma-1} \\ &= \frac{1}{2}\cdot\frac{4\pi}{3}\rho_0 R^3 \underbrace{\left(\frac{2u_0}{\gamma+1}\right)^2}_{①} + \frac{4\pi}{3}R^3\alpha \underbrace{\frac{2}{\gamma+1}\rho_0 u_0^2 \frac{1}{\gamma-1}}_{②} \\ &= \frac{4\pi}{3}R^3 u_0^2\left[\frac{1}{2}\rho_0\frac{4}{(\gamma+1)^2} + \alpha\rho_0\frac{2}{(\gamma+1)(\gamma-1)}\right], \end{aligned}$$

from which we conclude that

$$E \propto R^3 u_0^2 \propto t^{(6\alpha-3)/(4-3\alpha)}.$$

But E must be conserved. So we need $\alpha = 1/2$ to remove time dependence of E. Using $\alpha = 1/2$ we find

$$\boxed{R \propto t^{2/5}, \quad u_0 \propto t^{-3/5}, \quad p_1 \propto t^{-6/5}}$$

Similarity Solutions

The above problem only has 2 parameters, E and ρ_0. Look at their dimensions

$$[E] = \frac{ML^2}{T^2}, \quad [\rho_0] = \frac{M}{L^3}$$

These cannot be combined to give quantities with the dimension of length or time. So, there is no natural length scale or time scale in the problem!

Given some time t, the only way to combine E, ρ_0 and t to give a length scale is

$$\lambda = \left(\frac{Et^2}{\rho_0}\right)^{1/5}$$

We can define a dimensionless distance parameter

$$\xi \equiv \frac{r}{\lambda} = r\left(\frac{\rho_0}{Et^2}\right)^{1/5}$$

Then, for any variable in the problem $X(r,t)$, we will have

$$X = X_1(t)\tilde{X}(\xi)$$

i.e. X is a function of scaled distance ξ always has the same shaped scaled up/down by the time dependence factor $X_1(t)$.

So,

$$\begin{aligned}
\frac{\partial X}{\partial r} &= X_1 \frac{\mathrm{d}\tilde{X}}{\mathrm{d}\xi} \left.\frac{\partial \xi}{\partial r}\right|_t \\
\frac{\partial X}{\partial t} &= \tilde{X}(\xi)\frac{\mathrm{d}X_1}{\mathrm{d}t} + X_1 \frac{\mathrm{d}\tilde{X}}{\mathrm{d}\xi} \left.\frac{\partial \xi}{\partial t}\right|_r
\end{aligned}$$

ξ is neither a Lagrangian or an Eulerian coordinate. It labels a particular feature in the flow (e.g. shock wave) that can move through the fluid. So we can write

$$R_{\text{shock}} \propto \left(\frac{E}{\rho_0}\right)^{1/5} t^{2/5}$$

Let's put some numbers in for the case of supernova explosions,

$$\begin{aligned}
R(t) &= \xi_0 \left(\frac{E}{\rho_0}\right)^{1/5} t^{2/5} \quad \text{(we will assume } \xi_0 \sim 1\text{)} \\
u_0(t) &= \frac{\mathrm{d}R}{\mathrm{d}t} = \frac{2}{5}\xi_0 \left(\frac{E}{\rho_0 t^3}\right)^{1/5} = \frac{2}{5}\frac{R}{t}
\end{aligned}$$

In a supernova we have

$$\begin{aligned}
E &\approx 10^{44} \text{ J} = 10^{51} \text{ erg} \\
\rho_0 &= \rho_{\text{ISM}} \approx 10^{-21} \text{ kg m}^{-3}
\end{aligned}$$

So similarity solution gives

$$\left.\begin{aligned} R &\approx 0.3 t^{2/5} \text{ pc} \\ u_0 &\approx 10^5 t^{-3/5} \text{ km s}^{-1} \end{aligned}\right\} \text{where } t \text{ is measured in yrs}$$

The original explosion injects the stellar debris at about 10^4 km s^{-1}. So the above solution is valid for

$$\begin{aligned} t &\gtrsim 100 \text{ yr} \quad (\text{when } u_0 < u_{\text{inj}}) \\ t &\lesssim 10^5 \text{ yr} \quad (\text{after which energy losses become important}) \end{aligned}$$

Structure of the Blast Wave

We can, in principle, write each variable ρ, p, u, r in terms of separated functions of t and ξ. We can then substitute into the Eulerian equation of fluid dynamics (in spherical coordinates with $\partial/\partial\phi = \partial/\partial\theta = 0$, i.e. spherical symmetry).

The result is a set of ODE's where ξ is the only dependent variable — the time dependence cancels out! (Sedov 1946)

E.g. solution for $\gamma = 7/5$:

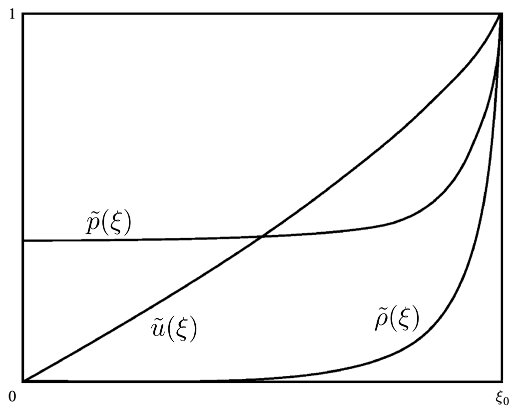

Figure F.7: Solution for $\gamma = 7/5$

These solutions tell us that

- Most of mass is swept up in a shell just behind the shock (from form of $\tilde{\rho}$);
- Post-shock pressure is indeed a multiple of p_{in} (from form of $\tilde{p}$ justifies $p_{\text{in}} = \alpha p_1$ assumption);
- Shell material is not really moving at a single velocity, but arguments above are restored by taking some weighted average (from form of $\tilde{u}$).

Breakdown of the Similarity Solution

The self similar solution breaks down when the surrounding medium pressure p_0 becomes significant, $p_1 \sim p_0$.

From the strong shock solution, we derived

$$p_1 = \frac{2}{\gamma+1}\rho_0 u_0^2, \quad c_s^2 = \frac{\gamma p_0}{\rho_0}$$

So if $p_1 \sim p_0$ then

$$\begin{aligned} & \frac{2}{\gamma+1}\rho_0 u_0^2 \sim \frac{\rho_0 c_s^2}{\gamma} \\ \Rightarrow \quad & u_0 \sim c_s \end{aligned}$$

i.e. shell not moving supersonically anymore.

The blast wave weakens to a sound wave:

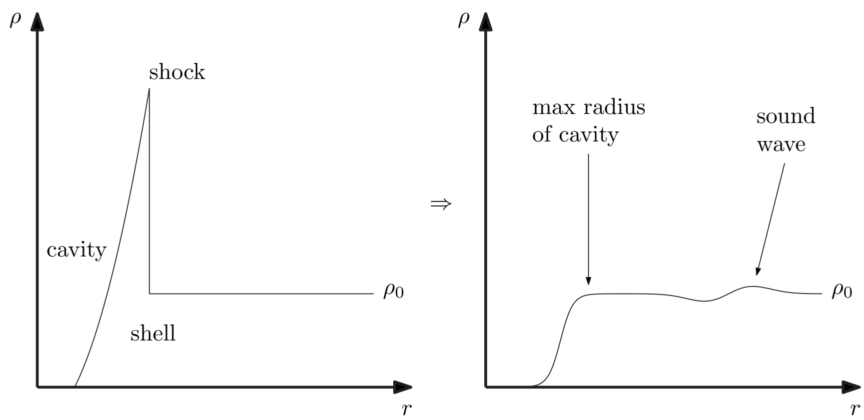

Figure F.8: Blast wave phase *vs* late phase

As a sound wave, disturbance passes into the undisturbed gas as a mild compression followed by a rarefaction. After the sound wave passes, gas returns to the original state.

For SN, the maximum bubble/cavity size is set by the radius when blast wave becomes sonic and $p_1 \sim p_0$. We've just shown that this implies

$$u_0^2 \sim \frac{\gamma+1}{2\gamma} c_s^2$$

We showed above that energy conservation gives

$$\begin{aligned}
E &= \frac{4\pi}{3}R^3\left[\frac{1}{2}\rho_0\left(\frac{2u_0}{\gamma+1}\right)^2 + \frac{\alpha}{\gamma-1}\frac{2\rho_0 u_0^2}{\gamma+1}\right] \quad \text{with } \alpha = \frac{1}{2} \\
&= \frac{4\pi}{3}R^3\rho_0 u_0^2\left[\frac{2}{(\gamma+1)^2} + \frac{1}{(\gamma-1)(\gamma+1)}\right] \\
&= \frac{4\pi}{3}R^3\rho_0 u_0^2\left[\frac{2(\gamma-1)+\gamma+1}{(\gamma+1)^2(\gamma-1)}\right] \\
&= \frac{4\pi}{3}R^3\rho_0 u_0^2\frac{3\gamma-1}{(\gamma+1)^2(\gamma-1)}
\end{aligned}$$

$$\Rightarrow \qquad u_0^2 = \frac{(\gamma+1)(\gamma^2-1)}{3\gamma-1}\cdot\frac{3E}{4\pi\rho_0 R^3} \quad \underbrace{\sim \frac{\gamma+1}{2\gamma}c_s^2}_{\text{when blast wave becomes sonic and } p_1\sim p_0}$$

$$\Rightarrow \qquad E \sim \frac{4\pi}{3}\rho_0 R_{\max}^3\frac{c_s^2}{2\gamma}\cdot\frac{3\gamma-1}{\gamma^2-1}$$

Internal energy initially contained within $R_{\max}$ is

$$E_{\text{init}} = \frac{4\pi}{3}R_{\max}^3\frac{p_0}{\gamma-1} = \frac{4\pi}{3}R_{\max}^3\rho_0\frac{c_s^2}{\gamma(\gamma-1)}$$

So, when $p_0 \sim p_1$, we have $E \sim E_{\text{init}}$. Therefore, blast wave propagates until the explosion energy is comparable to the internal energy in the sphere!

Some numbers:

- Timescale on which the bubble reaches $R_{\max}$ is roughly the sound crossing time
$$t_s \sim \frac{R_{\max}}{c_s}$$
For ISM: $T \sim 10^4$ K, $\rho \sim 10^{-21}$ kg m^{-3}, giving
$$\begin{aligned} R_{\max} &\sim \text{ few} \times 100 \text{ pc} \\ t_{\max} &\sim 10 \text{ Myr} \end{aligned}$$

- SN rate is about 10^{-7} Myr^{-1} pc^{-3}. So, over a duration $t_{\max}$, can find 1 SN in $\sim 10^6$ pc^3. But
$$\frac{4\pi}{3}R_{\max}^3 > 10^6 \text{ pc}^3$$
so filling factor of SN driven bubbles is > 1. This would seem to suggest that the entire ISM would be heated to SN to $> 10^6$ K. Not Observed!

  We need to account for cooling and the finite height of the Galactic disk (i.e. bubble "blow out"). After 10^5 yrs, when $R \sim 20$ pc, cooling losses become important and so the bubble grows more slowly than $R \propto t^{2/5}$. Simulations show that $R \propto t^{0.3}$ and $R_{\rm max} \sim 50$ pc, giving filling factor < 1. Thus, due to cooling, only a small fraction of E is deposited into ISM.

CHAPTER G

Bernoulli's Equation and Transonic Flows

G.1 Bernoulli's Equation

Let's start with the momentum equation:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p - \boldsymbol{\nabla}\Psi. \qquad (*)$$

eq.g.1.1

If the fluid is barotropic, then $p = p(\rho)$ and so

$$\begin{aligned} & \frac{\partial}{\partial x}\int \frac{\mathrm{d}p}{\rho} = \frac{\partial p}{\partial x}\frac{\mathrm{d}}{\mathrm{d}p}\int \frac{\mathrm{d}p}{\rho} = \frac{1}{\rho}\frac{\partial p}{\partial x} \\ \Rightarrow \qquad & \frac{1}{\rho}\boldsymbol{\nabla} p = \boldsymbol{\nabla}\left(\int \frac{\mathrm{d}p}{\rho}\right). \end{aligned}$$

Also, we have the vector identity

$$(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = \boldsymbol{\nabla}\left(\frac{1}{2}u^2\right) - \mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{u}). \qquad \text{(Ex. Sheet 1)}$$

Using these, the momentum equation $(*)$ becomes

$$\frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\nabla}\left(\frac{1}{2}u^2\right) - \mathbf{u} \times \mathbf{w} = -\boldsymbol{\nabla}\left[\int \frac{\mathrm{d}p}{\rho} + \Psi\right] \qquad (**)$$

eq.g.1.2

where we have defined the vorticity:

$$\boxed{\mathbf{w} = \boldsymbol{\nabla} \times \mathbf{u}} \qquad \text{Definition of Vorticity}$$

Now, assume a steady flow $(\partial \mathbf{u}/\partial t = 0)$ and take the dot product of $(**)$ with velocity $\mathbf{u}$. Since we have $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{w}) = 0$ always, the result is

$$\mathbf{u} \cdot \boldsymbol{\nabla}\left[\frac{1}{2}u^2 + \int \frac{\mathrm{d}p}{\rho} + \Psi\right] = 0$$

This gives us *Bernoulli's Principle*: For steady barotropic flows, the quantity

$$H = \frac{1}{2}u^2 + \int \frac{\mathrm{d}p}{\rho} + \Psi$$

is constant along a streamline. The quantity H is called Bernoulli's constant.

If $p = 0$, $H =$ constant is the statement that kinetic + potential energy is constant along streamlines.

If $p \neq 0$, pressure differences accelerate or decelerate the flow as it flows along the streamline.

Everyday examples of Bernoulli's equation at work:

Example (the apocryphal Aircraft wing)**.**

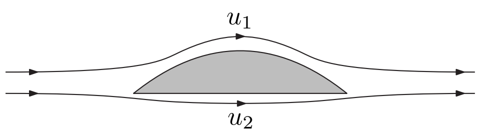

Figure G.1: Aircraft wing

$$\begin{array}{lcll} u_1 > u_2 & \Rightarrow & p_1 < p_2 & \text{from } H \\ & \Rightarrow & \text{pressure difference} & \\ & \Rightarrow & \text{lift force.} & \end{array}$$

Of course, this cannot be the whole story of how aircraft wings work or else inverted flight would be impossible!

Example (Shower curtain)**.**

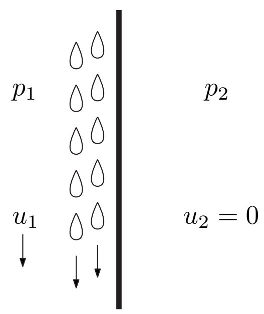

Figure G.2: Shower curtain

Downward flow of air on inside of curtain induced by falling water

$$\begin{aligned} &\Rightarrow \quad p_1 < p_2 \\ &\Rightarrow \quad \text{curtain blows inwards.} \end{aligned}$$

G.2 Rotational and Irrotational Flows

An *irrotational flow* is one in which $\boldsymbol{\nabla} \times \mathbf{u} = 0$ everywhere, i.e. the vorticity $\mathbf{w} = 0$ everywhere.

For a steady irrotational flow, $(**)$ gives that

$$\boldsymbol{\nabla} H = 0$$

so, $H =$ constant everywhere (not just along streamlines).

For a general (not necessarily irrotational or steady state) flow, we have

$$\frac{\partial \mathbf{u}}{\partial t} = -\boldsymbol{\nabla} H + \mathbf{u} \times \mathbf{w}$$

Take curl:

$$\frac{\partial}{\partial t} \underbrace{(\boldsymbol{\nabla} \times \mathbf{u})}_{\mathbf{w}} = \underbrace{-\boldsymbol{\nabla} \times (\boldsymbol{\nabla} H)}_{\equiv 0} + \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{w})$$

$$\Rightarrow \quad \boxed{\frac{\partial \mathbf{w}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{w})} \qquad \text{Helmholtz's Eqn.}$$

From Helmholtz's equation, we observe three results:

(i) If $\mathbf{w} = 0$ initially, it will stay zero thereafter. We will see later that this is no longer true once we include viscous terms.

