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METHODS OF THEORETICAL PHYSICS

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CHAPTER 9

Approximate Methods

Exact solutions of the equations of physics may be obtained for only a limited class of problems. For example, if the equation is the scalar Helmholtz equation, the method of separation of variables can be used in only 11 coordinate systems (see Sec. 5.1 for a discussion of this point). If the surface upon which boundary conditions are to be satisfied is not one of these coordinate surfaces, or if the boundary conditions are not the simple Dirichlet or Neumann types, the method of separation fails. If the Schroedinger equation is considered, the number of coordinate systems yielding separability may be sharply reduced because separability is possible for only particular types of dependence of the potential energy on the coordinates.

A similar situation exists if an integral equation formulation is employed. Here solutions may be obtained, through the use of transforms, for example, only if the kernel of the integral equation is of a particular form. To illustrate, the Fourier transform is appropriate if the kernel is a function of the difference of two variables and if the range of integration extends from $-\infty$ to $+\infty$ or from 0 to ∞. It is, of course a rather rare event for a problem to fall into the exactly soluble class, so that we are faced with the task of developing approximate techniques of sufficient power to handle the rest. Indeed, we shall find that, even for the problems which can be solved exactly, it may be more convenient to employ approximate methods, for the evaluation of the exact solution may be much too complicated.

Deviations from exactly soluble situations will be referred to as *perturbations*. *Surface perturbations* will refer to deviations in the boundary surface or boundary conditions (or both) from the exactly soluble form. For example, we may wish to determine the resonant frequencies in a room with a trapezoidal cross section, a surface which is not a coordinate surface of one of the 11 coordinate systems in which the scalar Helmholtz equation separates. Or the cross section may be rectangular, but the walls of the room may be acoustically treated so that the boundary conditions are no longer homogeneous Neumann conditions.

Volume perturbations will refer to deviations within the volume which take the problem outside the exactly solvable class. For example, the potential in the Schroedinger equation may be of such a nature that the equation is not separable. This generally occurs when two or more interacting particles move in a central field of force or, as a special case, a single particle moving in the fields generated by several centers of force. The Schroedinger equation for two electrons moving in the field of an atomic nucleus of charge Ze is

$$\left\{(\nabla_1^2 + \nabla_2^2) + \frac{2m}{\hbar^2}\left(E + \frac{ze^2}{r_1} + \frac{ze^2}{r_2} - \frac{e^2}{r_{12}}\right)\right\} \psi(\mathbf{r}_1,\mathbf{r}_2) = 0$$

where r_1 and r_2 are the distances of particles 1 and 2 from the nucleus, while r_{12} is the distance between the two particles. In the absence of the term e^2/r_{12} the equation is separable; the term e^2/r_{12} is a volume perturbation. Volume perturbation in acoustic problems involves the existence of regions of inhomogeneity in the volume, corresponding to a nonconstant index of refraction, which may arise from objects such as pillars, curtains, or nonuniform temperature distributions in the medium.

In this chapter, we shall consider three essentially different methods for obtaining approximate solutions: (1) the *perturbation* method, (2) the *variational* method, and (3) the V-I method, where V-I is short for *variation-iterational.* In the perturbation method, the volume or surface perturbations are assumed to be small and expansions in powers of a parameter measuring the size of the perturbation may be made, the leading term being the solution in the absence of any perturbation.

When the perturbation becomes large, this is an inconvenient procedure and then the variational method is more appropriate. As we showed in Chap. 3, the equations of physics may be placed in a variational form; *i.e.* it is possible to find a quantity involving the unknown function which is to be stationary upon variation of the function. In practical use a functional type, called the trial function, involving one or more parameters is inserted for the unknown function into the variational principle. The function may be varied by changing the value of the parameters. Improvements can be obtained by increasing the flexibility of the functional form, thereby involving the introduction of additional parameters.

Another method for improving the original trial function makes the trial function the first term in an expansion in a complete set of functions which are not necessarily mutually orthogonal. Neither of these procedures is as systematic as the iterative scheme employed in the V-I method. Moreover, the V-I method provides an estimate of error for each stage of the calculation, perhaps its most important attribute.

These methods may be applied to nearly all the problems we shall encounter in this book. The problems may involve determining the resonant frequencies for acoustical or electromagnetic vibrations within a cavity. We may wish to determine the capacity of a configuration of conductors or the flow of a fluid past an obstacle. Or we may be interested in the scattering and diffraction of waves by obstacles or, in quantum mechanics, in the scattering by a potential. All three methods are applicable to each situation but display, as we shall see, important differences in detail.

9.1 *Perturbation Methods*

Perturbation methods are particularly appropriate whenever the problem under consideration closely resembles one which is exactly solvable. It presumes that these differences are not singular in character, indeed, that one may change from the exactly solvable situation to the problem under consideration in a gradual fashion. This is expressed analytically by requiring that the perturbation be a continuous function of a parameter λ, measuring the strength of the perturbation.

If this is the case, it becomes possible to develop several formulas which describe the change in the physical situation as λ varies from zero and which differ principally in their rates of convergence; for as λ increases, more and more terms, involving higher and higher powers of λ, are required in order to attain an accurate result. Generally speaking these are just parts of infinite series in powers of λ, converging for λ smaller than λ_0, the radius of convergence, for which we shall develop both a qualitative meaning and a quantitative estimate. In addition, it is possible to give a practical method for the analytic continuation of the perturbation series beyond λ_0, so that it will become possible to solve strong as well as weak perturbation problems.

In the discussion to follow we shall first discuss perturbation formulas which are appropriate for problems having a discrete eigenvalue spectrum and where the perturbation is of the volume type. We then shall go on to apply these results to boundary perturbations in Sec. 9.2, to scattering and diffraction problems in Sec. 9.3.

The Usual Perturbation Formula. Perturbation formulas, as commonly developed, may be shown to be the consequence of the application of the iterative procedure to the integral equation formulation of the problem. As an example consider the Shroedinger equation describing the quantum mechanical motion of a particle in one dimension under the influence of a potential $\lambda V(x)$:

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \lambda V(x)\psi = E\psi$$

where E is the energy of the particle. Introducing the abbreviations

$$k^2 = (2M/\hbar^2)E; \quad U = (2M/\hbar^2)V$$

we obtain

$$(d^2\psi_n/dx^2) + [k^2 - \lambda U(x)]\psi = 0 \tag{9.1.1}$$

We shall assume that the motion is limited to the region $0 \leq x \leq L$ by an infinitely high potential barrier (not included in U) at $x = 0$ and $x = L$. As a consequence ψ must satisfy the boundary conditions

$$\psi(0) = 0 = \psi(L)$$

The exactly solvable problem will involve the same boundary conditions and will satisfy the equation

$$(d^2\varphi_n/dx^2) + k_n^2\varphi_n = 0 \tag{9.1.2}$$

The solutions are

$$\varphi_n = \sqrt{2/L}\,\sin(n\pi x/L); \quad n \text{ integer} \tag{9.1.3}$$
$$k_n^2 = \left(\frac{n\pi}{L}\right)^2; \quad \int_0^L \varphi_n\varphi_m\,dx = \delta_{nm}$$

The integral equation formulation of (9.1.1) may be obtained with the aid of the appropriate Green's function, a solution of

$$(d^2/dx^2)G_k(x|x_0) + k^2G_k(x|x_0) = -\delta(x - x_0)$$

satisfying the same boundary conditions as ψ. From Eq. (7.2.8)

$$\psi(x) = -\lambda \int_0^L G_k(x|x_0)U(x_0)\psi(x_0)\,dx_0 \tag{9.1.4}$$

The Green's function may be expressed in terms of φ_n [see Eq. (7.2.39)] as follows:

$$G_k(x|x_0) = \sum_p \frac{\varphi_p(x)\varphi_p(x_0)}{k_p^2 - k^2} \tag{9.1.5}$$

For one-dimensional cases G_k may be expressed in closed form:

$$G_k(x|x_0) = \frac{1}{k\sin(kL)}\begin{cases}\sin(kx)\sin[k(L - x_0)]; & (x \leq x_0)\\ \sin(kx_0)\sin[k(L - x)]; & (x \geq x_0)\end{cases} \tag{9.1.6}$$

Inserting series (9.1.5) into integral equation (9.1.4) yields

$$\psi(x) = \lambda \sum_p \frac{\int_0^L \varphi_p(x_0)U(x_0)\psi(x_0)\,dx_0}{k^2 - k_p^2}\,\varphi_p(x) \tag{9.1.7}$$

It is now convenient to separate from the series that mode which ψ approaches as λ goes to zero. Let this be φ_n, and let the corresponding ψ be ψ_n, so that

$$\psi_n \xrightarrow[\lambda \to 0]{} \varphi_n$$

We now rewrite the expansion above as follows:

$$\psi_n(x) = \varphi_n + \lambda \sum_{p \neq n} \frac{\int_0^L \varphi_p(x_0) U(x_0) \psi(x_0)\, dx_0}{k^2 - k_p^2} \varphi_p(x) \tag{9.1.8}$$

Here we have chosen the normalization of ψ_n so that the coefficient of φ_n is unity. This is always possible, since ψ_n, multiplied by an arbitrary constant, is still a solution of (9.1.1). The requirement that the coefficient of φ_n in Eq. (9.1.7) be unity leads to the condition

$$k^2 = k_n^2 + \int_0^L \varphi_n(x_0) U(x_0) \psi(x_0)\, dx_0 \tag{9.1.9}$$

We may verify that this is consistent with expansion (9.1.8) and the differential equation (9.1.1) as follows: Multiply (9.1.1) from the left by φ_n, and integrate from 0 to L, obtaining:

$$\int_0^L \varphi_n \left(\frac{d^2\psi}{dx^2}\right) dx + k^2 \int_0^L \psi\varphi_n\, dx - \lambda \int_0^L \varphi_n U \psi\, dx = 0$$

The first term may be evaluated by successive integration by parts and application of the differential equation satisfied by φ_n, so that

$$\int_0^L \varphi_n \left(\frac{d^2\psi}{dx^2}\right) dx = -k_n^2 \int_0^L \psi\varphi_n\, dx$$

We observe that $\int \psi\varphi_n\, dx = 1$, which follows from expansion (9.1.8) and the orthogonality condition in (9.1.3). Inserting these results leads immediately to Eq. (9.1.9).

We may now proceed to apply the iterative procedure to Eq. (9.1.8), inserting the results in Eq. (9.1.9) to obtain the value of k^2. As a zeroth approximation to ψ_n, $\psi_n^{(0)}$ we take φ_n and insert into the right-hand side of (9.1.8) for ψ_n to obtain the first iterate $\psi_n^{(1)}$

$$\psi_n^{(1)} = \varphi_n(x) + \lambda \sum_{p \neq n} \left(\frac{U_{pn}}{k^2 - k_p^2}\right) \varphi_p \tag{9.1.10}$$

where

$$U_{pn} = \int_0^L \varphi_p(x_0) U(x_0) \varphi_n(x_0)\, dx_0 \tag{9.1.11}$$

The second iterate $\psi_n^{(2)}$ is generated by inserting $\psi^{(1)}$ into (9.1.8)

$$\psi_n^{(2)} = \varphi_n(x) + \lambda \sum_{p \neq n} \left(\frac{U_{pn}}{k^2 - k_n}\right) \varphi_p + \lambda^2 \sum_{pq \neq n} \left[\frac{U_{pq} U_{qn}}{(k^2 - k_p^2)(k^2 - k_q^2)}\right] \varphi_p \tag{9.1.12}$$

where by $pq \neq n$ we mean that the terms $p = n$ and $q = n$ are to be omitted from the sum. Generally

$$\psi_n^{(a)} = \varphi_n + \lambda \sum_{p \neq n} \left[\frac{\int \varphi_p U \psi_n^{(a-1)}\, dx_0}{k^2 - k_p} \right] \varphi_p(x)$$

so that

$$\psi_n^{(a)} = \varphi_n + \lambda \sum_{p \neq n} \left(\frac{U_{pn}}{k^2 - k_p^2} \right) \varphi_p + \cdots + \lambda^a \sum_{pq \cdots \neq n} \left[\frac{U_{pq} U_{qr} \cdots U_{zn}}{(k^2 - k_p^2)(k^2 - k_q^2) \cdots (k^2 - k_z^2)} \right] \varphi_p \tag{9.1.13}$$

It should be noted that these formulas contain the unknown k^2. This is now to be obtained by inserting (9.1.13) into Eq. (9.1.9). The relation determining k^2 to this order is

$$k^2 = k_n^2 + \lambda U_{nn} + \lambda^2 \sum_{p \neq n} \frac{U_{np} U_{pn}}{k^2 - k_p^2} + \cdots + \lambda^{a+1} \sum_{pq \cdots \neq n} \frac{U_{np} U_{pq} U_{qr} U_{rs} \cdots U_{zn}}{(k^2 - k_p^2)(k^2 - k_q^2)(k^2 - k_r^2) \cdots (k^2 - k_z^2)} \tag{9.1.14}$$

This equation is now solved by successive approximation. If we call the ath approximation $(k^2)^{(a)}$:

$$(k^2)^{(1)} = k_n^2 + \lambda U_{nn}$$

$$(k^2)^{(2)} = k_n^2 + \lambda U_{nn} + \lambda^2 \sum_{p \neq n} \frac{U_{np} U_{pn}}{k_n^2 - k_p^2}$$

$$(k^2)^{(3)} = k_n^2 + \lambda U_{nn} + \lambda^2 \sum_{p \neq n} \frac{U_{np} U_{pn}}{k_n^2 + \lambda U_{nn} - k_p^2} + \lambda^3 \sum_{pq \neq n} \frac{U_{np} U_{pq} U_{qn}}{(k_n^2 - k_p^2)(k_n^2 - k_p^2)}$$

$$(k^2)^{(4)} = k_n^2 + \lambda U_{nn} + \lambda^2 \sum_{p \neq n} \frac{U_{np} U_{pn}}{k_n^2 + \lambda U_{nn} + \lambda^2 \sum_{q \neq n} \frac{U_{nq} U_{qn}}{k_n^2 - k_q^2} - k_p^2} + \lambda^3 \sum_{pq \neq n} \frac{U_{np} U_{pq} U_{qn}}{(k_n^2 + \lambda U_{nn} - k_p^2)(k_n^2 + \lambda U_{nn} - k_q^2)} + \lambda^4 \sum_{pqr \neq n} \frac{U_{np} U_{pq} U_{qr} U_{rn}}{(k_n^2 - k_p^2)(k_n^2 - k_q^2)(k_n^2 - k_r^2)}$$

More generally

$$(k^2)^{(a)} = k_n^2 + \lambda U_{nn} + \lambda^2 \sum_{p \neq n} \frac{U_{np}U_{pn}}{(k^2)^{(a-2)} - k_p^2} + \lambda^3 \sum_{pq \neq n} \frac{U_{np}U_{pq}U_{qn}}{[(k^2)^{(a-3)} - k_p^2][(k^2)^{(a-3)} - k_p^2]} + \cdots$$
$$+ \lambda^a \sum_{pq\cdots \neq n} \frac{U_{np}U_{pq}U_{qr}U_{rs} \cdots U_{zn}}{(k_n^2 - k_p^2)(k_n^2 - k_q^2)(k_n^2 - k_r^2) \cdots (k_n^2 - k_z^2)} \tag{9.1.15}$$

The corresponding wave functions are

$$\psi_n^{(1)} = \varphi_n(x) + \lambda \sum_{p \neq n} \left(\frac{U_{pn}}{k_n^2 - k_p^2}\right) \varphi_p(x)$$
$$\psi_n^{(2)} = \varphi_n(x) + \lambda \sum_{p \neq n} \left(\frac{U_{pn}}{k_n^2 + \lambda U_{nn} - k_p^2}\right) \varphi_p(x) + \lambda^2 \sum_{pq \neq n} \left[\frac{U_{pq}U_{qp}}{(k_n^2 - k_p^2)(k_n^2 - k_a^2)}\right] \varphi_p(x)$$
$$\psi_n^{(a)} = \varphi_n(x) + \lambda \sum_{p \neq n} \left[\frac{U_{pn}}{(k^2)^{(a-1)} - k_p^2}\right] \varphi_p + \lambda^2 \sum_{pq \neq n} \frac{U_{pq}U_{qn}}{[(k^2)^{(a-2)} - k_p^2][(k^2)^{(a-2)} - k_q^2]} \varphi_p + \cdots$$
$$+ \lambda^a \sum_{pq\cdots \neq n} \frac{U_{pq}U_{qr}U_{rs} \cdots U_{zn}}{(k_n^2 - k_p^2)(k_n^2 - k_q^2)(k_n^2 - k_r^2) \cdots (k_n^2 - k_z^2)} \varphi_p \tag{9.1.16}$$

the ath approximation in each case being accurate to λ^a [but it should be emphasized that (9.1.14) and (9.1.16) are not expansions in powers of λ, for the denominators contain λ]. The power series may be obtained by making approximate expansions of the denominators, a process which yields the Raleigh-Schroedinger[1] perturbation formulas. It is clear that the latter are not only considerably more complicated in appearance and application than Eqs. (9.1.15) and (9.1.16) but also will generally have a smaller radius of convergence than the above series. Also note that in evaluating Eq. (9.1.14) we have inserted for k^2 the approximation required to give λ^a accuracy. In practical applications it is often convenient for calculation to replace k^2 by $(k^2)^{(a-1)}$ everywhere.

Convergence of Series. The convergence of the series which occurs in the expression (9.1.16) for $\psi^{(a)}$ and (9.1.15) for $(k^2)^{(a)}$ is, of course,

[1] See any standard book on quantum mechanics, for example, Kemble, "The Fundamental Principles of Quantum Mechanics," pp. 380–388, McGraw-Hill, New York, 1937.

vital for their practical application. As far as the series for $\psi^{(a)}$ are concerned, we are interested in convergence in the mean (see Sec. 6.3) which in turn requires the convergence of the sum of the squares of the coefficients of φ_p in the expansion of an approximate ψ; or more explicitly, if $\psi^{(a)} = \Sigma A_p\varphi_p$, then we require that

$$\Sigma|A_p|^2 = \int_0^L |\psi^{(a)}|^2\, dx$$

converge. We can consider Eq. (9.1.16) directly, but it is more convenient to employ the original integral equation (9.1.4) itself, from which we see that

$$\psi^{(a)} = \varphi_n - \lambda \int_0^{\prime L} G_k(x|x_0)\, U(x_0)\psi^{(a-1)}(x_0)\, dx_0$$

where the prime on the integral indicates that the component of the integral proportional to φ_n should be dropped. Convergence in the mean of the expansions of $\psi^{(a)}$ requires that the integral above be quadratically integrable. In the one-dimensional problem being considered, G_k is piecewise continuous but has a discontinuous slope. Hence, presuming the convergence of $\psi^{(a-1)}$, a necessary condition for convergence of the integral is the integrability of U. Actually, we see that U can be rather singular and still have the integral converge; U can be a Dirac delta function or even the first derivative of a Dirac function. However, the second derivative of a delta function would be too singular, for the integral would then be proportional to the second derivative of $G_k\psi^{(a-1)}$, which is singular. (Note that the singularity in the delta function corresponds to the singularity of $1/R$, in one dimension; in n dimensions it corresponds to $1/R^n$.)

Stated in terms of the elements U_{pn}, convergence in the mean for series (9.1.16) will result if

$$|U_{pn}|^2 \xrightarrow[p\to\infty]{} k_p^{3-\epsilon}; \quad \epsilon > 0 \tag{9.1.17}$$

Of course if U involves a Dirac delta function or its derivative, the convergence of the series for $\psi^{(a)}$ will be slow, rendering practical application tedious and time consuming. Convergence may be speeded up by summing the series $\Sigma B_p\varphi_p$ in closed form, where A_p approaches B_p as p approaches infinity. Then this series may be subtracted from the original one, and the sum in closed form added on. The new series will converge more rapidly than the old. A procedure for obtaining such sums is suggested by the fact that the series representing $\int G_k U\psi^{(a)}\, dx_0$ and $\int G_0 U\psi^{(a)}\, dx_0$ converge in exactly the same way, since the slow rate of convergence arises from the discontinuous behavior of the slope of G_k.

We are thus led to consider the function

$$\chi^{(a)} = \lambda \int_0^L G_0(x|x_0)\, U(x_0)\psi^{(a-1)}\, dx_0$$

This function satisfies the differential equation

$$d^2\chi^{(a)}/dx^2 = \lambda U\psi^{a-1}$$

The function $\psi^{(a)}$ is then

$$\psi^{(a)} = \varphi_n - (\chi^{(a)})' - \lambda \int_0^{L'} (G_k - G_0)U(x_0)\psi^{(a-1)}\,dx_0$$
$$= \varphi_n - (\chi^{(a)})' - \lambda k^2 \sum_{p\neq n} \frac{\int_0^L \varphi_p U\psi^{(a-1)}\,dx_0}{k_p^2(k^2 - k_p^2)}\varphi_p$$

Where the primes on χ_a and the integral mean that the component of χ_a proportional to φ_n should be subtracted; *i.e.*,

$$(\chi_a)' = \chi_a - \varphi_n \int_0^L \chi_a\varphi_n\,dx$$

The important point that develops from this discussion is that convergence can be improved if the *inhomogeneous zero-energy problem can be solved* either in a closed form or in a more convergent form than (9.1.15).

Convergence in the mean of the series in (9.1.16) for $\psi^{(a)}$ does not ensure convergence of the corresponding series in Eq. (9.1.15) for $(k^2)^{(a)}$. To investigate this point substitute in Eq. (9.1.8) for (k^2), Eq. (9.1.16) for $\psi^{(a)}$. Then an approximate value of (k^2) is

$$k^2 \simeq k_n^2 + U_{nn} + \lambda \int_0^L dx \int_0^{L'} dx_0[\varphi_n(x)U(x)G_k(x|x_0)U(x_0)\psi^{(a-1)}(x_0)] \tag{9.1.18}$$

Again integrability of U is a necessary condition for convergence. U may be as singular as a Dirac delta function, but in contrast to the result for the series for ψ, it cannot be as singular as the first derivative of the Dirac delta function [for then G_k would be differentiated twice on evaluating the integral in (9.1.18)]. This does not invalidate our results for $\psi^{(a)}$ but indicates that in this case the convergence of the series for $\psi^{(a)}$ must be improved by some such technique as that discussed above before it can be substituted in (9.1.8) to obtain a value for k^2.

Multidimensional Problems. Formulas (9.1.7) to (9.1.15) apply to more general problems than (9.1.1). For example, if we are dealing with the motion of a particle in three dimensions in a potential $U_0 + \lambda U$, then ψ satisfies

$$\nabla^2\psi + (k^2 - U_0 - \lambda U)\psi = 0 \tag{9.1.19}$$

where the unperturbed problem, with exact solutions φ_n and eigenvalues k_n, is represented by the equation

$$\nabla^2\varphi_n + (k_n^2 - U_0)\varphi_n = 0$$

Or we may be dealing with the motion of two particles, which would require employing six dimensions (three coordinates for each particle). The equation is then

$$\nabla_1^2\psi + \nabla_2^2\psi + (k^2 - U_0 - \lambda U)\psi = 0 \tag{9.1.20}$$

where ψ is now a function of six variables, x_1, y_1, z_1, and x_2, y_2, z_2. In each of these cases the perturbation is given by λU and the results given in Eqs. (9.1.7) to (9.1.15) may be employed, where now

$$U_{pn} = \int \cdots \int \bar{\varphi}_p(x_1,x_2, \ldots ,x_N) U(x_1,x_2, \ldots ,x_N) \varphi_n(x_1,x_2, \ldots ,x_N)\, dx_1\, dx_2 \cdots dx_N \tag{9.1.21}$$

where the number of dimensions is N ($= 3$ or 6 in the examples above).

The convergence discussion must be modified, since the nature of the singularity of G_k changes with the number of dimensions. In one dimension, G_k has a discontinuous first derivative at $x = x_0$; in two dimensions, G_k has a logarithmic singularity; in three or more dimensions, $G_k \to AR^{-N+2}$, where R is the distance between source and observation point and N is the number of dimensions. We therefore consider the integral

$$\int \frac{U(\mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|^{N-2}}\, dv_0; \qquad N \geq 3$$
$$\int \ln|\mathbf{r} - \mathbf{r}_0| U(\mathbf{r}_0)\, dv_0; \quad N = 2$$

where $\mathbf{r}$ and $\mathbf{r}_0$ are vectors in a space of N dimensions.

A quadratically integrable result will be obtained after integration if $U(\mathbf{r}_0)$ is singular at $\mathbf{r}_1$ like $|\mathbf{r}_0 - \mathbf{r}_1|^{N-2}\delta(\mathbf{r}_0 - \mathbf{r}_1)$, or $|\mathbf{r}_0 - \mathbf{r}_1|^{-2}$, for all N except $N = 2$ where the form is $|\mathbf{r}_0 - \mathbf{r}_1|^{\epsilon}\delta$ ($\epsilon > 0$). These functions can be differentiated $N - 1$ times, the next differentiation yielding the delta function itself. Hence, they are functions which have discontinuities in their $(N - 1)$ gradient. We see that, as the number of dimensions increases, the weaker must be a possible singularity involving all the possible coordinates. Furthermore it must occur in a manner which reduces properly to the one-dimensional result if $U(r_0)$ involves only one rather than all the coordinates (a remark which may be employed to obtain the result just above).

In terms of the element U_{pn} convergence in the mean for series (9.1.15) will be obtained if

$$|U_{pn}|^2 \xrightarrow[p\to\infty]{} k_p^{4-N-\epsilon}; \quad \epsilon > 0 \tag{9.1.22}$$

An Example. We illustrate the use of the perturbation formulas by applying them to the Mathieu equation

$$(d^2\psi/d\theta^2) + [b - s\cos^2\theta]\psi = 0; \quad \psi(0) = \psi(2\pi)$$

This equation has been discussed earlier (see Chap. 5), where a continued-fraction method was employed to determine the values of b consistent with the periodic boundary conditions imposed on ψ. Physically the Mathieu equation is of interest in discussing wave propagation in elliptical cylinder coordinates [see Eqs. (11.2.70) *et seq.*]. The unperturbed eigenfunctions are

$$\varphi_n^{\pm} = \sqrt{\frac{\epsilon_n}{2\pi}}\,{\cos \atop \sin}(n\theta); \quad n = 0, 1, 2, \ldots$$

with the associated eigenvalues n^2. The (+) refers to the cosine solutions, which are even; the (−) to the sine solutions, which are odd. These eigenfunctions form a complete orthonormal set in the interval $(0,2\pi)$. The perturbation is $s \cos^2 \theta$, where s is now the parameter determining the size of the perturbation, so that in the perturbation formulas $\lambda = s$ and $U = \cos^2 \theta$. The next step is to evaluate the elements U_{nm} of the function $\cos^2 \theta$. Since $\cos^2 \theta$ is an even function of θ, it turns out that U_{nm} vanishes unless n and m are both odd or both even. For this example we shall focus our attention on the even case. Then

$$\begin{aligned} U_{nm} &= \tfrac{1}{2}\delta_{mn} + \tfrac{1}{4}[\delta_{m,n-2} + \delta_{m,n+2}]; \quad m \text{ and } n \neq 0 \\ U_{m0} &= U_{0m} = (1/\sqrt{8})\delta_{m2}; \quad m \neq 0; \quad U_{00} = \tfrac{1}{2} \end{aligned}$$

We consider two cases, corresponding to the unperturbed situation φ_0 with eigenvalue zero and φ_6 with eigenvalue $(6^2 = 36)$. The corresponding values of b_n are, according to (9.1.12), given by

$$b_n = n^2 + sU_{nn} + s^2 \sum_{p \neq n} \frac{U_{np}U_{pn}}{b_n - p^2} + s^3 \sum_{pq \neq n} \frac{U_{np}U_{pq}U_{qn}}{(b_n - p^2)(b_n - q^2)} + \cdots$$

In this example we shall work to fourth order. Then

$$\begin{aligned} b_0 = 0 + \frac{s}{2} + \frac{s^2}{8(b_0 - 4)} &+ \frac{s^3}{16(b_0 - 4)^2} \\ &+ \frac{s^4}{32(b_0 - 4)^2}\left[\frac{1}{b_0 - 4} + \frac{1}{4}\frac{1}{b_0 - 16}\right] + \cdots \end{aligned}$$

The value of b_0 has been computed for several values of s, the results being given in the table below, together with the exact value. Note that $b_0^{(a)}$ is the ath approximation for b_0. As expected, the convergence, which is excellent for $s < 1$, becomes increasingly poor above $s = 1$. At $s = 4$, there is no convergence at all, the successive approximations fluctuating about the correct value.

It is clear that this value of s is in excess of the radius of convergence of the perturbation series or at best is only slightly smaller.

Table 9.1

s	$b_0^{(0)}$	$b_0^{(1)}$	$b_0^{(2)}$	$b_0^{(3)}$	$b_0^{(4)}$	Exact
0.2	0	0.10000	0.09875			0.09875
1.0	0	0.50000	0.46430	0.46892		0.46896
2.0	0	1.00000	0.83333	0.89197	0.87367	0.87823
4.0	0	2.00000	1.00000	1.77778	1.15407	1.54486

As we shall see on page 1025, it is possible to provide analytic continuation beyond the radius of convergence and obtain an expression valid for all values of s. At this stage of the discussion, we may, however, employ the special technique of the Euler transformation [See Eq. (4.3.16)] which yields, for $s = 4$, the approximate value of 1.46231 for b_0.

The expression for b_6 is

$$b_6 = 36 + \frac{s}{2} + \frac{s^2}{16}\left[\frac{1}{b_6 - 16} - \frac{1}{64 - b_6}\right] + \frac{s^3}{32}\left[\frac{1}{(b_6 - 16)^2} + \frac{1}{(64 - b_6)^2}\right] + \frac{s^4}{64}\left\{\frac{1}{(b_6 - 16)^2}\left[\frac{1}{4(b_6 - 4)} + \frac{1}{b_6 - 16}\right] - \frac{1}{(64 - b_6)^2}\left[\frac{1}{4(100 - b_6)} + \frac{1}{64 - b_6}\right]\right\} + \cdots$$

The values of s and the successive approximations to b_6 are given in Table 9.2.

Table 9.2

s	$b_6^{(0)}$	$b_6^{(1)}$	$b_6^{(2)}$	$b_6^{(3)}$	Exact
0.5	36	36.25	36.25022		36.25022
2	36	37.00	37.00356		37.00357
30	36	51.00	48.27998	53.28531	51.82897

The convergence behavior is similar to the $n = 0$ case, except that the radius of convergence is considerably larger. This is to be expected, since the perturbing term $s \cos^2 \theta$ does not begin to approach the unperturbed value n^2 until s is fairly large. We shall later (page 1092) develop an approximation which is based upon this consideration. However, by $s = 30$, the perturbation series is no longer converging; Euler's transformation gives 48.89, still rather far from the exact answer.

It is clear from these examples that a method for improving the convergence with respect to increasing powers of λ would be highly desirable. We shall discuss two such methods, the Fredholm formula and the Feenberg formula. The first of these yields an expression valid for all values of the perturbation parameter λ.

Feenberg Perturbation Formula; Secular Determinant. Feenberg has emphasized that in the previous perturbation formulas elements U_{ab} may occur more then once in a given perturbation order, so that $(U_{ab})^2$ and higher powers of U_{ab} will appear. One may see this by examining the fifth-order term in (9.1.15):

$$\lambda^5 \sum_{pqrs\neq n} \frac{U_{np}U_{pq}U_{qr}U_{rs}U_{sn}}{(k^2 - k_p^2)(k^2 - k_q^2)(k^2 - k_r^2)(k^2 - k_s^2)}$$

It is clear that, in the course of summing over all the indices except n, both p and r will take on the value a while q and s will take on the value b so that the term $(U_{ab})^2$ will be present in the sum. Feenberg has then pointed out that it should be possible to obtain perturbation formulas in which such repetition of elements U_{ab} does not occur.

The reason for this remark can be seen from the following argument. Let the problem to be solved be

$$\mathfrak{L}(\psi) + k^2\psi = \lambda U\psi \tag{9.1.23}$$

where $\mathfrak{L}$ is an operator. In Eq. (9.1.1), $\mathfrak{L} = d^2/dx^2$; in (9.1.19), $\mathfrak{L} = \nabla^2 - U_0$; in (9.1.20), $\mathfrak{L} = \nabla_1^2 + \nabla_1^2 - U_0$. The unperturbed problem is

$$\mathfrak{L}(\varphi_n) + k_n^2\varphi_n = 0 \tag{9.1.24}$$

where, if $\mathfrak{L}$ is assumed to be a Hermitian operator, the functions φ_n form a complete orthonormal set:

$$\int \bar{\varphi}_n\varphi_m \, dv = \delta_{nm} \tag{9.1.25}$$

Let us now expand ψ in terms of this set:

$$\psi = \sum_p c_p\varphi_p$$

Substituting in Eq. (9.1.23) and using (9.1.24), we obtain

$$\sum_p c_p(k^2 - k_p^2)\varphi_p = \lambda \sum_p c_pU\varphi_p$$

Hence $$(k^2 - k_q^2)c_q = \lambda \sum_p c_p \int \bar{\varphi}_qU\varphi_p \, dv = \lambda \sum_p c_pU_{qp}$$

or $$\sum_p [(k^2 - k_q^2)\delta_{qp} - \lambda U_{qp}]c_p = 0 \tag{9.1.26}$$

This equation holds for each q, so that we have a set of linear simultaneous equations for the coefficients c_p, which are homogeneous. Such a set of equations will have a nonvanishing solution only if the determinant formed from the coefficients of the unknown c_p's vanishes. Therefore,

$$|(k^2 - k_q^2)\delta_{qp} - \lambda U_{qp}| = 0$$

or

$$\begin{vmatrix} k^2 - k_0^2 - \lambda U_{00} & -\lambda U_{01} & -\lambda U_{02} & \cdots \\ -\lambda U_{10} & k^2 - k_1^2 - \lambda U_{11} & -\lambda U_{12} & \cdots \\ -\lambda U_{20} & -\lambda U_{21} & k^2 - k_2^2 - \lambda U_{22} & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{vmatrix} = 0 \tag{9.1.27}$$

This is an equation determining k^2. The determinant is called the *secular determinant.* It was discussed in Chap. 1 (see page 59). It can now be seen that in expanding the secular determinant no repetition of a given element U_{ab} occurs. For example, suppose we are dealing with a problem in which only those U_{ab}'s for which $a \le 2$ and $b \le 2$ differ from zero. Then the determinant, upon expansion and rearrangement so as to resemble perturbation formula (9.1.15), becomes

$$k^2 = k_0^2 + \lambda U_{00} + \lambda^2 \left\{ \frac{U_{01}U_{10}}{k^2 - k_1^2 - \lambda U_{11} - \dfrac{\lambda^2 U_{21}U_{12}}{k^2 - k_2^2 - \lambda U_{22}}} + \frac{U_{02}U_{20}}{k^2 - k_2^2 - \lambda U_{22} - \dfrac{\lambda^2 U_{21}U_{12}}{k^2 - k_1^2 - \lambda U_{11}}} + \lambda^3 \frac{U_{02}U_{21}U_{10} + U_{20}U_{01}U_{12}}{(k^2 - k_1^2 - \lambda U_{11})(k^2 - k_2^2 - \lambda U_{22}) - \lambda^2 U_{12}U_{21}} \right\}$$

On the other hand if we write out Eq. (9.1.15) for this case, we do not obtain a finite number of terms but rather an infinite series; one can recognize that these additional terms arise from the expansion of the denominators in the above expression in a power series in λ:

$$k^2 = k_0^2 + \lambda U + \lambda^2 \left[\frac{U_{01}U_{10}}{k^2 - k_1^2} + \frac{U_{02}U_{20}}{k^2 - k_2^2} \right] + \lambda^3 \left[\frac{U_{01}U_{11}U_{10}}{(k^2 - k_1^2)^2} + \frac{U_{01}U_{12}U_{20} + U_{02}U_{21}U_{10}}{(k^2 - k_1^2)(k^2 - k_2^2)} + \frac{U_{02}U_{22}U_{20}}{(k^2 - k_2^2)^2} \right] + \cdots$$

The first and last terms in the λ^3 bracket are clearly the result of expanding the fractions $1/(k^2 - k_1^2 - \lambda U_{11})$ and $1/(k^2 - k_2^2 - \lambda U_{22})$, which appear in the exact expression. All such expansions do is to reduce the range of validity of the perturbation formula; usually it would be better to modify our development of perturbation theory so that they are not made.

For this purpose, consider Eqs. (9.1.26), determining the unknown coefficients c_p. Let us assume that we are considering that solution which would reduce to φ_n if the perturbation vanishes, so that $c_n = 1$. Then

$$(k^2 - k_p^2 - \lambda U_{pp})c_p = \lambda U_{pn} + \lambda \sum_{q \ne np} c_q U_{pq} \qquad (9.1.28)$$

(The subscript $q \ne np$ means that in the sum q cannot take on the values n or p.) To set up a method of successive approximations we must now write the equation determining c_q. To avoid any repetitive matrix elements it is necessary to separate out the terms in p and n in the equation for c_q:

$$(k^2 - k_q^2 - \lambda U_{qq})c_q = \lambda U_{qn} + \lambda c_p U_{qp} + \lambda \sum_{r \ne npq} c_r U_{qr} \quad (9.1.29)$$

Similarly for c_r:

$$(k^2 - k_r^2 - \lambda U_{rr})c_r = \lambda U_{rn} + \lambda c_p U_{rp} + \lambda c_q U_{rq} + \lambda \sum_{s \neq npqr} c_s U_{rs} \quad (9.1.30)$$

No approximations have been made as yet. Moreover, for finite secular determinants, involving therefore only a finite number of terms in the expansion for ψ, there are a finite number of equations of type (9.1.28) to (9.1.30). For example, if we were dealing with a third-order secular determinant equation, (9.1.29) would not contain the sum term; for a fourth-order case, the last equation would be (9.1.30) and would not contain the sum term; and so on. In each case the last equation would not contain a sum.

The solution of Eqs. (9.1.28) to (9.1.30) and the others, which would be developed as the process of successively separating coefficients is continued, is obtained by dropping the sum in the Nth equation, solving for the Nth c, and introducing this result in the $(N - 1)$st equation, etc. To obtain the formulas for an infinite secular determinant, mathematical induction is employed to determine a general solution for arbitrary N and then N is allowed to approach infinity.

We illustrate the first step in this procedure, by taking $N = 3$. From (9.1.29),

$$(k^2 - k_q^2 - \lambda U_{qq})c_q = \lambda U_{qn} + \lambda c_p U_{qp}$$

Substituting in Eq. (9.1.28) we obtain

$$\left[k^2 - k_p^2 - \lambda U_{pp} - \lambda^2 \sum_{q \neq np} \frac{U_{qp}U_{pq}}{k^2 - k_q^2 - \lambda U_{qq}}\right] c_p = \lambda U_{pn} + \lambda^2 \sum_{q \neq np} \frac{U_{pq}U_{qn}}{k^2 - k_q^2 - \lambda U_{qq}} \quad (9.1.31)$$

giving c_p correct to second order.

The equation determining k^2 is obtained by placing $q = n$ in (9.1.26) and substituting $c_n = 1$. Then

$$k^2 = k_n^2 + \lambda U_{nn} + \lambda \sum_{p \neq n} c_p U_{np} \quad (9.1.32)$$

Inserting the value of c_p from (9.1.31) gives k^2 to third order as

$$k^2 = k_n^2 + \lambda U_{nn} + \lambda^2 \sum_{p \neq n} \frac{U_{np}U_{pn}}{k^2 - k_p^2 - \lambda U_{pp} - \lambda^2 \displaystyle\sum_{q \neq pn} \frac{U_{pq}U_{qp}}{k^2 - k_q^2 - \lambda U_{qq}}}$$
$$+ \lambda^3 \sum_{\substack{p \neq n \\ q \neq pn}} \frac{U_{np}U_{pq}U_{qn}}{\left[k^2 - k_p^2 - \lambda U_{pp} - \lambda^2 \displaystyle\sum_{r \neq pn} \frac{U_{pr}U_{rp}}{k^2 - k_r^2 - \lambda U_{rr}}\right][k^2 - k_q^2 - \lambda U_{qq}]} \quad (9.1.33)$$

Comparison with the exact expression for a third-order secular determinant given above reveals that Eq. (9.1.33) is precise for that case and that for higher order determinants it makes an exact expansion of all the third-order determinants involving $k^2 - k_n^2 - \lambda U_{nn}$, approximations being introduced into the corresponding minors.

It is easy to follow the above argument for $N = 4$ or higher N and to see what the form of the solution for infinite N must be. We give the final results here.

Let

$$
\begin{aligned}
(\kappa^2)_{np} &= k_p^2 + \lambda U_{pp} + \lambda^2 \sum_{q \neq np} \frac{U_{pq}U_{qp}}{k^2 - (\kappa^2)_{npq}} \\
&\quad + \lambda^3 \sum_{\substack{q \neq np \\ r \neq npq}} \frac{U_{pq}U_{qr}U_{rp}}{[k^2 - (\kappa^2)_{npq}][k^2 - (\kappa^2)_{npqr}]} + \cdots \\
(\kappa^2)_{npq} &= k_q^2 + \lambda U_{qq} + \lambda^2 \sum_{r \neq npq} \frac{U_{qr}U_{rq}}{k^2 - (\kappa^2)_{npqr}} \\
&\quad + \lambda^3 \sum_{\substack{r \neq npq \\ s \neq npqr}} \frac{U_{qr}U_{rs}U_{sq}}{[k^2 - (\kappa^2)_{npqr}][k^2 - (\kappa^2)_{npqrs}]} + \cdots
\end{aligned}
\tag{9.1.34}
$$

etc. Each κ, with successively longer subscript, has successively fewer terms in each series multiplying the different powers of λ; more matrix elements are omitted, and if there are only a finite number of states, eventually the series are zero, since all the nonzero matrix elements are omitted. The general expression for c_p may then be given in terms of these constants $(\kappa^2)_{nq} \ldots$:

$$
\begin{aligned}
c_p &= \lambda \frac{U_{pn}}{k^2 - (\kappa^2)_{np}} + \lambda^2 \sum_{q \neq np} \frac{U_{pq}U_{qn}}{[k^2 - (\kappa^2)_{np}][k^2 - (\kappa^2)_{npq}]} \\
&\quad + \lambda^3 \sum_{\substack{q \neq np \\ r \neq npq}} \frac{U_{pq}U_{qr}U_{rn}}{[k^2 - (\kappa^2)_{np}][k^2 - (\kappa^2)_{npq}][k^2 - (\kappa^2)_{npqr}]} + \cdots
\end{aligned}
\tag{9.1.35}
$$

Inserting this result for c_p into eq. (9.1.32) for k^2 yields

$$
\begin{aligned}
k^2 &= k_n^2 + \lambda U_{nn} + \lambda^2 \sum_{p \neq n} \frac{U_{np}U_{pn}}{k^2 - (\kappa^2)_{np}} + \lambda^3 \sum_{\substack{p \neq n \\ q \neq pn}} \frac{U_{np}U_{pq}U_{qn}}{[k^2 - (\kappa^2)_{np}][k^2 - (\kappa^2)_{npq}]} \\
&\quad + \lambda^4 \sum_{\substack{p \neq n \\ q \neq pn \\ r \neq qpn}} \frac{U_{np}U_{pq}U_{qr}U_{rn}}{[k^2 - (\kappa^2)_{np}][k^2 - (\kappa^2)_{npq}][k^2 - (\kappa^2)_{npqr}]} + \cdots
\end{aligned}
\tag{9.1.36}
$$

It is clear that there are no repetitive elements U_{ab}, for the injunctions indicated under each summation sign do not permit any two indices to be equal. For any finite secular determinant, Eq. (9.1.36) will provide an exact expansion.

To solve Eq. (9.1.36) for k^2, we must adopt a successive approximation method. This yields the following formulation for the ath approximation:

$$\begin{aligned}(k^2)^{(a)} &= k_n^2 + \lambda U_{nn} + \lambda^2 \sum_{p\neq n} \frac{U_{np}U_{pn}}{[(k^2)^{(a-2)} - (\kappa^2{}_{np})^{(a-2)}]} \\ &+ \lambda^3 \sum_{\substack{p\neq n\\ q\neq np}} \frac{U_{np}U_{pq}U_{qn}}{[(k^2)^{(a-3)} - (\kappa^2)_{np}^{(a-3)}][(k^2)^{(a-3)} - (\kappa^2)_{npq}^{(a-3)}]} + \cdots \\ &+ \lambda^a \sum_{\substack{p\neq n\\ q\neq np\\ r\neq npq\\ \cdots}} \frac{U_{np}U_{pq}U_{qr}U_{rs}\cdots U_{zn}}{(k_n^2 - k_p^2)(k_n^2 - k_q^2)(k_n^2 - k_r^2)\cdots(k_n^2 - k_z^2)}\end{aligned} \tag{9.1.37}$$

Here $(\kappa^2)_{np\cdots}^{(a-2)}$ is the value of $(\kappa^2)_{np}\ldots$ in the $(a - 2)$ approximation, the error being proportional to λ^{a-1}. In the application of (9.1.37) one must proceed stepwise, building up to the desired order of approximation. One may readily evaluate $(k^2)^{(2)}$ as

$$(k^2)^{(2)} = k_n^2 + \lambda U_{nn} + \lambda^2 \sum_{p\neq n} \frac{U_{np}U_{pn}}{k_n^2 - k_p^2}$$

Then $(k^2)^{(3)}$ may be obtained from (9.1.37) and (9.1.34), $(k^2)^{(4)}$ from (9.1.37) and $(k^2)^{(3)}$, and so on. The results of this calculation may then be inserted into (9.1.35), and the corresponding value of the coefficients c_p determined.

The convergence of the Feenberg perturbation formula (9.1.37) is determined by the convergence of the pertinent determinantal ratio, which is

$$\frac{1}{M_{nn}} \left| \frac{(k^2 - k_q^2)\delta_{qp} - \lambda U_{qp}}{\delta_{qn} + (1 - \delta_{qn})[k^2 - k_q^2 - \lambda U_{qq}]} \right| \tag{9.1.38}$$

Where M_{nn} is the minor of the (n,n) element $(k^2 - k_n^2 - \lambda U_{nn})$ in the above determinant. In the ath approximation, all products in the expansion of the above determinants which contain more than a non-diagonal elements $U_{pq} (p \neq q)$ are dropped. The convergence of series (9.1.37) will break down if λ exceeds λ_0, the first value of λ for which M_{nn} is zero, or if the determinant fails to converge. The latter converges if the sum

$$\sum_{pq\neq n} \frac{U_{qp}}{k^2 - k_q^2 - \lambda U_{qq}} \tag{9.1.39}$$

converges. If the number of dimensions is N, then for convergence U_{pq} must go to zero for large p and q at least as rapidly as $(k_pk_q)^{-N-\epsilon}$, $\epsilon > 0$, if U is Hermitian so that $\bar{U}_{qp} = U_{pq}$. This condition is just met by a $U(r)$ whose $(2N - 1)$ derivative is discontinuous, so that the $2N$ derivative is singular. Any smoother dependence of U on r will, of course, yield a convergent sum (9.1.39). In one-dimensional problems where $2N - 1$ is unity, U must be at least piecewise continuous. Of course, the condition for the convergence in the mean of the series for ψ is less stringent. We may thus establish the required behavior of the elements U_{qp} by considering the convergence of the series Σc_p^2, where c_p is given by Eq. (9.1.35). Such an analysis shows that the growth of $|U_{pn}|^2$ as p increases must be like $k_p^{4-N-\epsilon}$, $\epsilon > 0$, as was obtained in Eq. (9.1.22) for the iterative-perturbation formula.

The above conditions are independent of the value of λ and, therefore, do not have any bearing on the convergence of (9.1.36) as a series in λ. The radius of convergence of this series is determined by the zeros of the minor M_{nn}. This is indicated by the form of Eq. (9.1.38). It may also be shown in a more obvious fashion by turning back to the secular determinant (9.1.27) and Eqs. (9.1.26) for the coefficient c_p for the purpose of determining c_p assuming that k^2 is known. As we shall see, these coefficients are infinite if the minor M_{nn} is zero. In Eq. (9.1.26) let $c_n = 1$; the equations now become a set of inhomogeneous linear simultaneous equations:

$$\sum_{p \neq n} [(k^2 - k_q^2)\delta_{qp} - \lambda U_{qp}]c_p = -[(k^2 - k_q^2)\delta_{qn} - \lambda U_{qn}]$$

The solutions for c_p are inversely proportional to the determinant formed by the coefficients of c_p. This determinant may be obtained from the secular determinant by removing the nth row and column and, therefore, just equals M_{nn}. The coefficients c_p thus are infinite if M_{nn} is zero and expansion (9.1.35) does not exist.

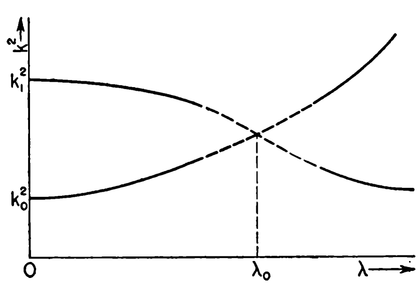

Fig. 9.1 Region of convergence of the sequence (9.1.37) to the correct value of k^2 is only for $\lambda < \lambda_0$.

The absolute value of λ_0, the smallest value of λ for which the minor M_{nn} considered as a function of λ is zero, therefore, is the radius of convergence of the series (9.1.35) for c_p. We shall now show that M_{nn} is zero whenever the solutions of the problem (9.1.23) are degenerate. The fact that M_{nn} is zero for those particular values of λ_0 and k^2 means that a solution of Eq. (9.1.23) exists which does not involve the function φ_n. Such a solution may, of course, be obtained by taking a linear combination of two degenerate solutions, thus proving the point. The converse of this theorem may be proved.

Our conclusion is then that the radius of convergence of the Feenberg perturbation formulas is determined by the λ of smallest absolute value for which the solutions become degenerate. In Fig. 9.1 we illustrate the convergent and divergent situations. The lines represent the variation of the first two eigenvalues k_1^2 and k_0^2 as λ increases from zero. The radius of convergence of the Feenberg series is λ_0 because of the degeneracy which occurs there. For $\lambda < \lambda_0$, the series will converge; for $\lambda \geq \lambda_0$, it will diverge.

An Example. The perturbation formula (9.1.36) will now be applied to the even periodic solutions of the Mathieu equation, discussed earlier on page 1009, with the aid of the iterative-perturbation formula (which was found to fail when $s = 4$ for $n = 0$ and $s = 30$ for $n = 6$).

To second order Eq. (9.1.36) reads (b_n in denominator $\simeq n^2 + \frac{1}{2}s$)

$$b_n = n^2 + \left(\frac{s}{2}\right) + \left(\frac{s^2}{16}\right)\left[\frac{1}{b_n - (n-2)^2 - (s/2)} + \frac{1}{b_n - (n+2)^2 - (s/2)}\right] + \cdots ; \quad n \neq 0$$

$$b_0 = \left(\frac{s}{2}\right) + \left(\frac{s^2}{8}\right)\left[\frac{1}{b_0 - 4 - (s/2)}\right] + \cdots ; \qquad n = 0$$

For the $n = 0$ case, $s = 4$, one obtains to this order $b = 2 - \frac{1}{2} = 1.5$, which is much closer to the precise value of 1.54486 than the iterative-perturbation value of 1.00 at this approximation. For the $n = 6$ case, $s = 30$, one obtains 51.8035 which agrees fairly well with the exact 51.8290, while the iterative-perturbation technique for the same approximation gives 48.2800. The difference between the two formulas arises from the assumption that s is small and the corresponding expansion made in the iterative-perturbation formula in powers of s. This is clearly not a practical procedure in the case just considered. For example, in the b_0 case $4 - b_0 = 2$, which is exactly the size of $\frac{1}{2}s$.

As a matter of fact, for the Mathieu equation, the Feenberg series yields the exact continued fraction (see page 564) which is employed to obtain the exact values quoted above. The terms in λ^3 and higher in (9.1.36) vanish. Consider, for example, the λ^3 term which involves $U_{np}U_{pq}U_{qn}(p \neq n,\ q \neq p)$. From the general expression for U_{nm} on page 1009 we see that p can be $n \pm 2$. Hence q can be only $n \pm 4$, whereupon U_{qn} vanishes and

$$b_n = n^2 + sU_{nn} + s^2 \sum_{p \neq n} \frac{U_{np}U_{pn}}{b_n - (\kappa^2)_{np}}$$

is exact. Now κ_{np}^2, from (9.1.34), is

$$\begin{aligned}\kappa_{np}^2 &= p^2 + sU_{pp} + s^2 \sum_{q\neq np} \frac{U_{pq}U_{qp}}{b_n - (\kappa^2)_{npq}} \\ &= p^2 + sU_{pp} \\ &\quad + s^2 \sum_{q\neq np} \cfrac{U_{pq}U_{qp}}{b_n - q^2 - sU_{qq} - s^2 \displaystyle\sum_{r\neq npq} \cfrac{U_{qr}U_{rq}}{b_n - r^2 - sU_{rr} - s^2 \displaystyle\sum_{t\neq npqr} \cfrac{U_{rt}U_{tr}}{b_n - t^2 - sU_{tt} - \cdots}}}\end{aligned}$$

We need now only to introduce the expression for U_{pq} given to obtain a continued fraction expression for b_n. For example, for $n = 0$

$$b_0 = \tfrac{1}{2}s + \cfrac{\frac{1}{8}s^2}{b_0 - 4 - \frac{1}{2}s - \cfrac{\frac{1}{16}s^2}{b_0 - 9 - \frac{1}{2}s - \cfrac{\frac{1}{16}s^2}{b - 16 - \frac{1}{2}s - \cdots}}}$$

If for $s = 4$ one places $b_0 = 1.5$ as obtained earlier and evaluates the right-hand side up to terms including $b_0 - 9 - \frac{1}{2}s$, one obtains $b_0 = 1.545$ to be compared with the exact value of 1.54486.

Fredholm Perturbation Formula. As we have shown above, the Feenberg perturbation formula may have a finite radius of convergence, the series diverging once λ exceeds that value for which a degeneracy in eigenvalues occurs. This result applies as well to the iterative-perturbation formula, for as has been indicated, these formulas may be obtained from the Feenberg result by a further set of expansions, which generally yield results with a slower rate of convergence.

In the present subsection, we shall develop a perturbation formula which may be applied for all values of λ; that is, it provides the analytic continuation of the Feenberg and iterative-perturbation formula beyond their radii of convergence. These formulas are most easily described if the perturbation procedure is discussed in terms of operators and vectors in abstract vector space. In addition, such a discussion will be general, thus extending the analysis to any perturbation problem.

Consider the eigenvalue problem

$$(\mathfrak{A} - E)\mathbf{e} = \lambda\mathfrak{B}\mathbf{e} \tag{9.1.40}$$

where E is the eigenvalue and $\mathfrak{A}$ and $\mathfrak{B}$ are Hermitian operators. The unperturbed problem, with eigenvalue ϵ_n and eigenvectors $\mathbf{f}_n$, is given by

$$\mathfrak{A}\mathbf{f}_n = \epsilon_n\mathbf{f}_n \tag{9.1.41}$$

The perturbation term is given by the operator $\lambda\mathfrak{B}$. Equations (9.1.40) and (9.1.41) are transcriptions in abstract vector space of Eqs. (9.1.1) and (9.1.2) or of the more complicated multidimensional problems (9.1.19) and (9.1.20) and, as emphasized above, apply as well to many other situations. The analogue of integral equation (9.1.4), from which the iterative-perturbation formulas are generated, is obtained here by introducing the inverse of $(\mathfrak{A} - E)$. Equation (9.1.40) becomes

$$\mathbf{e} = \lambda(\mathfrak{A} - E)^{-1}\mathfrak{B}\mathbf{e} \tag{9.1.42}$$

We again separate off the eigenvector which $\mathbf{e}$ approaches as λ approaches zero. Let this be $\mathbf{f}_n$, and let the corresponding $\mathbf{e}$ be $\mathbf{e}_n$, the value of E being E_n.

$$\mathbf{e}_n \xrightarrow[\lambda \to 0]{} \mathbf{f}_n$$

To describe the separation of $\mathbf{f}_n$ from the remainder of the right-hand side of the above equation, it is necessary to make use of the notion of the *projection operator* $\mathfrak{P}_n$. This operator is defined so that it picks out that part of any vector which is proportional to $\mathbf{f}_n$. It therefore has the following properties:

$$\mathfrak{P}_n\mathbf{f}_n = \mathbf{f}_n; \quad \mathfrak{P}_n\mathbf{f}_m = 0; \quad m \neq n$$

In the more concrete dyadic form

$$\mathfrak{P}_n = \mathbf{f}_n\mathbf{f}_n^*$$

Hence if $\mathbf{e} = \Sigma\alpha_p\mathbf{f}_p$, then $\mathfrak{P}_n\mathbf{e} = \alpha_n\mathbf{f}_n$. Thus in Eq. (9.1.42) for $\mathbf{e}_n$ above we may write

$$\mathbf{e}_n = \mathbf{f}_n + \lambda(1 - \mathfrak{P}_n)(\mathfrak{A} - E_n)^{-1}\mathfrak{B}\mathbf{e}_n \tag{9.1.43}$$

For convenience we introduce the abbreviation

$$\mathfrak{K} = (1 - \mathfrak{P}_n)(\mathfrak{A} - E_n)^{-1}\mathfrak{B} \tag{9.1.44}$$

so that (9.1.42) is now given by

$$\mathbf{e}_n = \mathbf{f}_n + \lambda\mathfrak{K}\mathbf{e}_n \tag{9.1.45}$$

an inhomogeneous equation for $\mathbf{e}_n$. The symbolic solution of this equation is

$$\mathbf{e}_n = \frac{1}{1 - \lambda\mathfrak{K}}\mathbf{f}_n = \mathbf{f}_n + \frac{\lambda\mathfrak{K}}{1 - \lambda\mathfrak{K}}\mathbf{f}_n \tag{9.1.46}$$

The iterative-perturbation formula for the vector $\mathbf{e}_n$ is obtained by expanding $1/(1 - \lambda\mathfrak{K})$ in a power series in λ:

$$\mathbf{e}_n = \mathbf{f}_n + \lambda\mathfrak{K}\mathbf{f}_n + \lambda^2(\mathfrak{K}^2\mathbf{f}_n) + \cdots \tag{9.1.47}$$

This series is identical with (9.1.11), for if we insert in the eigenvector expansion for $\mathfrak{K}$

$$\mathfrak{K} = \sum_{pq} \mathbf{f}_p\mathbf{f}_q^*(\mathbf{f}_p^*\mathfrak{K}\mathbf{f}_q)$$

the value

$$\mathbf{f}_p^*\mathfrak{R}\mathbf{f}_q = \mathbf{f}_p^*(1 - \mathfrak{P}_n)(\mathfrak{A} - E_n)^{-1}\mathfrak{B}\mathbf{f}_q = \frac{\mathbf{f}_p^*\mathfrak{B}\mathbf{f}_q}{\epsilon_p - E_n}; \quad p \neq n$$

then

$$\mathfrak{R} = \sum_{p \neq n} \frac{B_{pq}}{\epsilon_p - E_n} \mathbf{f}_p\mathbf{f}_q^*; \quad B_{pn} = (\mathbf{f}_p^*\mathfrak{B}\mathbf{f}_q)$$

The product $\mathfrak{R}\mathbf{f}_n$ is $\sum_{p \neq n} \frac{B_{pn}\mathbf{f}_p}{(\epsilon_p - E_n)}$ and is, of course, identical with the first-order term in (9.1.9). The product $\mathfrak{R}^2\mathbf{f}_n$ is

$$\mathfrak{R}^2\mathbf{f}_n = \sum_{q \neq n} \frac{B_{qn}}{\epsilon_q - E_n} \mathfrak{R}\mathbf{f}_q = \sum_{pq \neq n} \frac{B_{pq}B_{qn}}{(\epsilon_p - E_n)(\epsilon_q - E_n)} \mathbf{f}_p \qquad (9.1.48)$$

in agreement with the second term in (9.1.10), and so on.

The radius of convergence is given by the absolute magnitude of that value of λ for which the homogeneous form of (9.1.45) has a solution, for generally speaking $\mathfrak{R}^q\mathbf{f}_n$ will not be orthogonal to such a solution for all values of q. Thus the singularities of (9.1.46) are located at the eigenvalues λ_r of the eigenvalue problem

$$\mathbf{g}_r = \lambda_r\mathfrak{R}\mathbf{g}_r \qquad (9.1.49)$$

Substituting from Eq. (9.1.44) for $\mathfrak{R}$ and multiplying through by $(\mathfrak{A} - E)$, Eq. (9.1.49) becomes

$$(\mathfrak{A} - E)\mathbf{g}_r = \lambda_r(1 - \mathfrak{P}_n)\mathfrak{B}\mathbf{g}_r \qquad (9.1.50)$$

It is important to note that λ_r is not necessarily real, since $(1 - \mathfrak{P}_n)\mathfrak{B}$ is not necessarily Hermitian. From Eq. (9.1.50) it follows that $\mathbf{f}_n^* \cdot \mathbf{g}_r = 0$, so that $\mathbf{g}_r$ is a solution of the original eigenvalue problem (9.1.40) with $\lambda = \lambda_r$, which does not involve $\mathbf{f}_n$. It is possible to obtain such a solution if λ is such that Eq. (9.1.40) has degenerate solutions, that is, at least two solutions which have the same value of E and λ, for then linear combinations of these solutions may be found which do not contain $\mathbf{f}_n$. Hence, as in the Feenberg case, the radius of convergence of series (9.1.41) is determined by the λ of smallest absolute value for which the solutions of (9.1.40) are degenerate.

We turn now to the problem of developing a formulation of the perturbation theory which will provide an analytic continuation beyond the above radius of convergence. Let us rewrite Eq. (9.1.46), for the symbolic solution $\mathbf{e}_n$, as follows:

$$\mathbf{e}_n = \left[\frac{\chi(\lambda)/(1 - \lambda\mathfrak{R})}{\chi(\lambda)}\right]\mathbf{f}_n \qquad (9.1.51)$$

The function $\chi(\lambda)$ is to be chosen as follows. It is to be an entire function of λ; that is, it is to have no singularities in the complex plane of λ.

Its zeros are to be placed at the poles of $1/(1 - \lambda\mathfrak{K})$, that is, at $\lambda = \lambda_r$, and are to be of the same order. As a consequence $\chi(\lambda)/(1 - \lambda\mathfrak{K})$, as well as $\chi(\lambda)$, will be an entire function which may be expanded in a power series in λ, said series having an infinite radius of convergence (see page 382). Thus, the operator connecting $\mathbf{e}_n$ with $\mathbf{f}_n$ will be expressed as the ratio of two expressions, each of which may be expanded in a power series in λ with infinite radius of convergence. The singularity which is present in the original expression still remains in (9.1.51), for $\chi(\lambda)$ will be zero at $\lambda = \lambda_r$.

We now go on to determine the entire function χ. A similar problem has been discussed in Chap. 4, page 385. The procedure outlined there involves first evaluating the logarithmic derivative of χ, $\chi'(\lambda)/\chi(\lambda)$, where the prime represents differentiation. This function clearly has poles at the zeros of $\chi(\lambda)$, that is, at $\lambda = \lambda_r$. Thus to obtain χ we need to place $\chi'(\lambda)/\chi(\lambda)$ equal to a function having poles of the proper order at $\lambda = \lambda_r$. There are many such functions. We shall choose the simple form

$$\frac{\chi'(\lambda)}{\chi(\lambda)} = \text{Spur}\left[\frac{-\mathfrak{K}}{1 - \lambda\mathfrak{K}}\right] = \sum_r \frac{1}{\lambda - \lambda_r} \tag{9.1.52}$$

where the symbol Spur indicates the sum of the diagonal elements in the matrix for the argument. Since the diagonal elements of $\mathfrak{K}$ are $1/\lambda_n$, the second form results. This particular choice is made because it will lead to the Fredholm solution of an inhomogeneous integral equation of the second kind when $\mathfrak{K}$ corresponds to an integral operator, so that Eq. (9.1.40) corresponds to an integral equation. The most general choice for $\chi'(\lambda)/\chi(\lambda)$ is

$$\frac{\chi'(\lambda)}{\chi(\lambda)} = \text{Spur}\left[\frac{f(\lambda,\mathfrak{K})}{1 - \mathfrak{K}}\right] = \sum_r \frac{f(\lambda,1/\lambda_r)}{1 - (\lambda/\lambda_r)}$$

where f is any entire function of λ which does not have zeros at $\lambda = \lambda_r$ and for which $f(\lambda_r, 1/\lambda_r) = -(1/\lambda_r)$. The different possibilities for f weight the various perturbation orders differently.

Returning to (9.1.52) we may now integrate both sides of the equation from $\lambda = 0$ to $\lambda = \lambda$, assuming that $\lambda = 0$ is not one of the set $\{\lambda_r\}$; that is, the unperturbed problem (9.1.41) is not degenerate. Then

$$\chi(\lambda) = \exp\left[-\int_0^\lambda \text{Spur}\left(\frac{\mathfrak{K}}{1 - \lambda\mathfrak{K}}\right) d\lambda\right] \tag{9.1.53}$$

The next step in the program is to expand both $\chi(\lambda)$ and $\chi(\lambda)/(1 - \lambda\mathfrak{K})$ in a power series in λ. We shall need a notation for the Spurs of the various powers of $\mathfrak{K}$. Let

$$\kappa_p = \text{Spur } \mathfrak{K}^p \tag{9.1.54}$$

Consider first $\chi(\lambda)$, and let

$$\chi(\lambda) = \sum_{n=0}^{\infty} a_n\lambda^n; \quad a_0 = 1$$

Then
$$\chi'(\lambda)/\chi(\lambda) = \Sigma n a_n \lambda^{n-1}/\Sigma a_n\lambda^n$$

But from (9.1.52) and definition (9.1.54)

$$\chi'(\lambda)/\chi(\lambda) = -\Sigma\kappa_{n+1}\lambda^n$$

so that
$$\Sigma n a_n\lambda^{n-1} = -(\Sigma\kappa_{p+1}\lambda^p)(\Sigma a_s\lambda^s)$$

By equating the coefficients of like powers of λ on both sides of this equation we obtain recursion relations for the a_n's:

$$(n+1)a_{n+1} + \sum_{s=1}^{n} a_s\kappa_{n+1-s} = -\kappa_{n+1} \tag{9.1.55}$$

These equations form a set of linear inhomogeneous simultaneous equations for a_s the solutions of which may be expressed in determinantal form:

$$a_n = -\left(\frac{1}{n!}\right)\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & \kappa_1 \\ \kappa_1 & 2 & 0 & \cdots & 0 & \kappa_2 \\ \kappa_2 & \kappa_1 & 3 & \cdots & 0 & \kappa_3 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \kappa_{n-1} & \kappa_{n-2} & \kappa_{n-3} & \cdots & \kappa_1 & \kappa_n \end{vmatrix} \tag{9.1.56}$$

The first four of these coefficients a_n are given below:

$$\begin{aligned} a_1 &= -\kappa_1 \\ a_2 &= \tfrac{1}{2}(\kappa_1^2 - \kappa_2) \\ a_3 &= -\tfrac{1}{6}(2\kappa_3 - 3\kappa_1\kappa_2 + \kappa_1^3) \\ a_4 &= -\tfrac{1}{24}(6\kappa_4 - 8\kappa_1\kappa_3 + 6\kappa_2\kappa_1^2 - 3\kappa_2^2 - \kappa_1^4) \end{aligned} \tag{9.1.57}$$

We may employ a similar method to expand $\chi(\lambda)/(1 - \lambda\mathfrak{K})$ in a power series in λ. Take the logarithm and differentiate. Then if

$$\chi(\lambda)/(1 - \lambda\mathfrak{K}) = \sum_{n=0}^{\infty} b_n\lambda^n; \quad b_0 = 1 \tag{9.1.58}$$

$$-\text{Spur}\left(\frac{\mathfrak{K}}{1-\lambda\mathfrak{K}}\right) + \frac{\mathfrak{K}}{1-\lambda\mathfrak{K}} = (\Sigma n b_n\lambda^{n-1})/(\Sigma b_n\lambda^n)$$

Comparing with equations determining a_n we see that the only change is that each term κ_p is replaced by $\kappa_p - \mathfrak{K}^p$. Hence

$$b_n = -\left(\frac{1}{n!}\right)\cdot$$
$$\cdot\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & \kappa_1 - \mathfrak{K} \\ \kappa_1 - \mathfrak{K} & 2 & 0 & \cdots & 0 & \kappa_2 - \mathfrak{K}^2 \\ \kappa_2 - \mathfrak{K}^2 & \kappa_1 - \mathfrak{K} & 3 & \cdots & 0 & \kappa_1 - \mathfrak{K}^3 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \kappa_{n-1} - \mathfrak{K}^{n-1} & \kappa_{n-2} - \mathfrak{K}^{n-2} & \kappa_{n-3} - \mathfrak{K}^{n-3} & \cdots & \kappa_1 - \mathfrak{K} & \kappa_n - \mathfrak{K}^n \end{vmatrix}$$

If we write $b_n = b_n(\kappa,\mathfrak{K})$, signifying that b_n is a function of the $\mathfrak{K}$ and the Spurs of the powers of $\mathfrak{K}$, then

$$b_n(\kappa,0) = a_n$$

The first three of these coefficients are given below:

$$\left.\begin{aligned} b_1 &= \mathfrak{K} + a_1 \\ b_2 &= \mathfrak{K}^2 - \mathfrak{K}\kappa_1 + a_2 \\ b_3 &= \mathfrak{K}^3 - \mathfrak{K}^2\kappa_1 - \tfrac{1}{2}(\kappa_2 - \kappa_1^2)\mathfrak{K} + a_3 \\ b_n &= \mathfrak{K}^n + a_1\,\mathfrak{K}^{n-1} + a_2\,\mathfrak{K}^{n-2} + \cdots \end{aligned}\right\} \tag{9.1.59}$$

We may now fill in Eq. (9.1.51) and so obtain the Fredholm expansion for $\mathbf{e}_n$:

$$\mathbf{e}_n = \mathbf{f}_n + \left\{\frac{\sum_{p=1}^{\infty} c_p\lambda^p}{\sum_{p=0}^{\infty} a_p\lambda^p}\right\}\mathbf{f}_n; \quad c_p = b_p - a_p \tag{9.1.60}$$

Then to the third order

$$\mathbf{e}_n = \mathbf{f}_n + \lambda\mathfrak{K}\,\frac{1 + \lambda(\mathfrak{K} - \kappa_1) + \lambda^2[\mathfrak{K}^2 - \mathfrak{K}\kappa_1 - \frac{1}{2}(\kappa_2 - \kappa_1^2)]}{1 - \lambda\kappa_1 - \frac{1}{2}\lambda^2(\kappa_2 - \kappa_1^2) - \frac{1}{6}\lambda^3(2\kappa_3 - 3\kappa_1\kappa_2 + \kappa_1^3)}\,\mathbf{f}_n \tag{9.1.61}$$

The corresponding value of the eigenvalue E_n is obtained from

$$E_n = \epsilon_n - \lambda(\mathbf{f}_n^*\mathfrak{B}\mathbf{e}_n) \tag{9.1.62}$$

When Eq. (9.1.61) is inserted in this expression, a formula for E accurate to fifth order is obtained.

To evaluate (9.1.61) we require the value of the Spurs as well as $(\mathbf{f}_p^*\mathfrak{K}^a\mathbf{f}_n) = (\mathfrak{K}^a)_{pn}$ for $a \leq 4$. Recalling that $\mathfrak{K} = (1 - \mathfrak{P}_n)(\mathfrak{A} - E)^{-1}\mathfrak{B}$, we have

$$\begin{aligned} (\mathfrak{K})_{pn} &= \frac{B_{pn}}{\epsilon_p - E_n}; && p \neq n \\ (\mathfrak{K}^2)_{pn} &= \sum_{q\neq n} \frac{B_{pq}B_{qn}}{(\epsilon_p - E_n)(\epsilon_q - E_n)}; && p \neq n \\ (\mathfrak{K}^3)_{pn} &= \sum_{qr\neq n} \frac{B_{pq}B_{qr}B_{rn}}{(\epsilon_p - E_n)(\epsilon_q - E_n)(\epsilon_r - E_n)}; && p \neq n \\ (\mathfrak{K}^4)_{pn} &= \sum_{qrs\neq n} \frac{B_{pq}B_{qr}B_{rs}B_{sn}}{(\epsilon_p - E_n)(\epsilon_q - E_n)(\epsilon_r - E_n)(\epsilon_s - E_n)} && p \neq n \end{aligned} \tag{9.1.63}$$

$(\mathfrak{K}^a)_{nn}$ is zero because of the operator $(1 - \mathfrak{P}_n)$, though $(\mathfrak{K}^a)_{pp}$ is not. The Spurs κ_a are given by

$$\kappa_a = \sum_{p\neq n} (\mathfrak{K}^a)_{pp}$$

so that

$$\kappa_1 = \sum_{p\neq n} \frac{B_{pp}}{\epsilon_p - E_n}; \quad \kappa_2 = \sum_{pq\neq n} \frac{B_{pq}B_{qp}}{(\epsilon_p - E_n)(\epsilon_q - E_n)}; \quad \text{etc.} \tag{9.1.64}$$

We may now write

$$\begin{aligned}\mathbf{e}_n = \mathbf{f}_n + \sum_{p\neq n} \mathbf{f}_p \{\lambda(\mathfrak{R})_{pn} + \lambda^2[(\mathfrak{R}^2)_{pn} - \kappa_1(\mathfrak{R})_{pn}] + \lambda^3[(\mathfrak{R}^3)_{pn} - \kappa_1(\mathfrak{R}^2)_{pn} \\ - \tfrac{1}{2}(\kappa_2 - \kappa_1^2)(\mathfrak{R})_{pn}]\} \{1 - \lambda\kappa_1 - \tfrac{1}{2}\lambda^2(\kappa_2 - \kappa_1^2) \\ - \tfrac{1}{6}\lambda^3(2\kappa_3 - 3\kappa_1\kappa_2 + \kappa_1^3)\}^{-1}\end{aligned} \tag{9.1.65}$$

The equation determining E_n as given by Eq. (9.1.62) is

$$\begin{aligned}E_n = \epsilon_n - \lambda B_{nn} - \lambda^2 \sum_{p\neq n} B_{np} \{(\mathfrak{R})_{pn} + \lambda[(\mathfrak{R}^2)_{pn} - \kappa_1(\mathfrak{R})_{pn}] + \lambda^2[(\mathfrak{R}^3)_{pn} \\ - \kappa_1(\mathfrak{R}^2)_{pn} - \tfrac{1}{2}(\kappa_2 - \kappa_1^2)(\mathfrak{R})_{pn}]\} \{1 - \lambda\kappa_1 - \tfrac{1}{2}\lambda^2(\kappa_2 - \kappa_1^2) \\ - \tfrac{1}{6}\lambda^3(2\kappa_3 - 3\kappa_1\kappa_2 + \kappa_1^3)\}^{-1}\end{aligned} \tag{9.1.66}$$

Note that the energy equation does not show any change from the iterative-perturbation results until third order in λ is reached. Since the $(\mathfrak{R})_{pn}$, etc., in (9.1.66) involve E, (9.1.66) is not an explicit expression for E but rather an equation which is to be solved by a series of successive approximations already employed in the derivation of Eqs. (9.1.15) and (9.1.37).

Formulas (9.1.65) *and* (9.1.66) *may be employed in one-dimensional problems only* because κ_1 is infinite for any problem involving more dimensions. For this reason, it is important to obtain formulas which do not contain κ_1. This is most easily done by multiplying the expression (9.1.53) for $\chi(\lambda)$ by the entire function $e^{\lambda\kappa_1}$, assuming for the moment that κ_1 is finite so that

$$\chi(\lambda) = \exp\left\{-\int_0^\lambda \text{Spur}\left(\frac{\mathfrak{R}}{1 - \lambda\mathfrak{R}}\right) d\lambda + \lambda\kappa_1\right\} \tag{9.1.67}$$

The logarithmic derivative of the new $\chi(\lambda)$ is

$$\frac{\chi'(\lambda)}{\chi(\lambda)} = -\text{Spur}\left(\frac{\mathfrak{R}}{1 - \lambda\mathfrak{R}}\right) + \kappa_1 = -\sum_{p=1}^{\infty} \kappa_{p+1}\lambda^p$$

Comparing with the corresponding equation obtained for the $\chi(\lambda)$ of (9.1.53), we see that the present equation could be obtained from the earlier one by placing $\kappa_1 = 0$. We may then immediately write for the coefficients in the expansion of $\chi(\lambda)$

$$\chi(\lambda) = \Sigma a_n^1\lambda^n; \quad a_0^1 = 1; \quad a_1^1 = 0$$

$$a_n^1 = -\left(\frac{1}{n!}\right)\begin{vmatrix} 2 & 0 & 0 & 0 & \cdots & \kappa_2 \\ 0 & 3 & 0 & 0 & \cdots & \kappa_3 \\ \kappa_2 & 0 & 4 & 0 & \cdots & \kappa_4 \\ \kappa_3 & \kappa_2 & 0 & 5 & \cdots & \kappa_5 \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \kappa_{n-1} & \kappa_{n-2} & \kappa_{n-3} & \kappa_{n-4} & \cdots & \kappa_n \end{vmatrix} \tag{9.1.68}$$

Similarly if

$$\frac{\chi(\lambda)}{(1 - \lambda\mathfrak{K})} = \sum b_n^1 \lambda^n$$

then

$$b_n^1 = -\left(\frac{1}{n!}\right) \cdot \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & -\mathfrak{K} \\ -\mathfrak{K} & 2 & 0 & \cdots & 0 & \kappa_2 - \mathfrak{K}^2 \\ \kappa_2 - \mathfrak{K}^2 & -\mathfrak{K} & 3 & \cdots & 0 & \kappa_3 - \mathfrak{K}^3 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \kappa_{n-1} - \mathfrak{K}^{n-1} & \kappa_{n-2} - \mathfrak{K}^{n-2} & \kappa_{n-3} - \mathfrak{K}^{n-3} & \cdots & -\mathfrak{K} & \kappa_n - \mathfrak{K}^n \end{vmatrix} \tag{9.1.69}$$

so that

$$\begin{aligned} b_1 &= \mathfrak{K} \\ b_2 &= \mathfrak{K}^2 + a_2 \\ b_3 &= \mathfrak{K}^3 - \tfrac{1}{2}\kappa_2\mathfrak{K} + a_3 \\ b_4 &= \mathfrak{K}^4 - \tfrac{1}{2}\kappa_2\mathfrak{K}^2 - \tfrac{1}{3}\kappa_3\mathfrak{K} + a_4 \end{aligned}$$

Finally in formulas (9.1.65) and (9.1.66) we simply put $\kappa_1 = 0$. The expression for E now agrees with the iteration-perturbation results to third order, the Fredholm result differing in the fourth- and higher order terms.

An Example. Let us consider the Mathieu equation again. We may set up the following correlations $-(d^2/d\theta^2) \to \mathfrak{A}$, $b_n \to E_n$, $s \to -\lambda$, and $U \to \mathfrak{B}$, so that $U_{pq} = B_{pq}$. We now can employ Eq. (9.1.66) without modification, since the problem is one-dimensional. For the eigenvalue b_0, for which $E_n = 0$, the Fredholm expression carried out to fourth order is

$$\begin{aligned} b_0 = \tfrac{1}{2}s + \Bigg\{ & \frac{\frac{1}{8}s^2}{b_0 - 4} + \left[\frac{\frac{1}{16}s^3}{(b_0 - 4)^2} + \frac{\frac{1}{8}\kappa_1 s^3}{b_0 - 4}\right] + \Bigg[\frac{\frac{1}{32}s^4}{(b_0 - 4)^3} \\ & + \frac{\frac{1}{128}s^4}{(b_0 - 4)^2(b_0 - 16)} + \frac{\frac{1}{16}\kappa_1 s^4}{(b_0 - 4)^2} - \tfrac{1}{16}s^4\,\frac{\kappa_2 - \kappa_1^2}{b_0 - 4}\Bigg]\Bigg\} \cdot \\ & \cdot [1 + s\kappa_1 + \tfrac{1}{2}s^2(\kappa_2 - \kappa_1^2)]^{-1} \end{aligned}$$

This equation may be solved by successive approximations. The results are tabulated below for the case $s = 4$ with those obtained by the iteration-perturbation formula.

Table 9.3

Method	$b_0^{(0)}$	$b_0^{(1)}$	$b_0^{(2)}$	$b_0^{(3)}$	Exact
Fredholm	2.0000	1.0000	1.5553	1.5442	1.5449
I-P	2.0000	1.0000	1.7778	1.1541	

The superiority of the Fredholm results is obvious; the third-order result being in error by less than 1 per cent, the fourth approximation agreeing with the exact answer up to the fourth significant figure. An estimate of the radius of convergence of the iteration-perturbation formulas may be obtained by finding the smallest zero of the denominator in the expression for b_0 above. This yields $s \simeq 5$ as a value of the zero. It is thus not surprising that the convergence for $s = 4$ is so poor for the iteration-perturbation formulas.

Variation-Iteration Method. Each of the methods discussed above leads to an equation determining the eigenvalue, an equation which upon solution by the method of successive approximation [see, for example, Eq. (9.1.15)] yields a perturbation-type formula in which the eigenvalue is expressed in terms of the interaction parameter λ. Actually this dependence is rarely made explicit because of the unwieldy nature of the expressions which result.

It would, of course, be more convenient if a direct determination of E as an explicit function of λ could be obtained. It turns out to be possible to determine the inverse of this function, *i.e.*, to give λ as a function of E with limitations that will appear below. In this method we regard λ as the eigenvalue and E as given; *i.e.*, we ask for the interaction strength needed to obtain a particular value of E. This request corresponds to a situation often realized in practice in which E is known from experiment and the nature of the interaction is actually of greater interest. In the event that E for a particular value of λ is desired, it is necessary in this method to determine λ as a function of E in the neighborhood of this value of λ and then employ inverse interpolation.

Consider Eq. (9.1.40) as an eigenvalue equation for λ. The corresponding integral equation is given by (9.1.42) as

$$\mathbf{e} = \lambda(\mathfrak{A} - E)^{-1}\mathfrak{B}\mathbf{e}$$

We shall now solve this equation by iteration. Our procedure will differ from that employed in the iteration-perturbation method in that we shall not separate out a particular unperturbed eigenvector. Let us again assume that $\mathfrak{A}$ and $\mathfrak{B}$ are Hermitian. There is an eigenvalue λ_0 of smallest absolute value, corresponding to the smallest interaction strength for which E is an eigenvalue. The larger eigenvalues correspond to interaction strengths for which $\mathbf{e}$ is not the lowest state but the next state, and so on.

Let the zeroth approximation to $\mathbf{e}$ be $\mathbf{e}^{(0)}$, which we shall choose as one of the unperturbed set $\mathbf{f}_n$ for E close to ϵ_n. The first approximation to $\mathbf{e}$, $\mathbf{e}^{(1)}$, is obtained by inserting $\mathbf{e}^{(0)}$ for $\mathbf{e}$ in the right-hand side of the above integral equation for e so that

$$\mathbf{e}^{(1)} = (\mathfrak{A} - E)^{-1}\mathfrak{B}\mathbf{e}^{(0)}$$

Note that the factor λ may be dropped, since we are interested only in the form of $\mathbf{e}$. Its normalization is unimportant, since any constant multiple of a solution remains a solution of the linear homogeneous equation being considered. It is this possibility of omitting λ in the iteration process which permits the development of an explicit functional dependence of λ on E. Continuing the iteration process via the following recursion relation between the $(n - 1)$ approximation and the nth,

$$\mathbf{e}^{(n)} = (\mathfrak{A} - E)^{-1}\mathfrak{B}\mathbf{e}^{(n-1)} \tag{9.1.70}$$

yields

$$\mathbf{e}^{(n)} = [(\mathfrak{A} - E)^{-1}\mathfrak{B}]^n\mathbf{e}^{(0)} \tag{9.1.71}$$

The corresponding approximations for λ_0 are obtained from both the original equation

$$\lambda_0 = [\mathbf{e}^*(\mathfrak{A} - E)\mathbf{e}]/[\mathbf{e}^*\mathfrak{B}\mathbf{e}] \tag{9.1.72}$$

and the integral equation

$$\lambda_0 = [\mathbf{e}^*\mathfrak{B}\mathbf{e}]/[\mathbf{e}^*\mathfrak{B}(\mathfrak{A} - E)^{-1}\mathfrak{B}\mathbf{e}] \tag{9.1.73}$$

If now $\mathbf{e}^{(n)}$ is substituted for $\mathbf{e}$ in (9.1.72) we obtain the nth approximation for λ_0:

$$\lambda_0^{(n)} = [(\mathbf{e}^{(n)})^*(\mathfrak{A} - E)\mathbf{e}^{(n)}]/[(\mathbf{e}^{(n)})^*\mathfrak{B}\mathbf{e}^{(n)}]$$

or from (9.1.70)

$$\lambda_0^{(n)} = [(\mathbf{e}^{(n)})^*\mathfrak{B}\mathbf{e}^{(n-1)}]/[(\mathbf{e}^{(n)})^*\mathfrak{B}\mathbf{e}^{(n)}] \tag{9.1.74}$$

If $\mathbf{e}^{(n)}$ is substituted for $\mathbf{e}$ in (9.1.73), we obtain the $(n + \frac{1}{2})$ approximation, a notation which will be explained below:

$$\lambda_0{}^{(n+\frac{1}{2})} = [(\mathbf{e}^{(n)})^*\mathfrak{B}\mathbf{e}^{(n)}]/[(\mathbf{e}^{(n)})^*\mathfrak{B}\mathbf{e}^{(n+1)}] \tag{9.1.75}$$

It is obviously convenient to introduce the notation

$$[n,m] = (\mathbf{e}^{(n)})^*\mathfrak{B}\mathbf{e}^{(m)}$$

so that

$$\lambda_0^{(n)} = [n, n - 1]/[n,n]; \quad \lambda_0{}^{(n+\frac{1}{2})} = [n,n]/[n, n + 1] \tag{9.1.76}$$

The elements $[n,m]$ have the following properties:

$$[n,m] = [m,n] = [n + p, m - p]$$

If the above procedure converges, the goal toward which this approach was directed has been achieved, for the right-hand side of both (9.1.74) and (9.1.75) depends only upon E, so that an explicit dependence of λ on E has resulted. The discussion of the convergence belongs more properly in Sec. 9.4 on the variational method, for the rigorous proof of convergence leans heavily upon the variational principle. For the present we shall be satisfied with a summary of the results needed here and a rough discussion of the reasons which guarantee convergence; further details are to be found in Eqs. (9.4.93) to (9.4.110).

If both $\mathfrak{A} - E$ and $\mathfrak{B}$ are *positive definite*, then (with the conditions stated on page 1138)

$$\lambda_0^{(n)} \geq \lambda_0{}^{(n+\frac{1}{2})} \geq \lambda_0^{(n+1)} \cdots \geq \lambda_0 > 0 \tag{9.1.77}$$

In other words the successive approximations form a monotonic sequence converging to λ_0 from above [see Eq. (9.4.101)].

If $\mathfrak{A} - E$ is not positive definite while $\mathfrak{B}$ is, the eigenvalues λ_n are not necessarily positive. There is a corresponding weakening in the inequalities above. We have

$$|\lambda_0{}^{(n+\frac{1}{2})}| \geq |\lambda_0^{(n)}|; \quad |\lambda_0{}^{(n+\frac{1}{2})}| \geq |\lambda_0{}^{(n+\frac{3}{2})}| \cdot \cdot \cdot \geq |\lambda_0| \tag{9.1.78}$$

Hence the half-odd-integer approximations, *only*, approach $|\lambda_0|$ from above. The integer approximations may be either above or below $|\lambda_0|$.

Convergence of the Method. The rate of convergence toward the exact value of λ_0 depends most critically upon the ratio of the first two eigenvalues λ_0 and λ_1 for a given E. This may be demonstrated in rough fashion as follows: Let the eigenvalue λ_n of Eq. (9.1.40) have $\mathbf{g}_n$ as corresponding eigenvector, namely:

$$(\mathfrak{A} - E)\mathbf{g}_n = \lambda_n \mathfrak{B}\mathbf{g}_n \tag{9.1.79}$$

These form a complete set in terms of which we may expand the zeroth approximation:

$$\mathbf{e}^{(0)} = \Sigma a_m \mathbf{g}_m \tag{9.1.80}$$

where a_m are numerical coefficients. We may now evaluate $\mathbf{e}^{(1)}$ and then $\mathbf{e}^{(n)}$, since

$$\mathbf{e}^{(1)} = (\mathfrak{A} - E)^{-1}\mathfrak{B}(\Sigma a_m \mathbf{g}_m)$$

or

$$\mathbf{e}^{(1)} = \sum \left(\frac{a_m}{\lambda_m}\right) \mathbf{g}_m$$

and

$$\mathbf{e}^{(n)} = \sum \left(\frac{a_m}{\lambda_m^n}\right) \mathbf{g}_m \tag{9.1.81}$$

For Hermitian positive-definite operators the values of λ_m form a monotonic sequence $\lambda_0, \lambda_1, \lambda_m \cdot \cdot \cdot$. Hence in the absence of a degeneracy (*i.e.*, an equality of λ_0 and λ_1) the ratio of the amplitude of the lowest mode $\mathbf{g}_0$ to that of the first harmonic $\mathbf{g}_1$ will increase as the order of the iteration increases, for this ratio is

$$(a_0/a_1)(\lambda_1/\lambda_0)^n$$

so that, for a sufficiently high order of iteration, the expansion of $\mathbf{e}^{(n)}$ will involve principally $\mathbf{g}_0$ and the corresponding value of $\lambda_0^{(n)}$ will be very close to the exact λ_0. The rate of convergence of the $\lambda_0^{(n)}$ sequence is similar to that of $\mathbf{e}^{(n)}$, for it will be shown in Sec. 9.4 [see Eq. (9.4.106)] that

$$\left[\frac{\lambda_0^{(n+1)} - \lambda_0{}^{(n+\frac{1}{2})}}{\lambda_0{}^{(n+\frac{1}{2})} - \lambda_0^{(n)}}\right] \simeq \left(\frac{\lambda_0}{\lambda_1}\right)$$

If the ratio λ_0/λ_1 is close to unity, convergence is extremely slow. A special device is available for this case which permits extrapolation to the exact value in spite of the slow convergence. It may also, of course, be employed where convergence is satisfactory, but it is then not usually necessary.

Suppose that the iterations have progressed far enough so that only eigenvectors $\mathbf{g}_0$ and $\mathbf{g}_1$ remain. Then $\mathbf{e}^{(n)}$ as well as other quantities such as $\lambda_0^{(n)}$ may be decomposed into two terms, the exact and the error term. If the quantity of interest is denoted by F, then the ath approximation $F^{(a)}$ is given by

$$F^{(a)} = F + F_1$$

where F_1 is the error. Upon iterating once and evaluating $F_1^{(a+1)}$, we obtain

$$F^{(a+1)} = F + \epsilon F_1$$

where ϵ is the convergence factor, approximately equal to λ_0/λ_1 when $F = \lambda_0$ or $\mathbf{e}$. Iterating again yields

$$F^{(a+2)} = F + \epsilon^2 F_1$$

These three equations may now be solved for F:

$$F = F^{(a+2)} - [F^{(a+1)} - F^{(a+2)}]^2/[F^{(a)} - 2F^{(a+1)} + F^{(a+2)}] \quad (9.1.82)$$

and

$$\epsilon = [F^{(a+1)} - F^{(a+2)}]/[F^{(a)} - F^{(a+1)}] \quad (9.1.83)$$

In most applications ϵ, as given in (9.1.83), is obtained by placing $F^{(a)} = \lambda_0^{(n)}, F^{(a+1)} = \lambda_0^{(n+\frac{1}{2})}$, and $F^{(a+2)} = \lambda_0^{(n+1)}$. The value of ϵ so obtained is an estimate of λ_0/λ_1 and also may be employed in the extrapolation of other quantities, such as $\mathbf{e}$, which depend in a similar fashion on ϵ. If we denote such a quantity by G and its ath approximation by $G^{(a)}$, then the extrapolated value of G is

$$G = G^{(a+2)} - \frac{F^{(a+1)} - F^{(a+2)}}{F^{(a)} - 2F^{(a+1)} + F^{(a+2)}} [G^{(a+1)} - G^{(a+2)}] \quad (9.1.84)$$

The convergence of the $\lambda_0^{(n)}$ sequence for $\mathfrak{A} - E$ Hermitian but not positive definite, $\mathfrak{B}$, Hermitian and positive definite, also depends upon the ratio λ_0/λ_1. For operators of the above character the spectrum of eigenvalues λ_m extends in both directions, to $-\infty$ as well as to $+\infty$. This does not change the form of expansion (9.1.81) or the argument that immediately follows, except that in the latter λ_0/λ_1 must be replaced by its absolute value $|\lambda_0/\lambda_1|$. As in the positive-definite case, the convergence then depends upon the distribution of eigenvalues whose details rest on the precise nature of the operators $\mathfrak{A}$ and $\mathfrak{B}$. Then if λ_0 is approximately equal to either $+\lambda_1$ or $-\lambda_1$, convergence is poor. The remedy discussed above and contained in Eqs. (9.1.82) to (9.1.84) applies for the nonpositive definite case and is quite efficacious in practice after a sufficient number of iterations have been carried out.

Having indicated the existence of convergence for the iterative process, it now becomes possible to derive specific expressions for $\lambda_0^{(n)}$

and $\mathbf{e}^{(n)}$ of the perturbation type. Suppose we are concerned with the λ_0 and E relationship in the vicinity of E_m, an eigenvalue of the operator $\mathfrak{A}$ with the associated eigenvector $\mathbf{f}_m$. It is then appropriate to choose $\mathbf{f}_m$ as the first approximation $\mathbf{e}^{(0)}$. However, it is important to be certain that the corresponding value of λ is the smallest in absolute value for which the desired value of E can be obtained, for it is to this value that the iteration method converges. If $\mathbf{f}_m$ approximately equals the eigenvector for λ_1, the iteration method will not converge. This lack of convergence will be clear in the course of the calculation. Furthermore, results (9.1.77) and (9.1.78) will no longer hold.

Often it is possible to modify operators $\mathfrak{A}$ or $\mathfrak{B}$ so that the new λ is the lowest λ, as will be illustrated in an example given below. However, if this is impossible, then the necessary modifications in the variation-iteration method to ensure convergence are too cumbersome to permit the development of perturbation formulas here.

Assuming that it is possible to choose $\mathbf{e}^{(0)} = \mathbf{f}_m$, the necessary expressions for the evaluation of $\lambda_0^{(n)}$ and $\lambda_0^{(n+\frac{1}{2})}$ may be easily found. We need a general formula for $\mathbf{e}^{(n)}$. From (9.1.71)

$$\mathbf{e}^{(n)} = [\mathfrak{A} - E)^{-1}\mathfrak{B}]^n\mathbf{f}_n$$

so that

$$\mathbf{e}^{(1)} = \sum_p \left(\frac{B_{pm}}{\epsilon_p - E}\right)\mathbf{f}_p; \quad \mathbf{e}^{(2)} = \sum_{pq} \left[\frac{B_{pq}B_{qm}}{(\epsilon_p - E)(\epsilon_q - E)}\right]\mathbf{f}_p \tag{9.1.85}$$

from which the general formula may be easily ascertained. From these expressions for $\mathbf{e}^{(n)}$, it now becomes possible to evaluate the various approximations (9.1.74) and (9.1.75). From (9.1.76) and (9.1.85):

$$\begin{aligned} [0,0] &= B_{mm} \\ [0,1] &= \sum_p \frac{B_{mp}B_{pm}}{\epsilon_p - E} \\ [1,1] &= \sum_{pq} \frac{B_{mq}B_{qp}B_{pm}}{(\epsilon_q - E)(\epsilon_p - E)} \\ [1,2] &= \sum_{pqr} \frac{B_{mr}B_{rq}B_{qp}B_{pm}}{(\epsilon_r - E)(\epsilon_q - E)(\epsilon_p - E)} \end{aligned} \tag{9.1.86}$$

Hence we may now construct $\lambda_0^{(n)}$ and $\lambda_0^{(n+\frac{1}{2})}$ as given in (9.1.76). For example,

$$\lambda_0^{\frac{1}{2}} = B_{mm} \Big/ \sum_p \left(\frac{B_{mp}B_{pm}}{\epsilon_p - E}\right)$$

and

$$\lambda_0^{(1)} = \sum_p \left(\frac{B_{mp}B_{pm}}{\epsilon_p - E}\right) \Big/ \sum_{pq} \left[\frac{B_{mq}B_{qp}B_{pm}}{(\epsilon_q - E)(\epsilon_p - E)}\right] \tag{9.1.87}$$

and so on.

An Example. We return to the Mathieu equation following Eq. (9.1.22) and consider the case $b_0 = 1.54486$ ($s = 4$) for which the iteration-perturbation method failed to converge (see Table 9.1). The operator $\mathfrak{B}$ will be taken to correspond to $\cos^2\theta$, so that $\mathfrak{B}$ is positive definite. The eigenvalue $\lambda = +s$, while $\mathfrak{A} - E$ will correspond to $(d^2/d\theta^2) + b$. Note that $d^2/d\theta^2$ is a negative definite operator, since

$$\int_0^\pi \psi\left(\frac{d^2\psi}{d\theta^2}\right)d\theta = -\int_0^\pi \left(\frac{d\psi}{d\theta}\right)^2 d\theta$$

Hence $\mathfrak{A} - E$ is not positive definite, for there are functions ψ for which

$$\int\psi[(d^2/d\theta^2) + b]\psi\, d\theta < 0$$

Thus the eigenvalues λ, equal to s, will take on negative as well as positive values.

If now $\lambda_0^{(\frac{1}{2})}, \lambda_0^{(1)}, \lambda_0^{(\frac{3}{2})}$ are evaluated for $E = -b_0 = 1.54486$, the following is obtained:

$$\lambda_0^{(\frac{1}{2})} = 4.5080; \quad \lambda_0^{(1)} = 3.5847; \quad \lambda_0^{(\frac{3}{2})} = 4.9516$$

We note the complete lack of convergence to the correct value 4.0000 and the violation of the inequalities (9.1.78). This indicates that an eigenvalue of smaller magnitude than 4 exists for which $b_0 = 1.54486$, a statement which may be verified by examination of the table of the separation constants b. Indeed, taking λ_1 to be 4 and estimating $\lambda_1/\lambda_0 = \epsilon$ (in this case) from (9.1.83), we obtain $\lambda_0 \simeq 2.7$, which is not far from the actual value of 2.5. . . .

This difficulty may be overcome by redefining $\mathfrak{A}$ and $\mathfrak{B}$ as indicated by the following form of the Mathieu equation:

$$(d^2\psi/d\theta^2) + [b - s + s\sin^2\theta]\psi = 0$$

Then let

$$\mathfrak{B} \to \sin^2\theta; \quad \mathfrak{A} \to -(d^2/d\theta^2)$$
$$E \to (b - s); \quad \lambda \to s$$

It is clear that the eigenvalue λ_0 which is less than 4 will yield a value of $b - s$ which will be larger than the one of interest, $b - s = -2.45514$. To obtain this value of $b - s$ will, therefore, require an increase in λ_0 leading to the possibility it may exceed 4.

The values of the elements B_{pq} are now

$$B_{pq} = \tfrac{1}{2}\delta_{pq} - \tfrac{1}{4}(\delta_{p,\,q+2} + \delta_{p,\,q-2})$$
$$B_{p0} = B_{0p} = -(1/\sqrt{8})\delta_{p2}; \quad B_{00} = \tfrac{1}{2}$$

The resulting values of $\lambda_0^{(\frac{1}{2})}, \lambda_0^{(1)}, \lambda_0^{(\frac{3}{2})}$ are

$$\lambda_0^{(\frac{1}{2})} = 4.1257; \quad \lambda_0^{(1)} = 4.0230; \quad \lambda_0^{(\frac{3}{2})} = 4.0045$$

We note that there is convergence, the fourth approximation [$\lambda_0^{(0)}$ is the first] being accurate to 1 part in 1,000. This may be improved by using the extrapolation formula (9.1.82) which yields an extrapolated value of λ_0 equal to 4.0004, accurate to 1 part in 10,000.

The labor involved in calculating this series of approximations is equal to that of calculating the fourth-order iteration-perturbation result. The only difficulty, and we have overemphasized it here, arises from the requirement that the parameter λ converges toward the smallest possible value of λ which can have E_n as the value of E. Often this is clear from the physics of the problem being considered. If it is not, the absence of convergence and the violation of the associated inequalities will serve to indicate that the requirement has not been met. One must then adopt some device for changing the order around as was done in the above example.

Improved Perturbation Formulas. The results obtained for the iteration-perturbation and Fredholm methods may be substantially improved by means of a simple transformation, which results in a change in the energy denominators similar but not nearly so complete as that employed by Feenberg. In the present discussion, we shall take advantage of the fact that the above methods are solutions of a secular determinantal equation:

$$|(k^2 - k_q^2)\delta_{qp} - \lambda U_{qp}| = 0$$

Any determinantal equation which may be put into this form will have solutions, for example, in the iteration-perturbation method as given by (9.1.13) and (9.1.14). The energy denominators $k^2 - k_q^2$ appearing in these formulas are replaced in the Feenberg formulas by $k^2 - k_q^2 - \lambda U_{qq}$ plus higher order terms. It is possible to include the diagonal terms λU_{qq} in with $k^2 - k_q^2$ by means of a simple reordering of the above determinant as follows:

$$|(k^2 - k_q^2 - \lambda U_{qq})\delta_{qp} - \lambda U_{qp}(1 - \delta_{qp})| = 0 \qquad (9.1.88)$$

Then in the iteration-perturbation formulas, the combination $k^2 - k_q^2$ is to be replaced by $(k^2 - k_q^2 - \lambda U_{qq})$ and U_{qp} by $U_{qp}(1 - \delta_{qp})$ to obtain a solution of secular equation (9.1.88). One then obtains

$$\begin{aligned}\psi_n = \varphi_n &+ \lambda \sum_{p\neq n} \frac{U_{pn}}{(k^2 - k_p^2 - \lambda U_{pp})}\varphi_p \\ &+ \lambda^2 \sum_{\substack{pq\neq n\\ q\neq p}} \frac{U_{pq}U_{qn}}{(k^2 - k_p^2 - \lambda U_{pp})(k^2 - k_q^2 - \lambda U_{qq})}\varphi_p \\ &+ \lambda^3 \sum_{\substack{pqr\neq n\\ pr\neq q}} \frac{U_{pq}U_{qr}U_{rn}}{(k^2 - k_p^2 - \lambda U_{pp})(k^2 - k_q^2 - \lambda U_{qq})(k^2 - k_r^2 - \lambda U_{rr})}\varphi_p \\ &+ \cdots \qquad (9.1.89)\end{aligned}$$

while the determining equation for k^2 becomes

$$k^2 = k_n^2 + \lambda U_{nn} + \lambda^2 \sum_{p\neq n} \frac{U_{np}U_{pn}}{(k^2 - k_p^2 - \lambda U_{pp})}$$
$$+ \lambda^3 \sum_{\substack{pq\neq n \\ q\neq p}} \frac{U_{np}U_{pq}U_{qn}}{(k^2 - k_p^2 - \lambda U_{pp})(k^2 - k_q^2 - \lambda U_{qq})}$$
$$+ \lambda^4 \sum_{\substack{pqr\neq n \\ pr\neq q}} \frac{U_{np}U_{pq}U_{qr}U_{rn}}{(k^2 - k_p^2 - \lambda U_{pp})(k^2 - k_q^2 - \lambda U_{qq})(k^2 - k_r^2 - \lambda U_{rr})} + \cdots \quad (9.1.90)$$

We shall call these the *modified iteration-perturbation* formulas; they are a considerable improvement over the unmodified variety. Turning again to the Mathieu function example for which the iteration-perturbation formulas were so inadequate, for $s = 4$, $b_0(\text{exact}) = 1.5449$, the above formula to second order gives $b_0^{(2)} = 1.500$ as opposed to $b_0^{(2)} = 1.000$ obtained from the unmodified formulas. The above formulas are not so accurate as the Feenberg series but are considerably more convenient.

Similar modifications may be introduced into the Fredholm perturbation formula. The secular equation discussed there may be readily shown to be:

$$|(E - \epsilon_q)\delta_{qp} + \lambda B_{qp}| = 0$$

We reorder the terms as follows:

$$|(E - \epsilon_q + \lambda B_{qq})\delta_{qp} + \lambda B_{qp}(1 - \delta_{qp})| = 0 \quad (9.1.91)$$

Equations (9.1.63) giving the values of $(\mathfrak{K}^a)_{pn}$ may now be modified with $E - \epsilon_q + \lambda B_{qq}$ replacing $E - \epsilon_q$ and $B_{qp}(1 - \delta_{qp})$ replacing B_{qp}. The results are

$$(\mathfrak{K})_{pn} = \frac{B_{pn}}{\epsilon_p - E - \lambda B_{pp}}; \qquad p \neq n$$
$$(\mathfrak{K}^2)_{pn} = \sum_{q\neq np} \frac{B_{pq}B_{qn}}{(\epsilon_p - E - \lambda B_{pp})(\epsilon_q - E - \lambda B_{qq})}; \quad p \neq n \qquad (9.1.92)$$

etc. The Spurs given in (9.1.64) become

$$\kappa_1 = 0; \quad \kappa_2 = \sum_{\substack{pq\neq n \\ q\neq p}} \frac{B_{pq}B_{qp}}{(\epsilon_p - E - \lambda B_{pp})(\epsilon_q - E - \lambda B_{qq})} \quad (9.1.93)$$

etc. These results, (9.1.92) and (9.1.93), are to be substituted into (9.1.65) and (9.1.66) to obtain expressions for **e** and E. Since $\kappa_1 = 0$, no additional modifications need to be introduced when the number of dimensions exceeds one.

Nonorthogonal Functions. In the preceding discussion of the four perturbation methods we have always expanded the unknown eigenfunctions or eigenvectors in terms of a complete orthonormal set of which the unperturbed function is a member. It often happens, however (we shall see an example in the next section on boundary perturbations) that the unperturbed function is not a member of such a set but rather of one which is not orthogonal though it is complete. It is, of course, possible to orthogonalize or biorthogonalize such a set following the procedures discussed in Chap. 8. However, it is our purpose here to recast the perturbation methods so that they apply to expansions in nonorthogonal sets without resorting to the intermediate step of orthogonalization. As in the discussion on improved perturbation formulas we shall take advantage of the fact that the perturbation methods are solutions of certain secular equations. This is possible, since, as we shall see, a determinantal equation may be obtained even when the expansions are in a nonorthogonal set.

First let us derive such a determinantal equation just mentioned. Let the nonorthogonal set have members $\mathbf{f}_p$. Let the unperturbed solution be $\mathbf{f}_n$, and let the equation to be solved be

$$(\mathfrak{A} - E)\mathbf{e}_n = \lambda\mathfrak{B}\mathbf{e}_n$$

Let

$$\mathbf{e}_n = \Sigma c_p\mathbf{f}_p$$

Substituting this series into the equation for $\mathbf{e}_n$ yields

$$\Sigma(\mathfrak{A} - E - \lambda\mathfrak{B}) \cdot c_p\mathbf{f}_p = 0$$

We now take the product on the left with $\mathbf{f}_q^*$. In terms of the symbols

$$\begin{aligned} A_{qp} &= (\mathbf{f}_q^* \cdot \mathfrak{A} \cdot \mathbf{f}_p) \\ N_{qp} &= (\mathbf{f}_q^* \cdot \mathbf{f}_p) \\ B_{qp} &= (\mathbf{f}_q^* \cdot \mathfrak{B} \cdot \mathbf{f}_p) \end{aligned} \tag{9.1.94}$$

the following linear homogeneous simultaneous equations are obtained:

$$\sum_p c_p(A_{qp} - EN_{qp} - \lambda B_{qp}) = 0; \quad \text{for each } q \tag{9.1.95}$$

Solutions for these equations exist only if the determinant of the coefficients of c_p vanish:

$$|EN_{qp} + (\lambda B_{qp} - A_{qp})| = 0$$

(When orthogonal function expansions are employed, $N_{qp} = \delta_{qp}$.) It will be convenient to introduce the symbol

$$H_{qp} = \lambda B_{qp} - A_{qp}$$

so that the secular equation becomes

$$|EN_{qp} - H_{qp}| = 0 \tag{9.1.96}$$

Let us now compare this determinantal equation with that which holds for the case of an expansion in orthogonal functions, as given by Eq. (9.1.27):

$$|(k^2 - k_q^2)\delta_{qp} - \lambda U_{qp}| = 0$$

For comparison we rewrite (9.1.96) as follows:

$$|(EN_{qq} - H_{qq})\delta_{qp} - (H_{qp} - EN_{qp})(1 - \delta_{qp})| = 0$$

This suggests that we may take over the iteration-perturbation formulas (9.1.13) and (9.1.14) for the present case if the following substitutions are made:

$$\begin{aligned}(k^2 - k_q^2) &\to (EN_{qq} - H_{qq}) \\ \lambda U_{qp} &\to (H_{qp} - EN_{qp})(1 - \delta_{qp})\end{aligned} \tag{9.1.97}$$

The iteration-perturbation formulas, valid when the expansions are in a nonorthogonal set $\mathbf{f}_p$, are then

$$\mathbf{e}_n = \mathbf{f}_n + \sum_{p\neq n} \left(\frac{H_{pn} - EN_{pn}}{EN_{pp} - H_{pp}}\right)\mathbf{f}_p + \sum_{\substack{qp\neq n \\ q\neq p}} \frac{(H_{pq} - EN_{pq})(H_{qn} - EN_{qn})}{(EN_{pp} - H_{pp})(EN_{qq} - H_{qq})}\mathbf{f}_p + \cdots \tag{9.1.98}$$

while

$$EN_{nn} = H_{nn} + \sum_{p\neq n} \frac{(H_{np} - EN_{np})(H_{pn} - EN_{pn})}{EN_{pp} - H_{pp}} + \sum_{\substack{pq\neq n \\ q\neq p}} \frac{(H_{np} - EN_{np})(H_{pq} - EN_{pq})(H_{qn} - EN_{qn})}{(EN_{pp} - H_{pp})(EN_{qq} - H_{qq})} + \cdots \tag{9.1.99}$$

Substitutions (9.1.97) may be also employed in the Feenberg perturbation formulas (9.1.35) and (9.1.36) to obtain the corresponding results when the expansion of $\mathbf{e}_n$ is in terms of a nonorthogonal set. This yields

$$\begin{aligned}c_p = {} & \frac{H_{pn} - EN_{pn}}{EN_{pp} - H_{pp} - T_{np}} + \sum_{q\neq np} \frac{(H_{pq} - EN_{pq})(H_{qn} - EN_{qn})}{(EN_{pp} - H_{pp} - T_{np})(EN_{qq} - H_{qq} - T_{npq})} \\ & + \sum_{\substack{q\neq np \\ r\neq npq}} \frac{(H_{pq} - EN_{pq})(H_{qr} - EN_{qr})(H_{rn} - EN_{rn})}{(EN_{pp} - H_{pp} - T_{np})(EN_{qq} - H_{qq} - T_{npq})(EN_{rr} - H_{rr} - T_{npqr})} + \cdots\end{aligned} \tag{9.1.100}$$

where

$$T_{np} = \sum_{q \neq np} \frac{(H_{pq} - EN_{pq})(H_{qp} - EN_{qp})}{(EN_{qq} - H_{qq} - T_{npq})} + \sum_{\substack{q \neq np \\ r \neq npq}} \frac{(H_{pq} - EN_{pq})(H_{qr} - EN_{qr})(H_{rp} - EN_{rp})}{(EN_{qq} - H_{qq} - T_{npq})(EN_{rr} - H_{rr} - T_{npqr})} + \cdots$$

$$T_{npq} = \sum_{r \neq npq} \frac{(H_{qr} - EN_{qr})(H_{rq} - EN_{rq})}{EN_{rr} - H_{rr} - T_{npqr}} + \sum_{\substack{r \neq npq \\ s \neq npqr}} \frac{(H_{qr} - EN_{qr})(H_{rs} - EN_{rs})(H_{sq} - EN_{sq})}{(EN_{rr} - H_{rr} - T_{npqr})(EN_{ss} - H_{ss} - T_{npqrs})} + \cdots \quad (9.1.101)$$

The eigenvalue equation is given by appropriate substitution in Eq. (9.1.36):

$$EN_{nn} = H_{nn} + \sum_{p \neq n} \frac{(H_{np} - EN_{np})(H_{pn} - EN_{pn})}{EN_{pp} - H_{pp} - T_{np}} + \sum_{\substack{p \neq n \\ q \neq np}} \frac{(H_{np} - EN_{np})(H_{pq} - EN_{pq})(H_{qn} - EN_{qn})}{(EN_{pp} - H_{pp} - T_{np})(EN_{qq} - H_{qq} - T_{npq})} \quad (9.1.102)$$

We may also restate the Fredholm and the variation-iteration formulas for the expansion in nonorthogonal vectors by means of the above procedure. Expressions (9.1.65) (9.1.66), together with (9.1.63), represent solutions to the eigenvalue problem leading to the secular determinant

$$|(E - \epsilon_p)\delta_{qp} + \lambda B_{qp}| = 0$$

To obtain the proper formulas for the eigenvalue problem (9.1.96) being considered here, the following substitutions need to be made

$$E - \epsilon_p \to EN_{pp} - H_{pp}; \quad \lambda B_{qp} \to (EN_{qp} - H_{qp})(1 - \delta_{qp}) \quad (9.1.103)$$

Equation (9.1.63) then becomes

$$\lambda(\mathfrak{R})_{pn} = (H_{pn} - EN_{pn})/(EN_{pp} - H_{pp}); \qquad p \neq n$$

$$(\lambda^2\mathfrak{R}^2)_{pn} = \sum_{q \neq np} \frac{(H_{pq} - EN_{pq})(H_{qn} - EN_{qn})}{(EN_{pp} - H_{pp})(EN_{qq} - H_{qq})}; \quad p \neq n$$

and

$$\lambda^3(\mathfrak{R}^3)_{pn} = \sum_{\substack{q \neq np \\ r \neq npq}} \frac{(H_{pq} - EN_{pq})(H_{pr} - EN_{qr})(H_{rn} - EN_{rn})}{(EN_{pp} - H_{pp})(EN_{qq} - H_{qq})(EN_{rr} - H_{rr})}; \quad p \neq n$$

$$(9.1.104)$$

Equations (9.1.64) become

$$\begin{aligned}
\lambda\kappa_1 &= 0 \\
\lambda\kappa_2 &= \sum_{\substack{p\neq nq\\ q\neq np}} \frac{(H_{pq} - EN_{pq})(H_{qp} - EN_{qp})}{(EN_{pp} - H_{pp})(EN_{qq} - H_{qq})} \\
\lambda\kappa_3 &= \sum_{\substack{p\neq n\\ q\neq np\\ r\neq npq}} \frac{(H_{pq} - EN_{pq})(H_{qr} - EN_{qr})(H_{rp} - EN_{rp})}{(EN_{pp} - H_{pp})(EN_{qq} - H_{qq})(EN_{rr} - H_{rr})}
\end{aligned} \tag{9.1.105}$$

and so on. In using the above expressions (9.1.104) and (9.1.105) it is important to realize that $(\mathfrak{K}^a)_{pn}$ is *not* $(\mathbf{f}_p^* \cdot \mathfrak{K} \cdot \mathbf{f}_n)$ and κ_2 is *not* the Spur of $\mathfrak{K}^2$. Rather the equations state the values of these quantities to be substituted into Eqs. (9.1.65) and (9.1.66) in order to obtain expansions for $\mathbf{e}_n$ and E.

We finally turn to the variation-iteration method. Here λ was employed as the eigenvalue to be found and E assumed known, whereas in the formulas above the assumptions are just opposite. This means that the variation-iteration results can be generalized by the simple method of regrouping the terms of the secular determinant only if the values of N_{pq} and A_{pq} for $p \neq q$ are proportional to λ, as is the case for boundary-perturbation problems (see Sec. 9.2.). We shall content ourselves with writing down formulas following from this additional assumption, though presumably even if it were invalid, a variation-iteration formula would exist.

$$\begin{aligned}
N_{qp} &= \lambda n_{qp}; \quad q \neq p \\
A_{qp} &= \lambda a_{qp}; \quad q \neq p
\end{aligned} \tag{9.1.106}$$

and

$$En_{qp} + B_{qp} - a_{qp} = c_{qp}$$

the secular determinant (9.1.96) becomes

$$|(EN_{pp} - A_{pp})\delta_{pq} + \lambda c_{qp}| = 0$$

suggesting the following substitutions in (9.1.85) and (9.1.87):

$$(E - \epsilon_p) \to EN_{pp} - A_{pp}; \quad B_{qp} \to c_{qp}$$

The variation-iteration formulas, involving expansions in terms of nonorthogonal vectors and assumptions (9.1.106), are then

$$\mathbf{e}^{(1)} = \sum_p \left(\frac{c_{pn}}{A_{pp} - EN_{pp}}\right)\mathbf{f}_p;$$

$$\mathbf{e}^{(2)} = \sum_{pq} \left[\frac{c_{pq}c_{qn}}{(A_{pp} - EN_{pp})(A_{qq} - EN_{qq})}\right]\mathbf{f}_p \tag{9.1.107}$$

and

$$\lambda_0^{(\frac{1}{2})} = c_{00} \Big/ \sum_p \left(\frac{c_{np}c_{pn}}{A_{pp} - EN_{pp}} \right)$$

$$\lambda_0^{(1)} = \sum_p \left(\frac{c_{np}c_{pn}}{A_{pp} - EN_{pp}} \right) \Big/ \sum_{pq} \left[\frac{c_{np}c_{pq}c_{qn}}{(A_{pp} - EN_{pp})(A_{qq} - EN_{qq})} \right] \qquad (9.1.108)$$

and so on.

We conclude this section on expansions in nonorthogonal functions with a discussion of the limitations of the perturbation formulas for the various methods. We may expect that these will be rapidly convergent if the nondiagonal elements of the secular determinant (9.1.96) are small compared with the diagonal terms:

$$(EN_{qp} - H_{qp})/(EN_{qq} - H_{qq}) \ll 1; \quad q \neq p \qquad (9.1.109)$$

This inequality will hold (1) if λB_{qp} is small and (2) if the deviation from the diagonal form of the determinants A_{qp} and N_{qp} is small. In other words, for nonorthogonal functions the parameter λ no longer measures the size of the perturbation. If the values of N_{qp} and A_{qp} $(q \neq p)$ are comparable to N_{qq} and A_{qq}, we are not dealing with a small perturbation as far as these formulas are concerned. Part of this situation may be remedied by orthogonalization of the set $\mathbf{f}_p$, for then the nondiagonal values of N_{qp} will be zero. For a small perturbation we would still require that the nondiagonal elements of λB and A taken with respect to the orthogonal set be small compared with the diagonal ones.

9.2 *Boundary Perturbations*

In this section, we shall treat problems in which the deviation from an exactly solvable problem occurs at the boundaries, whereas in the preceding section the perturbations were confined to the interior of the bounded region. A perturbation which involves both types, volume and surface, may be handled by applying the results of 9.1 and 9.2 successively.

Boundary perturbations may include either a change in boundary conditions or a change in the shape of the boundary surface or both. We shall consider these two cases separately; when both are involved, we apply the results for each in turn. In our attack on this problem we shall replace the original differential equation and associated boundary conditions by an integral equation, which either may be solved by successive approximations or may be reduced to a secular determinant, to which the results of the preceding section may be applied. We shall restrict the discussion to the scalar Helmholtz equation; the generalization to most other linear equations is straightforward.

Perturbation of Boundary Conditions, f Small. Here we consider the eigenvalue problem which occurs when the boundary conditions are changed from homogeneous Neumann or homogeneous Dirichlet to the mixed boundary condition:

$$(\partial\psi/\partial n) + f(S)\psi = 0 \tag{9.2.1}$$

where f is, in general, a function of the surface coordinates. Exact solutions satisfying this boundary condition may be obtained for cavities (or rooms) in the form of a rectangular parallelepiped or circular cylinder or sphere, etc., if f is constant over each wall for which a given coordinate is constant. When f varies over a wall or when the wall shape is more complicated, the problem may not be solved by the method of separation of variables and we must have recourse to approximate methods. For a perturbation theory, these will naturally depend upon the size of f. We shall therefore develop two different results according as f is large or small.

We consider first f small. Then it is convenient to take as unperturbed solutions the functions which satisfy homogeneous Neumann conditions. Let these be φ_n, and let them form a complete orthonormal set, satisfying the equation

$$\nabla^2\varphi_n + k_n^2\varphi_n = 0$$

To obtain the integral equation satisfied by ψ, we employ the Green's function G_k satisfying Neumann conditions on the boundaries. Hence, G_k is a solution of the equation

$$\nabla_0^2 G_k(\mathbf{r}|\mathbf{r}_0) + k^2 G_k(\mathbf{r}|\mathbf{r}_0) = -4\pi\delta(\mathbf{r} - \mathbf{r}_0); \quad \partial G_k/\partial n_0 = 0$$

while ψ satisfies

$$\nabla_0^2\psi(\mathbf{r}_0) + k^2\psi(\mathbf{r}_0) = 0$$

and boundary condition (9.2.1). Multiplying the first of these equations by ψ and the second by G_k, subtracting the results, and integrating over the region inside the boundaries, we obtain

$$\int[G_k(\mathbf{r}|\mathbf{r}_0)\nabla_0^2\psi - \psi\nabla_0^2 G_k(\mathbf{r}|\mathbf{r}_0)]\,dV_0 = 4\pi\psi(\mathbf{r}) \tag{9.2.2}$$

Employing Green's theorem and the boundary condition satisfied by G_k

$$\psi(\mathbf{r}) = \frac{1}{4\pi}\int [G_k(\partial\psi/\partial n_0)]\,dS_0$$

where the integrand is to be evaluated on the boundary surface S. It should be recalled that the right-hand side is a representation of $\psi(\mathbf{r})$ only when $\mathbf{r}$ is inside the boundary surface and is zero when $\mathbf{r}$ is outside. The value for $\mathbf{r}$ on S is obtained as the limit as $\mathbf{r}$ approaches S from the inside of the region bounded by S. Introducing boundary condition (9.2.1) into (9.2.2) gives

$$\psi(\mathbf{r}) = -\frac{1}{4\pi}\int f(\mathbf{r}_0^S)G_k(\mathbf{r}|\mathbf{r}_0^S)\psi(\mathbf{r}_0^S)\,dS_0 \tag{9.2.3}$$

We see that, if the values of ψ on the surface can be determined, then ψ in the interior may be calculated as an integral over these surface values, $G_k(\mathbf{r}|\mathbf{r}_0^S)$ essentially giving the manner in which the values of ψ propagate from the surface into the interior. To determine these surface values we place $\mathbf{r}$ on S on both sides of side of (9.2.3) to obtain the integral equation for $\psi(\mathbf{r}^S)$:

$$\psi(\mathbf{r}^S) = -\frac{1}{4\pi}\int f(\mathbf{r}_0^S)G_k(\mathbf{r}^S|\mathbf{r}_0^S)\psi(\mathbf{r}_0^S)\,dS_0 \tag{9.2.4}$$

This is a homogeneous Fredholm integral equation of the second kind. It may be solved by successive approximations, or better still it may be reduced to a secular determinant for which the various expansions discussed in the Sec. 9.1 are available and which include the results obtained by the method of successive approximations.

To carry this program out, it is important to note that G_k may be expanded in the bilinear form [see Eq. (7.2.39)]

$$G_k = 4\pi \sum \frac{\bar{\varphi}_p(\mathbf{r}_0)\varphi_p(\mathbf{r})}{k_p^2 - k^2}$$

where the eigenfunctions φ satisfy homogeneous Neumann conditions. Then $\psi(\mathbf{r})$ may be expanded in the same fashion:

$$\psi = \Sigma c_p\varphi_p$$

Inserting the expansions into Eq. (9.2.3) yields

$$\sum_p c_p\varphi_p(\mathbf{r}) = \sum_{qp} c_q\varphi_p(\mathbf{r})\left[\frac{\int f(\mathbf{r}_0^S)\bar{\varphi}_p(\mathbf{r}_0^S)\varphi_q(\mathbf{r}_0^S)\,dS_0}{k^2 - k_p^2}\right]$$

Let

$$f_{pq} = \int f(\mathbf{r}_0^S)\bar{\varphi}_p(\mathbf{r}_0^S)\varphi_q(\mathbf{r}_0^S)\,dS_0 \tag{9.2.5}$$

We thus obtain the set of linear simultaneous equations

$$(k^2 - k_p^2)c_p = \sum_q f_{pq}c_q; \quad \text{for each } p \tag{9.2.6}$$

A nonzero solution of this homogeneous system exists only if the determinant of the coefficients vanishes, leading to the secular determinant

$$|(k^2 - k_p^2)\delta_{pq} - f_{pq}| = 0 \tag{9.2.7}$$

This becomes identical with the secular determinant (9.1.27), for which the iteration-perturbation formulas (including the improved version) and the Feenberg formulas are appropriate if λU_{pq} is placed equal to f_{pq}. For example, the modified iteration-perturbation formula for (9.2.7) is

$$k^2 = k_p^2 + f_{pp} + \sum_{q \neq p} \frac{f_{pq}f_{qp}}{k^2 - k_q^2 - f_{qq}} + \sum_{\substack{qr \neq p \\ r \neq q}} \frac{f_{pq}f_{qr}f_{rp}}{(k^2 - k_q^2 - f_{qq})(k^2 - k_r^2 - f_{rr})} + \cdots \quad (9.2.8)$$

The Fredholm and variation-perturbation formulas are given for the secular determinant

$$|(E - \epsilon_p)\delta_{pq} + \lambda B_{pq}| = 0$$

Hence, we may employ these formulas by placing $E - \epsilon_p = k^2 - k_p^2$ and $\lambda B_{pq} = -f_{pq}$. For the limitations on (9.2.8) arising from an f which has too strong a singularity, we may again turn to the discussion in Sec. 9.1, where we see that f may have discontinuities in value but may not be singular. We shall return to the singular case later on in this discussion.

Series (9.2.8) involve as many indices as there are dimensions to the problem. A considerable saving in effort may be achieved by reducing by one the number of indices to be summed over. This may be accomplished if the boundary is such that the unperturbed solutions are separable:

$$\varphi_p(\mathbf{r}) = \chi_\alpha(S)\Xi_\beta(\xi) \quad (9.2.9)$$

where ξ is the coordinate for which the boundary surface is constant and α and β describe the eigenfunction. The letter S stands for the other two coordinates, which specify the point on the boundary surface $\xi = \xi_S$ under consideration. The functions χ are complete and orthonormal on surface S. It may then be shown from formula (7.2.63) for the Green's function G_k that

$$G_k(\mathbf{r}^S|\mathbf{r}_0^S) = 4\pi \sum_\alpha A_\alpha(k)\chi_\alpha(S)\bar{\chi}_\alpha(S_0) \quad (9.2.10)$$

An example of such an expansion for cylindrical coordinates is given in Eq. (7.2.51), and we shall have further examples below. The dependence on k as contained in $A_\alpha(k)$ is considerably more complicated than that occurring in the bilinear expansion, $[1/(k^2 - k_p^2)]$. It therefore becomes inconvenient to employ k as eigenvalue. The form of the integral equation (9.2.4) suggests replacing the function f as follows:

$$f(S) = \mu F(S) \quad (9.2.11)$$

where μ is then a measure of the deviation of boundary conditions (9.2.1) from Neumann conditions. Then μ may be considered as an eigenvalue. The solution we shall obtain will give μ as a function of k^2; that is, it will answer the question how large should f be for a given distribution F so that k^2 has an assigned value. This is, of course, the question which is often asked in practice. Actually if the functional dependence of μ on k^2 is known (when f is complex, this may be represented by contour lines in the complex plane of k or f or vice versa), then the inversion to obtain k^2 as a function of μ is usually possible.

Inserting both (9.2.10) and (9.2.11) into integral equation (9.2.4) yields

$$\psi(\mathbf{r}^S) = -\mu \sum_\alpha A_\alpha(k)\chi_\alpha(S) \int \bar{\chi}_\alpha(S_0)F(S_0)\psi(S_0)\,dS_0$$

Expanding $\psi(\mathbf{r}^S)$ in the functions $\chi_\alpha(S)$

$$\psi(\mathbf{r}^S) = \sum_\gamma c_\gamma \chi_\gamma(S)$$

and inserting in both sides of the equation above yields

$$c_\alpha = -\mu A_\alpha(k) \sum_\gamma c_\gamma F_{\alpha\gamma}; \quad \text{for each } \alpha$$

where
$$F_{\alpha\gamma} = \int \bar{\chi}_\alpha F \chi_\gamma\,dS \tag{9.2.12}$$

Again nonzero solutions of the above system of equations can be obtained only if the following determinant vanishes:

$$|F_{\alpha\gamma} + (1/\mu A_\alpha)\delta_{\alpha\gamma}| = 0$$

A somewhat more symmetrical form can also be obtained:

$$|\sqrt{A_\alpha A_\gamma}\,F_{\alpha\gamma} + (1/\mu)\delta_{\alpha\gamma}| = 0 \tag{9.2.13}$$

This is again a secular determinant for $1/\mu$. Comparing with the standard forms employed in the previous section, we see that we may employ the iteration-perturbation (modified or unmodified) and the Feenberg formulas if the following identifications are made:

$$k^2 - k_p^2 - \lambda U_{pp} \to (1/\mu) + A_\alpha F_{\alpha\alpha}; \quad -\lambda U_{pq} = \sqrt{A_\alpha A_\gamma}\,F_{\alpha\gamma} \tag{9.2.14}$$

The Fredholm (modified and unmodified) and variation-perturbation formulas may be employed if $E - \epsilon_p + \lambda B_{pp}$ is replaced by $(1/\mu) + A_\alpha F_{\alpha\alpha}$ and λB_{pq} by $\sqrt{A_\alpha A_\gamma}\,F_{\alpha\gamma}$. For example, the modified iteration-perturbation formula yields for the αth eigenvalue $1/\mu_\alpha$

$$\frac{1}{\mu_\alpha} = -A_\alpha F_{\alpha\alpha} + \sum_{\gamma\neq\alpha} \frac{A_\alpha A_\gamma F_{\alpha\gamma}F_{\gamma\alpha}}{(1/\mu_\alpha) + A_\gamma F_{\gamma\gamma}} - \sum_{\substack{\gamma\delta\neq\alpha\\ \delta\neq\gamma}} \frac{A_\alpha A_\gamma A_\delta F_{\alpha\gamma}F_{\gamma\delta}F_{\delta\alpha}}{[(1/\mu_\alpha) + A_\gamma F_{\gamma\gamma}][(1/\mu_\alpha) + A_\delta F_{\delta\delta}]} + \cdots \tag{9.2.15}$$

This finishes (except for an example) our discussion of the boundary condition $(\partial\psi/\partial n) + f\psi = 0$, f small.

Perturbation of Boundary Conditions, f Large. We now consider the situation where f is large. The appropriate set of unperturbed solutions are those which satisfy Dirichlet conditions. We also choose a Green's function which satisfies these conditions. Under these conditions the reader may verify that Eq. (9.2.2) is replaced by

$$\psi(\mathbf{r}) = -\left(\frac{1}{4\pi}\right)\int\left(\frac{\partial G_k}{\partial n_0}\right)\psi(\mathbf{r}_0^S)\,dS_0$$

Introducing the boundary conditions satisfied by ψ yields the analogue of (9.2.3):

$$\psi(\mathbf{r}) = \frac{1}{4\pi}\int\left(\frac{\partial G_k}{\partial n_0}\right)\left(\frac{1}{f}\right)\left(\frac{\partial\psi}{\partial n_0}\right)dS_0 \tag{9.2.16}$$

Thus the value of ψ in the interior may again be obtained once the surface values of $\partial\psi/\partial n_0$ are known. The integral equation for the latter is obtained by taking the normal derivative on both sides and letting

$$\partial\psi(\mathbf{r}^S)/\partial n = V(S) \tag{9.2.17}$$

then
$$V(S) = \frac{1}{4\pi}\int\left(\frac{\partial^2 G_k}{\partial n\,\partial n_0}\right)\left(\frac{1}{f}\right)V(S_0)\,dS_0 \tag{9.2.18}$$

The kernel of this integral equation, $\partial^2 G_k/\partial n\,\partial n_0$, must be evaluated carefully in view of the discontinuities in G_k as a function of the observation point, when the source point is on the surface (see page 813). We therefore specify the meaning of $\partial^2 G_k/\partial n\,\partial n_0$ as follows:

$$\left(\frac{\partial^2 G_k}{\partial n\,\partial n_0}\right) = \lim_{r\to S}\left[\frac{\partial}{\partial n}\frac{\partial}{\partial n_0}G_k(\mathbf{r}|\mathbf{r}_0^S)\right]$$

where $\mathbf{r}$ approaches S from the interior. To evaluate this limit it is convenient first to examine the dyadic

$$\nabla G_k \nabla_0$$

the kernel of 9.2.18 being obtained as $\mathbf{n}\cdot\nabla G_k\nabla_0\cdot\mathbf{n}_0$. The expansion of this dyadic in terms of φ_n, satisfying Dirichlet conditions on S, is

$$4\pi\sum_n \frac{\nabla\varphi_n(\mathbf{r})\nabla_0\bar{\varphi}_n(r_0)}{k_n^2 - k^2}$$

We shall separate out the terms for large n in this expansion, since these are the terms determining singularities.

$$4\pi k^2\sum_n \frac{\nabla\varphi_n(\mathbf{r})\nabla_0\bar{\varphi}_n(\mathbf{r}_0)}{k_n^2(k_n^2 - k^2)} + 4\pi\sum_n \frac{\nabla\varphi_n(\mathbf{r})\nabla_0\bar{\varphi}_n(\mathbf{r}_0)}{k_n^2}$$

The second term can now be readily examined because the vector functions

$$\mathbf{v}_n = \nabla\varphi_n/k_n \tag{9.2.19}$$

form an orthonormal set which can be employed to describe any vector whose curl is zero (see Sec. 13.1). It may be readily verified through the use of Green's theorem that

$$\int \bar{\mathbf{v}}_n \cdot \mathbf{v}_m \, dV = \delta_{nm}$$

It is useful to introduce the singular dyadic $\mathfrak{D}_l$ defined by the following equation [see Eq. (13.1.30)]:

$$\mathfrak{D}_l(\mathbf{r}|\mathbf{r}_0) = \sum_n \mathbf{v}_n(\mathbf{r})\bar{\mathbf{v}}_n(\mathbf{r}_0) = \sum_n \mathbf{v}_n(\mathbf{r}) \int \bar{\mathbf{v}}_n(\mathbf{r}) \cdot \mathfrak{D}_l \, dV$$

so that the sum $\mathfrak{D}_l$ satisfies the condition

$$\int \bar{\mathbf{v}}_n(\mathbf{r}) \cdot \mathfrak{D}_l(\mathbf{r}|\mathbf{r}_0) \, dV = \bar{\mathbf{v}}_n(\mathbf{r}_0)$$

Since this relation holds for all n, it holds for any irrotational vector $\mathbf{A}$:

$$\int \mathbf{A}(\mathbf{r}) \cdot \mathfrak{D}_l(\mathbf{r}|\mathbf{r}_0) \, dV = \mathbf{A}(\mathbf{r}_0); \quad \text{curl } \mathbf{A} = 0 \tag{9.2.20}$$

Moreover, as $\mathbf{A}$ is any irrotational vector, it follows that $\mathfrak{D}_l$ must have a singularity at $\mathbf{r} = \mathbf{r}_0$ very much like that of the Dirac δ function, vanishing for $\mathbf{r} \neq \mathbf{r}_0$. This may be seen by picking a vector $\mathbf{A}(\mathbf{r})$ which is zero at $\mathbf{r}_0$ but is finite elsewhere. For an arbitrary $\mathbf{A}$, Eq. (9.2.20) will be satisfied only if $\mathfrak{D}_l(\mathbf{r}|\mathbf{r}_0)$ is zero for $\mathbf{r} \neq \mathbf{r}_0$.

This is all we shall need to evaluate $\partial^2 G_k/\partial n \, \partial n_0$, but we should add that $\mathfrak{D}_l(\mathbf{r}|\mathbf{r}_0)$ will appear again in Chap. 13, where it will be seen to be the curlless (there called longitudinal) part of the dyadic $\delta(\mathbf{r} - \mathbf{r}_0)\mathfrak{J}$, where $\mathfrak{J}$ is the idemfactor (see Eq. (13.1.31)). In addition

$$\mathfrak{G}_k = 4\pi \sum_n \frac{\nabla\varphi_n(\mathbf{r})\nabla_0\bar{\varphi}_n(\mathbf{r}_0)}{k_n^2(k_n^2 - k^2)} \tag{9.2.21}$$

is referred to in Chap. 13 as the longitudinal Green's dyadic.

Equations (9.2.16) and (9.2.18) may now be rewritten in terms of $\mathfrak{G}_k$, where we have dropped $\mathfrak{D}_l$, since it is zero when $\mathbf{r} \neq \mathbf{r}^S$:

$$\nabla\psi(\mathbf{r}) = \frac{k^2}{4\pi} \int \mathfrak{G}_k \cdot \mathbf{n}_0 \left(\frac{1}{f}\right)\left(\frac{\partial\psi}{\partial n_0}\right) dS_0 \tag{9.2.22}$$

and [see Eq. (9.2.18)]

$$V(S) = \frac{k^2}{4\pi} \int \mathbf{n} \cdot \mathfrak{G}_k \cdot \mathbf{n}_0 \left(\frac{1}{f}\right) V(S_0) \, dS_0 \tag{9.2.23}$$

We may obtain secular equations similar to (9.2.7) and (9.2.13). To obtain the first of these, expand $\nabla\psi$ in terms of the vectors (9.2.19):

$$\nabla\psi = \Sigma c_p \mathbf{v}_p = \Sigma(c_p/k_p)\nabla\varphi_p \tag{9.2.24}$$

and substitute in (9.2.22). Then

$$c_p = \sum_q c_q \frac{k^2}{k_p k_q(k_p^2 - k^2)} \int \left[\frac{\partial\bar{\varphi}_p}{\partial n_0}\left(\frac{1}{f}\right)\frac{\partial\varphi_q}{\partial n_0}\right] dS_0$$

The determinantal equation determining the eigenvalue k is then

$$|(k^2 - k_p^2)\delta_{pq} + (1/f)_{pq}| = 0 \tag{9.2.25}$$

where
$$\left(\frac{1}{f}\right)_{pq} = \frac{1}{k_p k_q} \int \left[\frac{\partial\bar{\varphi}_p}{\partial n_0}\left(\frac{1}{f}\right)\frac{\partial\varphi_q}{\partial n_0}\right] dS_0 \tag{9.2.26}$$

One can now obtain various perturbation formulas by appropriate identifications with the standard forms of Sec. 9.1. The iteration-perturbation formulas for this case may be developed by substituting $f_{pq} \to -(1/f)_{pq}$ in (9.2.8). The conditions on the convergence of the individual series limit the singularities of $1/f$ to finite discontinuities at the worst.

To obtain a secular determinant similar to (9.2.13) we must not only introduce definition (9.2.11), $f(S) = \mu F(S)$, but we must also find an expansion of $\mathbf{n}\cdot\mathfrak{G}_k\cdot\mathbf{n}_0$ similar to that of (9.2.10) for G_k. For this purpose we note that

$$\mathfrak{G}_k = \nabla G_k \nabla_0 - \nabla G_0 \nabla_0$$

Hence if, as is established in Eq. (7.2.63) [see also Eq. (9.2.10)] one may expand G_k for separable coordinates as follows:

$$G_k(S,\xi|S_0,\xi_0) = 4\pi \sum_\alpha \bar{\chi}_\alpha(S)\chi_\alpha(S_0)\bar{\Xi}_\alpha(\xi,k)\Xi_\alpha(\xi_0,k)$$

where ξ is the coordinate normal to S, the functions χ_α form a complete orthonormal set in the variables other than ξ. The form of the product of functions depending upon ξ and ξ_0 depends upon whether ξ_0 is greater or less than ξ. Let us assume that the value of ξ corresponding to the surface S is less than its values in the interior. In the application above we accordingly employ the combination appropriate to $\xi > \xi_0$. We may now write

$$\mathbf{n}\cdot\mathfrak{G}_k\cdot\mathbf{n}_0 = \sum_\alpha \chi_\alpha(S)\bar{\chi}(S_0)\left\{\left[\frac{\partial\Xi_\alpha(\xi,k)}{\partial\xi}\right]\left[\frac{\partial\bar{\Xi}_\alpha(\xi_0,k)}{\partial\xi_0}\right] - \left[\frac{\partial\Xi_\alpha(\xi,0)}{\partial\xi}\right]\left[\frac{\partial\bar{\Xi}(\xi_0,0)}{\partial\xi_0}\right]\right\}$$

where the derivatives are evaluated for ξ and ξ_0 at the boundary; or in a more abbreviated form,

$$\mathbf{n} \cdot \mathfrak{G}_k \cdot \mathbf{n}_0 = 4\pi \sum_\alpha \chi_\alpha(S)\bar{\chi}_\alpha(S_0)B_\alpha(k) \tag{9.2.27}$$

If now in (9.2.23) an expansion in χ_α is substituted for V, a determinantal equation is finally obtained:

$$|\mu\delta_{\alpha\gamma} - \sqrt{B_\alpha B_\gamma}\,(1/F)_{\alpha\gamma}| = 0 \tag{9.2.28}$$

where

$$(1/F)_{\alpha\gamma} = \int \bar{\chi}_\alpha(1/F)\chi_\gamma\, dS \tag{9.2.29}$$

The results of Sec. 9.1 may now be applied to the secular equation, a process which by now should be familiar to the reader.

We summarize the discussion to this point. Secular equations have been developed, the solution of which yields the solution of the scalar Helmholtz equation $\nabla^2\psi + k^2\psi = 0$, where ψ is subject to the mixed boundary condition $(\partial\psi/\partial n) + f\psi = 0$. Equations (9.2.7) and (9.2.13) apply when f is small, while (9.2.25) and (9.2.28) apply when f is large. Equations (9.2.13) and (9.2.28) are more convenient if the scalar Helmholtz equation can be solved by the method of separation of variables for homogeneous Neumann and Dirichlet conditions, respectively. If this is not the case, then (9.2.7) and (9.2.25) must be employed. Once the secular equation is obtained, the results of Sec. 9.1 may be applied. These formulas are useful as long as the individual series involved converge, as will occur in the present case if the function f is at most discontinuous. Indeed, if f has a delta function type of singularity, $\delta(S - S')$, then it is immediately clear from the original integral equation that no solution exists. This points up the fact that practical convergence will not be easy to obtain in the event that f is different from zero (or $1/f$) for the formulas (9.2.25) and (9.2.28) over only a small region.

As an example consider

$$\sum_{q\neq p} \frac{f_{pq}f_{qp}}{k^2 - k_q^2 - f_{qq}}$$

The integrals f_{pq} and f_{qp} will eventually diminish like $1/k_q$ for large q, so that convergence is ensured. However, this asymptotic dependence is not reached until the wavelength of the function φ_q is considerably smaller than the region in which f differs from zero. Hence if such a region be small, one will need to take many terms in the series before the factor $1/k_q^2$ will "take hold" and make the series converge. Of course, if the region is a point, as in the delta function case, this never happens and the series diverges. The manner in which this difficulty may be circumvented in the perturbation will be shown in the example which follows.

An Example. Consider a two-dimensional problem in which the boundary surface is a rectangle of dimensions a and b (see Fig. 9.2). We require the solution of the scalar Helmholtz equation satisfying homogeneous Neumann conditions everywhere except for a region on the boundary line $x = 0$ centered at y_0, that is, for $(y_0 + \frac{1}{2}w) > y > (y_0 - \frac{1}{2}w)$. In this region ψ satisfies $(\partial\psi/\partial n) + \mu\psi = 0$ or more specifically $-(\partial\psi/\partial x) + \mu\psi = 0$ with μ a constant. In other words, f in Eq. (9.2.1) is zero everywhere except in one region of width w centered at y_0, where it has the constant value μ.

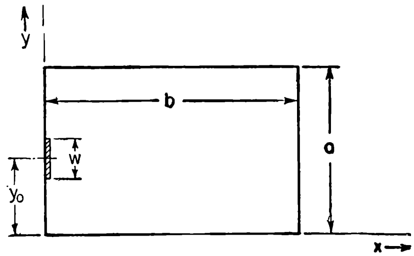

Fig. 9.2 Waves in a rectangle with special reflection properties along region of length w.

Since the unperturbed problem may be solved by separation, it is most convenient to employ the method developed in the material following Eq. (9.2.9) leading eventually to the secular equation (9.2.13). The solutions φ_n of the unperturbed problem are

$$\varphi_{\alpha\beta} = \sqrt{\frac{\epsilon_\alpha\epsilon_\beta}{ab}} \cos\left(\frac{\pi\beta x}{b}\right) \cos\left(\frac{\pi\alpha y}{a}\right); \quad \alpha \text{ and } \beta \text{ integers}$$

In terms of the notation of Eq. (9.2.9),

$$\chi_\alpha = \sqrt{\frac{\epsilon_\alpha}{a}} \cos\left(\frac{\pi\alpha y}{a}\right); \quad \Xi_\beta = \sqrt{\frac{\epsilon_\beta}{b}} \cos\left(\frac{\pi\beta x}{b}\right)$$

The Green's function $G_k(x,y|x_0y_0)$ may be obtained following the recipe given in Chap. 7:

$$G_k = -4\pi \sum_\alpha \chi_\alpha(y)\chi_\alpha(y_0) \frac{\csc[\sqrt{k^2 - (\pi\alpha/a)^2}\, b]}{\sqrt{k^2 - (\pi\alpha/a)^2}} \cdot$$
$$\cdot \begin{cases} \cos[\sqrt{k^2 - (\pi\alpha/a)^2}\, x] \cos[\sqrt{k^2 - (\pi\alpha/a)^2}\, (b - x_0)]; & x < x_0 \\ \cos[\sqrt{k^2 - (\pi\alpha/a)^2}\, (b - x_0)] \cos[\sqrt{k^2 - (\pi\alpha/a)^2}\, x]; & x > x_0 \end{cases}$$

Putting in the surface values $x = x_0 = 0$, we obtain

$$G_k(S|S_0) = -4\pi \sum_\alpha \chi_\alpha(y)\chi_\alpha(y_0) \frac{\cot[\sqrt{k^2 - (\pi\alpha/a)^2}\, b]}{\sqrt{k^2 - (\pi\alpha/a)^2}} \tag{9.2.30}$$

so that the factor $A_\alpha(k)$ which appears in Eq. (9.2.10) is

$$A_\alpha(k) = -\frac{\cot[\sqrt{k^2 - (\pi\alpha/a)^2}\, b]}{\sqrt{k^2 - (\pi\alpha/a)^2}}$$

The secular equation (9.2.13) requires, in addition to the value of A_α, the value of $F_{\alpha\gamma}$ defined in (9.2.12). For this example

$$F_{\alpha\gamma} = \int_{y_0-\frac{1}{2}w}^{y_0+\frac{1}{2}w} \chi_\alpha \chi_\gamma \, dy = \sqrt{\frac{\epsilon_\alpha \epsilon_\gamma}{a^2}} \int_{y_0-\frac{1}{2}w}^{y_0+\frac{1}{2}w} \cos\left(\frac{\pi\alpha y}{a}\right) \cos\left(\frac{\pi\gamma y}{a}\right) dy$$

or

$$F_{\alpha\gamma} = \tfrac{1}{2}\sqrt{\frac{\epsilon_\alpha\epsilon_\gamma}{a^2}} \left[\frac{\sin[(\pi/a)(\alpha+\gamma)y]}{(\pi/a)(\alpha+\gamma)} + \frac{\sin[(\pi/a)(\alpha-\gamma)y]}{(\pi/a)(\alpha-\gamma)}\right]_{y_0-\frac{1}{2}w}^{y_0+\frac{1}{2}w} \tag{9.2.31}$$

and

$$F_{\alpha\alpha} = \tfrac{1}{2}\left(\frac{\epsilon_\alpha}{a}\right)\left[y + \frac{\sin(2\pi\alpha y/a)}{(2\pi\alpha/a)}\right]_{y_0-\frac{1}{2}w}^{y_0+\frac{1}{2}w} \tag{9.2.32}$$

These results may now be substituted into the various perturbation formulas developed in Sec. 9.1. For the present we shall content ourselves with examining the iteration-perturbation formula up to second order, as given for this case by Eq. (9.2.15). One obtains

$$\frac{1}{\mu_\alpha A_\alpha} = -\tfrac{1}{2}\epsilon_\alpha\left[\frac{y}{a} + \frac{\sin(2\pi\alpha y/a)}{2\pi\alpha}\right] - \tfrac{1}{4}\sum_{\gamma\neq\alpha} \epsilon_\alpha\epsilon_\gamma \frac{\cot[\sqrt{(ka)^2-(\pi\gamma)^2}\,(b/a)]}{\sqrt{(ka)^2-(\pi\gamma)^2}} \cdot \\ \cdot \frac{\left\{\dfrac{\sin[\pi(\alpha+\gamma)(y/a)]}{\pi(\alpha+\gamma)} + \dfrac{\sin[\pi(\alpha-\gamma)(y/a)]}{\pi(\alpha-\gamma)}\right\}^2}{\left(\dfrac{1}{\mu_\alpha a}\right) + \tfrac{1}{2}\epsilon_\gamma\left\{\left(\dfrac{y}{a}\right) + \dfrac{\sin[2\pi\gamma(y/a)]}{2\pi\gamma}\right\}} \tag{9.2.33}$$

where each individual function of y has the limits $y_0 + \frac{1}{2}w$ and $y_0 - \frac{1}{2}w$ inserted for y as stated in (9.2.31) and (9.2.32). Note that in the numerator these substitutions are to be made before the indicated square is performed. Convergence of this series is good, since for large γ the summand approaches

$$\frac{-8\epsilon_\alpha}{(\pi\gamma)^3} \frac{\{\cos[\pi\alpha(y/a)]\,\sin[\pi\gamma(y/a)]\}^2}{(1/\mu_\alpha a) + (w/a)}$$

or inserting the limits

$$\frac{-8\epsilon_\alpha}{(\pi\gamma)^3} \frac{[\sin(\pi\gamma\eta_0^+)\,\cos(\pi\alpha\eta_0^+) - \sin(\pi\gamma\eta_0^-)\,\cos(\pi\alpha\eta_0^-)]^2}{(1/\mu_\alpha a) + (w/a)} \tag{9.2.34}$$

where we have introduced the abbreviations

$$\eta_0 = y_0/a; \quad \eta = y/a; \quad \eta_0^+ = (y_0 + \tfrac{1}{2}w)/a; \quad \eta_0^- = (y_0 - \tfrac{1}{2}w)/a$$

We next consider at what value of γ the asymptotic form $O(1/\gamma^3)$ of the summand in (9.2.33) becomes important, for then we shall know approximately how many terms of the series we shall have to evaluate in order to obtain an accurate value for the sum. It is, of course, necessary that $\mu \gg \alpha$, $\pi\gamma \gg ka$, and $\pi\gamma b/a \gg 1$. Another condition becomes more obvious if the limits are inserted into the terms $\{\sin[\pi(\alpha \pm \gamma)\eta]/\pi(\alpha \pm \gamma)\}$. One obtains

$$\frac{2\cos[\pi(\alpha\pm\gamma)\eta_0]\,\sin[\pi(\alpha\pm\gamma)(w/2a)]}{\pi(\alpha\pm\gamma)}$$

It is now clear that the asymptotic dependence will become valid only if γ is so large that

$$(\alpha \pm \gamma)(w/a) > 1 \tag{9.2.35}$$

If this condition is not satisfied, *i.e.*, if $(\alpha \pm \gamma)\, w \ll a$, then

$$\frac{\sin[\pi(\alpha + \gamma)(w/2a)]}{\pi(\alpha \pm \gamma)} \xrightarrow[w \to 0]{} (w/2a)$$

rather than the term $O(1/\gamma)$ which would be obtained if (9.2.35) were satisfied. Thus even for large γ, the terms in the series in (9.2.33) will decrease very slowly until (9.2.35) is satisfied. We see that the number of terms that would require accurate evaluation is approximately a/ω, so that series (9.2.33) is not convenient for small w/a.

Formulas for Small w. To overcome this difficulty, we shall subtract the series obtained by summing (9.2.34) from one to infinity. The terms of the remaining series will then be of the order $1/\gamma^5$ for large γ when (9.2.35) is satisfied and of the order of $1/\gamma^3$ when it is not, so that the series is manageable even when w approaches zero. The difficulties which were present in the original series will now be confined to the series subtracted out. To be more specific, let

$$S(\eta) = \sum_{\gamma=1}^{\infty} \frac{\cos(\pi\gamma\eta)}{\gamma^3} \quad \text{and} \quad \omega = \frac{w}{a} \tag{9.2.36}$$

Then

$$\begin{aligned}
\frac{1}{\mu_\alpha A_\alpha} = &-\tfrac{1}{2}\epsilon_\alpha\left[\omega + \frac{\sin(\pi\alpha\omega)\cos(2\pi\alpha\eta_0)}{\pi\alpha}\right] \\
&+ \frac{\epsilon_\alpha}{\pi^3[\omega + (1/\mu_\alpha a)]}\Big\{\cos^2(\pi\alpha\eta_0^+)[S(0) - S(2\eta_0^+)] \\
&+ \cos^2(\pi\alpha\eta_0^-)[S(0) - S(2\eta_0^-)] - 2\cos(\pi\alpha\eta_0^+)\cos(\pi\alpha\eta_0^-)[S(\omega) - S(\eta_0^+ + \eta_0^-)] \\
&- (2/\alpha^3)\sin^2(\pi\alpha\omega)\cos^2(2\pi\alpha\eta_0)\Big\} \\
&- \frac{1}{4}\sum_{\gamma\neq\alpha}\epsilon_\alpha\epsilon_\gamma\left\{\frac{\cot[\sqrt{(ka)^2 - (\pi\gamma)^2}\,(b/a)]}{\sqrt{(ka)^2 - (\pi\gamma)^2}}\cdot\right. \\
&\cdot\frac{\left\{\dfrac{\sin[\pi(\alpha+\gamma)\eta]}{\pi(\alpha+\gamma)} + \dfrac{\sin[\pi(\alpha-\gamma)\eta]}{\pi(\alpha-\gamma)}\right\}^2}{\left\{\dfrac{1}{\mu_\alpha a} + \tfrac{1}{2}\epsilon_\gamma\left[\eta + \dfrac{\sin(2\pi\gamma\eta)}{2\pi\gamma}\right]\right\}} \\
&\left. + \frac{4(1 - \delta_{0\gamma})}{(\pi\gamma)^3}\,\frac{[\cos(\pi\alpha\eta)\sin(\pi\gamma\eta)]^2}{(1/\mu_\alpha a) + \omega}\right\}
\end{aligned} \tag{9.2.37}$$

where the limits $\eta = \eta_0^+, \eta_0^-$ are still to be substituted in the latter part of this. The factor $(1 - \delta_{0\gamma})$ was inserted to indicate the omission of the $\gamma = 0$ term in the sum.

We now go on to determine the behavior of Eq. (9.2.37) for small ω.

We shall require some of the properties of $S(\eta)$. The sum obtained by taking the second derivative of $S(\eta)$ may be expressed in a closed form:

$$S''(\eta) = -\pi^2 \sum_{\gamma=1}^{\infty} [\cos(\pi\gamma\eta)]/\gamma = \left(\frac{\pi^2}{2}\right) \ln[2(1 - \cos \pi\eta)]$$

For small η,

$$S''(\eta) \xrightarrow[\eta \to 0]{} \pi^2 \ln(\pi\eta) \tag{9.2.38}$$

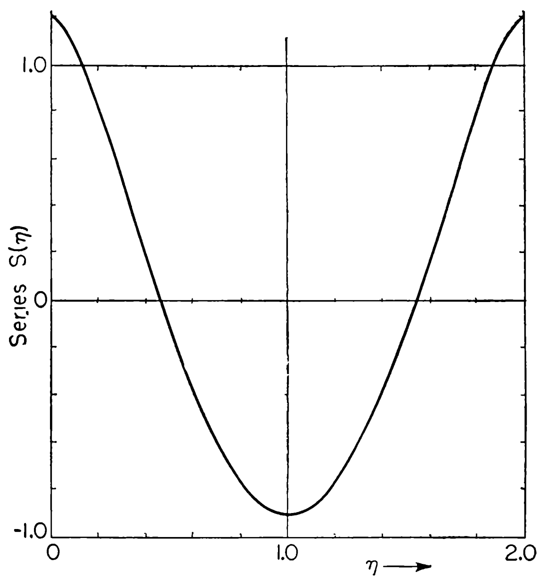

Fig. 9.3 Correction series $S(\eta)$, defined in Eq. (9.2.36), for effect of nonuniform boundary conditions.

By successive integrations we have

$$S'(\eta) = -\pi \sum_{\gamma=1}^{\infty} \left[\sin \frac{(\pi\gamma\eta)}{\gamma^2}\right] \to \pi^2\eta[\ln \eta + \ln \pi - 1] \tag{9.2.39}$$

and

$$S(\eta) \xrightarrow[\eta \to 0]{} S(0) + (\pi^2\eta^2/2)[\ln \eta - \tfrac{3}{2} + \ln \pi] \tag{9.2.40}$$

Graphs of the sums $S(\eta)$ and $-S'(\eta)/\pi$ are shown in Figs. 9.3 and 9.4. The approximate formulas given above hold over the range $0 \leq \eta \leq 0.5$. From Eq. (9.2.37) we may now discover the impossibility of dealing with a delta function type of singularity, *i.e.*, where $f(S)$ in (9.2.1) is $\delta(y - y_0)$. The results may be obtained from Eqs. (9.2.37) and (9.2.40) by letting $\omega \to 0$ and $\mu_\alpha \to \infty$ but $\mu_\alpha\omega \to C_\alpha$ a constant. Then because of the presence of the $S(\omega)$ term a logarithmic singularity appears, indicating perhaps a failure of perturbation theory.

This result is no accident, for we shall now show that, if $f(S) = C\delta(S - S')$, where S' is a point on the surface, then there is no finite solution of the integral equation determining ψ. Inserting the above expression for f into integral equation (9.2.4) yields

$$\psi(S) = -(C/4\pi)G_k(S|S')\psi(S') \tag{9.2.41}$$

To determine k^2 (or C) we should place $S = S'$ and find the value of k^2

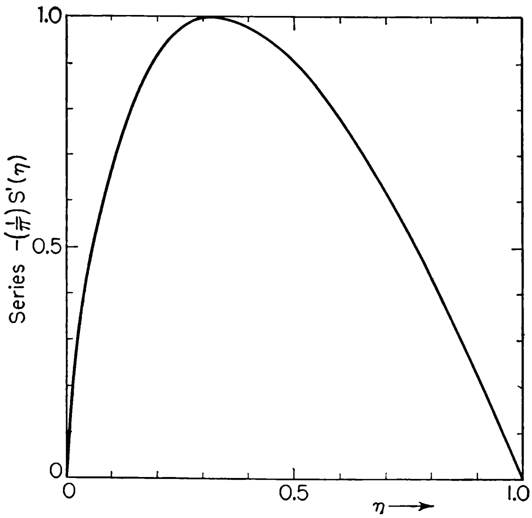

Fig. 9.4 Derivative of series $S(\eta)$ as function of η.

which makes the equation consistent:

$$4\pi/C = -\lim_{S'\to S} [G_k(S|S')]$$

For the two-dimensional problem considered here, $G_k(S|S')$ has a logarithmic singularity, independent of k, at $S = S'$, so that no solution of the integral equation exists for a nonzero C.

Long, Narrow Rectangle. Many terms of the series in the perturbation formulas also need to be evaluated when b/a is small, *i.e.*, when the rectangle is long and narrow. We can proceed to expand each term in a power series in b/a. However, it will turn out to be simpler if the expansion is made for the Green's function itself. From (9.2.30) and the series $\cot x = (1/x) - (x/3) + \cdots$ one obtains

$$G_k(0,y|0,y_0) \simeq \frac{4\pi}{b}\sum_\alpha \frac{\chi_\alpha(y)\chi_\alpha(y_0)}{(\pi\alpha/a)^2 - k^2} + \frac{4\pi b}{3}\sum_\alpha \chi_\alpha(y)\chi_\alpha(y_0)$$

The first sum is, except for the factor $1/b$, equal to the Green's function for a one-dimensional problem.

$$\Gamma_k(y|y_0) = 4\pi \sum_\alpha \frac{\chi_\alpha(y)\chi_\alpha(y_0)}{(\pi\alpha/a)^2 - k^2}$$

Then $$(d^2\Gamma_k/dy^2) + k^2\Gamma_k = -4\pi\delta(y - y_0)$$

The second sum in (9.2.42) is proportional to $\delta(y - y_0)$. Finally then

$$G_k(0,y|0,y_0) \simeq (1/b)\Gamma_k(y|y_0) + (4\pi b/3)\delta(y - y_0) \qquad (9.2.42)$$

If we had used more terms in the expansion of the cotangent, higher derivatives of the delta function would be added on to (9.2.42). Inserting (9.2.42) into integral equation (9.2.4) gives

$$\psi(y) = -(1/4\pi b)\int f(y_0)\Gamma_k(y|y_0)\psi(y_0)\,dy_0 - \tfrac{1}{3}bf(y)\psi(y)$$

This integral equation is equivalent to the following second-order differential equation and associated boundary conditions:

$$\begin{aligned}(d^2/dy^2)[1 + \tfrac{1}{3}bf]\psi + [k^2 - (f/b)]\psi &= 0 \qquad (9.2.43)\\ d[1 + \tfrac{1}{3}bf]\psi/dy = 0 \text{ at } y &= 0 \text{ and } y = a\end{aligned}$$

If f is a simple enough function, as it is in the case under discussion, this differential equation may be solved to obtain ψ on the surface and the value of the eigenvalue k^2. From ψ on the surface, ψ in the interior may be obtained from (9.2.3). Thus we see that if b is small enough, the effect of the changing boundary conditions is sufficiently localized to be represented by a differential rather than by an integral equation.

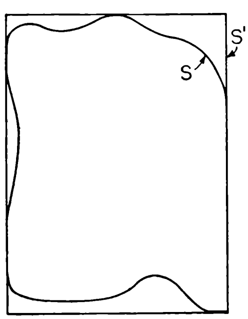

Fig. 9.5 Perturbation of boundary shape, from simple shape S' to distorted shape S.

Perturbation of Boundary Shape. Consider next the effect on the eigenvalue and eigenfunctions of a change in the boundary surface shape. Let the original surface be denoted S' while the surface to which it is distorted is S (see Fig. 9.5). Similarly the region enclosed by S' or S is denoted R' or R, respectively. In the figure we have drawn R as being contained within R'. This is essential if we are to express the eigenfunctions for region R in terms of those of R'. We again limit the discussion to the Helmholtz equation and to homogeneous Neumann or Dirichlet boundary conditions. It is presumed that the eigenfunction φ_n satisfying these boundary conditions on S' and their corresponding eigenvalues k_n^2 are known, so that

$$\nabla^2\varphi_n + k_n^2\varphi_n = 0 \qquad (9.2.44)$$

We now go on to obtain the integral equation for ψ satisfying boundary conditions on S and the equation

$$\nabla^2\psi + k^2\psi = 0$$

The integral equation follows from the integral representation of ψ in terms of a Green's function $G_k(\mathbf{r}|\mathbf{r}_0)$, satisfying boundary conditions on S', and of ψ itself, which may be obtained directly from applying Green's theorem to Eq. (9.2.2):

$$\psi(\mathbf{r}) = \frac{1}{4\pi}\int_S \left\{G_k(\mathbf{r}|\mathbf{r}_0^S)\left(\frac{\partial\psi}{\partial n_0}\right) - \psi\left[\frac{\partial G_k(\mathbf{r}|\mathbf{r}_0^S)}{\partial n_0}\right]\right\} dS_0 \qquad (9.2.45)$$

We emphasize that this integral represents ψ *inside* S *only*, being zero outside.

