Rotation I

Moment of inertia tensor

For a rigid body, the velocity of a point at $\underline{r}$ is given by $\underline{v} = \underline{\omega} \times \underline{r}$. Then,

$$\underline{J} = \sum \underline{r} \times \underline{f}$$
$$= \sum m \underline{r} \times (\underline{\omega} \times \underline{r})$$
$$= \sum m (r^2 \underline{\omega} - (\underline{\omega} \cdot \underline{r}) \underline{r})$$

$$\Rightarrow J_i = I_{ij} \omega_j$$

, where $I_{ij} = \sum m (r^2 \delta_{ij} - x_i x_j)$

Also, $T = \frac{1}{2} \sum m v^2$

$$= \frac{1}{2} \sum m (\underline{\omega} \times \underline{r}) \cdot (\underline{\omega} \times \underline{r})$$

$$= \frac{1}{2} \left(\sum m \underline{r} \times (\underline{\omega} \times \underline{r}) \right) \cdot \underline{\omega}$$

$$\Rightarrow T = \frac{1}{2} \underline{J} \cdot \underline{\omega}$$

Now I is a symmetric tensor, so can diagonalise:

$$I = \begin{pmatrix} I_1 & & 0 \\ & I_2 & \\ 0 & & I_3 \end{pmatrix}$$

$$\therefore \underline{J} = (I_1 \omega_1, I_2 \omega_2, I_3 \omega_3)$$

$$T = \frac{1}{2} (I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2)$$

Matrices recap

If A is a transformation matrix s.t

$$A\hat{e}_i = \hat{e}_i',$$

what is A_{ij}?

$$\Rightarrow A_{jk}(\hat{e}_i)_k = \hat{e}_i' \cdot \hat{e}_j$$

$$\Rightarrow A_{jk}\delta_{ik} = \hat{e}_i' \cdot \hat{e}_j$$

$$\Rightarrow A_{ji} = \hat{e}_i' \cdot \hat{e}_j$$

$$\Rightarrow A_{ij} = \hat{e}_j' \cdot \hat{e}_i$$

$$\Rightarrow A = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \hat{e}_1' & \hat{e}_2' & \cdots & \hat{e}_n' \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}$$

If A rotates axes, what is A_{ij}?

$$\underline{V} = V_i\hat{e}_i = V_j'\hat{e}_j'$$

$$\Rightarrow V_k' = V_i\hat{e}_i \cdot \hat{e}_k' = A_{ki}V_i$$

$$\Rightarrow A_{ki} = \hat{e}_k' \cdot \hat{e}_i$$

$$\Rightarrow A_{ij} = \hat{e}_i' \cdot \hat{e}_j$$

$$\Rightarrow A = \begin{pmatrix} \leftarrow & \hat{e}_1' & \rightarrow \\ \leftarrow & \hat{e}_2' & \rightarrow \\ & \vdots & \\ \leftarrow & \hat{e}_n' & \rightarrow \end{pmatrix}$$

Euler angles

3 rotations of coordinate axes:

1) Rotate coordinates about z by ϕ:

$xyz \to x'y'z'$

$$D = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2) Rotate about x' by θ:

$x'y'z' \to x''y''z''$

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

3) Rotate about z'' by ψ:

$x''y''z'' \to 123$

$$B = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Overall, from $xyz \to 123$, trans matrix is $A = BCD$:

$$A = \begin{pmatrix} \cos\phi\cos\psi - \sin\phi\sin\psi & \sin\phi\cos\psi + \sin\psi\cos\theta\cos\phi & \sin\psi\sin\theta \\ -\cos\phi\sin\psi - \sin\phi\cos\psi & -\sin\phi\sin\psi + \cos\psi\cos\theta\cos\phi & \sin\theta\cos\psi \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

Now a general rotation is given by

$\underline{\omega} = \dot\phi\,\hat\omega_\phi + \dot\theta\,\hat\omega_\theta + \dot\psi\,\hat\omega_\psi$

$\hat\omega_\phi = (0,0,1)$ in xyz

$\rightarrow \hat\omega_\phi = (\sin\theta\sin\psi, \sin\theta\cos\psi, \cos\theta)$ in 123