(ii) The flux of vorticity through a surface S that moves with the fluid is a constant, i.e.

$$\frac{\mathrm{D}}{\mathrm{D}t} \int_S \mathbf{w} \cdot \mathrm{d}\mathbf{S} = 0$$

Proof. We have

$$\frac{\mathrm{D}}{\mathrm{D}t} \int_S \mathbf{w} \cdot \mathrm{d}\mathbf{S} = \underbrace{\int_S \frac{\partial \mathbf{w}}{\partial t} \cdot \mathrm{d}\mathbf{S}}_{\text{intrinsic changes in } \mathbf{w}} + \underbrace{\int \mathbf{w} \cdot \frac{\mathrm{D}}{\mathrm{D}t} \mathrm{d}\mathbf{S}}_{\substack{\text{change in } S \\ \text{caused by flow}}}$$

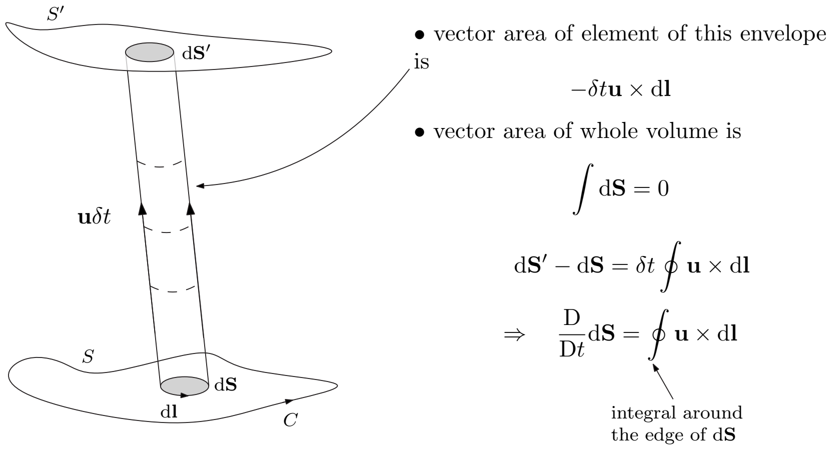

Figure G.3: Change of area element with time

So,

$$\begin{aligned}
\int_S \mathbf{w} \cdot \frac{\mathrm{D}}{\mathrm{D}t}\,\mathrm{d}\mathbf{S} &= \int_S \oint_{\partial \mathrm{d}\mathbf{S}} \mathbf{w} \cdot (\mathbf{u} \times \mathrm{d}\mathbf{l}) \\
&= \int_S \oint_{\partial \mathrm{d}\mathbf{S}} \mathbf{w} \times \mathbf{u} \cdot \mathrm{d}\mathbf{l} \\
&= \int_C \mathbf{w} \times \mathbf{u} \cdot \mathrm{d}\mathbf{l} \qquad \text{since “internal loops” cancel out} \\
&= \int_S \boldsymbol{\nabla} \times (\mathbf{w} \times \mathbf{u}) \cdot \mathrm{d}\mathbf{S}
\end{aligned}$$

$$\Rightarrow \qquad \frac{\mathrm{D}}{\mathrm{D}t}\int_S \mathbf{w} \cdot \mathrm{d}\mathbf{S} = \int_S \mathrm{d}\mathbf{S} \cdot \underbrace{\left(\frac{\partial \mathbf{w}}{\partial t} - \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{w})\right)}_{\substack{=0 \text{ from the} \\ \text{Helmholtz's eqn.}}}$$

$$\Rightarrow \qquad \frac{\mathrm{D}}{\mathrm{D}t}\int_S \mathbf{w} \cdot \mathrm{d}\mathbf{S} = 0$$

i.e. flux of vorticity is conserved and moves with the fluid. This is *Kelvin's vorticity theorem.*

□

(iii) For an irrotational flow, the fact that $\boldsymbol{\nabla} \times \mathbf{u} = 0$ everywhere implies that there exists a potential function Φ_u such that

$$\mathbf{u} = -\boldsymbol{\nabla}\Phi_u$$

If such a flow is also incompressible, then $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$ and so

$$\nabla^2 \Phi_u = 0$$

i.e. can reduce problem of finding velocity field to that of solving Laplace's equation.

G.3 The De Laval Nozzle

Consider steady flow in a tube with a variable cross-section $A(z)$;

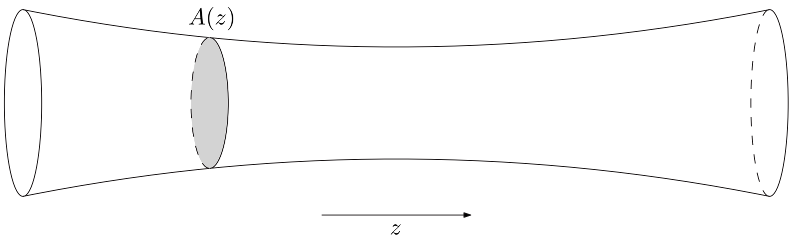

Figure G.4: Tube with variable cross-section

For a *steady flow*, mass conservation gives

$$\begin{aligned} & \rho u A = \text{constant } \dot{M} \qquad \text{(mass flow per second)} \\ \Rightarrow \quad & \ln \rho + \ln u + \ln A = \ln \dot{M} \\ \Rightarrow \quad & \frac{1}{\rho} \boldsymbol{\nabla} \rho + \boldsymbol{\nabla} \ln u + \boldsymbol{\nabla} \ln A = 0 \\ \Rightarrow \quad & \frac{1}{\rho} \boldsymbol{\nabla} \rho = -\boldsymbol{\nabla} \ln u - \boldsymbol{\nabla} \ln A \end{aligned}$$

and the momentum equation (with no gravity) gives

$$\mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} = -\frac{1}{\rho} \boldsymbol{\nabla} p$$

Let's further assume a *barotropic equation* of state. Then

$$\mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} = -\frac{1}{\rho} \frac{\mathrm{d}p}{\mathrm{d}\rho} \boldsymbol{\nabla} \rho$$

So, putting these pieces together gives

$$\mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} = \left[\boldsymbol{\nabla} \ln u + \boldsymbol{\nabla} \ln A \right] c_s^2 \qquad \left(c_s^2 = \mathrm{d}p / \mathrm{d}\rho \right) \qquad (\dagger)$$ eq.g.3.1

If flow is also irrotational, we have

$$(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = \boldsymbol{\nabla}\left(\frac{1}{2}u^2\right) = \frac{1}{2}u^2\boldsymbol{\nabla}\left(\ln u^2\right) = u^2\boldsymbol{\nabla}\ln u$$

and so, from (†) we have

$$\begin{aligned} & u^2\boldsymbol{\nabla}\ln u = [\boldsymbol{\nabla}\ln u + \boldsymbol{\nabla}\ln A]\, c_s^2 \\ \Rightarrow \qquad & (u^2 - c_s^2)\boldsymbol{\nabla}\ln u = c_s^2\boldsymbol{\nabla}\ln A \end{aligned}$$

This implies that an extremum of $A(z)$ must correspond to either

(a) Minimum or maximum in u, or

(b) $u = c_s$.

Thus, we see that there is the potential for a transition from subsonic to supersonic flow at a minimum or maximum of the cross-sectional area of the tube.

To make progress, we applying Bernoulli's equation

$$\frac{1}{2}u^2 + \int \frac{\mathrm{d}p}{\rho} = H, \text{ constant} \qquad \text{[no gravity, steady, irrotational]}$$

and examine the two standard barotropic cases.

Case I: Isothermal EoS

$$p = \frac{\mathcal{R}_* \rho T}{\mu}, \qquad T = \text{const.}$$

$$\begin{aligned} \Rightarrow \qquad \int \frac{\mathrm{d}p}{\rho} &= \int \frac{\mathcal{R}_* T}{\mu}\frac{\mathrm{d}\rho}{\rho} \\ &= \frac{\mathcal{R}_* T}{\mu}\ln\rho \\ &= c_s^2 \ln\rho \end{aligned}$$

Suppose that we have a minimum or maximum in $A(z)$ that allows a flow to have a sonic transition. Let $A = A_m$ at this location.

Then Bernoulli gives

$$\frac{1}{2}u^2 + c_s^2\ln\rho = \frac{1}{2}c_s^2 + c_s^2\ln\rho\Big|_{A=A_m}$$

$$\begin{aligned} \Rightarrow \qquad u^2 &= c_s^2\left[1 + 2\ln\left(\frac{\rho|_{A=A_m}}{\rho}\right)\right] \\ &= c_s^2\left[1 + 2\ln\left(\frac{uA}{c_s A_m}\right)\right] \end{aligned}$$

where this last step has used mass conservation, i.e. $\rho u A = \text{constant}$. Thus, given $A(z)$ we can determine $u(z)$ and $\rho(z)$, i.e. structure of the flow everywhere subject to given $\dot{M}$ and c_s.

Case II: Polytropic EoS

$$p = K\rho^{1+1/n}$$

Let's examine the case where the sonic transition occurs at $A = A_m$. But now we do not know the sound speed c_s since $c_s = c_s(\rho)$ and ρ varies. We need to solve for

$$c_s^2 = \frac{n+1}{n} K\rho^{1/n}$$

Now,

$$\begin{aligned}
\int \frac{\mathrm{d}p}{\rho} &= \int \frac{\mathrm{d}p}{\mathrm{d}\rho}\frac{\mathrm{d}\rho}{\rho} \\
&= \int K\frac{n+1}{n}\rho^{1/n}\frac{\mathrm{d}\rho}{\rho} \\
&= K\frac{n+1}{n}\int \rho^{1/n-1}\,\mathrm{d}\rho \\
&= K\frac{n+1}{n} n\rho^{1/n} \\
&= nc_s^2
\end{aligned}$$

Mass conservation:

$$\begin{aligned}
& \rho u A = \rho\Big|_{A_m} c_s\Big|_{A_m} A_m = \dot{M} \\
\Rightarrow \quad & \rho\Big|_{A_m} \left(\frac{n+1}{n}K\right)^{1/2} \rho^{1/2n}\Big|_{A_m} A_m = \dot{M} \\
\Rightarrow \quad & \rho^{2+1/n}\Big|_{A_m} \left(\frac{n+1}{n}K\right) A_m^2 = \dot{M}^2 \\
\Rightarrow \quad & \rho\Big|_{A_m} = \left[\left(\frac{\dot{M}}{A_m}\right)^2 \frac{n}{K(n+1)}\right]^{n/(2n+1)}
\end{aligned}$$

Knowing $\rho|_{A_m}$, we can now determine c_s and A_m. Bernoulli gives:

$$\frac{1}{2}u^2 + \int \frac{\mathrm{d}p}{\rho} = \text{const.}$$

$$\begin{aligned}
\Rightarrow \quad \frac{1}{2}\left(\frac{\dot{M}}{A\rho}\right)^2 + K(n+1)\rho^{1/n} &= \frac{1}{2}\, c_s^2\Big|_{A_m} + K(n+1)\,\rho^{1/n}\Big|_{A_m} \\
&= \frac{1}{2}\left(\frac{n+1}{n}\right) K\,\rho^{1/n}\Big|_{A_m} + K(n+1)\,\rho^{1/n}\Big|_{A_m} \\
&= \left(\frac{1}{2}+n\right)\left(\frac{n+1}{n}\right) K\,\rho^{1/n}\Big|_{A_m}
\end{aligned}$$

This is an implicit equation for the density structure through the flow.

General points of physical interpretation:

$$(u^2 - c_s^2)\boldsymbol{\nabla} \ln u = c_s^2 \boldsymbol{\nabla} \ln A$$

- In subsonic regime $u < c_s$

$$\begin{aligned} A \text{ decrease} \quad &\Rightarrow \quad \boldsymbol{\nabla} \ln u \text{ positive} \\ &\Rightarrow \quad u \text{ accelerates along streamline} \end{aligned}$$

  e.g. rivers flowing through narrows;

- In supersonic regime $u > c_s$

$$\begin{aligned} A \text{ increases} \quad &\Rightarrow \quad \boldsymbol{\nabla} \ln u \text{ positive} \\ &\Rightarrow \quad u \text{ accelerates along streamline} \end{aligned}$$

  Gas becomes very compressible. A increases, u increases, ρ is greatly reduced. $\dot{M} = A\rho u$ constant.

So, a nozzle that gets progressively narrower, reaches a minimum, and then widens again can be used to accelerate a flow from a subsonic to a supersonic regime.

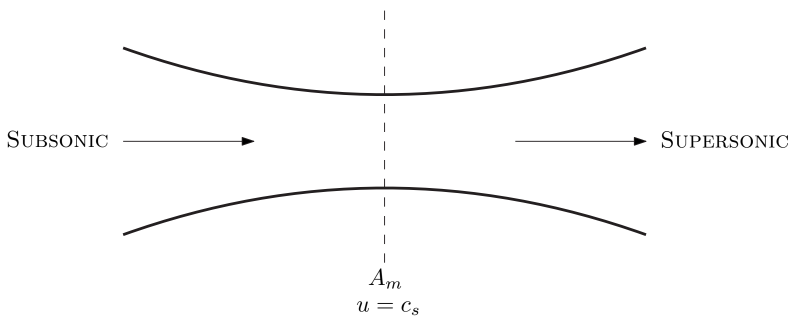

Figure G.5: De Laval nozzle

Recall momentum equation:

$$\begin{aligned} (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} &= -\frac{1}{\rho}\boldsymbol{\nabla}\rho\frac{\mathrm{d}p}{\mathrm{d}\rho} = -c_s^2 \boldsymbol{\nabla} \ln \rho \\ (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} &= u^2 \boldsymbol{\nabla} \ln u \\ \Rightarrow \qquad u^2 \boldsymbol{\nabla} \ln u &= -c_s^2 \boldsymbol{\nabla} \ln \rho \end{aligned}$$

- If $u \ll c_s, \nabla \ln u \gg \nabla \ln \rho$, this implies accelerations are important, pressure or density changes are small — almost incompressible;

- If $u \gg c_s, \nabla \ln u \ll \nabla \ln \rho$, $u \approx$ constant, pressure changes do not lead to much acceleration but there is change in ρ — compressible flow.

G.4 Spherical Accretion and Winds

We find flows with a mathematical structure when we consider steady-state and spherically-symmetric accretion flows or winds in the gravitational potential of a central body.

Consider the spherically-symmetric accretion of gas onto a star (described as a point of mass). We will assume

- gas is at rest at ∞ (reservior);
- steady state flow;
- barotropic EoS.

Mass conservation gives

$$\begin{aligned} & \rho u A = \text{constant } \dot{M} \\ \Rightarrow \qquad & 4\pi r^2 \rho u = \dot{M}, \end{aligned}$$

where, for convenience, we define u to be inward pointing.

Momentum equation gives

$$\begin{aligned} & u\frac{\mathrm{d}u}{\mathrm{d}r} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}r} - \frac{GM}{r^2} \\ \Rightarrow \qquad & u^2 \frac{\mathrm{d}\ln u}{\mathrm{d}r} = -c_s^2 \frac{\mathrm{d}\ln \rho}{\mathrm{d}r} - \frac{GM}{r^2}, \end{aligned} \tag{$*$}$$

eq.g.4.1

assuming self-gravity of the accretion gas is negligible.

Now, steady flow must have

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}r}\left(\ln \dot{M}\right) = 0 \\ \Rightarrow \qquad & \frac{\mathrm{d}}{\mathrm{d}r}\ln\rho + \frac{\mathrm{d}}{\mathrm{d}r}\ln u + \frac{\mathrm{d}}{\mathrm{d}r}\ln r^2 = 0 \\ \Rightarrow \qquad & \frac{\mathrm{d}}{\mathrm{d}r}\ln\rho = -\frac{\mathrm{d}}{\mathrm{d}r}\ln u - \frac{2}{r}. \end{aligned}$$

Substitute into ($*$) gives

$$u^2 \frac{\mathrm{d}}{\mathrm{d}r}\ln u = c_s^2\left(\frac{\mathrm{d}}{\mathrm{d}r}\ln u + \frac{2}{r}\right) - \frac{GM}{r^2}$$

Therefore

$$\boxed{(u^2 - c_s^2)\frac{\mathrm{d}}{\mathrm{d}r}\ln u = \frac{2c_s^2}{r}\left(1 - \frac{GM}{2c_s^2 r}\right)}.$$

There is a critical point in the flow at

$$r = r_s = \frac{GM}{2c_s^2} \qquad \textsc{Sonic Point}$$

where u is either a minimum/maximum or there is a sonic transition. This is called the *sonic point*, somewhat similar to De Laval nozzle, except no boundaries/tubes!

Can gain insight into the general structure of such flows by plotting possible solutions on the $(r/r_s, u)$ plane.

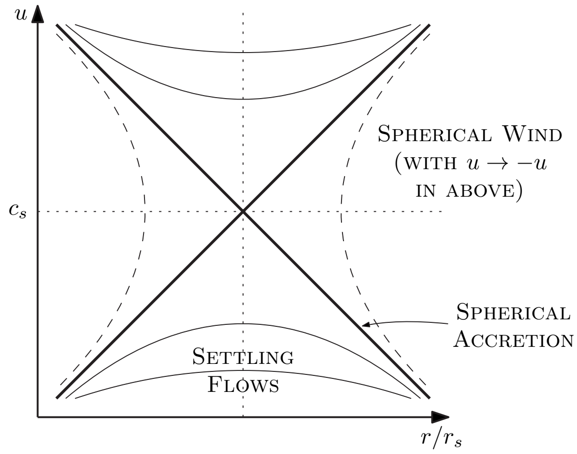

Figure G.6: Plot in $(r/r_s, u)$ plane

Back to accretion problem: progress requires the EoS.