We must now specify our boundary conditions. Let us first consider homogeneous Neumann conditions on S. We choose G_k and φ_n to satisfy homogeneous Neumann conditions on S'; hence (9.2.45) becomes

$$\psi(\mathbf{r}) = -\frac{1}{4\pi}\int_S \psi\left[\frac{\partial G_k(\mathbf{r}|\mathbf{r}_0^S)}{\partial n_0}\right] dS_0 \qquad (9.2.46)$$

We see again that, if ψ is known on S, it may be computed for $\mathbf{r}$ anywhere in the interior by means of (9.2.46). The integral equation is obtained by putting $\mathbf{r}$ on the surface S:

$$\psi(S) = -\frac{1}{4\pi}\int_S \psi(S_0)\left[\frac{\partial G_k(S|S_0)}{\partial n_0}\right] dS_0 \qquad (9.2.47)$$

We next reduce Eqs. (9.2.46) and (9.2.47) to their equivalent secular equations, to which we may apply the techniques described in Sec. 9.1. Because the function defined by the integral in (9.2.46) vanishes outside S and thus is discontinuous across S, its expansion in terms of φ_n does not converge very well. This suggests that an expansion in terms of a function which equals φ_n within S and is zero outside would be more appropriate. We shall denote this function by Φ_n. Then let

$$\psi(\mathbf{r}) = \sum_p c_p\Phi_p \qquad (9.2.48)$$

We insert (9.2.48) on both sides of (9.2.46) and employ the expansions (assuming φ_p to be real)

$$G_k = 4\pi \sum_q \frac{\varphi_q(\mathbf{r}_0)\varphi_q(\mathbf{r})}{k_q^2 - k^2}$$

and

$$\Phi_p = \sum_q N_{qp}\varphi_q; \quad N_{pq} = \int_R \varphi_p\varphi_q\, dv \qquad (9.2.49)$$

where the subscript R denotes integration over region R only. The simultaneous equations satisfied by c_q are

$$\sum_p c_p[N_{qp}(k^2 - k_q^2) - A_{qp}] = 0; \quad \text{for each } q \tag{9.2.50}$$

where
$$A_{qp} = \int \left(\frac{\partial \varphi_q}{\partial n}\right) \varphi_p \, dS \tag{9.2.51}$$

Equations (9.2.51) are exactly the same as Eqs. (9.1.95), obtained in the discussion of perturbation theory with nonorthogonal functions. We make the identifications $(EN_{pp} - H_{pp}) \to [(k^2 - k_p^2)N_{pp} - A_{pp}]$ and $(H_{pq} - EN_{pq}) \to [A_{pq} - (k^2 - k_p^2)N_{pq}]$. We may then employ expression (9.1.98) directly to obtain the form of the expansion for ψ. To second order ψ is

$$\begin{aligned}\psi = \Phi_n &+ \sum_{p \neq n} \left[\frac{A_{pn} - (k^2 - k_p^2)N_{pn}}{N_{pp}(k^2 - k_p^2) - A_{pp}}\right] \Phi_p \\ &+ \sum_{\substack{qp \neq n \\ q \neq p}} \frac{[A_{pq} - (k^2 - k_p^2)N_{pq}][A_{qn} - (k^2 - k_q^2)N_{qn}]}{[N_{pp}(k^2 - k_p^2) - A_{pp}][N_{qq}(k^2 - k_q^2) - A_{qq}]} \Phi_p\end{aligned} \tag{9.2.52}$$

It will not be possible to use Eq. (9.1.99) for the eigenvalue beyond second order, for as we shall see, the second series in (9.2.52) converges only in the mean, and therefore cannot be evaluated at the boundary surface by direct substitution. However, techniques do exist for improving the convergence; we shall discuss them later. The expression for k^2 to second order may be obtained from Eq. (9.1.99):

$$(k^2 - k_n^2)N_{nn} = A_{nn} + \sum_{p \neq n} \frac{[A_{np} - (k^2 - k_n^2)N_{np}][A_{pn} - (k^2 - k_p^2)N_{pn}]}{N_{pp}(k^2 - k_p^2) - A_{pp}} \tag{9.2.53}$$

Since it will prove impossible to obtain similar formulas for the higher order terms, it is important to have an expression for $k^2 - k_n^2$ in terms of ψ which can be obtained indirectly from (9.2.52). We start with the differential equations satisfied by ψ and φ_n:

$$\nabla^2\psi + k^2\psi = 0; \quad \nabla^2\varphi_n + k_n^2\varphi_n = 0$$

From this it immediately follows that

$$\psi\nabla^2\varphi_n - \varphi_n\nabla^2\psi = (k^2 - k_n^2)\varphi_n\psi$$

Integrating over region R, using Green's theorem, and inserting the boundary conditions satisfied by ψ yield

$$\int_S \psi \left(\frac{\partial \varphi_n}{\partial n}\right) dS = (k^2 - k_n^2) \int_R \psi\varphi_n \, dV$$

Let us write $\psi = \varphi_n + (\psi - \varphi_n)$. Then

$$(k^2 - k_n^2)N_{nn} = A_{nn} - (k^2 - k_n^2)\int_R \varphi_n(\psi - \varphi_n)\,dV + \int_S (\psi - \varphi_n)\left(\frac{\partial\varphi_n}{\partial n}\right)dS \quad (9.2.54)$$

If the zero and first-order terms in (9.2.52) are substituted in the above equation, Eq. (9.2.53) is obtained.

Evaluation of Integrals. In order to consider the important question of the convergence of (9.2.52) and (9.2.53), it is necessary to estimate the value of the integrals N_{pq} and A_{pq}. We shall first show that the volume integrals N_{pq} may be expressed in terms of surface integrals. From the equations satisfied by φ_p and φ_q we may readily obtain

$$[\varphi_q\nabla^2\varphi_p - \varphi_p\nabla^2\varphi_q] = (k_q^2 - k_p^2)\varphi_p\varphi_q$$

Integrating over region R, using Green's theorem, one finds

$$N_{pq} = \left[\frac{A_{pq} - A_{qp}}{k_q^2 - k_p^2}\right] \quad (9.2.55)$$

Turning now to N_{pp}, note that a relation between N_{pp} and the k^2 derivative of A_{pq} is suggested by Eq. (9.2.55) above. We therefore consider the equation satisfied by $\partial\varphi/\partial k^2$:

$$\nabla^2(\partial\varphi/\partial k^2) + k^2(\partial\varphi/\partial k^2) = -\varphi$$

Combining this and the equation for $\bar{\varphi}$ yields

$$-\bar{\varphi}\nabla^2(\partial\varphi/\partial k^2) + (\partial\varphi/\partial k^2)\nabla^2\bar{\varphi} = \bar{\varphi}\varphi$$

Integrating over the region R and evaluating at $k^2 = k_p^2$, the value of N_{pp} is

$$N_{pp} = \int_S \left[\left(\frac{\partial\varphi}{\partial k^2}\right)\left(\frac{\partial\bar{\varphi}}{\partial n}\right) - \left(\frac{\partial^2\varphi}{\partial n\partial k^2}\right)\bar{\varphi}\right]_{k^2=k_p{}^2} dS \quad (9.2.56)$$

This is a perfectly general result, applying as well to the case of separable coordinates, which may thus serve as an illustration. Incidentally, formula (9.2.56) is often the simplest way of evaluating a normalization integral for the separable case, for here it may be applied to each of the eigenfunctions in the product solution. For each of these the surface integral reduces to the value obtained by inserting the limits of the range of their dependent variable so that the normalization integral may be obtained without integration! To illustrate, let us find the normalization integral for the function

$$\varphi_p = \cos(p\pi x/b)$$

which satisfies Neumann boundary conditions at $x = 0$ and $x = b$. Let φ in (9.2.56) be $\cos kx$ so that $k_p = p\pi/b$. Then

$$[\partial\varphi/\partial k^2]_{k=k_p} = (-bx/2\pi p)\sin(p\pi x/b)$$

Noting that $\partial/\partial n = \partial/\partial x$ for this problem, Eq. (9.2.56) becomes

$$N_{pp} = \tfrac{1}{2}[x\sin^2(p\pi x/b) + x\cos^2(p\pi x/b) + (b/\pi p)\sin(p\pi x/b)\cos(p\pi x/b)]_0^b$$

so that $N_{pp} = \frac{1}{2}b$ as it should. A more complicated example will be worked out later when a specific problem on the perturbation of boundaries is considered.

Having now shown that we may restrict our considerations to surface integrals, we turn to the manner in which the surface is to be specified and the normal derivative and surface element is to be evaluated. Let us suppose that the unperturbed surface S' may be specified by fixing one (say ξ_1) of the three variables, ξ_1, ξ_2, ξ_3 in which the Helmholtz equation separates. Then the equation for the perturbed surface may be written as

$$S(\xi_1,\xi_2,\xi_3) = 0$$

where

$$S = \xi_1 - c - f(\xi_2,\xi_3)$$

where c is the value of ξ_1 on the unperturbed surface S; the function f then describes the perturbation. In terms of S the quantity $(\partial\varphi/\partial n)\,dS$ which occurs turns out to be

$$(\partial\varphi/\partial n)\,dS = h_2h_3\nabla\varphi\cdot\nabla S\,d\xi_2\,d\xi_3$$

and

$$\nabla\varphi\cdot\nabla S = (1/h_1^2)(\partial\varphi/\partial\xi_1) - (1/h_2^2)(\partial\varphi/\partial\xi_2)(\partial f/\partial\xi_2) - (1/h_3^2)(\partial\varphi/\partial\xi_3)(\partial f/\partial\xi_3) \quad (9.2.57)$$

where ξ_1 is to be replaced by $c + f(\xi_2,\xi_3)$ after the differentiations are performed. The range covered by the variables ξ_2 and ξ_3 is just that required to describe the original surface S'.

Convergence. The convergence of the various series in (9.2.52) and (9.2.53) is determined by the behavior of the elements A_{pq} for large k_p and k_q. This can be determined most readily by writing A_{pq} out explicitly. Let

$$\varphi_p(\xi_1,\xi_2,\xi_3) = \chi_p(\xi_2,\xi_3)X_p(\xi_1)$$

Then

$$A_{pq} = \iint h_2h_3d\xi_2\,d\xi_3\,\chi_q(\xi_2,\xi_3)\Bigg\{X_q[c + f(\xi_2,\xi_3)]\cdot \cdot\Bigg[\left(\frac{1}{h_1^2}\right)\chi_p(\xi_2,\xi_3)\left(\frac{dX_{1p}}{d\xi_1}\right)_{\xi_1=c+f} - \left(\frac{1}{h_2^2}\right)X_{1p}[c + f(\xi_2,\xi_3)]\left(\frac{\partial f}{\partial\xi_2}\right)\left(\frac{\partial\chi_p}{\partial\xi_2}\right) - \left(\frac{1}{h_3^2}\right)X_{1p}[c + f(\xi_2,\xi_3)]\left(\frac{\partial f}{\partial\xi_3}\right)\left(\frac{\partial\chi_p}{\partial\xi_3}\right)\Bigg]\Bigg\}$$

Thus A_{pq} is proportional to the coefficient in the expansion of the function of ξ_2 and ξ_3 in braces, in terms of the set $\chi_q(\xi_2,\xi_3)$. Generally speaking, we know that, if this function is sufficiently smooth, A_{pq} will decrease rapidly for large k_p and k_q. If the wavelength associated with χ_q were much smaller than the variation of the function, the rapid oscillation of χ_q would result in cancellation. Deviations from a smooth dependence may occur if $f(\xi_2,\xi_3)$ is not smooth and will occur because $X_{1q}[c + f(\xi_2,\xi_3)]$ oscillates more and more rapidly as q increases. We may consider each possibility separately if we assume that the singularities of f are isolated, so that f may be broken up into regions in which it is smooth, its singularities arising in the way these regions join. The effects arising from the oscillation of X mentioned above may then be evaluated for each region separately and lead immediately to the consideration of the following integral:

$$I = \iint h_2h_3\, d\xi_2\, d\xi_3\, \chi_q(\xi_2,\xi_3)X_{1q}[c + f(\xi_2,\xi_3)]$$

A_{pq} is proportional to I when q is so large that the remainder of the integrand hardly varies for many oscillations of χ_q and thus may be replaced approximately by a constant. Integral I will be estimated for the special case of cartesian coordinates. This involves no loss in generality; for sufficiently large values of q and therefore many oscillations of χ_q the curvature of the boundary surface should not be important. In fact, in Sec. 6.3 we demonstrate this theorem in detail in some comparisons between Fourier series and other orthogonal function series. Thus

$$I \simeq \iint d\xi_1\, d\xi_2 \cos\left(\frac{\pi q_3\xi_3}{a_3}\right) \cos\left(\frac{\pi q_2\xi_2}{a_2}\right) \cos\left\{\frac{\pi q_1}{a_1}\,[c + f(\xi_2,\xi_3)]\right\}$$

where $k_q^2 = \pi^2[(q_1/a_1)^2 + (q_2/a_2)^2 + (q_3/a_3)^2]$ and where we have taken for φ_q the functions satisfying Neumann conditions. By using the law of addition for cosines, I may be written as a sum of several terms of which we give a general term below:

$$I' = \iint d\xi_1\, d\xi_2 \cos\left[\left(\frac{\pi q_2}{a_2}\right)\xi_2 \pm \left(\frac{\pi q_3}{a_3}\right)\xi_3 \pm \left(\frac{\pi q_1}{a_1}\right)(c + f)\right]$$

where the plus and minus signs indicate that various possible combinations of terms will occur in I. The value of I' can be estimated by the method of steepest descents. In the first place

$$I' = \operatorname{Re}\left\{\iint d\xi_1\, d\xi_2 \exp i\left[\pm\left(\frac{\pi q_2}{a_2}\right)\xi_2 \pm \left(\frac{\pi q_3}{a_3}\right)\xi_3 + \left(\frac{\pi q_1}{a_1}\right)(c + f)\right]\right\}$$

We expand the exponent around those values of ξ_2 and ξ_3, ξ_2' and ξ_3', for which its gradient is zero. Therefore,

$$q_2/a_2 = \pm(q_1/a_1)(\partial f/\partial\xi_2'); \quad q_3/a_3 = \pm(q_1/a_1)(\partial f/\partial\xi_3')$$

It is always possible to find ratios q_2/q_1 and q_3/q_1 so that these equations are satisfied for each point on the surface. This will not exhaust all possible values of the ratios, so that an upper bound for I' is obtained if we assume that the above equalities may be always satisfied. We may now expand the exponent in a power series around ξ_2' and ξ_3':

$$\pm\left(\frac{\pi q_2}{a_2}\right)\xi_2' \pm \left(\frac{\pi q_3}{a_3}\right)\xi_3' + \left(\frac{\pi q_1}{a_1}\right)[c + f(\xi_2',\xi_3')] \\ + \left(\frac{i}{2}\right)\left(\frac{\pi q_1}{a_1}\right)\left|\left(\frac{\partial^2 f}{\partial \xi_2'^2}\right)(\xi_2 - \xi_2')^2 \right. \\ \left. + 2\left(\frac{\partial^2 f}{\partial \xi_2'\,\partial \xi_3'}\right)(\xi_2 - \xi_2')(\xi_3 - \xi_3') + \left(\frac{\partial^2 f}{\partial \xi_3'^2}\right)(\xi_3 - \xi_3')^2\right|$$

where the paths on the complex planes of ξ_2 and ξ_3 have been chosen so as to obtain "steepest descent." Then the integrals may be evaluated approximately by extending the range to plus and minus infinity, so that

$$|I'| \leq (2a_1/q_1)|(\partial^2 f/\partial \xi_2'^2)(\partial^2 f/\partial \xi_3'^2) - (\partial^2 f/\partial \xi_2'\partial \xi_3')^2|^{-\frac{1}{2}} \tag{9.2.58}$$

From this we may conclude that I and therefore A_{pq} is of the order of $1/q_1$ as q_1 goes to infinity or, in terms of k_q,

$$A_{pq} \xrightarrow[k_q \to \infty]{} O(1/k_q) \tag{9.2.59}$$

This result is no longer valid if the factor between the vertical lines in Eq. (9.2.58) should happen to vanish, as it will for particular points (which must be saddle points) on the boundary surface and therefore for only definite ratios of q_3 and q_2 to q_1. Because of the latter restriction, these exceptional cases will not be of sufficient importance to change the rate of convergence of (9.2.53) from that which would be predicted from Eq. (9.2.58).

It is possible to obtain the behavior of A_{pq} for large k_p in much the same fashion as the discussion above for large k_q. There is, however, an additional dependence arising from the derivatives in φ_p leading to a multiplicative factor of k_p. Hence

$$A_{pq} \xrightarrow[k_p \to \infty]{} O(1) \tag{9.2.60}$$

When both k_p and k_q are large, one obtains

$$A_{pq} \xrightarrow[k_p, k_q \to \infty]{} O\left(\frac{k_p}{k_p \pm k_q}\right) \tag{9.2.61}$$

where the plus and minus sign indicates that terms involving both combinations of k_p and k_q occur.

Equations (9.2.59) to (9.2.61) give the asymptotic dependence arising from the presence of terms like $X_q(c + f)$. We shall now discuss the limitations which arise from the singularities in f. The important terms here are the derivatives $\partial f/\partial \xi_1$ and $\partial f/\partial \xi_2$. Hence if f should have finite discontinuities, these derivatives would contain δ functions in variables ξ_1 and ξ_2, respectively, and therefore,

$$A_{pq} \xrightarrow[k_p,k_q \to \infty]{} O(k_p/k_p \pm k_q)$$

where this also holds if only one of the pair k_p, k_q becomes infinite. If f should be continuous but should have a discontinuous gradient, then

$$A_{pq} \xrightarrow[k_p,k_q \to \infty]{} O(k_p/(k_p \pm k_q)^2)$$

and so on. It is seen that the character of f never results in an asymptotic growth slower than that given by (9.2.61). Hence we may take (9.2.59) to (9.2.61) as the description of the asymptotic dependence of A_{pq}.

We may at last give the asymptotic dependence of the coefficients of φ_p in (9.2.52) and of the summand in (9.2.53). The individual factors behave as follows:

$$\begin{aligned}
&[A_{np} - (k^2 - k_n^2)N_{np}] \xrightarrow[k_p \to \infty]{} A_{np} \to O(1/k_p) \\
&[A_{pn} - (k^2 - k_p^2)N_{pn}] \xrightarrow[k_p \to \infty]{} A_{np} \\
&[(k^2 - k_p^2)N_{pp} - A_{pp}] \xrightarrow[k_p \to \infty]{} O(k_p^2) \\
&[A_{pq} - (k^2 - k_p^2)N_{pq}] \xrightarrow[k_p,k_q \to \infty]{} O(1)
\end{aligned}$$

Substituting these asymptotic values into Eq. (9.2.53) for $(k^2 - k_n^2)$, we see that the summand decreases as $1/k_p^4$ as k_p increases. This is sufficient for convergence of the series. The series for the next order may converge conditionally; the sum of the absolute value of its terms diverges logarithmically. This divergence arises because the series for the second-order term in the wavefunction converges only in the mean; *i.e.*, the sum of the squares of the coefficients of φ_p converges, which may be seen as follows. If the above asymptotic results are substituted into the second-order term of (9.2.52) we find that the coefficient of φ_p is $[\ln(k_p)/k_p^2]$ for large k_p; the sum $\Sigma[\ln^2(k_p)/k_p^4]$ converges.

Improving the Convergence. The conclusion we must draw from the above discussion is that it is impossible to obtain an explicit expansion of the eigenfunction ψ in terms of the unperturbed modes. In the case of the iteration-perturbation formula we must express the least convergent part of the series in a closed form, the remainder then converging sufficiently to make the next iteration possible. This may often be done by inspection. However, a somewhat more systematic procedure is based on integral equation (9.2.47). We first subtract out φ_n, since this is, of course, the principal term. This gives

$$\psi - \varphi_n = -\frac{1}{4\pi}\oint_S (\psi - \varphi_n)\frac{\partial}{\partial n_0} G_k(S|S_0)\, dS_0$$
$$-\frac{1}{4\pi}\oint_S G_k(S|S_0)\left(\frac{\partial \varphi_n}{\partial n_0}\right) dS_0 + (k^2 - k_n^2)\int_R \varphi_n G_k\, dV \quad (9.2.62)$$

We may obtain the first series in (9.2.52) by neglecting the surface integral involving $\psi - \varphi_n$ in the above expression. The iteration procedure involves essentially the subsequent substitution of this result for $\psi - \varphi_n$ on the right-hand side of (9.2.62). We see immediately that the poor convergence of $\partial G_k/\partial n_0$ is the source of our trouble. Let us therefore substitute for it

$$\frac{\partial G_k}{\partial n_0} = \frac{\partial G_0}{\partial n_0} + \frac{\partial}{\partial n_0}(G_k - G_0)$$

Here G_0 is the Green's function for the Laplace equation. If this can be expressed in closed form rather than in series form, we may then proceed. Series representation may be employed to represent $[\partial(G_k - G_0)/\partial n_0]$, since the convergence is now adequate. The results of iteration procedure may be expressed as follows. Let

$$\psi - \varphi_n = \Sigma \chi_p$$

where the χ_p satisfy the following recurrence relationship.

$$\chi_{p+1} = -\frac{1}{4\pi}\oint \chi_p \left(\frac{\partial G_0}{\partial n_0}\right) dS_0 - \left(\frac{1}{4\pi}\right)\oint \chi_p \left[\frac{\partial(G_k - G_0)}{\partial n_0}\right] dS_0 \quad (9.2.63)$$

The first term on the right-hand side must not be evaluated in terms of an expansion in φ_p. Direct integration or numerical techniques must be employed. In the second integral of (9.2.63) series representation may be used and the series integrated term by term.

Perturbation of Boundaries for Dirichlet Conditions. We employ the same notation as in the above discussion. The integral equation determining ψ may be obtained directly from Eq. (9.2.45) by placing $\psi = 0$ on the surface:

$$\psi(\mathbf{r}) = \frac{1}{4\pi}\oint G_k(\mathbf{r}|S_0)\left(\frac{\partial \psi}{\partial n_0}\right) dS_0 \quad (9.2.64)$$

We shall not attempt to obtain any more than a wave function correct to first order and our eigenvalue correct to second order because of the convergence troubles which arise. These are more intransigent in character than for Neumann conditions discussed above.

To obtain the first-order correction to ψ we split off, from the right-hand side of (9.2.64), the term proportional to the unperturbed eigenfunction φ_n:

$$\psi(\mathbf{r}) = \frac{\oint \varphi_n(\partial\psi/\partial n)\, dS}{k_n^2 - k^2}\varphi_n + \sum_{p\neq n}\frac{\oint \varphi_p(\partial\psi/\partial n)\, dS}{k_p^2 - k^2}\varphi_p$$

We may simplify the coefficient of φ_n by noting that

$$(k^2 - k_n^2)\int \varphi_n \psi \, dV = \int [\psi \nabla^2 \varphi_n - \varphi_n \nabla^2 \psi] \, dV = -\oint \varphi_n (\partial\psi/\partial n) \, dS \quad (9.2.65)$$

Hence

$$\psi(\mathbf{r}) = -\varphi_n \int \psi \varphi_n \, dV + \sum_{p \neq n} \frac{\oint \varphi_p (\partial\psi/\partial n) \, dS}{k_p^2 - k^2} \varphi_p$$

The first-order correction to ψ is obtained by inserting φ_n in the right-hand side for ψ:

$$\psi(\mathbf{r}) \simeq N_{nn}\varphi_n + \sum_{p \neq n} \frac{A_{np}}{k_p^2 - k^2} \varphi_p \quad (9.2.66)$$

where N_{nn} and A_{np} have been defined in Eqs. (9.2.49) and (9.2.51). The surface integral is of the order of $1/k_p$ for large k_p. The series in (9.2.66) thus converges rapidly enough to evaluate the volume integral $\int \psi \varphi_n \, dV$ in (9.2.65):

$$\int \psi \varphi_n \, dV = N_{nn}^2 + \sum_{p \neq n} \frac{A_{np} N_{np}}{k_p^2 - k^2} \simeq N_{nn}^2 \quad (9.2.67)$$

where we have indicated an approximate value for the integral, correct to first order, for that is all that is required in order to determine $(k^2 - k_n^2)$ to second order.

The series in (9.2.66) cannot be differentiated. We must therefore determine the value of $\partial\psi/\partial n$ in Eq. (9.2.65) in another way. Consider the integral equation determining $\nabla\psi$:

$$\nabla\psi(\mathbf{r}) = \frac{1}{4\pi} \oint \nabla G_k(\mathbf{r}|S_0) \left(\frac{\partial\psi}{\partial n_0}\right) dS_0 \quad (9.2.68)$$

This representation of $\nabla\psi$ vanishes outside S and is therefore discontinuous, since $\nabla\psi$ is not zero on S. We employ a similar representation for $\nabla\varphi_n$ and form $\nabla\psi - \nabla\varphi_n$, pushing the discontinuity into a higher order. From Green's theorem and the equations satisfied by φ_n and G_k we have

$$4\pi\nabla\varphi_n(\mathbf{r}) = (k_n^2 - k^2) \int \nabla G_k(\mathbf{r}|\mathbf{r}_0)\varphi_n(\mathbf{r}_0) \, dV_0 + \oint \left\{ \nabla G_k(\mathbf{r}|\mathbf{r}_0) \left(\frac{\partial\varphi_n}{\partial n_0}\right) - \varphi_n \nabla \left[\frac{\partial G_k(\mathbf{r}|S_0)}{\partial n_0}\right] \right\} dS_0$$

Hence

$$\nabla(\psi - \varphi_n) = \frac{1}{4\pi} \oint \varphi_n \nabla \left(\frac{\partial G_k}{\partial n_0}\right) dS_0 + \left[\frac{(k^2 - k_n^2)}{4\pi}\right] \int \nabla G_k(\mathbf{r}|\mathbf{r}_0)\varphi_n(\mathbf{r}_0) \, dV_0 + \left(\frac{1}{4\pi}\right) \oint \nabla G_k(\mathbf{r}|S_0) \left[\left(\frac{\partial\psi}{\partial n_0}\right) - \left(\frac{\partial\varphi_n}{\partial n_0}\right)\right] dS_0 \quad (9.2.69)$$

A first approximation for $\nabla(\psi - \varphi_n)$ is obtained by placing $\psi = \varphi_n$ on the right-hand side of (9.2.69). We must also make use of the replacement of $\nabla\nabla_0 G_k$ by $\nabla\nabla_0(G_k - G_0)$, a point which was made earlier in this chapter (page 1045). The series for $\nabla\psi$ is then, to the first order,

$$\nabla\psi \simeq \nabla\varphi_n + \sum_{p\neq n} \frac{[(k^2A_{pn}/k_p^2) + (k^2 - k_n^2)N_{pn}]}{(k_p^2 - k^2)} \nabla\varphi_p \qquad (9.2.70)$$

where N_{pn} is expressed in terms of surface integrals by Eq. (9.2.55). The coefficient of $\nabla\varphi_p$ is of the order of $1/k_p^4$ for large k_p and thus converges rapidly enough for substitution in Eq. (9.2.65). We finally obtain for $k^2 - k_n^2$ the following:

$$(k^2 - k_n^2)N_{nn} \simeq -A_{nn} + \sum_{p\neq n} \frac{A_{pn}[k^2A_{pn} + (k^2 - k_n^2)k_p^2N_{pn}]}{k_p^2(k_p^2 - k^2)} \qquad (9.2.71)$$

The series converges. In expressions (9.2.66) and (9.2.70), the wave function and its gradient are given by series which converge in the mean, while (9.2.71) for the eigenvalue converges absolutely.

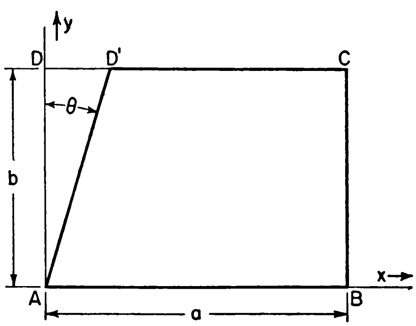

Fig. 9.6 Perturbation of boundary shape by inclination of side AD by angle θ.

A Special Class of Boundary Perturbation. Our discussion has demonstrated that, for a general boundary perturbation, convergence difficulties prevent an explicit expansion of the eigenvalue and eigenfunctions in terms of the unperturbed eigenfunction. There is, however, a specific type of boundary change for which ample convergence is achieved in all orders of the perturbation. To see what these surfaces must be, we return to the estimate of the behavior of A_{pq} for large k_p and k_q as determined from Eq. (9.2.58). We recall that the existence of a contour for which the phase

$$\pm\left(\frac{\pi q_2}{a_2}\right)\xi_2 \pm \left(\frac{\pi q_3}{a_3}\right)\xi_3 + \frac{\pi q_1}{a_1}[c + f(\xi_2,\xi_3)]$$

had a zero gradient led to estimates (9.2.59) to (9.2.61) which formed the basis of the discussion of convergence. Clearly, convergence would be much improved if $f(\xi_2,\xi_3)$ were a *linear function of* ξ_2 *and* ξ_3, for then there would be no point on the surface for which the gradient of the phase would be zero except for very special ratios of q_2/q_1 and q_3/q_1, which cannot influence the over-all convergence. For such a case it may be readily seen from (9.2.58) that $A_{pn} \to 0(1/k_p^2)$ as $k_p \to \infty$ and so on. This asymptotic behavior is enough to ensure convergence of the eigenvalue series for all orders of approximation. An illustration of a surface perturbation belonging to this special class is given in Fig. 9.6, where quadrilateral $ABCD'$ is perturbed from rectangle $ABCD$. In three dimensions, the rectangle becomes a parallelepiped and line AD' is replaced by a plane making a definite angle θ with one of the sides. Corresponding perturbations in other coordinate systems may be easily devised. We shall now carry out the evaluation of the elements A_{nn}, N_{rr} for the above case so as to verify this discussion as well as to illustrate the general theory. The boundary conditions will be taken as homogeneous Neumann.

The unperturbed eigenfunctions are

$$\begin{aligned} \varphi_{pq} &= \sqrt{\epsilon_p\epsilon_q/ab}\,\cos(\pi py/a)\,\cos(\pi qx/b); \quad p \text{ and } q \text{ integers} \\ k_n^2 &= (\pi p/a)^2 + (\pi q/b)^2 \end{aligned}$$

where the double index (pq) replaces the single as it occurs in the formula for A_{ps}. Similarly φ_r equals

$$\varphi_{st} = \sqrt{\epsilon_s\epsilon_t/ab}\,\cos(\pi sy/a)\,\cos(\pi tx/b); \quad s \text{ and } t \text{ integers}$$

The integral for A_{ps} may be obtained, the normal derivation being evaluated according to Eq. (9.2.57). For $p + s$ odd

$$A_{nr} = (\pi/ab)\sqrt{\epsilon_p\epsilon_q\epsilon_s\epsilon_t}\int_0^a \left\{\left[\left(\frac{q}{b}\right)\sin\left(\frac{\pi qy\tan\theta}{b}\right)\cos\left(\frac{\pi py}{a}\right) + \left(\frac{p}{a}\right)\tan\theta\,\sin\left(\frac{\pi py}{a}\right)\cos\left(\frac{\pi qy\tan\theta}{b}\right)\right]\cos\left(\frac{\pi sy}{a}\right)\cos\left(\frac{\pi ty\tan\theta}{b}\right)\right\}dy$$

The integrations are straightforward:

$$\begin{aligned} A_{nr} = \left(\frac{\frac{1}{2}\tan\theta}{ab}\right)\sqrt{\epsilon_p\epsilon_q\epsilon_s\epsilon_t}\,\Bigg\{&\sin^2\left[\frac{\pi a(q+t)\tan\theta}{2b}\right] \\ &\left[\frac{p(p+s) - q(q+t)(a/b)^2}{(p+s)^2 - [(q+t)(a/b)\tan\theta]^2} + \frac{p(p-s) - q(q+t)(a/b)^2}{(p-s)^2 - [(q+t)(a/b)\tan\theta]^2}\right] \\ &+ \sin^2\left[\frac{\pi a(q-t)\tan\theta}{2b}\right]\left[\frac{p(p+s) - q(q-t)(a/b)^2}{(p+s)^2 - [(q-t)(a/b)\tan\theta]^2}\right. \\ &\left.+ \frac{p(p-s) - q(q-t)(a/b)^2}{(p-s)^2 - [(q-t)(a/b)\tan\theta]^2}\right]\Bigg\} \end{aligned}$$

The expression for N_{nn} is

$$N_{nn} = \frac{\epsilon_p\epsilon_q}{2ab}\left\{\frac{ba}{2\epsilon_p\epsilon_q} - \frac{a^2\tan\theta}{4\epsilon_p\epsilon_q} + \frac{b^2\sin^2[\pi q(a/b)\tan\theta]}{4\pi^2q^2\tan\theta}\left[1 - \frac{(q/b)^2\tan^2\theta}{(b/a)^2 - (q/b)^2\tan^2\theta}\right]\right\}$$

We may immediately verify that $A_{nr} \xrightarrow[r\to\infty]{} O(1/k_r)$ whereas, for two-dimensional problems for the general surface, $A_{nr} \to O(1/\sqrt{k_r})$. We also note that, for the lowest mode for the rectangle (with $p = q = 0$), A_{nr}

vanishes; therefore, there is no frequency shift at all orders. This is, of course, correct, since φ = constant will still satisfy Neumann conditions on the perturbed surface.

9.3. *Perturbation Methods for Scattering and Diffraction*

In Secs. 9.1 and 9.2, we have emphasized the application of perturbation theory to the determination of eigenvalues and eigenfunctions. In Sec. 9.3 we consider problems in which the eigenvalues form a continuous spectrum. Since all eigenvalues are allowed, our attention will be focused on the wave function which is to be described over a domain usually extending to infinity (the continuous spectrum is a consequence

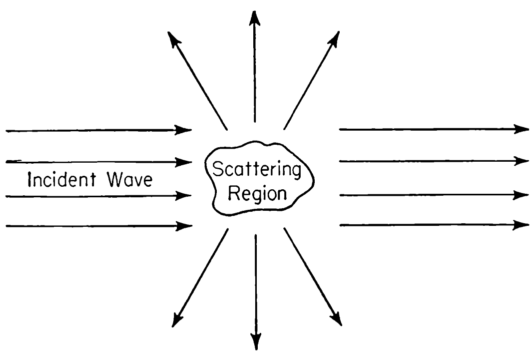

Fig. 9.7 Schematic representation of incident and scattered wave.

of this). It is the relation between the behavior of the wave function near infinity and the properties of the medium in which the wave propagates which is of greatest interest. The behavior at infinity is usually observed experimentally; the properties of the medium must often be deduced from the measurements by calculating the consequences of various assumed properties and comparing the theoretical predictions with experimental results (see further discussion in Secs. 11.3 and 12.3).

The actual experimental situation involved may be described as follows: Radiation from a source, usually placed at infinity, is directed toward a region, called the *scattering region*, whose properties differ from the surrounding medium. The *incident wave* may, for example, be a sound wave; the medium, air; and the inhomogeneity in the medium might be some solid object. Or the incident wave might represent a stream of electrons of a definite momentum, and the scattering region might be the electrostatic field associated with an atom. In either case it is clear that the scattering region will deflect some of the incident rays to form a *scattered wave*. At large distances from the scattering regions we observe both scattered and incident waves. The approximate prediction of the relative amounts of each is the problem to be considered in this section. Exact solutions of the scattering problem will be discussed in Chap. 11. We shall limit our inquiry to the Schroedinger equation and to the scalar Helmholtz equation; the procedures may be applied to other equations with little difficulty, in most cases.

Boundary Conditions for Scattering. The total wave function ψ may be decomposed into an incident wave ψ_i and a scattered wave ψ_s:

$$\psi = \psi_i + \psi_s \tag{9.3.1}$$

The part of the boundary condition which is common to all scattering problems is that, at large distances from the scattering region, ψ_s approaches a wave *diverging* from a point source located in the scattering region. In *three dimensions* we may take ψ_i as the incident plane wave

$$\psi_i = e^{ikz}$$

where ψ_i satisfies $\nabla^2\psi_i + k^2\psi_i = 0$. The boundary condition at infinity then requires that

$$\psi_s \to f(\vartheta,\varphi)(e^{ikr}/r); \quad r \to \infty \tag{9.3.2}$$

where r, ϑ, and φ form the spherical coordinates with an origin located in the scattering region. The function f is called the *scattering amplitude* or *angle-distribution factor*.

In two dimensions, an incident plane wave ψ_i is customarily written as

$$\psi_i = e^{ikx}$$

The scattered wave satisfies the condition

$$\psi_s \to f(\varphi)(e^{ikr}/\sqrt{r}); \quad r \to \infty \tag{9.3.3}$$

Here r and φ form polar coordinates, the origin being in the scattering region, the angle φ being taken with respect to the x axis.

Fig. 9.8 Schematic representation of one-dimensional transmission-reflection problem.

One-dimensional problems use a somewhat different nomenclature, since the scattering is now confined along the incident direction. We again take the incident wave to be

$$\psi_i = e^{ikx}$$

The boundary conditions at infinity are now stated in two sections. For large positive values of x, ψ approaches a transmitted plane wave:

$$\psi \to Ae^{ikx}; \quad x \to \infty \tag{9.3.4}$$

For large negative values of x, we have an incident plus a reflected wave:

$$\psi \to e^{ikx} + Be^{-ikx}; \quad x \to -\infty \tag{9.3.5}$$

Scattering Cross Section. The quantities measured at infinity are directly related to the function f or the coefficients A and B. For example, in acoustics the quantity $|f(\vartheta,\varphi)|^2\,d\Omega$ is the power scattered into a solid angle $d\Omega$ at (ϑ,φ) per unit incident intensity. On the other hand, if ψ is the Schroedinger wave function, $|f(\vartheta,\varphi)|^2\,d\Omega$ gives the current scattered into a solid angle $d\Omega$ per unit incident current density. Thus $|f(\vartheta,\varphi)|^2$ gives the angular distribution of the scattered radiation or particles. It is clear that the factor $|f(\vartheta,\varphi)|^2$ has the dimensions of an area and is often referred to as the *differential scattering cross section* σ:

$$\sigma(\vartheta,\varphi) = |f(\vartheta,\varphi)|^2; \quad \text{three dimensions} \tag{9.3.6}$$

In two dimensions $|f(\varphi)|^2\,d\varphi$ gives the power scattered into the angular range $d\varphi$ per unit incident flux length. It clearly has the dimensions of a length which is also customarily denoted by σ:

$$\sigma(\varphi) = |f(\varphi)|^2; \quad \text{two dimensions} \tag{9.3.7}$$

The total scattered power per unit incident intensity or the total scattered current per unit incident current density is obtained by integrating $\sigma(\vartheta,\varphi)$ over all solid angles. The result referred to as the *total scattering cross section* is denoted by the letter Q:

$$Q = \int \sigma(\vartheta,\varphi)\,d\Omega; \quad \text{three dimensions} \tag{9.3.8}$$

Similarly

$$Q = \int_0^{2\pi} \sigma(\varphi)\,d\varphi; \quad \text{two dimensions} \tag{9.3.9}$$

The total cross section has a simple physical interpretation. It is an area (in two dimensions a line segment) normal to the incident beam which intercepts an amount of incident power equal to the scattered power.

The physical meaning of the coefficients A and B in Eqs. (9.3.4) and (9.3.5) for the one-dimensional problem may be also determined. The reflected intensity per unit incident intensity is called the *reflection coefficient* R and is related to A as follows:

$$R = |A|^2 \tag{9.3.10}$$

The *transmission coefficient* T is the intensity of the transmitted wave:

$$T = |B|^2 \tag{9.3.11}$$

If there is no absorption, $|A|^2 + |B|^2$ must equal unity.

Scattering from a Spherically Symmetric Region—Phase Shifts. If in the Schroedinger equation the potential is a function of r only, or if in acoustics the scattering region is spherical in shape and isotropic, the three-dimensional equations may be reduced to a set of independent one-dimensional equations in r. They will differ from the one-dimensional problem discussed above, since $0 \leq r < \infty$ while $-\infty < x < \infty$. Consider then the Schroedinger equation

$$[\nabla^2 + (k^2 - \lambda U(r))]\psi = 0 \tag{9.3.12}$$

where $k^2 = (2m/h^2)E$ and $\lambda U(r) = (2m/h^2)\lambda V(r)$

E equals the energy, λV the potential energy of the particle of mass m. The method of separation of variables employing spherical coordinates is appropriate. We shall utilize the following elementary solutions:

$$P_l(\cos\vartheta)[u_l(kr)/kr]$$

where P_l are the Legendre polynomials and u_l satisfies.

$$[d^2u_l/dr^2] + \{k^2 - [l(l+1)/r^2] - \lambda U(r)\}u_l = 0 \tag{9.3.13}$$

This differential equation has two solutions. We pick the one which is zero at the origin, since ψ does not have a singularity there.

$$u_l(0) = 0 \tag{9.3.14}$$

The amplitude of u_l must now be chosen so that ψ satisfies boundary condition (9.3.2).

For this calculation we shall need the expansion of a plane wave in spherical coordinates. The elementary solutions are obtained by placing $U = 0$ in (9.3.13) and choosing that solution which is bounded at the origin. The elementary solutions are

$$P_l(\cos\vartheta)j_l(kr)$$

where [see Eq. (11.3.42)]

$$j_l(kr) = \sqrt{\pi/2kr}\, J_{l+\frac{1}{2}}(kr)$$

It may then be shown [see Eq. (11.3.45)] that

$$e^{ikz} = \sum_{l=0}^{\infty} (2l+1)i^lP_l(\cos\vartheta)j_l(kr) \tag{9.3.15}$$

This form for the plane wave suggests that ψ be written

$$\psi = \Sigma(2l+1)i^lP_l(\cos\vartheta)[u_l/kr] \tag{9.3.16}$$

and u_l be adjusted so that boundary condition (9.3.2) is satisfied. Therefore consider that

$$\psi - e^{ikz} = \psi_s = \Sigma(2l+1)i^lP_l(\cos\vartheta)[(u_l/kr) - j_l]$$

We must adjust u_l so that

$$[(u_l/kr) - j_l] \to A_l(e^{ikr}/r); \quad r \to \infty$$

Therefore u_l and j_l must be compared. The asymptotic $(r \to \infty)$ behavior of j_l is

$$j_l(kr) \to \frac{\cos[kr - \frac{1}{2}\pi(l+1)]}{kr}; \quad r \to \infty$$

If $U(r)$ does not extend out to infinity, u_l asymptotically satisfies the same equation as j_l. It may therefore be written as a linear combination of the two solutions of the free $(U = 0)$ wave equation. The asymptotic form of u_l is then

$$u_l \to e^{-i\eta_l} \cos[kr - \tfrac{1}{2}\pi(l+1) - \eta_l] \tag{9.3.17}$$

where η_l is the *phase shift*. The amplitude factor has been chosen so that

$$\left[\left(\frac{u_l}{kr}\right) - j_l\right] \to (e^{-2i\eta_l} - 1)\,\frac{e^{i[kr - \frac{1}{2}\pi(l+1)]}}{2kr}$$

Hence

$$\psi_s \to \left(\frac{1}{2ik}\right) \sum (2l+1) P_l(\cos\vartheta)[e^{-2i\eta_l} - 1]\left(\frac{e^{ikr}}{r}\right)$$

The function $f(\vartheta,\varphi)$ is the coefficient of e^{ikr}/r so that

$$\sigma(\vartheta,\varphi) = (1/4k^2)|\Sigma(2l+1)(e^{-2i\eta_l} - 1)P_l(\cos\vartheta)|^2 \tag{9.3.18}$$

The total cross section (assuming η_l to be real) is obtained by integration;

$$Q = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} \sin^2 \eta_l \tag{9.3.19}$$

The physical meaning of these results will be considered in detail in Chaps. 11 and 12. Here we are interested mainly in the mathematical problem which remains. To determine σ and Q we need the phase shift η_l. This is to be obtained by solving differential equation (9.3.13) for u_l, choosing the solution which has zero value at the origin and finally examining its asymptotic dependence, from which η_l may be determined according to Eq. (9.3.17) above.

It is instructive to compare the problem determining u_l and the standard one-dimensional problem. We note that

$$u_l \to \text{constant}[e^{-ikr} + (-1)^{l+1} e^{-2i\eta_l} e^{ikr}]; \quad r \to \infty$$

Recalling that $u_l(0) = 0$, we see that we may interpret $|(-1)^{l+1}\exp(-2i\eta_l)|^2$ as the reflection coefficient for a wave $\exp(-ikr)$ incident from the positive r side on a scattering region which ends at $r = 0$ with a perfect reflector.

Integral Equation for Scattering. As in the case of the discrete spectrum, the integral equation for the unknown function will be a convenient starting point for the development of the various approximations.

We shall first consider the full three-dimensional case, starting from the Schroedinger equation, and later shall turn to the integral equation for u_l. We rewrite (9.3.12) as an inhomogeneous equation:

$$\nabla_0^2\psi + k^2\psi = \lambda U(\mathbf{r}_0)\psi$$

where, as indicated, we are not necessarily dealing with central forces. Then, employing the free-space Green's function for the Helmholtz equation, $\exp(ikR)/R$, where $R = |\mathbf{r} - \mathbf{r}_0|$, we have

$$\psi(\mathbf{r}) = \psi_i(\mathbf{r}) - \frac{\lambda}{4\pi}\int \frac{e^{ikR}}{R} U(\mathbf{r}_0)\psi(\mathbf{r}_0)\, dV_0 \tag{9.3.20}$$

where we have added ψ_i, a solution of the homogeneous Helmholtz equation, to the particular solution given by the integral. It may now be shown that form (9.3.20) satisfies the boundary conditions at infinity. In that limit

$$R \to r - r_0 \cos\theta; \quad r \to \infty$$

where θ is the angle between the vectors $\mathbf{r}$ and $\mathbf{r}_0$. Since $\mathbf{r}$ is the vector to the point of observation, it is convenient to define the vector $\mathbf{k}_s$ in the direction of $\mathbf{r}$ but having the magnitude of k so that

$$\mathbf{k}_s \cdot \mathbf{r}_0 = kr_0 \cos\theta \tag{9.3.21}$$

Therefore, in the limit $r \to \infty$

$$\psi(\mathbf{r}) \to e^{i\mathbf{k}_i\cdot\mathbf{r}} - \frac{1}{4\pi}\frac{e^{ikr}}{r}\int e^{-i\mathbf{k}_s\cdot\mathbf{r}_0}U(\mathbf{r}_0)\psi(\mathbf{r}_0)\, dV_0$$

The scattered amplitude $f(\vartheta,\varphi)$ is the coefficient of $\exp(ikr)/r$;

$$f(\vartheta,\varphi) = -\frac{1}{4\pi}\int e^{-i\mathbf{k}_s\cdot\mathbf{r}_0}U(\mathbf{r}_0)\psi(\mathbf{r}_0)\, dV_0 \tag{9.3.22}$$

We thus have verified that the wave function satisfying Eq. (9.3.20) automatically fulfills the boundary conditions at infinity and incidentally have obtained a value for the scattering amplitude f in terms of ψ. Equation (9.3.20) is thus the integral equation for scattering. Before discussing the approximate solution of this integral equation, it is useful to discuss an important relation between the total cross section Q and f. We shall now show that for conservative systems (we could also prove it for the case when particles or energy is absorbed)

$$Q = (4\pi/k)\,\mathrm{Im}\, f(0) \tag{9.3.23}$$

where by $f(0)$ we mean the value of the amplitude $f(\vartheta,\varphi)$ for $\vartheta = 0$ or, referring to the definition of $\mathbf{k}_s$, for $\mathbf{k}_s$ in (9.3.22) equal to $\mathbf{k}_i$. In other words, the total cross section is related to the scattered amplitude evaluated in the direction of the incident wave.

To prove Eq. (9.3.23) we note that from Green's theorem and the equations satisfied by ψ and ψ_i it is easy to establish the relation

$$\lim_{r\to\infty} \int [\bar{\psi}_i(\partial\psi/\partial n) - \psi(\partial\bar{\psi}_i/\partial n)]\, dS = \int \bar{\psi}_i U \psi\, dV$$

The right-hand side of this equation may be rewritten by employing the expression for $f(0)$ given by Eq. (9.3.22). In addition let us take the surface to be a sphere of radius r. Then

$$\lim_{r\to\infty} \int [\bar{\psi}_i(\partial\psi/\partial r) - \psi(\partial\bar{\psi}_i/\partial r)]\, dS = -4\pi f(0) \tag{9.3.24}$$

It is possible to express the left-hand side of (9.3.24) in terms of Q by use of the conservation theorem:

$$\begin{aligned} \lim_{r\to\infty} \int [\bar{\psi}(\partial\psi/\partial r) - \psi(\partial\bar{\psi}/\partial r)]\, dS &= 0 \\ \lim_{r\to\infty} \int [\bar{\psi}_i(\partial\psi_i/\partial r) - \psi_i(\partial\bar{\psi}_i/\partial r)]\, dS &= 0 \end{aligned} \tag{9.3.25}$$

These simply say that, for either the total or just the incident wave alone, radiation (or in the Schroedinger case the number of particles) is conserved. If now in the first of these two equations we replace ψ by $\psi_i + \psi_s$ and employ the second, we obtain

$$\lim_{r\to\infty} \{2i \operatorname{Im} \int [\bar{\psi}_i(\partial\psi_s/\partial r) - \psi_s(\partial\bar{\psi}_i/\partial r)]\, dS\} + \lim_{r\to\infty} \int [\bar{\psi}_s(\partial\psi_s/\partial r) - \psi_s(\partial\bar{\psi}_s/\partial r)]\, dS = 0$$

Introducing (9.3.2) the second of these surface integrals just equals

$$2ik \int |f(\vartheta,\varphi)|^2\, d\Omega = 2ikQ$$

The first surface integral and the surface integral in (9.3.24) are nearly equal, as may be shown by inserting $\psi = \psi_i + \psi_s$ in (9.3.24) and employing the conservation condition for the plane wave equation (9.3.25). Theorem (9.3.23) immediately follows. The reader can verify Eq. (9.3.23) by comparing Eq. (9.3.19) and the expression $f(\vartheta,\varphi)$ given from the equations immediately above Eq. (9.13.18).

In this last discussion we have emphasized the scattering which occurs because of a volume perturbation. Similar results may be obtained when the scattering results from the introduction of a reflecting surface, *i.e.*, a surface perturbation. As an example suppose that ψ is to satisfy Dirichlet conditions on surface S. Then the integral equation for scattering becomes

$$\psi(\mathbf{r}) = \psi_i(\mathbf{r}) + \frac{1}{4\pi} \int \frac{e^{ikR}}{R} \left(\frac{\partial\psi}{\partial n_0}\right) dS_0 \tag{9.3.26}$$

By taking the limits $r \to \infty$ we find that the boundary conditions at infinity are automatically satisfied. The scattering amplitude is

$$f(\vartheta,\varphi) = (1/4\pi) \int e^{-i\mathbf{k}_s\cdot\mathbf{r}_0}(\partial\psi/\partial n_0)\, dS_0 \tag{9.3.27}$$

We may also show that (9.3.23) still holds and shall leave the proof for the reader.

Integral Equation for One-dimensional Problems. Consider the equation

$$(d^2\psi/dx^2) + [k^2 - \lambda U(x)]\psi = 0 \tag{9.3.28}$$

The boundary conditions are given by Eqs. (9.3.4) and (9.3.5). We employ the Green's function

$$G(x|x_0) = -(1/2ik)e^{ik|x-x_0|} \tag{9.3.29}$$

satisfying the equation

$$(d^2G/dx^2) + k^2G = -\delta(x - x_0)$$

In terms of G, ψ may be rewritten as an integral equation:

$$\psi(x) = e^{ikx} - \lambda \int_{-\infty}^{\infty} G(x|x_0)U(x_0)\psi(x_0)\,dx_0 \tag{9.3.30}$$

It is necessary again to verify that the boundary conditions at $x = \pm\infty$ are satisfied. For this purpose we write out Eq. (9.3.30) in detail:

$$\psi(x) = e^{ikx} + (\lambda/2ik)\int_{-\infty}^{x} e^{ik(x-x_0)}\,U(x_0)\psi(x_0)\,dx_0 \\ + (\lambda/2ik)\int_{x}^{\infty} e^{ik(x_0-x)}\,U(x_0)\psi(x_0)\,dx_0$$

In the limit $|x| \to \infty$, it follows that

$$\psi(x) \to e^{ikx}\left[1 + (\lambda/2ik)\int_{-\infty}^{\infty} e^{-ikx_0}\,U(x_0)\psi(x_0)\,dx_0\right];\quad x \to \infty$$

$$\psi(x) \to e^{ikx} + (\lambda/2ik)e^{-ikx}\int_{-\infty}^{\infty} e^{ikx_0}\,U(x_0)\psi(x_0)\,dx_0;\quad x \to -\infty$$

Boundary conditions (9.3.4) and (9.3.5) are satisfied. We may immediately obtain expressions for the reflection coefficient:

$$R = \frac{\lambda^2}{4k^2}\left|\int_{-\infty}^{\infty} e^{ikx_0}\,U(x_0)\psi(x_0)\,dx_0\right|^2 \tag{9.3.31}$$

The transmission coefficient is

$$T = \left|1 + \left(\frac{\lambda}{2ik}\right)\int_{-\infty}^{\infty} e^{-ikx_0}\,U(x_0)\psi(x_0)\,dx_0\right|^2 \tag{9.3.32}$$

We note in passing that the conservation condition

$$R + T = 1$$

follows from the equation below, which may be obtained directly from the differential equations satisfied by ψ:

$$\lim_{x\to\infty}[\bar{\psi}(d\psi/dx) - \psi(d\bar{\psi}/dx)] = \lim_{x\to-\infty}[\bar{\psi}(d\psi/dx) - \psi(d\bar{\psi}/dx)]$$

Integral Equation for Three Dimensions. We turn next to the integral equation satisfied by the function u_l defined by Eq. (9.3.13) and associated boundary conditions (9.3.14) and (9.3.17). This will differ even for the $l = 0$ case from (9.3.30) because of the different boundary conditions involved. We naturally employ a Green's function satisfying the equation

$$\frac{d^2G(r|r_0)}{dr^2} + \left[k^2 - \frac{l(l+1)}{r^2}\right]G(r|r_0) = -\delta(r - r_0)$$

To satisfy boundary conditions the proper choice for G is

$$G(r|r_0) = -krr_0 \begin{cases} j_l(kr)\, n_l(kr_0); & r \le r_0 \\ j_l(kr_0)\, n_l(kr); & r \ge r_0 \end{cases} \tag{9.3.33}$$

The functions j_l have been defined earlier (page 622), while n_l is the spherical Neumann function

$$n_l = \sqrt{\pi/2kr}\, N_{l+\frac{1}{2}}(kr)$$

defined in Eq. (11.3.42). The integral for u_l is then

$$u_l(r) = kr\, j_l(kr) - \lambda \int_0^\infty G(r|r_0)U(r_0)u_l(r_0)\, dr_0 \tag{9.3.34}$$

More explicitly

$$u_l(r) = kr\, j_l(kr) + \lambda kr\, n_l(kr) \int_0^r r_0 j_l(kr_0)U(r_0)u_l(r_0)\, dr_0 + \lambda kr\, j_l(kr) \int_r^\infty r_0 n_l(kr_0)U(r_0)u_l(r_0)\, dr_0$$

Recalling that krj_l approaches zero as r approaches zero, we see that the boundary condition $u_l(0) = 0$ is automatically satisfied. For large values of r

$$kr\, j_l(kr) \to \cos[kr - \tfrac{1}{2}(l+1)\pi]; \quad kr\, n_l(kr) \to \sin[kr - \tfrac{1}{2}(l+1)\pi]$$

Therefore

$$u_l \to \cos[kr - \tfrac{1}{2}(l+1)\pi] + \lambda \sin[kr - \tfrac{1}{2}(l+1)\pi] \int_0^\infty r_0 j_l(kr_0)U(r_0)u_l(r_0)\, dr_0; \quad r \to \infty \tag{9.3.35}$$

This agrees with the requirement (9.3.17) that u_l for large r be proportional to $\cos[kr - \frac{1}{2}(l+1)\pi - \eta_l]$. We may now express η_l in terms of the integral in Eq. (9.3.35):

$$\tan \eta_l = \lambda \int_0^\infty r_0 j_l(kr_0)U(r_0)u_l(r_0)\, dr_0 \tag{9.3.36}$$

The function u_l appearing here must be normalized to have the asymptotic behavior given by Eq. (9.3.35). Thus, all boundary conditions are properly satisfied by the solution of (9.3.34), and it is therefore the correct integral equation for u_l.

Born Approximation. The perturbation solutions of the integral equations for scattering follow the procedure developed in Sec. 9.1 for eigenvalues forming a discrete spectrum. The first of those discussed there was the iteration-perturbation method. This involves substituting the unperturbed function for the unknown function under the integral sign, to obtain the first approximation. The second approximation is obtained by substituting this first approximation for the unknown function, and so on. The first approximation for the scattered amplitude may be obtained by substituting the unperturbed wave function in the various integral expressions, Eqs. (9.3.22), (9.3.27), (9.3.31), and (9.3.36). For example, in Eq. (9.3.22) the appropriate unperturbed wave function is the incident wave $\exp(i\mathbf{k}_i \cdot \mathbf{r})$. Substituting yields the Born approximation

$$f_B(\vartheta,\varphi) = -(\lambda/4\pi)\int e^{i(\mathbf{k}_i-\mathbf{k}_s)\cdot\mathbf{r}}U(\mathbf{r})\,dV \tag{9.3.37}$$

The second approximation to f is obtained by substituting in the first approximation to ψ in Eq. (9.3.22), etc. The recursion formulas are then

$$\begin{aligned}\psi^{(n)}(\mathbf{r}) &= \psi_i(\mathbf{r}) - \frac{\lambda}{4\pi}\int\left(\frac{e^{ikR}}{R}\right)U(\mathbf{r}_0)\psi^{(n-1)}(\mathbf{r}_0)\,dV_0 \qquad (9.3.38)\\ f^{(n)}(\vartheta,\varphi) &= -\frac{\lambda}{4\pi}\int e^{-i\mathbf{k}_s\cdot\mathbf{r}}U(\mathbf{r})\psi^{(n-1)}(\mathbf{r})\,dV\end{aligned}$$

Similar results can be obtained for the other scattering integral equations discussed above and will be given in Chap. 12 for the functions u_l and their corresponding phase shifts. When applied to the integral equation arising from a surface perturbation [*e.g.*, Eq. (9.3.26)], the first approximation involving the substitution of the incident wave for the unknown ψ is often referred to as the *Kirchhoff approximation*. From Eq. (9.3.27) the corresponding scattering amplitude is

$$f_k(\vartheta,\varphi) = (i/4\pi)\int e^{i(\mathbf{k}_i-\mathbf{k}_s)\cdot\mathbf{r}_0}(\mathbf{n}_0\cdot\mathbf{k}_i)\,dS_0 \tag{9.3.39}$$

The total cross section can be evaluated from Eq. (9.3.23). However, it may be shown that, if $\psi^{(n)}$ is employed in that equation, the resulting Q is no better then the $(n-1)$th approximation;

$$Q \to \int|f^{(n-1)}|^2\,d\Omega$$

Therefore there is little point in employing (9.3.23) for this kind of perturbation scheme, though we shall find it very useful elsewhere. Just to illustrate,

$$f_B(0) = (-\lambda/4\pi)\int U(\mathbf{r})\,dV$$

The scattering amplitude is real, and therefore Eq. (9.3.23) yields zero for the cross section Q if function U is integrable.

We now turn to the question of the convergence of the sequence developed by Eq. (9.3.38) and the similar sequences developed by the iteration-perturbation method for the other scattering integral equations. As in the discrete eigenvalue distribution discussed in Sec. 9.1, the radius of convergence of the iteration-perturbation sequence is determined by the value of λ for which the homogeneous form of the scattering integral equation has a nonzero solution. For example, from Eq. (9.3.21), the iteration-perturbation sequence will fail to converge for λ's whose magnitudes exceed that value of λ, λ_0, for which the following equation has a solution:

$$\psi(\mathbf{r}) = -(\lambda_0/4\pi)\int (e^{ikR}/R)U(\mathbf{r}_0)\psi(\mathbf{r}_0)\,dV_0 \qquad (9.3.40)$$

We require, thus, solutions which for large values of r diverge from the scattering region, these to occur without the presence of the incident wave. It is clear physically that, for this to happen, the scattering region must contain sources or, in other words, that λ *must be complex*. Mathematically we see that in this case the kernel is not definite, so that the conclusion is not unexpected.

These special solutions of the homogeneous equation may be given a physical significance by noticing that, for $\lambda = \lambda_0$, the scattering amplitude is infinite. We may therefore expect that, as λ is varied along the real axis, the cross section will show a strong maximum or *resonance* when λ is close to the real part of λ_0.

Since λ_0 is a function of k, we may expect that, for *fixed* λ, resonances in the scattering will occur at those values of k for which Re λ_0 is near λ. One can get at this point more directly by asking for what complex values k_r of k, for λ fixed, does Eq. (9.3.40) have a nonzero solution. Again, at these values of k_r, the cross section is infinite. Hence we may expect resonances in the cross section when k is close to the real part of k_r. We note that the solutions for $k = k_r$ are very similar to the bound states of the system in their r dependence for large r. In the latter case,

$$\psi\ (\text{bound}) \to e^{-\kappa r}/r$$

while for the scattering case,

$$\psi\ (\text{free}) \to e^{ikr}/r$$

Hence these states of the system are logical extensions of the notion of bound states to the continuum. They are often called *virtual levels* and will be discussed in more detail in Chap. 12. For the present we remark that the radius of convergence of the iteration-perturbation series, as a function of the energy, is determined by the energy at which resonance occurs.

As an example of this discussion, we consider the one-dimensional Schroedinger equation for the potential energy illustrated in Fig. 9.9.

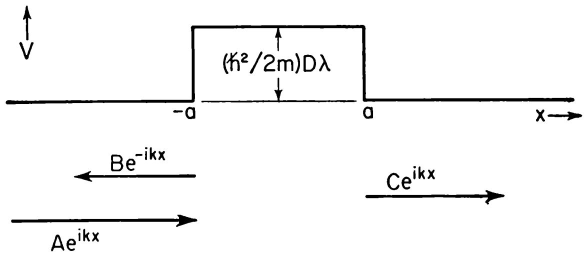

Fig. 9.9 Reflection and resonance for a one-dimensional scattering problem.

The waves present for $x < -a$ and $x > a$ are shown in the figure. For $-a < x < a$, we take $E \exp(i\kappa x) + F \exp(-i\kappa x)$, where

$$\kappa = \sqrt{k^2 - \lambda D}$$

Then, joining slope and value at $x = a$, we obtain

$$Ce^{ika} = Ee^{i\kappa a} + Fe^{-i\kappa a}; \quad kCe^{ika} = \kappa Ee^{i\kappa a} - \kappa Fe^{-i\kappa a}$$

The relation between C and E and F may now be obtained:

$$E = \frac{kC}{2\kappa}\left(\frac{\kappa}{k} + 1\right)e^{i(k-\kappa)a}; \quad F = \frac{kC}{2\kappa}\left(\frac{\kappa}{k} - 1\right)e^{i(k+\kappa)a}$$

We may again join slope and value at $x = -a$:

$$\begin{aligned} A &= -Be^{2ika} + Ee^{i(k-\kappa)a} + Fe^{i(k+\kappa)a} \\ A &= Be^{2ika} + [Ee^{i(k-\kappa)a} - Fe^{i(k+\kappa)a}](\kappa/k) \end{aligned}$$

Substituting for E and F in terms of C, we may solve directly for C:

$$C = A\,\frac{4k\kappa \exp(-2ika)}{(\kappa + k)^2 \exp(-2i\kappa a) - (\kappa - k)^2 \exp(2i\kappa a)} \tag{9.3.41}$$

Solutions for vanishing A (the condition for a virtual level) will exist only if the denominator of Eq. (9.3.41) is zero. Turning it about, C is infinite for a k (or λ) for which the denominator above vanishes as long as A is finite. It is, moreover, clear that any expansion of this expression in powers of λ or in powers of k will fail as soon as λ (or k^2) approaches the critical values for which the transmission amplitude C becomes infinite. We conclude the discussion by showing that critical values do exist. The determining equation is

$$i\tan(2\kappa a) = [2k\kappa/(k^2 + \kappa^2)]$$

If, for example, $\kappa a \ll 1$, that is, the energy of the incident particle is almost equal to the barrier potential energy, then this equation becomes

$$a(2k^2 - \lambda D) = -ik$$

from which an approximate critical value of either λ or k may be determined. The solutions are complex quantities.

Higher Born Approximations. Assuming that we are working in a range for which the iteration-perturbation series converges, the difficulty of evaluating the integrals involved still limits the usefulness of the series. In one-dimensional problems, it is often possible to carry out the first iteration analytically and so obtain the second Born approximation for η_l. For third and higher approximations, it is necessary to proceed numerically, which is possible here because of the one-dimensional nature of the integrals. However, numerical techniques are much more difficult and tedious to apply when two- or three-dimensional problems are being attacked. In such circumstances it is often useful to appeal to eigenfunction expansions of the Green's functions. In the present case, appropriate eigenfunctions are the plane waves

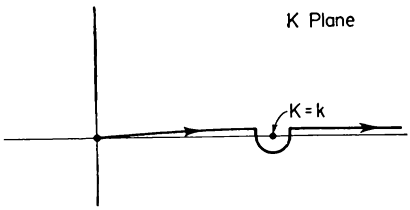

Fig. 9.10 Choice of contour for Fourier transform for Green's function.

$$[1/(2\pi)^{\frac{3}{2}}]\exp[i\mathbf{k}\cdot\mathbf{r}]$$

normalized so that

$$\frac{1}{(2\pi)^3}\int e^{i(\mathbf{k}-\mathbf{k}_0)\cdot\mathbf{r}}\,dV = \delta(\mathbf{k}-\mathbf{k}_0)$$

The expansion of the Green's function in terms of these eigenfunctions may be performed, following the general rules set up in Chap. 7. We obtain

$$\frac{e^{ikR}}{4\pi R} = \left(\frac{1}{2\pi}\right)^3 \int \frac{e^{i\mathbf{K}\cdot\mathbf{R}}}{K^2-k^2}\,dV_K \tag{9.3.42}$$

In order to understand this formula, and therefore the discussion following, it is important that we verify the above equality directly. Introducing spherical coordinates in K space, the integral may be quickly reduced to one over K:

$$\frac{e^{ikR}}{4\pi R} = \left(\frac{1}{2\pi^2 R}\right)\int_0^\infty \frac{K\sin(KR)}{K^2-k^2}\,dK$$

To continue further, it is necessary to choose the path of integration in the complex plane of K. Our choice, given in Fig. 9.10, is dictated by the requirement that we obtain a diverging wave. Since the integral is even in K, we may rewrite the above equation as

$$\frac{e^{ikR}}{4\pi R} = \left(\frac{1}{4\pi^2 R}\right)\int_{-\infty}^\infty \frac{K\sin(KR)}{K^2-k^2}\,dK$$

where the path of integration is given in Fig. 9.11. The integral may now be readily evaluated with the aid of the Cauchy integral formula [Eq. (4.2.9)], and thus Eq. (9.3.42) is verified.

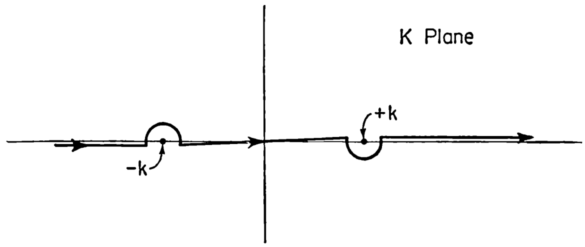

Fig. 9.11 Contour for extended integral for Green's function.

Substituting in integral equation (9.3.20) gives

$$\psi(\mathbf{r}) = \psi_i(\mathbf{r}) - \frac{\lambda}{(2\pi)^3} \iint \frac{e^{i\mathbf{K}\cdot\mathbf{R}}}{K^2 - k^2} U(\mathbf{r}_0)\psi(\mathbf{r}_0)\, dV_0\, dV_K \tag{9.3.43}$$

We now introduce the definitions

$$T(\mathbf{K}|\mathbf{k}_i) = \lambda \int e^{-i\mathbf{K}\cdot\mathbf{r}} U(\mathbf{r})\psi(\mathbf{r})\, dV_r \tag{9.3.44}$$

$$U(\mathbf{K}|\mathbf{k}) = \lambda \int e^{i(\mathbf{k}-\mathbf{K})\cdot\mathbf{r}} U(\mathbf{r})\, dV_r \tag{9.3.45}$$

It is clear from Eq. (9.3.22) that

$$f(\vartheta,\varphi) = -(1/4\pi) T(\mathbf{k}_s|\mathbf{k}_i) \tag{9.3.46}$$

while the first Born approximation, Eq. (9.3.37), is

$$f_B(\vartheta,\varphi) = -(1/4\pi) U(\mathbf{k}_s|\mathbf{k}_i) \tag{9.3.47}$$

We may now obtain the integral equation determining T by multiplying both sides of Eq. (9.3.43) by $\lambda U(\mathbf{r}) \exp(-i\mathbf{p}\cdot\mathbf{r})$ and integrating. Then

$$T(\mathbf{p}|\mathbf{k}_i) = U(\mathbf{p}|\mathbf{k}_i) - \left(\frac{1}{2\pi}\right)^3 \int \frac{U(\mathbf{p}|\mathbf{K}) T(\mathbf{K}|\mathbf{k}_i)}{K^2 - k^2}\, dV_K \tag{9.3.48}$$

This integral equation is, of course, equivalent to the original integral equation (9.3.20). It has the advantage over the former in that it deals directly with the scattering amplitude.

The application of the iterative-perturbation technique yields the first and higher Born approximations immediately. We use a superscript to indicate the order of the approximation.

$$T^{(1)}(\mathbf{k}_s|\mathbf{k}_i) = U(\mathbf{k}_s|\mathbf{k}_i)$$

$$T^{(2)}(\mathbf{k}_s|\mathbf{k}_i) = U(\mathbf{k}_s|\mathbf{k}_i) - \frac{1}{(2\pi)^3} \int \frac{U(\mathbf{k}_s|\mathbf{K}) U(\mathbf{K}|\mathbf{k}_i)}{K^2 - k^2}\, dV_K \tag{9.3.49}$$

$$T^{(3)}(\mathbf{k}_s|\mathbf{k}_i) = T^{(2)}(\mathbf{k}_s|\mathbf{k}_i) + \frac{1}{(2\pi)^6} \iint \frac{U(\mathbf{k}_s|\mathbf{K}_1) U(\mathbf{K}_1|\mathbf{K}_2) U(\mathbf{K}_2|\mathbf{k}_i)}{(K_1^2 - k^2)(K_2^2 - k^2)}\, dV_{K_1}\, dV_{K_2} \tag{9.3.50}$$

and so on. An example of the utility of these formulas will be given later in this section.

To obtain the corresponding integral equations for one- and two-dimensional problems, it is necessary to replace the factor $(2\pi)^{-3}$ by $(2\pi)^{-n}$, where n is the number of dimensions and, of course, the volume integrals in both $\mathbf{K}$ and $\mathbf{r}$ space by their lower dimensional forms.

Fredholm Series. Improvements on the iteration-perturbation results given above as the first and higher Born approximations may be obtained by adapting the Fredholm as well as the variation-perturbation methods discussed in Sec. 9.1 to the scattering problem. We shall not treat the Feenberg but only the Fredholm series; the variation-perturbation method will wait until Sec. 9.4, in which the variational method will be applied to scattering problems.

The Fredholm series, as given in Eqs. (9.1.60) and (9.1.65), may be directly applied to the scattering integral equations, since the Fredholm series is valid for any equation of the second kind. For one-dimensional problems, the solution of the equation

$$\mathbf{e} = \mathbf{f}_0 + \lambda\mathfrak{K}\mathbf{e}$$

is given by (and we give only the first few terms)

$$\mathbf{e} = \left\{1 + \lambda\mathfrak{K}\left[\frac{1 + \lambda(\mathfrak{K} - \kappa_1) + (\lambda^2/2)(2\mathfrak{K}^2 - 2\kappa_1\mathfrak{K} + \kappa_1^2 - \kappa_2) + \cdots}{1 - \lambda\kappa_1 + (\lambda^2/2)(\kappa_1^2 - \kappa_2) - \cdots}\right]\right\}\mathbf{f}_0 \tag{9.3.51}$$

where κ_n are the Spurs of the iterated operators $\mathfrak{K}$, as defined in Eq. (9.1.54). This expression may not be used for two- and three-dimensional problems, since κ_1 is not finite in these cases. The appropriate formula may be obtained from the above by formally placing $\kappa_1 = 0$, as was shown in Sec. 9.1. The series in the numerator and denominator converge for all values of λ and may, in principle, be employed to calculate the scattered amplitude to any order of approximation. To make the above equation more concrete, we write out the numerator of the expression for the scattered wave explicitly for the one-dimensional problem formulated in Eq. (9.3.30):

$$\begin{aligned} -\lambda \int_{-\infty}^{\infty} G(x|x_0)U(x_0)e^{ikx_0}\,dx_0 \\ + \lambda^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x|x_1)U(x_1)G(x_1|x_0)U(x_0)e^{ikx_0}\,dx_0\,dx_1 \\ - \lambda^2\left[\int_{-\infty}^{\infty} G(x|x_0)U(x_0)e^{ikx_0}\,dx_0\right]\left[\int_{-\infty}^{\infty} G(x_0|x_0)U(x_0)\,dx_0\right] + \cdots \end{aligned}$$

The denominator of the Fredholm series may be given a physical interpretation. For this purpose, we note that it is always possible to replace the scattering problem by an equivalent radiation problem in which the excitation of the radiating system is given by the incident wave, the resultant radiation forming the scattered wave. The amplitude of the radiated wave will contain a typical impedance denominator of the form $r - ix$. The resistive term r is related to the energy which the system radiates and reaches the detector at large distances from the system. The reactive terms do not involve any energy loss and arise from parts of the radiated wave which decay as distances from the radiating system increase. The value of x goes to zero at the resonance energies discussed earlier in the section on the convergence of the higher Born approximations. We should also expect that the resistive term would depend upon the total cross section, since this represents the radiated power.

To demonstrate the relation between the Fredholm denominator and the impedance concept, let us consider the former in the neighborhood of a resonance, *i.e.*, for λ close to λ_r. Then [see Eq. (9.1.53)]

$$\chi(\lambda) = \exp\left[-\int_0^\lambda \operatorname{Spur}\left(\frac{\mathfrak{K}}{1 - \lambda\mathfrak{K}}\right) d\lambda\right] \simeq (\text{constant})\ (\lambda - \lambda_r)$$

Now λ is a function of k^2 and at $k^2 = k_r^2 - (i\gamma/2)$, λ equals λ_r, where k_r is real. Then

$$\chi(\lambda) \simeq (\text{constant})\ [k^2 - k_r^2 + (i\gamma/2)](\partial\lambda/\partial k^2)_{k^2=k_r^2-\frac{1}{2}i\gamma}$$

The scattered amplitude f will contain $\chi(\lambda)$ in the denominator and thus will show a typical scattering resonance when $k^2 = k_r^2$. As stated above, this form recalls the response of a simple harmonic system with resonant angular frequency k_r and Q equal to $\gamma/2$, permitting the determination of the appropriate impedance.

We now go on to verify that the Fredholm denominator involves the total cross section. Consider the three-dimensional case where this denominator to second order is $(1 - \frac{1}{2}\lambda^2\kappa_2)$, where κ_2 is given by

$$\lambda^2\kappa_2 = \left(\frac{1}{2\pi}\right)^6 \iint \frac{U(\mathbf{p}|\mathbf{K})\,U(\mathbf{K}|\mathbf{p})}{(K^2 - k^2)(p^2 - k^2)}\, dV_K\, dV_p$$

Employing Eq. (9.3.49),

$$\lambda^2\kappa_2 = -\left(\frac{1}{2\pi}\right)^3 \int \frac{[T^{(2)}(\mathbf{p}|\mathbf{p}) - U(\mathbf{p}|\mathbf{p})]}{p^2 - k^2}\, dV_p$$

The $\mathbf{p}$ integration will be broken into two parts, one arising from the pole at $\mathbf{p} = \mathbf{k}$ and therefore permitting $\mathbf{p}$ to be a possible incident wave. We rewrite κ_2 as follows:

$$\lambda^2\kappa_2 = -\left(\frac{1}{2\pi}\right)^3 \left\{\frac{1}{2}\oint \frac{T^{(2)}(\mathbf{p}|\mathbf{p}) - U(\mathbf{p}|\mathbf{p})}{p^2 - k^2}\, dV_p + \mathcal{P}\int \frac{T^{(2)}(\mathbf{p}|\mathbf{p}) - U(\mathbf{p}|\mathbf{p})}{p^2 - k^2}\, dV_p\right\}$$

where the first integral is taken over a closed contour about $p = k$. The symbol $\mathcal{P}$ in the second integral refers to the principal value [see Eq. (4.2.9)] as obtained by taking the arithmetic average of the contribution from the two contours labeled 1 and 2 in Fig. 9.12. The first integral may be evaluated by Cauchy's integral theorem, so that finally

$$\lambda^2\kappa_2 = -\left(\frac{1}{2\pi}\right)^3 \left\{\frac{k\pi i}{2}\int [T^{(2)}(\mathbf{k}|\mathbf{k}) - U(\mathbf{k}|\mathbf{k})]\,d\Omega_k + \mathcal{P}\int \frac{[T^{(2)}(\mathbf{p}|\mathbf{p}) - U(\mathbf{p}|\mathbf{p})]}{p^2 - k^2}\,dV_p\right\} \quad (9.3.52)$$

Here **k** is a vector with the magnitude k. The quantities $U(\mathbf{k}|\mathbf{k})$ and $T^{(2)}(\mathbf{k}|\mathbf{k})$ are the first and second Born approximations for the scattering amplitudes for the direction **k** of a plane wave incident in the direction **k**.

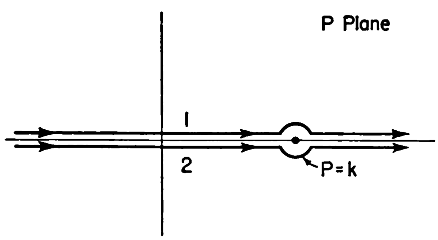

Fig. 9.12 Contours used in obtaining principal value of integral for κ_2.

These amplitudes are not independent of the direction of **k** if the scattering system is not spherically symmetric. In that event, we proceed by taking advantage of relation (9.3.23) from which we obtain

$$\mathrm{Im}[T^{(2)}(\mathbf{k}|\mathbf{k})] = -kQ_B(\mathbf{k}) \quad (9.3.53)$$

where Q_B is the first Born approximation for the total cross section. Hence the Fredholm denominator contains a term proportional to $k^2\bar{Q}_B$, where $\bar{Q}_B$ is the Born total cross section averaged over all angles of incidence.

If, on the other hand, the system is spherically symmetric, we may place

$$U(\mathbf{k}|\mathbf{k}) = -4\pi f_B(0);\quad T^{(2)}(\mathbf{k}|\mathbf{k}) = -4\pi f_B^{(2)}(0)$$

where f_B and $f_B^{(2)}$ are the first and second Born approximations for the scattering amplitude in the direction of the incident wave. Substituting in Eq. (9.3.52), we have

$$\lambda^2\kappa_2 = ik\,[f_B^{(2)}(0) - f_B(0)] - \left(\frac{1}{2\pi}\right)^3 \mathcal{P}\int \frac{T^{(2)}(\mathbf{p}|\mathbf{p}) - U(\mathbf{p}|\mathbf{p})}{p^2 - k^2}\,dV_p \quad (9.3.54)$$

An Example. Consider the transmission and reflection of matter waves by the potential barrier illustrated in Fig. 9.9. The exact answer for the transmitted amplitude C/A is given by Eq. (9.3.41). The equivalent integral equation and the exact expressions for the transmitted amplitude are given by Eqs. (9.3.29) to (9.3.31). Because $U(x_0)$ differs from zero only when $(-a < x_0 < a)$, it is clear that only the values of the nth approximation to ψ, $\psi^{(n)}(x_0)$, in this range need be known in order to obtain the $(n + 1)$ approximation to ψ and to the transmitted amplitude. Starting with the unperturbed incident wave as the zeroth approximation and employing recurrence relations (9.3.38), we find for $-a < x < a$ that

$$\psi^{(0)} = \exp(ikx)$$

$$\psi^{(1)} = \exp(ikx) + (\lambda D/2ik)\{[x + a - (1/2ik)]\exp(ikx) + (1/2ik)\exp[ik(2a - x)]\}$$

Employing Eq. (9.3.31), we may obtain the first and second Born approximations to the transmitted amplitude $f^{(1)}$ and $f^{(2)}$:

$$f^{(1)} = 1 + (\lambda D/2ik)(2a)$$

$$f^{(2)} = f^{(1)} + (\lambda D/2ik)^2\{2a[a - (1/2ik)] + [\exp(2ika)/2ik^2]\sin(2ka)\}$$

We compare with the exact expression (9.3.41) which we rewrite as follows:

$$f = 1\bigg/\left\{1 - \frac{(\kappa - k)^2}{4k\kappa}[\exp(4ika) - 1]\right\}$$

This must now be expanded in a power series in D:

$$f \simeq 1 + (i/2)(\lambda D/2k^2)^2[\exp(2ika)][\sin(2ka)]$$

We see that the Born approximations are hardly the correct expansions of the correct amplitude in powers of D. This might be expected, since the measure of the strength of the interaction depends upon a as well as D. Presumably the presence of the poles in Eq. (9.3.41) is responsible. However, if we evaluate the transmission coefficient $T = |f|^2$, we find

$$T \simeq 1 - (\lambda D \sin 2ka/2k^2)^2 \simeq T^{(2)}$$

where $T^{(2)}$ is computed from $f^{(2)}$ to order D^2. The second Born approximation gives, therefore, to this order a correct expression for the magnitude of the transmitted amplitude, the phase being incorrect.

Let us now turn to the Fredholm expansion. The Born approximations involve the expansion of f about $k = \kappa$. The Fredholm series, as we shall see, is to be compared with the separate expansion of the numerator (not required in this particular example, since the numerator is unity) and of the denominator. From Eq. (9.3.51) we immediately obtain

$$f \simeq 1 + \frac{(f^{(1)} - 1) + (f^{(2)} - f^{(1)}) - \lambda\kappa_1(f^{(1)} - 1)}{1 - \lambda\kappa_1 + (\lambda^2/2)(\kappa_1^2 - \kappa_2)}$$

where $f^{(1)}$ and $f^{(2)}$ are the Born amplitudes. The Spurs κ_1 and κ_2 are taken with respect to the operator $(1/2ik)\int \exp(ik|x - x_0|)U(x_0)$ — and are given by

$$\kappa_1 = \frac{1}{2ik}\int_{-\infty}^{\infty} U(x_0)\,dx_0$$

$$\kappa_2 = -\left(\frac{1}{4k^2}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(x_1)\exp(2ik|x_1 - x_0|)U(x_0)\,dx_0\,dx_1$$

These integrals may be easily evaluated:

$$\lambda\kappa_1 = (\lambda D/2ik)2a = f^{(1)} - 1$$

$$\frac{\lambda^2}{2}(\kappa_1^2 - \kappa_2) = \left(\frac{D^2\lambda^2}{8ik^3}\right)\left[\left(\frac{1}{k}\right)(\exp 2ika)(\sin 2ka) - 2a\right] - \left(\frac{D^2\lambda^2a^2}{2k^2}\right)$$
$$= -(f^{(2)} - f^{(1)}) + (f^{(1)} - 1)^2$$

Hence the Fredholm result is

$$f \simeq \frac{1}{1 - (f^{(2)} - 1) + (f^{(1)} - 1)^2}$$

The transmission coefficient T evaluated to second order in D in the denominator of the Fredholm f is

$$T \simeq \frac{1}{1 + [\lambda D \sin(2ka)/2k^2]^2}$$

We see that the Fredholm result is almost always better than the second Born approximation, since the former does not compound the error arising from the expansion of κ about k by performing a further expansion of the denominator.

A Three-dimensional Example. The higher Born approximations can be evaluated analytically for only a few cases. One of these is the "Yukawa" well

$$U = e^{-\mu r}/\mu r$$

We shall use the analysis in momentum space, leading to Eqs. (9.3.49) and (9.3.50). The matrix elements $U(\mathbf{k}_s|\mathbf{k}_i)$ involved are defined by Eq. (9.3.45) and for the above potential are given by

$$U(\mathbf{k}_s|\mathbf{k}_i) = \frac{4\pi\lambda}{\mu[\mu^2 + |\mathbf{k}_i - \mathbf{k}_s|^2]} = \frac{4\pi\lambda}{\mu[\mu^2 + 4k^2\sin^2\frac{1}{2}\vartheta]}$$

The first Born approximation for the amplitude $f(\vartheta,\varphi)$ is then, from Eq. (9.3.47),

$$f_B(\vartheta,\varphi) = -\frac{\lambda}{\mu[\mu^2 + |\mathbf{k}_i - \mathbf{k}_s|^2]}$$

The second Born approximation may be obtained from $(-1/4\pi)T^{(2)}(\mathbf{k}_s|\mathbf{k}_i)$ where

$$T^{(2)}(\mathbf{k}_s|\mathbf{k}_i) = U(\mathbf{k}_s|\mathbf{k}_i) - \frac{2\lambda^2}{\pi\mu^2}\int \frac{dV_K}{[\mu^2 + |\mathbf{K} - \mathbf{k}_s|^2][\mu^2 + |\mathbf{K} - \mathbf{k}_i|^2][K^2 - k^2]}$$

The evaluation of this integral can, as we shall see, be reduced to the evaluation of

$$J = \int \frac{dV_K}{[\tau^2 + |\mathbf{K} - \mathbf{p}|^2][K^2 - k^2]}$$

We take $\mathbf{p}$ as the z axis of spherical coordinates (K,α,β) in $\mathbf{K}$ space. The integration over β may be immediately performed so that

$$J = 2\pi \int_0^\pi \sin\alpha\, d\alpha \int_0^\infty \frac{K^2\, dK}{[\tau^2 + K^2 + p^2 - 2Kp\cos\alpha][K^2 - k^2]}$$

By introducing the transformations $\alpha = \pi - \psi$ and $K = -K'$, and averaging with the above expression, we obtain a more convenient expression:

$$J = \pi \int_0^\pi \sin\alpha\, d\alpha \int_{-\infty}^\infty \frac{K^2\, dK}{[\tau^2 + K^2 + p^2 - 2Kp\cos\alpha][K^2 - k^2]}$$

The contour for the K integration is shown in Fig. (9.11). We may close it by a semicircle of infinite radius in the upper half plane. Cauchy's integral formula may then be employed, contributions occurring at the poles of the denominator K_0 and K_1.

$$K_0 = k \quad \text{and} \quad K_1 = p\cos\alpha + i\sqrt{\tau^2 + p^2\sin^2\alpha}$$

Breaking up J into corresponding integrals, we have

$$J = J_0 + J_1$$

where
$$J_0 = \pi^2 ik \int_{-1}^1 \frac{dx}{\tau^2 + k^2 + p^2 - 2kpx}$$

and
$$J_1 = \left(\frac{\pi^2 i}{p}\right) \int_{i\tau-p}^{i\tau+p} \frac{K_1}{K_1^2 - k^2}\, dK_1$$

Hence, finally,

$$J = \left(\frac{\pi^2 i}{p}\right) \ln\left(\frac{k + p + i\tau}{k - p + i\tau}\right)$$

We shall also need

$$L = \int \frac{dV_K}{[\tau^2 + |\mathbf{K} - \mathbf{p}|^2]^2[K^2 - k^2]}$$

This integral may be obtained by differentiating J with respect to τ:

$$L = -\frac{\pi^2}{\tau[k^2 - \tau^2 - p^2 + 2ik\tau]}$$

The integral occurring in $T^{(2)}(\mathbf{k}_s|\mathbf{k}_i)$

$$M = \int \frac{dV_K}{[\mu^2 + |\mathbf{K} - \mathbf{k}_s|^2][\mu^2 + |\mathbf{K} - \mathbf{k}_i|^2][K^2 - k^2]}$$

may be related to L through the use of the integral representation

$$\frac{1}{ab} = \int_{-1}^1 \frac{2\, dz}{[a(1+z) + b(1-z)]^2}$$

Hence

$$M = 2\int_{-1}^1 dz \int \frac{dV_K}{[2\mu^2 + |\mathbf{K} - \mathbf{k}_s|^2(1+z) + |\mathbf{K} - \mathbf{k}_i|^2(1-z)]^2(K^2 - k^2)}$$

Rewriting the denominator to conform to the form of L, we obtain

$$M = \tfrac{1}{2}\int_{-1}^{1} dz \int \frac{dV_K}{K^2 - k^2}\,\{\mu^2 + k^2(1 - z^2)\sin^2(\tfrac{1}{2}\alpha) + |\tfrac{1}{2}(1 + z)\mathbf{k}_i + \tfrac{1}{2}(1 - z)\mathbf{k}_s - \mathbf{K}|^2\}^{-2}$$

The integral over K space is now in proper form, with

$$\tau^2 = \mu^2 + k^2(1 - z^2)\sin^2(\tfrac{1}{2}\alpha); \quad \mathbf{p} = \tfrac{1}{2}(1 + z)\mathbf{k}_i + \tfrac{1}{2}(1 - z)\mathbf{k}_s$$

Since the magnitude of $\mathbf{k}_i$ and $\mathbf{k}_s$ are both k,

$$\tau^2 + p^2 = \mu^2 + k^2$$

Therefore

$$M = -\frac{\pi^2}{2}\int_{-1}^{1} \frac{dz}{\tau[-\mu^2 + 2ik\tau]}$$

Since τ^2 is a simple algebraic function of z, the above integral can be performed:

$$M = \frac{\pi^2}{k \sin \frac{1}{2}\vartheta \sqrt{\mu^4 + 4k^2(\mu^2 + k^2 \sin^2 \frac{1}{2}\vartheta)}} \cdot \left\{\tan^{-1}\left[\frac{\mu k \sin \frac{1}{2}\vartheta}{\sqrt{\mu^4 + 4k^2(\mu^2 + k^2 \sin^2 \frac{1}{2}\vartheta)}}\right] + \frac{i}{2}\ln\left[\frac{\sqrt{\mu^4 + 4k^2(\mu^2 + k^2 \sin^2 \frac{1}{2}\vartheta)} + 2k^2 \sin \frac{1}{2}\vartheta}{\sqrt{\mu^4 + 4k^2(\mu^2 + k^2 \sin^2 \frac{1}{2}\vartheta)} - 2k^2 \sin \frac{1}{2}\vartheta}\right]\right\} \tag{9.3.55}$$

which is to be substituted into the equation for $T^{(2)}$ on page 1082. Convergence is limited here also, because of the "virtual levels."

We conclude this section by evaluating the Fredholm result to the second order. Since we are dealing with a three-dimensional problem, the Spur κ_1 no longer appears, so that the Fredholm expression for f may be written

$$f \simeq f^{(2)}/[1 - (\lambda^2/2)\kappa_2]$$

where all terms up to second-order in λ^2 in the numerator and denominator of the Fredholm expression for f have been retained. The integral for κ_2 is

$$\kappa_2 = \left(\frac{1}{4\pi^4\mu^2}\right)\iint \frac{dV_p\, dV_K}{(p^2 - k^2)(K^2 - k^2)[\mu^2 + |\mathbf{K} - \mathbf{k}|^2]^2}$$

The integral in K space is just proportional to the L integral discussed above, so that

$$\kappa_2 = \left(\frac{1}{4\pi^2\mu^3}\right)\int \frac{dV_p}{(p^2 - k^2)[p^2 - (k + i\mu)^2]}$$

This integral may be readily reduced to a contour integral, giving finally

$$\kappa_2 = \frac{1}{2\mu^3(\mu - 2ik)}$$

Hence, for small λ, the Fredholm second-order cross section will be larger than the corresponding Born approximation.

It is difficult to determine the accuracy of these results without either going on to higher approximations or carrying out an exact evaluation by the phase-shift method. However, the results obtained with the one-dimensional problem indicate that it is necessary to continue the calculation to one more order before accurate results will be obtained.

Long-wavelength Approximation. The Born approximation treats the potential $U(r)$ as the perturbation and employs the plane waves as the unperturbed wave functions. Other approximations consider k^2 or $1/k^2$ as the perturbation parameter. The first is the long-wavelength approximation, to be discussed immediately below; the second is appropriate for the short-wavelength limit.

The long-wavelength approximation is particularly useful for the scalar Helmholtz equation, since it relates its solution to the solution of the Poisson equation. The latter may be solved by the method of separation of variables and in two dimensions by the method of conformal mapping in many more than the 11 coordinate systems in which exact solutions of the scalar Helmholtz equation can be obtained. We shall discuss this application now and then go on to consider the long-wavelength approximation for the Schroedinger equation.

We expand the wave function ψ in a power series as follows:

$$\psi(\mathbf{r}) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \phi_n(\mathbf{r}) \tag{9.3.56}$$

This expansion assumes that ψ is an analytic function of k at $k = 0$. If this is not the case (example: two-dimensional problem with Dirichlet conditions on scatterer), the equations for ϕ_n will turn out not to have solutions. If we can determine the nature of the singularity, it is always possible to add it into Eq. (9.3.56). The form of the expansion is suggested by the expansion of the plane wave which $\psi(\mathbf{r})$ must, of course, approach as the distance from the scatterer becomes infinite.

Inserting expansion (9.3.56) into the scalar Helmholtz equation and equating the coefficient of a given power of k to zero yield:

$$\begin{aligned} \nabla^2\phi_0 &= 0; \quad \nabla^2\phi_1 = 0 \\ \nabla^2\phi_n &= n(n-1)\phi_{n-2} \end{aligned} \tag{9.3.57}$$

The boundary condition satisfied by each of these functions at the surface of the scatterer is that satisfied by ψ. The boundary condition at infinity is more complex. Since ψ approaches

$$\exp(ikz) + (1/r)\exp(ikr)f(\vartheta,\varphi)$$

it is clear that

$$\phi_0(\mathbf{r}) \to 1; \quad r \to \infty$$

To determine the boundary conditions satisfied by the other ϕ_i, it is convenient to turn to the integral equation satisfied by ψ, since it incorporates the boundary conditions exactly. To make the analysis more specific, let us take Dirichlet conditions $\psi = 0$. The integral equation is given in Eq. (9.3.26), which we repeat here:

$$\psi(\mathbf{r}) = \exp(ikz) + \left(\frac{1}{4\pi}\right) \int \left(\frac{e^{ikR}}{R}\right) \left(\frac{\partial\psi}{\partial n_0}\right) dS_0$$

We now insert expansion Eq. (9.3.56) and equate coefficients of powers of k again. (Note that, if we were not interested in a power series expansion in k, we could employ the above equation in an iterative function by inserting ϕ_0 for ψ, substituting the result for ψ to obtain the next approximation, etc.) We obtain

$$\phi_n = z^n + \frac{1}{4\pi} \int \frac{1}{R}\left(\frac{\partial\phi_n}{\partial n_0}\right) dS_0 + \frac{1}{4\pi} \sum_{p=1}^{n} \binom{n}{p} \int R^{p-1} \left(\frac{\partial\phi_{n-p}}{\partial n_0}\right) dS_0 \quad (9.3.58)$$

The first three of these are

$$\begin{aligned}
\phi_0 &= 1 + \frac{1}{4\pi} \int \frac{1}{R}\left(\frac{\partial\phi_0}{\partial n_0}\right) dS_0; \\
\phi_1 &= z + \frac{1}{4\pi} \int \frac{1}{R}\left(\frac{\partial\phi_1}{\partial n_0}\right) dS_0 + \frac{1}{4\pi} \int \left(\frac{\partial\phi_0}{\partial n_0}\right) dS_0 \\
\phi_2 &= z^2 + \frac{1}{4\pi} \int \frac{1}{R}\left(\frac{\partial\phi_2}{\partial n_0}\right) dS_0 + \frac{2}{4\pi} \int \left(\frac{\partial\phi_1}{\partial n_0}\right) dS_0 + \frac{1}{4\pi} \int R \left(\frac{\partial\phi_0}{\partial n_0}\right) dS_0
\end{aligned}$$

It may be verified directly, by applying the operator ∇^2 to the above equations, that Eqs. (9.3.58) and (9.3.57) are identical. The asymptotic dependence at r infinite is then obtained by expanding R^{p-1} in a decreasing power series starting at r^p. We take into account only the terms up to, but not including, the $1/r$ terms, since the latter is part of the scattered wave. For example,

$$\begin{aligned}
\phi_1 &\to z + \frac{1}{4\pi} \int \left(\frac{\partial\phi_0}{\partial n_0}\right) dS_0; \quad r \to \infty \\
\phi_2 &\to z^2 + \frac{r}{4\pi} \int \left(\frac{\partial\phi_0}{\partial n_0}\right) dS_0 + \frac{1}{4\pi} \int \left[2\left(\frac{\partial\phi_1}{\partial n_0}\right) - \mathbf{r}_0 \cdot \mathbf{a}_r \left(\frac{\partial\phi_0}{\partial n_0}\right)\right] dS_0
\end{aligned}$$

Here $\mathbf{a}_r$ is a unit vector in the $\mathbf{r}$ direction, *i.e.*, in the direction of scattering. These equations complete our specification of the ϕ's. Once they are determined, the series (9.3.56) may be inserted into Eq. (9.3.27) to obtain the scattered amplitude. If we expand the latter in a power series in k, one finds

$$f(\vartheta,\varphi) = \frac{1}{4\pi} \sum_n (-ik)^n \sum_{p=0}^{n} \frac{(-1)^p}{(n-p)!p!} \int (\mathbf{a}_r \cdot \mathbf{r}_0)^{n-p} \left(\frac{\partial\phi_p}{\partial n_0}\right) dS_0 \quad (9.3.59)$$

To second order in k,

$$f(\vartheta,\varphi) \simeq \frac{1}{4\pi}\left\{\int\left(\frac{\partial\phi_0}{\partial n_0}\right)dS_0 - ik\left[\int(\mathbf{a}_r\cdot\mathbf{r}_0)\left(\frac{\partial\phi_0}{\partial n_0}\right)dS_0 - \int\left(\frac{\partial\phi_1}{\partial n_0}\right)dS_0\right]\right.$$
$$\left. - \frac{k^2}{2}\left[\int(\mathbf{a}_r\cdot\mathbf{r}_0)^2\left(\frac{\partial\phi_0}{\partial n_0}\right)dS_0 - 2\int(\mathbf{a}_r\cdot\mathbf{r}_0)\left(\frac{\partial\phi_1}{\partial n_0}\right)dS_0 + \int\left(\frac{\partial\phi_2}{\partial n_0}\right)dS_0\right]\right\} \tag{9.3.60}$$

One sees immediately that, in the limit $k = 0$, the scattering is spherically symmetric. The corresponding scattered amplitude has the dimension of a length and is often referred to as the *scattering length* a. In terms of a, the scattering cross section $Q(k = 0)$ is $4\pi a^2$. In terms of a, ϕ_1 approaches $(z + a)$ as r approaches infinity.

To see how these formulas work out, consider the scattering from a sphere of radius A upon which Dirichlet conditions are satisfied. Then

$$\phi_0 = 1 - (A/r)$$

The scattering amplitude at $k = 0$ is

$$\frac{1}{4\pi}\int\left(\frac{\partial\phi_0}{\partial n_0}\right)dS_0 = -A^2\left(\frac{\partial\phi_0}{\partial r}\right)_{r=A} = -A$$

The wave function ϕ_1 must approach $(z - A)$ at large values of r. The first term z just represents a uniform flow at infinity, while the second is a constant pressure, the same case as that for ϕ_0. Then

$$\phi_1 = [r - (A^3/r^2)]\cos\vartheta - A[1 - (A/r)]$$

Inserting in Eq. (9.3.60), one obtains

$$f(\vartheta,\varphi) = -A + ikA^2 + \cdots$$

The function ϕ_2 satisfies the equation

$$\nabla^2\phi_2 = 2[1 - (A/r)]$$

and

$$\phi_2 \to z^2 - rA + 2A^2;\quad r \to \infty$$

The solution for ϕ_2 is then

$$\phi_2 = \tfrac{1}{3}[r^2 - (A^5/r^3)]P_2(\cos\vartheta) + \tfrac{1}{3}r^2 - Ar + 2A^2 - \tfrac{4}{3}(A^3/r)$$

To second order in k

$$f(\vartheta,\varphi) = -A + ikA^2 + (k^2A^3/3)(2 - 3\cos\vartheta) \tag{9.3.61}$$

We see that all that is needed to carry out the long-wavelength approximation is an adequate knowledge of the elementary solutions of the Laplace equation in the coordinate system appropriate to the scattering surface.

A three-dimensional problem was chosen to illustrate the general method. It, of course, applies as well to one-dimensional problems, as we shall see in the example below. We must always remember that the assumption that ψ is an analytic function of k may have to be modified in specific cases.

Long-wavelength Approximation for the Schroedinger Equation. We consider the motion of a particle of mass m, energy E in a spherically symmetric potential $V(r)$. Since wavelength λ and the energy are related by the equation

$$k/2\pi = 1/\lambda = (1/2\pi)\sqrt{2mE/\hbar^2}$$

long wavelengths correspond to low particle energies. At these long wavelengths, the isotropic component of the incident wave will prove to be most strongly affected, so that we shall concentrate here on the equation for u_0 given in (9.3.13) as

$$u_0'' + [k^2 - U(r)]u_0 = 0; \quad U = (2m/\hbar^2)V \tag{9.3.62}$$

However, the technique and some of the results apply to any l. It will be convenient to change our notation a little at this point. We replace the symbol u_0 by u_k to indicate that we are dealing with solutions of Eq. (9.3.62) with wave number k. In the *new* notation, u_0 will refer to the solution with $k = 0$.

The boundary conditions on $u_k(r)$ are

$$u_k(0) = 0; \quad u_k(r) \to \sin(kr - \eta); \quad r \to \infty$$

where we leave the amplitude at infinity unspecified. It shall be chosen later for reasons of convenience.

The problem we set here is to find $\eta(k)$, that is, the phase shift for a given wave number k in terms of the limiting phase shift $\eta(0)$. It is easy to establish a first-order perturbation result by comparing the equations for u_0 and u_k. This result is based upon the equality

$$u_0u_k'' - u_ku_0'' + k^2u_ku_0 = 0$$

The usual procedure of integrating immediately over the range of the independent variable is not appropriate here because the individual integrals diverge. We therefore introduce functions which have the same asymptotic dependence as u_0 and u_k. Let these be w_k and w_0, where

$$u_k(r) \to w_k(r); \quad r \to \infty$$

The function w_k is proportional to $\sin(kr - \eta)$ and satisfies the Schroedinger equation obtained from the u_k equation by setting the potential U equal to zero:

$$w_k'' + k^2w_k = 0$$

Comparing w_k and w_0, one obtains the equation

$$w_0w_k'' - w_kw_0'' + k^2w_kw_0 = 0$$

We now subtract this equation from the similar one satisfied by u_k and u_0. Integration over the entire range in r, zero to infinity, is now possible because the divergent terms at large r cancel each other out. Inserting the boundary conditions for u_k and u_0 yields the equation

$$[w_0 w_k' - w_k w_0']_{r=0} = k^2 \int_0^\infty (w_k w_0 - u_k u_0)\, dr \tag{9.3.63}$$

We must now insert the functions w_k and w_0. Choosing their amplitude so that

$$w_0(0) = w_k(0) = 1$$

one finds

$$w_k = -\sin(kr - \eta)/\sin\eta; \quad w_0 = 1 - (r/a) \tag{9.3.64}$$

where

$$a = \lim_{k\to 0}[k \cot \eta] \tag{9.3.65}$$

The value of a must be computed by solving the Schroedinger equation for $k = 0$.

Expressions (9.3.64) are now substituted in Eq. (9.3.63) with the following result:

$$-k\cot\eta = -\left(\frac{1}{a}\right) + k^2 \int_0^\infty [w_k w_0 - u_k u_0]\, dr \tag{9.3.66}$$

From this *exact* relation, one may obtain a formula for $(-k \cot \eta)$ correct to first order in k^2 by approximating w_k by w_0, u_k by u_0:

$$-k\cot\eta \simeq -\left(\frac{1}{a}\right) + k^2 \int_0^\infty [w_0^2 - u_0^2]\, dr \tag{9.3.67}$$

This is a very useful formula, not only from the calculational point of view but also from the point of view of correlating experimental data. From such data, $k \cot \eta$ may be determined. Its plot against k^2, that is, against the energy, is a straight line for small k. From its slope and intercept, the two parameters, a and the integral $\int(w_0^2 - u_0^2)\, dr$, are determined. Thus the experimental data can be fitted by any potential which has two free parameters.

To go on to higher order approximations requires substitution in Eq. (9.3.66) of more accurate expressions for w_k and u_k. Since the differential equation satisfied by u_k is a function of k^2, and since with proper normalization the "incident" wave is also a function of k^2, it follows that u_k may be expanded in powers of k^2 so that

$$u_k = u_0 + \sum_{n=1}^{\infty} k^{2n}\varphi_n \tag{9.3.68}$$

The differential equations satisfied by φ_n are

$$\varphi_n'' - U\varphi_n = -\varphi_{n-1}$$

where $\varphi_0 = u_0$. The functions φ_n are zero at r equal to zero. For their behavior at infinity, we expand the asymptotic form as given by Eq. (9.3.64) in a power series in k^2. We illustrate by obtaining the boundary conditions for φ_1, the first term in expansion (9.3.68). The function w_k is given by

$$w_k = \cos kr - (k \cot \eta)(\sin kr/k)$$

We may expand this equation to first order in k^2, where we replace $k \cot \eta$ by Eq. (9.3.67). Thus

$$w_k \simeq 1 - \left(\frac{r}{a}\right) + k^2\left[\left(\frac{r^3}{6a}\right) - \left(\frac{r^2}{2}\right) + r\int_0^\infty (w_0^2 - u_0^2)\,dr\right] \tag{9.3.69}$$

Hence

$$\varphi_1 \to \left(\frac{r^3}{6a}\right) - \left(\frac{r^2}{2}\right) + r\int_0^\infty (w_0^2 - u_0^2)\,dr; \quad r \to \infty \tag{9.3.70}$$

The specification of the function φ_1 is now complete. If the two independent solutions of the equation for u_0 are u_0 and v_0 with Wronskian equal to (-1), then

$$\varphi_1 = Cu_0 + v_0 \int_0^r u_0^2\,dr_0 - u_0 \int_0^r u_0v_0\,dr_0 \tag{9.3.71}$$

where C is to be adjusted so that boundary condition (9.3.70) is satisfied. To determine C, we need the behavior of the integrals for large r. The first of these may be written as

$$\int^r (u_0^2 - w_0^2)\,dr + \int_0^r w_0^2\,dr$$

For r infinite, the first integral is just the coefficient of k^2 in expansion (9.3.67) for $k \cot \eta$ (and also stays finite). Hence

$$\int_0^r u_0^2\,dr_0 \to \int_0^r w_0^2\,dr_0 - \int_0^\infty (w_0^2 - u_0^2)\,dr_0; \quad r \to \infty$$

The second integral in Eq. (9.3.71) also approaches a finite value for r infinite. To show this, we first note that, in order that $\int_0^\infty (w_0^2 - u_0^2)\,dr$ be finite, the expansion of u_0 for large r cannot contain any inverse first or inverse second powers of r. This fact, combined with the requirement that the Wronskian of u_0 and v_0 must equal (-1), is sufficient to show that the expansion of v_0 for large r is also without inverse first and second powers of r, from which the desired result follows. Hence

$$\begin{aligned}\varphi_1 \to C\left[1 - \left(\frac{r}{a}\right)\right] &\\ + a\int_0^\infty (w_0^2 - u_0^2)\,dr_0 &- \left[1 - \left(\frac{r}{a}\right)\right]\int_0^\infty (u_0v_0 + a - r_0)\,dr_0 \\ &+ \left(\frac{r^3}{6a}\right) - \left(\frac{r^2}{2}\right); \quad r \to \infty\end{aligned}$$

where we have inserted the explicit dependence of w_0 and the asymptotic form for v_0. Comparing with boundary condition (9.2.70), we see that

$$C = -a \int_0^\infty (w_0^2 - u_0^2)\, dr_0 + \int_0^\infty (u_0 v_0 + a - r_0)\, dr_0$$

Hence

$$\varphi_1 = u_0 \left\{ -a \int_0^\infty (w_0^2 - u_0^2)\, dr_0 + \int_r^\infty (u_0 v_0 + a - r_0)\, dr_0 + ar - \tfrac{1}{2}r^2 \right\} + v_0 \int_0^r u_0^2\, dr \quad (9.3.72)$$

Once φ_1 is known, the next term in the expansion of $k \cot \eta$ is determined:

$$-k \cot \eta \simeq \left(-\frac{1}{a}\right) + k^2 \int_0^\infty (w_0^2 - u_0^2)\, dr + k^4 \int_0^\infty [w_0 \chi_1 - u_0 \varphi_1]\, dr \quad (9.3.73)$$

where
$$w_k = w_0 + \sum_{n=1}^\infty k^{2n} \chi_n$$
so that
$$\varphi_1 \to \chi_1; \quad r \to \infty$$

The function χ_1 has been given in Eq. (9.3.69).

We have thus been able to determine the next term in the power series expansion of $k \cot \eta$ in terms of quadratures involving only the two independent solutions of the equation for u_0, the radial Schroedinger equation with $k = 0$. The procedure for obtaining this term may be readily continued to higher orders. It should also be noted that similar power series for $k \cot \eta$ about any value of k^2 may be obtained. However, the convergence of such series will in general be poorer than the one considered above.

Convergence. Let us first investigate the circumstances under which the energy dependence of the integral in Eq. (9.3.66) will be small. Since the integrand is a difference between asymptotic and exact forms, the integrand has a value only in the region where the potential U differs from zero. Note that U must go to zero rapidly with increasing r in order that the integral converge. This immediately limits the analysis to "short-range" potential and does not allow its application to long-range potentials such as the coulomb field. Indeed for this last case, the cross section at zero energy is infinite. The effective radius of interaction r_e is customarily taken as

$$r_e = 2 \int_0^\infty (w_0^2 - u_0^2)\, dr \quad (9.3.74)$$

because this value applies exactly to a potential constant for $(r < r_e)$ and zero for $(r > r_e)$.

Going back to the integral, we see that its energy dependence is determined by the k dependence of w_k and u_k within the region in which

U differs from zero, *i.e.*, for $r < r_e$. If in this region the potential is very large, changes in the energy of the incident particle which are small compared with this potential will have only a small effect on u_k, so that the replacement of u_k by u_0 involves a small error. Thus convergence will be good if

$$U_{av} \gg k^2 \tag{9.3.75}$$

where U_{av} is the measure of the average value of the potential in the range $0 < r < r_e$. If this is taken to be

$$\frac{1}{r_e} \int_0^\infty u_0^2 U \, dr$$

where the amplitude of u_0 is chosen so that its asymptotic form w_0 has the value unity at r equal to zero, then inequality (9.3.75) becomes approximately

$$k^2 r_e a \ll 1 \tag{9.3.76}$$

We consider now the k dependence of w_k in the region $r < r_e$. From the form (9.3.64) it is clear that this k dependence will be weak if

$$k r_e \ll 1 \tag{9.3.77}$$

Under these circumstances, replacing w_k by w_0 involves a small error. Thus we may conclude that convergence of the series for $k \cot \eta$ will be quick if the potential is strong and of short range.

The radius of convergence of the series is given by the value of k^2 at which $\eta = \pi$; for then, $k \cot \eta$ will be infinite and the series must diverge. At this value of the energy, the cross section would be zero. Hence it may be expected that the series for $k \cot \eta$ will converge up to the energy at which the first minimum in the total cross section occurs. To go beyond this energy requires some process of analytic continuation such as the Euler transformation.

Short-wavelength Approximation; WKBJ[1] Method. We shall concentrate our attention on the one-dimensional problems. The form most convenient for our discussion is that of the Schroedinger equation

$$(d^2\psi/dx^2) + [k^2 - U(x)]\psi = 0 \tag{9.3.78}$$

This restriction involves very little loss in generality, since any second-order differential equation, particularly the Sturm-Liouville equation, may be transformed into the Schroedinger form.

As the wavelength decreases, the variation of the potential $U(x)$ over a wavelength becomes smaller. This leads to the idea that in the limit

[1] The letters WKBJ stand for G. Wentzel, H. A. Kramers, L. Brillouin, and H. Jeffreys, who more or less independently rediscovered the procedure in connection with the solution of different problems.

one may consider $U(x)$ constant for several wavelengths about x and that in this region the effective wave number $q(x)$ is

$$q(x) = \sqrt{k^2 - U(x)}$$

The corresponding approximate solution of Eq. (9.3.78) is

$$\psi \simeq \exp[\pm iq(x)] \tag{9.3.79}$$

The condition, that q vary slowly over a wavelength, necessary for the validity of approximation (9.2.79) is given by

$$\frac{d(\ln q)}{q\,dx} \ll 1 \quad \text{or} \quad \left|\frac{(\partial U/\partial x)}{2(k^2 - U)^{\frac{3}{2}}}\right| \ll 1 \tag{9.3.80}$$

It is immediately clear that Eq. (9.3.79) will fail whenever $(\partial U/\partial x) \gg 1$, or at the zeros of $(k^2 - U)$.

The first type of failure is exemplified by the potential energy of an alpha particle in the field of an atomic nucleus. The force which, while the α particle is outside, is repulsive, being electrostatic in origin, changes very rapidly into a strong attractive force as the α particle enters the nucleus. The short-wavelength approximation may be applied only at those energies for which the wavelength of the alpha particle is small compared with the distance in which the change from repulsion to attraction takes place.

The second type of failure occurs if U is, at some point, greater than or equal to k^2. Then there will be a point at which $U = k^2$. Classically, this is the point at which the incident kinetic energy of the incident particle equals its potential energy, and therefore the point at which the kinetic energy of the particle is zero. It will, in the next instant, change its direction of motion; therefore these points, at which $k^2 = U$, are referred to as the *classical turning points*.

A more accurate version of these remarks will now be derived. Equation (9.3.79) suggests the substitution

$$\psi = \exp[\varphi(x)]$$

The function $\varphi(x)$ then satisfies the equation

$$(\varphi')^2 + \varphi'' + q^2 = 0 \tag{9.3.81}$$

This is a nonlinear, first-order equation for φ', the logarithmic derivative of ψ. This equation may now be solved by the iterative process in which it is assumed that φ' is a slowly varying function. Then

$$(\varphi')^2 \simeq -q^2 + (i/2q)[(\partial U/\partial x) + (i/2q)(\partial^2 U/\partial x^2) + (i/2q^2)(\partial U/\partial x)^2]$$

We again see the failure of this expansion at the zeros of $q^2 = (k^2 - U)$ and wherever U fluctuates too rapidly. Taking just the first two terms of the expansion leads to the result

$$\psi \simeq q^{-\frac{1}{2}} \exp[\pm\, i\textstyle\int q\, dx] \tag{9.3.82}$$

The general solution is given by the linear combination of these two solutions. The singularity at $q = 0$ is, of course, just a reflection of the failure of the series solution of Eq. (9.3.81).

Approximation (9.3.82) may be employed for the problem of the transmission and reflection of matter waves by a potential $U(x)$ if the energy of the incident particle is greater than $U(x)$ everywhere (see Fig. 9.13). If the particles are traveling in the positive x direction, with unit amplitude as $x \to -\infty$, then

$$\psi \simeq \sqrt{\left(\frac{k}{q}\right)} \exp\left[i \int_{-\infty}^{x} (q - k)\, dx\right] \exp(ikx) \tag{9.3.83}$$

Hence, in this approximation, the transmission coefficient is unity, only the phase of the wave changing. This corresponds to the classical result. We may obtain the next approximation for the reflection and transmis-

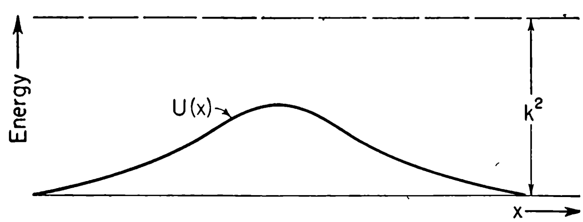

Fig. 9.13 Transmission of waves when the energy k^2 is everywhere greater than the potential function U.

sion coefficient by substituting expression (9.3.83) in Eqs. (9.3.31) and (9.3.32), ($\lambda = 1$). More accurate expressions for ψ may be obtained by employing (9.3.83) as the initial trial function in an iterative procedure based on integral equation (9.3.30). The convergence of such an iterative procedure might be expected to be best if k^2 is much greater than U everywhere. It can, of course, be improved by using the Fredholm method. A discussion of the situation where k^2 is close to the maximum value of U will be made later [see Eqs. (9.3.117) *et seq.*]. It should be noted that the iterative procedure discussed above does not yield an expansion of the reflection coefficient in inverse powers of k. It is entirely possible that the reflection coefficient is nonanalytic at k infinite, even though we know it to have the value zero there.

Relation to Integral Equation. Before going on to deal with situations other than that illustrated in Fig. 9.13, it is worth while to show how the results given above may be obtained directly from the integral equation. The derivation starts from the remark that, as k approaches infinity, the reflection coefficient goes to zero. Hence, if we consider waves incident from the left, as in Fig. 9.13, we should expect that in the integral equation for ψ

$$\psi = e^{ikx} + \left(\frac{1}{2ik}\right)\left[e^{ikx}\int_{-\infty}^{x} e^{ikx_0}\,U(x_0)\psi(x_0)\,dx_0 + e^{-ikx}\int_{x}^{\infty} e^{ikx_0}\,U(x_0)\psi(x_0)\,dx_0\right]$$

the term multiplying $\exp(ikx)$ should be dominant. Dropping the other integral term gives

$$\psi \simeq e^{ikx} + \left(\frac{1}{2ik}\right) e^{ikx}\int_{-\infty}^{x} e^{ikx_0}\,U(x_0)\psi(x_0)\,dx_0$$

Multiplying through by $\exp(-ikx)$ and differentiating with respect to x yields the differential equation

$$\frac{d}{dx}\,(e^{-ikx}\psi) = \frac{U}{2ik}\,(e^{-ikx}\psi)$$

This equation may be readily integrated:

$$\psi \simeq \exp i\left[kx - \left(\frac{1}{2k}\right)\int^{x} U\,dx\right] \tag{9.3.84}$$

We see that the phase in this expression contains just the first two terms in the expansion of $q(x)$ in inverse powers of k. Hence the above ψ is approximately equal to that given in Eq. (9.3.79). Improvement on Eq. (9.3.84) can be obtained by iteration, but this would yield a result completely equivalent to the iteration procedure discussed below Eq. (9.3.83).

Case of Well-separated Classical Turning Points. When k^2 is smaller than the maximum value of U, there will be at least one classical turning point. For the potential illustrated in Fig. 9.13 there will be two. The analysis we are about to describe will be useful if there is but one turning point or, if there are several, when these turning points are widely spaced.

Let us consider the case where q has one zero at $x = x_0$; as we have previously noted, approximate solution (9.3.82) is therefore singular at $x = x_0$. Actually, no such singularity exists, since x_0 is clearly a regular point of the differential equation (9.3.78) satisfied by ψ. We may readily obtain the solutions in the neighborhood of x_0 as follows: In the neighborhood of the zero we may approximate q^2 by

$$q^2 = a^2(x - x_0)$$

where a is a constant. The resulting differential equation

$$(d^2\psi/dx^2) + a^2(x - x_0)\psi = 0$$

may be solved in terms of Bessel functions of $\frac{1}{3}$ order. The two solutions are

$$\psi_+ = \sqrt{x - x_0}\, J_{\frac{1}{3}}[\tfrac{2}{3}a(x - x_0)^{\frac{3}{2}}] \tag{9.3.85}$$
$$\psi_- = \sqrt{x - x_0}\, J_{-\frac{1}{3}}[\tfrac{2}{3}a(x - x_0)^{\frac{3}{2}}] \tag{9.3.86}$$

The actual solution ψ must approach a linear combination of ψ_+ and ψ_- as x approaches x_0.

To find the proper combination, it is useful to change the independent variable from x to one which reduces to $\frac{2}{3}a(x - x_0)^{\frac{3}{2}}$ as x approaches x_0 and $\int q\,dx$ when x becomes large. Such a variable is

$$w = \int_{x_0}^{x} q\,dx \tag{9.3.87}$$

Then we see that the function P

$$P = \sqrt{w/q}\,[AJ_{\frac{1}{3}}(w) + BJ_{-\frac{1}{3}}(w)] \tag{9.3.88}$$

where A and B are constant, reduces to a linear combination of ψ_+ and ψ_- for x near x_0 and a linear combination of solutions (9.3.82) when x is large. The latter statement requires the use of asymptotic expressions (5.3.68) for the Bessel functions. The constants A and B are fixed by boundary conditions. The function P is thus an approximate solution of the differential equation satisfied by ψ in the neighborhood of x_0 and also as $x \to \pm\infty$. It deviates further from the exact solution elsewhere. To obtain an idea of the order of magnitude of the error, and also to suggest a procedure for improving on P, we determine the differential equation satisfied by P. A straightforward calculation gives

$$\frac{d^2P}{dx^2} + \left[q^2 - \frac{r''(x)}{r(x)}\right]P = 0 \tag{9.3.89}$$

where

$$r = w^{\frac{1}{6}}/\sqrt{q} \tag{9.3.90}$$

The ratio r''/r is thus a measure of the error. It is zero for x infinite. It is finite for $x = x_0$, being proportional to the second derivative of U at x_0, so that this error is small if the slope of the potential varies slowly. It will be large at any other zero of q or wherever the potential U changes sharply.

The form of differential equation (9.3.89) suggests that, when x_0 is the only zero of q, r''/r be treated as a perturbation. More explicitly, we write

$$(d^2\psi/dx^2) + [q^2 - (r''/r)]\psi = -(r''/r)\psi$$

This equation may be converted into an integral equation by employing the Green's function for the operator

$$d^2/dx^2 + q^2 - (r''/r)$$

which may be expressed in terms of two independent solutions which may be obtained from Eq. (9.3.88). The resulting integral equation may then be treated by iteration or by the Fredholm method, if this should be considered necessary.

We return now to approximation (9.3.88) in order to give an explicit statement of its asymptotic behavior. This is required in order to fit boundary conditions at $\pm\infty$. Consider first the case illustrated in Fig. 9.14, where $k^2 > U$ for $x > x_0$. The asymptotic dependence of solution (9.3.88) for $x \gg x_0$ may be readily found employing formulas

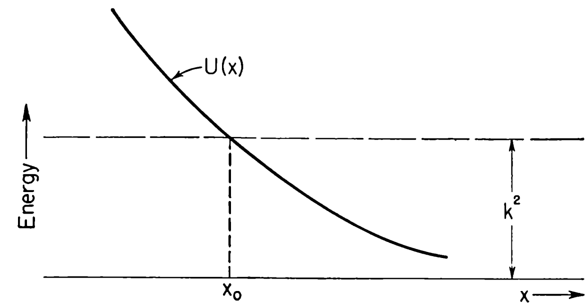

Fig. 9.14 Reflection of waves from potential barrier.

(5.3.68), giving the asymptotic behavior of Bessel functions. This gives

$$P \to \sqrt{2/\pi q}\,[A\cos(w - \tfrac{5}{12}\pi) + B\cos(w - \tfrac{1}{12}\pi)];\quad x \gg x_0;\quad k^2 > U \tag{9.3.91}$$

When $x \ll x_0$, q^2 is negative, so that both q and w are imaginary, having branch points at x_0. To continue our analysis, we must be careful to specify the branch of q which is to be used. Since function P is actually single-valued, either branch of q will do. We choose $q = \exp[\frac{1}{2}\pi i]|q|$ for $x < x_0$. Then, from definition (9.3.87) and the linear approximation for q^2, we find that, for $x < x_0$, $w = \exp[\frac{3}{2}\pi i]|w|$. Formula (5.3.68), giving the asymptotic dependence of Bessel functions whose argument has a phase between $\pi/2$ and $3\pi/2$, may be applied here with the result

$$P \to \tfrac{1}{2}\sqrt{2/\pi q}\,[(B - A)e^{|w|} + (Ae^{\pi i/6} + Be^{-\pi i/6})e^{-|w|}];\quad x \ll x_0;\quad k^2 < U \tag{9.3.92}$$

Because of the error in the asymptotic series beginning with the $\exp|w|$ term, the $\exp[-|w|]$ term is meaningful only when B is exactly equal to A.

Consider now the case illustrated in Fig. 9.15, where $k^2 < U$ for $x > x_0$. It is more convenient to employ the variable w' where

$$w' = \int_x^{x_0} q\,dx \tag{9.3.93}$$

since w' is positive and real where q is positive and real. The corresponding asymptotic solution P' is

$$P' = \sqrt{w'/q}\,[A'J_{\frac{1}{3}}(w') + B'J_{-\frac{1}{3}}(w')] \tag{9.3.94}$$

The asymptotic dependence of P' for $x \ll x_0$ may be obtained from Eq. (9.3.91) by replacing w by w', as well as A and B by A' and B'. In the range $x \gg x_0$, Eq. (9.3.92) should be employed where it should be noted that $|w| = |w'|$.

Let us now see how these formulas are used, by considering specific problems.

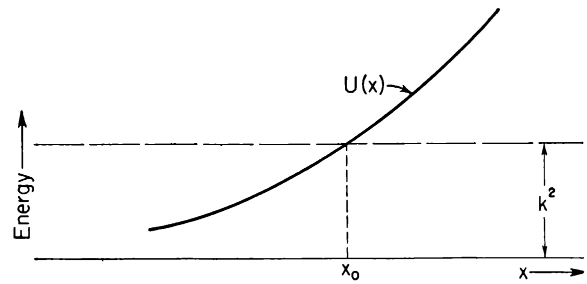

Fig. 9.15 Matching the solutions for a rising potential.

WKBJ Method for Bound Systems. We again take our example from quantum mechanics. The potential $U(x)$ is illustrated in Fig. 9.16. For the present application, it is essential only that the motion be classically bounded. The boundary conditions require that ψ go to zero at x equal to plus and minus infinity.

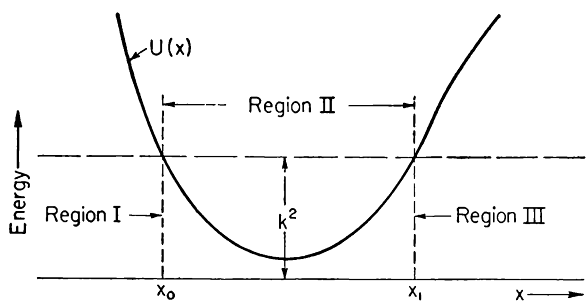

Fig. 9.16 Potential function for bound state.

Expression (9.3.92) is appropriate for region I. In order to satisfy boundary conditions, we must take $B = A$ so that

$$\psi \to A\,\sqrt{(2/\pi q)}\,\cos(\tfrac{1}{6}\pi)e^{-|w|}; \quad x \ll x_0$$
$$|w| = \int_x^{x_0} |q|\,dx \tag{9.3.95}$$

The corresponding result for region II is obtained by placing $A = B$ in Eq. (9.3.91):

$$\psi \to 2A\,\cos(\tfrac{1}{6}\pi)\,\cos(w - \tfrac{1}{4}\pi); \quad x \gg x_0$$
$$w = \int_{x_0}^{x} q\,dx \tag{9.3.96}$$

We may now apply a similar analysis based on Eq. (9.3.94) to the connection between the expression for the wave functions in region III and region II. Again, because of boundary conditions, $B' = A'$ so that

$$\psi \to A' \sqrt{(2/\pi q)} \cos(\tfrac{1}{6}\pi) e^{-|w'|}; \quad x \gg x_1$$
$$[w'| = \int_{x_1}^{x} |q| \, dx$$

In region II we find

$$\psi \to 2A' \sqrt{(2/\pi q)} \cos(\tfrac{1}{6}\pi) \cos(w' - \tfrac{1}{4}\pi); \quad x \ll x_1$$
$$w' = \int_{x}^{x_1} q \, dx \tag{9.3.97}$$

It is, of course, necessary that the solution of this problem be continuous, so expressions (9.3.96) and (9.3.97) should be identical. Rewriting the

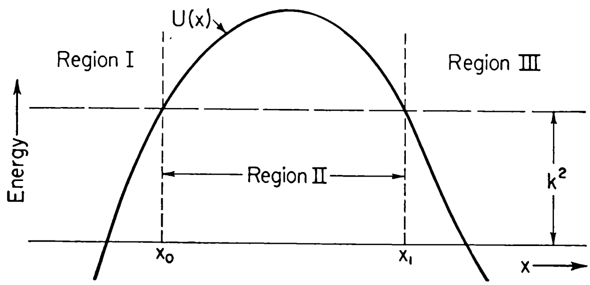

Fig. 9.17 Penetration of waves through potential barrier.

argument of the cosine in the latter as follows:

$$w' - (\tfrac{1}{4}\pi) = \int_{x_0}^{x_1} q \, dx - w - (\tfrac{1}{4}\pi)$$

we see that the continuity can be satisfied by placing

$$\int_{x_0}^{x_1} q \, dx - (\tfrac{1}{4}\pi) = (\tfrac{1}{4}\pi) + n\pi; \quad n \text{ an integer}$$

or
$$\int_{x_0}^{x_1} q \, dx = (n + \tfrac{1}{2})\pi \tag{9.3.98}$$

It is, of course, also necessary to choose the ratio of A/A' appropriately.

The derivation of (9.3.98) clearly assumes that x_0 and x_1 are so well separated that one may employ the asymptotic expressions for the Bessel function occurring in Eqs. (9.3.88) and (9.3.94). In other words, there must be many oscillations of the wave function in the region between x_0 and x_1 so that a necessary condition for the validity of Eq. (9.3.98) is

$$n \gg 1 \tag{9.3.99}$$

It is, of course, also necessary that the potential be slowly varying.

Penetration through a Barrier. We next consider a plane wave incident upon a region in which the potential takes the form illustrated in Fig. 9.17. Let the wave be incident from the left. Then in region III, there can be only a transmitted wave. This we take to be of the form

$$\psi \to \sqrt{(2/\pi q)}\,\exp\{i[w - \tfrac{1}{4}\pi]\};\quad x \gg x_1$$
$$w = \int_{x_1}^{x} q\,dx \tag{9.3.100}$$

This form may be obtained from Eq. (9.3.91) by appropriately choosing the constants A and B. In region II, Eq. (9.3.92) then gives

$$\psi \to \sqrt{2/\pi q}\; ie^{|w|} \quad (x \ll x_1) \tag{9.3.101}$$

We have dropped the term in Eq. (9.3.92) proportional to $\exp[-|w|]$, since it is generally smaller than the error in the $\exp|w|$ term. We see that ψ will increase rapidly when x is much smaller than x_1.

The form of ψ in region II is also given by Eq. (9.3.92) if w is replaced by w':

$$w' = \int_{x}^{x_0} q\,dx \tag{9.3.102}$$

We note that

$$|w| = -|w'| + \int_{x_0}^{x_1} |q|\,dx$$

so that Eq. (9.3.101) reads

$$\psi \to \sqrt{2/\pi q}\; ie^{\kappa}e^{-|w'|}$$

where

$$\kappa = \int_{x_0}^{x_1} |q|\,dx \tag{9.3.103}$$

Comparing with Eq. (9.3.92) (with w, A, B replaced by w', A', B') we see that $A' = B'$, so that

$$\psi \to \sqrt{(2/\pi q)}\; A' \cos(\tfrac{1}{6}\pi)e^{-|w'|}$$

Comparing with Eq. (9.3.101) yields

$$A' = ie^{\kappa}\cos(\tfrac{1}{6}\pi)$$

In region I, form (9.3.91) is appropriate with w replaced by w'. Inserting the above value of A' and rewriting (9.3.91) in the form

$$\sqrt{2/\pi q}\;\{\alpha \exp[i(w' - \tfrac{1}{4}\pi)] + \beta \exp[-i(w' - \tfrac{1}{4}\pi)]\}$$

we find that

$$\alpha = \beta = ie^{\kappa}$$

Comparing with Eq. (9.3.100), we see that the transmission coefficient T is given by $|1/\beta|^2$ so that

$$T \simeq e^{-2\kappa} \tag{9.3.104}$$

From the conservation condition, Eq. (9.3.11), the reflection coefficient R must be

$$R = 1 - T \simeq 1 - e^{-2\kappa} \tag{9.3.105}$$

Equations (9.3.104) and (9.3.105) should be valid whenever $\kappa \gg 1$, that is, $T \ll 1$, with the proviso, of course, that the potentials be slowly varying.

WKBJ Method for Radial Equations. It is not possible to apply the above results, particularly as given by Eqs. (9.3.91) and (9.3.92), directly to the radial equation (9.3.13),

$$u'' + [k^2 - U(r) - l(l+1)(1/r^2)]u = 0$$

because of its singularity at $r = 0$. The one-dimensional Schroedinger equation is singular at $x = \pm\infty$, and to bring the radial equation to the same form, it is appropriate that we change the independent variable in the radial equation to x so that $r = 0$ corresponds to $x = -\infty$. Accordingly, we introduce the transformation

$$r = e^x$$

The function u then satisfies

$$u'' - u' + e^{2x}[k^2 - U(e^x) - l(l+1)e^{-2x}]u = 0$$

where the primes now denote differentiation with respect to x. This is not in the one-dimensional Schroedinger form because of the presence of the first derivative term. This may, however, be eliminated by changing the dependent variable from u to χ as follows:

$$u = e^{\frac{1}{2}x}\chi(x) \tag{9.3.106}$$

The function χ satisfies

$$\chi'' + e^{2x}[k^2 - U(e^x) - (l + \tfrac{1}{2})^2 e^{-2x}]\chi = 0 \tag{9.3.107}$$

It is now possible to apply Eqs. (9.3.91) and (9.3.92) with q^2 being given by the coefficient of χ in Eq. (9.3.107). The function w is then

$$w = \int_{x_0}^{x} e^x \sqrt{k^2 - U(e^x) - (l + \tfrac{1}{2})^2 e^{-2x}}\, dx$$

Going back to the original independent variable r, w becomes

$$w = \int_{r_0}^{r} \sqrt{k^2 - U(r) - (l + \tfrac{1}{2})^2(1/r^2)}\, dr \tag{9.3.108}$$

Comparing with the differential equation for u in terms of r, we see that the equation may be treated by the WKBJ method for one-dimensional equations, provided that the factor $l(l+1)$ is replaced by $(l + \frac{1}{2})^2$. We shall accordingly call the integrand of Eq. (9.3.108) q_r. A simple application of this result is the calculation of the energies of a bound system. Following Eq. (9.3.98), these are determined by

$$\int_{r_0}^{r_1} q_r\, dr = (n + \tfrac{1}{2})\pi$$

The quantities r_1 and r_0 are the zeros of q_r. Again, a necessary condition for the validity of this formula is $n \gg 1$.

WKBJ Phase Shifts. The WKBJ aproximation may also be used to obtain an estimate of phase shifts η_l for scattering problems (see pages

1068 and 1072). We consider the case where q_r has just one zero, r_0, for positive values of k^2. A typical curve is shown in Fig. 9.18. We first consider the satisfaction of boundary conditions at $r = 0$. Here u must

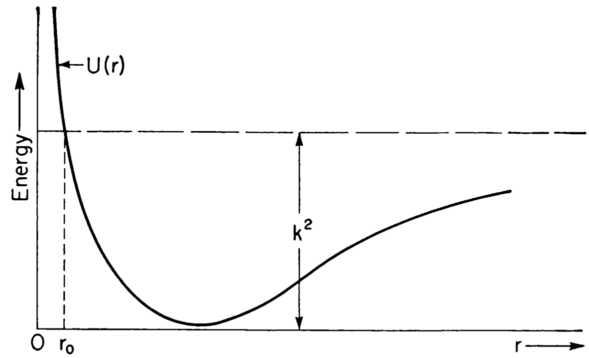

Fig. 9.18 Computation of phase shifts for scattering calculations.

be zero so that $\chi(x)$ must be zero at $x = -\infty$. Hence in Eq. (9.3.92) A must be placed equal to B so that

$$\chi \to \sqrt{\frac{2}{\pi q_r}}\, 2A \cos(\tfrac{1}{6}\pi) \cos[w - \tfrac{1}{4}\pi]; \quad r \gg r_0$$

We obtain the phase shift by comparing the argument of the cosine with and without the presence of the potential $U(r)$. We obtain

$$\eta_l = \int_{r_1}^{\infty} \sqrt{k^2 - (l + \tfrac{1}{2})^2(1/r^2)}\, dr - \int_{r_0}^{\infty} \sqrt{k^2 - U(r) - (l + \tfrac{1}{2})^2(1/r^2)}\, dr \quad (9.3.109)$$

where $r_1 = (l + \frac{1}{2})/k$. We may expect this result to be useful for large phase shifts. Small phase shifts may be estimated by the Born approximation [Eq. (9.3.36)] so that, by the use of the Born and WKBJ method, it is possible to obtain a rapid estimate of the phase shifts for all l. A comparison of some calculations of exact phase shifts with the WKBJ approximation Eq. (9.3.109) is given in the table below (taken from Mott and Massey, see Bibliography at the end of this chapter). We observe

Phase η

l	Exact	WKBJ
0	−9.696	−9.597
1	−7.452	−7.540
2	−4.469	−4.505
3	−1.238	−1.355
4	−0.445	−0.535
5	−0.143	−0.174

that the WKBJ approximation is excellent as long as the phase shift is of the order of 1 radian or larger. Actually the Born approximation does very well for those phases which are much less than unity.

Case of Closely Spaced Classical Turning-Points. The above analysis may be readily extended to situations in which there are more than two turning points as long as these are well separated, so that the appropriate asymptotic form of Eq. (9.3.88) may be employed. We now consider the situation in which the turning points are so close that this is impossible. The discussion is limited to situations in which only two turning points are close. This restriction does not in any way limit the generality of our results, since the more complex situation with more than two close turning points may be treated by considering each neighboring pair in turn.

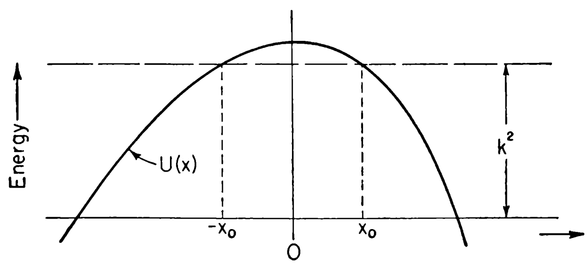

Fig. 9.19 Penetration of barrier of small height and width.

The attack will proceed as follows. For convenience, we take the origin halfway between the zeros, which then have coordinates $\pm x_0$ (see Fig. 9.19). We then find a solution of the differential equation valid in the region $(-x_0 < x < x_0)$. This solution may then be joined on to a linear combination of Bessel functions of order $\frac{1}{3}$, as given by Eq. (9.3.88), which is valid for $(x > x_0)$, and to another linear combination, valid for $(x < -x_0)$. Asymptotic forms may then be readily employed for each combination and the constants adjusted to satisfy boundary conditions.

In the region $(-x_0 < x < x_0)$ the function q^2 may be approximated by a parabola:

$$q^2 = b(x - x_0)(x + x_0)$$

so that the differential equation is

$$(d^2\psi/dx^2) + b(x^2 - x_0^2)\psi = 0 \qquad (9.3.110)$$

To reduce this equation to a standard form, we introduce the independent variable

$$\zeta = b^{\frac{1}{4}}x \qquad (9.3.111)$$

whereupon ψ satisfies

$$(d^2\psi/d\zeta^2) + (\zeta^2 - \sqrt{b}\, x_0^2)\psi = 0$$

This type of equation occurs when the scalar Helmholtz equation is separated in parabolic coordinates and is therefore discussed in some detail in Chap. 11. Referring to the tables at the end of that chapter, we see that the two independent solutions are $He(-\sqrt{b}\,x_0^2,\zeta)$ and $Ho(-\sqrt{b}\,x_0^2,\zeta)$. We shall employ the first two terms of the power series for these functions, since x_0 is assumed to be small. Hence

$$\psi \simeq \alpha[1 + \tfrac{1}{2}bx_0^2x^2 - \tfrac{1}{12}bx^4] + \beta x[1 + \tfrac{1}{6}bx_0^2x^2 - \tfrac{1}{20}bx^4] \quad (9.3.112)$$

where α and β are constants. To join on to the Bessel function approximation at $\pm x_0$, we need the value and slope of ψ at $x = \pm x_0$. For $x = x_0$

$$\begin{aligned} \psi(x_0) &\simeq \alpha[1 + \tfrac{5}{12}bx_0^4] + \beta x_0[1 + \tfrac{7}{60}bx_0^4] \\ \psi'(x_0) &\simeq \alpha(\tfrac{2}{3}bx_0^3) + \beta[1 + \tfrac{1}{4}bx_0^4] \end{aligned} \quad (9.3.113)$$

For $x = -x_0$, simply replace x_0 by $-x_0$ in the above expressions. (This remark will also hold for the equations given below.) Convergence of these expressions is roughly determined by the parameter bx_0^4. Now for $x > x_0$, let

$$\psi = \sqrt{(w/q)}\,[AJ_{\frac{1}{3}}(w) + BJ_{-\frac{1}{3}}(w)]$$

whose behavior near x_0 is indicated by Eqs. (9.3.85) and (9.3.86). Hence, for x near x_0,

$$\psi \to A[(a/3)^{\frac{1}{3}}/\Gamma(\tfrac{4}{3})](x - x_0) + B[(a/3)^{-\frac{1}{3}}/\Gamma(\tfrac{2}{3})]$$

where $q^2 = a^2(x - x_0)$ near x_0 and $a^2 = 2bx_0$. Joining these two forms (9.3.112) and (9.3.114) gives rise to the equations

$$\begin{aligned} B[(a/3)^{-\frac{1}{3}}/\Gamma(\tfrac{2}{3})] &= \alpha[1 + \tfrac{5}{12}bx_0^4] + \beta x_0[1 + \tfrac{7}{60}bx_0^4] \\ A[(a/3)^{\frac{1}{3}}/\Gamma(\tfrac{4}{3})] &= \alpha[\tfrac{2}{3}bx_0^3] + \beta[1 + \tfrac{1}{4}bx_0^4] \end{aligned} \quad (9.3.114)$$

When α and β are known, B and A may be determined from the above equation. When B and A are known, we must invert the equations, obtaining

$$\begin{aligned} \alpha &= x_0[B(a/3)^{-\frac{1}{3}}/\Gamma(\tfrac{2}{3})][1 + \tfrac{7}{60}bx_0^4] - [A(a/3)^{\frac{1}{3}}/\Gamma(\tfrac{4}{3})][1 + \tfrac{1}{4}bx_0^4] \\ \beta &= [A(a/3)^{\frac{1}{3}}/\Gamma(\tfrac{4}{3})][1 + \tfrac{5}{12}bx_0^4] - [B(a/3)^{-\frac{1}{3}}/\Gamma(\tfrac{2}{3})](\tfrac{2}{3}bx_0^3) \end{aligned} \quad (9.3.115)$$

To obtain the results appropriate near $x = -x_0$, we need only replace x_0 by $-x_0$ and A and B by, say, A' and B' in expressions (9.3.114) and (9.3.115).

We now have enough data to obtain approximate wave functions as well as reflection and transmission coefficients for k^2 near the maximum of $U(x)$. We shall outline the method, leaving the details to the reader. Suppose that a wave is incident from the left on the barrier illustrated in Fig. 9.19. Then the boundary condition for $(x \gg x_0)$ requires the presence there of a transmitted wave only. From the asymptotic form of ψ [Eq. (9.3.91) is valid there] the ratio of A to B is determined so that this boundary condition is satisfied. Then Eq. (9.3.115) is employed to determine α and β. Equation (9.3.114) can then be used to determine

A' and B' (remembering to replace x_0 by $-x_0$) and finally the asymptotic form of

$$\sqrt{(w/q)}\,[A'J_{\frac{1}{3}}(w) + B'J_{-\frac{1}{3}}(w)]$$

where
$$w = \int_x^{-x_0} q\,dx$$

yields the amplitude of the incident wave.

This procedure breaks down when x_0 and $-x_0$ actually coalesce, since then the linear approximation has no region of validity. However, the equation

$$(d^2\psi/dx^2) + cx^2\psi = 0$$

has a solution which may be extended to larger values of x by exactly the same procedure in which solutions (9.3.85) and (9.3.86) gave rise to Eq. (9.3.88). We obtain

$$\psi \simeq \sqrt{(w/q)}\,[AJ_{\frac{1}{4}}(w) + BJ_{-\frac{1}{4}}(w)] \tag{9.3.116}$$

The treatment of this approximation is not different in any essential respect from the $J_{\pm\frac{1}{3}}$ approximation, so that we shall not repeat it here.

This concludes our discussion of the WKBJ method. We should like to emphasize again the requirement that U be slowly varying, which applies throughout the discussion just completed. However, even when this condition is not satisfied, the WKBJ solutions often form convenient starting functions for an iterative process, which can be based, for example, on the integral equation satisfied by ψ.

Short-wavelength Approximation in Three Dimensions. The above analysis can be applied to two- and three-dimensional problems. It has been carried out, however, to only a limited extent. We shall content ourselves with showing that the Schroedinger-type equation

$$[\nabla^2 + q^2(\mathbf{r})]\psi = 0 \tag{9.3.117}$$

reduces for short wavelengths to an equation which, for optical and acoustical applications of Eq. (9.3.117), is appropriate for geometrical optics, *i.e.*, for ray theory. All diffraction effects therefore vanish in this limit, as might be expected. When (9.3.117) is a Schroedinger equation describing the motion of particles, the short-wavelength limit gives rise to the Hamilton-Jacobi equation [see Eqs. (3.2.12) *et seq.*], which is just a particular form of Newtonian mechanics. Our main application of these results will be found in Sec. 11.4, where the predictions of geometric optics are used as a first approximation in the wave theory.

Again, the fundamental assumption, that $q(\mathbf{r})$ does not vary rapidly within a wavelength, is made. Stated more positively, if q does vary rapidly in some region, diffraction effects will be important as long as the wavelength is not much smaller than the region. If q were a constant, the function

$$\exp[iq(\mathbf{a}_k \cdot \mathbf{r})]$$

where $\mathbf{a}_k$ is a unit vector in the direction of propagation, would be a solution of Eq. (9.3.117). For slowly varying q, this result suggests the following form for ψ:

$$\psi = e^{iw(\mathbf{r})} \tag{9.3.118}$$

Substituting in Eq. (9.3.117) leads to the following equation for the quantity w:

$$(\nabla w)^2 - i\nabla^2 w = q^2$$

Approximately, for slowly varying q,

$$(\nabla w)^2 = q^2 \tag{9.3.119}$$

This is just the Hamilton-Jacobi equation (3.2.12) when

$$q^2 = (2m/\hbar^2)(E - V)$$

and $w = S/\hbar$, where S is the action integral, thus proving the statement that in the short-wavelength limit the Schroedinger equation goes over to the equations of classical dynamics. As has been already shown in Chap. 3, the lines orthogonal to the constant w surfaces form the possible trajectories.

To show that Eq. (9.3.119) follows from geometric optics, we need only recall the principle of least action

$$\delta \int_{t_0}^{t_1} \sqrt{2mT}\, dt = \delta \int_{\mathbf{r}_0}^{\mathbf{r}_1} |\nabla S|\, ds = 0 \tag{9.3.120}$$

where S is the action integral, T the kinetic energy, ds a line element, and δ signifies variation. The variations are subject to the condition that energy be conserved. Equation (9.3.119) is a consequence of this variational principle. It is also a statement of Fermat's principle, as may be seen by replacing $|\nabla S|$ in Eq. (9.3.120) by q according to (9.3.119). We drop a constant factor to obtain

$$\delta \int_{\mathbf{r}_0}^{\mathbf{r}_1} q\, ds = 0$$

Hence q is proportional to the index of refraction. This equation is the statement that a correct ray trajectory is that one for which the optical path length is stationary.

9.4 *Variational Methods*

When the perturbation term is large, the perturbation methods described in the preceding sections become tedious and the physical meaning of the results is beclouded by the complexity of the expressions which are developed. In a case of this kind it is more fruitful to employ the variational method, for, as we shall see, it permits the exploitation of any information bearing on the problem such as might be available from purely intuitional considerations. There are, of course, limitations which we shall point out as the discussion proceeds.

In Chap. 3 we saw how the equations of physics follow from variational principles which are expressed in terms of the Lagrangian density $L[\psi, (\partial\psi/\partial x_1, \ldots, (\partial\psi/\partial x_n)]$:

$$\delta J = \delta \int \cdots \int L[\psi, (\partial\psi/\partial x_1), \ldots, (\partial\psi/\partial x_n)]\, dx_1 \cdots dx_n = 0 \quad (9.4.1)$$

The resulting equation of motion is

$$\frac{\partial L}{\partial \psi} = \sum_i \frac{\partial}{\partial x_i}\left[\frac{\partial L}{\partial(\partial\psi/\partial x_i)}\right] \quad (9.4.2)$$

The content of Eq. (9.4.1) may be stated in words as follows: Insert into the integral J in Eq. (9.4.1) all functions φ which satisfy the boundary and initial conditions satisfied by ψ. These functions are called *trial functions.* Then those trial functions which differ by a small quantity $\delta\varphi$ from the correct function ψ will have values of J which differ from the true value of J by amounts proportional to $(\delta\varphi)^2$, so that the differences δJ between these values and the true J will be small to the second order in $\delta\varphi$. If the trial function has parameters $(\alpha_1,\alpha_2, \ldots)$ which regulate its shape, then J for φ will be a function of the α's and will have a stationary value (*i.e.*, will be a maximum or minimum or saddle point) for those values of the α's for which φ is a solution of (9.4.2). In the neighborhood of the stationary values, J is naturally less sensitive to the details of the trial function than it is elsewhere.

The variational method attempts to carry out the program of evaluating J for all possible trial functions by employing a trial function containing one or more parameters referred to as *variational parameters,* with values which are to be determined. The particular form used is suggested by some a priori guesses as to the nature of the exact solution. Let these parameters be α_i. Then

$$J = J(\alpha_1 \cdots \alpha_s) \quad (9.4.3)$$

The condition that the integral be stationary leads then to the simultaneous equations

$$\partial J/\partial \alpha_i = 0 \qquad i = 1, \ldots, s \quad (9.4.4)$$

Solution of these equations determines the various possible values of α_i and therefore the best possible trial function of the form assumed. The accuracy of the result is increased by including more parameters in the trial function, thereby increasing its flexibility and thus its ability to represent the exact ψ.

It is very often possible to arrange the form of the variational principle so that the value of J for the exact ψ has a physical significance. For example, the Lagrangian density appropriate for the Laplace equation is

$$L = (\nabla\psi)^2$$

In electrostatics, $\nabla\psi = -\mathbf{E}$, where $\mathbf{E}$ is the electric field. Hence L is proportional to the electrostatic energy density and J to the total electrostatic energy. For two conductors with a fixed potential difference V_0, this energy is also $\frac{1}{2}CV_0^2$, where C is the capacity. More precisely,

$$\int(\nabla\psi)^2\,dV = 4\pi CV_0^2$$

The variational principle

$$\delta[(1/4\pi V_0^2)\int(\nabla\psi)^2\,dV] = 0$$

is referred to as the variational principle for C and written

$$\delta[C] = 0$$

where a bracket is placed around C to indicate that the quantity being varied is not C. It, of course, equals C only if the exact ψ is inserted into $[C]$.

Because of the stationary character of the variational integral (in this case, it has a minimum at the exact ψ) and its consequent insensitivity to the errors in the trial function, it is very often possible to obtain excellent estimates for C with a relatively crude trial function. This is clearly a property of great practical importance. Very often we are interested mainly in a single quantity, such as the resonant frequency, the binding energy, scattering phase shift, or reflection coefficient. It is possible to form variational principles for these quantities similar to that just discussed for the capacity. As in the first case, it is then possible to obtain accurate estimates of these quantities, employing fairly crude trial functions and thus without having to obtain a complete solution of the equation.

In most formulations, the original problem is, if necessary, reduced to an eigenvalue problem where the eigenvalue is the quantity whose value is desired. Therefore we shall first discuss the formation of variation principles for eigenvalue problems.

Variational Principle for Eigenvalue Problems. We shall employ the general operator notation, not only because of its generality but also because it reveals most clearly the technique employed in forming the variational principle. The eigenvalue equation to be discussed is

$$\mathfrak{L}(\psi) = \lambda\mathfrak{M}(\psi) \tag{9.4.5}$$

where $\mathfrak{L}$ and $\mathfrak{M}$ are differential or integral operators. We now assert that the following is a variational principle for λ:

$$\delta[\int\varphi\mathfrak{L}(\psi)\,dV/\int\varphi\mathfrak{M}(\psi)\,dV] = \delta[\lambda] = 0 \tag{9.4.6}$$

The function φ is to be defined in a moment. The integration is over all the volume determined by the independent variable upon which ψ and also φ depend. The manner in which this equation is obtained is obvious. The function φ, as yet arbitrary, was multiplied through Eq. (9.4.5); the integration performed. The resulting equation is then solved for λ. It is thus immediately clear that, if the exact ψ is inserted into $[\lambda]$, the exact λ will be obtained.

To show that Eq. (9.4.5) follows from (9.4.6), consider the equation

$$[\lambda]\int \varphi \mathfrak{M}(\psi)\, dV = \int \varphi \mathfrak{L}(\psi)\, dV$$

Now performing the variation, varying φ and λ,

$$\delta[\lambda]\int \varphi \mathfrak{M}(\psi)\, dV + [\lambda]\int \delta\varphi \mathfrak{M}(\psi)\, dV = \int \delta\varphi \mathfrak{L}(\psi)\, dV$$

Inserting the condition $\delta[\lambda] = 0$ and replacing $[\lambda]$ by λ elsewhere (since the effect of the variation is only calculated to first order), we obtain

$$\int \delta\varphi[\mathfrak{L}(\psi) - \lambda\mathfrak{M}(\psi)]\, dV = 0$$

Since $\delta\varphi$ is arbitrary, Eq. (9.4.5) follows.

We now turn to the equation satisfied by φ, which will also be determined from the variational principle. For this purpose we rewrite the integrals in Eq. (9.4.6) as follows:

$$\begin{aligned}\int \varphi \mathfrak{L}(\psi)\, dV &= \int \tilde{\mathfrak{L}}(\varphi)\psi\, dV + \oint P(\varphi,\psi)\, dS\\ \int \varphi \mathfrak{M}(\psi)\, dV &= \int \tilde{\mathfrak{M}}(\varphi)\psi\, dV + \oint Q(\varphi,\psi)\, dS\end{aligned}$$

Here $\tilde{\mathfrak{L}}$ and $\tilde{\mathfrak{M}}$ are the adjoints of $\mathfrak{L}$ and $\mathfrak{M}$, respectively; P and Q are the corresponding bilinear concomitants [see Eqs. (5.2.10) and (7.5.4)]; while the integrals over S are over the surface bounding V. We now choose φ so that it satisfies the boundary conditions adjoint to those satisfied by ψ [see Eq. (7.5.12)], *i.e.*, so that

$$P(\varphi,\psi) = 0;\quad \text{on } S \quad \text{and} \quad Q(\varphi,\psi) = 0;\quad \text{on } S$$

Under these conditions, Eq. (9.4.6) may be rewritten:

$$\delta[\lambda] = \delta\left[\int \psi\tilde{\mathfrak{L}}(\varphi)\, dV \Big/ \int \psi\tilde{\mathfrak{M}}(\varphi)\, dV\right] = 0$$

Therefore φ satisfies the equation

$$\tilde{\mathfrak{L}}(\varphi) = \lambda\tilde{\mathfrak{M}}(\varphi) \tag{9.4.7}$$

Thus φ satisfies the equation and boundary conditions which are the adjoints of those satisfied by ψ. Because it is the adjoint solution, we shall denote it by $\tilde{\psi}$, so that variational principle (9.4.6) reads

$$\delta\left[\int \tilde{\psi}\mathfrak{L}(\psi)\, dV \Big/ \int \tilde{\psi}\mathfrak{M}(\psi)\, dV\right] = 0$$

We note that, if $\mathfrak{L}$ is Hermitian, $(\bar{\tilde{\mathfrak{L}}}) = \mathfrak{L}$, then $\tilde{\psi} = \bar{\psi}$. If $\mathfrak{L}$ and $\mathfrak{M}$ are self-adjoint, it follows that $\varphi = \psi$ and the variational principle assumes the simpler form

$$\delta[\lambda] = \delta\left[\int \psi\mathfrak{L}(\psi)\, dV \Big/ \int \psi\mathfrak{M}(\psi)\, dV\right] = 0;\quad \mathfrak{L} \text{ and } \mathfrak{M} \text{ self-adjoint} \tag{9.4.8}$$

Other variational principles for λ may be obtained; there may be many equivalent ways of formulating the same problem. For example, for every differential equation plus boundary conditions, it is usually possible to formulate an equivalent integral equation. We shall illustrate this remark by setting up some additional variational principles which will be useful in our later discussions.

Consider first the case where $\mathfrak{L}$ is an operator such that no ψ exists for which $\mathfrak{L}\psi = 0$. This would be the case if $\mathfrak{L}$ were positive definite, for example. It is then possible to write

$$\psi = \lambda\mathfrak{L}^{-1}\mathfrak{M}(\psi) \tag{9.4.9}$$

where $\mathfrak{L}^{-1}$ is the inverse of $\mathfrak{L}$. If $\mathfrak{L}$ is a differential operator, this is the equivalent integral equation. For the self-adjoint case, *i.e.*, where $\mathfrak{L}$ and $\mathfrak{M}$ are self-adjoint, form (9.4.9) is not convenient, since $\mathfrak{L}^{-1}\mathfrak{M}$, whose adjoint is $\mathfrak{M}\mathfrak{L}^{-1}$, is not necessarily self-adjoint. This difficulty may be easily overcome by operating on both sides of Eq. (9.4.9) by $\mathfrak{M}$:

$$\mathfrak{M}(\psi) = \lambda\mathfrak{M}\mathfrak{L}^{-1}\mathfrak{M}(\psi) \tag{9.4.10}$$

For self-adjoint $\mathfrak{M}$ and $\mathfrak{L}$, the operator $\mathfrak{M}\mathfrak{L}^{-1}\mathfrak{M}$ is also self-adjoint.

The variational principle following from (9.4.10) is

$$\delta[\lambda] = \delta[\int\tilde{\psi}\mathfrak{M}(\psi)\,dV/\int\tilde{\psi}\mathfrak{M}\mathfrak{L}^{-1}\mathfrak{M}(\psi)\,dV] = 0 \tag{9.4.11}$$

for the general case where either $\mathfrak{L}$ or $\mathfrak{M}$ or both are not self-adjoint. The function $\tilde{\psi}$ satisfies, then,

$$\tilde{\mathfrak{M}}(\tilde{\psi}) = \tilde{\mathfrak{M}}\tilde{\mathfrak{L}}^{-1}\tilde{\mathfrak{M}}(\tilde{\psi}) \tag{9.4.12}$$

In the self-adjoint case

$$\delta[\lambda] = \delta[\int\psi\mathfrak{M}(\psi)\,dV/\int\psi\mathfrak{M}\mathfrak{L}^{-1}\mathfrak{M}(\psi)\,dV] = 0 \tag{9.4.13}$$

When solutions ψ_0 exist such that $\mathfrak{L}\psi_0 = 0$, Eq. (9.4.9) must be replaced by

$$\psi = A\psi_0 + \lambda\mathfrak{L}^{-1}\mathfrak{M}(\psi) \tag{9.4.14}$$

where A is a constant. This may be readily verified by operating on both sides of the equation with $\mathfrak{L}$. The constant A is different from zero if $\mathfrak{L}^{-1}$ has been chosen so that the term $\lambda\mathfrak{L}^{-1}\mathfrak{M}(\psi)$ cannot satisfy the boundary conditions which ψ must satisfy. Note the implication that $\mathfrak{L}^{-1}$ is not unique. This follows from the properties of ψ_0, since we may add to the eigenfunction representation of $\mathfrak{L}^{-1}$ terms containing ψ_0 without disturbing the relation

$$\mathfrak{L}\mathfrak{L}^{-1} = 1$$

Adding such terms will, of course, change the boundary conditions which $\mathfrak{L}^{-1}$ can meet. The statement that $\mathfrak{L}^{-1}$ is not unique has been pointed out earlier in our discussion of Green's functions, where it was shown that for a given differential operator there are many Green's functions, each satisfying different boundary conditions.

To form a variational principle leading to Eq. (9.4.14), which is similar to Eq. (9.4.13), we multiply Eq. (9.4.14) through by $\mathfrak{M}$ and also write down the adjoint equation:

$$\begin{aligned} \mathfrak{M}(\psi) &= A\mathfrak{M}(\psi_0) + \lambda\mathfrak{M}\mathfrak{L}^{-1}\mathfrak{M}(\psi) \\ \tilde{\mathfrak{M}}(\tilde{\psi}) &= A\tilde{\mathfrak{M}}(\tilde{\psi}_0) + \lambda\tilde{\mathfrak{M}}(\tilde{\mathfrak{L}})^{-1}\tilde{\mathfrak{M}}(\tilde{\psi}) \end{aligned} \qquad (9.4.15)$$

Here $\tilde{\psi}_0$ is the solution of $\tilde{\mathfrak{L}}\tilde{\psi}_0 = 0$ satisfying the boundary conditions adjoint to those satisfied by ψ_0. Our variational principle must yield both of these equations. It is

$$\delta[J] = \delta\{\int[\tilde{\psi}(\mathfrak{M} - \lambda\mathfrak{M}\mathfrak{L}^{-1}\mathfrak{M})\psi - A(\tilde{\psi}\mathfrak{M}\psi_0 + \tilde{\psi}_0\mathfrak{M}\psi)]\,dV\} = 0 \qquad (9.4.16)$$

This variational principle has the unhappy feature that it depends upon the amplitude of ψ and $\tilde{\psi}$. Since the original problem is homogeneous, the inhomogeneity of Eq. (9.4.14) is not real; it must be possible to formulate a variational principle which does not involve the amplitude of ψ. This is done by using the variational principle to determine the best value of the amplitude for a given trial function. Insert, therefore,

$$\psi = \alpha\chi; \quad \tilde{\psi} = \alpha\tilde{\chi} \qquad (9.4.17)$$

into $[J]$ and treat α as a variational parameter. $[J]$ becomes

$$[J] = \alpha^2\int[\tilde{\chi}(\mathfrak{M} - \lambda\mathfrak{M}\mathfrak{L}^{-1}\mathfrak{M})\chi]\,dV - A\alpha\int[\tilde{\chi}\mathfrak{M}\psi_0 + \tilde{\psi}_0\mathfrak{M}\chi]\,dV$$

The stationary values for J occur wherever $dJ/d\alpha = 0$, so that

$$\alpha = \{A\int(\tilde{\chi}\mathfrak{M}\psi_0 + \tilde{\psi}_0\mathfrak{M}\chi)\,dV/2\int[\tilde{\chi}(\mathfrak{M} - \lambda\mathfrak{M}\mathfrak{L}^{-1}\mathfrak{M})\chi]\,dV\} \qquad (9.4.18)$$

Inserting this value of α into J, we obtain

$$J = -(\tfrac{1}{4}A^2)\{[\int(\tilde{\chi}\mathfrak{M}\psi_0 + \tilde{\psi}_0\mathfrak{M}\chi)\,dV]^2/\int[(\tilde{\chi}\mathfrak{M} - \lambda\mathfrak{M}\mathfrak{L}^{-1}\mathfrak{M})\chi]\,dV\}$$

Since we have not yet carried out variations on the parameters in χ and $\tilde{\chi}$, this particular form for $[J]$ must still satisfy the condition $\delta J = 0$.

We may verify this by performing the variation and showing that the resultant equations for χ and $\tilde{\chi}$ may also be obtained by inserting Eqs. (9.4.17) and (9.4.18) in the original equation. For convenience, we replace the constant $-\frac{1}{4}A^2$ by $\frac{1}{2}$, calling the new quantity $[J']$:

$$[J'] = \{[\int(\tilde{\chi}\mathfrak{M}\psi_0 + \tilde{\psi}_0\mathfrak{M}\chi)\,dV]^2/2\int[\tilde{\chi}(\mathfrak{M} - \lambda\mathfrak{M}\mathfrak{L}^{-1}\mathfrak{M})\chi]\,dV\} \qquad (9.4.19)$$

The condition $\delta[J'] = 0$ leads to

$$J'[\mathfrak{M} - \lambda\mathfrak{M}(\mathfrak{L})^{-1}\mathfrak{M}]\chi = \mathfrak{M}\psi_0[\int(\tilde{\chi}\mathfrak{M}\psi_0 + \tilde{\psi}_0\mathfrak{M}\chi)\,dV] \qquad (9.4.20)$$

where J' in this equation is the value of expression (9.4.19) when the exact χ and $\tilde{\chi}$ are inserted. The equation for the adjoint $\tilde{\chi}$ is

$$J'[\tilde{\mathfrak{M}} - \lambda\tilde{\mathfrak{M}}(\tilde{\mathfrak{L}})^{-1}\tilde{\mathfrak{M}}]\tilde{\chi} = \tilde{\mathfrak{M}}\tilde{\psi}_0[\int(\tilde{\chi}\mathfrak{M}\psi_0 + \tilde{\psi}_0\mathfrak{M}\chi)\,dV] \qquad (9.4.21)$$

These equations are also obtained if the definitions of χ and $\tilde{\chi}$, Eq. (9.4.17), and the value of α, Eq. (9.4.18), are inserted into Eq. (9.4.15).

We note that both the variational principle based on $[J']$ and the equations for χ and $\tilde{\chi}$ are homogeneous; *i.e.*, if χ is a solution, so is any constant times χ. This has been achieved at the expense of introducing an additional parameter J' into the equation. Note that J' may be regarded as a parameter, since any pair of exact solutions of Eqs. (9.4.20) and (9.4.21) would automatically satisfy Eq. (9.4.19). Actually, in practice the inhomogeneous equation (9.4.14) often occurs for situations where the spectrum for λ is continuous. It then turns out that J' is the physically interesting quantity. For example, in the variational treatment of scattering, J' is a simple function of the phase shift (see page 1123).