$\hat\omega_\theta = (1,0,0)$ in $x''y''z''$

$\rightarrow \hat\omega_\theta = (\cos\psi, -\sin\psi, 0)$ in 123

$\hat\omega_\psi = (0,0,1)$ in 123

$$\rightarrow \underline{\omega} = \begin{pmatrix} \dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi \\ \dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi \\ \dot\phi\cos\theta + \dot\psi \end{pmatrix} \text{ in } 123 \quad \text{--- (1)}$$

<u>Euler's rotation eqs</u>

$$L = T - V = \frac{1}{2}\sum_{i=1}^{3} I_i\,\omega_i^2 - V$$

$$\text{E-L:}\quad \frac{d}{dt}\left(\frac{\partial T}{\partial\dot\psi}\right) - \frac{\partial T}{\partial\psi} = -\frac{\partial V}{\partial\psi} = \overset{\text{Ext torque}}{G_\psi}$$

$$\text{Now}\quad \frac{\partial T}{\partial\dot\psi} = \sum_{i=1}^{3}\frac{\partial T}{\partial\omega_i}\frac{\partial\omega_i}{\partial\dot\psi},\quad \frac{\partial T}{\partial\psi} = \sum_{i=1}^{3}\frac{\partial T}{\partial\omega_i}\frac{\partial\omega_i}{\partial\psi}$$

$$\frac{\partial\omega_1}{\partial\psi} = \omega_2 \qquad \frac{\partial\omega_1}{\partial\dot\psi} = 0$$

$$\frac{\partial\omega_2}{\partial\psi} = -\omega_1 \qquad \frac{\partial\omega_2}{\partial\dot\psi} = 0$$

$$\frac{\partial\omega_3}{\partial\psi} = 0 \qquad \frac{\partial\omega_3}{\partial\dot\psi} = 1$$

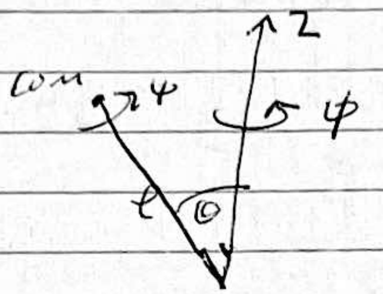
→ E-L become $I_3 \dot{\omega}_3 - (I_1 - I_2)\omega_1 \omega_2 = G_3$

Generalise: $I_1 \dot{\omega}_1 - (I_2 - I_3)\omega_2 \omega_3 = G_1$

$I_2 \dot{\omega}_2 - (I_3 - I_1)\omega_3 \omega_1 = G_2$

$I_3 \dot{\omega}_3 - (I_1 - I_2)\omega_1 \omega_2 = G_3$

<u>Symmetric top w/ point fixed</u>



$\dot{\phi}$ = 'Precession'

$\dot{\theta}$ = 'Nutation'

Using θ, can write

$$L = \frac{1}{2} I_1 (\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{1}{2} I_3 (\dot{\phi}\cos\theta + \dot{\psi})^2 - Mgl\cos\theta$$

$$p_\psi = I_3(\dot{\psi} + \dot{\phi}\cos\theta) = I_3 \omega_3 = I_1 a = \text{const} \quad \dots (2)$$

$$p_\phi = (I_1 \sin^2\theta + I_3 \cos^2\theta)\dot{\phi} + I_3 \dot{\psi}\cos\theta = I_1 b = \text{const} \quad \dots (3)$$

Can see that $\frac{\partial L}{\partial t} = 0$, so $H = E = T + V$ is const.

$$\frac{1}{2} I_1 (\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{1}{2} I_3 (\dot{\phi}\cos\theta + \dot{\psi})^2 + Mgl\cos\theta = E \text{ say} \quad \dots (4)$$

Now, solve for $\dot{\psi}, \dot{\phi}$ in (2), (3) to get

$$\dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta} \quad \dots (5)$$

$$\dot{\psi} = \frac{I_1}{I_3} a - \frac{\cos\theta(b - a\cos\theta)}{\sin^2\theta} \quad \dots (6)$$