Case I: Isothermal EoS

Equation of state is:

$$p = \frac{\mathcal{R}_* \rho T}{\mu}, \qquad T = \text{const.}$$

$$\Rightarrow \qquad c_s = \sqrt{\frac{\mathcal{R}_* T}{\mu}} = \text{const.}$$

and we know

$$r_s = \frac{GM}{2c_s^2}$$

Need to use Bernoulli's equation to constrain ρ and $\dot{M}$.

$$
\begin{aligned}
& H = \frac{1}{2}u^2 + \underbrace{\int \frac{\mathrm{d}p}{\rho}}_{c_s^2 \ln \rho} + \Psi = \text{const.} \\
\Rightarrow \quad & \frac{1}{2}u^2 + c_s^2 \ln \rho - \frac{GM}{r} = \frac{1}{2}c_s^2 + c_s^2 \ln \rho_s - \frac{GM}{r_s} \\
\Rightarrow \quad & \frac{1}{2}u^2 + c_s^2 \ln \rho - \frac{GM}{r} = c_s^2 \left(\ln \rho_s - \frac{3}{2} \right) \\
\Rightarrow \quad & u^2 = 2c_s^2 \left[\ln \left(\frac{\rho_s}{\rho} \right) - \frac{3}{2} \right] + \frac{2GM}{r}
\end{aligned}
$$

where ρ_s is the density at $r = r_s$.

Now,

as $r \to 0, u^2 \to 2GM/r$, i.e. free-fall;

as $r \to \infty$ and $u \to 0, \rho = \rho_s e^{-3/2}$, giving

$$\rho_s = \rho_\infty e^{3/2}$$

Thus, for a given ρ_∞, we know ρ_s and hence $\dot{M}$.

$$
\begin{aligned}
& \dot{M} = 4\pi r_s^2 \rho_s c_s \\
\Rightarrow \quad & \dot{M} = \frac{\pi G^2 M^2 e^{3/2} \rho_\infty}{c_s^3}
\end{aligned}
$$

Note.

– $\dot{M}$ proportional to M^2, more massive stars can accrete much more gas;

– $\dot{M}$ proportional to $1/c_s^3$, accretion very sensitive to temperature; can accrete more effectively from a colder medium.

Case II: Polytropic EoS

Equation of state is:

$$p = K\rho^{1+1/n}; \qquad \int \frac{\mathrm{d}p}{\rho} = K(n+1)\rho^{1/n} = nc_s^2$$

Bernoulli gives

$$\frac{1}{2}u^2 + (n+1)K\rho^{1/n} - \frac{GM}{r} = \frac{1}{2}c_s^2 + nc_s^2 - \frac{GM}{r_s}$$

with $r_s = GM/(2c_s^2)$. Using the mass accretion rate $\dot{M} = 4\pi r_s^2 \rho_s c_s$ we can then write

$$\begin{aligned} r_s &= \left(\frac{\dot{M}}{4\pi\rho_s c_s}\right)^{1/2} = \frac{GM}{2c_s^2} \\ \Rightarrow \qquad c_s &= \left(\frac{GM}{2}\right)^{2/3} \left(\frac{4\pi\rho_s}{\dot{M}}\right)^{1/3} \end{aligned}$$

Combine this with

$$c_s^2 = \frac{n+1}{n} K \rho_s^{1/n}$$

to get

$$\begin{aligned} \left(\frac{n+1}{n}\right) K \rho_s^{1/n} &= \left(\frac{GM}{2}\right)^{4/3} \left(\frac{4\pi\rho_s}{\dot{M}}\right)^{2/3} \\ \Rightarrow \qquad \rho_s^{1/n-2/3} = \rho_s^{(3-2n)/3n} &= \left(\frac{GM}{2}\right)^{4/3} \left(\frac{4\pi}{\dot{M}}\right)^{2/3} \frac{n}{(n+1)K} \\ \Rightarrow \qquad \rho_s &= \left(\frac{GM}{2}\right)^{4n/(3-2n)} \left(\frac{4\pi}{\dot{M}}\right)^{2n/(3-2n)} \left(\frac{n}{K(n+1)}\right)^{3n/(3-2n)} . \end{aligned}$$

Back to Bernoulli:

$$\begin{aligned} \frac{1}{2}u^2 + (n+1)K\rho^{1/n} - \frac{GM}{r} &= c_s^2 \left(n - \frac{3}{2}\right) \\ \Rightarrow \qquad \frac{1}{2}\left(\frac{\dot{M}}{4\pi r^2 \rho}\right)^2 + (n+1)K\rho^{1/n} &= c_s^2 \left(n - \frac{3}{2}\right) + \frac{GM}{r} \end{aligned}$$

As $r \to \infty, u \to 0$, we have

$$\rho_\infty = \left[\frac{c_s^2 \left(n - \frac{3}{2}\right)}{(n+1)K}\right]^n$$

$$c_{s,\infty}^2 = \frac{n+1}{n} K \rho_\infty^{1/n}$$

So, finally,

$$\begin{aligned} \dot{M} &= 4\pi r_s^2 \rho_s c_s \\ &= \frac{4\pi G^2 M^2}{4c_s^4} \cdot c_s \rho_\infty \left(\frac{n}{n - \frac{3}{2}}\right)^n \\ &= \frac{\pi G^2 M^2}{c_{s,\infty}^3} \rho_\infty \left(\frac{n}{n - \frac{3}{2}}\right)^{n-3/2} \end{aligned}$$

Therefore,

$$\boxed{\dot{M} = \frac{\pi (GM)^2 \rho_\infty}{c_{s,\infty}^3} \left(\frac{n}{n - \frac{3}{2}}\right)^{n-3/2}}$$

Same functional form as isothermal case, but now an additional coefficient related to polytropic index.

This is known as *Bondi Accretion*.

The generalisation to the case of a star accreting from a medium that it is moving through is called *Bondi-Hoyle-Lyttleton Accretion*. The result is

$$\dot{M} \sim \frac{(GM)^2 \rho_\infty}{(c_\infty^2 + v_\infty^2)^{3/2}}$$

where v_∞ is the velocity of gas relative to star at ∞.

CHAPTER H

Fluid Instabilities

Consider a fluid in a steady state ($\partial/\partial t = 0$). Thus it is in a state of equilibrium.

- If a small perturbation of this configuration grows with time, the configuration is *unstable* with respect to those perturbations;
- If a small perturbation decays with time or just oscillates around the equilibrium configuration, the configuration is *stable* with respect to those perturbations.

An awful lot of interesting astrophysics is due to the action of fluid instabilities!

Examples.

- Convection in stars;
- Multiphase nature of the ISM;
- Mixing of fluids that have relative motion;
- Turbulence in accretion disks;
- Formation of stars and galaxies.

In this chapter, we discuss some of the most important instabilities.

H.1 Convective Instability

This concerns the stability of a hydrostatic equilibrium. We can gain insight without doing a full perturbation analysis.

Consider following system:

- Ideal gas in hydrostatic equilibrium;

– Uniform gravitational field in $-\hat{\mathbf{z}}$ direction.

Now perturb a fluid element upwards, away from its equilibrium point.

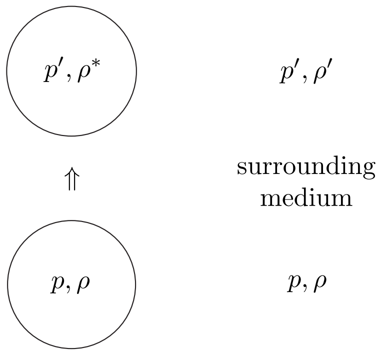

Figure H.1: Perturbing a fluid element upwards

We assume that any pressure imbalances are quickly removed by acoustic waves, but that heat exchange takes longer. This implies displaced element evolves adiabatically with a pressure p' equal to pressure at new location of atmosphere.

Since we assume heat transfer is slow, initially perturbations will change adiabatically. Stability depends on new value of density.

$$\begin{aligned}
\rho^* < \rho' \quad &\Rightarrow \quad \text{perturbed element buoyant} \\
&\Rightarrow \quad \text{system unstable;} \\
\rho^* > \rho' \quad &\Rightarrow \quad \text{perturbed element sinks back} \\
&\Rightarrow \quad \text{system stable.}
\end{aligned}$$

For adiabatic change,

$$\left.\begin{aligned} p &= K\rho^\gamma \\ p' &= K\rho^{*\gamma} \end{aligned}\right\} \quad \Rightarrow \quad \rho^* = \rho\left(\frac{p'}{p}\right)^{1/\gamma}.$$

To first order,

$$p' = p + \frac{\mathrm{d}p}{\mathrm{d}z}\delta z$$

$$\begin{aligned}
\Rightarrow \qquad \rho^* &= \rho\left(\frac{p + \frac{\mathrm{d}p}{\mathrm{d}z}\delta z}{p}\right)^{1/\gamma} \\
&= \rho\left(1 + \frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}z}\delta z\right)^{1/\gamma} \\
&\approx \rho + \frac{\rho}{p\gamma}\frac{\mathrm{d}p}{\mathrm{d}z}\delta z
\end{aligned}$$

In surrounding medium,

$$\rho' = \rho + \frac{\mathrm{d}\rho}{\mathrm{d}z}\delta z$$

and the system is unstable if $\rho^* < \rho'$. So instability needs

$$\begin{aligned}
& \rho + \frac{\rho}{p\gamma}\frac{\mathrm{d}p}{\mathrm{d}z}\delta z < \rho + \frac{\mathrm{d}\rho}{\mathrm{d}z}\delta z \\
\Rightarrow \quad & \frac{\rho}{p\gamma}\frac{\mathrm{d}p}{\mathrm{d}z} < \frac{\mathrm{d}\rho}{\mathrm{d}z} \\
\Rightarrow \quad & \frac{\mathrm{d}}{\mathrm{d}z}\ln p < \gamma\frac{\mathrm{d}}{\mathrm{d}z}\ln\rho \\
\Rightarrow \quad & \frac{\mathrm{d}}{\mathrm{d}z}\left(\ln p\rho^{-\gamma}\right) < 0 \\
\Rightarrow \quad & \frac{\mathrm{d}K}{\mathrm{d}z} < 0 \qquad \text{(instability)}
\end{aligned}$$

So, the system is unstable if the entropy of the atmosphere decreases with increasing height. This can also be related to temperature and pressure gradients.

$$\frac{\mathrm{d}K}{\mathrm{d}z} < 0 \qquad \Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}z}\ln K < 0$$

But

$$K = p\rho^{-\gamma} = (\text{const.})p^{1-\gamma}T^{\gamma} \qquad (p = \mathcal{R}_*\rho T/\mu)$$

so,

$$\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}z}\ln K = (1-\gamma)\frac{\mathrm{d}}{\mathrm{d}z}\ln p + \gamma\frac{\mathrm{d}}{\mathrm{d}z}\ln T < 0 \\
\Rightarrow \quad & \frac{\mathrm{d}T}{\mathrm{d}z} < \left(1 - \frac{1}{\gamma}\right)\frac{T}{p}\frac{\mathrm{d}p}{\mathrm{d}z} \qquad \text{(instability)}
\end{aligned}$$

Hence, we have the *Schwarzschild stability criterion* which reads

$$\boxed{\frac{\mathrm{d}T}{\mathrm{d}z} > \left(1 - \frac{1}{\gamma}\right)\frac{T}{p}\frac{\mathrm{d}p}{\mathrm{d}z}}$$

Since hydrostatic equilibrium requires $\mathrm{d}p/\mathrm{d}z < 0$, we see that (since $\gamma > 1$)

- Always stable to convection if $\mathrm{d}T/\mathrm{d}z > 0$;
- Otherwise, can tolerate a negative temperature gradient provided

$$\left|\frac{\mathrm{d}T}{\mathrm{d}z}\right| < \left(1 - \frac{1}{\gamma}\right)\frac{T}{p}\left|\frac{\mathrm{d}p}{\mathrm{d}z}\right|$$

  So convective instability develops when T declines too steeply with increasing height.

Examples (Convectively unstable systems)**.**

- Outer regions of low mass stars;
- Cores of high mass stars.

For stable configurations, we can example the dynamics of atmosphere: equation of motion is

$$
\begin{aligned}
&\rho^* \frac{\mathrm{d}^2}{\mathrm{d}t^2}\delta z = -g\left(\rho^* - \rho'\right) \\
\Rightarrow\quad &(\rho + \cancelto{\text{small}}{\delta\rho})\frac{\mathrm{d}^2}{\mathrm{d}t^2}\delta z = -g\left[\frac{\rho}{T}\frac{\mathrm{d}T}{\mathrm{d}z} - \left(1 - \frac{1}{\gamma}\right)\frac{\rho}{p}\frac{\mathrm{d}p}{\mathrm{d}z}\right]\delta z \\
\Rightarrow\quad &\frac{\mathrm{d}^2}{\mathrm{d}t^2}\delta z = -\frac{g}{T}\left[\frac{\mathrm{d}T}{\mathrm{d}z} - \left(1 - \frac{1}{\gamma}\right)\frac{T}{p}\frac{\mathrm{d}p}{\mathrm{d}z}\right]\delta z
\end{aligned}
$$

So, it is simple harmonic motion with angular frequency N where

$$
N^2 = \frac{g}{T}\left[\frac{\mathrm{d}T}{\mathrm{d}z} - \left(1 - \frac{1}{\gamma}\right)\frac{T}{p}\frac{\mathrm{d}p}{\mathrm{d}z}\right] \qquad \textsc{Brunt-Väisälä Frequency}
$$

These oscillations are *internal gravity waves*.

H.2 Jeans Instability

This concerns the stability of a self-gravitating fluid against gravitational collapse.

Consider following system:

- Uniform medium initially static;
- Barotropic EoS;
- Gravitational field generated by the medium itself.

So equilibrium is

$$
\begin{aligned}
&p = p_0, \ \text{const.} \\
&\rho = \rho_0, \ \text{const.} \\
&\mathbf{u} = \mathbf{0}
\end{aligned}
$$

and governing equations are

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0 \\
&\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla}p - \boldsymbol{\nabla}\Psi \\
&\nabla^2\Psi = 4\pi G\rho
\end{aligned}
$$

Introduce a perturbation:

$$\begin{aligned} p &= p_0 + \Delta p \\ \rho &= \rho_0 + \Delta \rho \\ \mathbf{u} &= \Delta \mathbf{u} \\ \Psi &= \Psi_0 + \Delta \Psi \end{aligned}$$

Note. There is an inconsistency between the assumption $\rho_0 = \text{constant} > 0$ and the assumption $\Psi_0 = \text{const.}$ We proceed anyways — this is the *Jeans swindle* (1902). This is closely tied to the fact that it is impossible to construct a model of a static infinite Universe. A more complete analysis of perturbations against a background of a (relativistic) homogenous expanding Universe recovers the same local instability as that found by Jeans, hence justifying the swindle.

Linearized equations are:

$$\frac{\partial \Delta \rho}{\partial t} + \rho_0 \boldsymbol{\nabla} \cdot (\Delta \mathbf{u}) = 0 \qquad ① \qquad \boxed{\texttt{eq.h.2.1}}$$

$$\frac{\partial \Delta \mathbf{u}}{\partial t} = -\frac{\mathrm{d}p}{\mathrm{d}\rho}\frac{1}{\rho_0}\boldsymbol{\nabla}(\Delta\rho) - \boldsymbol{\nabla}(\Delta\Psi) = -c_s^2 \frac{\boldsymbol{\nabla}(\Delta\rho)}{\rho_0} - \boldsymbol{\nabla}(\Delta\Psi) \qquad ② \qquad \boxed{\texttt{eq.h.2.2}}$$

$$\nabla^2(\Delta\Psi) = 4\pi G \Delta\rho \qquad ③ \qquad \boxed{\texttt{eq.h.2.3}}$$

Look for plane wave solutions

$$\begin{aligned} \Delta\rho &= \rho_1 e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x} - \omega t)} \\ \Delta\Psi &= \Psi_1 e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x} - \omega t)} \\ \Delta\mathbf{u} &= \mathbf{u}_1 e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x} - \omega t)}. \end{aligned}$$

Substitution into the linear equations gives

$$① \quad \Rightarrow \quad -\omega\rho_1 + \rho_0 \mathbf{k}\cdot\mathbf{u}_1 = 0 \qquad ④ \qquad \boxed{\texttt{eq.h.2.4}}$$

$$② \quad \Rightarrow \quad -\rho_0\omega\mathbf{u}_1 = -c_s^2\mathbf{k}\rho_1 - \rho_0\mathbf{k}\Psi_1 \qquad ⑤ \qquad \boxed{\texttt{eq.h.2.5}}$$

$$③ \quad \Rightarrow \quad -k^2\Psi_1 = 4\pi G\rho_1. \qquad ⑥ \qquad \boxed{\texttt{eq.h.2.6}}$$

Eliminating $\mathbf{u}_1$ and Ψ_1 from these

$$\begin{aligned} ④+⑤ \quad \Rightarrow \quad \rho_1\omega^2 &= k^2(\rho_1 c_s^2 + \rho_0\Psi_1) \\ &= k^2\rho_1 c_s^2 - 4\pi G\rho_0\rho_1 \qquad \text{from } ⑥ \end{aligned}$$

$$\Rightarrow \quad \omega^2 = c_s^2\left(k^2 - \frac{4\pi G\rho_0}{c_s^2}\right).$$

Introduce the Jeans wavenumber $k_J^2 = 4\pi G\rho_0/c_s^2$ so we have the dispersion relation

$$\boxed{\omega^2 = c_s^2(k^2 - k_J^2)}$$

Notes:

- For $k \gg k_J$, we have normal sound waves $\omega^2 = c_s^2 k^2$.
- For $k \gtrsim k_J$, we have modified sound waves. Gravity leads to dispersion of the wave and a slower group velocity.
- For $k < k_J$, ω is purely imaginary (for $k \in \mathbb{R}$), giving

  $$\omega = \mathrm{i}\tilde{\omega}, \qquad \tilde{\omega} \in \mathbb{R}$$

  and

  $$e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x} - \omega t)} = e^{\tilde{\omega} t} e^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$$

  leading to exponentially growing solution: *Gravitational Instability.*

The maximum stable wavelength is

$$\lambda_J = \frac{2\pi}{k_J} = \sqrt{\frac{\pi c_s^2}{G\rho_0}} \qquad \textsc{Jeans Length}$$

The associated mass is

$$M_J \sim \rho_0 \lambda_J^3 \qquad \textsc{Jeans Mass}$$

These are central concepts in the theory of

- *Star formation* (instability of giant molecular clouds);
- *Cosmological structure formation* (instability of the homogeneous primordial gas).