We conclude this discussion by giving the appropriate expressions for J' and Eq. (9.4.20) when $\mathfrak{L}$ is self-adjoint. $[J']$ is then

$$[J'] = \left\{[\int \chi \mathfrak{M} \psi_0 \, dV]^2 / \int [\chi(\mathfrak{M} - \lambda \mathfrak{M} \mathfrak{L}^{-1} \mathfrak{M})\chi] \, dV\right\} \tag{9.4.22}$$

The equation satisfied by χ is

$$J'(\mathfrak{M} - \lambda \mathfrak{M} \mathfrak{L}^{-1} \mathfrak{M})\chi = \mathfrak{M}\psi_0(\int \chi \mathfrak{M} \psi_0 \, dV) \tag{9.4.23}$$

We have been very abstract here in order to reveal the technique by means of which variational principles are formed for the various cases of interest. We turn now to specific examples whose concreteness may help to clarify the above discussion.

Variational Principles for Resonant Frequencies and Energy Levels. Consider the scalar Helmholtz equation

$$\nabla^2\psi + k^2\psi = 0$$

where ψ satisfies either *homogeneous Dirichlet* or *homogeneous Neumann* conditions on a surface S; the equation itself is to hold in the volume bounded by S. The operator ∇^2, together with the above boundary conditions, is self-adjoint, so that variational principle (9.4.8) is applicable. The eigenvalue λ is placed equal to k^2 so that the operator $\mathfrak{L}$ here equals ∇^2 and therefore Eq. (9.4.8) becomes

$$\delta[k^2] = \delta\left\{-\int \psi \nabla^2 \psi \, dV / \int \psi^2 \, dV\right\}$$

The character of the numerator is made more obvious upon application of Green's theorem:

$$[k^2] = [\int (\nabla\psi)^2 \, dV / \int \psi^2 \, dV] \tag{9.4.24}$$

To illustrate the derivation of variational principles, let us verify directly that $\delta[k^2] = 0$ leads to the scalar Helmholtz equation. We have

$$\delta[k^2]\int \psi^2 \, dV + k^2 \delta(\int \psi^2 \, dV) = \delta[\int (\nabla\psi)^2 \, dV]$$

Inserting $\delta[k^2] = 0$ and performing the variation yield

$$2k^2 \int \psi \, \delta\psi \, dV - 2\int \nabla\psi \cdot \delta(\nabla\psi) \, dV = 0$$

Employing the relation $\delta\nabla\psi = \nabla(\delta\psi)$ and Green's theorem, one finds

$$\int \delta\psi(k^2\psi + \nabla^2\psi)\,dV - \int(\partial\psi/\partial n)\,\delta\psi\,dS = 0 \qquad (9.4.25)$$

If homogeneous Neumann conditions are satisfied, $(\partial\psi/\partial n = 0)$ on S; if homogeneous Dirichlet conditions are satisfied, $\delta\psi$ must be equal to zero on S; hence, for both of these boundary conditions the surface integral above vanishes, and the result that ψ satisfies the scalar Helmholtz equation follows.

Turning back to Eq. (9.4.24), we see immediately that $[k^2]$ is always greater than or equal to zero. Hence, when all possible trial solutions are inserted, the values obtained for $[k^2]$ will have an absolute minimum. The trial function ψ_0 which gives this minimum is then the exact solution, and the value of $[k^2]$ at the minimum k_0^2, the corresponding eigenvalue. This minimum will be greater than zero for homogeneous Dirichlet conditions and equal to zero for Neumann conditions (ψ equal to a constant, giving $k^2 = 0$, is allowed for Neumann conditions but is obviously unsatisfactory for the Dirichlet case). As the theory in Chap. 6 reveals, the wave functions for the other resonant frequencies are all orthogonal to ψ_0 as well as to each other. One may therefore obtain a variational principle for k_1^2, the eigenvalue greater than k_0^2 but less than k_n^2, $(n > 1)$ by restricting the admissible trial functions to those orthogonal to ψ_0, satisfying therefore the condition

$$\int \psi\psi_0\,dV = 0$$

The minimum value of expression (9.4.24) for trial wave functions subject to the above condition is then k_1^2, the corresponding eigenfunction ψ_1. This process may be continued to obtain k_2^2 by further restricting the possible trial functions to those orthogonal to ψ_0 and ψ_1. It is clear that (see page 737)

$$k_0^2 \le k_1^2 \le k_2^2 \cdots$$

Variational principle (9.4.24) is no longer valid when mixed boundary conditions

$$(\partial\psi/\partial n) + f\psi = 0$$

are satisfied on S, for then the surface term in Eq. (9.4.25) does not vanish. This difficulty may be readily removed by adding a term to the numerator of Eq. (9.4.24) which cancels the surface term when the variation is performed. This term, when varied, must give $2\int f\psi\,\delta\psi\,dS$; hence

$$[k^2] = \left\{\int(\nabla\psi)^2\,dV + \int f\psi^2\,dS\right\}\Big/\int\psi^2\,dV \qquad (9.4.26)$$

Then Eq. (9.4.25) is replaced by

$$\int\delta\psi(k^2\psi + \nabla^2\psi)\,dV - \int[f\psi + (\partial\psi/\partial n)]\,\delta\psi\,dS = 0$$

The surface term now vanishes by virtue of the boundary condition satisfied by ψ, so that $\delta[k^2] = 0$ leads to the scalar Helmholtz equation for ψ.

Before leaving this subject, it is necessary to verify that, upon substitution of the exact ψ, Eq. (9.4.26) yields the exact value of k^2. This is, of course, a consistency condition, which is met by Eq. (9.4.26), as may be seen after the usual application of Green's theorem.

The variational principle for the Schroedinger equation

$$\nabla^2\psi + [k^2 - U(r)]\psi = 0$$

is very similar to that discussed above. It is to be noted that, although the operator in the above equation is self-adjoint, the boundary conditions are not generally real, so that ψ is often complex. Therefore the operator, together with the boundary conditions, is not self-adjoint. However, it is Hermitian, so that Eq. (9.4.6) applies with $\varphi = \bar{\psi}$. Hence

$$\delta[k^2] = \delta\{\int\bar{\psi}[U - \nabla^2]\psi\,dV/\int|\psi|^2\,dV\} = 0 \tag{9.4.27}$$

The integrals here, *e.g.*, the normalization integral in the denominator, are finite only for bound states of the system, for which ψ approaches zero as r approaches infinity. Another form which may be obtained by applying Green's theorem defines $[k^2]$ as follows:

$$[k^2] = \{\int[|\nabla\psi|^2 + U|\psi|^2]\,dV/\int|\psi|^2\,dV\} \tag{9.4.28}$$

In this last form it is immediately clear, as long as the trial functions ψ are chosen so that the integral over $U|\psi|^2$ is bounded, that $[k^2]$ will have an absolute minimum. The corresponding state of the system is called the *ground state*. We may again obtain the first state with greater k^2; the first *excited state*, by restricting the trial wave function to a form orthogonal to the ground-state wave function; the second excited state, by restricting the wave functions to those orthogonal to the ground state and first excited state wave function; and so on.

Vibration of a Circular Membrane. We now give a detailed example of the application of variational principle (9.4.24) for the scalar Helmholtz equation. The problem of the vibration of a clamped circular membrane is chosen because of its algebraic simplicity. Let the radius of the membrane be a. The boundary conditions satisfied by ψ are $\psi(a) = 0$ and the over-all requirement that ψ be continuous in value and gradient inside the boundary. We shall simplify our considerations further by investigating only the circularly symmetric modes so that ψ is a function of r only. With these simplifications, Eq. (9.4.24) becomes

$$[k^2] = \left\{\int_0^a (d\psi/dr)^2 r\,dr \Big/ \int_0^a \psi^2 r\,dr\right\}$$

It is clearly convenient to introduce the dimensionless independent variable $x = r/a$. Then

$$[(ka)^2] = \left\{ \int_0^1 \left(\frac{d\psi}{dx}\right)^2 x\,dx \Big/ \int_0^1 \psi^2 x\,dx \right\} \tag{9.4.29}$$

We now consider the possible trial functions. The simplest form which vanishes at $x = 1$ and at the same time has a continuous gradient at $(x = 0)$ is $(1 - x^2)$. [The function $(1 - x)$ is not suitable, since it would have a slope of (-1) at the origin, giving rise to a cusp at that point.] Upon inserting $(1 - x^2)$ into Eq. (9.4.29), we obtain $[(ka)^2] = 6$. The exact value is 5.7836, so that even with this exceedingly crude trial function we are already fairly close to the exact result. Note that the approximate result exceeds the exact one.

To obtain a closer approximation, we shall need to improve upon the trial function. We shall present two different attacks. In the first of these we take

$$\psi = A(1 - x^2) + B(1 - x^2)^2 \tag{9.4.30}$$

as the trial function, where A and B are variational parameters. The choice is based on the fact that any function which is zero at $x = 1$ and has zero slope at $(x = 0)$ may be expanded in a power series in $(1 - x^2)$. Inserting this new trial function into Eq. (9.4.29) yields

$$\begin{aligned}\Big[A^2 \int_0^1 (1 - x^2)^2 x\,dx &+ 2AB \int_0^1 (1 - x^2)^3 x\,dx \\ &+ B^2 \int_0^1 (1 - x^2)^4 x\,dx\Big] [k^2a^2] = A^2 \int_0^1 \left[\frac{d(1 - x^2)}{dx}\right]^2 x\,dx \\ + 2AB \int_0^1 \left[\frac{d(1 - x^2)}{dx}\right]&\left[\frac{d(1 - x^2)^2}{dx}\right] x\,dx + B^2 \int_0^1 \left[\frac{d(1 - x^2)^2}{dx}\right]^2 x\,dx\end{aligned}$$

We now differentiate with respect to A and B, place $(\partial[k^2a^2]/\partial A)$ and $(\partial[k^2a^2]/\partial B)$ equal to zero, in accordance with the requirements of the variational principle [see Eq. (9.4.4)]. One finds

$$\begin{aligned}A \left\{[k^2a^2] \int_0^1 (1 - x^2)^2 x\,dx - \int_0^1 \left[\frac{d(1 - x^2)}{dx}\right]^2 x\,dx\right\}& \\ + B \left\{[k^2a^2] \int_0^1 (1 - x^2)^3 x\,dx - \int_0^1 \left[\frac{d(1 - x^2)}{dx}\,\frac{d(1 - x^2)^2}{dx}\right] x\,dx\right\} &= 0 \\ A \left\{[k^2a^2] \int_0^1 (1 - x^2)^3 x\,dx - \int_0^1 \frac{d(1 - x^2)}{dx}\,\frac{d(1 - x^2)^2}{dx}\Big] x\,dx\right\}& \\ + B \left\{[k^2a^2] \int_0^1 (1 - x^2)^4 x\,dx - \int_0^1 \left[\frac{d(1 - x^2)^2}{dx}\right]^2 x\,dx\right\} &= 0\end{aligned} \tag{9.4.31}$$

This is a pair of linear homogeneous equations for A and B and has a nonzero solution only if the determinant formed from the coefficients of A and

B is zero. This determinant is just the sort that arises in perturbation theory when the unknown function has been expanded in a nonorthogonal set [see Eq. (9.1.96)]. We shall see later [see Eqs. (9.4.38) *et seq.*] that this is no accident. Inserting the values of the integrals in the above, the determinantal equation becomes

$$\begin{vmatrix} \frac{1}{6}[k^2a^2] - 1 & \frac{1}{8}[k^2a^2] - \frac{2}{3} \\ \frac{1}{8}[k^2a^2] - \frac{2}{3} & \frac{1}{10}[k^2a^2] - \frac{2}{3} \end{vmatrix} = 0 \tag{9.4.32}$$

This is a quadratic equation for $[k^2a^2]$. The smaller root is 5.7837, very close to the exact value 5.7832. The ratio B/A is, from the second of the equations in Eq. (9.4.31),

$$\frac{B}{A} = -\frac{\frac{2}{3} - \frac{1}{8}[k^2a^2]}{\frac{2}{3} - \frac{1}{10}[k^2a^2]} = 0.637$$

The wave function ψ is, from Eq. (9.4.30),

$$\psi = (1 - x^2) + 0.637(1 - x^2)^2 \tag{9.4.33}$$

A comparison between this approximate result, the first trial function $(1 - x^2)$, and the exact solution $J_0(2.4048x)$ is given in the table below. We have normalized ψ to be unity at $x = 0$ by multiplying the above expression by $1/1.637$. Note that the function $(1 - x^2)$ is in error by as much as 30 per cent in some regions; the eigenvalue it predicts, $k^2a^2 = 6$, is in error by only 3 per cent.

The second root of the determinantal equation (9.4.32) is also significant. The important point here is that the approximate wave function corresponding to this second root is orthogonal to the wave function determined by the first root and given in Eq. (9.4.33).

Eigenfunctions for Circular Membrane

x	Exact	Eq. (9.4.33)	$1 - x^2$
0.1	0.986	0.986	0.990
0.2	0.943	0.944	0.960
0.3	0.875	0.878	0.910
0.4	0.783	0.788	0.840
0.5	0.671	0.677	0.750
0.6	0.545	0.550	0.640
0.7	0.410	0.413	0.510
0.8	0.270	0.270	0.360
0.9	0.133	0.130	0.190
1.0	0	0	0

Hence, from the discussion following Eq. (9.4.28), we see that this second root and its corresponding trial function approximate the second solution of the original problem where, of course, we are still restricting the discussion to modes with circular symmetry. The first solution with the lowest eigenvalue has no nodal line; the second solution with the next highest eigenvalue has one nodal line. The exact value for this second eigenvalue is $k^2a^2 = 27.3$, while the second root of Eq. (9.4.32) is 36.9. The error is considerable, indicating that we shall require a much more complicated dependence than that provided by Eq. (9.4.30) before a satisfactory result can be obtained. The higher modes usually prove more difficult to compute by the variational method, because these modes oscillate more rapidly.

Nonlinear Variational Parameters. Trial wave function (9.4.30) is linear in its variational parameters. It is, of course, possible for the parameters to occur nonlinearly; *e.g.*, the following is a suitable trial function for the circular membrane:

$$\psi = \cos \alpha - \cos \alpha x$$

Here α is the variational parameter to be determined by the variational method. We shall not carry this calculation out, because the equation for α is rather complex. There is no particular advantage in using nonlinear parameters in this present problem, but in many cases it will be quite useful to do so. To illustrate, consider the Schroedinger equation

$$(d^2\psi/dx^2) + [-\alpha^2 + U_0 e^{-x^2}]\psi = 0; \quad -\infty < x < \infty$$

The variational principle reads

$$[\alpha^2] = -\left\{\int[(d\psi/dx)^2 - U_0 e^{-x^2}\psi^2]\,dx\Big/\int\psi^2 dx\right\}$$

We insert the trial wave function $e^{-(\beta x^2/2)}$ and find

$$[\alpha^2] = -[(\beta/2) - U_0\sqrt{\beta/(\beta+1)}]$$

The equation determining β, $(\partial[\alpha^2]/\partial\beta) = 0$, reduces to

$$\beta^{\frac{1}{2}}(\beta+1)^{\frac{3}{2}} = U_0$$

For this value of β

$$[\alpha^2] = \beta(\beta + \tfrac{1}{2})$$

This result is fairly simple; in this case the nonlinear form is easier to compute than some linear combination of trial functions.

Rayleigh-Ritz Method. The variational method yields a result which, for a positive definite operator, is always greater than the correct answer. In order to determine the accuracy of a variational calculation, it is necessary to have a systematic procedure for improving on the original trial function. The method of inserting additional nonlinear parameters will certainly increase the accuracy, but it is not certain that it will ultimately give the correct answer, since it is not clear that all possible functions, with all kinds of symmetry and behavior, will have been included. Another procedure which circumvents this difficulty involves the insertion of a linear combination of a complete set of functions, the coefficients forming a set of linear variational parameters. For example, in the case of the vibration of a circular membrane we should take

$$\psi = \sum_{n=0}^{\infty} A_n(1 - x^2)^n$$

For the Schroedinger equation discussed above

$$\psi = \sum_{n=0}^{\infty} A_n x^n e^{-(\beta x^2/2)}$$

would be appropriate.

We may, of course, prefer to employ orthogonal sets. For example, in the second case one could readily expand in Hermite polynomials:

$$\psi = \sum_{n=1}^{\infty} B_n H_n(\sqrt{\beta}\, x) e^{-(\beta x^2/2)} \tag{9.4.34}$$

These trial functions lead to an infinite system of equations for the coefficients A_n or B_n, which we shall now find explicitly. Let the trial wave function be

$$\psi = \Sigma A_n \phi_n \tag{9.4.35}$$

The equation to be solved is the general one (9.4.5), with $\mathfrak{L}$ and $\mathfrak{M}$ self-adjoint. We shall employ the variational principle for λ in the form

$$\delta J = \delta\{\int \psi \mathfrak{L}(\psi)\, dV - [\lambda] \int \psi \mathfrak{M}(\psi)\, dV\} = 0; \quad \delta[\lambda] = 0$$

Inserting trial wave function (9.4.34), we find for J

$$J = \sum_{nm} A_n A_m \{L_{nm} - [\lambda] M_{nm}\} = 0$$

where

$$L_{nm} = \int \phi_n \mathfrak{L}(\phi_m)\, dV; \quad M_{nm} = \int \phi_n \mathfrak{M}(\phi_m)\, dV \tag{9.4.36}$$

If the ϕ_n's are orthogonal with respect to the operator $\mathfrak{M}$, M_{nm} will be zero unless $n = m$. The variational method requires that $\partial J/\partial A_i = 0$. Hence

$$\sum_j A_j[L_{ji} - \lambda M_{ji}] = 0; \quad \text{for each } i \tag{9.4.37}$$

We have made use of the self-adjoint properties of $\mathfrak{L}$ and $\mathfrak{M}$ and have dropped the bracket about λ, since the solution of these equations must yield the exact value of λ. The solution of Eq. (9.4.37) is possible only if the determinant of the coefficients of A_j is zero:

$$|L_{ji} - \lambda M_{ji}| = 0 \tag{9.4.38}$$

Equation (9.4.38) is just the familiar secular determinantal equation discussed in Sec. 9.1. In that discussion, the close approximation of the first term is known in advance. In the variational method, the first term, as well as some of the characteristic parameters [*e.g.*, the parameter β in series (9.4.34)] are determined variationally.

The solution of secular equation (9.4.38) may proceed by the methods outlined in Sec. 9.1. However, often the first term ϕ_0 involves a sufficient number of variational parameters so that one expects it to be very close to the correct answer. Then the procedure which is generally followed involves taking the first few terms of the series (9.4.35) and solving the secular equation exactly, increasing the number of terms and solving again, and so on, stopping the calculation whenever the value of λ does not change with the addition of more terms. The final value of λ is then obtained by extrapolation. This procedure is, of course, often dictated by practical considerations. It is, of course, dangerous, in view of the absence of any rigorous establishment of convergence. The literature contains many examples of the appearance of false convergence limits occurring presumably because of the systematic omission of particular types of terms. It is therefore important to develop a precise estimate of the error in a variational calculation, as we shall do later in this section.

Application to Perturbation Theory. By employing the wave functions developed by perturbation theory as trial wave functions, some further insight into that theory may be obtained. Let χ_m be the normalized eigenfunctions of the operator $\mathfrak{L}_0$ satisfying

$$\mathfrak{L}_0\chi_m = \lambda_m\mathfrak{M}\chi_m; \quad \int\chi_n\mathfrak{M}\chi_m\,dV = \delta_{nm}$$

Then the solution ψ of the problem

$$\mathfrak{L}\psi = \lambda\mathfrak{M}\psi; \quad \mathfrak{L} = \mathfrak{L}_0 + \mathfrak{L}_1$$

is to first order

$$\psi = \chi_0 + \sum_{m\neq 0} \frac{L_{m0}^{(1)}}{\lambda - \lambda_m}\chi_m; \quad L_{m0}^{(1)} = \int \chi_m\mathfrak{L}_1\chi_0\,dV$$

The variational principle (9.4.8) requires the evaluation of the ratio

$$[\lambda] = [\int\psi\mathfrak{L}\psi\,dV/\int\psi\mathfrak{M}\psi\,dV]$$

If as trial wave function we employ just χ_0, we obtain

$$[\lambda] = \lambda_0 + L_{00}^{(1)}$$

This is just the first-order perturbation result. From the variational principle and for $\mathfrak{L}$ self-adjoint and positive definite, we may immediately conclude that

$$\lambda \leq (\lambda_0 + L_{00}^{(1)}) \tag{9.4.39}$$

Also note that this first-order estimate of λ is obtained with a trial function correct to zero order, bringing out again the relative lack of sensitivity of the variational expression for $[\lambda]$ to errors in the wave function.

Let us now insert the wave function which is correct to first order. Then

$$[\lambda] = \left\{ \frac{\lambda_0 + L_{00}^{(1)} + 2\sum_m \frac{(L_{m0}^{(1)})^2}{\lambda - \lambda_m} + \sum_m \frac{\lambda_m (L_{m0}^{(1)})^2}{(\lambda - \lambda_m)^2} + \sum_{mn} \frac{L_{0n}^{(1)} L_{nm}^{(1)} L_{m0}^{(1)}}{(\lambda - \lambda_m)(\lambda - \lambda_n)}}{1 + \sum_m \frac{(L_{m0}^{(1)})^2}{(\lambda - \lambda_m)^2}} \right\}$$

All the sums in the above expression omit the terms in which either or both m and n equal zero. Evaluating the above ratio to third order in the perturbation, we obtain

$$[\lambda] = \lambda_0 + L_{00}^{(1)} + \sum_{m \neq 0} \frac{(L_{m0}^{(1)})^2}{\lambda + L_{00}^{(1)} - \lambda_m} + \sum_{m,\, n \neq 0} \frac{L_{0n}^{(1)} L_{nm}^{(1)} L_{m0}^{(1)}}{(\lambda - \lambda_m)(\lambda - \lambda_n)} \tag{9.4.40}$$

This is just the perturbation result, Eq. (9.1.15), correct to third order. Note we have obtained it with a wave function correct to first order only. If we went on to use a wave function correct to second order, we should obtain a result correct to the fifth order, and so on. The odd-order perturbation results for λ may thus be obtained from the variational principle, and they must therefore give values which are larger than the exact value of λ. It is not possible to make a corresponding definite statement for the even orders.

Integral Equation and Corresponding Variational Principle. We shall now illustrate variational principle (9.4.13) and Eq. (9.4.9) for ψ, involving the inverse operator $\mathfrak{L}^{-1}$. We consider again the scalar Helmholtz equation, which we write in the more suggestive form

$$\nabla^2 \psi = -k^2 \psi$$

We consider a bounded domain upon which ψ satisfies homogeneous boundary conditions. An integral equation for ψ may be immediately obtained by employing the Green's function $G_0(\mathbf{r}|\mathbf{r}_0)$:

$$\nabla^2 G_0(\mathbf{r}|\mathbf{r}_0) = -4\pi\delta(\mathbf{r} - \mathbf{r}_0)$$

We choose that Green's function which satisfies the same homogeneous boundary conditions as those satisfied by ψ. Then,

$$\psi(\mathbf{r}) = \frac{k^2}{4\pi} \int G_0(\mathbf{r}|\mathbf{r}_0)\psi(\mathbf{r}_0)\, dv_0 \tag{9.4.41}$$

This is the equation which corresponds to Eq. (9.4.9). The inverse operator $\mathfrak{L}^{-1}$ is $-(1/4\pi)\int G(\mathbf{r}|\mathbf{r}_0)$. Variational principle (9.4.13) is then

$$\delta[k^2] = \delta\left[\frac{\int \psi^2(\mathbf{r})\,dv}{\dfrac{1}{4\pi}\displaystyle\iint \psi(\mathbf{r})G_0(\mathbf{r}|\mathbf{r}_0)\psi(\mathbf{r}_0)\,dv\,dv_0}\right] = 0 \tag{9.4.42}$$

Let us now verify directly that this variational principle leads to Eq. (9.4.41). For this purpose, it is simpler to consider the following variation:

$$\delta\left\{\frac{[k^2]}{4\pi}\iint \psi(\mathbf{r})G_0(\mathbf{r}|\mathbf{r}_0)\psi(\mathbf{r}_0)\,dv\,dv_0 - \int \psi^2(\mathbf{r})\,dv\right\} = 0$$

Recalling that $\delta[k^2] = 0$, then

$$\frac{k^2}{4\pi}\iint [\delta\psi(\mathbf{r})\,\psi(\mathbf{r}_0) + \psi(\mathbf{r})\,\delta\psi(\mathbf{r}_0)]G_0(\mathbf{r}|\mathbf{r}_0)\,dv\,dv_0 - 2\int \psi\,\delta\psi\,dv = 0$$

To obtain a common factor of $\delta\psi$, it is necessary to invert the order of integration for the term involving $\delta\psi(\mathbf{r}_0)$. For this to be possible without the appearance of additional terms, it is necessary that G_0 be a symmetrical function of $\mathbf{r}$ and $\mathbf{r}_0$ and that the limits of integration be definite. These conditions are, of course, met in the present problem, so that

$$\iint\psi(\mathbf{r})\,\delta\psi(\mathbf{r}_0)\,G_0(\mathbf{r}|\mathbf{r}_0)\,dv\,dv_0 = \iint\psi(\mathbf{r}_0)\,\delta\psi(\mathbf{r})\,G_0(\mathbf{r}|\mathbf{r}_0)\,dv\,dv_0$$

Hence

$$\int \delta\psi(\mathbf{r})\left[\frac{k^2}{4\pi}\int G_0(\mathbf{r}|\mathbf{r}_0)\psi(\mathbf{r}_0)\,dv_0 - \psi(\mathbf{r})\right]dv = 0$$

which, of course, leads to integral equation (9.4.41).

Variational principle (9.4.42) also applies to the Schroedinger equation, where G_0 is defined by

$$\nabla^2 G_0(\mathbf{r}|\mathbf{r}_0) - U(\mathbf{r})G_0(\mathbf{r}|\mathbf{r}_0) = -4\pi\delta(\mathbf{r} - \mathbf{r}_0)$$

It is not often used, because the finding of G_0 is much more difficult in the Schroedinger case. More often the integral equation for the Schroedinger equation is formulated in another way, in which the strength of the potential appears as a parameter. We therefore write the potential energy as $-\lambda f(r)$ and the Schroedinger equation as

$$\nabla^2\psi - \alpha^2\psi = -\lambda f(r)\psi$$

The corresponding integral equation is

$$\psi = (\lambda/4\pi)\int G_{i\alpha}(\mathbf{r}|\mathbf{r}_0)f(\mathbf{r}_0)\psi(\mathbf{r}_0)\,dv_0 \tag{9.4.43}$$

where

$$\nabla^2 G_{i\alpha} - \alpha^2 G_{i\alpha} = -4\pi\delta(\mathbf{r} - \mathbf{r}_0) \tag{9.4.44}$$

In many problems the $G_{i\alpha}$ appropriate to the infinite domain is required, and then

$$G_{i\alpha}(\mathbf{r}|\mathbf{r}_0) = e^{-\alpha|\mathbf{r}-\mathbf{r}_0|}/|\mathbf{r} - \mathbf{r}_0|$$

The variational principle for λ may now be obtained from Eq. (9.4.13):

$$\delta[\lambda] = \delta\left[\frac{\int f(\mathbf{r})\psi^2\,dv}{\int\int \psi(\mathbf{r})f(\mathbf{r})G_{i\alpha}(\mathbf{r}|\mathbf{r}_0)f(\mathbf{r}_0)\psi(\mathbf{r}_0)\,dv\,dv_0}\right] = 0 \qquad (9.4.45)$$

Note that this insertion of $f(\mathbf{r})$ was necessary in order that the function of $\mathbf{r}$ and $\mathbf{r}_0$ which appears in the denominator of Eq. (9.4.45) between $\psi(\mathbf{r})$ and $\psi(\mathbf{r}_0)$ be symmetric. Only then would the variation lead to the integral equation satisfied by ψ, as may be seen from the discussion following Eq. (9.4.42).

This variational principle assumes the value of α, the energy level of the system, and computes the strength, λ, of the potential required to yield the assumed α. Such a variational principle is particularly useful in situations where the law of force is as yet unknown, while the value of α is known from experiment. If, on the other hand, λ is known and α is desired, it would be necessary to employ Eq. (9.4.45) for a variety of values of α, so that the known value of λ is bracketed. The corresponding value of α would then be obtained by inverse interpolation.

Note the important practical feature of Eq. (9.4.45) that only the values of $\psi(\mathbf{r})$ within the range where f differs from zero occur. The correct dependence for very large values of r is furnished by the Green's function.

Similar variational principles may be set up for problems in the perturbation of boundary conditions. We have seen in Sec. 9.2 how that problem may be reduced to the solution of an integral equation in which only the values of ψ (or its normal derivative, depending on the type of boundary condition) on the perturbed surface occur. The variational principle would then be for a parameter measuring the size of the perturbation (corresponding to the size of the potential, in the case of the Schroedinger equation). If a variational principle for k^2 is desired, one must employ Eq. (9.4.26). We shall discuss examples of these techniques in Sec. 12.3, where the physical significance of the various terms will be discussed more fully.

An Example. We once more consider the vibration of a circular membrane and look for the resonant frequency for the lowest symmetrical mode. In that event, ψ is a function of r only. This may be inserted into Eqs. (9.4.41) and (9.4.42), or one may work directly with the ordinary differential equation satisfied by ψ. Either way one obtains the following integral equation and variational principle, which we have expressed in terms of the dimensionless variable $x = r/a$:

$$\psi = \frac{(ka)^2}{4\pi}\int_0^1 G_0(x|x_0)\psi(x_0)x_0\,dx_0 \qquad (9.4.46)$$

and

$$\delta[(ka)^2] = \delta\left\{\int_0^1 \psi^2 x\,dx \Big/ \int_0^1 \int_0^1 \psi(x)xG_0(x|x_0)x_0\psi(x)\,dx\,dx_0\right\} = 0 \quad (9.4.47)$$

Here
$$G_0(x|x_0) = -4\pi\begin{cases}\ln x; & x \ge x_0\\ \ln x_0; & x \le x_0\end{cases}$$

We now use the same trial function $(1 - x^2)$ employed in the earlier discussion. If we let ψ_1 be the function which is generated when $(1 - x^2)$ is inserted in the right-hand side of Eq. (9.4.46), we find, dropping a factor of k^2, since any constant factors drop out in the ratio in variational principle (9.4.47), that

$$\psi_1 = \tfrac{3}{16} - \tfrac{1}{4}x^2 + \tfrac{1}{16}x^4$$

The corresponding value of $(ka)^2$ from Eq. (9.4.47) is

$$[(ka)^2] = \left\{\int_0^1 \psi^2 x\,dx \Big/ \int_0^1 \psi(x)\psi_1(x)x\,dx\right\}$$

Inserting ψ and ψ_1, we find that $[(ka)^2]$ is approximately 5.8182, which is to be compared with the exact value 5.7832. Note that this approximation is considerably more accurate than the value 6 obtained when $(1 - x^2)$ is inserted directly into Eq. (9.4.29). This arises from the fact that ψ_1 is closer to the correct wave function than ψ as a consequence of the application of the Green's function operator to ψ. This procedure, as we shall see, has the effect of reducing the error in ψ and will become the basis of the variation-iteration method.

Variational Principle for Phase Shifts. The variational principles so far discussed are appropriate for problems for which the energy eigenvalues form a discrete spectrum. We have been dealing either with a finite domain upon whose surface the wave function satisfies boundary conditions or with the infinite domain, in which case the wave function must go to zero at infinity (see Sec. 12.3 for more details). We turn now to situations in which the eigenvalues form a continuous spectrum; solutions satisfying the appropriate boundary conditions exist for all values of the eigenvalue within some range of values. The wave functions are no longer quadratically integrable, so that the variational principles appropriate to bound systems are not directly applicable here.

As a first example of the modifications which must be introduced, we shall discuss the variational principle for the phase shift η_0. The radial differential equation involved has been discussed earlier [Eq. (9.3.12)]

$$(d^2u/dr^2) + [k^2 - U(r)]u = 0 \quad (9.4.48)$$

and will be discussed again in Sec. 12.3; here only the framework of the variational techniques of solution will be covered. The function u satisfies the boundary conditions

$$u(0) = 0; \quad u(r) \to \sin(kr - \eta); \quad r \to \infty$$

We note that the integral of u^2 is infinite. If this were not so, the form

$$\int_0^\infty [(u')^2 - (k^2 - U)u^2]\,dr$$

would, upon variation of u, lead to Eq. (9.4.48). In order to avoid this divergence of the integral at infinity, we must consider the difference between this integral and an integral which diverges in the same way at infinity. We therefore consider the function v which is a solution of

$$(d^2v/dr^2) + k^2v = 0$$

satisfying boundary condition

$$v(r) \to u(r); \quad r \to \infty \tag{9.4.49}$$

so that except for a normalizing factor v is $\sin(kr - \eta)$.

The variational principle for v, again disregarding the divergence mentioned above, would follow from the form

$$\int_0^\infty [(v')^2 - k^2v^2]\,dr$$

This expression diverges in exactly the same manner as the corresponding integral containing u, since u approaches v for large values of r.

This suggests that we consider the difference of the two integrals.

$$J = \int_0^\infty [(u')^2 - (v')^2 - k^2(u^2 - v^2) + Uu^2]\,dr \tag{9.4.50}$$

where the variation is to be performed on both u and v, subject, however, to the subsidiary condition (9.4.49). This integral is finite. By use of the equations for u and v, we can show that its value, for the exact u and v, is

$$J = \int_0^\infty \frac{d}{dr}[uu' - vv']\,dr = v'(0)v(0)$$

since $u \to 0$ when $r \to 0$ and $u \to v$ when $r \to \infty$. The variation of J will be written in a particular form, for a reason which will become clear shortly:

$$\delta J = \delta\left[\frac{v'(0)}{v(0)}v^2(0)\right] = v^2(0)\,\delta\left[\frac{v'(0)}{v(0)}\right] + 2v'(0)\,\delta[v(0)] \tag{9.4.51}$$

It is not difficult to show that the condition

$$\delta[v'(0)/v(0)] = 0$$

leads to the correct differential equations for u and v. Note first that

$$[v'(0)/v(0)] = -k\cot\eta$$

so that the variational principle is one for $k \cot \eta$. We first evaluate δJ from Eq. (9.4.50), varying both u and v and bearing in mind the relation (9.4.49) which holds between them:

$$\delta J = 2 \int_0^\infty \{\delta u[-u'' - k^2 u - Uu] + \delta v[v'' + k^2 v]\}\, dr + 2\delta v(0)v'(0)$$

Inserting expression (9.4.51), we obtain

$$v^2(0)\delta[k \cot \eta] = 2 \int_0^\infty \{\delta u[u'' + k^2 u + Uu] - \delta v[v'' + k^2 v]\}\, dr$$

From the condition $\delta[k \cot \eta] = 0$, the differential equations satisfied by u and v follow immediately.

We have chosen this particular form of the derivation in order to emphasize the independence of the variational principle for $k \cot \eta$ upon the amplitude of v. It is more convenient, however, in application to choose $v(0) = 1$. The variational integral J equals $(-k \cot \eta)$ for exact u and v and is to be varied, subject to conditions (9.4.49) and $v(0) = 1$. In summary,

$$\delta[-k \cot \eta] = 0$$

where

$$[-k \cot \eta] = \int_0^\infty [(u')^2 - (v')^2 - k^2(u^2 - v^2) + Uu^2]\, dr \qquad (9.4.52)$$
$$u(r) \to v(r) \quad \text{as} \quad r \to \infty; \quad v(0) = 1$$

As a simple application of this formula, we shall obtain the k^2 derivative of $-k \cot \eta$, evaluated at a particular value of k (k_0, for example). For this purpose, we insert the solutions corresponding to $k = k_0$ into variational principle (9.4.52) as trial wave functions. Then the dependence of $[-k \cot \eta]$ on k^2 is explicit, and we may evaluate the k^2 derivative:

$$\frac{d(-k \cot \eta)}{dk^2} = \int^\infty (v_k^2 - u_k^2)\, dr \qquad (9.4.53)$$

where u_k and v_k are the functions u and v for the wave number k. If we place k equal to zero, the result is equivalent to value (9.3.67) obtained by perturbation methods.

This variational principle [Eq. (9.4.52)] for the phase shift may be used in much the same way as the corresponding one [Eq. (9.4.28)] for k^2. One inserts in Eq. (9.4.52) trial wave functions for u and v, satisfying the supplementary conditions, and then determines the values of the variational parameters contained in u and v by the condition that the variational integral must be stationary. Note that, since the integrand of Eq. (9.4.52) is not definite, the stationary value may be either a maximum or a minimum or even a point of inflection.

The trial function for v may be taken to be $[\sin(kr - \eta)/\sin(-\eta)]$ where, however, η is to be treated as a variational parameter, or as is often done, a first guess is inserted for η. It would clearly be more convenient if a variational principle for $k \cot \eta$ could be found in which there is no explicit dependence on η such as is now contained in v. Toward this end we now introduce the difference function w.

$$w = v - u; \quad v = -[\sin(kr - \eta)/\sin(\eta)] \qquad (9.4.54)$$
$$w \to 0 \quad \text{as} \quad r \to \infty; \quad w(0) = 1$$

The differential equation satisfied by w is

$$w'' + [k^2 - U]w = -Uv \qquad (9.4.55)$$

Inserting w into the variational integral and making use of the properties of v, we find

$$AR^2 + BR + C = 0; \quad R = -k \cot \eta$$
$$A = \int_0^\infty \left[\frac{U \sin^2(kr)}{k^2}\right] dr$$
$$B = 1 + \left(\frac{2}{k}\right) \int_0^\infty U \sin(kr) \cos(kr)\, dr - \left(\frac{2}{k}\right) \int_0^\infty Uw \sin(kr)\, dr \qquad (9.4.56)$$
$$C = \int_0^\infty [(w')^2 - (k^2 - U)w^2]\, dr + \int_0^\infty U[\cos^2(kr) - 2w \cos(kr)]\, dr$$

We now solve the above quadratic equation for R, choosing the root which behaves properly in the limit of small U:

$$R = -(B + \sqrt{B^2 - 4AC})/2A \qquad (9.4.57)$$

This is an exact expression for R which need not, of course, be suitable for a variational principle for R. However, we are fortunate here; we shall now show that the requirement $\delta(R) = 0$ for variations in w, subject to the boundary conditions given in Eq. (9.4.54), leads to the proper differential equation for w, Eq. (9.4.55).

Employing the right-hand side of Eq. (9.4.57) as $[R]$, we find that

$$\delta[R] = 0 = \delta C + R\, \delta B \qquad (9.4.58)$$

where we have dropped some constant factors. Now

$$\delta B = -\left(\frac{2}{k}\right) \int_0^\infty \delta w\, U \sin(kr)\, dr$$

and $$\delta C = -2 \int_0^\infty \delta w\, U \cos(kr)\, dr - 2 \int_0^\infty \delta w[w'' + k^2 w - Uw]\, dr$$

Inserting these expressions for δB and δC into Eq. (9.4.58) leads immediately to the correct differential equation for w.

The variational principle based on Eq. (9.4.57) is often easier to use than that based on Eq. (9.4.50). One inserts a trial wave function for w which is zero at infinity and unity at the origin, involving one or more variational parameters such as the γ in $\exp(-\gamma r)$. One then locates the values of the parameters for which R is stationary, either by tabulation of R directly or by placing the differential of R with respect to these parameters equal to zero, whichever happens to be most convenient.

This variational principle yields the Born approximation Eq. (9.3.37) when $w = \cos(kr)$ is used as a trial wave function, for then C is zero and B is unity. The corresponding u is $\sin(kr)$, the incident wave. Actually, $\cos(kr)$ does not satisfy the proper boundary condition at infinity, so that we must consider this result as being obtained by a limiting process. We could, for example, employ trial wave functions $e^{-\epsilon r}\cos kr$ and determine the behavior of R as ϵ tends to zero.

Variational principles of the above type for the higher phase shifts η_l, $l \geq 1$, which are as convenient or as accurate as Eq. (9.4.57), have been developed and are stated in Prob. 9.8.

Variational Principle for the Phase Shift, Based on an Integral Equation. The integral satisfied by u_l was given earlier, in Eq. (9.3.34):

$$u_l = Akr\,j_l(kr) - \left(\frac{1}{4\pi}\right)\int_0^\infty G_k^{(l)}(r|r_0)U(r_0)u_l(r_0)\,dr_0 \qquad (9.4.59)$$

where A is a constant, j_l is the spherical Bessel function [see Eq. (5.3.67)], and $G_k^{(l)}$ is given in Eq. (9.3.33). Equation (9.4.59) is an inhomogeneous integral equation, so the method giving rise to Eq. (9.4.22) applies. We shall, however, employ a direct approach to bring out the physical implications.

The method consists in replacing Eq. (9.4.59) by a homogeneous equation, in which the phase shift η_l appears explicitly, and A is eliminated. The relation between A and η_l is obtained by examining Eq. (9.4.59) for r large. Following the analysis leading to Eq. (9.3.36), we obtain the exact relation

$$\tan\eta_l = \left(\frac{1}{A}\right)\int_0^\infty j_l(kr_0)U(r_0)u_l(r_0)r_0\,dr_0$$

Eliminating A in Eq. (9.4.59), we now obtain a homogeneous equation:

$$u_l(r) = (kr)j_l(kr)\cot(\eta_l)\int_0^\infty j_l(kr_0)U(r_0)u_l(r_0)r_0\,dr_0 - \left(\frac{1}{4\pi}\right)\int_0^\infty G_k^{(l)}(r|r_0)U(r_0)u_l(r_0)\,dr_0 \qquad (9.4.60)$$

The corresponding variational integral is

$$[\cot\eta_l] = k\,\frac{\displaystyle\int_0 Uu_l^2\,dr + \left(\frac{1}{4\pi}\right)\int_0^\infty\int_0^\infty u_l(r)U(r)G_k^{(l)}(r|r_0)U(r_0)u_l(r_0)\,dr_0}{\left[\displaystyle\int_0^\infty krj_lUu_l\,dr\right]^2} \qquad (9.4.61)$$

Varying this expression will lead to the integral equation above. Note that, since $G_k^{(l)}$ is not a definite function, the stationary value of [cot η_l] will not necessarily be either a minimum or a maximum. By inserting the specific representation (9.3.33) for $G_k^{(l)}$, Eq. (9.4.61) may be simplified somewhat:

$$[\cot \eta_l] = \frac{k \int_0^\infty U u_l^2 \, dr - 2 \int_0^\infty kr n_l(kr) U(r) u_l(r) \int_0^r kr_0 j_l(kr_0) U(r_0) u_l(r_0) \, dr_0 \, dr}{\left[\int_0^\infty kr j_l U u_l \, dr \right]^2} \tag{9.4.62}$$

The Born approximation for η_l may be immediately obtained by inserting the incident wave $[krj_l(kr)]$ as a trial wave function and dropping the second term in the numerator, since it is of higher order in U than the first. Actually, a very considerable improvement over the Born approximation is often obtained if the second term is not omitted:

$$[\cot(\eta_l)] \simeq \cot[\eta_l^B] \left\{ 1 - 2 \frac{\int_0^\infty (kr\, n_l) U (krj_l) \int_0^r (kr_0 j_l)^2 U \, dr_0 \, dr}{k \int_0^\infty U (krj_l)^2 \, dr} \right\} \tag{9.4.63}$$

where η_l^B is the Born approximation phase shift.

We shall not give any specific examples here of the use of this variational principle, as it will be discussed in detail in Chap. 12. As is usual, trial functions with variational parameters are inserted for u_l; the best values of these parameters are those for which [cot η_l] is stationary. Note that only the values of u_l are needed within the region where U does not vanish. We may be quite cavalier with our assumed form elsewhere.

Variational Principle for the Transmission Amplitude. The problem of transmission and reflection of waves by a potential barrier proceeds in much the same way as the foregoing discussion for phase shifts. There is, however, one important difference, which requires detailed discussion. The equation is

$$\psi'' + [k^2 - U(x)]\psi = 0$$

where U is supposed to have its largest values for $|x|$ small and to go to zero for $|x| \to \infty$. The corresponding integral equation, for an incident wave of amplitude A coming from $-\infty$, is

$$\psi = Ae^{ikx} + \left(\frac{1}{4\pi}\right) \int_{-\infty}^{\infty} G_k(x|x_0) U(x_0) \psi(x_0) \, dx_0 \tag{9.4.64}$$

We shall convert this inhomogeneous equation into a homogeneous form by expressing A in terms of the transmission amplitude f. To determine this relation, it is necessary to insert G_k explicitly [see Eq. (9.3.29)]:

$$\psi = Ae^{ikx} + \left(\frac{i}{2k}\right)\int_{-\infty}^{\infty} e^{ik|x-x_0|}U(x_0)\psi(x_0)\,dx_0$$

For large positive values of x, we obtain the transmitted wave

$$\psi \to e^{ikx}\left[A + \left(\frac{i}{2k}\right)\int_{-\infty}^{\infty} e^{-ikx_0}U(x_0)\psi(x_0)\,dx_0\right];\quad x \to \infty$$

Hence the amplitude of the wave transmitted through the potential barrier U, per unit incident amplitude, is

$$f = 1 + \left(\frac{i}{2kA}\right)\int_{-\infty}^{\infty} e^{-ikx_0}U(x_0)\psi(x_0)\,dx_0$$

Solving for A and eliminating it in Eq. (9.4.64) give a homogeneous equation:

$$\psi = \left[\frac{i}{2k(f-1)}\right]e^{ikx}\int_{-\infty}^{\infty} e^{-ikx_0}U(x_0)\psi(x_0)\,dx_0 + \left(\frac{1}{4\pi}\right)\int_{-\infty}^{\infty} G_k(x|x_0)U(x_0)\psi(x_0)\,dx_0 \quad (9.4.65)$$

Because of the presence of complex quantities, we shall need to insert the adjoint function $\tilde{\psi}$ whose properties we shall determine from the variational principle for f:

$$\frac{1}{(f-1)} = \left(\frac{2k}{i}\right)\frac{\displaystyle\int_{-\infty}^{\infty}\tilde{\psi}U\psi\,dx - \left(\frac{1}{4\pi}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{\psi}(x)U(x)G_k(x|x_0)U(x_0)\psi(x_0)\,dx_0\,dx}{\left[\displaystyle\int_{-\infty}^{\infty}\tilde{\psi}Ue^{ikx}\,dx\right]\left[\displaystyle\int_{-\infty}^{\infty}e^{-ikx}U\psi\,dx\right]} \quad (9.4.66)$$

On performing the variation with respect to $\tilde{\psi}$, one obtains the correct equation for ψ. The equation for $\tilde{\psi}$ is obtained by performing the variation with respect to ψ and is

$$\tilde{\psi} = \left[\frac{i}{2k(f-1)}\right]e^{-ikx}\int_{-\infty}^{\infty}\tilde{\psi}Ue^{ikx}\,dx + \left(\frac{1}{4\pi}\right)\int_{-\infty}^{\infty}\tilde{\psi}U(x_0)G_k(x|x_0)\,dx_0 \quad (9.4.67)$$

Upon comparing wivh Eq. (9.4.65), we see that $\tilde{\psi}$ is a solution of the problem in which the wave is incident in the negative k direction. The incident wave is of the form e^{-ikx}, and as x approaches large negative values, we find waves only of the form e^{-ikx}. We also note that f is the transmission amplitude for waves incident from either direction upon the barrier. This is, of course, a particular case of the reciprocity principle.

In using variational principle (9.4.66), a trial function is inserted for ψ, which we shall call ψ_k. For $\tilde{\psi}$ we insert ψ_{-k}, that is, the same trial function with reversal of sign for the direction of incidence of the wave motion. The usual method for determining the variational parameters may then be employed. As a simple example, we obtain the Born approximation by inserting $\psi = e^{ikx}$ and $\psi_{-k} = e^{-ikx}$. Then

$$\left[\frac{1}{f-1}\right] \simeq \left[\frac{\dfrac{2k}{i}}{\displaystyle\int_{-\infty}^{\infty} U\,dx}\right]\left\{1 - \frac{\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ikx}U(x)e^{ik|x-x_0|}U(x_0)e^{ikx_0}\,dx_0\,dx}{\dfrac{2k}{i}\displaystyle\int_{-\infty}^{\infty} U\,dx}\right\} \tag{9.4.68}$$

Variational Principle for Three-dimensional Scattering Problems. Following the above procedure, we shall be able to derive a variational principle for the scattering amplitude. We start with the Schroedinger equation

$$\nabla^2\psi + (k^2 - U)\psi = 0$$

where U is large only in a region near the origin, is zero at $r \to \infty$. The corresponding integral equation, for an incident plane wave, is

$$\psi = Ae^{i\mathbf{k}_i\cdot\mathbf{r}} - \frac{1}{4\pi}\int \frac{e^{ikR}}{R}\,U(\mathbf{r}_0)\psi(\mathbf{r}_0)\,dV_0$$

where $R = |\mathbf{r} - \mathbf{r}_0|$ and $\mathbf{k}_i$ is a vector of magnitude k in the direction of incidence. By eliminating A, this equation may be made homogeneous. For large distances from the origin

$$\psi \to Ae^{i\mathbf{k}_i\cdot\mathbf{r}} - \left(\frac{e^{ikr}}{4\pi r}\right)\int e^{-i\mathbf{k}_s\cdot\mathbf{r}_0}U(\mathbf{r}_0)\psi(\mathbf{r}_0)\,dV_0$$

where $\mathbf{k}_s$ is a vector of magnitude $\mathbf{k}$ in the direction of observation, *i.e.*, in the direction of the scattered wave [see also Eq. (12.3.68)]. If the scattered amplitude is written $-(1/4\pi)T(\mathbf{k}_s|\mathbf{k}_i)$, then

$$T(\mathbf{k}_s|\mathbf{k}_i) = \frac{1}{A}\int e^{-i\mathbf{k}_s\cdot\mathbf{r}_0}U(\mathbf{r}_0)\psi(\mathbf{r}_0)\,dV_0$$

and the integral equation is

$$\psi = \left[\frac{1}{T(\mathbf{k}_s|\mathbf{k}_i)}\right]e^{i\mathbf{k}_i\cdot\mathbf{r}}\int e^{-i\mathbf{k}_s\cdot\mathbf{r}_0}U(\mathbf{r}_0)\psi(\mathbf{r}_0)\,dV_0 - \left(\frac{1}{4\pi}\right)\int\left(\frac{e^{ikR}}{R}\right)U(\mathbf{r}_0)\psi(\mathbf{r}_0)\,dV_0$$

The corresponding variational principle for $T(\mathbf{k}_s|\mathbf{k}_i)$ is based on the variation of

$$[T(\mathbf{k}_s|\mathbf{k}_i)] = \left\{\frac{[\int \tilde{\psi} U e^{i\mathbf{k}_i\cdot\mathbf{r}}\, dV][\int e^{-i\mathbf{k}_s\cdot\mathbf{r}} U\psi\, dV]}{\int \tilde{\psi} U\psi\, dV + \left(\frac{1}{4\pi}\right)\iint \tilde{\psi}(\mathbf{r})U(\mathbf{r})\left(\frac{e^{ikR}}{R}\right)U(\mathbf{r}_0)\psi(\mathbf{r}_0)\, dV_0\, dV}\right\} \tag{9.4.69}$$

where $\tilde{\psi}$ is again the adjoint to ψ. Varying $[T]$ with respect to $\tilde{\psi}$ leads to the correct equation for ψ, while varying $[T]$ with respect to ψ leads to similar equation for $\tilde{\psi}$ where, however, the incident plane wave moves in the direction $(-\mathbf{k}_s)$. If we therefore indicate the direction of incidence in ψ more explicitly by the notation $\psi(\mathbf{k}_i)$, then $\tilde{\psi} = \psi(-\mathbf{k}_s)$. The reciprocity theorem

$$T(\mathbf{k}_s|\mathbf{k}_i) = T(-\mathbf{k}_i|-\mathbf{k}_s)$$

is an immediate consequence of Eq. (9.4.69).

Applications of this variational principle will be discussed in detail in Sec. 12.3. The procedure again involves choosing a trial function for $\psi(\mathbf{k}_i)$ and the corresponding $\psi(-\mathbf{k}_s)$, inserting both into Eq. (9.4.69) and performing the variation. We shall note here only that the Born approximation is automatically obtained if ψ is put equal to $\exp(i\mathbf{k}_i \cdot \mathbf{r})$ and $\tilde{\psi}$ to $\exp(-i\mathbf{k}_s \cdot \mathbf{r})$ and the second term in the denominator is omitted.

We could also have obtained a variational principle for T by employing the integral equation for T developed in Sec. 9.3, Eq. (9.3.48). This integral equation is just the Fourier transform of the integral equation for ψ, and the corresponding variational principle may be obtained directly from Eq. (9.4.69) by introducing the Fourier transforms of the functions appearing therein.

Variational Principles for Surface Perturbations. The variational principles developed so far in this section have been concerned with volume perturbation (appropriate for the Schroedinger equation or for the scattering of waves by variations of index of refraction). We now turn to problems in which either the boundary or the boundary conditions are such that the usual methods of separation of variables fail. We consider the variational principles based upon the differential form first, limiting the initial discussion to problems in which the boundary shape has been perturbed but the boundary conditions are still either homogeneous Dirichlet or Neumann. We have already written down a variational principle for k^2 in which $[k^2]$ is varied:

$$[k^2] = \{\int (\nabla\psi)^2\, dV / \int \psi^2\, dV\} \tag{9.4.70}$$

where we have particularized to the Helmholtz equation. Our remarks may, however, be easily extended to more general cases. The type of trial functions which may be inserted into $[k^2]$ may need to be restricted, as we may see upon forming the variation of $[k^2]$:

$$\int \delta\psi[\nabla^2\psi + k^2\psi]\,dV - \oint \left(\frac{\partial\psi}{\partial n}\right)\delta\psi\,dS = 0$$

We note that, if $\partial\psi/\partial n = 0$, then the scalar Helmholtz equation for ψ follows immediately. For Dirichlet conditions, it will follow only if $\delta\psi = 0$ on S, that is, if the trial functions inserted in $[k^2]$ are zero on S. This restriction may be rather difficult to meet in practice if the boundary is rather complicated. We therefore shall devise a variational principle in which the trial functions are unrestricted, as in the Neumann case. This is done by adding a term to variational principle (9.4.70) of such a form as to convert the surface term into one whose integrand is proportional to ψ rather than $\partial\psi/\partial n$. It is easy to verify that the appropriate expression for $[k^2]$ is

$$[k^2] = \left\{\int (\nabla\psi)^2\,dV - 2\oint \psi\left(\frac{\partial\psi}{\partial n}\right)dS\right\}\Big/\textstyle\int\psi^2\,dV \qquad (9.4.71)$$

We may then insert in (9.4.71), as well as in (9.4.70), any trial function. Equation (9.4.70) is appropriate for homogeneous Neumann conditions, while (9.4.71) applies to the Dirichlet case. These two forms take on a more symmetrical appearance if Green's theorem is employed. They become

$$[k^2] = \left\{-\int \psi\nabla^2\psi\,dV + \oint \psi\left(\frac{\partial\psi}{\partial n}\right)dS\right\}\Big/\int\psi^2\,dV;\quad \text{Neumann} \qquad (9.4.72)$$

and

$$[k^2] = \left\{-\int \psi\nabla^2\psi\,dV - \oint \psi\left(\frac{\partial\psi}{\partial n}\right)dS\right\}\Big/\int\psi^2\,dV;\quad \text{Dirichlet} \qquad (9.4.73)$$

These expressions may now be used in the customary variational manner, already described many times. For example, in the one-dimensional problem in which Dirichlet conditions are satisfied at $x = 0$ and $x = 1$, if the trial wave function $\sin(\kappa x)$ (κ a variational parameter) is inserted into Eq. (9.4.73) and the resultant expression differentiated with respect to κ, then the best values of κ are given by solutions of $\sin\kappa$ equal zero. We may also obtain the first-order perturbation formula derived in Sec. 9.2 by employing normalized solutions φ of the Helmholtz equation in Eqs. (9.4.72) and (9.4.73):

$$k^2 \simeq k_0^2 + \left\{\oint \varphi\left(\frac{\partial\varphi}{\partial n}\right)dS\Big/\int\varphi^2\,dV\right\} \simeq k_0^2 + \oint \varphi\left(\frac{\partial\varphi}{\partial n}\right)dS;\quad \text{Neumann}$$

$$k^2 \simeq k_0^2 - \oint \varphi\left(\frac{\partial\varphi}{\partial n}\right)dS;\quad \text{Dirichlet}$$

For the situation in which the boundary conditions are mixed, $(\partial\psi/\partial n) + f\psi = 0$, then the suitable variational quantity in which the trial functions are unrestricted is

$$[k^2] = \left\{\int(\nabla\psi)^2\,dV + 2\oint f\psi^2\,dS\right\}/\int\psi^2\,dV$$

Upon variation this gives

$$\int \delta\psi[\nabla^2\psi + k^2\psi]\,dV - \oint \delta\psi\left[\left(\frac{\partial\psi}{\partial n}\right) + f\psi\right]dS = 0$$

In the symmetrical form, similar to Eq. (9.4.72),

$$[k^2] = \left\{-\int\psi\nabla^2\psi\,dV + \oint f\psi^2\,dS\right\}/\int\psi^2\,dV \tag{9.4.74}$$

By inserting the unperturbed normalized wave function φ, Eq. (9.4.74) becomes

$$[k^2] \simeq k_0^2 + \int f\varphi^2\,dS$$

Variational Principle Based on the Integral Equation for Boundary Perturbations. The integral equations for ψ for the various kinds of boundary conditions have been derived in Sec. 9.2 [see Eqs. (9.2.4), (9.2.16), (9.2.22), and (9.2.47)]. The simplest case for the present considerations is furnished by the mixed boundary condition. We take one of these to illustrate. The equation is

$$\psi(S) = -\frac{1}{4\pi}\oint f(S_0)\psi(S_0)G_k(S|S_0)\,dS_0 \tag{9.4.75}$$

where G_k is the Green's function satisfying homogeneous Neumann conditions. The parameter k is too thoroughly buried in G_k and f to have an explicit variational principle for it based on Eq. (9.4.75). It is rather more convenient to take k^2 as known and employ the size of f as the eigenvalue. To make this attack more explicit, let

$$f(S) = \mu F(S)$$

Then the variational principle for μ places the variation of $[\mu]$ equal to zero, where

$$[\mu] = -\left\{\oint \psi^2(S)F(S)\,dS\Big/\frac{1}{4\pi}\oint\oint \psi(S)F(S)G_k(S|S_0)F(S_0)\psi(S_0)\,dS\,dS_0\right\} \tag{9.4.76}$$

It is, of course, essential that G_k be symmetric in S and S_0 for the adjoint function to equal ψ. This principle will give μ as a function of k so that it answers the problem in acoustics of how much material of a given type is required to obtain a given absorption. To obtain the resonant frequency and absorption for a given distribution of material, it would be necessary to invert—by interpolation, for example—so as to obtain the inverse function $k = k(\mu)$ from $\mu = \mu(k)$.

The integral equation for boundary-shape perturbations, say in the case where ψ satisfies homogeneous Neumann conditions, is

$$\psi = -\frac{1}{4\pi}\oint \psi \left(\frac{\partial G_k}{\partial n_0}\right) dS_0 \tag{9.4.77}$$

The kernel $\partial G_k/\partial n_0$ is not symmetric. To make it so, operate on both sides with $\partial/\partial n$, obtaining

$$\oint \psi \left(\frac{\partial^2 G_k}{\partial n\, \partial n_0}\right) dS_0 = 0$$

We note again [see Eq. (9.2.18) and below] that $\partial^2 G_k/\partial n\, \partial n_0$ must be evaluated with care by keeping the variable $\mathbf{r}$ away from S while the derivative with respect to the other variable is evaluated. This, according to our earlier discussion, may be accomplished directly if the substitution below is made:

$$G_k \rightarrow \Gamma_k = (G_k - G_0) \tag{9.4.78}$$

Then the integral equation becomes

$$\oint \psi \left(\frac{\partial^2 \Gamma_k}{\partial n\, \partial n_0}\right) dS_0 = 0 \tag{9.4.79}$$

The quantity whose variation is zero is

$$J = \oint\oint \psi(S) \left(\frac{\partial^2 \Gamma_k}{\partial n\, \partial n_0}\right) \psi(S_0)\, dS_0\, dS \tag{9.4.80}$$

In use k^2 (or the perturbation size) is left arbitrary. The variational parameters in the trial function are adjusted so that $\delta J = 0$. Then Eq. (9.4.79) is solved for k^2. In the more usual cases, discussed earlier, the solution for k^2 may be made explicitly so that the variational integral does not depend explicitly on k^2. Unfortunately, this is not the case in the present problem. It is instead necessary to solve the equation $\delta J = 0$ and Eq. (9.4.79) simultaneously. We shall not discuss the Dirichlet case, since the integral equation (9.2.64) is not essentially different from Eq. (9.4.79).

Scattering from Surfaces. A variational method may be set up for scattering from an object upon which ψ satisfies homogeneous boundary conditions. Consider Neumann conditions first. Then the integral equation for ψ is

$$\psi = Ae^{i\mathbf{k}_i\cdot\mathbf{r}} - \left(\frac{1}{4\pi}\right)\oint \frac{\partial}{\partial n_0}\left(\frac{e^{ikR}}{R}\right)\psi(S_0)\, dS_0$$

where A is a constant, $\mathbf{k}_i$ is a vector of magnitude k in the incident direction, and $R = |\mathbf{r} - \mathbf{r}_0|$, where $\mathbf{r}_0$ is on the surface of the scatterer. We may make this equation homogeneous by expressing A in terms of the scattering amplitude. For large r,

$$\psi \to Ae^{i\mathbf{k}_i\cdot\mathbf{r}} + \left(\frac{1}{4\pi}\right)(e^{ikr}/r) \oint i(\mathbf{n}_0 \cdot \mathbf{k}_s)\psi(S_0)e^{-i\mathbf{k}_s\cdot\mathbf{r}_0}\, dS_0$$

where $\mathbf{k}_s$ is a vector of magnitude k in the direction of scattering. The scattered amplitude f is

$$f(\vartheta,\varphi) = -\left(\frac{1}{4\pi}\right) T(\mathbf{k}_s|\mathbf{k}_i) = +\frac{1}{4\pi A}\oint i(\mathbf{n}_0 \cdot \mathbf{k}_s)\psi(S_0)e^{-i\mathbf{k}_s\cdot\mathbf{r}_0}\, dS_0$$

Solving for A and inserting into the equation for ψ gives

$$\psi = -\left(\frac{1}{T}\right) e^{i\mathbf{k}_i\cdot\mathbf{r}} \oint i(\mathbf{n}_0 \cdot \mathbf{k}_s)\psi(S_0)e^{-i\mathbf{k}_s\cdot\mathbf{r}_0}\, dS_0 - \left(\frac{1}{4\pi}\right)\oint \frac{\partial}{\partial n_0}\left(\frac{e^{ikR}}{R}\right)\psi(S_0)\, dS_0 \tag{9.4.81}$$

This equation is not in a convenient form for the formation of a variational principle for ψ, since the kernel of the integral equation is not symmetric. As before, we take the normal derivative of both sides of Eq. (9.4.81) and evaluate on S. Then

$$0 = \left(\frac{1}{T}\right)(\mathbf{n} \cdot \mathbf{k}_i)e^{i\mathbf{k}_i\cdot\mathbf{r}} \oint (\mathbf{n}_0 \cdot \mathbf{k}_s)\psi(S_0)e^{-i\mathbf{k}_s\cdot\mathbf{r}_0}\, dS_0 - \left(\frac{1}{4\pi}\right)\oint \frac{\partial^2}{\partial n_0\,\partial n}\left(\frac{e^{ikR}}{R}\right)\psi(S_0)\, dS_0 \tag{9.4.82}$$

We again remark that the evaluation of $(\partial^2/\partial n\,\partial n_0)(e^{ikR}/R)$ must be made by evaluating the normal derivative with respect to, say, $\mathbf{n}_0$, with $\mathbf{r}$ not on S, and then taking the $\mathbf{n}$ derivative. The variational principle for T can now be obtained. We require that $\delta[T] = 0$ where

$$[T] = \left\{\frac{\oint (\mathbf{n} \cdot \mathbf{k}_i)\tilde{\psi}e^{i\mathbf{k}_i\cdot\mathbf{r}}\, dS \oint e^{-i\mathbf{k}_s\cdot\mathbf{r}_0}(\mathbf{n}_0 \cdot \mathbf{k}_s)\psi(S_0)\, dS_0}{\left(\frac{1}{4\pi}\right)\oint\oint \tilde{\psi}(S)\,\frac{\partial^2}{\partial n\,\partial n_0}\left(\frac{e^{ikR}}{R}\right)\psi(S_0)\, dS_0\, dS}\right\} \tag{9.4.83}$$

Upon performing the variation on ψ, the integral equation for $\tilde{\psi}$ is obtained. As expected,

$$\tilde{\psi} = \psi(-\mathbf{k}_s) \tag{9.4.84}$$

In words, the adjoint solution is the solution of the original problem, with the wave incident on the scatterer in the direction $(-\mathbf{k}_s)$. It again follows that $T(\mathbf{k}_s|\mathbf{k}_i) = T(-\mathbf{k}_i|-\mathbf{k}_s)$.

A similar procedure leads to a variational principle for T when Dirichlet conditions are satisfied on the surface of the scatterer. We find that $[T]$ is given by

$$[T] = -\left\{\frac{\oint e^{-i\mathbf{k}_s\cdot\mathbf{r}_0}\left(\frac{\partial\psi}{\partial n_0}\right) dS_0 \oint e^{i\mathbf{k}_i\cdot\mathbf{r}}\left(\frac{\partial\tilde{\psi}}{\partial n}\right) dS}{\frac{1}{4\pi}\oint\oint\left(\frac{\partial\tilde{\psi}}{\partial n}\right)\left(\frac{e^{ikR}}{R}\right)\left(\frac{\partial\psi}{\partial n_0}\right) dS\, dS_0}\right\} \tag{9.4.85}$$

Again $\tilde{\psi}$ is given by Eq. (9.4.84).

We note that both variational principles (9.4.83) and (9.4.85) involve the values of ψ and $\partial\psi/\partial n$, respectively, on the surface only. It is therefore necessary to obtain a trial wave function which approximates ψ or $\partial\psi/\partial n$ on the surface S only. Examples involving these variational principles, as well as the very similar ones for diffraction problems, are carried out in Sec. 11.4.

A Variational Principle for Radiation Problems. The physical situation here may be translated into an inhomogeneous boundary condition wherein either the normal derivative of ψ or the value of ψ on the surface of the radiating system is specified. We shall consider here an example in which ψ satisfies the Helmholtz equation and where the normal derivative of ψ is given on a surface S. In acoustics this corresponds to an impressed normal velocity which is, of course, sinusoidal with time. The integral equation may be readily obtained. We start from the general relation

$$\psi = \frac{1}{4\pi} \oint \left[\frac{e^{ikR}}{R} \left(\frac{\partial\psi}{\partial n_0} \right) - \psi(S_0) \frac{\partial}{\partial n_0} \left(\frac{e^{ikR}}{R} \right) \right] dS_0$$

Inserting the prescribed value of $\partial\psi/\partial n$, which we shall call $N(S)$, yields

$$\psi = \frac{1}{4\pi} \oint \left[\frac{e^{ikR}}{R} N - \psi(S_0) \frac{\partial}{\partial n_0} \left(\frac{e^{ikR}}{R} \right) \right] dS_0 \tag{9.4.86}$$

For simplicity, we choose N real. To obtain an integral equation with a symmetric kernel, we evaluate the normal derivative of ψ so that

$$-N(S) + \left(\frac{1}{4\pi} \right) \oint N(S_0) \frac{\partial}{\partial n} \left(\frac{e^{ikR}}{R} \right) dS_0 = \oint \psi(S_0) \frac{\partial^2}{\partial n\, \partial n_0} \left(\frac{e^{ikR}}{R} \right) dS_0 \tag{9.4.87}$$

The left-hand side is a known function which we shall call ϕ, and Eq. (9.4.87) is then an inhomogeneous integral equation.

The function ϕ is the negative of the difference between the actual normal derivative N and the normal derivative obtained from a first approximation for ψ obtained from Eq. (9.4.86) by dropping the second term. Calling this latter normal derivative $N^{(0)}$, we have

$$\phi = N^{(0)} - N \tag{9.4.88}$$

The physical quantity of interest is the radiated power, which is proportional to the imaginary part of $\oint \bar{\psi} N\, dS$, as indicated in Eqs. (3.3.15) and (11.2.27). For real N, however, this is also proportional to $\oint \psi N\, dS$. The real part of $\oint \bar{\psi} N\, dS$ may also be given an interpretation, for $[\psi/(\partial\psi/\partial n)]$ is proportional to the acoustic impedance. Hence the real part of

$$\oint N\psi\, dS = \oint N^2 \left[\frac{\psi}{(\partial\psi/\partial n)} \right] dS$$

is proportional to the "reactive" power, *i.e.*, the power which does not leave the radiating system permanently by going to infinity but returns to the radiator and thereby affects the manner in which it radiates.

The variational quantity J, which is to be stationary, may be found by the general method for treating inhomogeneous integral equations, as discussed at the beginning of this section. From Eq. (9.4.8) we have

$$[J] = \left\{ \frac{\left[\oint \psi\phi \, dS \right]^2}{\frac{1}{4\pi} \oint\oint \psi(S) \frac{\partial^2 G_k}{\partial n \, \partial n_0} \psi(S_0) \, dS \, dS_0} \right\} \tag{9.4.89}$$

The variational principle is then $\delta[J] = 0$. We now investigate the value of J for the exact ψ. From Eqs. (9.4.87) and (9.4.88)

$$J = \oint \psi\phi \, dS; \quad J = \oint \psi(N^{(0)} - N) \, dS \tag{9.4.90}$$

We may reduce this expression further by making use of the definition of $N^{(0)}$ for

$$\oint \psi N^{(0)} \, dS = \left(\frac{1}{4\pi}\right) \oint\oint \psi(S) \frac{\partial}{\partial n} \left(\frac{e^{ikR}}{R}\right) N(S_0) \, dS_0 \, dS$$

Employing Eq. (9.4.86), we have

$$\oint \psi N^{(0)} \, dS = \left(\frac{1}{4\pi}\right) \oint\oint N(S) \left(\frac{e^{ikR}}{R}\right) N(S_0) \, dS \, dS_0 - \oint \psi N \, dS$$

Hence

$$J = \left(\frac{1}{4\pi}\right) \oint\oint N(S) \left(\frac{e^{ikR}}{R}\right) N(S_0) \, dS \, dS_0 - 2 \oint \psi N \, dS \tag{9.4.91}$$

We may therefore conclude that Eq. (9.4.89) is the basis of a variational principle for $\oint \psi N \, dS$. This result prompts us to introduce the quantity $[K]$ to be varied, such that K equals $\oint \psi N \, dS$:

$$[K] = -\tfrac{1}{2} \left\{ \frac{\left[\oint \psi\phi \, dS \right]^2}{\left(\frac{1}{4\pi}\right) \oint\oint \psi(S) \left(\frac{\partial^2 G_k}{\partial n \, \partial n_0}\right) \psi(S_0) \, dS \, dS_0} \right\} + \left(\frac{1}{8\pi}\right) \oint\oint N(S) \left(\frac{e^{ikR}}{R}\right) N(S_0) \, dS \, dS_0 \tag{9.4.92}$$

where $\delta[K] = 0$.

Variation-Iteration Method. The variational methods described above do not provide any simple method of estimating the accuracy of their results. Generally, one simply extends the flexibility of the trial wave function by inserting additional variational parameters and observing the convergence of the quantity of interest with the number of such parameters. In principle, such a method must involve an infinite number of parameters, for one is certain of the answer only when all of a complete set of functions has been employed as trial wave functions in the variational integrals. The variation-iteration technique which we have discussed in Sec. 9.1 (pages 1026 to 1030) not only provides an estimate of the error, by giving both an upper and a lower bound to quantities being varied, but also results in a method for systematically improving upon the trial wave function. Since we have already discussed the technique, we shall only review its salient features and give proofs of some theorems which were postponed until we had discussed variational problems.

We shall again employ the formal operator language to describe our results, first limiting our considerations to positive-definite self-adjoint operators. Let

$$\mathfrak{L}\psi = \lambda\mathfrak{M}\psi \tag{9.4.93}$$

The iteration process is described as follows: Let φ_0 be the initial trial wave function. This may be obtained by any of the various variational methods described in this section. The first *iterate* φ_1 is given by

$$\varphi_1 = \mathfrak{L}^{-1}\mathfrak{M}\varphi_0$$

and the nth in terms of the $(n-1)$st iterate by

$$\varphi_n = \mathfrak{L}^{-1}\mathfrak{M}\varphi_{n-1}$$

We are tacitly assuming here that $\mathfrak{L}^{-1}\mathfrak{M}$ is an integral operator, for the process of integration, depending as it does on many values of φ_{n-1}, generally acts so as to improve the wave function, so that φ_n is closer to ψ than φ_{n-1}. In more general terms, $\mathfrak{L}$ and $\mathfrak{M}$ must be such that the possible values of λ are bounded from below and extend to plus infinity. If we had been rather clumsy in our arrangements, so that $1/\lambda$ occurred rather than λ, its spectrum would have gone from zero up to some maximum value, and the convergence of the iterative process would have been correspondingly poorer.

An insight into the qualitative aspects of the iterative process may be obtained as follows. Let the eigenfunctions of Eq. (9.4.93) be χ_n and the corresponding eigenvalues λ_n:

$$\mathfrak{L}\chi_n = \lambda_n\mathfrak{M}\chi_n; \quad \lambda_0 \le \lambda_1 \le \lambda_2 \cdots$$

These functions form a complete, orthogonal set in terms of which we may expand φ_0:

$$\varphi_0 = \sum_{p=0}^{\infty} a_p\chi_p$$

We immediately find that

$$\varphi_n = \sum_{p=0}^{\infty} \left(\frac{a_p}{\lambda_p^n}\right)\chi_p$$

It may now be seen that the set φ_n converges to χ_0 if λ_0 is smaller than λ_{p+1}. The convergence is more rapid the greater the ratio λ_1/λ_0. If there is a degeneracy so that $\lambda_0 = \lambda_1$, then φ_n converges to a linear combination of φ_0 and φ_1.

The various iterates may be inserted into the two variational expressions for λ given in Eqs. (9.4.8) and (9.4.13). These give the following approximations to λ_0 when φ_n is employed as a trial function:

$$\lambda_0^{(n)} = \left\{\frac{\int \varphi_n \mathfrak{L} \varphi_n \, dV}{\int \varphi_n \mathfrak{M} \varphi_n \, dV}\right\} = \left\{\frac{\int \varphi_n \mathfrak{M} \varphi_{n-1} \, dV}{\int \varphi_n \mathfrak{M} \varphi_n \, dV}\right\} \tag{9.4.94}$$

$$\lambda_0^{(n+\frac{1}{2})} = \left\{\frac{\int \varphi_n \mathfrak{M} \varphi_n \, dV}{\int \varphi_n \mathfrak{M} \mathfrak{L}^{-1} \mathfrak{M} \varphi_n \, dV}\right\} = \left\{\frac{\int \varphi_n \mathfrak{M} \varphi_n \, dV}{\int \varphi_n \mathfrak{M} \varphi_{n+1} \, dV}\right\} \tag{9.4.95}$$

The reason for the particular choice of superscript for λ_0 in Eq. (9.4.95) will be made plain shortly.

Because of the form of Eqs. (9.4.94) and (9.4.95) it is convenient to introduce the notation $[n,m]$:

$$[n,m] = \int \varphi_n \mathfrak{M} \varphi_m \, dV \tag{9.4.96}$$

The value of $\lambda_0^{(n)}$ and $\lambda_0^{(n+\frac{1}{2})}$ may be expressed in terms of these matrix elements:

$$\lambda_0^{(n)} = [n, n-1]/[n,n]; \quad \lambda_0^{(n+\frac{1}{2})} = [n,n]/[n, n+1] \tag{9.4.97}$$

Incidentally, we note that

$$[n,m] = [m,n] = [n-s, m+s] \tag{9.4.98}$$

where s is an integer.

We may now show that the set $\lambda_0^{(a)}$, including both integral and half-integral values of a, form a monotonic sequence of decreasing values, approaching the exact eigenvalue λ_0 from above. In symbols,

$$\lambda_0^{(n)} \geq \lambda_0^{(n+\frac{1}{2})} \geq \lambda_0^{(n+1)} \cdots \geq \lambda_0 \tag{9.4.99}$$

$$\left.\begin{matrix} \lambda_0^{(n)} \\ \lambda_0^{(n+\frac{1}{2})} \end{matrix}\right\} \xrightarrow[n \to \infty]{} \lambda_0; \quad \text{if } \int \chi_0 \mathfrak{M} \varphi_0 \, dV \neq 0 \tag{9.4.100}$$

The condition in Eq. (9.4.100) is inserted in order to ensure the presence of some amount of χ_0 in the original trial function. If none is present, the $\lambda_0^{(n)}$ sequence will converge to λ_1, the second eigenvalue of Eq. (9.4.93). The inequalities (9.4.99) follow from the Schwarz inequality Eq. (1.6.31) which may be obtained from the positive-definite character of $\mathfrak{L}$ and $\mathfrak{M}$. We have

$$\left(\int \psi \mathfrak{M} \chi \, dV\right)^2 \leq \left(\int \psi \mathfrak{M} \psi \, dV\right)\left(\int \chi \mathfrak{M} \chi \, dV\right)$$

and a similar inequality with $\mathfrak{L}$ replacing $\mathfrak{M}$. The functions ψ and χ are arbitrary. If we let ψ be φ_n and χ be φ_{n-1}, the Schwarz inequality, based on the positive-definite character of $\mathfrak{M}$, leads to the inequality

$$\lambda_0^{(n-\frac{1}{2})} \geq \lambda_0^{(n)} \tag{9.4.101}$$

Similarly, from the Schwarz inequality, based on the positive-definite character of $\mathfrak{L}$, one finds

$$\lambda_0^{(n-1)} \geq \lambda_0^{(n-\frac{1}{2})}$$

Combining these two inequalities results immediately in (9.4.99).

The proof of Eq. (9.4.100) will be based on the method of *reductio ad absurdum*. Note that it is not an immediate consequence of the variational principle, unless we have a proof that the set φ_n is complete. Suppose that we assume the contrary of the theorem; *i.e.*, suppose that the $\lambda_0^{(a)}$ sequence is bounded from below by ν, not equal to λ_0 and (from the variational principle) greater than λ_0. We now show that this assumption is inconsistent with the specified requirement that $\int \chi_0 \mathfrak{M} \varphi_0 \, dV \neq 0$. Consider the quantity

$$f = \sum_{n=0}^{\infty} (\lambda_0)^n \varphi_{n+1}$$

This series converges in the mean if $\nu > \lambda_0$. This follows because $\int f \mathfrak{M} f \, dV$ is bounded, for

$$\begin{aligned}\left[\int f \mathfrak{M} f \, dV\right]^{\frac{1}{2}} &= \left[\int \left(\sum_n (\lambda_0)^n \varphi_{n+1}\right) \mathfrak{M} \left(\sum_p (\lambda_0)^p \varphi_{p+1}\right) dV\right]^{\frac{1}{2}} \\ &\leq \sum_n (\lambda_0)^n [n+1, n+1]^{\frac{1}{2}}\end{aligned}$$

The inequality is a case of the Bessel inequality Eq. (1.6.32). From the Cauchy convergence criterion, this last series converges if

$$\frac{(\lambda_0)^{n+1}[n+2, n+2]^{\frac{1}{2}}}{(\lambda_0)^n[n+1, n+1]^{\frac{1}{2}}} < 1; \quad \text{as } n \to \infty$$

or

$$\frac{\lambda_0^2}{\lambda_0^{(n+\frac{3}{2})}\lambda_0^{(n+2)}} < 1; \quad \text{as } n \to \infty$$

According to hypothesis, this condition is met, since $\lambda_0^{(n+\frac{3}{2})}$ and $\lambda_0^{(n+2)}$ are both greater than ν, which in turn is greater than λ_0. Hence, series (9.4.101) converges in the mean.

Because of this convergence, it is now possible to evaluate

$$(\mathfrak{L} - \lambda_0 \mathfrak{M}) f = \mathfrak{M} \varphi_0$$

by series rearrangement. We see that

$$\int \chi_0 \mathfrak{M} \varphi_0 \, dV = \int \chi_0 (\mathfrak{L} - \lambda_0 \mathfrak{M}) f \, dV = 0$$

which contradicts the hypothesis in Eq. (9.4.100). Hence the assumption that ν is different from λ_0 is incorrect, so that the sequence $\lambda_0^{(a)}$

approaches λ_0 from above and has λ_0 as its limit, thus actually demonstrating that in principle (*i.e.*, if all the integrations can be carried out and if enough iterations are made) the variation-iteration method will give the correct answer.

Our next task is to estimate the rate of convergence and to determine a lower bound for λ_0 which, together with the upper bounds provided by the $\lambda_0^{(a)}$ sequence, indicates the error in the method, since λ_0 must be between the upper and lower bound. We are already aware that convergence will be better if the ratio λ_0/λ_1 is small. A more quantitative estimate will follow from consideration of the inequality

$$\int\{\varphi[1 - \mathfrak{M}\mathfrak{L}^{-1}\lambda_1]\mathfrak{M}[1 - \mathfrak{L}^{-1}\mathfrak{M}\lambda_0]\varphi\}\, dV \geq 0 \tag{9.4.102}$$

where φ is arbitrary. This inequality is a consequence of the variational principle, as we shall now show. Note that the operator

$$(1 - \lambda_0\mathfrak{L}^{-1}\mathfrak{M})$$

is definite, for

$$\int\varphi\mathfrak{M}(1 - \lambda_0\mathfrak{L}^{-1}\mathfrak{M})\varphi\, dV \geq 0$$

an inequality which follows from the variational principle for λ_0,

$$\lambda_0 \leq \{\int\varphi\mathfrak{M}\varphi\, dV / \int\varphi\mathfrak{M}\mathfrak{L}^{-1}\mathfrak{M}\varphi\, dV\}$$

We are therefore permitted to introduce a nonsingular operator equal to the square root of $(1 - \lambda_0\mathfrak{L}^{-1}\mathfrak{M})$. We note, moreover, that the function $\sqrt{1 - \lambda_0\mathfrak{L}^{-1}\mathfrak{M}}\,\varphi$ is orthogonal to χ_0 and is therefore a suitable trial function in the variational principle for λ_1. Hence

$$\lambda_1 \leq \left\{\frac{\int[\sqrt{1 - \lambda_0\mathfrak{L}^{-1}\mathfrak{M}}\,\varphi]\mathfrak{M}[\sqrt{1 - \lambda_0\mathfrak{L}^{-1}\mathfrak{M}}\,\varphi]\, dV}{\int[\sqrt{1 - \lambda_0\mathfrak{L}^{-1}\mathfrak{M}}\,\varphi]\mathfrak{M}\mathfrak{L}^{-1}\mathfrak{M}[\sqrt{1 - \lambda_0\mathfrak{L}^{-1}\mathfrak{M}}\,\varphi]\, dV}\right\}$$

We may reduce both the numerator and denominator of this ratio considerably. We illustrate with the numerator, which, because of the self-adjoint character of $\mathfrak{L}$ and $\mathfrak{M}$, may be written

$$\int\varphi[\sqrt{1 - \lambda_0\mathfrak{M}\mathfrak{L}^{-1}}\,\mathfrak{M}\,\sqrt{1 - \lambda_0\mathfrak{L}^{-1}\mathfrak{M}}]\varphi\, dV$$

In addition,

$$\sqrt{1 - \lambda_0\mathfrak{M}\mathfrak{L}^{-1}}\,\mathfrak{M} = \mathfrak{M}\,\sqrt{1 - \lambda_0\mathfrak{L}^{-1}\mathfrak{M}} \tag{9.4.103}$$

This is readily shown from the power series expansion of either side; *e.g.*, a typical term in the power series expansion of the left side is

$$\begin{aligned}(\mathfrak{M}\mathfrak{L}^{-1})^n\mathfrak{M} &= \mathfrak{M}\mathfrak{L}^{-1}\mathfrak{M}\mathfrak{L}^{-1} \cdots \mathfrak{M}\mathfrak{L}^{-1}\mathfrak{M} \\ &= \mathfrak{M}(\mathfrak{L}^{-1}\mathfrak{M}\mathfrak{L}^{-1}\mathfrak{M} \cdots) = \mathfrak{M}(\mathfrak{L}^{-1}\mathfrak{M})^n\end{aligned}$$

Inserting Eq. (9.4.103) into the numerator gives

$$\int\varphi\mathfrak{M}(1 - \lambda_0\mathfrak{L}^{-1}\mathfrak{M})\varphi\, dV$$

while the inequality for λ_1 becomes

$$\lambda_1 \le \{\int \varphi \mathfrak{M}(1 - \lambda_0 \mathfrak{L}^{-1}\mathfrak{M})\varphi\, dV / \int \varphi \mathfrak{M}\mathfrak{L}^{-1}\mathfrak{M}(1 - \lambda_0\mathfrak{L}^{-1}\mathfrak{M})\varphi\, dV\}$$

which is just inequality (9.4.102).

A somewhat simpler and more suggestive proof, which, however, involves infinite processes, is obtained by inserting the expansion of φ in terms of the complete, orthogonal set χ_m, which we take to be normalized for the present purposes:

$$\int \chi_n \mathfrak{M} \chi_m\, dV = \delta_{nm}$$

If

$$\varphi = \sum_{p=0}^{\infty} C_p \chi_p$$

then

$$\int \{\varphi[1 - \mathfrak{M}\mathfrak{L}^{-1}\lambda_1]\mathfrak{M}[1 - \mathfrak{L}^{-1}\mathfrak{M}\lambda_0]\varphi\}\, dV = \Sigma C_p^2 \left[1 - \left(\frac{\lambda_1}{\lambda_p}\right)\right]\left[1 - \left(\frac{\lambda_0}{\lambda_p}\right)\right]$$

which is obviously greater than zero.

If we now insert φ_n for φ in inequality (9.4.102) and employ definition (9.4.96), we find that

$$\{[n,n] - (\lambda_1 + \lambda_0)[n + 1, n] + \lambda_0\lambda_1[n + 1, n + 1]\} \ge 0$$

or dividing by $[n + 1, n + 1]$,

$$[\lambda_0^{(n+\frac{1}{2})}\lambda_0^{(n+1)} - (\lambda_1 + \lambda_0)\lambda^{(n+1)} + \lambda_0\lambda_1] \ge 0 \qquad (9.4.104)$$

The same inequality, with all the superscripts increased by $\frac{1}{2}$, follows from

$$\int \{\varphi[1 - \mathfrak{M}\mathfrak{L}^{-1}\lambda_1]\mathfrak{M}\mathfrak{L}^{-1}\mathfrak{M}[1 - \lambda_0\mathfrak{L}^{-1}\mathfrak{M}]\varphi\}\, dV \ge 0 \qquad (9.4.105)$$

This may be proved in much the same way as inequality (9.4.102). For example, the expression for the integral in terms of the expansion of φ in χ_p is

$$\sum_p \left(\frac{C_p^2}{\lambda_p}\right)\left[1 - \left(\frac{\lambda_1}{\lambda_p}\right)\right]\left[1 - \frac{\lambda_0}{\lambda_p}\right] \ge 0$$

Our first use of inequality (9.4.104) is to obtain an estimate of the rate of convergence, for the inequality may be regrouped as follows:

$$\lambda_0^{(n+1)}[\lambda_0^{(n+\frac{1}{2})} - \lambda_0] - \lambda_1[\lambda_0^{(n+1)} - \lambda_0] \ge 0$$

or

$$\frac{\lambda_0^{(n+1)} - \lambda_0}{\lambda_0^{(n+\frac{1}{2})} - \lambda_0} \le \frac{\lambda_0^{(n+1)}}{\lambda_1} \simeq \frac{\lambda_0}{\lambda_1} \qquad (9.4.106)$$

This shows directly that $\lambda_0^{(n+1)}$ is closer to λ_0 than is $\lambda_0^{(n+\frac{1}{2})}$, the ratio of their distances from λ_0 being less than $\lambda_0^{(n+1)}/\lambda_1$, which for sufficiently great n is about λ_0/λ_1. Thus the ratio λ_0/λ_1 is an upper bound to the rate of convergence which is approached as n increases. This may be seen directly from the series expansion of φ_n and finally of $\lambda_0^{(a)}$. When

λ_1 is much larger than λ_0, the convergence is then very rapid. It is poor in the case of a near degeneracy when $\lambda_0 \simeq \lambda_1$. In that event the following special technique has been found to be efficacious (it is also useful even when $\lambda_0 \ll \lambda_1$, for it provides a method for extrapolation to the exact answer).

An Extrapolation Method. Suppose that the iterations have proceeded so far that φ_n is a mixture of the eigenfunctions χ_0 and χ_1, so that further iterations would result simply in the gradual elimination of χ_1. This, of course, is a slow process if $\lambda_0 \simeq \lambda_1$. Taking, then,

$$\varphi_n = \chi_0 + b\chi_1$$

we may calculate the successive iterates as

$$\varphi_{n+1} = (\chi_0/\lambda_0) + b(\chi_1/\lambda_1); \quad \varphi_{n+2} = (\chi_0/\lambda_0^2) + b(\chi_1/\lambda_1^2)$$

From these wave functions we may obtain $\lambda_0^{(n+\frac{1}{2})}$, $\lambda_0^{(n+1)}$, $\lambda_0^{(n+\frac{3}{2})}$, which we now give under the assumption that $b^2 \ll 1$:

$$\begin{aligned}\lambda_0^{(n+\frac{1}{2})} &\simeq \lambda_0 + b^2\lambda_0[1 - (\lambda_0/\lambda_1)] \\ \lambda_0^{(n+1)} &\simeq \lambda_0 + b^2\lambda_0[1 - (\lambda_0/\lambda_1)](\lambda_0/\lambda_1) \\ \lambda_0^{(n+\frac{3}{2})} &\simeq \lambda_0 + b^2\lambda_0[1 - (\lambda_0/\lambda_1)](\lambda_0/\lambda_1)^2\end{aligned}$$

This shows in an explicit fashion that the convergence of $\lambda_0^{(a)}$ to λ_0 is given by λ_0/λ_1. We also note that the above three equations may be considered as three equations in the three unknowns λ_0, $b^2\lambda_0[1 - (\lambda_0/\lambda_1)]$, and λ_0/λ_1 and so could be solved for λ_0. Since these remarks hold for any quantity which is the ratio of two bilinear functions of ψ, we shall write the result of this extrapolation in a rather general way.

Let $F^{(0)}$, $F^{(1)}$, and $F^{(2)}$ be the values of a sequence approaching, as a limit, a quantity F, like λ_0, obtained by successive iterations; for example, $F^{(0)} = \lambda_0^{(n+\frac{1}{2})}$, $F^{(1)} = \lambda_0^{(n+1)}$, $F^{(2)} = \lambda_0^{(n+\frac{3}{2})}$. Then following the above, if the iterations at stage $F^{(0)}$ have proceeded far enough, we may assume that

$$F^{(0)} = F + f; \quad F^{(1)} = F + \epsilon f; \quad F^{(2)} = F + \epsilon^2 f$$

where f is the error at stage $F^{(0)}$, while ϵ, the parameter giving the rate of convergence, is λ_0/λ_1. We may now solve for F:

$$F = F^{(0)} - \left\{[F^{(0)} - F^{(1)}]^2/[F^{(0)} - 2F^{(1)} + F^{(2)}]\right\} \qquad (9.4.107)$$

If two sets of numbers $F^{(n)}$ and $G^{(n)}$ depend upon ϵ in the same way, then it is possible to obtain ϵ from one sequence and use it to extrapolate the other so that

$$F = F^{(0)} - \left\{[F^{(0)} - F^{(1)}][G^{(0)} - G^{(1)}]/[G^{(0)} - 2G^{(1)} + G^{(2)}]\right\} \qquad (9.4.108)$$

with

$$\epsilon = [G^{(2)} - G^{(1)}]/[G^{(1)} - G^{(0)}] \qquad (9.4.109)$$

As we shall see when we illustrate with a specific example, the extrapolation method is remarkably accurate.

It may be also used to extrapolate φ_n to ψ_0. Here, however, $F^{(0)}$, $F^{(1)}$, and $F^{(2)}$ are given by φ_n, $\lambda_0\varphi_{n+1}$, and $\lambda_0^2\varphi_{n+2}$ where λ_0 can be obtained by the extrapolation method.

Lower Bounds for λ_0. Inequality (9.4.104) may also be used to obtain a lower bound on λ_0. We rearrange the terms as follows:

$$\{\lambda_0[\lambda_1 - \lambda_0^{(n+1)}] - \lambda_0^{(n+1)}[\lambda_1 - \lambda_0^{(n+\frac{1}{2})}]\} \geq 0$$

If the iterations have proceeded far enough so that

$$\lambda_1 > \lambda_0^{(n+1)}$$

then

$$\lambda_0^{(n+1)} \geq \lambda_0 \geq \lambda_0^{(n+1)} \left[1 - \frac{\lambda_0^{(n+\frac{1}{2})} - \lambda_0^{(n+1)}}{\lambda_1 - \lambda_0^{(n+1)}}\right] \tag{9.4.110}$$

We remark again that this inequality holds if n is replaced everywhere by $(n + \frac{1}{2})$. It is thus possible to state that λ_0 lies between two numbers, the accuracy of this determination depending upon the agreement between them, *i.e.*, on the smallness of the ratio $[\lambda_0^{(n+\frac{1}{2})} - \lambda_0^{(n+1)}]/[\lambda_1 - \lambda_0^{(n+1)}]$. We still must find a way to estimate λ_1, but before we do so, let us point out that many inequalities of the form of Eq. (9.4.104) exist and therefore many lower bounds other than given by Eq. (9.4.110) can be found. Which of these gives the closest lower bound depends on rather special circumstances arising from the particular values of $\lambda_0^{(a)}$ involved. For example, a lower bound may be obtained from the inequality

$$\int\{\varphi[1 - \lambda_1^2(\mathfrak{M}\mathfrak{L}^{-1})^2]\mathfrak{M}[1 - \lambda_0\mathfrak{L}^{-1}\mathfrak{M}]\varphi\}\,dV \geq 0 \tag{9.4.111}$$

The proof is much the same as the one for inequality (9.4.102). Inserting φ_n for φ in (9.4.111) yields

$$\{[n,n] - \lambda_0[n + 1, n] - \lambda_1^2[n + 1, n + 1] + \lambda_0\lambda_1^2[n + 1, n + 2]\} \geq 0$$

or

$$\{\lambda_0^{(n+\frac{1}{2})}\lambda_0^{(n+1)} - \lambda_0\lambda_0^{(n+1)} - \lambda_1^2 + [\lambda_0\lambda_1^2/\lambda_0^{(n+\frac{3}{2})}]\} \geq 0$$

The lower bound follows if $\lambda_1^2 > [\lambda_0^{(n+1)}\lambda_0^{(n+\frac{3}{2})}]$:

$$\lambda_0 \geq \lambda_0^{(n+\frac{3}{2})}\left\{1 - \frac{\lambda_0^{(n+1)}[\lambda_0^{(n+\frac{1}{2})} - \lambda_0^{(n+\frac{3}{2})}]}{\lambda_1^2 - \lambda_0^{(n+\frac{3}{2})}\lambda_0^{(n+1)}}\right\} \tag{9.4.112}$$

Application of this lower bound requires the values of three successive iterates for λ_0, while lower bound (9.4.110) requires only two.

We now turn to the problem of estimating λ_1. It is important to note that this estimate must be less than or equal to λ_1 but must be greater than $\lambda_0^{(n+1)}$. The customary procedure makes use of the following relation:

$$\text{Spur }(\mathfrak{L}^{-1}\mathfrak{M})^2 = \sum_m \int \chi_m\mathfrak{M}(\mathfrak{L}^{-1}\mathfrak{M})^2\chi_m\,dV = \sum_m \left(\frac{1}{\lambda_m^2}\right) \tag{9.4.113}$$

This relation is of value, because it is often possible to evaluate the Spur directly, as we shall see in an example which will be discussed soon (see also Sec. 12.3. From Eq. (9.4.113), it follows that

$$\left[\mathrm{Spur}(\mathcal{L}^{-1}\mathfrak{M})^2 - \left(\frac{1}{\lambda_0^2}\right)\right] = \left(\frac{1}{\lambda_1^2}\right) + \sum_{m=2}^{\infty}\left(\frac{1}{\lambda_m^2}\right)$$

Therefore

$$\lambda_1^2 > \{1/[\mathrm{Spur}(\mathcal{L}^{-1}\mathfrak{M})^2 - (1/\lambda_0^2)]\} > \{1/[\mathrm{Spur}(\mathcal{L}^{-1}\mathfrak{M})^2 - (1/\lambda_0^{(n+1)})^2]\} \tag{9.4.114}$$

Hence the right side of this inequality may be used in place of λ_1 in either Eq. (9.4.110) or (9.4.112). We can, of course, use Spurs of higher powers of $\mathcal{L}^{-1}\mathfrak{M}$, and these would converge better, but they are considerably more difficult to evaluate. We must use the Spur of at least the second power because the Spur of $\mathcal{L}^{-1}\mathfrak{M}$ is infinite for two or three dimensions.

Other methods of obtaining rough estimates of λ_1 (and the word *rough* should be noted here) take advantage of some special character of the problem involved. It is, for example, possible to employ experimental information. In the following we shall discuss other analytic methods for obtaining lower bounds which require less, or even no, information concerning λ_1.

The first of these that we shall discuss starts out from the quantity $[\beta^2]$.

$$[\beta^2] = \{\textstyle\int[(\mathfrak{M}^{-1}\mathcal{L} - \alpha)\varphi]\mathfrak{M}[(\mathfrak{M}^{-1}\mathcal{L} - \alpha)\varphi]\,dV/\int\varphi\mathfrak{M}\varphi\,dV\}$$

where α is an arbitrary parameter. $[\beta^2]$ is positive. We find its minimum value for a given value of α, setting $\delta[\beta^2] = 0$. We find that the φ for which β^2 is stationary satisfies the equation

$$(\mathfrak{M}^{-1}\mathcal{L} - \alpha)^2\varphi = \beta^2\varphi \quad \text{or} \quad (\mathfrak{M}^{-1}\mathcal{L} - \alpha - \beta)(\mathfrak{M}^{-1}\mathcal{L} - \alpha + \beta)\varphi = 0$$

We see that β^2 equals $(\lambda_n - \alpha)^2$ for those φ's for which β^2 is stationary. In order that the least of these stationary values occur for $\lambda_n = \lambda_0$, we require that

$$\lambda_0 \le \alpha \le \tfrac{1}{2}(\lambda_0 + \lambda_1) \tag{9.4.115}$$

and the least value of β^2 is then $(\alpha - \lambda_0)^2$. Hence, for α satisfying (9.4.115),

$$(\alpha - \lambda_0)^2 \le \left\{\textstyle\int[(\mathfrak{M}^{-1}\mathcal{L} - \alpha)\varphi]\mathfrak{M}[(\mathfrak{M}^{-1}\mathcal{L} - \alpha)\varphi]\,dV/\int\varphi\mathfrak{M}\varphi\,dV\right\}$$

If we let φ equal φ_{n+1}, this inequality becomes

$$(\alpha - \lambda_0)^2 \le \alpha^2 - 2\alpha\lambda_0^{(n+1)} + \lambda_0^{(n+\frac{1}{2})}\lambda_0^{(n+1)} \tag{9.4.116}$$

or

$$\lambda_0 \ge \alpha - \sqrt{\alpha^2 - 2\alpha\lambda_0^{(n+1)} + \lambda_0^{(n+1)}\lambda_0^{(n+\frac{1}{2})}}$$

or

$$\lambda_0 \geq \lambda_0^{(n+1)} - [\sqrt{(\alpha - \lambda_0^{(n+1)})^2 + \lambda_0^{(n+1)}(\lambda_0^{(n+\frac{1}{2})} - \lambda_0^{(n+1)})} - (\alpha - \lambda_0^{(n+1)})] \tag{9.4.117}$$

Inequality (9.4.116) and lower bound (9.4.117) are generalizations of inequalities (9.4.104) and (9.4.110), as may be seen by inserting $\alpha = \frac{1}{2}(\lambda_0 + \lambda_1)$ in (9.4.116). Moreover, since the lower bound (9.4.117) is a steadily increasing function of α, the best lower bound estimate is obtained by taking the maximum value of α, which is just $\frac{1}{2}(\lambda_0 + \lambda_1)$ according to Eq. (9.4.115). Hence the lower bound (9.4.110) is the best lower bound one may obtain from the analysis we have just gone through. Its chief value lies in the absence of any explicit dependence of λ_1. We just need to know where λ_1 is roughly in order to be certain to satisfy inequality (9.4.115).

Before going on to the next type of lower bound, which is rather different in nature from the one discussed above, we should point out one other difficulty in the use of the lower bounds given above in Eqs. (9.4.110) and (9.4.117). These seem to require the evaluation of $\mathfrak{L}^{-1}\mathfrak{M}\varphi_n$. It is not always an easy matter to obtain the inverse operator for $\mathfrak{L}$, and for this reason it is important to show that it is not essential to do so except in so far as a closer estimate of λ_0 will result.

It is, however, only possible to replace the problem of finding $\mathfrak{L}^{-1}$ by that of finding $\mathfrak{M}^{-1}$. The latter is often considerably simpler, because $\mathfrak{M}$ is often a constant or a coordinate function rather than a differential operator. The procedure is as follows: Suppose that we find (by a variational or another method) an estimate of χ_0. Let us call this estimate φ_{n+1}. Then φ_{n+1} is related to φ_n, from Eq. (9.4.94), as follows:

$$\mathfrak{L}\varphi_{n+1} = \mathfrak{M}\varphi_n$$

It is now possible to express $\lambda_0^{(n+\frac{1}{2})}$ and $\lambda_0^{(n+1)}$ in terms of φ_{n+1} and the operators $\mathfrak{L}$ and $\mathfrak{M}$ without the occurrence of the inverse operator $\mathfrak{L}^{-1}$:

$$\lambda_0^{(n+\frac{1}{2})} = \frac{[n,n]}{[n, n+1]} = \frac{\int \varphi_n \mathfrak{M}\varphi_n \, dV}{\int \varphi_n \mathfrak{M}\varphi_{n+1} \, dV} = \frac{\int \varphi_{n+1}\mathfrak{L}\mathfrak{M}^{-1}\mathfrak{L}\varphi_{n+1} \, dV}{\int \varphi_{n+1}\mathfrak{L}\varphi_{n+1} \, dV} \tag{9.4.118}$$

$$\lambda_0^{(n+1)} = \frac{[n+1, n]}{[n+1, n+1]} = \frac{\int \varphi_{n+1}\mathfrak{L}\varphi_{n+1} \, dV}{\int \varphi_{n+1}\mathfrak{M}\varphi_{n+1} \, dV} \tag{9.4.119}$$

In these expressions, only the known function φ_{n+1} appears, as promised. This particular form of the lower bound embodied in Eq. (9.4.117), together with Eqs. (9.4.118) and (9.4.119), has been employed to obtain a lower bound to the energy of the helium atom in the ground state.

Comparison Method for Lower Bounds. A rather different method of obtaining lower bounds is based on comparison theorems very similar to those employed in the Sturmian theory of differential equations (see Sec. 6.3). The physical notion behind this method is quite simple.

Suppose that we wish to solve a Schroedinger equation with a potential $-U(x)$. We now compare the eigenvalues of this problem with one which we can solve exactly and for which the potential $-U'(x)$ is always larger than U, as illustrated in Fig. 9.20, where we show two possible comparison potentials $U_1'(x)$, $U_2'(x)$. Physically, it is clear that the energy of the ground state for either U_1' or U_2' will be below that of $U(x)$ and is therefore a lower bound for the latter case.

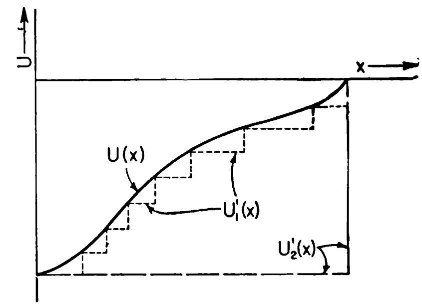

Fig. 9.20 Potential function $U(x)$ and comparison functions U_1' and U_2', where $U_2' \le U_1' \le U(x)$ for all x.

Let us now formulate this comparison principle in a more precise fashion. Let

$$\mathfrak{L}\chi = \lambda\mathfrak{M}\chi \quad \text{and} \quad \mathfrak{L}\varphi = \lambda'\mathfrak{M}'\varphi$$

where

$$\int\psi\mathfrak{M}'\psi\, dV \ge \int\psi\mathfrak{M}\psi\, dV \qquad (9.4.120)$$

for any ψ. We wish to obtain a lower bound to λ_0, the first eigenvalue of the unprimed problem, with eigenfunction χ_0. From the variational principle for λ_0'

$$\lambda_0' \le \left[\int\chi_0\mathfrak{L}\chi_0\, dV \Big/ \int\chi_0\mathfrak{M}'\chi_0\, dV\right]$$

or

$$\lambda_0' \le \lambda_0\left[\int\chi_0\mathfrak{M}\chi_0\, dV \Big/ \int\chi_0\mathfrak{M}'\chi_0\, dV\right]$$

From inequality (9.4.120)

$$\lambda_0 \ge \lambda_0' \qquad (9.4.121)$$

The general usefulness of this method of obtaining lower bounds depends upon the possibility of formulating exactly solvable problems satisfying inequality (9.4.120). The closer the comparison problem to the original problem, the closer the lower bound to λ_0. In Fig. 9.20 we have illustrated two possible comparison potentials for the one-dimensional Schroedinger equation. Similar possibilities exist for three-dimensional problems. For the many-particle Schroedinger equation, comparison problems may be obtained by throwing out the interparticle interactions if they are repulsive or replacing them by oscillator potentials if they are attractive. This technique may also be used to derive the following rough bounds on λ_0:

$$\min(\mathfrak{M}\varphi_{n-1}/\mathfrak{M}\varphi_n) < \lambda_0 < \max(\mathfrak{M}\varphi_{n-1}/\mathfrak{M}\varphi_n) \qquad (9.4.122)$$

where by min and max we mean the minimum and maximum values of the ratio of these functions.

Suppose that we designate the minimum value of $(\mathfrak{L}\varphi_n/\mathfrak{M}\varphi_n)$ by $\underline{\lambda}$ and the maximum value by $\bar{\lambda}$. Then φ_n must satisfy the variational principles $\delta[\underline{\lambda}]$ and $\delta[\bar{\lambda}] = 0$, where

$$[\underline{\lambda}] = \int \varphi \mathcal{L} \varphi \, dV / \int \varphi f \mathfrak{M} \varphi \, dV; \quad [\bar{\lambda}] = \int \varphi \mathcal{L} \varphi \, dV / \int \varphi g \mathfrak{M} \varphi \, dV$$

where it is assumed that f and g are positive functions which necessarily are respectively bigger and less than unity everywhere. They respectively equal $\mathcal{L}\varphi_n / \underline{\lambda} \mathfrak{M} \varphi_n$ and $\mathcal{L}\varphi_n / \bar{\lambda} \mathfrak{M} \varphi_n$.

Since f is greater than unity, it immediately follows, if χ_0 is used as a trial wave function, that

$$\underline{\lambda} \le \int \chi_0 \mathcal{L} \chi_0 \, dV / \int \chi_0 f \mathfrak{M} \chi_0 \, dV \le \int \chi_0 \mathcal{L} \chi_0 \, dV / \int \chi_0 \mathfrak{M} \chi_0 \, dV = \lambda_0$$

proving the theorem, since $\mathcal{L}\varphi_n / \mathfrak{M}\varphi_n$ is just $\mathfrak{M}\varphi_{n+1} / \mathfrak{M}\varphi_n$.

The second leg of Eq. (9.4.122), relating to $\bar{\lambda}$, may be proved from the variation principle satisfied by λ_0:

$$\lambda_0 \le \int \varphi_n \mathcal{L} \varphi_n \, dV / \int \varphi_n \mathfrak{M} \varphi_n \, dV \le \int \varphi_n \mathcal{L} \varphi_n \, dV / \int \varphi_n g \mathfrak{M} \varphi_n \, dV = \bar{\lambda}$$

Note that this theorem, as proved, holds only when $\mathfrak{M}\varphi_{n+1} / \mathfrak{M}\varphi_n$ is a positive function. This is nearly always the case for the lowest mode of vibration, which is generally without nodes.

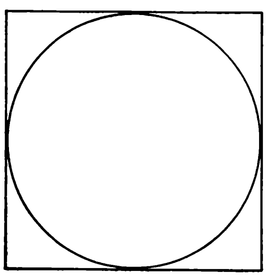

Fig. 9.21 Comparison of resonance frequencies for two different boundaries.

A similar type of lower bound may be obtained for problems involving the perturbation of boundaries. For example, one would expect the resonant frequency for the circular region shown in Fig. 9.21 to be larger than the corresponding frequency for the circumscribing pair. This remark may be made more precise for Dirichlet conditions by making use of variational principle (9.4.70). Let the eigenvalue which applies to the circumscribed boundary be k^2 and its corresponding wave function ψ. Let the eigenvalue for the circumscribing region be $(k')^2$. Then, in the variational principle for $(k')^2$, insert ψ as a trial function, placing it equal to zero in the region between the two boundaries. Then from Eq. (9.4.70) it immediately follows that

$$(k')^2 \le \left[\int (\nabla\psi)^2 \, dV / \int \psi^2 \, dV\right] = k^2$$

or

$$k^2 \ge (k')^2 \tag{9.4.123}$$

It might be added that, since k^2 itself must be less than the corresponding quantity for a boundary which is contained within the original boundary, the k^2 for a region having the same volume as the original volume will come rather close to the exact value. For example, if a boundary circle has a radius a, the length of the side of a square of the same area would be $\sqrt{\pi}\, a$. The value of k^2 for a mode of the same symmetry as the circularly symmetric mode for the circular region is

$$k^2 = 2(\pi / \sqrt{\pi}\, a)^2 \quad \text{or} \quad (ka)^2 = 2\pi$$

which is a fair approximation to the correct value of 5.7832.

An Example. Before going on to the application of the variation-iteration method to other types of problems, we shall work through an illustrative example. Let us consider once more the vibration of a circular membrane of radius a, particularizing again to the lowest mode of circular symmetry. The differential equation is

$$(d^2\psi/dx^2) + (1/x)(d\psi/dx) + \lambda\psi = 0$$

where $x = (ka)$ and $\lambda = (ka)^2$. The positive-definite operators $\mathfrak{L}$ and $\mathfrak{M}$ are

$$\mathfrak{L} = -(xd^2/dx^2 + d/dx); \quad \mathfrak{M} = x$$

and the operator $\mathfrak{L}^{-1}\mathfrak{M}$ is

$$\mathfrak{L}^{-1}\mathfrak{M} = \frac{1}{4\pi} \int G_k(x|x_0)x_0$$

as may be seen from integral equation (9.4.46). The Green's function is given below Eq. (9.4.47). We take $\varphi_0 = (1 - x^2)$ and iterate, obtaining

$$\begin{aligned}
\varphi_0 &= 1 - x^2 \\
\varphi_1 &= \tfrac{1}{16}[3 - 4x^2 + x^4] \\
\varphi_2 &= [1/(16)(576)][19 - 27x^2 + 9x^4 - x^6]
\end{aligned}$$

The elements $[n,m]$ may now be readily evaluated, and the following values of $\lambda_0^{(a)}$ obtained:

$$\lambda_0^{(0)} = 6; \quad \lambda_0^{(\frac{1}{2})} = 5.818182; \quad \lambda_0^{(1)} = 5.789474; \quad \lambda_0^{(\frac{3}{2})} = 5.784355 \tag{9.4.124}$$

These are to be compared with the exact value of 5.783186; $\lambda_0^{(\frac{3}{2})}$ has an error of about 1 part in 5,000. Of course in practice we must rely on the lower bound (9.4.110) for an estimate of the error. We need a lower bound for λ_1, which may be obtained from formula (9.1.114). The integral giving $\mathrm{Spur}(\mathfrak{L}^{-1}\mathfrak{M})^2$ is

$$\mathrm{Spur}(\mathfrak{L}^{-1}\mathfrak{M})^2 = \int_0^1 \int_0^1 G_k(x_0|x)xG_k(x|x_0)x_0\, dx_0\, dx = \tfrac{1}{32}$$

Hence λ_1 is greater than 27.2126. Actually it equals 30.4713. Inserting this lower bound for λ_1 into lower bound (9.4.110), we obtain

$$5.782972 < \lambda_0 < 5.784355$$

We may improve upon this answer by using the extrapolation method of Eq. (9.4.107). This gives 5.783244, which is now in error by only one part in a million.

Other examples of the variation-iteration method are given in Chap. 12.

$\mathfrak{L}$ Not Positive-Definite. We retain the assumption that $\mathfrak{M}$ is positive-definite and that $\mathfrak{L}$ is self-adjoint. Problems of this type come up in the application of the variation-iteration method to the determination of scattering phase shifts. As may be seen from Eq. (9.3.35), the kernels of the integral equation involve sinusoidal functions which take on both positive and negative values. In some problems, *e.g.*, the binding energy of the deuteron with tensor forces included, $\mathfrak{L}$ is positive-definite and $\mathfrak{M}$ is not. The results for this case are very similar to those reported here.

Under the foregoing assumptions the eigenvalues λ_n are real, but assume both positive and negative values, the eigenvalue spectrum being now unbounded in both directions. Let us order the eigenvalues according to their absolute value, $|\lambda_0|$ being the smallest, $|\lambda_1|$ the next largest, and so on. Moreover, let us choose the sign of $\mathfrak{L}$ so that λ_0 is positive. The iteration procedure (9.4.94) and the definitions of $\lambda_0^{(n)}$, $\lambda_0^{(n+\frac{1}{2})}$ in Eq. (9.4.97) may be set up as before. It is, however, necessary now to revise the theorems regarding the sequence $\lambda_0^{(a)}$ as given in Eqs. (9.4.99), (9.4.100), and so on.

Some information may be obtained by operating on the original eigenvalue problem (9.4.93) so as to obtain one in which all the operators are positive-definite. The simplest of these is

$$[\mathfrak{L}\mathfrak{M}^{-1}\mathfrak{L}]\psi = \lambda^2\mathfrak{M}\psi$$

The theorems of the preceding discussion for the positive-definite case may now be taken over *in toto*. The successive approximations to χ_0, starting with φ_0, are $\varphi_0, \varphi_2, \ldots, \varphi_{2n}$. Hence the quantity corresponding to $\lambda_0^{(n)}$ and $\lambda_0^{(n+\frac{1}{2})}$ obtained by inserting φ_{2n} into the variational principles for λ_0 corresponding to Eq. (9.4.94) are

$$\lambda_0^{(n)} \longrightarrow \lambda_0^{(2n-\frac{1}{2})}\lambda_0^{(2n)}; \quad \lambda_0^{(n+\frac{1}{2})} \longrightarrow \lambda_0^{(2n+\frac{1}{2})}\lambda_0^{(2n+1)}$$

These may now be inserted into Eqs. (9.4.99) and (9.4.100) and yield

$$[\lambda_0^{(2n-\frac{1}{2})}\lambda_0^{(2n)}] \geq [\lambda_0^{(2n+\frac{1}{2})}\lambda_0^{(2n+1)}] \geq \lambda_0^{(2n+\frac{3}{2})}\lambda_0^{(2n+2)} \geq \cdots \geq \lambda_0^2 \tag{9.4.125}$$

$$\left.\begin{matrix} \lambda_0^{(2n-\frac{1}{2})}\lambda_0^{(2n)} \\ \lambda_0^{(2n+\frac{1}{2}}\lambda)_0^{(2n+1)} \end{matrix}\right\} \xrightarrow[n\to\infty]{} \lambda_0^2; \quad \text{if } \int \chi_0\mathfrak{M}\varphi_0\,dV \neq 0$$

The analogue of inequality (9.4.104) may now be established, convergence criteria and a lower bound being consequences thereof. The convergence is governed by the ratio $(\lambda_0/\lambda_1)^2$ as follows:

$$\frac{\lambda_0^{(2n+\frac{3}{2})}\lambda_0^{(2n+2)} - \lambda_0^2}{\lambda_0^{(2n+\frac{1}{2})}\lambda_0^{(2n+1)} - \lambda_0^2} \leq \frac{\lambda_0^{(2n+\frac{3}{2})}\lambda_0^{(2n+2)}}{\lambda_1^2} \simeq \left(\frac{\lambda_0}{\lambda_1}\right)^2$$

while the lower bound (9.4.110) becomes

$$\lambda_0^2 \geq \lambda_0^{(2n+\frac{3}{2})}\lambda_0^{(2n+2)}\left[1 - \frac{\lambda_0^{(2n+\frac{1}{2})}\lambda_0^{(2n+1)} - \lambda_0^{(2n+\frac{3}{2})}\lambda_0^{(2n+2)}}{\lambda_1^2 - \lambda_0^{(2n+\frac{3}{2})}\lambda_0^{(2n+2)}}\right]$$

It is important to note that the convergence depends upon the absolute value of the ratio λ_0/λ_1. Thus, if two eigenvalues exist which are equal in magnitude but opposite in sign, there will be no convergence because the iterated problem (9.4.124) has a degeneracy.

It is clear that the above relations do not give us complete information. This may be most readily seen if Eqs. (9.4.125) and (9.4.126) are expressed directly in terms of $\lambda_0^{(a)}$ (using $\varphi_1, \varphi_3 \ . \ . \ .$ as well as $\varphi_0, \varphi_2 \ . \ . \ .$):

$$\lambda_0^{(n-\frac{1}{2})}\lambda_0^{(n)} \geq \lambda_0^{(n+\frac{1}{2})}\lambda_0^{(n+1)} \geq \cdot \cdot \cdot \geq \lambda_0^2 \qquad (9.4.126)$$
$$\lambda_0^{(n-\frac{1}{2})}\lambda_0^{(n)} \xrightarrow[n\to\infty]{} \lambda_0^2; \quad \text{if } \int \chi_0 \mathfrak{M} \varphi_0 \, dV \neq 0$$

The behavior of the individual $\lambda_0^{(a)}$ or equivalently of the product $\lambda_0^{(n)}\lambda_0^{(n+\frac{1}{2})}$ is not stated. We therefore turn to the properties of the individual $\lambda_0^{(a)}$.

From the fact that $\mathfrak{M}$ is positive-definite, inequality (9.4.101) is still valid:

$$\lambda_0^{(n-\frac{1}{2})} \geq \lambda_0^{(n)}$$

Combining this with Eq. (9.4.126), we obtain

$$\lambda_0^{(n-\frac{1}{2})} \geq \lambda_0 \qquad (9.4.127)$$

We shall now show that the sequence $\lambda_0^{(n-\frac{1}{2})}$ is monotonically approaching λ_0 from above if the iterations have proceeded far enough so that

$$\lambda_1^2 > \lambda_0^{(n+1)}\lambda_0^{(n+\frac{3}{2})}$$

When both operators were positive-definite, both sequences approached λ_0 monotonically from above. In the present case this is true of one sequence only. The proof is based on inequality (9.4.111), which holds whether $\mathfrak{L}$ is positive-definite or not. It leads to the inequality

$$[\lambda_0^{(n+\frac{1}{2})}\lambda_0^{(n+1)} - \lambda_0\lambda_0^{(n+1)} - \lambda_1^2 + (\lambda_0\lambda_1^2/\lambda_0^{(n+\frac{3}{2})})] \geq 0$$

Rearranging and making use of our assumption on λ_1^2, we obtain

$$[\lambda_0^{(n+\frac{3}{2})} - \lambda_0]/[\lambda_0^{(n+\frac{1}{2})} - \lambda_0] \leq [\lambda_0^{(n+1)}\lambda_0^{(n+\frac{3}{2})}/\lambda_1^2] < 1 \qquad (9.4.128)$$

For this inequality to be true, it is necessary that

$$\lambda_0^{(n+\frac{1}{2})} > \lambda_0^{(n+\frac{3}{2})} \qquad (9.4.129)$$

proving the monotonicity of the $\lambda_0^{(n+\frac{1}{2})}$ sequence.

No such statement may be made for the $\lambda_0^{(n)}$ sequence. Its behavior may be quite erratic; $\lambda_0^{(n)}$ may be greater than λ_0 for one value of n and smaller than λ_0 for another. A more definite statement may be made if the iterations have proceeded so far that the main contaminant in φ_n is χ_1. Then if $\varphi_n = \chi_0 + b\chi_1$

$$\lambda_0^{(n+1)} \simeq \lambda_0 + b^2\lambda_0[1 - (\lambda_0/\lambda_1)](\lambda_0/\lambda_1)$$
$$\lambda_0^{(n+2)} \simeq \lambda_0 + b^2\lambda_0[1 - (\lambda_0/\lambda_1)](\lambda_0/\lambda_1)^3$$

We see that in this limit (*i.e.*, as n goes to infinity) the $\lambda_0^{(n)}$ sequence will approach λ_0 monotonically from above or below, depending on the sign of

λ_0/λ_1. If λ_0/λ_1 is positive, $\lambda_0^{(n)}$ will approach λ_0 from above; if negative, from below.

We may now go on to discuss convergence and lower bounds. The rate of convergence is given by Eq. (9.4.28) and is just as predicted, being given by the ratio $(\lambda_0/\lambda_1)^2$. Various lower bounds similar to Eq. (9.4.110) may be devised. A few of these will now be given.

Inequality (9.4.110) still applies when λ_1 is replaced by its absolute value. This follows from the validity of the inequality

$$\int \{\varphi[1 - \mathfrak{M}\mathfrak{L}^{-1}|\lambda_1|]\mathfrak{M}[1 - \mathfrak{L}^{-1}\mathfrak{M}\lambda_0]\varphi \, dV \geq 0$$

This may be most readily seen by inserting a general expansion for φ in terms of χ_p. From it one obtains

$$\lambda_0 \geq \lambda_0^{(n+1)} \left[1 - \frac{\lambda_0^{(n+\frac{1}{2})} - \lambda_0^{(n+1)}}{|\lambda_1| - \lambda_0^{(n+1)}}\right] \tag{9.4.130}$$

Inequality (9.4.111) is obviously appropriate for the present problem, since λ_1^2 only enters and thus the sign of λ_1 is not important. As a consequence, lower bound (9.4.112) is valid for nonpositive definite $\mathfrak{L}$.

The indefinite character of $\mathfrak{L}$ proves to be an advantage as well as a disadvantage, as noted above. For example, the possibility that $\lambda_0^{(n)}$, n integer, may be smaller than λ_0 permits the derivation of an upper bound which can be smaller than $\lambda_0^{(n+\frac{1}{2})}$. We start from the inequality

$$\int \{\varphi[1 + |\lambda_1|\mathfrak{M}\mathfrak{L}^{-1}]\mathfrak{M}[1 - \lambda_0\mathfrak{L}^{-1}\mathfrak{M}]\varphi\} \, dV \geq 0$$

Letting φ equal φ_n, we obtain an upper bound to λ_0:

$$\lambda_0 \leq \lambda_0^{(n+1)} \left[1 + \frac{\lambda_0^{(n+\frac{1}{2})} - \lambda_0^{(n+1)}}{|\lambda_1| + \lambda_0^{(n+1)}}\right] \tag{9.4.131}$$

In a second example we shall show that it is possible to improve upon inequality (9.4.130) if by some means it may be shown that λ_1 is less than zero. Then the inequality below holds.

$$\int \{\varphi[1 - \mathfrak{M}\mathfrak{L}^{-1}|\lambda_2|]\mathfrak{M}[1 - \mathfrak{L}^{-1}\mathfrak{M}\lambda_0]\varphi\} \, dV \geq 0 \tag{9.4.132}$$

If a general expansion for φ in terms of χ_p is substituted for φ, this inequality is equivalent to

$$\sum a_p^2 \left[1 - \left(\frac{|\lambda_2|}{\lambda_p}\right)\right]\left[1 - \left(\frac{\lambda_0}{\lambda_p}\right)\right] \geq 0$$

This inequality leads to the following lower bound:

$$\lambda_0 \geq \lambda_0^{(n+1)} \left[1 - \frac{\lambda_0^{(n+\frac{1}{2})} - \lambda_0^{(n+1)}}{|\lambda_2| - \lambda_0^{(n+1)}}\right] \tag{9.4.133}$$

It is, of course, necessary to obtain a lower bound for λ_2. The following formula, which is a simple extension of Eq. (9.4.114), will do this:

$$\lambda_2^2 > \{1/[\text{Spur}(\mathfrak{L}^{-1}\mathfrak{M})^2 - (1/\lambda_0^2) - (1/\lambda_1^2)]\} \qquad (9.4.134)$$

If now upper bounds are inserted for λ_0^2 and λ_1^2, a lower bound for $|\lambda_2|$ is obtained. Of course, the somewhat more special procedures described after (9.4.120) may also be used.

These theorems have been employed in only a few problems as yet. Perhaps their most important application might be to integral equations for scattering, such as Eq. (9.4.60), in which the depth of the potential U is used as a variational parameter and η_l is assumed. Specific cases have not yet been investigated sufficiently, however.

Variational Methods for the Higher Eigenvalues. The variational methods described so far are useful only for the calculation of the eigenvalue of least absolute value. We now turn to the problem of employing similar methods for the other eigenvalues. Most of the procedures which have been developed depend upon the fact that the wave functions for the higher modes are orthogonal to the lowest one. If the wave function for the lowest mode is available, then one employs trial wave functions which are orthogonal to it. This will lead to a determination of the next eigenvalue λ_1 and its corresponding eigenfunction χ_1. If λ_2 is desired, trial wave functions which are orthogonal to χ_0 and χ_1 are used, and so on. In practice, the exact χ_0 is usually not known and the requirement of orthogonality to an approximation of χ_0 does not completely eliminate χ_0 from the trial function. This leads to a certain unavoidable error in estimating λ_1, which we shall now consider.

The variation-iteration method is most suitable for this purpose. We begin with an initial trial function ψ_0 which is orthogonal to χ_0', the approximation we have obtained for χ_0, the eigenfunction for the lowest mode. We then iterate on ψ_0, reorthogonalize, iterate again, and so on. If χ_0' is expanded in terms of the eigenfunction χ_n, the value of the coefficients of all terms but χ_0 should be small:

$$\chi_0' = \chi_0 + \sum_{n=1}^{\infty} \epsilon_n \chi_n; \quad \epsilon_n \ll 1$$

The approximate value of λ_0 obtained from inserting this function into the variational principle differs from λ_0 by $\Delta\lambda_0$:

$$\Delta\lambda_0 \simeq \sum_{n=1}^{\infty} \epsilon_n^2(\lambda_n - \lambda_0) \qquad (9.4.135)$$

A trial function orthogonal to χ_0' to first order in ϵ_n is given by

$$\psi_0 = -\left(\sum_{n=1}^{\infty} \epsilon_n a_n\right)\chi_0 + \sum_{n=1}^{\infty} a_n(1 - \epsilon_n)\chi_n$$

If we now iterate, we obtain ψ_0':

$$\psi_0' = -\left(\sum_{n=1}^{\infty} \epsilon_n a_n\right)\left(\frac{\chi_0}{\lambda_0}\right) + \sum_{n=1}^{\infty} a_n(1 - \epsilon_n)\left(\frac{\chi_n}{\lambda_n}\right)$$

Reorthogonalizing to χ_0' gives

$$\psi_1 = -\sum_{n=1}^{\infty}\left(\frac{\epsilon_n a_n}{\lambda_n}\right)\chi_0 + \sum_{n=1}^{\infty} a_n(1 - \epsilon_n)\left(\frac{\chi_n}{\lambda_n}\right)$$

It is clear that the pth iterate orthogonal to χ_0' is

$$\psi_n = -\sum_{n=1}^{\infty}\left(\frac{\epsilon_n a_n}{\lambda_n^p}\right)\chi_0 + \sum_{n=1}^{\infty} a_n(1 - \epsilon_n)\left(\frac{\chi_n}{\lambda_n^p}\right) \qquad (9.4.136)$$

We note the following very significant feature: The ratio of the coefficient of χ_0 to the coefficient of χ_1 tends toward a constant as the number of iterations p increases, while the ratio of the coefficients of the other χ's to that of χ_1 tends to zero. Inserting Eq. (9.4.136) into the variational principle for the eigenvalue yields

$$\frac{\lambda_0\left[\sum_n (a_n\epsilon_n/\lambda_n^p)\right]^2 + \sum_n\left[a_n^2(1 - \epsilon_n)^2/\lambda_n^{2p-1}\right]}{\left[\sum_n (a_n\epsilon_n/\lambda_n^p)\right]^2 + \sum_n\left[a_n^2(1 - \epsilon_n)^2/\lambda_n^{2p}\right]}$$

If we now take the limit of this ratio as the number of iterations goes to infinity, *i.e.*, as p becomes infinite, it approaches λ_1', which will differ from λ_1:

$$\lambda_1' = \lambda_1\{1 - [1 - (\lambda_0/\lambda_1)]\epsilon_1^2\} \qquad (9.4.137)$$

Hence there is an unavoidable error in the determination of λ_1 which is present even after an infinite number of iterations. The final value of λ_1' is less than λ_1, the deviation being proportional to the amount of χ_1 present in χ_0'. An upper bound to the error in terms of $\Delta\lambda_0$ may be obtained, since from Eq. (9.4.135)

$$\epsilon_1^2 \le [\Delta\lambda_0/(\lambda_1 - \lambda_0)]$$

Therefore

$$\lambda_1' \ge \lambda_1\{1 - (\Delta\lambda_0/\lambda_1)\} \qquad (9.4.138)$$

so that $\Delta\lambda_0/\lambda_1$ measures the maximum value of the fractional error, which result is the main conclusion of this discussion. Hence, if the error in the determination of λ_0 is sufficiently small, there is no difficulty in obtaining the value of λ_1 by the process of iteration and orthogonalization described above. The convergence of the iterative scheme is fixed by the ratio λ_1/λ_2. In conclusion, we recall the linear combination of an iterate of ψ_n, ψ'_n, and χ'_0 which is orthogonal to χ'_0:

$$\psi_{n+1} = \psi'_n - \left(\int \psi'_n \mathfrak{M} \chi'_0 \, dV \Big/ \int \chi'_0 \mathfrak{M} \chi'_0 \, dV\right)\chi'_0 \tag{9.4.139}$$

The above procedure may be readily adapted for the calculation of λ_2 and higher eigenvalues. Once λ_1 (subject to the error described above) and its corresponding eigenfunction χ'_1 are determined, we now choose a trial function orthogonal to χ'_0 and χ'_1, iterate, reorthogonalize, and so on. It is clear that, as we go on to higher and higher eigenvalues, the unavoidable errors will accumulate. Moreover, in order to obtain a particular higher eigenvalue, it becomes necessary to determine all the smaller eigenvalues and eigenfunctions. We now turn to a method which does not have these difficulties but develops an equation by which any particular eigenvalue can be calculated.

Method of Minimized Iterations. The iterative schemes described above are designed to emphasize one eigenvalue at the expense of the remainder, so that after several iterations the trial function consists mainly of one of the eigenfunctions and thus contains very little information on the other eigenfunctions. The iterative method we shall develop here will consist in the iterative construction of a set of orthogonal functions each of which will contain fairly complete information on all or some of the eigenfunctions. This set then forms the basis for an expansion of the solution of the problem and leads, as we shall see, to a continued-fraction equation for the eigenvalues.

We shall designate the members of the set by ψ_n, the first iterate of ψ_n by ψ'_n:

$$\mathfrak{L}\psi'_n = \mathfrak{M}\psi_n \tag{9.4.140}$$

The following abbreviations for integrals involving $\mathfrak{M}$ will be used:

$$\{n,m\} = \int \psi_n \mathfrak{M} \psi_p \, dV \tag{9.4.141}$$

$$\{n',m\} = \int \psi'_n \mathfrak{M} \psi_p \, dV \tag{9.4.142}$$

Let the initial trial function be ψ_0. The function ψ_1 is determined by the condition that it is that linear combination of ψ_0 and ψ'_0 which is orthogonal to ψ_0:

$$\psi_1 = \psi'_0 - [\{0',0\}/\{0,0\}]\psi_0 \tag{9.4.143}$$

The function ψ_2 is determined by the condition that it must consist of ψ'_1, ψ_1, ψ_0 and be orthogonal to ψ_1 and ψ_0. Hence

$$\psi_2 = \psi'_1 - [\{1',1\}/\{1,1\}]\psi_1 - [\{1',0\}/\{0,0\}]\psi_0 \tag{9.4.144}$$

Similarly, one is tempted to write a like expression for ψ_3. However, as we shall show, this and all the remaining ψ_n's contain only three terms. For example,

$$\psi_3 = \psi'_2 - [\{2',2\}/\{2,2\}]\psi_2 - [\{2',1\}/\{1,1\}]\psi_1 \tag{9.4.145}$$

The ψ_0 term is multiplied by the coefficient $[\{2',0\}/\{0,0\}]$, which is zero, since

$$\{2',0\} = \int (\mathcal{L}^{-1}\mathfrak{M}\psi_2)\mathfrak{M}\psi_0\,dV = \int \psi_2\mathfrak{M}\mathcal{L}^{-1}\mathfrak{M}\psi_0\,dV = \{2,0'\}$$

However, the function ψ_0' may, from Eq. (9.4.143), be expressed in terms of ψ_1 and ψ_0, both of which are orthogonal to ψ_2 so that $(2,0')$ is zero. In general, we find

$$\psi_{n+1} = \psi_n' - \left[\frac{\{n',n\}}{\{n,n\}}\right]\psi_n - \left[\frac{\{n', n-1\}}{\{n-1, n-1\}}\right]\psi_{n-1} \qquad (9.4.146)$$

The coefficients of ψ_{n-2} and so on are zero. For example, the coefficient of ψ_{n-2} is proportional to

$$\{n', n-2\} = \{n, (n-2)'\}$$

However, ψ_{n-2}' may be expressed in terms of ψ_{n-1}, ψ_{n-2}, and so on, each of which is orthogonal to ψ_n, so that $\{n, (n-2)'\}$ is zero.

Note that Eq. (9.4.146) applies to the ψ_1 equation (9.4.143) if we adopt the convention that a ψ with negative subscript, therefore, ψ_{-1}, is zero. We have thus developed a scheme for generating an orthogonal set by means of a comparatively simple three-term recurrence formula. One small simplification should be noted:

$$\{n', n-1\} = \{n,n\} \qquad (9.4.147)$$

If we were dealing with a problem involving a finite number, say N, of eigenfunctions and eigenvalues, then ψ_{N+1} would be zero. Since clearly all the eigenfunctions χ_0 to χ_N of $\mathcal{L}\chi = \lambda\mathfrak{M}\chi$ must be a linear combination of the orthogonal set ψ_0 to ψ_N, and since ψ_{N+1} is expandable in the χ's, it must be a linear combination of the ψ's. Since it must also be orthogonal to each of them, it must be zero. (This is true, of course, only if ψ_0, the initial trial function, involves all of the eigenfunctions; if not, the first ψ_n which is zero would occur for n less than $N + 1$.) Barring this accident, the set ψ_n is then complete and orthogonal.

We may then expand ψ, the solution of

$$\mathcal{L}\chi = \lambda\mathfrak{M}\chi$$

in terms of ψ_n:

$$\chi = \sum_n a_n\psi_n$$

Substituting this expansion into the equation gives

$$\begin{aligned}\sum a_n\psi_n &= \lambda \sum a_n\psi_n' \\ &= \lambda \sum a_n \left[\psi_{n+1} + \frac{\{n',n\}}{\{n,n\}}\psi_n + \frac{\{n', n-1\}}{\{n-1, n-1\}}\psi_{n-1}\right]\end{aligned}$$

We therefore find the following three-term recursion formula for a_n:

$$a_{n-1} + a_n \left[\frac{\{n',n\}}{\{n,n\}} - \left(\frac{1}{\lambda}\right) \right] + \left[\frac{\{(n+1)', n\}}{\{n,n\}} \right] a_{n+1} = 0 \qquad (9.4.148)$$

The equation determining λ may be obtained by application of the technique developed for handling three-term recursion formula developed in Chap. 5 in the section on Mathieu function [Eq. (5.2.78) and following]. One introduces the new dependent variable

$$R_n = a_{n-1}/a_n \qquad (9.4.149)$$

so that Eq. (9.4.148) becomes

$$R_n = \left[\left(\frac{1}{\lambda}\right) - \frac{\{n',n\}}{\{n,n\}} \right] + \left[\frac{\{(n+1), (n+1)\}}{\{n,n\}} \right] \frac{1}{R_{n+1}}$$

By successive substitution we obtain

$$R_1 = (1/\lambda) - [\{1',1\}/\{1,1\}] + \cfrac{\{2,2\}/\{1,1\}}{\cfrac{1}{\lambda} - \cfrac{\{2',2\}}{\{2,2\}} + \cfrac{\{3,3\}/\{2,2\}}{\cfrac{1}{\lambda} - \cfrac{\{3',3\}}{\{3,3\}} + \cfrac{\{4,4\}/\{3,3\}}{\cfrac{1}{\lambda} - \cfrac{\{4',4\}}{\{4,4\}} + \cdots}}}$$

However, from the relation

$$a_0[(\{0',0\}/\{0,0\}) - (1/\lambda)] + a_1(\{1',0\}/\{0,0\}) = 0$$

we also have

$$R_1 = \frac{(\{1',0\}/\{0,0\})}{(1/\lambda) - (\{0',0\}/\{0,0\})} \qquad (9.4.150)$$

Substituting this result for R_1 in Eq. (9.4.149) gives an equation for $1/\lambda$. The various methods for solving such a continued-fraction equation of this kind have been discussed in Chap. 5 following Eq. (5.2.78). This continued-fraction expansion is an exact representation of the secular equation for $1/\lambda$ which may be obtained from Eq. (9.4.148). Moreover, it terminates if the operators $\mathfrak{L}$ and $\mathfrak{M}$ are such that there is a finite number of eigenvalues and eigenfunctions.

Upon comparing the method of minimized iterations and the variation-iteration method as described earlier, we note that the functions ψ_n are simply mutually orthogonal linear combinations of the iterates φ_n. These combinations could have been determined directly by the Schmidt process (see Chap. 8). The present procedure is, however, superior in numerical applications where the Schmidt method would rapidly become inaccurate because of round-off errors. In the method of minimized iterations this may be corrected for by checking the orthogonality of a new ψ_n with all preceding ψ_n's. Any accumulated error may then be corrected for by subtracting off the appropriate amount of each of the preceding ψ_n's as determined by Eq. (9.4.139).

As a corollary, we can note that the ratios $[\{n',n\}/\{n,n\}]$ and also $\{n,n\}/\{n+1, n+1\}]$ may be expressed in terms of the $\lambda_0^{(n)}, \lambda_0^{(n+\frac{1}{2})}$. For example,

$$\frac{\{0',0\}}{\{0,0\}} = \frac{1}{\lambda_0^{(\frac{1}{2})}}; \quad \frac{\{1',0\}}{\{0,0\}} = \frac{1 - (\lambda_0^{(1)}/\lambda_0^{(\frac{3}{2})})}{\lambda_0^{(1)}\lambda_0^{(\frac{1}{2})}}$$

$$\frac{\{1',1\}}{\{1,1\}} = \frac{1}{\lambda_0^{(\frac{3}{2})}}\left[\frac{1 - (\lambda_0^{(\frac{3}{2})}/\lambda_0^{(\frac{1}{2})})}{1 - (\lambda_0^{(1)}/\lambda_0^{(\frac{1}{2})})}\right] - \frac{1}{\lambda_0^{(\frac{1}{2})}} \tag{9.4.151}$$

Using the data given earlier [Eq. (9.4.124)] on the values $\psi_0^{(n)}$ in the application of the variation-iteration method to the circularly symmetric vibrations of a membrane, the various ratios given in Eq. (9.4.151) may be evaluated and inserted into Eqs. (9.4.150) and (9.4.149), dropping all terms in the latter involving the index 2 or higher. We then obtain a quadratic equation for λ which may be solved for λ_0 and λ_1. We obtain 5.78325 for λ_0 (the exact value is 5.78319), while for λ_1 we find 32.5, which is to be compared with the exact value of 30.471. At least one more term of the continued-fraction equation must be included before results for λ_1 as accurate as those for λ_0 will become available.

In conclusion, we should like to point out one other advantage of the method of minimized iterations. If we are dealing with a given interval of integration, then for a given weight function $\mathfrak{M}$ there is only one set of orthogonal functions satisfying the appropriate boundary conditions. If these should be known in advance, it is no longer necessary to carry out orthogonalization (9.4.146). One need only determine the constant which must multiply each orthogonal function in order that it satisfy Eq. (9.4.146).

Problems for Chapter 9

9.1 An elliptical membrane of mass σ per unit area under a tension T is rigidly clamped along its edge, an ellipse of major axis a and minor axis b. Show that a solution of its free transverse vibration must be expressible in terms of the series

$$\psi = \sum_{m,\,n=1}^{\infty} A_{mn} x^{2m} y^{2n}$$

Obtain the secular determinant, giving the wave number k, by substituting in Eq. (9.4.71) and performing the required variations. Obtain a perturbation solution for the lowest mode, using the dominant term A_{11}.

Obtain the variational solution if only the terms A_{11}, A_{02}, A_{20} are included in the trial wave function. Obtain a lower bound from Eq. (9.4.17).

9.2 A particle of mass m moves in a potential $-V_0e^{-\beta r}$, where β and V_0 are constants. Show that the spherically symmetric bound state solution of the Schroedinger equation may be put in the form $u(x)$, where u is a solution of

$$(d^2u/dx^2) + [-1 + \lambda e^{-\frac{1}{2}x}]u = 0; \quad u = 0 \text{ at } x = 0 \text{ and } \infty$$

where a particular value has been taken for β. Starting with the initial trial function $u_0 = e^{-x}$, obtain u_1 and u_2 according to the method of minimized iterations, considering λ as the eigenvalue parameter. Solve the resultant continued fraction for λ. Compare your answer with the exact one, given in Sec. 12.3.

9.3 A uniform electric field E in the z direction acts upon an electric dipole of moment p which is free to rotate only. Set up the Schroedinger equation in spherical coordinates (ϑ,φ), where ϑ is the angle between the dipole axis and z. Show that, when the motion is independent of φ, the equation becomes

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\psi}{d\mu}\right] + \left(\frac{2I}{\hbar^2}\right)[W + Ep\mu]\psi = 0$$

where $\mu = \cos\vartheta$ and I is the moment of inertia. Solve for the state of lowest energy employing (*a*) iteration-perturbation method, (*b*) variation-iteration method starting with a constant as the initial trial function. Perform two iterations, and obtain upper and lower bounds. Note that in the variation method the eigenvalue parameter is Ep. Compare the results of the two methods.

9.4 A plane sound wave is incident upon a disk of radius a at an angle ϑ. Obtain the cross section for scattering in the long-wavelength limit (see Sec. 10.3 for solutions of the corresponding Laplace equation). Find the pressure and torque exerted on the disk.

9.5 A sound wave is propagated down a duct with sudden change of cross section. The duct is enclosed by the following four lines: $y = a$, the positive x axis, the y axis between $y = 0$ and $y = a - b$, and the line $y = a - b$ for $x \le 0$. We consider modes independent of z. Show that, under a conformal transformation $w = w(\zeta)$ $\zeta = x + iy$, $w = u + iv$, the Helmholtz equation becomes

$$\frac{\partial^2\psi}{\partial u^2} + \frac{\partial^2\psi}{\partial v^2} + k^2\left|\frac{d\zeta}{dw}\right|^2\psi = 0$$

Determine the value of $|d\zeta/dw|$ for the transformation from the step channel to one without a step (see Prob. 10.21). Solve the problem of the reflection of a wave traveling in the duct in the positive x direction by using the Born approximation in the (u,v) coordinate system.

9.6 In calculating scattering problems we often wish to obtain the asymptotic phase of the radial factor of the solution [see Eq. (9.3.13) and Secs. 11.3 and 12.3]. In such a problem, suppose that the equation for the radial factor is

$$(d^2u/dr^2) + [k^2 + \lambda U]u = 0; \quad u(0) = 0; \quad u \to \sin(kr - \eta)$$

Show that the solution u_+ of the above equation, which approaches e^{ikr} for large r, satisfies the following integral equation

$$u_+(r) = e^{ikr}\left(1 - \frac{i\lambda}{2k}\int_0^\infty e^{-ikr_0}Uu\,dr_0\right) + \frac{i\lambda}{2k}\int_0^\infty e^{ik(r-r_0)}U(r_0)u(r_0)\,dr_0$$

Find the integral equation for the function $u_-(r)$ which approaches e^{-ikr}. Show that

$$u = u_-(r) + S(k)u_+(r); \quad S(k) = -[u_-(0)/u_+(0)]$$

Relate S to the phase shift η, and show that $|S| = 1$ for real U. Show that $S(\alpha) = 0$ determines the possible bound states of the system. Evaluate u_+ and u_- and therefore S to second order in λ for a potential well which is constant out to $r = a$ after which it is zero (employing e^{ikr} as a starting approximation for u_+ and e^{-ikr} for u_-). Compare with the Fredholm solution.

9.7 Show that the variational integral [Eq. (9.4.61)] for $\cot\eta_0$ becomes, in the limit of k small,

$$k\cot\eta_0 = \left[\int_0^\infty Uu_0^2\,dr - 2\int_0^\infty Uu_0\int_0^{r_0} rUu_0\,dr_0\,dr\right]\Big/\left[\int_0^\infty Uu_0r\,dr\right]^2$$

Insert the trial wave function $\alpha + \beta r$ in this expression. The potential U is attractive, having a constant depth λ out to a radius a and is zero for $r > a$. Determine α, β, r, and finally $k\cot\eta_0$ by the variational method.

9.8 The radial Schroedinger equation for a particle of angular momentum l is

$$(d^2u/dr^2) + [k^2 + U(r) - l(l+1)/r^2]u = 0$$
$$u(0) = 0; \quad u \to \cos[kr - \tfrac{1}{2}(l+1)\pi - \eta]; \quad \text{as } r \to \infty$$

Show that the function $v(r)$

$$y(r) = u(r) + w_1\cot\eta - w_{2\infty}; \quad w_1 = krj_l(kr); \quad w_{2\infty} = \cos(kr - \tfrac{1}{2}l\pi)$$

satisfies the equation

$$\frac{d^2y}{dr^2} + k^2y - \frac{l(l+1)}{r^2}(y + w_{2\infty}) + U(y - \cot\eta w_1 + w_{2\infty}) = 0$$
$$y(0) = w_{2\infty}(0) + \text{terms of order } r^{l+1}; \quad y(\infty) = 0$$

Show that a variational principle may be found for $k \cot \eta$:

$$[k \cot \eta] = \int_0^\infty dr \left\{ \left(\frac{dy}{dr}\right)^2 - k^2y^2 + \frac{l(l+1)}{r^2}(y + w_{2\infty})^2 \right.$$
$$\left. + U(y - \cot \eta w_1 + w_{2\infty})^2 \right\}$$

This may be written

$$A(k \cot \eta)^2 - 2B(k \cot \eta) + C = 0$$

Show that a variational principle equivalent to the above is given by

$$[k \cot \eta] = (B + \sqrt{B^2 - AC})/A$$

Determine A, B, and C.

9.9 A plane wave of number k is incident upon an infinite rigid cylinder of radius a. Show that in the limit $ka \gg 1$,

$$\psi = 0 \text{ on the shadow side of the cylinder}$$
$$\psi = 2\psi_i \text{ on the illuminated side of the cylinder}$$

where ψ_i is the incident wave. Prove that the exact expression for the scattered amplitude is given by

$$|f(\varphi)|^2 = \frac{1}{8\pi k}\left|\int e^{-i\mathbf{k}_s\cdot\mathbf{r}}\left[\frac{\partial\psi_i}{\partial n} - i\mathbf{k}_s\cdot\mathbf{n}\psi_s\right]ds\right|^2$$

where ψ_s is the scattered wave. Insert the above short-wavelength approximations into the above integral. Show that the integral over the shadow of the cylinder yields the contribution

$$\frac{ka^2}{2\pi}\cos^2\left(\frac{\varphi}{2}\right)\frac{\sin^2(ka\sin\varphi)}{k^2a^2\sin^2(\varphi/2)}$$

Evaluate the integral over the illuminated side approximately. Its contribution to the cross section is

$$(a/2)\sin(\varphi/2) + \text{rapidly oscillating terms}$$

Show that the total cross section is $4a$. Discuss.

9.10 A particle of mass m moves in an attractive spherical potential $-V_0 \exp[-(r/a)^2]$. Employ the variational trial function $\sin \kappa r$ (with κ a variational parameter) in the variational principle [Eq. (9.4.61)] for the $l = 0$ phase shift. Compare over a range of values of ka with the Born approximation for the V_0 given by

$$2mV_0a^2/\hbar^2 = \tfrac{1}{4}\pi^2$$

9.11 In the differential equation

$$(d^2\psi/dx^2) + a^2q(x)\psi = 0; \quad q = a_1x + a_2x^2 + \cdots ; \quad a_1 \neq 0$$

make the change in independent variable to $z = \int^x \sqrt{q(x)}\, dx$ and introduce the functions

$$Q = q^{-\frac{1}{4}}(d^2q^{\frac{1}{4}}/dz^2) = -[(5/36z^2) + \lambda z^{-\frac{2}{3}} + \lambda_0 + \lambda_1 z^{\frac{2}{3}} + \cdots]$$

and
$$Q_1(z) = \int_0^z Q(y)\, dy$$

Show that the two solutions of the equation for ψ are

$$\psi_{1,2} \simeq q^{-\frac{1}{4}} \exp\{\pm i[az - (Q_1/2a) \mp (i/4a^2Q)]\}; \quad z \neq 0$$

$$\psi_{1,2} \simeq \sqrt{\tfrac{1}{2}\pi a}\; i^{\pm\frac{5}{6}} \left\{\exp \mp \left[\frac{1}{2ia}\int_0^\infty \left(Q + \left(\frac{5}{36z^2}\right)\right) dz\right]\right\} z^{\frac{1}{6}}\zeta H_{\frac{1}{3}}^{(1,2)}(\eta)$$

where

$$\zeta = \sqrt{z^{\frac{2}{3}} + \lambda\kappa^{-2} - \tfrac{1}{5}\lambda_1\kappa^{-2}\xi^2}; \quad \eta = \kappa[\xi + \tfrac{1}{5}\lambda_1\kappa^{-2}\xi^2]^{\frac{3}{2}}$$
$$\kappa^2 = a^2 + \lambda_0; \quad \xi = z^{\frac{2}{3}} + \lambda\kappa^{-2}$$

Apply the formulas to the Hankel function $H_p(p \sec \lambda)$. (Imai)

Tabulation of Approximation Methods

Bound States; Volume Perturbation. The equation to be solved is

$$\nabla^2\psi + (k^2 - U_0 - \lambda U)\psi = 0$$

The unperturbed problem is $\nabla^2\varphi_n + (k_n^2 - U_0)\varphi_n = 0$, which can be solved exactly, having eigenfunctions φ_n and eigenvalues k_n^2 (assumed here nondegenerate; for the effects of degeneracy see page 1673). The various approximation series for ψ involve the matrix elements of the perturbation potential

$$U_{mn} = \int \cdots \int \bar{\varphi}_m U \varphi_n \, dx_1 \cdots dx_N$$

Iterative-perturbation Series. The first, second, and ath approximations for ψ and its eigenvalue k^2 are [see Eq. (9.1.15)]

$$\psi^{(1)} = \varphi_n + \lambda \sum_{p\neq n} \left(\frac{U_{pn}}{k_n^2 - k_p^2}\right) \varphi_p$$

$$(k^2)^{(1)} = k_n^2 + \lambda U_{nn}$$

$$\psi^{(2)} = \varphi_n + \lambda \sum_{p\neq n} \frac{U_{pn}\varphi_p}{(k_n^2 + \lambda U_{nn} - k_p^2)} + \lambda^2 \sum_{pq\neq n} \frac{U_{pq}U_{qn}\varphi_p}{(k_n^2 - k_p^2)(k_n^2 - k_q^2)}$$

$$(k^2)^{(2)} = k_n^2 + \lambda U_{nn} + \lambda^2 \sum_{p\neq n} \frac{U_{np}U_{pn}}{k_n^2 - k_p^2}$$

$$\psi^{(a)} = \varphi_n + \lambda \sum_{p \neq n} \frac{U_{pn}\varphi_p}{[(k^2)^{(a-1)} - k_p^2]} + \cdots + \lambda^a \sum_{pq \cdots \neq n} \frac{U_{pq} \cdots U_{zn}\varphi_p}{(k_n^2 - k_p^2)(k_n^2 - k_q^2) \cdots (k_n^2 - k_z^2)}$$

$$(k^2)^{(a)} = k_n^2 + \lambda U_{nn} + \lambda^2 \sum_{p \neq n} \frac{U_{np}U_{pn}}{[(k^2)^{(a-2)} - k_p^2]} + \cdots + \lambda^a \sum_{pq \cdots \neq n} \frac{U_{np}U_{pq} \cdots U_{zn}}{(k_n^2 - k_p^2)(k_n^2 - k_q^2) \cdots (k_n^2 - k_z^2)}$$

See pages 1005 and 1008 for discussion of convergence.

Feenberg Series. Formulas which eliminate duplication of matrix components in the successive series do not differ from the above for the first two orders. The ath order is [see Eq. (9.1.37)]

$$\psi^{(a)} = \varphi_n + \lambda \sum_{p \neq n} \frac{U_{pn}\varphi_p}{[(k^2)^{(a-1)} - (\kappa^2)_{np}^{(a-1)}]} + \cdots + \lambda^a \sum_{\substack{p \neq n \\ q \neq np \\ \cdots}} \frac{U_{pq} \cdots U_{zn}\varphi_p}{(k_n^2 - k_p^2)(k_n^2 - k_q^2) \cdots (k_n^2 - k_z^2)}$$

$$(k^2)^{(a)} = k_n^2 + \lambda U_{nn} + \lambda^2 \sum_{p \neq n} \frac{U_{np}U_{pn}}{(k^2)^{(a-2)} - (\kappa^2)_{np}^{(a-2)}} + \cdots + \lambda^a \sum_{\substack{p \neq n \\ q \neq np \\ \cdots}} \frac{U_{np}U_{pq} \cdots U_{zn}}{(k_n^2 - k_p^2)(k_n^2 - k_q^2) \cdots (k_n^2 - k_z^2)}$$

where

$$(\kappa^2)_{np}^{(m)} = k_p^2 + \lambda U_{pp} + \lambda^2 \sum_{q \neq np} \frac{U_{pq}U_{qp}}{[(k^2)^{(m-2)} - (\kappa^2)_{npq}^{(m-2)}]} + \cdots + \lambda^m \sum_{\substack{q \neq np \\ r \neq npq \\ \cdots}} \frac{U_{pq}U_{qr} \cdots U_{zp}}{(k^2 - k_q^2)(k^2 - k_r^2) \cdots (k^2 - k_z^2)}$$

$$(\kappa^2)_{npq}^{(m)} = k_q^2 + \lambda U_{qq} + \lambda^2 \sum_{r \neq npq} \frac{U_{qr}U_{rq}}{(k^2)^{(m-2)} - (\kappa^2)_{npqr}^{(m-2)}} + \cdots$$

These formulas usually converge for values of λ smaller than that which produces a degeneracy.

Fredholm Formula. The following formulas converge for all values of λ. Again there is no difference between this and the earlier formulas for the first two orders of approximation. The next order, for two- or three-dimensional problems, is

$$\psi^{(3)} = \varphi_n + \sum_{p\neq n} \varphi_p \left\{ \frac{\lambda U_{pn}}{[(k^2)^{(2)} - k_p^2]} + \lambda^2 \sum_{q\neq n} \frac{U_{pq}U_{qn} - U_{pn}U_{qq}}{[(k^2)^{(1)} - k_p^2][(k^2)^{(1)} - k_q^2]} \right.$$
$$\left. + \lambda^2 \sum_{qr\neq n} \frac{U_{pq}U_{qr}U_{rn} - U_{qq}U_{pr}U_{rn} - \frac{1}{2}U_{pn}(U_{qr}U_{rq} - U_{qq}U_{rr})}{(k_n^2 - k_p^2)(k_n^2 - k_q^2)(k_n^2 - k_r^2)} \right\} \cdot$$
$$\cdot \left\{ 1 - \lambda \sum_{q\neq n} \frac{U_{qq}}{[(k^2)^{(1)} - k_q^2]} - \tfrac{1}{2}\lambda^2 \sum_{qr\neq n} \frac{U_{qr}U_{rq} - U_{qq}U_{rr}}{(k_n^2 - k_q^2)(k_n^2 - k_r^2)} \right\}^{-1}$$
$$(k^2)^{(3)} = k_n^2 + \lambda U_{nn} + \sum_{p\neq n} U_{np} \left\{ \frac{\lambda^2 U_{pn}}{[(k^2)^{(1)} - k_p^2]} \right.$$
$$\left. + \lambda^3 \sum_{r\neq n} \frac{U_{pr}U_{rn} - U_{rr}U_{pn}}{(k_n^2 - k_r^2)(k_n^2 - k_p^2)} \right\} \left\{ 1 - \lambda \sum_{p\neq n} \frac{U_{qq}}{(k_n^2 - k_p^2)} \right\}^{-1}$$

The general formulas are given on pages 1023 to 1025.

These formulas diverge in one-dimensional cases, where the Spur $\Sigma U_{rr}(1 - \delta_{rn})/(k_n^2 - k_r^2)$ diverges, although the higher Spurs converge. In this case [see Eq. (9.1.67)] we simply remove the divergent series and use the formulas as though it were zero.

Variational Principles for Bound States. A variational formulation of the equation $(\nabla^2 + k^2 - U_0 - \lambda U)\psi = 0$ for k^2 is

$$[k^2] = \left\{ \int \tilde{\varphi}(-\nabla^2 + U_0 + \lambda U)\varphi \, dv \Big/ \int \tilde{\varphi}\varphi \, dv \right\}; \quad \delta[k^2] = 0$$

[see Eq. (9.4.8)]. Function φ is a trial function, satisfying the same boundary conditions as ψ, equipped with parameters to modify its shape between the boundaries; $\tilde{\varphi}$ is the adjoint function, satisfying equation and boundary conditions adjoint to φ (see page 880). Considered as a function of the parameters, $[k^2]$ has a stationary value (we shall call it a "minimum," though it can be a maximum or just a turning point) at which $\varphi = \psi_0$ and $[k^2] = k_0^2$, the lowest eigenfunction and eigenvalue of the equation. To obtain k_1^2 and ψ_1 we add the condition that φ be orthogonal to ψ_0 and find the new stationary value of $[k^2]$, and so on.

Instead of "minimizing" $[k^2]$ we can use the variational expression for the magnitude of the perturbing potential λ,

$$[\lambda] = \left\{ \int \tilde{\varphi}(\nabla^2 - U_0 + k^2)\varphi \, dv \Big/ \int \tilde{\varphi}U\varphi \, dv \right\}; \quad \delta[\lambda] = 0$$

We then find the least value of λ which has k^2 as the eigenvalue.

We can also use the Green's function G_u, a solution of

$$(\nabla^2 - U_0 - \lambda U)G_u(\mathbf{r}|\mathbf{r}_0) = -4\pi\delta(\mathbf{r} - \mathbf{r}_0)$$

and "minimize" the expression

$$[k^2] = \left\{ \int \tilde{\varphi}\varphi \, dv \Big/ \iint \tilde{\varphi}(\mathbf{r}) G_u(\mathbf{r}|\mathbf{r}_0) \varphi(\mathbf{r}_0) \, dv \, dv_0 \right\}$$

Or we can use the Green's function G_k, solution of

$$(\nabla^2 + k^2 - U_0) G_k(\mathbf{r}|\mathbf{r}_0) = -4\pi\delta(\mathbf{r} - \mathbf{r}_0)$$

and "minimize" the expression for λ,

$$[\lambda] = \left\{ \int \tilde{\varphi} U \varphi \, dv \Big/ \iint \tilde{\varphi}(\mathbf{r}) U(\mathbf{r}) G_k(\mathbf{r}|\mathbf{r}_0) U(\mathbf{r}_0) \varphi(\mathbf{r}_0) \, dv \, dv_0 \right\}$$

If a solution ψ_0 of $(\nabla^2 + k^2 - U_0)\psi = 0$ exists, for the value of k^2 chosen, we must then "minimize" the expression

$$[J] = \left\{ \left[\int \tilde{\varphi} U \psi_0 \, dv + \int \tilde{\psi}_0 U \varphi \, dv \right]^2 \Big/ \left[\int \tilde{\varphi} U \varphi \, dv - \lambda \iint \tilde{\varphi} U G_k U \varphi \, dv \, dv_0 \right] \right\}$$

[see Eq. (9.4.19)].

Variation-iteration Method. We transform the differential equation into an integral equation by means of the Green's function G_k given above. The integral equation for the sequence of successive approximations is

$$\psi^{(n)}(\mathbf{r}) = \lambda \int G_k(\mathbf{r}|\mathbf{r}_0) \psi^{(n-1)}(\mathbf{r}_0) U(\mathbf{r}_0) \, dv_0$$

If desirable, one can set $\psi^{(0)} = \varphi_n$.

The quantity λ is then to be the eigenvalue, to be computed as a function of k^2. If necessary, after the calculations are finished, the results can be inverted to find k^2 as a function of λ. Setting

$$[m,n] = \int \psi^{(m)} U \psi^{(n)} \, dv$$

we can show that, if both $(\nabla^2 - k^2 - U_0)$ and U are positive-definite operators, then the sequence $\lambda^{(n)} \geq \lambda^{(n+\frac{1}{2})} \geq \lambda^{(n+1)} \cdots$, where

$$\lambda^{(n)} = [n, n-1]/[n,n] \quad \text{and} \quad \lambda^{(n+\frac{1}{2})} = [n,n]/[n, n+1]$$

converges uniformly to the lowest eigenvalue λ_0 and the corresponding $\psi^{(n)}$ to the corresponding eigenfunction ψ_0 [formulas for speeding convergence are given in (9.1.84) and (9.4.108)].

If the parameter k^2 is near k_n^2, we can set $\psi^{(0)} = \varphi_n$, in which case

$$\psi^{(1)} = \sum_p \frac{U_{pn}\varphi_p}{k^2 - k_p^2}; \quad \psi^{(2)} = \sum_{pq} \frac{U_{pq} U_{qn} \varphi_p}{(k^2 - k_p^2)(k^2 - k_q^2)}$$

$$\lambda^{(\frac{1}{2})} = \left[U_{nn} \Big/ \sum_p \frac{U_{np} U_{pn}}{k^2 - k_p^2} \right];$$

$$\lambda^{(1)} = \left[\sum_p \frac{U_{np} U_{pn}}{k^2 - k_p^2} \Big/ \sum_{pq} \frac{U_{np} U_{pq} U_{qn}}{(k^2 - k_p^2)(k^2 - k_q^2)} \right]$$

and so on. See pages 1031 and 1149 for applications.

Perturbation of Boundary Conditions. If the unperturbed problem is the solution of $(\nabla^2 + k_n^2)\varphi_n = 0$ in a region R, bounded by the surface S, on which homogeneous Neumann conditions $(\partial\varphi/\partial n = 0)$ are required, then the solution of $(\nabla^2 + k^2)\psi = 0$ in R, having boundary conditions

$$(\partial\psi/\partial n) + \mu F(S)\psi = 0; \quad \text{on surface } S$$

may be solved in terms of φ_n and k_n^2, if μ is small. We set $\mu = \lambda$ and simply substitute for λU_{mn} in the foregoing approximation formulas the matrix element

$$f_{mn} = \oint \bar{\varphi}_m(S)F(S)\varphi_n(S)\, dS$$

where the integration is over the whole surface S bounding R.

If μ is large, the most appropriate unperturbed system is the set of solutions of $(\nabla^2 + k^2)\chi = 0$ satisfying Dirichlet conditions $(\chi = 0)$ on S, with eigenfunctions χ_n and eigenvalues k_n^2. We let $1/\mu = \lambda$ and substitute

$$q_{mn} = \frac{1}{k_m k_n} \oint \left(\frac{\partial \chi_m}{\partial n}\right)\left[\frac{1}{F(S)}\right]\left(\frac{\partial \chi_n}{\partial n}\right) dS$$

for λU_{mn} in the various perturbation formulas. For special cases and discussion of convergence see Sec. 9.2.

We may also use a variational principle to obtain ψ and k^2; we "minimize" the expression

$$[k^2] = \left\{[\textstyle\int(\nabla\bar{\varphi})\cdot(\nabla\varphi)\, dv - \mu\oint\bar{\varphi}F\varphi\, dS]/\int\bar{\varphi}\varphi\, dv\right\}$$

[see (Eq. 9.4.26)] where the surface integral is over all of S.