(5), (6) into (4):

$$\frac{1}{2}\dot{\theta}^2 + \frac{Mgl\cos\theta}{I_1} + \frac{1}{2}\frac{(b-a\cos\theta)^2}{\sin^2\theta} = \frac{E'}{I_1} \quad \cdots (7)$$

where $E' = E - \frac{1}{2}I_3\omega_3^2$

Now, make dimless. Choose $t \sim \frac{1}{b}$ & define

$$p = \frac{Mgl}{I_1 b^2} \cdots (8), \quad \xi = \frac{a}{b} \cdots (9), \quad \varepsilon = \frac{E'}{I_1 b^2} \cdots (10)$$

$$(7): \frac{1}{2}\left(\frac{d\theta}{dt'}\right)^2 + p\cos\theta + \frac{(1-\xi\cos\theta)^2}{2\sin^2\theta} = \varepsilon \quad \cdots (11)$$

Let $u = \cos\theta$

$$\Rightarrow (11): \left(\frac{du}{dt'}\right)^2 = 2(1-u^2)(\varepsilon - pu) - (1-\xi u)^2 = f(u) \cdots (12)$$

Note, allowed values of u are $-1 \leq u \leq 1$.

$f(-1) = -(\xi+1)^2 < 0$
$f(1) = -(\xi-1)^2 < 0$

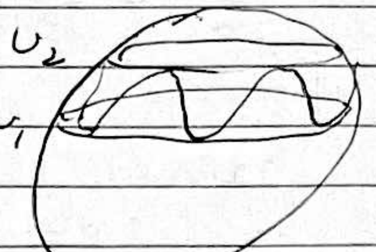
$\Rightarrow$ So, must have one real root for $u > 1$
$\Rightarrow$ must have 0, 1 or 2 real roots for $-1 < u < 1$.
0 roots $\Rightarrow f(u) < 0 \ \forall u \in [-1,1] \Rightarrow$ not allowed
so 1 or 2 roots.

Locus of figure axis
Let u_1 and u_2 be roots of $f(u)$ in $[-1,1]$ and $u' = \frac{1}{\xi}$. From ⑫

$$\frac{d\phi}{dt'} = \frac{1 - \frac{u}{u'}}{1-u^2} \quad \text{... ⑬}$$

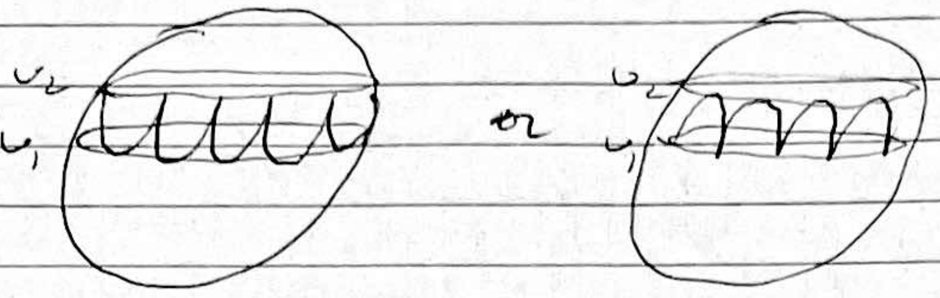
Remember, $f(u)$ must be nonnegative always, so $u_1 < u < u_2 \Rightarrow$ figure axis remains between two bounding circles always.

1) If $u' > u_2$, then since $u \in [u_1, u_2]$, $u < u'$ always, so from ⑬, $\frac{d\phi}{dt'} > 0$ always $\Rightarrow$ unidirectional precession



2) If $u_1 < u' < u_2$, at $u = u_2$, $\frac{d\phi}{dt'} < 0$ and at $u = u_1$, $\frac{d\phi}{dt'} > 0$:

3) If $u' = u_1$ or u_2, at u_1 or u_2, both $\dot{\theta} = 0$ and $\dot{\phi} = 0$:



~~Ex.~~ First pic above is actually $u_0 = u_2$ and is $\dot\theta(0) = \dot\psi(0) = 0$.