H.3 Rayleigh-Taylor and Kelvin-Helmholtz Instability

This concerns the stability of an interface with a discontinuous change in tangential velocity and/or density.

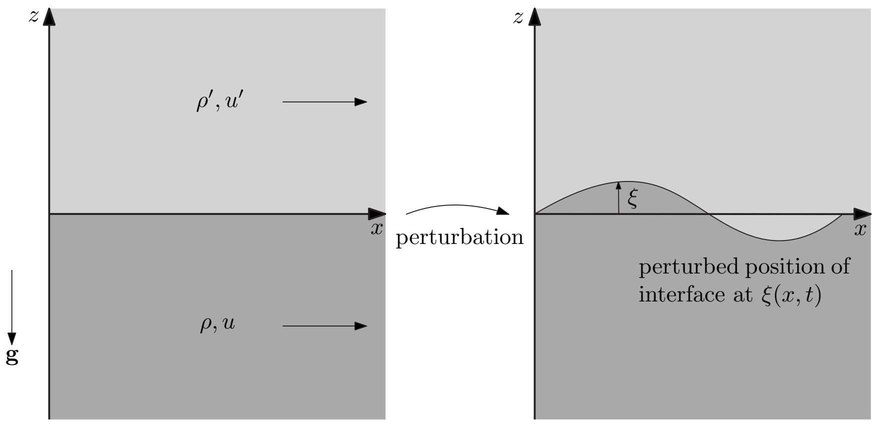

Figure H.2: Perturbation of interface of discontinuity

For convenience, let's assume:

- Constant gravity, ideal fluid;
- Pressure continuous across the interface;
- Incompressible flow $\boldsymbol{\nabla}\cdot\mathbf{u}=0$;
- Irrotational flow $\boldsymbol{\nabla}\times\mathbf{u}=0 \Rightarrow \mathbf{u}=-\boldsymbol{\nabla}\Phi$;
- 2D problem (symmetry direction into the page of Fig. H.2).

The momentum equation (for either upper or lower fluid) is

$$
\begin{aligned}
& \frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}=-\frac{1}{\rho}\boldsymbol{\nabla}p+\mathbf{g} \\
\Rightarrow\quad & -\boldsymbol{\nabla}\frac{\partial \Phi}{\partial t}+\boldsymbol{\nabla}\left(\frac{1}{2}u^2\right)=-\underbrace{\boldsymbol{\nabla}\left(\frac{p}{\rho}\right)}_{\substack{\text{since } \rho=\text{const.}\\ \text{within each fluid}}}-\boldsymbol{\nabla}\Psi \\
\Rightarrow\quad & \boldsymbol{\nabla}\left[-\frac{\partial \Phi}{\partial t}+\frac{1}{2}u^2+\frac{p}{\rho}+\Psi\right]=0 \\
\Rightarrow\quad & -\frac{\partial \Phi}{\partial t}+\frac{1}{2}u^2+\frac{p}{\rho}+\Psi=F(t) \qquad (*)
\end{aligned}
$$

eq.h.3.1

where $F(t)$ is a function that is constant in space but not in time.

Now consider a perturbation at the interface of these two fluids. Let us study the evolution of the perturbed position of the interface $\xi(x,t)$.

The velocity potential $\mathbf{u} = -\boldsymbol{\nabla}\Phi$, so if the unperturbed velocities in the fluids are U and U' we have

$$\begin{aligned}\Phi_{\text{low}} &= -Ux + \phi \\ \Phi_{\text{up}} &= -U'x + \phi'\end{aligned}$$

$$\Rightarrow \qquad \nabla^2\phi = \nabla^2\phi' = 0 \qquad (\text{since } \boldsymbol{\nabla}\cdot\mathbf{u} = 0) \tag{①}$$ eq.h.3.2

ϕ and ϕ' are sourced by displacements of the interface. Consider an element of the lower fluid that is at the interface. Then

$$u_z = \frac{\mathrm{D}\xi}{\mathrm{D}t}$$

giving

$$\left.\begin{aligned}-\frac{\partial\phi}{\partial z} &= \frac{\partial\xi}{\partial t} + U\frac{\partial\xi}{\partial x} \\ -\frac{\partial\phi'}{\partial z} &= \frac{\partial\xi}{\partial t} + U'\frac{\partial\xi}{\partial x}\end{aligned}\right\} \qquad \text{to first order} \tag{②}$$ eq.h.3.3

Now look for plane wave solutions

$$\begin{aligned}\xi &= Ae^{\mathrm{i}(kx-\omega t)} \\ \phi &= Ce^{\mathrm{i}(kx-\omega t)+k_z z} \\ \phi' &= C'e^{\mathrm{i}(kx-\omega t)+k_z' z}\end{aligned}$$

where extra terms on the exponents $k_z z$ and $k_z' z$ are there to seek solutions where perturbed potential decays at large $|z|$.

But we know that

$$\begin{aligned}\nabla^2\phi = 0 \qquad &\Rightarrow \qquad -k^2 + k_z^2 = 0 \\ &\Rightarrow \qquad k_z = |k|\end{aligned}$$

so $\phi \to 0$ as $z \to -\infty$.

$$\begin{aligned}\nabla^2\phi' = 0 \qquad &\Rightarrow \qquad -k^2 + {k_z'}^2 = 0 \\ &\Rightarrow \qquad k_z' = -|k|\end{aligned}$$

since $\phi' \to 0$ as $z \to \infty$.

For now, let's stipulate $k > 0$. So

$$\begin{aligned}\phi &= Ce^{\mathrm{i}(kx-\omega t)+kz} \\ \phi' &= C'e^{\mathrm{i}(kx-\omega t)-kz}\end{aligned}$$

From ②, we have

$$-kC = -\mathrm{i}\omega A + \mathrm{i}UkA = \mathrm{i}(kU - \omega)A \qquad ③ \qquad \boxed{\texttt{eq.h.3.4}}$$
$$kC' = \mathrm{i}(kU' - \omega)A \qquad ④ \qquad \boxed{\texttt{eq.h.3.5}}$$

We need one more equation if we're to solve for A, C, C'. We get that from pressure balance across the interface.

$$p = -\rho\left(-\frac{\partial\phi}{\partial t} + \frac{1}{2}u^2 + g\xi\right) + \rho F(t)$$
$$p' = -\rho'\left(-\frac{\partial\phi'}{\partial t} + \frac{1}{2}u'^2 + g\xi\right) + \rho' F'(t)$$

and equality at $z = 0$:

$$\rho\left(-\frac{\partial\phi}{\partial t} + \frac{u^2}{2} + g\xi\right) = \rho'\left(-\frac{\partial\phi'}{\partial t} + \frac{u'^2}{2} + g\xi\right) + K(t) \qquad ⑤ \qquad \boxed{\texttt{eq.h.3.6}}$$

where

$$K \equiv \rho F(t) - \rho' F'(t)$$

The perturbation vanishes for $z \to \pm\infty$ at all times, so we can look at equation ($*$) for each fluid in the limit $|z| \to \infty$, taking limit carefully so that Ψ terms cancel, to get

$$\rho F(t) - \rho' F'(t) = \underbrace{\frac{1}{2}U^2\rho - \frac{1}{2}U'^2\rho'}_{\substack{\text{conditions at } \infty \\ \text{and so a constant}}}$$

Therefore, $K(t)$ is actually a constant.

Next in our attempt to use ⑤ to match across boundary, we need to determine u and u'. Now

$$\mathbf{u} = -\boldsymbol{\nabla}\Phi = -\boldsymbol{\nabla}(-Ux + \phi) = U\hat{\mathbf{x}} - \boldsymbol{\nabla}\phi$$
$$\Rightarrow \qquad u^2 = U^2 - 2U\frac{\partial\phi}{\partial x} \qquad \text{(dropping 2nd order terms)}$$

and similarly

$$u'^2 = U'^2 - 2U'\frac{\partial\phi'}{\partial x}$$

So, ⑤ reads

$$\rho\left(-\frac{\partial\phi}{\partial t} + \cancel{\frac{1}{2}U^2} - U\frac{\partial\phi}{\partial x} + g\xi\right) = \rho'\left(-\frac{\partial\phi'}{\partial t} + \cancel{\frac{1}{2}U'^2} - U'\frac{\partial\phi'}{\partial x} + g\xi\right) + \underbrace{\cancel{\frac{1}{2}U^2}\rho - \cancel{\frac{1}{2}U'^2}\rho'}_{K}$$

$$
\begin{aligned}
\Rightarrow \qquad & \rho\left(-\frac{\partial\phi}{\partial t} - U\frac{\partial\phi}{\partial x} + g\xi\right) = \rho'\left(-\frac{\partial\phi'}{\partial t} - U'\frac{\partial\phi'}{\partial x} + g\xi\right) \\
\Rightarrow \qquad & \rho\mathrm{i}\omega C - \rho U\mathrm{i}kC + \rho gA = \rho'\mathrm{i}\omega C' - \rho' U'\mathrm{i}kC' + \rho' gA \\
\Rightarrow \qquad & \rho(kU-\omega)C + \mathrm{i}\rho gA = \rho'(kU'-\omega)C' + \mathrm{i}\rho' gA
\end{aligned}
$$

Now eliminate C and C' from ③ and ④ to give

$$
\boxed{\rho(kU-\omega)^2 + \rho'(kU'-\omega)^2 = kg(\rho-\rho')}
$$

This is the dispersion relation for our system. Let's now look at some specific applications.

(A) *Surface gravity waves*: two fluids at rest initially with $\rho' < \rho$ (i.e. denser fluid on bottom). The dispersion relation gives

$$
\begin{aligned}
& \omega^2(\rho+\rho') = kg(\rho-\rho') \\
\Rightarrow \qquad & \omega^2 = k\frac{g(\rho-\rho')}{\rho+\rho'}
\end{aligned}
$$

So, for $k \in \mathbb{R}$, we have that $\omega \in \mathbb{R}$ and hence system displays oscillations/waves. Phase speed is

$$
\frac{\omega}{k} = \pm\sqrt{\frac{g}{k}\frac{\rho-\rho'}{\rho+\rho'}} = \underbrace{f(k)}_{\substack{\text{waves are} \\ \text{dispersive}}}
$$

If $\rho' \ll \rho$, then $\omega/k = \pm\sqrt{g/k}$.

Example. Surface waves on ocean.

(B) *Static stratified fluid*: two fluids at rest initially with $\rho' > \rho$ (i.e. denser fluid on top). Then

$$
\omega^2 = k\frac{g(\rho-\rho')}{\rho+\rho'}
$$

So, for $k \in \mathbb{R}$ we have $\omega^2 < 0$ and so ω is purely imaginary.

$$
\frac{\omega}{k} = \pm\mathrm{i}\sqrt{\frac{g}{k}\frac{\rho'-\rho}{\rho+\rho'}}
$$

The positive root of this gives us exponentially growing solutions. This is the *Rayleigh-Taylor Instability.*

(C) *Fluids in motion*: two fluids with $\rho > \rho'$ (so stable to Rayleigh-Taylor) but different velocities non-zero U and U'. Take full dispersion relation:

$$\rho(kU - \omega)^2 + \rho'(kU' - \omega)^2 = kg(\rho - \rho')$$

divide by k^2 and solve the quadratic in ω/k,

$$\Rightarrow \qquad \frac{\omega}{k} = \frac{\rho U + \rho' U'}{\rho + \rho'} \pm \sqrt{\frac{g}{k}\frac{\rho - \rho'}{\rho + \rho'} - \frac{\rho\rho'(U - U')^2}{(\rho + \rho')^2}}$$

There is instability if

$$\frac{g}{k}\frac{\rho - \rho'}{\rho + \rho'} - \frac{\rho\rho'(U - U')^2}{(\rho + \rho')^2} < 0 \qquad \text{(instability)}$$

If $g = 0$, then any relative motion gives instability, i.e. *Kelvin-Helmholtz Instability*;

If $g \neq 0$, then unstable modes are those with

$$k > \frac{(\rho^2 - \rho'^2)g}{\rho\rho'(U - U')^2}$$

i.e. gravity is stabilising influence.

H.4 Thermal Instability

This concerns the stability of a medium in thermal equilibrium (heating = cooling) to perturbations in temperature. Consider the following system:

- No gravitational field;
- Static thermal equilibrium

$$\mathbf{u}_0 = \mathbf{0}, \dot{Q}_0 = 0, \underbrace{\boldsymbol{\nabla} p_0 = 0, \boldsymbol{\nabla}\rho_0 = 0}_{\boldsymbol{\nabla} K_0 = 0} \qquad \text{where } p = K\rho^\gamma$$

Let's start by deriving an alternative form of the energy equation that involves the entropy-like variable K; this will be well suited to problems of thermal instability.

$$\begin{aligned} p = K\rho^\gamma \qquad \Rightarrow \qquad \mathrm{d}p &= \rho^\gamma\,\mathrm{d}K + K\gamma\rho^{\gamma-1}\,\mathrm{d}\rho \\ &= \rho^\gamma\,\mathrm{d}K + \frac{\gamma p}{\rho}\,\mathrm{d}\rho \qquad ① \end{aligned}$$

eq.h.4.1

$$p = \frac{\mathcal{R}_*}{\mu}\rho T \qquad \Rightarrow \qquad \mathrm{d}p = \frac{\mathcal{R}_*}{\mu}T\,\mathrm{d}\rho + \frac{\mathcal{R}_*}{\mu}\rho\,\mathrm{d}T$$
$$= \frac{p}{\rho}\,\mathrm{d}\rho + \frac{\mathcal{R}_*}{\mu}\rho\,\mathrm{d}T \qquad ② \qquad \boxed{\texttt{eq.h.4.2}}$$

Equate ① and ② to give

$$\rho^\gamma\,\mathrm{d}K + \gamma\frac{p}{\rho}\,\mathrm{d}\rho = \frac{p}{\rho}\,\mathrm{d}\rho + \frac{\mathcal{R}_*}{\mu}\rho\,\mathrm{d}T$$
$$\Rightarrow \qquad \rho^\gamma\,\mathrm{d}K = (1-\gamma)\frac{p}{\rho}\,\mathrm{d}\rho + \frac{\mathcal{R}_*}{\mu}\rho\,\mathrm{d}T$$
$$\Rightarrow \qquad \mathrm{d}K = \rho^{1-\gamma}(1-\gamma)\underbrace{\left[\frac{p}{\rho^2}\,\mathrm{d}\rho + \frac{\mathcal{R}_*}{\mu(1-\gamma)}\,\mathrm{d}T\right]}_{-\text{đ}Q}$$

First law of thermodynamics:

$$\text{đ}Q = p\,\mathrm{d}V + \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}T}\,\mathrm{d}T\,, \qquad \text{(unit mass)}$$

and so

$$\text{đ}Q = p\,\mathrm{d}(1/\rho) + C_V\,\mathrm{d}T$$
$$= -\frac{p}{\rho^2}\,\mathrm{d}\rho - \frac{\mathcal{R}_*}{\mu(1-\gamma)}\,\mathrm{d}T\,. \qquad \text{since we have } (\gamma-1)C_V = \mathcal{R}_*/\mu$$

Then we have

$$\mathrm{d}K = -(1-\gamma)\rho^{1-\gamma}\text{đ}Q \qquad \text{for fluid element}$$