Perturbation of Boundary Shape. Suppose that region R, bounded by S, is wholly contained in region R', bounded by surface S', for which exact solutions φ_n and eigenvalues k_n^2 can be computed, for Neumann conditions on S'. The perturbation series for a solution of $(\nabla^2 + k^2)\psi = 0$, satisfying homogeneous Neumann conditions on S, is obtained by use of the secular determinant

$$|N_{mn}(k^2 - k_m^2) - A_{mn}| = 0$$

where $\quad A_{mn} = \oint \left(\frac{\partial\bar{\varphi}_m}{\partial n}\right)\varphi_n\, dS; \quad$ over S

and $\quad N_{mn} = \int \bar{\varphi}_m\varphi_n\, dv \text{ (over } R) = \frac{A_{mn} - A_{nm}}{k_n^2 - k_m^2}$

$$N_{nn} = \int \left[\left(\frac{\partial\varphi_n}{\partial k^2}\right)\left(\frac{\partial\bar{\varphi}_n}{\partial n}\right) - \bar{\varphi}_n\left(\frac{\partial^2\varphi_n}{\partial n\, \partial k^2}\right)\right]_{k=k_n} dS; \quad \text{over } S$$

Then, to second order, ψ is zero outside R; inside R it is

$$\psi \simeq \varphi_n + \sum_{p\neq n}\left[\frac{A_{pn} - (k^2 - k_p^2)N_{pn}}{(k_n^2 - k_p^2 + A_{nn})N_{pp} - A_{pp}}\right]\varphi_p + \sum_{pq\neq n}\frac{[A_{pq} - (k_n^2 - k_p^2)N_{pq}][A_{qn} - (k_n^2 - k_q^2)N_{qn}]}{[(k_n^2 - k_p^2)N_{pp} - A_{pp}][(k_n^2 - k_q^2)N_{qq} - A_{qq}]}\varphi_p$$

and the corresponding eigenvalue is

$$k^2 \simeq k_n^2 + \left(\frac{A_{nn}}{N_{nn}}\right) + \sum_{p\neq n} \frac{A_{np}[A_{pn} - (k_n^2 - k_p^2)N_{pn}]}{N_{nn}[(k_n^2 - k_p^2)N_{pp} - A_{pp}]}$$

The higher order expressions usually diverge (see page 1056). When the boundary conditions are homogeneous Dirichlet ($\psi = 0$ on S, $\varphi_n = 0$ on S'), usually only the first-order expression for ψ and the second-order one for k^2 are convergent:

$$\psi \simeq N_{nn}\varphi_n + \sum_{p\neq n} \frac{A_{np}\varphi_p}{k_p^2 - k_n^2};\quad \text{in } R; \quad = 0; \quad \text{outside } R$$

$$k^2 \simeq k_n^2 - \left(\frac{A_{nn}}{N_{nn}}\right) + \sum_{p\neq n} \frac{k_n^2 A_{pn}^2}{N_{nn}k_p^2(k_p^2 - k_n^2)}$$

Perturbation Formulas for Scattering. An incident plane wave, $e^{i\mathbf{k}_i\cdot\mathbf{r}}$ is scattered from a finite region R near the origin; the asymptotic form of the scattered wave at distance $\mathbf{r}$, in the direction of $\mathbf{k}_s$ from the region R, is

$$\psi_s \rightarrow -T(\mathbf{k}_s|\mathbf{k}_i)(e^{ikr}/4\pi r); \quad r \rightarrow \infty$$

where k is the magnitude of the incident wave vector $\mathbf{k}_i$ and also of the scattered wave vector $\mathbf{k}_s$.

If the scattering is caused by a volume perturbation, such as the potential function U in the Schroedinger equation $(\nabla^2 + k^2 - U)\psi = 0$ then the integral equation for ψ is

$$\psi(\mathbf{r}) = e^{i\mathbf{k}_i\cdot\mathbf{r}} - \int \left(\frac{e^{ikR}}{4\pi R}\right) U(\mathbf{r}_0)\psi(\mathbf{r}_0)\,dv_0$$

and $$T(\mathbf{k}_s|\mathbf{k}_i) = \int e^{-i\mathbf{k}_s\cdot\mathbf{r}_0}\, U(\mathbf{r}_0)\psi(\mathbf{r}_0)\,dv_0; \quad R = |\mathbf{r} - \mathbf{r}_0|$$

The successive *Born* approximations [Eq. (9.3.38)] are obtained by carrying out the following sequence of integrations:

$$\psi^{(n)}(\mathbf{r}) = e^{i\mathbf{k}_i\cdot\mathbf{r}} - \int \left(\frac{e^{ikR}}{4\pi R}\right) U(\mathbf{r}_0)\psi^{(n-1)}(\mathbf{r}_0)\,dv_0; \quad \psi^{(0)} = e^{i\mathbf{k}_i\cdot\mathbf{r}}$$

$$T^{(n)}(\mathbf{k}_s|\mathbf{k}_i) = \int e^{-i\mathbf{k}_s\cdot\mathbf{r}}U(\mathbf{r})\psi^{(n-1)}(\mathbf{r})\,dv; \quad T^{(1)} = \int e^{i(\mathbf{k}_i - \mathbf{k}_s)\cdot\mathbf{r}}U\,dv$$

The successive Born approximations for the angle-distribution factor T may be given in terms of the Fourier transform of the potential function

$$U(\mathbf{k}_s|\mathbf{k}_i) = \int e^{i(\mathbf{k}_i - \mathbf{k}_s)\cdot\mathbf{r}}U(\mathbf{r})\,dv = T^{(1)}(\mathbf{k}_s|\mathbf{k}_i); \quad \text{see Eq. (9.3.45)}$$

$$T^{(2)}(\mathbf{k}_s|\mathbf{k}_i) = U(\mathbf{k}_s|\mathbf{k}_i) - \int \left[\frac{U(\mathbf{k}_s|\mathbf{K})U(\mathbf{K}|\mathbf{k}_i)}{(2\pi)^3(K^2 - k^2)}\right] dv_K; \quad \text{etc.}$$

If the scattering is caused by some perturbing object with surface S, near the origin, on which boundary conditions must be satisfied, the integral equation for ψ is

$$\psi(\mathbf{r}) = e^{i\mathbf{k}_i\cdot\mathbf{r}} \begin{cases} + \displaystyle\int \left(\frac{e^{ikR}}{4\pi R}\right)\left(\frac{\partial\psi}{\partial n_0}\right) dS_0; & \psi = 0 \text{ and } S \\ - \displaystyle\int \psi(S_0)\frac{\partial}{\partial n_0}\left(\frac{e^{ikR}}{4\pi R}\right) dS_0; & \dfrac{\partial\psi}{\partial n} = 0 \text{ on } S \end{cases}$$

where the normal gradient is into the surface. The successive approximations to ψ and for T, starting with

$$T^{(1)}(\mathbf{k}_s|\mathbf{k}_i) = i \oint \begin{matrix}(\mathbf{k}_i\cdot\mathbf{r}^s)\\(\mathbf{k}_i\cdot\mathbf{r}^s)\end{matrix} e^{i(\mathbf{k}_i-\mathbf{k}_s)\cdot\mathbf{r}^s}\, dS$$

where the integration is over surface S (the top scalar product is for $\psi = 0$; the lower one for $\partial\psi/\partial n = 0$ on S), are called successive *Kirchhoff* approximations [Eq. (9.3.39)].

Variational Principles for Scattering. For a volume perturbation caused by potential function $U(\mathbf{r})$ present near the origin, the variational principle for the angle-distribution factor T is obtained by "minimizing" the expression

$$[T(\mathbf{k}_s|\mathbf{k}_i)] = \left\{\frac{\int \tilde\varphi U e^{i\mathbf{k}_i\cdot\mathbf{r}}\, dv \int \varphi U e^{-i\mathbf{k}_s\cdot\mathbf{r}}\, dv}{\int \tilde\varphi U\varphi\, dv + \int\int \tilde\varphi U (e^{ikR}/4\pi R) U\varphi\, dv\, dv_0}\right\}$$

[see (Eq. 9.4.69)] where $\tilde\varphi$ is the adjoint wave function to φ, corresponding to a plane wave incident in the direction $-\mathbf{k}_s$, being scattered in the direction $-\mathbf{k}_i$. For scattering from a surface S, with $\partial\psi/\partial n = 0$ on S, the variational expression to "minimize" is

$$[T(\mathbf{k}_s|\mathbf{k}_i)] = \left\{\frac{\oint (\mathbf{n}\cdot\mathbf{k}_i)\tilde\varphi e^{i\mathbf{k}_i\cdot\mathbf{r}^s}\, dS \oint (\mathbf{n}_0\cdot\mathbf{k}_s)\varphi e^{-i\mathbf{k}_s\cdot\mathbf{r}_0^s}\, dS_0}{\oint\oint \tilde\varphi(S)\dfrac{\partial^2}{\partial n\,\partial n_0}\left(\dfrac{e^{ikR}}{4\pi R}\right)\varphi(S_0)\, dS\, dS_0}\right\}$$

[see Eq. (9.4.83)]. For $\psi = 0$ on S we "minimize"

$$[T(\mathbf{k}_s|\mathbf{k}_i)] = -\left\{\frac{\oint \left(\dfrac{\partial\tilde\varphi}{\partial n}\right) e^{i\mathbf{k}_i\cdot\mathbf{r}^s}\, dS \oint e^{-i\mathbf{k}_s\cdot\mathbf{r}_0^s}\left(\dfrac{\partial\varphi}{\partial n_0}\right) dS_0}{\oint\oint \left(\dfrac{\partial\tilde\varphi}{\partial n}\right)\left(\dfrac{e^{ikR}}{4\pi R}\right)\left(\dfrac{\partial\varphi}{\partial n_0}\right) dS\, dS_0}\right\}$$

[see Eq. (9.4.84)]. Each of these integrations is over surface S.

Scattering from Spherically Symmetric Object. The incident-plus-scattered wave may be expanded in spherical harmonics;

$$\psi = e^{ikr\cos\varphi} + \psi_s = \sum (2m+1)i^m P_m(\cos\vartheta)\left[\frac{u_m(r)}{kr}\right]$$

where $$e^{ikr\cos\varphi} = \Sigma(2m+1)i^m P_m(\cos\vartheta) j_m(kr)$$

Functions j_m are the spherical Bessel functions [see Eq. (11.3.43)]. If the scatterer is represented by a potential $U(r)$, the function $u_m(r)$ must satisfy the equation $u_m'' + \{k^2 - [m(m+1)/r^2] - U(r)\}u_m = 0$ and must go to zero as $r \to 0$. For large values of r its asymptotic form is

$$u_m(r) \to e^{-i\eta_m} \cos[kr - \tfrac{1}{2}\pi(m+1) - \eta_m]$$

where η_m is called the phase shift. Therefore the scattered wave has the asymptotic form (ϑ is the angle between $\mathbf{k}_s$ and $\mathbf{k}_i$)

$$\psi_s(\mathbf{r}) \to f(\vartheta)(e^{ikr}/r); \quad f(\vartheta) = -(1/4\pi)T(\mathbf{k}_s|\mathbf{k}_i)$$

$$f(\vartheta) = -\left(\frac{1}{k}\right) \sum (2m+1)e^{-i\eta_m} \sin(\eta_m) P_m(\cos\vartheta)$$

The differential and total cross section for scattering [see Eqs. (9.3.6) and (9.3.8)] are then $\sigma = |f(\vartheta)|^2$ and

$$Q = \int \sigma\, d\Omega = \left(\frac{4\pi}{k^2}\right) \sum \sin^2 \eta_m$$

The integral equation satisfied by u_m is

$$u_m(r) = krj_m(kr) + kr\left[n_m(kr) \int_0^r r_0 j_m(kr_0) U(r_0) u_m(r_0)\, dr_0 + j_m(kr) \int_r^\infty r_0 n_m(kr_0) U(r_0) u_m(r_0)\, dr_0\right]$$

[see Eq. (9.3.34)], and the phase shift is determined by the equation

$$\tan \eta_m = \int_0^\infty r_0 j_m(kr_0) U(r_0) u_m(r_0)\, dr_0$$

The first Born approximation is obtained by placing $j_m(kr)$ for $u_m(r)$ in these equations; the higher approximations are obtained by iteration.

Variational principles for u_m and η_m may also be found. We first write down two for u_0 and η_0; corresponding expressions for $m > 0$ are not difficult to set up. The first utilizes a comparison function v_0, a solution of $(v_0'' + k^2 v_0) = 0$ which has the same phase shift as u as $r \to \infty$ and which approaches 1 as $r \to 0$. In other words,

$$v_0(r) = -\sin(kr - \eta_0)/\sin(\eta_0); \quad u_0(r) \to -\csc(\eta_0)\sin(kr - \eta_0); \quad r \to \infty$$

The first principle then requires the "minimizing" of

$$[-k \cot \eta_0] = \int_0^\infty \left[\left(\frac{d\varphi}{dr}\right)^2 - \left(\frac{dv}{dr}\right)^2 - k^2(\varphi^2 - v^2) + \varphi^2 U(r)\right] dr$$

where φ is a trial function which must satisfy the following boundary conditions

$$\varphi \to \begin{cases} 0; & r \to 0 \\ -\csc(\eta_0)\sin(kr - \eta_0); & r \to \infty \end{cases}$$

A variational principle in which η does not occur implicitly inside the integral utilizes the trial function w, where $w = v - u$; $w(0) = 1$; $w \to 0$ $(r \to \infty)$. We find the form of w which "minimizes"

$$[k \cot \eta_0] = (1/2A)[B + \sqrt{B^2 - 4AC}]$$

where

$$A = \int_0^\infty U\left[\frac{\sin^2(kr)}{k^2}\right] dr; \quad \text{see Eq. (9.4.56)}$$

$$B = 1 + \left(\frac{1}{k}\right)\int_0^\infty U \sin(2kr)\, dr - \left(\frac{2}{k}\right)\int_0^\infty Uw \sin(kr)\, dr$$

$$C = \int_0^\infty \left[\left(\frac{dw}{dr}\right)^2 - (k^2 - U)w^2\right] dr + \int_0^\infty U[\cos^2(kr) - 2w\cos(kr)]\, dr$$

Another variational principle may be written down for any value of m; the trial function φ_m equals u_m when

$$[k \cot \eta_m] = \left\{\frac{\int_0^\infty U\varphi_m^2\, dr - 2k\int_0^\infty rn_m(kr)\varphi_m U \int_0^r r_0 j_m(kr_0)\varphi_m U\, dr\, dr_0}{\left[\int_0^\infty rj_m(kr)U(r)\varphi_m(r)\, dr\right]^2}\right\}$$

is "minimized," goes to zero when $r \to 0$ as r^{m+1}, and is finite at $r \to \infty$; the exact form of φ_m for r large is not important, for the integrands vanish there because of the presence of the factor $U(r)$. A correct form for u_m may then be obtained (if required) by inserting the best form of φ_m into the integral equation satisfied by u_m, given on the preceding page.

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CHAPTER 10

Solutions of Laplace's and Poisson's Equations

We have now completed our general study of fields and their behavior. We have inspected the various kinds of fields of interest in physics, have investigated the various equations and boundary conditions which make the field correspond to one or more aspect of the physical situation, and have discussed the various mathematical techniques by which the equations may be solved and the boundary conditions satisfied. The rest of this work will be devoted to a more or less systematic application of these general techniques and results to specific physical problems.

In the course of our previous investigations we have already discussed many applications as examples to clarify a point; so most of the kinds of problems which arise and most of the ways they can be solved are already familiar to us. What is needed now is a systematic presentation to show how the techniques should be applied to new problems, which techniques to apply first in which cases, and so on. This aspect will occupy us for the remainder of the book.

It should be emphasized again that this is a book on the methods of theoretical physics, on the mathematical tools which may be used to solve problems in many branches of physics. Consequently we shall be concerned here with the physical content of the problems only enough to show the relation to the physics but not enough to give a connected picture of the physical aspect in all its ramifications. We shall often have to jump from one field of physics to another with scant regard for logical unity on the side of content, showing (we hope) how a given technique may be used to solve problems from a large number of fields, with a wide variety of content.

Moreover, we shall be primarily concerned with the application of the more advanced techniques of computation. The simpler tools of theoretical physics have been touched on, here and there, in the earlier parts of this work, and they are elucidated in many well-known treatises. It is felt that the less familiar methods need the exposition here. As a result, of course, the remaining chapters of this work will be rather heavy going for the casual reader, and in many places, it will be hard to see the forest for the trees. It is hoped that, from the very mass of details, the less casual reader will eventually sense the pattern of technique and will begin to acquire that semi-intuitive "feel" for a new problem which is so difficult to teach but which is so useful a faculty for the theoretical physicist.

We start with the simplest equation of those under consideration, the Laplace equation $\nabla^2\psi = 0$ and its related inhomogeneous equation, the Poisson equation $\nabla^2\psi = -4\pi\rho$. These are elliptic equations and therefore require Dirichlet or Neumann conditions on a closed boundary. Referring to pages 7 and 690 we recall that the Laplace equation is equivalent to the requirement that there be no maximum or minimum of ψ inside the boundary and that the value of ψ at a point is equal to the average value of the ψ's for neighboring points.

We have developed two methods for fitting the boundary conditions: the use of eigenfunctions and the use of Green's functions. For the most part we shall use both, expanding the Green's functions in eigenfunctions for the suitable coordinate system. In the special case of the Laplace equation for two dimensions, the properties of functions of a complex variable and of conformal transformation provide us with several special methods of considerable power. Unfortunately these methods do not extend to three dimensions, at least in their full generality.

The Laplace equation, as we saw in Chaps. 2 and 3, arises in gravitational or electrostatic problems, the solution being the electrostatic or gravitational potential. The boundaries are usually the surfaces of conductors, and the boundary conditions are Dirichlet conditions, the potential of the conductors being given. Laplace's equation also arises in the hydrodynamical problem of the steady, irrotational flow of an incompressible fluid, the solution being the velocity potential (or the flow function), the boundary conditions being Neumann conditions, arising from the simple requirement that the fluid cannot flow into the solid boundaries. It also comes up in describing the steady-state flow of heat, the steady diffusion of a solute (or of neutrons) and in a large number of other steady-state cases. The scalar field satisfying the equation and the boundary conditions is a *potential* of some sort; its gradient is proportional to the steady-state flow or force.

When "sources" or "sinks" of the flow, or "charges" producing the force, are present we must use the Poisson equation, with the quantity ρ the density of source or of charge. Here we use the Green's functions for space, not for the boundary, adjusted with enough of the solution of Laplace's equation to fit the boundary conditions.

These problems will be considered in sequence for the various separable coordinate systems, each useful for differently shaped boundary surfaces. We first take up the two-dimensional systems, with their special techniques based on the properties of analytic functions, and then go on to three-dimensional cases, where available techniques for solution are more limited.

10.1 *Solutions in Two Dimensions*

Referring to the discussion on pages 499 to 504, a coordinate system in two dimensions may be related to the cartesian coordinates x, y by means of a function $w(z)$ of the complex variable $z = x + iy$. The real and imaginary parts of $w = \xi_1 + i\xi_2$ are orthogonal coordinates which have the special property that the corresponding scale factors h_1 and h_2 *are equal* (which is another way of saying that the transformation $w \to z$ is *conformal*). Therefore the Laplace equation has the particularly simple form

$$\nabla^2\psi = \left|\frac{dw}{dz}\right|^2 \left[\frac{\partial^2\psi}{\partial\xi_1^2} + \frac{\partial^2\psi}{\partial\xi_2^2}\right] = 0 \qquad (10.1.1)$$

and the eigenfunction solutions may be found for a large number of different coordinate systems at the same time.

The separated solutions of this Laplace equation can take on the various forms

$$\xi_1,\ \xi_2,\ e^{ik\xi_1 \pm k\xi_2},\ \sin(k\xi_1)\sinh(k\xi_2),\ \cosh(k\xi_1)\sin(k\xi_2);\ \text{etc.} \qquad (10.1.2)$$

The solution of the particularly simple Poisson equation

$$\nabla^2\psi = \left|\frac{dw}{dz}\right|^2 \left[\frac{\partial^2\psi}{\partial\xi_1^2} + \frac{\partial^2\psi}{\partial\xi_2^2}\right] = -4\pi\delta(x - x_0)\delta(y - y_0) \qquad (10.1.3)$$

corresponding to a unit "charge" concentrated at the point (x_0,y_0) is according to Eq. (7.1.9), the Green's function

$$G(x,y|x_0,y_0) = \mathrm{Re}[-2\ln(z - z_0)] = \ln(1/|z - z_0|^2) \qquad (10.1.4)$$

which, of course, must be transformed into dependence on the new variables w, ξ_1, and ξ_2. Further discussion is more fruitful when related to a particular coordinate system.

Cartesian Coordinates. These simplest coordinates are suitable for infinite plane boundaries (parallel to each other) and for the inside of rectangular prismatic boundaries. For instance, above an infinite plane boundary (placed at the plane $y = 0$) with boundary potential on the plane depending only on x, we use the unit solutions $e^{-ky\pm ikx}$, because the y factor goes to zero at $y \to \infty$ and the x factors can be combined to obtain eigenfunctions for the range $-\infty < x < \infty$. Since this is an infinite range, there is a continuous allowed range of the eigenvalue k^2 from 0 to ∞. We have worked out this case already in Chap. 6. Equation (6.3.3) gives us the expression for the potential in the positive y

plane when the potential is specified as being $\psi_0(x)$ along the boundary $y = 0$.

As an example, we might calculate the potential distribution between the arrangement of plates shown in Fig. 10.1. The strip of width d is held at unit potential, and the rest of the $y = 0$ plane is held at zero

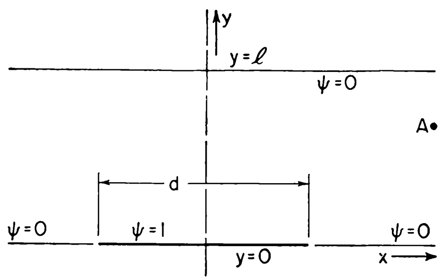

Fig. 10.1 Potential between two planes, with Dirichlet conditions depending on x only.

potential, while the plate at $y = l$ is also zero potential. We then use a modification of Eq. (6.3.3):

$$\begin{aligned}\psi(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sinh[k(l-y)]}{\sinh(kl)}\,dk\int_{-d/2}^{d/2}\cos[k(\xi - x)]\,d\xi \\ &= \frac{1}{2\pi}\operatorname{Re}\left\{\int_{-d/2}^{d/2}d\xi\int_{-\infty}^{\infty}\frac{\sinh[k(l-y)]}{\sinh(kl)}\,e^{ik(\xi-x)}\,dk\right\}\end{aligned}$$

The integral over k may be evaluated by contour integration. If $(\xi - x)$ is positive, we close the contour by a semicircle of infinite radius around the upper half plane of k, and the integral is therefore $(2\pi i)$ times the sum of the residues at all the poles of the integrand in the upper half plane (since as long as $y < l$ and $\xi > x$, the integral around the semicircle is zero). The poles are at the zeros of $\sinh(kl)$, which are at $k = i(n\pi/l)$ $(n = 1, 2, \ldots)$. Some algebraic drudgery gives us eventually

$$\begin{aligned}\psi(x,y) &= \sum_{n=1}^{\infty}\sin\left(\frac{n\pi y}{l}\right)\int_{-d/2}^{d/2}e^{-(n\pi/l)|\xi-x|}\,d\xi \\ &= \sum_{n=1}^{\infty}\left(\frac{2l}{n\pi}\right)\sin\left(\frac{n\pi y}{l}\right)\left[1 - e^{-(n\pi d/2l)}\cosh\left(\frac{n\pi x}{l}\right)\right]; \quad |x| < \tfrac{1}{2}d \\ &= \sum_{n=1}^{\infty}\left(\frac{2l}{n\pi}\right)\sin\left(\frac{n\pi y}{l}\right)e^{-(n\pi|x|/l)}\sinh\left(\frac{n\pi d}{2l}\right); \qquad |x| > \tfrac{1}{2}d\end{aligned}$$

Naturally, if the potential of the strip is V instead of 1, we should multiply this result by V.

Thus our integral over eigenfunctions along the x axis has resulted in a series over eigenfunctions along the y axis. The series including the first terms in the bracket turns out to be the expansion of a simple function

$$f(y) = \sum_{n=1}^{\infty} \left(\frac{2}{n\pi}\right) \sin\left(\frac{n\pi y}{l}\right) = \left(1 - \frac{y}{l}\right); \quad 0 < y \le l$$

This part of the series is conditionally convergent and so cannot be differentiated (which is due to the fact that it is quite a "strain" on the sine series to give us a function which has unit value at $y = 0$). But if we use the closed form for this, we can write

$$\psi(x,y) = \begin{cases} l\left(1 - \frac{y}{l}\right) - \sum_{n=1}^{\infty} \left(\frac{2l}{n\pi}\right) e^{-(n\pi d/2l)} \cosh\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi y}{l}\right); \\ \qquad\qquad\qquad\qquad |x| < \frac{1}{2}d \\ \frac{1}{2}l\left(1 - \frac{y}{l}\right) - \sum_{n=1}^{\infty} \left(\frac{l}{n\pi}\right) e^{-(n\pi d/l)} \sin\left(\frac{n\pi y}{l}\right); \quad |x| = \frac{1}{2}d \\ \sum_{n=1}^{\infty} \left(\frac{2l}{n\pi}\right) e^{-(n\pi|x|/l)} \sinh\left(\frac{n\pi d}{2l}\right) \sin\left(\frac{n\pi y}{l}\right); \quad |x| > \frac{1}{2}d \end{cases} \tag{10.1.5}$$

The series can be differentiated except at $x = \pm\frac{1}{2}d$, $y = 0$, where there is a discontinuity in the potential. We note that this closed form, $[1 - (y/l)]$, is of itself a solution of Laplace's equation, satisfying the boundary condition at $y = 0$ and $y = l$ for a portion of the range of x.

We notice that, for $|x|$ considerably larger than $\frac{1}{2}d$ (that is, at point A in Fig. 10.1), the third series converges so rapidly that only the first term is appreciable and

$$\psi(x,y) \simeq \left(\frac{2l}{\pi}\right) e^{-(\pi|x|/l)} \sinh\left(\frac{\pi d}{2l}\right) \sin\left(\frac{\pi y}{l}\right); \quad |x| \gg \frac{1}{2}d$$

The charge density on the strip at unit potential is obtained by using the formula that $\sigma = -(1/4\pi)(\partial\psi/\partial y)$ at $y = 0$. Now that we have removed the nondifferentiable part of the series, we can obtain for the charge density

$$\sigma(y) = \frac{1}{4\pi} + \frac{1}{2\pi} \sum_{n=1}^{\infty} e^{-(n\pi d/2l)} \cosh\left(\frac{n\pi x}{l}\right); \quad |x| < \frac{1}{2}d$$

By use of the series $\ln(1 - x) = -\sum \left(\frac{x^n}{n}\right)$, we can obtain a closed expression for the charge density per unit length on the strip between $-x$ and $+x$ ($|x| < \frac{1}{2}d$):

$$Q = \int_{-x}^{x} \sigma\, dx = \frac{x}{2\pi} + \frac{l}{\pi^2} \ln\left[\frac{e^{\pi d/2l} - e^{-\pi x/l}}{e^{\pi d/2l} - e^{+\pi x/l}}\right]; \quad 0 < x < \frac{1}{2}d \tag{10.1.6}$$

This has a logarithmic infinity when $x \to \frac{1}{2}d$ (that is, when we include all the charge on the strip) because the charge density on the outside edge of the strip goes to infinity owing to the close proximity of the conducting plane, at zero potential, making up the rest of the $y = 0$ plane. When d is large compared with l, the logarithmic term is small compared with the first term, except near the edge ($x \to \frac{1}{2}d$).

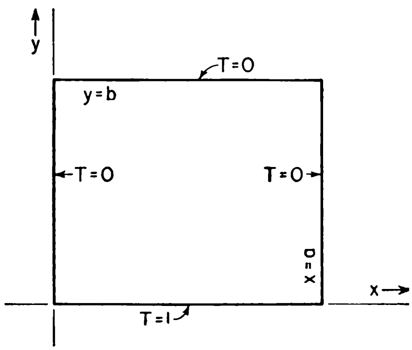

Fig. 10.2 Boundary conditions for steady flow of heat in rectangular bar.

Rectangular Prism Heated on One Side. Many other similar calculations may be made for potential distributions above a plane of infinite extent. When we confine ourselves, however, to the inside of a rectangular prism, we use the Fourier series instead of the integral. For instance, if we had a bar of rectangular cross section, as shown in Fig. (10.2), heated to temperature $T = 1$ along the bottom and kept at zero temperature along the sides and top, the temperature at (x,y) inside the bar would be [from Eq. (6.3.2)]

$$T(x,y) = \frac{4}{\pi} \sum_{n=0}^{\infty} \left(\frac{1}{2n+1}\right) \frac{\sinh[(\pi/a)(2n+1)(b-y)]}{\sinh[(2n+1)(\pi b/a)]} \sin\left[(2n+1)\frac{\pi x}{a}\right] \tag{10.1.7}$$

which is fairly rapidly convergent except when $y \to 0$.

We can improve the convergence of this series still more by noting that for large values of n the ratio of hyperbolic sines reduces to $e^{-(\pi y/a)/(2n+1)}$. Such a series can be summed; so we add and subtract it from each term:

$$\frac{4}{\pi}\frac{1}{2n+1}\left\{e^{-(\pi y/a)/(2n+1)} \sin\left[\frac{\pi x}{a}(2n+1)\right]\right.$$
$$+ \frac{\sinh\left[\frac{\pi}{a}(b-y)(2n+1)\right] - e^{-(\pi y/a)(2n+1)} \sinh\left[\frac{\pi b}{a}(2n+1)\right]}{\sinh[(2n+1)(\pi b/a)]} \cdot$$
$$\left.\cdot \sin\left[\frac{\pi x}{a}(2n+1)\right]\right\} = \frac{4}{\pi}\frac{1}{2n+1}\left\{\mathrm{Im}[e^{i(\pi/a)(2n+1)(x+iy)}]\right.$$
$$\left.+ \frac{\sinh[(\pi y/a)(2n+1)]}{\sinh[(\pi b/a)(2n+1)]} e^{-(\pi b/a)(2n+1)} \sin\left[\left(\frac{\pi x}{a}\right)(2n+1)\right]\right\}$$

Since $\tanh^{-1} u = u + \frac{1}{3}u^3 + \cdots$, we can sum the first (and least convergent) terms, obtaining ($z = x + iy$):

$$T(x,y) = \frac{4}{\pi}\Bigg\{\mathrm{Im}[\tanh^{-1}(e^{i\pi z/a})]$$

$$- \sum_{n=1}^{\infty}\left[\frac{e^{-(\pi b/a)(2n+1)}}{2n+1}\right]\frac{\sinh\left[\frac{\pi y}{a}(2n+1)\right]}{\sinh\left[\frac{\pi b}{a}(2n+1)\right]}\sin\left[\frac{\pi x}{a}(2n+1)\right]\Bigg\}$$

$$= \frac{2}{\pi}\tan^{-1}\left[\frac{\sin(\pi x/a)}{\sinh(\pi y/a)}\right]$$

$$- 4\sum_{n=1}^{\infty}\left[\frac{e^{-(\pi b/a)(2n+1)}}{\pi(2n+1)}\right]\frac{\sinh\left[\frac{\pi y}{a}(2n+1)\right]}{\sinh\left[\frac{\pi b}{a}(2n+1)\right]}\sin\left[\frac{\pi x}{a}(2n+1)\right]$$

where the series (and its derivative) is small and so rapidly convergent that usually only the first term needs to be included. In fact the first term is everywhere less than 0.1 if b is about the same size as a. Therefore, to very good approximation,

$$T(x,y) \simeq \frac{2}{\pi}\left\{\tan^{-1}\left[\frac{\sin(\pi x/a)}{\sinh(\pi y/a)}\right] - 2e^{-(\pi b/a)}\frac{\sinh(\pi y/a)}{\sinh(\pi b/a)}\sin(\pi x/a)\right\} \tag{10.1.8}$$

When b goes to infinity and the prism becomes a half-infinite slab, heated at its bottom edge, the first term alone gives the temperature distribution. From its derivation, this first term is the real part of the function $-(4i/\pi)\tanh^{-1}(e^{i\pi z/a}) = \psi + i\chi$:

$$\psi = \frac{2}{\pi}\tan^{-1}\left[\frac{\sin(\pi x/a)}{\sinh(\pi y/a)}\right];\quad \chi = -\frac{2}{\pi}\tanh^{-1}\left[\frac{\cos(\pi x/a)}{\cosh(\pi y/a)}\right]$$

Therefore, in this case, we have closed forms for both the potential (the temperature in this case) and the flow function (see page 156). The rest of the series, in Eq. (10.1.8) and before, can be considered as a correction term to this basic solution in order to bring the potential down to zero at $y = b$ rather than at $y = \infty$.

The gradient of the temperature is proportional to the flow of heat; as in the earlier example there is a singularity in this gradient near $y = 0$, $x = 0$ or a.

Green's Function. According to Eq. (10.1.4) the Green's function in cartesian coordinates is $-\ln[(x - x_0)^2 + (y - y_0)^2]$, having an infinity at the point (x_0,y_0), the position of the "source," and diminishing in value continuously at points farther and farther away from this. When the boundary is an infinite plane, such as the one at $y = 0$, the resulting solution may be obtained by the method of images (see page 812).

For instance, the potential ψ due to a unit line charge at point (x_0,y_0) (a long thin wire, perpendicular to the x, y plane, of unit charge per unit length of wire) and an infinite conducting plane at $y = 0$ kept at zero potential is

$$\psi(x,y) = \ln\left[\frac{(x - x_0)^2 + (y + y_0)^2}{(x - x_0)^2 + (y - y_0)^2}\right] \tag{10.1.9}$$

The charge density induced on the grounded plane is

$$\sigma = -\frac{1}{4\pi}\left[\frac{\partial\psi}{\partial y}\right]_{y=0} = \frac{-(y_0/\pi)}{(x - x_0)^2 + y_0^2}$$

having a maximum at $x = x_0$ and a total charge in a strip between z and $z + 1$ (from $x \to -\infty$ to $x \to \infty$) of 1, as it must have to balance the unit charge per unit length on the wire.

The lines of force constitute a coordinate system $\chi =$ constant which is everywhere orthogonal to the equipotential lines $\psi =$ constant (see page 14). In other words grad χ is perpendicular to grad ψ, which can be achieved by setting $\partial\psi/\partial x = \partial\chi/\partial y$ and $\partial\psi/\partial y = -(\partial\chi/\partial x)$ for the two-dimensional case we are considering here. But according to Eq. (4.1.10), this corresponds to the requirement that the potential function ψ be the real part and χ be the imaginary part of some function of the complex variable $z = x + iy$. In the present case the function, for charge q per unit length, is

$$\begin{aligned}
F(z) &= \psi + i\chi = 2q\ln\left(\frac{z - \bar{z}_0}{z - z_0}\right); \quad z_0 = x_0 + iy_0; \quad \bar{z}_0 = x_0 - iy_0 \\
\psi &= q\ln\left[\frac{(x - x_0)^2 + (y + y_0)^2}{(x - x_0)^2 + (y - y_0)^2}\right] \\
\chi &= 2q\tan^{-1}\left(\frac{y + y_0}{x - x_0}\right) - 2q\tan^{-1}\left(\frac{y - y_0}{x - x_0}\right) \\
&= 2q\tan^{-1}\left[\frac{2y_0(x - x_0)}{(x - x_0)^2 + y^2 - y_0^2}\right]
\end{aligned} \tag{10.1.10}$$

The lines $\psi =$ constant, $\chi =$ constant are shown in Fig. 10.3a (they are the bipolar coordinate lines). The function χ is called the *flow function* (see page 13) or the line-of-force function. Unfortunately this simple method of computing the lines of force applies only to two-dimensional potential problems.

As we showed on page 810, if the "source" at (x_0,y_0) is a uniform line charge (q per unit length) parallel to the z axis, then ψ represents the electrostatic potential and χ the lines of force, with the electric intensity equal to $-$grad ψ. On the other hand, as the earlier discussion shows, if the "source" is a thin wire carrying current q in the positive z direction, the function ψ represents the lines of *magnetic force* and the function χ is the magnetic "potential." This potential is multivalued, for $\tan^{-1} z$ is a multivalued function (depending on the number of times one goes around the wire), but its gradient, grad χ, is equal to the magnetic intensity at the point in question.

Therefore, if we have a thin wire at (x_0,y_0) carrying current q in the positive z direction and another wire at $(x_0,-y_0)$ with current q in the opposite direction, the function ψ of Eq. (10.1.10) will represent the

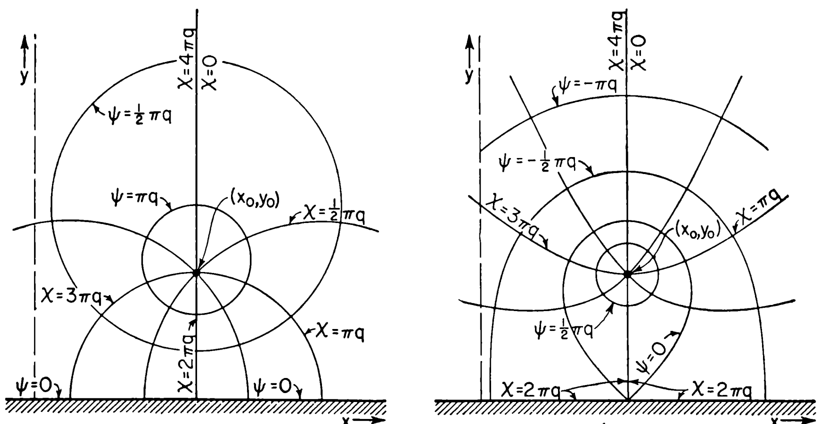

Fig. 10.3 Potential and stream function for source line and plane.

resulting lines of magnetic force, and the magnitude of grad χ (which equals $|dF/dz|$) is equal to the magnitude of the magnetic intensity at (x,y).

We can show that the equipotential lines $\psi = V$ are circles by setting up the equation for $\psi = V$ in terms of the coordinates $\xi = x - x_0$, $\eta = y - y_0$, using the center x_0, y_0 as an origin. Taking the exponential of both sides and completing the squares, we obtain for the line corresponding to the potential V

$$\xi^2 + \left(\eta - \frac{2y_0}{e^{V/q} - 1}\right)^2 = \left(\frac{2y_0}{e^{V/q} - 1}\right)^2 e^{V/q}$$

This is the equation of a circle of radius $2y_0e^{V/2q}/(e^{V/q} - 1)$ with the center at x_0, $y_0 + [2y_0/(e^{V/q} - 1)]$, somewhat above the focus x_0, y_0 of the lines of flow χ = constant.

For sufficiently large values of V we can neglect powers of $e^{-V/2q}$ larger than the first. To this approximation the radius of the circle of potential V is $2y_0e^{-V/2q}$, with center at (x_0,y_0). Therefore if a wire of this radius, with center at (x_0,y_0), is charged with q units per unit length, it will be at potential V with respect to the grounded plane at $y = 0$. Turning this around, we are now in a position to find the amount of charge required to raise a wire, of radius ρ, with center a distance y_0 from a grounded plane, to a potential V above the ground potential. To the first approximation in $e^{-V/2q}$, we can set

$$\rho \simeq 2y_0 e^{-V/2q} \quad \text{or} \quad q \simeq \frac{V}{2\ln(2y_0/\rho)} \qquad (\rho \ll y_0)$$

Therefore the capacitance per unit length of a wire of radius ρ with center a distance $y_0(y_0 \gg \rho)$ away from a conducting plate is

$$C = q/V \simeq [2\ln(2y_0/\rho)]^{-1} \quad \text{esu} \tag{10.1.11}$$

For wires of larger radius we must use the exact expressions for radius and center. This will be discussed later.

For the opposite sort of boundary condition, we can use the image method to calculate the velocity potential for a line source of fluid at (x_0,y_0) and a solid plane at $y = 0$. In this case the velocity potential is to have a zero normal gradient at $y = 0$, which is obtained by adding the image potential instead of subtracting it:

$$\begin{aligned} F &= -2q\ln[(z - z_0)(z - \bar{z}_0)] = \psi + i\chi \\ \psi &= -q\ln\{[(x - x_0)^2 + (y - y_0)^2][(x - x_0)^2 + (y + y_0)^2]\} \\ \chi &= -2q\tan^{-1}\left[\frac{y - y_0}{x - x_0}\right] - 2q\tan^{-1}\left[\frac{y + y_0}{x - x_0}\right] \\ &= -2q\tan^{-1}\left[\frac{2y(x - x_0)}{(x - x_0)^2 + y_0^2 - y^2}\right] \end{aligned} \tag{10.1.12}$$

These potential and flow lines are shown in Fig. 10.3*b*. The function ψ corresponds to the lines of magnetic force about a wire carrying current q at point (x_0,y_0) above a soft iron plate at $x = 0$. The permeability of the iron is supposed to be so large that the magnetic field is everywhere normal to the surface.

Referring to Eq. (2.4.2), we see that this problem also corresponds to that of a thin pipe, pierced with holes so it serves as a source of fluid, buried a depth y_0 below an impervious plane surface in a porous soil with flow resistance R to the fluid flow. The total flow out of the pipe per unit length is q, which percolates through the soil along flow lines given by the function χ. According to Eqs. (2.4.2) *et seq.*, the pressure at any point is equal to minus the gravity potential plus R times the potential ψ.

By this definition the pressure would go negative at large enough distances from the pipe. Presumably we should have to have some sort of "sinks" at large distances, where the fluid would be absorbed and the pressure is zero. Suppose this to occur on a semicircular surface, centered at point $(x_0,0)$ and of radius $D(D \gg y_0)$. Then for ψ to be zero on this surface, we must add $q \ln(D^4)$ to the expression given in Eq. (10.1.12), and the potential at the surface of a pipe of radius $\rho(\rho \ll y_0)$ is approximately $\psi \simeq -2q \ln(2\rho y_0/D^2)$. The pressure in the pipe required to produce a flow out q per unit length is then

$$P \simeq y_0 g - 2qR \ln(2\rho y_0/D^2); \quad \rho \ll y_0 \ll D$$

where y_0 is the depth below the surface, g is the rate of pressure increase due to gravity, R the flow resistivity of the soil, and ρ the radius of the pipe (assumed small compared with y_0).

We note that, the *smaller* y_0 is, the larger the overpressure at the pipe to cause a given flow. This is not surprising, for making y_0 smaller brings the impervious layer closer to the pipe and thereby obstructs the flow somewhat more. The pressure at point (x,y) is

$$P = gy - 2qR \ln[(1/D^4)(x^2 + y^2 + y_0^2)^2 - 4(y_0 y/D^2)^2]$$

where y is the distance below the impervious layer and where we have set $x_0 = 0$, since it serves no useful purpose here.

Polar Coordinates. The potential about an isolated charged wire can be used to set up the useful polar coordinates $w = \ln(x + iy) = \zeta + i\phi$, where $\zeta = \frac{1}{2}\ln(x^2 + y^2)$ and $\phi = \tan^{-1}(y/x)$. The scale factors for these coordinates are $h_\zeta = h_\phi = |dz/dw| = e^\zeta$, and the expression for the Laplacian is [see Eq. (10.1.1)]

$$\nabla^2\psi = e^{-2\zeta}\left[\frac{\partial^2\psi}{\partial\zeta^2} + \frac{\partial^2\psi}{\partial\phi^2}\right]; \quad e^\zeta = r = \sqrt{x^2 + y^2}$$

Therefore possible solutions of the Laplace equation are ζ, ϕ, $e^{m\zeta}\cos m\phi$, and so on. Any solution can be expressed as the real part of some function of the complex variable $w = \zeta + i\phi = \ln z$.

If the boundary surfaces are concentric circular cylinders these coordinates are obviously the ones to use. For instance, the potential outside a cylinder of radius a must be represented by a series of functions of the sort $e^{-m\zeta}\cos(m\phi) = r^{-m}\cos(m\phi)$ or $r^{-m}\sin(m\phi)$, where m must be an integer in order that there be continuity in ψ at $\phi = 0, 2\pi$. By using the usual Fourier series expressions, we find that the potential outside the cylinder, when the potential on the cylinder is specified as $\psi_0(\phi)$ and when $\psi = 0$ at $r \to \infty$, is

$$\psi(r,\phi) = \sum_{n=0}^{\infty}\left(\frac{a}{r}\right)^n \left\{\frac{\epsilon_n}{2\pi}\int_0^{2\pi}\psi_0(\alpha)\cos[n(\alpha - \phi)]\,d\alpha\right\} \tag{10.1.13}$$

where $\epsilon_0 = 1$, $\epsilon_n = 2$ for $n = 1, 2, 3, \ldots$.

For instance this potential, when one side of the cylinder ($0 < \phi < \pi$) is at potential $+1$ and the other side ($\pi < \phi < 2\pi$) is at potential -1, is

$$\psi(r,\phi) = \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{1}{2m+1} \left(\frac{a}{r}\right)^{2m+1} \sin[(2m+1)\phi] = \frac{2}{\pi} \tan^{-1}\left[\frac{2ar \sin \phi}{r^2 - a^2}\right]$$

Referring to page 1179, we see that the flow function corresponding to this potential is

$$\chi(r,\phi) = -\frac{2}{\pi} \tanh^{-1}\left[\frac{2ar \cos \phi}{r^2 + a^2}\right] \qquad (10.1.14)$$

On the other hand the potential between two concentric cylinders, the inner (radius a) at zero potential and the outer (radius b) at potential V, is

$$\psi = V \frac{\zeta - \zeta_0}{\zeta_1 - \zeta_0} = V \frac{\ln(r/a)}{\ln(b/a)}$$

Cylinders Placed in Fields. In many potential problems we are interested in determining the effect of a boundary of one shape on the field produced by boundaries of another shape. For instance, we are interested in computing the field about a cylinder inserted in a uniform field, presumably produced by parallel planes a large distance away. It is useful in such cases to be able to express the solutions of Laplace's equation, suitable for one coordinate system, in terms of solutions suitable for another system.

A very simple example of this is that the solution $\psi = -Ex$, suitable for rectangular boundaries and representing a uniform field of force ($-$ grad $\psi = E\mathbf{i}$), is $-r \cos \phi$ when expressed in polar coordinates $r = e^\zeta$ and ϕ. An instructive example of the use of such relations is given by the problem of the dielectric cylinder in a uniform field. The dielectric constant will be chosen as ϵ, and the boundary conditions are that, at the surface, the normal potential gradient outside must equal ϵ times the normal gradient inside while the tangential potential gradients are equal (or, what is the same thing, the inner and outer potentials are equal at the surface).

The potential of the uniform field $-Er \cos \phi$ has a $\cos \phi$ factor, which will be present in all the terms, for the fit must be made at the surface, $r = a$, ϕ varying. Inside the cylinder the only possible term which can have the right ϕ dependence is $Ar \cos \phi$, since the other type, $(B/r) \cos \phi$, does not stay finite at $r = 0$. The possible form outside can be $Ar \cos \phi + (B/r) \cos \phi$, which becomes $Ar \cos \phi$ for r sufficiently large. But for large values of r the field is supposed to reduce to $-Er \cos \phi$, corresponding to the uniform field. Consequently the forms to be used to fit at the boundary $r = a$ are $\psi = -Er \cos \phi + (B/r) \cos \phi$ for $r > a$ and $\psi = Ar \cos \phi$ for $r < a$. To fit the boundary conditions we must have that $-Ea + (B/a) = Aa$ and $-E - (B/a^2) = \epsilon A$. Solving these we finally obtain

$$\psi(r,\phi) = \begin{cases} -Er \cos \phi + E\left(\dfrac{\epsilon - 1}{\epsilon + 1}\right)\dfrac{a^2}{r} \cos \phi \\ -[2E/(\epsilon + 1)]r \cos \phi \end{cases}$$

Another, still simpler case is that of a cylinder immersed in a nonviscous, incompressible fluid flowing past at velocity v_0. Here the boundary conditions are that the velocity, grad ψ, is v_0, directed along the x axis, for $r \to \infty$, and is tangential to the surface of the cylinder ($\partial\psi/\partial r = 0$) at $r = a$. The solution satisfying these requirements is

$$\psi = v_0 r \cos \phi + v_0(a^2/r) \cos \phi; \quad \chi = v_0 r \sin \phi - v_0(a^2/r) \sin \phi$$

The equipotential and flow lines are shown in Fig. 10.4.

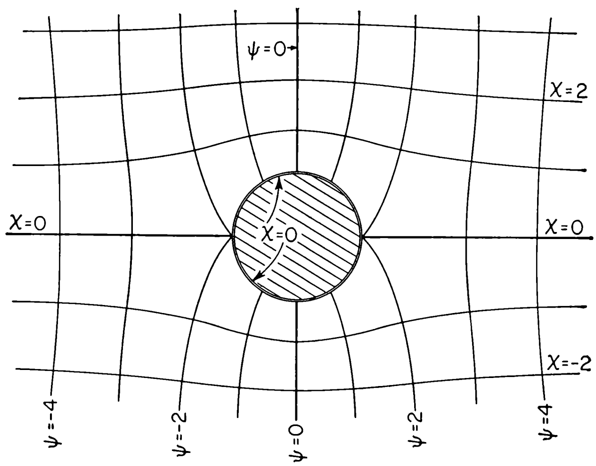

Fig. 10.4 Velocity potential and stream function for irrotational flow about cylinder.

Flow of Viscous Liquids. Referring to Eqs. (2.3.14) and (2.3.15), we note that the relation between pressure and fluid velocity in the slow, steady flow of a viscous, incompressible liquid is

$$\text{grad } p = -2\eta \text{ curl } \boldsymbol{\omega}$$

where p is the pressure in the liquid, η its coefficient of viscosity, $\boldsymbol{\omega} = \frac{1}{2}$ curl $\mathbf{v}$ is the vorticity, and $\mathbf{v}$ the velocity of the liquid. We neglect volume forces $\mathbf{F}$, such as gravity; we insist that div $\mathbf{v} = 0$, for the liquid is to be incompressible; and we neglect the term $\mathbf{v} \cdot \nabla\mathbf{v}$, since $\mathbf{v}$ is to be small.

In two-dimensional flow, $\mathbf{v} = \mathbf{i}v_x + \mathbf{j}v_y$ and the vorticity $\boldsymbol{\omega}$ is in the z direction with magnitude

$$\omega = \tfrac{1}{2}\frac{\partial v_y}{\partial x} - \tfrac{1}{2}\frac{\partial v_x}{\partial y}$$

The equation relating pressure p with magnitude of vorticity ω is, for two dimensions,

$$\frac{1}{2\eta}\frac{\partial p}{\partial x} = -\frac{\partial \omega}{\partial y}; \quad \frac{1}{2\eta}\frac{\partial p}{\partial y} = \frac{\partial \omega}{\partial x}$$

which are the Cauchy relations between real and imaginary parts of a function of a complex variable. Therefore if we can find a suitable function $W(z)$ of the complex variable $z = x + iy$, its real part can be $p/2\eta$ and its imaginary part can be $-\omega$. From this one function we therefore can find pressure and vorticity.

Moreover, if we can find a flow function $U(x,y)$, the derivatives of which are related to the velocity by the equations

$$\partial U/\partial y = v_x; \quad \partial U/\partial x = -v_y$$

(in other words the curl of $U\mathbf{k}$ equals $\mathbf{v}$) we see that the Laplacian of U is related to the vorticity by the Poisson equation

$$\nabla^2 U = -2\omega$$

Therefore if we can find U (or else W) and make it fit the boundary conditions, we can find ω and p (or else U and p) and solve a problem in viscous flow.

A very simple case is that of two-dimensional flow (parallel to the x, y plane) of a viscous fluid between two parallel planes, one at $y = +b$ and the other at $y = -b$. The flow will all be parallel to the x axis, and the fluid velocity at the point (x,y) will be $v_x = (3Q/4b^3)(b^2 - y^2)$, where Q is the volume flow between the two planes per unit distance perpendicular to the x, y plane. This indicates that $U = (Q/b^3)[\frac{3}{4}b^2y - \frac{1}{4}y^3]$. The Laplacian of U is $-(3Q/2b^3)y$, which is the imaginary part of the analytic function

$$W = -\frac{3Q}{2b^3}[x + iy - x_0] = \frac{p}{2\eta} - i\omega$$

Therefore the pressure is $(3\eta Q/b^3)(x_0 - x)$, where x_0 is the edge of the plates (where the pressure is zero) and the vorticity is $(3Q/2b^3)y$, being largest numerically near the plates at $y = \pm b$, which is not surprising.

These relationships can also be expressed in terms of curvilinear coordinates in two dimensions. We set $f(\zeta + i\phi) = x + iy$ where ζ and ϕ are the new coordinates. The scale factors h_ζ and h_ϕ are both equal to $|df/dw|$, where $w = \zeta + i\phi$, and the Laplacian in the new coordinates is [by Eq. (4.1.11)]

$$\nabla^2 U = 4\left(\frac{\partial^2 U}{\partial z\, \partial \bar{z}}\right) = \frac{4}{h^2}\left(\frac{\partial^2 U}{\partial w\, \partial \bar{w}}\right)$$

where $h^2 = |df/dw|^2$ and w and $\bar{w}$ are considered to be independent variables.

Now let us take the flow function U to be the sum of two functions

$$U = \frac{i}{8}[f(w)F(\bar{w}) - f(\bar{w})F(w)] + U_0 = \tfrac{1}{4}\,\mathrm{Im}[f(\bar{w})F(w)] + U_0 \qquad (10.1.15)$$

where $f(w)$ is the function defining the coordinates ζ and ϕ, F is some arbitrary function of w or $\bar{w}$, and U_0 is a solution of Laplace's equation $\nabla^2 U_0 = 0$; F and U_0 are chosen to fit the conditions of the problem. The term in brackets is not an analytic function, for it depends on both w and $\bar{w}$. From it, however, we can find the velocity by differentiation:

$$v_\zeta = \frac{1}{h}\frac{\partial U}{\partial \phi}; \quad v_\phi = -\frac{1}{h}\frac{\partial U}{\partial \zeta}$$

Since the term in brackets is not the imaginary part of an analytic function, it will not necessarily be a solution of Laplace's equation. We had counted on this for, according to the discussion above, the vorticity

$$\omega = -\tfrac{1}{2}\nabla^2 U = -\frac{i}{4h^2}[f'(w)F'(\bar{w}) - f'(\bar{w})F'(w)]$$
$$= \frac{i}{4}\left[\frac{F'(w)}{f'(w)} - \frac{F'(\bar{w})}{f'(\bar{w})}\right] = -\tfrac{1}{2}\,\mathrm{Im}\left[\frac{F'(w)}{f'(w)}\right]$$

where the primes indicate differentiation with respect to w or $\bar{w}$.

But this result shows that, if we choose U to have the form given in Eq. (10.1.15), then the vorticity will be the imaginary part of a function of $w = \zeta + i\phi$. Therefore, from the earlier discussion, for slow flow the pressure and vorticity will be given by the complex function

$$W = \tfrac{1}{2}[F'(w)/f'(w)] = (p/2\eta) + \text{constant} - i\omega \qquad (10.1.16)$$

As an example, which is not quite so trivial as the parallel-plate one, suppose that we tilt the plates so they are at an angle 2α to each other. Expressed in polar coordinates, one of the boundary planes is at $\phi = \alpha$, the other at $\phi = -\alpha$. Suppose that the fluid is being forced from the narrowest region, say from $r = r_0$, where the plates are a distance $2r_0 \sin\alpha$ apart, out to the wider opening at $r = r_1$, where the plates are $2r_1 \sin\alpha$ apart. The suitable coordinates are the polar coordinates ζ and ϕ, where $z = f(w) = e^w$, $w = \zeta + i\phi$. The scale factor is $h = |f'| = e^\zeta$ and $x = e^\zeta \cos\phi$, $y = e^\zeta \sin\phi$.

We should like a function F such that the flow function U is a function of ϕ but not of ζ, in which case the flow will be all radial. To obtain this we try $F(w) = Ae^{-w}$. Then from Eq. (10.1.15)

$$U = \tfrac{1}{8}iA[e^{2i\phi} - e^{-2i\phi}] + B\phi = B\phi - \tfrac{1}{4}A\sin(2\phi)$$

The function $B\phi$, which is a solution of Laplace's equation, is added in order to fit the boundary conditions. Computing v, we find

$$v_r = e^{-\zeta}[B - \tfrac{1}{2}A\cos(2\phi)]; \quad v_\phi = 0$$

which must be zero at $\phi = \pm\alpha$. Therefore $B = \frac{1}{2}A\cos(2\alpha)$. The value of A is then determined by computing the total flow

$$Q = \int_{-\alpha}^{\alpha} v_r e^{\xi}\, d\phi = \tfrac{1}{2}A[2\alpha\cos(2\alpha) - \sin(2\alpha)]$$

so that $A = Q/(\alpha\cos 2\alpha - \frac{1}{2}\sin 2\alpha)$. Consequently

$$F(w) = \frac{Qe^{-w}}{\alpha\cos 2\alpha - \frac{1}{2}\sin 2\alpha} \quad\text{and}\quad W = \frac{-Qe^{-2w}}{2\alpha\cos 2\alpha - \sin 2\alpha}$$

The final solution is, therefore,

$$\begin{aligned} v_r &= \frac{(Q/r)}{\sin 2\alpha - 2\alpha\cos 2\alpha}[\cos(2\phi) - \cos(2\alpha)]; \quad v_\phi = 0 \\ p &= \frac{-(2\eta Q)}{\sin 2\alpha - 2\alpha\cos 2\alpha}\left[\frac{1}{r_0^2} - \frac{\cos(2\phi)}{r^2}\right] \\ \omega &= -\frac{Q}{\sin 2\alpha - 2\alpha\cos 2\alpha}\left[\frac{\sin(2\phi)}{r^2}\right] \end{aligned} \qquad (10.1.17)$$

where we have made the pressure zero at $r = r_0$, $\phi = 0$. The pressure drop from r_0 to r_1 along the axis is then

$$[2\eta Q/(\sin 2\alpha - 2\alpha\cos 2\alpha)]\,[(r_1^2 - r_0^2)/r_1^2 r_0^2].$$

As one can see, this method is not too satisfactory because one must "work backward," trying various functions F until one is found (if ever) which corresponds to a case of physical interest for some coordinate system. When such a case is found, the results are useful, but it is not a very satisfactory way to do theoretical physics.

Green's Function in Polar Coordinates. The potential due to a unit line charge at (r_0,ϕ_0) is given by the real part of the complex function [see Eq. (10.1.4)]

$$\begin{aligned} -\ln(z - z_0)^2 &= -2\ln z + 2\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{z_0}{z}\right)^n; \quad |z| > |z_0| \\ &= -2\ln z_0 + 2\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{z}{z_0}\right)^n; \quad |z| < |z_0| \end{aligned}$$

where $z = re^{i\phi}$ and $z_0 = r_0e^{i\phi_0}$. Therefore the potential, which is the Green's function expressed in polar coordinates, is

$$\begin{aligned} G(r,\phi|r_0,\phi_0) &= -2\ln r + 2\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r_0}{r}\right)^n\cos[n(\phi - \phi_0)]; \quad r > r_0 \\ &= -2\ln r_0 + 2\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{r_0}\right)^n\cos[n(\phi_0 - \phi)]; \quad r < r_0 \end{aligned} \qquad (10.1.18)$$

A problem which can be solved by the use of this Green's function is that of the potential distribution, about a grounded cylinder of radius a with center at the origin, due to a line charge of q per unit length at the point $(r_0,0)(r_0 > a)$. In terms of the coordinates (r,ϕ) about the origin, the potential of the line charge, if the grounded cylinder were not there, would be by Eq. (10.1.18)

$$-2q \ln r_0 + 2q \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{r}{r_0}\right)^n \cos(n\phi); \quad r < r_0$$

We must add to this a combination of terms of the sort $(1/r)^n \cos(n\phi)$, which are solutions of Laplace's equation, so that the potential will be zero at $r = a$. It is not difficult to see that this combination must be

$$\begin{aligned}\psi(r,\phi) &= 2q \ln\left(\frac{r}{a}\right) - 2q \sum_{n=1}^{\infty} \frac{1}{n} \left[\left(\frac{a^2}{r_0 r}\right)^n - \left(\frac{r}{r_0}\right)^n\right] \cos(n\phi); \quad a < r < r_0 \\ &= 2q \ln\left(\frac{r_0}{a}\right) - 2q \sum_{n=1}^{\infty} \frac{1}{n} \left[\left(\frac{a^2}{r_0 r}\right)^n - \left(\frac{r_0}{r}\right)^n\right] \cos(n\phi); \quad r > r_0\end{aligned} \tag{10.1.19}$$

the second expression being due to the fact that the expression for the Green's function has a different form when $r > r_0$ from when $r < r_0$

What we have done is to add the series

$$2q \ln\left(\frac{rr_0}{a}\right) - 2q \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{a^2}{r_0 r}\right) \cos(n\phi)$$

to the Green's function $qG(r,\phi|r_0,0)$. But this series is (aside from a constant) really the potential due to a line of charge $-q$ per unit length at the point $r = a^2/r_0$, $\phi = 0$, which is the *image* of the point $(r_0,0)$ in the circle $r = a$. Therefore we can write the potential of Eq. (10.1.19) as

$$\begin{aligned}\psi(r,\phi) &= qG(r,\phi|r_0,0) - qG\left(r,\phi \left| \frac{a^2}{r_0}, 0\right.\right) + 2q \ln\left(\frac{r_0}{a}\right) \\ &= q \ln\left\{\frac{(r/a)^2 + (a/r_0)^2 - 2(r/r_0) \cos \phi}{1 + (r/r_0)^2 - 2(r/r_0) \cos \phi}\right\}\end{aligned} \tag{10.1.20}$$

This form could, of course, have been derived directly from the discussion of Chap. 7 on image potentials from planes, cylinders, and spheres.

This function is the real part of the complex function

$$2q \ln\left[r_0 \left(z - \frac{a^2}{r_0}\right) \Big/ a(z - r_0)\right]$$

The imaginary part is the flow function

$$\chi(r,\phi) = 2q \tan^{-1}\left[\frac{(a^2 - r_0^2)(r/r_0)\sin\phi}{r^2 + a^2 - (r/r_0)(r_0^2 + a^2)\cos\phi}\right]$$

which gives the lines of force or of flow. This solution represents not only the distribution of electric potential and lines of force about a grounded cylinder and a line charge but also the steady-state diffusion of solute originating at the line $r = r_0$, $\phi = 0$ and being taken out of solution at the cylindrical surface $r = a$, the density of solute at any point being proportional to ψ. The potential and flow lines are shown in Fig. 10.5.

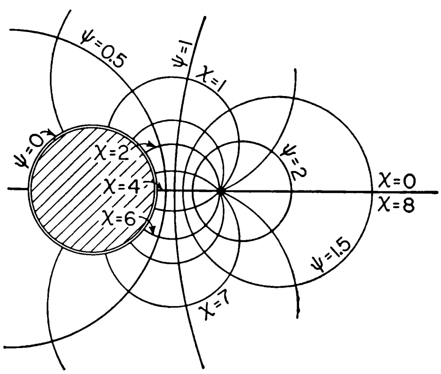

Fig. 10.5 Potential and flow function for source outside a grounded circular cylinder.

Very close to the point $r = r_0$, $\phi = 0$ the equipotential lines are nearly circular, the potential of the circle of radius ρ being obtained most easily by computing, to the first order in ρ/r_0, the potential ψ for $z = r_0 + \rho e^{i\theta}$:

$$V = \mathrm{Re}\left\{2q \ln\left[r_0\left(z - \frac{a^2}{r_0}\right)\Big/ a(z - r_0)\right]\right\}$$

$$\simeq -2q\,\mathrm{Re}\left\{\ln\left[a\rho e^{i\theta}\Big/ r_0\left(r_0 - \frac{a^2}{r_0}\right)\right]\right\} \simeq -2q \ln\left[\frac{\rho a}{r_0^2 - a^2}\right]; \qquad r_0 > a \gg \rho$$

From this the capacitance between a small wire and a cylinder can be computed. One can also compute the density of charge induced on the cylinder (or if it represents a diffusion problem, the rate of deposition on the cylinder) by differentiation of the result.

Internal Heating of Cylinders. The Green's function may also be useful in solving Poisson's equation for a distribution of charge or sources inside a circular cylinder. A typical problem of this sort arises in the calculation of the distribution of temperature inside cylindrical bars of uranium which are being heated internally by fission. At any point in the cylinder the rate of generation of heat is proportional to the rate of fission at that point, which is in turn proportional to the density of slow neutrons there. According to Eq. (2.4.4), if the amount of heat generated per unit volume is $q(x,y)$ and the conductivity of the uranium is K, the equation for the temperature T inside the cylinder is the Poisson equation $\nabla^2 T = -(q/K)$.

For a point source of heat at the point (r_0,ϕ_0) inside the cylinder, the steady-state distribution in temperature is given by the Green's function of Eq. (10.1.18), modified by the solution of Laplace's equation which makes the whole fit the boundary conditions at the surface of the cylinder $r = a$. If the boundary condition is that $T = 0$ at $R = a$, then the temperature at r, ϕ is

$$T(r,\phi|r_0,\phi_0) = \frac{q(r_0,\phi_0)}{4\pi K}\left\{G(r,\phi|r_0,\phi_0) + 2\ln a - 2\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r_0 r}{a^2}\right)^n \cos[n(\phi_0 - \phi)]\right\}$$

and if the rate of generating heat is distributed over the interior with density $q(r_0,\phi_0)$ at (r_0,ϕ_0) (independent of z, the distance along the cylinder axis), the resulting temperature will be the integral of $T(r,\phi|r_0,\phi_0)$ over the interior of the cylinder:

$$\begin{aligned}T(r,\phi) = \frac{1}{2\pi K}\int_0^{2\pi} d\phi_0 \Bigg\{\int_0^r r_0\,dr_0\Bigg[\ln\left(\frac{a}{r}\right) &- \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r_0^n r^n}{a^{2n}} - \frac{r_0^n}{r^n}\right)\cos n(\phi_0-\phi)\Bigg] q(r_0,\phi_0) + \int_r^a r_0\,dr_0\Bigg[\ln\left(\frac{a}{r_0}\right) \\ &- \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r_0^n r^n}{a^{2n}} - \frac{r^n}{r_0^n}\right)\cos n(\phi_0-\phi)\Bigg] q(r_0,\phi_0)\Bigg\} \qquad (10.1.21)\end{aligned}$$

If, in addition, the density of neutrons is isotropic, the heat generation q will be independent of ϕ_0 and the expression for the temperature will have the simpler form

$$T(r) = \frac{1}{K}\left\{\ln\left(\frac{a}{r}\right)\int_0^r q(r_0)r_0\,dr_0 + \int_r^a \ln\left(\frac{a}{r_0}\right) q(r_0)r_0\,dr_0\right\}$$

Simplifying still further, we can assume that the rate of fission is greatest near the outer surface of the cylinder, reducing toward the center roughly according to the equation $q(r_0) = (Q/\pi a^2)[(b^2 + r_0^2)/(b^2 + \frac{1}{2}a^2)]$, where Q is the total heat generated per second per unit length of cylinder. In this case we can perform the integrations in terms of simple functions, and find that

$$T(r) = \left(\frac{Q}{4\pi K}\right)\left[\frac{b^2(a^2 - r^2) + \frac{1}{4}(a^4 - r^4)}{a^2b^2 + \frac{1}{2}a^4}\right]$$

It is not surprising to find that the temperature is a maximum at the center of the cylinder and that, the smaller b is (the more the heat generation is concentrated near the surface), the smaller is the maximum temperature at the center and the steeper is the temperature gradient near the outer surface.

Potential Near a Slotted Cylinder. The Green's function technique may be used to compute the potential around a grounded, metallic, cylindrical shell of radius $r = a$, which has a slot cut in it from $\phi_1 < \phi < \phi_2$. (In other words, the surface $r = a$, $\phi_2 < \phi < 2\pi + \phi_1$ for all values of z is a metal shell; the rest of the cylinder, from ϕ_1 to ϕ_2, is an open gap.) We return to Eq. (7.2.7) for our starting point.

$$\psi(\mathbf{r}) = (1/4\pi) \oint [G(\mathbf{r}|\mathbf{r}_0^s) \operatorname{grad}_0 \psi(\mathbf{r}_0^s) - \psi(\mathbf{r}_0^s) \operatorname{grad}_0 G(\mathbf{r}|\mathbf{r}_0^s)] \cdot d\mathbf{A}_0$$

when there is no free charge. We shall use this in two ways.

In the first place we consider the metallic part of the slotted cylinder to be the surface S_0 and the Green's function the one for free space, given in Eq. (10.1.18). Since the potential ψ is supposed to be zero at this surface, only the first term in the integrand is present, and we have

$$\psi(r,\phi) = \frac{a}{4\pi} \int_{\phi_2}^{2\pi+\phi_1} G(r,\phi|a,\beta) \left[\frac{\partial}{\partial r} \psi_i(r,\beta) - \frac{\partial}{\partial r} \psi_0(r,\beta) \right]_{r=a} d\beta$$

where ψ_i is the potential just inside the shell and ψ_0 that just outside. As we said, $G(r,\phi|a,\beta)$ is that given in Eq. (10.1.18). This equation has a very simple interpretation; the difference of gradient of ψ between inside and outside the surface, divided by 4π, is the charge density induced in the shell by the potential. Naturally the potential is that caused by this charge, except that we can add to the integral any solution of Laplace's equation which does not have a discontinuity in the finite region in order to satisfy boundary conditions at infinity.

An exact solution of this integral equation is not easy, but we can use it to obtain a first approximation to the field. Suppose that the slotted cylinder is placed in a uniform field of intensity E at large values of r. To the zeroth approximation, therefore, the potential outside the cylinder is $\psi_0^0 = -Er \cos \phi + E(a^2/r) \cos \phi$ and inside the cylinder $\psi_i^0 = 0$. The quantity in brackets in the integral is then $2E \cos \phi$, which is 4π times the charge induced on the cylinder if it has no slot. According to our discussion, the first approximation to the field inside or outside the slotted cylinder is

$$\psi(r,\phi) \simeq -Er \cos \phi + \frac{Ea}{2\pi} \int_{\phi_2}^{2\pi+\phi_1} G(r,\phi|a,\beta) \cos \beta \, d\beta$$

where the first term is added to fit the requirements at infinity.

We note that, when $\phi_1 = \phi_2$ (no slot), $\psi_0 = -Er \cos \phi + E(a^2/r) \cos \phi$ and $\psi_i = 0$, as it must, and if there is no cylinder (*i.e.*, if it is all slot), then $\psi_0 = \psi_i = -Er \cos \phi$. For intermediate values we insert series (10.1.18), obtaining for the internal potential, for instance,

$$\psi_i^1 \simeq \frac{Ea}{\pi} \ln a \int_{\phi_1}^{\phi_2} \cos \beta \, d\beta - \frac{Ea}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{r}{a} \right)^n \int_{\phi_1}^{\phi_2} \cos \beta \cos n(\phi - \beta) \, d\beta$$

where we have utilized the orthogonality of the cosine functions to change from integration over the metal surface ($\phi_2 < \phi < 2\pi + \phi_1$) to integration over the slot ($\phi_1 < \phi < \phi_2$). This potential is, of course, just that produced by the charge which has been removed when the slot was cut, the charge which is *not* there, so to speak. This potential is not exactly equal to zero along the rest of the cylinder, but after all, this is only the first approximation.

Therefore we set about computing, for the next approximation, a potential which *is* zero at $r = a$, $\phi_2 < \phi < 2\pi + \phi_1$ and which has the values given in the first approximation expression, for $r = a$, $\phi_1 < \phi < \phi_2$. To do this we use the Green's function, given in Eq. (10.1.19), which goes to zero at $r = a$. Using Eq. (7.2.7) again, we have

$$\psi(r,\phi) = -\frac{a}{4\pi}\int_{\phi_1}^{\phi_2} \psi^1(a,\gamma)\left[\pm\frac{\partial}{\partial r_0}G(r,\phi|r_0,\gamma)\right]_{r_0=a} d\gamma$$

where the Green's function used is zero at $r = a$ and the positive gradient is used for $r < a$, the negative for $r > a$.

At $r = a$, the series for ψ_i^1 is

$$\psi^1(a,\gamma) = Ea\ln(a)X_{10} - \sum_{n=1}^{\infty}\frac{Ea}{n}[X_{1n}\cos(n\phi) + Y_{1n}\sin(n\phi)]$$

where

$$X_{mn} = \frac{1}{\pi}\int_{\phi_1}^{\phi_2}\cos(m\gamma)\cos(n\gamma)\,d\gamma = \left[\frac{\sin(m-n)\gamma}{2\pi(m-n)} + \frac{\sin(m+n)\gamma}{2\pi(m+n)}\right]_{\phi_1}^{\phi_2}$$

$$Y_{mn} = \frac{1}{\pi}\int_{\phi_1}^{\phi_2}\cos(m\gamma)\sin(n\gamma)\,d\gamma = \left[\frac{\cos(m-n)\gamma}{2\pi(m-n)} - \frac{\cos(m+n)\gamma}{2\pi(m+n)}\right]_{\phi_1}^{\phi_2}$$

When $\phi_2 = \phi_1$ (no slot), all the X's and Y's are zero, and when $\phi_2 = \phi_1 + 2\pi$ (all slot, no cylinder), $X_{mm} = 2/\epsilon_m$ and all others are zero ($\epsilon_0 = 1$, $\epsilon_m = 2$ for $m > 0$).

The potential inside the slotted cylinder to the second approximation is obtained by using the Green's function, like Eq. (10.1.19), for $r < a$:

$$\left[\frac{\partial}{\partial r_0}G(r,\phi|r_0,\gamma)\right] = \frac{\partial}{\partial r_0}\left\{2\ln\left(\frac{a}{r_0}\right) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left[\left(\frac{r}{r_0}\right)^m - \left(\frac{rr_0}{a^2}\right)^m\right]\cos[m(\phi-\gamma)]\right\}$$

$$\xrightarrow[r_0\to a]{} -\frac{2}{a}\sum_{m=0}^{\infty}\epsilon_m\left(\frac{r}{a}\right)^m\cos[m(\phi-\gamma)]$$

Inserting all this in the equation for the second approximation, we have

$$\psi_1^2(r,\phi) \simeq \tfrac{1}{2}aE \sum_{m=0}^{\infty} \epsilon_m \left(\frac{r}{a}\right)^m [A_m \cos(m\phi) + B_m \sin(m\phi)]$$

where

$$A_m = X_{10}X_{m0} \ln a - \sum_{n=1}^{\infty} \frac{1}{n} (X_{1n}X_{mn} + Y_{1n}Y_{mn})$$

$$B_m = - \sum_{n=1}^{\infty} \frac{1}{n} (X_{1n}Y_{nm} + Y_{1n}Z_{mn})$$

and where

$$Z_{mn} = \frac{1}{\pi} \int_{\phi_1}^{\phi_2} \sin(m\gamma) \sin(n\gamma)\, d\gamma = \left[\frac{\sin(m-n)\gamma}{2\pi(m-n)} - \frac{\sin(m+n)\gamma}{2\pi(m+n)}\right]_{\phi_1}^{\phi_2}$$

The potential at the center of the cylinder is thus

$$\psi(0) = \tfrac{1}{2}aE \left\{(\sin\phi_2 - \sin\phi_1)(\phi_2 - \phi_1)(\ln a/\pi^2) \right.$$
$$\left. - \sum_{n=1}^{\infty} \left[\frac{1}{2\pi^2 n^2}\right] (\sin n\phi_2 - \sin n\phi_1) \left[\frac{\sin(n+1)\gamma}{n+1} + \frac{\sin(n-1)\gamma}{n-1}\right]_{\phi_1}^{\phi_2}\right\}$$

The potential outside the cylinder, to the same approximation, is

$$\psi_0^2(r,\phi) \simeq -Er\cos\phi + E\left(\frac{a^2}{r}\right)\cos\phi$$
$$+ \tfrac{1}{2}aE \sum_{m=0}^{\infty} \epsilon_m \left(\frac{a}{r}\right)^m [A_m \cos(m\phi) + B_m \sin(m\phi)]$$

where the first two terms are required by the conditions at infinity. They are, of course, zero at $r = a$, and the series is zero at $r = a$, $\phi_2 < \phi < 2\pi + \phi_1$, just as the series for ψ_i is. The two expressions, therefore, fit in value over the slot, though they are not exactly continuous in gradient. A still better approximation may be computed by taking the difference in gradient of these solutions and integrating over the metal surface, as we did for ψ_0^0 and ψ_i^0, and so on.

Another method of calculation of the field assumes that we know the potential $\psi_s(\phi)$ in the slot, which we can always center about $\phi = 0$ (*i.e.*, the slot can be in the range $-\tfrac{1}{2}\Delta < \phi < \tfrac{1}{2}\Delta$) with the angular width of slot equal to Δ. The potential inside and out is then

$$\psi(r,\phi) = \begin{cases} \displaystyle\sum_{m=0}^{\infty} [A_m \cos(m\phi) + B_m \sin(m\phi)] \left(\frac{r}{a}\right)^m; \quad r < a \\ \displaystyle E\left[\left(\frac{a^2}{r}\right) - r\right] \cos(\phi - \alpha) + \sum_{m=0}^{\infty} [A_m \cos(m\phi) \\ \qquad\qquad\qquad\qquad + B_m \sin(m\phi)] \left(\frac{a}{r}\right)^m; \quad r > a \end{cases}$$

where
$$\left.\begin{matrix} A_m \\ B_m \end{matrix}\right\} = \left(\frac{\epsilon_m}{2\pi}\right) \int_{-\frac{1}{2}\Delta}^{\frac{1}{2}\Delta} \psi_s(u) \begin{matrix}\cos\\ \sin\end{matrix}(mu)\, du \qquad (10.1.22)$$

We shall see on page 1206 that in certain cases it is possible to make a fairly accurate assumption as to the dependence of $\psi_s(u)$ on u in the range $\pm\frac{1}{2}\Delta$, when we can obtain an expression for the potential directly. Another use of the formula would be to assume a form for ψ_s and then to adjust this form so that it produces the best fit. As given, $\psi(r,\phi)$ has a discontinuity in slope at $r = a$, $-\frac{1}{2}\Delta < \phi < \frac{1}{2}\Delta$. We could adjust ψ so that the average discontinuity across the gap is zero or so that the mean square of the discontinuity is a minimum.

Elliptic Coordinates. Referring to Eq. (5.1.15) we see that the transformation $w = \mu + i\vartheta = \cosh^{-1}(2z/a)$ gives rise to the elliptic coordinates

$$\begin{aligned} x &= \tfrac{1}{2}a \cosh\mu \cos\vartheta; \quad y = \tfrac{1}{2}a \sinh\mu \sin\vartheta \\ h_\mu &= h_\vartheta = |dz/dw| = \tfrac{1}{2}a \sqrt{\sinh^2\mu + \sin^2\vartheta} = \tfrac{1}{2}a \sqrt{\cosh^2\mu - \cos^2\vartheta} \\ r &= \tfrac{1}{2}a \sqrt{\cosh^2\mu - \sin^2\vartheta}; \quad \phi = \tan^{-1}[\tanh\mu \tan\vartheta] \qquad (10.1.23) \\ r_1 &= \sqrt{(x + \tfrac{1}{2}a)^2 + y^2} = \tfrac{1}{2}a(\cosh\mu + \cos\vartheta) \\ r_2 &= \sqrt{(x - \tfrac{1}{2}a)^2 + y^2} = \tfrac{1}{2}a(\cosh\mu - \cos\vartheta) \end{aligned}$$

where the curves μ = constant are ellipses and the ϑ curves are hyperbolas, all with foci at $x = \pm\frac{1}{2}a$. They are shown in Fig. 10.6.

As before the Laplace equation has elementary solutions μ, ϑ, $e^\mu \cos\vartheta$, $\cosh(n\mu)\sin(n\vartheta)$, etc. The surfaces suitable for these coordinates are elliptic cylinders with major axis $a\cosh\mu$ and minor axis $a\sinh\mu$ (the surfaces μ = constant) and, in particular, the limiting case $\mu = 0$, which corresponds to the plane strip of width a with center along the z axis, or else the hyperbolic cylinders ϑ = constant and, in particular, the limiting case $\vartheta = 0, \pi$, corresponding to the (x,z) plane *except* for the strip of width a, centered on the z axis.

For instance, the velocity potential for an incompressible, nonviscous fluid flowing through a slit of width a in a plane barrier, which is filled on both sides with fluid, is just

$$\psi = A\mu = A \cosh^{-1}[(r_1 + r_2)/a] \qquad (10.1.24)$$

where r_1 is the distance from the point where the potential is measured to one edge of the slit (the one at $x = -\frac{1}{2}a$, $y = 0$) and r_2 is the distance to the other edge (at $x = \frac{1}{2}a$, $y = 0$). For very large values of $r = \sqrt{x^2 + y^2}$, the two distances become

$$r_1 \to r + \tfrac{1}{2}a \cos\phi; \quad r_2 \to r - \tfrac{1}{2}a \cos\phi$$

so that, to the second order in a/r, we have

$$\psi \xrightarrow[r\to\infty]{} A \ln(4r/a); \quad \mathbf{v} = \operatorname{grad}\psi \to A/r$$

The stream function is, of course,

$$\chi = A\vartheta = A \cos^{-1}[(r_1 - r_2)/a] \xrightarrow[r\to\infty]{} A|\phi|$$

In this example we let μ go from $-\infty$ to $+\infty$, thereby using the change of sign of μ to give the change of sign of y. The angle ϑ needs

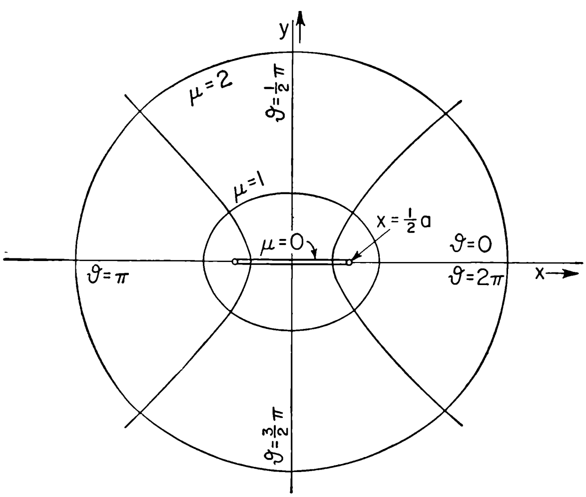

Fig. 10.6 Elliptic coordinates, μ and ϑ.

to go only from 0 to π; we do not need to have complete continuity **in** ϑ, since the half planes $\vartheta = 0$ and $\vartheta = \pi$ are barriers.

The fluid velocity across the line $\mu = 0$ is

$$\begin{aligned}\operatorname{grad}\psi &= \frac{(2/a)}{\sqrt{\cosh^2\mu - \cos^2\vartheta}}\left[\mathbf{a}_\mu \frac{\partial\psi}{\partial\mu} + \mathbf{a}_\vartheta \frac{\partial\psi}{\partial\vartheta}\right] \\ &= \frac{A}{\sqrt{(a/2)^2 - x^2}}\,\mathbf{j}; \quad \text{when } \mu = 0\end{aligned}$$

and the total flow across the x axis from $x = x_0$ to $x = +x_1$

$$(|x_0|,|x_1| < \tfrac{1}{2}a) \text{ is } A(\vartheta_{x_1} - \vartheta_{x_0}) = A[\cos^{-1}(2x_1/a) - \cos^{-1}(2x_0/a)]$$

The total flow through the slit is therefore πA.

If we wish to talk of the electrostatic potential about a strip of width a, kept at potential V with respect to a grounded, confocal, elliptic cylinder of major axis $a \cosh \mu_0$, minor axis $a \sinh \mu_0$, we let ϑ go from 0 to 2π and keep μ positive. The negative y half of the plane then corresponds to $\pi < \vartheta < 2\pi$ rather than to $\mu < 0$. The potential for the case mentioned is

$$\psi = V[1 - (\mu/\mu_0)]$$

the charge density at point $\vartheta = \cos^{-1}(2x/a)$ on the strip is

$$\sigma = \frac{(V/\mu_0)}{2\pi a|\sin \vartheta|}$$

and the total charge on the strip, per unit length, is

$$Q = \tfrac{1}{2}a \int_0^{2\pi} \sigma|\sin \vartheta|\, d\vartheta = \frac{V}{2\mu_0}$$

where we have included the charge on "both sides" of the strip (the upper, $0 < \vartheta < \pi$, and the lower side, $\pi < \vartheta < 2\pi$). The capacitance per unit length of this combination is therefore

$$C = Q/V = 1/2\mu_0$$

When μ_0 is large, the elliptic cylinder becomes almost equal to a circular cylinder, concentric with the strip, of radius $r_0 \simeq (a/4)e^{\mu_0}$. Therefore the limiting capacitance between a circular cylinder of radius r_0 and a small concentric strip of width a is

$$C \simeq [\tfrac{1}{2} \ln(4r_0/a)] \quad \text{esu per unit length}$$

If the inner strip were replaced by a concentric inner cylinder of diameter a, equal to the width of the strip, the capacitance would be $[\frac{1}{2} \ln(2r_0/a)]$, a somewhat larger value.

Viscous Flow through a Slit. Returning to the analysis of page 1186 for viscous flow, we can compute, by using elliptic coordinates, the case of an incompressible, viscous fluid passing through a slit of width a. We wish the velocity to be along the coordinate lines ϑ = constant, so that the stream function U, defined in Eq. (10.1.15), should depend only on ϑ. Since $f(\mu + i\vartheta) = z = \frac{1}{2}a \cosh w$, we can obtain an expression with imaginary part independent of μ, by setting $F(w) = A \sinh w$ and by setting $U_0 = B\vartheta$ (which is a solution of Laplace's equation). We then can compute values of velocity $\mathbf{v}$ and adjust A and B so that $\mathbf{v}$ is zero at the boundaries $\vartheta = 0$ and $\vartheta = \pi$ and so that the total flow through the slit, per unit length of slit, is Q.

The resulting calculations show that the correct expressions for $F(w)$ and U are, for $f(w) = \frac{1}{2}a \cosh w = z$,

$$F(w) = -(2Q/\pi a) \sinh w$$

$$U = \left(\frac{Q}{\pi}\right)(\vartheta - \tfrac{1}{2}\sin 2\vartheta); \quad v_\mu = \frac{4Q}{\pi a}\frac{-\sin^2 \vartheta}{\sqrt{\sinh^2 \mu + \sin^2 \vartheta}}; \quad v_\vartheta = 0$$

The velocity is greatest at the center of the slit and falls off to zero at the edges of the slit and also at large distances from the slit. In the plane of the slit, a distance x from the center line, the velocity is upward, of magnitude

$$v_y = (4Q/\pi a)\sqrt{1 - (2x/a)^2}; \quad \mu = 0$$

which goes to zero at $x = \pm\frac{1}{2}a$, the edge of the slit.

We next use Eq. (10.1.16) to compute the complex function W, from which we compute the pressure and the vorticity:

$$W = -\frac{2Q}{\pi a^2}\coth w; \quad \omega = \frac{Q}{\pi a^2}\frac{-\sin 2\vartheta}{\sinh^2\mu + \sin^2\vartheta}$$

$$p = \frac{2\eta Q}{\pi a^2}\left[2 - \frac{\sinh 2\mu}{\sinh^2\mu + \sin^2\vartheta}\right] \to \begin{cases} 0; & \mu \to \infty \\ 4\eta Q/\pi a^2; & \mu \to -\infty \end{cases}$$

The *flow resistance* of the slot, therefore, the ratio of pressure drop to total flow per unit length of slit, Q, is

$$R = p/Q = 4\eta/\pi a^2$$

proportional to the coefficient of viscosity and inversely proportional to the square of the slit width.

It should be emphasized again that these calculations are valid only when the viscosity is large enough so that the kinetic energy term in Eq. (2.3.15) is negligible compared with the term representing viscous rate of deformation.

The potential distribution outside an elliptic cylinder $\mu = \mu_0$ which is held at a specified potential $\psi_0(\vartheta)$ on its surface is, as before,

$$\psi(\mu,\vartheta) = \left[\frac{1}{2\pi}\int_0^{2\pi}\psi_0(\beta)\,d\beta\right](\mu - \mu_0) + \sum_{n=1}^{\infty}\left[\frac{1}{\pi}\int_0^{2\pi}\psi_0(\beta)\cos m(\beta - \vartheta)\,d\beta\right]e^{m(\mu_0-\mu)} \quad (10.1.25)$$

which stays finite for $\mu > \mu_0$, except for the first term. Similarly the potential inside such a surface is

$$\psi(\mu,\vartheta) = \left[\frac{1}{2\pi}\int_0^{2\pi}\psi_0(\beta)\,d\beta\right] + \sum_{n=1}^{\infty}\left[\frac{1}{\pi}\int_0^{2\pi}\psi_0(\beta)\cos(m\beta)\,d\beta\right]\cos(m\vartheta)\frac{\cosh(m\mu)}{\cosh(m\mu_0)} + \sum_{n=1}^{\infty}\left[\frac{1}{\pi}\int_0^{2\pi}\psi_0(\beta)\sin(m\beta)\,d\beta\right]\sin(m\vartheta)\frac{\sinh(m\mu)}{\sinh(m\mu_0)}$$

where the $\cosh(m\mu)$ factor is used with $\cos(m\vartheta)$ and the $\sinh(m\mu)$ with $\sin(m\vartheta)$ in order to have continuity at $\mu = 0$, where there is no boundary. Here for even functions of ϑ (cosine terms) the gradient of the μ term must be zero at $\mu = 0$, and for the odd functions of ϑ (sine terms) the value of the μ factor must be zero at $\mu = 0$.

For instance, the potential and flow functions outside an elliptic cylinder defined by $\mu = \mu_0$, the right half of which is at potential $+1$ and the left half at potential -1, is, after reducing the series to a closed form,

$$\psi = \frac{2}{\pi}\tan^{-1}\left[\frac{\cos\vartheta}{\sinh(\mu - \mu_0)}\right];\quad \chi = -\frac{2}{\pi}\tanh^{-1}\left[\frac{\sin\vartheta}{\cosh(\mu - \mu_0)}\right] \qquad (10.1.26)$$

which is but another variant of Eq. (10.1.14) for the new coordinates.

Elliptic Cylinders in Uniform Fields. Continuing our previous schedule, we next inquire as to the form of the expression for a uniform field in the γ direction:

$$-E(x\cos\gamma + y\sin\gamma) = -\tfrac{1}{2}Ea[\cosh\mu\cos\vartheta\cos\gamma + \sinh\mu\sin\vartheta\sin\gamma]$$

Another set of solutions, having the same ϑ dependence as this but vanishing at infinity, is $e^{-\mu}\cos\vartheta$ or $e^{-\mu}\sin\vartheta$, which can be used to help fit boundary conditions.

For instance, if the elliptic cylinder, of major axis $a\cosh\mu_0$ and minor axis $a\sinh\mu_0$, in a uniform field, has a dielectric constant ϵ, the potential turns out to be

$$\psi = -Ee^{\mu_0}\left[\frac{x\cos\gamma}{\cosh\mu_0 + \epsilon\sinh\mu_0} + \frac{y\sin\gamma}{\epsilon\cosh\mu_0 + \sinh\mu_0}\right];\quad \mu < \mu_0$$

$$(10.1.27)$$

$$= -E(x\cos\gamma + y\sin\gamma) + \tfrac{1}{4}Ea(\epsilon - 1)e^{\mu_0}\sinh(2\mu_0)e^{-\mu}\left[\frac{\cos\vartheta\cos\gamma}{\cosh\mu_0 + \epsilon\sinh\mu_0} + \frac{\sin\vartheta\sin\gamma}{\epsilon\cosh\mu_0 + \sinh\mu_0}\right];\quad \mu > \mu_0$$

The field inside the elliptic cylinder is a uniform field, but with a different magnitude *and direction* from the applied field. The added field outside goes to zero at large distances and is just sufficient to make the potential continuous in value at $\mu = \mu_0$ and the normal gradient outside equal ϵ times the normal gradient just inside the surface. If ψ is the magnetic potential and ϵ is the permeability, the solutions represent a ferro-magnetic bar in a magnetic field. For iron, the permeability is very large and ψ at the surface is practically equal to zero.

Another simpler problem is that of irrotational flow past an elliptic cylinder. The velocity potential for uniform flow with velocity v_0 in the γ direction is, of course,

$$v_0(x\cos\gamma + y\sin\gamma) = \tfrac{1}{2}av_0[\cosh\mu\cos\vartheta\cos\gamma + \sinh\mu\sin\vartheta\sin\gamma]$$

To this must be added enough of the solutions $e^{-\mu}\cos\vartheta$, $e^{-\mu}\sin\vartheta$ (which go to zero at $\mu\to\infty$) to make the normal gradient of ψ zero at $\mu=\mu_0$, the surface of the cylinder. The final expressions for the velocity potential and the flow function are

$$\begin{aligned}\psi &= \tfrac{1}{2}av_0[\cos\gamma\cos\vartheta(\cosh\mu + e^{\mu_0-\mu}\sinh\mu_0)\\ &\qquad + \sin\gamma\sin\vartheta(\sinh\mu + e^{-\mu}\cosh\mu_0)]\\ \chi &= \tfrac{1}{2}av_0[\cos\gamma\sin\vartheta(\sinh\mu - e^{\mu_0-\mu}\sinh\mu_0)\\ &\qquad - \sin\gamma\cos\vartheta(\cosh\mu - e^{\mu_0-\mu}\cosh\mu_0)]\end{aligned} \tag{10.1.28}$$

which are the real and imaginary parts, respectively, of the function

$$F(\mu+i\vartheta) = \tfrac{1}{2}av_0[e^{-i\gamma}\cosh(\mu+i\vartheta) + e^{\mu_0-\mu-i\vartheta}\sinh(\mu_0+i\gamma)]$$

When $\mu_0=0$, the elliptic cylinder reduces to a strip of width a, perpendicular to the y axis, and therefore at an angle γ with respect to the steady flow.

The velocity of the fluid past the surface at $\mu=0$ is, of course,

$$v = |\text{grad}\,\psi|_{\mu=\mu_0} = \frac{2/a}{\sqrt{\sinh^2\mu_0+\sin^2\vartheta}}\left[\frac{\partial\psi}{\partial\vartheta}\right]_{\mu=\mu_0} = \frac{v_0e^{\mu_0}\sin(\gamma-\vartheta)}{\sqrt{\sinh^2\mu_0+\sin^2\vartheta}} \tag{10.1.29}$$

According to Bernoulli's equation the pressure at the point (μ_0,ϑ) at the surface is

$$p = p_0 + \tfrac{1}{2}\rho v_0^2\left[1 - \frac{e^{2\mu_0}\sin^2(\gamma-\vartheta)}{\sinh^2\mu_0+\sin^2\vartheta}\right] \tag{10.1.30}$$

where p_0 is the pressure in the fluid a large distance away from the cylinder and where we have neglected the effect of gravity on the fluid (which would add a term ρg times the depth of the fluid from the upper surface).

We shall deal with this problem of pressures and total forces, due to the fluid motion, in more detail in the next section. Here we shall anticipate the results obtained there and state that the net force on the elliptic cylinder is zero, though there is a net torque. The fact that there is no net force is at first surprising. However, for completely irrotational flow and when viscosity is neglected, the flow pattern is completely symmetric about the cylinder; for each spot on the bottom half with a given velocity, and therefore pressure, there is a symmetrical spot on the top half with the same surface velocity and pressure, so that the vertical forces all cancel—likewise with the horizontal components.

The pressure is smallest near the points of greatest curvature of the cylinder, $\vartheta=0$ or π, where the fluid velocity is greatest. For very flat elliptic cross sections, μ_0 small, this velocity will become so large there that the pressure will become negative. Consequently, of course, the fluid will not "stick" to the cylinder and steady, irrotational flow will not be possible. What happens is that vortices slide off the sharp, trailing edge, which set up a counterrotation of fluid around the cylinder, so that the velocity at the sharp edge is as small as possible. The flow in the vortices need not be gone into here; all we need to take into account is the counterrotation of the fluid which they cause.

To illustrate the principle, we take the case of the flat strip, for $\mu_0 = 0$. In this case the surface velocity and pressure without the rotation are

$$v = \frac{v_0 \sin(\gamma - \vartheta)}{|\sin \vartheta|}; \quad p = p_0 - \tfrac{1}{2}\rho v_0^2 \left[\frac{\sin^2(\gamma - \vartheta) - \sin^2 \vartheta}{\sin^2 \vartheta}\right]$$

At the trailing edge ($\vartheta = 0$) the velocity is infinite and the pressure is $-\infty$. The (physically impossible) flow lines are shown in Fig. 10.7*a*.

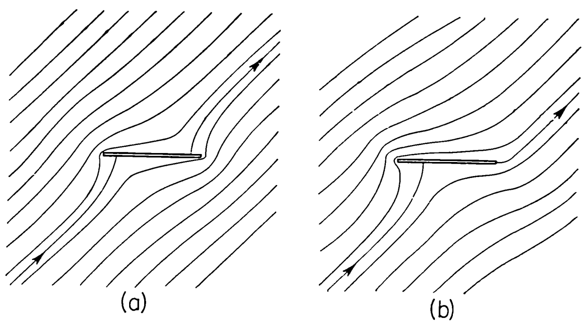

Fig. 10.7 Flow lines past rigid strip: (*a*) without circulation, (*b*) with sufficient circulation to allow smooth flow off trailing edge.

Of course, rotational flow cannot be represented by a single-valued potential, but simple rotation about the elliptic cylinder can be represented by the multivalued potential $A\vartheta$, which is a solution of Laplace's equation and which corresponds to a velocity $[2A/a \sqrt{\sinh^2 \mu + \sin^2 \vartheta}]$ in the ϑ direction at the point (μ,ϑ). What must be done is to add enough of this rotational potential to the potential of Eq. (10.1.28) to make v zero (or at least finite) at $\vartheta = 0$. The results are

$$\begin{aligned}
\psi &= \tfrac{1}{2}av_0[\cos\gamma \cos\vartheta \cosh\mu + \sin\gamma \sin\vartheta \cosh\mu - \vartheta \sin\gamma] \\
\chi &= \tfrac{1}{2}av_0[\cos\gamma \sin\vartheta \sinh\mu - \sin\gamma \cos\vartheta \sinh\mu + \mu \sin\gamma] \\
v_{\mu=0} &= v_0 \frac{\sin(\gamma - \vartheta) - \sin\gamma}{|\sin\vartheta|} = -v_0[\cos\gamma + \sin\gamma \tan(\tfrac{1}{2}\vartheta)] \qquad (10.1.31) \\
p &= p_0 + \tfrac{1}{2}\rho v_0^2\{\sin^2\gamma\, [1 - \tan^2(\tfrac{1}{2}\vartheta)] - \sin 2\gamma \tan(\tfrac{1}{2}\vartheta)\}
\end{aligned}$$

The pressure on the strip is now "lopsided," there being a preponderance of pressure on the underside of the strip ($\pi < \vartheta < 2\pi$) because of the term tan $(\frac{1}{2}\vartheta)$ in the expression for the pressure. This is owing to the fact that, because of the rotation, the velocity on the underside of the strip is less, in general, than the velocity over the top of the strip and the reduction in pressure, due to the velocity, is consequently not so large there. We thus obtain a net *lift* on the strip because of the circulation.

We have deliberately evaded two troublesome questions: What becomes of the rotation at large distances from the strip, and what happens at the leading edge, $\vartheta = \pi$, where the velocity is still infinite? The rotational term, proportional to ϑ, extends out to infinity, and it is difficult to see why there should be such circulation far from the strip, where the flow should be undisturbed by it. Presumably the answer is that somehow the vortices thrown off by the trailing edge combine with the rotation to cancel it outside some radius which is not absurdly large but is sufficiently large so that the vortex-circulation combination has little distorting influence on the circulation close to the strip; hence it is not necessary to go into the details of the combining of circulation and vortex, and we need not be distracted from our concentration on the flow near the strip.

The neglect of the infinite velocity about the leading edge is not quite so easy to explain away. Possibly only the trailing edge manufactures vortices which leave the edge in such a manner as to cause circulation; the vortex at the leading edge may cling to the edge, forming a "dead air" space and altering the form of the lines of flow. In usual air-flow calculations, the leading edge is not sharp, but rounded, so that the velocity is not so high as that around the trailing edge and the tendency for vortex formation is presumably not as great. We shall discuss this problem further in the next section.

Green's Function for Elliptic Coordinates. The potential due to a unit line charge at (x_0,y_0), obtained from the usual complex function $-2\ln(z - z_0)$, can also be expanded in terms of the elliptic coordinate function $w = \mu + i\vartheta$, for $z = \frac{1}{2}a\cosh w$. We have

$$\begin{aligned}
-&\ln[(a^2/16)(e^w + e^{-w} - e^{w_0} - e^{-w_0})^2] \\
&= -2\ln\{a\sinh[\tfrac{1}{2}(w + w_0)]\sinh[\tfrac{1}{2}(w - w_0)]\} \\
&= -2\ln(a/4) - 2w - 2\ln(1 - e^{w_0-w}) - 2\ln(1 - e^{-w_0-w}); \quad \mu > \mu_0 \\
&= -2\ln(a/4) - 2w_0 - 2\ln(1 - e^{w-w_0}) - 2\ln(1 - e^{-w-w_0}); \quad \mu < \mu_0
\end{aligned}$$

Therefore the Green's function is

$$\begin{aligned}
G(\mu,\vartheta|\mu_0,\vartheta_0) &= -2\left[\mu + \ln\left(\frac{a}{4}\right)\right] + \sum_{n=1}^{\infty}\frac{4}{n}e^{-n\mu}[\cosh n\mu_0\cos n\vartheta\cos n\vartheta_0 \\
&\qquad + \sinh n\mu_0\sin n\vartheta\sin n\vartheta_0]; \quad \mu > \mu_0 \qquad (10.1.32) \\
&= -2\left[\mu_0 + \ln\left(\frac{a}{4}\right)\right] + \sum_{n=1}^{\infty}\frac{4}{n}e^{-n\mu_0}[\cosh n\mu\cos n\vartheta\cos n\vartheta_0 \\
&\qquad + \sinh n\mu\sin n\vartheta\sin n\vartheta_0]; \quad \mu < \mu_0
\end{aligned}$$

As with all these Green's function expansions, the series is only conditionally convergent and should not be differentiated unless the poorly convergent part can be condensed into a closed function.

For instance, the capacitance of a thin wire of radius ρ, which is inside a hollow elliptic cylinder of major axis $a \cosh \mu_1$ and minor axis $a \sinh \mu_1$, is obtained by first obtaining the potential due to a charge q per unit length at the point μ_0, ϑ_0, when the potential at $\mu = \mu_1(\mu_1 > \mu_0)$ is kept zero. The potential distribution is

$$\begin{aligned}\psi(\mu,\vartheta) = -2q\,\mathrm{Re}[\ln(z - z_0)] + 2q\left[\mu_1 + \ln\left(\frac{a}{4}\right)\right] \\ - \sum_{n=1}^{\infty} \frac{4q}{n} e^{-n\mu_1} \left[\frac{\cosh n\mu_0 \cos n\vartheta_0}{\cosh n\mu_1} \cosh n\mu \cos n\vartheta \right. \\ \left. + \frac{\sinh n\mu_0 \sin n\vartheta_0}{\sinh n\mu_1} \sinh n\mu \sin n\vartheta\right]\end{aligned} \tag{10.1.33}$$

When $(z - z_0) = \rho e^{i\phi}$, with ρ much smaller than a, then the potential at the cylindrical surface ρ = constant is, to the first order in ρ/a,

$$\begin{aligned}V \simeq 2q \ln(a/4\rho) + 2q\mu_1 \\ - 4q \sum_{n=1}^{\infty} \frac{1}{n} e^{-n\mu_1} \left[\frac{\cosh^2 n\mu_0 \cos^2 n\vartheta_0}{\cosh n\mu_1} + \frac{\sinh^2 n\mu_0 \sin^2 n\vartheta_0}{\sinh n\mu_1}\right]\end{aligned}$$

The ratio q/V is the capacitance per unit length in electrostatic units.

The Green's function may also be modified to fit the Neumann boundary condition of zero slope at the boundary $\mu = \mu_1$. For instance, the magnetic field between two wires, carrying opposing current of magnitude q, placed at the foci of the ellipses and surrounded by an elliptic cylinder of iron (permeability ϵ) with inner surface at $\mu = \mu_1$, can be computed. The potential function for opposing "sources" at the two foci is the real part of the function, which has infinities at $z = \pm \frac{1}{2}a$,

$$F(\mu,\vartheta) = -4q \ln\left[\frac{1 - e^{-w}}{1 + e^{-w}}\right] = 8q \tanh^{-1}(e^{-w})$$

which is

$$\begin{aligned}\psi_0(\mu,\vartheta) &= 8q \sum_{n=0}^{\infty} \frac{1}{2n+1} e^{-(2n+1)\mu} \cos[(2n+1)\vartheta] \\ &= 4q \tanh^{-1}[\cos\vartheta/\cosh\mu]\end{aligned} \tag{10.1.34}$$

and the "flow function" is the imaginary part, which is

$$\chi_0(\mu,\vartheta) = -8q \sum_{n=0}^{\infty} \frac{1}{2n+1} e^{-(2n+1)\mu} \sin[(2n+1)\vartheta] = 4q \tan^{-1}\left[\frac{\sin\vartheta}{\sinh\mu}\right]$$

This is, of course, discontinuous along the line $\mu = 0$, as are all flow functions representing circulation around points (for instance, the flow function around a single line is proportional to the angle ϑ, which is discontinuous at $\vartheta = 0, 2\pi$) although the gradient of χ is continuous.

As we pointed out earlier, the function χ represents the magnetic potential around the two wires, and the function ψ the lines of magnetic force. The effect of the iron surface at $\mu = \mu_1$ is to add a series of terms of the type $A \sinh(m\mu)\sin(m\vartheta)$ which *is* continuous at $\mu = 0$, so that the boundary conditions at $\mu = \mu_1$ are satisfied. In the iron ($\mu > \mu_1$) the terms are of the type $e^{-m\mu}\sin(m\vartheta)$, which converges at $\mu \to \infty$ (we do not need continuity at $\mu = 0$ here where $\mu > \mu_1$). The boundary condition is that the normal gradient of χ just inside the surface $\mu = \mu_1$ equals ϵ times the normal gradient just outside (in the iron) and that χ itself be continuous at $\mu = \mu_1$.

The resulting solution is

$$\chi = \begin{cases} -8q \displaystyle\sum_{n=0}^{\infty} \frac{e^{(2n+1)(\mu_1-\mu)}\sin(2n+1)\vartheta}{\epsilon\sinh(2n+1)\mu_1 + \cosh(2n+1)\mu_1}\frac{1}{(2n+1)}; & \mu > \mu_1 \\ 4q\tan^{-1}\left[\dfrac{\sin\vartheta}{\sinh\mu}\right] & \\ \quad + 8q \displaystyle\sum_{n=0}^{\infty} \frac{(\epsilon-1)e^{-(2n+1)\mu_1}\sinh(2n+1)\mu}{\epsilon\sinh(2n+1)\mu_1 + \cosh(2n+1)\mu_1}\frac{\sin(2n+1)\vartheta}{(2n+1)}; & \mu < \mu_1 \end{cases}$$

When the permeability ϵ is very large, the boundary condition reduces to the requirement that the surface $\mu = \mu_1$ correspond to the equipotential surface $\chi = 0$ and the lines of constant ψ (the lines of magnetic force) be normal to this surface. Then the magnetic potential inside an iron elliptic cylinder, caused by opposing currents at the foci, is given by the imaginary part of the function ($w = \mu + i\vartheta$)

$$F(w) = 8q\tanh^{-1}(e^{-w}) + 8q\sum_{n=0}^{\infty}\frac{e^{-(2n+1)\mu_1}}{(2n+1)\sinh[(2n+1)\mu_1]}\cosh[(2n+1)w]$$

The real part of F, as we have stated above, represents the lines of magnetic force. We note that the field between the two wires is nearly uniform. Calculations of this type are of use in the design of large electronuclear machines.

The series of Eq. (10.1.33) can also be used to find the distribution of temperature inside an elliptic cylinder with longitudinal distribution of sources of heat. For instance, if the source is a strip of resistive metal of width a, which is surrounded by isotropic insulating material with outer boundary an elliptic cylinder having foci at the two edges of the strip, the series of Eq. (10.1.33) may be used, with q the rate of heat production per unit area of strip and with ψ the temperature (we assume that T is held fixed at zero at the surface $\mu = \mu_1$). Since $\frac{1}{2}a|\sin\vartheta|\,d\vartheta$ is the element of width across the strip, we integrate $\frac{1}{2}\psi$ over ϑ_0 with $\mu_0 = 0$ (we use the factor $\frac{1}{2}$ because we integrate from 0 to 2π, which goes over the strip twice). For this case, therefore, the temperature distribution is

$$T(\mu) = \tfrac{1}{4}a\int_0^{2\pi}\psi|\sin\vartheta|\,d\vartheta = 2aq(\mu_1 - \mu) = 2aq\left[\mu_1 - \cosh^{-1}\left(\frac{r_1 + r_2}{a}\right)\right]$$

where the equation for the outer surface is $(r_1 + r_2) = a\cosh\mu_1$ [see Eq. (10.1.23)].

Another useful calculation is the one which obtains the magnetic field due to two wires symmetrically placed at points (μ_0,ϑ_0) and $(\mu_0,-\vartheta_0)$ with currents q in opposing directions in the presence of an iron elliptic cylinder with outer surface at $\mu = \mu_1$ $(\mu_1 < \mu_0)$. We assume that the iron has large enough permeability so that the magnetic potential can be zero everywhere over the surface μ_1.

The potential and stream functions due to a $+q$ "source" at (μ_0,ϑ_0) and a $-q$ "source" at $(\mu_0,-\vartheta_0)$ are, from Eq. (10.1.32), the real and imaginary parts of the function

$$F(w) = 2q\ln\left[\frac{e^w + e^{-w} - e^{\bar{w}_0} - e^{-\bar{w}_0}}{e^w + e^{-w} - e^{w_0} - e^{-w_0}}\right]$$
$$= 2q\ln\left[\frac{\sinh\frac{1}{2}(w + \bar{w}_0)\sinh\frac{1}{2}(w - \bar{w}_0)}{\sinh\frac{1}{2}(w + w_0)\sinh\frac{1}{2}(w - w_0)}\right]$$

which can be expanded in the series

$$F = \begin{cases} 8iq\displaystyle\sum_{n=1}^{\infty}\frac{e^{-nw}}{n}\sinh(n\mu_0)\sin(n\vartheta_0); & \mu > \mu_0 \\ -8iq\displaystyle\sum_{n=1}^{\infty}\frac{e^{-n\mu_0}}{n}\sin(n\vartheta_0)\cosh(nw) - 4iq\vartheta_0; & \mu < \mu_0 \end{cases}$$

To this we must add enough of a series in e^{-nw} to make the imaginary part of the combination zero at $\mu = \mu_1 < \mu_0$. The results, for the real part ψ which represents the magnetic lines of force and for χ, the imaginary part, which represents the magnetic potential, are

$$\psi = \mathrm{Re}[F(w)] + 8q\sum_{n=1}^{\infty}\frac{1}{n}e^{n(\mu_1-\mu_0-\mu)}\sin(n\vartheta_0)\cosh(n\mu_1)\sin(n\vartheta)$$

$$\chi = \mathrm{Im}[F(w)] + 8q\sum_{n=1}^{\infty}\frac{1}{n}e^{n(\mu_1-\mu_0-\mu)}\sin(n\vartheta_0)\cosh(n\mu_1)\cos(n\vartheta) + 4q\vartheta_0$$

or

$$\psi + i\chi = 2q \ln \left\{ \frac{\sinh[\frac{1}{2}(w - \bar{w}_0)] \sinh[\mu_1 - \frac{1}{2}(w + w_0)]}{\sinh[\frac{1}{2}(w - w_0)] \sinh[\mu_1 - \frac{1}{2}(w + w_0)]} \right\}$$
$$= 2q \ln \left\{ \frac{\cosh(\mu_1 - \mu_0) - \cosh(w + i\vartheta_0 - \mu_1)}{\cosh(\mu_1 - \mu_0) - \cosh(w - i\vartheta_0 - \mu_1)} \right\} \qquad (10.1.35)$$

When the two wires are at the ends of the minor axis of the elliptic cylinder of iron ($\mu_0 = \mu_1$, $\vartheta_0 = \frac{1}{2}\pi$), this reduces to

$$\psi = 4q \ln \left[\frac{\cosh(\mu - \mu_1) + \sin\vartheta}{\cosh(\mu - \mu_1) - \sin\vartheta} \right]; \quad \chi = 4q \tan^{-1}\left[\frac{\cos\vartheta}{\sinh(\mu - \mu_1)} \right]$$

The lines of constant ψ (magnetic lines of force) and of constant χ (magnetic potential) are shown in Fig. 10.8.

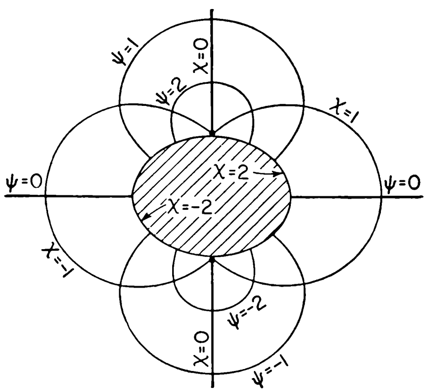

Fig. 10.8 Magnetic lines of force ψ and potential lines χ around elliptic cylinder of iron.

Potential Inside a Cylinder with Narrow Slot. Having now available the results with elliptic coordinates, we can obtain a more accurate solution of the problem discussed on page 1192 in a simpler manner than that given earlier. We start with the expression for the penetration of potential through a slit of width c in a grounded metallic plane. The potential

$$\psi = -\tfrac{1}{4}cEe^{\mu} \sin\phi$$

approaches $-\frac{1}{2}cE \sinh\mu \sin\phi = -Ey$ for μ large and positive, *i.e.*, for points above the x axis and some distance from the slit. In other words this corresponds to an applied field of intensity E, normal to the x, z plane. The potential is zero along the metallic surfaces, $\phi = 0, \pi$. For μ negative (below the x axis) the potential rapidly goes to zero, as it should.

We have thus shown that, for an applied field of intensity E normal to a grounded, plane conductor, having a slit in it of width c, the potential within the slot is

$$\psi = -\tfrac{1}{4}cE \sin\vartheta = -\tfrac{1}{4}cE\sqrt{1 - \cos^2\vartheta} = -\tfrac{1}{2}E\sqrt{(\tfrac{1}{2}c)^2 - x^2}$$

for $y = 0$, $-\frac{1}{2}c < x < \frac{1}{2}c$, the origin of coordinates being at the center of the slit.

For a grounded cylinder of radius a in an applied field of intensity E at an angle α with the x axis, the potential for no slit in the cylinder is $-E[r - (a^2/r)]\cos(\phi - \alpha)$. The normal gradient at the surface of the cylinder at $\phi = 0$ is then $-2E \cos\alpha$, and if the slit in the cylinder is between $-\frac{1}{2}\Delta < \phi < \frac{1}{2}\Delta$ (slit width $a\Delta$, centered at $r = a$, $\phi = 0$), the potential in the slit is, approximately,

$$\psi_s \simeq -\tfrac{1}{2}E\cos\alpha\sqrt{(\tfrac{1}{2}\Delta a)^2 - (a\phi)^2} = -\tfrac{1}{4}Ea\Delta\cos\alpha\sqrt{1 - (2\phi/\Delta)^2}$$

for $-\frac{1}{2}\Delta < \phi < \frac{1}{2}\Delta$, $r = a$. The approximation is better the smaller is the slit width Δa compared with the cylinder diameter $2a$.

Returning now to Eq. (10.1.22) we can substitute directly for the coefficients A_m (the B_m's are zero to this approximation). We have, using the integral representation (5.3.65) for the Bessel functions,

$$\begin{aligned}
A_m &\simeq -\left(\frac{\epsilon_m}{16\pi}\right) Ea\Delta^2\cos\alpha \int_{-1}^{1}\cos(\tfrac{1}{2}mt\Delta)\sqrt{1 - t^2}\,dt \\
&= -\tfrac{1}{32}Ea\Delta^2\cos\alpha; \qquad\qquad\qquad\qquad\qquad\qquad m = 0 \\
&= -(1/4m)Ea\Delta\cos\alpha J_1(\tfrac{1}{2}m\Delta) \xrightarrow[\Delta\to 0]{} -\tfrac{1}{16}Ea\Delta^2\cos\alpha; \quad m > 0
\end{aligned}$$

To this approximation, therefore, the potential along the axis of the slotted cylinder is $-(Ea/32)\Delta^2\cos\alpha$. When Δ is very small, the potential inside the cylinder is

$$\begin{aligned}
\psi &\simeq -\frac{Ea\Delta^2}{16}\cos\alpha\,\mathrm{Re}\left[\sum_m \left(\frac{r}{a}\right)^m e^{im\phi} - \tfrac{1}{2}\right] \\
&= -\frac{Ea\Delta^2}{32}\cos\alpha\,\mathrm{Re}\left[\frac{1 + (r/a)e^{i\phi}}{1 - (r/a)e^{i\phi}}\right] \\
&= -\tfrac{1}{32}Ea\Delta^2\cos\alpha\left[\frac{1 - (r/a)^2}{1 + (r/a)^2 - 2(r/a)\cos\phi}\right]
\end{aligned}$$

which is the surface Green's function for the Dirichlet condition that ψ be zero at $r = a$ except at $r = a$, $\phi = 0$.

Parabolic Coordinates. As was shown on pages 499 to 503, the two-dimensional coordinates for which the wave equation separates are obtained by conformal transformations giving $z = x + iy$ as a simple function of the new coordinates $z = f(w)$, $w = \xi_1 + i\xi_2$. For polar coordinates, $z = e^w$, and for elliptic coordinates, $z = \frac{1}{2}a\cosh w$. Of course the Laplace equation separates for any coordinate system, belonging to any conformal transformation corresponding to any analytic function f Nevertheless the cases for which the wave equation also is separable are useful ones, and we should explore all of them. The case we have not discussed as yet is that of the parabolic coordinates, corresponding to $z = \frac{1}{2}w^2$, $w = \lambda + i\eta$ [see Eq. (5.1.9)]:

$$\begin{aligned}
x &= \tfrac{1}{2}(\lambda^2 - \eta^2);\quad y = \lambda\eta;\quad h_\lambda = h_\eta = |w| = \sqrt{\lambda^2 + \eta^2} \\
r &= \sqrt{x^2 + y^2} = \tfrac{1}{2}(\lambda^2 + \eta^2);\quad \lambda^2 = r + x;\quad \eta^2 = r - x
\end{aligned} \tag{10.1.36}$$

The point $z = 0$ is a branch point for the transformation, and only one-half of the w plane needs to be used to cover the z plane completely. The branch cut can be any line going from $z = 0$ to $z = \infty$ and, for convenience, can be placed inside the boundary surface so that the physical requirements forbid crossing the cut. For instance, the field outside the surface $\lambda = \lambda_0$, held at constant potential, is

$$\psi = A\lambda + B = A(r + x) + B$$

In this case we need use only positive values of λ, larger than λ_0, and η must then be allowed to range from $-\infty$ to $+\infty$.

In general, the potential outside the boundary $\lambda = \lambda_0$, which is held at potential $\psi_0(\eta)$ at the point (λ_0,η), is

$$\psi(\lambda,\eta) = \frac{1}{\pi}\int_0^\infty e^{k(\lambda_0-\lambda)}\,dk\int_{-\infty}^\infty \psi_0(\beta)\cos[k(\eta - \beta)]\,d\beta;\quad \lambda > \lambda_0 \qquad (10.1.37)$$

If the boundary is the half plane given by $\lambda_0 = 0$ (the negative x axis) and the boundary condition is that the strip from $x = 0$ to $x = -a$ is at potential V whereas the rest of the negative x axis is at zero potential, this general formula reduces to

$$\psi(\lambda,\eta) = \frac{V}{\pi}\left\{\tan^{-1}\left[\frac{\sqrt{2a} + \eta}{\lambda}\right] + \tan^{-1}\left[\frac{\sqrt{2a} - \eta}{\lambda}\right]\right\} \qquad (10.1.38)$$

The corresponding function χ, giving the lines of force, is

$$\chi(\lambda,\mu) = \frac{V}{2\pi}\ln\left[\frac{\lambda^2 + (\eta + \sqrt{2a})^2}{\lambda^2 + (\eta - \sqrt{2a})^2}\right]$$

The Green's function in parabolic coordinates is obtained, as usual, by taking the real part of the function

$$F = -2\ln[\tfrac{1}{2}(w^2 - w_0^2)] = -2\ln[\tfrac{1}{2}(w - w_0)(w + w_0)] \qquad (10.1.39)$$

An interesting example of the method of images, however, can be devised by using polar coordinates, with center at the point $\lambda = \eta = 0$. Suppose that the positive x axis (the surface $\eta = 0$) is to be kept at zero potential. The solutions in polar coordinates which satisfy the boundary condition that $\psi = 0$ at $\phi = 0$ and $\phi = 2\pi$ are the set

$$r^{\frac{1}{2}n}\sin(\tfrac{1}{2}n\phi) \quad\text{and}\quad r^{-\frac{1}{2}n}\sin(\tfrac{1}{2}n\phi);\quad \text{where } n = 1, 2, 3, \ldots$$

The Green's function for this set of functions is [see Eq. (7.2.63)]

$$G = \left\{\begin{array}{ll}\displaystyle\sum \frac{4}{n}\left(\frac{r}{r_0}\right)^{\frac{1}{2}n}\sin(\tfrac{1}{2}n\phi_0)\sin(\tfrac{1}{2}n\phi); & r < r_0 \\ \displaystyle\sum \frac{4}{n}\left(\frac{r_0}{r}\right)^{\frac{1}{2}n}\sin(\tfrac{1}{2}n\phi_0)\sin(\tfrac{1}{2}n\phi); & r_0 < r\end{array}\right\} \qquad (10.1.40)$$

The half integers in the exponents and sine functions correspond to the fact that $w = \lambda + i\eta = \sqrt{2z} = \sqrt{2r}\, e^{\frac{1}{2}i\phi}$ and that we need only use one-half of the w plane to cover the z plane.

The other half of the w plane may be used to set up image sources to fit the boundary conditions at $\eta = 0$. For instance, all the z plane, outside the boundary line of the positive x axis, may be covered by letting λ range from $-\infty$ to $+\infty$ but requiring that η range only from 0 to $+\infty$. The factor $(w + w_0)$ in Eq. (10.1.39), corresponding to a "source" at $\lambda = -\lambda_0$, $\eta = -\eta_0$, should thus not be included when the boundary is along the positive x axis, and the Green's function should be the real part of the function

$$F = -2\ln[\tfrac{1}{2}(w - w_0)] = -2\ln[\tfrac{1}{2}(\lambda + i\eta)] + \sum \frac{2}{n}\left(\frac{r_0}{r}\right)^{\frac{1}{2}n} e^{\frac{1}{2}ni(\phi_0-\phi)}; \qquad r > r_0$$

$$= -2\ln[\tfrac{1}{2}(\lambda_0 + i\eta_0)] + \sum \frac{2}{n}\left(\frac{r}{r_0}\right)^{\frac{1}{2}n} e^{\frac{1}{2}ni(\phi-\phi_0)}; \qquad r < r_0$$

The image source should be in the part of the w plane which is not used to represent the z plane, in other words, in the part for η negative. As represented in Fig. 10.9, the range of ϕ from 0 to 2π (shown by solid lines) is in the "real" part of the z plane and corresponds to η positive; the range of ϕ from 0 to -2π (shown by dotted lines) corresponds to η negative and is to be used for the image sources.

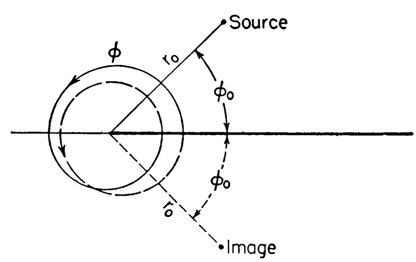

Fig. 10.9 Image of source in second Riemann surface around branch point at edge of half plane $\phi = 0$.

For the surface $\eta = 0$ to be zero potential, we put a negative image source at $\lambda = \lambda_0$, $\eta = -\eta_0$ (in the "image" region), resulting in

$$F = 2\ln\left[\frac{w - \bar{w}_0}{w - w_0}\right] = \sum_{n=1}^{\infty} \frac{4i}{n}\left(\frac{r_0}{r}\right)^{\frac{1}{2}n} \sin\left(\frac{n\phi_0}{2}\right) e^{-\frac{1}{2}ni\phi}$$

The real part of this is identical with the series given in Eq. (10.1.40). Thus, for once, we have found a "practical" use for the concept of Riemann surfaces around a branch point. Here one surface corresponds to "real space" and the other to "image space."

This completes the roster of coordinates for which the wave equation separates. Other systems, for which the Laplace equation separates, but not the wave equation, merit discussion. One such system is the *hyperbolic system*, for which $z = \sqrt{2w} = \sqrt{2\mu + 2i\kappa}$, $r^2 = 2|w|$, $h_\mu = h_\kappa = (1/r)$, which will be discussed in several problems.

Bipolar Coordinates. A more useful coordinate system, for which the wave equation does not separate (but the Laplace equation does) is the one suitable to two parallel cylinders or a cylinder (of finite radius) parallel to a plane. We can obtain it by considering the complex function from which we obtain the potential for two opposite line charges a distance $2a$ apart:

$$w = \ln[(a + z)/(a - z)] = 2 \tanh^{-1}(z/a)$$

We set $w = \xi + i\theta$ and $z = x + iy = a \tanh(w/2)$, obtaining

$$\begin{gathered} x = \frac{a \sinh \xi}{\cosh \xi + \cos \theta}; \quad y = \frac{a \sin \theta}{\cosh \xi + \cos \theta}; \quad h_\xi = h_\theta = \frac{a}{\cosh \xi + \cos \theta} \\ \xi = \tanh^{-1}\left[\frac{2ax}{a^2 + x^2 + y^2}\right]; \quad \theta = \tan^{-1}\left[\frac{2ay}{a^2 - x^2 - y^2}\right] \qquad (10.1.41) \\ r = a\sqrt{\frac{\sinh^2 \xi + \sin^2 \theta}{(\cosh \xi + \cos \theta)^2}}; \quad \phi = \tan^{-1}\left[\frac{\sin \theta}{\sinh \xi}\right] \end{gathered}$$

The coordinate θ is an angular one, going from 0 to 2π; coordinate ξ is a "radial" one, the range 0 to ∞ covering the positive half of the x, y plane, the negative range of ξ corresponding to the negative range of x. The portion of the x axis from $x = -a$ to $x = +a$ corresponds to the line $\theta = 0$, $-\infty < \xi < \infty$; the rest of the x axis corresponds to $\theta = \pi$, ($a < x < \infty$ corresponding to $\infty > \xi > 0$ and $-\infty < x < -a$ to $0 > \xi > -\infty$). The line $\xi =$ constant is a circle of radius $a \operatorname{csch} \xi$ with center at $x = a \coth \xi$, $y = 0$; the line $\theta =$ constant is a circle of radius $a \csc \theta$ with center at $x = 0$, $y = a \cot \theta$ [every θ circle goes through the points $(\pm a,0)$]. The coordinate system is shown in Fig. 10.10. We note that the parts of the circles $\theta =$ constant above the x axis are labeled with a value of θ which differs by π from the label for the part of the same circle below the x axis. The reasons for this are apparent.

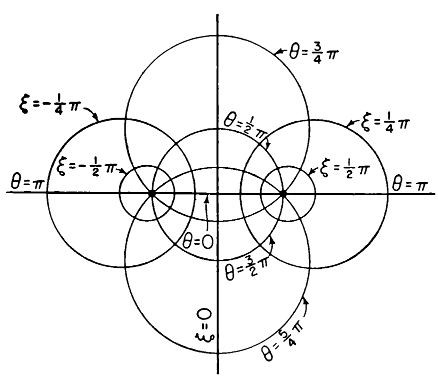

Fig. 10.10 Bipolar coordinates ξ and θ.

The potential distribution outside two cylinders can be computed in terms of these coordinates. If one cylinder corresponds to the surface ξ_0, its radius is $b = a \operatorname{csch} \xi_0$ and its axis is a distance $a \coth \xi_0$ from the y axis; if the other cylinder corresponds to the surface $-\xi_1$, its radius is $c = a \operatorname{csch} \xi_1$ and its axis is a distance $a \coth \xi_1$ on the opposite side of the y axis. Therefore if the distance between centers of the cylinders is

d, the proper interfocal distance a can be determined by solving the equation

$$d = \sqrt{b^2 + a^2} + \sqrt{c^2 + a^2} \tag{10.1.42}$$

The potential is proportional to ξ, and if the cylinder of radius $b(\xi = \xi_0)$ is at zero potential and the cylinder of radius $c(\xi = -\xi_1)$ is at potential V, the potential distribution is

$$\psi = V\frac{\xi_0 - \xi}{\xi_0 + \xi_1} = V\frac{\sinh^{-1}(a/b) - \xi}{\sinh^{-1}(a/b) + \sinh^{-1}(a/c)} \tag{10.1.43}$$

The electric intensity at the first surface is

$$E = \frac{\cosh \xi_0 + \cos \theta}{a}\,\frac{V}{\xi_0 + \xi_1}$$

and the total charge per unit length on the first cylinder is

$$q = \int_0^{2\pi} \frac{E}{4\pi}\,\frac{a\,d\theta}{\cosh \xi_0 + \cos \theta} = \frac{V/2}{\xi_0 + \xi_1} = \frac{V/2}{\sinh^{-1}(a/b) + \sinh^{-1}(a/c)} \tag{10.1.44}$$

The capacitance between the cylinders, per unit length, is q/V, as usual. If both b and c are small compared with d, then $a \simeq d/2$ to the first order in b/a and c/a, and the capacitance per unit length is approximately

$$\frac{\frac{1}{2}}{\ln(d/b) + \ln(d/c)} \quad \text{esu per unit length}$$

which checks with Eq. (10.1.11) when $d = 2y_0$ and $b = c = \rho$. Equation (10.1.44) gives the correct value of the capacitance for any values of b, c, and a which are geometrically possible.

Two Cylinders in a Uniform Field. Our program here is as before: to express solutions of the Laplace equation in other coordinate systems in terms of the elementary solutions in bipolar coordinates. Here again, as with all two-dimensional systems, we utilize the properties of analytic functions of a complex variable to aid us. The task is somewhat more arduous this time, for even the expressions for the uniform field come out to be infinite series. We have

$$z = x + iy = a\tanh(\tfrac{1}{2}w) = a\frac{1 - e^{-w}}{1 + e^{-w}} = -a\frac{1 - e^{w}}{1 + e^{w}}$$

For $\xi > 0$ we expand in powers of e^{-w}, and for $\xi < 0$ we use powers of e^w; the resulting series for z is

$$z = x + iy = \begin{cases} a + 2a\displaystyle\sum_{n=1}^{\infty} (-1)^n e^{-n\xi - in\theta}; & \xi > 0 \\ -a - 2a\displaystyle\sum_{n=1}^{\infty} (-1)^n e^{n\xi + in\theta}; & \xi < 0 \end{cases} \tag{10.1.45}$$

from which we can obtain series for a uniform field in any direction. The series converge except for $\xi = 0$.

For instance, the expression for a potential field corresponding to an intensity (or a velocity) at an angle ϕ with respect to the interpolar axis is

$$(x\cos\phi + y\sin\phi) = \begin{cases} b + 2a\sum_{n=1}^{\infty}(-1)^n e^{-n\xi}\cos(n\theta+\phi); & \xi > 0 \\ -b - 2a\sum_{n=1}^{\infty}(-1)^n e^{n\xi}\cos(n\theta-\phi); & \xi < 0 \end{cases} \tag{10.1.46}$$

where $b = a\cos\phi$.

To this we must add solutions which stay finite outside the cylinder boundaries so as to satisfy the boundary conditions at the boundaries. The solutions which do stay finite outside the cylinders ($\xi < \xi_0$ or $\xi > -\xi_1$) and which are periodic in θ are the set $\sinh(m\xi)\sin(m\theta)$, $\sinh(n\xi)\cos(n\theta)$, $\cosh(m\xi)\sin(m\theta)$, etc. If the extra terms are to go to zero at infinity (which corresponds to the point $\xi = 0$, $\theta = \pi$) we can use $\sinh(m\xi)\cos(m\theta)$ or $\cosh(m\xi)\sin(m\theta)$, but not $\cosh(m\xi)\cos(m\theta)$.

For an incompressible, nonviscous fluid, flowing past the cylinders with uniform velocity v, the boundary conditions are that $\partial\psi/\partial\xi$ be zero at the cylinder boundaries, $\xi = \xi_0$ and $\xi = -\xi_1$, as given before in connection with Eq. (10.1.42). We finally obtain

$$\begin{aligned}\psi &= v_0(x\cos\phi + y\sin\phi) \\ &\quad + 2av_0\sum_{n=1}^{\infty}(-1)^n\left\{\frac{e^{n\xi}}{e^{2n\xi_0} - e^{-2n\xi_1}}[\cos(n\theta+\phi) - e^{-2n\xi_1}\cos(n\theta-\phi)]\right. \\ &\qquad \left. - \frac{e^{-n\xi}}{e^{2n\xi_1} - e^{-2n\xi_0}}[\cos(n\theta-\phi) - e^{2n\xi_0}\cos(n\theta+\phi)]\right\}\end{aligned} \tag{10.1.47}$$

The extra series stays finite at infinity ($\xi = 0$, $\theta = \pi$). This correction series converges everywhere unless ξ_0 or ξ_1 is zero, *i.e.*, unless one or the other cylinder is infinite in radius, its surface coinciding with the $y - z$ plane. In this case the infinite plane boundary would be expected to distort the field even at infinity, and it is not surprising that the series does not converge everywhere.

When the two cylinders are the same size ($\xi_1 = \xi_0$), the potential function next to the right-hand cylinder ($\xi = \xi_0$) reduces to

$$v_0 a\cos\phi - 2v_0 a\sum_{n=1}^{\infty}(-1)^n\left[\frac{\cos\phi\cos(n\theta)}{\cosh(n\xi_0)} - \frac{\sin\phi\sin(n\theta)}{\sinh(n\xi_0)}\right]$$

and the fluid speed next to this cylinder is

$$v = \frac{\cosh \xi_0 + \cos \theta}{a} \frac{\partial \psi}{\partial \theta}$$
$$= 2v_0(\cosh \xi_0 + \cos \theta) \sum_{n=1}^{\infty} n(-1)^n \left[\frac{\cos \phi \sin(n\theta)}{\cosh(n\xi_0)} + \frac{\sin \phi \cos(n\theta)}{\sinh(n\xi_0)} \right]$$

The average speed is larger when $\theta = 0$, that is, on the side nearest the other cylinder. The pressure is therefore smaller here than on the side away from the other cylinder, and the net force tends to push (or, rather, to pull) the two cylinders together.

When the diameter of the cylinders is small compared with a, only the first term in this series need be considered. Moreover the radius of each cylinder b is approximately equal to $2ae^{-\xi_0}$, and the angle θ is approximately equal to the polar angle about the cylindrical axis at $x = a$, $y = 0$. The pressure at the point (ξ_0,θ) on the surface is, approximately,

$$p = p_0 - \tfrac{1}{2}\rho v^2 \simeq p_0 - 2\rho v_0^2(\cosh \xi_0 + \cos \theta)^2 \left[\frac{\cos \phi \sin \theta}{\cosh(\xi_0)} + \frac{\sin \phi \cos \theta}{\sinh(\xi_0)} \right]^2$$
$$\simeq -2\rho v_0^2 \left(1 + \frac{2b}{d} \cos \theta\right)^2 \sin^2(\theta + \phi) + p_0$$

where, if $b \ll a$, $\sinh \xi_0 \simeq \cosh \xi_0 = d/2b$ and d, the distance between the centers of the cylinders, is approximately equal to $2a$. The net force on the right-hand cylinder is, approximately, the integral of $[ap(\mathbf{i} \cos \theta - \mathbf{j} \sin \theta)/(\cosh \xi_0 + \cos \theta)]\, d\theta$. This net force is, then,

$$\mathbf{F} \simeq \pi\rho v_0^2(b^2/d)[-\mathbf{i}(1 + 2 \sin^2 \phi) + \mathbf{j} \sin 2\phi] \tag{10.1.48}$$

The x component is always negative, pushing the right-hand cylinder toward the other one; this mutual attraction is strongest when the asymptotic flow is at right angles to the line between the cylinders (when $\phi = \frac{1}{2}\pi$ or $\frac{3}{2}\pi$). The y component of the force on the right-hand cylinder is upward when ϕ is between 0 and $\frac{1}{2}\pi$ or between π and $\frac{3}{2}\pi$ and is zero when $\phi = 0$, $\frac{1}{2}\pi$, π, or $\frac{3}{2}\pi$. The net force on the left-hand cylinder is minus the $\mathbf{F}$ given in Eq. (10.1.48). We note that $\mathbf{F}$ decreases as the cylinders are pulled farther apart (d is increased).

Green's Function in Bipolar Coordinates. As with the other coordinate systems, we express the Green's function in terms of the elementary solutions for this system:

$$F = -2 \ln(z - z_0) = -2 \ln a - 2 \ln[\tanh(\tfrac{1}{2}w) - \tanh(\tfrac{1}{2}w_0)]$$
$$= -2 \ln 2a - 2 \ln[(e^{-w_0} - e^{-w})/(1 + e^{-w})(1 + e^{-w_0})] \tag{10.1.49}$$

This is to be expanded in different ways, depending on whether ξ or ξ_0 or $(\xi - \xi_0)$ is positive or negative. For instance,

$$F = 2\ln\left(\frac{-e^w}{2a}\right) + \sum_{n=1}^{\infty} \frac{2}{n} \{e^{n(w-w_0)} - (-1)^n[e^{-nw_0} + e^{-nw}]\}$$

for both ξ and ξ_0 positive and ξ_0 larger than ξ, but

$$F' = 2\ln(1/2a) + \sum_{n=1}^{\infty} \frac{2}{n} \{e^{n(w_0-w)} - (-1)^n[e^{nw_0} + e^{-nw}]\}$$

for ξ positive and ξ_0 negative, and so on. These series converge in their specified ranges, though they diverge at the limit of ξ or $\xi_0 \to 0$, which corresponds to the y axis and to the circle at infinity.

For instance, the magnetic lines of force and potential outside two parallel, cylindrical, conducting shells, each of radius a csch(ξ_0), with axes a distance $2a \coth \xi_0$ apart, carrying a total current I in opposite directions is given by the real and imaginary parts of the series

$$\frac{I}{2\pi}\int_0^{2\pi} \frac{aF\,d\theta_0}{\cosh\xi_0 + \cos\theta_0} - \frac{I}{2\pi}\int_0^{2\pi} \frac{aF'\,d\theta_0}{\cosh\xi_0 + \cos\theta_0}$$

where F is the series given above, with $w_0 = \xi_0 + i\theta_0$, and F' is the one given above, with $w_0 = -\xi_0 + i\theta_0$.

In order to calculate this (and other cases) we must set up the Fourier series expansion of the function $1/(\cosh \xi_0 + \cos \theta_0)$.

$$\frac{1}{\cosh\xi_0 + \cos\theta_0} = \sum_{n=0}^{\infty} A_n \cos(n\theta_0); \quad A_n = \frac{\epsilon_n}{2\pi}\int_0^{2\pi} \frac{\cos(n\theta_0)\,d\theta_0}{\cosh\xi_0 + \cos\theta_0}$$

where $\epsilon_0 = 1$, $\epsilon_n = 2$ $(n > 0)$. But to the integral for A_n we can add the integral

$$i\frac{\epsilon_n}{2\pi}\int_0^{2\pi} \frac{\sin(n\theta_0)\,d\theta_0}{\cosh\xi_0 + \cos\theta_0}$$

which is zero, due to the antisymmetry of the integrand. Consequently

$$A_n = \frac{\epsilon_n}{2\pi}\int_0^{2\pi} \frac{e^{in\theta_0}\,d\theta_0}{\cosh\xi_0 + \cos\theta_0}$$

Changing integration variables by letting $e^{i\theta_0} = z$, we change to a contour integral for z:

$$A_n = \frac{\epsilon_n}{\pi i}\oint \frac{z^n\,dz}{(z + e^{\xi_0})(z + e^{-\xi_0})}$$

where the contour is a circle of unit radius about the origin.

The integrand has one simple pole inside the contour, at $z = -e^{-\xi_0}$ (assuming ξ_0 is positive, if ξ_0 is negative the included pole is the one at $z = -e^{\xi_0}$). The residue at this pole is

$$\frac{\epsilon_n}{\pi i}\left[\frac{(-e^{-\xi_0})^n}{e^{\xi_0} - e^{-\xi_0}}\right]; \quad \xi_0 > 0 \quad \text{or} \quad \frac{\epsilon_n}{\pi i}\left[\frac{(-e^{\xi_0})^n}{e^{-\xi_0} - e^{\xi_0}}\right]; \quad \xi_0 < 0$$

Therefore the required Fourier series is

$$\frac{1}{\cosh \xi_0 + \cos \theta_0} = \frac{1}{\sinh|\xi_0|} + \sum_{m=1}^{\infty} \frac{2(-1)^m e^{-m|\xi_0|}}{\sinh|\xi_0|} \cos(m\theta_0) \qquad (10.1.50)$$

for either sign of ξ_0 (however, ξ_0 is supposed to be real).

Going back to the series for F and F' we combine the two series, making use of the orthogonality of $\cos(n\theta_0)$ and $\cos(m\theta_0)$ and $\sin(n\theta_0)$ to obtain the function

$$\begin{aligned}\psi + i\chi &= \frac{2Ia}{\sinh \xi_0}\left\{\ln(-e^w) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-n\xi_0}[e^{n(w-\xi_0)} - e^{-n(w+\xi_0)}]\right\} \\ &= \frac{2Ia}{\sinh \xi_0}\left\{\xi + i(\theta - \pi) + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-2n\xi_0} \sinh[n(\xi + i\theta)]\right\}\end{aligned} \qquad (10.1.51)$$

when ξ is positive. The real part of this function corresponds to the lines of magnetic force and the imaginary part to the magnetic potential. Because of the factors $e^{-2n\xi_0}$ the series converges quite rapidly and is, in fact, usually a small correction term to the main term $(2Ia/\sinh \xi_0) \cdot \cdot [\xi + i(\theta - \pi)]$. If the current were not uniformly distributed about the cylinder but had a density which varied with θ_0 according to the function $(\cosh \xi_0 + \cos \theta_0)$ (so that the current is somewhat less on the side away from the other cylinder than on the adjacent side), then there would be no correction series, the lines of force would be the lines of constant ξ, and the potential would be proportional to θ. As it is, the presence of the other cylinder distorts the field around the uniformly distributed current sheet so that the lines of magnetic force near the cylinder are not parallel to the cylinder surface.

10.2 *Complex Variables and the Two-dimensional Laplace Equation*

In the preceding section we have investigated the solutions of Laplace's equation in two dimensions by methods which will, in general, be useful for three dimensions and for other equations; we have found the eigenfunction solutions for various separable coordinates and have set up the series for the Green's function in these coordinates, indicating by examples how the various boundary conditions are satisfied. But during the investigation it must have become apparent that for the Laplace equation in two dimensions there is available a very powerful special technique, the use of the complex variable, which makes it possible to simplify our calculations and to obtain solutions otherwise inaccessible. In this section we shall detour from the main course of our investigations in order to bring out what this special technique can do for us in the case of the two-dimensional Laplace equation.

We start with a function $F(z) = \psi + i\chi$ of the complex variable $z = x + iy$, where x and y are the cartesian coordinates for the two-dimensional problem under study (all quantities by definition being independent of the third cartesian coordinate). The discussion in Chap. 4 has shown that both the real part ψ and the imaginary part χ are solutions of the Laplace equation in two dimensions and are related by the Cauchy-Riemann equations

$$\frac{\partial \psi}{\partial x} = \frac{\partial \chi}{\partial y}; \quad \frac{\partial \psi}{\partial y} = -\frac{\partial \chi}{\partial x} \tag{10.2.1}$$

In other words the function $F(z)$ is an analytic function of z (except for a discrete number of singularities), but a combination of $F(z)$ and $F(\bar{z})$ or $\bar{F}(z)$ is *not* an analytic function of z. However, the combinations $\frac{1}{2}(F + \bar{F})$ and $\frac{1}{2}(F - \bar{F})$ (and, indeed, any *linear* combination of F and $\bar{F}$), viewed as functions of x and y, are solutions of the Laplace equation in x and y.

It is possible to transform to a new coordinate system by means of a conformal transformation defined by the functional relationship $z = f(\zeta)$, $\zeta = \xi + i\eta$, with new coordinates ξ and η. The scale factors are equal, and the system is orthogonal, since it is a conformal transformation:

$$\begin{gathered} h_\xi = h_\eta = h = |f'| = 1/|\zeta'|; \quad f' = dz/d\zeta; \quad \zeta' = d\zeta/dz \\ f(\zeta) = x + iy; \quad \zeta(z) = \xi + i\eta \end{gathered} \tag{10.2.2}$$

Since f is a function of ζ alone, the complex conjugate $\bar{f} = x - iy$ is a function of $\bar{\zeta} = \xi - i\eta$ alone.

Since F is a function of z, it may also be written as a function of ζ, so that any linear combination of ψ and χ will also be a solution of Laplace's equation in the coordinates ξ, η and ψ and χ will be related to each other by equations analogous to Eqs. (10.2.1), with ξ, η substituted for x, y. Likewise the complex conjugate $\bar{F} = \psi - i\chi$ is a function of $\bar{\zeta}$ alone (independent of ζ) *if* F is a function of z and $z = f$ is a function of ζ.

We remember, of course, that complex numbers have direction as well as magnitude. In terms of the unit vectors **i**, **j**, **k**, the function $\Phi = R + iJ$ may be written as $R\mathbf{i} + J\mathbf{j}$. As shown in Sec. 4.1, the differential operators grad, div, curl in two dimensions may be written in complex notation. For instance, for a scalar function R, the vector, grad R, may be written as the complex number

$$\nabla R = \frac{\partial R}{\partial x} + i\frac{\partial R}{\partial y}; \quad \text{with } \bar{\nabla} R = \frac{\partial R}{\partial x} - i\frac{\partial R}{\partial y} \tag{10.2.3}$$

as the complex conjugate. In general, if R is any arbitrary real function of the real variables x and y, the complex numbers ∇R and $\bar{\nabla} R$ are *not* analytic functions of z.

If the scalar function R of x and y is the magnitude of a vector $R\mathbf{k}$, normal to the x, y plane, then the curl

$$\operatorname{curl}(R\mathbf{k}) = \mathbf{i}\frac{\partial R}{\partial y} - \mathbf{j}\frac{\partial R}{\partial x} \to \frac{\partial R}{\partial y} - i\frac{\partial R}{\partial x} = -i\nabla R \tag{10.2.4}$$

On the other hand, if $\Phi = R + iJ$ is a complex function of x and y (not necessarily an analytic function of $z = x + iy$) which represents the vector $\boldsymbol{\Phi}$ in the x, y plane, we obtain [see Eq. (4.1.6)]

$$\bar{\nabla}\Phi = \frac{\partial R}{\partial x} + \frac{\partial J}{\partial y} + i\left(\frac{\partial J}{\partial x} - \frac{\partial R}{\partial y}\right) = \operatorname{div} \boldsymbol{\Phi} + i \operatorname{curl}_z \boldsymbol{\Phi} \tag{10.2.5}$$

the real part of $\bar{\nabla}\Phi$ being the divergence of vector $\boldsymbol{\Phi}$ and the imaginary part being the z component of its curl. If Φ is an analytic function $F(z)$, Eq. (10.2.1) ensures that $\bar{\nabla} F = 0$ and therefore that div $\boldsymbol{\Phi}$ and curl $\boldsymbol{\Phi}$ are zero.

This last statement may be much more easily understood if we express these quantities, not in terms of real and imaginary parts, R, J, x, and y, but (if we can) in terms of the functions Φ, z, and their conjugates $\bar{\Phi}$, $\bar{z}$. Since $x = \frac{1}{2}(\bar{z} + z)$ and $y = \frac{1}{2}i(\bar{z} - z)$, a complex function $\Phi(z,\bar{z})$ of x and y may be expressed as a function of z and $\bar{z}$ or of ζ and $\bar{\zeta}$ [see Eq. (10.2.2)]:

$$\nabla\Phi = 2\frac{\partial\Phi}{\partial\bar{z}} = \frac{2}{\bar{f}'}\frac{\partial\Phi}{\partial\bar{\zeta}} \quad \text{and} \quad \bar{\nabla}\Phi = 2\frac{\partial\Phi}{\partial z} = \frac{2}{f'}\frac{\partial\Phi}{\partial\zeta} \tag{10.2.6}$$

and the Laplacian is

$$\nabla^2\Phi = \left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}\right) = \nabla\bar{\nabla}\Phi = 4\frac{\partial^2\Phi}{\partial z\,\partial\bar{z}} = \frac{4}{|f'|^2}\frac{\partial^2\Phi}{\partial\zeta\,\partial\bar{\zeta}}$$

In these equations Φ may be everywhere real ($\Phi = R$) or everywhere imaginary ($\Phi = iJ$) as special cases.

If, however, Φ *is a function F of z alone* (*i.e.*, if F is an analytic function of z except for a discrete number of singularities), then $\bar{F} = \psi - i\chi$ is a function of $\bar{z}$ alone, and we have

$$\nabla^2 F = \nabla^2\bar{F} = \nabla^2\psi = \nabla^2\chi = 0; \quad F \text{ analytic} \tag{10.2.7}$$

except at the singularities of F. The complex numbers representing the vectors obtained by taking the gradients of the real part or the imaginary part of $F(z)$ are then [from Eqs. (10.2.6)]

$$\begin{aligned} \text{grad}\,\psi &= \tfrac{1}{2}\nabla(\bar{F}+F) = \overline{\left(\frac{dF}{dz}\right)} = \overline{\left(\frac{1}{f'}\frac{dF}{d\zeta}\right)};\quad F(z) = \psi + i\chi \\ \text{grad}\,\chi &= \tfrac{1}{2}i\nabla(\bar{F}-F) = i\,\text{grad}\,\psi;\quad z = x + iy = f(\zeta) \end{aligned} \tag{10.2.8}$$

where we can take the derivative with respect to z or ζ and then take the complex conjugate or else take the derivative of $\bar{F}$ with respect to $\bar{z}$ or $\bar{\zeta}$. Correspondingly, we have

$$\text{curl}(\mathbf{k}\psi) = -i\nabla\psi = -i\overline{\left(\frac{dF}{dz}\right)} = -i\,\text{curl}(\mathbf{k}\chi)$$

when F is a function of z, not $\bar{z}$.

Fields, Boundary Conditions, and Analytic Functions. With these conventions in mind, we can quickly recapitulate the connection between various suitable analytic functions $F(z)$, their real and imaginary parts ψ and χ, their derivatives with respect to z (or another variable $\zeta = \xi + i\eta$, representing another coordinate system), and the physical quantities related to various problems involving solutions of the Laplace or Poisson equation.

For instance, for two-dimensional *electrostatic* (or gravitational) *problems* we choose a suitable analytic function $F(z)$ such that its real part $\psi(x,y)$ is the electrostatic (or gravitational) potential. Then its imaginary part $\chi(x,y)$ represents the lines of force. The complex number $-\text{grad}\,\psi = -\overline{(dF/dz)}$ represents the intensity vector, and the value of $(1/4\pi)|dF/dz|$ at a point on the surface of a conductor is equal to the charge density at that point. The problem usually is to find a function $F(z)$ such that the boundaries (at constant potential) coincide with one or more of the family of curves $\psi =$ constant.

An extremely powerful technique can be evolved for going from one potential problem involving Dirichlet boundary conditions along a given boundary line to one involving similar conditions along a boundary line of different shape. We determine a conformal transform $\zeta = \zeta(z)$ [or $z = f(\zeta)$] such that the old boundary changes into the new one; then the new potential is the real part of $F(\zeta)$, the imaginary part corresponds to the new lines of force, the intensity is represented by the complex function $-(1/\bar{f}')\overline{(dF/d\zeta)}$, and the charge density at a given point on a conducting surface in the new coordinates is equal to $(1/4\pi|f'|)|dF/d\zeta|$ at that point. Several examples of this technique will be discussed later in this section.

For the *magnetostatic case* we usually choose χ to be the magnetic potential and ψ to represent the magnetic lines of force. Then the magnetic intensity is $\text{grad}\,\chi = i\overline{(dF/dz)} = (i/\bar{f}')\overline{(dF/d\zeta)}$. For *steady-state diffusion* ψ is usually the density of the diffusing substance (or the temperature) and the flow density is $-a^2\overline{(dF/dz)}$ where a is the diffusion constant [see (Eq. 2.4.3)]. The total flow of material between the two flow lines χ_0 and χ_1 and between x, y planes a unit distance apart can be obtained by integrating $-a^2$ grad $\psi = -a^2\overline{(dF/dz)}$ along a line of constant ψ (which is everywhere perpendicular to the flow) from χ_0 to χ_1. The integration is along a path for which ψ does not change, so that the integral of $-\overline{(dF/dz)}$ is just the change in χ, and the total flow is $a^2(\chi_1 - \chi_0)$. Similarly the total magnetic flux between the lines of force ψ_0 and ψ_1 and within unit distance along the cylindrical axis is just $(\psi_1 - \psi_0)$.

For the *steady flow of an incompressible fluid* Eq. (2.3.14) reduces to

$$\operatorname{grad}(p + V + \tfrac{1}{2}\rho v^2) + 2\eta \operatorname{curl} \mathbf{w} - 2\rho \mathbf{v} \times \mathbf{w} = 0 \qquad (10.2.9)$$

where p is the pressure, V the gravitational potential (or other body potential), ρ the fluid density, η its coefficient of viscosity, $\mathbf{v}$ its velocity, and $\mathbf{w} = \frac{1}{2}$ curl $\mathbf{v}$ its vorticity. Since the fluid is incompressible, we also have div $\mathbf{v} = 0$, which means that $\mathbf{v}$ is the sum of the gradient of a scalar potential and the curl of a vector potential.

For *irrotational flow* the vector potential is zero and $\mathbf{v}$ is obtainable from a scalar velocity potential, which is a solution of Laplace's equation. For two-dimensional flow we find this potential is equal to the real part of some analytic function of z, $F(z) = \psi + i\chi$, where χ is the corresponding flow function. The fluid velocity is given by the complex number $\nabla\psi = -\overline{(dF/dz)}$, and the total flux of fluid between the flow lines χ_0 and χ_1, per unit thickness perpendicular to the x, y plane, is $(\chi_1 - \chi_0)$. In this case $(p + V + \frac{1}{2}\rho v^2)$ is a constant, so that the pressure at any point is

$$p = \text{constant} - V - \tfrac{1}{2}\rho|dF/dz|^2 \qquad (10.2.10)$$

which is the two-dimensional form of Bernouilli's equation. The problem usually is to find an analytic function $F(z)$ having a flow function χ such that the boundary surface coincides with one or more of the lines χ = constant. When this is done, the net force on the boundary can be computed by means of Eq. (10.2.10).

If the *flow is not irrotational but is small*, so that the terms $2\rho\, \mathbf{v} \times \mathbf{w}$ and $\frac{1}{2}\rho v^2$ may be neglected compared with the other terms in Eq. (10.2.9), then, as we have shown on page 1186, we find an analytic function $W(z)$, with real part $(p + V)/2\eta$ determining the pressure and imaginary part equal to minus the magnitude $w = \frac{1}{2}\operatorname{curl}_z \mathbf{v}$ of the vorticity, which in the two-dimensional case is normal to the x, y plane. Here the velocity cannot be the gradient of a scalar potential; one must also use a vector potential, normal to the x, y plane. Suppose that the scalar potential for the velocity is ψ and the vector potential is $\mathbf{A} = A\mathbf{k}$; then the velocity is

$$\mathbf{v} = \operatorname{curl} \mathbf{A} - \operatorname{grad} \psi = \mathbf{i}\left(\frac{\partial A}{\partial y} - \frac{\partial \psi}{\partial x}\right) - \mathbf{j}\left(\frac{\partial A}{\partial x} + \frac{\partial \psi}{\partial y}\right)$$

To translate this into the language of complex variables we note that the scalar potential may be the real part of an analytic function $F(z) = \psi + i\chi$ (since $\nabla^2\psi$ must be zero) but that A is, in general, a function of both z and $\bar{z}$ (*i.e.*, an arbitrary function of x and y). We translate $\psi(x,y)$ into the function $\frac{1}{2}[F(z) + \overline{F(z)}]$ of z and $\bar{z}$ and then use Eqs. (10.2.4) and (10.2.6) to obtain the complex notation for the vector $\mathbf{v}$:

$$v = v_x + iv_y = -(\partial/\partial\bar{z})[\overline{F(z)} + 2iA(z,\bar{z})] \qquad (10.2.11)$$

where $F(z)$ is a complex function of $z = x + iy$, $\bar{F}$ is a function of z, and A is a real function of z and $\bar{z} = x - iy$.

The connection between the functions F and A and the pressure-vorticity function $W(z) = [(p + V)/2\eta] - iw$ is obtained from the definition of the vorticity, $\mathbf{w} = \frac{1}{2}$ curl $\mathbf{v}$. Since the divergence of $\mathbf{v}$ is zero, we can use Eqs. (10.2.5) and (10.2.6) to obtain

$$-i\frac{\partial v}{\partial z} = -2\frac{\partial^2 A}{\partial z\,\partial\bar{z}} = \tfrac{1}{2}\,\mathrm{curl}_z\,\mathbf{v} = w = \tfrac{1}{2}i[W(z) - \overline{W(z)}]$$

Therefore A and W can be related as follows: We find the analytic function of z, $U(z)$, such that $dU/dz = W(z)$, and then set

$$A = \tfrac{1}{2}\,\mathrm{Im}[\bar{z}U(z)] = -\tfrac{1}{4}i[\bar{z}U(z) - z\overline{U(z)}] \qquad (10.2.12)$$

which results in

$$4i\frac{\partial^2 A}{\partial z\,\partial\bar{z}} = U'(z) - \overline{U'(z)} = -2iw$$

if $U'(z) = W(z)$. In terms of the two basic analytic functions U and F, the expressions for the velocity, the vorticity, and the pressure for slow viscous flow are therefore

$$\begin{aligned} v &= -\overline{F'(z)} - \tfrac{1}{2}U(z) + \tfrac{1}{2}z\overline{U'(z)} = v_x + iv_y \\ w &= -\mathrm{Im}[U'(z)];\quad p = -V + 2\eta\,\mathrm{Re}[U'(z)] \end{aligned} \qquad (10.2.13)$$

where V is the gravitational potential for the fluid.

The boundary conditions can be satisfied by adjusting U and F so that the flow function

$$\Omega = \mathrm{Im}[F(z) - \tfrac{1}{2}\bar{z}U(z)]$$

gives lines of flow Ω = constant which coincide with the boundaries. It is not difficult to show that grad $\Omega = -iv$, so that the velocity is everywhere parallel to the lines Ω = constant and so that the difference $\Omega_1 - \Omega_0$ equals the total flux of liquid between the flow lines Ω_0 and Ω_1, per unit thickness perpendicular to the x, y plane (which is the definition of a flow function). What is needed, in each coordinate system given by $z = f(\zeta)$, is to find a function $U(z)$ such that the imaginary part of $\overline{f(\zeta)}U[f(\zeta)]$ is zero or a constant at the boundary surface $\xi = \xi_0$ or $\eta = \eta_0$ (see page 1186).

Finally, one wishes to find expressions for the force of the fluid on the boundary surface. This should also be given in complex form; it may be best expressed in terms of the force on an element of area. Suppose that this element is of unit length normal to the x, y plane and is of width $dz = dx + i\,dy$, the quantity dz giving not only the magnitude of the width but its direction as well. The vector "representing" the area element is perpendicular to dz but equal to dz in magnitude. If we assume that, as we face the direction of increasing z, the "outside" is to the right, then the vector area element is $-i\,dz = dy - i\,dx$, a vector normal to dz, equal to it in magnitude.

The force on this area element due to the fluid pressure is $ip\,dz$, where p is a scalar, given by either Eq. (10.2.10) or (10.2.13). The force on it due to fluid viscosity can be obtained by using Eq. (2.3.10). Translating into complex notation for two dimensions, the additional force, due to viscosity, is

$$dP = \eta\left[2\frac{\partial v_x}{\partial x}\,dy - \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)dx\right] + i\eta\left[\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)dy - 2\frac{\partial v_y}{\partial y}\,dx\right]$$

using the fact that $\partial v_x/\partial x = -(\partial v_y/\partial y)$, since $\operatorname{div}\mathbf{v} = 0$. We can reduce this to

$$\begin{aligned} dP &= \eta(dx - i\,dy)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(iv_x - v_y) = 2i\eta\,d\bar{z}\,\frac{\partial v}{\partial \bar{z}} \\ &= -i\eta[2\overline{F''(z)} - z\overline{U''(z)}]\,d\bar{z} \end{aligned} \tag{10.2.14}$$

when the vector v is given by Eq. (10.2.13).

This expression for slow viscous flow may be used whether the flow is irrotational ($U = 0$) or has vorticity ($U \neq 0$). However, if the flow is pure potential flow, the total force on the boundary, from point 0 to point 1, is

$$-2i\eta\int_0^1 \overline{\frac{dF'(z)}{dz}}\,d\bar{z} = 2i\eta[\overline{F'(z_0)} - \overline{F'(z_1)}]$$

and if the boundary is a closed cylinder or prism, $z_1 = z_0$, so that the net force due to viscosity is zero. This simply means that, since potential flow assumes negligible viscosity, we should not be surprised if the viscous forces cancel out. If U is not zero, then v is not a function of just $\bar{z}$ and the integration around a closed boundary is not necessarily zero.

Some Elementary Solutions. In accordance with the program discussed on page 1216, we should set up a few extremely simple solutions for different physical conditions and solve them in detail. Other solutions may then be manufactured by the simple(?) process of devising conformal transformations to carry the simple boundaries into the new boundaries.

The simplest case is, of course, for parallel boundaries, one at $y = y_0$ and the other at $y = y_1$. Here the potential is usually proportional to

z itself. For instance, for an electrostatic problem, if the boundary at $y = y_0$ is at zero potential and the upper boundary is at potential V, then the potential between the plates is given by the real part of the complex function

$$F(z) = \psi + i\chi = \frac{V}{y_1 - y_0}(-iz - y_0) = V\frac{y - y_0 - ix}{y_1 - y_0} \quad (10.2.15)$$

The electric intensity at the point z between the plates is $-\overline{F'(z)} = iV/(y_1 - y_0)$, pointed in the y direction and (in this simple case) independent of z. The charge density on either of the boundary conductors is $V/4\pi(y_1 - y_0)$, so that the capacitance per unit area of plate is $1/4\pi(y_1 - y_0)$ esu.

If the potential is to represent irrotational flow between the plates at velocity v_0, the analytic function is

$$F(z) = -v_0 z; \quad \psi = -v_0 x; \quad \chi = -v_0 y$$

The velocity $-F'(\bar{z}) = v_0$ is everywhere constant, and the total flow between the boundaries, per unit depth, is the difference between values of χ at the boundaries, $v_0(y_1 - y_0)$. If the pressure at the top plate is zero, that at the bottom plate is $\rho g(y_1 - y_0)$, where g is the acceleration of gravity.

For viscous flow between the parallel plates the function $U(z)$ discussed on page 1186 is proportional to z or z^2, depending on the boundary conditions. If the upper plate is moving with velocity v_0 in the x direction and the bottom one is stationary, we set

$$F(z) = \frac{v_0}{y_1 - y_0}(\tfrac{1}{4}iz^2 + y_0 z); \quad U(z) = \frac{(iv_0 z/2)}{y_1 - y_0} = \tfrac{1}{2}v_0\frac{ix - y}{y_1 - y_0} \quad (10.2.16)$$

Since $\overline{U'(z)} = -\frac{1}{2}i[v_0/(y_1 - y_0)]$, we have that the velocity at the point z is

$$v = \frac{v_0}{y_1 - y_0}[\tfrac{1}{2}(ix + y - 2y_0) + \tfrac{1}{4}(y - ix) + \tfrac{1}{4}(y - ix)] = v_0\frac{y - y_0}{y_1 - y_0}$$

Similarly the vorticity and pressure at point z are

$$w = -\frac{(v_0/2)}{y_1 - y_0}; \quad p = \rho g(y_1 - y)$$

the vorticity being negative (clockwise) and independent of position and the pressure depending only on the gravitational force. Using Eq. (10.2.14), we find that the force on a unit area of the bottom (stationary) plate is

$$-2i\eta\overline{F''} = \eta v_0/(y_1 - y_0)$$

which is in the positive x direction.

If, on the other hand, both plates are stationary and the fluid is being forced through between the plates, we find that setting

$$F(z) = \frac{(Q/2)}{(y_1 - y_0)^3}[-(z - iy_0)^3 + 3i(y_1 - y_0)(z - iy_0)^2]$$
$$U(z) = \frac{3Q}{(y_1 - y_0)^3}[-(z - iy_0)^2 + i(y_1 - y_0)(z - iy_0)]$$

gives us

$$v = \frac{6Q}{(y_1 - y_0)^3}[(y_1 - y_0)(y - y_0) - (y - y_0)^2] = \frac{6Q}{(y_1 - y_0)^3}(y - y_0)(y_1 - y)$$
$$w = \frac{6Q}{(y_1 - y_0)^3}(2y - y_0 - y_1); \quad p = \rho g(y_1 - y) + \frac{12\eta Q}{(y_1 - y_0)^3}(x_0 - x) \tag{10.2.17}$$

where Q is the total flux of fluid between the plates, per unit thickness normal to the x, y plane. The vorticity is zero at the midline $y = \frac{1}{2}(y_0 + y_1)$, being positive above the line (counterclockwise) and negative (clockwise) below. The pressure is assumed zero at the point $z = x_0 + iy_1$. It increases with decrease of y because of the force of gravity, and it increases with decrease of x because it requires force to push the viscous fluid between the plates.