~~which~~ Also, $\dot\theta(0) = \dot\psi(0) = 0$ into (8):

$$b = \frac{I_3}{I_1}\dot\psi\, u_0$$

But (2): $a = \dfrac{I_3}{I_1}\dot\psi$

$$\Rightarrow u' = \frac{1}{\Sigma} = \frac{b}{a} = u_0 = u_2 \leftarrow$$

<u>Fast top and $\dot\theta(0) = \dot\phi(0) = 0$</u>

Assume $\dot\theta(0) = \dot\phi(0) = 0$ and

$$\frac{1}{2} I_3 \omega_3^2 \gg mgl \quad \ldots (14) \quad \text{(Fast top)}$$

Putting $u = u_0$ into (12) and using $f(u_0) = 0$ and $\Sigma = \frac{1}{u_0}$, get

$$\Sigma = p\, u_0 \quad \ldots (15)$$

(15) into (12),

$$f(u) = 2p(u - u_0)\left(u^2 - \frac{1}{2pu_0^2}u + \frac{1}{2pu_0} - 1\right)$$

Get roots of quadratic - define

$$I = \frac{1}{2pu_0^2} = \frac{I_1 a^2}{2mgl} = \frac{I_3}{I_1}\,\frac{\frac{1}{2} I_3 \omega_3^2}{mgl}$$

(14) means $I \gg 1$. Quadratic is then

$$\nu^2 - I\nu - (1 - I\nu_0) = 0$$

$$\rightarrow \nu_\pm = \frac{I}{2}\left(1 \pm \left(1 - \frac{4\nu_0}{I} + \frac{4}{I^2}\right)^{\frac{1}{2}}\right)$$

$$\rightarrow \nu_\pm \simeq \frac{I}{2}\left(1 \pm \left(1 - \frac{2\nu_0}{I} + \frac{2(1-\nu_0^2)}{I^2}\right)\right)$$

$$\nu_+ \simeq I \gg 1 \text{ so not allowed}$$

$$\nu_- \simeq \nu_0 - \frac{1-\nu_0^2}{I}$$

so $\quad \nu_1 = \nu_- \simeq \nu_0 - \sin^2\theta_0 \dfrac{I_1}{I_3} \dfrac{2Mgl}{I_3\omega_3^2} \quad \cdots$ (16)

1) $|\nu_1 - \nu_0|$ gives extent of nutation. If ω_3 increases, nutation decreases

Frequency of nutation: Energy eq:

$$\dot{\nu}^2 = 2p(\nu - \nu_0)\left(\nu^2 - \frac{1}{2p\nu_0^2}\nu + \frac{1}{2p\nu_0} - 1\right) \quad (17)$$

If (14) is satisfied, ν is close to ν_0 i.e.

$$\nu = \nu_0 + x, \text{ where } x \ll \nu_0$$

means

$\frac{d}{dt'} \rightarrow$ (17): $\quad \dot{x}^2 = -2p\sin^2\theta_0 x + 2p(2\nu_0 - I)x^2 + O(x^3)$

$$\rightarrow \frac{d^2x}{dt'^2} + 2p(I - 2\nu_0)x = -p\sin^2\theta_0$$

But $t' = bt$

$$\rightarrow \ddot{x} + 2b^2 p I x = -b^2 p \sin^2\theta_0 \quad \cdots (18)$$

$$\Rightarrow u^2 = 2b^2 / I = a^2$$
$$\Rightarrow u = a = \frac{I_3}{I_1}\omega_3 \quad \ldots (19)$$

2) So larger ω_3 means faster nutation.

Precession: $\dot\phi = \frac{b(1-\xi u)}{1-u^2}$ (13)

let $u = u_0 + x$, $x \ll u_0$

$$\Rightarrow \dot\phi \approx -\frac{a x}{\sin^2\theta_0}$$

Solve (18) w B.C: $x(0)=0$, $\dot x(0)=0$

$$\Rightarrow x(t) = -\frac{\sin^2\theta_0}{2I}(1-\cos(at))$$

$$\Rightarrow \dot\phi \approx \frac{a}{2I}(1-\cos(at)) = \frac{a}{I}\sin^2\left(\frac{at}{2}\right)$$

$$\Rightarrow \langle\dot\phi\rangle = \frac{a}{2I} = \frac{Mgl}{I_3\omega_3} \quad \ldots (20)$$

3) So larger ω_3 means slower precession.