Turn this into Lagrangian energy equation by noting that $\dot{Q} = -\text{đ}Q/\mathrm{d}t$,

$$\Rightarrow \qquad \frac{\mathrm{D}K}{\mathrm{D}t} = -(\gamma-1)\rho^{1-\gamma}\dot{Q}$$
$$\Rightarrow \qquad \frac{1}{K}\frac{\mathrm{D}K}{\mathrm{D}t} \equiv \frac{\mathrm{D}}{\mathrm{D}t}(\ln K) = -(\gamma-1)\frac{\rho}{p}\dot{Q}$$

$$\boxed{\frac{1}{K}\frac{\mathrm{D}K}{\mathrm{D}t} = -(\gamma-1)\frac{\rho\dot{Q}}{p}} \qquad \text{ENTROPY FORM OF ENERGY EQN} \qquad ③ \qquad \boxed{\texttt{eq.h.4.3}}$$

This joins our usual continuity and momentum equations

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0 \qquad ④ \qquad \boxed{\texttt{eq.h.4.4}}$$
$$\rho\frac{\partial\mathbf{u}}{\partial t} + \rho\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} = -\boldsymbol{\nabla}p \qquad ⑤ \qquad \boxed{\texttt{eq.h.4.5}}$$

Now we look at thermal stability. Consider small perturbations to the equilibrium

$$\begin{aligned} \rho &\to \rho_0 + \Delta\rho \\ p &\to p_0 + \Delta p \\ \mathbf{u} &\to \Delta\mathbf{u} \\ K &\to K_0 + \Delta K. \end{aligned}$$

Linearize the equations:

$$④ \quad \Rightarrow \quad \frac{\partial \Delta\rho}{\partial t} + \rho_0 \boldsymbol{\nabla} \cdot (\Delta\mathbf{u}) = 0 \qquad ⑥ \qquad \boxed{\texttt{eq.h.4.6}}$$

$$⑤ \quad \Rightarrow \quad \rho_0 \frac{\partial \Delta\mathbf{u}}{\partial t} = -\boldsymbol{\nabla}(\Delta p) \qquad ⑦ \qquad \boxed{\texttt{eq.h.4.7}}$$

$$③ \quad \Rightarrow \quad \frac{\partial \Delta K}{\partial t} = -\frac{\gamma - 1}{\rho_0^{\gamma-1}} \Delta\dot{Q},$$

where we can write

$$\Delta\dot{Q} = \left.\frac{\partial \dot{Q}}{\partial p}\right|_\rho \Delta p + \left.\frac{\partial \dot{Q}}{\partial \rho}\right|_p \Delta\rho$$

so that

$$\frac{\partial \Delta K}{\partial t} = -A^* \Delta p - B^* \Delta\rho \qquad ⑧ \qquad \boxed{\texttt{eq.h.4.8}}$$

with

$$A^* = \frac{\gamma - 1}{\rho_0^{\gamma-1}} \left.\frac{\partial \dot{Q}}{\partial p}\right|_\rho, \qquad B^* = \frac{\gamma - 1}{\rho_0^{\gamma-1}} \left.\frac{\partial \dot{Q}}{\partial \rho}\right|_p$$

We also have

$$p = K\rho^\gamma \qquad \Rightarrow \qquad \Delta p = \rho_0^\gamma \Delta K + \gamma \frac{p_0}{\rho_0} \Delta\rho \qquad ⑨ \qquad \boxed{\texttt{eq.h.4.9}}$$

We seek solutions of the form

$$\begin{aligned} \Delta p &= p_1 e^{\mathbf{i}\mathbf{k}\cdot\mathbf{x} + qt} \\ \Delta\rho &= \rho_1 e^{\mathbf{i}\mathbf{k}\cdot\mathbf{x} + qt} \\ \Delta\mathbf{u} &= \mathbf{u}_1 e^{\mathbf{i}\mathbf{k}\cdot\mathbf{x} + qt} \\ \Delta K &= K_1 e^{\mathbf{i}\mathbf{k}\cdot\mathbf{x} + qt} \end{aligned}$$

so, instability if $\mathrm{Re}(q) > 0$. Substituting into linearized equations gives

$$\begin{aligned} ⑥ \quad &\Rightarrow \quad q\rho_1 + \rho_0 \mathbf{i}\mathbf{k} \cdot \mathbf{u}_1 = 0 \\ ⑦ \quad &\Rightarrow \quad q\rho_0 \mathbf{u}_1 = -\mathbf{i}\mathbf{k} p_1 \\ ⑧ \quad &\Rightarrow \quad qK_1 = -A^* p_1 - B^* \rho_1 \\ ⑨ \quad &\Rightarrow \quad p_1 = \rho_0^\gamma K_1 + \frac{\gamma p_0}{\rho_0} \rho_1 \end{aligned}$$

We can combine these to obtain the dispersion relation:

$$\begin{aligned} \frac{A^* q}{k^2} - \frac{B^*}{q} &= -\left(\frac{q^2}{k^2} + \gamma\frac{p_0}{\rho_0}\right)\frac{1}{\rho_0^\gamma} \\ \Rightarrow \qquad & \underbrace{q^3 + A^*\rho_0^\gamma q^2 + k^2\gamma\frac{p_0}{\rho_0}q - B^* k^2 \rho_0^\gamma}_{\text{cubic in } q \text{, call } E(q)} = 0 \end{aligned}$$

This has at least one real root — system is unstable if that real root is positive, $q > 0$.

Now $E(\infty) = \infty, E(0) = -B^* k^2 \rho_0^\gamma$. So the system is unstable if $B^* > 0$.

$$\begin{aligned} \therefore \qquad & B^* = \frac{\gamma - 1}{\rho_0^{\gamma-1}} \left.\frac{\partial \dot{Q}}{\partial \rho}\right|_p > 0 \qquad \text{(condition for instability)} \\ \Rightarrow \qquad & \left.\frac{\partial \dot{Q}}{\partial\left(\frac{\mu p}{\mathcal{R}_* T}\right)}\right|_p > 0 \\ \Rightarrow \qquad & -\frac{T^2}{p}\left.\frac{\partial \dot{Q}}{\partial T}\right|_p > 0 \end{aligned}$$

$$\therefore \qquad \boxed{\text{unstable if} \qquad \left.\frac{\partial \dot{Q}}{\partial T}\right|_p < 0} \qquad \textsc{Field Criterion}$$

The system is always unstable if it's Field unstable (named after George Field who wrote the classice paper on thermal instability in 1965).

However, even a Field stable system can be unstable if $A^* < 0 \Rightarrow \partial\dot{Q}\big/\partial T\,\big|_\rho < 0$. From the dispersion relation, we see that this can happen for long wavelength modes, i.e. k small. Then

$$q^2(q + A^*\rho_0^\gamma) \approx 0 \qquad \Rightarrow \qquad q \approx -A^*\rho_0^\gamma$$

Interpretation:

- Short wavelength perturbations are readily brought into pressure equilibrium by the action of sound waves, therefore, thermal instability proceeds at fixed pressure;

- Long wavelength perturbations: there is insufficient time for sound waves to equalise pressure with surroundings, so they tend to develop at constant density.

Example. Let's assume a specific form for $\dot{Q}$

$$\begin{aligned}\dot{Q} &= A\rho T^{\alpha} - H \\ &= \frac{A\mu}{\mathcal{R}_*} p T^{\alpha-1} - H\end{aligned}$$

$$\Rightarrow \qquad \left.\frac{\partial \dot{Q}}{\partial T}\right|_p = (\alpha - 1)\frac{A\mu p}{\mathcal{R}_*} T^{\alpha-2}$$

This is Field unstable, $\left.\partial \dot{Q} \middle/ \partial T \right|_p < 0$ if $\alpha < 1$.

Bremsstrahlung has $\alpha = 0.5 \Rightarrow$ *Field unstable.*

CHAPTER I

Viscous Flows

Thus far, we have been assuming that changes in the momentum of a fluid element are due entirely to pressure forces (acting normal to the surface of the element) or gravity (acting on the bulk).

This assumption is justified in the limit $\lambda \to 0$, i.e. the the particles composing the fluid have vanishingly small collisional mean-free-path.

For finite-λ, momentum can diffuse through the fluid. This brings us to a discussion of *viscosity*.

I.1 Basics of Viscosity

In a viscous flow, momentum can be transferred if there are velocity differences between fluid elements.

Continuity equation unchanged

$$\frac{\partial \rho}{\partial t} + \partial_j(\rho u_j) = 0$$

But momentum equation needs to be changed

$$\frac{\partial}{\partial t}(\rho u_i) = -\partial_j \sigma_{ij} + \rho g_i, \qquad g_i = -\partial_i \Psi$$

with

$$\sigma_{ij} = \rho u_i u_j + p\delta_{ij} - \underbrace{\sigma'_{ij}}_{\substack{\text{viscous} \\ \text{stress tensor}}}$$

As we'll see later, σ'_{ij} is related to velocity gradients.

The connection between the viscous stress tensor and the microphysics (i.e. the mean-free-path) is uncovered by considering a simple linear shear flow:

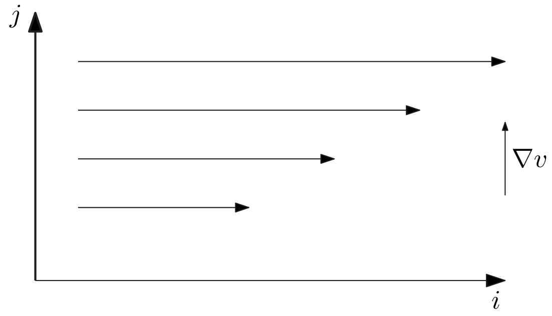

Figure I.1: Linear shear flow

Microscopically, thermal/random motion of the particles can allow momentum to "diffuse" across streamlines. Becomes more important as gas gets less collisional.

Let's analyse the microscopic behaviour: assume the typical (thermal) velocity in j-direction is u_j. So the momentum flux associated with this is

$$\underbrace{\rho u_i u_j}_{\substack{i\text{-cpt of momentum} \\ \text{carried in } j\text{-direction}}}$$

Typical thermal velocity is $\sim \sqrt{kT/m}$. So, flux of ith-component of momentum in the upward jth direction is

$$\rho u_i \alpha \sqrt{\frac{kT}{m}}, \qquad \alpha \sim 1$$

For the element on the other side of the surface in the j-direction, the corresponding momentum flux across surface is

$$-\rho u_i^* \alpha \sqrt{\frac{kT}{m}}$$

where u_i^* is i-velocity of *that* element. For a j-separation of δl we have

$$u_i^* = u_i + \delta l (\partial_j u_i)$$

So,

$$\text{net momentum flux} = -\rho(\partial_j u_i)\delta l \sqrt{\frac{kT}{m}}$$

The relevant scale δl is the mean-free-path

$$\delta l \sim \lambda = \frac{1}{n\sigma},$$

where σ is the collision cross section of the particles. If we treat the particles as hard spheres of radius a (decent approximation for neutral gas), then

$$\sigma = \pi a^2$$

So,

$$\text{net momentum flux} = -\rho(\partial_j u_i)\frac{m}{\rho\pi a^2}\alpha\sqrt{\frac{kT}{m}}$$

Putting this into momentum equation:

$$\frac{\partial}{\partial t}(\rho u_i) = -\partial_j(\rho u_i u_j + p\delta_{ij}) + \partial_j\left[\underbrace{\frac{\alpha}{\pi a^2}\sqrt{mkT}}_{\equiv\eta,\text{ shear viscosity}}\partial_j u_i\right] + \rho g_i$$

A rigorous derivation shows that, for this hard-sphere model, $\alpha = 5\sqrt{\pi}/64$.

Observations about the shear viscosity:

- η is independent of density (a denser gas has more particles to transport the momentum but the mean-free-path is shorter);
- η increases with T;
- Isothermal system has $\eta = \text{const.}$

For a fully ionized plasma (e.g. the ICM), the mean-free-path is set by Coulomb collisions. Then

$$\lambda \propto T^2, \qquad v_{\text{th}} \propto \sqrt{T} \qquad \Rightarrow \qquad \eta \propto T^{5/2}.$$

Thus the viscosity has a stronger temperature dependence than found for hard-sphere collisions.

I.2 Navier-Stokes Equation

The most general form of σ'_{ij} which is

- Galilean invariant;
- Linear in velocity components;
- Isotropic

is given by

$$\sigma'_{ij} = \eta\left(\partial_j u_i + \partial_i u_j - \frac{2}{3}\delta_{ij}\partial_k u_k\right) + \zeta\delta_{ij}\partial_k u_k$$

with η and ζ independent of velocity. This is a *symmetric* tensor which ensures that there aren't unbalanced torques on fluid elements.

The term associated with η relates to momentum transfer in *shear flows* (this term has zero trace).

The term associated with ζ relates to momentum transfer due to bulk compression ($\partial_k u_k \equiv \boldsymbol{\nabla}\cdot\mathbf{u}$).

Putting this into the momentum equation gives

$$\begin{aligned}\frac{\partial(\rho u_i)}{\partial t} =& - \partial_j(\rho u_i u_j) - \partial_j p\delta_{ij} \\ &+ \partial_j\left[\eta\left(\partial_j u_i + \partial_i u_j - \frac{2}{3}\delta_{ij}\partial_k u_k\right) + \zeta\delta_{ij}\partial_k u_k\right] + \rho g_i\end{aligned}$$

which we can combine with continuity equation to give

$$\begin{aligned}\rho\left(\frac{\partial u_i}{\partial t} + u_j\partial_j u_i\right) =& - \partial_j p\delta_{ij} \\ &+ \partial_j\left[\eta\left(\partial_j u_i + \partial_i u_j - \frac{2}{3}\delta_{ij}\partial_k u_k\right) + \zeta\delta_{ij}\partial_k u_k\right] + \rho g_i\end{aligned}$$

This is the general form of the *Navier-Stokes equation.*

Outside of shocks ($\zeta \approx 0$) and for isothermal fluid ($\eta =$ constant) we have

$$\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla}p - \boldsymbol{\nabla}\Psi + \underbrace{\frac{\eta}{\rho}}_{\substack{\equiv\nu \\ \text{kinematic} \\ \text{viscosity}}}\left[\nabla^2\mathbf{u} + \frac{1}{3}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u})\right]$$

I.3 Vorticity in Viscous Flows

Start with the Navier-Stokes equation with $\zeta = 0$ and $\eta =$ const.. Take the curl of this, recalling definition of the vorticity $\mathbf{w} = \boldsymbol{\nabla}\times\mathbf{u}$:

$$\begin{aligned}&\frac{\partial\mathbf{w}}{\partial t} + \boldsymbol{\nabla}\times(\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}) \\ &\qquad= \boldsymbol{\nabla}\times\left(-\frac{1}{\rho}\boldsymbol{\nabla}p - \boldsymbol{\nabla}\Psi + \frac{\eta}{\rho}\left[\nabla^2\mathbf{u} + \frac{1}{3}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u})\right]\right)\end{aligned}$$

To tidy up LHS, use the vector identity and definition of vorticity:

$$\begin{aligned}&\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} = \frac{1}{2}\boldsymbol{\nabla}u^2 - \mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{u}) \\ \Rightarrow\quad &\boldsymbol{\nabla}\times(\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}) = -\boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{w}).\end{aligned}$$

To tidy up RHS, assume a barotropic fluid, $p = p(\rho)$:

$$\begin{aligned}\Rightarrow\quad \boldsymbol{\nabla}\times\left(\frac{1}{\rho}\boldsymbol{\nabla}p\right) &= \boldsymbol{\nabla}\left(\frac{1}{\rho}\right)\times\boldsymbol{\nabla}p + \frac{1}{\rho}\boldsymbol{\nabla}\times\boldsymbol{\nabla}p \\ &= -\frac{1}{\rho^2}\underbrace{\boldsymbol{\nabla}\rho\times\boldsymbol{\nabla}p}_{\substack{=0 \text{ since surfaces of} \\ \text{constant } \rho \text{ and } p \text{ align}}}\end{aligned}$$

Putting pieces together, we get

$$\frac{\partial \mathbf{w}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{w}) + \boldsymbol{\nabla} \times \left[\frac{\eta}{\rho}\nabla^2 \mathbf{u}\right]$$

$$\Rightarrow \qquad \boxed{\frac{\partial \mathbf{w}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{w}) + \frac{\eta}{\rho}\nabla^2 \mathbf{w}}$$

where, in the last step, we have ignored gradients of $\nu = \eta/\rho$ (so strictly assumed uniform density). So, vorticity is carried with flow but also diffuses through flow due to action of vorticity.

I.4 Energy Dissipation in Incompressible Viscous Flows

Viscosity leads to dissipation of kinetic energy into heat — an irreversible process.