Having obtained elementary solutions of physical problems for rectangular coordinates, we should now show how to carry these over to other coordinates. One of the simple coordinate changes, which are of use in many problems, is the one (shown in Fig. 10.11)

$$w = \xi + i\phi = \ln(z + a); \quad z = f(w) = e^w - a \tag{10.2.18}$$

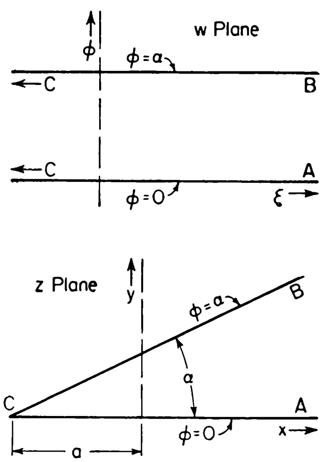

Fig. 10.11 Conformal transformation from rectangular to polar coordinates.

which changes a pair of parallel plates CA and CB, in the w plane, into a pair of plates at an angle α, meeting at the point $z = -a$. The transformation has been chosen so that the origin ($w = 0$) on the w plane is also the origin ($z = 0$) on the z plane.

If plate CA ($\phi = 0$) is at zero potential and plate CB ($\phi = \alpha$) is at potential V_0, then the potential is obtained from Eq. (10.2.15) by changing z into w and letting $y_0 = 0$, $y_1 = \alpha$:

$$F(z) = F(w) = \psi + i\chi = -iV_0 w/\alpha = (-iV_0/\alpha)\ln(z + a)$$
$$= \frac{V_0}{\alpha}\tan^{-1}\left(\frac{y}{x + a}\right) - \frac{iV_0}{2\alpha}\ln[(x + a)^2 + y^2]$$

The equipotential lines (ϕ = constant) are straight lines through the center at $x = -a$, at various angles ϕ to the x axis; the lines of force (ξ = constant) are arcs of circles of various radii $r = e\xi$, with center at

$x = -a$. The intensity at the point z (or w, depending on the coordinate system used) is

$$-\overline{F'(z)} = \left[\overline{\frac{(iV_0/\alpha)}{z+a}}\right] = \frac{V_0}{\alpha}\left[\frac{y - i(x+a)}{y^2 + (x+a)^2}\right] = -\left[\overline{\frac{1}{f'}\frac{dF}{dw}}\right] = -\frac{iV_0}{\alpha}e^{-\xi+i\theta}$$

The last form shows most clearly that the intensity is inversely proportional to $r(= e^\xi)$, the distance from $z = -a$, and is at right angles to the radius vector pointed toward the real axis (the direction factor is $-ie^{i\phi}$, rotated $-90°$, $-i$, from the direction $e^{i\phi}$, along the radius). The charge density at the point x on the lower plate is $(1/4\pi)|F'(z)| = V_0/4\pi\alpha r$, falling off as one goes away from the branch point $z = -a$.

If F is to represent fluid flowing out between the plates, it has the form $-v_0 w = -v_0 \ln[(z+a)/a]$. Here the lines of flow radiate out from the center $z = -a$, and the velocity potential lines are parts of concentric circles. The velocity is

$$-\overline{F'(z)} = [\overline{v_0 e^{-w}}] = v_0 e^{-\xi+i\theta}$$

having a magnitude v_0/r and pointed in the radial direction.

For viscous flow between the slanted plates we need to choose complex functions $F(z)$ and $U(z)$ so that the flow function Ω, given on page 1220, is a function of ϕ alone. This turns out to be easier than for rectangular coordinates; it was done on page 1187. Since $\bar{z} = e^{\xi - i\phi} - a$, we can choose

$$U = Ce^{-w};\quad F = -\tfrac{1}{2}Ce^{-w} + Bw$$

The function A defined in Eq. (10.2.12) is

$$A = -\tfrac{1}{4}i[Ce^{\bar{w}-w} - Ce^{-w} - \bar{C}e^{w-\bar{w}} + \bar{C}e^{-\bar{w}}]$$

and the velocity defined in Eq. (10.2.11) is

$$\begin{aligned} v &= -\frac{1}{2\bar{f}'}\frac{d}{d\bar{w}}[-\bar{C}e^{-\bar{w}} + 2B\bar{w} + Ce^{\bar{w}-w} - Ce^{-w} - \bar{C}e^{w-\bar{w}} + \bar{C}e^{-\bar{w}}] \\ &= -e^{-\bar{w}}[B + |C|\cos(2\phi - \beta)];\quad \text{where } C = |C|e^{i\beta} \end{aligned}$$

To have $v = 0$ at $\phi = 0$ and $\phi = \alpha$, we set $\beta = \alpha$ and $B = -|C|\cos\alpha$. The flow function is then

$$\Omega = \tfrac{1}{2}|C|\sin(2\phi - \alpha) - |C|\phi\cos\alpha$$

The rest of the solution is given in Eqs. (10.1.17) *et seq.*

A transformation suitable for cylindrical surfaces may be obtained by setting $a = 0$ in Eq. (10.2.18). The electrostatic case is obtained by setting $F = [V_0/(\xi_1 - \xi_0)](w - \xi_0)$, the real part being a potential going to zero at $r = e^{\xi_0}$ and to V_0 at $r = e^{\xi_1}$. The charge density on the inner cylinder is

$$\frac{1}{4\pi}\left|\frac{dF}{dz}\right| = \frac{1}{4\pi|f'|}\left|\frac{dF}{dw}\right| = \frac{V_0 e^{-\xi_0}}{4\pi(\xi_1 - \xi_0)}$$

and the total area of a unit length of the cylinder is $2\pi e^{\xi_0}$, so that the charge per unit length is $V_0/2(\xi_1 - \xi_0)$, from which the capacitance may be computed.

For a viscous liquid between an inner cylinder of radius $r_0 = e^{\xi_0}$ at rest and an outer cylinder of radius $r_1 = e^{\xi_1}$ rotating with angular velocity ω, we can set the two functions F and U such that the flow function

$$\Omega = \mathrm{Im}[F(w) - \tfrac{1}{2}e^{\bar{w}}U(w)]$$

is a function of ξ alone (not ϕ). This means that U can be $iCz = iCe^w$ and F can perhaps be iBw, where C and B are real.

The amplitude of the vector potential is then $A = -\frac{1}{2}Ce^{w+\bar{w}}$, and the velocity is

$$v = -(1/\bar{f}')(d/d\bar{w})[-iB\bar{w} - iCe^{w+\bar{w}}] = ie^{-\phi}[Be^{-\xi} + Ce^{\xi}]$$

which is pointed in the positive ϕ direction. B and C must be adjusted so that $|v| = 0$ at $\xi = \xi_0 = \ln r_0$ and is $v_0 = \omega r_1$ at $\xi = \xi_1 = \ln r_1$; the result is that $B = -Ce^{2\xi_0}$ and $C = [v_0e^{-\xi_0}/2\sinh(\xi_1 - \xi_0)]$, so that, using Eqs. (10.2.13), we have

$$v = iv_0e^{i\phi}\left[\frac{\sinh(\xi - \xi_0)}{\sinh(\xi_1 - \xi_0)}\right]; \quad w = -C = \left[\frac{v_0e^{-\xi_0}}{2\sinh(\xi_1 - \xi_0)}\right]; \quad p = -\rho gy$$

The net force on the inner cylinder may be computed by using Eq. (10.2.14). Since $U = iCz$, $U'' = 0$; since $F = -iCe^{2\xi_0}\ln z$, $F'' = iCe^{2\xi_0}/z^2 = iCe^{-2i\phi}$ at $\xi = \xi_0$; and since $dz = ie^{\xi_0+i\phi}\,d\phi$ at $\xi = \xi_0$, therefore

$$dP = 2i\eta Ce^{\xi_0+i\phi}\,d\phi = ie^{i\phi}\left[\frac{v_0\eta}{\sinh(\xi_1 - \xi_0)}\right]d\phi$$

which points in the positive ϕ direction, tangential to the cylinder surface. The total force on the inner cylinder is just the buoyancy of the displaced fluid, since the integral of dP is zero. However, the torque on the inner cylinder due to the force dP is just $r_0|dP| = e^{\xi_0}|dP|$, so that the total torque due to viscosity is

$$\frac{2\pi\eta v_0e^{\xi_0}}{\sinh(\xi_1 - \xi_0)} = \frac{4\pi\eta v_0r_0^2r_1}{r_1^2 - r_0^2}$$

Transformation of Solutions. Many times the application of successive transformations produces the boundary surface we desire. For instance, the transformation

$$w = u + iv = \cosh(\pi z/a); \quad z = x + iy \qquad (10.2.19)$$

shown in Fig. 10.12, changes the real axis $v = 0$ into the broken line: positive x axis, y axis from 0 to ia, positive part of the line $y = a$ (later in this section we shall show how to obtain such transformations more or less at will). If now we have a solution corresponding to boundary conditions along the real axis, we can, by this transformation, change to solutions along the boundary $ABCDE$ as shown in Fig. 10.12.

For instance, the potential arising when the plane $v = 0$ from $u = 0$ to $+\infty$ is at zero potential and the rest, from $u = 0$ to $-\infty$, is at potential V_0 is the real part of

$$F = \psi + i\chi = -i\frac{V_0}{\pi}\ln w = \frac{V_0}{\pi}\tan^{-1}\left(\frac{v}{u}\right) - i\frac{V_0}{2\pi}\ln(u^2 + v^2)$$

Substituting the transformation of Eq. (10.2.19), we obtain

$$F = -i\frac{V_0}{\pi}\ln\left[\cosh\left(\frac{\pi z}{a}\right)\right]$$
$$= \frac{V_0}{\pi}\tan^{-1}\left[\tanh\left(\frac{\pi x}{a}\right)\tan\left(\frac{\pi y}{a}\right)\right] - i\frac{V_0}{2\pi}\ln\left[\cosh^2\left(\frac{\pi x}{a}\right) - \sin^2\left(\frac{\pi y}{a}\right)\right]$$

the real part of which is the potential arising when the part of the z plane marked ABC in Fig. 10.12 is at zero potential and the part marked CDE is at potential V_0.

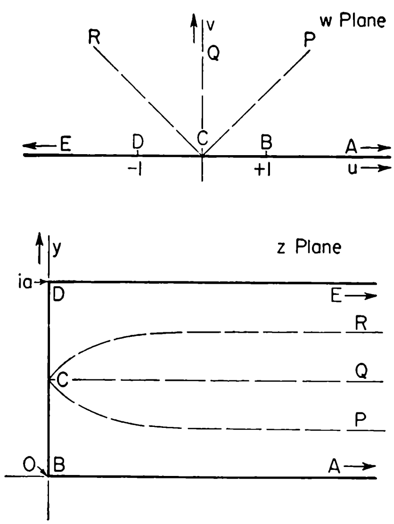

Fig. 10.12 Conformal transform of boundary ACE from w to z plane. Elliptic coordinates in w plane go to rectangular coordinates in z plane.

On the other hand, the magnetic potential about a wire at $w = ib$, carrying current q into the paper, with the $v = 0$ plane the upper boundary of a high-permeability iron plate, is the imaginary part of the function [see Eq. (10.2.24)]

$$F = -2q\ln(w^2 + b^2)$$

Therefore the magnetic potential due to a wire carrying current q into the paper, at the point $x = (a/\pi)\sinh^{-1} b$ in a slot in a mass of iron of the sort shown in Fig. 10.12, is the imaginary part of the function

$$F = -2q\ln[\cosh^2(\pi z/a) + b^2]$$

The magnetic field strength along the boundary surface at the positive x axis is

$$\left|\frac{dF}{dz}\right|_{y=0} = \left|\frac{dF}{dw}\right|\frac{1}{|dz/dw|} = \frac{4\pi q}{a}u\frac{\sqrt{u^2-1}}{u^2+b^2};\quad u = \cosh\left(\frac{\pi x}{a}\right)$$

for example.

For a third example, we can show that the steady-state temperature in the positive half of the w plane, when the portion $-1 < w < +1$ of the u axis is kept at temperature T_0 and the rest of the u axis is at zero temperature, turns out to be the real part of

$$F = -i\left(\frac{T_0}{\pi}\right)\ln\left[\frac{w-1}{w+1}\right] = \left(\frac{2iT_0}{\pi}\right)\coth^{-1} w$$

Consequently the temperature distribution in a semi-infinite slab of material of thickness a, with its flat end kept at temperature T_0 and its sides at zero temperature, is the real part of

$$F = (2iT_0/\pi)\coth^{-1}[\cosh(\pi z/a)]$$

a result of which we have already obtained by other methods [see Eq. (10.1.8)].

We can put transformation in series with transformation to obtain other useful cases. For instance, if transformation (10.2.19) is to the $\zeta = \xi + i\eta$ plane instead of the z plane, where $w = \cosh\zeta$, we can go on to the transformation $\zeta = \ln(z/a)$ or $z = ae^\zeta$, where the point B on the ζ plane ($\zeta = 0$) corresponds to $z = a$, the line BCD on the ζ plane ($0 \le \eta \le \pi$) corresponds to the semicircle about $z = 0$ of radius a, and the line DE corresponds to the part of the x axis to the left of $x = -a$. Consequently the transformation

$$w = 2a\cosh[\ln(z/a)] = z + (a^2/z)$$

changes the real w axis into the line plus semicircle shown in Fig. 10.13.

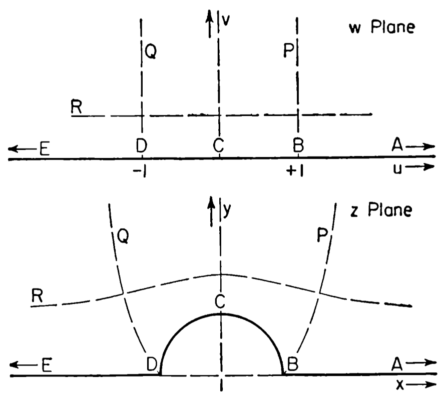

Fig. 10.13 Conformal transform from elliptic coordinates in w plane to polar coordinates in z plane.

We have inserted the factor $2a$, so that $dw/dz \to 1$ for large values of z (that is, so that the w and z plane coincide for z large).

If the problem is to depict the irrotational flow of a nonviscous fluid past a cylinder, we can start with the simple potential $F = -v_0 w$ corresponding to uniform flow to the right with velocity v_0 in the w plane. Making the transformation, we have

$$F = \psi + i\chi = -v_0\left[z + \left(\frac{a^2}{z}\right)\right] = -v_0 x - \frac{v_0 a^2 x}{x^2+y^2} - iv_0 y + \frac{iv_0 a^2 y}{x^2+y^2}$$

the real part of which is the velocity potential (see page 1219).

If, on the other hand, the portion of the z plane above the line $ABCDE$ in Fig. 10.13 is filled with heat-conducting material, the semicircular part BCD is kept at temperature T_0, and the rest of the x axis kept at zero temperature, the temperature distribution will be the real part of

$$F = (2iT_0/\pi)\coth^{-1}[z + (a^2/z)]$$

and so on.

Circulation and Lift. The compactness of the complex notation makes it possible to compute a number of things about fields much more easily than we were able to do in Sec. 10.1. This is particularly evident in the discussion of nonviscous flow about cylinders of various shapes. On page 1200 we discussed a case where the presence of a sharp trailing edge requires a *circulation* of fluid about the cylinder, in addition to the irrotational flow past it. Here we shall take advantage of the complex variable notation to study the simplest case in more detail than before and to show how other cases may be computed fairly easily by use of conformal transformation.

The simplest case is that of flow plus circulation about a circular cylinder of radius a in the ζ plane. As indicated on pages 19 and 451, the velocity potential for such flow is the real part of the function

$$F = -v_0[\zeta + (a^2/\zeta)] - iv_r \ln \zeta$$

where v_0 is the velocity (along the ξ axis) of the fluid at infinity and v_r is the *circulation* of the fluid around the cylinder.

The velocity of the fluid at the surface of the cylinder is $(-\bar{F}')$ at $\zeta = ae^{i\theta}$, which is

$$v_0 - v_0 e^{2i\theta} - i(v_r/a)e^{i\theta} = -ie^{i\theta}[2v_0 \sin\theta + (v_r/a)]$$

This is pointed along the surface, as it should be. Along the upper surface it is pointed clockwise (in the negative θ direction), and the magnitude of the fluid speed along the upper surface is in general larger than it is along the lower surface if v_r is positive (in other words, the circulation aids the flow about the top surface and hinders it along the lower surface).

The force on the cylinder (neglecting the frictional force due to viscosity) is that due to the fluid pressure, as computed by Bernouilli's equation (2.3.16). This consists of two terms: a vertical buoyancy equal to the weight of the displaced fluid and a term which is the integral of the reduction in pressure, $-\frac{1}{2}\rho v^2$, due to fluid velocity, over the whole surface. This part of the force, per unit element of cylinder of unit length parallel to the cylinder axis and width $dz = iae^{i\theta}\,d\theta$, is

$$p_r(i\,dz) = -\tfrac{1}{2}\rho|dF/dz|^2 i\,dz = \tfrac{1}{2}\rho a[2v_0 \sin\theta + (v_r/a)]^2 e^{i\theta}\,d\theta$$

the quantity $i\,dz$ being perpendicular to dz and pointing *in* to the cylinder when the integration is counterclockwise about the cylinder (in the direction of increasing θ). The total Bernouilli force, in direction and magnitude, is therefore

$$\tfrac{1}{2}\rho a \int_0^{2\pi} \left[4v_0 \sin^2\theta + 4\left(\frac{v_0 v_r}{a}\right)\sin\theta + \left(\frac{v_r}{a}\right)^2\right] e^{i\theta}\,d\theta = 2\pi i \rho v_0 v_r$$

The *lift* on the cylinder, per unit length of cylinder, caused by the combined flow and circulation is therefore

$$L = 2\pi\rho v_0 v_r \tag{10.2.20}$$

This is a very interesting equation. In the first place it indicates that the lift is *independent of the size of the cylinder;* evidently for a given v_0 and v_r, the smaller the cylinder, the greater the differential force per unit

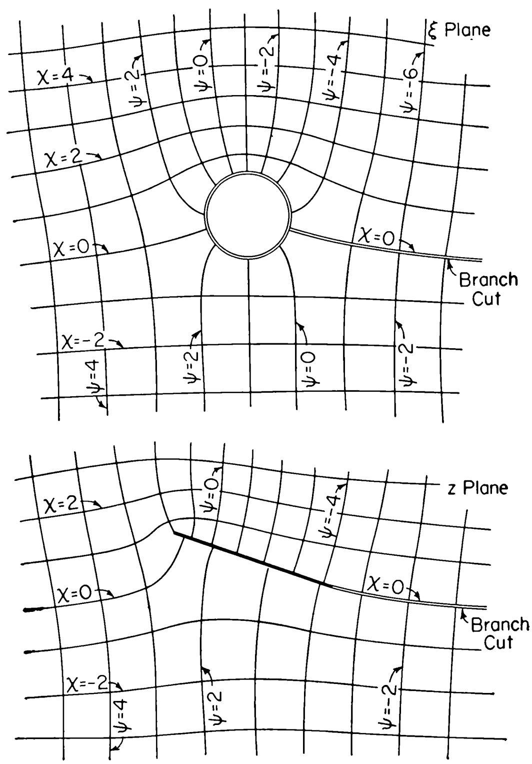

Fig. 10.14 Velocity potential and flow lines for flow plus circulation about cylinder and strip. See Fig. 10.7.

area, so that the net force is independent of a. In the second place it shows that the lift is proportional to the *product of flow and circulation;* if *either* v_0 *or* v_r is zero, the lift is zero; the combination of the two is needed to give a difference of v^2 between the top and bottom of the cylinder.

We now are in a position to modify the flow lines and boundary shape by a conformal transformation in order to find the lift on cylinders of other cross section. It can be shown that, if the transformation is one which coincides with the ζ plane at large distances from the cylinder, then *the lift is the same on the new cylinder as it is on the circular cylinder,* as given in Eq. (10.2.20).

For instance, the transformation $z = \zeta + (a^2e^{-2i\alpha}/\zeta)$ changes the circular cylinder $\zeta = ae^{i\theta}$ into the strip of cross section

$$z = 2ae^{-i\alpha}\cos(\theta + \alpha)$$

of width $4a$, inclined at an *angle of attack* α to the x axis, as shown in Fig. 10.14. The speed of the fluid at the surface of the strip is $|dF/dz| = |dF/d\zeta|[1/|dz/d\zeta|]$ for $\zeta = ae^{i\theta}$, which is

$$v_s = \frac{2v_0 \sin\theta + (v_r/a)}{2\sin(\theta + \alpha)}$$

This is now to be adjusted, according to page 1201, so that the circulation v_r is just enough to make v_s finite at the trailing edge, $\theta = -\alpha$. Therefore $v_r = 2av_0 \sin\alpha$ and

$$v_s = v_0[\sin\theta + \sin\alpha]/\sin(\theta + \alpha)$$

at the surface of the strip.

The Bernouilli force per element of strip dz is again $-\frac{1}{2}\rho v^2 i\, dz$ so that the net force is

$$\begin{aligned} -\tfrac{1}{2}\rho v_0^2 \int_0^{2\pi} \left[\frac{\sin\theta + \sin\alpha}{\sin(\theta + \alpha)}\right]^2 [-2iae^{-i\alpha}\sin(\theta + \alpha)\, d\theta] \\ = 2\rho a v_0^2 i e^{-i\alpha} \int_0^{\pi} [\cos^2\alpha + 2\sin\alpha\cos\alpha\tan\phi + \sin^2\alpha\tan^2\phi]\sin\phi\cos\phi\, d\phi \end{aligned}$$

where $2\phi = \theta + \alpha$. The first term in the square brackets integrates to zero. The second integrates to $\pi \sin\alpha\cos\alpha$. The third term has an infinity at $\phi = \frac{1}{2}\pi$, corresponding to the infinite velocity at the leading edge. If we assume that the actual contour is a bit *outside* the sharp-edged strip, however, we can integrate *around* the simple pole and, after some difficulty, obtain $i\pi \sin^2\alpha$. Therefore the combination is

$$\pi \sin\alpha\,(\cos\alpha + i\sin\alpha) = \pi e^{i\alpha}\sin\alpha$$

The net force is thus perpendicular to the asymptotic flow v_0 and is equal to $2\pi\rho a v_0^2 \sin\alpha$.

But we could have saved this trouble by remembering that the net force on the cylinder can be calculated by reckoning the force across flow lines at large distances from the cylinder, and since we have chosen our transformation so that at large distances the flow for the strip is the same as the flow for the circular cylinder, therefore the lift is the same, that given in Eq. (10.2.20). All we needed to do with the strip was to compute the velocity about it in order to fix the circulation so that the flow off the trailing edge would be finite. To do this v_r had to be $2av_0 \sin \alpha$, and substituting in Eq. (10.2.20) we should have found that

$$L = 2\pi\rho a v_0^2 \sin \alpha$$

a much easier way to arrive at the same result. Lift for many other cross sections may be computed in the same way.

We should note that the lift seems to increase continuously with increase of α, up to $\alpha = \frac{1}{2}\pi$. In actual practice, if the angle of attack α is too large, the front edge begins to "throw off" vortices, the flow lines do not follow the top surface, and the strip loses lift entirely, being said to "stall." When ϕ is near π, the leading and trailing edges must be reversed, v_r must be set equal to $-2av_0 \sin \alpha$, and the "lift" reverses sign.

On the other hand the torque on the strip, due to the Bernouilli forces, must be computed anew for each form. For the strip, we multiply the integrand in the expression for the force by $|z| = 2a \cos(\theta + \alpha)$. After a number of reductions, we find that only the middle term, with $\sin \alpha \cos \alpha$, contributes and that the net torque about the center of the strip, per unit length of strip, is

$$T = \pi\rho a^2 v_0^2 \sin(2\alpha)$$

Many other transformations of coordinates yield interesting solutions for useful boundary shapes. Several of them will be discussed later in this section when we come to utilize the Schwarz-Christoffel technique for computing transformations to fit arbitrary prismatic boundaries. Before this, however, we shall discuss the types of fields formed by various distributions of line sources.

Fields Due to Distributions of Line Sources. The fundamental field due to a unit line source is

$$F = -2 \ln z = -\ln(x^2 + y^2) - 2i \tan^{-1}(y/x) \qquad (10.2.21)$$

If this field is due to a unit electrostatic line charge (unit charge per length), then $\psi = -\ln(x^2 + y^2)$ is the potential function and $-\bar{F}' = 2/\bar{z}$ is the intensity vector (pointed along the radius vector z and inversely proportional to $|z| = r$). If it is due to a unit current in a wire perpendicular to the x, y plane, through the origin, then $\chi = -2 \tan^{-1}(y/x)$ is the magnetic potential, $\psi =$ constant gives its lines of force, and $i\bar{F}' = -i(2/\bar{z})$ is the magnetic intensity (pointed perpendicular to the radius vector in the clockwise direction, since positive current is assumed to be going *into* the paper).

If the field is due to a line source of fluid, sending out fluid at a unit rate per unit length of source line, then ψ is the velocity potential, χ the flow function, and the velocity vector is $-\bar{F}' = 2/\bar{z}$. If, on the other hand, F represents the circulatory flow of fluid about a *vortex line*, ψ will be the flow function, χ the velocity potential, and $-i\bar{F}' = i(2/\bar{z})$ will be the velocity vector. This sort of flow could be caused by a long wire placed along the rectangular axis perpendicular to the x, y plane and rotated counterclockwise with angular velocity equal to 2 divided by the square of the radius of the wire (assuming that there is enough viscosity so that the fluid next to the wire surface will move with the surface and that the motion has gone on long enough so that steady, irrotational flow has been set up).

What we have done is to find the *meromorphic* (see page 382) function, (in this case z) which has a simple zero at $z = 0$ and no other zero or pole in the finite part of the plane. We then take -2 times the natural logarithm of this function to find F. If the source is of strength q, we use a function with branch point z^q at $z = 0$. If we wish to find the solution for several source lines, one at $z = z_1$ with strength q_1, another at $z = z_2$ with strength q_2, and so on, we take the natural logarithm of the product $(z - z_1)^{q_1}(z - z_2)^{q_2} \cdots$ and the potential, flow function, velocity, and so on are obtained from the function

$$F = -2 \ln\left[\prod_n (z - z_n)^{q_n}\right] \tag{10.2.22}$$

For instance, the function

$$F = -2 \ln\left[\frac{z-a}{z+a}\right] = \psi + i\chi; \quad \psi = \ln\left[\frac{(x+a)^2 + y^2}{(x-a)^2 + y^2}\right]$$
$$\chi = 2\tan^{-1}\left(\frac{y}{x+a}\right) - 2\tan^{-1}\left(\frac{y}{x-a}\right) = -2\tan^{-1}\left[\frac{2ay}{x^2+y^2-a^2}\right] \tag{10.2.23}$$

corresponds to a unit positive line source at $z = a$ and a unit negative one at $z = -a$. This could be due to unit line of charge at $z = a$ and conducting plane at zero potential at $x = 0$, in which case ψ is the electrostatic potential and

$$-\bar{F}' = \frac{2}{\bar{z}-a} - \frac{2}{\bar{z}+a} = \frac{4a}{\bar{z}^2 - a^2} = 4a\left[\frac{(x^2-y^2-a^2) + 2ixy}{(x^2+y^2)^2 - 2a^2(x^2-y^2) + a^4}\right]$$

is the electrostatic intensity. The charge density on the conducting plane at $x = 0$ is the value at $x = 0$ of

$$\frac{1}{4\pi}|F'| = \frac{a}{\pi}\sqrt{\frac{1}{(z^2-a^2)(\bar{z}^2-a^2)}} = \frac{(a/\pi)}{\sqrt{(x^2+y^2)^2 - 2a^2(x^2-y^2) + a^4}}$$

which is $[a/\pi(y^2 + a^2)]$. If the charge is not on a line of zero thickness but is on a thin wire of radius ρ, with center at $z = a$, the potential at the surface of the wire is given by the real part of the function

$$2\ln\left[\frac{a + \rho e^{i\theta} + a}{a + \rho e^{i\theta} - a}\right] = 2\ln(2a/\rho) - 2i\theta + 2\ln[1 + (\rho/2a)e^{i\theta}]$$

where we have set $z = a + \rho e^{i\theta}$ at the surface of the wire. Due to the last term, the equipotential surfaces are not quite circles with centers at $z = a$, and thus the real part of this quantity is not exactly constant for ρ constant. But when ρ is small enough compared with a, we can neglect the last term. Therefore to the first approximation in the small quantity ρ/a, the potential of the surface of the wire is $2\ln(2a/\rho)$. From this we can compute the capacitance of the wire with respect to the plane conductor (see page 1182).

The function F corresponding to the irrotational flow of viscous fluid around two spinning rods a distance $2a$ apart is

$$\begin{aligned} F &= 2iA\ln(z^2 - a^2) = \psi + i\chi \\ \psi &= -2A\tan^{-1}\left[\frac{2xy}{x^2 - y^2 - a^2}\right] \\ \chi &= A\ln[(x^2 + y^2)^2 - 2a^2(x^2 - y^2) + a^4] \end{aligned} \tag{10.2.24}$$

if the two rods are spinning in the same direction. The velocity vector is

$$\begin{aligned} -\overline{F'} &= \frac{+4i\bar{z}A}{\bar{z}^2 - a^2} = -4A\left[\frac{y(x^2 + y^2 - a^2) - ix(x^2 + y^2 + a^2)}{(x^2 + y^2)^2 - 2a^2(x^2 - y^2) + a^4}\right] \\ &\xrightarrow[\rho\to 0]{} (2iA/\rho)e^{i\theta};\quad \text{for } z = a + \rho e^{i\theta} \end{aligned}$$

which must equal the velocity $i\rho\omega e^{i\theta}$ of the surface of the rod. At the surface of one of the rods ($z = a + \rho e^{i\theta}$) the force due to viscosity, as given by Eq. (10.2.14), is obtained by setting $\bar{z} = a + \rho e^{-i\theta}$ in the formula $-2i\eta\bar{F}'' = 8\eta A(\bar{z}^2 + a^2)/(\bar{z}^2 - a^2)^2$. To the first order in ρ/a, it is $-(4i\eta A/\rho)e^{i\theta}\,d\theta$, so that the total retarding torque per unit length is $8\pi\eta A = 4\pi\rho^2\eta\omega$.

If the two sources are inside a circular cylinder of radius $b > a$, then we can use the method of images again, placing the image a distance b^2/a away from the origin. For instance, the magnetic field due to two wires, carrying equal and opposite current inside an iron cylinder, is obtained from the function

$$\begin{aligned} F &= -2q\ln\left\{\frac{[z - a][z - (b^2/a)]}{[z + a][z + (b^2/a)]}\right\} = \psi + i\chi \\ \psi &= 2q\ln\left\{\frac{[x + a]^2 + y^2}{[x - a]^2 + y^2}\,\frac{[x + (b^2/a)]^2 + y^2}{[x - (b^2/a)]^2 + y^2}\right\} \\ \chi &= 2q\left\{\tan^{-1}\left[\frac{2ay}{x^2 + y^2 - a^2}\right] + \tan^{-1}\left[\frac{2b^2y/a}{x^2 + y^2 - (b^4/a^2)}\right]\right\} \end{aligned} \tag{10.2.25}$$

The magnetic potential χ goes to zero at the circle $x^2 + y^2 = b^2$, and at this surface ($z = be^{i\phi}$) the magnetic intensity is given by the vector

$$H = i\overline{F'} = 8aq\left[\frac{(a^2 + b^2)\sin\phi}{(a^2 + b^2)^2 - 4a^2b^2\cos^2\phi}\right]e^{i\varphi}$$

which is everywhere normal to the surface. This intensity is largest at $\phi = \frac{1}{2}\pi$, where it has the magnitude $[8aq(a^2 + b^2)/(b^2 - a^2)^2]$. If there had been no iron cylinder at $r = b$, the intensity at the same point would have had the magnitude $[4aq/(a^2 + b^2)]$, a much smaller value.

The function representing the flow of heat in a cylinder of D-shaped cross section, made up of the y axis and the right-hand half of the circle of radius b, can be set up in similar manner. If the cylinder is heated by a small tube, of radius ρ with center at $z = a$ $(0 < a < b)$, which produces q units of heat per unit time per unit length of tube, the temperature of the cylindrical solid is given by the real part of the function

$$
\begin{aligned}
F &= -\frac{q}{2\pi K}\ln\left\{\frac{[z-a][z+(b^2/a)]}{[z+a][z-(b^2/a)]}\right\} = \psi + i\chi \\
\psi &= \frac{q}{2\pi K}\ln\left\{\frac{[x+a]^2+y^2}{[x-a]^2+y^2}\,\frac{[x-(b^2/a)]^2+y^2}{[x+(b^2/a)]^2+y^2}\right\} \\
\chi &= \frac{q}{2\pi K}\left\{\tan^{-1}\left[\frac{2ay}{x^2+y^2-a^2}\right] - \tan^{-1}\left[\frac{2b^2y/a}{x^2+y^2-(b^4/a^2)}\right]\right\}
\end{aligned}
\tag{10.2.26}
$$

where K is the conductivity of the material in the cylinder and where we have assumed that the temperature $T = 0$ on the D-shaped boundary of the cylinder.

At the surface of the heating rod ($z = a + \rho e^{i\theta}$) the function F becomes

$$
\begin{aligned}
F &= -\frac{q}{2\pi K}\ln\left\{\frac{\rho e^{i\theta}}{2a+\rho e^{i\theta}}\,\frac{(a^2+b^2)+a\rho e^{i\theta}}{(a^2-b^2)+a\rho e^{i\theta}}\right\} \\
&\to \frac{q}{2\pi K}\left\{\ln\left[\frac{2a(b^2-a^2)}{\rho(b^2+a^2)}\right] - i\theta\right\};\quad \frac{\rho}{2a}\to 0
\end{aligned}
$$

The total flow of heat is the conductivity K times the increase in χ as θ is changed from 0 to 2π, which is q (as it should be). The temperature of the surface of the heating rod is

$$(q/2\pi K)\ln[2a(b^2 - a^2)/\rho(b^2 + a^2)]$$

which is smaller the closer the heating rod is to the surface (a small or $b - a$ small) or the more conductive is the cylinder, and is larger the smaller is the radius of the heating rod.

Grid Potentials and Amplification Factors. A potential distribution of some interest in connection with triode vacuum tubes involves a hollow conducting anode cylinder of inside radius b, an equally spaced grid of wires a distance a $(a < b)$ out from the cylinder axis, and a cathode wire along the axis. If there are n grid wires, with axes all parallel to the cathode, one can be placed at the point $z = a$, the next at $ae^{2\pi/in}$, and so on, z taking on all the n roots of the equation $z^n - a^n = 0$. The potential due to all n wires, equally charged, is then $-(2q_g/n)\ln(z^n - a^n)$, where q_g is the total charge per unit length on all n wires. The combined potential, due to charge $-q_c$ on the cathode and $-q_g$ on the grid, when the potential of the anode is kept at zero potential, is given by the real part of

$$F = 2q_c \ln\left[\frac{z}{b}\right] + \frac{2}{n} q_g \ln\left[\frac{z^n - a^n}{z^n - (b^2/a)^n}\right] \tag{10.2.27}$$

If the radius of the cathode is ρ_c, its potential to the first order in ρ_c/a is

$$-2q_c \ln(b/\rho_c) - 4q_g \ln(b/a)$$

and if the radius of each grid wire is ρ_g, the grid potential is approximately

$$-2q_c \ln(b/a) + (2q_g/n) \ln[n\rho_g a^{2n-1}/(b^{2n} - a^{2n})]$$

Now suppose we adjust potentials so that the cathode is grounded, the grid potential is V_g, and the anode potential is V_a. Referring to the above expressions we see that

$$V_a \simeq 2q_c \ln(b/\rho_c) + 4q_g \ln(b/a)$$

$$V_g \simeq 2q_c \ln(a/\rho_c) + (2q_g/n) \ln\left[\frac{(n\rho_g/a)}{1 - (a/b)^{2n}}\right]$$

or

$$q_c \simeq \tfrac{1}{2}\frac{V_g \ln(b/a) + (V_a/2n) \ln\{(a/n\rho_g)[1 - (a/b)^{2n}]\}}{\ln(a/\rho_c) \ln(b/a) + (1/2n) \ln(b/\rho_c) \ln\{(a/n\rho_g)[1 - (a/b)^{2n}]\}}$$

$$q_g \simeq \tfrac{1}{4}\frac{V_a \ln(a/\rho_c) - V_g \ln(b/\rho_c)}{\ln(a/\rho_c) \ln(b/a) + (1/2n) \ln(b/\rho_c) \ln\{(a/n\rho_g)[1 - (a/b)^{2n}]\}} \tag{10.2.28}$$

From these equations we can calculate several properties of the system of electrodes. The capacitance of the cathode with respect to grid and anode is the ratio of q_c to V_a when we set $V_g = V_a$; the capacitance of the grid with respect to anode and cathode is the ratio of q_g to V_g when $V_a = 0$; and so on.

The electric intensity at the surface of the cathode ($z = \rho_c e^{i\theta}$) is to the first approximation in the small quantities ρ_c/a and $(a/b)^{2n}$,

$$E \simeq \frac{2q_c}{\rho_c} \simeq \frac{1}{\rho_c}\frac{V_g \ln(b/a) + V_a(1/2n)[\ln(a/n\rho_g) - (a/b)^{2n}]}{\ln(a/\rho_c) \ln(b/a) + (1/2n) \ln(b/\rho_c)[\ln(a/n\rho_g) - (a/b)^{2n}]}$$

If n is reasonably large (4 or more), the coefficient of V_a in this expression is considerably smaller than the coefficient of V_g. This means that the grid *screens* the field of the anode from the cathode, the effect on E of a certain change in V_a being considerably smaller than the effect of a similar change in V_g. The ratio of the effect on the cathode field of the grid to that of the anode is called the *amplification constant* of the vacuum tube. In this simple case it is

$$\mu \simeq \frac{2n \ln(b/a)}{\ln(a/n\rho_g) - (a/b)^{2n}} \tag{10.2.29}$$

Linear Arrays of Source Lines. In case we have an infinite array of equal, parallel source lines, spaced along the x axis a distance a apart, the potential distribution can be obtained from the logarithm of a function which goes to zero at the points 0, $\pm a$, $\pm 2a$, . . . as $[z - (na)]$. Such a function is, of course, $A \sin(\pi z/a)$. Therefore the potential due to a uniform grid of equal source lines, with spacing a along the x axis, is the real part of

$$
\begin{aligned}
F_s &= -2 \ln[2 \sin(\pi z/a)] = \psi_s + i\chi_s \xrightarrow[|y| \to \infty]{} \pm[(2\pi i/a)z - i\pi] \\
\psi_s &= -\ln 4[\sin^2(\pi x/a) + \sinh^2(\pi y/a)] \xrightarrow[|y| \to \infty]{} -(2\pi/a)|y| \\
\chi_s &= -2 \tan^{-1}[\cot(\pi x/a) \tanh(\pi y/a)] \xrightarrow[|y| \to \infty]{} \pm[(2\pi/a)x - \pi]
\end{aligned}
\tag{10.2.30}
$$

where the plus sign, for the limiting values, is used when y is positive and the negative sign when y is negative and large.

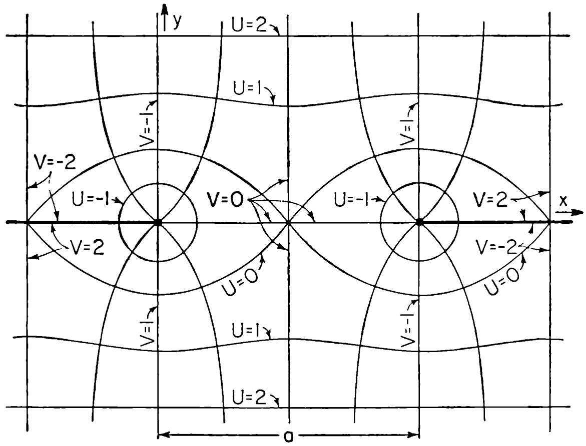

Fig. 10.15 Potential and flow lines near grid of charged, parallel wires: $\pi U = \psi - \ln 4$ and $\pi V = \chi$, where ψ and χ are given in Eq. (10.2.30).

At large distances from the grid ($|y|$ large), the field is uniform, as though it were caused by a plane sheet of charge of uniform density $1/a$ per unit area. A small distance ρ from any one line ($z = \rho e^{i\theta}$, say) the function F_s is, approximately,

$$F_s \simeq -2 \ln(2\pi\rho e^{i\theta}/a) = 2 \ln(a/2\pi\rho) - 2i\theta$$

If the source lines alternate in sign, the potential may be obtained from the function

$$
\begin{aligned}
F_a &= -2 \ln[\tan(\pi z/a)] = 2 \ln[\cot(\pi z/a)] \xrightarrow[|y| \to \infty]{} \mp i\pi \\
\psi_a &= \ln\left[\frac{\cosh(2\pi y/a) + \cos(2\pi x/a)}{\cosh(2\pi y/a) - \cos(2\pi x/a)}\right] \xrightarrow[|y| \to \infty]{} 0 \\
\chi_a &= -2 \tan^{-1}\left[\frac{\sinh(2\pi y/a)}{\sin(2\pi x/a)}\right] \xrightarrow[|y| \to \infty]{} \mp\pi
\end{aligned}
\tag{10.2.31}
$$

If a uniformly charged grid of parallel wires is placed a distance b from a grounded, plane conductor, the potential field can be computed from the function

$$F = -2a\sigma \ln\left\{\frac{\sin[(\pi/a)(z - ib)]}{\sin[(\pi/a)(z + ib)]}\right\} \tag{10.2.32}$$

where a is the wire spacing, σ the grid charge per unit area (so that $a\sigma$ is the charge per unit length of an individual wire); the grounded plate is along the real axis, the grid is at $y = b$, and the image grid is at $y = -b$.

The potential of the grid wires (at $z = ib + \rho e^{i\theta}$, for instance) is, approximately,

$$V \simeq 2a\sigma \ln(a/2\pi\rho) + 4\pi\sigma b; \quad \rho \ll a \ll b$$

Therefore the capacitance per unit area of the grid-plate system is

$$C = \frac{\sigma}{V} = \frac{1}{4\pi b + 2a \ln(a/2\pi\rho)}$$

Suppose that the grid were grounded and the plate at $y = 0$ were at potential $-V$; then the potential at point (x,y) would be

$$\psi = \frac{V}{2 \ln(a/2\pi\rho) + (4\pi b/a)} \ln\left\{\frac{\sin^2[\pi x/a] + \sinh^2[(\pi/a)(y + b)]}{\sin^2[\pi x/a] + \sinh^2[(\pi/a)(y - b)]}\right\} - V \tag{10.2.33}$$

Above the grid the asymptotic value of the potential is

$$\psi \to -\frac{2 \ln(a/2\pi\rho)}{(4\pi b/a) + 2 \ln(a/2\pi\rho)} V; \quad y \gg b$$

which is an indication of the shielding power of the grid. If the grid were a perfect shield, this asymptotic potential would be zero. If the grid spacing a is considerably smaller than the plate-grid distance b, the actual asymptotic potential is quite small, though not zero.

For a finite sequence of equally spaced, equally charged wires, we can call on the properties of the gamma function to help us. For instance, for a sequence of $N + 1$ wires placed at $z = 0, -a, -2a, \ldots, -Na$, each charged with σa units per unit length (average of σ per unit area of the plane normal to the y axis), we have

$$F = -2\sigma a \ln[z(z + a) \cdots (z + Na)] = 2\sigma a \ln\left[a^{-N-1} \frac{\Gamma(z/a)}{\Gamma\left(\dfrac{z}{a} + N + 1\right)}\right] \tag{10.2.34}$$

If N is small, we can use the product form to obtain our results, as we have done earlier, but if N is fairly large, it is better to use the properties of the gamma function. Referring to Chap. 4 we collect the following properties:

$$\ln[\Gamma(w)] = -\gamma w - \ln w - \sum_{n=1}^{\infty} \left[\ln\left(1 + \frac{w}{n}\right) - \frac{w}{n}\right];$$
$$\xrightarrow[|w| \to \infty]{} (w - \tfrac{1}{2}) \ln w - w + \ln \sqrt{2\pi} \tag{10.2.35}$$
$$\frac{d}{dw} \ln[\Gamma(w)] = -\gamma + \sum_{n=0}^{\infty} \left[\frac{1}{n+1} - \frac{1}{n+w}\right] = \psi(w);$$
$$\xrightarrow[|w| \to \infty]{} \ln w; \quad \gamma = 0.5772 \ldots$$

where both of the asymptotic formulas are valid except when w is a negative real number.

For instance, if the source lines are wires carrying current (and $\sigma a = I$, the current carried by each wire), then the magnetic intensity at point z is

$$H = i\bar{F}' = \frac{2iI}{a}\left[\bar{\psi}\left(\frac{\bar{z}}{a}\right) - \bar{\psi}\left(\frac{\bar{z}}{a} + N + 1\right)\right] = -2iI \sum_{n=0}^{N} \frac{1}{\bar{z} + an}$$
$$\to -i(2I/a) \ln[1 + (a/\bar{z})(N + 1)]$$

For z real and positive this field is pointed downward, whereas for a large imaginary part for z the field is horizontal. When $z = Re^{i\theta}$, with $R \gg Na$, the intensity becomes

$$H \to [2(N + 1)I/R](-ie^{i\theta})$$

which is the field that would result if there was but one wire carrying a total current of $(N + 1)I$.

Two-dimensional Array of Line Sources. Finally, we should study the potential distribution about an array of line sources, regularly spaced over the whole x, y plane. This potential is generated by taking the logarithm of an elliptic function (see Sec. 4.5), for such a function has zeros and simple poles regularly spaced on the complex plane. As we saw earlier, the array is periodic, consisting of an endless repetition of *unit cells*, which are parallelograms or, in most applications, are rectangles of sides a (parallel to the x axis) and b (parallel to the y axis). The elliptic functions are doubly periodic, repeating their values over and over again, the value for each congruent point, in each unit cell, being the same.

As was stated in (4.5.66), the number of zeros of the elliptic function must equal its number of simple poles in a unit cell. This corresponds to the requirement that the net charge in each unit cell must be zero, or else the potential would be infinite everywhere. The spacing and distribution of the line charges in each cell will determine the particular elliptic function to be used.

For instance the function $\operatorname{sn}(u,k)$ has simple zeros at $u = 2mK + 2inK'$ (where m, $n = 0$, ± 1, ± 2, . . .) and has simple poles at $u =$

$2mK + i(2n + 1)K'$, a row of zeros being along the real axis, spaced a distance $2K$ apart; a row of poles along the line $y = K'$, each vertically above a zero on the real axis; and so on. Consequently the potential due to an array of line sources of the sort shown in Fig. 10.16a is the real part of the function

$$\begin{aligned} F_a &= 2q \ln\left\{\mathrm{sn}\left[\frac{K}{a}z, k\right] \cdot \mathrm{sn}\left[\frac{K}{a}(z + a), k\right]\right\} \\ &= 2q \ln\{\mathrm{sn}(u,k)\ \mathrm{cn}(u,k)/\mathrm{dn}(u,k)\};\quad u = Kz/a \qquad (10.2.36) \end{aligned}$$

where k is chosen so that $K'/K = b/a$, from the numerical table on page 487.

This table includes only the cases for $K' \geq K$ (that is $b \geq a$). For the cases where $b < a$, we can rotate the figure by 90°, interchange a for b (and therefore K for K' and k for k'), and use the transformations $\mathrm{sn}(iu,k') = [i\ \mathrm{sn}(u,k)/\mathrm{cn}(u,k)]$, etc., obtaining for the potential the real part of

$$F'_a = 2q \ln\{\mathrm{sn}(u,k)/\mathrm{cn}(u,k) \cdot \mathrm{dn}(u,k)\};\quad u = Kz/a \qquad (10.2.37)$$

where now the new b is not smaller than the new a as it was before. [Or else we can exchange a for b, K for K', and k for k' in the table and use Eq. (10.2.36) for the larger values.]

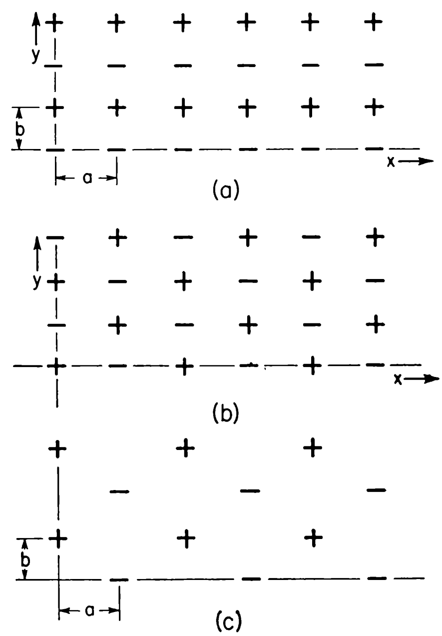

Fig. 10.16 Distribution of parallel line sources corresponding to various elliptic functions.

On the other hand, if the array is of the sort given in Fig. 10.16b, the potential is the real part of

$$F_b = 2q \ln\left\{\frac{\mathrm{sn}(u,k)}{\mathrm{sn}(u + K, k)}\right\} = 2q \ln\left\{\frac{\mathrm{sn}(u,k)\ \mathrm{dn}(u,k)}{\mathrm{cn}(u,k)}\right\} \qquad (10.2.38)$$

Finally, if the array is that shown in Fig. 10.16c, we can use the function $\mathrm{cn}(u,k)$, which has zeros at $(2m + 1)K + 2inK'$ and poles at $2mK + i(2n + 1)K'$. The potential will be the real part of

$$F_c = 2q \ln[\mathrm{cn}(u,k)];\quad u = Kz/a \qquad (10.2.39)$$

In case a the potential at the surface of a wire $z = \rho e^{i\theta}$ ($\rho \ll a$) is obtained from the properties of the elliptic functions:

$$\mathrm{sn}(u,k) \to u;\quad \mathrm{cn}(u,k) \to 1;\quad \mathrm{dn}(u,k) \to 1;\quad \text{for } u \to 0$$

Therefore the potential of the negatively charged wires is approximately $-2q \ln(a/K\rho)$. At the surface $z = iK' + \epsilon e^{i\theta}$ of the positively charged wire, we use the expressions

$$\mathrm{sn}(u + iK', k) \to \frac{1}{ku}; \quad \mathrm{cn}(u + iK', k) \to \frac{1}{iku}; \quad \mathrm{dn}(u + iK', k) \to \frac{1}{iu}$$

for $u \to 0$. Consequently the approximate expression for the potential of the positively charged wires is $2q \ln(a/k^2K\epsilon)$, and the potential difference between the positively and negatively charged wires in case a is

$$V_a = 2a \ln(a^2/k^2K^2\epsilon\rho) \quad (10.2.40)$$

By similar means we can show that the corresponding potential differences in cases b and c of Fig. 10.16c are

$$V_b = 2q \ln(a^2/K^2\epsilon\rho);$$
$$V_c = 2q \ln(a^2/kk'K^2\epsilon\rho) \quad (10.2.41)$$

A little study of Fig. 10.16 will indicate why $V_a > V_c > V_b$ when $b > a$.

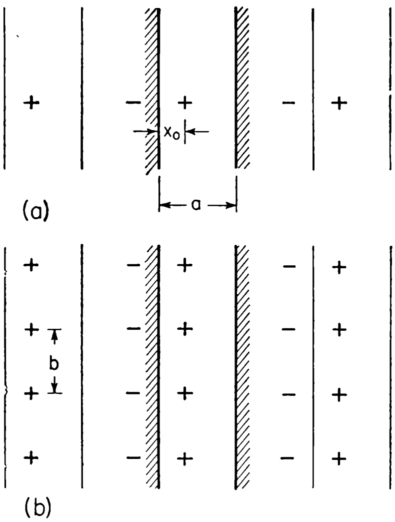

Fig. 10.17 Periodic distribution of images.

Periodic Distributions of Images. When the distribution of line charges lies between two conducting planes, parallel to each other and to the line charges, the image of each line charge in one plane produces an image in the other plane, and so on, until, to satisfy completely the pair of boundary conditions, we must use an infinite sequence of symmetrically placed images. For instance, if the two conducting planes are perpendicular to the x axis, one cutting through the origin and the other at $x = a$, and if the line charge is at $x = x_0$ ($x_0 < a$), then there must be a set of images at the points $x = x_0 + 2ma$ ($m = \pm 1, \pm 2, \ldots$), all of the same sign as the original, plus another set of opposite sign at the points $x = -x_0 + 2na$ ($n = 0, \pm 1, \pm 2, \ldots$), as in Fig. 10.17$a$.

The potential due to this arrangement of line charge plus images is the real part of the logarithm of a function which has poles at $x = x_0 + 2ma$ and zeros at $x = -x_0 + 2na$.

$$F = 2q \ln\left\{\frac{\cot[\pi x_0/a] + \cot[(\pi/2a)(z - x_0)]}{\cot[\pi x_0/a] - \cot[\pi x_0/2a]}\right\}$$
$$= 2q \ln\left\{\frac{\sin[(\pi/2a)(x_0 + z)]}{\sin[(\pi/2a)(x_0 - z)]}\right\} \quad (10.2.42)$$

which is adjusted so that the function in the braces is unity at $x = 0$ (whence, by periodicity, it will be ± 1 at $x = na$). Therefore the potential of both plates is zero, and the potential of a wire of radius ρ, carrying the charge ($z = x_0 + \rho e^{i\theta}$), is, approximately,

$$V \simeq 2q \ln\left[\frac{\sin(\pi x_0/a)}{(\pi\rho/2a)}\right]; \quad \rho \ll x_0 \tag{10.2.43}$$

The capacitance between the wire and the two plates is therefore

$$\frac{q}{V} \simeq \frac{1}{2\ln[\sin(\pi x_0/a)/(\pi\rho/2a)]}$$

Next we can set up the potential due to a grid of parallel wires, spaced a distance $2b$ apart, placed between two parallel plates, as in Fig. 10.17*b*. We look for an elliptic function which is infinite at the points $x_0 + 2am + 2ibn$, which turns out to be $\mathrm{cn}[(K/a)(z - x_0), k]/\mathrm{sn}[(K/a)(z - x_0), k]$ where k is adjusted so that $K' = K(b/a)$. The image points where the function is to be zero are to be at $-x_0 + 2am + 2ibn$. The function which satisfies this is the equivalent of the difference $\cot[\pi x_0/a] + \cot[(\pi/2a)(z - x_0)]$, so that the required potential is the real part of

$$F = 2q \ln\left\{\frac{\dfrac{\mathrm{cn}(2Kx_0/a)}{\mathrm{sn}(2Kx_0/a)} - \dfrac{\mathrm{cn}[(K/a)(x_0 - z)]}{\mathrm{sn}[(K/a)(x_0 - z)]}}{\dfrac{\mathrm{cn}(2Kx_0/a)}{\mathrm{sn}(2Kx_0/a)} - \dfrac{\mathrm{cn}(Kx_0/a)}{\mathrm{sn}(Kx_0/a)}}\right\} \tag{10.2.44}$$

which has been adjusted so that the potential of the plates is zero.

At the surface of the conductor, $z = x_0 + \rho e^{i\theta}$, this potential is, approximately,

$$\begin{aligned} V &\simeq 2q \ln\left\{\frac{-(a/K\rho)}{\dfrac{\mathrm{cn}(2Kx_0/a)}{\mathrm{sn}(2Kx_0/a)} - \dfrac{\mathrm{cn}(Kx_0/a)}{\mathrm{sn}(Kx_0/a)}}\right\} \\ &= 2q \ln\left\{\frac{(2a/K\rho)\,\mathrm{sn}(\alpha,k)\,\mathrm{cn}(\alpha,k)\,\mathrm{dn}(\alpha,k)}{\mathrm{cn}^2(\alpha,k)[2\,\mathrm{dn}(\alpha,k) - 1] + \mathrm{sn}^2(\alpha,b)\,\mathrm{dn}^2(\alpha,k)}\right\} \end{aligned}$$

where $\alpha = Kx_0/a$. From this expression for the potential we can calculate the capacitance, per wire, between the grid and the plates. We note that, when the grid spacing $2b$ becomes much larger than the plate spacing a, the parameter k goes to zero and the expression for V reduces to Eq. (10.2.43), since, for $k \to 0$, we have

$$\mathrm{sn}(u,k) \to \sin(\pi u/2K); \quad \mathrm{cn}(u,k) \to \cos(\pi u/2K); \quad \mathrm{dn}(u,k) \to 1$$

We have not, of course, shown that the potential given in Eq. (10.2.44) is zero all along the imaginary axis and along the axis $x = a$. However, as we have seen on page 429, specifying the nature and position of the zeros and poles of an elliptic function inside its unit cell uniquely specifies the elliptic function, except for a constant factor. Similarly we know that, once we have specified the position and magnitude of all the line charges, real and image, we have uniquely specified the potential field. Consequently a formula which fits all the image charges must satisfy the boundary conditions which gave rise to the image charges. This corresponds to the mathematical statement that for $z = iy$ ($x = 0$) the function

$$\frac{\mathrm{cn}(2Kx_0/a,k)}{\mathrm{sn}(2Kx_0/a,k)} - \frac{\mathrm{cn}[(K/a)(x_0 - z),\, k]}{\mathrm{sn}[(K/a)(x_0 - z),\, k]} = Ae^{i\vartheta} \tag{10.2.45}$$

has a constant amplitude (that is, A is independent of y when $x = 0$). A tedious sequence of manipulations is required to prove this.

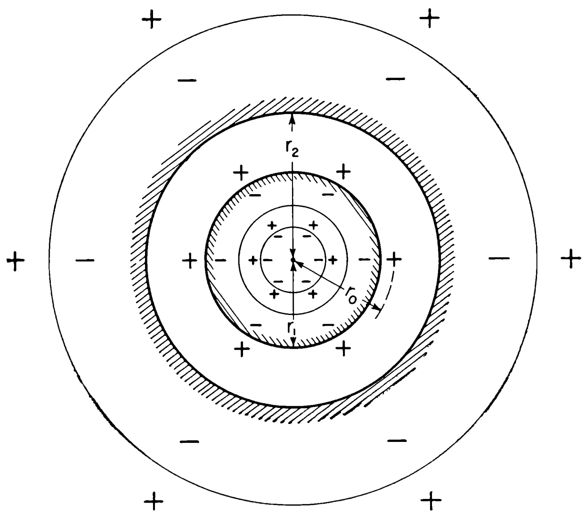

Fig. 10.18 Distribution of images for line source between concentric cylinders.

We may now use the transformation

$$z = r_1 e^w; \quad w = \ln(z/r_1)$$

to change the distribution of Fig. 10.17*b* (which we now call the w plane) into the distribution of Fig. 10.18, corresponding to a grid of n wires between two cylinders, the inner one of radius r_1 and the outer one of radius r_2. To find k, K, and K' we require that $r_2 = r_1 e^a$ and that $r_0 e^{2\pi i/n}$ corresponds to $w = x_0 + 2ib$. Therefore

$$a = \ln(r_2/r_1); \quad x_0 = \ln(r_0/r_1); \quad b = \pi/n$$

and the potential is the real part of

$$F = 2q \ln \left\{ \frac{\dfrac{\mathrm{cn}[2\gamma \ln(r_0/r_1)]}{\mathrm{sn}[2\gamma \ln(r_0/r_1)]} - \dfrac{\mathrm{cn}[\gamma \ln(r_0/z)]}{\mathrm{sn}[\gamma \ln(r_0/z)]}}{\dfrac{\mathrm{cn}[2\gamma \ln(r_0/r_1)]}{\mathrm{sn}[2\gamma \ln(r_0/r_1)]} - \dfrac{\mathrm{cn}[\gamma \ln(r_0/r_1)]}{\mathrm{sn}[\gamma \ln(r_0/r_1)]}} \right\} \tag{10.2.46}$$

where $\gamma = K/a = [K/\ln(r_2/r_1)]$ and where K, K', and k are chosen so that $K'/K = b/a = [\pi/n \ln(r_2/r_1)]$.

Finally, when we have a single line charge inside a rectangular prism held at zero potential, as shown in Fig. 10.19, the images of the single charge repeat themselves in a doubly periodic manner over the whole z plane. From what we have already done, it is not difficult to see that the potential inside the region $0 < x < a$, $0 < y < b$ is the real part of the function

$$F = 2q \ln \left\{ \frac{\dfrac{\mathrm{cn}(2Kx_0/a)}{\mathrm{sn}(2Kx_0/a)} - \dfrac{\mathrm{cn}[(K/a)(z_0 - z)]}{\mathrm{sn}[(K/a)(z_0 - z)]}}{\dfrac{\mathrm{cn}(2Kx_0/a)}{\mathrm{sn}(2Kx_0/a)} - \dfrac{\mathrm{cn}[(K/a)(\bar{z}_0 - z)]}{\mathrm{sn}[(K/a)(\bar{z}_0 - z)]}} \right\} \tag{10.2.47}$$

where $z_0 = x_0 + iy_0$ and $\bar{z}_0 = x_0 - iy_0$.

From this latter formula one can compute the capacitance of a wire in a rectangular wave guide or the distribution of heat in a rectangular bar enclosing any two-dimensional distribution of heat sources.

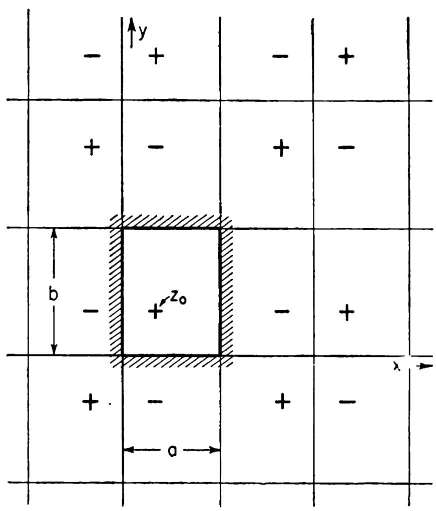

Fig. 10.19 Distribution of images for line source inside rectangular prism of sides a, b.

Potentials about Prisms. As a final example of the use of complex functions in solving the two-dimensional Laplace equation, we consider the distribution of potential about prismatic surfaces having a cross section, in the x, y plane, which is a polygon (simple or degenerate) and which extends indefinitely in a direction perpendicular to the x, y plane. What is done is to solve an analogous problem with relation to the real w axis as boundary and then to use the Schwarz-Christoffel transformation to convert the real axis of the w plane into the required polygon in the z plane.

As we saw in Sec. 4.6, if the polygon has exterior angles $\alpha_1, \alpha_2, \ldots,$ α_N at successive vertices corresponding to the points $a_1, a_2, \ldots, a_{N-1},$ ∞ along the real axis in the w plane, then the transformation which changes the upper half of the w plane into the interior of the polygon in the z plane is defined by the differential equation

$$\frac{dz}{dw} = \prod_{n=1}^{N-1} \frac{A}{(w - a_n)^{\alpha_n/\pi}} \tag{10.2.48}$$

The interior of the polygon, to the left as we go around its perimeter, plots onto the upper half of the w plane, and α_n is the leftward (counterclockwise) turn at the nth vertex. Usually the a_n's and A and the constant of integration z_0 are adjusted, after the integration is accomplished, so that the vertices of the polygon are placed properly in the z plane. This transformation and some of its uses have been discussed in Sec. 4.6; we shall consider a few further examples here to illustrate a few points.

A degenerate prism is the one shown in Fig. 10.20. The point A at $z = -\infty$ corresponds to $w = -\infty$. At B ($z = ia$ and $w = -1$) the

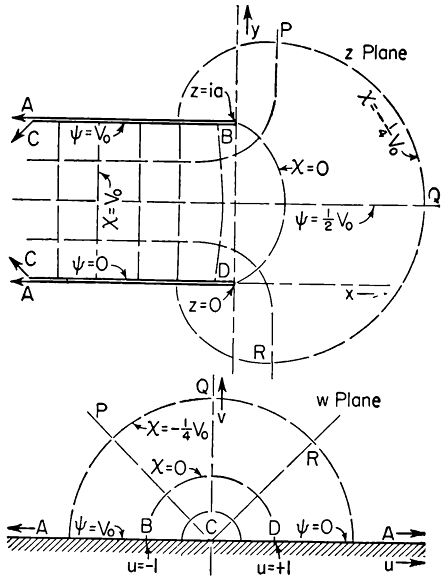

Fig. 10.20 Schwarz-Christoffel transformation for parallel-plate condenser ABC-CDA.

exterior angle is $\alpha_1 = -\pi$, at C ($z = -\infty$, $w = 0$) the angle is $\alpha_2 = +\pi$, and at D ($z = 0$, $w = +1$) the angle is again $-\pi$. Therefore the transformation is given by the equation

$$z - z_0 = A \int \frac{w^2 - 1}{w} \, dw = A[\tfrac{1}{2}w^2 - \ln w]$$

We have chosen this direction to go around the "polygon" $ABCD$ so that we have the "interior" of the "polygon" the whole of the z plane, which then corresponds to the upper half plane of w. If we had reversed the order by interchanging the positions of B and D in the z plane and thereby reversing the signs of the three angles, the upper half plane of w would have corresponded to the infinitesimal strips between AB and BC and between CD and DA.

To have $w = 1$ correspond to $z = 0$, we set $z_0 = -\frac{1}{2}A$. To have $w = -1 = e^{i\pi}$ correspond to $z = ia$ requires that $-i\pi A = ia$ or $A =$

$-(a/\pi)$. Therefore the proper transformation is

$$z = \frac{a}{\pi}[\ln w + \tfrac{1}{2}(1 - w^2)]; \quad \frac{dz}{dw} = \frac{a}{\pi}\left(\frac{1}{w} - w\right) \tag{10.2.49}$$

Parallel-plate Condenser. We now choose further transformations of the w plane to correspond to desired distributions of potential. For instance, the potential distribution

$$F = \psi + i\chi = (V_0/i\pi)\ln w; \quad w = e^{(i\pi F/V_0)}$$

gives a potential ψ corresponding to the positive half of the real w axis being at zero potential and the negative half being at potential V_0. Transforming to the z plane, the lines ψ = constant correspond to the equipotential lines about two plane parallel condenser plates, the lower being at potential zero and the upper at potential V_0. The relation between z and $F = \psi + i\chi$,

$$\begin{aligned} z &= \left(\frac{a}{\pi}\right)\left[\frac{i\pi}{V_0}F + \tfrac{1}{2}(1 - e^{2\pi iF/V_0})\right]; \quad \frac{dz}{dF} = i\left(\frac{a}{V_0}\right)(1 - e^{2\pi iF/V_0}) \\ x &= -\left[\frac{a}{V_0}\right]\chi + \frac{a}{2\pi}[1 - e^{-2\pi\chi/V_0}\cos(2\pi\psi/V_0)] \\ y &= \left[\frac{a}{V_0}\right]\psi - \frac{a}{2\pi}e^{-2\pi\chi/V_0}\sin(2\pi\psi/V_0) \end{aligned} \tag{10.2.50}$$

allows the potential lines and lines of force to be computed parametrically.

The region for $|w| < 1$, given by $\chi > 0$, corresponds to the region between the two plates. Here the exponential is small and the electrical intensity ($\bar{F}$) is approximately equal to iV_0/a, a constant pointed from lower to upper plate. The magnitude of the electric intensity at the point given by ψ, χ is

$$\left|\frac{dF}{dz}\right| = \frac{V_0/a}{\sqrt{1 + e^{-4\pi\chi/V_0} - 2e^{-2\pi\chi/V_0}\cos(2\pi\psi/V_0)}} \tag{10.2.51}$$

and the charge density on the lower plate ($\psi = 0$) is

$$\sigma = \frac{1}{4\pi}\left|\frac{dF}{dz}\right|_{\psi=0} = \left|\frac{V_0/4\pi a}{1 - e^{-2\pi\chi/V_0}}\right|; \quad x = -\frac{a\chi}{V_0} + \frac{a}{2\pi}[1 - e^{-2\pi\chi/V_0}]$$

The part for $\chi > 0$ is on the upper side of this plate; the part for $\chi < 0$ on the lower.

For χ positive (on the inside between the plates) the charge density soon becomes $V_0/4\pi a$, independent of x. For χ negative (outside), σ soon becomes equal to $(V_0/4\pi a)e^{2\pi\chi/V_0} \simeq -(V_0/2x)$, an amount which diminishes as one goes away from the edge at $x = 0$. At the edge the charge density becomes infinite.

We remember that the flow function χ can help us compute total charge on a conducting surface; in fact the charge (per unit length perpendicular to x, y) between the point for which χ is χ_1 and the point for which $\chi = \chi_2$ is just $(\chi_2 - \chi_1)/4\pi$. To find the total charge on the inside and outside of one of the plates, within a distance l of the edge, we merely have to find χ for $x = -l$ along the outside and inside of the plate. If l is larger than a, this is not difficult. From Eq. (10.2.50), we see that for the inside, for χ positive, to a good approximation $\chi_2 \simeq (V_0 l/a) + (V_0/2\pi)$, whereas for the outside, for χ negative, $\chi_1 \simeq -(V_0/2\pi)\ln(2\pi l/a)$.

Consequently the total charge in a strip of the plate of unit width, extending a length l in from the edge, is

$$Q \simeq \frac{V_0 l}{4\pi a}\left[1 + \left(\frac{a}{2\pi l}\right)\ln\left(\frac{2\pi e l}{a}\right)\right]; \quad \ln(e) = 1$$

The second term is an "edge correction" for the additional charge concentrated near the edge of the plate. The capacitance of a pair of long strips $2l$ wide and a distance a apart, per unit length of strip, would thus be

$$C \simeq \frac{l}{2\pi a}\left[1 + \left(\frac{a}{2\pi l}\right)\ln\left(\frac{2\pi e l}{a}\right)\right]$$

The larger l is compared with a, the smaller is this correction term.

Along an equipotential line close to the lower plate (ψ small) the electric intensity is uniform inside the gap (χ positive); it rises to a maximum value near the edge of the plate (χ zero) and then falls off rapidly for χ negative. For equipotential surfaces between $\psi = \frac{1}{4}V_0$ and $\psi = \frac{3}{4}V_0$ there is no maximum value of intensity, $|dF/dz|$ being a monotonously decreasing function of χ as χ goes from $-\infty$ to $+\infty$. Consequently if the conducting surface is along the lines $\psi = \frac{1}{4}V_0$ and $\psi = \frac{3}{4}V_0$, as shown in Fig. 10.20 (lines CR and CP), there would be no point of maximum surface intensity where sparking might initiate a breakdown. Because of the reduced distance between conductors the potential difference is only half that between the thin plates for the same uniform intensity between the plates, but the peak surface intensity, near the sharp edge, for the thin plates is considerably larger than this, so the rounded conductor can support a larger potential difference before breaking down.

Other transformations are also possible, corresponding to other boundary conditions. For instance, the Green's function for a source line at $w = u_0 + iv_0$ is

$$F = -2q\ln\left[\frac{w - u_0 - iv_0}{w - u_0 + iv_0}\right] = -4iq\tan^{-1}\left[\frac{w - u_0}{v_0}\right] = \psi + i\chi$$

$$w = u_0 + iv_0\tanh\left(\frac{F}{4q}\right); \quad \frac{dF}{dw} = \frac{-4iqv_0}{(w - u_0)^2 + v_0^2} \qquad (10.2.52)$$

The complete transformation from ψ, χ to x, y is then obtained by inserting the expression for w, given directly above, into Eq. (10.2.49). The source line in the z plane is at z_0, obtained by setting $w_0 = u_0 + iv_0$ into Eq. (10.2.49). Near the point z_0, the relation between z and F is

$$z = z_0 + \left(\frac{dz}{dw}\right)_{w_0}(w - w_0) + \cdots$$
$$= z_0 + \frac{a}{\pi}\left(\frac{1}{w_0} - w_0\right)[-2iv_0 e^{-F/2q}] + \cdots$$

or, approximately

$$F = \psi + i\chi \simeq -2q \ln\left[\frac{\pi w_0(z - z_0)}{a(1 - w_0^2)}\right]; \quad z \to z_0$$

For instance, if $w_0 = i$, $z_0 = (a/\pi) + \frac{1}{2}ia$. If a wire of radius ρ ($\rho \ll a$) is placed at this point, charged with q units per length of wire, the potential of its surface is approximately

$$V \simeq 2q \ln(2a/\pi\rho)$$

and the potential of the two plates is, of course, zero. The charge density induced on the surface of either plate, at the point corresponding to w (on the real w axis), is

$$\frac{1}{4\pi}\left|\frac{dF}{dw}\right|\left|\frac{dw}{dz}\right| = \frac{qv_0/a}{(w - u_0)^2 + v_0^2}\left|\frac{w}{1 - w^2}\right|$$

This density is zero at the far end ($w = 0$ or $w = \pm\infty$); it increases to a maximum at the edges of the plates ($w = \pm 1$) nearest the line charge. It is, in general, smaller on the outside of the plates ($|w| > 1$) than on the inside ($|w| < 1$).

Variable Condenser. Another degenerate prism is that of a semi-infinite plate midway between two infinite plates, as shown in Fig. 10.21. Here we are to plot the region between the parallel plates onto the upper half plane in w. Starting at A, the angles are $+\pi$ at B, $+\pi$ at C, and $-\pi$ at D. Therefore the equation for the transformation is

$$z - z_0 = A\int\frac{(u - 1)\,du}{u(u + 1)} = A\left[2\ln\left(\frac{w + 1}{2}\right) - \ln(w)\right]$$

Since $z = 0$ at $w = 1$, z_0 turns out to be zero. Since z is to be $-ia+$ (real quantity) when $0 > w > -1$, we must have $A = a/\pi$; therefore

$$z = \frac{a}{\pi}\ln\left[\frac{w}{4} + \frac{1}{2} + \frac{1}{4w}\right] = \frac{2a}{\pi}\ln\left[\tfrac{1}{2}\sqrt{w} + \tfrac{1}{2}\frac{1}{\sqrt{w}}\right] \quad (10.2.53)$$

The problem of the distribution of potential about the edge of a plate in a variable condenser can be investigated by relating F and w so that the plate ADC is at zero potential and the plates AB and BC are at potential V_0. This is done by setting

$$w = e^{(\pi iF/V_0)}; \quad dF/dw = V_0/i\pi w$$

which eventually gives us

$$z = \frac{2a}{\pi} \ln\left[\cosh\left(\frac{\pi i F}{2V_0}\right)\right]; \quad F = \frac{2V_0}{\pi i} \cosh^{-1}[e^{\pi z/2a}]$$
$$x = \frac{a}{\pi} \ln\left[\cosh^2\left(\frac{\pi\chi}{2V_0}\right)\cos^2\left(\frac{\pi\psi}{2V_0}\right) + \sinh^2\left(\frac{\pi\chi}{2V_0}\right)\sin^2\left(\frac{\pi\psi}{2V_0}\right)\right] \quad (10.2.54)$$
$$y = -\frac{2a}{\pi} \tan^{-1}\left[\tan\left(\frac{\pi\psi}{2V_0}\right)\tanh\left(\frac{\pi\chi}{2V_0}\right)\right]$$

The equipotential lines are shown in Fig. 10.21; we see again that the field is uniform in the regions near A and C, that it vanishes rapidly as

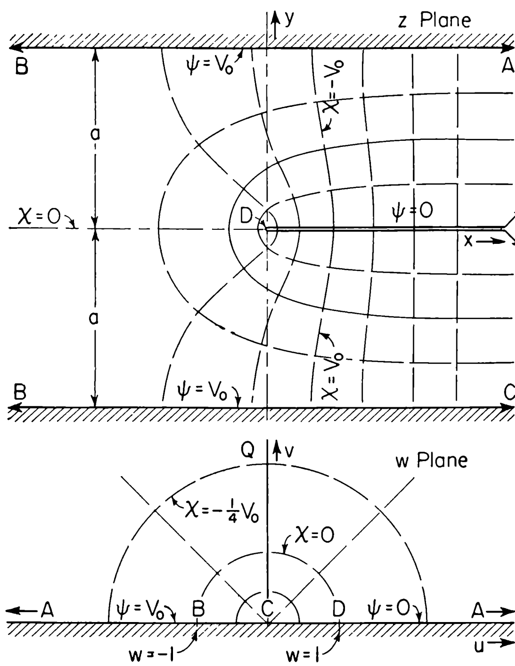

Fig. 10.21 Schwarz-Christoffel transformation for condenser AB-BC-CDA. Optimum shape for inner conductor shown by solid line $CQ(\psi = \frac{1}{2}V_0)$.

we go from D to B, and that it concentrates at the edge D. The magnitude of the electric intensity is, of course,

$$\left|\frac{dF}{dz}\right| = \frac{V_0/a}{\sqrt{1 + e^{-2\pi x/a} - 2e^{-\pi x/a}\cos(\pi y/a)}} = \frac{V_0}{a}\left|\cot\left(\frac{\pi F}{aV_0}\right)\right| \quad (10.2.55)$$

which emphasizes these points again: for positive x, $|dF/dz| \to (V_0/a)$, and for negative x, $|dF/dx| \to (V_0/a)e^{\pi x/a}$, which vanishes rapidly as x gets large negative.

The charge density on the central plate ($y = 0$) is

$$\frac{2}{4\pi}\left|\frac{dF}{dz}\right|_{y=0} = \frac{(V_0/2\pi a)}{1 - e^{-\pi x/a}}$$

where we have included the charge on both sides of the plate.

A plate of infinitesimal thickness along the positive real z axis has a surface field intensity V_0/a far from the edge, rising to infinity at the edge. A plate of finite thickness, with shape corresponding to the curve $\psi = \psi_0$ ($\psi_0 < V_0$) has a rounded edge instead of a sharp one and has a thickness, well away from the edge ($|\chi|$ large), of $2\psi_0/V_0$, so that the air gap is only $a[1 - (\psi_0/V_0)]$ instead of a. The field intensity at this surface is

$$\left|\frac{dF}{dz}\right|_{\psi=\psi_0} = \frac{V_0}{a}\sqrt{\frac{\cosh^2(\pi\chi/2V_0) - \sin^2(\pi\psi_0/2V_0)}{\sinh^2(\pi\chi/2V_0) + \sin^2(\pi\psi_0/2V_0)}} \qquad (10.2.56)$$

which equals V_0/a at some distance from the edge ($|\chi|$ large) and which, if $\psi_0 < \frac{1}{2}V_0$, has a maximum value of $(V_0/a)\cot(\pi\psi_0/2V_0)$ at the edge, $\chi = 0$. If we wish to design a condenser plate so that there is least chance of breakdown of the dielectric, we wish to adjust ψ_0 so that the intensity at the edge is no larger than it is away from the edge.

Suppose that we try a number of plates of different shapes, corresponding to different values of ψ_0 and asymptotic thicknesses $2b = (2\psi_0/V_0)$, so that the maximum intensity anywhere on the surface is as small as possible. We shall adjust V_0, the potential of the outer planes, so that the potential *difference* $V_0 - \psi_0$ is always V, a constant. We have $\psi_0 = V_0 b/a$ and therefore $V_0 = aV/(a - b)$, where $a - b$ is the asymptotic gap width between central conductor and upper or lower plate. The intensity at the edge of the central conductor ($\chi = 0$) is $[V/(a - b)]\cot(\pi b/2a)$, whereas the asymptotic value of the surface intensity ($|\chi|$ large) is $[V/(a - b)]$. In Fig. 10.22 are plotted these two limiting intensities as functions of b/a, the fraction of the gap which is filled with central conductor. We see that, if the central plate is too narrow, the edge field is too high and, if the central plate is too wide, the asymptotic field is too large; but if the plate just fills half the gap, *i.e.*, if $\psi_0 = \frac{1}{2}V_0$, both fields are equal and the intensity is $2V/a$ every-

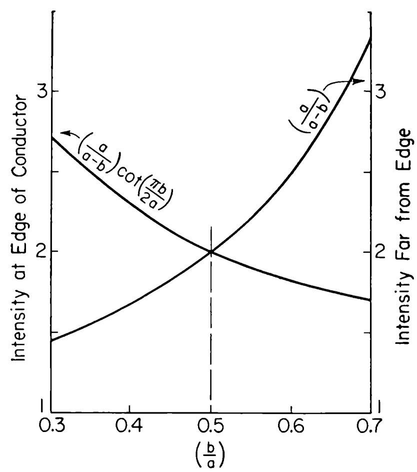

Fig. 10.22 Determination of optimum shape of condenser plate.

where on the surface of the central plate. This optimum cross section is shown in Fig. (10.21); it is the shape to use to minimize the possibility of breakdown of the dielectric.

Other Rectangular Shapes. Another "polygon" of use in microwave calculations (for "elbows" in a wave guide, for instance) is that shown in Fig. 10.23. Starting from A and keeping the "interior" on the left as we go around, as usual, the angles are $\frac{1}{2}\pi$ at $w = -1$, π at $w = 0$, and $-\frac{1}{2}\pi$ at $w = \alpha^2$. The transformation is then

$$z - z_0 = A \int \sqrt{\frac{\alpha^2 - w}{1 + w}} \frac{dw}{w} = 2A \int \left[\frac{1}{t^2 + 1} - \frac{\alpha^2}{\alpha^2 - t^2}\right] dt$$
$$= -2A\alpha \tanh^{-1} \sqrt{\frac{\alpha^2 - w}{\alpha^2 + \alpha^2 w}} + 2A \tan^{-1} \sqrt{\frac{\alpha^2 - w}{1 + w}}$$

where $t = \sqrt{(\alpha^2 - w)/(w + 1)}$. Setting $w = \alpha^2$ in this, we see that $z_0 = 0$. Going back along the real w axis from α^2 makes the hyperbolic

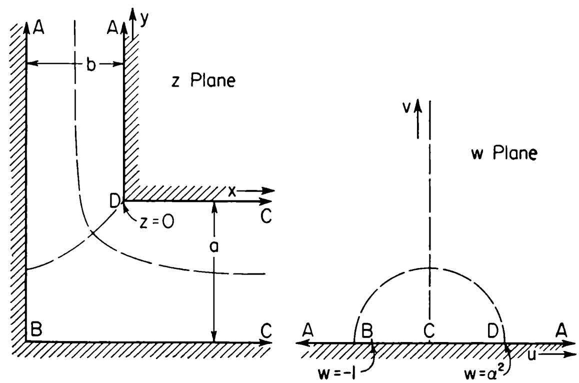

Fig. 10.23 Schwarz-Christoffel transformation for "elbow bend."

tangent increase along its real axis, until at $w = 0$, $\tanh^{-1}(1) = \infty$. Decreasing w just below zero makes the inverse hyperbolic tangent suddenly add a term $-\frac{1}{2}i\pi$, and its real part begins again to diminish, until, at $w = -1$, $\tanh^{-1}(\infty) = -\frac{1}{2}i\pi$ and $\tan^{-1}(\infty) = \frac{1}{2}\pi$. Consequently at $w = 1$, which is supposed to correspond to $z = -b - ia$, we have $z = + i\pi A\alpha + \pi A$. Therefore $A = -(b/\pi)$ and $A\alpha = -(a/\pi)$, and our transformation is

$$z = \frac{2a}{\pi} \tanh^{-1} \sqrt{\frac{a^2 - wb^2}{a^2(1 + w)}} - \frac{2b}{\pi} \tan^{-1} \sqrt{\frac{a^2 - wb^2}{b^2(1 + w)}}$$
$$\frac{dz}{dw} = -\frac{1}{\pi w} \sqrt{\frac{a^2 - wb^2}{1 + w}} \tag{10.2.57}$$

We next make the transformation

$$w = e^{\pi i F/V_0}; \quad F = \psi + i\chi = \frac{V_0}{\pi i} \ln w; \quad \frac{dF}{dw} = \frac{V_0}{\pi i w}$$

which corresponds to the potential-force distribution for the surface ADC at zero potential and the surface ABC at potential V_0. The field strength at the surface is

$$\left|\frac{dF}{dz}\right| = \left|\frac{dF/dw}{dz/dw}\right| = V_0\left|\sqrt{\frac{1+w}{a^2 - b^2w}}\right|; \quad \text{for } w \text{ real}$$

For instance, well away from the corner, toward C, the charge density is $V/4\pi a$; toward A, it is $V/4\pi b$. At $w = a^2/b^2 = \alpha^2$, the charge density becomes infinite, and at $w = -1$ (point B), the charge density is zero.

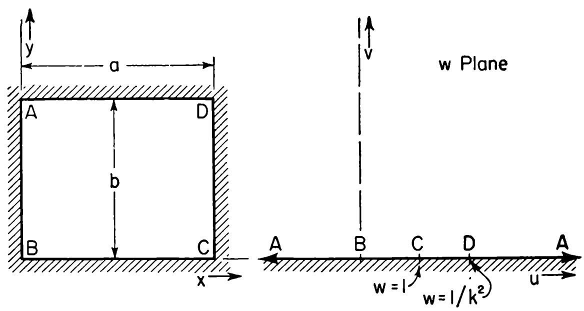

Fig. 10.24 Schwarz-Christoffel transformation for inside of rectangle.

For irrotational flow of an incompressible fluid along the channel we use the auxiliary transformation

$$w = e^{\pi F/Q}; \quad F = \psi + i\chi = (Q/\pi)\ln w$$

where Q is the total flow of fluid between the boundaries, ψ is the velocity potential, and χ the flow function. For the case $a = b$, we have

$$z = \frac{2a}{\pi}\left[\tanh^{-1}\frac{\pi}{\sqrt{\tanh(F/2Q)}} - \tan^{-1}\sqrt{\tanh(\pi F/2Q)}\right] \tag{10.2.58}$$

and the magnitude of the velocity at the point designated by $F = \psi + i\chi$ is

$$v = (Q/a)\left|\sqrt{\coth(\pi F/2Q)}\right|$$

Other physical problems concerned with similar boundaries may also be solved once the basic transformation (10.2.57) is known.

Finally, we can use the Jacobi elliptic functions to generate the transformation of the inside of a rectangle into the upper half of the w plane. Referring to Fig. 10.24, we see that

$$z - z_0 = kA\int\frac{dw}{\sqrt{w(1-w)(1-k^2w)}} = 2kA\,\mathrm{sn}^{-1}(\sqrt{w},k)$$

Since sn is zero, according to page 488, when w is zero, z_0 is zero. Also, when $w = 1$, sn^{-1} equals K and z is supposed to be a, so that $A = a/2kK$

and we can write the transformation as

$$w = [\mathrm{sn}(Kz/a,k)]^2 \tag{10.2.59}$$

When $z = a + ib$, w is supposed to be i/k^2, but sn $= 1/k$ when its argument is $K + iK'$; consequently k must be chosen so that $Kb/a = K'$ or $K'/K = b/a$, as given in the table on page 487.

As before, the potential distribution inside the rectangle when the portion AB is at potential V_0 and the rest at zero potential is given by

$$w = e^{\pi iF/V_0};\quad F = \psi + i\chi = \frac{2V_0}{\pi i}\ln[\mathrm{sn}(Kz/a,k)] \tag{10.2.60}$$

The lines of force of this problem are equivalent to the potential lines of a problem where opposite line charges are placed at the two corners A and B. Incidentally, this solution should be compared with Eq. (10.1.7).

On the other hand, the potential due to a line charge at the point z_0, inside the rectangle, with the whole boundary at zero potential, is obtained from the real part of the function

$$F = 2q\ln\left(\frac{w - \bar{w}_0}{w - w_0}\right) = 2q\ln\left\{\frac{\mathrm{sn}^2(Kz/a,k) - \mathrm{sn}^2(K\bar{z}_0/a,k)}{\mathrm{sn}^2(Kz/a,k) - \mathrm{sn}^2(Kz_0/a,k)}\right\} \tag{10.2.61}$$

which is to be compared with Eq. (10.2.47).

After this interesting excursion into special methods for the solution of potential problems in two dimensions, we must return to the main theme of our discussion, the development of eigenfunction solutions appropriate to various coordinate solutions and the solution of particular problems by appropriate series of these eigenfunctions.

10.3 *Solutions for Three Dimensions*

It is with some regret that we turn now to the solution of Laplace's and Poisson's equations in three dimensions. The extraordinarily apt way in which the properties of functions of a complex variable fit our needs for solutions in two dimensions has perhaps been too easy an introduction to our task. For more dimensions and other equations the solutions will nearly always turn out to be infinite series or integrals, which only seldom will be rapidly convergent, and in general our task will be more difficult. As we saw in Chap. 5, even the Laplace equation separates in but a few three-dimensional coordinate systems, and most solutions which may be calculated out to give a numerical answer are limited to these systems.

In the case of the three-dimensional Laplace equation

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = 0$$

separation is achieved for all the coordinates, listed on pages 656 to 664, for which the wave equation separates and, in addition, for the cyclidal coordinates, of which the bispherical and toroidal coordinates are the most useful. These systems fall into three general classes: cylindrical coordinates, rotational coordinates, and a third, less symmetric class. The cylindrical coordinates are generated by taking a two-dimensional coordinate system in the $x - y$ plane and translating it along the z axis, thus generating cylindrical surfaces with axes parallel to the z axis. The particular solutions for such systems, which are independent of z, were studied in the first two sections of this chapter; more general solutions can be devised but are not of very great interest. Consequently we shall study solutions for cartesian and circular cylindrical coordinates only and leave the other systems to be discussed in connection with the wave equation, where they are of more importance.

Rotational coordinates are formed by taking two-dimensional coordinates, in a plane passing through the z axis, and rotating this plane about the z axis. Circular cylinder coordinates (the only system which is at once cylindrical and rotational) are formed by rotating rectangular coordinates, spherical coordinates by rotating polar coordinates, spheroidal coordinates by rotating elliptic coordinates, and so on. Toroidal and bispherical coordinates may also be included in this class.

The third, more general, forms include paraboloidal and ellipsoidal coordinates, which have neither cylindrical nor rotational symmetry.

We shall cling to our useful complex variables as long as possible. For instance, any function of a complex variable X, corresponding to a plane parallel to the z axis and at an angle u with respect to the x axis,

$$X = z + i(x \cos u + y \sin u) \tag{10.3.1}$$

is a solution of Laplace's equation, and any combination of such solutions

$$\psi(x,y,z) = \int_0^{2\pi} F(X)f(u)\,du \tag{10.3.2}$$

is also a solution of Laplace's equation. We could, of course, try different functions for F and f and see what sort of solutions we obtain. But we are trying to avoid the process of finding solutions by backing into them, so to speak, and so we shall, instead, obtain solutions for the various separable coordinates by solving the separated equations. In many cases, however, we shall find that the resulting solution can be expressed in terms of an integral of the general form of (10.3.2), and this integral representation will turn out to be exceedingly useful in helping us sum series of eigenfunctions and in expressing one eigenfunction in terms of others.

The integral form of Eq. (10.3.2) is a generalization of the integral representations which were discussed in Sec. 5.3 and partakes of the same unifying power possessed by the integrals discussed there. The form is particularly simple and useful for rotational coordinate systems, where the function $f(u)$ turns out to be either $\cos(mu)$ or $\sin(mu)$, with m an integer. The functions F, for a sequence of eigenfunctions suitable for a given coordinate system, then turn out to be a sequence of functions suitable for a series expansion, such as the various integral powers of X, or a sequence of one-dimensional eigenfunctions of X, etc. The relations between the functions F for different coordinate systems constitute a means of establishing relationships between solutions for the different coordinates.

Integral Form for the Green's Function. As an example of the integral form of Eq. (10.3.2) and also to provide us with a few formulas which will be useful all through this section, we shall perform a few maneuvers with the Green's function for the Laplace equation, suitable for unlimited three dimensions. We have already seen, in Chap. 7, that the Green's function for this case is

$$G(x,y,z|x_0,y_0,z_0) = 1/R; \quad R^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 \tag{10.3.3}$$

but this turns out to be a particularly obdurate form for G when we come to expand it in terms of various eigenfunction solutions.

In Chap. 7, of course, we developed general techniques for obtaining eigenfunction series for Green's functions, and our series could be thus obtained simply by turning an algebraic crank. It is probably more useful here, however, to go through the "crank assembly" once more, in this simple (and rather specialized) case, particularly since we shall be able to introduce a different point of view from that used in Chap. 7.

We start with a solution of the equation

$$\nabla^2\psi = -4\pi\delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \tag{10.3.4}$$

in a rectangular enclosure of finite size of sides a, b, and c. If the boundary conditions are simply that the eigenfunctions be periodic at the boundaries (*i.e.*, that ψ at one wall perpendicular to the x axis equal ψ at the other parallel wall, etc.), then the eigenfunctions are

$$\psi = \sum_{l,m,n} A_{lmn}e^{2\pi i\Omega_{lmn}}; \quad \Omega_{lmn} = \left(\frac{lx}{a}\right) + \left(\frac{my}{b}\right) + \left(\frac{nz}{c}\right)$$

where the coefficients A are adjusted so that ψ satisfies Eq. (10.3.4).

To calculate the coefficients, we first put the series into the equation

$$-4\pi^2 \sum_{\lambda\mu\nu} A_{\lambda\mu\nu}\left[\left(\frac{\lambda}{a}\right)^2 + \left(\frac{\mu}{b}\right)^2 + \left(\frac{\nu}{c}\right)^2\right] e^{2\pi i\Omega_{\lambda\mu\nu}}$$
$$= -4\pi\delta(x - x_0)\delta(y - y_0)\delta(z - z_0)$$

Multiplying both sides by $\exp(-2\pi i\Omega_{lmn})$ and integrating over the volume of the enclosure give us

$$-4\pi^2 abcA_{lmn}\left[\left(\frac{l}{a}\right)^2+\left(\frac{m}{b}\right)^2+\left(\frac{n}{c}\right)^2\right] = -4\pi e^{-2\pi i\Omega'_{lmn}}$$

where $\Omega'_{lmn} = (lx_0/a) + (my_0/b) + (nz_0/c)$. The solution of Eq. (10.3.4) which satisfies the periodicity boundary conditions is therefore

$$\psi = \sum_{l,m,n} \frac{(1/\pi abc)}{[(l/a)^2 + (m/b)^2 + (n/c)^2]} \exp\left\{2\pi i\left[\frac{l}{a}(x-x_0) + \frac{m}{b}(y-y_0) + \frac{n}{c}(z-z_0)\right]\right\} \tag{10.3.5}$$

We now let a, b, c go to infinity, pausing long enough to remark that a related solution for a finite enclosure will be discussed somewhat later. We set $2\pi l/a = k_x$, $2\pi m/b = k_y$, $2\pi n/c = k_z$. As a, b, and c go to infinity, the number of integral values of l corresponding to the element between k_x and $k_x + dk_x$ will be $(a/2\pi)\,dk_x$ on the average, and similarly for dk_y and dk_z. Consequently in the limit the Green's function changes from the sum of Eq. (10.3.5) to the integral

$$\frac{1}{R} = \frac{1}{2\pi^2}\int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \int_{-\infty}^{\infty} dk_z \left\{\frac{e^{i[k_x(x-x_0)+k_y(y-y_0)+k_z(z-z_0)]}}{[k_x^2+k_y^2+k_z^2]}\right\} \tag{10.3.6}$$

This is a perfectly satisfactory integral representation for $1/R$, but still more useful forms may be obtained by carrying out some of these integrations in "k space." For instance, k_x, k_y, k_z may be considered as the components of a vector $\mathbf{K}$, and $(x - x_0)$, etc., as components of a vector $\mathbf{R}$. The quantity in braces is then $e^{i\mathbf{K}\cdot\mathbf{R}}/K^2$, and the volume element is $K^2\,dK \sin\vartheta\, d\vartheta\, d\varphi$, where ϑ is the angle between $\mathbf{R}$ and $\mathbf{K}(\mathbf{K}\cdot\mathbf{R} = KR\cos\vartheta)$. After integrating over the axial angle φ, we have

$$\frac{1}{R} = \frac{1}{\pi}\int_0^{\infty} dk \int_{-1}^{1} e^{iKRw}\,dw = \frac{2}{\pi R}\int_0^{\infty} \sin(KR)\frac{dK}{K} = \frac{1}{R};\quad w = \cos\vartheta \tag{10.3.7}$$

But a more interesting form is obtained if we use circular cylindrical coordinates in "k space," setting $k_x = k\cos u$ and $k_y = k\sin u$ and integrating over k_z. The integrand

$$\left[\frac{(1/\pi)}{k^2+k_z^2}\right] e^{ikG+ik_z(z-z_0)};\quad G = (x-x_0)\cos u + (y-y_0)\sin u$$

has simple poles at $k_z = \pm ik$, with residue $(1/2\pi ik)e^{ikG-k(z-z_0)}$ at $k_z = ik$ and residue $-(1/2\pi ik)e^{ikG+k(z-z_0)}$ at $k_x = ik$. The infinite integral over k_z may be changed, by the usual methods, into a contour integral about

ik if $(z - z_0)$ is positive or about $-ik$ (in a negative direction) if $(z - z_0)$ is negative (see page 415). Consequently another integral representation for the Green's function is

$$\begin{aligned}\frac{1}{R} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dk_x}{k}\int_{-\infty}^{\infty} dk_y\, e^{-k|z-z_0|+ik[(x-x_0)\cos u+(y-y_0)\sin u]} \\ &= \frac{1}{2\pi}\int_0^{2\pi} du \int_0^{\infty} dk\, e^{-k[X_0-X]} = \frac{1}{2\pi}\int_0^{2\pi}\frac{du}{[X_0 - X]}; \\ & \qquad k_x = k\cos u;\quad k_y = k\sin u \end{aligned} \tag{10.3.8}$$

where, if $z < z_0$, $X = z + i(x\cos u + y\sin u)$, $X_0 = z_0 + i(x_0\cos u + y_0\sin u)$ and, if $z > z_0$, $X = -z + i(x\cos u + y\sin u)$, $X_0 = -z_0 + i(x_0\cos u + y_0\sin u)$.

Elementary integration of this last integral, over u, will verify that it equals $1/R$. In fact, we could as well have obtained Eq. (10.3.8) by a lucky guess, aided by looking at tables of integrals, but it is somewhat more satisfactory to obtain the result by straightforward reasoning. We could, of course, have obtained the first line of Eq. (10.3.8) by using the general formulas of Chap. 7, in particular Eq. (7.2.42), but it is probably more enlightening to go over the whole process of reasoning again in this particularly useful case.

At any rate we have shown that the Green's function $1/R$ can be expressed as a function of the type of Eq. (10.3.2), with quantities of the sort defined in Eq. (10.3.1). Integral (10.3.8) will be useful when we come to calculating the expansion of $1/R$ in series of eigenfunctions suitable for different coordinates.

Solutions in Rectangular Coordinates. Solutions suitable for cartesian coordinates are of the general type, $e^{-kx}\sin(\pi my/a)\sin(\pi nz/b)$, $\sin(\pi mx/a)\sinh(ky)\sin(\pi nz/b)$, etc., where $k^2 = (\pi m/a)^2 + (\pi n/b)^2$. For instance, the potential distribution inside a rectangular enclosure of sides a, b, c, when the side $z = 0$ is at potential $\psi_0(x_0,y_0)$ and the other five sides are at zero potential, is

$$\begin{aligned}\psi = \frac{4}{ab}\sum_{m,n}\left[\int_0^a dx_0\int_0^b dy_0\,\psi_0\sin\left(\frac{\pi m x_0}{a}\right)\sin\left(\frac{\pi n y_0}{b}\right)\right]\cdot \\ \cdot\left\{\frac{\sinh[k_{mn}(c-z)]}{\sinh[k_{mn}c]}\right\}\sin\left(\frac{\pi m x}{a}\right)\sin\left(\frac{\pi n y}{b}\right)\end{aligned} \tag{10.3.9}$$

where $k_{mn}^2 = (\pi m/a)^2 + (\pi n/b)^2$. As a comparison with Eq. (10.1.7) we compute the case when $\psi_0 = 1$ (the base is kept at unit potential, the other sides at zero):

$$\psi = \frac{16}{\pi^2}\sum_{m,n}\frac{\sinh[k_{mn}(c-z)]}{\sinh[k_{mn}c]}\,\frac{\sin\left[\frac{\pi x}{a}(2m+1)\right]\sin\left[\frac{\pi y}{b}(2n+1)\right]}{(2m+1)(2n+1)} \tag{10.3.10}$$

where, this time, $k_{mn} = (\pi/a)^2(2m + 1)^2 + (\pi/b)^2(2n + 1)^2$. Finally, when c is very much larger than a or b, this series reduces to

$$\psi = \frac{16}{\pi^2} \sum_{m,n} \exp\left[-z \sqrt{\left(\frac{\pi}{a}\right)^2 (2m+1)^2 + \left(\frac{\pi}{b}\right)^2 (2n+1)^2} \right] \cdot \\ \cdot \frac{\sin[(\pi x/a)(2m+1)] \sin[(\pi y/b)(2n+1)]}{(2m+1)(2n+1)} \tag{10.3.11}$$

which is also the temperature distribution in a long bar of rectangular cross section, heated to unit temperature at the flat end and kept at zero temperature at the sides. The exponential ensures that each term diminishes as z increases, *i.e.*, as we go away from the heated end. The terms for higher m and n die out more rapidly, and for sufficiently large values of z, the temperature is approximately

$$\psi \simeq (16/\pi^2)e^{-\pi z\sqrt{a^2+b^2}} \sin(\pi x/a) \sin(\pi y/b)$$

Series (10.3.11) is to be compared with the analogous two-dimensional series, given in Eq. (10.1.8). In the two-dimensional case the series could be summed to give a closed form for the solution. In the three-dimensional case, because of the square-root term in the exponential, the series does not represent any known, simple function of x, y, and z.

Another example of a solution appropriate to rectangular coordinates is the case of the steady-state flow of a nonviscous fluid inside a duct of rectangular cross section. The solution representing steady flow with velocity v_0 in the direction (parallel to the axis of the duct) is the velocity potential

$$\psi = -v_{\text{av}} z$$

However, the velocity distribution at the beginning of the duct ($z = 0$) may not be uniform, and we must correct the formula for uniform flow to allow for these initial irregularities. Suppose that the z component of the velocity at $z = 0$ is $v_0(x,y)$, which is a function of the point (x,y) of the initial, cross-sectional plane. The solution of Laplace's equation which fits the boundary conditions:

$$-(\partial\psi/\partial z) = v_0(x,y), \quad \text{at } z = 0; \quad \partial\psi/\partial x = 0, \quad \text{at } x = 0, \quad x = a \\ \partial\psi/\partial y = 0, \quad \text{at } y = 0, \quad y = b; \quad -(\partial\psi/\partial z) \to v_{\text{av}}, \quad \text{for } z \to \infty$$

must be a sum of eigenfunctions of the sort

$$\psi = \sum_{m,n=0}^{\infty} A_{mn} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \exp\left[-\pi z \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \right] \tag{10.3.12}$$

where the term for $m = n = 0$ is just $-A_{00}z$. This fits all but the boundary condition at $z = 0$. To fit this we set

$$
\begin{aligned}
A_{00} &= v_{\text{av}} = \frac{1}{ab}\int_0^a dx \int_0^b dy\, v_0(x,y) \\
A_{m0} &= \frac{2}{\pi m b}\int_0^a dx \cos\left(\frac{m\pi x}{a}\right) \int_0^b dy\, v_0(x,y) \\
A_{mn} &= \frac{(4/\pi ab)}{\sqrt{(m/a)^2 + (n/b)^2}} \int_0^a dx \cos\left(\frac{\pi m x}{a}\right) \int_0^b dy \cos\left(\frac{\pi n y}{b}\right) v_0(x,y)
\end{aligned}
\tag{10.3.13}
$$

with a formula for A_{0n} similar to A_{m0}, with y substituted for x and n for m.

We notice that all the correction terms for m or $n \neq 0$ have an exponential factor ensuring their eventual vanishing for large z. Thus the irregularities in the flow, caused by the initial conditions, gradually "even out," so that, far from the origin, the flow is uniform across the duct, having the average velocity v_0.

For instance, if the air is admitted into the duct at $z = 0$ through a small hole in a flat plate which caps the $z = 0$ end, the hole being at the point (x_0,y_0) on the plate, then $v_0 = Q\delta(x - x_0)\delta(y - y_0)$ and

$$
\begin{aligned}
A_{00} &= \frac{Q}{ab};\quad A_{m0} = \frac{2Q}{\pi m b}\cos\left(\frac{\pi m x_0}{a}\right); \\
A_{mn} &= \frac{(4Q/\pi ab)}{\sqrt{(m/a)^2 + (n/b)^2}} \cos\left(\frac{\pi m x_0}{a}\right) \cos\left(\frac{\pi n y_0}{b}\right)
\end{aligned}
$$

Cases where a duct of one size is joined to a duct of another size at $z = 0$ (or where a septum with a hole of arbitrary shape is placed across the duct at $z = 0$) may be solved by setting up a series expression for the flow for $z < 0$ and another series for the flow for $z > 0$ and joining the series at $z = 0$. We shall postpone discussions of such joining problems until the chapter on wave motion, where similar problems are encountered.

The Green's function for the interior of a rectangular enclosure, with boundary condition that ψ be zero at the walls, is a three-dimensional equivalent of an elliptic function. The series may be computed just as Eq. (10.3.5) was computed, except that sine functions are used. The result is

$$
\begin{aligned}
G(x,y,z|x_0,y_0,z_0) = \sum_{l,m,n} \frac{(8/\pi abc)}{(l/a)^2 + (m/b)^2 + (n/c)^2} \sin\left(\frac{\pi l x_0}{a}\right) \sin\left(\frac{\pi m y_0}{b}\right) \cdot \\
\cdot \sin\left(\frac{\pi n z_0}{c}\right) \sin\left(\frac{\pi l x}{a}\right) \sin\left(\frac{\pi m y}{b}\right) \sin\left(\frac{\pi n z}{c}\right)
\end{aligned}
\tag{10.3.14}
$$

The temperature distribution for a rectangular block of metal with a uniform distribution of heat sources inside and the surface kept at zero is then proportional to G, integrated over the whole interior (in the x_0, y_0, z_0 coordinates). The result eventually becomes

$$\sum_{\substack{l,m,n\\ \text{odd}}} \frac{(64/\pi^4 lmn)}{(l/a)^2 + (m/b)^2 + (n/c)^2} \sin\left(\frac{\pi l x}{a}\right) \sin\left(\frac{\pi m y}{b}\right) \sin\left(\frac{\pi n z}{c}\right)$$

which converges well enough so that values may be computed without undue complication (the sum only includes odd values of l,m,n).

Solutions in Circular Cylindrical Coordinates. In the coordinates r, ϕ, z, the Laplace equation has the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} = 0$$

This separates into the equations

$$\psi = \Phi(\phi)R(r)Z(z); \quad (d^2\Phi/d\phi^2) + m^2\Phi = 0$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left(k^2 - \frac{m^2}{r^2}\right)R = 0; \quad \frac{d^2Z}{dz^2} - k^2 Z = 0$$

The solution of the first equation, for the ϕ factor, is either $\sin(m\phi)$ or $\cos(m\phi)$. If there are no boundaries for ϕ = constant but ϕ is allowed to go from 0 to 2π, then m must be zero or an integer. The equation for Z also results in trigonometric or hyperbolic functions, depending on whether k is imaginary or real. If both boundaries perpendicular to z (the ends of the cylinders) have homogeneous boundary conditions (zero value, for instance, or zero slope) and boundary values are to be fitted along the curved surface, then k must be imaginary. If the values are specified at one or both ends, k should be real, for the factor Z may then be a hyperbolic function.

The equation for R is a Bessel equation with solutions $J_m(kr)$ and $N_m(kr)$ [see Eqs. (5.3.63) and (5.3.76) and page 1321] if k is real. The Bessel function J_m is finite at the origin, so it must be used if the cylindrical axis is inside the boundary surface, where the field is to be computed. The Neumann function N_m is the second solution, being infinite at the origin.

When k is imaginary, we use the hyperbolic Bessel functions K_m and I_m (see page 1323). I_m is finite at the origin but infinite at infinity; K_m is infinite at the origin but zero at infinity.

As an example of the use of these solutions, we consider the steady-state distribution of temperature inside a cylindrical bar, having the two ends ($z = 0$ and $z = l$) kept at zero temperature and the circular side ($r = a$) kept at temperature $T_a(z)$. The series representation for this solution is

$$T = \sum_{n=1}^{\infty} A_n \frac{I_0(\pi n r/l)}{I_0(\pi n a/l)} \sin\left(\frac{\pi n z}{l}\right); \quad A_n = \frac{2}{l}\int_0^l T_a(z) \sin\left(\frac{\pi n z}{l}\right) dz$$

If the surface temperature varies with ϕ as well as with z, the series involves values of m larger than zero and the integrals for the coefficients involve integrals over ϕ as well as z.

If the curved sides ($r = a$) are kept at zero temperature and the variable boundary conditions are to be applied to one or both flat ends ($z = 0$, $z = l$), we must use Bessel functions with real values of k. We set up an orthogonal set of eigenfunctions for each value of m by solving the equation $J_m(ka) = 0$. The eigenvalues, roots of the equation $J_m(\pi\beta) = 0$, can be labeled by the subscripts m and n, the first subscript indicating the order m of the Bessel function and the second labeling the particular root, $n = 1$ being the lowest and $\beta_{m,n+1} > \beta_{mn}$. Values for the first few roots are given on page 1565.

The sequence $J_m(\pi\beta_{mn}r/a)$, for different n's, is a complete set of eigenfunctions, with all the corresponding properties of orthogonality, etc. For instance, since $J_m(\pi\beta_{mn}) = 0$, we have from page 1322

$$\begin{aligned}\int_0^a J_m\left(\frac{\pi\beta_{mn}r}{a}\right) J_m\left(\frac{\pi\beta_{ms}r}{a}\right) r\,dr &= 0; \qquad s \neq n \\ &= -\tfrac{1}{2}a^2 J_{m-1}(\pi\beta_{mn})J_{m+1}(\pi\beta_{mn}) = \tfrac{1}{2}a^2[J_{m+1}(\pi\beta_{mn})]^2; \quad s = n \qquad (10.3.15)\end{aligned}$$

Therefore any function $\psi_0(r,\phi)$, in the range $(0 \leq r \leq a;\ 0 \leq \phi \leq 2\pi)$, may be represented by the series

$$\psi_0(r,\phi) = \sum_{m,n} \frac{\epsilon_m}{\pi} \left\{ \int_0^{2\pi} d\theta \cos[m(\phi - \theta)] \int_0^a u\,du\,\psi_0(u,\theta) J_m\left(\frac{\pi\beta_{mn}u}{a}\right)\right\} \cdot [aJ_{m+1}(\pi\beta_{mn})]^{-2} J_m(\pi\beta_{mn}r/a)$$

where $\epsilon_0 = 1$, $\epsilon_m = 2$ for $m = 1, 2, 3, \ldots$.

The potential inside a cylinder of radius a and of length l, with potential zero at all faces except the end $z = 0$, where it is $\psi_0(r,\phi)$, is therefore given by the double series

$$\psi(r,\phi,z) = \sum_{m,n} \left\{ \int_0^{2\pi} \cos[m(\phi - \theta)]\,d\theta \int_0^a \psi_0(u,\theta) J_m\left(\frac{\pi\beta_{mn}u}{a}\right) u\,du \right\} \cdot \left\{ \frac{\epsilon_m \sinh[(\pi\beta_{mn}/a)(l - z)]}{\pi[aJ_{m+1}(\pi\beta_{mn})]^2 \sinh[\pi\beta_{mn}l/a]} \right\} J_m\left(\frac{\pi\beta_{mn}r}{a}\right) \qquad (10.3.16)$$

For an infinite spread in the r direction this series becomes an integral, as shown on page 765. The Fourier-Bessel integral

$$F(r) = \int_0^\infty u\,du \int_0^\infty F(w) J_m(ur) J_m(uw)\,w\,dw \qquad (10.3.17)$$

is useful, and the extension of this to polar coordinates (r,ϕ), by means of Eq. (5.3.66), is also important:

$$\psi_0(r,\phi) = \frac{1}{2\pi} \int_0^\infty u\,du \int_0^\infty w\,dw \int_0^{2\pi} d\theta\,\psi_0(w,\theta) J_0(uR) \qquad (10.3.18)$$

where $R^2 = r^2 + w^2 - 2rw\cos(\phi - \theta)$.

For instance, the potential above the plane $z = 0$, which is held at zero potential except for the circular region $(0 \le r < a)$, which is held at potential V_0, is

$$\begin{aligned}\psi(r,\phi,z) &= V_0 \int_0^\infty e^{-uz} u\,du \int_0^a J_0(ur)J_0(uw)w\,dw \\ &= V_0 a \int_0^\infty e^{-uz} J_1(ua)J_0(ur)\,du \qquad (10.3.19)\end{aligned}$$

The asymptotic expansion for large values of z can be obtained by using the first terms in the series expansions for J_0 and J_1:

$$\psi \simeq \tfrac{1}{2}V_0 a^2 \int_0^\infty e^{-uz} u\,du = \tfrac{1}{2}V_0\left(\frac{a^2}{z^2}\right); \quad z \gg r,\, a$$

A more general expression, good for $z \gg a$, but for any value of r, is obtained from the use of the formulas

$$\int_0^\infty e^{-uz}J_0(ur)\,du = \frac{1}{\sqrt{z^2+r^2}}; \quad \int_0^\infty e^{-uz}J_0(ur)u\,du = \frac{z}{(z^2+r^2)^{\frac{3}{2}}} \qquad (10.3.20)$$

The first is obtained from the integral representation of J_0, and the second by differentiating the first with respect to z. This second formula enables one to show that

$$\psi \simeq [zV_0a^2/2(z^2+r^2)^{\frac{3}{2}}]; \quad z \gg a$$

Solutions for Neumann conditions are needed to solve the problem of irrotational flow of an incompressible fluid through a circular duct of radius a. The solution for uniform flow is, of course, $\psi = -v_0 z$; the solutions for nonuniform flow in the tube are of the sort

$$e^{\pm(\pi\alpha_{mn}z/a)}\cos(m\phi)J_m(\pi\alpha_{mn}r/a)$$

where α_{mn} is the nth root of the equation

$$(d/d\alpha)[J_m(\pi\alpha)] = 0$$

Values of the first few α's are given on page 1565; we note that $\alpha_{0n} = \beta_{1n}$ but that all other α's differ from all other β's.

If the flow initiates at the $z = 0$ end, where the z component of the velocity is $v_0(r,\phi)$, the velocity potential along the tube, for $z > 0$, is

$$\begin{aligned}\psi = \psi_{00} + \sum_{m,n} &\left\{\int_0^{2\pi}\cos[m(\phi-\theta)]\,d\theta \int_0^a v_0(u,\theta)J_m\left(\frac{\pi\alpha_{mn}u}{a}\right)u\,du\right\} \cdot \\ &\cdot \left\{\frac{\epsilon_m \exp[-(\pi\alpha_{mn}z/a)]}{\pi^2 a\alpha_{mn}[J_m(\pi\alpha_{mn})]^2}\right\} J_m\left(\frac{\pi\alpha_{mn}r}{a}\right) \qquad (10.3.21)\end{aligned}$$

which is to be compared with Eq. (10.3.16). For large values of z all the terms but the one for $m = 0$, $n = 0$ become negligibly small:

$$\psi \xrightarrow[z\to\infty]{} \psi_{00} = \frac{-z}{\pi a^2}\left\{\int_0^{2\pi} d\theta \int_0^a v_0(u,\theta)u\,du\right\}$$

For instance, if the flow all comes from a small hole in the center of the plate at $z = 0$, the series becomes

$$\psi = -\frac{zQ}{\pi a^2} + \sum_{n=1}^{\infty}\left(\frac{Q}{\pi^2 a\alpha_{0n}}\right) e^{-\pi\alpha_{0n}z/a}[J_0(\pi\alpha_{0n})]^{-2}J_0\left(\frac{\pi\alpha_{0n}r}{a}\right)$$

The net flow down the tube is Q at any point along the tube, for each term in the series for $n > 0$ is orthogonal to the lowest eigenfunction $J_0(0) = 1$ and therefore the net flow represented by the series is zero, no matter what the value of z.

Integral Representation for the Eigenfunction. Since the eigenfunctions $\cos(m\phi)e^{kz}J_m(kr)$ are solutions of the Laplace equation, we should be able to express them in terms of an integral of the form of Eq. (10.3.2). A hint of the general form for $F(X)$ and $f(u)$ may be obtained from Eq. (5.3.65) and the discussion of page 1253. We have

$$\int_0^{2\pi} e^{ikr\cos u}\cos(mu)\,du = 2\pi i^m J_m(kr)$$

If we set $u = w - \phi$ and multiply both sides by e^{kz}, the exponent in the integral becomes

$$k[z + ir\cos(w - \phi)] = k[z + i(x\cos\phi + y\sin\phi)] = kX$$

according to Eq. (10.3.1). We therefore have that

$$\begin{aligned}\int_0^{2\pi} e^{kX}\cos(mu)\,du &= e^{kz}\int_0^{2\pi} e^{ikr\cos w}[\cos mw\cos m\phi + \sin mw\sin m\phi]\,dw \\ &= 2\pi i^m e^{kz}\cos(m\phi)J_m(kr) \qquad (10.3.22)\end{aligned}$$

the integral with $\sin(mw)$ being zero because of the antisymmetry of the sine function. We also have

$$\int_0^{2\pi} e^{kX}\sin(mu)\,du = 2\pi i^m e^{kz}\sin(m\phi)J_m(kr)$$

where, in both cases, $X = z + i(x\cos u + y\sin u)$.

These turn out to be extremely useful expressions, for from them we can express other elementary solutions in terms of the cylindrical eigenfunctions. For instance, according to Eq. (10.3.8), the Green's function

$$1/R = [(z - z_0)^2 + r^2 + r_0^2 - 2rr_0\cos(\phi - \phi_0)]^{-\frac{1}{2}}$$

is

$$\frac{1}{R} = \frac{1}{2\pi}\int_0^{\infty} dk \int_0^{2\pi} e^{-k|z-z_0| + ik[r\cos(u-\phi) - r_0\cos(u-\phi_0)]}\,du$$

This must be expandable in a Fourier series in $\cos[m(\phi - \phi_0)]$, where the coefficients of the series are

$$A_m = \frac{\epsilon_m}{4\pi^2}\int_0^\infty dk \int_0^{2\pi} dw \int_0^{2\pi} dt\, \cos(mt) e^{-k|z-z_0|+ik[r\cos w - r_0\cos(t+w)]}$$

where we have set $w = u - \phi$ and $t = \phi - \phi_0$. Next, letting $t + w = v$, we obtain

$$\begin{aligned} A_m &= \frac{\epsilon_m}{4\pi^2}\int_0^\infty dk\, e^{-k|z-z_0|}\int_0^{2\pi} e^{ikr\cos w}\cos(mw)\,dw \int_0^{2\pi} e^{-ikr_0\cos v}\cos(mv)\,dv \\ &= \epsilon_m i^{2m}\int_0^\infty dk\, e^{-k|z-z_0|}J_m(kr)(-1)^m J_m(kr_0) \end{aligned}$$

and so, finally, the series for the Green's function turns out to be

$$\frac{1}{R} = \sum_{m=0}^{\infty} \epsilon_m \cos[m(\phi - \phi_0)] \int_0^\infty J_m(kr)J_m(kr_0)e^{-k|z-z_0|}\,dk \qquad (10.3.23)$$

This expression is useful for calculations involving fields outside cylinders.

Green's Function for Interior Field. For the Green's function inside a cylinder we use a series similar to Eq. (10.3.14), a series solution of $\nabla^2\psi = -4\pi\delta$, where δ is zero except at the point r_0, ϕ_0, z_0 inside the cylinder and integrates to unity. We set

$$\psi = \sum_{m,n,s} A_{mns}\cos[m(\phi - \phi_0)]\sin\left(\frac{\pi n z}{l}\right) J_m\left(\frac{\pi\beta_{ms} r}{a}\right)$$

which goes to zero at $z = 0$, $z = l$, $r = a$, the boundaries of the cylinder. Inserting this in the equation for ψ we obtain

$$\pi^2 \sum A_{mns}\left[\left(\frac{n}{l}\right)^2 + \left(\frac{\beta_{ms}}{a}\right)^2\right]\cos[m(\phi - \phi_0)]\sin\left(\frac{\pi n z}{l}\right) J_m\left(\frac{\pi\beta_{ms} r}{a}\right) = 4\pi\delta$$

Multiplication by one of the eigenfunctions and integration of both sides of the equation over the volume of the cylinder produce an expression for the interior Green's function

$$\begin{aligned} G(r,\phi,z|r_0,\phi_0,z_0) = \sum_{mns} &\frac{(8\epsilon_m/\pi^2 l a^2)\cos[m(\phi - \phi_0)]}{J_{m+1}^2(\pi\beta_{ms})[(n/l)^2 + (\beta_{ms}/a)^2]}\,\cdot \\ &\cdot \sin\left(\frac{\pi n z}{l}\right)\sin\left(\frac{\pi n z_0}{l}\right) J_m\left(\frac{\pi\beta_{ms} r}{a}\right) J_m\left(\frac{\pi\beta_{ms} r_0}{a}\right) \end{aligned} \qquad (10.3.24)$$

The field inside a grounded cylinder containing a uniform distribution of charge of unit density is obtained by integrating this over the volume of the cylinder in the r_0, ϕ_0, z_0 coordinates.

$$\psi = \sum_{n,s} \frac{[32/\pi^3\beta_{0s}(2n+1)]}{J_1(\pi\beta_{0s})\{[(2n+1)/l]^2 + (\beta_{0s}/a)^2\}} \sin\left[\frac{\pi z}{l}(2n+1)\right] J_0\left[\frac{\pi\beta_{0s}r}{a}\right]$$

The series is also proportional to the temperature distribution inside a cylinder of uranium, with uniform distribution of fissions, kept at zero temperature at its surface.

Solutions in Spherical Coordinates. The spherical coordinates $r = \sqrt{x^2 + y^2 + z^2}$, $\vartheta = \tan^{-1}[(1/z)\sqrt{x^2 + y^2}]$, $\phi = \tan^{-1}(y/x)$, are an extremely useful system and well deserve detailed study. The Laplace equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2 \sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\psi}{\partial\vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2\psi}{\partial\phi^2} = 0$$

separates as follows: $\psi = R(r)\Theta(\vartheta)\Phi(\phi)$,

$$(d^2\Phi/d\phi^2) + m^2\Phi = 0$$

$$\frac{1}{\sin\vartheta}\frac{d}{d\vartheta}\left(\sin\vartheta\frac{d\Theta}{d\vartheta}\right) + \left[n(n+1) - \frac{m^2}{\sin^2\vartheta}\right]\Theta = 0$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{n(n+1)}{r^2}R = 0$$

Solutions of the first equation are $\cos(m\phi)$ and $\sin(m\phi)$, and if no boundaries are along the planes $\phi =$ constant, requirements of continuity and periodicity of Φ require that m be zero or an integer. Solutions of the second equation are the Legendre functions

$$P_n^m(\cos\vartheta) = \sin^m\vartheta\, T_{n-m}^m(\cos\vartheta)$$

discussed on pages 601 and 782. We have shown that these functions are finite over the range $0 \le \vartheta \le \pi$ only when n is an integer equal to m or larger. Solutions of the third equation are r^n or $1/r^{n+1}$. Therefore the unit solutions are

$$r^n Y_{mn}^e; \quad r^n Y_{mn}^0; \quad r^{-n-1}Y_{mn}^e; \quad r^{-n-1}Y_{mn}^0$$

where

$$Y_{mn}^e = \cos(m\phi)P_n^m(\cos\vartheta); \quad Y_{mn}^0 = \sin(m\phi)P_n^m(\cos\vartheta) \qquad (10.3.25)$$

The functions Y are called *spherical harmonics*, the ones for $m = 0$ being *zonal harmonics* (since these functions depend only on ϑ, the nodal lines divide the sphere into zones), the ones for $m = n$ being *sectoral harmonics* (since these functions depend only on ϕ, the nodal lines divide the sphere into sectors), and the rest, for $0 < m < n$, being *tesseral harmonics.* These functions are eigenfunctions for the two-dimensional surface of the sphere, being mutually orthogonal and having a normalization constant

$$\int_0^{2\pi} d\phi \int_0^\pi [Y_{mn}(\vartheta,\phi)]^2 \sin\vartheta\, d\vartheta = \frac{4\pi}{\epsilon_m(2n+1)}\left[\frac{(n+m)!}{(n-m)!}\right]$$

where the superscript of the Y can be either e (even) or 0 (odd) (except that Y^0_{0n} does not exist) and where (as before) $\epsilon_0 = 1$, $\epsilon_n = 2(n = 1, 2, 3 \ldots)$. Consequently any function $V_0(\phi,\vartheta)$, specified over the surface of a sphere, may be expressed in terms of the series

$$\begin{aligned} V_0(\phi,\vartheta) &= \sum_{m,n} [A_{mn}Y^e_{mn}(\vartheta,\phi) + B_{mn}Y^0_{mn}(\vartheta,\phi)] \\ A_{mn} &= \frac{(2n+1)\epsilon_m}{4\pi}\left[\frac{(n-m)!}{(n+m)!}\right]\int_0^{2\pi} d\phi \int_0^\pi V_0 Y^e_{mn} \sin\vartheta\, d\vartheta \end{aligned} \tag{10.3.26}$$

with the integral for B_{mn} similar to that for A_{mn}, except that Y^0_{mn} is substituted for Y^e_{mn} and the terms for $m = 0$ are omitted.

We therefore see that the potential inside a sphere of radius a, having potential $V_0(\phi,\vartheta)$ specified on its surface, is given by the series

$$\psi = \sum_{m,n} [A_{mn}Y^e_{mn}(\vartheta,\phi) + B_{mn}Y^0_{mn}(\vartheta,\phi)]\left(\frac{r}{a}\right)^n \tag{10.3.27}$$

where the coefficients A and B are given by Eq. (10.3.26). The terms for larger values of n, representing the "finer details" of the variation of $V_0(\phi,\vartheta)$ with angle, are large only near $r = a$; near the center of the sphere the potential is nearly uniform, equal to the first term,

$$\frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^\pi V_0(\phi,\vartheta) \sin\vartheta\, d\vartheta = V_{\text{av}}$$

which is the average value of V_0 over the surface of the sphere.

For instance, if the spherical surface is at potential V for ϑ between 0 and ϑ_0 and is zero for ϑ between ϑ_0 and π, we use the formula for the integral of P_n given on page 1326. All the B's are zero and also all the A's, except for $A_{0n} = A_n$, which are

$$A_n = \tfrac{1}{2}(2n+1)\int_{\cos\theta_0}^1 VP_n(x)\,dx = \tfrac{1}{2}V[P_{n-1}(\cos\vartheta_0) - P_{n+1}(\cos\vartheta_0)]$$

where, for $n = 0$, we set $P_{-1} = 1$. Therefore the potential inside the sphere is

$$\psi = \tfrac{1}{2}V\sum_{n=0}^{\infty} [P_{n-1}(\cos\vartheta_0) - P_{n+1}(\cos\vartheta_0)]P_n(\cos\vartheta)\left(\frac{r}{a}\right)^n$$

The potential outside a sphere of radius a, with potential $V_0(\phi,\vartheta)$ specified at its surface, is given by a series similar to that of Eq. (10.3.27), except that the factors $(r/a)^n$ are replaced by factors $(a/r)^{n+1}$, thus ensuring that the potential goes to zero at infinity. We note that, at very large distances from the sphere ($r \gg a$), the potential is proportional to $1/r$ (Coulomb field), with a factor a times the average potential V_{av} of the sphere's surface. Viewed from a distance, the sphere acts as though it had a net charge equal to its capacitance a (in electrostatic units) times its average potential.

Fields of Charged Disks and from Currents in Wire Loops. A uniform field in the z direction has a potential proportional to $-z = -rP_1(\cos\vartheta)$. When a sphere of radius a, with center at the origin, disturbs this field, we must add a term $(1/r^2)P_1(\cos\vartheta)$ to satisfy the boundary conditions at $r = a$. For instance, if the field represents flow of an incompressible nonviscous fluid, the boundary condition is that $\partial\psi/\partial r = 0$ at $r = a$. The resulting velocity potential is then

$$\psi = -vP_1(\cos\vartheta)[r + \tfrac{1}{2}(a^3/r^2)] \tag{10.3.28}$$

The velocity at the surface of the sphere is in the ϑ direction and is

$$-\frac{1}{a}\left(\frac{\partial\psi}{\partial\vartheta}\right)_a = -\tfrac{3}{2}v\sin\vartheta$$

the negative sign indicating that it is in the direction of *decreasing* ϑ. The maximum fluid velocity is thus at the equator of the sphere and is $\frac{3}{2}$ of the uniform velocity at infinity. If the fluid is at rest at infinity and the sphere is moved with velocity v through the fluid, the fluid immediately ahead of the sphere will move with the sphere with velocity v, whereas the fluid next to the equator will move in the opposite direction with velocity $\frac{1}{2}v$, in order for the displaced fluid to get around the sphere from front to back. Viscosity will, of course, modify this flow; we shall discuss its effect in a later chapter.

If the uniform field is an electric one, of intensity E at infinity, and if the sphere is of dielectric material with dielectric constant ϵ, then the potential must be adjusted so that its value just inside the surface is equal to its value just outside and so that its radial derivative just outside equals ϵ times its normal slope just inside. A little algebraic manipulation will show that the required pair of functions is

$$\psi = \begin{cases} -[3E/(2+\epsilon)]r\cos\vartheta; & r < a \\ -Er\cos\vartheta + E[(\epsilon-1)/(\epsilon+2)](a^3/r^2)\cos\vartheta; & r > a \end{cases}$$

indicating that the field is uniform at infinity and also inside the sphere, the intensity inside the sphere being less than the asymptotic intensity when the dielectric constant is larger than unity.

Next we compute the field about a flat disk of radius a, uniformly charged with total charge Q and charge density $Q/\pi a^2$. We set the center of the disk at the origin and the z axis normal to the surface of the disk. The potential a distance r along the z axis from the front of the disk is

$$\begin{aligned}\psi_{\vartheta=0} &= \frac{Q}{\pi a^2}\,2\pi \int_0^a \frac{y\,dy}{\sqrt{r^2+y^2}} = \frac{2Q}{a^2}\left[\sqrt{a^2+r^2}-r\right] \\ &= \frac{2Q}{a}\left[1-\left(\frac{r}{a}\right)+\tfrac{1}{2}\left(\frac{r}{a}\right)^2-\tfrac{1}{8}\left(\frac{r}{a}\right)^4+\tfrac{1}{16}\left(\frac{r}{a}\right)^6-\cdots\right]; \quad r<a \\ &= \frac{2Q}{a}\left[\tfrac{1}{2}\left(\frac{a}{r}\right)-\tfrac{1}{8}\left(\frac{a}{r}\right)^3+\tfrac{1}{16}\left(\frac{a}{r}\right)^5-\cdots\right]; \quad r>a\end{aligned}$$

We know that the potential is symmetric about the z axis and therefore that it must be expandable in terms of the zonal harmonics $r^nP_n(\cos\vartheta)$ or $r^{-n-1}P_n(\cos\vartheta)$ for values of ϑ different from zero (*i.e.*, off the z axis). We also know that $P_n(1) = 1$. From these facts we can deduce that the expansion of the potential of the disk, at the point r, ϑ, ϕ is

$$\begin{aligned}\psi &= \frac{2Q}{a}\left[1-\left(\frac{r}{a}\right)|P_1(\cos\vartheta)|+\tfrac{1}{2}\left(\frac{r}{a}\right)^2 P_2(\cos\vartheta)\right. \\ &\qquad\qquad \left.-\tfrac{1}{8}\left(\frac{r}{a}\right)^4 P_4(\cos\vartheta)+\cdots\right]; \quad r<a \\ &= \frac{2Q}{a}\left[\tfrac{1}{2}\left(\frac{a}{r}\right)-\tfrac{1}{8}\left(\frac{a}{r}\right)^3 P_2(\cos\vartheta)\right. \\ &\qquad\qquad \left.+\tfrac{1}{16}\left(\frac{a}{r}\right)^5 P_4(\cos\vartheta)-\cdots\right]; \quad r>a\end{aligned} \tag{10.3.29}$$

This potential is independent of ϕ, as it should be. At very large distances the potential reduces to Q/r, as it should. We note that there is a discontinuity in gradient at the surface of the disk because of the term $(aQ/a^2)r|\cos\vartheta| = (2Q/a^2)|z|$. The discontinuity in gradient is therefore $4Q/a^2 = (4\pi\sigma)$, where $\sigma = Q/\pi a^2$ is the surface density of charge on the disk. We also note, by remembering that the two series must be equal at $r = a$, that

$$|\cos\vartheta| = \tfrac{1}{2}+\tfrac{5}{8}P_2(\cos\vartheta)-\tfrac{3}{16}P_4(\cos\vartheta)+\tfrac{13}{128}P_6(\cos\vartheta)-\cdots$$

To find the potential caused by a dipole disk of radius a, at the origin, normal to the z axis, we remember from elementary electromagnetic theory that the magnetic potential caused by a loop of wire carrying current I is equal to that of a magnetic dipole surface of dipole density I, with its boundary the wire. Consequently the potential of a dipole disk is equal to the magnetic potential of a circular loop of wire coincident with the edge of the disk; it is also equal to I times the solid angle subtended by the disk at the point in question. The magnetic intensity a distance r from the plane of the disk, along the axis of the disk, is

$$H = I(2\pi a)\frac{1}{(r^2+a^2)}\frac{a}{\sqrt{r^2+a^2}} = 2\pi I\left[\frac{a^2}{(r^2+a^2)^{\frac{3}{2}}}\right]$$

Integrating this from r to infinity gives us the magnetic potential along the axis:

$$\begin{aligned}\psi &= 2\pi I\left[1 - \frac{r}{\sqrt{r^2+a^2}}\right] = 2\pi I\left[\tfrac{1}{2}\left(\frac{r}{a}\right)^2 - \frac{1\cdot 3}{2\cdot 4}\left(\frac{a}{r}\right)^4 \right. \\ &\qquad\qquad\left. + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(\frac{a}{r}\right)^6 - \cdots\right]; \quad r > a \\ &= 2\pi I\left[1 - \frac{r}{a} + \tfrac{1}{2}\left(\frac{r}{a}\right)^3 - \frac{1\cdot 3}{2\cdot 4}\left(\frac{r}{a}\right)^5 \right. \\ &\qquad\qquad\left. + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(\frac{r}{a}\right)^7 - \cdots\right]; \quad r < a\end{aligned}$$

Along the negative z axis the potential is minus this.

Again fitting our spherical harmonics to the power series, we find that the magnetic potential from a circular loop of wire carrying current I or the potential of a disk of radius a having dipole density I per unit area of disk is

$$\begin{aligned}\psi &= 2\pi I\left[\frac{\cos\vartheta}{|\cos\vartheta|} - \left(\frac{r}{a}\right)P_1(\cos\vartheta) \right. \\ &\qquad\qquad\left. + \tfrac{1}{2}\left(\frac{r}{a}\right)^3 P_3(\cos\vartheta) - \cdots\right]; \quad r < a \\ &= 2\pi I\left[\tfrac{1}{2}\left(\frac{a}{r}\right)^2 P_1(\cos\vartheta) \right. \\ &\qquad\qquad\left. - \frac{1\cdot 3}{2\cdot 4}\left(\frac{a}{r}\right)^4 P_3(\cos\vartheta) + \cdots\right]; \quad r > a\end{aligned} \tag{10.3.30}$$

where the first term in the first bracket is just an easy way to write the discontinuous function which is $+1$ for $\vartheta < \frac{1}{2}\pi$ and is -1 for $\vartheta > \frac{1}{2}\pi$. This discontinuity, as one goes through the disk (as ϑ goes from $\frac{1}{2}\pi - \epsilon$ to $\frac{1}{2}\pi + \epsilon$ for $r < a$), makes a discontinuity in ψ of the amount $4\pi I$. This is just the amount expected for a disk with dipole density I. This discontinuity in the magnetic potential for a wire loop is caused by the fact that the current in the loop produces a vorticity in the field at the wire, and any attempt to represent a field with vortices by a potential function would be expected to have discontinuities in the potential (see the discussion on page 1228 of the circulation about a cylinder).

We note, from Eq. (10.3.30), that the potential at large distances from the disk (or loop of wire) reduces to $\pi a^2 I(1/r^2)\cos\vartheta$, which, as we shall see later in this section, is the potential of a simple dipole of "strength" $\pi a^2 I$. We also note that the discontinuous step function $(\cos\vartheta/|\cos\vartheta|)$ is expressed by the series

$$\tfrac{3}{2}P_1(\cos\vartheta) - \tfrac{7}{8}P_3(\cos\vartheta) + \tfrac{11}{16}P_5(\cos\vartheta) - \cdots = \begin{cases} 1; & 0 < \vartheta < \frac{1}{2}\pi \\ -1; & \frac{1}{2}\pi < \vartheta < \pi \end{cases}$$

Fields of Charged Spherical Caps. As the previous example indicated, moving through a dipole layer of surface density I produces a discontinuity in value of the potential of an amount $4\pi I$, but no discontinuity in normal gradient. By this means we can calculate the potential of a dipole layer on the surface of a sphere of radius a. For instance, the function

$$\psi_n(r,\vartheta) = \begin{cases} -\left(\dfrac{n+1}{2n+1}\right)\left(\dfrac{r}{a}\right)^n P_n(\cos\vartheta); & r < a \\ \dfrac{n}{2n+1}\left(\dfrac{a}{r}\right)^{n+1} P_n(\cos\vartheta); & r > a \end{cases}$$

is finite everywhere, has a discontinuity of value $P_n(\cos\vartheta)$ at $r = a$, but has no discontinuity in normal gradient there. To find the potential of an arbitrary distribution of surface dipoles on the surface $r = a$, we build up a series of ψ_n's to correspond to the required density on the sphere.

For instance, the dipole surface can be a spherical cap; the dipole density on the surface of the sphere of radius a can be equal to I for ϑ from zero to ϑ_0 and be zero for from ϑ_0 to π. The potential of such a distribution of charge must be $\Sigma A_n P_n(\cos\vartheta)$, where

$$A_n = 4\pi I\left(\frac{2n+1}{2}\right)\int_0^{\vartheta_0} \psi_n(r,\vartheta)P_n(\cos\vartheta)\sin\vartheta\,d\vartheta$$

Therefore

$$\psi = 2\pi I \sum_{n=0}^{\infty} [P_{n-1}(\cos\vartheta_0) - P_{n+1}(\cos\vartheta_0)]P_n(\cos\vartheta) \begin{cases} -\left(\dfrac{n+1}{2n+1}\right)\left(\dfrac{r}{a}\right)^n; & r < a \\ \dfrac{n}{2n+1}\left(\dfrac{a}{r}\right)^{n+1}; & r > a \end{cases} \tag{10.3.31}$$

is the potential of a dipole cap of angular radius ϑ_0 and of uniform surface density I. It is also the magnetic potential of a wire loop, placed at the circle $r = a$, $\vartheta = \vartheta_0$, carrying current I. Here we have replaced the current loop by a spherical dipole shell instead of a plane disk with the same perimeter, as in Eq. (10.3.30). The two formulas should coincide (except for the position of the discontinuity) when the loop is at the equatorial plane, *i.e.*, when $\vartheta_0 = \frac{1}{2}\pi$. In this case, using one of the formulas on page 1325,

$$\begin{aligned}\psi &= 2\pi I\left[-1 - \left(\frac{r}{a}\right)P_1(\cos\vartheta) + \tfrac{1}{2}\left(\frac{r}{a}\right)^3 P_3(\cos\vartheta) - \frac{1\cdot 3}{2\cdot 4}\left(\frac{r}{a}\right)^5 P_5(\cos\vartheta) + \cdots\right]; \quad r < a \\ &= 2\pi I\left[\tfrac{1}{2}\left(\frac{a}{r}\right)^2 P_1(\cos\vartheta) - \frac{1\cdot 3}{2\cdot 4}\left(\frac{a}{r}\right)^4 P_3(\cos\vartheta) + \cdots\right]; \quad r > a\end{aligned} \tag{10.3.32}$$

We notice that the series for $r > a$ is identical with that for Eq. (10.3.30) whereas the series for $r < a$ is equivalent except that we have moved the discontinuity from the equatorial plane $\vartheta = \frac{1}{2}\pi$ to one surface of the sphere $r = a$ for $\vartheta < \frac{1}{2}\pi$.

Since a surface charge causes a jump in value of ψ rather than in gradient, the corresponding series for the potential of a uniformly charged spherical cap:

$$\psi = 2\pi\sigma \sum_{n=0}^{\infty} [P_{n-1}(\cos\vartheta_0) - P_{n+1}(\cos\vartheta_0)]P_n(\cos\vartheta) \begin{cases} \dfrac{1}{2n+1}\left(\dfrac{r}{a}\right)^n; & r < a \\ \dfrac{1}{2n+1}\left(\dfrac{a}{r}\right)^{n+1}; & r > a \end{cases} \tag{10.3.33}$$

is related to (but not equivalent to) Eq. (10.3.29).

Integral Representation of Solutions. Before we discuss other problems, it is important that we find the integral representation, in the form of Eq. (10.3.2), for the eigenfunction solutions in spherical coordinates. Since

$$X = z + ix\cos u + iy\sin u = r[\cos\vartheta + i\sin\vartheta\cos(u - \phi)]$$

it would appear that a solution involving a power of r must be expressed by an integral with $F(X)$ being a simple power of X. We try

$$\begin{aligned}
\int_0^{2\pi} X^n \cos(mu)\,du
&= r^n \int_0^{2\pi} [\cos\vartheta + i\sin\vartheta\cos w]^n \cos[m(w+\phi)]\,dw \\
&= r^n \cos(m\phi) \int_0^{2\pi} [\cos\vartheta + i\sin\vartheta\cos w]^n e^{imw}\,dw \\
&= 2^{-n} i^{m-1} \sin^m\vartheta \left[\oint (t^2 - 1)^n (t - \cos\vartheta)^{-n-m-1}\,dt\right] r^n \cos(m\phi) \\
&= 2\pi i^m \frac{r^n n!}{(n+m)!} \cos(m\phi) \sin^m\vartheta\, T^m_{n-m}(\cos\vartheta) = \frac{2\pi i^m n!}{(n+m)!} r^n Y^e_{mn}(\vartheta,\phi)
\end{aligned} \tag{10.3.34}$$

where we have set $t = \cos\vartheta + i\sin\vartheta\, e^{iw}$ and have used Eq. (5.3.37). Likewise, we have

$$\int_0^{2\pi} X^n \sin(mu)\,du = 2\pi i^m \frac{n!}{(n+m)!} r^n Y^0_{mn}(\vartheta,\phi)$$

and by similar methods,

$$\int_0^{2\pi} X^{-n-1} \begin{matrix}\cos\\ \sin\end{matrix}(mu)\,du = 2\pi i^{-m} \frac{(n-m)!}{n!} r^{-n-1} \begin{cases} Y^e_{mn}(\vartheta,\phi) \\ Y^0_{mn}(\vartheta,\phi) \end{cases} \tag{10.3.35}$$

These formulas will turn out to be exceedingly useful in our future work. They enable us to go relatively easily from the spherical coordinates to other systems. For instance, by expressing X in terms of x, y, and z and then carrying out the integration, we find that

$$\begin{aligned}
&Y_{00} = 1; \quad Y_{01} = \frac{z}{r}; \quad Y_{02} = \frac{1}{2r^2}(2z^2 - x^2 - y^2) \\
&\qquad\qquad\qquad\qquad\qquad Y_{03} = \frac{z}{2r^3}(2z^2 - 3x^2 - 3y^2) \\
&Y^e_{11} = \frac{x}{r}; \quad Y^e_{12} = 3\frac{xz}{r^2}; \quad Y^e_{13} = \frac{3}{2}\frac{x}{r^3}(4z^2 - x^2 - y^2) \\
&Y^0_{11} = \frac{y}{r}; \quad Y^0_{12} = 3\frac{yz}{r^2}; \quad Y^0_{13} = \frac{3}{2}\frac{y}{r^3}(4z^2 - x^2 - y^2) \qquad (10.3.36) \\
&Y^e_{22} = 3\frac{x^2 - y^2}{r^2}; \quad Y^e_{23} = 15\frac{x^2 - y^2}{r}z \\
&Y^0_{22} = 6\frac{xy}{r^2}; \quad Y^0_{23} = 30\frac{xyz}{r^3}; \quad Y^e_{33} = 15x\frac{x^2 - 3y^2}{r^3}; \quad Y^0_{33} = 15y\frac{3x^2 - y^2}{r^3}
\end{aligned}$$

where $r^2 = x^2 + y^2 + z^2$. It is not difficult to see from this that the $(2n + 1)$ functions

$$r^nY_{0n}; \quad r^nY^e_{1n}; \quad \cdots; \quad r^nY^e_{nn}; \quad r^nY^0_{1n}; \quad \cdots; \quad r^nY^0_{nn}$$

are all homogeneous, rational polynomials of x, y, z of degree n and also are all solutions of Laplace's equation. In fact all possible homogeneous polynomials of x, y, z of degree n which are solutions of Laplace's equation can be formed by a suitable linear combination of the $(2n + 1)$ independent ones listed. They form a complete subset of the totality of solutions; from the point of view of group theory they are representations of an n-fold symmetry group (for $n = 0$, the completely symmetric identity of rotation by 2π about any axis; for $n = 1$, the threefold symmetry of 90° rotation about the three cartesian axes; for $n = 2$, the symmetry of 45° rotations about the corresponding axes; and so on).

These same functions, of degree n, when divided by r^{2n+1}, result in a new set of functions which are also all solutions of Laplace's equation.

The solutions given in Eqs. (10.3.34) may be expressed in terms of solutions about a new origin by using the integral representation. For instance, one of the solutions about a center at the point $(0,0,a)$ may be expressed in terms of solutions about the point $(0,0,0)$ as follows:

$$\begin{aligned}
(r')^nY^e_{mn}(\vartheta',\phi) &= \frac{(n+m)!}{2\pi i^m n!}\int_0^{2\pi}(X - a)^n\cos(mu)\,du \\
&= \frac{(n+m)!}{2\pi i^m n!}\int_0^{2\pi}[X^n - naX^{n-1} \\
&\qquad + \tfrac{1}{2}n(n-1)a^2X^{n-2} - \cdots]\cos(mu)\,du \\
&= a^n\sum_{s=m}^{n}\frac{(-1)^{n-s}(n+m)!}{(n-s)!(m+s)!}\left(\frac{r}{a}\right)^s Y^e_{ms}(\vartheta,\phi)
\end{aligned}$$

For the solutions with negative powers of r we obtain an infinite series when we expand $(X - a)^{-n-1}$, and we must use a different form to obtain convergence, depending on whether r is larger or smaller than a. For r larger than a we have

$$
\begin{aligned}
(r')^{-n-1}Y^e_{mn}(\vartheta',\phi) &= \frac{n!i^m}{2\pi(n-m)!}\int_0^{2\pi}(X-a)^{-n-1}\cos(mu)\,du \\
&= \frac{i^m n!}{2\pi(n-m)!}\int_0^{2\pi}\frac{1}{X^{n+1}}\left[1+(n+1)\frac{a}{X}\right. \\
&\qquad\left. + \frac{(n+1)(n+2)}{2!}\left(\frac{a}{X}\right)^2 + \cdots\right]\cos(mu)\,du \\
&= a^{-n-1}\sum_{s=n}^{\infty}\frac{(s-m)!}{(n-m)!(s-n)!}\left(\frac{a}{r}\right)^{s+1}Y^e_{ms}(\vartheta,\phi)
\end{aligned}
$$

The formulas giving the potentials for the unprimed coordinates in terms of the primed coordinates are obtained by changing a to $(-a)$ in these equations.

For instance, one can obtain an approximate solution of the following problem: A grounded sphere of outer radius b is inside a sphere of inner radius $c(c > b)$ which is at potential V; find the distribution of potential between the spheres when the center of the inner sphere is displaced a distance a from the center of the outer $(a < c - b)$. We shall assume that the line between centers is along the z axis, that the coordinates with respect to the large sphere are r, ϑ, ϕ and those with respect to the small sphere (with center at $r = a$, $\vartheta = 0$) are r', ϑ', ϕ.

Since the potential at $r' = b$ is to be zero, the most general expression must be

$$
\psi = \sum_{n=0}^{\infty} A_n\left[\left(\frac{r'}{b}\right)^n - \left(\frac{b}{r'}\right)^{n+1}\right]P_n(\cos\vartheta')
$$

where we have used only the $m = 0$ harmonics, since we have arranged things to be symmetric about the z axis. We note that $A_0 b$ is equal to Q, the total charge on the inner sphere (can you prove this?). By using the expansions above and by reversing the order of summation, we find that this same potential, when expressed in the coordinates r, ϑ, referred to the center of the large sphere is

$$
\begin{aligned}
\psi = \sum_{s=0}^{\infty}\Bigg\{&\left(\frac{r}{a}\right)^s\sum_{n=s}^{\infty}A_n\left(\frac{a}{b}\right)^n\left[\frac{(-1)^{n-s}n!}{s!(n-s)!}\right] \\
&- \left(\frac{a}{r}\right)^{s+1}\sum_{n=0}^{s}A_n\left(\frac{b}{a}\right)^{n+1}\left[\frac{s!}{n!(s-n)!}\right]\Bigg\}P_s(\cos\vartheta)
\end{aligned}
$$

Since the large sphere at $r = c$ is to be at uniform potential V, the equations for the coefficients A are

$$\sum_{n=0}^{\infty} A_n \left(\frac{-a}{b}\right)^n - \left(\frac{b}{c}\right) A_0 = V$$

$$\sum_{n=0}^{s} A_n \left(\frac{b}{a}\right)^n \left[\frac{s!}{n!(s-n)!}\right] - \left(\frac{c}{a}\right)^{2s+1} \sum_{n=s}^{\infty} A_n \left(\frac{a}{b}\right)^{n+1} \left[\frac{(-1)^{n-s} n!}{s!(n-s)!}\right] = 0; \quad s > 0$$

When a is small compared with b or c, these simultaneous equations may be solved by successive approximation methods. We assume that $A_n = (a/b)^n B_n$, where B_n is neither very large nor very small. Then

$$\frac{c-b}{c} B_0 - \left(\frac{a}{b}\right)^2 B_1 + \left(\frac{a}{b}\right)^4 B_2 - \cdots = V$$

$$\left[B_0 + sB_1 + \frac{s(s-1)}{2} B_2 + \cdots + sB_{s-1} + B_s\right] - \left(\frac{c}{b}\right)^{2s+1} \left[B_s - (s+1)\left(\frac{a}{b}\right)^2 B_{s+1} + \frac{(s+1)(s+2)}{2!}\left(\frac{a}{b}\right)^4 B_{s+2} - \cdots\right] = 0$$

When a/b is small, the equations for $s > 0$ can be solved to give

$$B_1 \simeq \frac{b^3}{c^3 - b^3} B_0; \quad B_2 \simeq \frac{b^5(c^3 + b^3)}{(c^5 - b^5)(c^3 - b^3)} B_0; \quad \cdots$$

The value of B_0 is obtained from the first equation. When b is very much smaller than c, we need use only B_1 and may neglect the higher B's. In this case

$$B_0 = A_0 \simeq \frac{cV}{c-b}\left[1 - \frac{(ca^4/b)}{(c^3 - b^3)(c-b)}\right]$$

and the capacitance of the combination, to this approximation, is

$$C = \left(\frac{bA_0}{V}\right) \simeq \frac{cb}{c-b}\left[1 - \frac{(ca^4/b)}{(c^3 - b^3)(c-b)}\right]$$

Better ways of discussing this problem will be discussed later.

The Green's Function Expansion. The solution $1/r$ is the Green's function for the source at the origin. When the source is at the point $r = r_0$, $\vartheta = 0$, the Green's function is

$$\frac{1}{R} = \frac{1}{\sqrt{r^2 + a^2 - 2ra\cos\vartheta}} = \frac{1}{r_0} \sum_{s=0}^{\infty} \left(\frac{r_0}{r}\right)^{s+1} P_s(\cos\vartheta); \quad r > a$$

as could have been shown from Eq. (5.3.28). When the source is at the general point r_0, ϑ_0, ϕ_0, we can best obtain the expansion from the general formula (7.2.63).

The normalized eigenfunctions for the ϑ coordinate are

$$\sqrt{(n + \tfrac{1}{2}) \frac{(n-m)!}{(n+m)!}}\, P_n^m(\cos\vartheta)$$

The normalized eigenfunctions for the ϕ coordinate are either

$$\sqrt{\frac{1}{2\pi}}\, e^{\pm im\phi}$$

or $\sqrt{\epsilon_m/2\pi}\ {\cos \atop \sin}\,(m\phi)$, the degeneracy requiring both functions to be used for each value of m. The radial functions are not eigenfunctions, but the quantity

$$L = \frac{1}{2n+1} \begin{cases} r^n/r_0^{n+1}; & r < r_0 \\ r_0^n/r^{n+1}; & r > r_0 \end{cases}$$

has a discontinuity $1/r^2$ in slope at $r = r_0$ and is therefore the proper Green's function for the r coordinate (see page 831). Consequently the proper series for the Green's function for the three-dimensional Laplace equation in spherical coordinates is

$$\begin{aligned} \frac{1}{R} &= \sum_{n=0}^{\infty} P_n(\cos\omega) \begin{cases} r^n/r_0^{n+1}; & r_0 > r \\ r_0^n/r^{n+1}; & r < r_0 \end{cases} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \epsilon_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\vartheta_0) P_n^m(\cos\vartheta) \cdot \\ &\qquad \cdot \cos[m(\phi - \phi_0)] \begin{cases} r^n/r_0^{n+1}; & r_0 > r \\ r_0^n/r^{n+1}; & r < r_0 \end{cases} \end{aligned} \tag{10.3.37}$$

where $R^2 = r^2 + r_0^2 - 2rr_0 \cos\omega$ and

$$\cos\omega = \cos\vartheta_0 \cos\vartheta + \sin\vartheta_0 \sin\vartheta \cos(\phi - \phi_0)$$

Angles ω, ϑ, and ϑ_0 are three sides of a spherical triangle.

These formulas enable us to obtain several useful formulas for the Legendre function of the angle ω in terms of the other two sides of the spherical triangle ϑ and ϑ_0 and the included angle $(\phi - \phi_0)$. For instance, by equating coefficients of r^n/r_0^{n+1} in both series, we obtain the *addition theorem* for spherical harmonics:

$$P_n(\cos\omega) = \sum_{m=0}^{n} \epsilon_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\vartheta) P_n^m(\cos\vartheta_0) \cos[m(\phi - \phi_0)] \tag{10.3.38}$$

where ϵ_m is the Neumann factor; $\epsilon_0 = 1$, $\epsilon_n = 2$ $(n = 1, 2, 3, \ldots)$.

But we could have obtained the first series of Eqs. (10.3.37) by using Eq. (10.3.8). When $r_0 > r$ we have

$$\frac{1}{R} = \frac{1}{2\pi}\int_0^{2\pi} \frac{du}{X_0 - X} = \sum_{n=0}^{\infty} \frac{1}{2\pi}\int_0^{2\pi} \frac{X^n}{X_0^{n+1}}\,du$$
$$= \sum_{n=0}^{\infty} \left(\frac{r^n}{r_0^{n+1}}\right) \frac{1}{2\pi}\int_0^{2\pi} \frac{[\cos\vartheta + i\sin\vartheta\cos(\phi - u)]^n}{[\cos\vartheta_0 + i\sin\vartheta_0\cos(\phi_0 - u)]^{n+1}}\,du$$

Therefore we obtain an integral representation for the Legendre function of $\cos\omega$,

$$P_n(\cos\omega) = \frac{1}{2\pi}\int_0^{2\pi} \frac{[\cos\vartheta + i\sin\vartheta\cos(\phi - u)]^n}{[\cos\vartheta_0 + i\sin\vartheta_0\cos(\phi_0 - u)]^{n+1}}\,du \quad (10.3.39)$$

which is to be compared with Eqs. (10.3.34) and (10.3.35). The integral representations for $P_n(\cos\vartheta)$ or $P_n(\cos\vartheta_0)$ may be obtained by setting ϑ or $\vartheta_0 = 0$ in Eq. (10.3.39).

The Green's function for an interior problem, for a unit charge at r_0, ϑ_0, ϕ_0 inside a grounded sphere of radius a ($a > r_0$), is then

$$G_i(r,\vartheta,\phi|r_0,\vartheta_0,\phi_0) = \frac{1}{R}$$
$$- \sum_{n,m} \epsilon_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\vartheta_0)P_n^m(\cos\vartheta)\cos[m(\phi - \phi_0)]\,\frac{r_0^n r^n}{a^{2n+1}} \quad (10.3.40)$$

where the expansion for $1/R$ is given in Eq. (10.3.37).

Suppose that a uniformly charged wire is stretched along the z axis from $+a$ to 0 inside the grounded hollow sphere. The potential of a length dz of the wire, for positive z, is

$$q\sum_{n=0}^{\infty} P_n(\cos\vartheta)\left[A_n(z,r) - \left(\frac{z^n r^n}{a^{2n+1}}\right)\right]dz;\quad A_n = \begin{cases} r^n/z^{n+1}; & r < z \\ z^n/r^{n+1}; & r > z\end{cases}$$

where q is the charge per unit length. The series for negative values of z have an additional factor $(-1)^n$ multiplying the nth term $[= P_n(-1)]$. Therefore the potential at the point r, ϑ, ϕ inside the grounded sphere, due to the charged wire, is

$$q\sum_{n=0}^{\infty} P_n(\cos\vartheta)\left\{\frac{1}{r^{n+1}}\int_0^r z^n\,dz + r^n\int_r^a \frac{dz}{z^{n+1}} - \frac{r^n}{a^{2n+1}}\int_0^a z^n\,dz\right\}$$
$$= q\sum_{n=0}^{\infty}\left\{\frac{1}{n+1} + \frac{1}{n} - \frac{r^n}{na^n} - \frac{r^n}{(n+1)a^n}\right\}P_n(\cos\vartheta)$$
$$= q\sum_{n=0}^{\infty} \frac{2n+1}{n(n+1)}\left[1 - \left(\frac{r}{a}\right)^n\right]P_n(\cos\vartheta)$$

The convergence of the first terms is poor, owing to the discontinuity in potential at $r = a$, $\vartheta = 0, \pi$.

The Green's function for an exterior problem, the potential of a charge at r_0, ϑ_0, ϕ_0 in the presence of a grounded sphere of radius a ($a < r_0$), is

$$G_0(r,\vartheta,\phi|r_0,\vartheta_0,\phi_0) = \frac{1}{R} - \sum_{m,n} \epsilon_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\vartheta_0) P_n^m(\cos\vartheta) \cos[m(\phi - \phi_0)] \frac{a^{2n+1}}{r_0^{n+1} r^{n+1}} \tag{10.3.41}$$

This additional term, which ensures the potential going to zero at $r = a$, is identical with that which would arise from an image charge of magnitude a/r_0 at the point $r = a^2/r_0$, $\vartheta = \vartheta_0$, $\phi = \phi_0$. Comparison with Eq. (10.3.40) shows that the image of a point charge q at point (r_0,ϑ_0,ϕ_0) in a grounded spherical surface at $r = a$ (with a either larger or smaller than r_0) is a charge $q(a/r_0)$ at the point a^2/r_0, ϑ_0, ϕ_0. If the charge is outside the surface ($r_0 > a$), the image is inside the surface and is smaller than the original; if the charge is inside, the image is outside and larger than the original.

Dipoles, Quadrupoles, and Multipoles. The potential from an arbitrary distribution of charge density $\rho(r_0,\vartheta_0,\phi_0)$ which is inside a sphere $r = a$, at a point outside this sphere, is given by the series

$$\psi = \sum_{n=0}^{\infty} \sum_{m=0}^{n} [A_{nm} Y_{mn}^e(\vartheta,\phi) + B_{nm} Y_{mn}^o(\vartheta,\phi)] \frac{1}{r^{n+1}} \tag{10.3.42}$$

where

$$A_{nm} = \epsilon_m \frac{(n-m)!}{(n+m)!} \int_0^{2\pi} \cos(m\phi_0)\, d\phi_0 \int_0^{\pi} P_n^m(\cos\vartheta_0) \sin\vartheta_0\, d\vartheta_0 \cdot \cdot \int^a \rho(r_0,\vartheta_0,\phi_0) r_0^{n+2}\, dr_0$$

and the coefficient B_{nm} is a similar volume integral, with $\sin(m\phi_0)$ substituted for $\cos(m\phi_0)$ in the integrand ($B_{n0} = 0$).

We notice that the coefficient A_{00} is just the total charge inside the sphere ($r = a$); in other words the magnitude of the $1/r$ term outside the sphere, in charge-free space, is just this total charge, independent of the distribution of charge inside the sphere. Each of the other terms in the series are measures of different aspects of the charge distribution, as indicated by the corresponding integral.

But there is another way of computing this potential. According to Eq. (1.4.8) the potential at (x,y,z) is

$$\psi = \iiint\limits_V \frac{\rho(x_0,y_0,z_0)}{R}\, dx_0\, dy_0\, dz_0$$

where $R^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$ and where the integration is over the volume inside the sphere $r = a$, within which all the charge is supposed to reside. If r is larger than a, we can obtain a reasonably convergent series for the integral by expanding $1/R$ in a Taylor's series expansion for x_0, y_0, z_0, which never get larger than a. It is not difficult to see that, symbolically, we can write

$$\frac{1}{R} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left[x_0 \frac{\partial}{\partial x} + y_0 \frac{\partial}{\partial y} + z_0 \frac{\partial}{\partial z} \right]^n \left(\frac{1}{r} \right)$$

where $r^2 = x^2 + y^2 + z^2$. Therefore, we can expand $1/R$ into a multiple series, each of which is a factor $(x_0^\lambda y_0^\mu z_0^\nu)$, times a numerical factor, times $(\partial^{\lambda+\mu+\nu}/\partial x^\lambda \, \partial y^\mu \, \partial z^\nu)(1/r)$, which is a function of x, y, z (or of r, ϑ, ϕ). The first of these factors, times $\rho(x_0,y_0,z_0)$, is to be integrated over the interior of the sphere $r_0 = a$, to give the final numerical constant in front of the function of x, y, z.

By using the multinomial theorem

$$(\alpha + \beta + \gamma)^n = \sum_{\lambda,\mu,\nu} \frac{n!}{\lambda!\mu!\nu!} \alpha^\lambda \beta^\mu \gamma^\nu$$

(where the sum is over all different values of λ, μ, ν for which $\lambda + \mu + \nu = n$) we can write out the expansion for ψ explicitly:

$$\psi = \sum_{n=0}^{\infty} \sum_{l=0}^{n} \sum_{k=0}^{n-l} \frac{(-1)^n}{l!k!(n-l-k)!} \left\{ \iiint x_0^l y_0^k z_0^{n-l-k} \rho(x_0,y_0,z_0) \, dx_0 \, dy_0 \, dz_0 \right\} \cdot \\ \cdot \frac{\partial^n}{\partial x^l \, \partial y^k \, \partial z^{n-l-k}} \left[\frac{1}{r} \right]; \quad \text{for } r > a \qquad (10.3.43)$$

where the density ρ (and all its derivatives) is zero on and outside the sphere of radius a. This corresponds to the description of the previous paragraph. We can, if we wish, integrate the quantity in braces by parts so as to substitute derivatives of ρ with respect to x_0, y_0, z_0 instead of the factor containing powers of x_0, y_0, z_0.

This series (in either form) shows that all that we can find out about the charge distribution ρ inside the sphere of radius a, by measurements of potential *outside the sphere*, are the properties represented by the magnitudes of the various integrals:

$$\frac{(-1)^n}{l!k!(n-l-k)!} \iiint x_0^l y_0^k z_0^{n-l-k} \rho \, dx_0 \, dy_0 \, dz_0 \qquad \textbf{(10.3.44)}$$

or

$$\iiint \frac{\partial^n}{\partial x_0^l \, \partial y_0^k \, \partial z_0^{n-l-k}} [\rho(x_0,y_0,z_0)] \, dx_0 \, dy_0 \, dz_0$$

depending on the form of the integral we choose to use for our calculations. In fact we can redistribute ρ inside the sphere in any complicated way we wish *as long as the values of the integrals* (10.3.44) *are unchanged*, and we should not be able to detect the redistribution outside the sphere. Of course, when we come down to it, the requirement that *all* the integrals be unchanged puts a pretty severe restriction, in theory, on our freedom to redistribute ρ. In fact the second form of the integrals seems to show that, if we also require that ρ be expressible in terms of a convergent power series in x_0, y_0, z_0 within the sphere, then no redistribution in theory is possible at all, because all the coefficients of the power series are fixed by fixing the values of the integrals. However, the matter is not quite so simply determinate as this, as we shall shortly see.

In practice, in addition, we have a considerable freedom of redistribution, because if r is appreciably larger than a, the potentials (given by the derivatives of $1/r$) for the higher values of n become so small as to be impossible of measurement, and all we can obtain are the values of the integrals (10.3.44) for n less than some finite value, which is smaller the larger is r, the distance at which we measure the potential.