1), 2), 3) are the relevant conclusions.

<u>Initial conds for pure precession</u>
Want to find initial conds which lead to $\dot\theta = 0\ \forall t$ and just pure precession.

$\Rightarrow$ u_0 must be a double root, so $f(u_0)=0$ and $\left.\frac{df}{du}\right|_{u_0} = 0$.

Using $f(u) = 2(1-u^2)(\epsilon - p u) - (1-\xi u)^2$;

$f(u_0)=0$: $\epsilon - p u_0 = \dfrac{(1-\xi u_0)^2}{2(1-u_0^2)}$ ... (21)

$\dfrac{df}{du}\Big|_{u_0} = 0$: $2u_0(\epsilon - p u_0) + p(1-u_0^2) - \xi(1-\xi u_0) = 0$ ... (22)

(21) into (22):

$$(1-\xi u_0)\left(u_0(1-\xi u_0) - \xi(1-u_0^2)\right) + p(1-u_0^2)^2 = 0$$

Use $1-\xi u_0 = \dfrac{\dot\phi \sin^2\theta_0}{b}$

$$\to \frac{\dot\phi \sin^2\theta_0}{b}\left(\frac{\dot\phi \sin^2\theta_0 \cos\theta_0}{b} - \xi \sin^2\theta_0\right) + p \sin^4\theta_0 = 0$$

$$\to \frac{I_3\omega_3}{I_1}\dot\phi - \dot\phi^2 \cos\theta_0 = \frac{Mgl}{I_1} \quad \text{... (22)}$$

(22) is a general eq for $\dot\phi$.

There are real solutions to (22), i.e. pure precession is possible if

$$\omega_3 > \omega_{3c} = \frac{2}{I_3}\sqrt{Mgl I_1 \cos\theta_0} \quad \text{... (23)}$$

~~For ex~~ If $\omega_3 > \omega_{3c}$, can choose ω_3 and θ_0 to set desired precession rate.

If precession slow, neglect $\dot\phi^2$ term in (21)

$$\to \dot\phi = \frac{Mgl}{I_3\omega_3}$$

Can also get this result if $\omega_3 \gg \omega_{3c}$.

Sol of 22:

$$\dot\phi = \frac{I_3\omega_3}{2I_1\cos\theta_0}\left(1 \pm \left(1 - \frac{4mgl I_1\cos\theta_0}{I_3^2\omega_3^2}\right)^{\frac{1}{2}}\right)$$

If $\omega_3 \gg \omega_{3c}$,

$$\dot\phi \simeq \frac{I_3\omega_3}{2I_1\cos\theta_0}\left(1 \pm \left(1 - \frac{2mgl I_1\cos\theta_0}{I_3^2\omega_3^2}\right)\right)$$

$$\rightarrow \dot\phi_- = \frac{mgl}{I_3\omega_3}$$

$$\dot\phi_+ = \frac{I_3}{I_1\cos\theta_0}\omega_3 \quad (24)$$

$u=1$ is a root of $f(u)$

(11) at $t=0$: $u(0)=1$, ~~$\dot u(0)=0$~~

$\rightarrow \varepsilon = p \quad (25)$

$\rightarrow E - \frac{1}{2}I_3\omega_3^2 = mgl$

(25) into (12): $\varepsilon = 1$ and so

$$f(u) = (1-u)^2\left[2p(1+u) - 1\right]$$

$\rightarrow u=1$ is a double root, other root is

$$u_1 = \frac{1}{2p} - 1 = \frac{I_1 b^2}{2mgl} - 1 = \frac{I_1 a^2}{2mgl} - 1$$

$$\rightarrow u_1 = \frac{I_3^2\omega_3^2}{2mglI_1} - 1$$

If $u_1 > 1$, i.e if $\omega_3 > \omega_{3c} = \frac{2}{I_3}\sqrt{mglI_1}$, top remains vertical

If $\omega_3 < \omega_{3c}$, top nutates between $u_2 = 0$ and $u_1 = \frac{1}{2p} - 1 \neq u_2$.

If $\omega_3 > \omega_{3c}$ initially and $\theta_0 = 0$, friction slows ω_3 to $< \omega_{3c}$, so top starts to nutate (wobble).