Let's analyse this in the case of an incompressible flow so that we don't need to about about $p\,dV$ work. Then the total kinetic energy is

$$E_{\text{kin}} = \frac{1}{2}\int \rho u^2 \,\mathrm{d}V$$

Let's consider the rate of change of E_{kin} with time

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2\right) &= u_i \frac{\partial}{\partial t}(\rho u_i) \\
&= -u_i \partial_j(\rho u_i u_j) - u_i \partial_j \delta_{ij} p + u_i \partial_j \sigma'_{ij} \\
&= -u_i \partial_j(\rho u_i u_j) - u_i \partial_i p + \partial_j(u_i \sigma'_{ij}) - \sigma'_{ij}\partial_j u_i
\end{aligned}$$

Look at the first term of RHS:

$$u_i \partial_j(\rho u_i u_j) = u_i \left(u_j \partial_j(\rho u_i) + \rho u_i \cancelto{0}{\partial_j u_j}\right)$$

where last term is zero due to incompressible assumption,

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0 \qquad \Rightarrow \qquad \partial_j u_j = 0$$

Also note that

$$\begin{aligned}
\partial_j \left(\rho u_j \cdot \frac{1}{2} u_i u_i\right) &= \frac{1}{2}\rho u_i u_i \cancelto{0}{\partial_j u_j} + u_j \partial_j \left(\frac{1}{2}\rho u_i u_i\right) \\
&= u_j u_i \partial_j(\rho u_i)
\end{aligned}$$

$$\therefore \qquad u_i \partial_j(\rho u_i u_j) = \partial_j \left(\rho u_j \cdot \frac{1}{2} u_i u_i\right).$$

So,

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2\right) &= -\partial_j\left(\rho u_j \cdot \frac{1}{2}u_i u_i\right) - \partial_i(u_i p) + p\partial_i u_i + \partial_j(u_i\sigma'_{ij}) - \sigma'_{ij}\partial_j u_i \\ &= -\partial_i\left(\rho u_i\left[\frac{1}{2}u^2 + \frac{p}{\rho}\right] - u_j\sigma'_{ij}\right) - \sigma'_{ij}\partial_j u_i.\end{aligned}$$

Integrating over the volume,

$$\begin{aligned}\frac{\partial E_{\text{kin}}}{\partial t} &= \frac{\partial}{\partial t}\int_V \frac{1}{2}\rho u^2 \,\mathrm{d}V \\ &= -\int \partial_i\left(\rho u_i\left[\frac{1}{2}u^2 + \frac{p}{\rho}\right] - u_j\sigma'_{ij}\right)\mathrm{d}V - \int \sigma'_{ij}\partial_j u_i \,\mathrm{d}V \\ &= \underbrace{-\oint\left(\rho\mathbf{u}\left[\frac{1}{2}u^2 + \frac{p}{\rho}\right] - \mathbf{u}\cdot\underline{\underline{\boldsymbol{\sigma}}}'\right)\cdot\mathrm{d}\mathbf{S}}_{\substack{\text{Energy flux into volume including} \\ \text{work done by viscous forces } \mathbf{u}\cdot\underline{\underline{\boldsymbol{\sigma}}}}} \quad \underbrace{-\int \sigma'_{ij}\partial_j u_i \,\mathrm{d}V}_{\substack{\text{Rate of change of } E_{\text{kin}} \text{ due} \\ \text{viscous dissipation}}}\end{aligned}$$

Let's take the volume V to be the whole fluid so that the surface integral is zero (e.g. $\mathbf{v}$ at bounding surface $= 0$, or $\mathbf{v}$ at $\infty = 0$).

Then

$$\begin{aligned}\frac{\partial E_{\text{kin}}}{\partial t} &= -\int \sigma'_{ij}\partial_j u_i \,\mathrm{d}V \\ &= -\frac{1}{2}\int \sigma'_{ij}(\partial_j u_i + \partial_i u_j)\,\mathrm{d}V \qquad \text{since } \sigma' \text{ is symmetric}\end{aligned}$$

But $\sigma'_{ij} = \eta(\partial_j u_i + \partial_i u_j)$ for an incompressible fluid. So,

$$\frac{\partial E_{\text{kin}}}{\partial t} = -\frac{1}{2}\int \eta(\partial_j u_i + \partial_i u_j)^2 \,\mathrm{d}V$$

We see that η needs to be positive in order for us to obey the 2nd law of thermodynamics.

Note. The course book [**pafd**] together with its correction posted on the course website have adopted a different sign convention, choosing to set

$$\sigma_{ij} = \rho u_i u_j + p\delta_{ij} \underbrace{+}_{\substack{\text{sign}\\\text{change}}} \sigma'_{ij}$$

$$\sigma'_{ij} = \underbrace{-}_{\substack{\text{sign}\\\text{change}}} \eta\left(\partial_j u_i + \partial_i u_j - \frac{2}{3}\delta_{ij}\partial_k u_k\right) \underbrace{-}_{\substack{\text{sign}\\\text{change}}} \zeta\delta_{ij}\partial_k u_k$$

The alternative convention adopted in these notes is more standard.

I.5 Viscous Flow through a Pipe

Now consider flow through a long pipe with a constant circular cross-section

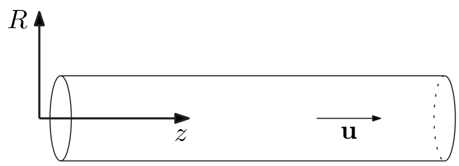

Figure I.2: A pipe

Assume

- Steady flow with $u_R = u_\phi = 0, u_z \neq 0$;
- Incompressible, uniform density fluid;
- Neglect gravity.

Navier-Stokes equation reads

$$\underbrace{\cancelto{0}{\frac{\partial \mathbf{u}}{\partial t}}}_{\substack{\text{steady}\\\text{state}}} + \underbrace{\cancelto{0}{\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}}}_{\text{symmetry}} = -\frac{1}{\rho}\boldsymbol{\nabla}p + \nu\left[\nabla^2\mathbf{u} + \underbrace{\frac{1}{3}\boldsymbol{\nabla}(\cancelto{0}{\boldsymbol{\nabla}\cdot\mathbf{u}})}_{\text{incompressible}}\right]$$

$$\Rightarrow \qquad \nu\nabla^2\mathbf{u} = \frac{1}{\rho}\boldsymbol{\nabla}p$$

By symmetry we have

$$u_R = u_\phi = 0 \qquad \Rightarrow \qquad \frac{\partial p}{\partial R} = \frac{\partial p}{\partial \phi} = 0$$

For the z-component

$$\underbrace{\frac{1}{\rho}\frac{\partial p}{\partial z}}_{\substack{\text{function of}\\ z \text{ only}}} = \underbrace{\nu\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial u_z}{\partial R}\right)}_{\text{function of } R \text{ only}} = \underbrace{-\frac{1}{\rho}\frac{\Delta p}{l}}_{\substack{\text{constant, written}\\ \text{in terms of global}\\ \text{pressure gradient}}}$$

Integrating gives

$$u = -\frac{\Delta p}{4\rho\nu l}R^2 + a\ln R + b$$

where a and b are constants. Apply boundary conditions:

- At $R = 0, u$ finite $\Rightarrow a = 0$;
- At $R = R_0, u = 0$ (no slip BC at wall).

$$\Rightarrow \qquad u = \frac{\Delta p}{4\nu\rho l}(R_0^2 - R^2)$$

So velocity profile is *parabolic*.

The mass flux passing through an annular element $2\pi R\,\mathrm{d}R$ is $2\pi R\rho u\,\mathrm{d}R$. So, the total mass flow rate is

$$Q = \int_0^{R_0} 2\pi\rho u R\,\mathrm{d}R = \frac{\pi}{8}\frac{\Delta p}{\nu l}R_0^4$$

As $\eta \to 0$, i.e. $\nu \to 0$, the flow rate $\to \infty$ (or, in other words, an inviscid flow can't be in steady state in this pipe if there is a non-zero pressure gradient).

If Δp increases sufficiently, it becomes unstable and irregular, giving turbulent motions above critical speed.

The actual transition to turbulence is usually phrased in terms of the *Reynolds number*

$$\mathrm{Re} \equiv \frac{LV}{\nu}$$

where L and V are "characteristic" length and velocity scales of the system. We have turbulence when

$$\mathrm{Re} > \mathrm{Re}_{\mathrm{crit}}$$

I.6 Accretion Disks

Accretion disks are one of the most important applications of the N-S equation in astrophysics.

Consider some gas flowing towards some central object (star, planet, black hole...). Almost always, the gas will have significant angular momentum about that object. If gravitationally bound to the object, the gas will settle into a plane defined by the mean angular momentum vector. Residual motions in other directions will be damped out on a free-fall timescale.

The gas will settle into circular orbits — the lowest energy configuration for a given angular momentum. In the vertical direction (parallel to angular momentum vector) the system will come into hydrostatic equilibrium with internal vertical pressure gradient balancing the vertical component of gravity. In the radial direction (along direction towards central object), the system will achieve a state where the centripetal force is supplied by gravity and the radial pressure gradient.

Very important special case is when the disk is "thin", meaning that scale-height in vertical direction h is much less than radius r. Then, radial pressure gradients are negligible and we can just write

$$\Omega^2 R = \frac{GM}{R^2} \qquad \Rightarrow \qquad \Omega = \sqrt{\frac{GM}{R^3}}$$

where Ω is the angular velocity of the flow around the central object. This means that

$$\frac{\mathrm{d}\Omega}{\mathrm{d}R} \neq 0 \qquad \Rightarrow \qquad \text{shear flow}$$

Viscosity will allow angular momentum to be transferred from the fast moving inner regions to the more slowly moving outer regions. This means the inner disk fluid elements lose angular momentum. We have

$$J = R^2\Omega = \sqrt{GMR} \qquad \text{per unit mass}$$

meaning that inner disk fluid elements drift inwards.

Ultimately, most of the mass flows inwards; a small amount of the mass carries all of the angular momentum out to large radius.

Let's set up a simple model for a geometrically-thin accretion disk. We assume:

- Cylindrical polar coordinates (R, ϕ, z);
- Axisymmetric, $\partial/\partial\phi = 0$;
- Hydrostatic equilibrium in z-direction, $u_z = 0$;
- u_ϕ close to Keplerian velocity (i.e. thin disk);
- u_R small and set by action of viscosity;
- Bulk viscosity zero.

Continuity equation in cylindrical polars is

$$\frac{\partial\rho}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\rho u_R) + \underbrace{\cancelto{0}{\frac{1}{R}\frac{\partial}{\partial\phi}(\rho u_\phi)}}_{\text{axisymmetry}} + \underbrace{\cancelto{0}{\frac{\partial}{\partial z}(\rho u_z)}}_{\text{hydrostatic eqm.}} = 0$$

$$\Rightarrow \qquad \frac{\partial\rho}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\rho u_R) = 0$$

Define the surface density Σ by

$$\Sigma \equiv \int_{-\infty}^{\infty} \rho \, \mathrm{d}z$$

Then, integrating above form of continuity equation over z we have

$$\frac{\partial\Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\Sigma u_R) = 0$$ ① eq.i.6.1

We can get the same result by thinking of the disk as a set of rings/annuli:

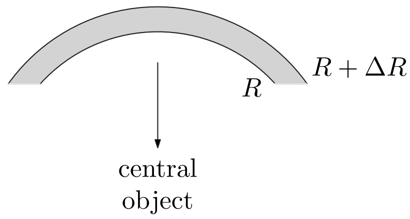

Figure I.3: Infinitesimal annulus element of disk

$$\begin{array}{c}\text{rate of change}\\ \text{of mass in}\\ \text{the annulus}\end{array} = \begin{array}{c}\text{flux into}\\ \text{annulus}\end{array} + \begin{array}{c}\text{flux out}\\ \text{of annulus}\end{array}$$

$$\begin{aligned}
\Rightarrow \quad & \frac{\partial}{\partial t}(2\pi R \Delta R \Sigma) = 2\pi R \Sigma(R) u_R(R) - 2\pi(R+\Delta R)\Sigma(R+\Delta R)u_R(R+\Delta R) \\
\Rightarrow \quad & R\frac{\partial \Sigma}{\partial t} = -\left[\frac{(R+\Delta R)\Sigma(R+\Delta R)u_R(R+\Delta R) - R\Sigma(R)u_R(R)}{\Delta R}\right] \\
\Rightarrow \quad & R\frac{\partial \Sigma}{\partial t} = -\frac{\partial}{\partial R}(R\Sigma u_R) \qquad \text{taking } \Delta R \to 0
\end{aligned}$$

Now we look at conservation of angular momentum. Here we use the ring/annulus approach (but we could also start with the Navier-Stokes equation in cylindrical polars). Clearly,

$$\begin{array}{c}\text{rate of change}\\ \text{of ang. mtm.}\end{array} = \begin{array}{c}\text{ang. mtm.}\\ \text{of mass}\\ \text{entering ring}\end{array} - \begin{array}{c}\text{ang. mtm.}\\ \text{of mass}\\ \text{leaving ring}\end{array} + \begin{array}{c}\text{net torque on ring}\\ \text{(viscous, magnetic, etc.)}\end{array}$$

$$\Rightarrow \quad \frac{\partial}{\partial t}(2\pi R \Delta R \Sigma R^2 \Omega) = f(R) - f(R+\Delta R) + G(R+\Delta R) - G(R)$$

where

$$f(R) \equiv 2\pi R \Sigma u_R \Omega R^2$$

and $G(R)$ is torque exerted by disk outside of radius R on the disk inside of radius R:

$$G(R) = 2\pi R \nu \Sigma R \frac{\partial \Omega}{\partial R} R = 2\pi R^3 \nu \Sigma \frac{\mathrm{d}\Omega}{\mathrm{d}R}$$

$$\therefore \quad \frac{\partial}{\partial t}(R\Sigma u_\phi) = -\frac{1}{R}\frac{\partial}{\partial R}(\Sigma R^2 u_\phi u_R) + \frac{1}{R}\frac{\partial}{\partial R}\left(\nu \Sigma R^3 \frac{\mathrm{d}\Omega}{\mathrm{d}R}\right) \qquad (2)$$

eq.i.6.2

Now assume $\partial u_\phi/\partial t = 0$ since gas is on Keplerian orbits. Then

$$
\begin{aligned}
& Ru_\phi\frac{\partial\Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(\Sigma R^2 u_\phi u_R) = \frac{1}{R}\frac{\partial}{\partial R}\left(\nu\Sigma R^3\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)\\
\Rightarrow\quad & -u_\phi\frac{\partial}{\partial R}(R\Sigma u_R) + \frac{1}{R}\frac{\partial}{\partial R}\left(\Sigma R^2 u_\phi u_R\right) = \frac{1}{R}\frac{\partial}{\partial R}\left(\nu\Sigma R^3\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)\\
\Rightarrow\quad & \cancel{-u_\phi\frac{\partial}{\partial R}(R\Sigma u_R)} + \cancel{\frac{u_\phi R}{R}\frac{\partial}{\partial R}(R\Sigma u_R)} + \Sigma u_R\frac{\partial}{\partial R}(u_\phi R) = \frac{1}{R}\frac{\partial}{\partial R}\left(\nu\Sigma R^3\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)\\
\Rightarrow\quad & R\Sigma u_R\frac{\partial}{\partial R}(R^2\Omega) = \frac{\partial}{\partial R}\left(\nu\Sigma R^3\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)\\
\Rightarrow\quad & u_R = \frac{\frac{\partial}{\partial R}\left(\nu\Sigma R^3\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)}{R\Sigma\frac{\partial}{\partial R}(R^2\Omega)}
\end{aligned}
$$

Substitute this into ① and specialise to the case of a Newtonian point source gravitational field $\Omega = \sqrt{GM/R^3}$ gives

$$
\boxed{\frac{\partial\Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}(\nu\Sigma R^{1/2})\right]}
$$

So the surface density $\Sigma(R,t)$ obeys a diffusion equation.