For instance, for r very large, we can only measure the coefficient of the $1/r$ term, for $n = 0$. The value of this integral is just the *total charge* q. This quantity would be unchanged if we compressed all the charge into the origin, *i.e.*, if we substituted, instead of the actual distribution ρ, the simple distribution $q\delta(x)\delta(y)\delta(z)$. For r somewhat smaller we could also measure the magnitudes $D_x = \iiint x_0\rho\, dx_0\, dy_0\, dz_0$ and the similar integrals for D_y and D_z. The vector having components D_x, D_y, D_z is called the *dipole strength* of the charge distribution ρ. Its magnitude is unchanged if we replace ρ by a *dipole* charge symbolized by the derivative delta functions (see page 837)

$$D_x\delta'(x)\delta(y)\delta(z) + D_y\delta(x)\delta'(y)\delta(z) + D_z\delta(x)\delta(y)\delta'(z)$$

corresponding to the charge $+D/\epsilon$ ($D^2 = D_x^2 + D_y^2 + D_z^2$) at the point $x = \epsilon D_x/2D$, $y = \epsilon D_y/2D$, $z = \epsilon D_z/2D$ and the charge $-D/\epsilon$ on the other side of the origin, with ϵ vanishingly small. A distribution of this simple sort will, of course, give only the three $n = 1$ terms in series (10.3.43); all other terms will be zero.

The simplest distribution of charge which will give the potentials of order two involves combinations of three or four point charges. For instance, the integral

$$\iiint x_0y_0\rho\, dx_0\, dy_0\, dz_0 = Q_{xy}$$

can be duplicated by using a "dipole of dipoles," a dipole along the x axis, with elements, each of which are dipoles along the y axis, corresponding to charges $+(Q_{xy}/\epsilon^2)$ placed at $x = y = \pm\frac{1}{2}\epsilon$, $z = 0$ and charges of $-(Q_{xy}/\epsilon^2)$ at $x = -y = \pm\frac{1}{2}\epsilon$, $z = 0$, with ϵ made vanishingly small. In terms of the delta function notation this can be expressed by

$$Q_{xy}\delta'(x)\delta'(y)\delta(z)$$

On the other hand the integral

$$\iiint z_0^2\rho\, dx_0\, dy_0\, dz_0 = Q_{zz}$$

can be duplicated by placing a charge $-2(Q_{zz}/\epsilon^2)$ at the origin; a charge $+(Q_{zz}/\epsilon^2)$ at the point $x = y = 0$, $z = \epsilon$; and another one $+(Q_{zz}/\epsilon^2)$ at the point $x = y = 0$, $z = -\epsilon$ and then letting ϵ go to zero. The delta function representation of this is

$$Q_{zz}\delta(x)\delta(y)\delta''(z)$$

Distributions of three and four charges described in the previous paragraphs are called *quadrupoles*, and we can say that our second-order potentials may be successfully duplicated by proper choice of a symmetric *quadrupole dyadic*, with components Q_{xx}, $Q_{xy} = Q_{yx}$, etc., fixed by the six second-order integrals.

Perhaps we can now bring in the phrase "and so on." The next simplest distribution of charge is called an octopole, and the still higher ones are called simply multipoles.

And now we can reword the statement made on page 1278. *The potential distribution outside the sphere of radius a is unchanged if we substitute for the actual charge* ρ (which is all inside the sphere) *the following "simplified" distribution:*

A point charge at the origin of magnitude q.

A dipole at the origin of magnitude and direction to give the components D_x, D_y, D_z.

A quadrupole at the origin of components Q_{xx}, Q_{xy}, etc.

As far as the potential distribution outside the charge goes, *any charge distribution may be replaced by a suitable collection of multipoles.*

Perhaps by this time the reader is wondering why we bother about the spherical harmonic expansion when the multipole expansion is so much more "picturable"; why we bother with series (10.3.42) when we have series (10.3.43). The answer is twofold: First, the multipole series is redundant, and second, aside from the redundancies, the two series are not very different after all.

Comparing series (10.3.43) and (10.3.42) we note that the Taylor's series expansion has $(n + 1)(n + 2)$ terms of the nth order whereas the spherical harmonic series has $(2n + 1)$ terms of the nth order. Both series have one "zeroth" order term and three first-order ones. But the Taylor's series has six second-order terms, whereas the spherical harmonic series has five; either the Taylor's series has a redundant term or else the spherical harmonic series neglects to include one "degree of freedom" of the charge distribution ρ, or both.

The correct answer is "both." There is a degree of freedom of the charge included in the Taylor's series which *does not give rise to any potential outside the sphere* and which is omitted from the spherical harmonic series *because* it does not give rise to a potential. The symmetric charge distribution which would give $Q_{xx} = Q_{yy} = Q_{zz} = Q$ and $Q_{xy} = Q_{xz} = Q_{xz} = 0$ will give rise, by Eq. (10.3.43), to the potential

$$Q\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\left(\frac{1}{r}\right) = Q\nabla^2\left(\frac{1}{r}\right)$$

which is zero because $1/r$ *is a solution of Laplace's equation for* $r > 0$. This distribution corresponds to a negative charge at the origin, surrounded by a symmetric, spherical shell of equal positive charge, which has zero potential outside the outer shell. The spherical harmonic series does not bother to include a term for such a distribution and, therefore, has one less second-order term than does the Taylor series.

This additional requirement, that those "degrees of freedom" of the charge density be omitted which do not give rise to any potential outside the charge, is more easily applied if we change from the variables x, y, z in Eq. (10.3.43) to the variables $w = x + iy$, $\bar{w} = x - iy$ and z (see page 352). The second-order terms then are

$$\tfrac{1}{2}\iiint \rho(x_0y_0z_0)\,dx_0\,dy_0\,dz_0 \left\{w_0^2\frac{\partial^2}{\partial w^2} + \bar{w}_0^2\frac{\partial^2}{\partial \bar{w}^2} + z_0^2\frac{\partial^2}{\partial z^2} + 2w_0\bar{w}_0\frac{\partial^2}{\partial w\,\partial \bar{w}} + 2w_0z_0\frac{\partial^2}{\partial w\,\partial z} + 2w_0z_0\frac{\partial^2}{\partial \bar{w}\,\partial z}\right\}\left(\frac{1}{r}\right)$$

where $r^2 = w\bar{w} + z^2$. However, direct differentiation will show that

$$\frac{\partial^2}{\partial w\,\partial \bar{w}}\left(\frac{1}{r}\right) = \frac{1}{4}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{1}{r}\right) = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y}\right)\left(\frac{1}{r}\right) = -\frac{1}{4}\frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right)$$

because $1/r$ satisfies Laplace's equation. Therefore the fourth term produces the same potential as the third, and we can write the expression

$$\tfrac{1}{2}\iiint \rho(x_0,y_0,z_0)\,dx_0\,dy_0\,dz_0 \left\{w_0^2\frac{\partial^2}{\partial w^2} + \bar{w}_0^2\frac{\partial^2}{\partial \bar{w}^2} + (z_0^2 - \tfrac{1}{2}w_0\bar{w}_0)\frac{\partial^2}{\partial z^2} + 2w_0z_0\frac{\partial^2}{\partial w\,\partial z} + 2\bar{w}_0z\frac{\partial^2}{\partial \bar{w}\,\partial z}\right\}\left(\frac{1}{r}\right)$$

Performing all the differentiations and comparing terms with the x, y, z equivalents of the spherical harmonics, given in Eqs. (10.3.36), shows that the expressions above are exactly equal to

$$\iiint \rho(x_0y_0z_0)\,dx_0\,dy_0\,dz_0 \left(\frac{r_0^2}{r^3}\right) \{Y_{02}(\vartheta_0,\phi_0)Y_{02}(\vartheta,\phi) + \tfrac{1}{3}Y_{12}^e(\vartheta_0,\phi_0)Y_{12}^e(\vartheta,\phi)$$
$$+ \tfrac{1}{3}Y_{12}^0(\vartheta_0,\phi_0)Y_{12}^0(\vartheta,\phi) + \tfrac{1}{12}Y_{22}^e(\vartheta_0,\phi_0)Y_{22}^e(\vartheta,\phi) + \tfrac{1}{12}Y_{22}^0(\vartheta_0,\phi_0)Y_{22}^0(\vartheta,\phi)\}$$

which are just the second-order terms in the spherical harmonic series.

To show the very close connection between the two series for all orders, we use Eq. (10.3.35) to show that

$$\begin{aligned}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)^m \frac{\partial^{n-m}}{\partial z^{n-m}}\left(\frac{1}{r}\right) &= \frac{2^m}{2\pi}\frac{\partial^n}{\partial \bar{w}^m\,\partial z^{n-m}}\int_0^{2\pi}\frac{du}{X} \\ &= \frac{(-1)^n i^m}{2\pi}\,n!\int_0^{2\pi}\frac{e^{imu}\,du}{X^{n+1}} = (-1)^n(n-m)!e^{im\phi}P_n^m(\cos\vartheta)r^{-n-1} \\ &= (-1)^n(n-m)!r^{-n-1}[Y_{mn}^e(\vartheta,\phi) + iY_{mn}^0(\vartheta,\phi)]\end{aligned} \tag{10.3.45}$$

where $X = z + i(x\cos u + y\sin u) = z + \frac{1}{2}i(we^{-iu} + \bar{w}e^{iu})$. This formula, together with the one for the mixed derivatives

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)^{m+l}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)^l \frac{\partial^{n-m-2l}}{\partial z^{n-m-2l}}\left(\frac{1}{r}\right) = (-1)^l\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)^m \frac{\partial^{n-m}}{\partial z^{n-m}}\left(\frac{1}{r}\right)$$

allows one, after considerable algebraic drudgery, to show that series (10.3.43) is exactly equal to series (10.3.42). Thus we can justify the

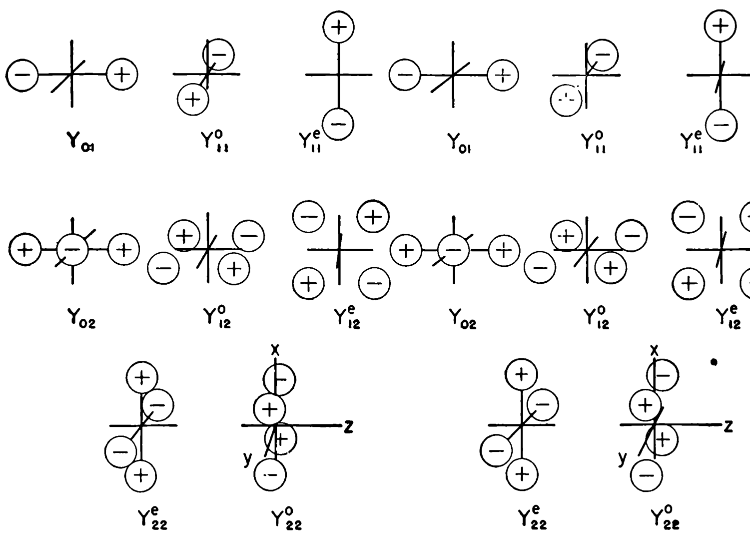

Fig. 10.25 Arrangements of dipoles and quadrupoles generating spherical harmonic fields as indicated.

statements made on page 1279, and thus also we can relate the multiple potentials with the spherical harmonic notation. Figure 10.25 shows the arrangements of dipoles and quadrupoles which correspond to the first two orders of spherical harmonics.

Now we can return to the question of how much we can tell about the charge density $\rho(x_0,y_0,z_0)$ inside the sphere $r = a$ by measurements of the potential, caused by ρ, outside $r = a$. The coefficients of the nth order spherical harmonics in series (10.3.42) are the volume integrals, over the sphere, of r_0^n times the same nth spherical harmonic of ϑ_0 and ϕ_0 times ρ. *Any modification of ρ which does not change these coefficients will produce the same potential outside $r = a$.* In particular we can add or subtract from ρ any amount of a charge distribution of the form $\chi_n(r_0)Y_{mn}(\vartheta_0,\phi_0)$ for which

$$\int_0^a \chi_n(r_0)r_0^{n+2}\,dr_0 = 0$$

for it is not difficult to see that any such charge distribution gives zero potential outside $r = a$.

A particular resolution of ρ,

$$\rho(r_0,\vartheta_0,\phi_0) = \sum_{m,n} [\rho_{mn}^e(r_0)Y_{mn}^e(\vartheta_0,\phi_0) + \rho_{mn}^0(r_0)Y_{mn}^0(\vartheta_0,\phi_0)]$$

where

$$\rho_{mn}^e(r_0) = \frac{\epsilon_m(2n+1)}{4\pi}\,\frac{(n-m)!}{(n+m)!}\int_0^{2\pi} d\phi_0 \int_0^{\pi} \sin\vartheta_0\,d\vartheta_0\,\rho(r_0,\vartheta_0,\phi_0)Y_{mn}^e(\vartheta_0,\phi_0)$$

with a similar expression for ρ^0, is useful here. Because of the completeness of the sequence of spherical harmonics this series completely corresponds to ρ itself, within the least-squares definition for fit of eigenfunction series, as given in Sec. 6.3. We note that ρ_{00} represents the total charge inside $r = a$ and that all other ρY's have zero net charge.

But having obtained the radial distributions ρ_{mn}, we may now change ρ without changing the potential ψ by changing any ρ_{mn} in the series in any way we wish as long as the integral of the changed ρ_{mn}, times r_0^{n+2}, over r_0 is not different from the same integral for the unchanged ρ_{mn}. In particular, we can change each ρ_{mn} (except for $m = n = 0$) into

$$q_{mn}^e(r_0) = \epsilon^{-n-3}\rho_{mn}^e(r_0/\epsilon);\quad q_{mn}^0(r_0) = \epsilon^{-n-3}\rho_{mn}^0(r_0/\epsilon);\quad 0 < \epsilon < 1$$

for then

$$\int_0^{a\epsilon} q_{mn}^e(r_0)r_0^{n+2}\,dr_0 = \epsilon^{-n-3}\int_0^{a\epsilon} \rho_{mn}^e(r_0/\epsilon)r_0^{n+2}\,dr_0$$
$$= \int_0^a \rho_{mn}^e(x)x^{n+2}\,dx = \frac{2n+1}{4\pi}A_{mn}$$

and similarly for q^0. Thus we can compress each partial density ρ_{mn} into a smaller and smaller sphere of radius $a\epsilon$, as ϵ is reduced in magnitude, increasing the charge magnitude by an appropriate factor $1/\epsilon^{n+3}$ at the same time without changing the potential outside $r = a$ in the slightest.

In the limit of ϵ vanishingly small, the equivalent charge distribution reduces to a sequence of multipoles, the multipole of order n being

$$\lim_{\epsilon\to 0}\left\{\sum_{m=0}^{n}[q^e_{mn}Y^e_{mn}+q^0_{mn}Y^0_{mn}]\right\}$$

The components of these multipoles are really all that we can measure concerning the charge distribution ρ by measuring potential ψ outside of $r = a$.

Spherical Shell with Hole. Having the Green's functions, we can now go back to the method used for deriving Eq. (10.1.22), the potential around a slotted cylinder, to find the potential around a spherical shell of metal, having a circular hole in it. Suppose that the surface $r = a$, $\vartheta_1 < \vartheta < \pi$ is metal, the region $0 < \vartheta < \vartheta_1$ is open, and we wish to compute the potential inside and outside when the metal shell is at a potential V with respect to infinity. In this case, we set the potential at infinity equal to $-V$, so that the potential of the shell is zero.

When no hole is present ($\vartheta_1 = 0$), the potential outside is $\psi_0^0 = -V + (Va/r)$ and that inside is $\psi_i^0 = 0$. The difference in normal gradients at $r = a$ is thus V/a, and the integral expression for the first approximation to the potential with a hole present is (see page 806)

$$\begin{aligned}\psi(r,\vartheta) &\simeq \frac{Va}{4\pi}\int_0^{2\pi}d\phi_0\int_{\vartheta_1}^{\pi}G(r,\vartheta,\phi|a,\vartheta_0,\phi_0)\sin\vartheta_0\,d\vartheta_0 - V\\ &\simeq -\tfrac{1}{2}V\sum_{n=0}^{\infty}A_n\left(\frac{r}{a}\right)^n P_n(\cos\vartheta); \qquad r<a\\ &\simeq -V+\frac{Va}{r}-\tfrac{1}{2}V\sum_{n=0}^{\infty}A_n\left(\frac{a}{r}\right)^{n+1}P_n(\cos\vartheta); \quad r>a\end{aligned}$$

where G is the series for $1/R$ given in Eq. (10.3.37) and

$$A_n = \int_0^{\vartheta_1}P_n(\cos\vartheta)\sin\vartheta\,d\vartheta = \frac{1}{2n+1}[P_{n-1}(\cos\vartheta_1) - P_{n+1}(\cos\vartheta_1)]$$

(for $n = 0$, $A_0 = 1 - \cos\vartheta_1$).

This expression is, of course, minus the potential of the cap which has been removed to make the hole. It is not exactly equal to zero along the metal shell, but it is continuous in value and slope across the opening from inside to outside. A better approximation would be to have

$$\psi(a,\vartheta) \simeq \begin{cases} -\tfrac{1}{2}V\displaystyle\sum_{n=0}^{\infty}A_nP_n(\cos\vartheta); & 0<\vartheta<\vartheta_1\\ 0; & \vartheta_1<\vartheta<\pi\end{cases}$$

Such a potential is easily computed; it is

$$\psi(r,\vartheta) \simeq -\tfrac{1}{4}V \sum_{n=0}^{\infty} \left[\sum_{m=0}^{\infty} A_{0m}A_{nm}\right] P_n(\cos\vartheta)\left(\frac{r}{a}\right)^n; \qquad r < a$$

$$\simeq -V + \frac{Va}{r} - \tfrac{1}{4}V \sum_{n=0}^{\infty} \left[\sum_{m=0}^{\infty} A_{0m}A_{nm}\right] P_n(\cos\vartheta)\left(\frac{a}{r}\right)^{n+1}; \quad r > a$$

where $$A_{nm} = (2n+1)\int_{\cos\vartheta_1}^{1} P_n(x)P_m(x)\,dx$$

This potential, of course, is not quite continuous in slope over the opening, but otherwise it is a good fit for the boundary conditions. The potential at the center of the sphere is then

$$\psi(0) \simeq -\tfrac{1}{4}V \sum_{m=0}^{\infty} \frac{1}{(2n+1)^2}\,[P_{n-1}(\cos\vartheta_1) - P_{n+1}(\cos\vartheta_1)]^2$$

a quantity which is quite small if ϑ_1 is small, is equal to $-V$ if $\vartheta_1 = \pi$ (that is, if there is no metallic surface at all).

We could also utilize the methods of page 1206 for computing the potential if its values in the opening are known.

Prolate Spheroidal Coordinates. Spherical coordinates are formed by rotating polar coordinates about a diameter; prolate spheroidal coordinates are formed by rotating the elliptic coordinates of page 1195 about the major axis of the ellipses. For purposes of variety we shall modify the scale factors for the coordinates. Suppose that the foci of the spheroids are at the points $x = y = 0$, $z = \pm\frac{1}{2}a$, and suppose that the distances from the point (x,y,z) to these foci are

$$r_1 = \sqrt{(z+\tfrac{1}{2}a)^2 + x^2 + y^2};\quad r_2 = \sqrt{(z-\tfrac{1}{2}a)^2 + x^2 + y^2}$$

Then the prolate spheroidal coordinate system ξ, η, ϕ is defined as follows:

$$\xi = (r_1 + r_2)/a;\quad \eta = (r_1 - r_2)/a;\quad \phi = \tan^{-1}(y/x)$$
$$z = \tfrac{1}{2}a\xi\eta;\quad x = \tfrac{1}{2}a\sqrt{(\xi^2-1)(1-\eta^2)}\cos\phi$$
$$y = \tfrac{1}{2}a\sqrt{(\xi^2-1)(1-\eta^2)}\sin\phi$$
$$h_\xi = \tfrac{1}{2}a\sqrt{\frac{\xi^2-\eta^2}{\xi^2-1}};\quad h_\eta = \tfrac{1}{2}a\sqrt{\frac{\xi^2-\eta^2}{1-\eta^2}};\quad h_\phi = \tfrac{1}{2}a\sqrt{(\xi^2-1)(1-\eta^2)}$$
$$\nabla^2\psi = \frac{4}{a^2(\xi^2-\eta^2)}\left[\frac{\partial}{\partial\xi}(\xi^2-1)\frac{\partial\psi}{\partial\xi} + \frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial\psi}{\partial\eta} + \frac{\xi^2-1+1-\eta^2}{(\xi^2-1)(1-\eta^2)}\frac{\partial^2\psi}{\partial\phi^2}\right] \tag{10.3.46}$$

where ξ goes from 1 to ∞, η goes from -1 to $+1$, and ϕ from 0 to 2π. The surface ξ = constant is a prolate spheroid with interfocal distance a, major axis $\frac{1}{2}\xi a$, and minor axis $\frac{1}{2}a\sqrt{\xi^2-1}$; the surfaces η = constant are the two sheets of a hyperboloid of revolution with foci at $z = \pm\frac{1}{2}a$, asymptotic to the cone which has its generating line at an angle $\vartheta = \cos^{-1}\eta$ to the z axis, and the surface $\phi =$ constant is a plane through the z axis at an angle ϕ to the x, z plane.

The separated equations, for $\psi = \Phi(\phi)X(\xi)H(\eta)$, are

$$\frac{d^2\Phi}{d\phi^2} = -m^2\Phi$$

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{dH}{d\eta}\right] + n(n+1)H - \frac{m^2}{1-\eta^2}H = 0$$

$$\frac{d}{d\xi}\left[(1-\xi^2)\frac{dX}{d\xi}\right] + n(n+1)X - \frac{m^2}{1-\xi^2}X = 0$$

The allowed solutions of the first equation, for periodic boundary conditions on ϕ, are $\cos(m\phi)$, $\sin(m\phi)$, where m is zero or a positive integer. The second and third equations are those solved by the spherical harmonics P_n^m and their second solutions Q_n^m (see page 1327). For the second equation, η goes from -1, which value it has along the negative z axis, to $+1$ which it has along the positive z axis. In order that H be finite over this range, n must be zero or a positive integer and H must be proportional to the Legendre function of the first kind, $P_n^m(\eta)$. The variable ξ goes from $+1$, along the line between foci, to infinity. There is no solution which is finite over the whole of this range for most values of n and m, so we use whatever combination of $P_n^m(\xi)$ and $Q_n^m(\xi)$ will stay finite inside the boundaries of the problem.

For instance, the potential distribution outside a prolate spherical surface $\xi = \xi_0$, which is held at potential $\psi_0(\eta,\phi)$, is

$$\psi = \sum_{n=0}^{\infty}\sum_{m=0}^{n}[A_{mn}\cos(m\phi) + B_{mn}\sin(m\phi)]P_n^m(\eta)\left[\frac{Q_n^m(\xi)}{Q_n^m(\xi_0)}\right] \tag{10.3.47}$$

where

$$\left.\begin{matrix}A_{mn}\\ B_{mn}\end{matrix}\right\} = \frac{\epsilon_m}{4\pi}(2n+1)\frac{(n-m)!}{(n+m)!}\int_0^{2\pi}\begin{matrix}\cos\\ \sin\end{matrix}(m\phi)\,d\phi\int_{-1}^{1}\psi_0(\eta,\phi)P_n^m(\eta)\,d\eta$$

If ψ_0 is constant over the surface, this reduces to

$$\psi = \left[\frac{\psi_0}{Q_0(\xi_0)}\right]Q_0(\xi) = \psi_0\frac{\ln[(\xi+1)/(\xi-1)]}{\ln[(\xi_0+1)/(\xi_0-1)]}$$

At large distances from the spheroid $\xi = \xi_0$, the coordinate ξ becomes equal to $2r/a$, where r is the distance from the center of the spheroid. Therefore the potential from a prolate spheroid $\xi = \xi_0$ at a uniform potential ψ_0, at large distances from the spheroid, is the usual inverse first-power potential

$$\psi \simeq \frac{\psi_0}{\ln[(\xi_0+1)/(\xi_0-1)]}\left(\frac{a}{r}\right); \quad r \gg a$$

from which we can compute the charge on the spheroid and thus its capacitance.

If a grounded, conducting spheroid, $\xi = \xi_0$, is in a uniform field of intensity E at large distance from the spheroid and at a direction θ with respect to the z axis in the x, z plane, then the potential must have a uniform-field part

$$\begin{aligned} -Ex\sin\theta - Ez\cos\theta &= -\tfrac{1}{2}aE[\xi\eta\cos\theta + \sqrt{(\xi^2-1)(1-\eta^2)}\cos\phi\sin\theta] \\ &= -\tfrac{1}{2}aE[P_1^0(\xi)P_1^0(\eta)\cos\theta + iP_1^1(\xi)P_1^1(\eta)\cos\phi\sin\theta] \end{aligned}$$

plus enough of a solution, which vanishes at infinity, to cancel this at $\xi = \xi_0$. The final result is

$$\begin{aligned} \psi = \tfrac{1}{2}aE\Bigg\{&\cos\theta P_1^0(\eta)\left[\frac{P_1^0(\xi_0)}{Q_1^0(\xi_0)}Q_1^0(\xi) - P_1^0(\xi)\right] \\ &+ i\sin\theta\cos\phi\, P_1^1(\eta)\left[\frac{P_1^1(\xi_0)}{Q_1^1(\xi_0)}Q_1^1(\xi) - P_1^1(\xi)\right]\Bigg\} \end{aligned} \tag{10.3.48}$$

We have had to introduce a factor i because our definition of spherical harmonics

$$P_n^m(z) = (1 - z^2)^{\frac{1}{2}m}T_{n-m}^m(z)$$

is contrived to be real for z less than unity (this is usually called the *Ferrer's spherical harmonic*). For the coordinate ξ, which is always larger than unity, we perhaps should redefine our function to be

$$p_n^m(z) = (z^2 - 1)^{\frac{1}{2}m}T_{n-m}^m(z) = e^{\frac{1}{2}im\pi}P_n^m(z)$$

(which is usually called the *Hobson's spherical harmonic*) so as to keep imaginary quantities out of the formulas. Rather than introduce another symbol, however, it is easier to remember that

$$i^mP_n^m(\xi) = (\xi^2 - 1)^{\frac{1}{2}m}T_{n-m}^m(\xi)$$

is a real function of ξ for $\xi > 1$. The second solutions,

$$Q_n^m(\xi) = (-1)^m(\xi^2 - 1)^{\frac{1}{2}m}V_{n-m}^m(\xi)$$

(see page 603 for definition of V) may as well be defined to be real for $\xi > 1$, since they are used only for the ξ coordinate.

The charge induced on the surface of the spheroid by the field is given by the quantity

$$\sigma = -\frac{1}{4\pi}\left(\frac{1}{h_\xi}\frac{\partial\psi}{\partial\xi}\right)_{\xi=\xi_0} = -\frac{1}{2\pi a}\sqrt{\frac{\xi_0^2-1}{\xi_0^2-\eta^2}}\left(\frac{\partial\psi}{\partial\xi}\right)_{\xi=\xi_0}$$

Because of the factor $\sqrt{\xi_0^2 - \eta^2}$ in the denominator, the surface density σ tends to be greater at the two ends of the spheroid ($\eta = \pm 1$) than it is elsewhere.

Another problem of interest is that of a half spheroid ($\xi = \xi_0$ for $0 \le \eta \le 1$) kept at potential V_0, with respect to the plane ($\eta = 0$ for $\xi_0 \le \xi$) which is grounded. In the limit of values of ξ_0 near unity this corresponds to the case of a charged vertical antenna or a probe pipe driven into the ground to be used as a source or sink of current or of seepage water. To fit the boundary conditions at $\eta = 0$, we must use zonal harmonics of odd order, which have nodes at $\eta = 0$. We set up the series

$$\psi = \sum_{n=0}^{\infty} A_n P_{2n+1}(\eta) \left[\frac{Q_{2n+1}(\xi)}{Q_{2n+1}(\xi_0)} \right]$$

with the functions P_{2n+1} being a complete set for the range $0 \le \eta \le 1$. Therefore, in order that $\psi = V_0$ for $\xi = \xi_0$ and for positive values of η,

$$\begin{aligned} A_n &= (4n + 3) \int_0^1 V_0 P_{2n+1}(\eta)\, d\eta = V_0[P_{2n}(0) - P_{2n+2}(0)] \\ &= \frac{(-1)^n (2n)!(4n + 3)}{2^{2n+1} n!(n + 1)!} V_0 \end{aligned}$$

from the table at the end of this chapter. Consequently

$$\psi = V_0 \left\{ \tfrac{3}{2} P_1(\eta) \frac{Q_1(\xi)}{Q_1(\xi_0)} - \frac{1 \cdot 7}{2 \cdot 4} P_3(\eta) \frac{Q_3(\xi)}{Q_3(\xi_0)} + \frac{1 \cdot 3 \cdot 11}{2 \cdot 4 \cdot 6} P_5(\eta) \frac{Q_5(\xi)}{Q_5(\xi_0)} - \cdots \right\} \tag{10.3.49}$$

At large distances the first term predominates, and

$$Q_1(\xi) = \tfrac{1}{2}\xi \ln\left(\frac{\xi + 1}{\xi - 1}\right) - 1 \simeq \frac{1}{3\xi^2}; \quad \xi \gg 1$$

so that

$$\psi \simeq \tfrac{1}{8} V_0 [a^2/r^2 Q_1(\xi_0)] \cos \vartheta; \qquad r \gg a$$

where r is the radius vector from the base of the half spheroid to the point where ψ is measured and ϑ is the angle between r and the axis of the half spheroid.

When the half spheroid is very long and thin, its height above the grounded plane is approximately equal to $\frac{1}{2}a$, half the interfocal distance. If its radius at the ground ($\eta = 0$) is ρ, then $\rho = \frac{1}{2}a \sqrt{\xi_0^2 - 1}$ or $\xi_0^2 = 1 + (2\rho/a)^2$. Since

$$Q_n(\xi) \simeq \tfrac{1}{2} \ln\left(\frac{\xi + 1}{\xi - 1}\right) \quad \text{and} \quad \frac{d}{d\xi} Q_n(\xi) \simeq \frac{-1}{\xi^2 - 1}$$

as ξ approaches unity, we have the potential at the surface and the charge density at the surface equaling

$$\psi = V_0\left[\tfrac{3}{2}P_1(\eta) - \frac{1\cdot 7}{2\cdot 4}P_3(\eta) + \cdots\right] = V_0; \quad \eta > 0$$

$$\begin{aligned}\sigma &= -\frac{(2/a)}{4\pi}\sqrt{\frac{\xi_0^2 - 1}{\xi_0^2 - \eta^2}}\left[\frac{\partial\psi}{\partial\xi}\right]_{\xi=\xi_0} \\ &\simeq \frac{1}{2\pi a}\left(\frac{2\rho}{a}\right)\frac{V_0}{\sqrt{1-\eta^2}}\frac{(a/2\rho)^2}{\ln(a/\rho)}\left[\tfrac{3}{2}P_1(\eta) - \frac{1\cdot 7}{2\cdot 4}P_3(\eta) + \cdots\right] \\ &\simeq \frac{1}{4\pi\rho}\frac{V_0}{\ln(a/\rho)}\frac{1}{\sqrt{1-\eta^2}}; \quad (\eta > 0); \quad \text{when } \xi_0^2 = 1 + \left(\frac{2\rho}{a}\right)^2 \to 1\end{aligned}$$

The total charge on the surface of the half spheroid in this limiting case is

$$Q = \int_0^{2\pi} d\phi \int_0^1 \sigma\frac{a^2}{4}\sqrt{(\xi_0^2 - 1)(1 - \eta^2)}\, d\eta \simeq \frac{V_0 a}{4\ln(a/\rho)};$$
$$\xi_0^2 = 1 + \left(\frac{2\rho}{a}\right)^2 \to 1$$

The capacitance of this long half spheroid, with respect to the grounded plane, is therefore $[a/4\ln(a/\rho)]$, which is equal to the capacitance of a cylinder of radius ρ and length $\frac{1}{2}a$ with respect to a grounded, concentric cylinder of radius a (see page 1184).

Integral Representations for Spheroidal Solutions. The expression for X in prolate spheroidal coordinates is

$$X = z + i(x\cos u + y\sin u) = \tfrac{1}{2}a[\eta\xi - \sqrt{(\xi^2 - 1)(\eta^2 - 1)}\cos(u - \phi)]$$

Referring back to Eq. (10.3.39) we see that

$$P_n[\eta\xi - \sqrt{(\xi^2-1)(\eta^2-1)}\cos(\theta)] = \frac{1}{2\pi}\int_0^{2\pi}\frac{[\eta + \sqrt{\eta^2-1}\cos(\theta - w)]^n}{[\xi + \sqrt{\xi^2 - 1}\cos(w)]^{n+1}}\,dw$$

and that

$$\begin{aligned}P_n^m(\eta) &= \frac{n!}{2\pi i^m(n-m)!}\int_0^{2\pi}\frac{\cos(mu)\,du}{[\eta + \sqrt{\eta^2 - 1}\cos(u)]^{n+1}} \\ &= \frac{i^m(n+m)!}{2\pi n!}\int_0^{2\pi}\cos(mu)[\eta^2 + \sqrt{\eta^2 - 1}\cos u]^n\,du\end{aligned}$$

We therefore can see that the integral representation for solutions of the first kind is

$$\begin{aligned}&\int_0^{2\pi} P_n\left(\frac{2X}{a}\right)\frac{\cos}{\sin}(mu)\,du \\ &\quad = \frac{1}{2\pi}\int_0^{2\pi}\frac{\cos}{\sin}(mu)\,du\int_0^{2\pi}\frac{[\eta + \sqrt{\eta^2 - 1}\cos(u - \phi - w)]^n}{[\xi + \sqrt{\xi^2 - 1}\cos(w)]^{n+1}}\,dw \\ &\quad = 2\pi i^{-m}\frac{(n-m)!}{(n+m)!}\frac{\cos}{\sin}(m\phi)[(\xi^2 - 1)(1 - \eta^2)]^{\frac{1}{2}m}T_{n-m}^m(\xi)T_{n-m}^m(\eta) \\ &\quad = 2\pi\frac{(n-m)!}{(n+m)!}\frac{\cos}{\sin}(m\phi)P_n^m(\xi)P_n^m(\eta) \qquad (10.3.50)\end{aligned}$$

Also by methods similar to those by which we obtained Eqs. (10.3.34) and (10.3.39) we can obtain

$$Q_n^m(\xi) = \frac{in!}{(n-m)!}\int_0^{i\infty} \frac{\cos(mu)\,du}{[\xi + \sqrt{\xi^2-1}\cos u]^{n+1}}$$

and

$$Q_n[\xi\eta - \sqrt{(\xi^2-1)(\eta^2-1)}\cos\theta] = i\int_0^{i\infty} \frac{[\eta + \sqrt{\eta^2-1}\cos(\theta - w)]^n}{[\xi + \sqrt{\xi^2-1}\cos w]^{n+1}}\,dw$$

Therefore the integral representation for the solutions of the second kind is

$$\begin{aligned}\int_0^{2\pi} Q_n\left(\frac{2X}{a}\right) {\cos \atop \sin}(mu)\,du &= 2\pi\frac{(n-m)!}{(n+m)!}{\cos \atop \sin}(m\phi)P_n^m(\eta)Q_n^m(\xi) \\ &= 2\pi(-1)^m\frac{(n-m)!}{(n+m)!}{\cos \atop \sin}(m\phi)[(\xi^2-1)(1-\eta^2)]^{\frac{1}{2}m}T_{n-m}^m(\eta)V_{n-m}^m(\xi)\end{aligned} \tag{10.3.51}$$

These integral representations enable us to calculate the potential distribution outside a prolate spheroid of interfocal distance a, major semiaxis $\frac{1}{2}a\xi_0$, and minor semiaxis $\frac{1}{2}a\sqrt{\xi_0^2 - 1} = \rho$, at potential V_0, inside a grounded, concentric, hollow sphere of inside diameter c, larger than $a\xi_0$. To the "zeroth" approximation the potential near the spheroid is

$$\begin{aligned}V_0\frac{Q_0(\xi)}{Q_0(\xi_0)} &= \frac{V_0}{\ln\left[\dfrac{\sqrt{1+(2\rho/a)^2}+1}{\sqrt{1+(2\rho/a)^2}-1}\right]}\ln\left[\frac{\xi+1}{\xi-1}\right] \\ &= \frac{V_0}{2\pi Q_0(\xi_0)}\int_0^{2\pi} Q_0\left(\frac{2X}{a}\right)du\end{aligned}$$

But for $|X|$ larger than a, we have

$$Q_0\left(\frac{2X}{a}\right) = \left(\frac{a}{2X}\right)\left[1 + \tfrac{1}{3}\left(\frac{a}{2X}\right)^2 + \tfrac{1}{5}\left(\frac{a}{2X}\right)^4 + \cdots\right]$$

Therefore, using Eqs. (10.3.35), the zeroth approximation, for r larger than $\frac{1}{2}a$, can be written

$$V_0\frac{Q_0(\xi)}{Q_0(\xi_0)} = \frac{V_0}{Q_0(\xi_0)}\left[\frac{a}{2r} + \tfrac{1}{3}\left(\frac{a}{2r}\right)^3 P_2(\cos\vartheta) + \tfrac{1}{5}\left(\frac{a}{2r}\right)^5 P_4(\cos\vartheta) + \cdots\right]$$

where r, ϑ, ϕ, are the spherical coordinates with center at the center of the spheroid and sphere.

At the inner surface of the sphere ($r = c$), the potential should be zero. To the second approximation in the quantity $a/2c$, which is assumed smaller than unity, we can obtain this by subtracting the solution of Laplace's equation

$$\frac{V_0}{Q_0(\xi_0)}\left[\left(\frac{a}{2c}\right) + \tfrac{1}{3}\left(\frac{a}{2c}\right)^3\left(\frac{r}{c}\right)^2 P_2(\cos\vartheta)\right]$$

which cancels the first two terms of the zeroth approximation at $r = c$. We should also modify the constant $[V_0/Q_0(\xi_0)]$ to the noncommittal A, which can then be adjusted to make the spherical surface have the potential V_0.

Therefore, if $a/2c$ is less than $\frac{1}{2}$, a fairly good approximation to the true potential near the spherical surface is

$$\psi \simeq A\left\{\left[\left(\frac{a}{2r}\right) - \left(\frac{a}{2c}\right)\right] + \tfrac{1}{3}\left[\left(\frac{a}{2r}\right)^3 - \left(\frac{a}{2c}\right)^3\left(\frac{r}{c}\right)^2\right] P_2(\cos\vartheta)\right\}$$

which is zero at $r = c$ and should be constant, to the second order in $a/2c$, at $\xi = \xi_0$.

Near the spheroidal surface we need to transform back to spheroidal solutions. We use Eq. (10.3.34) to obtain $r^2P_2(\cos\vartheta) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} X^2\,du$, which with Eq. (6.3.40), $(2X/a)^2 = \frac{2}{3}P_2(2X/a) + \frac{1}{3}P_0(2X/a)$, allows us to convert the additional r^2P_2 term into spheroidal solutions by means of Eq. (10.3.50):

$$r^2P_2(\cos\vartheta) = \frac{a^2}{8\pi}\int_0^{2\pi}\left[\tfrac{2}{3}P_2\left(\frac{2X}{a}\right) + \tfrac{1}{3}P_0\left(\frac{2X}{a}\right)\right]du$$
$$= [\tfrac{2}{3}P_2(\eta)P_2(\xi) + \tfrac{1}{3}]\left(\frac{a}{2}\right)^2$$

Therefore the potential function near the spheroid is, approximately,

$$\psi \simeq A\left\{Q_0(\xi) - \left(\frac{a}{2c}\right) - \left(\frac{a}{2c}\right)^5[\tfrac{2}{3}P_2(\eta)P_2(\xi) + \tfrac{1}{3}]\right\}$$

which is nearly constant at $\xi = \xi_0$, with the exception of the term to the fifth order in the small quantity $a/2c$. In order that ψ be V_0 at $\xi = \xi_0$, to the fourth order, we set $A = V_0/[Q_0(\xi_0) - (a/2c)]$.

Consequently, if the diameter of the spheroid at its mid-section 2ρ is considerably smaller than its length, which is approximately a, and if a is smaller than the radius of the hollow, grounded sphere, then the potential between spheroid and sphere, to the fourth order in $a/2c$, is

$$\psi \simeq V_0\frac{\ln[(\xi-1)/(\xi+1)] - (a/c)}{\ln[(\xi_0-1)/(\xi_0+1)] - (a/c)};\quad \xi_0^2 = 1 + (2\rho/a)^2 \tag{10.3.52}$$

$$\psi \simeq \frac{2V_0}{\{\ln[(\xi_0+1)/(\xi_0-1)] - (a/c)\}}\left\{\left(\frac{a}{2r}\right) - \left(\frac{a}{2c}\right) + \tfrac{1}{3}\left[\left(\frac{a}{2r}\right)^3 - \left(\frac{a}{2c}\right)^3\left(\frac{r}{c}\right)^2\right]P_2(\cos\vartheta)\right\}$$

The total charge on the inside of the hollow sphere, $-(1/4\pi)(\partial\psi/\partial r)$ at $r = c$, turns out to be

$$Q \simeq \frac{aV_0}{\{\ln[(\xi_0 + 1)/(\xi_0 - 1)] - (a/c)\}} \simeq \frac{aV_0}{2\ln(a/\rho) + (\rho/a)^2 - (a/c)}; \quad \rho \ll a$$

The capacitance of the system is, of course, Q/V_0.

Green's Function for Prolate Spheroids. The proper expansion to obtain the Green's function can be obtained from Eq. (6.3.44). We use Eq. (10.3.8) to get

$$\frac{1}{R} = \frac{1}{2\pi}\int_0^{2\pi} \frac{du}{X - X_0} = \frac{1}{\pi a}\sum_{n=0}^{\infty} (2n+1)\int_0^{2\pi} P_n\left(\frac{2X_0}{a}\right) Q_n\left(\frac{2X}{a}\right) du; \qquad r > r_0$$

Each term of this series should be a Fourier series in $\cos(\phi - \phi_0)$. In computing the coefficients we find we can use Eqs. (10.3.50) and (10.3.51) to obtain the final answer:

$$\frac{1}{R} = \frac{2}{a}\sum_{n=0}^{\infty}(2n+1)\sum_{m=0}^{n}\epsilon_m i^m \left[\frac{(n-m)!}{(n+m)!}\right]^2 \cos[m(\phi - \phi_0)] \cdot \\ \cdot P_n^m(\eta_0)P_n^m(\eta)\begin{cases} P_n^m(\xi_0)Q_n^m(\xi); & \xi > \xi_0 \\ P_n^m(\xi)Q_n^m(\xi_0); & \xi < \xi_0 \end{cases} \tag{10.3.53}$$

The potential caused by a charge Q at the point ξ_0, η_0, ϕ_0, outside a grounded conducting spheroid $\xi = \xi_1$, is

$$\psi = \frac{2Q}{a}\sum_{n=0}(2n+1)\sum_{m=0}^{n}\epsilon_m i^m\left[\frac{(n-m)!}{(n+m)!}\right]^2\cos[m(\phi-\phi_0)]P_n^m(\eta_0) \cdot \\ \cdot P_n^m(\eta)\left[\begin{Bmatrix} P_n^m(\xi_0)Q_n^m(\xi) \\ P_n^m(\xi)Q_n^m(\xi_0)\end{Bmatrix} - \frac{Q_n^m(\xi_0)}{Q_n^m(\xi_1)}P_n^m(\xi_1)Q_n^m(\xi)\right] \tag{10.3.54}$$

where the upper term in the braces is used when $\xi > \xi_0$ and the lower when $\xi < \xi_0$.

The charge density induced on the spheroid is

$$\begin{aligned}\sigma &= \frac{-1}{2\pi a}\sqrt{\frac{\xi_1^2 - 1}{\xi_1^2 - \eta^2}}\left[\frac{\partial}{\partial\xi}\psi\right]_{\xi=\xi_1} \\ &= -\frac{Q}{2\pi a}\sqrt{\frac{\xi_1^2-1}{\xi_1^2-\eta^2}}\sum_{n,m}(2n+1)i^m\epsilon_m\left[\frac{(n-m)!}{(n+m)!}\right]^2\cos[m(\phi-\phi_0)] \cdot \\ &\qquad \cdot P_n^m(\eta_0)P_n^m(\eta)[Q_n^m(\xi_0)/Q_n^m(\xi_1)]\Delta(P_n^m,Q_n^m) \\ &= \frac{Q}{\pi a^2\sqrt{(\xi_1^2-1)(\xi_1^2-\eta^2)}}\sum_{n,m}(2n+1)(-1)^m\epsilon_m\frac{(n-m)!}{(n+m)!} \\ &\qquad P_n^m(\eta_0)\frac{Q_n^m(\xi_0)}{Q_n^m(\xi_1)}P_n^m(\eta)\cos[m(\phi-\phi_0)]\end{aligned}$$

where Δ is the Wronskian of P and Q, which reduces to a simple form. The integral of this, $\sigma h_\eta \, d\eta \, h_\phi \, d\phi$, integrated over the surface $\xi = \xi_1$, is, of course, just equal to Q.

Oblate Spheroids. If we rotate confocal elliptic coordinates about their minor axes, we obtain oblate spheroidal coordinates:

$$\begin{aligned}
z &= a\xi\eta; \quad x = a\sqrt{(\xi^2+1)(1-\eta^2)}\cos\phi \\
y &= a\sqrt{(\xi^2+1)(1-\eta^2)}\sin\phi \\
h_\xi &= a\sqrt{\frac{\xi^2+\eta^2}{\xi^2+1}}; \quad h_\eta = a\sqrt{\frac{\xi^2+\eta^2}{1-\eta^2}} \\
h_\phi &= a\sqrt{(\xi^2+1)(1-\eta^2)} \\
\nabla^2\psi &= \frac{1}{a^2(\xi^2+\eta^2)}\left[\frac{\partial}{\partial\xi}(\xi^2+1)\frac{\partial\psi}{\partial\xi} + \frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial\psi}{\partial\eta} \right.\\
&\qquad\qquad \left. + \frac{\xi^2+1-1+\eta^2}{(\xi^2+1)(1-\eta^2)}\frac{\partial^2\psi}{\partial\phi^2}\right]
\end{aligned} \tag{10.3.55}$$

where ξ goes from 0 to ∞ and η goes from -1 to $+1$. The surface $\xi = 0$ is a disk of radius a, in the x, y plane, with center at the origin. The surface $\eta = 1$ is the positive z axis, the surface $\eta = -1$ the negative z axis, and the surface $\eta = 0$ is the x, y plane *except* for the part inside a circle of radius a, centered at the origin (which portion is the surface $\xi = 0$). The surfaces $\xi = $ constant > 0 are flattened spheroids of thickness, through the axis, of $(2\xi a)$ and of radius, at the equator, of $a\sqrt{\xi^2+1}$. The surfaces $\eta = $ constant are hyperboloids of one sheet, asymptotic to the cone of angle $\cos^{-1}\eta$ with respect to the z axis, which is the axis of the cone.

The separated equations for ϕ and for η are the same as for the prolate spheroids, the factors being

$$\sin(m\phi),\ \cos(m\phi),\ \text{and } P_n^m(\eta)$$

where m and n are positive integers (or zero). The equation for the ξ factor is that for the functions

$$P_n^m(i\xi) \quad \text{and} \quad Q_n^m(i\xi)$$

The potential of a disk of radius a ($\xi = 0$) held at potential V_0 with respect to infinity is, thus,

$$\begin{aligned}
\psi &= \left(\frac{V_0}{Q_0(0)}\right) Q_0(i\xi) = \left(\frac{V_0}{\ln(-1)}\right)\ln\left(\frac{i\xi+1}{i\xi-1}\right) \\
&= \left(\frac{iV_0}{\pi}\frac{2}{i}\right)\tan^{-1}\left(\frac{1}{\xi}\right) = \left(\frac{2V_0}{\pi}\right)\tan^{-1}\left(\frac{1}{\xi}\right)
\end{aligned} \tag{10.3.56}$$

when ξ is very large, it becomes approximately equal to r/a, where r is the radial distance from the center of the disk. Therefore the asymptotic form of the potential is $2V_0a/\pi r$, indicating that the total charge is $2V_0a/\pi$ and that the capacitance of a disk of radius a is just $2a/\pi$.

A velocity potential representing steady flow in the direction normal to the y axis and at angle θ to the z axis is

$$\begin{aligned}\psi &= -v_0(z\cos\theta + x\sin\theta)\\ &= -av_0[\xi\eta\cos\theta + \sqrt{(\xi^2+1)(1-\eta^2)}\cos\phi\sin\theta]\\ &= iav_0[\cos\theta\, P_1^0(\eta)P_1^0(i\xi) + \sin\theta\cos\phi\, P_1^1(\eta)P_1^1(i\xi)]\end{aligned}$$

If now we insert an oblate spheroid $\xi = \xi_0$ with boundary condition that the flow be tangential to the surface, we must add terms of the sort

$$AP_1^0(\eta)Q_1^0(i\xi) + BP_1^1(\eta)Q_1^1(i\xi)$$

with A and B adjusted so that the ξ derivative of the sum, at $\xi = \xi_0$, is zero. For the case of a flat disk, $\xi_0 = 0$, we have $A = 2av_0/\pi$ and $B = 0$, so that

$$\psi = -av_0\left\{\eta\cos\theta\left[\xi + \frac{2}{\pi} - \frac{2}{\pi}\xi\tan^{-1}\left(\frac{1}{\xi}\right)\right] + \sin\theta\cos\phi\sqrt{(\xi^2+1)(1-\eta^2)}\right\} \qquad (10.3.57)$$

At the surface of the disk, at the point $\xi = 0$, η, ϕ, the two components of velocity are

$$v_\eta = \frac{2v_0}{\pi|\eta|}\sqrt{1-\eta^2}\cos\theta - \frac{|\eta|}{\eta}\sin\theta\cos\phi;\quad v_\phi = -v_0\sin\theta\sin\phi$$

and the excess pressure, because of the fluid motion, is

$$-\tfrac{1}{2}\rho v^2 = -\tfrac{1}{2}\rho v_0^2\left[\frac{4}{\pi^2}\cos^2\theta\left(\frac{1-\eta^2}{\eta^2}\right) + \sin^2\theta - \frac{4}{\pi\eta}\sqrt{1-\eta^2}\cos\theta\sin\theta\cos\phi\right]$$

This is minus infinity at the sharp disk edges ($\eta = 0$) of course; it is also not surprising that, when $\theta = 90°$, *i.e.*, when the flow is parallel to the plane of the thin disk, the excess pressure is $-\frac{1}{2}\rho v_0^2$, the same as it would be if the disk were not there. We note also that the velocity at the surface of the disk ($\xi = 0$) is zero at the point $\phi = 0$; $\eta = [\sqrt{1 + (\pi/4)^2\tan^2\theta} - (\pi/4)\tan\theta]$, corresponding to

$$z = y = 0;\quad x = a\left(\frac{\pi}{2}\right)\tan\theta\left[1 + \left(\frac{\pi}{2}\right)^2\tan^2\theta\right]^{-\frac{1}{2}}$$

on the side of the disk pointing in the positive z direction and at the point $\phi = \pi$; $\eta = -[\sqrt{1 + (\pi/4)\tan^2\theta} - (\pi/4)\tan\theta]$ corresponding to

$$z = y = 0;\quad x = -a\left(\frac{\pi}{2}\right)\tan\theta\left[1 + \left(\frac{\pi}{2}\right)^2\tan^2\theta^2\right]^{-\frac{1}{2}}$$

on the negative side of the disk.

We are not yet considering three-dimensional vortex flow, so we shall not here compute the circulational flow about the disk to remove the infinite velocity along the trailing edge, à *la* page 1200. Hence also we cannot now discuss the net forces on the disk.

Flow through an Orifice. Another problem is, of course, the steady irrotational flow through a circular aperture of radius a. The boundary here is the surface $\eta = 0$, and the solution must be continuous at $\xi = 0$. Because of the axial symmetry we use the $n = 0$ solutions. The η factor is just $P_0(\eta) = 1$, whereas the ξ factor is a combination of $P_0(i\xi)$ and $Q_0(i\xi)$ which fits smoothly together at the opening $\xi = 0$. The solution is

$$\psi = \begin{cases} -\dfrac{1}{4a}\dot{Q}\left[1 - \dfrac{2}{\pi}\tan^{-1}\left(\dfrac{1}{\xi}\right)\right]; & 1 \geq \eta > 0 \\ \dfrac{1}{4a}\dot{Q}\left[1 - \dfrac{2}{\pi}\tan^{-1}\left(\dfrac{1}{\xi}\right)\right]; & 0 > \eta \geq -1 \end{cases} \tag{10.3.58}$$

This function has zero value at $\xi = 0$, and the slope is continuous across this surface. At large distances from the opening, $\xi \to r/a$ and the potential becomes

$$\psi \simeq \begin{cases} -(\dot{Q}/4a) + (\dot{Q}/2\pi r); & \eta \text{ positive} \\ +(\dot{Q}/4a) - (\dot{Q}/2\pi r); & \eta \text{ negative} \end{cases}$$

The velocity is everywhere perpendicular to the ξ = constant surfaces and is

$$v_\xi = \begin{cases} \dfrac{(\dot{Q}/2\pi a^2)}{\sqrt{(\xi^2 + \eta^2)(\xi^2 + 1)}}; & \eta \text{ positive} \\ \dfrac{-(\dot{Q}/2\pi a^2)}{\sqrt{(\xi^2 + \eta^2)(\xi^2 + 1)}}; & \eta \text{ negative} \end{cases}$$

The total flow in cubic centimeters per second out of the positive side (which equals the total flow inward on the negative side) is then

$$\int_0^{2\pi} d\phi \int_0^1 d\eta\, v_\xi\, h_\eta\, h_\phi = \iint v_\xi\, dS = \dot{Q}$$

which is, of course, why we chose the constant $\dot{Q}$ in Eq. (10.3.58) the way we did.

If we chose ξ_0 large enough, the kinetic energy of the fluid outside the surface $\xi = \xi_0$ is negligible, and to obtain the total kinetic energy of the fluid we compute the volume integral of $\frac{1}{2}\rho v^2$ inside this surface: $T = \frac{1}{2}\rho\iiint |\text{grad}\,\psi|^2\, dv$. This may be integrated directly, but we can use Green's theorem to change to a surface integral over the surface S, which is the surface $\xi = \xi_0$ for η positive, and S', which is the surface $\xi = \xi_0$ for η negative. Thus we have

$$T = \tfrac{1}{2}\rho\left[\iint \psi\, \text{grad}\,\psi\, dS' - \iint \psi\, \text{grad}\,\psi\, dS\right]$$

Since the surfaces S and S' are far enough away from the orifice so that

ψ is practically constant over the surfaces, this becomes

$$T = \tfrac{1}{2}\rho[\psi_S \textstyle\iint v_\xi \, dS - \psi_{S'} \iint v_\xi \, dS'] = \tfrac{1}{2}\rho(\psi_{s'} - \psi_s)\dot{Q} = \tfrac{1}{2}(\rho/2a)\dot{Q}^2 \qquad (10.3.59)$$

by Eq. (10.3.58).

The potential energy of the flow through the orifice for an incompressible fluid can be given in terms of the pressure difference ΔP between the surface S and the surface S'. If a total volume dQ cc is pushed through the opening against this pressure difference, the change in potential energy is

$$dV = -\Delta P \, dQ$$

Utilizing Lagrange's equations (3.2.4) we have for the acceleration of the flow

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Q}}\right) = \left(\frac{\rho}{2a}\right)\ddot{Q} = -\frac{\partial V}{\partial Q} = \Delta P$$

This ratio between pressure drop across the hole and acceleration of flow through the hole, $\rho/2a$, may be called the *inertance* of the orifice. It is related to an effective mass of the "plug" of fluid in and near the hole, which has to be accelerated by the pressure drop. Since Q is total flow, $Q/\pi a^2$ is an average displacement, in centimeters, of the fluid through the hole, and $\pi a^2 \, \Delta P$ is the average force, in dynes, across the hole. Therefore the equation

$$(\tfrac{1}{2}\pi^2 a^3 \rho)(\ddot{Q}/\pi a^2) = (\pi a^2 \, \Delta P)$$

is analogous to Newton's equation, and the quantity

$$M = \tfrac{1}{2}\pi^2 a^3 \rho \qquad (10.3.60)$$

is the effective mass of the "plug" of fluid which has to be accelerated when we initiate flow through the circular orifice in the plane. This expression will be useful when we come to calculate the transmission of waves through an orifice in the next chapter.

Integral Representations and Green's Functions. It is not difficult to modify Eqs. (10.3.50) and (10.3.51) to correspond to the oblate spheroidal coordinates. The new expression for X is

$$X = a[\xi\eta - \sqrt{(\xi^2 + 1)(\eta^2 - 1)} \cos(u - \phi)]$$

and the integral representations are

$$\begin{aligned}
\int_0^{2\pi} P_n\left(\frac{X}{a}\right) \frac{\cos}{\sin}(mu)\, du
&= \frac{i^{n+1}}{2\pi} \int^{2\pi} \frac{\cos}{\sin}(mu)\, du \int_0^{2\pi} \frac{[\eta + \sqrt{\eta^2 - 1}\cos(u - \phi - w)]^n}{[i\xi + \sqrt{-\xi^2 - 1}\cos(w)]^{n+1}}\, dw \\
&= 2\pi i^{n+1} \frac{(n-m)!}{(n+m)!} \frac{\cos}{\sin}(m\phi)[(\xi^2 + 1)(1 - \eta^2)]^{\frac{1}{2}m} T^m_{n-m}(i\xi) T^m_{n-m}(\eta) \\
&= 2\pi i^{n+1} \frac{(n-m)!}{(n+m)!} \frac{\cos}{\sin}(m\phi) P^m_n(\eta) P^m_n(i\xi) \qquad (10.3.61)
\end{aligned}$$

and the one for the second solution,

$$\int_0^{2\pi} Q_n\left(\frac{X}{a}\right) \frac{\cos}{\sin}(mu)\, du = 2\pi i^{n+1} \frac{(n-m)!}{(n+m)!} \frac{\cos}{\sin}(m\phi) P_n^m(\eta) Q_n^m(i\xi) \tag{10.3.62}$$

In like manner, comparing with Eq. (10.3.54), we find that

$$\frac{1}{R} = \frac{2}{a} \sum_{n=0}^{\infty} (2n+1) \sum_{m=0}^{n} \epsilon_m i^{m+1} \left[\frac{(n-m)!}{(n+m)!}\right]^2 \cos[m(\phi - \phi_0)] P_n^m(\eta_0) \cdot \\ \cdot P_n^m(\eta) \begin{cases} P_n^m(i\xi_0) Q_n^m(i\xi); & \xi > \xi_0 \\ P_n^m(i\xi) Q_n^m(i\xi_0); & \xi_0 > \xi \end{cases} \tag{10.3.63}$$

which enables us to solve Poisson's equation in these coordinates.

Parabolic Coordinates. The coordinate system formed by rotating two-dimensional parabolic coordinates about their axis is also called the parabolic system:

$$z = \tfrac{1}{2}(\lambda^2 - \mu^2); \quad x = \lambda\mu \cos\phi; \quad y = \lambda\mu \sin\phi; \quad r = \tfrac{1}{2}(\lambda^2 + \mu^2)$$
$$h_\lambda = \sqrt{\lambda^2 + \mu^2} = h_\mu; \quad h_\phi = \lambda\mu \tag{10.3.64}$$
$$\nabla^2\psi = \frac{1}{\lambda\mu(\lambda^2+\mu^2)}\left[\mu \frac{\partial}{\partial\lambda}\left(\lambda \frac{\partial\psi}{\partial\lambda}\right) + \lambda \frac{\partial}{\partial\mu}\left(\mu \frac{\partial\psi}{\partial\mu}\right) + \frac{\lambda^2+\mu^2}{\lambda\mu}\left(\frac{\partial^2\psi}{\partial\phi^2}\right)\right]$$

where ϕ goes from 0 to 2π and λ, μ go from 0 to ∞. The equations for the separated factors $\psi = L(\lambda)M(\mu)\Phi(\phi)$ are

$$\frac{d^2\Phi}{d\phi^2} = -m^2\Phi; \quad \frac{1}{\lambda}\frac{d}{d\lambda}\left(\lambda \frac{dL}{d\lambda}\right) - \frac{m^2}{\lambda^2} L = -k^2 L;$$
$$\frac{1}{\mu}\frac{d}{d\mu}\left(\mu \frac{dM}{d\mu}\right) - \frac{m^2}{\mu^2} M = k^2 M$$

If ϕ is allowed to go from zero to 2π, m must be zero or a positive integer and Φ is either sine or cosine of $(m\phi)$. The equations for L and M are Bessel functions of the nth order (see page 619), the one for L for a Bessel function of real argument and the one for M for a Bessel function of imaginary argument. These quantities

$$I_m(z) = (-i)^m J_m(iz) \to e^z/\sqrt{2\pi z}$$
$$K_m(z) = (\pi/2)(i)^{m+1} H_m(iz) \to (\sqrt{\pi/2z})e^{-z}$$

are defined, with some of their properties, in the table at the end of this chapter. They are related to the ordinary Bessel functions in a manner analogous to the relation between the hyperbolic functions $\sinh(z)$, e^{-z} and the trigonometric functions $\sin(z)$, e^{iz}

A typical solution of Laplace's equation in parabolic coordinates is

$$\frac{\cos}{\sin}(m\phi) J_m(k\lambda)[a_m(k) I_m(k\mu) + b_m(k) K_m(k\mu)]$$

where we have chosen μ to have the hyperbolic Bessel functions because we assume that the boundary conditions are on a surface μ = constant.

For instance, if the surface $\mu = \mu_0$ had a rotationally symmetric distribution of potential, $\psi = \psi_0(\lambda)$, specified, and if the potential is to go to zero at $\mu \to \infty$, then we use the Fourier-Bessel integral to obtain

$$\psi(\lambda,\mu) = \int_0^\infty J_0(k\lambda)[K_0(k\mu)/K_0(k\mu_0)]\left[\int_0^\infty \psi_0(u)J_0(ku)u\,du\right]k\,dk \tag{10.3.65}$$

If the boundary is $\lambda = \lambda_0$, we simply interchange μ and λ in this expression.

If the problem is an interior one, we use I's instead of K's; the I's go to zero at $z = 0$, except for I_0, which goes to unity with zero slope. For instance, if the region is the space inside the two paraboloids $\mu = \mu_0$ and $\lambda = \lambda_0$, with potential $\psi_0(\lambda)$ on μ_0 and $\psi_1(\mu)$ on λ_0, we use the Fourier-Bessel series (page 765) to obtain

$$\begin{aligned}\psi(\lambda,\mu) \\ &= \sum_{n=1}^{\infty}\left[\frac{2}{\lambda_0^2 J_1^2(\pi\beta_{0n})}\int_0^{\lambda_0}\psi_0(u)J_0\left(\frac{\pi\beta_{0n}u}{\lambda_0}\right)u\,du\right]J_0\left(\frac{\pi\beta_{0n}\lambda}{\lambda_0}\right)\frac{I_0(\pi\beta_{0n}\mu/\lambda_0)}{I_0(\pi\beta_{0n}\mu_0/\lambda_0)} \\ &+ \sum_{n=1}^{\infty}\left[\frac{2}{\mu_0^2 J_1^2(\pi\beta_{0n})}\int_0^{\mu_0}\psi_1(w)J_0\left(\frac{\pi\beta_{0n}w}{\mu_0}\right)w\,dw\right]J_0\left(\frac{\pi\beta_{0n}\mu}{\mu_0}\right)\frac{I_0(\pi\beta_{0n}\lambda/\mu_0)}{I_0(\pi\beta_{0n}\lambda_0/\mu_0)}\end{aligned} \tag{10.3.66}$$

The integral representation for these solutions is not easy to obtain in a straightforward manner. Since the coordinates λ, μ have dimensions of the square root of length and since the solutions are Bessel functions, we might expect to have the integrand of Eq. (10.3.2) be $J_n(k\sqrt{2X})$ times a cosine or sine of (mu). Excising large sequences of algebraic detail, the major steps in the derivation are as follows: By use of Eq. (5.3.65) we have

$$\begin{aligned}J_0(k\sqrt{2X}) \\ &= \frac{1}{2\pi}\int_0^{2\pi}\exp\{ik\cos v\sqrt{[\lambda\cos(u-\phi)+i\mu]^2+\lambda^2\sin^2(u-\phi)}\}\,dv\end{aligned}$$

But

$$\sqrt{[\lambda\cos(u-\phi)+i\mu]^2+\lambda^2\sin^2(u-\phi)}\cos v = \lambda\cos(u-t-\phi)+i\mu\cos t$$

where
$$t = v + \tan^{-1}\left[\frac{\lambda\sin(u-\phi)}{\lambda\cos(u-\phi)+i\mu}\right]$$

Therefore,

$$\begin{aligned}\int_0^{2\pi} J_0(k\sqrt{2X})\,{}^{\cos}_{\sin}(mu)\,du \\ &= \frac{1}{2\pi}\int_0^{2\pi}dw\int_0^{2\pi}dt\,e^{ik\lambda\cos w - k\mu\cos t}\,{}^{\cos}_{\sin}[m(w+t+\phi)] \\ &= 2\pi i^{-m}\,{}^{\cos}_{\sin}(m\phi)J_m(k\lambda)I_m(k\mu)\end{aligned} \tag{10.3.67}$$

The integral expression with $K_m(k\mu)$ instead of $I_m(k\mu)$ has $H_0(k\sqrt{2X})$ instead of $J_0(k\sqrt{2X})$ in the integral.

To find the expression for the Green's function, we use the Fourier-Bessel integral and an integral given on page 1324 for

$$K_0(-iz) = \left(\frac{i\pi}{2}\right) H_0(z)$$

to obtain

$$\frac{\frac{1}{2}}{X - X_0} = \int_0^\infty J_0(k\sqrt{2X})k\,dk \int^\infty \frac{J_0(ky)}{y^2 - 2X_0}\,y\,dy$$
$$= 2\pi^2 i \int_0^\infty J_0(k\sqrt{2X})H_0(k\sqrt{2X_0})k\,dk$$

and finally,

$$\frac{1}{R} = 2\sum_{m=0}^\infty \epsilon_m \cos[m(\phi - \phi_0)]\cdot$$
$$\cdot \int_0^\infty J_m(k\lambda)J_m(k\lambda_0)I_m(k\mu)K_m(k\mu_0)k\,dk; \quad \mu_0 > \mu \quad (10.3.68)$$

It is hardly remunerative to discuss this coordinate system in more detail.

Bispherical Coordinates. A more rewarding set of coordinates is obtained by rotating bipolar axes about the line between the two poles:

$$\begin{aligned}
z &= \frac{a\sinh\mu}{\cosh\mu - \cos\eta}; \quad x = \frac{a\sin\eta\cos\phi}{\cosh\mu - \cos\eta} \\
y &= \frac{a\sin\eta\sin\phi}{\cosh\mu - \cos\eta}; \quad h_\mu = h_\eta = \frac{a}{\cosh\mu - \cos\eta} \\
h_\phi &= \frac{a\sin\eta}{\cosh\mu - \cos\eta}; \quad r = a\sqrt{\frac{\cosh\mu + \cos\eta}{\cosh\mu - \cos\eta}} \\
\nabla^2\psi &= \frac{1}{h_\mu^3}\left[\frac{\partial}{\partial\mu}\left(h_\mu\frac{\partial\psi}{\partial\mu}\right) + \frac{1}{\sin\eta}\frac{\partial}{\partial\eta}\left(h_\mu\sin\eta\frac{\partial\psi}{\partial\eta}\right) + \frac{h_\mu}{\sin^2\eta}\frac{\partial^2\psi}{\partial\phi^2}\right]
\end{aligned} \quad (10.3.69)$$

where μ goes from $-\infty$ to ∞, η goes from 0 to π, and ϕ goes from 0 to 2π. The surface $\mu = \mu_0$ is a sphere of radius $a\,|\mathrm{csch}\,\mu_0|$ with center at $z = a\coth\mu_0$, $x = y = 0$. The two poles are at $\mu = \pm\infty$ ($z = +a$, $x = y = 0$), and the central plane $z = 0$ is the surface $\mu = 0$. The surface $\eta = \eta_0$ is a fourth-order surface formed by rotating about the z axis that part of the circle, in the $x - z$ plane, of radius $a\csc\eta_0$ with center at $z = 0$, $x = a\cot\eta_0$, which is in the positive part of the $x - z$ plane. All these surfaces, of constant η, go through the two poles. Those for $\eta < \frac{1}{2}\pi$ have "dimples" at each pole; those for $\eta > \frac{1}{2}\pi$ have sharp points there. The surface $\eta = 0$ is the z axis for $|z| > a$ plus the sphere at infinity, the surface $\eta = \frac{1}{2}\pi$ is the sphere of radius a with center at the origin, and the surface $\eta = \pi$ is the z axis for $|z| < a$.

As indicated on page 665, the Laplace equation separates only by setting $\psi = \sqrt{\cosh\mu - \cos\eta}\,F$. The equation $\nabla^2\psi = 0$ then transforms to

$$\frac{\partial^2 F}{\partial\mu^2} + \frac{1}{\sin\eta}\frac{\partial}{\partial\eta}\left(\sin\eta\frac{\partial F}{\partial\eta}\right) + \frac{1}{\sin^2\eta}\frac{\partial^2 F}{\partial\phi^2} - \tfrac{1}{4}F = 0$$

which separates into the three equations, $F = M(\mu)H(\eta)\Phi(\phi)$,

$$(d^2\Phi/d\phi^2) = -m^2\Phi;\quad (d^2M/d\mu^2) = (n + \tfrac{1}{2})^2M$$
$$\frac{1}{\sin\eta}\frac{d}{d\eta}\left(\sin\eta\frac{dH}{d\eta}\right) - \frac{m^2H}{\sin^2\eta} = -n(n+1)H$$

All these equations should be familiar by now. In order that Φ be continuous as ϕ goes from 0 around to 2π, m must be zero or an integer, and in order that H be finite both at $\eta = 0$ and $\eta = \pi$, n must be an integer equal to or larger than m. Therefore, typical solutions are

$$e^{-(n+\frac{1}{2})\mu}P_n^m(\cos\eta)\cos(m\phi);\quad \sinh[(n+\tfrac{1}{2})\mu]P_n^m(\cos\eta)\sin(m\phi);\quad \text{etc.}$$

and typical solutions of the original Laplace equation are

$$\sqrt{\cosh\mu - \cos\eta}\; e^{\pm(n+\frac{1}{2})\mu}P_n^m(\cos\eta)\,{\cos \atop \sin}(m\phi);\quad \text{etc.}$$

The factor $\sqrt{\cosh\mu - \cos\eta}$ causes some trouble when we come to fix boundary conditions. For instance, to compute the potential outside a sphere of radius ρ, with center a distance b from the origin [sphere $\mu = \mu_0 = \cosh^{-1}(b/\rho)$ for interpolar distance $2a = 2\sqrt{b^2 - \rho^2}$], which is at uniform potential V_0, we have to compute the series

$$\sqrt{\cosh\mu - \cos\eta}\sum_{n=0}^{\infty}[A_ne^{(n+\frac{1}{2})\mu} + B_ne^{-(n+\frac{1}{2})\mu}]P_n(\cos\eta)$$

which reduces to the constant V_0 at $\mu = \mu_0$. In other words we must compute the expansion of $(\cosh\mu - \cos\eta)^{-\frac{1}{2}}$ in terms of zonal harmonics of $\cos\eta$. This can be done by taking the expansion of $(1 + h^2 - 2h\cos\eta)^{-\frac{1}{2}}$, given in Eq. (5.3.27), for $h = e^{-\mu}$. We have, finally,

$$\frac{1}{\sqrt{\cosh\mu - \cos\eta}} = \sqrt{2}\sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})|\mu|}P_n(\cos\eta) \tag{10.3.70}$$

With this result achieved, it is not difficult to show that the potential distribution between the conducting sphere $\mu = \mu_0$ (radius $\rho = a\,\mathrm{csch}\,\mu_0$, center at $x = y = 0$, $z = b = a\coth\mu_0$), held at potential V_0, and the grounded plane $\mu = 0$ ($z = 0$) is given by the series

$$\psi = \sqrt{2}\,V_0\sqrt{\cosh\mu - \cos\eta}\sum_{n=0}^{\infty}\frac{e^{-(n+\frac{1}{2})\mu_0}}{\sinh[(n+\frac{1}{2})\mu_0]}\sinh[(n+\tfrac{1}{2})\mu]P_n(\cos\eta) \tag{10.3.71}$$

The charge density on the surface of the plane $\mu = 0$ is

$$\sigma = \frac{1 - \cos\eta}{4\pi a}\left(\frac{\partial\psi}{\partial\mu}\right)_{\mu=0}$$
$$= \frac{V_0}{\sqrt{8}\pi a}(1 - \cos\eta)^{\frac{3}{2}} \sum_n^\infty (n + \tfrac{1}{2}) \frac{e^{-(n+\frac{1}{2})\mu_0}}{\sinh[(n + \frac{1}{2})\mu_0]} P_n(\cos\eta)$$

and the total charge induced on this plane is

$$\iint \frac{\sigma a^2 \sin\eta\, d\eta\, d\phi}{(1 - \cos\eta)^2}$$
$$= \frac{aV_0}{\sqrt{8}\pi}\int_0^{2\pi} d\phi \int_0^\pi \sum_n \frac{(n + \frac{1}{2})}{\sqrt{1 - \cos\eta}} \frac{e^{-(n+\frac{1}{2})\mu_0}}{\sinh[(n + \frac{1}{2})\mu_0]} P_n(\cos\eta)\sin\eta\, d\eta$$

Using Eq. (10.3.70) again to obtain a double sum of products of zonal harmonics and then using the orthogonal properties of these functions, we finally obtain

$$Q = aV_0 \sum_{n=0}^\infty \frac{2}{e^{(2n+1)\mu_0} - 1} = \sum_{n=0}^\infty \frac{2V_0\sqrt{b^2 - \rho^2}}{[(b/\rho) + \sqrt{(b/\rho)^2 - 1}]^{2n+1} - 1} \tag{10.3.72}$$

from which we can obtain the capacitance of a sphere of radius ρ with respect to a plane a distance b from its center.

By differentiating Eq. (10.3.70) with respect to μ or η, we obtain expressions for x, y, and z:

$$|z| = \frac{a\sinh|\mu|}{\cosh\mu - \cos\eta}$$
$$= \sqrt{2}\, a\sqrt{\cosh\mu - \cos\eta}\sum_{n=0}^\infty (2n + 1)e^{-(n+\frac{1}{2})|\mu|}P_n(\cos\eta) \tag{10.3.73}$$
$$\begin{matrix} x \\ y \end{matrix} = \frac{a\sin\eta}{\cosh\mu - \cos\eta}\begin{matrix}\cos \\ \sin\end{matrix}\phi$$
$$= \sqrt{8}\, a\sqrt{\cosh\mu - \cos\eta}\sum_{n=1}^\infty e^{-(n+\frac{1}{2})|\mu|}P'_n(\cos\eta)\begin{matrix}\cos \\ \sin\end{matrix}\phi$$

These series, of course, do not converge at $\mu = 0$, as evidenced by the fact that $|\mu|$ appears. From these formulas we can show that the potential around two grounded, conducting spheres, at $\mu = \pm\mu_0$, in a uniform field E along the z axis, is

$$\psi = -Ez + \sqrt{2}\, Ea\sqrt{\cosh\mu - \cos\eta}\, \cdot$$
$$\cdot \sum_{n=0}^\infty \frac{(2n + 1)e^{-(n+\frac{1}{2})\mu_0}}{\sinh(n + \frac{1}{2})\mu_0}\sinh(n + \tfrac{1}{2})\mu P_n(\cos\eta)$$

which holds for both positive and negative values of μ, such that $|\mu| \leq \mu_0$.

The Green's function for this coordinate system can be obtained by the general techniques of page 892. The normalized eigenfunctions for η and ϕ are

$$\sqrt{\epsilon_m \frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}\,P_n^m(\cos\eta)\,{\cos \atop \sin}(m\phi) = \Psi_{nm}(\phi,\eta)$$

The equation to be solved is

$$\left[\frac{\partial^2 F}{\partial\mu^2} + \frac{1}{\sin\eta}\frac{\partial}{\partial\eta}\left(\sin\eta\,\frac{\partial F}{\partial\eta}\right) + \frac{1}{\sin^2\eta}\frac{\partial^2 F}{\partial\phi^2} - \tfrac{1}{4}F\right]$$
$$= -4\pi\,\frac{\sqrt{\cosh\mu - \cos\eta}}{a\sin\eta}\,\delta(\mu-\mu_0)\delta(\eta-\eta_0)\delta(\phi-\phi_0)$$

Inserting the series

$$F = \sum_{n,m} A_{nm}e^{-(n+\frac{1}{2})|\mu-\mu_0|}\Psi_{nm}(\phi,\eta);\quad G = \sqrt{\cosh\mu - \cos\eta}\,F$$

[which has a discontinuity in slope for $\mu = \mu_0$ equal to $(2n + 1)$ times the (n,m)th term] into the equation, multiplying both sides by $\Psi_{nm}\sin\eta$, and integrating over ϕ and η show that

$$A_{nm} = \frac{1}{a}\sqrt{\cosh\mu_0 - \cos\eta_0}\,\frac{4\pi}{2n+1}\,\Psi_{nm}(\phi_0,\eta_0)$$

Consequently the Green's function is

$$G = \frac{1}{a}\sqrt{(\cosh\mu - \cos\eta)(\cosh\mu_0 - \cos\eta_0)}\;\cdot$$
$$\cdot\sum_{n=0}^{\infty}\sum_{m=0}^{n}\epsilon_m\frac{(n-m)!}{(n+m)!}\cos[m(\phi-\phi_0)]\;\cdot$$
$$\cdot\,P_n^m(\cos\eta_0)P_n^m(\cos\eta)e^{-(n+\frac{1}{2})|\mu-\mu_0|} \quad (10.3.74)$$

which is useful for solving many problems.

Toroidal Coordinates. If we rotate bipolar axes about the perpendicular bisector of the line between the two poles, we obtain toroidal coordinates:

$$z = \frac{a\sin\eta}{\cosh\mu - \cos\eta};\quad x = \frac{a\sinh\mu\cos\phi}{\cosh\mu - \cos\eta};\quad y = \frac{a\sinh\mu\sin\phi}{\cosh\mu - \cos\eta}$$
$$h_\mu = h_\eta = \frac{a}{\cosh\mu - \cos\eta};\quad h_\phi = \frac{a\sinh\mu}{\cosh\mu - \cos\eta} \quad (10.3.75)$$
$$\nabla^2\psi = \frac{1}{h_\mu^3}\left[\frac{1}{\sinh\mu}\frac{\partial}{\partial\mu}\left(h_\mu\sinh\mu\,\frac{\partial\psi}{\partial\mu}\right) + \frac{\partial}{\partial\eta}\left(h_\eta\frac{\partial\psi}{\partial\eta}\right) + \frac{h_\mu}{\sinh^2\mu}\frac{\partial^2\psi}{\partial\phi^2}\right]$$

where μ ranges from 0 to ∞, η from 0 to 2π, and ϕ from 0 to 2π. The surface $\mu = \mu_0$ is a torus with axial circle in the x, y plane, centered at the origin, of radius $a\coth\mu_0$, having a circular cross section of radius

$a \operatorname{csch} \mu_0$. The surface $\eta = \eta_0$ (for $\eta_0 < \pi$) is that part of the sphere of radius $a \csc \eta_0$, with center at $x = y = 0$, $z = a \cot \eta_0$, which is above the x, y plane, and the rest of the same sphere, below the x, y plane, is the surface $\eta = 2\pi - \eta_0$. The dividing line is the circle in the x, y plane, with center at the origin, of radius a, which is also the surface $\mu = \infty$. The z axis corresponds to $\mu = 0$, the part of the x, y plane inside the circle $\mu = \infty$ corresponds to $\eta = \pi$, and the rest of the x, y plane outside the circle corresponds to $\eta = 0$ or 2π.

We again set $\psi = \sqrt{\cosh \mu - \cos \eta}\, F(\mu,\eta,\phi)$ and find that the Laplace equation reduces to

$$\frac{1}{\sinh \mu} \frac{\partial}{\partial \mu}\left(\sinh \mu \frac{\partial F}{\partial \mu}\right) + \frac{\partial^2 F}{\partial \mu^2} + \frac{1}{\sinh^2 \mu} \frac{\partial^2 F}{\partial \phi^2} + \tfrac{1}{4}F = 0$$

The ϕ factor is $\cos(m\phi)$ or $\sin(m\phi)$, and since ϕ is a periodic coordinate, m is zero or a positive integer to ensure continuity. Coordinate η is also periodic, so that the η factor must also be $\cos(n\eta)$ or $\sin(n\eta)$ with n zero or an integer. The μ factor, on the other hand, is a solution of

$$\frac{1}{\sinh \mu} \frac{d}{d\mu}\left(\sinh \mu \frac{dM}{d\mu}\right) - \frac{m^2 M}{\sinh^2 \mu} - (n^2 - \tfrac{1}{4})M = 0$$

which is a half-order spherical harmonic, either

$$\begin{aligned}
P^m_{n-\frac{1}{2}}(\cosh \mu) &= i^m \sinh^m \mu \, T^m_{n-m-\frac{1}{2}}(\cosh \mu) \\
&= \frac{i^m \Gamma(n + m + \frac{1}{2})}{2^m m! \Gamma(n - m + \frac{1}{2})} \sinh^m \mu \cdot \\
&\qquad \cdot F(m - n + \tfrac{1}{2}, m + n + \tfrac{1}{2} \,|m + 1|\, - \sinh^2 \tfrac{1}{2}\mu) \\
&= \frac{i^m \Gamma(n + m + \frac{1}{2})}{2^m m! \Gamma(n - m + \frac{1}{2})} \cosh^{n-\frac{1}{2}} \mu \tanh^m \mu \cdot \\
&\qquad \cdot F\left(\frac{m - n + \frac{1}{2}}{2}, \frac{m - n + \frac{3}{2}}{2} \,|m + 1|\, \tanh^2 \mu\right) \\
&= \frac{i^m \Gamma(n + m + \frac{1}{2})}{2^m m! \Gamma(n - m + \frac{1}{2})} \frac{\tanh^m(\mu)}{\cosh^{n+\frac{1}{2}}(\mu)} \cdot \\
&\qquad \cdot F\left(\frac{m + n + \frac{1}{2}}{2}, \frac{m + n + \frac{3}{2}}{2} \,|m + 1|\, \tanh^2 \mu\right) \qquad (10.3.76)
\end{aligned}$$

[using Eqs. (5.2.52), (5.3.17) and those on page 668] or else

$$\begin{aligned}
Q^m_{n-\frac{1}{2}}(\cosh \mu) &= (-1)^m \sinh^m \mu \, V^m_{n-m-\frac{1}{2}}(\cosh \mu) \\
&= \frac{\Gamma(\frac{1}{2})\Gamma(n + m + \frac{1}{2}) \tanh^m \mu}{\Gamma(n + 1) 2^{n+\frac{1}{2}} \cosh^{n+\frac{1}{2}} \mu} \cdot \\
&\qquad \cdot F\left(\frac{n + m + \frac{1}{2}}{2}, \frac{n + m + \frac{3}{2}}{2} \,|n + 1|\, \operatorname{sech}^2 \mu\right) \qquad (10.3.77)
\end{aligned}$$

Since both $m + 1$ and $n + 1$ are integers, P has a logarithmic singularity at $\mu \to \infty$ and Q has one at $\mu \to 0$, as discussed on pages 598 and 1329. For instance, the behavior of the function $Q_{n-\frac{1}{2}}(\cosh \mu)$ at $\mu \to 0$ is obtained by calculating it for small values of m and then letting $m \to 0$. For small values of m compared with unity, we have, from Eqs. (5.2.49) and (10.3.77),

$$
\begin{aligned}
Q^m_{n-\frac{1}{2}}(\cosh \mu) &= \frac{\sqrt{\pi}\,\Gamma(n + m + \frac{1}{2})}{2^{n+\frac{1}{2}}m} \cdot \\
&\cdot \frac{\Gamma(1 + m)\Gamma(1 - m)\cosh^{-n-\frac{1}{2}}\mu}{\Gamma\left(\frac{n + m + \frac{1}{2}}{2}\right)\Gamma\left(\frac{n - m + \frac{1}{2}}{2}\right)\Gamma\left(\frac{n + m + \frac{3}{2}}{2}\right)\Gamma\left(\frac{n - m + \frac{3}{2}}{2}\right)} \cdot \\
&\cdot \left\{ e^{-m\ln[\tanh\mu]} \sum_{s=0}^{\infty} \frac{\Gamma(\frac{1}{2}n + \frac{1}{4} - \frac{1}{2}m + s)\Gamma(\frac{1}{2}n + \frac{3}{4} - \frac{1}{2}m + s)}{s!\Gamma(s + 1 - m)} [\tanh^2\mu]^s \right. \\
&\left. - e^{m\ln[\tanh\mu]} \sum_{s=0}^{\infty} \frac{\Gamma(\frac{1}{2}n + \frac{1}{4} + \frac{1}{2}m + s)\Gamma(\frac{1}{2}n + \frac{3}{4} + \frac{1}{2}m + s)}{s!\Gamma(s + 1 + m)} [\tanh^2\mu]^s \right\} \\
&\xrightarrow[m\to 0]{} -\frac{1}{2\cosh^{n+\frac{1}{2}}\mu} \left\{ 2\ln[\tanh\mu]\, F(\tfrac{1}{2}n + \tfrac{1}{4}, \tfrac{1}{2}n + \tfrac{3}{4}|1|\tanh^2\mu) \right. \\
&\qquad + \frac{2^n}{\sqrt{2\pi}\,\Gamma(n + \frac{1}{2})} \sum_{s=0}^{\infty} \frac{\Gamma(s + \frac{1}{2}n + \frac{1}{4})\Gamma(s + \frac{1}{2}n + \frac{3}{4})}{[s!]^2} \cdot \\
&\left. \cdot \tanh^{2s}(\mu)[\psi(s + \tfrac{1}{2}n + \tfrac{1}{4}) + \psi(s + \tfrac{1}{2}n + \tfrac{3}{4}) - 2\psi(s + 1)] \right\} \qquad (10.3.78) \\
&\xrightarrow[\mu\to 0]{} -\ln\mu
\end{aligned}
$$

where ψ is the logarithmic derivative of Γ, defined in Eq. (4.5.43). A similar expression applies for the behavior of $P_{n-\frac{1}{2}}(\cosh \mu)$ at $\mu \to 0$ (see table at end of this chapter).

The solutions of Laplace's equation in toroidal coordinates are therefore given in terms of the series

$$
\begin{aligned}
\psi = \sqrt{\cosh\mu - \cos\eta} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} [a_m \cos(m\phi) + b_m \sin(m\phi)] \cdot \\
\cdot [c_n \cos(n\eta) + d_n \sin(n\eta)][A_{mn}P^m_{n-\frac{1}{2}}(\cosh\mu) + B_{mn}Q^m_{n-\frac{1}{2}}(\cosh\mu)]
\end{aligned}
$$

As with the bispherical coordinates, we need an expression for $[\cosh \mu - \cos \eta]^{-\frac{1}{2}}$. This can be obtained from the integral expression for the spherical harmonic of the second kind. By a complicated sequence of integrations by parts, starting from Eq. (5.3.29), it may be shown that

$$\sqrt{2}\,Q_{n-\frac{1}{2}}(\cosh\mu) = e^{-(n+\frac{1}{2})\mu}\int_0^\pi \frac{\sin^{2n}\theta\,d\theta}{(1-2e^{-\mu}\cos\theta+e^{-2\mu})^{n+\frac{1}{2}}}$$
$$= \int_0^\pi \frac{\cos(n\theta)\,d\theta\,e^{-\frac{1}{2}\mu}}{\sqrt{1-2e^{-\mu}\cos\theta+e^{-2\mu}}} = \int_0^\pi \frac{\cos(n\theta)\,d\theta}{\sqrt{\cosh\mu-\cos\theta}}$$

Consequently,

$$\frac{1}{\sqrt{\cosh\mu-\cos\eta}} = \frac{\sqrt{2}}{\pi}\sum_{n=0}^{\infty} Q_{n-\frac{1}{2}}(\cosh\mu)\cos(n\eta) \qquad (10.3.79)$$

Therefore the series for the potential outside the torus $\mu = \mu_0$ (axial circle radius $a \coth \mu_0$, cross-sectional radius $a \operatorname{csch} \mu_0$), which is held at uniform potential V_0, is

$$\psi = \frac{V_0}{\pi}\sqrt{2(\cosh\mu-\cos\eta)}\sum_{n=0}^{\infty}\left[\frac{Q_{n-\frac{1}{2}}(\cosh\mu_0)}{P_{n-\frac{1}{2}}(\cosh\mu_0)}\right]P_{n-\frac{1}{2}}(\cosh\mu)\cos(n\eta) \qquad (10.3.80)$$

Many other interesting problems may be computed by various methods by now familiar. For instance, by the methods of Chap. 7, we obtain for the Green's function expansion

$$\frac{1}{R} = \frac{1}{\pi}\sqrt{(\cosh\mu-\cos\eta)(\cosh\mu_0-\cos\eta_0)}\sum_{m,n=0}^{\infty}\epsilon_m\epsilon_n(-i)^m\frac{\Gamma(n-m+\frac{1}{2})}{\Gamma(n-m+\frac{1}{2})}\cdot$$
$$\cdot\cos[m(\phi-\phi_0)]\cos[n(\eta-\eta_0)]\begin{cases}P^m_{n-\frac{1}{2}}(\cosh\mu)Q^m_{n-\frac{1}{2}}(\cosh\mu_0); & \mu_0>\mu\\ P^m_{n-\frac{1}{2}}(\cosh\mu_0)Q^m_{n-\frac{1}{2}}(\cosh\mu); & \mu_0<\mu\end{cases} \qquad (10.3.81)$$

This series may be modified to fit boundary conditions and may be integrated to produce the potential of a toroidal distribution of charge.

Ellipsoidal Coordinates. We come, finally, to the most general form of separable coordinates (most general for separability of the wave equation; for the Laplace equation the general cyclidal coordinates are more general but are also still less useful), the ellipsoidal coordinates tabulated on pages 511 and 663. The coordinate ξ_1 ranges from a to ∞, the range of ξ_2 is from b to a ($b < a$), and that of ξ_3 is from $-b$ to $+b$. The surfaces ξ_1 = constant are ellipsoids, approximating spheres for ξ_1 large, with ξ_1 approaching r for large values; the surfaces ξ_2 = constant are hyperboloids of one-sheet, and the surfaces ξ_3 = constant are hyperboloids of two sheets.

The solution of the Laplace equation separates, without the confusing factor required in bispherical or toroidal coordinates:

$$\psi = F(\xi_1)G(\xi_2)H(\xi_3)$$

The equations for the factors are all similar:

$$\begin{aligned}
&\sqrt{(\xi_1^2 - a^2)(\xi_1^2 - b^2)}\,\frac{d}{d\xi_1}\left[\sqrt{(\xi_1^2 - a^2)(\xi_1^2 - b^2)}\,\frac{dF}{d\xi_1}\right] \\
&\qquad\qquad = [m(m+1)\xi_1^2 - \kappa]F \\
&\sqrt{(a^2 - \xi_2^2)(\xi_2^2 - b^2)}\,\frac{d}{d\xi_2}\left[\sqrt{(a^2 - \xi_2^2)(\xi_2^2 - b^2)}\,\frac{dG}{d\xi_2}\right] \\
&\qquad\qquad = -[m(m+1)\xi_2^2 - \kappa]G \\
&\sqrt{(a^2 - \xi_3^2)(b^2 - \xi_3^2)}\,\frac{d}{d\xi_3}\left[\sqrt{(a^2 - \xi_3^2)(b^2 - \xi_3^2)}\,\frac{dH}{d\xi_3}\right] \\
&\qquad\qquad = [m(m+1)\xi_3^2 - \kappa]H
\end{aligned} \tag{10.3.82}$$

except for the reversals in order of some factors so as to maintain a positive sign over the proper range of the variable. The parameters m and κ are the separation constants; we note that they occur in all three equations, that the equations are separated but the separation constants are not (see pages 518 and 757).

These equations are all of the same form:

$$\frac{d^2F}{dz^2} + \left[\frac{z}{z^2 - a^2} + \frac{z}{z^2 - b^2}\right]\frac{dF}{dz} + \left[\frac{\kappa - m(m+1)z^2}{(z^2 - a^2)(z^2 - b^2)}\right]F = 0 \tag{10.3.83}$$

$$[z^4 - (a^2 + b^2)z^2 + a^2b^2]F'' + z[2z^2 - (a^2 + b^2)]F' + [\kappa - m(m+1)z^2]F = 0$$

where the solutions in the range $z > a$ are for the coordinate ξ_1, those in the range $a > z > b$ are for ξ_2, and those in the range $b > z > -b$ are for ξ_3. This equation is called the *Lamé equation.* It has the following regular singular points:

At $z = \pm b$; with indices 0 and $\frac{1}{2}$
At $z = \pm a$; with indices 0 and $\frac{1}{2}$
At $z = \infty$; with indices $m + 1$ and $-m$

We remark here, in review, that all the other equations which we have been struggling with in all of this section may be derived from the Lamé equation by suitable confluence of its singularities.

For the solutions suitable for boundary conditions at $z = 0$, we use an expansion in powers of z. Setting $F = \Sigma d_n z^n$, we find that the three-term recursion formula (see page 540) for the d's is

$$a^2b^2n(n-1)d_n = [(n-2)^2(a^2 + b^2) - \kappa]d_{n-2} + [m(m+1) - (n-3)(n-4)]d_{n-4} \tag{10.3.84}$$

indicating that the series for F can be separated into two sorts of series, one having only even powers of z and the other having only odd powers. Since all three factors of the solution must correspond to the same values of m and κ, we must see whether we can find solutions which will be valid for the whole range $0 < z < \infty$ and can be used for all three factors. The series for most values of m and of κ are infinite ones, and we should see for what values of z they converge before we set about using them.

To test convergence, we find the limiting value of the ratio $z^2\gamma_n$, where $\gamma_n = d_n/d_{n-2}$, as n approaches infinity, and find for what values of z, m, and κ this ratio is less than unity. By dividing Eq. (10.3.84) by $n(n-1)\,a^2b^2d_{n-4}$ and neglecting terms in $1/n^2$ and smaller, we have

$$\gamma_n\gamma_{n-2} - \left(1 - \frac{3}{n}\right)\left[\frac{a^2+b^2}{a^2b^2}\right]\gamma_{n-2} + \left(1 - \frac{6}{n}\right)\frac{1}{a^2b^2} = 0$$

As n is made large, presumably $\gamma_n \to \gamma_{n-2} \to \gamma$, so that for very large n the above equation becomes a quadratic equation for γ, giving

$$\gamma \simeq \left(1 - \frac{3}{n}\right)\frac{1}{b^2} \quad \text{or} \quad \gamma \simeq \left(1 - \frac{3}{n}\right)\frac{1}{a^2}$$

Therefore $z^2\gamma$ is less than unity for z less than b or a, and the series, *if it is infinite*, converges only out to $z = b$ or $z = a$. Consequently, such series, for an arbitrary value of κ and m, would give convergent results for two of the factors but not for all three. If we try to expand about any other point, we shall find the same limitation; a series can be found convergent at two of the singular points, but not at three or more.

However, for integral values of m and for certain values of κ, one or the other of the series breaks off, giving a polynomial solution which is finite at four of the five singular points. For instance, Eq. (10.3.84) shows that two possible solutions of Eq. (10.3.83) are

$$\begin{aligned} E_0^0(z) &= 1;\quad \text{for } m = 0;\quad \kappa = \kappa_0 = 0;\quad d_1 = 0 \\ E_1^0(z) &= z;\quad \text{for } m = 1;\quad \kappa = \kappa_1 = a^2 + b^2;\quad d_0 = 0 \end{aligned} \tag{10.3.85}$$

These solutions converge clear out to infinity.

Therefore a possible solution of the Laplace equation is a constant, $E_0^0(\xi_1)E_0^0(\xi_2)E_0^0(\xi_3)$. This is one of the two *ellipsoidal harmonics of the first species* (others will be discussed shortly). Likewise another solution of the Laplace equation is

$$AE_1^0(\xi_1)E_1^0(\xi_2)E_1^0(\xi_3) = abAz$$

which is one of the *ellipsoidal harmonics of the second species.*

To obtain the next solution in powers of z we must solve a quadratic equation. Setting $m = 2$ and $d_1 = 0$ in Eqs. (10.3.84), we see that

$$d_2 = \frac{-\kappa}{2a^2b^2}\,d_0;\quad d_4 = \frac{4a^2 + 4b^2 - \kappa}{12a^2b^2}\,d_2 + \frac{1}{2a^2b^2}\,d_0$$
$$d_6 = \frac{16a^2 + 16b^2 - \kappa}{30a^2b^2}\,d_4$$

Therefore if d_4 can be made zero, all the higher d's will be zero and the solution will be a polynomial of second order. This requires that $[(4a^2 + 4b^2 + \kappa)](\kappa/2a^2b^2) = 6$ or that

$$\kappa = 2[(a^2 + b^2) \mp \sqrt{(a^2 + b^2)^2 - 3a^2b^2}]$$

with corresponding solutions

$$\begin{aligned} E_0^2 &= z^2 - \tfrac{1}{3}[(a^2 + b^2) + \sqrt{(a^2 + b^2)^2 - 3a^2b^2}] \\ E_2^1 &= z^2 - \tfrac{1}{3}[(a^2 + b^2) - \sqrt{(a^2 + b^2)^2 - 3a^2b^2}] \end{aligned} \tag{10.3.86}$$

One may continue, for larger integral values of m, obtaining polynomials of higher order, solving successively higher order equations for κ. But one may also obtain solutions having a factor $\sqrt{z^2 - a^2}$ in them. We set $F = \sqrt{z^2 - a^2}\, B(z)$, obtaining from Eq. (10.3.83)

$$\begin{aligned} [z^4 - (a^2 + b^2)z^2 + a^2b^2]B'' + z[4z^2 - (a^2 + 3b^2)]B' \\ - [\kappa + b^2 + (m + 3)(m - 1)]B = 0 \end{aligned}$$

which has polynomial solutions giving further convergent forms for solutions of Eq. (10.3.83)

$$\begin{aligned} E_1^1(z) &= \sqrt{z^2 - a^2}; \quad m = 1; \quad \kappa = b^2 \\ E_2^2(z) &= z\sqrt{z^2 - a^2}; \quad m = 2; \quad \kappa = a^2 + 4b^2 \end{aligned} \tag{10.3.87}$$

The general form of the solution

$$CE_1^1(\xi_1)E_1^1(\xi_2)E_1^1(\xi_3) = C\sqrt{a^2(a^2 - b^2)}x$$

is another of the ellipsoidal harmonics of the second species, whereas

$$DE_2^2(\xi_1)E_2^2(\xi_2)E_2^2(\xi_3) = Dab\sqrt{a^2(a^2 - b^2)}xz$$

is one of the *ellipsoidal harmonics of the third species.* By replacing a with b in Eq. (10.3.86) we obtain solutions $E_1^2 = \sqrt{z^2 - b^2}$ and $E_2^3 = z\sqrt{z^2 - b^2}$.

Finally there are solutions of the form of a polynomial in z multiplied by a factor $\sqrt{(z^2 - a^2)(z^2 - b^2)}$, the first of which is

$$E_2^4(z) = \sqrt{(z^2 - a^2)(z^2 - b^2)}; \quad m = 2; \quad \kappa = a^2 + b^2 \tag{10.3.88}$$

A product of these produces a solution xy which is still another harmonic of the third species. The next solution, $z\sqrt{(z^2 - a^2)(z^2 - b^2)}$, results in (xyz), which is an *ellipsoidal harmonic of the fourth species.*

All these solutions go to zero at 0, a, or b and have poles at infinity going as z^m for large z. Second solutions are often useful, which go as z^{-m-1} for large z. These may be obtained from the E solutions already obtained by use of Eq. (5.2.6). We define

$$F_m^p(z) = (2m + 1)E_m^p(z) \int_z^\infty \frac{dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)}\, [E_m^p(x)]^2} \tag{10.8.89}$$

which is a solution of Eq. (10.3.83) for the same values of κ and m as is $E_m^p(z)$. These solutions are various kinds of elliptic integrals (see page 432 *et seq.*). The first one is (see page 486)

$$F_0^0(z) = \int_z^\infty \frac{dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)}} = \frac{1}{a} \operatorname{sn}^{-1}\left(\frac{a}{z}, \frac{b}{a}\right) \tag{10.3.90}$$

For instance, the potential outside the conducting ellipsoid $\xi_1 = c$, which is held at potential V_0 with respect to infinity, is

$$\psi = AE_0^0(\xi_3)E_0^0(\xi_2)F_0^0(\xi_1) = V_0 \left[\operatorname{sn}^{-1}\left(\frac{a}{\xi_1}, \frac{b}{a}\right) \Big/ \operatorname{sn}^{-1}\left(\frac{a}{c}, \frac{b}{a}\right)\right] \tag{10.3.91}$$

When ξ_1 is very large, F_0^0 approaches the function $1/\xi_1$ and the inverse sn function approaches a/ξ_1. Consequently, $\psi \to \left[V_0 a/\xi_1 \operatorname{sn}^{-1}\left(\frac{a}{c}, \frac{b}{a}\right)\right]$ and the total charge on the ellipsoid must be $[V_0 a/\operatorname{sn}^{-1}]$. Therefore the capacitance of an isolated ellipsoid $\xi_1 = c$, with three semiaxes c, $\sqrt{c^2 - a^2}$, $\sqrt{c^2 - b^2}$ is

$$C = \frac{a}{\operatorname{sn}^{-1}\left(\frac{a}{c}, \frac{b}{a}\right)}$$

When $c = a$, the surface is a flat elliptic disk of semiaxes a and $\sqrt{a^2 - b^2}$ and the inverse sn function reduces to the period constant K, so that the capacitance is $a/K(b/a)$. When $b = 0$, K goes to $\frac{1}{2}\pi$, so that the capacitance of a circular disk of radius a is $2a/\pi$, corresponding to the results of Eq. (10.3.56).

Other more complicated problems may be solved by the use of these functions. We notice that the products $E_m^p(\xi_2)E_m^p(\xi_3)$ form a complete set of functions, in terms of which we may fit boundary conditions on ellipsoidal surfaces $\xi_1 =$ constant. A combination of $E_m^p(\xi_1)$ and $F_m^p(\xi_1)$ will then express the dependence on ξ_1.

Finally, we notice that, if we change the scale of the coordinates from ξ_1, ξ_2, ξ_3 to λ, μ, ν, so that

$$\lambda = a \int_a^{\xi_1} \frac{dt}{\sqrt{(t^2 - a^2)(t^2 - b^2)}}; \quad \mu = a \int_{\xi_2}^a \frac{dt}{\sqrt{(a^2 - t^2)(t^2 - b^2)}};$$

$$\nu = a \int_0^{\xi_3} \frac{dt}{\sqrt{(a^2 - t^2)(b^2 - t^2)}}$$

then Eqs. (10.3.82) will take on a particularly simple form:

$$\frac{d^2F}{d\lambda^2} = \frac{1}{a^2}[m(m + 1)\xi_1^2 - \kappa]F; \quad \text{etc.}$$

But a reinvestigation of the discussion of elliptic functions in Chap. 4 (or in treatises on the subject) shows that these equations correspond to

$$\xi_1 = a\,\frac{\mathrm{dn}(\lambda,k)}{\mathrm{cn}(\lambda,k)};\quad \xi_2 = a\,\mathrm{dn}(\mu,k');\quad \xi_3 = b\,\mathrm{sn}(\nu,k) \qquad (10.3.92)$$
$$k = b/a;\quad k' = (1/a)\sqrt{a^2 - b^2} = \sqrt{1 - k^2}$$

This makes it a little easier to write down equations but helps very little in simplifying computations. It does allow one to show that the integral representation for the Lamé functions involves Legendre function kernels. For instance, for some types of function

$$E_m^p[b \sin \nu] = h \int_{-2K}^{2K} P_m[k \,\mathrm{sn}\, \nu \,\mathrm{sn}\, u] E_m^p[b \,\mathrm{sn}\, u]\, du$$

which is equivalent to Eq. (10.3.2). In this case, however, the equation is not an integral *representation,* giving E in terms of some simpler functions, already known, but is an *integral equation,* giving E in terms of an integral over itself. This particular example is not of enough practical interest to warrant further study, but reference to page 634 for the Mathieu functions and to the next chapter for the spheroidal wave functions will show that even an integral equation can be of use, though not so useful as a simple integral representation.

Problems for Chapter 10

10.1 A bar of rectangular cross section, of height b and width a, is held at zero degrees temperature on the three sides $y = 0$, $x = 0$, $x = a$. The side $y = b$ is heated by radiation, so that

$$\beta(\partial/\partial b)T(x,b) = T_0 - T(x,b)$$

where $T(x,y)$ is the temperature of the point x, y in the bar, T_0 is the temperature of the radiant heater, and β is a constant. Set up the Fourier series expression for $T(x,y)$ which satisfies these boundary conditions. Show that the total heat flowing into the face $y = b$ per second is

$$\left(\frac{8T_0\kappa}{\pi}\right) \sum_{n=0}^{\infty} \frac{1/(2n + 1)}{[(\pi\beta/a)(2n + 1)] + \tanh[(2n + 1)(\pi b/a)]}$$

where κ is the heat conductivity of the material of the bar.

10.2 Show that the Green's function for a unit line source inside a slot bounded by the planes $y = 0$, $x = 0$, and $x = a$, with homogeneous Neumann conditions at the boundary, is

$$G_0(x,y|x_0,y_0) = -\frac{4\pi}{a}\begin{Bmatrix} y \\ y_0 \end{Bmatrix} + \sum_{n=1}^{\infty} \frac{8}{n}\cos\left(\frac{\pi n x_0}{a}\right)\cos\left(\frac{\pi n x}{a}\right)\begin{cases} \cosh\left(\frac{\pi n y_0}{a}\right)\exp\left(-\frac{\pi n y}{a}\right) \\ \cosh\left(\frac{\pi n y}{a}\right)\exp\left(-\frac{\pi n y_0}{a}\right) \end{cases}$$

where the upper terms in the braces are used when $y \geq y_0$, the lower when $y \leq y_0$. Show that this is the real part of the function

$$F = -2\ln\left\{\left[\cos\left(\frac{\pi z}{a}\right) - \cos\left(\frac{\pi z_0}{a}\right)\right]\left[\cos\left(\frac{\pi z}{a}\right) - \cos\left(\frac{\pi \bar{z}_0}{a}\right)\right]\right\}$$

where $z = x + iy$, $z_0 = x_0 + iy_0$, $\bar{z}_0 = x_0 - iy_0$. Show that the imaginary part of F is the magnetic potential of a wire at (x_0,y_0), carrying unit current, when the slot is surrounded by iron of very large permeability.

10.3 Two parallel wires a distance $2b$ apart are embedded in insulating material the outer surface of which is a circular cylinder of radius a ($a > b$), the axis of the cylinder being midway between the wires. Current in each wire generates U units of heat per second per unit length in each wire. If the heat conductivity of the insulating material is κ, and if the radius ρ of the cross section of the wires is much smaller than b, what will be the steady-state temperature of the surface of each wire when the outer surface of the insulator is kept at zero temperature?

10.4 In a region of space of permeability $\mu = 1$, a magnetic field can be represented by the magnetic potential

$$\psi = H_0[-x + (1/6R^2)(x^3 - 3xy^2)]; \quad r^2 = x^2 + y^2 < R^2$$

where $H = -\operatorname{grad}\psi$. This field is distorted by placing in it a cylinder of material of permeability μ, of radius of cross section $a \ll R$ and with its axis at $r = 0$. What is the modified magnetic potential, both inside and outside the cylinder?

10.5 A long strip of conductor of width a is inside an insulating sheath whose outer surface is a cylinder with cross section an ellipse having foci at the edges of the strip and with minor axis $2b$. The strip is raised to a potential V higher than the potential of the outer surface of the sheath. Current leaks through this sheath, whose resistivity is R. What current is lost per centimeter length of the strip after the steady state has been reached?

10.6 High-frequency current of frequency f cycles per sec is passed through a copper wire whose cross section is a circle of radius a. In such a case the current density at a distance r from the axis of the wire is approximately

$$i = \frac{3I}{\pi a^2}\left[\frac{1 - k(1 - r^4/a^4)}{3 - 2k}\right]$$

where $k = f^2a^4/31{,}000$ (for Cu) and I is the rms value of the current in amperes. The approximation is valid as long as k is less than unity. If the surface of the wire is kept at 0°C, what will be the steady-state distribution of temperature inside the wire? For $a = 1/4$ cm and $I = 10$ amp, plot the temperature at the center of the wire as a function of the frequency f from $f = 0$ to $f = 2{,}000$ cycles per sec.

10.7 A line charge of density q per centimeter is parallel to and a distance c from the axis of a dielectric cylinder of radius a and dielectric constant ϵ $(c > a)$. What is the distribution of potential inside and outside the cylinder? What is the electric intensity at the center of the cylinder?

10.8 The surfaces $\vartheta = 0$, $\vartheta = \pi$ in elliptic coordinates represent two coplanar half planes separated by a slit of width a. Show that the Green's function for the Laplace equation for the boundary condition that $G = 0$ when $\vartheta = 0$ or $\vartheta = \pi$ is

$$G_0(\mu,\vartheta|\mu_0,\vartheta_0) = \sum_{n=1}^{\infty} \left(\frac{4}{n}\right) e^{-n|\mu-\mu_0|} \sin(n\vartheta_0) \sin(n\vartheta)$$

where $0 \leq \vartheta, \vartheta_0 \leq \pi$ and $-\infty < \mu, \mu_0 < \infty$. Show that this is the real part of the function

$$F = 2 \ln \left\{ \frac{\sinh[\frac{1}{2}(\mu + i\vartheta - \mu_0 - i\vartheta_0)]}{\sinh[\frac{1}{2}(\mu + i\vartheta - \mu_0 + i\vartheta_0)]} \right\}$$

Calculate the capacitance of a wire of radius ρ with center at the point x_0, y_0 $(\rho \ll y_0)$ with respect to the surfaces $\vartheta = 0$, $\vartheta = \pi$.

10.9 A number N of conductors are mutually insulated from each other. The nth one is at potential V_n with respect to infinity and has total charge Q_n. Show that

$$Q_n = \sum_{m=1}^{N} c_{nm}V_m \quad \text{and} \quad V_n = \sum_{m=1}^{N} a_{nm}Q_m; \quad c_{mn} = c_{nm}$$

and obtain the relationship between the c's and the a's. Coefficients c_{nn} are called *coefficients of capacitance* or simply the capacitance of conductor n; coefficients c_{nm} are called *coefficients of induction* or simply the mutual capacitance between conductors m and n. Show, from the results of Chap. 3, that the field about the conductors must adjust itself so that

$$W = \tfrac{1}{2} \sum_{mn} V_m c_{mn} V_n = \tfrac{1}{2} \sum_{mn} Q_m a_{mn} Q_n$$

is a minimum for the boundary conditions applied. Obtain the coefficients c for the situation given in Eq. (10.2.28), and express the amplification constant in terms of the c's.

10.10 The half plane $y = 0$, $x = 0$ is kept at zero potential and two wires, parallel to this plane, one of radius ρ_1 with its axis at x_1, y_1 ($\rho_1 \ll y_1$) and the other of radius ρ_2 at x_2, y_2 ($\rho_2 \ll y_2$), are at potentials V_1 and V_2, respectively. Show that the potential is approximately given by the real part of the expression

$$F = 2Q_1 \ln\left(\frac{w - \bar{w}_1}{w - w_1}\right) + 2Q_2 \ln\left(\frac{w - \bar{w}_2}{w - w_2}\right); \quad w = \sqrt{2(x + iy)}$$

From this obtain values of the capacitance coefficients c_{11}, c_{22}, c_{01}, c_{02}, c_{12} where the half plane is the conductor 0. (See Prob. 10.9 for definition of the c's.)

10.11 Show that the Green's function in the two-dimensional hyperbolic coordinate system, defined by $z = (x + iy) = \sqrt{2w}$, $w = \mu + i\kappa$, for the Laplace equation is

$$G(\mu,\kappa|\mu_0,\kappa_0) = \frac{1}{\sqrt{\mu_0^2 + \kappa_0^2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\frac{e^{i\mu u + i\kappa v}}{u^2 + v^2}\right] du\, dv$$

Show that the Green's function for the region of positive κ (first quadrant) which is zero at the surface $\kappa = 0$ (positive x axis, positive y axis) can be expanded in polar coordinates r, φ as

$$\sum_{n=1}^{\infty} \frac{4}{m} \sin(2n\varphi_2) \sin(2n\varphi) \begin{Bmatrix} (r/r_0)^{2n}; & r < r_0 \\ (r_0/r)^{2n}; & r > r_0 \end{Bmatrix} = 2 \operatorname{Re}\left[\ln\left(\frac{w - \overline{w_0}}{w - w_0}\right)\right]$$

Calculate the capacitance between the surface $\kappa = 0$ and a wire of radius ρ, with axis parallel to the z axis, located at x_0, y_0 ($\rho \ll x_0,y_0$).

10.12 Two dielectric cylinders, each of radius c, having parallel axes a distance $2b$ apart, both have dielectric constants ϵ (for the rest of space $\epsilon = 1$). A uniform electric field E is pointed perpendicular to the cylinder axes and at an angle φ to the plane between them. Prove that the resulting potential distribution may be written

$$\psi = Ea \cos\varphi + 2aE \sum_{n=1}^{\infty} (-1)^n \left[\frac{1 + \coth(\eta\xi_0)}{\epsilon + \coth(n\xi_0)}\right] e^{-n\xi} \cos(n\theta + \varphi); \qquad \xi > \xi_0$$

$$= Ea \cos\varphi + 2aE \sum_{n=1}^{\infty} (-1)^n \left\{e^{-n\xi} - \frac{(\epsilon - 1)e^{-n\xi_0}}{\epsilon + \coth(n\xi_0)} \frac{\sinh(n\xi)}{\sinh(n\xi_0)}\right\} \cos(n\theta + \varphi); \quad 0 < \xi < \xi_0$$

in bipolar coordinates. What are the expressions for $\xi < 0$? Express ξ_0 in terms of b and c. Discuss the case for $b \gg c$, showing the first-order effect of the presence of one cylinder on the field in the other.

10.13 Two grounded metallic cylinders of radius c with parallel axes a distance $2b$ apart are placed in a uniform field E, perpendicular to the axes and at angle ϕ to the interaxial plane. Calculate the potential distribution outside the cylinders in terms of the bipolar coordinates θ, ξ given in Eq. (10.1.41), and compute the charge density on their surface. To the first order in the quantity c/b, calculate the force on each element of surface of the cylinders and thus obtain an approximate formula, analogous to Eq. (10.1.48) for the induced force between the cylinders, valid when $b \gg c$.

10.14 A conducting cylinder of radius c with its axis parallel to and a distance b from the yz plane carries current I distributed uniformly over its interior. Assuming that the heat conductivity of the wire is the same as that of the rest of space to the right of the yz plane and that the yz plane is kept at zero temperature, calculate the temperature distribution inside and outside the cylinder.

10.15 A grid of wires of radius ρ, with axes at $y = 0$, $x = \ldots,$ $-2a, -a, 0, a, 2a, \ldots$, at potential V_g, is placed between an infinite plate at $y = c$, at potential V_p, and another plate at $y = -b$, at zero potential. Show that the potential in the intermediate space is, approximately,

$$\psi = \frac{1}{4\pi cb + 2a(c+b)\ln(a/2\pi\rho)} \left\{ 4\pi cbV_g + 2abV_p \ln\left(\frac{a}{2\pi\rho}\right) + y\left[2\pi V_g(c-b) + 2\pi bV_p + 2aV_p \ln\left(\frac{a}{2\pi\rho}\right)\right] - [aV_g(c+b) - abV_p]\ln\left[4\sin^2\left(\frac{\pi x}{a}\right) + 4\sinh^2\left(\frac{\pi y}{a}\right)\right]\right\}$$

What are the coefficients of capacitance c_{gg}, c_{ga}, c_{gp} (see Prob. 10.9) between the grid and the two plates? What is the amplification constant relating grid and plate (at $y = c$) with the cathode (at $y = -b$)?

10.16 The transformation $z = (w + iae^{i\varphi}\tan\psi) + \dfrac{a^2e^{2i\varphi}}{(w + iae^{i\varphi}\tan\psi)}$ changes the circle of radius $a\sec\psi$ centered at the origin on the w plane into an arc of a circle whose chord is of length $4a$, inclined at an angle φ to the horizontal, on the z plane. Set up the equation for flow plus circulation about this arc, and show that, for there to be no turbulence at the rear end of the arc, the circulatory velocity must be $V_r = 2aV_0\sec\psi\cdot\sin(\varphi+\psi)$. If the lift and drag of an arc-shaped strip per unit length of strip are given by the formula $L + iD = 2\pi\rho V_0V_r$, show that the lift is $L = 4\pi\rho aV_0^2\sec\psi\sin(\varphi+\psi)$ [see Eq. (10.2.20)].

10.17 The regions R_1, above and to the right of the planes ($x = 0$, $y \geq 0$), ($y = 0$, $x \geq 0$), and R_2, below and to the right of the planes ($x = 0$, $y \leq -a$) ($y = -a$, $x \geq 0$), are filled with iron of very large permeability μ; the remaining region R_0 to the left of $x = 0$ and between the "pole pieces" R_1 and R_2 is at permeability $\mu = 1$. Use the Schwarz-Christoffel transformation to show that, when the magnetic potential of R_1 and R_2 differs by V, the lines of magnetic potential χ and force ψ are given by the conformal transformation

$$z = \left(\frac{a}{\pi}\right)\cosh^{-1}\left\{\exp\left[\left(\frac{-\pi}{V}\right)(\psi + i\chi)\right]\right\} - \left(\frac{a}{\pi}\right)\left\{1 - \exp\left[\left(\frac{2\pi}{V}\right)(\psi + i\chi)\right]\right\}^{\frac{1}{2}}$$

Compute and plot the magnitude of the magnetic field along the midline $y = -\frac{1}{2}a$ to show how this field falls off near the edge of the pole pieces. Show that near the corner $z = 0$ the magnetic field intensity is approximately $H = -i(\overline{dF/dz}) \simeq -i(V/a)(a/3\pi\bar{z})^{\frac{1}{3}}$.

10.18 The two pole pieces, separated by a slot of width a, described in Prob. 10.17 have, in addition, a wire of negligible radius, carrying current I, at the point $x = b$, $y = -\frac{1}{2}a$. Assuming that for this problem the only source of magnetomotive force is the current in the wire, show that the relation between $z = x + iy$ and the magnetic force-potential function $F = \psi + i\chi$ is

$$z = \left(\frac{a}{\pi}\right)\{\cosh^{-1}[1/\sqrt{e^{-F/2I} - \beta^2}] - \sqrt{1 + \beta^2 - e^{-F/2I}}\}$$

where $\pi b/a = \operatorname{csch}^{-1}\beta - \sqrt{1 + \beta^2}$. Plot the magnitude of the magnetic field along the midline $y = \frac{1}{2}a$, for $\beta = 0.5, 2$.

10.19 By means of the Schwarz-Christoffel transformation, transform the interior of the "polygon" bounded by the lines $y = 0$, $y = -a$ and the part of the y axis between $y = -b$ and $y = -a$ (the channel between $y = 0$ and $y = -a$, partially blocked by a perpendicular septum, leaving a gap of width b) into the upper half of the w plane. Show that the velocity potential ψ and flow function χ for fluid motion along this channel are related to the variable $z = x + iy$ by the equations

$$z = \left(\frac{a}{\pi}\right)\ln\left\{\frac{\sqrt{[(1/w) + \alpha][(1/w) + (1/\alpha)]} + (1/w) + \frac{1}{2}[\alpha + (1/\alpha)]}{\sqrt{[w + \alpha][w + (1/\alpha)]} + w + \frac{1}{2}[\alpha + (1/\alpha)]}\right\}$$
$$F = \psi + i\chi = (aU/\pi)\ln w; \quad \alpha = \cot^2(\pi b/4a)$$

where U is the fluid velocity at $x = \pm\infty$. Show that the fluid velocity at the point corresponding to w is

$$v = U\left\{\frac{\sqrt{[\bar{w} + \alpha][\bar{w} + (1/\alpha)]}}{\bar{w} + 1}\right\}$$

in direction and magnitude.

10.20 For the conditions of the preceding problem, show that, when $b \ll a$, then to the third order in the small quantity $\pi b/4a$, the flow function χ along the negative y axis (*i.e.*, in the gap) is related to y by the approximate equation

$$-\tan\left(\frac{\pi y}{4a}\right) \simeq \tan\left(\frac{\pi b}{4a}\right)\sin\left(\frac{\pi\chi}{2aU}\right)\left[1 - \tan^2\left(\frac{\pi b}{4a}\right)\cos^2\left(\frac{\pi\chi}{2aU}\right)\right]$$

and that the fluid velocity normal to this line is, to the second order,

$$v \simeq \frac{2aU}{\pi b}\,\frac{1}{\sqrt{1-(y^2/b^2)}}\left[1 + \tfrac{2}{3}\left(\frac{\pi b}{4a}\right)^2\left(1 - 2\frac{y}{b^2}\right)\right]$$

10.21 By means of the Schwarz-Christoffel transformation show that the transformation of the interior of the "step channel" enclosed by the following four lines, $y = a$, the positive x axis, the portion of the y axis between $y = 0$ and $y = a - b$, and the line $y = (a - b)$ for $x \le 0$, into the upper half of the w plane is given by

$$z = \frac{a}{\pi}\left\{\cosh^{-1}\left[\frac{2w - \alpha - 1}{\alpha - 1}\right] - \frac{1}{\sqrt{\alpha}}\cosh^{-1}\left[\frac{(\alpha+1)w - 2\alpha}{(\alpha - 1)w}\right]\right\}$$

where $\alpha = (a/b)^2$. By use of the auxiliary formula $F = \psi + i\chi = (aU/\pi)\ln w$ show that the velocity vector is

$$v = U\sqrt{(\bar{w} - \alpha)/(\bar{w} - 1)}$$

and that close to the point $z = i(a - b)$ the velocity is

$$v \simeq U[2a(a^2 - b^2)/3\pi b^2]^{\frac{1}{3}} e^{-\frac{1}{3}\pi i}(\bar{z} + ia - ib)^{-\frac{1}{3}}$$

10.22 Show that the equation

$$z = a/[\cos(n\tan^{-1} w)]^{2/n}$$

transforms the real w axis into the n radial lines ($z = re^{i\varphi}$) $r \ge a$, $\varphi = 0$, $1/n, 2/n, \ldots, (n-1)/n$. Where are the points on the w plane corresponding to $r = a$, $\varphi = m/n$; to $z = 0$? Show that the capacitance per unit length between a wire centered at $z = 0$ with radius $\rho \ll a$ and the system of fins, with traces on the x, y plane corresponding to the above radial lines, is approximately

$$C \simeq n/4[\ln(2a/\rho)]^{\frac{1}{2}n}$$

10.23 A condenser is made up of an outer surface S_0, at potential V, completely enclosing an inner surface S_i, at zero potential, the intermediate region R having dielectric constant ϵ. Show that the integral equation for the potential in region R is

$$\psi(\mathbf{r}) = V + \frac{1}{4\pi}\int G(\mathbf{r}|\mathbf{r}_i')\left[\frac{\partial}{\partial n'}\psi(r_i')\right]dS_i'$$

where G is the Green's function (for the Laplace equation) for the whole interior bounded by S_0 (when S_i is not present) which goes to zero on S_0, where $\mathbf{r}_i$ is the radius vector of a point on S_i and where $\partial/\partial n_i$ represents the gradient normal to S_i, pointing away from $R(\partial\psi/4\pi\ \partial n_i$ is the charge density on $S_i)$. Show that this integral equation may be converted into the following variational principle:

$$[C] = \left\{[\int \chi(\mathbf{r}_i)\, dS_i]^2 / \int\int \chi(\mathbf{r}_i) G(\mathbf{r}_i|\mathbf{r}'_i)\chi(\mathbf{r}'_i)\, dS_i\, dS'_i\right\}$$

the charge distribution function χ which makes $[C]$ stationary being proportional to the correct charge density on S_i ($\sigma = V\chi \int \chi\, dS_i / \int\int \chi G \chi\, dS_i\, dS'_i$) and this stationary value of $[C]$ being the value of the capacitance $C = (1/V)\int \sigma\, dS_i$.

10.24 Use the variational principle of Prob. 10.23 to compute the capacitance of a sphere of radius b inside a cylindrical can of radius a and axial length l ($b < a, \frac{1}{2}l$), the center of the sphere being at the center of the can. Use the Green's function of Eq. (10.3.24) and use the formula

$$\int_0^\pi \cos(\alpha \cos \vartheta) J_0(\beta \sin \vartheta) P_{2n}(\cos \vartheta) \sin \vartheta\, d\vartheta$$
$$= (-1)^n 2P_{2n}(\alpha/\sqrt{\alpha^2 + \beta^2})\, j_{2n}(\sqrt{\alpha^2 + \beta^2})$$

to compute the integrals. First calculate C, assuming $\chi = 1$ (uniform charge density over sphere), then use $\chi = 1 + \gamma P_2(\cos \vartheta)$, varying γ to obtain the best value, for minimum C. How much better is the second result?

10.25 A cylindrical cup of radius a and axial length b has its circumference and one end made of metal at zero potential. The other end is covered by a grid at potential V_0, fine enough so that the space just outside the grid is all at potential V_0. The gas inside the cup is at a pressure low enough so that the mean free path is large compared with a and b. At some instant the gas is uniformly ionized, after which the electrons created will be pulled toward the grid and most of them emerge from the cup. Show that the average energy of the ions emerging is

$$eV_0 \left[1 - \frac{4a}{b} \sum_{n=0}^{\infty} \frac{\tanh(\nu_n b/2a)}{\nu_n^3}\right]$$

where ν_n is the nth root of the equation $J_0(\nu) = 0$ and e is the electronic charge. Compute this average energy when $a = b$.

10.26 A probe conductor in the shape of half a prolate spheroid of maximum radius b and length a is driven into the earth with its axis perpendicular and its median plane tangential to the surface. It is held at a potential V_0 with respect to the earth a great distance from the probe. Show that, if a current I flows out from the probe when the steady state is reached, then the conductivity of the earth (assume it is uniform) will be

$$\frac{I}{4\pi\sqrt{a^2 - b^2}\, V_0} \ln \frac{a + \sqrt{a^2 - b^2}}{a - \sqrt{a^2 - b^2}}$$

10.27 A body has a shape nearly that of a sphere, so that its surface can be represented by the equation $r = a + F(\vartheta,\varphi)$, where F is always small enough so that F^2/a^2 can be neglected. Show that, if

$$F(\vartheta,\varphi) = \sum_{m,n} (A_{mn} \cos m\varphi + B_{mn} \sin m\varphi) P_n^m(\cos\vartheta)$$

then to this approximation the potential due to this body is

$$\psi = V_0 \frac{a}{r} + \sum_{m,n} (A_{mn} \cos m\varphi + B_{mn} \sin m\varphi) \left(\frac{a^n}{r^{n+1}}\right) P_n^m(\cos\vartheta)$$

if the surface of the body is kept at potential V_0.

Suppose that a sphere is flattened a little on one side, so that the surface is

$$r = \begin{cases} (a \cos\vartheta_0/\cos\vartheta) \xrightarrow[\vartheta_0 \text{ small}]{} a(2 - \cos\vartheta)/(2 - \cos\vartheta_0); & \theta \le \theta_0 \\ a; & \theta \ge \theta_0 \end{cases}$$

Show that

$$F = \frac{a/2}{2 - \cos\vartheta_0} \sum_{n=0}^{\infty} \frac{P_n(\cos\vartheta)}{(2n+3)(2n-1)} [-(2n-1)P_{n+2}(\cos\vartheta_0) + 2(2n+1)P_n(\cos\vartheta_0) - (2n+3)P_{n-2}(\cos\vartheta_0)]$$

[in the terms $n = 0$ and 1, where negative subscripts occur, we use the relation $P_{-\lambda}(z) = P_{\lambda-1}(z)$]. Show that the capacity of such a flattened sphere is approximately

$$C = 4\pi\epsilon a \left\{1 - \frac{a}{6(2 - \cos\vartheta_0)} [3P_1(\cos\vartheta_0) - P_2(\cos\theta_o) - 2]\right\}$$

What is the value of the correction when $\vartheta_0 = 10°$? What is the increase in the field strength at the surface, at $\vartheta = \vartheta_0$, when $\vartheta_0 = 10°$?

10.28 Show that, if $\psi(r,\vartheta,\varphi)$ is a solution of the Laplace equation in spherical coordinates, then $(a^2/r)\psi(a^2/r,\vartheta,\varphi)$ is also a solution of the Laplace equation. If ψ satisfies the Poisson equation, what is the charge density for the Poisson equation satisfied by $(a^2/r)\psi(a^2/r,\vartheta,\varphi)$? [This transformation $r \to \rho = a^2/r$ is called an *inversion* of the point r, ϑ, φ into (ρ,ϑ,φ) with respect to a sphere of radius a whose center is at the origin.] Show that the inversion of a sphere with respect to a point on its surface is a plane; that of a spherical cap of angle ϑ_0, a disk. By the method of inversion find the surface charge density and total charge on a sphere, from which a cap of angle ϑ_0 has been removed, when placed in an electric field which is uniform at large distances from the cap and directed along the axis of the cap. Compare the results with those of page 1269.

10.29 An electric charge distribution of density $\rho(r,\vartheta,\varphi)$ is zero outside $r = a$. Show that the field outside $r = a$ corresponds to an effective charge $q = A_{00}$, an effective dipole moment with components $D_x = A_{11}$, $D_y = B_{11}$, $D_z = A_{10}$ and an effective quadrupole moment with components

$$\begin{aligned} Q_{xx} &= 2A_{22} - \tfrac{1}{3}A_{20}; \quad Q_{xy} = 2B_{22} = Q_{yx} \\ Q_{xz} &= A_{21} = Q_{zx}; \quad Q_{yy} = -2A_{22} - \tfrac{1}{3}A_{20} \\ Q_{yz} &= B_{21} = Q_{zy}; \quad Q_{zz} = \tfrac{2}{3}A_{20} \end{aligned}$$

when one requires that $Q_{xx} + Q_{yy} + Q_{zz} = 0$ (see page 1276 for definition of the A's and B's).

10.30 By use of Eqs. (10.3.22) and (10.3.34), show that

$$e^{kz}\cos(m\varphi)J_m(k\rho) = \sum_{n=m}^{\infty} \frac{(kr)^n}{(n+m)!}\cos(m\varphi)P_n^m(\cos\vartheta)$$

and consequently that

$$r^n\cos(m\varphi)P_n^m(\cos\vartheta) = \lim_{k\to 0}\left\{\frac{(n+m)!}{n!}\frac{d^n}{dk^n}\left[e^{kz}J_m(k\rho)\cos(m\varphi)\right]\right\}$$

where r, ϑ, φ are spherical coordinates and ρ, z, φ are the coaxial circular cylindrical coordinates. The interior of a circular cylinder of finite length l ($\rho \le a$, $|z| \le \frac{1}{2}l$) is filled with uniform charge density. By using the above relations calculate the first three terms ($n = 0, 1, 2, 3$) in the multipole expansion (10.3.42) for the potential at large distances from the cylinder. What are the elements of the quadrupole dyadic for this charge distribution (see Prob. 10.29)?

10.31 A diatomic molecule can be approximately represented by a nucleus of charge $+3$ at $z = \frac{1}{2}a$, a nucleus of charge $+1$ at $z = -\frac{1}{2}a$, and a negative charge distribution

$$\rho = -\frac{16}{\pi a^3}\frac{(\xi_0 - \xi)(1 + \frac{1}{3}\eta)}{(\xi_0 - 1)^2(\xi^2 - \eta^2)}; \quad \xi < \xi_0; \quad \rho = 0; \quad \xi \ge \xi_0$$

where ξ, η, φ are prolate spheroidal coordinates with foci at $z = \pm a$. Compute the electrostatic potential, for $\xi > \xi_0$, for this charge distribution. By use of Eqs. (10.3.35) and (10.3.51) compute the effective dipole and quadrupole moments of the molecule (see Prob. 10.29).

10.32 A disk of radius a, at potential V_0, is inside a concentric sphere of radius c ($c > a$) which is at zero potential. Use the method of pages 1289 *et seq.* to compute the potential distribution between disk and sphere and the capacitance of the system, neglecting terms to the fifth order and higher in a/c.

10.33 A grounded, conducting plane has a circular orifice in it of radius a. A point charge q is placed on the axis of the orifice, a distance b out from the center of the orifice. What is the potential distribution produced? How fast does the potential go to zero on the other side of the plane from the charge for large distances along the axis of the orifice charge?

10.34 Using bispherical coordinates, compute the capacitance of a sphere of outer radius b inside a sphere of inner radius c, the center of the inner sphere being a distance d from that of the outer one ($c > b + d$). What is the charge distribution on the inner sphere? Compare these results with those given on page 1273.

10.35 A rigid sphere of radius ρ has its center a distance $b > \rho$ from an infinite plane. Incompressible fluid flows past sphere and plane, the asymptotic velocity being v_0, in a direction parallel to the plane (see page 1300). Calculate the η and φ components of the fluid velocity at the surface of the sphere, $\mu = \mu_0$. Obtain an approximate expression for the net force on the sphere.

10.36 The fourth-order surface defined by the parametric equations

$$z = \frac{a \sinh \mu}{\cosh \mu - \cos \eta_0}; \quad x = \frac{a \sin \mu_0 \cos \varphi}{\cosh \mu - \cos \eta_0}; \quad y = \frac{a \sin \eta_0 \sin \varphi}{\cosh \mu - \cos \eta_0}$$

for $-\infty < \mu < \infty$, $0 \leq \varphi \leq 2\pi$, η_0 fixed, is kept at potential V_0 with respect to infinity. Show that the potential, in bispherical coordinates, outside this surface can be written in the form

$$\psi = \frac{V_0}{2\pi} \sqrt{\cosh \mu - \cos \eta} \int_{-\infty}^{\infty} e^{ik\mu} \frac{P_{ik-\frac{1}{2}}(\cos \eta)}{P_{ik-\frac{1}{2}}(\cos \eta_0)} \, dk \cdot \\ \cdot \int_{-\infty}^{\infty} \frac{e^{-ik\lambda} \, d\lambda}{\sqrt{\cosh \mu - \cos \eta_0}}$$

where

$$P_{ik-\frac{1}{2}}(\cos \eta) = F(\tfrac{1}{2} - ik, \tfrac{1}{2} + ik|1| \sin^2 \tfrac{1}{2}\eta)$$

10.37 Show that, in toroidal coordinates [see Eq. (10.3.75)], the series for z is

$$z = \frac{\sqrt{8}}{\pi} a \sqrt{\cosh \mu - \cos \eta} \sum_{n=0}^{\infty} \eta Q_{n-\frac{1}{2}}(\cosh \mu) \sin (n\eta)$$

By use of this series compute the expression for the velocity potential for the flow of an incompressible fluid about the torus $\mu = \mu_0$, when the asymptotic flow vector has magnitude v_0, pointed along the axis of the toroid.

10.38 A conducting ellipsoid of axes λ_0, $\sqrt{\lambda_0^2 - b^2}$, $\sqrt{\lambda_0^2 - c^2}$ is kept at zero potential and is placed in a field, originally uniform with intensity E, in such a manner that its longest axis is in line with the field.

Show that the potential outside the ellipsoid is

$$\psi = -Ez + \frac{\lambda\mu\nu}{bc}\frac{D(b/c, \sin^{-1} c/\lambda)}{D(b/c, \sin^{-1} c/\lambda_0)}$$

where λ, μ, ν are ellipsoidal coordinates (see Sec. 2.64) and where

$$D(k,\psi) = \int_0^\infty \frac{\sin^2 \psi \, d\psi}{\sqrt{1 - k^2 \sin^2 \psi}}$$

Find the induced charge on the ellipsoid. Find the velocity potential for the corresponding hydrodynamical problem.

Trigonometric and Hyperbolic Functions

Trigonometric Functions (see Tables I to V, Appendix):

$$\cos z = 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - \cdots = \tfrac{1}{2}(e^{iz} + e^{-iz}) = \cosh(iz)$$

$$\sin z = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \cdots = \frac{1}{2i}(e^{iz} - e^{-iz}) = -i \sinh(iz)$$

$$\sin^2 z + \cos^2 z = 1; \quad \sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\cos x \cos y = \tfrac{1}{2}\cos(x + y) + \tfrac{1}{2}\cos(x - y)$$

$$\sin x \cos y = \tfrac{1}{2}\sin(x + y) + \tfrac{1}{2}\sin(x - y)$$

$$\sin x \sin y = -\tfrac{1}{2}\cos(x + y) + \tfrac{1}{2}\cos(x - y)$$

$$\frac{d}{dz}\cos z = -\sin z; \quad \frac{d}{dz}\sin z = \cos z$$

$$\int_{-\infty}^{\infty} e^{izt}\,dt \int_{-\infty}^{\infty} f(u)e^{-itu}\,du = 2\pi f(z); \quad \text{Fourier integral}$$

For $m = 2, 4, 6, \ldots$

$$\cos^m z = \frac{1}{2^{m-1}}\left[\cos mz + m\cos(m - 2)z + \frac{m(m - 1)}{2!}\cos(m - 4)z + \cdots + \frac{m(m - 1)\cdots(\frac{1}{2}m + 1)}{2(\frac{1}{2}m)!}\right]$$

$$\sin^m z = \frac{1}{2^{m-1}}\left[\frac{m(m - 1)\cdots(\frac{1}{2}m + 1)}{2(\frac{1}{2}m)!} - \frac{m(m - 1)\cdots(\frac{1}{2}m + 2)}{(\frac{1}{2}m - 1)!}\cos 2z + \cdots - (-1)^{\frac{1}{2}m} m\cos(m - 2)z + (-1)^{\frac{1}{2}m}\cos mz\right]$$

$$\cos(mz) = \cos^m z - \frac{m(m - 1)}{2!}\cos^{m-2} z \sin^2 z + \cdots + (-1)^{\frac{1}{2}m}\sin^m z$$

$$\sin(mz) = m\cos^{m-1} z \sin z - \frac{m(m - 1)(m - 2)}{3!}\cos^{m-3} z \sin^3 z + \cdots - (-1)^{\frac{1}{2}m} m \cos z \sin^{m-1} z$$

For $m = 1, 3, 5, 7, \ldots$

$$\cos^m z = \frac{1}{2^{m-1}}\left[\cos mz + m\cos(m-2)z + \frac{m(m-1)}{2!}\cos(m-4)z + \cdots + \frac{m(m-1)\cdots(\frac{1}{2}m+\frac{1}{2})}{(\frac{1}{2}m-\frac{1}{2})!}\cos z\right]$$

$$\sin^m z = \frac{1}{2^{m-1}}\left[\frac{m(m-1)\cdots(\frac{1}{2}m+\frac{1}{2})}{(\frac{1}{2}m-\frac{1}{2})!}\sin z - \frac{m(m-1)\cdots(\frac{1}{2}m+\frac{1}{2})}{(\frac{1}{2}m-\frac{1}{2})!}\sin 3z + \cdots - (-1)^{\frac{1}{2}m-\frac{1}{2}}m\sin(m-2)z + (-1)^{\frac{1}{2}m-\frac{1}{2}}\sin mz\right]$$

$$\cos(mz) = \cos^m z - \frac{m(m-1)}{2!}\cos^{m-2} z\sin^2 z + \cdots + (-1)^{\frac{1}{2}m-\frac{1}{2}}m\cos z\sin^{m-1} z$$

$$\sin(mz) = m\cos^{m-1} z\sin z - \frac{m(m-1)(m-2)}{3!}\cos^{m-3} z\sin^3 z + \cdots + (-1)^{\frac{1}{2}m-\frac{1}{2}}\sin^m z$$

Hyperbolic Functions

$$\cosh z = 1 + \frac{1}{2!}z^2 + \frac{1}{4!}z^4 + \cdots = \tfrac{1}{2}(e^z + e^{-z}) = \cos(iz)$$

$$\sinh z = z + \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \cdots = \tfrac{1}{2}(e^z - e^{-z}) = -i\sin(iz)$$

$$\cosh^2 z - \sinh^2 z = 1; \quad \frac{d}{dz}\cosh z = \sinh z$$

$$\frac{d}{dz}\sinh z = \cosh z$$

Generating Functions Relating Hyperbolic and Trigonometric Functions

$$\frac{\sinh u}{\cosh u - \cos z} = \sum_{n=0}^{\infty} \epsilon_n e^{-nu}\cos(nz); \quad \frac{e^{-u}\sin z}{\cosh u - \cos z} = 2\sum_{n=1}^{\infty} e^{-nu}\sin(nz)$$

Bessel Functions

[See Eq. (5.3.63) *et seq.*, and Tables X and XI, Appendix]:

$$J_m(z) = \frac{1}{m!}\left(\frac{z}{2}\right)^m\left[1 - \frac{(z/2)^2}{1!(m+1)} + \frac{(z/2)^4}{2!(m+1)(m+2)} - \cdots\right]$$

$$\xrightarrow[z\to\infty]{} \sqrt{\frac{2}{\pi z}}\cos[z - \tfrac{1}{2}\pi(m+\tfrac{1}{2})]; \quad \operatorname{Re} z > 0$$

$$N_n(z) = \frac{1}{\pi}\left[2\ln\left(\frac{z}{2}\right) + \gamma - \psi(n+1)\right]J_n(z) - \frac{1}{\pi}\sum_{m=0}^{n-1}\frac{(n-m-1)!}{m!(z/2)^{n-2m}}$$
$$- \frac{1}{\pi}\sum_{m=1}^{\infty}(-1)^m\frac{(z/2)^{n+2m}}{m!(n+m)!}\left[\sum_{r=1}^{m}\left(\frac{1}{r}+\frac{1}{r+n}\right)\right]$$
$$\xrightarrow[z\to 0]{} \frac{2}{\pi}[\ln z - 0.11593];\quad n = 0$$
$$\xrightarrow[z\to 0]{} -\frac{(n-1)!}{\pi}\left(\frac{2}{z}\right)^n;\quad n = 1, 2, 3, \ldots$$
$$\xrightarrow[z\to\infty]{} \sqrt{\frac{2}{\pi z}}\sin[z - \tfrac{1}{2}\pi(n+\tfrac{1}{2})];\quad \operatorname{Re} z > 0$$
$$J_{-m}(z) = (-1)^m J_m(z);\quad N_{-m}(z) = (-1)^m N_m(z)$$
$$N_{m-1}(z)J_m(z) - N_m(z)J_{m-1}(z) = \Delta(J_m,N_m) = 2/\pi z$$
$$e^{iz\cos\phi} = \sum_{m=-\infty}^{\infty} e^{im(\phi+\frac{1}{2}\pi)}J_m(z) = \sum_{m=0}^{\infty}\epsilon_m i^m\cos(m\phi)J_m(z)$$
$$H_m(z) = J_m(z) + iN_m(z) \xrightarrow[z\to\infty]{} \sqrt{\frac{2}{\pi z}}\, e^{iz-\frac{1}{2}\pi i(m+\frac{1}{2})}$$

See also tables at end of Chap. 11.

General Formulas Relating Bessel Functions. If $Z_m(z) = aJ_m(z) + bN_m(z)$ and $Y_m(z) = cJ_m(z) + dN_m(z)$ with a, b, c, d independent of m and z, then

$$\frac{1}{z}\frac{d}{dz}\left(z\frac{dZ_m}{dz}\right) + \left(1 - \frac{m^2}{z^2}\right)Z_m = 0;\quad \text{Bessel equation}$$
$$(2m/z)Z_m(z) = Z_{m-1}(z) + Z_{m+1}(z)$$
$$2\frac{d}{dz}Z_m(z) = Z_{m-1}(z) - Z_{m+1}(z)$$
$$\int z^{m+1}Z_m\,dz = z^{m+1}Z_{m+1}(z);\quad \int z^{-m+1}Z_m\,dz = -z^{-m+1}Z_{m-1}(z)$$
$$(\alpha^2 - \beta^2)\int Z_m(\alpha z)Y_m(\beta z)z\,dz = \beta z Z_m(\alpha z)Y_{m-1}(\beta z) - \alpha z Z_{m-1}(\alpha z)Y_m(\beta z)$$
$$\int[Z_0(\alpha z)]^2 z\,dz = \tfrac{1}{2}z^2[Z_0^2(\alpha z) + Z_1^2(\alpha z)]$$
$$\int[Z_m(\alpha z)]^2 z\,dz = \tfrac{1}{2}z^2[Z_m^2(\alpha z) - Z_{m-1}(\alpha z)Z_{m+1}(\alpha z)]$$
$$\frac{d}{dz}[z^{m+1}Z_{m+1}(z)] = z^{m+1}Z_m(z);\quad \frac{d}{dz}[z^{-m+1}Z_{m-1}(z)] = -z^{-m+1}Z_m(z)$$

Series Relations. If r, r_0, $R = \sqrt{r^2 + r_0^2 - 2rr_0\cos\phi}$ are the sides of a triangle with ϕ the angle between r and r_0 and ψ the angle between r_0 and R, then

$$e^{i\nu\psi}Z_\nu(kR) = \sum_{m=-\infty}^{\infty} e^{im\phi}J_m(kr)Z_{\nu+m}(kr_0)$$

where $0 < r < r_0$; $0 < \psi < \frac{1}{2}\pi$; ν real; $e^{2i\psi} = \dfrac{r_0 - re^{-i\phi}}{r_0 - re^{i\phi}}$

$$Z_0(kR) = \sum_{n=0}^{\infty} \epsilon_n \cos(n\phi) \begin{cases} J_n(kr)Z_n(kr_0); & r_0 > r \\ J_n(kr_0)Z_n(kr); & r_0 < r \end{cases}$$

$$\frac{Z_\nu(kR)}{R^\nu} = \frac{\sqrt{2\pi}}{(krr_0)^\nu} \sum_{m=0}^{\infty} (\nu + m)T_m^{\nu-\frac{1}{2}}(\cos\phi)J_{\nu+m}(kr)Z_{\nu+m}(kr_0); r_0 > r > 0$$

$$\sum_{n=0}^{\infty} B_n \cos\left(\frac{\pi nx}{a}\right) = \begin{cases} [1 - (x/b)^2]^\sigma; & 0 < |x| < b \\ 0; & b < |x| < a \end{cases}$$

where

$$B_0 = \tfrac{1}{2}\sqrt{\pi}\left(\frac{b}{a}\right)\frac{\Gamma(\sigma + 1)}{\Gamma(\sigma + \frac{3}{2})}; \quad B_n = \sqrt{\pi}\left(\frac{b}{a}\right)\frac{\Gamma(\sigma + 1)}{(\pi nb/2a)^{\sigma+\frac{1}{2}}} J_{\sigma+\frac{1}{2}}\left(\frac{\pi nb}{a}\right)$$

Hyperbolic Bessel Functions:

$$I_m(z) = (1/i)^m J_m(iz) \xrightarrow[z\to\infty]{} \frac{1}{\sqrt{2\pi z}} e^z$$

$$K_m(z) = \tfrac{1}{2}\pi i^{m+1}[J_m(iz) + iN_m(iz)] \xrightarrow[z\to\infty]{} \sqrt{\frac{\pi}{2z}}\, e^{-z}$$

$$I_{-m}(z) = I_m(z); \quad K_m(z) = K_{-m}(z)$$

$$I_m(z)K_{m-1}(z) + I_{m-1}(z)K_m(z) = -\Delta(I_m, K_m) = (1/z)$$

$$I_{m-1}(z) - I_{m+1}(z) = \left(\frac{2m}{z}\right) I_m(z); \quad I_{m-1}(z) + I_{m+1}(z) = 2\frac{d}{dz} I_m(z)$$

Definite Integrals Involving Bessel Functions:

$$J_m(z) = \frac{1}{2\pi i^m}\int_0^{2\pi} e^{iz\cos u}\cos(mu)\,du$$

$$= \frac{1}{\sqrt{\pi}}\frac{(z/2)^m}{\Gamma(m + \frac{1}{2})}\int_0^\pi \cos(z\cos u)\sin^{2m} u\,du$$

$$J_0(z) = \frac{2}{\pi}\int_0^1 \frac{\cos(zu)}{\sqrt{1 - u^2}}\,du; \quad N_0(z) = -\frac{2}{\pi}\int_0^\infty \cos(z\cosh u)\,du$$

$$H_0(kx) = \frac{-i}{\pi}\int_{-\infty}^{\infty} \frac{e^{ik\sqrt{x^2+t^2}}}{\sqrt{x^2 + t^2}}\,dt; \quad K_0(kx) = \int_0^\infty \frac{\cos(xt)}{\sqrt{k^2 + t^2}}\,dt$$

$$= \int_1^\infty \frac{e^{-kxu}}{\sqrt{u^2 - 1}}\,du$$

$$J_m(z) = \frac{2(2/z)^m}{\Gamma(\frac{1}{2} - m)\Gamma(\frac{1}{2})}\int_1^\infty \frac{\sin(zt)}{(t^2 - 1)^{m+\frac{1}{2}}}\,dt$$

$$N_m(z) = \frac{-2(2/z)^m}{\Gamma(\frac{1}{2} - m)\Gamma(\frac{1}{2})}\int_1^\infty \frac{\cos(zt)}{(t^2 - 1)^{m+\frac{1}{2}}}\,dt$$

$$K_{m-n}(kx) = \frac{2^n\Gamma(n+1)}{k^n x^{m-n}} \int_0^\infty \frac{J_m(kt)t^{m+1}}{(t^2+x^2)^{n+1}}\,dt$$

$$I_{\frac{1}{2}m}(\tfrac{1}{2}kx)K_{\frac{1}{2}m}(\tfrac{1}{2}kx) = \int_0^\infty \frac{J_\nu(kt)}{\sqrt{t^2+x^2}}\,dt$$

$$\int_0^\infty e^{-at}J_m(bt)\,\frac{dt}{t} = \frac{1}{mb^m}[\sqrt{a^2+b^2}-a]^m; \quad a \text{ and } b \text{ real and positive}$$

$$\int_0^\infty e^{-at}J_m(bt)t^m\,dt = \frac{(2b)^m\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})(a^2+b^2)^{m+\frac{1}{2}}}$$

$$J_{m-n-1}(ax) = \frac{a^{n+1}x^{m-n-1}}{2^n\Gamma(n+1)} \int_0^\infty \frac{J_m(a\sqrt{t^2+x^2})}{(t^2+x^2)^{\frac{1}{2}m}}\,t^{2n+1}\,dt \qquad \tfrac{1}{2}m - \tfrac{1}{4} > n > -1$$

$$[e^{iuv}/v] = \int_0^\infty \frac{J_0(tv)}{\sqrt{t^2-u^2}}\,t\,dt$$

$$K_{m-n-1}(ax) = \frac{a^{n+1}x^{m-n-1}}{2^n\Gamma(n+1)} \int_0^\infty \frac{K_m(a\sqrt{t^2+x^2})}{(t^2+x^2)^{\frac{1}{2}m}}\,t^{2n+1}\,dt; \quad n > -1$$

$$\int_0^\infty J_{m+1}(at)J_n(bt)t^{n-m}\,dt = \begin{cases} 0; & b > a \ge 0 \\ \dfrac{(a^2-b^2)^{m-n}b^n}{2^{m-n}a^{m+1}\Gamma(m-n+1)}; & a \ge b \ge 0 \end{cases}$$

$$a,\ b,\ m,\ n \text{ real}; \quad m+1 > n > -1$$

$$\int_0^\infty J_m(bt)J_n(a\sqrt{t^2+x^2})(t^2+x^2)^{-\frac{1}{2}n}t^{m+1}\,dt$$

$$= \begin{cases} 0; & b > a \ge 0 \\ \dfrac{b^m}{a^n}\left[\dfrac{1}{x}\sqrt{a^2-b^2}\right]^{n-m-1} J_{n-m-1}(x\sqrt{a^2-b^2}); & a > b \ge 0 \end{cases}$$

$$x,\ a,\ b,\ m,\ n \text{ real}; \quad n > m > -1$$

$$\int_0^\infty \frac{[J_m(\sqrt{t^2+z^2})]^2}{(t^2+z^2)^m}\,t^{2m-2}\,dt = \int_z^\infty \frac{[J_m(u)]^2}{u^{2m-1}}(u^2-z^2)^{m-\frac{3}{2}}\,du$$

$$= \frac{\Gamma(m-\frac{1}{2})}{2z^{m+1}\Gamma(\frac{1}{2})}S_m(2z); \quad \text{Re } m > \tfrac{1}{2}; \quad \text{Re } z > 0$$

$$\text{where } S_m(z) = \frac{2(z/2)^m}{\Gamma(m+\frac{1}{2})\Gamma(\frac{1}{2})}\int_0^{\frac{1}{2}\pi} \sin(z\cos u)\sin^{2m}u\,du$$

$$= \sum_{n=0}^\infty \frac{(-1)^n(z/2)^{m+2n+1}}{\Gamma(n+\frac{3}{2})\Gamma(m+n+\frac{3}{2})}$$

$$\xrightarrow[z\to\infty]{} \frac{(z/2)^{m-1}}{\Gamma(m+\frac{1}{2})\Gamma(\frac{1}{2})} + \sqrt{\frac{2}{\pi z}}\sin[z-\tfrac{1}{2}\pi(m+\tfrac{1}{2})]; \quad \text{Re } z > 0$$

$$\int_0^{\frac{1}{2}\pi}[J_1(x\sin\theta)]^2\frac{d\theta}{\sin\theta} = \int_0^1 \frac{[J_1(xu)]^2}{u\sqrt{1-u^2}}\,du = \tfrac{1}{2} - \left(\frac{1}{2x}\right)J_1(2x)$$

$$\int_0^\infty e^{-at}N_0(bt)\,dt = -\frac{(2/\pi)}{\sqrt{a^2+b^2}}\ln\left[\frac{a+\sqrt{a^2+b^2}}{a}\right];$$

a, b real and positive

$$\int_0^\infty J_m(tz)t\,dt\int_0^\infty J_m(tu)F(u)u\,du = F(z);\quad \text{Fourier-Bessel integral}$$

$$\int_0^{\frac{1}{2}\pi} J_m(z\sin\vartheta)\sin^{m+1}\vartheta\cos^{2n+1}\vartheta\,d\vartheta = \frac{2^n\Gamma(n+1)}{z^{n+1}}J_{m+n+1}(z)$$

$$\int_0^{\frac{1}{2}\pi} J_m(a\sin\theta)J_n(b\cos\theta)\sin^{m+1}\theta\cos^{n+1}\theta\,d\theta = a^m b^n\frac{J_{m+n+1}(\sqrt{a^2+b^2})}{(a^2+b^2)^{\frac{1}{2}(m+n+1)}}$$

$$\int_0^{\frac{1}{2}\pi}\cos(a\cos\theta)\cos(b\sin\theta)\,d\theta = \tfrac{1}{2}\pi J_0(\sqrt{a^2+b^2})$$

$$\int_0^\pi e^{iz\cos\vartheta\cos u}J_m(z\sin\vartheta\sin u)P_n^m(\cos u)\sin u\,du$$

$$= i^{n-m}\sqrt{\frac{2\pi}{z}}P_n^m(\cos\vartheta)J_{n+\frac{1}{2}}(z)$$

Legendre Functions

[See Eq. (5.3.36) and page 782; also Tables VI to IX and XIII, Appendix.]

$$P_n^m(z) = (1-z^2)^{\frac{1}{2}m}\frac{d^m}{dz^m}P_n(z) = (1-z^2)^{\frac{1}{2}m}T_{n-m}^m(z)$$

$$= \frac{(1-z^2)^{\frac{1}{2}m}}{2^n n!}\frac{d^{m+n}}{dz^{m+n}}(z^2-1)^n;\quad m, n = 0, 1, 2, 3, \ldots;\quad m \le n$$

$$= \frac{(n+m)!(1-z^2)^{\frac{1}{2}m}}{2^m m!(n-m)!}F\left(m-n, m+n+1|m+1|\frac{1-z}{2}\right)$$

$$= \frac{i^m(2n)!}{2^n n!(n-m)!}(z^2-1)^{\frac{1}{2}m}z^{n-m}F\left(\frac{m-n}{2}, \frac{m-n+1}{2}|\tfrac{1}{2}-n|\frac{1}{z^2}\right)$$

$$P_{m+2l+1}^m(0) = 0;\quad P_{m+2l}^m(0) = \frac{(-1)^l(2m+2l)!}{2^{m+2l}l!(m+l)!}$$

For $z = \cos\vartheta$, we have

$$P_0^0 = 1;\quad P_1^0 = z = \cos\vartheta;\quad P_2^0 = \tfrac{1}{2}(3z^2-1) = \tfrac{1}{4}(3\cos 2\vartheta+1);$$

$$P_3^0 = \tfrac{1}{2}(5z^3-3z) = \tfrac{1}{8}(5\cos 3\vartheta + 3\cos\vartheta);\ \cdots$$

$$P_1^1 = \sqrt{1-z^2} = \sin\vartheta;\quad P_2^1 = 3z\sqrt{1-z^2} = \tfrac{3}{2}\sin 2\vartheta;$$

$$P_3^1 = \tfrac{3}{2}\sqrt{1-z^2}(5z^2-1) = \tfrac{3}{8}(\sin\vartheta + 5\sin 3\vartheta);\ \cdots$$

$$P_2^2 = 3(1-z^2) = \tfrac{3}{2}(1-\cos 2\vartheta);$$

$$P_3^2 = 15z(1-z^2) = \tfrac{15}{4}(\cos\vartheta - \cos 3\vartheta);\ \cdots$$

$$P_3^3 = 15(1-z^2)^{\frac{3}{2}} = \tfrac{15}{4}(3\sin\vartheta - \sin 3\vartheta);\ \cdots$$

$$(2n+1)\sqrt{1-z^2}P_n^m(z) = P_{n+1}^{m+1}(z) - P_{n-1}^{m+1}(z)$$

$$= (n+m)(n+m-1)P_{n-1}^{m-1}(z)$$

$$- (n-m+1)(n-m+2)P_{n+1}^{m-1}(z)$$

$$(2n+1)zP_n^m(z) = (n-m+1)P_{n+1}^m(z) + (n+m)P_{n-1}^m(z)$$

$$(1-z^2)\frac{d}{dz}P_n^m(z) = (n+1)zP_n^m(z) - (n-m+1)P_{n+1}^m(z)$$

$$\frac{d}{dz}[(1-z^2)^{\frac{1}{2}m}P_n^m(z)] = -(n-m+1)(n+m)(1-z^2)^{\frac{1}{2}m-\frac{1}{2}}P_n^{m-1}(z)$$

$$\int_{-1}^{1} P_n^m(z)P_l^m(z)\,dz = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{nl}$$

$$\int_{-1}^{1}\frac{P_n^m(z)P_n^k(z)}{1-z^2}\,dz = \frac{1}{m}\frac{(n+m)!}{(n-m)!}\delta_{mk}$$

$$\frac{2^m\Gamma(m+\frac{1}{2})\sin^m\vartheta}{\Gamma(\frac{1}{2})[1+h^2-2h\cos\vartheta]^{m+\frac{1}{2}}} = \sum_{n=0}^{\infty} h^nP_{n+m}^m(\cos\vartheta)$$

$$P_n^m(z) = \frac{(n+m)!}{(n-m)!}(1-z^2)^{-\frac{1}{2}m}\int_z^1 du_1\int_{u_1}^1 du_2\cdots\int_{u_{m-1}}^1 du_mP_n(u_m)$$

Zonal Harmonics: $P_n(z) = P_n^0(z) = T_n^0(z)$ (see page 748).

$$P_{2n}(z) = (-1)^n\frac{(2n)!}{[2^nn!]^2}F(-n, n+\tfrac{1}{2}|\tfrac{1}{2}|z^2)$$

$$P_{2n+1}(z) = (-1)^n\frac{(2n+1)!}{[2^nn!]^2}zF(-n, n+\tfrac{3}{2}|\tfrac{3}{2}|z^2)$$

$$(2n+1)z^{2n} = P_0(z) + \frac{5\cdot 2n}{2n+3}P_2(z) + \frac{9\cdot 2n(2n-2)}{(2n+3)(2n+5)}P_4(z) + \cdots$$

$$(2n+3)z^{2n+1} = 3P_1(z) + \frac{7\cdot 2n}{2n+5}P_3(z) + \frac{11\cdot 2n(2n-2)}{(2n+5)(2n+7)}P_5(z) + \cdots$$

$$\frac{d}{dz}P_n(z) = (2n-1)P_{n-1}(z) + (2n-5)P_{n-3}(z) + (2n-9)P_{n-5}(z) + \cdots$$

$$\frac{(2/\pi)}{\sin\vartheta} = P_0(\cos\vartheta) + 5(\tfrac{1}{2})^2P_2(\cos\vartheta) + 9\left(\frac{1\cdot 3}{2\cdot 4}\right)^2P_4(\cos\vartheta) + 13\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2P_6(\cos\vartheta) + \cdots$$

$$\int(2n+1)P_n(z)\,dz = [P_{n+1}(z) - P_{n-1}(z)];$$

when $n = 0$, $P_{-1}(z)$ is taken to be 1

$$\int_0^\pi P_{2n}(\cos\vartheta)\,d\vartheta = \pi\left\{\frac{(2n)!}{[2^nn!]^2}\right\}^2;$$

$$\int_0^\pi P_{2n+1}(\cos\vartheta)\cos\vartheta\,d\vartheta = \pi\frac{(2n)!(2n+2)!}{[2^nn!2^{n+1}(n+1)!]^2}$$

$$\int_0^\pi P_n(\cos\vartheta)\sin(m\vartheta)\,d\vartheta = \begin{cases} 2\dfrac{(m+n-1)(m+n-3)\cdots(m-n+1)}{(m+n)(m+n-2)\cdots(m-n)}; & \text{if } n<m \text{ and } (n+m) \text{ is odd} \\ 0; & \text{otherwise}\end{cases}$$

$$P_n[\cos\vartheta\cos\vartheta_0 + \sin\vartheta\sin\vartheta_0\cos(\phi-\phi_0)] = P_n(\cos\vartheta)P_n(\cos\vartheta_0)$$
$$+ 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\vartheta)P_n^m(\cos\vartheta_0)\cos[m(\phi-\phi_0)]$$

if $Z_n(\vartheta_0,\phi_0) = a_0P_n(\cos\vartheta_0) + \sum_{m=1}^{n} a_m\cos(m\phi_0+\alpha_m)P_n^m(\cos\vartheta_0)$, then

$$\int_0^{2\pi}d\phi_0\int_0^{2\pi}Z_n(\vartheta_0,\phi_0)P_n[\cos\vartheta\cos\vartheta_0$$
$$+\sin\vartheta\sin\vartheta_0\cos(\phi-\phi_0)]\sin\vartheta_0\,d\vartheta_0 = \frac{4\pi}{2n+1}Z_n(\vartheta,\phi)$$

$$\sqrt{2}\sum_{n=0}^{\infty}\cos[(n+\tfrac{1}{2})\beta]P_n(\cos\vartheta) = \begin{cases}1/\sqrt{\cos\beta-\cos\vartheta}; & 0\le\beta<\vartheta\le\pi\\ 0; & 0\le\vartheta<\beta\le\pi\end{cases}$$

$$\sqrt{2}\sum_{n=0}^{\infty}\sin[(n+\tfrac{1}{2})\beta]P_n(\cos\vartheta) = \begin{cases}0; & 0\le\beta<\vartheta\le\pi\\ 1/\sqrt{\cos\vartheta-\cos\beta}; & 0\le\vartheta<\beta\le\pi\end{cases}$$

Legendre Functions of the Second Kind [see Eqs. (5.3.41) *et seq.*]:

$$Q_n^m = (-1)^m(z^2-1)^{\frac{1}{2}m}\frac{d^m}{dz^m}Q_n(z) = (-1)^m(z^2-1)^{\frac{1}{2}m}V_{n-m}^m(z)$$
$$= (z^2-1)^{\frac{1}{2}m}\frac{2^m\Gamma(\frac{1}{2})(n+m)!}{\Gamma(n+\frac{3}{2})(2z)^{n+m+1}}F\left(\frac{n+m+1}{2},\frac{n+m+2}{2}\middle|n+\tfrac{3}{2}\middle|\frac{1}{z^2}\right)$$

$$Q_n(z) = \frac{1}{2^nn!}\frac{d^n}{dz^n}\left[(z^2-1)^n\ln\left(\frac{z+1}{z-1}\right)\right] - \tfrac{1}{2}P_n(z)\ln\left(\frac{z+1}{z-1}\right)$$

In order that Q be single-valued, we make a cut on the real axis between $+1$ and -1.