Note (Notes on accretion disks)**.**

- In general, $\nu = \nu(R,\Sigma)$ and so this is a non-linear diffusion equation for Σ. It reduces to linear if $\nu = \nu(R)$;

- Solutions of this equation show that an initial ring of matter will broaden and then "slump" inwards towards the central object;

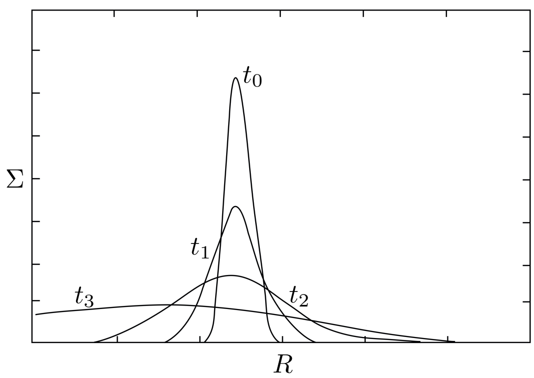

Figure I.4: The viscous evolution of a ring

- Timescale for evolution is t_ν where

$$\begin{aligned} \frac{\Sigma}{t_\nu} &\sim \frac{1}{R}\frac{1}{R}R^{1/2}\frac{1}{R}\nu\Sigma R^{1/2} \sim \frac{\nu\Sigma}{R^2} \\ \Rightarrow \qquad t_\nu &\sim \frac{R^2}{\nu} = \frac{R}{u_\phi}\frac{Ru_\phi}{\nu} = \Omega^{-1}\,\mathrm{Re} \end{aligned}$$

  where Re is the Reynolds number;

- If viscosity is due to particle thermal motions, typical values would suggest that $\mathrm{Re} \sim 10^{14}$! This means

$$t_\nu \ll \text{age of universe}$$

  There must be another source of effective viscosity: we now know that there is an *effective viscisty* due to MHD turbulence driven by the *magnetorotational instability.*

I.7 Steady-State, Geometrically-Thin Disks

Consider a steady state such that $\partial/\partial t = 0$. Then

$$\begin{aligned} &\frac{\partial\Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\Sigma u_R) = 0 \\ \Rightarrow \qquad &R\Sigma u_R = C_1 = -\frac{\dot{m}}{2\pi} \end{aligned}$$

where $\dot{m} = -2\pi R\Sigma u_R$ is the steady state mass inflow rate. Now recall that

$$\begin{aligned} u_R &= \frac{\frac{\partial}{\partial R}\left(\nu\Sigma R^3\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)}{R\Sigma\frac{\partial}{\partial R}(R^2\Omega)} \\ \Rightarrow \qquad -\frac{\dot{m}}{2\pi R\Sigma} &= -\frac{3}{\Sigma R^{1/2}}\frac{\partial}{\partial R}(\nu\Sigma R^{1/2}) \qquad \text{for } \Omega^2 = GM/R^3 \\ \Rightarrow \qquad \nu\Sigma &= \frac{\dot{m}}{3\pi}\left(1 - \sqrt{\frac{R_*}{R}}\right) \end{aligned}$$

where we have taken as a boundary condition that $\nu\Sigma = 0$ at $R = R_*$. This amounts to saying that there are no viscous torques at $R = R_*$. Physically R_* can be:

- Surface of accreting star;

- Innermost circular orbit around a black hole.

Let's now calculate the viscous dissipation neglecting $p\,\mathrm{d}V$ work and bulk viscosity. Specifically, we will calculate the viscous dissipation per unit surface area of the disk:

$$\begin{aligned} F_{\text{diss}} &= -\int \sigma'_{ij}\partial_j u_i \frac{\mathrm{d}V}{2\pi R\,\mathrm{d}R\,\mathrm{d}\phi} \\ &= \frac{1}{2}\int \eta(\partial_j u_i + \partial_i u_j)^2\,\mathrm{d}z \\ &= \int \eta R^2 \left(\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)^2 \mathrm{d}z \\ &= \nu\Sigma R^2 \left(\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)^2 \end{aligned}$$

Combining with our previous result for $\nu\Sigma$ and recalling that $\Omega^2 = GM/R^3$, we have

$$F_{\text{diss}} = \frac{3GM\dot{m}}{4\pi R^3}\left(1 - \sqrt{\frac{R_*}{R}}\right)$$

Note (Notes on dissipation in disk)**.**

- Total energy emitted is

  $$L = \int_{R_*}^{\infty} F_{\text{diss}} 2\pi R\,\mathrm{d}R = \frac{GM\dot{m}}{2R_*}$$

  Here, $-GM/R_*$ is gravitational potential at R_*. Therefore, $GM\dot{m}/R_*$ is the rate of gravitational energy loss of flow. $GM\dot{m}/2R_*$ is radiated, other half stays in flow as kinetic energy and is dissipated in boundary layer on star, or carried into the black hole;

- At given location far from inner edge ($R > R_*$) we have

  $$F_{\text{diss}} \approx \frac{3GM\dot{m}}{4\pi R^3}$$

  But an elementary estimate based on loss of gravitational potential energy would give

  $$F_{\text{diss,est}} = \underbrace{\frac{1}{2\pi R\,\mathrm{d}R}}_{\substack{\text{area of}\\ \text{annulus}}} \cdot \underbrace{\left|\frac{\partial}{\partial R}\left(\frac{GM\dot{m}}{R}\right)\right|}_{\substack{\text{change in grav.}\\ \text{potential of } \dot{m}\\ \text{over annulus}}} \cdot \underbrace{\frac{1}{2}}_{\substack{\text{half converts}\\ \text{to radiation,}\\ \text{rest to kinetic}}} = \frac{GM\dot{m}}{4\pi R^3}$$

  The extra factor of "3" in the correct formula is due to the transport of energy through the disk by viscous torques.

Radiation from Steady-State Thin Disks

If disk is optically-thick, all radiation is thermalised and it radiates locally as a black body

$$\underbrace{2}_{\substack{\text{top and}\\\text{bottom of disk}}} \cdot \sigma_{\mathrm{SB}} T_{\mathrm{eff}}^4 = \frac{3GM\dot{m}}{4\pi R^3}\left(1-\sqrt{\frac{R_*}{R}}\right)$$

$$\Rightarrow \qquad T_{\mathrm{eff}} = \left[\frac{3GM\dot{m}}{8\pi\sigma R^3}\left(1-\sqrt{\frac{R_*}{R}}\right)\right]^{1/4}$$

So, for $R \gg R_*, T_{\mathrm{eff}} \propto R^{-3/4}$.

The radiation emitted at a frequency f is

$$F_f = \int_{R_*}^{\infty} \frac{2h}{c^2}\frac{f^3}{e^{hf/kT_{\mathrm{eff}}}-1} 2\pi R \,\mathrm{d}R$$

So, we see that all of the observables from a steady-state disk are independent of viscosity ν (provided it is large enough to provide necessary angular momentum transport). In order to study/constrain ν, we need to study non-steady disks.

CHAPTER J

Plasmas

Plasmas are fluids composed of charged particles, thus, electromagnetic fields become important for both microphysics and large scale dynamics.

J.1 Magnetohydrodynamic (MHD) Equations

Consider a fully ionised hydrogen plasma, so contains only protons (number density n^+, bulk velocity $\mathbf{u}^+$) and electrons $(n^-, \mathbf{u}^-)$. Mass conservation for each of the proton and electron fluids is

$$\frac{\partial n^+}{\partial t} + \boldsymbol{\nabla} \cdot \left(n^+ \mathbf{u}^+\right) = 0$$
$$\frac{\partial n^-}{\partial t} + \boldsymbol{\nabla} \cdot \left(n^- \mathbf{u}^-\right) = 0$$

The mass density is $\rho = m^+ n^+ + m^- n^-$ and the center-of-mass velocity is

$$\mathbf{u} = \frac{m^+ n^+ \mathbf{u}^+ + m^- n^- \mathbf{u}^-}{m^+ n^+ + m^- n^-}$$

So, we can combine these to give

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0. \qquad \text{Continuity Equation}$$

The continuity equation is the same as found before.

The charge density is $q = n^+ e^+ + n^- e^-$ and the current density is $\mathbf{j} = e^+ n^+ \mathbf{u}^+ + e^- n^- \mathbf{u}^-$. So, the above information also gives conservation of charge equation:

$$\frac{\partial q}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} = 0, \qquad \text{Charge Conservation}$$

When we formulate the momentum equation, we have to consider the Lorentz force on each particle

$$\mathbf{F} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

So, for the two species of particles:

$$m^+n^+\left(\frac{\partial \mathbf{u}^+}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}^+\right)=e^+n^+(\mathbf{E}+\mathbf{u}^+\times\mathbf{B})-f^+\boldsymbol{\nabla}p$$
$$m^-n^-\left(\frac{\partial \mathbf{u}^-}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}^-\right)=e^-n^-(\mathbf{E}+\mathbf{u}^-\times\mathbf{B})-f^-\boldsymbol{\nabla}p$$

where $f^\pm$ is fraction of pressure gradient that accelerates the protons/electrons.

Summing these equations gives

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}\right)=q\mathbf{E}+\mathbf{j}\times\mathbf{B}-\boldsymbol{\nabla}p$$

Ohm's law lets us relate $\mathbf{j}$ to $\mathbf{E}$ and $\mathbf{B}$:

$$\mathbf{j}=\sigma(\mathbf{E}+\mathbf{u}\times\mathbf{B})$$

where σ is the electrical conductivity. This equation is needed to close the set of equations.

So, recapping the current set of equations

$$\frac{\partial \rho}{\partial t}+\boldsymbol{\nabla}\cdot(\rho\mathbf{u})=0 \tag{①}$$ eq.j.1.1

$$\frac{\partial q}{\partial t}+\boldsymbol{\nabla}\cdot\mathbf{j}=0 \tag{②}$$ eq.j.1.2

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}\right)=q\mathbf{E}+\mathbf{j}\times\mathbf{B}-\boldsymbol{\nabla}p \tag{③}$$ eq.j.1.3

$$\mathbf{j}=\sigma(\mathbf{E}+\mathbf{u}\times\mathbf{B}) \tag{④}$$ eq.j.1.4

We need to relate $q,\mathbf{j},\mathbf{E}$ and $\mathbf{B}$ — Maxwell's equations!

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{B}&=0\\
\boldsymbol{\nabla}\cdot\mathbf{E}&=\frac{q}{\epsilon_0}\\
\boldsymbol{\nabla}\times\mathbf{B}&=\mu_0\mathbf{j}+\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\\
\boldsymbol{\nabla}\times\mathbf{E}&=-\frac{\partial \mathbf{B}}{\partial t}
\end{aligned}$$

where we note $\epsilon_0\mu_0=1/c^2$.

Simplifying MHD

Let us simplify in the case of a non-relativistic, highly conducting plasma. Suppose fields are varying over length scales l and timescales τ. Then

(i)

$$\boldsymbol{\nabla}\times\mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t}\qquad\Rightarrow\qquad\frac{E}{B}\sim\frac{l}{\tau}\sim u$$

(ii)

$$\left|\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\right| \Big/ |\boldsymbol{\nabla}\times\mathbf{B}| \sim \frac{1}{c^2}\left(\frac{l}{\tau}\right)^2 \sim \frac{u^2}{c^2} \ll 1$$

for non-relativistic flows. Therefore, displacement current can be ignored in non-relativistic MHD;

(iii) Look at two terms from ③:

$$\frac{|q\mathbf{E}|}{|\mathbf{j}\times\mathbf{B}|} \sim \frac{qE}{jB} \sim \frac{\epsilon_0 E/l}{B/l\mu_0}\frac{E}{B} \sim u^2\epsilon_0\mu_0 \sim \left(\frac{u}{c}\right)^2 \ll 1$$

Therefore, charge neutrality is preserved to a high approximation due to strength of electrostatic forces. If there is a charge imbalance, it will oscillate with a characteristic frequency, the *plasma frequency*

$$\omega_p = \sqrt{\frac{ne^2}{\epsilon_0 m_e}}$$

(iv) Neglecting displacement current in relevant Maxwell equation, we get

$$\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{j} = \mu_0\sigma(\mathbf{E}+\mathbf{u}\times\mathbf{B})$$

Take curl:

$$\begin{aligned}
&\underbrace{\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{B})}_{=-\nabla^2\mathbf{B}-\boldsymbol{\nabla}(\cancelto{0}{\boldsymbol{\nabla}\cdot\mathbf{B}})} = \mu_0\sigma(\underbrace{\boldsymbol{\nabla}\times\mathbf{E}}_{-\frac{\partial\mathbf{B}}{\partial t}}+\boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{B}))\\
\Rightarrow\qquad &\frac{\partial\mathbf{B}}{\partial t} = \underbrace{\boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{B})}_{\substack{\text{advection of}\\\text{the field}\\\text{by the flow}}} + \underbrace{\frac{1}{\mu_0\sigma}\nabla^2\mathbf{B}}_{\substack{\text{dissipation of}\\\text{the field}\\\text{through the flow}}}
\end{aligned}$$

If the fluid is a good conductor, i.e. σ is very large, then we can ignore the diffusion term and we have an equation that is analogous to the Helmholtz equation/Kelvin's theorem:

$$\frac{\partial\mathbf{B}}{\partial t} = \boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{B})$$

We talk about the "freezing" of the magnetic flux into the plasma. In the high σ limit we must also have

$$\begin{aligned}
&\mathbf{j} = \sigma(\mathbf{E}+\mathbf{u}\times\mathbf{B}) &&\text{is finite}\\
\Rightarrow\quad &\mathbf{E}+\mathbf{u}\times\mathbf{B} = 0 &&\text{as } \sigma\to\infty\\
\Rightarrow\quad &\mathbf{E}\cdot\mathbf{B} = 0\\
\text{i.e.}\quad &\mathbf{E}\perp\mathbf{B}
\end{aligned}$$

So, the full set of *ideal MHD equations*, i.e., equations describing a non-relativistic, perfectly conducting, charge neutral plasma are:

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0\\
&\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}\right) = \mathbf{j}\times\mathbf{B} - \boldsymbol{\nabla}p\\
&\left.\begin{aligned}
&\mathbf{E} + \mathbf{u}\times\mathbf{B} = 0\\
&\boldsymbol{\nabla}\cdot\mathbf{B} = 0\\
&\boldsymbol{\nabla}\cdot\mathbf{E} = 0\\
&\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{j}\\
&\boldsymbol{\nabla}\times\mathbf{E} = -\,\partial\mathbf{B}/\partial t\\
&p = K\rho^\gamma
\end{aligned}\right\} \quad \Rightarrow \quad
\begin{aligned}
&\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{B})\\
&\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{j}\\
&\boldsymbol{\nabla}\cdot\mathbf{B} = 0
\end{aligned}
\end{aligned}$$

J.2 The Dynamical Effects of Magnetic Fields

The magnetic force density appearing in the above ideal MHD equations is

$$\mathbf{f}_{\text{mag}} = \mathbf{j}\times\mathbf{B} = \frac{1}{\mu_0}(\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B}$$

So using vector identity this is

$$\mathbf{f}_{\text{mag}} = \frac{1}{\mu_0}\Bigg[\underbrace{-\boldsymbol{\nabla}\left(\frac{B^2}{2}\right)}_{\substack{\text{magnetic pressure}\\ \text{term with}\\ p_{\text{mag}}=B^2/2\mu_0}} + \underbrace{(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{B}}_{\substack{\text{magnetic tension}\\ \text{term (vanishes for}\\ \text{straight field lines)}}}\Bigg]$$

Since there are new force terms in the momentum equation, this will change the nature of the waves that are possible.

J.3 Waves in Plasmas

We can repeat the perturbation analysis that we conducted for sound waves but now include the effects of a magnetic field. We will perturb about an equilibrium consisting of a static ($\mathbf{u} = 0$) plasma with uniform density ρ_0, uniform pressure p_0, and uniform magnetic field $\mathbf{B}_0$.

We start by writing down the governing equations of ideal MHD, assuming a barotropic equation of state:

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0 \\
&\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla}) \mathbf{u} \right) = \frac{1}{\mu_0} (\boldsymbol{\nabla} \times \mathbf{B}) \times \mathbf{B} - \boldsymbol{\nabla} p \qquad \text{(J.1)} \\
&\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{B}) \\
&\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \\
&p = p(\rho)
\end{aligned}
\tag{J.2}
$$

We now introduce perturbations and linearize the equations:

$$
\begin{aligned}
&\frac{\partial \delta\rho}{\partial t} + \rho_0 \boldsymbol{\nabla} \cdot (\delta \mathbf{u}) = 0 \\
&\rho_0 \frac{\partial \delta \mathbf{u}}{\partial t} = \frac{1}{\mu_0} (\boldsymbol{\nabla} \times \delta \mathbf{B}) \times \mathbf{B}_0 - c_s^2 \boldsymbol{\nabla} \delta\rho \\
&\frac{\partial \delta \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\delta \mathbf{u} \times \mathbf{B}_0) = -\mathbf{B}_0 (\boldsymbol{\nabla} \cdot \delta \mathbf{u}) + (\mathbf{B}_0 \cdot \boldsymbol{\nabla}) \delta \mathbf{u} \\
&\boldsymbol{\nabla} \cdot \delta \mathbf{B} = 0
\end{aligned}
$$

We now adopt our usual plane wave form for the perturbations,

$$
\begin{aligned}
\delta\rho &= \delta\rho_1 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \\
\delta p &= \delta p_1 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \\
\delta \mathbf{u} &= \delta \mathbf{u}_1 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \\
\delta \mathbf{B} &= \delta \mathbf{B}_1 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}.
\end{aligned}
$$

The continuity equation becomes,

$$
\begin{aligned}
&-i\omega\delta\rho + i\rho_0 \mathbf{k} \cdot \delta \mathbf{u} = 0 \\
\Rightarrow \qquad &\omega \delta\rho = \rho_0 \mathbf{k} \cdot \delta \mathbf{u}.
\end{aligned}
$$

The momentum equation becomes,

$$
\begin{aligned}
&-i\omega\rho_0 \delta \mathbf{u} = \frac{i}{\mu_0} (\mathbf{k} \times \delta \mathbf{B}) \times \mathbf{B}_0 - i c_s^2 \delta\rho \mathbf{k} \\
\Rightarrow \qquad &\omega \rho_0 \delta \mathbf{u} = \frac{1}{\mu_0} \left([\mathbf{B}_0 \cdot \delta \mathbf{B}) \mathbf{k} - (\mathbf{B}_0 \cdot \mathbf{k}) \delta \mathbf{B} \right] + c_s^2 \delta\rho\, \mathbf{k}.
\end{aligned}
$$

Finally, the flux-freezing (induction) equation becomes,

$$
\begin{aligned}
&-i\omega \delta \mathbf{B} = -i \mathbf{B}_0 (\mathbf{k} \cdot \delta \mathbf{u}) + i (\mathbf{B}_0 \cdot \mathbf{k}) \delta \mathbf{u} \\
\Rightarrow \qquad &\omega \delta \mathbf{B} = \mathbf{B}_0 (\mathbf{k} \cdot \delta \mathbf{u}) - (\mathbf{B}_0 \cdot \mathbf{k}) \delta \mathbf{u}.
\end{aligned}
$$

The full dispersion relation for MHD waves is then derived from eliminating the perturbation amplitudes from these expressions. Here, we are going to gain insight for the physics by just focusing on some special cases.