For $z = \cosh\mu$; $\coth(\frac{1}{2}\mu) = \sqrt{\dfrac{z+1}{z-1}}$;

$$\tfrac{1}{2}\ln\left(\frac{z+1}{z-1}\right) = \ln[\coth(\tfrac{1}{2}\mu)] = 2\tanh^{-1}(e^{-\mu}):$$

$$Q_0^0 = \tfrac{1}{2}\ln\left(\frac{z+1}{z-1}\right) = \ln[\coth(\tfrac{1}{2}\mu)];$$

$$Q_1^0 = \tfrac{1}{2}z\ln\left(\frac{z+1}{z-1}\right) - 1 = \cosh\mu\ln[\coth(\tfrac{1}{2}\mu)] - 1$$

$$Q_2^0 = \tfrac{1}{4}(3z^2-1)\ln\left(\frac{z+1}{z-1}\right) - \tfrac{3}{2}z$$
$$= \tfrac{1}{8}[3\cosh(2\mu)+1]\ln[\coth(\tfrac{1}{2}\mu)] - \tfrac{3}{2}\cosh\mu$$

$$Q_1^1 = \sqrt{z^2-1}\left[\frac{z}{z^2-1} - \tfrac{1}{2}\ln\left(\frac{z+1}{z-1}\right)\right] = \coth\mu - \sinh\mu\ln[\coth(\tfrac{1}{2}\mu)]$$

$$Q_2^1 = \sqrt{z^2-1}\left[\frac{3z^2-2}{z^2-1} - \tfrac{3}{2}z\ln\left(\frac{z+1}{z-1}\right)\right]$$
$$= \operatorname{csch}\mu + 3\sinh\mu - \tfrac{3}{2}\sinh(2\mu)\ln[\coth(\tfrac{1}{2}\mu)]$$

$$Q_2^2 = \tfrac{3}{2}(z^2 - 1)\ln\left(\frac{z+1}{z-1}\right) - \frac{3z^3 - 5z}{z^2 - 1}$$
$$= \tfrac{3}{2}[\cosh(2\mu) - 1]\ln[\coth(\tfrac{1}{2}\mu)] - \cosh\mu\,[3 - 2\,\mathrm{csch}^2\,\mu$$
$$Q_n = \tfrac{1}{2}P_n(z)\ln\left(\frac{z+1}{z-1}\right) - \frac{2n-1}{1\cdot n}P_{n-1}(z) - \frac{2n-5}{3(n-1)}P_{n-3}(z)\cdots$$
$$(2n+1)\sqrt{z^2-1}\,Q_n^m(z) = Q_{n-1}^m(z) - Q_{n+1}^m(z)$$
$$(2n+1)zQ_n^m(z) = (n-m+1)Q_{n+1}^m(z) + (n+m)Q_{n-1}^m(z)$$
$$(z^2-1)\frac{d}{dz}Q_n^m(z) = (n-m+1)Q_{n+1}^m(z) - (n+1)zQ_n^m(z)$$

For $|h| < 1$, we have

$$\frac{\cosh^{-1}[h\,\mathrm{csch}\,\mu - \coth\mu]}{\sqrt{1 + h^2 - 2h\cosh\mu}} = \sum_{n=0}^{\infty} h^n Q_n(\cosh\mu)$$

For a point $z = x + iy$ which is inside an ellipse, drawn through the point t and having foci at the points ± 1, the following expression converges:

$$\frac{1}{t-z} = \sum_{n=0}^{\infty}(2n+1)P_n(z)Q_n(t)$$

When $\mu > \mu_0$ the following expansion holds:

$$Q_n[\cosh\mu\cosh\mu_0 + \sinh\mu\sinh\mu_0\cos(\phi - \phi_0)] = Q_n(\cosh\mu)P_n(\cosh\mu_0)$$
$$+ 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}i^m Q_n^m(\cosh\mu)P_n^m(\cosh\mu_0)\cos[m(\phi - \phi_0)]$$

Note: $P_n^m(\cosh\mu_0) = i^m \sinh^m\mu_0 T_{n-m}^m(\cosh\mu_0)$

$$\Delta(P_n^m, Q_n^m) = \frac{(n-m+1)}{1-z^2}[P_{n+1}^m(z)Q_n^m(z) - P_n^m(z)Q_{n+1}^m(z)]$$
$$= \frac{i^m(n+m)!}{(n-m)!(1-z^2)}$$

Functions of imaginary argument:

$$P_n^m(iz)$$
$$= \frac{(2n)!}{2^n n!(n-m)!}(z^2+1)^{\frac{1}{2}m}(iz)^{n-m}F\left(\frac{m-n}{2}, \frac{m-n+1}{2}\,\middle|\,\tfrac{1}{2} - n\,\middle|\, -\frac{1}{z^2}\right)$$
$$Q_n^m(iz) = i^m\frac{2^n n!(n+m)!}{(2n+1)!}\cdot$$
$$\cdot(z^2+1)^{\frac{1}{2}m}(iz)^{-n-m-1}F\left(\frac{n+m+1}{2}, \frac{n+m+2}{2}\,\middle|\, n + \tfrac{3}{2}\,\middle|\, -\frac{1}{z^2}\right)$$
$$Q_0^0(iz) = -i\tan^{-1}(1/z); \quad Q_1^0(iz) = z\tan^{-1}(1/z) - 1$$
$$Q_1^1(iz) = \frac{z}{\sqrt{z^2+1}} - \sqrt{z^2+1}\tan^{-1}(1/z)$$

Toroidal Harmonics:

$$P^m_{n-\frac{1}{2}}(\cosh\mu) = i^m \sinh^m(\mu) T^m_{n-m-\frac{1}{2}}(\cosh\mu)$$
$$= \frac{i^m\Gamma(n+m+\frac{1}{2})}{2^m m!\Gamma(n-m+\frac{1}{2})} \frac{\tanh^m\mu}{\cosh^{-n+\frac{1}{2}}\mu} F\left(\frac{m-n+\frac{1}{2}}{2}, \frac{m-n+\frac{3}{2}}{2} \middle| m+1 \middle| \tanh^2\mu\right)$$
$$= \frac{i^m\Gamma(n+m+\frac{1}{2})}{m!\Gamma(n-m+\frac{1}{2})} \frac{\tanh^m(\frac{1}{2}\mu)}{\cosh^{1-2n}(\frac{1}{2}\mu)} F(\tfrac{1}{2}-n, \tfrac{1}{2}+m-n|m+1|\tanh^2\tfrac{1}{2}\mu)$$
$$= \frac{i^m(n-1)!2^n}{\sqrt{2\pi}\,\Gamma(n-m+\frac{1}{2})} \tanh^m\mu\cosh^{n-\frac{1}{2}}\mu\cdot$$
$$\cdot\left[1 + \frac{(m-n+\frac{1}{2})(m-n+\frac{3}{2})}{1!(1-n)}(\tfrac{1}{2}\operatorname{sech}\mu)^2 + \cdots\right.$$
$$\cdots + \frac{(m-n+\frac{1}{2})(m-n+\frac{3}{2})(m-n+\frac{5}{2})\cdots(n+m-\frac{1}{2})}{(n-1)!(1-n)(2-n)\cdots(-2)(-1)}\cdot$$
$$\left.\cdot(\tfrac{1}{2}\operatorname{sech}\mu)^{2n-2}\right]$$
$$-(-i)^m\frac{\Gamma(n+m+\frac{1}{2})}{2^{n-\frac{1}{2}}\pi^{\frac{1}{2}}n!}\frac{\tanh^m\mu}{\cosh^{n+\frac{1}{2}}\mu}\ln[\operatorname{sech}\mu]\cdot$$
$$\cdot F\left(\frac{n+m+\frac{1}{2}}{2}, \frac{n+m+\frac{3}{2}}{2}\middle| n+1\middle|\operatorname{sech}^2 z\right)$$
$$+(-i)^m\frac{2^{m-1}}{\pi^2}\frac{\tanh^m\mu}{\cosh^{n+\frac{1}{2}}\mu}\sum_{s=0}^{\infty}\frac{\Gamma\left(s+\frac{n+m+\frac{1}{2}}{2}\right)\Gamma\left(s+\frac{n+m+\frac{3}{2}}{2}\right)}{s!(n+s)!}\cdot$$
$$\cdot\left[\psi(s+1)+\psi(n+s+1)-\psi(\tfrac{1}{2}n+\tfrac{1}{2}m+s+\tfrac{1}{4})\right.$$
$$\left.-\psi(\tfrac{1}{2}n+\tfrac{1}{2}m+s+\tfrac{3}{4})\right]\operatorname{sech}^{2s}\mu$$

(When $n = 0$ the first of the three series in the last expression vanishes.)

$$Q^m_{n-\frac{1}{2}}(\cosh\mu) = (-1)^m\sinh^m(\mu)V^m_{n-m-\frac{1}{2}}(\cosh\mu)$$
$$= \frac{\sqrt{\pi}\,\Gamma(n+m+\frac{1}{2})}{2^{n+\frac{1}{2}}n!}\frac{\coth^m\mu}{\cosh^{n+\frac{1}{2}}\mu}\cdot$$
$$\cdot F\left(\frac{n-m+\frac{1}{2}}{2}, \frac{n-m+\frac{3}{2}}{2}\middle| n+1\middle|\operatorname{sech}^2\mu\right)$$
$$= 2^{m-1}(m-1)!\frac{\operatorname{sech}^{n+\frac{1}{2}}\mu}{\tanh^m\mu}\left[1+\frac{(n-m+\frac{1}{2})(n-m+\frac{3}{2})}{1!(1-m)}(\tfrac{1}{2}\tanh\mu)^2\right.$$
$$+\cdots+\frac{(n-m+\frac{1}{2})(n-m+\frac{3}{2})(n-m+\frac{5}{2})\cdots(n+m-\frac{1}{2})}{(m-1)!(1-m)(2-m)\cdots(-2)(-1)}\cdot$$
$$\left.\cdot(\tfrac{1}{2}\tanh\mu)^{2m-2}\right]$$

$$- (-1)^m \frac{\Gamma(n+m+\frac{1}{2})}{2^m m!\Gamma(n-m+\frac{1}{2})} \tanh^m \mu \operatorname{sech}^{n+\frac{1}{2}} \mu \ln[\tanh \mu] \cdot$$
$$\cdot F\left(\frac{n+m+\frac{1}{2}}{2}, \frac{n+m+\frac{3}{2}}{2} \middle| m+1 \middle| \tanh^2 \mu\right)$$
$$- (-1)^m \frac{2^{n-\frac{1}{2}} \operatorname{sech}^{n+\frac{1}{2}} \mu}{\sqrt{\pi}\,\Gamma(n-m+\frac{1}{2})} \sum_{s=0}^{\infty} \frac{\Gamma\left(s+\frac{n+m+\frac{1}{2}}{2}\right)\left(s+\frac{n+m+\frac{3}{2}}{2}\right)}{s!(m+s)!} \cdot$$
$$\cdot \left[\psi(\tfrac{1}{2}n+\tfrac{1}{2}m+\tfrac{1}{4}+s) + \psi(\tfrac{1}{2}n+\tfrac{1}{2}m+\tfrac{3}{4}+s) - \psi(s+1) - \psi(m+s+1)\right] \tanh^{2s} \mu$$

(When $m = 0$ the first of the three series in this last expression vanishes.)

The Wronskian for these solutions is

$$\Delta(P,Q) = P^m_{n-\frac{1}{2}} \frac{d}{d\mu} Q^m_{n-\frac{1}{2}} - Q^m_{n-\frac{1}{2}} \frac{d}{d\mu} P^m_{n-\frac{1}{2}} = - \frac{i^m\Gamma(n+m+\frac{1}{2})}{\Gamma(n-m+\frac{1}{2}) \sinh \mu}$$

$$\frac{1}{\sqrt{\cosh \mu - \cos \eta}} = \frac{\sqrt{2}}{\pi} \sum_{n=0}^{\infty} Q_{n-\frac{1}{2}}(\cosh \mu) \cos(n\eta)$$

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CHAPTER 11

The Wave Equation

In Chap. 10 there was a certain lack of logical unity in our development of the various solutions of the Laplace equation, because of the inclusion of the special techniques of the functions of a complex variable. These techniques were too valuable to be omitted, but their use was limited to two dimensions, so that the attack on two-dimensional problems had to be fundamentally different from the attack on three-dimensional cases. A connection, of sorts, was provided by the use of the integral representation (10.3.2).

In the present chapter there will again be a certain lack of close, logical unity. The special case, where the dependence on time is simple harmonic, is so prevalent that one must discuss it in detail. And since techniques for the solution of this special case are, many of them, inapplicable to solutions for more general time dependence, there will again be an interruption of logical continuity, in the interests of utility. There will emerge, however, a connection between the special and the general cases, via the Fourier integral.

Simple harmonic motion of a "wavy" medium can arise in two ways: during free or during forced vibration. It is possible to start the system, by giving it an impulse of just the right sort at just the right places, so that its subsequent free motion is periodic, and *usually* if it is periodic it is simple harmonic. Such frequencies of free vibration are eigenvalues for the system, only certain values are allowed. On the other hand, a simple harmonic source or a simple harmonic boundary condition will produce forced motion of the system at any frequency (though when the forcing frequency equals one of the frequencies of free vibration, the response may be infinite).

In either case, when the dependence on time is simple harmonic, the time dependence may be separated off, $\Psi = \psi_k(x,y,z)e^{-i\omega t}$, and the resulting equation for ψ is the Helmholtz equation $\nabla^2\psi + k^2\psi = 0$, $k = \omega/c$. In other words, only when every part of the solution has a sinusoidal dependence on time, with a single frequency everywhere, is it possible to separate off the time term as a factor and deal with a space-dependent solution of the Helmholtz equation. This space factor, as we have seen, is to satisfy homogeneous Dirichlet/Neumann conditions on the boundary, except for the parts of the boundary which are acting as sources, where inhomogeneous conditions are prescribed.

If homogeneous boundary conditions apply everywhere over a finite closed surface, the corresponding set of solutions ψ_k of the homogeneous Helmholtz equation are eigenfunctions, the eigenvalues k^2 form a discrete set of allowed values and the free vibrations of the system may all be represented in terms of a series of these eigenfunction solutions. Solutions for forced, simple-harmonic motion are solutions of an inhomogeneous Helmholtz equation or for inhomogeneous boundary conditions (or both); the inhomogeneities correspond to the distribution of "sources," regions where energy interchange between the system and some "driving system" can take place. Such solutions can be found for every value of k.

Once the forced simple harmonic motion solutions are found for all values of $k = \omega/c$, it is then possible to compute the response of the system to forces which are any arbitrary function of time, by use of the Fourier integral technique.

Consequently, we shall spend a great deal of our time, in this chapter, in calculations of solutions of the Helmholtz equation. From these solutions one can obtain expressions for free vibration of the system or, alternately, for response to an arbitrary forcing function of time. The chapter first takes up eigenfunction techniques for solution in one, two, and three dimensions. It turns out that these techniques are most satisfactory for low frequencies (long wavelengths). The last section will concentrate on Green's function techniques, which are also useful (when they can be applied) for the high-frequency limit of very short wavelength. In the first section some interesting variations of the usual wave equation will be discussed.

11.1 *Wave Motion in One Space Dimension*

We have already discussed the motion of a perfectly flexible string, under tension between rigid supports, in Secs. 2.1 and 6.3. Here we shall discuss variations of the simple case, which arise in various physical problems. Then we turn our attention to the transmission of sound inside tubes, where certain useful analogies with electric transmission lines will be discussed. In both of these examples we shall study the use of the Fourier transform method, which connects the steady-state response of a system with its free vibrations and with its transient response.

In extending our study of the simple string we can investigate a wide variety of effects which serve to differentiate an actual string from our idealized string. For instance (as mentioned in Sec. 2.1) the force on each element of an actual string is caused by many other things beside just the restoring force of the tension, as was assumed in the simple case. The string vibrates in air, or in some other medium, and this medium reacts on the string to give each element an additional load, reactive as well as resistive. Most strings have stiffness and internal friction, and no end support is perfectly rigid.

In a great number of cases these additional complications have each a relatively small effect on the motion of the string, compared with the tension (which is, after all, the reason the problem of the simple string is of interest). It is therefore possible to study the effects of each of the complicating factors separately, or a few at a time, since the effects are additive to the first order of approximation. In most actual cases, too, the additional factors enter, to an appreciable extent, for only one or two of them at a time. So we shall study the motion of strings with distributed friction *or* with yielding supports *or* with additional stiffness, etc., rather than indiscriminately adding all the complications and obtaining an indigestibly complicated answer.

Fourier Transforms. Before we take up these additional complications in sequence, it will be well to outline the connection, by means of the Fourier transform, between forced, simple harmonic vibrations and transient motions and free vibrations. Force may be applied to a string (or to a membrane or an elastic medium or an electromagnetic field, etc.) in some region within the boundary surface or on the surface. If the force is simple-harmonic, with frequency $\omega/2\pi$, the equation to be satisfied is the inhomogeneous one

$$\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = -4\pi F_\omega(x,y,z)e^{-i\omega t}$$

(with homogeneous space boundary conditions) for an internal force, or else the homogeneous equation

$$\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = 0$$

(for a force on the boundary) with inhomogeneous boundary conditions $\alpha\Psi(\mathbf{r}^s) + \beta B(\mathbf{r}^s) = F_\omega(\mathbf{r}^s)e^{-i\omega t}$ over some part of the boundary. (B is the normal gradient of Ψ at the surface, as before.) In either case we may set $\Psi = \psi(x,y,z)e^{-i\omega t}$ for the steady-state solution, and have ψ satisfy either the inhomogeneous Helmholtz equation

$$\nabla^2\psi + k^2\psi = -4\pi F_\omega; \quad k = \omega/c$$

with homogeneous boundary conditions or else have ψ satisfy the homogeneous equation

$$\nabla^2\psi + k^2\psi = 0$$

with the inhomogeneous boundary condition $\alpha\psi(\mathbf{r}^s) + \beta N(\mathbf{r}^s) = F_\omega(\mathbf{r}^s)$ over part of the boundary ($B = Ne^{-i\omega t}$).

But we need not include such complications in our first survey of the method. It is sufficient to say that a certain simple-harmonic applied force $F(\omega)e^{-i\omega t}$ produces a response in the system which is a product of the "amplitude" $F(\omega)$ of the driving force and of a function $\psi(x,y,z)$, which is the amplitude of response for a "unit" force.

Force $F(\omega)e^{-i\omega t}$ produces steady-state response $F(\omega)\psi(x,y,z|\omega)e^{-i\omega t}$ where the quantity ψ, the response per unit force at frequency $\omega/2\pi$, is called the *admittance* of the system for the force, by analogy with a-c circuit theory.

If we are interested only in steady-state response to simple harmonic forces, there would be nothing more to say. But we also wish to calculate the response to an arbitrary nonperiodic force $f(t)$. The way we do this, of course, is to break up $f(t)$ into its simple harmonic components by means of the Fourier integral [see Eq. (4.8.2)]

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega t}\,d\omega; \quad F(\omega) = \int_{-\infty}^{\infty} f(t)e^{+i\omega t}\,dt \tag{11.1.1}$$

Then, having solved the steady-state problem to obtain the admittance, $\psi(x,y,z|\omega)$, of the system, we put the components of the response together to obtain the response of the system to the original, nonperiodic force.

$$\Psi(x,y,z|t) = \int_{-\infty}^{\infty} \psi(x,y,z|\omega)F(\omega)e^{-i\omega t}\,d\omega \tag{11.1.2}$$

This is all quite straightforward as long as the integral for $F(\omega)$, containing $f(t)$, is absolutely convergent. But there are times, for example when the admittance ψ has singularities on the real axis of ω, when the integral of (11.1.2) is not absolutely convergent. Of course we can still compute a value for the integral if we consider it to be a contour integral for which we are allowed to "by-pass" the singularities. But the results will be radically different depending on whether we by-pass the singularities by circling around above them (in the positive imaginary direction) or below them (in the negative imaginary direction). So now we must go back to physics to find a rule for by-passing which corresponds to "reality."

Only a nondissipative system has an admittance with singularities on the real ω axis. For, as we shall see shortly, the poles of the admittance, considered as a function of the complex quantity $\omega = 2\pi\nu + i\kappa$, give the frequencies ν_n and corresponding damping constants κ_n for the free vibrations of the system. If friction is involved, these poles are all *below* the real ω axis ($\omega_n = 2\pi\nu_n - i\kappa_n$) for then the time factor for free vibration of the system will be $\exp(-i\omega_n t) = \exp(-2\pi i\nu_n t - \kappa_n t)$, the amplitude *diminishing* as time increases. Since we cannot have negative friction, it appears that there cannot be poles of the admittance above the real ω axis, for κ_n negative.

We now see why we have trouble with Eq. (11.1.2) when the admittance ψ has poles on the real axis. For this is the case only when the system has no friction and when, therefore, its free vibration does not damp out. In this case, however, we cannot, in principle, set up a steady-state forced motion of the system, for a steady-state vibration is one for which the transient, free vibration has damped out and only the forced motion is left, and with no friction the transient does not damp out. What is done in cases of this sort is to remind ourselves that no system is absolutely without friction and that, even though the damping is exceedingly small, nevertheless if we wait long enough the free vibrations will damp out, leaving the steady-state vibration. This means that we assume that, even in this limiting case, the poles of the admittance are a small amount *below* the real axis of ω. Thus we see that our contour should be drawn *over the top* of all poles of the admittance ψ.

Alternatively, in order to avoid going through this verbiage each time we consider a system with "negligible" friction, we could just as well say that the line of integration of Eq. (11.1.2) is an infinitesimal amount *above* the real axis of ω, so as to ensure that the contour goes *above* all poles of $\psi(x,y,z|\omega)$, even if we assume that the system has "negligible" friction and try placing the poles of ψ on the real axis. This convention works well, leading to convergent integrals, only when $t > 0$. This is not always a serious handicap, however, for many driving forces do not have an infinite past history, being zero before some time which might as well be called $t = 0$. Whenever the system is at rest before $t = 0$, then this simplifying convention may be used. We shall see its power in a few pages.

String with Friction. As an example of our procedure we choose a flexible string under tension T, subject to a small amount of frictional retardation by the medium in which it moves. Referring to Sec. 2.1, we find that the equation of motion of the string is

$$\rho \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2} - R \frac{\partial y}{\partial t} \tag{11.1.3}$$

where ρ is the mass per unit length of the string, R is the frictional resistance of the medium per unit length of string, per unit velocity of this part, and the velocity of wave motion is $c = \sqrt{T/\rho}$. For actual media the resistance R depends on frequency, but to simplify the discussion we shall first take it to be constant.

If we apply a simple-harmonic transverse driving force at the point $x = x_0$, the equation to be satisfied is the inhomogeneous one

$$\frac{\partial^2 y}{\partial x^2} - \frac{2\kappa}{c^2}\frac{\partial y}{\partial t} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = -F(x_0|\omega)\delta(x - x_0)e^{-i\omega t}$$

where $\kappa = R/2\rho$, $c^2 = T/\rho$ and $F(x_0|\omega)$ is the ratio of the amplitude of the applied force to the tension T of the string. To obtain the steady-state response we set $y(x,t) = \psi(x|\omega)e^{-i\omega t}$ and obtain the inhomogeneous Helmholtz equation

$$\frac{d^2\psi}{dx^2} + \frac{1}{c^2}(\omega^2 + 2i\kappa\omega)\psi = -F(x_0|\omega)\delta(x - x_0)$$

As was shown in Secs. (2.1) and (7.2) the solution of this equation is one which has a discontinuity in slope equal to F at $x = x_0$.

We assume that the string supports are perfectly rigid and, if the length is l, that they are located at $x = 0$ and $x = l$. A combination of solutions of the homogeneous equation, which goes to zero at $x = 0$, $x = l$ and which has unit discontinuity at $x = x_0$ is the Green's function

$$G(x|x_0|\omega) = \begin{cases} \dfrac{\sin[kx]\sin[k(l - x_0)]}{k\sin(kl)}; & x < x_0 \\[2ex] \dfrac{\sin[kx_0]\sin[k(l - x)]}{k\sin(kl)}; & x > x_0 \end{cases} \tag{11.1.4}$$

where $k = (\omega/c)\sqrt{1 + (2i\kappa/\omega)}$. The solution of Eq. (11.1.3) is then $\psi = F(\omega)G(x|x_0|\omega)$. This ratio between amplitude of motion and amplitude of force, $\psi/F = G$, is the *admittance* of the string for the pair of points x, x_0. As a function of ω, it has poles wherever $\sin(kl)$ is zero, *i.e.*, whenever $kl = n\pi$ ($n = 0, \pm1, \pm2, \ldots$), or whenever $\omega^2 + 2i\kappa\omega - (n\pi c/l)^2 = 0$. The roots of this equation are

$$\omega_n = 2\pi\nu_n - i\kappa; \quad \nu_n = (nc/2l)\sqrt{1 - (\kappa l/n\pi c)^2} \tag{11.1.5}$$

all on a line a distance κ below the real axis of ω, as predicted. The larger the resistance factor R, the farther below the real axis are the poles.

Having solved the problem of steady-state vibration of the string for frequency $\omega/2\pi$, we are now in a position to apply Eqs. (11.1.1) and (11.1.2) to obtain the response to a force $Tf(t)\delta(x - x_0)$ applied to the point x_0. For example, suppose the force $Tf(t)$ to be a one-directional push on the point $x = x_0$, such that $f(t) = (P_0 a/2T)e^{-a|t|}$, rising to a maximum at $t = 0$ and falling off exponentially on both sides of $t = 0$. The total transverse impulse given to point x_0 on the string is thus

$$T\int_{-\infty}^{\infty} f(t)\,dt = P_0 \quad \text{dyne-sec}$$

(since F and f are given in units of tension T, we multiply by T to get force in dynes). The Fourier transform of this transient force is

$$
\begin{aligned}
F(x_0|\omega) &= \int_{-\infty}^{\infty} f(t)e^{i\omega t}\,dt \\
&= \frac{P_0a}{2T}\left\{\int_{-\infty}^{0} e^{at+i\omega t}\,dt + \int_{0}^{\infty} e^{-at+i\omega t}\,dt\right\} \\
&= \frac{P_0a}{2T}\left[\frac{1}{a+i\omega} + \frac{1}{a-i\omega}\right] = \frac{P_0a^2/T}{\omega^2+a^2}
\end{aligned}
$$

This function has two simple poles, at $\omega = \pm ia$.

To obtain the transient response of the string we now use Eq. (11.1.2). The integral to be evaluated is

$$
\frac{P_0a^2}{2\pi T}\int_{-\infty}^{\infty} G(x|x_0|\omega)\,\frac{e^{-i\omega t}}{\omega^2+a^2}\,d\omega
$$

The poles of the integrand are at $\omega = \pm ia$, for $F(x_0|\omega)$, and at $2\pi\nu_n - i\kappa$ ($n = 0, \pm 1, \pm 2, \ldots$), for G. We can change the infinite integral into a contour integral if we can show that the integrand vanishes either on the infinitely large semicircle around the upper half of the ω plane or on the similar semicircle around the lower half. Substitution of $k = \pm(iw/c)$ ($\omega = \pm iw$) in Eq. (11.1.4) shows that G goes exponentially to zero both on the upper and lower semicircle except when $x = x_0$, when $|G|$ is $|1/2k|$ on both semicircles. The exponential $e^{-i\omega t}$ is zero on the upper semicircle (and infinite on the lower one) when t is negative and is zero on the lower semicircle (and infinite on the upper one) when t is positive. Therefore the contour encircles the upper half of the ω plane when t is negative (and the contour is followed in a positive direction), and it encircles the lower half of the ω plane when t is positive (when the contour is traversed in a negative direction).

For negative values of t the only pole of the integrand enclosed is the one at $\omega = ia$. The residue at this pole is

$$
\frac{P_0a^2}{2\pi T}\,G(x|x_0|ia)\,\frac{e^{at}}{2ia} = \frac{P_0ae^{at}}{4\pi iTk_a}
\begin{cases}
\dfrac{\sinh[k_ax]\,\sinh[k_a(l-x_a)]}{\sinh(k_al)}; & x < x_0 \\[2ex]
\dfrac{\sinh[k_ax_0]\,\sinh[k_a(l-x)]}{\sinh(k_al)}; & x > x_0
\end{cases}
$$

where $k_a = (a/c)\sqrt{1+(2\kappa/a)}$. Since the value of the contour integral (in the counterclockwise direction) is $2\pi i$ times the residue, we have, for the shape of the string for $t < 0$

$$
y(x|t) = \frac{P_0ae^{at}}{2Tk_a}
\begin{cases}
\dfrac{\sinh(k_ax)\,\sinh[k_a(l-x_0)]}{\sinh(k_al)}; & x < x_0 \\[2ex]
\dfrac{\sinh(k_ax_0)\,\sinh[k_a(l-x)]}{\sinh(k_al)}; & x > x_0
\end{cases}
\tag{11.1.6}
$$

For positive values of t, the lower half of the ω plane is enclosed and the contour is traversed in the negative (clockwise) direction. In addition to the pole at $-ia$, producing a term similar to Eq. (11.1.6), there is the sequence of poles at $\omega_n = 2\pi\nu_n - i\kappa$. At the nth such pole, the term $\sin(kl)$ goes to zero as $(-1)^n(kl - n\pi) = (-1)^n(2l^2\nu_n/nc^2)(\omega - \omega_n)$ so the residue at this pole is

$$-\frac{P_0a^2}{2\pi T}\left[\frac{\sin(\pi nx/l)\sin(\pi nx_0/l)}{2\pi(l\nu_n/c^2)}\right]\frac{e^{-2\pi i\nu_nt-\kappa t+2i\phi_n}}{W_n^2}$$

where $(\omega_n^2 + a^2) = (2\pi\nu_n)^2 - \kappa^2 + a^2 - (2i\kappa)(2\pi\nu_n) = W_n^2e^{-2i\phi_n}$. For every pole with positive n there is one with negative n, having a ν_n with reversed sign and, therefore, a ϕ_n with reversed sign. Therefore the residues may be added in pairs. The final result, for $t > 0$, is

$$y = \frac{2P_0a^2c^2}{Tl}\sum_{n=1}^{\infty}\left[\frac{\sin(\pi nx/l)\sin(\pi nx_0/l)}{2\pi\nu_nW_n^2}\right]e^{-\kappa t}\sin(2\pi\nu_nt - 2\phi_n)$$
$$+\frac{P_0ae^{-at}}{2Tk_a'}\begin{cases}\dfrac{\sinh(k_a'x)\sinh[k_a'(l - x_0)]}{\sinh(k_a'l)}; & x < x_0\\[2ex] \dfrac{\sinh(k_a'x_0)\sinh[k_a'(l - x)]}{\sinh(k_a'l)}; & x > x_0\end{cases} \qquad (11.1.7)$$

where $k_a' = (a/c)\sqrt{1 - (2\kappa/a)}$.

This expression, together with the one for $t < 0$, shows interesting properties which are typical of most transient responses: It consists of a term having the behavior of the forcing function (the terms in $e^{\pm at}$) plus a series representing the free vibration of the system "caused" by the discontinuity in the forcing function at time $t = 0$. The frequencies ν_n and the damping factor κ correspond to the natural frequencies and damping factor for free vibration of the string with friction. By means of the Fourier transform, we have thus utilized the expressions for steady-state response to give us the transient behavior and to obtain the properties of the free vibration of the system. Consequently, in this chapter, our first task will always be to find the steady-state response to a unit, simple-harmonic driving force, knowing that from this solution we can find expressions for both transient and free vibrations.

Before leaving this example we shall simplify the procedure still further. By letting a become infinitely large we reduce the force function to an instantaneous impulse P_0 at $t = 0$ at the point x_0. The response, for $P_0 = 1$, is therefore a double Green's function, for a force concentrated both in time and in space. When $a \to \infty$, the second term vanishes and $W_n^2 \to a^2$, $\phi_n \to 0$. The "impulse function" for an instantaneous impulse at $t = \tau$, concentrated at $x = x_0$ is then, simply

$$g(x|x_0|t - \tau) = \frac{2c^2}{T} \sum_{n=1}^{\infty} \frac{\sin(\pi n x/l) \sin(\pi n x_0/l)}{2\pi \nu_n} \cdot \\ \cdot e^{-\kappa(t-\tau)} \sin[2\pi\nu_n(t - \tau)] \quad (11.1.8)$$

for $t > \tau$, where $\nu_n = (nc/2l) \sqrt{1 - (\kappa l/\pi nc)^2}$. If now a force $f(x|t)$ per unit length is applied along the string, with an arbitrary dependence on time, we can find the resulting response by calculating the double integral

$$y(x|t) = \int_0^l dx_0 \int_{-\infty}^t d\tau\, f(x_0|\tau) g(x|x_0|t - \tau) \qquad (11.1.9)$$

which is a sum of all the responses for unit impulse at $t = \tau$ and $x = x_0$, multiplied by the force amplitude at $x = x_0$ and $t = \tau$, for every τ *before* time t.

This formulation shows the solution to be made up of all the responses of the system to the forces imposed before time t. The result comes out in terms of a Fourier series in x, for both the term representing the free vibrations and that having the time dependence of the forcing function $f(t)$. Thus the term given in Eq. (11.1.6) plus the second term of Eq. (11.1.7) would come out as the Fourier series equivalent of these single terms. Although the series for the impulse response, g, is only conditionally convergent, the series resulting from the integration of Eq. (11.1.9) is convergent if $f(x|t)$ is piecewise continuous and if the integral of $|f|$ over t from $-\infty$ to $+\infty$ converges.

Laplace Transform. We have been rather glib in some of our assumptions in the previous discussion. We assumed, for instance, that if $F(\omega)$ was the Fourier transform for the forcing function $f(t)$, and if $\psi(\omega)$ was the steady-state solution for $F(\omega)$, then the response to the forcing function $f(t)$ would be the Fourier transform of $\psi(\omega)$. This is certainly a plausible assumption, but it would be well to scrutinize the derivation in some detail in order to see whether there are special exceptions or limits to the formula.

We shall specialize our proof to some extent, but not sufficiently to handicap us in future calculations. What we assume is that the system has not been subjected to an applied force for an indefinitely remote time in the past, that for all times earlier than a certain time the system was not disturbed in any way. This, of course, is equivalent to saying that the system was at rest before this time (which might as well be taken as $t = 0$) or, in certain cases, was in motion with uniform velocity. Therefore specifying the displacement and velocity of the system just before $t = 0$, and specifying the nature of the force applied during and after $t = 0$ should specify the motion of the system.

As soon as we limit the applied force to times $t > 0$, we can use the second convention mentioned on page 1335 that ω in our Fourier integrals always has an imaginary part larger than zero (or, better yet, has an argument greater than zero and less than π). The integral changing from $f(t)$ to $F(\omega)$ then becomes

$$F(\omega) = \int_0^\infty e^{i\omega t} f(t)\,dt; \quad 0 < \text{argument } \omega < \pi$$

where the imaginary part of ω provides a certain amount of negative real exponential to ensure convergence of the integral. As a matter of fact, we might as well perform our actual integration with ω made pure imaginary, $\omega = ip$, so that

$$F(\omega) = \int_0^\infty e^{-pt} f(t)\,dt; \quad p = -i\omega \qquad (11.1.10)$$
$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} F(\omega)\,d\omega$$

where the integration in the second integral is taken along a line a distance ϵ ($\epsilon > 0$) above the real axis in the ω plane.

Since $f(t)$ is zero for $t < 0$, the second integral shows that $F(\omega)$ has no poles in the upper half of the ω plane, *above* the line $\operatorname{Im} \omega = \epsilon > 0$. And this means that the function $F(\omega)$ defined in the first integral is an analytic function of ω for all values of ω above the line $\operatorname{Im} \omega = \epsilon > 0$. In most cases we can then obtain the behavior of F by analytic continuation, in the lower half of the ω plane, where F has singularities.

The interrelation between F and f, p and ω given in Eqs. (11.1.10) is known as the *Laplace transformation* (see Secs. 4.8 and 5.3) and the function F is known as the *Laplace transform* of f. If the following conditions are satisfied: (1) $f(t)$ is defined for all real values of t and is zero for $t < 0$; (2) the integral

$$\int_0^\infty e^{-ct} f(t)\,dt$$

converges for all real values of c greater than or equal to some $\epsilon > 0$; (3) the function $F(\omega)$ is analytic for all finite values of ω above the line $\operatorname{Im} \omega = \epsilon$; and (4) the integral

$$\int_{-\infty}^{\infty} |F(x + ic)|\,dx$$

converges for all real values of c greater than or equal to ϵ; then, *if one of the relations of* Eq. (11.1.10) *holds, both relations hold.* In particular the integral defining $F(\omega)$ in terms of an integral of f over t is a valid representation for all values of $\omega = ip$ for which $\operatorname{Im} \omega \geq \epsilon$. In most cases, by analytic continuation, we can determine the nature of $F(\omega)$ for the lower half of the ω plane, where the integral representation is not valid.

Several general formulas relating Laplace transforms will be of interest in our future calculations. If f and F are related as given in Eqs. (11.1.10) and in the last paragraph, then by integrating by parts

$$\int_0^\infty e^{-pt} \frac{d}{dt}[f(t)]\,dt = [e^{-pt}f(t)]_0^\infty + p\int_0^\infty e^{-pt}f(t)\,dt$$
$$= -f(0) - i\omega F(\omega); \quad p = -i\omega$$

Consequently the Laplace transforms of df/dt and of (d^2f/dt^2) are:

$$\begin{aligned} &\text{Laplace transform of } f(t) && \text{is } F(\omega) \\ &\text{Laplace transform of } f'(t) && \text{is } pF(\omega) - f(0) \\ &\text{Laplace transform of } f''(t) && \text{is } p^2F(\omega) - pf(0) - f'(0) \end{aligned} \tag{11.1.11}$$

if the derivatives satisfy the convergence requirements laid down on the preceding page.

Likewise by reversing the order of integration and then letting $t - \tau = u$, we obtain [see Eq. (4.8.33)]

$$\int_0^\infty dt\, e^{-pt} \int_0^t f_1(\tau)f_2(t-\tau)\,d\tau = \int_0^\infty d\tau\, f_1(\tau) \int^\infty e^{-pt}f_2(t-\tau)\,dt$$
$$= \int_0^\infty e^{-pt}f_1(\tau)\,d\tau \int_0^\infty e^{-pu}f_2(u)\,du$$

Consequently, if

$$\begin{aligned} &\text{Laplace transform of } f_1(t) && \text{is } F_1(\omega) \\ \text{and} \qquad &\text{Laplace transform of } f_2(t) && \text{is } F_2(\omega) \end{aligned} \tag{11.1.12}$$

then Laplace transform of $\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau$ is $F_1(\omega)F_2(\omega)$

String with Friction. Now let us apply this machinery to the string, with friction, between two rigid supports, introduced on page 1335. The equation of motion of the string, with a force $Tf(t)$ applied to the point $x = x_0$ (where f is zero for $t < 0$) is, from Eq. (11.1.3) or (11.1.4),

$$\frac{\partial^2 y}{\partial t^2} + 2\kappa\frac{\partial y}{\partial t} - c^2\frac{\partial^2 y}{\partial x^2} = c^2\delta(x - x_0)f(x_0|t) \tag{11.1.13}$$

where $\kappa = R/2\rho$, $c^2 = T/\rho$, T is the tension in dynes, ρ the mass per unit length, R the resistance of the medium per unit length, and fT is measured in dynes. The displacement y of the string is a function of x and t.

We now take the Laplace transform f both sides, by multiplying by e^{-pt} and integrating over t from 0 to ∞;

$$[p^2 + 2\kappa p]Y - c^2\frac{\partial^2 Y}{\partial x^2} = c^2\delta(x - x_0)F(x_0|\omega) + (p + 2\kappa)y(0) + y'(0)$$

where $Y(x,\omega)$ is the Laplace transform of $y(x,t)$, F is the Laplace transform of f, $y(0)$ and $y'(0)$, the initial displacement and velocity of the string (functions of x but not of t or ω) and $p = -i\omega$. Setting a Fourier series $\Sigma A_n \sin(\pi nx/l)$ for Y and assuming, for the time being, that the string was at rest in equilibrium before the force was applied, we solve for A_n and eventually obtain

$$Y(x,\omega) = \frac{2c^2}{l} \sum_{n=1}^{\infty} \frac{\sin(\pi n x_0/l)\,\sin(\pi n x/l)}{p^2 + 2\kappa p + (\pi n c/l)^2} F(x_0|\omega)$$

But this steady-state amplitude is equal to the force amplitude $TF(x_0|\omega)$, times a Green's function for a force of unit amplitude and frequency $\omega/2\pi$, applied at $x = x_0$,

$$G(x|x_0|\omega) = \frac{2c^2}{lT} \sum_{n=1}^{\infty} \frac{\sin(\pi n x_0/l)\,\sin(\pi n x/l)}{p^2 + 2\kappa p + (\pi n c/l)^2} \tag{11.1.14}$$

This quantity is the Laplace transform of the response to a unit pulse at $t = 0$ at $x = x_0$. Since

$$\int_0^\infty e^{-pt-bt} \sin(at)\, dt = \frac{a}{(p+b)^2 + a^2}$$

we see that G is the Laplace transform of the Fourier series $g(x|x_0|t)$ given in Eq. (11.1.8). And finally, since Y [which is the product of $G(x|x_0|\omega)$, the Laplace transform of $g(x|x_0|t)$, and of $F(x_0|\omega)$, the Laplace transform of the driving force $f(x_0|t)$] is the Laplace transform of $y(x|x_0|t)$, then the actual displacement of the string under the action of force $f(x_0|t)$, turns out to be, by Eq. (11.1.12),

$$y(x|x_0|t) = \int_0 f(x_0|\tau) g(x|x_0|t - \tau)\, d\tau \tag{11.1.15}$$

and, if the force is distributed over the string, we can eventually obtain Eq. (11.1.9) for $y(x|t)$.

Therefore, by the Laplace transform method, we have avoided long discussions of convergence and long calculations of residues about poles. In exchange for this increased safety, we lose somewhat in flexibility and straightforwardness. For to go from G to g we have had to work backward, to guess a g such that $\int e^{-pt} g\, dt$ was equal to G. The problem is very like that of finding the integral of some special function, a process which is done by looking the answer up in a table of integrals. Today there are similar tables of Laplace transforms, and one can often find the functions needed in the table. (A short table of transforms is given at the end of this chapter.) If one cannot, one is then reduced to using the second of Eqs. (11.1.10) with the attendant complications of contour integration and residues.

One can summarize the procedure for calculating the response of a system, originally at rest in equilibrium, to a force $f(x,y,z|t)$, applied at the points x, y, z (either within or on the boundary of the system) at times $t > 0$:

$$\left.\begin{array}{l}\textit{First compute the Green's function } G(x|x_0|\omega) \textit{ for the steady-state response of}\\ \textit{the system to a force of unit amplitude and frequency } \omega/2\pi, \textit{ applied at point}\\ (x_0,y_0,z_0) \textit{ within or on the boundary, by solving an inhomogeneous Helm-}\\ \textit{holtz equation or one with inhomogeneous boundary conditions. Find the}\\ \textit{impulse function } g(x|x_0|t), \textit{ for which } G \textit{ is the Laplace transform, either by}\\ \textit{contour integration of the second of Eqs. (11.1.10) or by inverting the first.}\\ \textit{The response to } f(x,t) \textit{ is then given by Eq. (11.1.15).}\end{array}\right\} \quad (11.1.16)$$

This applies when the system is at rest and $f(x,t)$ is zero for $t < 0$. If the system is not at rest for $t < 0$, we must include the Laplace transforms of the initial position and velocity, as shown in Eqs. (11.1.11), in our calculations. The rest of this section will display a number of applications of this technique. Incidentally, it may be of interest to ponder the close connection between Laplace and Fourier transforms and dynamically conjugate variables in quantum mechanics, as discussed in Sec. 2.6.

String with Elastic Support. As an example of a system driven from one part of the boundary, we return to the case of the flexible string embedded in an elastic medium, discussed in Eqs. (2.1.27) *et seq.* The equation of motion is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - c^2\mu^2 y; \quad c^2 = \frac{T}{\rho}; \quad \mu^2 = \frac{K}{T} \qquad (11.1.17)$$

Suppose the string to be infinite in length and to be driven by a transverse force applied to the end at $x = 0$. The transverse force required to displace this end was shown to be $-T(\partial y/\partial x)$, where T is the tension on the string. As shown in Eq. (2.1.29), a solution corresponding to a simple-harmonic wave traveling in the positive x direction, away from the driving force at $x = 0$, is

$$y = A \exp[-\sqrt{\mu^2c^2 - \omega^2}\,(x/c) - i\omega t]$$

Consequently a solution of the Helmholtz equation with the inhomogeneous boundary condition $-T(\partial\psi/\partial x)_0 = 1$ (unit force amplitude at $x = 0$) is

$$G(x|0|\omega) = (1/T\sqrt{p^2 + \mu^2c^2}) \exp[-(x/c)\sqrt{p^2 + \mu^2c^2}]; \quad p = i\omega \qquad (11.1.18)$$

Referring to the table of Laplace transforms at the end of this chapter, we find the impulse function, corresponding to the response to a unit impulse, applied transversely to the end of the string at $x = 0$,

$$g(x|0|t) = \begin{cases} (1/T)J_0(\mu c\sqrt{t^2 - (x/c)^2}); & t > x/c \\ 0; & t < x/c \end{cases} \qquad (11.1.19)$$

The shape of this wave is shown in Fig. 2.7, where it is compared with the impulse function for the simple elastic string and for one with friction. One notices that, although the wave front remains sharp as it moves forward with velocity c, the wave leaves a "wake" behind it which changes shape as time goes on. In this case the leading part of the wave gets sharper and narrower the farther it travels.

If the end of the string is pushed by a transverse force $f(t)$, then the shape of the string, as function of x and t, is, according to Eq. (11.1.15),

$$y(x,t) = \frac{1}{T}\int_0^{t-(x/c)} f(\tau)J_0\left[\mu c\sqrt{(t-\tau)^2 - \left(\frac{x}{c}\right)^2}\right] d\tau \qquad (11.1.20)$$

which, if necessary, may be integrated numerically.

String with Nonrigid Supports. It might seem rather roundabout to derive the free vibrations of a string from its steady-state forced vibration, when we could perhaps more easily derive it directly. Another example will show that the apparently roundabout way is sometimes the most effective and that at times the supposedly direct method involves us in difficulties.

We suppose that a string of length l is under tension T and is supported by a rigid support at $x = 0$ and a nonrigid support at $x = l$. This latter support has enough longitudinal strength to support the tension T, but it yields a little to transverse force imparted to it by the string. Suppose this yielding involves both friction and stiffness of the support for sidewise motion, so that the relation between the transverse force transmitted by the string, which is $-T(\partial y/\partial x)_l$, is equal to R_s times the transverse velocity of the support, $(\partial y/\partial t)_l$, plus K_s times the displacement of the support $y(l)$:

$$\begin{aligned} -T(\partial y/\partial x) &= R_s(\partial y/\partial t) + K_s y; \quad &&\text{at } x = l \\ y &= 0; &&\text{at } x = 0 \end{aligned} \qquad (11.1.21)$$

We could, of course, have assumed that the support at $x = 0$ also yielded to transverse force and we could also have included the transverse inertia of the supports; but we are here including only enough complication to bring out the method involved, not so much as to confuse the results with additional complexity, which may be added later if desired. For the same reason we again neglect the reaction of the medium around the string.

If we desire to compute the free vibrations of this system directly, we would find the eigenfunction solutions of the wave equation, $y = [A\sin(kx) + B\cos(kx)]e^{-ikct}$ which satisfy the boundary conditions of Eqs. (11.1.21). They require that B be zero, so that $y = \sin(kx)e^{-ikct}$, where

$$-Tk\cos(kl) = -ikcR_s\sin(kl) + K_s\sin(kl)$$

or

$$\tan(kl) = -Tk/(K_s - ikcR_s) \qquad (11.1.22)$$

Roots of this transcendental equation for k are the eigenvalues k_n of the system. The lowest value is $k_0 = 0$, but this does not need inclusion, since the corresponding eigenfunction is zero. The other roots may be computed, with sufficient trouble, to whatever accuracy is desired.

The difficulty of this problem is not with the calculation of eigenvalues, however; it is with the use of the eigenfunctions. The trouble is that the eigenfunctions are *not mutually orthogonal.* And they are not mutually orthogonal because the *boundary conditions depend on the eigenvalue k.* In Chap. 6 we proved the orthogonality of eigenfunctions by integrating a combination of solutions which, in the present case, reduces to

$$\left[y_1 \frac{dy_2}{dx} - y_2 \frac{dy_1}{dx}\right]_0^l = (k_1^2 - k_2^2) \int_0^l y_1 y_2 \, dx$$

Since both eigenfunctions y_1 and y_2 are zero at $x = 0$, the lower limit on the left results in zero as before. But since at $x = l$

$$dy/dx = (K_s - ikcR_s/T)y$$

the upper limit results in

$$(icR_s/T)(k_1 - k_2)y_1(l)y_2(l)$$

which is *not* zero unless $k_1 = k_2$. Therefore the integral on the right is not zero in the present case when $k_1 \neq k_2$ and, therefore, the eigenfunctions are not mutually orthogonal. (In Chap. 6 this integral was zero when $k_1 \neq k_2$ because we assumed the ratio between y and $\partial y/\partial x$ at the boundary *did not depend* on k.) This does not mean that the eigenfunctions are not a complete set. They are. But it does mean that it is not too easy to compute the coefficients of the eigenfunction series which would be equivalent to an arbitrarily chosen function of x in the range $0 < x < l$.

Suppose we now attack the problem from the other direction, via the steady-state forced motion first. To compute the Green's function for a force of unit amplitude and frequency $\omega/2\pi$, applied at $x = x_0$, we must solve the Helmholtz equation

$$\frac{d^2G}{dx^2} + k^2G = \frac{-1}{T}\,\delta(x - x_0); \quad k = \frac{\omega}{c} \tag{11.1.23}$$

subject to the boundary conditions of Eqs. (11.1.21), where now the dependence of G on time is specifically through the factor $e^{-i\omega t}$. Therefore, the boundary conditions are:

$$G = 0 \quad \text{at } x = 0; \quad T(\partial G/\partial x) = [ikcR_s - K_s]G \quad \text{at } x = l \tag{11.1.24}$$

with $k = \omega/c$ and ω given by the frequency of the driving force.

We can set up a solution of (11.1.23) in terms of eigenfunctions satisfying (11.1.24). These eigenfunctions *will* be mutually orthogonal, for the k and ω in these equations is determined by the driving force and is the same for all eigenfunctions. We can alternately satisfy (11.1.23) by a solution with the necessary discontinuity at $x = x_0$. Thus, by methods used before:

$$G(x|x_0|\omega) = \begin{cases} \dfrac{\sin(kx)\sin[k(l - x_0) - \theta]}{kT\sin(kl - \theta)}; & x < x_0 \\ \dfrac{\sin(kx_0)\sin[k(l - x) - \theta]}{kT\sin(kl - \theta)}; & x > x_0 \end{cases}$$

$$= 2l\sum_{n=0}^{\infty} \frac{(2\pi\beta_n/T)}{2\pi\beta_n - \sin(2\pi\beta_n)} \frac{\sin(\pi\beta_n x_0/l)\sin(\pi\beta_n x/l)}{[(kl)^2 - (\pi\beta_n)^2]} \qquad (11.1.25)$$

where $\theta = \tan^{-1}[Tk/(ikcR_s - K_s)]$ and $\beta_n(k)$ is the nth root of the equation

$$\tan(\pi\beta_n) = kT/(ikcR_s - K_s) \qquad (11.1.26)$$

The eigenfunctions $\sin(\pi\beta_n x/l)$ in the series expansion of G are mutually orthogonal, since they all satisfy the boundary condition at $x = l$ *for the same value of* $k = \omega/c$. In other words the boundary conditions of Eq. (11.1.24) are independent of the eigenvalue, so that our proof of orthogonality, given in Chap. 6, is valid. Both eigenfunctions and eigenvalues β_n, roots of Eq. (11.1.26), are functions of k and thus of the driving frequency $\omega/2\pi$. See also the discussion on page 727.

To find the impulse function g, for which the function G is the Laplace transform, either we compute the contour integral given by the second of Eqs. (11.1.10) or else we look g up in a table of Laplace transforms. We choose the former method this time:

$$g(x|x_0|t) = \frac{c}{\pi l}\int_{-\infty}^{\infty}\sum_n \frac{(2\pi\beta_n/T)}{2\pi\beta_n - \sin(2\pi\beta_n)}\left[\frac{\sin(\pi\beta_n x_0/l)}{k - (\pi\beta_n/l)}\right]\left[\frac{\sin(\pi\beta_n x/l)}{k + (\pi\beta_n/l)}\right] e^{-ickt}\,dk$$

The simple poles of this integrand are at the solutions of the transcendental equation

$$lk = \pm\pi\beta_n(k) = \mp\cot^{-1}\left[\frac{(K_s/k) - icR_s}{T}\right] \quad \text{or}$$

$$\left(\frac{K_s}{k}\right) - icR_s \pm T\cot(kl) = 0 \qquad (11.1.27)$$

which is to be compared with Eq. (11.1.22). For each value of n in β_n there is a pair of roots, below the real k axis and symmetrically placed with respect to the imaginary k axis. For large values of n, approximate values are

$$\left.\begin{aligned} k_n &\simeq \frac{\pi}{l}\,(n + \tfrac{1}{2}) + \frac{(K_s/T)}{\pi(n + \frac{1}{2})} - i\,\frac{cR_s}{lT} \\ k_n' &\simeq -\frac{\pi}{l}\,(n + \tfrac{1}{2}) - \frac{K_s/T}{\pi(n + \frac{1}{2})} - i\,\frac{cR_s}{lT} \end{aligned}\right\} \quad Tn \gg K_s; \quad cR_s \gg lT$$

and for any value of n and of cR_s/lT the roots may be written

$$k_n = (\pi\alpha_n/l) - i\kappa_n(cR_s/lT); \quad k_n' = -(\pi\alpha_n/l) - i\kappa_n(cR_s/lT) \tag{11.1.28}$$

where α_n and κ_n are dimensionless numbers arising from the solution of Eq. (11.1.27). As shown above, for very large values of n, $\alpha_n \to n + \frac{1}{2}$ and $\kappa_n \to 1$.

Computing the residues about each pole, we see that

$$g(x|x_0|t) = \begin{cases} 0 & t < 0 \\ \dfrac{4c}{T} \displaystyle\sum_{n=1}^{\infty} \operatorname{Im}\left\{\left[\frac{\sin(k_n x_0)\sin(k_n x)}{2k_n l - \sin(2k_n l)}\right] e^{-ik_nct}\right\}; & t > 0 \end{cases} \tag{11.1.29}$$

where k_n is the nth root of Eq. (11.1.27) as given in Eq. (11.1.28). This solution gives the shape of the string when a unit impulse is given the point $x = x_0$ at $t = 0$.

Going on with our analysis, we can compute the shape of the string when a force TF_0 is suddenly applied at $x = x_0$ at $t = 0$. The Laplace transform of $F_0u(t)$ is $F_0/p = iF_0/\omega$. According to the table at the end of this chapter if G is the transform of g, then G/p is the transform of $\int g\,dt$. Therefore, the required shape of the string for the force $TF_0u(t)$ applied at $x = x_0$ is

$$\begin{aligned} \psi_u &= F_0T\int_0^t g(x|x_0|\tau)\,d\tau \\ &= 4F_0 \sum_{n=1}^{\infty} \operatorname{Re}\left\{\left[\frac{\sin(k_n x_0)\sin(k_n x)}{2k_n l - \sin(2k_n l)}\right]\left[\frac{1 - e^{-ik_nct}}{k_n}\right]\right\} \end{aligned}$$

The terms independent of time in this series correspond to the ultimate shape of the string, after the transient has died out. It is equal to the limiting form of the steady-state response, given in Eq. (11.1.25), for $k \to 0$:

$$TF_0G(x|x_0|0) = \begin{cases} F_0 \dfrac{x[l + (T/K_s) - x_0]}{l + (T/K_s)}; & x < x_0 \\ F_0 \dfrac{x_0[l + (T/K_s) - x]}{l + (T/K_s)}; & x > x_0 \end{cases} \tag{11.1.30}$$

Therefore, the shape of the string, which is originally held aside at the point $x = x_0$ by a steady force TF_0 and then is released at $t = 0$, is

$$\psi = TF_0G(x|x_0|0) - \psi_u = F_0 \sum_{n=1}^{\infty} \operatorname{Re}\left\{\left[\frac{\sin(k_n x_0)\sin(k_n x)}{k_n l - \frac{1}{2}\sin(2k_n l)}\right]\frac{e^{-ik_nct}}{k_n}\right\} \tag{11.1.31}$$

This series represents the free, damped vibration of the string started with an initial shape, given in Eq. (11.1.30). The series is in terms of eigenfunctions, $\sin(k_n x)$, which are not mutually orthogonal, as we showed on page 1345. But we were able to obtain the series by starting with the orthogonal set of eigenfunctions, $\sin(\pi\beta_n x/l)$, for steady-state, forced motion and using the Laplace transform to compute the transient behavior.

Reflection from a Frictional Support. When $K_s = 0$, the transverse impedance of the end support is purely resistive, and both θ and β_n, used in Eqs. (11.1.25), are independent of k. In this case we can find the impulse function $g(x|x_0|t)$, more easily by using the table of Laplace transforms at the end of this chapter. Using the closed form for $(x < x_0)$ we have

$$G(x|x_0|\omega) = c\,\frac{\sinh(px/c)\sinh[(p/c)(l - x_0) + b]}{Tp\sinh[(pl/c) + b]}$$
$$= \frac{(c/4)}{pT\sinh[(pl/c) + b]}\left\{e^{(p/c)(l+x-x_0)+b} - e^{(p/c)(l-x-x_0)+b} + e^{(p/c)(x_0-x-l)-b} - e^{(p/c)(x+x_0-l)-b}\right\}$$

where $b = \tanh^{-1}(T/cR_s)$. Then, using the table, we find that

$$g(x|x_0|t) = \tfrac{1}{2}\frac{c}{T}\left\{\sum_{n=0}^{\infty} e^{-2nb}[u(ct - |x - x_0| - 2nl) - u(ct - x - x_0 - 2nl)] + \sum_{n=1}^{\infty} e^{-2nb}[u(ct + |x - x_0| - 2nl) - u(ct + x + x_0 - 2nl)]\right\} \qquad (11.1.32)$$

This is a very interesting representation for the impulsive wave. The term $u(ct - |x - x_0|)$ $(n = 0)$ corresponds to a sharp-edged wave spreading out from the point of impact $x = x_0$ with velocity c. The term $-u(ct - x - x_0)$ corresponds to the first reflection of this wave at the rigid support $x = 0$, without change of amplitude but with change of sign. The term $-e^{-2b}u(ct + x + x_0 - 2l)$ corresponds to the first reflection at the nonrigid support. Here it suffers a change in amplitude by a factor

$$|e^{-2b}| = |e^{-2\tanh^{-1}(T/cR_s)}| = \left|\frac{T - cR_s}{T + cR_s}\right|$$

with a reversal of sign if $cR_s > T$ (no reversal if $cR_s < T$). The reflections continue, with further reduction in amplitude whenever the wave strikes the nonrigid support at $x = x_0$.

If the force on the string is distributed in length and time, we integrate according to Eq. (11.1.9). It is interesting to notice that this solution comes out in terms of the elementary solutions $f(x - ct)$ and $g(x + ct)$, representing waves in opposite directions, plus their reflections at the ends. For a resistive support ($K_s = 0$), the reflected wave suffers a change in direction and magnitude, but is similar in shape.

Sound Waves in a Tube. Another common example of the one-dimensional wave equation is the transmission of sound inside a tube (refer to Sec. 2.3). If the wavelength $2\pi c/\omega$ is considerably longer than the circumference of the tube, whatever wave motion there is, is along the axis. If the tube has uniform cross section the fluid displacement ξ, velocity v, velocity potential ψ and excess pressure P are all functions of the distance x along the tube and of time t only. The interrelations between these quantities are as follows:

$$v = \frac{\partial \xi}{\partial t} = -\frac{\partial \psi}{\partial x}; \quad P = \rho \frac{\partial \psi}{\partial t}; \quad \frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} \tag{11.1.33}$$

Since the velocity potential is a solution of the wave equation, so also are ξ, v and P.

The far end of the tube, $x = l$, may be open, or it may be closed with a termination which yields to some extent to the pressure inside the tube. In either case the nature of the termination is sufficiently characterized, for our purposes here, by specifying its *acoustic impedance*, the ratio z_l between the pressure p and the fluid velocity out of the tube, v, at the termination, for a simple harmonic pressure fluctuation of frequency $\omega/2\pi$. Thus z_l is a function of ω as well as of the nature of the termination.

The value of the complex quantity z_l fixes the position and the relative magnitude of the pressure amplitude maxima and minima (nodes and loops) of a sinusoidal wave along the tube, from $x = l$ to $x = 0$, at the near end.

A one-dimensional sinusoidal wave, plus its reflection, may be expressed in a variety of ways:

$$\begin{aligned} \psi &= A_i e^{i\omega[(x/c)-t]} + A_r e^{-i\omega[(x/c)+t]} \\ &= A \sinh[\pi(\alpha - i\beta) + i(\omega x/c)]e^{-i\omega t} \\ A &= 2A_i e^{-\pi(\alpha - i\beta)} = -2A_r e^{\pi(\alpha - i\beta)} \end{aligned} \tag{11.1.34}$$

where $|A_i|$ is the amplitude of the wave incident on the termination, $|A_r|$ the amplitude of the reflected wave and the ratio between, giving both amplitude ratio and phase difference, is $A_r/A_i = e^{-\pi[2\alpha - i(2\beta+1)]}$. The excess pressure and fluid velocity at a distance x along the tube are then (we use β_0 as being the phase relationship at $x=0$)

$$\begin{aligned} P &= -i\rho\omega A \sinh[\pi(\alpha - i\beta_0) + i(\omega x/c)]e^{-i\omega t} \\ v &= -i(\omega/c)A \cosh[\pi(\alpha - i\beta_0) + i(\omega x/c)]e^{-i\omega t} \end{aligned}$$

and their ratio is

$$z(x) = \rho c \tanh[\pi(\alpha - i\beta_0) + i(\omega x/c)] \tag{11.1.35}$$

We thus see that fixing the value of $z = P/v$ at $x = l$ fixes the values of α and β,

$$\alpha - i\beta_l = (1/\pi)\tanh^{-1}(z_l/\rho c);\quad \beta_0 = \beta_l + (\omega l/\pi c)$$

and therefore fixes the amplitude ratio and relative phase of incident and reflected waves in the tube. If z_l is pure imaginary (purely reactive), then $\alpha = 0$ and the reflected wave has the same amplitude as the incident wave. If z_l is real (purely resistive) and is equal to ρc, then $\alpha = \infty$ and there is no reflected wave (this would be the terminal impedance if an equal tube of infinite length were attached at $x = l$; there would then be no reflected wave). Returning to the formula for P, we see that the pressure amplitude at x is

$$|P| = \rho\omega|A|\sqrt{\sinh^2[\pi\alpha] + \sin^2[(\omega/c)(l - x) + \pi\beta_l]}$$

Thus whenever $(\omega/c)(l - x) = -\pi\beta_l, \pi(1 - \beta_l), \pi(2 - \beta_l), \ldots$, the pressure amplitude has a minimum, equal to $\rho\omega|A|\sinh(\pi\alpha)$ and whenever $(\omega/c)(l - x) = \pi(\frac{1}{2} - \beta_l), \pi(\frac{3}{2} - \beta_l), \ldots$ it has a maximum equal to $\rho\omega|A|\cosh(\pi\alpha)$. The position of these minima and maxima fix the value of β_l and therefore of β_0, whereas the ratio of the pressure amplitude at minima with that at maxima is equal to $\tanh(\pi\alpha)$, so that this ratio determines the value of α.

Sound is introduced into the $x = 0$ end of the tube. In order to measure the response of the tube to excitation, we can imagine this end capped by a flat piston of negligible impedance, which can move back and forth along the axis, generating the waves. To produce a wave of the sort written in Eq. (11.1.34), the displacement and velocity of the piston and the pressure at its surface must be

$$\xi_0 = (A/c)\cosh(\pi\alpha - i\pi\beta_0)e^{-i\omega t};\quad v_0 = -i(\omega/c)A\cosh(\pi\alpha - i\pi\beta_0)e^{-i\omega t}$$
$$P_0 = -i\rho\omega A\sinh(\pi\alpha - i\pi\beta_0)e^{-i\omega t}$$

If the piston is driven by a simple harmonic force of unit amplitude and frequency $\omega/2\pi$, the wave amplitude A must be adjusted so that the pressure at the piston, times the cross-sectional area of the tube, S, is equal to the applied force, $e^{-i\omega t}$ dynes. Therefore,

$$A = \frac{i}{S\rho\omega\sinh[\pi\alpha - i\pi\beta_l - i(\omega l/c)]}$$

and the resulting velocity-potential, Green's function, is

$$G_f(x|0|\omega) = \frac{i\sinh[\pi\alpha - i\pi\beta_l - i(\omega/c)(l - x)]}{S\rho\omega\sinh[\pi\alpha - i\pi\beta_l - i(\omega l/c)]} \qquad (11.1.36)$$

To find the response to a unit impulse given the piston we have to find the function g, of which G is the Laplace transform.

When the termination impedance z_l is independent of ω, then α and β_l are also independent of ω, and the calculation of g, by means of the table of Laplace transforms, is quite easy.

$$g_f(x|0|t) = \frac{i}{S\rho} \sum_{n=0}^{\infty} \left\{ e^{-2\pi n(\alpha - i\beta_l)} u\left[t - \frac{x}{c} - 2n\frac{l}{c}\right] \right.$$
$$\left. - e^{-2\pi(n+1)(\alpha - i\beta_l)} u\left[t + \frac{x}{c} - 2(n+1)\frac{l}{c}\right] \right\} \quad (11.1.37)$$

The particle velocity and pressure, being derivatives of g, involve delta functions. The first term, for $n = 0$, is the initial impulsive wave traveling away from the piston, the second term, for $n = 0$, is the first reflected wave, returning to the origin with reduced amplitude ($e^{-2\pi\alpha}$) and possible change in sign (if β_l is constant it is either 0 or $\frac{1}{2}\pi$), and so on, just as for Eq. (11.1.32) for the string. As with the formulas for the string, the response for any sort of force $f(t)$ applied to the piston may be obtained by integration. For instance the fluid velocity is

$$v(x,t) = \frac{1}{S\rho c} \sum_{n=0}^{\infty} \left\{ e^{-2\pi n(\alpha - i\beta)} f\left(t - \frac{x + 2nl}{c}\right) \right.$$
$$\left. + e^{-2\pi(n+1)(\alpha - i\beta_l)} f\left(t + \frac{x - 2(n+1)l}{c}\right) \right\} \quad (11.1.38)$$

where $f(t) = 0$ for $t < 0$.

On the other hand, if the piston's displacement, $\xi_0(t)$, is specified as a function of time, the fluid displacement may be obtained by going back to Eq. (11.1.35). If the piston displacement is simple-harmonic with unit amplitude, the steady-state fluid displacement is

$$G_d(x|0|\omega) = \frac{\cosh[\pi(\alpha - i\beta_l) - i(\omega/c)(l - x)]}{\cosh[\pi(\alpha - i\beta_l) - i(\omega l/c)]}$$

and the corresponding impulse function is

$$g_d(x|0|t) = \sum_{n=0}^{\infty} (-1)^n \left\{ e^{-2\pi n(\alpha - i\beta_l)} \delta\left[t - \frac{x + 2nl}{c}\right] \right.$$
$$\left. + e^{-2\pi(n+1)(\alpha - i\beta_l)} \delta\left[t - \frac{x + 2(n+1)l}{c}\right] \right\}$$

and, finally, the fluid displacement for displacement $\xi_0(t)$ of the piston is

$$\xi(x,t) = \sum_{n=0}^{\infty} (-1)^n \left\{ e^{-2\pi n(\alpha - i\beta_l)} \xi_0\left(t - \frac{x + 2nl}{c}\right) \right.$$
$$\left. + e^{-2\pi(n+1)(\alpha - i\beta_l)} \xi_0\left(t + \frac{x - 2(n+1)l}{c}\right) \right\} \quad (11.1.39)$$

Tube with a Varying Cross Section. When the cross-sectional area S of the tube varies with x, the distance along the axis, but when $\sqrt{S}$ still is everywhere small compared to the wavelength of the sound, then, to the first approximation in the ratio of $\sqrt{S}$ to wavelength, the wave is still a one-dimensional one. The equation for propagation is not the simple wave equation, of course. With a tube of varying cross section the fluid velocity v will change from point to point in conformity with the change in S. We must assume, however, that v is reasonably uniform over S, for each value of x.

Modifying the derivation of Sec. 2.3 to our present needs, we find that the acceleration of any element of volume of the fluid is proportional to the rate of change of pressure,

$$\rho \frac{\partial v}{\partial t} = -\frac{\partial P}{\partial x} \tag{11.1.40}$$

as before. However, as the fluid is displaced by an amount ξ, the proportional change in volume of a thin sheet of the fluid, dx thick and stretched across the tube, is

$$dV/V = \left[S\xi + dx \frac{\partial}{\partial x}(S\xi) - S\xi \right] \Big/ S\,dx = (1/S)(\partial S\xi/\partial x)$$

But according to the relation between the elasticity of the fluid and the velocity of sound, c, we have that $-(dV/V) = P/\rho c^2$. Differentiating with respect to time we have

$$\frac{1}{S}\frac{\partial}{\partial x}(Sv) = -\frac{1}{\rho c^2}\frac{\partial P}{\partial t} \tag{11.1.41}$$

Combining this with Eq. (11.1.40) we have the final equation for the excess pressure, and also for the velocity potential ψ,

$$\frac{1}{S}\frac{\partial}{\partial x}\left(S\frac{\partial \psi}{\partial x}\right) = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}; \quad v = -\frac{\partial \psi}{\partial x}; \quad P = \rho \frac{\partial \psi}{\partial t} \tag{11.1.42}$$

Both pressure and fluid velocity obey this modified wave equation, which approximately takes into account the variation of cross-sectional size with x. The equation is a good approximation as long as the magnitude of the rate of change of $\sqrt{S}$ with x is much smaller than unity (as long as the tube "flares" slowly). As long as this is so, actual fluid velocity will be nearly parallel to the x axis and nearly uniform across the tube, so that the net flow down the tube, I, may be considered to be equal to vS.

The effect of this slight variation of cross section on the wave amplitude is not difficult to estimate. According to the discussion of page 309, the product of pressure and fluid velocity is equal to the energy flow density, the *intensity*. The total flow of energy down the tube, averaged over time, must be independent of distance along the tube for a steady-state wave; otherwise the energy would pile up or be drained away from some portion, and the wave would not be steady state. A steady-state wave should have the following general form,

$$\psi = A(x)e^{\pm i\varphi(x)-i\omega t} \tag{11.1.43}$$

where A is the amplitude of the wave and φ is often called the *phase* of the wave. Since both p and v are proportional to A, the intensity is proportional to A^2. What we have said about the total flow of energy indicates that SA^2 must be independent of x (to the degree of approximation dealt with here) and therefore that A is inversely proportional to $\sqrt{S}$.

But the dependence of both A and φ on x is more clearly seen by substituting Eq. (11.1.43) in Eq. (11.1.42) and equating real and imaginary parts to zero separately:

$$\left(\frac{d\varphi}{dx}\right)^2 = \left(\frac{\omega}{c}\right)^2 + \frac{1}{SA}\frac{d}{dx}\left(S\frac{dA}{dx}\right); \quad \frac{d}{dx}\left[SA^2\frac{d\varphi}{dx}\right] = 0 \tag{11.1.44}$$

Remembering that dS/dx is small compared to $\sqrt{S}$ for Eq. (11.1.42) to be valid, we can obtain a successive approximation solution of these two equations. For the "zeroth" approximation we could set A constant and $d\varphi/dx = \omega/c$. This would pay no attention at all to the variation of S with x and would give us the familiar solution $Ae^{\pm(i\omega/c)x-i\omega t}$, valid for a uniform tube. The next approximation gives $A = C/\sqrt{S}$ and $\varphi = \omega x/c$, satisfying the second equation exactly, but leaving the small term $-(1/\sqrt{S})(d^2\sqrt{S}/dx^2)$ uncanceled in the first equation.

The second approximation gives

$$A = \frac{C}{\sqrt{S}}; \quad \varphi = \int_a^x \left[\left(\frac{\omega}{c}\right)^2 - \frac{1}{\sqrt{S}}\frac{d^2\sqrt{S}}{dx^2}\right]^{\frac{1}{2}} dx$$

which satisfies the first equation but leaves a still smaller term uncanceled in the second equation, and so on. To this approximation the pressure and fluid velocity for the waves in each direction are, approximately,

$$\begin{aligned}
P &\simeq (i\rho\omega/\sqrt{S})[C_+e^{i\varphi(x)} + C_-e^{-i\varphi(x)}]e^{-i\omega t} \\
&\simeq -(i\rho\omega/\sqrt{S})C\sinh[\pi(\alpha - i\beta) + i\varphi(x)]e^{-i\omega t} \\
v &\simeq -\frac{i\omega}{c\sqrt{S}}\sqrt{1 - \frac{c^2}{\omega^2}\frac{1}{\sqrt{S}}\frac{d^2}{dx^2}\sqrt{S}}\ [C_+e^{i\varphi} - C_-e^{-i\varphi}]e^{-i\omega t} \\
&\simeq -\frac{i\omega}{c\sqrt{S}}\sqrt{1 - \frac{c^2}{\omega^2}\frac{1}{\sqrt{S}}\frac{d^2}{dx^2}\sqrt{S}}\ C\cosh[\pi(\alpha - i\beta) + i\varphi(x)]e^{-i\omega t}
\end{aligned}$$

The case of the *exponential horn* may be analyzed in detail. Here $S = S_0e^{2x/h}$, so that $(1/\sqrt{S})(d^2\sqrt{S}/dx^2) = 1/h^2$, a constant. For this case, for a wave entirely in the positive direction ($C_- = 0$),

$$P \simeq -(i\rho\omega/\sqrt{S_0})C_+ \exp[ix\sqrt{(\omega/c)^2 - (1/h)^2} - i\omega t - (x/h)] \quad (11.1.45)$$
$$v \simeq -\frac{i}{\sqrt{S_0}}\sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{1}{h}\right)^2}\,C_+ \exp\left[ix\sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{1}{h}\right)^2} - i\omega t - \left(\frac{x}{h}\right)\right]$$

The pressure amplitude in the exponential horn when the piston at $x = 0$ is displaced at unit amplitude and frequency $\omega/2\pi(\xi = -v/i\omega)$ is

$$G_d(x|0|\omega) = [\rho c p^2/\sqrt{p^2 + (c/h)^2}]\exp[-(x/h) - (x/c)\sqrt{p^2 + (c/h)^2}]$$
$$p = -i\omega$$

The table at the end of this chapter shows that the pressure wave in the horn, produced by a unit impulsive displacement of the piston, is

$$g_d(x|0|t) = \rho c e^{-(x/h)} u\left(t - \frac{x}{c}\right)\frac{d^2}{dt^2}J_0\left[\frac{c}{h}\sqrt{t^2 - \left(\frac{x}{c}\right)^2}\right] \quad (11.1.46)$$

Finally, the pressure wave produced down an infinite exponential horn (*i.e.*, long enough so no reflected wave comes back from the far end) by the displacement of the piston by an amount $\xi_0(t)(t > 0)$ is from Eq. (11.1.15)

$$P(x,t) = \rho c e^{-(x/h)} \int_{x/c}^{t} \xi_0(t - \tau)\frac{d^2}{d\tau^2}J_0\left[\frac{c}{h}\sqrt{\tau^2 - \left(\frac{x}{c}\right)^2}\right]d\tau \quad (11.1.47)$$

This equation shows that waves change shape as they travel along an exponential tube [Eq. (11.1.39) shows they do not change shape in a uniform tube]. The pressure at a point a distance x from the piston at time t depends on the displacement of the piston from the start of motion of the piston ($t = 0$) to a time x/c ago, $t - (x/c)$, instead of depending only on the displacement at time $t - (x/c)$, as it does with the uniform tube. Since J_0'' is largest when its argument is zero, the contribution from the displacement at $t - (x/c)$ is greater than the contributions from earlier times, especially when c/h is small, but the contributions from earlier times are not negligible in this case.

Acoustic Circuit Elements. When the cross-sectional area, S, varies slowly with x, Eqs. (11.1.44) are usually accurate enough to depict the behavior of sound in such tubes. Whenever S changes its functional behavior abruptly (*i.e.*, in a space short compared to a wavelength), reflection occurs and the amount of the reflected waves may be computed (approximately) by equating pressures and total flow of fluid on both sides of the discontinuity.

For such calculations the analogy to electric transmission-line analysis is useful. The pressure is the analogue of the voltage and the total flow of fluid, Sv, is the analogue of the current, and may be labeled I, to remind us of the analogy. The matching at junctions may be then performed by matching *analogous impedances*

$$Z = \frac{P}{I} = \frac{1}{S}\frac{P}{v} = \frac{z}{S} \tag{11.1.48}$$

since both P and I are continuous across such junctions.

For instance the analogous input impedance of one end of an infinite, uniform tube of cross-sectional area S is $\rho c/S$, and the input impedance at the $x = 0$ end of a uniform tube of length l, terminated by an analogous impedance Z_l, is

$$Z_0 = (\rho c/S) \tanh[\pi(\alpha - i\beta_l) - i(\omega l/c)] \tag{11.1.49}$$

where $$\tanh(\pi\alpha - i\pi\beta_l) = SZ_l/\rho c$$

A change of cross-sectional size produces a reflection. If the tube from $x = l$ to $x = \infty$ has area S_l and from $x = 0$ to $x = l$ has area S_0, the analogous impedance at $x = l$ (looking in the positive direction) is $\rho c/S_l$ and the analogous impedance at $x = 0$ is

$$(\rho c/S_0) \tanh[\tanh^{-1}(S_0/S_l) - i(\omega l/c)]$$

By working through the equations, we see that the wave reflected from the junction suffers a change of sign if $S_l > S_0$, no change in sign if $S_l < S_0$, and that there is no reflected wave if $S_l = S_0$.

A short, wider portion of the tube will be analogous to a capacitance across a transmission line. The additional volume V will "store" additional air displacement. If the net flow of fluid into the additional volume is $Q = \int I\,dt$ the relation between Q and P is the usual $P = \rho c^2 Q/V$, so that the *analogous capacitance* is

$$C = V/\rho c^2 \tag{11.1.50}$$

On the other hand, a short, narrower portion of the tube is analogous to an inductance. A pressure difference across the constriction will produce an acceleration of the fluid in the constriction, inversely proportional to the mass of air contained.

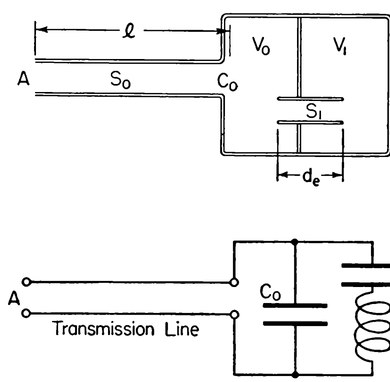

Fig. 11.1 Acoustical circuit and equivalent electric circuit.

The force on this "slug" of fluid is $(\Delta P)S$, where ΔP is the pressure difference and S, the cross-sectional area of the constriction. The mass of fluid to be accelerated is ρ times the volume of the "slug," Sd_e, where d_e is the effective length of the constriction (we shall define "effective" shortly). The acceleration is, of course, dv/dt, where Sv is the volume flow I through the constriction. The equation of motion for the fluid in the constriction is therefore

$$S\,\Delta P = \rho S d_e \frac{dv}{dt} \quad \text{or} \quad \Delta P = \left(\frac{\rho d_e}{S}\right)\left(\frac{dI}{dt}\right)$$

and, by analogy with electric circuits, the analogous *inductance* is

$$L = \rho d_e/S \tag{11.1.51}$$

which may be inserted in the equivalent electric circuit to compute the acoustic behavior. For instance the acoustical circuit shown in Fig. 11.1 is equivalent to the electric circuit shown below it, with the transmission line from A to C_0 (of length l) having a characteristic impedance of $\rho c/S_0$, the capacitance C_0 being $V_0/\rho c^2$, the inductance L_1 being $\rho d_e/S_1$ and the capacitance C_1 being $V_1/\rho c^2$. The *acoustic* impedance (ratio of pressure to velocity) at A is, therefore,

$$-i\rho c \tan\left\{\left(\frac{\omega l}{c}\right) - \tan^{-1}\left[\frac{S_0}{\rho c\omega}\left(C_0 + \frac{C_1}{1 - \omega^2 L_1 C_1}\right)^{-1}\right]\right\}$$

We have used effective length of the constriction, instead of actual length, because the picture of the air in the constriction being equivalent to an inductance is only an approximate one, and we can improve the approximation by correcting the length for "end effects." From another point of view, the reason we can consider a constriction to be an inductance is because in and near the constriction the fluid speeds up and there is, therefore, *a concentration of kinetic energy there* (just as there is a concentration of potential energy in an additional volume). The fluid in the narrow part (cross section S_1) is going at a velocity $v(S_0/S_1)$ when the fluid in the wider part (cross section S_0) is going at speed v. If S_1/S_0 is small enough, the kinetic energy per unit length of tube in the constriction, $\frac{1}{2}\rho v^2 S_0(S_0/S_1)$ is considerably larger than the same quantity in the main part of the tube, $\frac{1}{2}\rho v^2 S_0$.

Just as the magnetic energy in an inductance is $\frac{1}{2}LI^2$, so here we can define our analogous impedance by calculating the total kinetic energy of the "speeded up" part of the fluid. We see that this is $\frac{1}{2}\rho d_e(v^2 S_0^2/S_1) = \frac{1}{2}(\rho d_e/S_1)I^2$, when $I = vS_0$ is the net flow through the tube. We have again used d_e instead of the actual length of the constriction, because the fluid starts speeding up before it enters the constriction and does not slow down immediately after leaving it, so that more fluid than is in the constriction is speeded up. The quantity d_e is thus the actual length, d, plus a correction term to allow for the extra fluid speeding up and slowing down at the two ends.

What should be done, of course, is to calculate the irrotational steady flow of the fluid through the constriction by the methods of Chap. 10, and then to compute the additional kinetic energy produced by the constriction. When this is expressed in terms of $\frac{1}{2}LI^2$, where I is the total flow through the constriction, then L is the analogous inductance. This was done, on page 1295, for a circular hole of radius a in a plate of negligible thickness, and the result was that the kinetic energy was equal to $\frac{1}{2}(8\rho/3\pi^2 a)I^2$. Therefore, the analogous impedance, for a circular constriction of "zero length" is $8\rho/3\pi^2 a$, and the "end correction" to use in Eq. (11.1.51) in this limiting case is $8a/3\pi$. One is therefore tempted to use this same correction for a circular constriction of actual length d, so that Eq. (11.1.51) would become

$$L = (\rho/\pi a^2)[d + (8a/3\pi)] \tag{11.1.52}$$

which actually turns out to be a good approximation as long as a is small compared to the radius of cross section of the rest of the tube and as d is considerably smaller than a wave length. These restrictions are only natural since, first, the derivation of page 1294 was for a hole in an infinite plate, and the main part of the tube should thus be large enough so as to be remote compared to radius a and, second, the whole concept of lumped-circuit elements loses validity as soon as an appreciable phase difference develops between the two ends of one of the elements, either in acoustic or electric circuits.

This technique, of joining short portions of solutions of Laplace's equation onto solutions of the wave equation, to obtain approximate solutions for boundary conditions too complicated to solve the wave equation exactly (but not too complicated to solve the Laplace equation), is a very useful one and will be invoked from time to time later.

Free-wave Representations. Finally, it might be well to remind ourselves that the most general solutions of the simple wave equation in one space dimension may be made up of a combination of free waves in the two directions

$$\psi = f(ct - x) + F(ct + x) \tag{11.1.53}$$

and that boundary conditions may be satisfied by their use, instead of by eigenfunction solutions [see Eq. (7.3.18), for instance].

As an example, we consider the case of a simple string of infinite length, extending in the positive direction from $x = 0$, where there is a support having transverse stiffness and resistance. In other words, the transverse displacement of the support, y_0, is related to the transverse force from the string, $T(\partial y/\partial x)_0$, by the equation

$$R_0(\partial y/\partial t) + K_0 y = T(\partial y/\partial x); \quad x = 0 \tag{11.1.54}$$

Substituting Eq. (11.1.53) in Eq. (11.1.54) we obtain

$$(cR_0 + T)\frac{d}{dz}f(z) + (cR_0 - T)\frac{d}{dz}F(z) + K_0 f(z) + K_0 F(z) = 0 \quad (11.1.55)$$

If the incident wave, $F(ct + x)$, is supposed to be known, this is an equation for f, with solution

$$f(z) = e^{-\kappa_0 z}\int_a^z \left[\gamma_0 \frac{d}{du}F(u) - \kappa_0 F(u)\right] e^{\kappa_0 u}\,du \quad (11.1.56)$$

where $$\kappa_0 = \frac{K_0}{cR_0 + T} \quad \text{and} \quad \gamma_0 = \frac{T - cR_0}{T + cR_0}$$

and a is any convenient constant. When $K_0 = 0$, the support being resistive only, then $f(z) = \gamma_0 F(z)$ and

$$y = F(ct + x) + [(T - cR_0)/(T + cR_0)]F(ct - x) \quad (11.1.57)$$

the reflected wave being identical with the incident one, but reduced in size by the *reflection factor* γ_0. If R_0 is infinite, the support is rigid ($\gamma_0 = -1$), the reflected wave is equal but opposite in sign to the incident one; if R_0 is zero, the support has no transverse impedance ($\gamma_0 = 1$) and the reflected wave is equal in magnitude and sign to the incident one.

To fit initial conditions we adjust f and F so that $y = y_0(x)$ and $\partial y/\partial t = v_0(x)$ at $t = 0$. This gives

$$\begin{aligned} y &= \tfrac{1}{2}y_0(x + ct) + \tfrac{1}{2}y_0(x - ct) \\ &\qquad + \tfrac{1}{2}U_0(x + ct) - \tfrac{1}{2}U_0(x - ct); \quad x > ct \\ &= \tfrac{1}{2}y_0(x + ct) + \tfrac{1}{2}U_0(x + ct) + \tfrac{1}{2}[y_0(0) - U_0(0)]e^{\kappa_0(x-ct)} \\ &\quad + \tfrac{1}{2}e^{\kappa_0(x-ct)}\int_0^{ct-x}\left[\gamma_0\frac{d}{du}y_0(u) + \frac{\gamma_0}{c}v_0(u) \right. \\ &\qquad\qquad \left. - \kappa_0 y_0(u) - \kappa_0 U_0(u)\right] e^{\kappa_0 u}\,du; \quad x < ct \end{aligned} \quad (11.1.58)$$

where $U_0(z) = \dfrac{1}{c}\displaystyle\int_0^z v_0(u)\,du$. The third term in the expression for $x < ct$ represents the relaxation of the support, if it was initially displaced, and the fourth term, the integral, corresponds to the part of the wave reflected from the support.

One can also calculate, by the same method, the wave motion in a string supported at both ends by yielding supports, a distance l apart. To avoid overcomplex formulas we write out the solution for resistive supports, the transverse resistance of the one at $x = 0$ being R_0 and that for the one at $x = l$ being R_l. Taking into account the successive reflections at both ends, we have

$$y = f(x - ct) + F(x + ct)$$

$$f(z) = \begin{cases} \frac{1}{2}[y_0(z) - U_0(z)]; & l > z > 0 \\ \frac{1}{2}\gamma_0[y_0(-z) + U_0(-z)]; & 0 > z > -l \\ \frac{1}{2}\gamma_0\gamma_l[y_0(z - 2l) - U_0(z - 2l)]; & -l > z > -2l \\ \frac{1}{2}\gamma_0^2\gamma_l[y_0(-z - 2l) + U_0(-z - 2l)]; & -2l > z > -3l \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \end{cases}$$

$$F(z) = \begin{cases} \frac{1}{2}[y_0(z) + U_0(z)]; & 0 < z < l \\ \frac{1}{2}\gamma_l[y_0(2l - z) - U_0(2l - z)]; & l < z < 2l \\ \frac{1}{2}\gamma_l\gamma_0[y_0(z + 2l) + U_0(z + 2l)]; & 2l < z < 3l \\ \frac{1}{2}\gamma_l^2\gamma_0[y_0(4l - z) - U_0(4l - z)]; & 3l < z < 4l \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \end{cases}$$

where $\gamma_0 = (T - cR_0)/(T + cR_0)$; $\gamma_l = (T - cR_l)/(T + cR_l)$

When R_0 and R_l are infinite both supports are rigid and, since $\gamma_0 = \gamma_l = -1$, each reflection results in a change of sign and the motion is periodic with period $2l/c$.

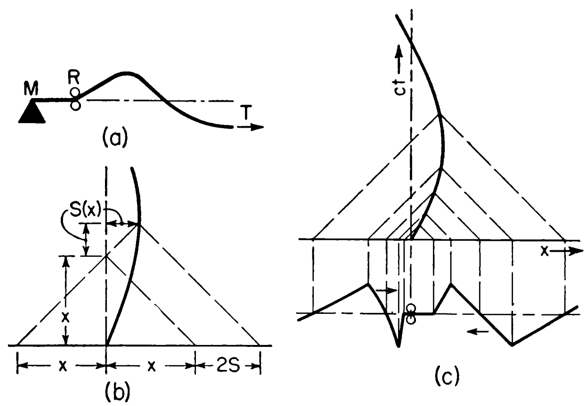

Fig. 11.2 Reflection of waves in a string from a moving end clamp R. Drawing (c) shows how reflected wave may be constructed; (b) defines functions $S(x)$.

Movable Supports. One other case merits discussion: where one or both supports move longitudinally a finite amount. As shown in Fig. 11.2*a* this can be realized in the case of the string by having a rigid support at M which takes up the tension and, in addition, a pair of rollers R, which force the displacement y of the string to be zero at the position of the rollers. Now suppose the rollers are moved back and forth. The boundary conditions to be satisfied are those illustrated in Fig. 11.2*b*, where the motion is plotted on the (x,ct) plane. The displacement y must be zero along the wavy boundary.

To solve this, for the case where the boundary is at $x = A \sin(\omega t)$, we must use the function $S(x)$, defined as follows (see Fig. 11.2*b*):

$$S(x) = A \sin k[x + S(x)] \qquad (11.1.59)$$

For small values of A, S may be expressed in a power series in A:

$$S(x) = (A - \tfrac{1}{8}k^2A^3 + \cdots) \sin(kx) + (\tfrac{1}{2}kA^2 - \tfrac{1}{6}k^3A^4 + \cdots) \sin(2kx) + (\tfrac{3}{8}k^2A^3 - \cdots) \sin(3kx) + (\tfrac{1}{3}k^3A^4 - \cdots) \sin(4kx) + \cdots$$

Study of Fig. 11.2c will show that the equation for the displacement of a string with one support at $x = A \sin(\omega t)$ (constant tension T) and the other at $x = \infty$, having initial displacement $y_0(x)$ and initial velocity $v_0(x)$, is

$$\begin{aligned} y(x,t) &= \tfrac{1}{2}y_0(x + ct) + \tfrac{1}{2}y_0(x - ct) + \tfrac{1}{2}U_0(x + ct) - \tfrac{1}{2}U_0(x - ct); \quad \text{for } x > ct \\ &= \tfrac{1}{2}y_0(x + ct) + \tfrac{1}{2}U_0(x + ct) + \tfrac{1}{2}y_0[ct - x + 2S(ct - x)] \\ &\qquad - \tfrac{1}{2}U_0[ct - x + 2S(ct - x)]; \quad \text{for } x < ct \end{aligned}$$

Here the reflected waves are distorted because of the motion of the support. We note that the speed of the rollers must be always less than c, the wave velocity, for this analysis to hold.

Similar formulas may be devised for boundaries a finite distance apart, with one or both moving. If the support motion is periodic, the resulting string motion is not periodic *except* when the support frequency is an integral multiple of the frequency $\pi c/l$ of free vibration of the string. Formulas may also be obtained for the motion of air in a long thin tube, when the piston at $x = 0$ is moved back and forth a finite amplitude.

11.2 *Waves in Two Space Dimensions*

As soon as we begin to consider waves in more than one space dimension, we encounter new phenomena, not exhibited by the one-dimensional waves. Solutions of the wave equation in one dimension, as discussed in the previous section, can be represented by the general formula $f(x - ct) + F(x + ct)$, corresponding to the superposition of two waves in opposite directions. Each wave moves in one or the other direction with velocity c and with no change of form as it moves. Aside from the scale factor c, the dependence on time is the same as the dependence on space and each partial wave, a second later, looks the same as before except that it has moved a distance c. The track of any point on the partial wave, as a function of x and t, is a *characteristic*, $x \pm ct =$ constant (see page 682).

In two or more space dimensions waves travel in many more than just two directions. There are one-dimensional solutions of the sort $F[x \cos u + y \sin u - ct]$ which are already familiar to us but there are other, more complicated forms, where the wave travels in several directions at once, the wave changing shape as it travels along [see the discussion of Eq. (6.1.16)]. These waves must be investigated before we are able to satisfy two- and three-dimensional boundary conditions.

Referring back to Eq. (10.3.2), we should expect that an integral expression of the sort

$$\psi(x,y,t) = \int_0^{2\pi} F_u[x \cos u + y \sin u - ct]\, du \qquad (11.2.1)$$

could represent any wave in two space dimensions (three total dimensions). A specialization of this to the Helmholtz equation $(\nabla^2 + k^2)\psi = 0$, for waves of frequency $\omega/2\pi = kc/2\pi$, having the time factor $e^{-i\omega t}$, is

$$\psi(x,y|\omega) = \int_0^{2\pi} \Phi(u) e^{ik(x \cos u + y \sin u)}\, du \qquad (11.2.2)$$

which is a special case of the general integral formulation of Eq. (5.3.86). The exponent in the integrand, aside from the i factor, is just the scalar product between the two-dimensional vector $\mathbf{r} = \mathbf{i}x + \mathbf{j}y$ and a vector $\mathbf{k} = k\mathbf{a}_u$ with magnitude $k = \omega/c$ and at an angle u with respect to the x axis. The wave $e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}$ is simple-harmonic, and the wave $F_u(\mathbf{r} \cdot \mathbf{a}_u - ct)$ is a general, linear wave in the direction of the unit vector $\mathbf{a}_u$ (at an angle u to the x axis). A combination of all such waves, in all directions u, with variable amplitude and phase for each direction, constitutes the most general solution of the wave equation (or the Helmholtz equation) in two space dimensions (it is sometimes necessary to extend the integral to complex values of u).

It will be useful to express all our eigenfunction and Green's function solutions in this form, for the interrelations between them will be thus more apparent.

Fourier Transforms and Green's Functions. A discussion of the interrelation between Fourier integral solutions, Green's functions and the representations of Eqs. (11.2.1) and (11.2.2) will provide a useful introduction to this section. Suppose we wish to solve the following inhomogeneous equation in two space dimensions:

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -4\pi\rho(x,y,t) \qquad (11.2.3)$$

where there are no boundaries in the finite part of the x, y plane, and the sole boundary condition is that the radiation is outward at infinity.

One way to solve this would be to use the Green's function for the Helmholtz equation, for a unit source at (x_0,y_0) of frequency $\omega/2\pi$, $G(\mathbf{r}|\mathbf{r}_0|\omega) = i\pi H_0^{(1)}(kR)$, where $k = \omega/c$ and $R^2 = [(x - x_0)^2 + (y - y_0)^2]$ [see Eq. (7.2.18)]. This steady-state solution for a unit simple-harmonic source is, according to (11.1.16), the Laplace transform of $g(\mathbf{r}|\mathbf{r}_0|t)$, the solution of Eq. (11.2.3) for a unit pulse at $t = 0$ at (x_0,y_0), that is, for $\rho = \delta(x - x_0)\delta(y - y_0)\delta(t)$.

But suppose we start by taking the Fourier transform of Eq. (11.2.3) for all three dimensions. We always have (subject to earlier discussed limitations of convergence)

$$\begin{aligned}\psi(x,y|t) &= \frac{1}{(2\pi)^3}\iiint_{-\infty}^{\infty} F(\xi,\eta|\omega)e^{i\xi x+i\eta y-i\omega t}\,d\xi\,d\eta\,d\omega \\ F(\xi,\eta|\omega) &= \iiint_{-\infty}^{\infty} \psi(x,y|t)e^{-i\xi x-i\eta y+i\omega t}\,dx\,dy\,dt\end{aligned} \tag{11.2.4}$$

where F is the general Fourier transform to ψ. For instance, the general Fourier transform for the inhomogeneous part for a unit impulsive source, $-4\pi\delta(x - x_0)\delta(y - y_0)\delta(t - t_0)$, is $-4\pi e^{-i\xi x_0-i\eta y_0+i\omega t_0}$. Inserting the integral for ψ into Eq. (11.2.3) and equating integrands on both sides, we obtain the expression for the general Fourier transform for the solution

$$F(\xi,\eta|\omega) = \frac{4\pi e^{-i\xi x_0-i\eta y_0+i\omega t_0}}{\xi^2 + \eta^2 - (\omega/c)^2} \tag{11.2.5}$$

The general solution of Eq. (11.2.3) for unlimited space may be built up from this elementary solution by integration. For instance, the solution of the inhomogeneous Helmholtz equation for a unit (simple-harmonic) line source at (x_0,y_0) is the Fourier transform of F for the space coordinates,

$$\begin{aligned}G(\mathbf{r}|\mathbf{r}_0|\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta\,\frac{e^{i\mathbf{k}\cdot\mathbf{R}}}{k^2 - (\omega/c)^2} \\ &= \frac{1}{\pi}\int_0^{\infty} k\,dk \int_0^{2\pi} \frac{e^{ikR\cos(u-\theta)}}{k^2 - (\omega/c)^2}\,du\end{aligned}$$

where $\mathbf{k}$ is the vector with components ξ, η, at an angle u to the x axis and $\mathbf{R}$ is the vector with components $(x - x_0)$, $(y - y_0)$, at an angle θ to the x axis. The integration over u results in the Bessel function $2\pi J_0(kR)$, as reference to Eq. (5.3.65) or the table at the end of Chap. 10 will show. The integration over k utilizes the following equation:

$$K_0(\alpha R) = \int_0^{\infty} \frac{J_0(kR)}{k^2 + \alpha^2}\,k\,dk; \quad K_0(z) = \tfrac{1}{2}\pi i H_0^{(1)}(iz)$$

which may be obtained by juggling integral representations and which is listed at the end of the previous chapter. By letting $\omega = ip$ (see page 1340), we finally obtain

$$G(\mathbf{r}|\mathbf{r}_0|\omega) = 2K_0(pR/c) = i\pi H_0^{(1)}(\omega R/c)$$

which is just what we found in Eq. (7.2.18) for the Green's function for the two-dimensional Helmholtz equation.

The final solution of Eq. (11.2.3) for an impulsive source involves one further integration, over ω. Or else we can use the methods of page 1343 and find the function having G as its Laplace transform. A slight rearrangement of the integral representation for H_0 gives

$$K_0\left(\frac{pR}{c}\right) = \int_1^\infty \frac{e^{-pRu/c}}{\sqrt{u^2-1}}\,du = \int_{R/c}^\infty \frac{e^{-pt}\,dt}{\sqrt{t^2-(R/c)^2}}$$

Consequently $2K_0(pR/c)$ is the Laplace transform of $[2/\sqrt{t^2-(R/c)^2}]$ for $t > R/c$ and the solution of Eq. (11.2.3) for an impulsive source at (x_0,y_0) is [see Eq. (7.3.15)]

$$g(\mathbf{r}|\mathbf{r}_0|t) = \begin{cases} 0; & t < R/c \\ 2/\sqrt{t^2-(R/c)^2}; & t > R/c \end{cases} \tag{11.2.6}$$

Therefore, in two dimensions, the wave response at a distance R away from an impulsive disturbance, occurring at $t = 0$, is zero until $t = R/c$, the time required for the wave front to traverse the distance R; thereafter it is proportional to $[t^2 - (R/c)^2]^{-\frac{1}{2}}$. This wave has a sharp front but leaves a "wake" behind it, only slowly returning to zero, as was mentioned in connection with Eq. (6.1.16). The behavior of the wave caused by a general source function $\rho(r,\varphi,t)$ (which is assumed zero for $t < 0$) may be most easily seen by taking the observation point (r,φ) to be the origin and by using the polar coordinates (R,θ) for the source point. By using Eq. (11.1.15) we see that the solution of (11.2.3) is

$$\begin{aligned}\psi(0,0,t) &= 2\int_0^{2\pi} d\theta \int_0^\infty R\,dR \int_{R/c}^t \frac{\rho(R,\theta,t-\tau)}{\sqrt{\tau^2-(R/c)^2}}\,d\tau \\ &= 2c\int_0^t d\tau \int_0^c R\,dR \int_0^{2\pi} \frac{\rho(R,\theta,t-\tau)}{\sqrt{c^2\tau^2-R^2}}\,d\theta \end{aligned} \tag{11.2.7}$$

The second form shows that a time t after the source began "transmitting" the solution ψ depends only on the behavior of the source function ρ, within a maximum distance ct from the point of observation (the origin here). The effect of sources farther away is felt only after a longer time.

This formula should be compared with that of Eq. (7.3.19), which gives the solution for a specified initial condition at $t = 0$. The relationship should be obvious.

We can write our solution in still another way, which emphasizes the relation with Eq. (11.2.1). To write the general formula for the source density function $\rho(x,y,t)\delta(t - t_0)$, we first write the Fourier transform of this density function as

$$P(\xi,\eta|t_0)e^{i\omega t_0};\quad \text{where } P = \iint\limits_{-\infty}^{\infty} \rho(x,y,t_0)e^{-i\xi x - i\eta y}\,dx\,dy$$

The equation for F, the general Fourier transform of ψ, is obtained by inserting the first of Eqs. (11.2.4) into (11.2.3) for the above density, and we obtain

$$F = \frac{4\pi P(\xi,\eta|t_0)e^{i\omega t_0}}{\xi^2 + \eta^2 - (\omega/c)^2}$$

The Fourier transform of this is

$$\begin{aligned}\psi(\mathbf{r}|t|t_0) &= \frac{1}{2\pi^2}\iiint\limits_{-\infty}^{\infty} \frac{P(\xi,\eta|t_0)}{\xi^2 + \eta^2 - (\omega/c^2)} e^{i\xi x + i\eta y - i\omega(t-t_0)}\, d\xi\, d\eta\, d\omega \\ &= -\frac{c^2}{2\pi^2}\int_0^{2\pi} du \int_0^{\infty} k\, dk\, P(k,u|t_0)e^{i\,\cdot\mathbf{r}} \int_{-\infty}^{\infty} \frac{e^{-i\omega(t-t_0)}}{\omega^2 - k^2c^2}\, d\omega\end{aligned}$$

The integral over ω is a standard form for reduction to contour integrals. The two poles are close to the contour however, and we must choose our path to fit physical requirements and boundary conditions. For instance the residue at $\omega = kc$ results in an outgoing wave, which is wanted, but the residue at $-kc$ is not wanted as it is incoming. Therefore, we take the residue about the one pole for $t > t_0$ which results in $(\pi i/kc)e^{-ikc(t-t_0)}$. Consequently the solution for ψ is, for $t > t_0$

$$\psi(\mathbf{r}|t|t_0) = \frac{c}{2\pi i}\int_0^{2\pi} du \left\{\int_0^{\infty} P(k,u|t_0)e^{i\mathbf{k}\cdot\mathbf{r} - ikc(t-t_0)}\, dk\right\}$$

and the solution of Eq. (11.2.3) for a general source density ρ which is zero for $t < 0$ but equal to $\rho(x,y,t)$ for $t > 0$ is

$$\psi(x,y,t) = \int_0^{\infty} dk \left\{\int_0^{2\pi} \Phi(k,u,t)e^{ik(x\cos u + y\sin u - ct)}\, du\right\} \tag{11.2.8}$$

where

$$\begin{aligned}(2\pi i/c)\Phi(k,u|t) &= \int_0^t P(k,u|t_0)e^{ikct_0}\, dt_0 \\ &= \iint\limits_{-\infty}^{\infty} dx_0\, dy_0 \int_0^t \rho(x_0,y_0,t_0)e^{ik(x_0\cos u + y_0\sin u - ct_0)}\, dt_0\end{aligned}$$

The formula for ψ is an integral over k of an integral expression of the sort given in Eq. (11.2.2) and the whole function is of the general form given in Eq. (11.2.1).

Rectangular Coordinates. When boundaries are rectangular, the integrals of Eqs. (11.2.1) and (11.2.2) degenerate into sums. For instance for the case of a flexible membrane stretched at tension T dynes per cm from a rigid frame at the lines $x = 0$, $x = a$, $y = 0$, $y = b$, the eigenfunctions are $\sin(\pi mx/a)\sin(\pi ny/b)$, with m, n integers, and the eigenvalues of the frequency are $\omega_{mn}/2\pi$ where

$$\omega_{mn}^2 = \pi^2c^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]$$

If the boundary conditions at the same rectangular boundary are that the normal gradient be zero (instead of the value) then the eigenfunctions are $\cos(\pi m x/a)\cos(\pi n y/b)$; the eigenvalues are unchanged.

The Green's function for a unit simple-harmonic driving force concentrated at (x_0,y_0) inside the boundary is, by methods by now quite familiar,

$$G(\mathbf{r}|\mathbf{r}_0|\omega) = \frac{16}{\pi ab}\sum_{m,n}\frac{\sin\left(\frac{\pi m x_0}{a}\right)\sin\left(\frac{\pi m x}{a}\right)\sin\left(\frac{\pi n y_0}{b}\right)\sin\left(\frac{\pi n y}{b}\right)}{(m/a)^2+(n/b)^2-(\omega/\pi c)^2} \tag{11.2.9}$$

From this, by the methods of the Laplace transform, we may obtain series expressions, for any distribution of sources and for any dependence on time. The impulse function, corresponding to a unit impulse being given to the point (x_0,y_0) at $t_0 = 0$ is

$$g(\mathbf{r}|\mathbf{r}_0|t) = \frac{16c^2}{ab}\sum_{m,n}\frac{\sin\left(\frac{\pi m x_0}{a}\right)\sin\left(\frac{\pi m x}{a}\right)\sin\left(\frac{\pi n y_0}{b}\right)\sin\left(\frac{\pi n y}{b}\right)}{\sqrt{(m/a)^2+(n/b)^2}}\sin(\omega_{mn}t) \tag{11.2.10}$$

where $\omega_{mn}^2 = (\pi c m/a)^2 + (\pi c n/b)^2$.

Another formulation of the steady-state Green's function G may be obtained, as previously, by taking a Fourier series in y but then solving the inhomogeneous equation in x in terms of a single term, with a discontinuity at x_0. Assuming that $G = \sum F_n(x)\sin\left(\frac{\pi n y}{b}\right)$ we have, for F_n, the equation,

$$\frac{d^2F_n}{dx^2} + k_n^2 F_n = -\left(\frac{8\pi}{b}\right)\sin\left(\frac{\pi n y_0}{b}\right)\delta(x - x_0)$$

where $k_n^2 = (\omega/c)^2 - (\pi n/b)^2$. The solutions of these, which go to zero at $x = 0$ and $x = a$, may be found easily and we finally arrive at the formula,

$$G(\mathbf{r}|\mathbf{r}_0|\omega) = \frac{8\pi}{b}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{\pi n y_0}{b}\right)\sin\left(\frac{\pi n y}{b}\right)}{k_n\sin(k_n a)}\begin{cases}\sin(k_n x_0)\sin[k_n(a-x)]; & x > x_0\\ \sin(k_n x)\sin[k_n(a-x_0)]; & x < x_0\end{cases} \tag{11.2.11}$$

for the Green's function. This formula is, of course, equivalent to that of Eq. (11.2.9) and results in Eq. (11.2.10) on carrying through the Laplace or Fourier transformation. We can obtain still another form for G by interchanging (x,x_0,a) for (y,y_0,b) in Eq. (11.2.11).

Other Boundary Conditions. This last form of the Green's function is particularly useful when we come to deal with less simple boundary conditions. For instance, the side $x = a$ may not be rigid but may move an amount proportional to the transverse force, as was assumed for the string on page 1344. Since the transverse force per unit length of boundary is $T(\partial\psi/\partial n)$ where n is the distance normal to the boundary ($-T\,\partial\psi/\partial x$ for $x = a$), this is equivalent to saying that some linear combination of ψ, $\dot\psi$, $\ddot\psi$ is proportional to $\partial\psi/\partial n$ at the boundary; and *this* is equivalent, for simple harmonic motion, to the general homogeneous boundary condition

$$\psi = Y(\mathbf{r}^s,\omega)(\partial\psi/\partial n) = -Y(\partial\psi/\partial x);\quad \text{for the boundary } x = a$$

The ratio Y, the complex ratio between displacement and normal slope at a point on the boundary for simple harmonic motion of frequency $\omega/2\pi$, is a function of ω and of the point on the boundary under consideration. It may be called the *specific admittance* of the boundary. For a simple mass-resistance-stiffness reaction $Y = [-M\omega^2 - i\omega R + K]^{-1}$, but in many cases the dependence on ω is more complicated: in all physically realizable cases, however, the imaginary part of Y is positive (as long as we assume that the time factor is $e^{-i\omega t}$).

If Y is a constant over the surface $x = a$ and is zero over the rest of the boundary (rigid support), the Green's function is

$$G = \frac{8\pi}{b}\sum_{n=1}^{\infty}\frac{\sin(\pi n y_0/b)\sin(\pi n y/b)}{k_n\sin(k_n a + \theta_n)}\begin{cases}\sin(k_n x_0)\sin[k_n(a - x) + \theta_n]; & x > x_0\\ \sin(k_n x)\sin[k_n(a - x_0) + \theta_n]; & x < x_0\end{cases} \tag{11.2.12}$$

where $k_n^2 = (\omega/c)^2 - (\pi n/b)^2$ and $\tan\theta_n = Yk_n$. The phase angle θ is complex if Y is complex; in general it is a function of ω.

As we saw in the last section, to find the natural frequencies of vibration of the system we apply the Fourier (or Laplace) transform to this Green's function to obtain the response of the system when given an impulse at $t = 0$. The values of ω where the integrand has a pole are the "natural frequencies" and the resulting dependence on x of the residue at this pole gives the shape of the "standing wave" having this natural frequency. Indeed, as we saw in the last section, we can compute the transient response of the system to any initial conditions, even though the eigenfunctions in the resulting series are not mutually orthogonal. The "natural frequencies" are the values of ω (divided by 2π) for which the denominators of the various terms of Eq. (11.2.12) go to zero,

$$\sin(k_n a + \theta_n) = 0;\quad k_n a + \tan^{-1}(k_n Y) = \pi m \tag{11.2.13}$$

where m is a positive integer. Since both k_n and Y are functions of ω, this constitutes a transcendental equation relating ω, m, n, a, b and the physical constants of the boundary $x = a$ which enter into the expression for the boundary admittance Y. If Y is complex (*i.e.*, if the admittance is resistive as well as reactive) then the roots for ω are complex, and the time dependence of the free vibrations, $e^{-i\omega t}$, has an exponential damping factor. The value of ω which satisfies Eq. (11.2.13) for a given value of m and n can be called $\omega_{mn} = 2\pi\nu_{mn} - i\kappa_{mn}$.

Incidentally, it may be remarked that we can calculate ω if we know Y and, vice versa, we may compute Y if we know the various roots ω_{mn}. At times it is useful to be able to deduce boundary conditions by a measurement of natural frequencies and damping constants.

Variable Boundary Admittance. The Green's function form of Eq. (11.2.11) may also be used to calculate the free-field oscillations when the admittance Y for the boundary $x = a$ changes from point to point along the side, that is, Y is a function of y, as well as ω. To do this we go back to Eq. (7.1.15) relating the solution ψ of the boundary-value problem and the Green's function.

As an example of this, we can take the acoustic case instead of the membrane. The rectangular space between $x = 0$ and $x = a$, $y = 0$ and $y = b$ is filled with some fluid which is vibrating only in the x, y plane. The solution $\psi(x,y,t)$ is the velocity potential, $-$ grad ψ is the fluid velocity and $+\rho(\partial\psi/\partial t)$ the excess pressure. We assume that the three walls, $x = 0$, $y = 0$, $y = b$ are rigid so that the normal gradient of ψ at these walls is zero. Along the wall $x = a$, however, the wall is porous, or else the whole structure is not quite rigid, so that the ratio between normal velocity and pressure at this surface is not zero. We define the *normal acoustic impedance* z of this wall by the equation

$$z = \frac{\text{pressure}}{\text{normal velocity}} = \frac{i\omega\rho\psi}{(\partial\psi/\partial x)}; \quad x = a$$

if the motion is simple harmonic with frequency $\omega/2\pi$ so the time factor in ψ is $e^{-i\omega t}$. The ratio between normal gradient and value of ψ at $x = a$ is then $Y(y,\omega)$, where

$$Y(y,\omega)\psi(a,y) = (\partial\psi/\partial x)_{x=a}$$

where $Y = i\omega\rho/z$ may be called the *deflection admittance* of the wall. (It is proportional to the ratio between normal displacement and pressure at the wall.)

To solve this boundary-value problem (normal gradient equal to zero on walls $x = 0$, $y = 0$, $y = b$, equal to Y times ψ on $x = a$) we use the Green's function

$$G(x,y|x_0,y_0|\omega) = -\frac{4\pi}{b}\sum_{n=0}^{\infty}\epsilon_n\left[\frac{\cos(\pi n y_0/b)}{k_n\sin(k_n a)}\right]\cdot \cos\left(\frac{\pi n y}{b}\right)\begin{cases}\cos(k_n x_0)\cos[k_n(a-x)]; & x > x_0\\ \cos(k_n x)\cos[k_n(a-x_0)]; & x < x_0\end{cases} \tag{11.2.14}$$

where $k_n^2 = (\omega/c)^2 - (\pi n/b)^2$ as before. Since this function has zero normal gradient at all the boundary surfaces, Eq. (7.1.15) becomes

$$\psi(x,y) = \frac{1}{4\pi}\int_0^b G(x,y|a,y_0)\left[\frac{\partial}{\partial x_0}\psi(x_0,y_0)\right]_{x_0=a} dy_0$$
$$= -\frac{1}{b}\sum_{n=0}^{\infty}\epsilon_n\left[\frac{\cos(\pi ny/b)\cos(k_nx)}{k_n\sin(k_na)}\right]\int_0^b Y(y_0,\omega)\psi(a,y_0)\cos\left(\frac{\pi ny_0}{b}\right)dy_0 \tag{11.2.15}$$

which is an integral equation for ψ, at some point inside the boundary, in terms of the same ψ at the boundary where Y is not zero (*i.e.*, at $x = a$). We note that G has a discontinuity of value at this boundary.

Except for discontinuities at the boundary $x = a$, we can represent the solution ψ by the series

$$\psi(x,y) = \sum_{m=0}^{\infty} F_m(x)\cos\left(\frac{\pi my}{b}\right)e^{-i\omega t} \tag{11.2.16}$$

Insertion of this in Eq. (11.2.15) results in equations for the F's

$$F_n(x) = -\frac{\cos(k_nx)}{k_n\sin(k_na)}\sum_{m=0}^{\infty} Y_{nm}F_m(a) \tag{11.2.17}$$

where $$Y_{mn}(\omega) = \frac{\epsilon_n}{b}\int_0^b Y(y,\omega)\cos\left(\frac{\pi my}{b}\right)\cos\left(\frac{\pi ny}{b}\right)dy$$

is the *matrix element* of Y in terms of the eigenfunctions $\cos(\pi ny/b)$.

We still have not determined the resonance frequencies and damping factors of the system. In principle these are obtained by setting $x = a$ in Eq. (11.2.17),

$$\sum_{m=0}^{\infty} Y_{mn}F_m(a) + k_n\tan(k_na)F_n(a) = 0 \tag{11.2.18}$$

This can be solved for the constants $F_n(a)$ if the determinant

$$D = \begin{vmatrix} Y_{00} + k_0\tan(k_0a) & Y_{10} & Y_{20} & \cdots \\ Y_{01} & Y_{11} + k_1\tan(k_1a) & Y_{21} & \cdots \\ Y_{02} & Y_{12} & Y_{22} + k_2\tan(k_2a) & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

is equal to zero. Both Y_{mn} and $k_n\tan(k_na)$ are functions of ω. The values of ω for which this determinant is zero are the "natural frequencies" of the system and the corresponding values of the k_n's and F's allow one to calculate the corresponding eigenfunctions ψ. The impedance z may, of course, be complex, in which case the matrix components Y_{mn} will have imaginary parts as well as real parts and the roots ω will be complex. The real parts are the *natural frequencies* (times 2π) (see pages 1334 and 1338) and the imaginary parts (which must be negative for an absorptive wall) are the *damping factors*.

The procedure of solution and the interrelation between the various factors will become clearer if we compute an approximate solution for the case when b times the matrix elements Y_{mn} are small compared to unity. When all the Y's are zero, the eigenfunctions take on the simple form $\psi^0_{\sigma\tau} = \cos(\pi\sigma x/a)\cos(\pi\tau y/b)$ (σ,τ integers) and the eigenvalues are

$$\omega^0_{\sigma\tau} = c\sqrt{(\pi\sigma/a)^2 + (\pi\tau/b)^2}$$

Naturally, when the Y's are small, the ψ's and allowed values of ω differ from these by small amounts, and this may be used to guide our calculations. Chapter 9 gives details of the procedure.

For instance, we should expect that F_τ in series (11.2.16) would be large and all the other F's small. In Eqs. (11.2.18), the terms $Y_{mn}F_m(a)$, for $m \neq \tau$, would be smaller than $Y_{\tau n}F_\tau(a)$ and could be discarded as second-order terms. Consequently an approximate equation for the small F's, in terms of F_τ, would be

$$F_n(a) \simeq -\frac{Y_{\tau n}F_\tau(a)}{k_n\tan(k_n a)};\quad n \neq \tau$$

The equation to determine ω is then the one for $n = \tau$:

$$k_\tau\tan(k_\tau a) \simeq -Y_{\tau\tau} + \sum_{m\neq\tau}\frac{Y_{\tau m}Y_{m\tau}}{k_m\tan(k_m a)} \tag{11.2.19}$$

Referring to the definition for k_n and to the limiting value of the eigenvalue, $\omega^0_{\sigma\tau}$, we see that $(k_\tau a)$ should be equal to $\pi\sigma + e_{\sigma\tau}$ where σ is the other integer and e is small compared to π. The eigenvalue for ω is, of course, given by the equation

$$\omega^2_{\sigma\tau} = \pi^2c^2\left[\left(\frac{\sigma}{a} + \frac{e_{\sigma\tau}}{\pi a}\right)^2 + \left(\frac{\tau}{b}\right)^2\right]$$

and the quantities k_m in the series are, to the approximation necessary,

$$k_m \simeq \sqrt{(\pi\sigma/a)^2 + (\pi\tau/b)^2 - (\pi m/b)^2};\quad m \neq \tau$$

Therefore $k_\tau\tan(k_\tau a) \simeq (\pi\sigma e_{\sigma\tau}/a) + (e^2_{\sigma\tau}/a)$, and the solution of Eq. (11.2.19) is, to the second order in the small quantities Y_{mn},

$$e_{0\tau} \simeq \sqrt{-aY_{\tau\tau}} + \sqrt{\frac{a}{-4Y_{\tau\tau}}}\sum_{m\neq\tau}\frac{Y_{\tau m}Y_{m\tau}}{k_m\tan(k_m a)}$$

$$e_{\sigma\tau} \simeq -\frac{a}{\pi\sigma}Y_{\tau\tau} - \frac{a^2}{(\pi\sigma)^3}(Y_{\tau\tau})^2 + \frac{a}{\pi\sigma}\sum_{m\neq\tau}\frac{Y_{\tau m}Y_{m\tau}}{k_m\tan(k_m a)};\quad \sigma \neq 0$$

From this one can obtain the eigenvalues $\omega_{\sigma\tau}$ to the first or second order. To the first order in the Y's the eigenfunctions are

$$\psi \simeq F_\tau(a) \left\{ \cos\left[(\pi\sigma + e_{\sigma\tau}) \frac{x}{a} \right] \cos\left(\frac{\pi\tau y}{b}\right) - \sum_{n \neq \tau} \frac{Y_{n\tau} \cos(k_n x)}{k_n \sin(k_n a)} \cos\left(\frac{\pi n y}{b}\right) \right\} e^{-i\omega_{\sigma\tau} t} \tag{11.2.20}$$

where the constant $F_\tau(a)$ may be set equal to unity or adjusted so that ψ is normalized.

The first term in this expression adjusts the boundary conditions along the boundary $x = a$ in an average sort of way, the second term (the series) adjusts the solution to allow for the variation of Y with y, the distance along the wall. If the wall is uniformly covered so that z and Y do not depend on y, then the terms $Y_{n\tau}$ are all zero, because of the orthogonality of $\cos(\pi\tau y/b)$ with $\cos(\pi n y/b)$, and only the first term remains.

At low frequencies many walls exhibit a stiffness reactance, together with a resistive portion, so that

$$z(y,\omega) = \rho c[\gamma(y) + (i/\omega)\kappa(y)] = (i\omega\rho/Y); \quad 0 < y < b$$

where ρc is the characteristic impedance of the fluid (see page 313) so that γ and κ/ω are dimensionless. For low frequencies, κ is considerably larger than $\gamma\omega$. Therefore, the matrix components are

$$Y_{mn} \simeq \left(\frac{\omega^2}{c}\right) S_{mn} + i\left(\frac{\omega^3}{c}\right) K_{mn}$$

where $$S_{mn} = \frac{\epsilon_n}{b} \int_0^b \left[\frac{1}{\kappa(y)}\right] \cos\left(\frac{\pi n y}{b}\right) \cos\left(\frac{\pi n y}{b}\right) dy$$

and $$K_{mn} = \frac{\epsilon_n}{b} \int_0^b \left[\frac{\gamma(y)}{\kappa^2(y)}\right] \cos\left(\frac{\pi m y}{b}\right) \cos\left(\frac{\pi n y}{b}\right) dy$$

The matrix S might be called the susceptance matrix and K the conductance matrix.

To the first approximation the eigenfunction is given by Eq. (11.2.20) with the requisite values inserted for the Y's. The eigenvalues $\omega_{\sigma\tau}$, to the first approximation, are

$$\omega_{\sigma\tau} = 2\pi\nu_{\sigma\tau} - i\kappa_{\sigma\tau}; \quad \kappa_{\sigma\tau} = \frac{\epsilon_\sigma c}{2a} [\omega^0_{\sigma\tau}]^2 K_{\tau\tau}$$

$$2\pi\nu_{\sigma\tau} = \omega^0_{\sigma\tau}\left[1 - \frac{\epsilon_\sigma c}{2a} S_{\tau\tau}\right]; \quad [\omega^0_{\sigma\tau}]^2 = \pi^2 c^2 \left[\left(\frac{\sigma}{a}\right)^2 + \left(\frac{\tau}{b}\right)^2\right]$$

where $\epsilon_0 = 1$, $\epsilon_1 = \epsilon_2 = \epsilon_3 = \cdots = 2$. Each "natural frequency" ν is thus reduced slightly by the effective susceptance S. The damping factors κ, measuring the speed at which the standing wave damps out with time, are proportional to the conductance constants K, the average of the wall conductance over the wall at $x = a$, weighted by the factor $\cos^2(\pi\tau y/b)$ before averaging, as indicated by the formula defining $K_{\tau\tau}$.

Polar Coordinates. The wave equation separates in the polar coordinates r, ϕ (see page 503), the separated equations being $\psi = R(r)\Phi(\phi)e^{-i\omega t}$,

$$\frac{d^2\Phi}{d\phi^2} = -m^2\Phi; \quad \frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left[\left(\frac{\omega}{c}\right)^2 - \left(\frac{m}{r}\right)^2\right]R = 0$$

The solution of the first is sine or cosine of $m\phi$, and if ϕ has no barriers to periodicity, m must be an integer (we shall later take up a case where m is not necessarily an integer). The solution of the second is a Bessel function of order m, with argument $\omega r/c$, a linear combination of the functions $J_m(\omega r/c)$ and $N_m(\omega r/c)$ which have been discussed in Sec. 5.3. Many of the properties of these functions are given in the tables at the end of this chapter and Chap. 10. We note that the J's are finite at $r = 0$, the N's are not.

Equation (5.3.65) gives one of the most important properties of the Bessel function, its integral representation. It is not difficult to modify it to give an expression of the general form of Eq. (11.2.2) for the polar coordinate factors:

$$\begin{matrix}\cos\\ \sin\end{matrix}(m\phi)J_m(kr) = \frac{1}{2\pi i^m}\int_0^{2\pi} e^{-ikr\cos(\phi-u)} \begin{matrix}\cos\\ \sin\end{matrix}(mu)\,du \qquad (11.2.21)$$

when m is an integer. This equation is, of course, one for the coefficients of a Fourier series in ϕ, so that we also have an expression relating a plane wave solution, $e^{ikx} = e^{ikr\cos\phi}$, with the polar solutions,

$$e^{ikr\cos\phi} = e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_{m=0}^{\infty} \epsilon_m i^m \cos(m\phi)J_m(kr)$$

which has already been discussed (see page 620). We have also obtained, in Eq. (5.3.66), the useful formula

$$J_0(kR) = \sum_{m=0}^{\infty} J_m(kr_0)J_m(kr)\cos[m(\phi - \phi_0)]$$

where $R^2 = r^2 + r_0^2 - 2rr_0\cos(\phi - \phi_0)$ is the distance between the two points (r,ϕ) and (r_0,ϕ_0).

For outgoing radiation, we will use the Hankel function of the first kind

$$H_m^{(1)}(kr) = J_m(kr) + iN_m(kr) = \frac{2}{\pi i^{m+1}}\int_0^{i\infty} e^{ikr\cos u}\cos(mu)\,du \qquad (11.2.22)$$

(in many cases, when there is not likely to be confusion with the second Hankel function $H_m^{(2)} = J_m - iN_m$, for incoming waves, we will omit the superscript for $H_m^{(1)}$ and write simply H_m). The integral representation differs from that for J_m by the upper limit of integration (and a factor $2/i$). Consequently many of the formulas relating the J's may be transformed quickly to ones for the H's. For instance, the Green's function for the two-dimensional, unbounded domain (see page 811) may be expressed in terms of a series of the type of Eq. (5.3.66) [see Eq. (7.2.51)]

$$\begin{aligned} G(\mathbf{r}|\mathbf{r}_0|\omega) &= i\pi H_0(kR) \\ &= i\pi \sum_{m=0}^{\infty} \cos[m(\phi - \phi_0)] \begin{cases} J_m(kr)H_m(kr_0); & r < r_0 \\ J_m(kr_0)H_m(kr); & r > r_0 \end{cases} \end{aligned} \tag{11.2.23}$$

which, of course, could have been obtained from the general formula (7.2.63).

Waves inside a Circular Boundary. For waves inside a boundary at $r = a$, we use the Bessel functions J_m, which are finite at $r = 0$. For a circular membrane ($\psi = 0$ at $r = a$), we use a combination of

$$\begin{matrix}\cos \\ \sin\end{matrix}(m\phi) J_m(\pi\beta_{mn}r/a); \quad J_m(\pi\beta_{mn}) = 0$$

Values of some of the roots β are given at the end of the chapter. The functions $J_m(\pi\beta_{mn}r/a)$, for different n, are mutually orthogonal. Their normalizing constant is

$$\int_0^a [J_m(\pi\beta_{mn}r/a)]^2 r\,dr = \tfrac{1}{2}a^2[J_{m-1}(\pi\beta_{mn})]^2$$

where we remember, for $m = 0$, that $J_{-1}(z) = -J_1(z)$. The sine, cosine functions of ϕ are orthogonal for different m's, so that the products given above constitute a complete, orthogonal set of eigenfunctions, with which to expand any piecewise continuous function of r and ϕ for $r < a$. For instance, if a circular membrane, initially at rest, is given a velocity $v_0(r,\phi)$ at $t = 0$, the subsequent displacement of the membrane is

$$\begin{aligned} \psi &= \sum_{m,n} [A_{mn} \cos m\phi + B_{mn} \sin m\phi] J_m(\pi\beta_{mn}r/a) \sin(\omega_{mn}t) \\ \left.\begin{matrix} A_{mn} \\ B_{mn} \end{matrix}\right\} &= \frac{\epsilon_m}{\pi^2 ac\beta_{mn} J_{m-1}^2(\pi\beta_{mn})} \int_0^{2\pi} \begin{matrix}\cos \\ \sin\end{matrix}(m\phi)\,d\phi \int_0^a v_0 J_m\left(\frac{\pi\beta_{mn}r}{a}\right) r\,dr \\ \omega_{mn} &= (\pi c/a)\beta_{mn} \end{aligned} \tag{11.2.24}$$

The Green's function for a simple harmonic source of unit amplitude at (r_0,ϕ_0) is

$$G(\mathbf{r}|\mathbf{r}_0|\omega) = \frac{c^2}{\pi a^2} \sum_{m,n} \frac{\epsilon_m \cos[m(\phi - \phi_0)]}{(\omega_{mn}^2 - \omega^2) J_{m-1}^2(\pi\beta_{mn})} J_m\left(\frac{\pi\beta_{mn}r_0}{a}\right) J_m\left(\frac{\pi\beta_{mn}r}{a}\right) \tag{11.2.25}$$

which is to be compared with the expansion of Eq. (11.2.23) for the Green's function for an unbounded domain. In the present case the Hankel functions are not used because the boundary at $r = a$ reflects the outgoing waves and gives standing waves. Since the reflection at $r = a$ is "built in" to the present series, its limiting form for a very large is *not* the same as series (11.2.23). From Eq. (11.2.25), by use of the Laplace transform, we can calculate the transient motion of a membrane.

If the boundary condition at $r = a$ is that the normal gradient be zero, as for sound waves, the solutions are built up from the products

$$\frac{\cos}{\sin}(m\phi)J_m(\pi\alpha_{mn}r/a); \quad \frac{d}{d\alpha}J_m(\pi\alpha_{mn}) = 0$$

Values of the α's are also given at the end of the chapter. Formulas similar to Eqs. (11.2.24) and (11.2.25) can be set up, with α_{mn} instead of β_{mn} and with $[J_m(\pi\alpha_{mn})]^2$ instead of $[J_{m-1}(\pi\beta_{mn})]^2$. For instance, the impulse function for the velocity potential ψ arising when a unit pressure pulse is set off at (r_0,ϕ_0) at $t = 0$ is obtained by means of the Laplace transform. The pressure pulse is the function which has the G of Eq. (11.2.25) as Laplace transform (with α instead of β, etc.). Since the Laplace transform of $(1/\omega_{mn}) \sin(\omega_{mn}t)$ is $(p^2 + \omega_{mn}^2)^{-1}$, where $p = i\omega$, the pressure pulse is easy to compute. The velocity potential is $-1/p$ times the integral of this from $t = 0$ onward, so that, finally,

$$g(\mathbf{r}|\mathbf{r}_0|t) = \frac{c^2}{\pi a^2} \sum_{m,n} \epsilon_m \frac{\cos[m(\phi - \phi_0)]}{\omega_{mn}^2 J_m^2(\pi\alpha_{mn})} J_m\left(\frac{\pi\alpha_{mn}r_0}{a}\right) J_m\left(\frac{\pi\alpha_{mn}r}{a}\right) \cdot \cdot [\cos(\omega_{mn}t) - 1]$$

where $\omega_{mn} = (\pi c/a)\alpha_{mn}$. The gradient of this is the velocity.

The case where the boundary condition at $r = a$ is the more general one, $Y(\phi,\omega)\psi = \partial\psi/\partial r$, may be computed by the procedure demonstrated for the rectangular boundary.

Radiation from a Circular Boundary. When dealing with waves outside a circular boundary, we are able to use both Bessel and Neumann functions, in the combination satisfying boundary conditions at infinity. For instance if the surface of the circle is "causing" the wave motion, the waves should proceed outward, and we should use the Hankel function of the first kind $H_m(kr) = J_m(kr) + iN_m(kr)$, where we omit the superscript (1) because we are not going to need the Hankel function of the second kind $H_m^{(2)}(kr)$ (we practically never have ingoing waves). Using the notation given at the end of the chapter

$$H_m(z) = -iC_m(z)e^{i\delta_m(z)}; \quad (d/dz)H_m(z) = iC'_m(z)e^{i\delta_m'(z)}$$

we can fit the boundary conditions at $r = a$ in terms of C and δ.

For instance, suppose the circle is the cross section of a long cylinder which is vibrating in some manner so that the radial velocity of its surface at the point given by the angle ϕ is $V(\phi)e^{-i\omega t}$ [that is, the equation for the surface at time t is $r = a + (iV/\omega)e^{-i\omega t}$, where V/ω is small compared to a]. We assume that the velocity potential for the fluid outside the cylinder is

$$\psi = \sum_{m=0}^{\infty} A_m H_m(kr) \cos(m\phi) e^{-i\omega t}; \quad \omega = kc$$

if $V(\phi)$ is symmetric in ϕ, so that $V(2\pi - \phi) = V(\phi)$ (otherwise we add a sine series), and the radial velocity at the surface of the cylinder is the gradient of this, so that we have the Fourier series relationship,

$$V(\phi) = ik \sum_{m=0}^{\infty} A_m C'_m(ka) e^{i\delta_m'(ka)} \cos(m\phi)$$

(Of course we can also add a series in $\sin(m\phi)$ if V is not symmetric.) Consequently the coefficients of the series for ψ are

$$A_m = \frac{V_m e^{-i\delta_m'(ka)}}{ikC'_m(ka)}; \quad V_m = \frac{\epsilon_m}{2\pi} \int_0^{2\pi} V(\phi) \cos(m\phi)\, d\phi$$

At large distances from the cylinder the asymptotic form of the Hankel function may be used, and the solution has the following simpler form:

$$\psi \simeq \sqrt{\frac{ac^2}{\pi r k^2}}\, e^{ikr - i\omega t} f(\phi); \quad f(\phi) = \sqrt{\frac{2k}{iac^2}} \sum_{m=0}^{\infty} A_m i^{-m} \cos(m\phi) \tag{11.2.26}$$

For a simple-harmonic sound wave the intensity of the sound (see page 1353) is the product of pressure and velocity (when we use the real part of the complex expressions for both p and v). The average intensity over a time cycle is equal to one-half the real part of the product of the complex representation of p times the complex conjugate of v. For large values of kr the intensity is radial, with magnitude

$$S = (\rho c^3 a/2\pi r)F(\phi); \quad F(\phi) = |f(\phi)|^2 \tag{11.2.27}$$

$$F(\phi) = \frac{2}{c^2 ka} \sum_{m,n} \frac{V_m V_n}{C'_m C'_n} \cos[\delta'_m - \delta'_n + \tfrac{1}{2}\pi(m - n)] \cos(m\phi) \cos(n\phi)$$

where the argument ka is understood for the quantities C'_m and δ'_m. The dimensionless function $F(\phi)$ is called the *angular distribution factor*, since it is proportional to the intensity radiated in the ϕ direction from the cylinder. It indicates that the intensity distribution is, in general, quite complicated, unless all but one of the V_m's are zero. In fact, one can say that only if the radial velocity distribution around the cylinder $r = a$ is proportional to a linear combination of $\cos(m\phi)$ and $\sin(m\phi)$, with m a single integer, will the intensity have a simple dependence on angle, for all values of the radius.

The total energy radiated per second by a unit length of the vibrating cylinder is the integral of $Sr\,d\phi$ over ϕ, which is

$$P = \frac{\rho c}{k} \sum_{m=0}^{\infty} \frac{2}{\epsilon_m} \left[\frac{V_m}{C'_m}\right]^2$$

which is $(\rho c^2 a)$ times a dimensionless function of (ka) and $V(\phi)$.

Referring to the tables at the end of this chapter, we see that when $ka = \omega a/c = 2\pi a/\lambda$ is very small (*i.e.*, when the wavelength λ of the radiated wave is much longer than the perimeter of the radiating cylinder) then C'_0 and δ'_0 are much larger than the C's and δ's for $m > 0$. Therefore the long-wavelength limits of the angle dependence of wave amplitude, intensity, and total power radiated are given by Eqs. (11.2.26) and (11.2.27) with

$$f(\phi) \simeq -\sqrt{\frac{ika}{8c^2}} \int_0^{2\pi} V(\phi)\,d\phi; \quad P \simeq \tfrac{1}{8}\rho\omega a^2 \left[\int_0^{2\pi} V(\phi)\,d\phi\right]^2; \quad ka \ll 1$$

unless $V_0 = 0$. For the low frequencies, therefore, the radiation spreads out uniformly in all directions, with an amplitude proportional to the average radial velocity of the surface.

For the high-frequency limit, $ka = 2\pi a/\lambda$ is large and the wavelength λ is much smaller than the perimeter $2\pi a$. In this case a number of values of m in the summation for f of Eq. (11.2.26) are considerably smaller than (ka). For these lower values of m we have $\delta'_m \simeq ka - \frac{1}{2}\pi(m + \frac{1}{2})$ and $C'_m \simeq \sqrt{2/\pi ka}$. For values of m near (ka) this relationship breaks down and for values of m considerably larger than (ka), δ'_m is quite small and C'_m quite large. Therefore, to fairly good approximation, the series for the velocity potential is

$$\psi \simeq \frac{1}{ik}\sqrt{\frac{a}{r}}\, e^{ik(r-a)-i\omega t} \left[\sum_{m=0}^{M} V_m \cos(m\phi)\right]; \quad M \doteq ka \gg 1$$

The radial velocity of the fluid, a fairly large distance from the cylindrical surface, is thus

$$v_r = \frac{\partial\psi}{\partial r} \simeq \sqrt{\frac{a}{r}}\, e^{ik(r-a)-i\omega t} \sum_{m=0}^{M} \frac{\epsilon_m}{2\pi} \int^{2\pi} V(\alpha)\cos(m\alpha)\cos(m\phi)\,d\alpha$$

If $V(\phi)$ were not symmetrical with respect to ϕ but were a general function, this approximate solution for the asymptotic form for radial velocity, for short wavelengths, would be

$$v(r,\phi) \simeq \sqrt{\frac{a}{r}}\, e^{ik(r-a)-i\omega t} \int_0^{2\pi} V(\alpha) \left\{ \sum_{m=0}^{M} \frac{\epsilon_m}{2\pi} \cos[m(\alpha - \phi)] \right\} d\alpha$$

As we showed on page 747, this equals

$$\sqrt{\frac{a}{r}}\, e^{ik(r-a)-i\omega t} \int_0^{2\pi} V(\alpha) \left\{ \frac{\sin[(M + \frac{1}{2})(\alpha - \phi)]}{2\pi \sin[\frac{1}{2}(\alpha - \phi)]} \right\} d\alpha; \quad M \simeq ka$$

The quantity in braces approaches the delta function $\delta(\alpha - \phi)$ if M is allowed to go to infinity (*i.e.*, if ka becomes large enough). Therefore, when the wavelength $\lambda = 2\pi c/\omega = 2\pi/k$ is vanishingly small compared to $(2\pi a)$, the fluid velocity at large distances from the cylinder is

$$v(r,\phi) \rightarrow \sqrt{\frac{a}{r}}\, e^{ik(r-a)} \, [V(\phi)e^{-i\omega t}]$$

where the quantity in the brackets is the radial velocity of the surface radially "below" the spot measured.

Consequently for extremely short wavelengths the fluid velocity amplitude at the point (r,ϕ) is equal to the surface velocity amplitude at the point (a,ϕ) just "below" (r,ϕ), diminished by the factor $\sqrt{a/r}$. If the surface velocity is concentrated along an elementary line of the cylinder [*i.e.*, if $V = \delta(\phi - \phi_0)$], then the radiation will extend radially outward from this line and there will be no radiation in any other direction. This is just what would be expected of corpuscular radiation rather than waves. If ka is large but not infinite, then the quantity in braces above will have a peak at $\alpha = \phi$, but the peak will not be infinitesimally narrow; it will have a half width approximately equal to $\pi/ka = \lambda/2a$. Therefore the wave pattern at some distance from the surface will not exactly reproduce all the fine details of the velocity pattern of the cylindrical surface; there will be a "blurring" of details with angular dimensions finer than $\lambda/2a$ radians. In other words, diffraction will be noticeable.

Thus as we change relative size of wavelength and cylinder, we change from uniform radiation in all directions (long-wavelength limit) to radiation which follows the distribution of radial velocity on the surface (short-wave limit, geometrical optics). In the intermediate range, when λ is about the size of $2\pi a$, the radiated wave is complicated in form, and must be computed out from the series (11.2.27).

Scattering of Plane Wave from Cylinder. Sometimes the wave is not generated by the cylindrical surface but is generated elsewhere, the cylinder affecting the wave only by the boundary conditions at its surface. The simplest example of this sort arises when the wave is generated at infinity, resulting in a plane wave, which is scattered by the cylinder. A typical plane wave, perpendicular to the cylinder axis, is

$$\psi = Ae^{ikx-i\omega t} = Ae^{-i\omega t} \sum_{m=0}^{\infty} \epsilon_m i^m J_m(kr) \cos(m\phi)$$

The wave, by itself, does not usually fit the boundary conditions at the surface of the cylinder, $r = a$ (in fact, if it did fit them, the cylinder would be "transparent"). An additional wave must therefore be present, which will serve to fit the conditions at $r = a$. This wave, since it is caused by the presence of the cylinder, is termed the *scattered wave*. It must radiate outward from the cylinder and so must be made up of Hankel functions of the first kind, which represent waves going outward.

To each term of the plane wave expansion we must therefore add another term of the form $B_m H_m(kr) \cos(m\phi)$, such that $A\epsilon_m i^m J_m(kr) + B_m H_m(kr)$ satisfy the boundary conditions at $r = a$. If the wave is an electromagnetic plane wave with the electric vector parallel to the cylinder axis, and the cylinder is a perfect conductor, the boundary condition is that the wave function be zero at $r = a$ (see page 218). On the other hand, if the magnetic vector is parallel to the axis, or if the wave is an acoustic one and the cylinder surface is rigid, the requirement is that the normal gradient be zero at $r = a$. Intermediate cases, with value proportional to the gradient, will occur when the surface is not perfectly conductive or is not perfectly rigid.

As an example of the calculations, we shall work through the former case, where the value of the wave function is zero at $r = a$. Expressing both Bessel and Hankel functions in terms of their amplitudes and phase angles, as indicated in the tables at the end of this chapter, we have that

$$0 = A\epsilon_m i^m C_m(ka) \sin[\delta_m(ka)] - iB_m C_m(ka) e^{i\delta_m(ka)}$$

or

$$B_m = A\epsilon_m i^{m-1} e^{-i\delta_m} \sin(\delta_m)$$

where the argument, ka, of the phase angle δ_m will be understood for the rest of the discussion. Therefore the complete wave, satisfying the dual requirement that at large distances from the cylinder it consist of a plane wave in the positive x direction plus an outgoing radial wave and that it go to zero at the surface of the cylinder, has the general form

$$\Psi = Ae^{ikx-i\omega t} + \psi_s(r,\phi)e^{-i\omega t}$$

where

$$\psi_s = -iA \sum_{m=0}^{\infty} \epsilon_m e^{\frac{1}{2}m\pi i - i\delta_m} \sin(\delta_m) H_m(kr) \cos(m\phi) \tag{11.2.28}$$

is the scattered wave. The scattered wave combines with the plane wave, at places reinforcing it and at other places (such as at the surface of the cylinder) interfering with it.

To separate out the scattered wave, we may imagine the "plane wave" to be limited at the sides (by passing through a slit, for instance) so that it extends only over a region from $y = -b$ to $y = +b$, covering the cylinder completely, but not covering all space. Of course we know, from the discussion on page 227, that b must be considerably larger than the wavelength $2\pi/k$ if the wave, after passing through the slit, is to remain a plane wave, with substantially parallel wave fronts. In fact we can say that the plane wave cannot be exactly confined between the two parallel lines at $y = \pm b$, but must diverge slightly as it goes to the right, the angle of divergence being approximately $\lambda/2b = \pi/kb$, which is to be kept small.

To measure the scattered wave we confine ourselves to the region outside the lines $y = \pm b$, which means that we cannot measure ψ_s for $\phi = 0$ or $\phi = \pi$, exactly in the direction of the plane wave or opposite to it. If we go to large enough distances R away from the cylinder, we can make the angle $\varphi \simeq b/R$, at which we can just begin to separate ψ_s from the plane wave, quite small, but we can never make it zero. Consequently we can never completely compare a calculated ψ_s with measurements. If ψ_s varies reasonably smoothly with ϕ near $\phi = 0$, we can extrapolate to $\phi = 0$ with reasonable certainty, but if ψ_s varies rapidly with ϕ near $\phi = 0$, we cannot be sure whether the extrapolated measurements correspond to the computed values or not.

For large values of r, where the scattered wave is to be measured, Eq. (11.2.28) has the asymptotic form

$$\psi_s \xrightarrow[r \to \infty]{} -A \sqrt{\frac{2i}{\pi kr}}\, e^{ikr} \sum_{m=0}^{\infty} \epsilon_m e^{-i\delta_m(ka)} \sin[\delta_m(ka)] \cos(m\phi) \tag{11.2.29}$$

where the series gives the angular dependence of the scattered wave. For either electromagnetic or acoustic waves the intensity is proportional to the square of the absolute magnitude of the wave function. For instance the intensity of the plane wave part is $K|A|^2$ (where the constant K is the proportionality constant) pointed in the positive x direction. Similarly the intensity of the scattered wave for large r is K times the square of the magnitude of ψ_s given in Eq. (11.2.29), with the intensity pointed, in this case along r, away from the center. Consequently the ratio of the scattered intensity at the point (r,ϕ) (r very large) to the intensity of the incident plane wave is the square of the magnitude of ψ_s, divided by $|A|^2$:

$$S(\phi) = \frac{2}{\pi kr} \sum_{m,n} \epsilon_m \epsilon_n \sin(\delta_m) \sin(\delta_n) \cos(\delta_m - \delta_n) \cos(m\phi) \cos(n\phi) \tag{11.2.30}$$

The actual intensity at (r,ϕ) ($kr \gg 1$) is therefore $S(\phi)$ times the intensity of the incident plane wave, pointed away from the center of the cylinder.

The total energy scattered, per second, per unit incident intensity, is then the integral of $S(\phi)r\,d\phi$ over ϕ,

$$Q = \int_0^{2\pi} S(\phi)r\,d\phi = \frac{4}{k}\sum_{m=0}^{\infty} \epsilon_m \sin^2[\delta_m(ka)] \tag{11.2.31}$$

This has the dimensions of length (since k has dimension of inverse length), because it is the ratio of energy flow outward per unit length of cylinder, divided by the energy flow per unit *area* of the incident wave. To obtain the actual energy scattered per second per unit length of cylinder, we multiply Q by the intensity (energy flow per unit area) of the incident wave. It is as though the energy falling on a strip of width Q, parallel to the cylinder axis, were transferred from the incident plane wave to the scattered wave. Therefore Q is often called the *effective width* of the cylinder for scattering a plane wave of wavelength $2\pi/k$.

Scattered and Reflected Wave. To understand some aspects of the phenomenon of scattering, it is useful to rearrange our terms still further. We have just shown that the procedure of separating our solution into a pure plane wave and a scattered wave leads to comparison with measurement only for angles sufficiently different from 0 to π so that we can find the scattered wave unmixed with plane wave. For angles sufficiently close to straight ahead and straight behind, we are able to measure only the combination of plane-plus-scattered wave. In these regions the combination produces interference in some places and reinforcement in others.

The asymptotic form of the plane wave expansion, by itself, is

$$Ae^{ikx-i\omega t} \xrightarrow[r\to\infty]{} A\sqrt{\frac{1}{2\pi ikr}}\left\{e^{ikr}\sum_{m=0}^{\infty}\epsilon_m\cos[m\phi] + ie^{-ikr}\sum_{m=0}^{\infty}\epsilon_m\cos[m(\phi-\pi)]\right\}e^{-i\omega t}$$

But we have seen, a few pages ago, that

$$\sum_{m=0}^{M}\epsilon_m\cos(m\phi) = e^{-iM\phi}\left[\frac{1-e^{(2M+1)i\phi}}{1-e^{i\phi}}\right] = \frac{\sin[(M+\frac{1}{2})\phi]}{\sin(\frac{1}{2}\phi)}$$

which, for M large, approaches 2π times the delta function $\delta(\phi)$. Consequently, at very large distances from the origin, the incoming wave (with e^{-ikr}) is all coming from the left ($\phi = \pi$) and the outgoing wave (with e^{ikr}) is all going to the right ($\phi = 0$). This is, of course, what we would expect of a plane wave along the x axis.

Next let us see how the presence of the scattered wave modifies the asymptotic behavior of the combined plane-plus-scattered wave. We have

$$Ae^{ikx-i\omega t} + \psi_s e^{-i\omega t} \xrightarrow[r\to\infty]{} A\sqrt{\frac{1}{2\pi ikr}}\left\{ie^{-ikr-i\omega t}\sum_{m=0}^{\infty}\epsilon_m \cos[m(\phi-\pi)] + e^{ikr-i\omega t}\sum_{m=0}^{\infty}\epsilon_m e^{-2i\delta_m(ka)}\cos(m\phi)\right\} \quad (11.2.32)$$

In this case the incoming wave is the same, but the outgoing wave is distorted, as is evidenced by the factor $e^{-2i\delta_m}$ in each term. This, in general, destroys the coordinated reinforcement present in the plane wave, reduces the amount going parallel to the x axis and sends some out at other angles. An integration of the square of the absolute magnitude of the second series is the same as that of the first series (as long as all δ_m's are real) so that, unless the boundary conditions are absorptive and some δ_m's are complex, the outgoing energy is equal to the incoming.

When the cylinder becomes very large, it is practically a plane reflector and the wave is reflected back to the left again. We can see this by inserting the asymptotic value of $\delta_m(\to z - \frac{1}{2}\pi[m - \frac{1}{2}])$ for infinitely large values of a (a not as large as r, however!) into the second series of Eq. (11.2.32). This term then becomes

$$\frac{A}{i}\sqrt{\frac{1}{2\pi ikr}}\,e^{ik(r-2a)-i\omega t}\lim_{M\to\infty}\left\{\frac{\sin[(M+\frac{1}{2})(\phi-\pi)]}{\sin[\frac{1}{2}(\phi-\pi)]}\right\}$$

The angle factor ensures that all of the outgoing wave goes to the left again ($\phi = \pi$).

Short- and Long-wavelength Limits. When ka is large but not infinite, the outgoing wave is not all reflected backward; some of it goes on to form a plane wave, continuing to the right, with a *shadow*, a reduction in intensity, behind the cylinder, and the rest of it is reflected in other directions. To put it another way, for wavelengths short compared to $2\pi a$, the scattered wave, ψ_s, has two parts; one going to the right, which interferes with the undistorted plane wave, e^{ikr}, to form a shadow, and the rest radiating out in other directions to form the *reflected wave*.

This separation of the *scattered wave* into two parts, the *reflected* and the *shadow-forming* wave, may be seen by inserting the asymptotic value of $\delta_m(ka)$ in Eqs. (11.2.31) and (11.2.30). First, for Eq. (11.2.31), for the effective width for scattering Q, we note that δ_m is approximately equal to $ka - \frac{1}{2}\pi(m - \frac{1}{2})$ for m less than ka and is approximately zero for m greater than ka. Consequently

$$Q \to \frac{4}{k} \sum_{m=0}^{ka} \epsilon_m \sin^2[ka + \tfrac{1}{4}\pi - \tfrac{1}{2}m\pi] \simeq 4a \tag{11.2.33}$$

which is *just twice the width of the cylinder,* $2a$. This would be a quite confusing result if it were not for our previous discussion. To obtain the scattered wave, ψ_s, we have separated off an absolutely untouched plane wave. This is a natural sort of separation for long wavelengths for here the plane wave is little altered; but for short wavelengths the plane wave is not unaltered, a part behind the cylinder has been cut completely out (the shadow). Evidently a part of the scattered wave is used to interfere with the plane wave to form the shadow and the rest is reflected in other directions. Since, for real values of δ_m no energy is lost, half of the scattered wave must be the shadow-forming wave, producing a cancellation of the plane wave for a width $2a$ behind the cylinder and the other half must be the reflected wave with effective width $(2a)$ likewise. Thus what is lost from the plane wave (the shadow) turns up in the reflected wave.

To compute the details of this dual role of the scattered wave, we must substitute the asymptotic values of $\delta_m(ka)$ into Eq. (11.2.30). It turns out that a higher order of accuracy is needed to obtain all the details than was necessary for the limiting value of the effective width Q. The final result,

$$S(\phi) \simeq \left(\frac{a}{2r}\right) \sin(\tfrac{1}{2}\phi) + \left(\frac{1}{2\pi kr}\right) \cot^2(\tfrac{1}{2}\phi) \sin^2(ka\phi) \tag{11.2.34}$$

for $ka \gg 1$, will be obtained by another technique, later in this chapter. The first term represents the reflected intensity and is the intensity one would compute for geometrical optics, with elastic reflection of the "wavicles" from each portion of the half-cylinder exposed to the incident wave. The second term has a very high peak in the direction of the plane wave ($\phi = 0$) and very little intensity elsewhere. This, of course, corresponds to the shadow-forming portion of the scattered wave. For the degree of accuracy used and the geometrical assumptions involved ($r \gg a \gg \lambda$), the situation corresponds to the so-called Fraunhofer diffraction situation in optics. The Fresnel diffraction situation, closer to the cylinder, would have to be worked out by using the full series of Eq. (11.2.28), not using the asymptotic expressions for the functions $J_m(kr)$ or $H_m(kr)$.

The long-wavelength limit, $ka \ll 1$, is much simpler to compute. Here, according to the formulas at the end of this chapter, $\delta_0 \to [\pi/2 \ln(1/ka)]$ and δ_m, for $m > 0$, goes to zero as $(ka)^{2m}$. Consequently only the first term in series (11.2.30) and (11.2.31) need to be included, and

$$S(\phi) \to \left[\frac{\pi}{2kr \ln^2(1/ka)}\right]; \quad ka \to 0 \qquad Q \to [\pi^2 a/ka \ln^2(1/ka)] \tag{11.2.35}$$

for the case where ψ is to be zero at the surface of the cylinder. Here the scattered wave spreads out uniformly in all directions, there is no sharp shadow-forming peak near $\phi = 0$, and the total scattered intensity is easy to compare with measurement. We see that the effective width Q becomes quite large for large wavelengths. This is because the requirement that $\psi = 0$ at $r = a$ affects the wave no matter how small a is compared to $\lambda = 2\pi/k$, the region within which it is affected being of the order of a wavelength in diameter.

If we had taken the boundary condition that the normal gradient of ψ be zero at $r = a$, the wave would be affected much less strongly at long wavelengths. One of the problems at the end of this chapter will show that for zero gradient at $r = a$ the scattered intensity and effective scattering width Q will be given by formulas similar to (11.2.30) and (11.2.31), except that δ'_m will be substituted for δ_m. The limiting values in these cases are then

$$S(\phi) \to \left(\frac{\pi a}{8r}\right)(ka)^3(1 - 2\cos\phi)^2 \qquad Q \to \tfrac{3}{4}\pi^2 a(ka)^3; \quad ka \ll 1 \tag{11.2.36}$$

for $\partial\psi/\partial r = 0$ at $r = a$. In this case the scattered intensity is not uniform in direction and is much smaller in total amount than for the case of Eq. (11.2.35). On the other hand the short-wave limits for the two cases are essentially the same, being given approximately by Eqs. (11.2.33) and (11.2.34). Starting from the long-wavelength limit and increasing the frequency gradually (decreasing the wavelength) produces a gradual increase in complexity of the dependence of the scattered intensity S on angle ϕ. A peak gradually develops in the forward direction, but for a time it is wide enough so that measurements away from the main beam may be extrapolated in to $\phi = 0$ with fair accuracy, to compare S and Q with experiment. With still further decrease of wavelength, however, the forward peak is confined more and more to those directions ϕ near zero, where separation between incident wave and scattered wave is not complete. As a consequence the measured values of Q will tend to fall below the values predicted by Eq. (11.2.31) and in the limit the measured amount will correspond to the total *reflected* intensity, which corresponds to $\frac{1}{2}Q$, rather than total *scattered* intensity, since the forward peak in S will not be measured at all. The exact way in which this will take place will depend on the details of the method of measurement.

Scattering of Plane Wave from Knife Edge. Another scattering problem which utilizes the solutions in polar coordinates is that of scattering from the half plane $x = 0$, $y < 0$. The angle functions which fit Dirichlet or Neumann conditions are sine or cosine of $(\frac{1}{2}m\phi)$, where m is an integer. We thus arrive at an expansion of solutions of the wave equation in terms of half-integral trigonometric and Bessel functions. For instance, a function proportional to the Green's function

$$\Gamma = \tfrac{1}{2}\sum_{m=0}^{\infty} \epsilon_m \cos[\tfrac{1}{2}m(\phi - \phi_0)] \begin{cases} J_{\frac{1}{2}m}(kr)H_{\frac{1}{2}m}(kr_0); & r < r_0 \\ J_{\frac{1}{2}m}(kr_0)H_{\frac{1}{2}m}(kr); & r > r_0 \end{cases} \tag{11.2.37}$$

is analogous to Eq. (10.1.40) for Laplace's equation. In both cases the full range of ϕ is from 0 to 4π (or, if we wish, from $-\frac{1}{2}\pi$ to $\frac{7}{2}\pi$) and we shall have to explore its behavior over all this range.

Of still more interest is the "plane wave" expansion obtained by letting $\phi_0 = 0$ and $r_0 \to \infty$,

$$\sqrt{\frac{\pi i k r_0}{2}}\, e^{-ikr_0}\Gamma \to \tfrac{1}{2}\sum_{m=0}^{\infty} \epsilon_m(-i)^{\frac{1}{2}m}\cos(\tfrac{1}{2}m\phi)J_{\frac{1}{2}m}(kr) = u(r,\phi) \tag{11.2.38}$$

This function does not repeat itself before ϕ has increased by an amount 4π, rather than 2π. Therefore $u(r, \phi + 2\pi)$ is not necessarily the same as $u(r,\phi)$. The usefulness of this peculiar property was discussed in connection with Eq. (10.1.40). For a barrier present along the negative y axis, the region $-\frac{1}{2}\pi < \phi < \frac{3}{2}\pi$ is "real space" and the range $\frac{3}{2}\pi < \phi < \frac{7}{2}\pi$ is "image space," to adjust as we wish in order to fit boundary conditions at $\phi = -\frac{1}{2}\pi$ and $\phi = \frac{3}{2}\pi$.

After all, in order to fix up solutions to fit boundary conditions, by means of reflected waves or images, we need an "image space" outside the region of measurement in order to place images or sources there. When the boundary is the whole y axis and the region of interest is the part for x positive, then the negative x region is the "image space" and we fit boundary conditions along the y axis by setting up sources in image space to cancel the value or slope of the waves started at the real sources in "real space." When the boundary is only the negative half of the y axis, the whole range $-\frac{1}{2}\pi < \phi < \frac{3}{2}\pi$ is used for "real space" so we have had to add an image space in the range $\frac{3}{2}\pi < \phi < \frac{7}{2}\pi$ where we can put our image sources. At first, of course, we are using plane waves (sources a long way off) but the principle is the same. The "actual" source is at infinity for $\phi = 0$, the image source will be at infinity for $\phi = 3\pi$.

It is not difficult to see that $u(r,\phi)$ is more (or less, depending on how one looks at it) than just a plane wave. In the first place $u(r, \phi + 2\pi)$ is not equal to $u(r,\phi)$, yet the sum of the two is just a plane wave:

$$u(r,\phi) + u(r, \phi + 2\pi) = \tfrac{1}{2} \sum_{m=0}^{\infty} \epsilon_m[1 + (-1)^m](-i)^{\frac{1}{2}m} \cos(\tfrac{1}{2}\phi)J_{\frac{1}{2}m}(kr)$$
$$= \sum_{n=0}^{\infty} \epsilon_n(-i)^n \cos(n\phi)J_n(kr) = e^{-ikr\cos\phi} \quad (11.2.39)$$

according to Eq. (11.2.22). Next let us calculate the asymptotic behavior of $u(r,\phi)$ for $r \to \infty$:

$$u(r,\phi) \to \sqrt{\frac{1}{2\pi ikr}}\, [e^{ikr}F(-\tfrac{1}{2}\pi,\phi) + ie^{-ikr}F(0,\phi)]$$

where
$$F(\alpha,\phi) = \sum_{m=0}^{\infty} \epsilon_m e^{im\alpha} \cos(\tfrac{1}{2}m\phi)$$

also the asymptotic behavior of $u(r,\phi) - e^{-ikr\cos\phi} = -u(r, \phi + 2\pi)$:

$$u(r,\phi) - e^{-ikr\cos\phi} = -\sqrt{\frac{1}{2\pi ikr}}\, [e^{ikr}F(\tfrac{1}{2}\pi,\phi) + ie^{-ikr}F(\pi,\phi)]$$

The cosine series $F(\alpha,\phi)$ in these expressions are not convergent series as they stand; they come from asymptotic expansions of Bessel functions, and they should be handled with the care required of such series. In particular, we should expect a sort of Stokes' phenomenon (see page 609) to come in as ϕ goes over the whole range $0 < \phi < 4\pi$. We have, of course, looked at such series a few pages back and found they were related to delta functions.

To compute the series, we consider first the sum

$$2\sum_m e^{im\alpha+\frac{1}{2}im\phi} - 1 = \frac{2}{1 - e^{i\alpha+\frac{1}{2}i\phi}} - 1 = \frac{1 + e^{i\alpha+\frac{1}{2}i\phi}}{1 - e^{i\alpha+\frac{1}{2}i\phi}}; \quad \phi \text{ real}$$

where the imaginary part of α must be positive to ensure convergence. Combining this with the series for negative ϕ gives us

$$F(\alpha,\phi) = \sum_{m=0}^{\infty} \epsilon_m e^{im\alpha} \cos(\tfrac{1}{2}m\phi) = \frac{-i\sin\alpha}{\cos\alpha - \cos(\frac{1}{2}\phi)}; \quad \mathrm{Im}\,\alpha > 0 \quad (11.2.40)$$

By setting $\alpha = i\delta$ and letting δ approach zero we see that $F(0,\phi)$ is proportional to a periodic delta function, being zero, except exactly at $\phi = 0$, 4π, ... Similarly, by setting $\alpha = \pi + i\delta$ we see that $F(\pi,\phi)$ is zero everywhere except at $\phi = -2\pi, 2\pi$, . . . , where it is infinite like a delta function. On the other hand

$$F(\tfrac{1}{2}\pi,\phi) = \frac{i}{\cos(\frac{1}{2}\phi)}; \quad F(-\tfrac{1}{2}\pi,\phi) = \frac{-i}{\cos(\frac{1}{2}\phi)}$$

Returning to the asymptotic series for u, we see that the delta functions represent the undisturbed plane wave, for $u(r,\phi)$ has a term in e^{-ikr} times a delta function with a peak at $\phi = 0$; this corresponds to a plane wave e^{-ikx}, coming in from the right along the x axis. We check this by noting that $u(r,\phi)$ may also be written as e^{-ikx} plus a series which does not have delta function at $\phi = 0$ (but does have one at $\phi = 2\pi$, etc.). Working out details, we finally find that the function u has the following asymptotic behavior for the different ranges of ϕ:

$$\begin{aligned} u(r,\phi) &\simeq e^{-ikr\cos\phi} - \sqrt{\frac{i}{8\pi kr}}\,\frac{e^{ikr}}{\cos(\frac{1}{2}\phi)}; & -\pi < \phi < \pi \\ &\simeq -\sqrt{\frac{i}{8\pi kr}}\,\frac{e^{ikr}}{\cos(\frac{1}{2}\phi)}; & \pi < \phi < 3\pi \qquad (11.2.41) \\ &\simeq e^{-ikr\cos\phi} - \sqrt{\frac{i}{8\pi kr}}\,\frac{e^{ikr}}{\cos(\frac{1}{2}\phi)}; & 3\pi < \phi < 5\pi \end{aligned}$$

and so on, for $r \to \infty$.

Returning to the exact series for $u(r,\phi)$, we see that u, by itself, cannot satisfy boundary conditions at $\phi = -\frac{1}{2}\pi$ or $\frac{3}{2}\pi$. We can add or subtract $u(r, 3\pi - \phi)$, however, to fit Neumann or Dirichlet conditions halfway between 0 and 3π. For Neumann conditions, for instance, we take as a solution the sum, which has zero normal gradient on both sides of the barrier, $\phi = -\frac{1}{2}\pi$ and $\phi = \frac{3}{2}\pi$. The asymptotic behavior of this function in the different parts of "real space" is

$$\begin{aligned} \psi &= u(r,\phi) + u(r, 3\pi - \phi) \\ &\simeq e^{-ikr\cos\phi} + e^{ikr\cos\phi} + f(r,\phi); & -\tfrac{1}{2}\pi < \phi < 0 \\ &\simeq e^{-ikr\cos\phi} + f(r,\phi); & 0 < \phi < \pi \\ &\simeq f(r,\phi); & \pi < \phi < \tfrac{3}{2}\pi \end{aligned}$$

where $$f(r,\phi) = \sqrt{\frac{i}{8\pi kr}}\, e^{ikr}\left[\frac{1}{\sin(\frac{1}{2}\phi)} - \frac{1}{\cos(\frac{1}{2}\phi)}\right]$$

The behavior in "image space," $\frac{3}{2}\pi < \phi < \frac{7}{2}\pi$, is not of interest to us, of course, though it may easily be worked out.