Firstly, we consider the case of modes with wavevectors orthogonal to the background magnetic field direction, $\mathbf{k} \perp \mathbf{B}_0$. The linearized equations then become

$$
\begin{aligned}
\omega\delta\rho &= \rho_0 \mathbf{k} \cdot \delta\mathbf{u} \\
\omega\rho_0\delta\mathbf{u} &= \frac{1}{\mu_0}(\mathbf{B}_0 \cdot \delta\mathbf{B})\mathbf{k} + c_s^2 \delta\rho\, \mathbf{k} \\
\omega\delta\mathbf{B} &= \mathbf{B}_0(\mathbf{k} \cdot \delta\mathbf{u})
\end{aligned}
$$

We can immediately notice from the second of these relations that the velocity perturbations are aligned with the wavevector, $\delta\mathbf{u} \parallel \mathbf{k}$, i.e. these are longitudinal modes. Eliminating $\delta\rho$ and $\delta\mathbf{B}$ from this set of equations in favour of $\delta\mathbf{u}$, we get

$$
\omega^2\rho_0\delta\mathbf{u} = \frac{1}{\mu_0} B_0^2(\mathbf{k} \cdot \delta\mathbf{u})\mathbf{k} + c_s^2\rho_0(\mathbf{k} \cdot \delta\mathbf{u})\mathbf{k}
$$

Take the dot product of this last equation with $\mathbf{k}$ and then cancel $\mathbf{k} \cdot \delta\mathbf{u}$ throughout (since we know that this must be non-zero since modes are longitudinal),

$$
\begin{aligned}
\omega^2\rho_0 &= \frac{k^2 B_0^2}{\mu_0} + c_s^2\rho_0 k^2 \\
\Rightarrow \qquad \omega^2 &= \left(c_s^2 + \frac{B^2}{\mu_0\rho_0}\right) k^2 \\
\omega^2 &= \left(c_s^2 + v_A^2\right) k^2,
\end{aligned}
$$

where we have defined the Alfvén speed,

$$
v_A = \sqrt{\frac{B_0^2}{\mu_0\rho_0}},
$$

This describes dispersion-free longitudinal waves with a phase speed $\sqrt{c_s^2 + v_A^2}$. The restoring force comes from both the gas pressure and magnetic pressure acting in phase. This is known as the fast magnetosonic wave.

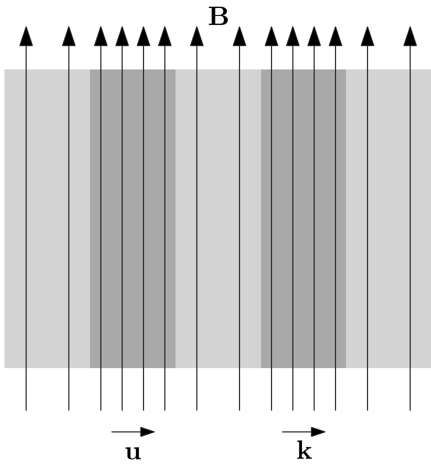

Figure J.1: Fast magnetosonic wave

We now consider the case of modes with $\mathbf{k} \parallel \mathbf{B}_0$. The linearized equations become,

$$\begin{aligned} \omega\delta\rho &= \rho_0 \mathbf{k}\cdot\delta\mathbf{u} \\ \omega\rho_0\delta\mathbf{u} &= \frac{1}{\mu_0}\left[(\mathbf{B}_0\cdot\delta\mathbf{B})\mathbf{k} - B_0 k\,\delta\mathbf{B}\right] + c_s^2\delta\rho\,\mathbf{k} \\ \omega\delta\mathbf{B} &= \mathbf{B}_0(\mathbf{k}\cdot\delta\mathbf{u}) - B_0 k\delta\mathbf{u}. \end{aligned}$$

Eliminating $\delta\rho$ and $\delta\mathbf{B}$ from this set of equations in favour of $\delta\mathbf{u}$, we get

$$\omega^2\rho_0\delta\mathbf{u} = \frac{1}{\mu_0}(B_0^2k^2\delta\mathbf{u} - (\mathbf{B}_0\cdot\delta\mathbf{u})B_0k\mathbf{k}) + c_s^2(\mathbf{k}\cdot\delta\mathbf{u})\mathbf{k}.$$

There are actually two distinct wave modes wrapped up in these expression, a longitudinal mode and a transverse mode. To extract the longitudinal mode, take the dot product with $\mathbf{k}$

$$\omega^2\rho_0(\mathbf{k}\cdot\delta\mathbf{u}) = \frac{1}{\mu_0}(B_0^2k^2(\mathbf{k}\cdot\delta\mathbf{u}) - (\mathbf{B}_0\cdot\delta\mathbf{u})B_0k^3) + c_s^2(\mathbf{k}\cdot\delta\mathbf{u})k^2$$

and cancel factor of $\mathbf{k}\cdot\delta\mathbf{u}$ to get

$$\Rightarrow \omega^2 = c_s^2k^2.$$

These are simply sound waves, with the magnetic field not playing a role since the velocity perturbations are directed along the magnetic field.

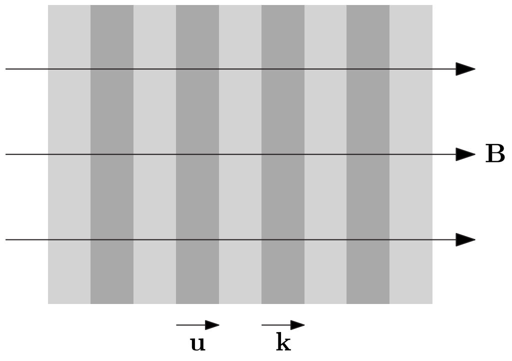

Figure J.2: Longitudinal wave with $\mathbf{k} \parallel \mathbf{B}$

Return to the more general expression for the case $\mathbf{k} \parallel \mathbf{B}_0$;

$$\omega^2 \rho_0 \delta \mathbf{u} = \frac{1}{\mu_0}(B_0^2 k^2 \delta \mathbf{u} - (\mathbf{B}_0 \cdot \delta \mathbf{u}) B_0 k \mathbf{k}) + c_s^2 (\mathbf{k} \cdot \delta \mathbf{u})\mathbf{k}.$$

Taking the cross product with $\mathbf{k}$, we get

$$\omega^2 = \frac{B_0^2}{\mu_0 \rho_0} k^2 = v_A^2 k^2.$$

This describes transverse waves with phase speed v_A where the restoring force is provided by magnetic tension. These are Alfvèn waves.

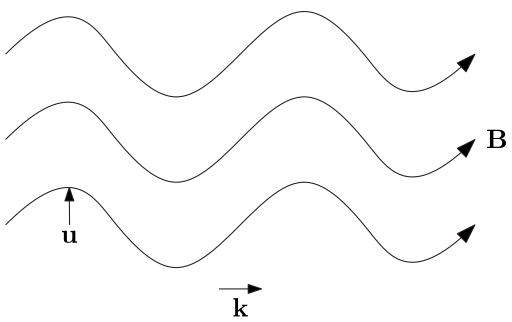

Figure J.3: Alfvén wave

J.4 Instabilities in Plasmas

The presence of magnetic forces can profoundly affect the nature of instabilities in plasmas. For example, we can repeat the derivation of the Rayleigh-Taylor instability including a magnetic field aligned with the interface.

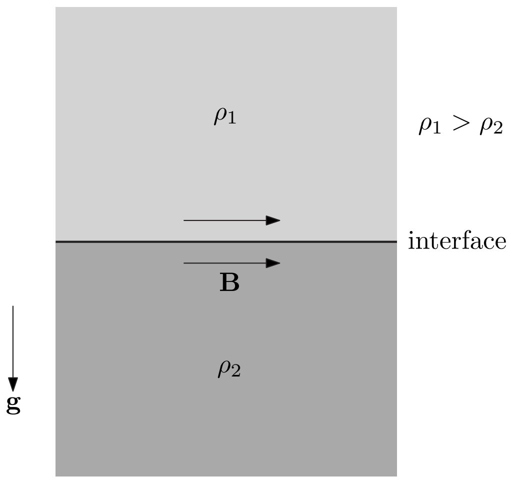

Figure J.4: Configuration of fluid interface

We will not repeat the analysis here, but we find the new dispersion relation is

$$\omega^2 = -kg\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} + \frac{2}{\mu_0}\frac{(\mathbf{k} \cdot \mathbf{B})^2}{\rho_1 + \rho_2}$$

For sufficiently small wavelength (high $|\mathbf{k}|$), the second term always wins, giving stable oscillations (Alfvén waves in this case). The interpretation is that magnetic tension forces tend to *stabilise* R-T modes.

J.5 Magnetorotational Instability

We end with a discussion of an MHD instability which is extremely important for accretion disks. We examine the stability of a plasma which is in orbit about a central object.

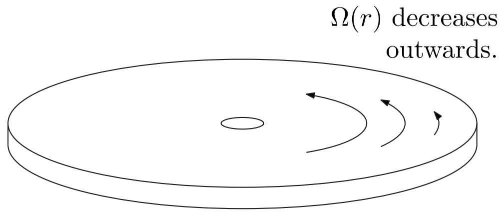

Figure J.5: Shear flow in an accretion disk

To uncover the essence of the instability, we simplify as much as possible. We conduct a "local analysis" meaning that we consider the dynamics in some small patch of the rotating flow at $\mathbf{R} = \mathbf{R}_0$, working in the comoving reference frame of the equilibrium flow. We assume that the equilibrium flow has an angular velocity about the central body $\Omega(R)$. We let our local frame of reference rotate at $\Omega(R_0)$ and set up a Cartesian coordinate system with $\hat{\mathbf{z}}$ pointing "upwards" (meaning aligned with the angular velocity $\boldsymbol{\Omega}$) and $\hat{\mathbf{x}}$ pointing outwards (i.e. away from the central body axis). Working in a Lagrangian picture, the momentum equation is:

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \frac{1}{\mu_0\rho}(\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B} + 2\mathbf{u}\times\boldsymbol{\Omega} - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}) - R\Omega(R)^2\hat{\mathbf{R}},$$

where the last term is an expression of gravity. Further simplifying, let us assume that the flow is cold so that pressure forces are negligible (this assume can be readily relaxed but will make the analysis more involved). Introducing perturbations and assuming a plane-wave form, we have

$$\begin{aligned} & \frac{D\,\Delta\mathbf{u}}{Dt} - 2\Delta\mathbf{u}\times\boldsymbol{\Omega} = \frac{1}{\mu_0\rho}(\mathbf{B}_0\cdot\boldsymbol{\nabla})\Delta\mathbf{B} - \Delta x\, R\frac{d\Omega^2}{dR}\hat{\mathbf{R}} \\ \Rightarrow \quad & -i\omega\Delta\mathbf{u} - 2\Delta\mathbf{u}\times\boldsymbol{\Omega} = \frac{i}{\mu_o\rho}B_0 k\,\Delta\mathbf{B} - \Delta x\, R\frac{d\Omega^2}{dR}\hat{\mathbf{R}} \end{aligned}$$

The induction equation gives

$$\begin{aligned} & \frac{\partial\Delta\mathbf{B}}{\partial t} = \boldsymbol{\nabla}\times(\Delta\mathbf{u}\times\mathbf{B}_0) = (\mathbf{B}_0\cdot\boldsymbol{\nabla})\Delta\mathbf{u} \\ \Rightarrow \quad & -i\omega\,\Delta\mathbf{B} = ikB_0\,\Delta\mathbf{u} \\ \Rightarrow \quad & \Delta\mathbf{B} = -\frac{kB_0}{\omega}\Delta\mathbf{u}, \end{aligned}$$

and we can easily relate Δx and Δu_x;

$$\begin{aligned} & \frac{D\,\Delta x}{Dt} = \Delta u_x \\ \Rightarrow \quad & -i\omega\,\Delta x = \Delta u_x \\ \Rightarrow \quad & \Delta x = \frac{i\,\Delta u_x}{\omega}. \end{aligned}$$

So eliminating in favour of $\Delta\mathbf{u}$ in our perturbed form of the momentum equation, we have

$$-i\omega\Delta\mathbf{u} - 2\Delta\mathbf{u}\times\boldsymbol{\Omega} = \frac{i}{\mu_o\rho}B_0 k\,\frac{kB_0}{\omega}\Delta\mathbf{u} - \frac{i\,\Delta u_x}{\omega}\,R\frac{d\Omega^2}{dR}\hat{\mathbf{R}}$$

Writing this out in components and noting that $B_0^2/\rho_0\mu_0 = v_A^2$ gives,

$$\begin{aligned} \omega^2\Delta u_x - 2i\Delta u_y\Omega\omega &= (kv_A)^2\Delta u_x + \Delta u_x\frac{d\Omega^2}{d(\ln R)} \\ \omega^2\Delta u_y + 2i\Delta u_x\Omega\omega &= (kv_A)^2\Delta u_y, \end{aligned}$$

or in matrix form.

$$\begin{pmatrix} \omega^2 - (kv_A)^2 - \frac{d\Omega^2}{d(\ln R)} & -2i\omega\Omega \\ 2i\omega\Omega & \omega^2 - (kv_A)^2 \end{pmatrix} \begin{pmatrix} \Delta u_x \\ \Delta u_y \end{pmatrix} = 0.$$

We obtain the dispersion relation by setting the determinant of the matrix to zero. This gives

$$\left[\omega^2 - (kv_A)^2 - \frac{d\Omega^2}{d(\ln R)}\right]\left[\omega^2 - (kv_A)^2\right] - 4\Omega^2\omega^2 = 0$$

Writing as a quadratic in ω^2 gives our final form of the dispersion relation:

$$\omega^4 - \omega^2\left[4\Omega^2 + \frac{d\Omega^2}{d(\ln R)} + 2(kv_A)^2\right] + (kv_A)^2\left[(kv_A)^2 + \frac{d\Omega^2}{d(\ln R)}\right] = 0.$$

If we "turn off" magnetic forces by setting $v_A = 0$, the dispersion relation gives

$$\begin{aligned} \omega^2 &= 4\Omega^2 + \frac{d\Omega^2}{d(\ln R)} \\ &= \frac{1}{R^3}\frac{d}{dR}(R^4\Omega^2) \equiv \kappa_R^2 \\ &= \Omega^2. \qquad \text{(Keplerian)} \end{aligned}$$

For a Keplerian profile $\Omega^2 = GM/R^3$, or indeed any profile in which the specific angular momentum $R^2\Omega$ increases with radius, this describes local radial oscillations of the flow at the radial epicyclic frequency κ_R.

Now turn on magnetic forces, so $v_A > 0$. There will be instability if $\omega^2 < 0$. Considering the basic properties of the dispersion relation, viewed as a quadratic in ω^2, we see that there will be instability if

$$(kv_A)^2 + \frac{d\Omega^2}{d(\ln R)} < 0.$$

This is the magneto-rotational instability (MRI). For sufficiently weak magnetic field or long wavelength (small k) modes, there will be instability if the angular velocity decreases outwards,

$$\frac{d\Omega^2}{dR} < 0 \qquad \text{(instability)}.$$

Magnetic tension will stabilize modes with $k > k_{\text{crit}}$ where

$$(k_{\text{crit}}v_A)^2 = -\frac{d\Omega^2}{d(\ln R)} \quad (= 3\Omega^2 \text{ for Keplerian})$$

Specializing to the Keplerian case, we find that the fastest growing mode has a growth rate

$$|\omega_{\max}| = \frac{3}{2}\Omega$$

and wavenumber given by

$$k_{\max} v_A \approx \Omega.$$

The instability has an interesting property — while the magnetic field is essential for its existence, the maximum growth rate is independent of the magnetic field. Formally, within ideal hydrodynamics, the instability exists as $B_0 \to 0$ but not at $B_0 = 0$. Of course, the wavelength of the mode with the maximum growth rate $k_{\max} \to \infty$ as $B_0 \to 0$ and so in practice finite viscosity or finite conductivity effects will kill the MRI for sufficiently small B_0.

The MRI is central to the modern theory of accretion disks. MRI drives the turbulence that, as we have described previously, is essential for the transport of angular momentum in an accretion disk.

Bibliography

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