The function ψ satisfies all our expectations of the physical problem: a plane wave e^{-ikx} is coming from the right to strike the knife edge; in the region $-\frac{1}{2}\pi < \phi < 0$, the wave strikes the flat surface and is reflected directly backward, $(e^{ikr\cos\phi})$; in the range $0 < \phi < \pi$, the plane wave travels unimpeded to the left; and in the "shadow region" $\pi < \phi < \frac{3}{2}\pi$, there is no plane wave. In each of these regions, however, there is a scattered wave of asymptotic intensity

$$S(\phi) = \frac{1}{8\pi kr}\left[\frac{1}{\sin(\frac{1}{2}\phi)} - \frac{1}{\cos(\frac{1}{2}\phi)}\right]^2; \quad \phi \neq 0, \pi$$

times the incident intensity radiating out from the edge. This expression, of course, holds only for very large distances from the edge. All it shows is that the knife edge will appear bright no matter from what angle ϕ it is viewed (except for $\phi = \frac{1}{2}\pi$).

Fresnel Diffraction from Knife Edge. A more accurate expression for the function u may be obtained by playing around with the exact series, (11.2.38). Differentiating with respect to ϕ, we have

$$-\frac{1}{r}\frac{\partial u}{\partial \phi} = \frac{1}{4}k\sum_{m=0}^{\infty} \epsilon_m(-i)^{\frac{1}{2}m}\left(\frac{m}{kr}\right)J_{\frac{1}{2}m}(kr)\sin(\tfrac{1}{2}m\phi)$$

Using the formulas $(2n/z)J_n(z) = [J_{n-1}(z) + J_{n+1}(z)]$

$$J_{-\frac{1}{2}}(z) = \sqrt{2/\pi z}\cos z; \quad J_{\frac{1}{2}}(z) = \sqrt{2/\pi z}\sin z$$

we can convert this into the differential equation

$$\frac{\partial u}{\partial \phi} - ikr\sin\phi\, u = -\tfrac{1}{2}kr\sqrt{\frac{2}{\pi ikr}}\,e^{ikr}\sin(\tfrac{1}{2}\phi)$$

which has as a solution

$$u(r,\phi) = \frac{e^{-ikr\cos\phi}}{\sqrt{i\pi}}\,\Phi[\sqrt{2kr}\cos(\tfrac{1}{2}\phi)] \tag{11.2.42}$$

where $\Phi(z) = \int_{-\infty}^{z} e^{it^2}\,dt$ is called the *Fresnel integral* (or, rather, the real and imaginary parts of Φ are the Fresnel integrals). The lower limit to the integral is chosen so that in the shadow region, where $\cos(\frac{1}{2}\phi)$ is negative, u goes to zero as r approaches infinity.

The solution satisfying Neumann conditions on the boundary $\phi = -\frac{1}{2}\pi, \frac{3}{2}\pi$ is therefore

$$\psi = u(r,\phi) + u(r, 3\pi - \phi) = (1/\sqrt{i\pi})\,\{e^{-ikr\cos\phi}\,\Phi[\sqrt{2kr}\cos(\tfrac{1}{2}\phi)] + e^{ikr\cos\phi}\,\Phi[-\sqrt{2kr}\sin(\tfrac{1}{2}\phi)]\}$$

The function $\Phi(z)$ is zero for z large and negative; as z is brought from $-\infty$ to zero, it swings around zero in the complex plane in a spiral, gradually widening its swing until, at $z = 0$ it moves over to swing about $\sqrt{\pi i}$ in a gradually diminishing spiral as z goes to $+\infty$. This oscillation near $z = 0$, together with the jump from oscillation about zero to oscillation about $\sqrt{\pi i}$, defines the shadow and the reflection for any value of r. For $0 < \phi < \pi$, for large r, only the first term is large and, $\psi \simeq e^{-ikr\cos\phi}$. For $-\frac{1}{2}\pi < \phi < 0$ and r large, both terms are large and we have both incident and reflected waves. As we go from $\phi = \pi - \delta$ to $\phi = \pi + \delta$, the first term drops in magnitude from about unity to near zero, dropping more rapidly the larger (kr) is. Since the second term is here negligible, the line $\phi = \pi$ represents the shadow line, below which the intensity is small, above which it is large, near which diffraction fluctuations occur.

Equation (11.2.42) illustrates, at the same time, the power and the difficulties of the integral representation technique. The closed form for u is a deceptively simple form in which to obtain an exact solution of the wave equation for these boundary conditions; and it certainly is easier to obtain limiting behavior from it than from the series expression of Eq. (11.2.38). But it is very difficult to see how one could have gone directly to the integral expression as being *the* desired solution (or even *a* solution) of the wave equation. Presumably it was originally found by analogy with the Fresnel integrals, which were obtained by considerations of interference, though they will also turn up later in connection with parabolic coordinates. Once found, of course, it is not difficult to work backward to show that is indeed a solution of the wave equation, or to show the equivalence of series and integral forms, as was done here. More will be said on this subject when we come to discuss integral equation (Green's function) techniques for solving diffraction problems.

Scattering from a Cylinder with Slit. Approximate solutions, for the scattering of waves from a cylindrical shell with slit in it, may be obtained by the methods used in obtaining Eq. (10.1.22) (see also page 1206). Suppose the cylinder is of radius a, the slit is from $\phi = -\frac{1}{2}\Delta$ to $+\frac{1}{2}\Delta$ and the boundary condition is that ψ is zero. We set up the outside solution to be a combination of the solution for a complete cylinder plus a series to fit the potential in the slot; and the inside solution the same series only with standing wave solutions (Bessel functions) instead of outgoing wave solutions (Hankel functions):

$$\psi(r,\phi) = \begin{cases} \displaystyle\sum_{m=0}^{\infty} \left[\frac{J_m(kr)}{J_m(ka)}\right] [A_m \cos(m\phi) + B_m \sin(m\phi)]e^{-i\omega t}; \quad r < a \\ \displaystyle A \sum_{m=0}^{\infty} \epsilon_n i^n [J_n(kr) - ie^{-i\delta_n} \sin \delta_n H_n(kr)] \cos[m(\phi - \alpha)]e^{-i\omega t} \\ \displaystyle\quad + \sum_{m=0}^{\infty} \left[\frac{H_m(kr)}{H_m(ka)}\right] [A_m \cos(m\phi) + B_m \sin(m\phi)]e^{-i\omega t}; \quad r > a \end{cases} \tag{11.2.43}$$

The first series for $r > a$ is the plane-plus-scattered wave of Eq. (11.2.28), which is zero at $r = a$. Consequently the two expressions, for $r < a$ and for $r > a$, are equal at $r = a$. Since, at $r = a$, ψ is zero for $\frac{1}{2}\Delta < \phi < 2\pi - \frac{1}{2}\Delta$ and is equal to $\psi(a,\phi)$ for $-\frac{1}{2}\Delta < \phi < \frac{1}{2}\Delta$, the coefficients A_m, B_m are

$$A_m = \frac{\epsilon_m}{2\pi} \int_{-\frac{1}{2}\Delta}^{\frac{1}{2}\Delta} \psi(a,w) \cos(mw)\, dw; \quad B_m = \frac{1}{\pi} \int_{-\frac{1}{2}\Delta}^{\frac{1}{2}\Delta} \psi(a,w) \sin(mw)\, dw$$

An exact solution of the problem would require the solution of the infinite set of equations for the unknowns, A_m, B_m, obtained by requiring that ψ and $\partial\psi/\partial r$ be continuous in the slit. But this is too much to solve exactly; a successive approximation technique is needed.

On page 1206 we solved a similar problem for the solution of the Laplace equation. In that case we *assumed a form* for $\psi(a,\phi)$ which was an exact solution for a slit in a plane, was approximately correct for a slit in a cylinder as long as the slit width was small compared to the circumference of the cylinder. We could do this because the potential "leaking through" the slit is small and does not extend very far on the "other side" of the slit (for $r < a$ for the cylinder, for $y < 0$ for the plane).

This technique requires modification for solutions of the wave equation. The new complication is the possibility of resonance. The penetration of the static field into a closed space is always small, dying out rapidly the farther it penetrates. Waves, however, can have resonances in a closed space and, near the resonance frequency, a large-amplitude standing wave may be set up by a very small "driving force" coming through the slit.

Returning to the static case, we saw there that the potential in the slit was $B\sqrt{1 - (2\phi/\Delta)^2}$ where B was a constant related to the gradient of the applied field. The series for the potential inside the cylinder was then $\Sigma A_m(r/a)^m \cos(m\phi)$, where

$$A_m = \left(\frac{\epsilon_m\Delta}{4\pi}\right) B \int_{-1}^{1} \cos(\tfrac{1}{2}\Delta m u)\sqrt{1-u^2}\,du = BD_m \qquad (11.2.44)$$
$$D_0 = \tfrac{1}{8}\Delta;\quad D_m = (1/m)J_1(\tfrac{1}{2}m\Delta);\quad m > 0$$

In other words we assume we know, approximately, the ratio between the A_m's, so we reduce our problem to one of solving for a *single unknown*, the quantity B. This expression for the potential came from the formula for potential in elliptic coordinates. We set the center of the slit at $\mu = \vartheta = 0$ and imagine that, near the slit, the cylinder is practically equal to a plane. The elliptic coordinates are then $(r - a) \simeq (\frac{1}{2}a\Delta)\cdot$ $\cdot \sinh\mu \sin\vartheta$, $\phi \simeq (\frac{1}{2}\Delta)\cosh\mu\cos\vartheta$, where $(a\Delta)$ is the width of the slit.

The potential which dies away exponentially for $r < a$ is then $\psi = Be^{\mu}\sin\vartheta$, which is $(4B/a\Delta)(r - a) + Be^{-\mu}\sin\vartheta$ for $r > a(\mu > 0)$ and which dies out exponentially for $r < a(\mu > 0)$. As we saw above, the series expansion $B\Sigma D_m\left(\dfrac{r}{a}\right)^m \cos(m\phi)$ is equal to $Be^{\mu}\sin\vartheta$ in the slit ($\mu = 0$), is approximately equal to $Be^{\mu}\sin\vartheta$ close to the slit, and also satisfies the boundary conditions $\psi = 0$ for $r = a$ far from the slit, where we could not expect $Be^{\mu}\sin\vartheta$ to be a valid solution. It is therefore a good approximation to the correct solution for the static case, as soon as we adjust B to fit the radial gradient of the applied field at the slit. This was done on page 1192.

In the present case, we can take ψ again to have the form

$$V\sqrt{1-\left(\frac{2\phi}{\Delta}\right)^2} = V\sin\vartheta$$

within the slit ($\mu = 0$), but we cannot consider the solution to be just $Ve^{\mu}\sin\vartheta$. There is the possibility that the potential is also large inside: in other words, we must assume that $\psi = \frac{1}{2}(V + R)e^{\mu}\sin\vartheta + \frac{1}{2}(V - R)e^{-\mu}\sin\vartheta$, which is still equal to $V\sin\vartheta$ at $\mu = 0$ but which approaches $(2/a\Delta)(V + R)(r - a)$ some distance outside the slit and which approaches the function $(2/a\Delta)(R - V)(r - a)$ at points, inside the cylinder, not too close to the slit yet not too far away [*i.e.*, distance greater than $(a\Delta)$ but smaller than a].

We can distinguish between the two cases by computing the radial gradient of the potential in the slit. For the static case this is

$$\frac{2}{a\Delta\sin\vartheta}\left[\frac{\partial}{\partial\mu}(Ve^{\mu}\sin\vartheta)\right]_{\mu=0} = \left(\frac{2V}{a\Delta}\right) \simeq \frac{V}{a}\sum_m mD_m\cos(m\phi)$$

where the series is obtained by differentiating the series for ψ for $r < a$. This series has very poor convergence, but we know from the analysis above that it is nearly equal to $2V/a\Delta$ for $(-\frac{1}{2}\Delta < \phi < \frac{1}{2}\Delta)$ and is zero for the other values of ϕ.

For the nonstatic case the ψ, for $r < a$, involves Bessel functions, as shown in Eq. (11.2.43), and not just powers of r. The actual radial slope at $r = a$ is

$$-\left(\frac{V}{a}\right)\sum_m D_m\tan(\alpha_m)\cos(m\phi);\quad \tan(\alpha_m) = -\frac{a}{J_m(ka)}\left[\frac{d}{dr}J_m(kr)\right]_{r=a}$$

which differs from the static value $2V/a\Delta$ by the amount

$$-\frac{V}{a}\sum_m D_m[m + \tan\alpha_m]\cos(m\phi)$$

Reference to the tables at the end of this chapter indicates that $\tan(\alpha_m) \to -m$ for $m \gg ka$, so that this series converges much more rapidly than either series for slopes.

We thus have that the radial slope in the slit ($r = a$, $\phi = 0$) is

$$v_i = \left(\frac{2V}{a\Delta}\right)\left\{1 - \tfrac{1}{2}\Delta\sum_{m=0}^{\infty} D_m[m + \tan\alpha_m]\right\}$$

This value of the radial slope in the slit has been obtained by use of the interior solution, which we can call ψ_i. We could, of course, obtain it by adding the radial slope v_f of the applied plane-plus-scattered wave to the slope v_0 of the last series of Eq. (11.2.43). The slope of the applied field at the center of the slit is

$$v_f = Ak \sum_{m=0}^{\infty} \epsilon_m i^m \left[J'_m(ka) - \frac{J_m(ka)}{H_m(ka)} H'_m(ka) \right]$$

where the primes indicate differentiation with respect to the argument (ka) and where we have substituted back for δ_m in terms of J_m and H_m. This series can be simplified by using the expression for the Wronskian for the Bessel functions,

$$\Delta(J_m,H_m) = J_m(ka)H_{m-1}(ka) - J_{m-1}(ka)H_m(ka) = 2i/\pi ka \quad (11.2.45)$$

The radial gradient of the applied field at the center of the slit is therefore $v_f = (2A/\pi a)F(\alpha)$, where

$$F(\alpha) = \sum_{m=0}^{\infty} \epsilon_m i^m \left[\frac{e^{-i\delta_m(ka)}}{C_m(ka)} \right] \cos(m\alpha) \quad (11.2.46)$$

The negative gradient of the external driven field at $r = a$, $\phi = 0$ is

$$v_0 = -Vk \sum_{m=0}^{\infty} D_m \left[\frac{H'_m(ka)}{H_m(ka)} \right]$$

and the equation expressing the continuity of slope at the slit is $v_i = v_f - v_0$, or $v_f = v_i + v_0$, which serves to determine the value of the unknown constant V, the value of ψ at $r = a$, $\phi = 0$.

$$v_i + v_0 = Vk \sum_{m=0}^{\infty} D_m \left[\frac{J'_m(ka)}{J_m(ka)} - \frac{H'_m(ka)}{H_m(ka)} \right] = -Vk \sum_{m=0}^{\infty} D_m \frac{\Delta(J_m,H_m)}{J_m(ka)H_m(ka)}$$

$$= \frac{2V}{\pi a} \sum_{m=0}^{\infty} D_m \frac{e^{-i\delta_m(ka)}}{[C_m(ka)]^2 \sin[\delta_m(ka)]} = \frac{2V}{\pi a \Delta} Y(\Delta,ka)$$

$$Y(\Delta,ka) = \tfrac{1}{8} \frac{\Delta e^{-i\delta_0}}{C_0^2 \sin \delta_0} + \sum_{m=1}^{\infty} \frac{\Delta e^{-i\delta_m}}{C_m^2 \sin \delta_m} \frac{J_1(\tfrac{1}{2}m\Delta)}{m}$$

This series also does not converge well. As can be seen from the tables at the end of this chapter, when m is larger than (ka), we have

$$\frac{e^{-i\delta_m}}{C_m^2 \sin \delta_m} \to \frac{\pi^2(ka/2)^{2m}(m!)^2}{[(m-1)!]^2 \pi m (ka/2)^{2m}} = \pi m$$

so that the limiting form of the sum is $\left(\frac{2V}{\pi a}\right)\sum \pi m D_m$. However, we have seen earlier that $\Sigma m D_m \simeq \frac{2}{\Delta}$, so by subtracting and adding we obtain the much more convergent expression

$$Y(\Delta,ka) = 2\pi + \Delta^2\left\{\frac{1}{8}\frac{e^{-i\delta_0}}{C_0^2 \sin\delta_0} + \sum_{m=1}^{\infty}\left[\frac{e^{-i\delta_m}}{C_m^2\sin\delta_m} - \pi m\right]\frac{J_1(\frac{1}{2}m\Delta)}{m\Delta}\right\} \tag{11.2.47}$$

But $v_i + v_0$ must equal v_f in order to have continuity of slope at the slit. Therefore the equation

$$V = A\Delta[F(\alpha)/Y(\Delta,ka)]$$

expresses the amplitude of the field in the slit in terms of the slope of the applied field and the "admittance" Y of the interior and exterior driven fields.

This admittance goes to infinity for those values of (ka) for which $\sin\delta_m$, for some m, is zero. This is the case when $J_m(ka) = 0$, in other words when there would be *resonance* inside the cylinder if the cylinder were closed and the field inside were "driven" from inside. At these frequencies V, the field in the slot, is zero, as indeed it must be. Therefore at the frequencies of internal resonance the cylinder behaves as though it had *no slot*. Close to one of these resonance points, the admittance

$$Y \to \begin{cases} \left[\dfrac{e^{-i\delta_0}}{C_0}\right]\left[\dfrac{\Delta^2/8}{J_0(ka)}\right] & J_0(ka)\to 0 \\ \left[\dfrac{J_1(\frac{1}{2}m\Delta)}{m\Delta}\right]\left[\dfrac{e^{-i\delta_m}}{C_m}\right]\left[\dfrac{\Delta^2}{J_m(ka)}\right]; & J_m(ka)\to 0 \end{cases}$$

We have thus obtained approximate expressions for $\psi(a,\phi)$, $V\sqrt{1-(2\phi/\Delta)^2}$ with V as given below Eq. (11.2.47), to insert in Eq. (11.2.43) to compute the scattering and the resonance inside the cylinder. As was indicated in Eq. (11.2.44), the coefficients A_m are equal to VD_m, and the coefficients B_m are zero. The wave potential at the center of the cylinder is then

$$\psi(0) \simeq \left[\frac{A\Delta^2}{8J_0(ka)}\right]\left[\frac{F(\alpha)}{Y(\Delta,ka)}\right]e^{-i\omega t}$$

which is ordinarily small because of the Δ^2 factor. When there is resonance for the $m = 0$ standing waves, when $J_0(ka) = 0$, $\psi(0)$ is large but not infinite, however, for as $J_0(ka)$ goes to zero, Y becomes infinite. Taking the limiting values, $\psi(0) \to iH_0(ka)AF(\alpha)$ when $J_0(ka) = 0$, which is larger than the usual form by the factor $1/\Delta^2$. At these frequencies the correct solution outside the cylinder is the plane-plus-scattered wave for the slitless cylinder.

Of course when the frequency is such that $J_m(ka) = 0$ for $m \neq 0$, the function Y becomes infinite and since, here, $J_0(ka)$ does not at the same time become zero, the amplitude of the wave potential at $r = 0$ becomes zero. This is just because the other standing wave, for $m \neq 0$, which is excited, has zero amplitude at $r = 0$; at other positions in the cylinder this other wave would be detected. At each resonance frequency, each root of $J_m(2\pi\nu a/c) = 0$, the standing wave inside the cylinder has the form of a single eigenfunction, the other waves having zero amplitude at that frequency.

The scattered wave, per unit amplitude of the incident wave ($A = 1$), is

$$\psi_s \xrightarrow[r\to\infty]{} -\sqrt{\frac{2i}{\pi kr}}\, e^{ikr-i\omega t} \sum_{m=0}^{\infty} \epsilon_m e^{-i\delta_m} \left\{ \sin\delta_m \cos[m(\phi-\alpha)] - \left[\frac{\Delta J_1(\frac{1}{2}m\Delta)}{2mi^m C_m Y(\Delta,ka)}\right] F(\alpha)\cos(m\phi) \right\} \qquad (11.2.48)$$

which is to be compared with Eq. (11.2.29). The effective width of the slotted cylinder for scattering is

$$Q = \frac{4}{k}\sum_{m=0}^{\infty} \epsilon_m \left\{ \sin^2\delta_m - \left[\frac{\sin\delta_m \cos(\frac{1}{2}m\pi) J_1(\frac{1}{2}m\Delta)}{mC_m Y(\Delta,ka)}\right] F(\alpha)\cos(m\alpha) + \left[\frac{\Delta J_1(\frac{1}{2}m\Delta)}{2mC_m}\right]^2 \frac{F^2(\alpha)}{Y^2(\Delta,ka)} \right\} \qquad (11.2.49)$$

where the functions $F(\alpha)$ and $Y(\Delta,ka)$ are given in Eqs. (11.2.46) and (11.2.47). These formulas are for the boundary condition $\psi = 0$ at $r = a$, $\frac{1}{2}\Delta < \phi < 2\pi - \frac{1}{2}\Delta$ and for a unit plane wave incident at an angle α with respect to the axis of the slit ($\phi = 0$). We make the convention that $(\Delta/m)J_1(\frac{1}{2}m\Delta) = \frac{1}{4}\Delta^2$ when $m = 0$.

We note again that, when $J_m(ka) = 0$, the interior wave is large and there is no external evidence of the slit. There is no infinite response, because we have taken account, not only of the resonant load inside the cylinder, but of the radiation load outside, and the radiation admittance is never zero.

Slotted Cylinder, Neumann Conditions. The case for normal gradient zero at the cylinder differs from the one just discussed chiefly in the fact that the limiting case of potential function ($k \to 0$) does not give a solution involving penetration within the cylinder. The case of Neumann conditions corresponds to fluid motion and that for $k = 0$ that of incompressible fluid flow. We certainly cannot have a steady-state flow of an incompressible fluid into or out of a slit without having some complicated circulatory motion, which we rule out when we say we wish to solve for potential-directed flow. As soon as we consider oscillatory flow of an elastic fluid, of course, we can have motion inside the cylinder, indeed we can have resonance. But we are deprived of the aid of a steady-flow solution.

Nevertheless one can use some of the results of the study of solutions of Laplace's equation in elliptic coordinates. For a slit of width $a\Delta$ in a plane barrier, steady flow through the slit is given by the velocity potential $\psi = \kappa + \frac{1}{2}(a\Delta)v_0\mu$ where the elliptic coordinates μ, ϑ are defined on page 1195.

Here ψ is uniform across the slit ($\mu = 0$), but the velocity of the fluid through the slit is not uniform across the slit.

$$v = \frac{2}{(a\Delta)\sin\vartheta}\left[\frac{\partial\psi}{\partial\mu}\right]_{\mu=0} = \frac{v_0}{\sin\vartheta} = \frac{v_0}{\sqrt{1-(2\phi/\Delta)^2}} \qquad (11.2.50)$$

having a minimum value v_0 at the middle of the slit ($\phi = 0$) and rising to infinity next to both edges, where the fluid is flowing around the sharp edges. As long as the slit width is small compared to the cylinder circumference ($2\pi a$) and also to the wavelength $2\pi/k = 2\pi c/\omega$, we can expect the distribution of potential ψ and distribution of velocity v through the slit to conform fairly closely to these expressions, even though we cannot expect the behavior of the potential and velocity very far away from the slit, to conform.

Accordingly, we can write the velocity potential for wave motion to be

$$\psi(r,\phi) = \begin{cases} \displaystyle\sum_{m=0}^{\infty} A_m\left[\frac{J_m(kr)}{kJ'_m(ka)}\right]\cos(m\phi)e^{-i\omega t}; \quad r < a \\ \displaystyle\sum_{m=0}^{\infty} \epsilon_m i^m\left[J_m(kr) - \frac{J'_m(ka)}{H'_m(ka)}H_m(kr)\right]\cos[m(\phi-\alpha)]e^{-i\omega t} \\ \qquad + \displaystyle\sum_{m=0}^{\infty} A_m[H_m(kr)/kH'_m(ka)]\cos(m\phi)e^{-i\omega t}; \quad r > a \end{cases} \qquad (11.2.51)$$

where, to fit the simple distribution of velocity, we have

$$A_m = \frac{v_0\epsilon_m}{2\pi}\int_{-\frac{1}{2}\Delta}^{\frac{1}{2}\Delta}\frac{\cos(mu)\,du}{\sqrt{1-(2u/\Delta)^2}} = \tfrac{1}{4}v_0\,\Delta\epsilon_m\,J_0(\tfrac{1}{2}m\Delta)$$

Parenthetically, we can point out that the series of Eq. (11.2.51) may be considered to be the eigenfunction series for fitting the boundary condition at $r = a$, or it may be considered as the Green's function solution corresponding to the requirement that at $r = a$, $(\partial/\partial a)\psi(a,\phi) = 0$ for

$\frac{1}{2}\Delta < \phi < 2\pi - \frac{1}{2}\Delta$ and is $v_0/\sqrt{1 - (2\phi/\Delta)^2}$ for $-\frac{1}{2}\Delta < \phi < \frac{1}{2}\Delta$. For instance, for the interior Green's function satisfying Neumann conditions at $r = a$, we have, from Eq. (7.2.51),

$$G(r,\phi|r_0,\phi_0) = i\pi \sum_{m=0}^{\infty} \epsilon_m \cos[m(\phi - \phi_0)] J_m(kr) \left\{ H_m(kr_0) - \frac{H'_m(ka) J_m(kr_0)}{J'_m(ka)} \right\}; \quad r < r_0$$

The interior solution, fitting the requirement mentioned, is according to Eq. (7.2.10),

$$\psi(r,\psi) = \frac{a}{4\pi} \int_{-\frac{1}{2}\Delta}^{\frac{1}{2}\Delta} \frac{v_0}{\sqrt{1 - (2\phi_0/\Delta)^2}} G(r,\phi|a,\phi_0)\, d\phi_0$$

whence, using the expression for the Wronskian, $\Delta(J_m,H_m)$, we obtain the first part of Eq. (11.2.51). What we have done by this approximation is to substitute for the infinity of unknowns A_m a single unknown v_0, which can, of course, be more easily determined.

We solve for the magnitude of v_0 compared to A by balancing impedances. The pressure in the fluid is related to the velocity potential ψ by the relation $p = -i\omega\rho\psi$, whereas the velocity is $-\text{grad}\,\psi$. The ratio of pressure to velocity amplitude is called the *acoustic impedance* (see pages 310 and 1367). The ratio of ψ to v may be called the *field impedance;* for constant frequency it is proportional to the acoustic impedance and is somewhat simpler for us to use here. The magnitude of the plane-plus-scattered wave at the center of the slit,

$$\psi_f = A \sum_{m=0}^{\infty} \epsilon_m i^m [J_m(ka) H'_m(ka) - J'_m(ka) H_m(ka)] \frac{\cos(m\alpha)}{H'_m(ka)}$$
$$= 2A\, F'(\alpha); \quad F'(\alpha) = \sum_{m=0}^{\infty} \epsilon_m i^m [e^{-i\delta_m'}/\pi k a C'_m] \cos(m\alpha) \qquad (11.2.52)$$

may be considered to be the "driving force." It drives the standing wave inside the cylinder and also the extra scattered wave outside the cylinder, and to determine the amplitude of the driven motion (the velocity v_0) we first find the field impedance of the interior and exterior waves.

The field impedance of the interior standing wave is the ratio of the interior ψ at the center of the slit to v_0, the corresponding inward velocity there,

$$z_i = \tfrac{1}{4}\Delta a \sum_{m=0}^{\infty} \epsilon_m \left[\frac{J_m(ka)}{kaJ'_m(ka)}\right] J_0(\tfrac{1}{2}m\Delta)$$
$$= -\frac{a}{2\pi} \sum_{m=0}^{\infty} \epsilon_m \cot(\alpha_m) \int_{-\frac{1}{2}\Delta}^{\frac{1}{2}\Delta} \frac{\cos(mu)}{\sqrt{1-(2u/\Delta)^2}}\, du$$

This series is very poorly convergent, and a few dodges are needed to simplify the calculations. We note from the tables at the end of this chapter, that $\cot \alpha_m \to -(1/m)$ for $m \gg ka$. But, by simple limiting processes, we can show that

$$\sum_{m=1}^{\infty} \frac{a}{\pi m} \cos(mu) = -\frac{a}{\pi} \mathrm{Re}[\ln(1-e^{iu})] = -\frac{a}{2\pi} \ln[2 - 2\cos(u)]$$
$$\simeq -(a/\pi) \ln|u| \quad \text{for } u \ll \pi$$

Therefore

$$\frac{a}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \int_{-\frac{1}{2}\Delta}^{\frac{1}{2}\Delta} \frac{\cos(mu)}{\sqrt{1-(2u/\Delta)^2}}\, du \simeq -\frac{\Delta a}{2\pi} \int_{-1}^{1} \frac{\ln(\frac{1}{2}|w|\Delta)}{\sqrt{1-w^2}}\, dw$$
$$= -\frac{a\Delta}{\pi} \int_0^{\frac{1}{2}\pi} \ln(\tfrac{1}{2}\Delta\cos z)\, dz = \tfrac{1}{2}a\Delta \ln(4/\Delta)$$

and
$$z_i \simeq \tfrac{1}{2}a\Delta[\ln(4/\Delta) - \tfrac{1}{2}f'(\Delta,ka)] \tag{11.2.53}$$

where
$$f'(\Delta,ka) = \cot(\alpha_0) + 2\sum_{m=1}^{\infty} \left[\frac{1}{m} + \cot(\alpha_m)\right] J_0(\tfrac{1}{2}m\Delta)$$

where the series for f' converges fairly rapidly. The actual acoustic impedance is, of course, $i\omega\rho z_i$, a pure reactance. We note that f' is a function of Δ but also of (ka), since the angles α_m are functions of (ka). These angles are tabulated at the back of the book. Because of the terms $\cot(\alpha_m)$, the impedance becomes infinite for those values of frequency at which $J'_m(ka) = 0$.

The field impedance of the extra external wave, at the center of the slit, is the extra exterior field divided by $-v_0$,

$$z_0 = -\frac{a}{2\pi} \sum \epsilon_m \left[\frac{H_m(ka)}{kaH'_m(ka)}\right] \int_{-\frac{1}{2}\Delta}^{\frac{1}{2}\Delta} \frac{\cos(mu)}{\sqrt{1-(2u/\Delta)^2}}\, du$$
$$= \tfrac{1}{2}a\Delta \left\{\ln\left(\frac{4}{\Delta}\right) + \left[\frac{e^{i(\delta_0-\delta_0')}}{2ka}\right]\left(\frac{C_0}{C'_0}\right) + \sum_{m=1}^{\infty} \left[\frac{e^{i(\delta_m-\delta_m')}}{2ka}\left(\frac{C_m}{C'_m}\right) - \frac{1}{m}\right] J_0(\tfrac{1}{2}m\Delta)\right\}$$

The acoustic impedance $(i\omega\rho z_0)$, unlike $(i\omega\rho z_i)$, is not purely imaginary, but has some real terms, the *radiation resistance*. The sum of the two impedances is the total impedance opposing the "driving force" ψ_f,

$$z_0 + z_i = -\frac{a}{4}\sum_{m=0}^{\infty} \epsilon_m \left[\frac{H_m(ka)}{kaH'_m(ka)} - \frac{J_m(ka)}{kaJ'_m(ka)}\right] J_0(\tfrac{1}{2}m\Delta) = 2a\Delta\, Z(\Delta,ka)$$

where
$$Z(\Delta,ka) = -\frac{1}{4\pi(ka)^2}\sum_{m=0}^{\infty} \frac{\epsilon_m e^{-i\delta_m'} J_0(\frac{1}{2}m\Delta)}{(C'_m)^2 \sin(\delta'_m)}$$

This also converges poorly, for large m the series acts like

$$\frac{1}{4\pi(ka)^2}\sum_{m=1}^{\infty} \epsilon_m \left[\frac{4\pi^2(ka/2)^{2m+2}}{(m!)^2}\right]\left(\frac{(m!)^2}{\pi m(ka/2)^{2m}}\right)\left(\frac{2}{\pi\Delta}\right)\int_{-\frac{1}{2}\Delta}^{\frac{1}{2}\Delta} \frac{\cos(mu)\,du}{\sqrt{1-(2u/\Delta)^2}}$$
$$= \frac{1}{\pi\Delta}\int \sum \left(\frac{1}{m}\right)\cos(mu)\,\frac{du}{\sqrt{1-(2u/\Delta)^2}} \simeq \tfrac{1}{2}\ln\left(\frac{4}{\Delta}\right)$$

Consequently we can write

$$Z(\Delta,ka) = \tfrac{1}{2}\ln\left(\frac{4}{\Delta}\right) - \frac{e^{-i\delta_0'}/k^2a^2}{4\pi(C'_0)^2 \sin\delta'_0}$$
$$- \tfrac{1}{2}(ka)^{-2}\sum_{m=1}^{\infty}\left[\frac{e^{-i\delta_m'}}{\pi(kaC'_m)^2 \sin\delta'_m} + \frac{1}{m}\right] J_0(\tfrac{1}{2}m\Delta) \qquad (11.2.54)$$

This is proportional to the impedance of the motion of the fluid through the slit, including the reactive load of the air inside the cylinder and the partly resistive load of the radiation field outside the cylinder. Comparison with Eq. (11.2.47) shows that the ratio between the first term and the rest is larger here, when Δ is small, than for the previous case. This arises because the penetration of a potential through a slit is much greater when the boundary conditions are zero gradient than when they are zero value.

We can now compute the fluid velocity at the center of the slit:

$$v_0 \simeq \frac{\psi}{z_0 + z_i} = \frac{A}{a\Delta}\,\frac{F'(\alpha)}{Z(\Delta,ka)}$$

which can now be inserted back in Eq. (11.2.51) to obtain the velocity potential inside and outside the cylinder. The value of the velocity potential in the slit is obtained from the equation

$$\psi_i = v_0 z_i = \psi_f\left[\frac{z_i}{(z_0 + z_i)}\right]$$
$$\psi_i \simeq \frac{AF'(\alpha)}{2Z(\Delta,ka)}\left[\ln\left(\frac{4}{\Delta}\right) - \tfrac{1}{2}f'(\Delta,ka)\right] \qquad (11.2.55)$$

for $-\frac{1}{2}\Delta < \phi < \frac{1}{2}\Delta$.

When there is resonance inside the cylinder, $J'_m(ka) = 0$ for some m, so that $\sin \delta'_m$ is zero and z_i is infinite. Consequently the velocity through the slit, v_0, is zero at the resonance frequencies (as long as there is no energy absorption) and the potential in the slit is then, of course, just ψ_f.

The expressions for scattered wave and effective width of cylinder for scattering are quite similar to those given in Eq. (11.2.49). In this case, also, Q never becomes infinite, for $Z(\Delta,ka)$ never becomes zero, being complex, though the two additional terms in the series for Q may become zero, when $Z(\Delta,ka)$ becomes infinite (when $\sin \delta'_m = 0$).

The relation between the two cases may become clearer when viewed from a different standpoint. The values of the wave functions at the slit may be considered as analogues to voltages, the normal gradients through the slit as currents. In the present case the driving force is a voltage, ψ_f; the plane-plus-scattered wave has no gradient at the slit. The gradient of the inner function, ψ_i, must equal that of the outer driven function, ψ_0, both being equal to v_0. For continuity in value, however, we must have $\psi_i = \psi_f + \psi_0$. If we consider ψ_i to be the voltage across the interior load and $-\psi_0$ that across the exterior load, the equations are those of a series circuit; the voltages across the two loads add, the current is common, and the forcing voltage is applied across both loads. Naturally when either load has infinite impedance, the current is zero, but when either load has zero impedance, the current is not infinite, nor will the current become infinite unless the two impedances just cancel each other at some frequency, which does not usually happen. The external load does not have ups and downs of value as frequency is changed, but the internal load does fluctuate considerably, being infinite at the internal resonance frequencies and being zero at intermediate frequencies. The largest value of v_0, and therefore the largest amount of additional scattered wave, comes at these intermediate frequencies.

Returning to the earlier case, for Dirichlet conditions, we see the complete reverse. Here the "driving force" is a constant current generator, of magnitude v_f, and for continuity the internal slope (or current) v_i must equal the sum of the external ones, $v_f - v_0$, so that the forcing current v_f divides into two parts, v_i and v_0. On the other hand the value of the potential (the voltage) at the slit is V, common to both interior and exterior. This situation is typical of a shunt circuit, a constant current generator driving the internal and external load in parallel. The admittance of the whole is the sum of the two admittances, and the voltage V is the ratio of v_f to the sum of the admittances. Whenever the internal arm has a resonance, when $J'_m(ka) = 0$, V is zero and the scattered wave is the same as if there were no slit. For other frequencies the additional wave generated by the fluid motion through the slit reinforces and, in places, interferes with the usual scattered wave.

A few numerical calculations will suffice to show the difference between the two cases. For instance, when $(ka) = 2\pi a/\lambda$ is quite small we have for the two cases:

Case I: ($\psi = 0$ at $r = a$, $\frac{1}{2}\Delta < \phi < 2\pi - \frac{1}{2}\Delta$, incident plane wave amplitude A at angle α).

Potential in slit:

$$(r = a,\ -\tfrac{1}{2}\Delta < \phi < \tfrac{1}{2}\Delta) \simeq -[A\Delta/4\ln(ka)]\sqrt{1 - (2\phi/\Delta)^2}$$

Gradient in slit: $v_i \simeq -[A/2a\ln(ka)]$

Potential at center of cylinder: $\psi(0) \simeq -[A\Delta^2/32\ln(ka)]$

Effective width for scattering: $Q \simeq \dfrac{\pi^2 a}{ka\ln^2(ka)}\left[1 + \dfrac{\Delta^2}{16\ln(ka)}\right]$

Case II: ($\partial\psi/\partial r = 0$ at $r = a$, $\frac{1}{2}\Delta < \phi < 2\pi - \frac{1}{2}\Delta$)

Potential in slit: $\psi_i \simeq [\pi kaA/2\ln(4/\Delta)]\,[\ln(4/\Delta) - (1/ka)^2]$

Gradient in slit $\simeq \left[\dfrac{\pi Aka}{2\Delta a\ln(4/\Delta)}\right]\dfrac{1}{\sqrt{1 - (2\phi/\Delta)^2}}$

Potential at center of cylinder: $\psi(0) \simeq -[A\pi/2ka\ln(4/\Delta)]$

Effective width for scattering: $Q \simeq \pi^2(ka)^3\left[1 - \dfrac{\pi}{ka\ln(4/\Delta)}\right]$

These quantities are functions of two small quantities, (ka) and Δ. As functions of Δ, as Δ goes to zero, the quantities in case I go to zero more rapidly than those of case II. This is because a field satisfying Dirichlet conditions penetrates through a slit much less easily than one satisfying Neumann conditions. The central potential $\psi(0)$ for case II goes to infinity when (ka) goes to zero, but we have seen that case II has its difficulties for the static case. For any finite (but small) value of (ka), for Δ small, we see that the first term of Q (that due to the cylinder without slit) is smaller for case II than for case I, but that the second term in the brackets is smaller for case I than for case II. A cylinder requiring zero gradients at its surface disturbs long waves less than does a cylinder requiring zero value, but a slit in the former makes a greater change in the scattering than it does with the latter.

Problems of this general type, where solutions must be joined in a gap in a boundary, will be solved by integral equation and variational techniques in Sec. 11.4.

Waves in Parabolic Coordinates. Referring to Eq. (5.1.10), we see that in the parabolic coordinates

$$x = \tfrac{1}{2}(\lambda^2 - \mu^2);\quad y = \lambda\mu;\quad h_\mu = h_\nu = \sqrt{\mu^2 + \lambda^2} = \sqrt{2r}$$
$$\lambda = \sqrt{r + x};\quad \mu = \sqrt{r - x}$$

the Helmholtz equation $\nabla^2\psi + k^2\psi = 0$ separates into

$$\psi = M(\mu)L(\lambda);\quad (d^2M/d\mu^2) + (ka + k^2\mu^2)M = 0$$
$$(d^2L/d\lambda^2) + (-ka + k^2\lambda^2)L = 0$$

[see Eqs. (10.1.36) *et seq.* for a discussion of the Laplace equation for these coordinates]. The solutions for both μ and λ factors may be expressed in terms of the functions $H(a,\mu\sqrt{k})$ and $H(-a,\lambda\sqrt{k})$, where $H(a,x)$ is a solution of the equation

$$(d^2H/dx^2) + (a + x^2)H = 0; \quad x^2 = k(\lambda^2,\mu^2) \tag{11.2.56}$$

This equation has an irregular singular point at infinity, and no singular point in the finite plane. The singularity at infinity, however, is of a higher species than that for the simple exponential of x. The basic fundamental set for $x = 0$ is a symmetric function, $H_e(a,x)$, represented by a series of even powers of x, and an antisymmetric function $H_0(a,x)$, represented by a series of odd powers. By changing to $z = \frac{1}{2}x^2$, we can show that Eq. (11.2.56) may be changed to a confluent hypergeometric equation, of the type of Eq. (5.3.44). In fact we can show that

$$\begin{aligned} H_e(a,x) &= e^{-\frac{1}{2}ix^2}F(\tfrac{1}{4} + \tfrac{1}{4}ia|\tfrac{1}{2}|ix^2) \\ &= 1 - \tfrac{1}{2}ax^2 + \tfrac{1}{24}(a^2 - 2)x^4 - \tfrac{1}{720}(a^3 - 14a)x^6 + \cdots \\ H_0(a,x) &= xe^{-\frac{1}{2}ix^2}F(\tfrac{3}{4} + \tfrac{1}{4}ia|\tfrac{3}{2}|ix^2) \\ &= x - \tfrac{1}{6}ax^3 + \tfrac{1}{120}(a^2 - 6)x^5 - \tfrac{1}{5040}(a^3 - 26a)x^7 + \cdots \end{aligned} \tag{11.2.57}$$

Even though the expression in terms of the confluent hypergeometric function involves imaginary quantities, the series for H_e and H_0 has all real coefficients for the various powers of x, so that H_e and H_0 are real when x is real.

Reference to Eq. (5.3.63) shows that when $a = 0$,

$$H_e(0,x) = \Gamma(\tfrac{3}{4})\sqrt{\tfrac{1}{2}x}\,[J_{-\frac{1}{4}}(\tfrac{1}{2}x^2)]; \quad H_0(0,x) = \Gamma(\tfrac{5}{4})\sqrt{2x}\,[J_{\frac{1}{4}}(\tfrac{1}{2}x^2)]$$

The solutions for $a \neq 0$ may be expressed in terms of a series of Bessel functions. For instance we find that if we set $H(a,x) = \sqrt{x}\,G(a,z)$ where $z = \frac{1}{2}x^2$, then G satisfies the equation

$$\frac{d^2G}{dz^2} + \frac{1}{z}\frac{dG}{dz} + \left[1 + \frac{a}{2z} - \frac{1}{16z^2}\right]G = 0$$

Setting $G_e = \sum_{n=0}^{\infty} b_n J_{n-\frac{1}{4}}(z)$ for the even function, into the equation for G, we obtain

$$\begin{aligned} 0 &= 2z\sum b_n J_{n-\frac{1}{4}}(z)\left[\frac{1}{2}\frac{a}{z} + \frac{(n - \frac{1}{4})^2 - (\frac{1}{4})^2}{z^2}\right] \\ &= \sum b_n J_{n-\frac{1}{4}}(z)\left[a + \frac{n(n - \frac{1}{2})}{z}\right] \\ &= \sum_{n=0}^{\infty} J_{n-\frac{1}{4}}(z)\left[\frac{(n-1)(n-\frac{3}{4})}{n - \frac{5}{4}}\,b_{n-1} + ab_n + \frac{(n+1)(n+\frac{1}{2})}{n + \frac{3}{4}}\,b_{n+1}\right] \end{aligned}$$

Equating coefficients of the various J's equal to zero in turn, we find that $ab_0 + \frac{3}{2}b_1 = 0$, $\frac{12}{7}b_2 + ab_1 = 0$, $\frac{30}{11}b_3 + ab_2 + \frac{10}{3}b_1 = 0$, etc. Solving these recursion formulas (see page 539) in turn and adjusting a_0 so that the new series fits the series of Eq. (11.2.57), we finally have

$$H_e(a,x) = \Gamma(\tfrac{3}{4}) \sqrt{\tfrac{1}{2}x}\, [J_{-\frac{1}{4}}(\tfrac{1}{2}x^2) - \tfrac{3}{2}aJ_{\frac{3}{4}}(\tfrac{1}{2}x^2) + \tfrac{7}{8}a^2J_{\frac{7}{4}}(\tfrac{1}{2}x^2) + \tfrac{11}{6}(a - \tfrac{7}{40}a^3)J_{\frac{11}{4}}(\tfrac{1}{2}x^2) + \cdots]$$

Similarly

$$H_0(a,x) = \Gamma(\tfrac{5}{4}) \sqrt{2x}\, [J_{\frac{1}{4}}(\tfrac{1}{2}x^2) - \tfrac{5}{6}aJ_{\frac{5}{4}}(\tfrac{1}{2}x^2) + \tfrac{3}{8}a^2J_{\frac{9}{4}}(\tfrac{1}{2}x^2) + \tfrac{13}{42}(a - \tfrac{3}{8}a^3)J_{\frac{13}{4}}(\tfrac{1}{2}x^2) + \cdots] \quad (11.2.58)$$

These series expressions converge satisfactorily for $\frac{1}{2}x^2$ not larger than about 10 and for a less than about $\frac{1}{2}$.

Eigenfunctions for Interior Problems. These functions may be used to find eigenfunctions and eigenvalues for the interior of a closed area bounded by confocal, coaxial parabolas. As soon as we begin trying to solve actual problems, a few properties of parabolic coordinates appear which turn out to have a considerable influence on the choice of functions. An examination of the actual plot of the coordinate system (see Fig. 5.1) shows that one or the other of the coordinates λ, μ may go from $-\infty$ to $+\infty$, but not both. If μ has negative as well as positive values, then the range for λ is from 0 to $+\infty$. There is no complication with an exterior problem, for the field outside the parabola $\lambda = \lambda_0$, for instance; here the obvious range is, for λ between λ_0 and ∞, and for μ between $-\infty$ and $+\infty$. But for an interior problem, such as for the area included in the parabolas $\lambda = \lambda_0$ and $\mu = \mu_0$, we could use either the range $0 < \lambda < \lambda_0$, $-\mu_0 < \mu < \mu_0$ or $-\lambda_0 < \lambda < \lambda_0$, $0 < \mu < \mu_0$. A little thought will indicate that if the λ factor in the solution for the interior problem is odd, having a sign, for negative μ, opposite to that for positive μ, then the μ factor must also be an odd function, for it must go to zero at $\mu = 0$ (and vice versa). In other words, if the function has a node for $\lambda = 0$ (along the positive x axis), this same node must continue along $\mu = 0$ (along the negative x axis). Conversely, if the λ factor is even, having a maximum (or minimum) at $\lambda = 0$, the μ factor must also be even, for an interior problem, so that the maximum (or minimum) continues over $\mu = 0$. The symmetry of the boundaries ($\mu = \mu_0$, $\lambda = \lambda_0$) in the interior problem requires this pairing up of factors; the whole x axis, positive and negative, must be either a node or a loop for the eigenfunction.

If the bounded area is also symmetric with respect to the y axis, then $\lambda_0 = \mu_0$ and it turns out that some of the eigenfunctions correspond to the separation constant a being zero. In these cases the solutions reduce to Bessel functions, as indicated by Eqs. (11.2.58). Some of the allowed values of k are then obtained by using the roots of the equations,

$$J_{-\frac{1}{4}}(\gamma_{2m}) = 0; \quad J_{\frac{1}{4}}(\gamma_{2m+1}) = 0; \quad m = 0, 1, 2, \ldots \tag{11.2.59}$$

if the boundary condition at $\mu = \mu_0 = \lambda = \lambda_0$ is that the function goes to zero. The roots γ_n ($n = 0, 1, 2, \ldots$) alternate between even and odd solutions and each λ factor is similar to the corresponding μ factor. These solutions correspond to the solutions, for a square membrane, where n_x is equal to n_y, in the solution $\sin(\pi n_x x/a)\sin(\pi n_y y/a)$. Recapitulating, we have shown that for the symmetric case $\mu_0 = \lambda_0$, some of the eigenfunctions and eigenvalues, for $n_\mu = n_\lambda = n$, have the following forms:

$$\begin{aligned}
\psi_{nn}(\lambda,\mu) &= \begin{cases} \frac{1}{2}[\Gamma(\frac{3}{4})]^2\sqrt{\lambda\mu}\, J_{-\frac{1}{4}}(\gamma_n\lambda^2/\lambda_0^2)J_{-\frac{1}{4}}(\gamma_n\mu^2/\mu_0^2); & n \text{ even} \\ 2[\Gamma(\frac{5}{4})]^2\sqrt{\lambda\mu}\, J_{\frac{1}{4}}(\gamma_n\lambda^2/\lambda_0^2)J_{\frac{1}{4}}(\gamma_n\mu^2/\mu_0^2); & n \text{ odd} \end{cases} \\
k_{nn} &= 2(\gamma_n/\lambda_0^2) = 2(\gamma_n/\mu_0^2); \quad a_{nn} = 0 \\
\gamma_0 &= 2.006; \quad \gamma_1 = 2.781; \quad \gamma_2 = 5.123; \quad \gamma_3 = 5.906; \quad \ldots
\end{aligned} \tag{11.2.60}$$

where the allowed values of γ are obtained by solving Eqs. (11.2.59).

For the cases $\mu_0 \neq \lambda_0$ and for the nonsymmetrical solutions for $\mu_0 = \lambda_0$, a_{mn} is not zero and the problem is more difficult. The equation $H_e(a, \lambda_0\sqrt{k}) = 0$, for a given value of λ_0, defines a family of curves relating a and k (plotting k as a function of a). The curve nearest the a axis may be labeled $m = 0$, the next $m = 2$, and so on. The equation $H_e(-a, \mu_0\sqrt{k}) = 0$ produces another set of curves on the (a,k) plane, the lowest of which may be labeled $n = 0$, the next $n = 2$, etc. Wherever these two families of curves cross defines an allowed value of a (called a_{mn}) and an allowed value of k (called k_{mn}) and the corresponding eigenfunction $\psi_{mn} = H_e(a_{mn}, \lambda\sqrt{k_{mn}})H_e(-a_{mn}, \mu\sqrt{k_{mn}})$.

These functions are only the even ones, and the quantum numbers (n,m) are correspondingly even. To obtain the odd functions, we carry out the same procedure for the odd functions H_0. The equation $H_0(a, \lambda_0\sqrt{k}) = 0$ defines a family of curves for k against a, the lowest of which is for $m = 1$, the next for $m = 3$, etc. The other family, for $n = 1, 3, \ldots$, is obtained from $H_0(-a, \mu_0\sqrt{k}) = 0$. The intersections of these two families produce the odd functions. By this means we obtain all the allowed eigenfunctions and eigenvalues (the fact that no solution is obtained for m odd and n even, or vice versa, simply reflects the fact that the λ factor cannot be odd if the μ factor is even, and vice versa).

For the case $\lambda_0 = \mu_0$, the lowest mode, for k_{00}, and the next lowest, for k_{11}, are given by Eqs. (11.2.60). The next one, for k_{02} (and k_{20}, which in this case equals k_{02}) is not included in Eqs. (11.2.60) and must be computed by the methods of the preceding two paragraphs. The next modes, for k_{13} and k_{31} must also be computed by the more complicated methods. The sixth mode, for k_{22}, is again given by Eqs. (11.2.60) and so on.

There are few cases, in actuality, that correspond particularly closely to a set of parabolic boundaries, so it is hardly worth our while to pursue this matter of interior solutions further. If necessary, the general methods of Chaps. 6 and 7 may be applied to the solutions H_e and H_0 without great difficulty.

Waves outside Parabolic Boundaries. There is more excuse for pursuing the exterior problem in parabolic coordinates. A knife edge may be considered to be equivalent to the boundary $\lambda = \lambda_0$, where λ_0 tends toward zero. It would be useful to obtain expressions for the Green's function and its limit, the plane wave, for parabolic wave functions. We have, of course, solved the problem of the scattering of a plane wave by a knife edge by use of cylindrical wave functions (see page 1383), but a use of the new wave functions may make the solution easier or may make other modifications possible.

To deal with exterior problems, we should examine our solutions H_e and H_0 with respect to their asymptotic behavior, to see whether other combinations of these two independent solutions would not be better suited for use with exterior problems. Referring to Eqs. (5.3.51) we see that, if $x = |x|e^{i\theta}$ ($x = \sqrt{k}\,\lambda$ or $\sqrt{k}\,\mu$), then the asymptotic behavior of the F's, for x and a real, is

$$\begin{aligned} H_e(a,x) &\to \frac{2\Gamma(\frac{1}{2})e^{-\pi a/8}}{|\Gamma(\frac{1}{4} + \frac{1}{4}ia)|\sqrt{x}} \cos[\tfrac{1}{2}x^2 + \tfrac{1}{2}a \ln x - \tfrac{1}{8}\pi - \sigma(a)] \\ H_0(a,x) &\to \frac{2\Gamma(\frac{3}{2})e^{-\pi a/8}}{|\Gamma(\frac{3}{4} + \frac{1}{4}ia)|\sqrt{x}} \cos[\tfrac{1}{2}x^2 + \tfrac{1}{2}a \ln x - \tfrac{3}{8}\pi - \tau(a)] \end{aligned} \tag{11.2.61}$$

where

$$\Gamma(\tfrac{1}{4} + \tfrac{1}{4}ia) = |\Gamma(\tfrac{1}{4} + \tfrac{1}{4}ia)|e^{i\sigma(a)};\quad \Gamma(\tfrac{3}{4} + \tfrac{1}{4}ia) = |\Gamma(\tfrac{3}{4} + \tfrac{1}{4}ia)|e^{i\tau(a)}$$

and where we have used the formula $x^{\frac{1}{2}ia} = e^{\frac{1}{2}ia \ln x}$.

These formulas do not look to be too satisfactory in their behavior at infinity, even though they are real for x real. The trouble is the term $\frac{1}{2}a \ln x$ in the cosine arguments. There seems to be no way to make these fit to the usual asymptotic wave forms for other coordinates. The difficulty, of course, is that a has been taken real; if a were imaginary, the factor $x^{\frac{1}{2}ia}$ would be a real power of x and the logarithmic term would disappear from the cosine arguments. There is, of course, no reason that a should be real for the exterior problem. It turned out to be real for the interior problem, and must be real for the functions F_e and F_0 to be real. But for the exterior problem it is not necessary that our functions be real; in polar coordinates we use the complex Hankel functions $H_m(kr)$ for exterior problems, in preference to the real functions $J_m(kr)$, because of their simple asymptotic behavior.

Suppose we make a imaginary, let it be $(2m + 1)i$, where m is at present any real quantity, positive or negative. Our separated solution of the Helmholtz equation is then

$$H_e[(2m+1)i, \lambda\sqrt{k}]\, H_e[-(2m+1)i, \mu\sqrt{k}]$$

or

$$H_0[(2m+1)i, \lambda\sqrt{k}]\, H_0[-(2m+1)i, \mu\sqrt{k}]$$

where

$$\begin{aligned} H_e[(2m+1)i, x] &= e^{-\frac{1}{2}ix^2}F(-\tfrac{1}{2}m|\tfrac{1}{2}|ix^2) \\ H_0[(2m+1)i, x] &= xe^{-\frac{1}{2}ix^2}F(\tfrac{1}{2}-\tfrac{1}{2}m|\tfrac{3}{2}|ix^2) \end{aligned} \tag{11.2.62}$$

in terms of the confluent hypergeometric functions. The asymptotic behavior of these functions is again obtained by the use of Eqs. (5.3.51). For m and x real, we have

$$H_e[(2m+1)i, x] \to \frac{\Gamma(\frac{1}{2})x^{-m-1}}{\Gamma(-\frac{1}{2}m)}\, e^{\frac{1}{2}ix^2-\frac{1}{4}i\pi(m+1)} + \frac{\Gamma(\frac{1}{2})x^m}{\Gamma(\frac{1}{2}+\frac{1}{2}m)}\, e^{-\frac{1}{2}ix^2-\frac{1}{4}i\pi m}$$

$$H_0[(2m+1)i, x] \to \frac{\Gamma(\frac{1}{2})x^{-m-1}}{\Gamma(\frac{1}{2}-\frac{1}{2}m)}\, e^{\frac{1}{2}ix^2-\frac{1}{4}i\pi(m+2)} + \frac{\Gamma(2)x^m}{\Gamma(1+\frac{1}{2}m)}\, e^{-\frac{1}{2}ix^2-\frac{1}{4}i\pi(m-1)}$$

An examination of these expressions indicates that it would be simpler for the exterior problem to choose a pair of solutions which would have simple asymptotic behavior of $x^{-m-1}e^{\frac{1}{2}ix^2}$ or $x^m e^{-\frac{1}{2}ix^2}$, instead of the combinations represented by the H's. This leads us to consider the confluent hypergeometric functions of the third kind, U_1 and U_2, defined in Eqs. (5.3.52). The function which is useful is called the *Weber function*, defined as follows, for $z = |z|e^{i\phi}$, $-\frac{1}{2}\pi < \phi < +\frac{1}{2}\pi$

$$\begin{gathered} D_m(z) = 2^{\frac{1}{2}m}e^{-\frac{1}{4}z^2+\frac{1}{2}i\pi m}U_2(-\tfrac{1}{2}m|\tfrac{1}{2}|\tfrac{1}{2}z^2) \to z^m e^{-\frac{1}{4}z^2} \\ (d^2/dz^2)D_m(z) + [m+\tfrac{1}{2}-\tfrac{1}{4}z^2]D_m(z) = 0 \end{gathered} \tag{11.2.63}$$

In terms of this function, a pair of suitable solutions for exterior problems in parabolic coordinates are

$$\begin{aligned} D_m(x\sqrt{2i}) &= 2^{\frac{1}{2}m}i^m e^{-\frac{1}{2}ix^2}U_2(-\tfrac{1}{2}m|\tfrac{1}{2}|ix^2) \to (2ix^2)^{\frac{1}{2}m}e^{-\frac{1}{2}ix^2} \\ &= 2^{\frac{1}{2}m}\Gamma(\tfrac{1}{2})\left\{\frac{1}{\Gamma(\frac{1}{2}-\frac{1}{2}m)}H_e[(2m+1)i, x] \right. \\ &\qquad \left. - \frac{2\sqrt{i}}{\Gamma(-\frac{1}{2}m)}H_0[(2m+1)i, x]\right\} \qquad (11.2.64) \\ D_{-m-1}(x\sqrt{-2i}) &= 2^{-\frac{1}{2}(m+1)}i^{m+1}e^{-\frac{1}{2}ix^2}U_1(-\tfrac{1}{2}m|\tfrac{1}{2}|ix^2) \to \frac{e^{+\frac{1}{2}ix^2}}{(-2ix^2)^{\frac{1}{2}(m+1)}} \\ &= \frac{\Gamma(\frac{1}{2})}{2^{\frac{1}{2}(m+1)}}\left\{\frac{1}{\Gamma(1+\frac{1}{2}m)}H_e[(2m+1)i, x] \right. \\ &\qquad \left. + \frac{2i^{\frac{3}{2}}}{\Gamma(\frac{1}{2}+\frac{1}{2}m)}H_0[(2m+1)i, x]\right\} \end{aligned}$$

for x and m real.

Thus the change from one solution with one behavior at infinity, to the other, with the opposite behavior at infinity, is effected, in this notation, by changing from m to $-m - 1$ and by simultaneously taking the complex conjugate. The functions both satisfy the differential equation

$$(d^2y/dx^2) + [(2m + 1)i + x^2]y = 0$$

which corresponds to Eq. (11.2.56) with $a = (2m + 1)i$, as required. If the solution for the λ factor is a combination of $D_m(\lambda \sqrt{2ik})$ and $D_{-m-1}(\lambda \sqrt{-2ik})$, with $a = (2m + 1)i$, the corresponding solution for the μ factor must be a solution of the equation for $-a$,

$$(d^2y/dx^2) + [-(2m + 1)i + x^2]y = 0$$

which is a combination of the complex conjugates of the solutions of Eq. (11.2.63), *i.e.*,

$$\begin{aligned} D_m(x \sqrt{-2i}) &= 2^{\frac{1}{2}m}i^{-m}e^{-\frac{1}{2}ix^2}U_1(\tfrac{1}{2} + \tfrac{1}{2}m|\tfrac{1}{2}|ix^2) \to (-2ix^2)^{\frac{1}{2}m}e^{\frac{1}{2}ix^2} \\ &= 2^{\frac{1}{2}m}\Gamma(\tfrac{1}{2}) \left\{\frac{1}{\Gamma(\frac{1}{2} - \frac{1}{2}m)} H_e[-(2m + 1)i, x] \right. \\ &\qquad \left. - \frac{2\sqrt{-i}}{\Gamma(-\frac{1}{2}m)} H_0[-(2m + 1)i, x]\right\} \qquad (11.2.65) \end{aligned}$$

$$\begin{aligned} D_{-m-1}(x \sqrt{2i}) &= 2^{\frac{1}{2}m}i^{-m-1}e^{-\frac{1}{2}ix^2}U_2(\tfrac{1}{2} + \tfrac{1}{2}m|\tfrac{1}{2}|ix^2) \to \frac{e^{-\frac{1}{2}ix^2}}{(2ix^2)^{\frac{1}{2}(m+1)}} \\ &= 2^{\frac{1}{2}m}\Gamma(\tfrac{1}{2}) \left\{\frac{1}{\Gamma(1 + \frac{1}{2}m)} H_e[-(2m + 1)i, x] \right. \\ &\qquad \left. + \frac{2(-i)^{\frac{3}{2}}}{\Gamma(\frac{1}{2} + \frac{1}{2}m)} H_0[-(2m + 1)i, x]\right\} \qquad (11.2.66) \end{aligned}$$

where we have used Eqs. (5.2.62) and (11.2.62) to make the change.

In order to have the exterior solution of the Helmholtz equation in parabolic coordinates be single-valued over the (λ,μ) plane, we should use integral values of m, though both positive and negative integers are allowed (the negative integers corresponding to the second solutions); if m were not an integer, the solution would be multivalued and some branch cut would be needed. It thus simplifies the expressions for the Weber functions considerably for m to be an integer. For instance, for $m = 0$

$$D_0(z) = e^{-\frac{1}{4}z^2}F(0|\tfrac{1}{2}|\tfrac{1}{2}z^2) = e^{-\frac{1}{4}z^2} \qquad (11.2.67)$$

By use of the integral representation for U_2 or by using the recurrence formulas for the confluent hypergeometric function, we can show that, for any value of m,

$$(d/dz)D_m(z) = \tfrac{1}{2}zD_m(z) - D_{m+1}(z) = -\tfrac{1}{2}zD_m(z) + mD_{m-1}(z)$$

or

$$D_{m+1}(z) = -e^{\frac{1}{4}z^2} \frac{d}{dz} [e^{-\frac{1}{4}z^2}D_m(z)]; \quad D_m(z) = e^{\frac{1}{4}z^2} \int_z^\infty e^{-\frac{1}{4}u^2}D_{m+1}(u)\, du$$

where we have chosen the limits of integration to satisfy the asymptotic behavior of D.

Consequently, for integral values of m, we have

$$\begin{aligned} D_m(z) &= (-1)^m e^{\frac{1}{4}z^2}[(d^m/dz^m)e^{-\frac{1}{2}z^2}] = 2^{\frac{1}{2}m}e^{-\frac{1}{4}z^2}H_m(z\sqrt{\tfrac{1}{2}}) \\ D_{-m-1}(z) &= e^{\frac{1}{4}z^2}\int_z^\infty du_0 \int_{u_0}^\infty du_1 \cdots \int_{u_{m-1}}^\infty du_m e^{-\frac{1}{2}u_m^2} \end{aligned} \tag{11.2.68}$$

We notice that the function $H_m(z)$ is the Hermite polynomial discussed in the tables at the end of Chap. 6. We also note that

$$D_{-1}(z\sqrt{2i}) = e^{\frac{1}{2}iz^2}\int_{z\sqrt{2i}}^\infty e^{-\frac{1}{2}u_0^2}\,du_0 \tag{11.2.69}$$

which is closely related to the Fresnel integrals discussed on page 1386.

Green's Function and Plane Wave Expansions. If the functions D_m and D_{-m-1} are the ones to use for exterior problems, we should be able to express the free-space, two-dimensional Green's function $i\pi H_0(kR)$ and the plane wave $e^{ikr\cos(\phi-u)}$ as series using them. That this is indeed true may be shown in several ways. We can, of course, apply the general formula of Eq. (7.2.63) to find that

$$\begin{aligned} i\pi H_0(kR) = \frac{\sqrt{8\pi}}{i}\sum_{m=0}^\infty \frac{i^m}{m!} D_m(\lambda_0\sqrt{-2ik})D_m(\lambda\sqrt{-2ik}) \cdot \\ \cdot \begin{cases} D_m(\mu\sqrt{2ik})D_{-m-1}(\mu_0\sqrt{-2ik}); & \mu_0 > \mu \\ D_m(\mu_0\sqrt{2ik})D_{-m-1}(\mu\sqrt{-2ik}); & \mu > \mu_0 \end{cases} \end{aligned} \tag{11.2.70}$$

though this is not too easy, because of the fact that both factors depend on both m and k in much the same way. We make the choice as to which is "eigenfunction" and which auxiliary factor by remembering that the variable for the Hermite polynomial goes from $-\infty$ to $+\infty$. Consequently, in Eq. (11.2.70), we are assuming that $-\infty < \lambda < +\infty$ and $0 < \mu < \infty$. This means that the "cut" comes along the positive x axis, $\mu = 0$. Complications arise as we cross this line.

From this equation we can obtain the plane wave expansion by letting the source point (λ_0,μ_0) go to infinity. But we can also obtain the same series by referring to one of the formulas in the table at the end of this chapter, an expansion theorem obtained (after much struggle) from the integral representation for the D's. The required expansion is

$$\begin{aligned} e^{ik(x\cos u + y\sin u)} &= e^{-\frac{1}{2}ik(\lambda^2+\mu^2)+ik[\lambda\cos(\frac{1}{2}u)+\mu\sin(\frac{1}{2}u)]^2} \\ &= \sec(\tfrac{1}{2}u)\sum_{m=0}^\infty \frac{i^m\tan^m(\frac{1}{2}u)}{m!} D_m(\lambda\sqrt{-2ik})D_m(\mu\sqrt{2ik}) \\ &= \csc(\tfrac{1}{2}u)\sum_{m=0}^\infty \frac{\cot^m(\frac{1}{2}u)}{i^m m!} D_m(\lambda\sqrt{2ik})D_m(\mu\sqrt{-2ik}) \end{aligned} \tag{11.2.71}$$

where the first series converges well for values of u near zero and the second for values near π.

When $u = \pi$ and the plane wave is moving to the left, parallel to the x axis, the expression takes on the particularly simple form

$$e^{-ikx} = D_0(\lambda \sqrt{2ik})D_0(\mu \sqrt{-2ik}) = e^{\frac{1}{2}ik(\mu^2-\lambda^2)} \qquad (11.2.72)$$

We use $u = \pi$ in order to have the "cut," $\mu = 0$, come in the oncoming wave region, so it will not confuse the shadow region.

This simple form may be used as a start to calculate the diffraction of a plane wave from a knife edge perpendicular to the wave-vector. Such a "knife edge" would be the negative y axis, or the line $\lambda = -\mu$. We need only to add to the product for the plane wave some other solution in parabolic coordinates which will give the combination zero value (for Dirichlet conditions) along the line $\lambda = -\mu$ but nowhere else in the plane. The latter requirement makes it somewhat harder, for our first thought would be to use

$$D_0(\lambda \sqrt{2ik})D_0(\mu \sqrt{-2ik}) - D_0(\lambda \sqrt{-2ik})D_0(\mu \sqrt{2ik})$$

But this is zero also for $\lambda = +\mu$ (the positive u axis), and the expression, which is $(e^{-ikx} - e^{+ikx})$, represents reflection from the whole y axis, which is not what is required.

We therefore try other combinations of $D_0(\lambda \sqrt{\pm 2ik})$, $D_{-1}(\lambda \sqrt{\pm 2ik})$ with the corresponding factors for μ, using for check purposes the relationships

$$\begin{aligned} D_0(-w) &= D_0(w) \\ D_{-1}(-x \sqrt{-2i}) &= -D_{-1}(x \sqrt{-2i}) + \sqrt{2\pi}\, D_0(x \sqrt{2i}) \end{aligned} \qquad (11.2.73)$$

The expression which finally emerges is

$$\begin{aligned} \psi(\lambda,\mu) = D_0(\lambda \sqrt{2ik})&D_0(\mu \sqrt{-2ik}) \\ &- (1/\sqrt{2\pi})[D_0(\mu \sqrt{-2ik})D_{-1}(\lambda \sqrt{-2ik}) \\ &\qquad + D_0(\lambda \sqrt{-2ik})D_{-1}(\mu \sqrt{-2ik})] \end{aligned} \qquad (11.2.74)$$

which goes to zero for $\lambda = -\mu$ but not for $\lambda = +\mu$, thus being the solution we want.

Analysis of this solution follows very much the same line as was already given on page 1386.

The function

$$D_{-1}(w \sqrt{-2ik}) = \sqrt{-2ik}\, e^{-\frac{1}{2}ikw^2} \int_w^{\infty\sqrt{i}} e^{ikv^2}\, dv \qquad (11.2.75)$$

is small for w large and positive, reducing asymptotically to the required $[e^{\frac{1}{2}ikw^2}/w \sqrt{-2ik}]$. For w large and negative, D_{-1} does not vanish, but has the asymptotic form $\sqrt{2\pi}\, e^{-\frac{1}{2}ikw^2} - [e^{+\frac{1}{2}ikw^2}/w \sqrt{-2ik}]$. For w near zero, D_{-1} has the oscillations typical of a Fresnel integral. For all values of w, $D_0(w \sqrt{-2ik})$ is simply $e^{\frac{1}{2}ikw^2}$.

With these values in mind, let us see what formula (11.2.74) gives us some distance behind the knife edge, where μ is large and, near the edge of the shadow, λ is small. Putting in the exponential expressions for the D_0's and the asymptotic expression for $D_{-1}(\mu \sqrt{-2ik})$, we have

$$\psi \xrightarrow[x\to-\infty]{} e^{-ikx} - \frac{e^{ikr}}{2\sqrt{-i\pi kr}} - \frac{e^{\frac{1}{2}ik\mu^2}}{\sqrt{2\pi}} D_{-1}(\lambda \sqrt{-2ik}) \qquad (11.2.76)$$

where $r = \frac{1}{2}(\lambda^2 + \mu^2)$ is the radial distance from the knife edge. The first term is the incident wave, the second term is the radially outgoing wave scattered from the knife edge, and the third term is the diffraction and shadow-producing term. When λ is positive ($y > 0$), this term is small and contributes a little more to the scattered wave. As λ increases in the negative direction, the term approaches $-e^{-ikx} + [e^{ikr}/2\lambda \sqrt{-i\pi k}]$, the first term of which cancels the incident wave (causing the shadow) and the second term adding to the scattered wave. For λ near zero, this term oscillates, producing the diffraction effects.

Elliptic Coordinates. A more useful set of coordinates is the elliptic set, defined by [see also Eq. (10.1.23)]

$$\begin{aligned} x &= \tfrac{1}{2}a \cosh\mu \cos\vartheta; \quad y = \tfrac{1}{2}a \sinh\mu \sin\vartheta \\ h_\mu &= h_\vartheta = \tfrac{1}{2}a \sqrt{\sinh^2\mu + \sin^2\vartheta}; \quad r = \tfrac{1}{2}a\sqrt{\cosh^2\mu - \sin^2\vartheta} \\ r_1 &= \tfrac{1}{2}a(\cosh\mu + \cos\vartheta); \quad r_2 = \tfrac{1}{2}a(\cosh\mu - \cos\vartheta) \end{aligned} \qquad (11.2.77)$$

where r is the distance to point (x,y) from the origin, r_2 its distance from the right-hand focus, at $(\frac{1}{2}a,0)$, and r_1 its distance from the left-hand focus, at $(-\frac{1}{2}a,0)$. Potential solutions for such coordinates were discussed in Sec. 10.1. The coordinate system is pictured in Fig. 5.3. The Helmholtz equation is

$$\frac{\partial^2\psi}{\partial\mu^2} + \frac{\partial^2\psi}{\partial\vartheta^2} + \tfrac{1}{4}a^2k^2[\cosh^2\mu - \cos^2\vartheta]\psi = 0 \qquad (11.2.78)$$

which separates into $\psi = M(\mu)H(\vartheta)$, where

$$\begin{aligned} (d^2H/d\vartheta^2) + (b - h^2\cos^2\vartheta)H &= 0 \\ -(d^2M/d\mu^2) + (b - h^2\cosh^2\mu)M &= 0 \end{aligned} \qquad (11.2.79)$$

The first of these equations is Mathieu's equation, given in Eq. (5.2.67) and discussed in Chap. 5. The second equation is also Mathieu's equation but for an imaginary argument. If we consider $z = \cos\vartheta$ to be the independent variable, then the range $-1 \le z \le 1$ is used for the coordinate ϑ and the range $1 \le z < \infty$ is used for the coordinate μ. As indicated in Eqs. (5.2.77) and (5.3.87), it is desirable to use the variables ϑ, μ when computing the coefficients for the series expansions of the solutions but, when considering the general mathematical properties of the solutions, it is best to use the variable

$$z = \begin{cases} \cos\vartheta = (1/a)(r_2 - r_1); & -1 \le z \le 1 \\ \cosh\mu = (1/a)(r_2 + r_1); & 1 \le z < \infty \end{cases} \qquad (11.2.80)$$

When the boundary line is an ellipse, μ = constant, and ϑ is free to vary from 0 to 2π (which is usually the case), then the ϑ factor must be periodic, with period π or 2π. This occurs only for discrete values of b, and there results the set of eigenfunctions, Se, So, called the *Mathieu functions*, defined on page 565. When h goes to zero, these functions reduce to the trigonometric functions and b reduces to the square of an integer:

$$h \to 0;\quad Se_m(h, \cos\vartheta) \to \cos(m\vartheta);\quad So_m(h, \cos\vartheta) \to \sin(m\vartheta)$$
$$be_m \to bo_m \to m^2$$

The parameter $h = \frac{1}{2}ak = \frac{1}{2}(a\omega/c) = \pi a/\lambda$ is proportional to the frequency and is equal to the ratio between π times the interfocal distance a and the wavelength λ.

We have discussed many of the properties of these functions in Secs. 5.2 and 5.3 and some numerical values are given in the tables at the back of the book. They are the only ones needed for coordinate ϑ, as long as ϑ is free to vary from 0 to 2π. For the μ coordinate we need also the second solutions, which are not periodic in ϑ, but this disadvantage is not a handicap for the μ coordinate, which is not periodic itself. Therefore for the work of this chapter we need the periodic eigenfunctions for ϑ, together with two solutions for μ, one proportional to Se, So and a second one, independent of Se, So, defined in Eqs. (5.3.84) and (5.3.91).

The functions have been discussed in Chap. 5 and are tabulated at the end of this chapter. We confine our formulas here to the minimum, though this minimum is not insignificant, owing to the unfortunate propensity of Mathieu function formulas for coming in fours, being different for even and odd (about $\vartheta = 0$) and also different depending on whether the functions are periodic in π or 2π. The eigenfunction solutions for the angle factors are:

$$\begin{aligned} Se_{2m}(h, \cos\vartheta) &= \sum_n B^e_{2n}(h,2m)\cos(2n\vartheta);\quad \sum_n B^e_{2n} = 1 \\ Se_{2m+1}(h, \cos\vartheta) &= \sum_n B^e_{2n+1}(h, 2m+1)\cos[(2n+1)\vartheta]; \\ &\qquad\qquad \sum_n B^e_{2n+1} = 1 \end{aligned} \tag{11.2.81}$$

$$\begin{aligned} So_{2m}(h, \cos\vartheta) &= \sum_n B^0_{2n}(h,2m)\sin(2n\vartheta);\quad \sum_n (2n)B^0_{2n} = 1 \\ So_{2m+1}(h, \cos\vartheta) &= \sum_n B^0_{2n+1}(h, 2m+1)\sin[(2n+1)\vartheta]; \\ &\qquad\qquad \sum_n (2n+1)B^0_{2n+1} = 1 \end{aligned} \tag{11.2.82}$$

All these functions, for any one given value of h, form a complete set of eigenfunctions which are mutually orthogonal. The Se_m functions are even about $\vartheta = 0, \pi$ and the So functions are odd. The functions with even order $(2m)$ are even about $\vartheta = \frac{1}{2}\pi, \frac{3}{2}\pi$, and the functions of odd order $(2m + 1)$ are odd there. For example $Se_{2m}(\vartheta) = Se_{2m}(\pi \pm \vartheta) = Se_{2m}(-\vartheta)$ and $So_{2m+1}(\vartheta) = -So_{2m+1}(\pi - \vartheta) = So_{2m+1}(\pi + \vartheta) = -So_{2m+1}(-\vartheta)$, etc. The normalization constants for these functions are

$$\begin{aligned}
\int_0^{2\pi} [Se_{2m}(h, \cos\vartheta)]^2\, d\vartheta &= 2\pi \sum_{n=0}^{\infty} \left(\frac{1}{\epsilon_n}\right) [B^e_{2n}(h,2m)]^2 = M^e_{2m}(h) \\
\int_0^{2\pi} [Se_{2m+1}]^2\, d\vartheta &= \pi \sum_{n=0}^{\infty} [B^e_{2n+1}(h, 2m+1)]^2 = M^e_{2m+1}(h) \\
\int_0^{2\pi} [So_{2m}]^2\, d &= \pi \sum_{n=1}^{\infty} [B^0_{2n}(h,2m)]^2 = M^0_{2m}(h) \\
\int_0^{2\pi} [So_{2m+1}]^2\, d &= \pi \sum_{n=0}^{\infty} [B^0_{2n+1}(h, 2m+1)]^2 = M^0_{2m+1}(h)
\end{aligned} \tag{11.2.83}$$

As discussed on page 566 we have computed the coefficients B so that the value of Se and the slope of So are unity at $\vartheta = 0$. This is done in order to make some of our formulas come out more simply and to make the functions correspond as closely as possible to the trigonometric functions, to which they reduce as $h \to 0$. There is some resulting embarrassment because of this choice, for as h increases, the amplitude of Se_m (for instance) near $\vartheta = \frac{1}{2}\pi$ becomes much larger than its value at $\vartheta = 0$.

Another method of normalization would be to keep $B_m(h,m)$ equal to unity for all h, which again would make $Se_m \to \cos(m\vartheta)$ as $h \to 0$; but this normalization makes $Se_m \to \infty$ for some h. Another "safe" way is to fix the value of the normalization constant M, so that the integral of the square of Se_m over ϑ from 0 to 2π is $(2\pi/\epsilon_m)$, as it is for the limiting form, $\cos(m\vartheta)$. This also ensures a finite eigenfunction for every real value of h, but it introduces constants equal to the value of Se_m at $\vartheta = 0$, into a large number of expressions we shall use. Everything considered, therefore, it is best to set $Se_m = 1$ for $\vartheta = 0$, for all h. Over the range of h usually encountered, there is no trouble, and since the normalization constants M are computed and tabulated, it is easy to obtain the function which is normalized to $(2\pi/\epsilon_m)$ by multiplying Se_m by $[2\pi/\epsilon_m M^2_m(h)]^{\frac{1}{2}}$; similarly with the odd functions, So_m.

All the constants, the coefficients B, the separation constants b, and the normalization constants M, are functions of h, of the index m, and of the label e, o which distinguishes between the even and odd solutions about $\vartheta = 0$. When it is necessary to emphasize this dependence, the full regalia of superscripts and parentheses will be included, as we have in Eqs. (11.2.82) and (11.2.83); otherwise we shall trim as many off as we can, leaving them as be_m, B_{2m} or M, for instance.

The Radial Solutions. Returning to the solutions for μ, we note again the results reached on page 634, that the coefficients for expansion of the Bessel function series are the same as the coefficients for the Fourier series expansion. We could, of course, simply change from ϑ to $i\mu$ in Eq. (11.2.82) and use the resulting series in hyperbolic functions. This may be more elegantly represented by reverting to the variable $z = \cos\vartheta = \cosh\mu$ and, correspondingly, changing to the Tchebyscheff polynomials defined in Eq. (5.3.43) and in the tables at the end of Chap. 6. For instance, we have

$$\begin{aligned} So_{2m+1}(h,z) &= \sqrt{\tfrac{1}{2}\pi(z^2-1)} \sum_n B^o_{2n+1} T^{\frac{1}{2}}_{2n}(z) \\ Se_{2m}(h,z) &= \sqrt{\tfrac{1}{2}\pi} \sum_n (2n) B^e_{2n} T^{-\frac{1}{2}}_{2n}(z) \end{aligned} \tag{11.2.84}$$

We can then follow the behavior of the series by using the analytic properties of the polynomials T, which are related to the hypergeometric function.

But none of these efforts will produce as simple a set of expressions for the asymptotic behavior of the Se, So function, for large values of z, as will the expansion in terms of Bessel functions, discussed on pages 639 *et seq.* It has been shown [Eq. (5.3.83)] that the function

$$Je_{2m}(h, \cosh\mu) = \sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} (-1)^{n-m} B^e_{2n}(h,2m) J_{2n}(h\cosh\mu)$$

is proportional to the function $Se_{2m}(h, \cosh\mu)$. We could make the two functions equal by dividing out the proportionality factor, but it is preferable to keep it in, since Je_{2m}, as defined, has the simple asymptotic form

$$Je_{2m} \to \frac{1}{\sqrt{h\cosh\mu}} \cos[h\cosh\mu - \tfrac{1}{2}\pi(2m+\tfrac{1}{2})]$$

The tables at the end of this chapter give the complete formulas defining the *radial functions of the first kind*, Je, Jo. We note that as h goes to zero these functions reduce to a single Bessel function and, therefore, that the limiting form of the product solution $Se_m(h, \cos\vartheta)$ $Je_m(h, \cosh\mu)$ is $\sqrt{\frac{1}{2}\pi}\cos(m\vartheta)J_m(h\cosh\mu)$, as it should be (in the limit the wave reduces to a circular wave). We should also note the expansions in products of Bessel functions; these expansions are by far the most rapidly convergent of the series shown. Since Je, Jo are proportional to the Se, So, they share their behavior at $z = 1$ ($\mu, \vartheta = 0$); in other words the value of Jo_m is zero at $\mu = 0$ and the slope of Je_m is zero there. The value of Je_m and the slope of Jo_m is obtainable in terms of the coefficients B as given in the formulas at the end of this chapter.

Second solutions of the Mathieu function, for the same values of b as for the Se, So eigenfunctions, are needed for the radial solution in μ. This may be most easily obtained by substituting the second Bessel function N_m for J_m in the expressions for Je, Jo. For instance the *radial function of the second kind*, corresponding to Je_{2m}, is

$$Ne_{2m}(h, \cosh\mu) = \sqrt{\frac{\pi}{2}}\sum_{n=0}^{\infty}(-1)^{n-m}B^e_{2n}(h,2m)N_{2n}(h\cosh\mu)$$

The relationship between Je and Ne is thus simple and the asymptotic behavior of Ne

$$Ne_{2m} \to \frac{1}{\sqrt{h\cosh\mu}}\sin[h\cosh\mu - \tfrac{1}{2}\pi(2m + \tfrac{1}{2})]$$

is just 90° out of phase with that of Je. The particular series quoted above is not very convergent, in fact it does not converge absolutely for any finite value of μ. But the alternative series

$$Ne_{2m}(h, \cosh\mu) = \frac{1}{B^e_0}\sqrt{\frac{\pi}{2}}\sum_{n=0}^{\infty}(-1)^{n-m}B^e_{2n}N_n(\tfrac{1}{2}he^{\mu})J_n(\tfrac{1}{2}he^{-\mu})$$

[where we have left the label $(h,2m)$ off the coefficients B^e_{2n} and B^e_0, to save space] does converge satisfactorily clear in to $\mu = 0$.

The Wronskian, relating a solution of the first kind with the corresponding solution of the second kind, is just equal to unity (which is, of course, the reason we have chosen the coefficients the way we have). This has certain implications with regard to the value and slope of Ne, No at $\mu = 0$. For instance, since the slope of Je_m is zero at $\mu = 0$, the slope of Ne_m (*i.e.*, the derivative with respect to μ) at $\mu = 0$ must just be the reciprocal of the value of Je_m at $\mu = 0$. Vice versa, the value of No_m at $\mu = 0$ must equal minus the reciprocal of the slope of Jo_m at $\mu = 0$. Values of Ne_m and slopes of No_m must be computed by the use of the series using products of Bessel functions. In addition the fact that $\Delta(Je,Ne) = 1$ simplifies the calculation of the Green's function.

Approximations for Small Values of h and m. There are times when it is useful to have approximate expressions for some of the eigenfunctions, rather than more accurate numerical tables. For long wavelengths, for instance, expressions in powers of h will allow us to compute limiting behavior. At the other extreme, calculation of the function for h extremely large is sometimes required, and an approximate form is desirable.

For the long-wave approximation, we start with the modified Mathieu equation

$$(d^2y/d\vartheta^2) + (b - \tfrac{1}{2}h^2 - \tfrac{1}{2}h^2 \cos 2\vartheta)y = 0$$

Evidently all dependence on h may be expressed in terms of series in powers of h^2. As an example of the method of calculation, we consider the lowest function

$$y = Se_0(h, \cos\vartheta) = B_0 + B_2\cos(2\vartheta) + B_4\cos(4\vartheta) + \cdots$$

Since this must go to 1 when h is zero, we see that, as functions of h, the coefficients B must have power series of the sort

$$B_0 = 1 + a_0h^2 + b_0h^4 + \cdots; \quad B_2 = a_2h^2 + b_2h^4 + \cdots$$
$$B_4 = a_4h^2 + b_4h^4 + \cdots$$

Since $be_0(h)$ goes to zero as h goes to zero, we must also have

$$be_0 - \tfrac{1}{2}h^2 = \alpha h^2 + \beta h^4 + \cdots$$

Substituting the Fourier series for Se_0 into the equation we obtain

$$\begin{aligned} y'' &= -4B_2\cos(2x) - \cdots \\ (b - \tfrac{1}{2}h^2)y &= (b - \tfrac{1}{2}h^2)B_0 + (b - \tfrac{1}{2}h^2)B_2\cos(2x) + \cdots \\ -\tfrac{1}{2}h^2\cos(2x)y &= -\tfrac{1}{4}h^2B_2 - \tfrac{1}{4}h^2(2B_0 + B_4)\cos(2x) - \cdots \end{aligned}$$

Equating coefficients of the various cosine terms to zero gives us

$$(b - \tfrac{1}{2}h^2)B_0 = \tfrac{1}{4}h^2B_2; \quad [(b - \tfrac{1}{2}h^2) - 4]B_2 = \tfrac{1}{4}h^2(2B_0 + B_4); \quad \cdots$$

Substituting the assumed series forms for $(b - \frac{1}{2}h^2)$ and the B's gives us $\alpha = 0; \beta = \frac{1}{4}a_2; a_2 = -\frac{1}{8}; b_2 = -\frac{1}{8}a_0 - \frac{1}{16}a_4$; etc., from which we obtain, eventually, $\alpha = 0$, $\beta = -\frac{1}{32}$; $a_2 = -\frac{1}{8}$; $a_4 = 0$; $b_4 = \frac{1}{512}$ so that we can write

$$Se_0 = (1 + a_0h^2 + b_0h^4) - (\tfrac{1}{8}h^2 + \tfrac{1}{8}a_0h^4)\cos(2\vartheta) + (h^4/512)\cos(4\vartheta) + \cdots$$

Since Se_0 must be unity at $\vartheta = 0$, we have $a_0 = \frac{1}{8}$ and $b_0 = \frac{7}{512}$. Consequently, to the fourth power of h, we have

$$Se_0 = (1 + \tfrac{1}{8}h^2 + \tfrac{7}{512}h^4) - (\tfrac{1}{8}h^2 + \tfrac{1}{64}h^4)\cos(2\vartheta) + (h^4/512)\cos(4\vartheta) - \cdots$$

and the separation constant $be_0 = \frac{1}{2}h^2 - \frac{1}{32}h^4 + \cdots$.

Carrying out the same calculations for the other functions, we finally obtain, to the fourth power in the small quantity h:

$$\begin{aligned}
Se_0(h, \cos\vartheta) &= \left(1 + \frac{h^2}{8} + \frac{7h^4}{512}\right) - \left(\frac{h^2}{8} + \frac{h^4}{64}\right)\cos(2\vartheta) \\
&\qquad + \frac{h^4}{512}\cos(4\vartheta) - \cdots \\
be_0(h) &= \tfrac{1}{2}h^2 - \tfrac{1}{32}h^4 + \cdots \\
So_1(h, \cos\vartheta) &= \left(1 + \frac{3h^2}{32} + \frac{13h^4}{3072}\right)\sin\vartheta \\
&\qquad - \left(\frac{h^2}{32} + \frac{h^4}{512}\right)\sin(3\vartheta) + \frac{h^4}{3072}\sin(5\vartheta) - \cdots \\
bo_1(h) &= 1 + \tfrac{1}{4}h^2 - \tfrac{1}{128}h^4 + \cdots \\
Se_1(h, \cos\vartheta) &= \left(1 + \frac{h^2}{32} + \frac{5h^4}{3072}\right)\cos\vartheta \\
&\qquad - \left(\frac{h^2}{32} + \frac{h^4}{512}\right)\cos(3\vartheta) + \frac{h^4}{3072}\cos(5\vartheta) - \cdots \\
be_1(h) &= 1 + \tfrac{3}{4}h^2 - \tfrac{1}{128}h^4 + \cdots \\
So_2(h, \cos\vartheta) &= \left(\tfrac{1}{2} + \frac{h^2}{48} + \frac{23h^4}{36864}\right)\sin(2\vartheta) \\
&\qquad - \left(\frac{h^2}{96} + \frac{h^4}{2304}\right)\sin(4\vartheta) + \frac{h^4}{12288}\sin(6\vartheta) - \cdots \\
bo_2(h) &= 4 + \tfrac{1}{2}h^2 - \tfrac{1}{192}h^4 + \cdots
\end{aligned} \tag{11.2.85}$$

We note that the eigenvalues b increase with the square of h for small values of h. The corresponding normalization constants are

$$\begin{aligned}
M_0^e(h) &= \pi(2 + \tfrac{1}{2}h^2 + \tfrac{13}{128}h^4 + \cdots) \\
M_1^o(h) &= \pi(1 + \tfrac{3}{16}h^2 + \tfrac{7}{384}h^4 + \cdots) \\
M_1^e(h) &= \pi(1 + \tfrac{1}{16}h^2 + \tfrac{1}{192}h^4 + \cdots) \\
M_2^o &= \tfrac{1}{4}\pi(1 + \tfrac{1}{12}h^2 + \tfrac{43}{9216}h^4 + \cdots)
\end{aligned}$$

Referring to the tables at the end of this chapter, we see that, once the values of the coefficients B are determined, it is possible to compute all the radial functions. The most important of these quantities are the values of these functions and of their derivatives with respect to μ, which are necessary for fitting boundary conditions at $\mu = 0$. Expressions for these quantities are:

$$Je_0(h,1) = \sqrt{\frac{\pi}{2}}\left[1 - \frac{h^2}{8} + \frac{7h^4}{512} - \cdots\right];\quad Je_0'(h,1) = 0$$

$$Jo_1(h,1) = 0;\quad Jo_1'(h,1) = \tfrac{1}{2}h\sqrt{\frac{\pi}{2}}\left[1 - \frac{h^2}{32} + \frac{5h^4}{3{,}072} - \cdots\right]$$

$$Je_1(h,1) = \tfrac{1}{2}h\sqrt{\frac{\pi}{2}}\left[1 - \frac{3h^2}{32} + \frac{13h^4}{3{,}072} - \cdots\right];\quad Je_1'(h,1) = 0$$

$$Jo_2(h,1) = 0;\quad Jo_2'(h,1) = \tfrac{1}{8}h^2\sqrt{\frac{\pi}{2}}\left[1 - \tfrac{1}{24}h^2 + \cdots\right]$$

$$Ne_0(h,1) \simeq \sqrt{\frac{2}{\pi}}\,(1 - \tfrac{1}{8}h^2)\ln\left(\frac{\gamma h}{4}\right);\quad \gamma = 1.781$$

$$Ne_0'(h,1) = \sqrt{\frac{2}{\pi}}\left[1 + \frac{h^2}{8} + \frac{h^4}{512} + \cdots\right] \qquad (11.2.86)$$

$$No_1(h,1) = -\frac{2}{h}\sqrt{\frac{2}{\pi}}\left[1 + \frac{h^2}{32} - \frac{h^4}{1{,}536} + \cdots\right]$$

$$No_1'(h,1) \simeq \frac{2}{h}\sqrt{\frac{2}{\pi}}\left[1 - \tfrac{3}{32}h^2 + \tfrac{1}{4}h^2\ln\left(\frac{\gamma h}{4}\right)\right]$$

$$Ne_1(h,1) \simeq -\frac{2}{h}\sqrt{\frac{2}{\pi}}\,[1 + \tfrac{3}{32}h^2]$$

$$Ne_1'(h,1) = \frac{2}{h}\sqrt{\frac{2}{\pi}}\left[1 + \frac{3h^2}{32} + \frac{7h^4}{1{,}536} + \cdots\right]$$

$$No_2(h,1) \simeq -\frac{8}{h^2}\sqrt{\frac{2}{\pi}}\,(1 + \tfrac{1}{24}h^2);\quad No_2'(h,1) \simeq \frac{16}{h^2}\sqrt{\frac{2}{\pi}}\,(1 - \tfrac{1}{24}h^2)$$

where the formulas for Ne_0, No_1', Ne_1, No_2, No_2' are to the first order in h^2, the others are to the second order.

Approximations for h Small and m Large. For larger values of m, in particular, for values of b larger than h^2, we can use the WKBJ approximation, discussed in Sec. 9.3, to give us an approximate solution. According to this approximation, a solution of Eq. (11.2.79) is

$$H(\vartheta) \simeq [b - h^2\cos^2\vartheta]^{-\frac{1}{4}}\,{\cos \atop \sin}\left[\int_0^\vartheta \sqrt{b - h^2\cos^2 u}\,du\right]$$

as long as $(b - h^2\cos^2\vartheta)$ is nowhere zero along the real axis. The integral in the brackets is related to the *elliptic integral* of the second kind;

$$\int_0^\vartheta \sqrt{b - h^2\cos^2 u}\,du = \sqrt{b}\left[E\left(\tfrac{1}{2}\pi, \frac{h}{\sqrt{b}}\right) + E\left(\vartheta - \tfrac{1}{2}\pi, \frac{h}{\sqrt{b}}\right)\right]$$

where

$$E(x,k) = \int_0^x \sqrt{1 - k^2\sin^2\phi}\,d\phi$$

and $$E(\tfrac{1}{2}\pi,k) = \tfrac{1}{2}\pi[1 - \tfrac{1}{4}k^2 - \tfrac{3}{64}k^4 - \tfrac{5}{256}k^6 \cdots]$$

In order that $H(\vartheta)$ be periodic in ϑ, we must have the argument of the cosine or sine be an integral multiple of π when ϑ is π. This is equivalent to the equation

$$2\sqrt{b}\,E(\tfrac{1}{2}\pi,h/\sqrt{b}) = m\pi$$

relating the complete integral of the second kind to m and b. Using the power series for $E(\frac{1}{2}\pi,k)$, we can obtain approximate expressions for the separation constants

$$be_m \simeq bo_m \simeq m^2 + \tfrac{1}{2}h^2 + \tfrac{1}{32}(h^4/m^2) + \cdots$$

To this approximation there is no difference between the values of be and bo for the same value of m. An exact expansion of the higher eigenfunctions in powers of h shows that this is so to an order in powers of h^2 just less than the order m of the eigenfunction. We find the general formula

$$be_m \simeq bo_m \simeq m^2 + \tfrac{1}{2}h^2 + \frac{h^4}{32(m^2-1)} + \frac{(5m^2+7)h^8}{2048(m^2-1)^3(m^3-4)} + \cdots \tag{11.2.87}$$

which is correct for $m = 1$ only as far as the constant term ($m^2 = 1$), is correct for $m = 2$ only as far as the h^2 term, for $m = 3$ only as far as the h^4 term, and so on. For each value of m, the difference between be_m and bo_m has the order h^{2m} for small values of h. For $m > 4$ we can use all the terms given in Eq. (11.2.87).

Turning now to the eigenfunctions themselves, we can expand the integrand in the argument of the cosine or sine in powers of h^2 and integrate, using the expansion for the allowed values of b. This gives us, to the same order of accuracy as is valid for Eq. (11.2.87),

$$\begin{aligned} Se_m(h,\cos\vartheta) &\simeq \left[\frac{be_m - h^2}{be_m - h^2\cos^2\vartheta}\right]^{\frac{1}{4}} \cdot \\ &\quad\cdot \cos\left[m\vartheta - \left(\frac{h^2}{8m}\right)\sin(2\vartheta) - \left(\frac{h^4}{128m^3}\right)\sin(4\vartheta) - \cdots\right] \\ So_m(h,\cos\vartheta) &\simeq \left[\frac{bo_m - h^2}{bo_m - h^2\cos^2\vartheta}\right]^{\frac{1}{4}}\left[m - \left(\frac{h^2}{4m}\right) - \left(\frac{h^4}{32m^3}\right) - \cdots\right]^{-1} \cdot \\ &\quad\cdot \sin\left[m\vartheta - \left(\frac{h^2}{8m}\right)\sin(2\vartheta) - \left(\frac{h^4}{128m^3}\right)\sin(4\vartheta) - \cdots\right] \end{aligned} \tag{11.2.88}$$

which may be expanded in a Fourier series as in Eq. (11.2.85). Knowing the coefficients of the Fourier series, one can then calculate the series for the radial functions as before.

We notice that the larger the value of m, the larger must h become before Se_m, So_m depart much from simple trigonometric functions and before be_m, bo_m differ much from each other. As h increases in size, $(be_m - bo_m)$ increases from zero, at first as h^{2m}, until finally be_m becomes nearly equal to bo_{m+1}, as we shall demonstrate shortly. For small values of h, the b's increase from their initial value of m^2 by an amount proportional to h^2; for large values of h, the b's vary linearly with h, as the next discussion will show.

Expressions for h Large. When h is very large, the solutions turn out to be large only at $\vartheta = \frac{1}{2}\pi$ and $-\frac{1}{2}\pi$, as a review of the Sturm-Liouville analysis will indicate. Referring to the Liouville equation (6.3.12), we see that the Mathieu equation has $p = 1$, $q = -h^2 \cos^2 \vartheta$, $r = 1$, and $\lambda = b$. As indicated on page 723, wherever $(b - h^2 \cos^2 \vartheta)$ is positive, solution y has sinusoidal behavior, and when $(b - h^2 \cos^2 \vartheta)$ is negative, y there has exponential behavior. If b were negative, y would have exponential behavior over the whole range of ϑ, and therefore could not be periodic. If b is positive but smaller than h^2, y can be sinusoidal only near $\vartheta = \frac{1}{2}\pi$ or $-\frac{1}{2}\pi$, where $\cos^2 \vartheta$ is zero. Consequently, the only large values y can have must be near these points, with y dropping to very small values at $\vartheta = 0$ and π, the curvature in these regions being away from the axis, *i.e.*, exponential-like.

For large values of h, the functions Se, So must, therefore, have their predominant values near $\vartheta = \pm\frac{1}{2}\pi$ and must drop off rapidly on either side to take on related small values near $\vartheta = 0, \pi$. Perhaps we should try solving Eq. (11.2.79) near $\vartheta = \pm\frac{1}{2}\pi$, with the boundary condition that the allowed solutions drop off exponentially (like a negative exponential) on either side of these regions.

For large values of h, close to the point $\vartheta = \frac{1}{2}\pi$, Eq. (11.2.79) is, approximately,

$$(d^2y/dx^2) + (b - h^2x^2)y = 0; \quad x = \vartheta - \tfrac{1}{2}\pi$$

This is the equation for Hermite polynomial eigenfunctions, if the allowed range of x were from $-\infty$ to ∞. These polynomials are tabulated at the end of Chap. 6. The eigenfunction solutions, which fall exponentially to zero for large values of z are $\psi(z) = e^{-\frac{1}{2}z^2}H_n(z)$ which satisfy the differential equation

$$(d^2\psi_n/dz^2) + [(2n + 1) - z^2]\psi_n = 0; \quad \psi_n(z) = e^{-\frac{1}{2}z^2}H_n(z)$$

If we require that, near $x = 0$ ($\vartheta = \frac{1}{2}\pi$), y be equal to $A\psi_n(x\sqrt{h})$, then y would satisfy the equation

$$(d^2y/dx^2) + [h(2n + 1) - h^2x^2]y = 0$$

and would fall off exponentially away from $x = 0$.

The actual solution must, therefore, be nearly equal to $A\psi_n(x\sqrt{h})$ near $x = 0$, but it must depart from this behavior for other ranges of x because, of course, y must be periodic in x with period 2π, whereas ψ_n is not periodic, going to zero as $|x|$ increases. We are now in a position, however, to solve Eq. (11.2.79) approximately by setting in a combination of ψ's which solves (11.2.79) approximately in the important regions, near $\vartheta = \frac{1}{2}\pi$ and $-\frac{1}{2}\pi$. We can use one of the combinations $A\{\psi_n[\sqrt{h}\,(\vartheta - \frac{1}{2}\pi)] \pm \psi_n[\sqrt{h}\,(\vartheta + \frac{1}{2}\pi)]\}$. The combination with the plus sign is symmetric about $\vartheta = 0$, so this must correspond to the Se functions; the combination with the minus sign is antisymmetric, so it must be an So function. If the "overlap" between the two ψ functions is small, the solution is probably a good approximation.

We still have to make the solution periodic, but this can be done by adding similar solutions with ϑ increased by 2π, 4π, . . . , -2π, -4π, ... We finally have, for an approximate solution for large h,

$$\begin{aligned} Se_m(h, \cos\vartheta) &\simeq A_m \sum_{n=-\infty}^{\infty} \{\psi_m[\sqrt{h}\,(\vartheta - \tfrac{1}{2}\pi + 2n\pi)] \\ &\qquad\qquad + \psi_m[\sqrt{h}\,(\vartheta + \tfrac{1}{2}\pi + 2n\pi)]\} \\ \psi_m(z) &= e^{-\frac{1}{2}z^2}H_m(z) \\ So_m(h, \cos\vartheta) &\simeq B_m \sum_{n=-\infty}^{\infty} \{\psi_{m-1}[\sqrt{h}\,(\vartheta - \tfrac{1}{2}\pi + 2n\pi)] \\ &\qquad\qquad - \psi_{m-1}[\sqrt{h}\,(\vartheta + \tfrac{1}{2}\pi + 2n\pi)]\} \end{aligned} \tag{11.2.89}$$

where the constants A_m, B_m are determined by our requirement on Se, So at $\vartheta = 0$. The first few eigenfunctions may be expressed in terms of elliptic functions. For instance the lowest one is

$$\begin{aligned} Se_0(h, \cos\vartheta) &\simeq A \sum_{m=-\infty}^{\infty} [e^{-\frac{1}{2}h(\vartheta - \frac{1}{2}\pi + 2m\pi)^2} + e^{-\frac{1}{2}h(\vartheta + \frac{1}{2}\pi + 2m\pi)^2}] \\ &= A \sum_{n=-\infty}^{\infty} e^{-\frac{1}{2}h(\vartheta + n\pi + \frac{1}{2}\pi)^2} = A e^{-\frac{1}{2}h\vartheta^2}\vartheta_2(\tfrac{1}{2}ih\pi\vartheta, e^{-\frac{1}{2}h\pi^2}) \end{aligned}$$

where ϑ_2 is one of the elliptic theta functions defined in Eqs. (4.5.70). Since $Se_0(h,1)$ must be unity, the coefficient A must be $[1/\vartheta_2(0,e^{-\frac{1}{2}h\pi^2})]$. By a fairly well-known transformation of the theta functions (see Prob. 4.51) we can show that

$$Se_0(h, \cos\vartheta) \simeq [\vartheta_4(\vartheta, e^{-2/h})/\vartheta_4(0, e^{-2/h})]$$

which demonstrates the periodicity of the function Se_0.

We therefore have

$$Se_0(h, \cos\vartheta) \simeq \frac{\sqrt{\frac{1}{2}h\pi}}{\vartheta_4(0, e^{-2/h})} \sum_{n=-\infty}^{\infty} e^{-\frac{1}{2}h(\vartheta + n\pi + \frac{1}{2}\pi)^2} = \frac{\vartheta_4(\vartheta, e^{-2/h})}{\vartheta_4(0, e^{-2/h})}$$

Likewise

$$So_1(h, \cos\vartheta) \simeq \frac{\sqrt{\frac{1}{2}\pi h}}{\vartheta_2(0,e^{-2/h})\vartheta_3(0,e^{-2/h})\vartheta_4(0,e^{-2/h})} \sum_{n=-\infty}^{\infty} (-1)^n e^{-\frac{1}{2}h(\vartheta+n\pi+\frac{1}{2}\pi)^2}$$

$$= \frac{\vartheta_1(\vartheta,e^{-2/h})}{\vartheta_2(0,e^{-2/h})\vartheta_3(0,e^{-2/h})\vartheta_4(0,e^{-2/h})} \tag{11.2.90}$$

These are extremely interesting results, relating the solutions of the wave equation in elliptic coordinates to elliptic functions. To this approximation the functions Se_1, So_2, turn out to be related to the derivatives of Se_0, So_1 with respect to ϑ.

To see over what range these approximations hold and also to compute approximate values of the separation constant b, we consider the approximate expressions for y to be trial functions for a variational calculation. According to Eq. (6.2.20) the quantity

$$\Omega = \frac{\left\{\int_0^{2\pi} \psi\left[-\left(\frac{d^2\psi}{d\vartheta^2}\right) + h^2\psi\cos^2\vartheta\right]d\vartheta\right\}}{\int_0^{2\pi}\psi^2\,d\vartheta}$$

is not smaller than b_0^e, and approaches the correct value of b_0^e as ψ approaches the true shape of Se_0.

To see how these integrals go, we compute the normalization constant for the approximate form of Se_0. By "spreading out" one of the sums for Se_0 and correspondingly "piecing out" the range of integration, we obtain

$$M_0^e = A^2\int_{-\infty}^{\infty} e^{-\frac{1}{2}hx^2} \sum_{n=-\infty}^{\infty} e^{-\frac{1}{2}h(x+n\pi)^2}\,dx = A^2\sqrt{\frac{\pi}{h}}\sum_{n=0}^{\infty}\epsilon_n e^{-\frac{1}{4}h\pi^2n^2}$$

$$= A^2\sqrt{\frac{\pi}{h}}\,\vartheta_3(0,e^{-\frac{1}{4}h\pi^2}) = A^2\frac{2}{h}\,\vartheta_3(0,e^{-4/h})$$

The numerator is simplified by using the relations $-\psi'' = (h - h^2x^2)\psi$ and $\cos^2\vartheta = \frac{1}{2} - \frac{1}{2}\cos(2x)$. The calculations give

$$b_0^e \le h + \tfrac{1}{2}h^2 - \frac{h^2\sqrt{h/\pi}}{\vartheta_3(0,e^{-\frac{1}{4}h\pi^2})}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}hx^2}[x^2 + \tfrac{1}{2}\cos(2x)]e^{-\frac{1}{2}h(x+n\pi)^2}\,dx$$

$$= \tfrac{1}{2}h + \tfrac{1}{2}h^2 - \tfrac{1}{2}h^2e^{-1/h}\left[\frac{\vartheta_4(0,e^{-\frac{1}{4}h\pi^2})}{\vartheta_3(0,e^{-\frac{1}{4}h\pi^2})}\right] - \frac{h}{\vartheta_3}\sum_{n=0}^{\infty}\epsilon_n(\tfrac{1}{4}h\pi^2n^2)e^{-\frac{1}{4}h\pi^2n^2}$$

$$\xrightarrow[h\to\infty]{} h - \frac{1}{4} + \frac{1}{6h} - \frac{1}{24h^2} + \cdots - (\tfrac{1}{2}\pi^2 - 2)h^2e^{-\frac{1}{4}h\pi^2} + \cdots$$

On the other hand the approximate value of b_1^0, the separation constant for So_1, turns out to be

$$b_1^0 \leq \tfrac{1}{2}h + \tfrac{1}{2}h^2 - \tfrac{1}{2}h^2 e^{-1/h}\left[\frac{\vartheta_3(0,e^{-\frac{1}{4}h\pi^2})}{\vartheta_4(0,e^{-\frac{1}{4}h\pi^2})}\right] - \frac{h}{\vartheta_4}\sum_{n=1}^{\infty}\epsilon_n(-1)^n(\tfrac{1}{4}h\pi^2 n^2)e^{-\frac{1}{4}h\pi^2 n^2} \qquad (11.2.91)$$

$$\xrightarrow[h\to\infty]{} h - \frac{1}{4} + \frac{1}{6h} - \frac{1}{24h^2} + \cdots + (\tfrac{1}{2}\pi^2 - 2)h^2 e^{-\frac{1}{4}h\pi^2} + \cdots$$

the sole difference being in the last terms. We note that both eigenvalues increase proportional to h for large h.

The factor $e^{-\frac{1}{4}h\pi^2}$ measures the "overlap" of one term with the next in the series. If it is small, we can be sure that the formulas above will be good approximations to the correct solutions. We see that as long as $h > 2$ this overlap term is less than $\frac{1}{100}$, and we should expect the formulas to be correct to within less than a per cent, roughly speaking.

Waves inside an Elliptic Boundary. To solve the interior problem, we must find the values of h for which the radial function Je, Jo satisfies certain boundary conditions at $\mu = \mu_0$, the boundary ellipse. We must use the Je, Jo functions here, for the Ne, No functions have a discontinuity in value or slope at $\mu = 0$ and are used only in exterior problems.

For an elliptic membrane the boundary condition is that $\psi = 0$ along the boundary. Suppose this boundary is an ellipse of major axis A and minor axis B. Referring to the equations for the coordinates, we see that $a \cosh \mu_0 = A$ and $a \sinh \mu_0 = B$, so that $a^2 = A^2 - B^2$ and $\mu_0 = \tanh^{-1}(B/A)$. Consequently we can calculate a and μ_0, given A and B, or given the interfocal distance a and the eccentricity $e = \sqrt{1 - (B/A)^2} = \operatorname{sech} \mu_0$ (or any other pair of the quantities A, B, a, e). The boundary condition, which fixes the allowed values of the frequency, is

$$Je_m(h, \cosh \mu_0) = 0 \quad \text{or} \quad Jo_m(h, \cosh \mu_0) = 0$$

where $h = a\omega/2c = \pi a\nu/c = \pi a/\lambda$. We adjust ω, and thus h, until Je (or Jo) is zero for $\mu = \mu_0$.

When μ_0 is large (*i.e.*, when the eccentricity $e = \operatorname{sech} \mu_0$ is small), there will be roots of these equations for small values of h. In this case we can use the approximate forms given in Eqs. (11.2.85), to obtain the allowed frequencies. For instance, for the lowest mode of vibration ($m = 0$), the radial function, to the second order in the small quantity h, is

$$Je_0(h, \cosh \mu) \simeq \sqrt{\frac{\pi}{2}}\,\{(1 + \tfrac{1}{8}h^2)J_0(h \cosh \mu) + \tfrac{1}{8}h^2 J_2(h \cosh \mu) + \cdots\}$$

If the higher terms were not present, this function would be zero when $h \cosh \mu_0$ were equal to $\pi\beta_{01}$ (see table on page 1565), the lowest root of $J_0(\pi\beta) = 0$. If h is small, the root of $Je = 0$ will not be far from this. Near $\pi\beta_{01}$, we have $J_0(x) \simeq (\pi\beta_{01} - x)J_1(\pi\beta_{01})$ so that the equation $Je_0 = 0$ becomes, approximately,

$$(1 + \tfrac{1}{8}h^2)(\pi\beta_{01} - h \cosh \mu_0)J_1(\pi\beta_{01}) + \tfrac{1}{8}h^2 J_2(\pi\beta_{01}) = 0$$

with a solution, accurate to the second order in $e = (1/\cosh \mu_0)$

$$h_{01} \simeq \pi\beta_{01}e + \tfrac{1}{8}(\pi\beta_{01}e)^2[J_2(\pi\beta_{01})/J_1(\pi\beta_{01})] \tag{11.2.92}$$

Inserting numerical values $\beta_{01} = 0.7655$, $J_2(\pi\beta_{01}) = 0.4318$, $J_1(\pi\beta_{01}) = 0.5191$ and expressing things in terms of directly measurable quantities, we have for the lowest mode of the nearly circular membrane

$$\nu_{01} \simeq (c/A)[0.7655 + 0.1914e]; \quad B \to A$$

where $c = \sqrt{T/\rho}$ is the speed of transverse waves along the membrane, A is the major axis of the boundary, and $e = \sqrt{1 - (B/A)^2}$ is the eccentricity of the boundary. The first term is the same as for a circular membrane of diameter A. The second term is a first-order correction to this frequency to take into account the effect of eccentricity. It obviously is not valid except for very small values of eccentricity e. The corresponding eigenfunction is

$$\psi \simeq \sqrt{\frac{\pi}{2}}\,[(1 + \tfrac{1}{8}h^2) - \tfrac{1}{8}h^2 \cos(2\vartheta)][(1 + \tfrac{1}{8}h^2)J_0(h \cosh \mu) + \tfrac{1}{8}h^2 J_2(h \cosh \mu)]$$

with the value of h given by Eq. (11.2.92).

When the eccentricity is large, μ_0 is small and, to the second order in μ_0, $B \simeq a\mu_0$ and $A \simeq a(1 + \tfrac{1}{2}\mu_0^2)$ or, to the third order in B/A, $a \simeq A - \tfrac{1}{2}(B^2/A)$, $\mu_0 \simeq (B/A) + \tfrac{1}{3}(B/A)^3$. In this case even the lowest allowed value of h is large, and we should use the formulas preceding Eq. (11.2.90). To fit the boundary conditions, we substitute $i\mu$ for ϑ in the theta-function expression for Se_0. Setting this equal to zero gives us

$$\vartheta_2(-\tfrac{1}{2}h\pi\mu_0, e^{-\frac{1}{2}h\pi^2}) = 2e^{-\frac{1}{8}\pi^2 h}[\cos(\tfrac{1}{2}h\pi\mu_0) + e^{-h\pi^2}\cos(\tfrac{3}{2}h\pi\mu_0) + \cdots] = 0$$

The lowest root is $h\mu_0 = 1$ or

$$h_{01} \simeq \left(\frac{A}{B}\right) - \tfrac{1}{3}\left(\frac{B}{A}\right) \quad \text{or} \quad \nu_{01} \simeq \frac{c}{\pi B}\left[1 + \tfrac{1}{6}\left(\frac{B}{A}\right)^2\right]; \quad B \ll A$$

In this limit the frequency is more nearly determined by the magnitude of the minor axis B than by the major axis $A (A \gg B)$.

The corresponding expression for the eigenfunction for the lowest mode for the very eccentric elliptic membrane is

$$\psi_{01} \simeq K\left[\sum_{n=0}^{\infty} \epsilon_n(-1)^n e^{-2n^2/h} \cos(2n\vartheta)\right] \cdot \\ \cdot \left\{e^{\frac{1}{2}h\mu^2} \sum_{m=0}^{\infty} e^{-\frac{1}{2}h\pi^2(m+\frac{1}{2})^2} \cos[(m+\tfrac{1}{2})\pi h\mu]\right\}$$

where the value of h is that given above.

For intermediate ranges of eccentricity for the boundary, or for more accurate answers, we must use the exact solutions, as tabulated at the back of the book. The solutions are $Se_m(h, \cos\vartheta)Je_m(h, \cosh\mu)$ with h a root of $Je_m(h, \cosh\mu_0) = 0$, or the corresponding combination with So and Jo.

Green's Function and Plane Wave Expansions. The general techniques leading to Eq. (7.2.63) are easy to apply here to obtain the expansion in elliptic coordinates of the two-dimensional Green's function for the whole plane:

$$\begin{aligned} G(\mathbf{r}|\mathbf{r}_0|\omega) &= G_k(\mathbf{r}|\mathbf{r}_0) = i\pi H_0^1(kR) \\ &= 4\pi i\left\{\sum_{m=0}^{\infty}\left[\frac{Se_m(h,\cos\vartheta_0)}{M_m^e(h)}\right] Se_m(h,\cos\vartheta)\cdot\right. \\ &\qquad \cdot\begin{cases} Je_m(h,\cosh\mu_0)He_m(h,\cosh\mu); & \mu > \mu_0 \\ Je_m(h,\cosh\mu)He_m(h,\cosh\mu_0); & \mu_0 > \mu \end{cases} \\ &\quad + \sum_{m=1}^{\infty}\left[\frac{So_m(h,\cos\vartheta_0)}{M_m^0(h)}\right] So_m(h,\cos\vartheta)\cdot \\ &\qquad \left.\cdot\begin{cases} Jo_m(h,\cosh\mu_0)Ho_m(h,\cosh\mu); & \mu > \mu_0 \\ Jo_m(h,\cosh\mu)Ho_m(h,\cosh\mu_0); & \mu_0 > \mu \end{cases}\right\} \end{aligned} \quad (11.2.93)$$

where $h = ak/2 = a\omega/2c$ and where

$$\begin{aligned} He_m &= Je_m + iNe_m = -iC_m^e(h,\mu)e^{i\delta_m{}^e(h,\mu)} \\ Ho_m &= Jo_m + iNo_m = -iC_m^0(h,\mu)e^{i\delta_m{}^0(h,\mu)} \end{aligned}$$

are the solutions corresponding to outgoing waves.

The derivation of the plane wave expansion from this is straightforward, but for this once we will calculate it out. We set the source point a large distance from the origin in a direction $u + \pi$ to the positive x axis. The quantity R then becomes $r_0 + r\cos(u - \phi)$, with $r_0 \gg r$, $\vartheta_0 \to \phi + \pi$ and $kr_0 \to h\cosh\mu_0$. The asymptotic form for $i\pi H_0(kR)$ is thus $\sqrt{2\pi i/kr_0}\, e^{ikr_0+ikr\cos(u-\phi)}$. Using the relation $Se_m(h, -\cos u) = (-1)^m Se_m(h, \cos u)$ and the asymptotic form for He_m, and similarly for Ho_m, we have

$$\sqrt{\frac{2\pi i}{kr_0}}\, e^{ikr_0+ikr\cos(u-\phi)} \xrightarrow[r_0\to\infty]{} \frac{4\pi\sqrt{i}}{\sqrt{h\cosh\mu_0}}\, e^{ih\cosh\mu_0} \sum_m (-1)^m e^{-\frac{1}{2}im\pi} \cdot \left\{\left[\frac{Se_m(u)}{M_m^e}\right] Se_m(\vartheta)Je_m(\mu) + \left[\frac{So_m(u)}{M_m^0}\right] So_m(\vartheta)Jo_m(\mu)\right\}$$

Dividing out as many things as we can, we finally obtain

$$e^{ikr\cos(u-\phi)} = e^{ik[x\cos u+y\sin u]} = \sqrt{8\pi}\sum_m i^m \left\{\left[\frac{Se_m(h,\cos u)}{M_m^e(h)}\right] Se_m(h,\cos\vartheta)Je_m(h,\cosh\mu) + \left[\frac{So_m(h,\cos u)}{M_m^0(h)}\right] So_m(h,\cos\vartheta)Jo_m(h,\cosh\mu)\right\} \tag{11.2.94}$$

giving the expansion of a plane wave proceeding at an angle u with respect to the positive x axis (*i.e.*, with respect to the major axis of the ellipse).

From this last formula we can obtain the integral expression for the eigenfunction solutions in elliptic coordinates, having the form of Eq. (11.2.2):

$$\begin{aligned} Se_m(h,\cos\vartheta)Je_m(h,\cosh\mu) &= \frac{1}{\sqrt{8\pi}\, i^m}\int_0^{2\pi} e^{ikX} Se_m(h,\cos u)\,du \\ So_m(h,\cos\vartheta)Jo_m(h,\cosh\mu) &= \frac{1}{\sqrt{8\pi}\, i^m}\int_0^{2\pi} e^{ikX} So_m(h,\cos u)\,du \end{aligned} \tag{11.2.95}$$

where $X = r\cos(u-\phi) = x\cos u + y\sin u$. And from this formula we can obtain expressions for Se_mJe_m in terms of the polar coordinate solutions r^mJ_m. Expanding $Se_m(h, \cos u)$ in Fourier series (11.2.81) and using Eq. (11.2.21), we have

$$Se_{2m}(h,\cos\vartheta)Je_{2m}(h,\cosh\mu) = \sqrt{\tfrac{1}{2}\pi}\sum_{n=0}^{\infty} B_{2n}^e(h,2m)\cos(2n\phi)J_{2n}(kr) \tag{11.2.96}$$

and similarly with Se_{2m+1} and the So, Jo functions.

Since Se_m is an eigenfunction, we can expand a trigonometric function of u in a series of Se, So's, as

$$\cos(2mu) = 2\pi\sum_{n=0}^{\infty}\left(\frac{1}{M_{2n}^e\epsilon_m}\right) B_{2m}^e(h,2n)Se_{2n}(h,\cos u)$$

Therefore, we can write

$$\cos(2m\phi)J_{2m}(kr) = \frac{\sqrt{8\pi}}{\epsilon_m}\sum_{n=0}^{\infty}\left[\frac{B_{2m}^e(h,2n)}{M_{2n}^e(h)}\right] Se_{2n}(h,\cos\vartheta)Je_{2n}(h,\cosh\mu)$$

which is the inverse of Eq. (11.2.96). There are obvious extensions to other symmetries for ϕ or ϑ.

Radiation from a Vibrating Strip. The simplest external problem is the one dealing with the radiation from a surface which is just the strip $\mu = 0$, of zero thickness and width a. If the strip is long enough and the motion is uniform along its axis, then the problem becomes two-dimensional and elliptic axes may be used, the focal points of the ellipse being at the two edges of the strip. Two examples will be given: one the acoustic radiation from a vibrating strip; the other electromagnetic radiation from a strip carrying current.

In the case of the vibrating strip we assume the motion to be normal to the surface of the strip, with a velocity $v_0e^{-i\omega t}$ over the part of the surface from $\vartheta = 0$ to $\vartheta = \pi$ and $-v_0e^{-i\omega t}$ from $\vartheta = \pi$ to $\vartheta = 2\pi$. Therefore, if ψ is the velocity potential, the boundary condition at $\mu = 0$ is that $-\operatorname{grad}_\mu \psi = +v_0(0 < \vartheta < \pi) = -v_0(\pi < \vartheta < 2\pi)$ or, since $\operatorname{grad}_\mu \psi$ at $\mu = 0$ is $|2/a \sin \vartheta|(\partial\psi/\partial\mu)$,

$$(\partial\psi/\partial\mu)_{\mu=0} = -\tfrac{1}{2}av_0 \sin \vartheta$$

which means that ψ will be an odd function of ϑ, so we must use So, Jo, No. The boundary condition at infinity is that the radiation is going outward, so we use the analogues of the Hankel functions, Ho_m, for the radial functions. The slope of this function is

$$\frac{d}{d\mu} Ho_m(h, \cosh \mu) = iC_m^0(h,\mu)e^{i\delta_m{}^0(h,\mu)} = \frac{d}{d\mu}[Jo_m + iNo_m]$$

which defines the quantities C_m^0 and δ_m^0, analogous to the polar coordinate case.

What is needed in this problem is the value of the slope at $\mu = 0$. Calling the values of C and δ at $\mu = 0$ just C_m^0 and δ_m^0 and using the formulas at the end of this chapter, we have

$$\begin{aligned}[(d/d\mu)Ho_m(h, \cosh \mu)]_{\mu=0} &= iC_m^0\, e^{i\delta_m{}^0} \\ &= \sqrt{\tfrac{1}{2}\pi}\left\{\frac{\frac{1}{4}h^2(-1)^{\frac{1}{2}m}B_2^0(h,m)}{So'_m(h,0)} + \frac{i}{B_2^0(h,m)}\sum_{n=1}^{\infty}(-1)^{n-\frac{1}{2}m}\cdot\right. \\ &\quad \left.\cdot\,(2n)B_{2n}^0(h,m)[2J_nN_n - J_{n-1}N_{n+1} - J_{n+1}N_{n-1}]\right\}; \quad m \text{ even}\end{aligned}$$

where the arguments for the J_n, M_n, etc., are all $\frac{1}{2}h$. There is a similar formula for m odd. By using Eqs. (11.2.86) we see that, for h small,

$$C_1^0 \simeq \frac{2}{h}\sqrt{\frac{2}{\pi}}\,(1 - \tfrac{3}{32}h^2); \quad \delta_1^0 \simeq -\frac{\pi}{8}h^2(1 + \tfrac{1}{16}h^2)$$

Next we expand $\sin \vartheta$ in a series of the eigenfunctions So. Only odd values of m enter (why?), and we eventually find that

$$\sin \vartheta = \pi \sum_{m=0}^{\infty} \left[\frac{B_1^0(h, 2m+1)}{M_{2m+1}^0(h)} \right] So_{2m+1}(h, \cos \vartheta)$$

Consequently, the expression for the radiated wave is

$$\psi = \tfrac{1}{2} i\pi a v_0 \sum_{m=0}^{\infty} \left[\frac{B_1^0(h, 2m+1)}{M_{2m+1}^0(h)} \right] \left[\frac{e^{-i\delta_{2m+1}{}^0}}{C_{2m+1}^0} \right] \cdot$$
$$\cdot\, So_{2m+1}(h, \cos \vartheta) Ho_{2m+1}(h, \cosh \mu)$$

From this equation we can compute the reaction of the fluid back of the strip (and thus the acoustic impedance load on the strip) and also the amount of energy radiated by the strip. To find the impedance, we first compute the pressure at the surface of the strip, $\mu = 0$. Since

$$\begin{aligned} Ho_{2m+1}(h,1) &= Jo_{2m+1}(h,1) + iNo_{2m+1}(h,1) \\ &= 0 - [i/Jo'_{2m+1}(h,1)] = [i/C_{2m+1}^0 \sin(\delta_{2m+1}^0)] \end{aligned}$$

and since the pressure p at any point is $-i\omega\rho$ times the velocity potential ψ there, we have

$$p = -\tfrac{1}{2}\pi a \omega \rho v_0 \sum_m \left[\frac{B_1^0(h, 2m+1)}{M_{2m+1}^0} \right] \left[\frac{i e^{-i\delta_{2m+1}{}^0}}{(C_{2m+1}^0)^2 \sin(\delta_{2m+1}^0)} \right] So_{2m+1}(h, \cos \vartheta)$$

The total force F per unit length on the strip is $\frac{1}{2}a\int p \sin \vartheta \, d\vartheta$ in the negative y direction

$$F = -\tfrac{1}{2} a \pi^2 h \rho c v_0 \sum_m \left[\frac{B_1^0(h, 2m+1)}{C_{2m+1}^0(h,0)} \right]^2 \left[\frac{1 + i \cot(\delta_{2m+1}^0)}{M_{2m+1}^0} \right]$$

The ratio between F and the strip velocity v_0 is the acoustic impedance of the radiation load, per unit length of strip, $Z = R - iX$, where

$$\begin{aligned} R &= \tfrac{1}{2} a \pi^2 h \rho c \sum_m \left[\frac{B_1^0(h, 2m+1)}{C_{2m+1}^0} \right]^2 \frac{1}{M_{2m+1}^0} \\ X &= -\tfrac{1}{2} a \pi^2 h \rho c \sum_m \left[\frac{B_1^0(h, 2m+1)}{C_{2m+1}^0} \right]^2 \frac{\cot(\delta_{2m+1}^0)}{M_{2m+1}^0} \end{aligned} \qquad (11.2.97)$$

where R is the radiation resistance and X the radiation reactance per unit length of strip.

For low frequencies ($\omega \ll 2c/a$) or long wavelengths ($\lambda \gg \pi a$), h is small compared with unity and one can use Eqs. (11.2.85) and (11.2.86) to compute approximate values of the impedance terms:

$$R \simeq \frac{\pi^2}{16}\,a\rho c h^3 = \left(\frac{\pi^2 a^4 \omega^3}{128c^3}\right)\rho c;\quad X \simeq \frac{\pi}{2}\,a\rho c h = \omega(\tfrac{1}{4}\pi a^2 \rho)$$

The radiation resistance at low frequencies thus increases with the cube of the frequency, whereas the reactance increases linearly with the frequency. At very low frequencies, therefore, the fluid load is a masslike reactance ($X = M\omega$) with an effective mass equal to ($\frac{1}{4}\pi\rho a^2$) per unit length of strip.

At very high frequencies, the radiation is a plane wave moving out from both sides of the strip, normal to its surface. Since the acoustic impedance in a plane wave is, according to page 313, ρc per unit area, the limiting value of R at very high frequencies ($\omega \gg 2c/a$) is $2\rho ca$ and that of X is zero. For intermediate frequencies, for ω of the order of magnitude of $2c/a$, we must use the tables and the complete formulas (11.2.97).

Going back to the equation for ψ, the limiting value at large distances from the strip, where $\cosh\mu \to 2r/a$, $\vartheta \to \phi$ (where r and ϕ are polar coordinates, using the center of the strip as origin), is

$$\psi \to \frac{i\pi a v_0}{\sqrt{2ikr}}\,e^{ikr} \sum_m \left[\frac{B_1^0(h,\,2m+1)e^{-i\delta_{2m+1}^0}}{(-1)^m M_{2m+1}^0 C_{2m+1}^0}\right] So_{2m+1}(h,\cos\phi)$$

At these distances the wave is practically a plane wave, so the intensity (see page 1374) $P = \frac{1}{2}\bar{p}u = \frac{1}{2}\rho c k^2|\psi|^2$,

$$S = \frac{\pi^2 \rho c k a^2 v_0^2}{4r} \sum_{m,n} \left[\frac{B_1^0(h,\,2m+1)B_1^0(h,\,2n+1)}{M_{2m+1}^0 M_{2n+1}^0 C_{2m+1}^0 C_{2n+1}^0}\right]\cdot$$
$$\cdot\cos[\delta_{2m+1}^0 - \delta_{2n+1}^0 + \pi(m-n)]So_{2m+1}(h,\cos\phi)So_{2n+1}(h,\cos\phi) \quad (11.2.98)$$

The total energy radiated per unit length of strip is this, times r, integrated over ϕ. Because of the orthogonality of the So's, this is

$$P = \tfrac{1}{4}a\pi^2 h\rho c v_0^2 \sum_{m=0}^{\infty} \left[\frac{B_1^0(h,\,2m+1)}{C_{2m+1}^0}\right]^2 \left[\frac{1}{M_{2m+1}^0}\right]$$

which is equal to $\frac{1}{2}v_0^2$ times the radiation resistance R, as it should.

Radiation from a Strip Carrying Current. To discuss the radiation from a conducting strip carrying current, we need to review the electromagnetic equations discussed in Sec. 2.5. We start with the vector potential **A**. If there is no free charge, we can set the scalar potential equal to zero by requiring the vector potential to have zero divergence.

Then the electric intensity is $-(1/c)(\partial \mathbf{A}/\partial t) = (i\omega/c)\mathbf{A}$ for a simple harmonic field (with $\epsilon = \mu = 1$) and the magnetic intensity is $\mathbf{H} = \text{curl } \mathbf{A}$. The equation for $\mathbf{A}$ is

$$\Box^2 \mathbf{A} = -\text{ curl curl } \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{J}$$

where $\mathbf{J}$ is the current density.

If the potential drop, driving the current through the strip, is impressed uniformly along the length of the strip, $\mathbf{A}$ is everywhere in the z direction, perpendicular to the $(x,y)(\mu,\vartheta)$ plane, having the form $\mathbf{a}_z\psi e^{-i\omega t}$, where ψ is a scalar. To have $\text{div } \mathbf{A} = 0$, ψ must be a function of x, y or μ, ϑ and not of z, which reduces the whole problem to a two-dimensional one. In spite of the fact that $\mathbf{A}$, and $\mathbf{E}$ are in the z direction, the space dependence of ψ, $\mathbf{A}$, $\mathbf{E}$, and $\mathbf{H}$ involves x and y and not z, and we can deal entirely with the scalar ψ. This amplitude of the vector potential satisfies the Helmholtz equation, the homogeneous one outside the strip, the inhomogeneous one inside the strip.

The boundary conditions for ψ arise from the requirement that the potential drop be uniform along the strip and that the wave be an outgoing one. If the potential drop per unit length of strip is $E_0e^{-i\omega t}$, the first boundary condition requires that $\psi = -i(cE_0/\omega)$ at $\mu = 0$, the second boundary condition requires that ψ be made up of the radial functions He, Ho.

The boundary condition on ψ at $\mu = 0$ requires that ψ be symmetric about $\vartheta = 0$, which restricts the radial functions to the He type, and also that ψ be symmetric about $\vartheta = \frac{1}{2}\pi, \frac{3}{2}\pi$, which further restricts us to the He_{2m} type function. Consequently $J(\vartheta)$ is symmetric about $\vartheta = 0$, π and also about $\vartheta = \frac{1}{2}\pi, \frac{3}{2}\pi$. The series for ψ is thus

$$\Sigma c_m Se_{2m}(h, \cos\vartheta) He_{2m}(h, \cosh\mu)$$

From the resulting solution we can compute the current along the strip (and thus the impedance) as well as the intensity of radiation at large distances. The use of the functions He_{2m} ensures that there is a discontinuity in slope of ψ as we go through the strip from one side to the other. This discontinuity is equal to twice the slope of ψ at $\mu = 0$, so that the current density in the strip ($\mu = 0$) at the point $(0,\vartheta)$ is $-(1/2\pi)$ times the slope of ψ at $\mu = 0$:

$$\frac{2}{a|\sin\vartheta|}\left[\frac{\partial\psi}{\partial\mu}\right]_{\mu=0} = -2\pi J(\vartheta)$$

The total current along the strip in the positive z direction is then $\frac{1}{2}a\int J(\vartheta)\,|\sin\vartheta|\,d\vartheta$.

Using the orthogonal properties of the Se_{2m} functions to calculate the coefficients of the series from the boundary condition at $\mu = 0$, we have for the amplitude of the vector potential at the point (μ,ϑ)

$$\psi(\mu,\vartheta) = 2\pi\left(\frac{cE_0}{\omega}\right)\sum_{m=0}^{\infty}\left[\frac{B_0^e(h,2m)}{M_{2m}^e(h)}\right]\left[\frac{e^{-i\delta_{2m}^e}}{C_{2m}^e}\right]Se_{2m}(h,\cos\vartheta)He_{2m}(h,\cosh\mu)$$

where $He_{2m}(h,1) = -iC_{2m}^e e^{i\delta_{2m}^e} = [Je_{2m}(h,1) + iNe_{2m}(h,1)]$. The current density along the surface of the strip at the point $\mu = 0$, [where we count the two "sides" separately, the points $0 < \vartheta < \pi$ being different from the points $\pi < \vartheta < 2\pi$, and the total current density in the strip at the point $x = \frac{1}{2}a\cos\vartheta$ being the sum of $J(\vartheta)$ and $J(2\pi - \vartheta)$] is

$$J(\vartheta) = \frac{E_0}{h|\sin\vartheta|}\sum_m\left[\frac{B_0^e(h,2m)}{M_{2m}^e(h)}\right]\left[\frac{1 + i\cot(\delta_{2m}^e)}{(C_{2m}^e)^2}\right]Se_{2m}(h,\cos\vartheta)$$

We see that the current is not uniform over the strip, for it goes to infinity at each edge (as did the velocity of flow through a slit). The total current I is not infinite, however. The admittance of the strip per unit length, the ratio $I/E_0 = Y = G - iS$, turns out to be

$$\begin{aligned} G(\omega) &= \left(\frac{\pi a}{h}\right)\sum_m\left[\frac{B_0^e(h,2m)}{C_{2m}^e(h)}\right]^2\left[\frac{1}{M_{2m}^e}\right]; \\ S(\omega) &= -\left(\frac{\pi a}{h}\right)\sum_m\left[\frac{B_0^e(h,2m)}{C_{2m}^e}\right]^2\left[\frac{\cot(\delta_{2m}^e)}{M_{2m}^e}\right] \end{aligned} \tag{11.2.99}$$

For very low frequencies ($\lambda \gg \pi a$), the limiting dependence on ω and a is given by the formulas

$$G \simeq \left(\frac{\pi c}{2\omega}\right)\frac{1}{\ln^2(4/\gamma h)}; \quad S = -\frac{(c/\omega)}{\ln(4/\gamma h)}; \quad h \ll 1$$

which corresponds, to this approximation, to a radiation resistance R of the strip and a radiation inductance L per unit length, where

$$R \simeq (\pi\omega/2c); \quad L \simeq [(1/c)\ln(0.717\lambda/a)]$$

as long as $\ln(0.717\lambda/a) \gg 1$. We have, in usual notation, $1/Y = Z = R - i\omega L$. The radiation resistance for very low frequencies is thus independent of the size (or shape) of the strip. As long as the conductor cross section is small compared to the wavelength, the real part of the impedance does not depend on the shape of the conductor cross section. The effective inductance, for low frequencies, does depend on the strip width a, but only logarithmically. Because of the logarithms, convergence is slow and the approximate formulas hold for a much narrower range of ω than do the ones for acoustic radiation.

It is not difficult to calculate the intensity of radiation from the strip, at large distances, by calculating Poynting's vector [see Eq. (2.5.28)]. The total energy radiated per unit length of strip is, of course, $\frac{1}{2}GE_0^2$.

Scattering of Waves from Strips. The scattering of waves from a strip is calculated by methods quite similar to those used for obtaining Eq. (11.2.28), for scattering from a cylinder. We have two limiting cases: case I, where the wave is zero at $\mu = 0$, corresponding to electromagnetic waves with the electric vector polarized along the axis of the strip and case II, where the normal gradient is zero at $\mu = 0$, corresponding to acoustic waves and also electromagnetic waves with the magnetic vector in the z direction. In both cases we start with the plane wave expansion of Eq. (11.2.94) and add enough outgoing waves (with functions He, Ho) to satisfy the boundary conditions.

We shall simplify the formulas by using the following relations:

$$\begin{aligned}
Je_m &= C_m^e \sin \delta_m^e; & Ne_m &= -C_m^e \cos \delta_m^e; & He_m &= -iC_m^e e^{i\delta_m^e} \\
Jo'_m &= -C_m^0 \sin \delta_m^0; & No'_m &= C_m^0 \cos \delta_m^0; & Ho'_m &= iC_m^0 e^{i\delta_m^0} \\
Je'_m &= -C_m^{e'} \sin \delta_m^{e'}; & Ne'_m &= C_m^{e'} \cos \delta_m^{e'}; & He'_m &= iC_m^{e'} e^{i\delta_m^{e'}} \\
Jo_m &= C_m^{0''} \sin \delta_m^{0''}; & No_m &= -C_m^{0''} \cos \delta_m^{0''}; & Ho_m &= -iC_m^{0''} e^{i\delta_m^{0''}}
\end{aligned} \tag{11.2.100}$$

where the prime on the Je, Ne, etc., functions indicates the derivative with respect to μ. The reason for the lack of prime on C_m^0 and δ_m^0 and the double prime on $C_m^{0''}$ and $\delta_m^{0''}$ will be apparent shortly. The parameter h has been omitted to simplify the equations. So has the argument μ, which can take on any positive, real value, though in most cases considered μ will be zero for these formulas. When $\mu = 0$, two of the four sets of definitions simplify:

$$\begin{aligned}
\delta_m^{0''} &= 0; & C_m^{0''} &= -[1/C_m^0 \sin \delta_m^0] \\
\delta_m^{e'} &= 0; & C_m^{e'} &= [1/C_m^e \sin \delta_m^e]; \quad \text{for } \mu = 0
\end{aligned}$$

so that for the $\mu = 0$ case we need only use the unprimed amplitudes and phase factors. Unless we specify otherwise, the quantities C and δ will be considered to be for $\mu = 0$ in the following discussion, so that these modifications may be used.

For small values of h, we have

$$\begin{aligned}
C_0^e &\simeq \sqrt{\frac{2}{\pi}}\left(1 - \frac{1}{8}h^2\right)\left[\ln\left(\frac{4}{\gamma h}\right) + \frac{\pi^2}{8\ln(4/\gamma h)}\right] \\
\delta_0^e &\simeq \frac{(\pi/2)}{\ln(4/\gamma h)}\left[1 - \frac{(\pi/12)}{\ln^2(4/\gamma h)}\right] \\
C_1^e &\simeq \sqrt{\frac{2}{\pi}}\left(\frac{2}{h} + \frac{3}{16}h\right); \quad \delta_1^e \simeq \frac{\pi h^2}{8} \\
C_1^0 &\simeq \sqrt{\frac{2}{\pi}}\left(\frac{2}{h} - \frac{3}{16}h\right); \quad \delta_1^0 \simeq -\left(\frac{\pi h^2}{8}\right)\left(1 + \frac{1}{16}h^2\right) \\
C_2^0 &\simeq \sqrt{\frac{2}{\pi}}\left(\frac{16}{h^2} - \frac{2}{3}\right); \quad \delta_2^0 \simeq -\left(\frac{\pi h^4}{256}\right); \quad \gamma = 1.7811 \cdots
\end{aligned} \tag{11.2.101}$$

For case I ($\psi = 0$ at $\mu = 0$) the functions Jo_m are already zero at $\mu = 0$ so only the Je terms in Eq. (11.2.94) need corrections added to them. The series for the scattered wave, in this case, may be obtained by an obvious variant of that used to obtain Eq. (11.2.28),

$$\psi_s^{\mathrm{I}} = -i\sqrt{8\pi}\sum_{m=0}^{\infty} i^m \left[\frac{Se_m(h, \cos u)}{M_m^e(h)}\right] e^{-i\delta_m^e} \sin(\delta_m^e) \cdot \\ \cdot Se_m(h, \cos \vartheta) He_m(h, \cosh \mu) \tag{11.2.102}$$

for a unit-amplitude, incident wave. The scattered wave for case II, on the other hand, involves only the So, Ho functions, since Je'_m is zero at $\mu = 0$:

$$\psi_s^{\mathrm{II}} = -i\sqrt{8\pi}\sum_{m=1}^{\infty} i^m \left[\frac{So_m(h, \cos u)}{M_m^0(h)}\right] e^{-i\delta_m^0} \sin(\delta_m^0) \cdot \\ \cdot So_m(h, \cos \vartheta) Ho_m(h, \cosh \mu) \tag{11.2.103}$$

The intensity scattered at large distances from the strip, per unit incident intensity, is just the value of $|\psi_s|^2$ for large μ,

$$S^{\mathrm{I}} \to \frac{8\pi}{kr}\sum_{m,n}\left[\frac{\sin(\delta_m^e)\sin(\delta_n^e)}{M_m^e M_n^e}\right]\cos(\delta_m^e - \delta_n^e) \cdot \\ \cdot Se_m(h, \cos u) Se_n(h, \cos u) Se_m(h, \cos \vartheta) Se_n(h, \cos \vartheta) \tag{11.2.104}$$

with a similar equation, using M_m^0, δ_m^0, So_m for case II. The total energy scattered per unit length of strip, per unit intensity of incident wave (*i.e.*, the effective scattering width of the strip) for case I is then

$$Q^{\mathrm{I}} = \frac{4\pi a}{h}\sum_{m=0}^{\infty}\left[\frac{1}{M_m^e(h)}\right]\sin^2 \delta_m^e [Se_m(h, \cos u)]^2 \tag{11.2.105}$$

where u is the angle of incidence of the plane wave. The parallel formula for case II is obvious.

Several interesting (but obvious) facts are apparent from these formulas. In the first place Q^{II} is zero when $u = 0$, whereas Q^{I} is not zero there. When the plane wave comes parallel to the strip's cross section (along the x axis), then the strip does not disturb the wave at all if the boundary condition is that normal gradient be zero, but it is disturbed if the requirement is that ψ be zero at the strip. Likewise, the scattered intensity for case II is zero for $\vartheta = 0, \pi$, no matter what the angle of incidence, this is not true for case I.

For very low frequencies, *i.e.*, for wavelengths λ very long compared to (πa), the scattered intensities and effective widths are

$$
\begin{aligned}
S^{\text{I}} &\to \frac{(\pi a/4r)}{h \ln^2(4/\gamma h)} = \frac{\lambda}{4r \ln^2(0.717\lambda/a)} \\
Q^{\text{I}} &\to \frac{\pi^2 a}{2h \ln^2(4/\gamma h)} = \frac{\pi\lambda}{2 \ln^2(0.717\lambda/a)} \\
S^{\text{II}} &\to \frac{\pi a h^3}{16r} \sin^2 u \sin^2 \vartheta = \frac{\pi^4 a^4}{16 r\lambda^3} \sin^2 u \sin^2 \vartheta \qquad (11.2.106) \\
Q^{\text{II}} &\to \frac{\pi^2 a h^3}{16} \sin^2 u = \frac{\pi^5 a^4}{16\lambda^3} \sin^2 u
\end{aligned}
$$

when $\lambda \gg (\pi a)$ or $h \ll 1$. We note that, as the frequency goes to zero $(\lambda \to \infty)$, the scattering for case I $(\psi = 0)$ increases, whereas for case II $(\partial\psi/\partial\mu = 0)$, it decreases. This same effect was noticed in the calculations of scattering from a circular cylinder. The $\psi = 0$ boundary condition affects the wave even for very long wavelengths, whereas a long wave hardly notices the $\partial\psi/\partial\mu = 0$ condition.

Diffraction through a Slit, Babinet's Principle. These same functions may be used to compute the penetration of a plane wave through a slit in a plane. The boundary here is the two half-infinite planes $\vartheta = 0$, $\vartheta = \pi$, with edges a distance a (the slit width) apart. Here the region is divided into two parts $(0 < \vartheta < \pi)$ and $(\pi < \vartheta < 2\pi)$ with a junction through the slit, along $\mu = 0$, across which the solution must be continuous in value and gradient. In the first region $(\pi < \vartheta < 2\pi)$, there must be present the incident wave, in a direction u with respect to the positive x axis, the wave reflected from the plane surfaces, moving in a direction $-u$, plus a scattered wave, caused by the absence of boundary at the slit. In the second region $(0 < \vartheta < \pi)$, there is only the diffracted wave which has penetrated through the slit. In either region, a complete set of eigenfunctions is *either* the set Se_m *or* the set So_m (not both).

If the boundary conditions require zero gradient along the surfaces $\vartheta = 0, \pi$ (case II), we can use only angle functions in each region, which are even functions about $\vartheta = 0, \pi$. These functions are the set $Se_m(h, \cos \vartheta)$. The incident plus reflected wave, in region 1 $(\pi < \vartheta < 2\pi)$ is, from Eq. (11.2.94),

$$
e^{-ikr \cos(\phi - u)} + e^{ikr \cos(\phi + u)} = 2\sqrt{8\pi} \sum i^m \left[\frac{Se_m(h, \cos u)}{M^e_m(h)} \right] Se_m(h, \cos \vartheta) Je_m(h, \cosh \mu)
$$

To this, of course, we must add an outgoing wave to help make continuity across the boundary $\mu = 0$. In region 1 $(0 < \vartheta < \pi)$, only the outgoing wave is present,

$$\psi = \begin{cases} \sqrt{8\pi} \sum_{m=0}^{\infty} i^m \left[\frac{Se_m(h, \cos u)}{M_m^e(h)} \right] \cdot \\ \quad \cdot Se_m(h, \cos \vartheta) D_m He_m(h, \cosh \mu); \quad 0 < \vartheta < \pi \\ \sqrt{8\pi} \sum_{m=0}^{\infty} i^m \left[\frac{Se_m(h, \cos u)}{M_m^e(h)} \right] Se_m(h, \cos \vartheta) \cdot \\ \quad \cdot [2Je_m(h, \cosh \mu) + A_m He_m(h, \cosh \mu)]; \quad \pi < \vartheta < 2\pi \end{cases} \tag{11.2.107}$$

where we must adjust the coefficients D_m, A_m so that, for each value of m, we have, at $\mu = 0$,

$$[\text{Value in region } 1]_{\mu=0} = [\text{value in region } 2]_{\mu=0}$$
$$[\mu \text{ derivative in region } 1]_{\mu=0} = -[\mu \text{ derivative in region } 2]_{\mu=0}$$

Since the slope of Je_m at $\mu = 0$ is zero, we must have that $D_m = -A_m$, and finally, using the definitions (11.2.100), we have

$$-A_m = D_m = ie^{-i\delta_m^e} \sin \delta_m^e$$

Consequently, the diffracted wave in region 2 ($0 < \vartheta < \pi$) for boundary condition II on the surfaces $\vartheta = 0, \pi$, is exactly the same as $-\psi_s^{\mathrm{I}}$ of Eq. (11.2.102), which is the wave scattered by a strip having boundary condition I. This interesting result should be restated to savor its full flavor: The wave scattered by a strip of width a, for boundary condition I is the same (except for change of sign) as the wave penetrating through a slit of width a, for boundary condition II, for the same angle of incidence and intensity of the initial plane wave. We interchange barrier for open space, and vice versa, and at the same time interchange Dirichlet for Neumann conditions, and the result is the same diffracted wave.

Put in still another way, for boundary condition II on the surfaces $\vartheta = 0, \pi$, the wave in region 2 ($0 < \vartheta < \pi$) is just minus the ψ_s^{I} of Eq. (11.2.102); for boundary condition I on the surface $\mu = 0$, the wave in region 2 (beyond the strip or slit) is $e^{ikr\cos(\phi-u)} + \psi_s^{\mathrm{I}}$; the sum of these two is just the undistorted plane wave $e^{ikr\cos(\phi-n)}$ in this region. The statement, that the sum of the wave diffracted from one boundary and the wave diffracted from the "inverse" boundary (in the sense indicated above) is just the undistorted plane wave, is called *Babinet's principle.* It holds for more general boundaries than the slit-strip system dealt with here, as will be shown in Sec. 11.4.

It of course holds for the alternative pair: boundary condition II (zero slope) on the strip $\mu = 0$ and boundary condition I (zero value) on the planes $\vartheta = 0, \pi$ defining the slit. In region 2 ($0 < \vartheta < \pi$) the wave for the former is $e^{ikr\cos(\phi-n)} + \psi_s^{\mathrm{II}}$ [defined in Eq. (11.2.103)] and the wave for the latter is $-\psi_s^{\mathrm{II}}$. The total energy penetrating the slit, per unit incident intensity, is just one-half the effective width Q for the "inverse" strip [the factor $\frac{1}{2}$ coming from the fact that for the penetration we integrate over one-half the plane ($0 < \vartheta < \pi$) instead of the whole plane ($0 < \vartheta < 2\pi$) as for Q].

11.3 *Waves in Three Space Dimensions*

Waves in three space dimensions exhibit the same sort of complications as do two-dimensional waves, only "more so." There are plane waves, with the wave fronts parallel planes, perpendicular to the direction of motion. These have the functional form $f(\mathbf{k} \cdot \mathbf{r} - \omega t)$, where $\mathbf{r}$ is the radius vector to the observation point and $\mathbf{k}$ is the *propagation vector*, with magnitude* $k = \omega/c$ and direction that of the wave motion [see the discussion of Eq. (2.2.4)].

There are also cylindrical waves, with the motion and space dependence in planes perpendicular to the cylinder axis. In addition there are waves of more general sort, where the variation in space is neither linear nor planar.

All these waves may be built up by superposition of plane waves in a manner related to that of Eq. (11.2.1). If we express the direction of the propagation vector $\mathbf{k}$ in terms of the spherical angles u (the angle between the vector $\mathbf{k}$ and the z axis) and v (the angle between the k-z plane and the x-z plane), the general expression for a three-dimensional wave is

$$\Psi = \int_0^{2\pi} dv \int_0^{\pi} \sin u \, du \, F(u,v|\mathbf{k} \cdot \mathbf{r} - kct) \tag{11.3.1}$$

The wave may also be expressed as a threefold Fourier integral by integrating over the three rectangular components of k,

$$\Psi = \frac{1}{8\pi^3} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} d\zeta \, f(\xi,\eta,\zeta) e^{i(\mathbf{k}\cdot\mathbf{r} - kct)} \tag{11.3.2}$$

where ξ, η, ζ are the x, y, z components of the vector $\mathbf{k}$.

For simple harmonic waves, $\Psi = \psi e^{-i\omega t}$, with ψ a solution of the three-dimensional Helmholtz equation, the integration must be over the surface of a sphere in "k space" since the magnitude of k is fixed:

$$\psi_k(x,y,z) = \int_0^{2\pi} dv \int \sin u \, du \, A(u,v) e^{i\mathbf{k}\cdot\mathbf{r}} \tag{11.3.3}$$

where
$$\begin{aligned} \mathbf{k} \cdot \mathbf{r} &= x \sin u \cos v + y \sin u \sin v + z \cos u \\ &= r[\cos \vartheta \cos u + \sin \vartheta \sin u \cos(v - \phi)] \end{aligned}$$

* Unfortunately custom has decreed that the propagation vector have the same symbol as the unit vector along the z axis. For this and the next chapters we will use $\mathbf{k}$ to mean the propagation vector and $\mathbf{a}_x$, $\mathbf{a}_y$ and $\mathbf{a}_z$, the unit vectors, when they are needed.

with r, ϑ, ϕ being the spherical coordinates of the observation point. The integration over u may be from 0 to π or it may be in the complex u plane. We shall find integral representations of the eigenfunction solutions in various three-dimensional coordinates, of the form of Eq. (11.3.3), to be quite useful [see, for example, Eq. (11.4.49)].

Green's Function for Free Space. We can also generalize our discussion of Fourier transforms and Green's function, on pages 823 and 1361, to three dimensions. This time we shall start with the expression for the radiation from a unit point pulse at (x_0,y_0,z_0,t_0), given in Eq. (7.3.8),

$$g(\mathbf{r}|\mathbf{r}_0|t - t_0) = \delta[t - t_0 - (R/c)]/R \tag{11.3.4}$$

where $R^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$. The radiation from a unit simple harmonic source is then the Laplace transform of this

$$\begin{aligned} G(\mathbf{r}|\mathbf{r}_0|\omega) &= \int_0^\infty g(\mathbf{r}|\mathbf{r}_0|t)e^{-pt}\,dt; \qquad p = -i\omega \\ &= [e^{-pR/c}/R] = [e^{ikR}/R]; \quad k = \omega/c \end{aligned} \tag{11.3.5}$$

and, finally, the complete Fourier transform for g is

$$F(\mathbf{r}_0,t_0|\mathbf{k},\omega) = 4\pi e^{-i\mathbf{k}\cdot\mathbf{r}_0+i\omega t_0}/[k^2 - (\omega/c)^2] \tag{11.3.6}$$

where the components of $\mathbf{k}$ are ξ, η, ζ. Then

$$G(\mathbf{r}|\mathbf{r}_0|\omega) = \frac{1}{8\pi^3}\iiint\limits_{-\infty}^{\infty} Fe^{i\mathbf{k}\cdot\mathbf{r}-i\omega t_0}\,d\xi\,d\eta\,d\zeta$$

$$g(\mathbf{r}|\mathbf{r}_0|t - t_0) = \frac{1}{16\pi^4}\iiint\limits_{-\infty}^{\infty} Fe^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\,d\xi\,d\eta\,d\zeta\,d\omega$$

These equations should be compared with the quantum mechanical relation (2.6.24) between wave functions for specified position and wave functions for specified momentum.

The solutions of the inhomogeneous equation,

$$\square^2\psi = \begin{cases} 0; & t < 0 \\ -4\pi\rho(\mathbf{r},t); & t > 0 \end{cases}$$

subject to no boundary conditions except that the wave be outgoing in all directions at infinity, are given by the various expressions

$$\begin{aligned} \psi(\mathbf{r},t) &= \iiiint\limits_{-\infty}^{\infty} \rho(\mathbf{r}_0,t_0)g(\mathbf{r}|\mathbf{r}_0|t - t_0)\,dx_0\,dy_0\,dz_0\,dt_0 \\ &= \iiiint\limits_{-\infty}^{\infty} P(\mathbf{r}_0|\omega)G(\mathbf{r}|\mathbf{r}_0|\omega)e^{-i\omega t}\,dx_0\,dy_0\,dz_0\,d\omega \\ &= \frac{1}{4\pi^3}\iiiint\limits_{-\infty}^{\infty} Q(\mathbf{k}|\omega)\,\frac{e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}}{k^2 - (\omega/c)^2}\,d\xi\,d\eta\,d\zeta\,d\omega \end{aligned} \tag{11.3.7}$$

where $$P(\mathbf{r}_0|\omega) = \int_0^\infty e^{-pt_0}\,\rho(\mathbf{r}_0,t_0)\,dt_0 \quad (p = -i\omega)$$

and $$Q(\mathbf{k}|\omega) = \iiint_{-\infty}^{\infty} dx_0\,dy_0\,dz_0 \int_0^\infty dt_0 e^{-i\mathbf{k}\cdot\mathbf{r}_0 - pt_0}\rho(\mathbf{r}_0,t_0)$$

and the expressions for g and G are those of Eqs. (11.3.4) and (11.3.5) if there is no boundary a finite distance from the volume within which ρ differs from zero.

When there are boundaries nearby, the functions g, G, and F are modified, in ways which can be computed when the boundaries are simple in shape, as was indicated in Sec. 7.3. We shall come back to this when we have acquired further technical skill.

Rectangular Enclosure. The eigenfunction solutions for standing waves in a rectangular enclosure having sides l_x, l_y, l_z are either

$$\begin{aligned} \psi_n(r)e^{-i\omega_n t} &= \cos(\pi n_x x/l_x)\,\cos(\pi n_y y/l_y)\,\cos(\pi n_z z/l_z)e^{-i\omega_n t} \\ \text{or} \qquad &= \sin(\pi n_x x/l_x)\,\sin(\pi n_y y/l_y)\,\sin(\pi n_z z/l_z)e^{-i\omega_n t} \end{aligned} \tag{11.3.8}$$

with n_x, n_y, n_z integers. The origin of coordinates is supposed to be at one corner of the enclosure. The first form is for homogeneous Neumann conditions at the walls, the second for homogeneous Dirichlet conditions. In either form, the allowed value of the angular velocity is

$$\omega_n = \pi c\sqrt{(n_x/l_x)^2 + (n_y/l_y)^2 + (n_z/l_z)^2} \tag{11.3.9}$$

except that the cases where any n is zero are omitted in the second form.

If the boundary conditions are more complex, we can compute the eigenfunctions, natural frequencies, and damping factors by the methods used to derive Eq. (11.2.20). The behavior of the wave under transient excitation may then be calculated by methods used in obtaining Eq. (11.1.29).

In these and other calculations we need the expression for the wave generated by a simple-harmonic, unit source at point (x_0,y_0,z_0) in the room. By methods which should be quite familiar by now, the Green's function for the case of Neumann conditions (normal gradient zero) is

$$\begin{aligned} G(\mathbf{r}|\mathbf{r}_0|\omega) &= \frac{4\pi c^2}{l_x l_y l_z}\sum_n \epsilon_n \left[\frac{\psi_n(r_0)}{\omega_n^2 - \omega^2}\right]\psi_n(r) \\ &= -\frac{4\pi}{l_x l_y}\sum_{m,n}\epsilon_m\epsilon_n\left[\frac{\cos(\pi m x_0/l_x)\,\cos(\pi n y_0/l_y)}{k_{mn}\,\sin(k_{mn}l_z)}\right]\cdot \\ &\cdot\cos\left(\frac{\pi m x}{l_x}\right)\cos\left(\frac{\pi n y}{l_y}\right)\begin{cases}\cos(k_{mn}z_0)\,\cos[k_{mn}(l_z - z)]; & z > z_0 \\ \cos(k_{mn}z)\,\cos[k_{mn}(l_z - z_0)]; & z < z_0\end{cases} \end{aligned} \tag{11.3.10}$$

where n in the first form stands for the trio of integers n_x, n_y, n_z; ψ_n is the first of the functions defined in (11.3.8); where $\epsilon_n = \epsilon_{n_x}\epsilon_{n_y}\epsilon_{n_z}$ and where ω_n

is given by Eq. (11.3.9). In the second form, analogous to Eq. (11.2.14), $k_{mn}^2 = (\omega/c)^2 - (\pi m/l_x)^2 - (\pi n/l_y)^2$; this second form may of course be modified by interchanging x, y, and z. For the Green's function for Dirichlet conditions, we interchange cosines with sines, leave out the zero of each integer in the sums, change ϵ_n in the first form to 8, and $\epsilon_m\epsilon_n$ in the second to 4.

For cases where one or more of the boundary walls has a more general boundary condition than either simple Dirichlet or Neumann conditions, the formulas of Eqs. (11.2.12) may be extended to three dimensions. In particular, if the boundary impedance has a real (resistive) component, the allowed "frequencies" ω are complex, with an imaginary part, $-i\kappa$, which produces exponential damping of the waves with time.

Distortion of Standing Wave by Strip. The Green's function may also be used to compute the effect on the standing waves and on the

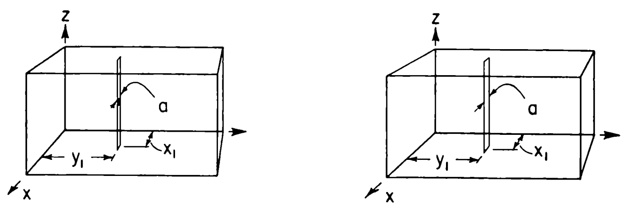

Fig. 11.3 Diffraction of standing waves by strip of width a.

resonance frequencies of small modifications of the boundary shapes. An example which does not involve too much apparent complexity is that where a strip of width a is stretched from floor to ceiling of the enclosure. We assume that the center line of the strip is the line $x = x_1$, $y = y_1$ and that the strip surface is parallel to the $x - z$ plane. If the strip is small enough and far enough from the walls, we could consider it to be in an infinite region and would use the functions discussed in the last section to compute the answer.

The standing wave in the room without the strip present could be $\psi_0 = \cos(\pi\nu x/l_x)\cos(\pi\sigma y/l_y)\cos(\pi\tau z/l_z)$ (ν, σ, $\tau = 0, 1, 2, 3, \ldots$). This may be considered to be a properly phased combination of eight plane waves, one with a propagation vector $\mathbf{k}$, with components $+(\pi\nu/l_x)$, $+(\pi\sigma/l_y)$, $+(\pi\tau/l_z)$, one with components $+(\pi\nu/l_x)$, $+(\pi\sigma/l_y)$, $-(\pi\tau/l_z)$, and so on, over the eight possible combinations of plus and minus signs for the components; all eight waves having the same magnitude of the vector,

$$k_0 = \pi\sqrt{(\nu/l_x)^2 + (\sigma/l_y)^2 + (\tau/l_z)^2} \tag{11.3.11}$$

From this point of view the standing wave is a combination of all the plane waves reflected from each of the six walls. All these waves strike the strip and are scattered by it.

Returning to page 1428 where we discussed the scattering of plane waves from a strip, we see that the z component of the solution $\cos(\pi\tau z/l_z)$ can be taken as a modulation factor for the solution of a two-dimensional Helmholtz equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \pi^2\left[\left(\frac{\nu}{l_x}\right)^2 + \left(\frac{\sigma}{l_y}\right)^2\right]\psi = 0$$

which corresponds to the scattering of plane waves from the strip centered at x_1, y_1. Changing to coordinates centered at the strip, $x = x_1 + \xi$, $y = y_1 + \eta$ and then to elliptic coordinates, $\xi = \frac{1}{2}a \cosh\mu \cos\vartheta$, $\eta = \frac{1}{2}a \sinh\mu \sin\vartheta$, we ask for the distortion, due to the strip, of the combination of plane waves

$$\begin{aligned}\psi = \cos(\pi\tau z/l_z)[&e^{(i\pi\nu/l_x)x_1+(i\pi\sigma/l_y)y_1}e^{(2ih/a)\rho\cos(u-\phi)} \\ &+ e^{(i\pi\nu/l_x)x_1-(i\pi\sigma/l_y)y_1}e^{(2ih/a)\rho\cos(-u-\phi)} \\ &+ e^{-(i\pi\nu/l_x)x_1-(\pi\sigma/l_y)y_1}e^{(2ih/a)\rho\cos(\pi+u-\phi)} \\ &+ e^{-(i\pi\nu/l_x)x_1+(i\pi\sigma/l_y)y_1}e^{(2ih/a)\rho\cos(\pi-u-\phi)}]\end{aligned} \tag{11.3.12}$$

where $\rho^2 = \xi^2 + \eta^2$; $\tan\phi = \eta/\xi$; $\tan u = \sigma l_x/\nu l_y$, and $h^2 = (\pi a/2)^2[(\nu/l_x) + (\sigma/l_y)^2]$.

From Eqs. (11.2.102) *et seq.*, we could compute the effect of the strip on the four plane waves, if these waves were out in the open with no room walls about. The presence of the walls modifies these results profoundly at some distance from the strip but, unless the strip is close to one wall, they should have little effect on the value of the solution at the surface of the strip. Consequently, we know with some accuracy the behavior of the modified standing wave near the strip. To find its behavior in the rest of the room, we can use the Green's function technique.

From Eq. (7.2.7) we have

$$\psi(\mathbf{r}) = -\frac{1}{4\pi}\oint[\psi(\mathbf{r}_0^s)\,\mathrm{grad}_0\,G(\mathbf{r}|\mathbf{r}_0^s|\omega)]\cdot d\mathbf{A}_0$$

for a solution ψ which has zero normal gradient at all boundary surfaces. If we use the G of Eq. (11.3.10), the integral over the surface of the room vanishes, because G has zero normal gradient there, and the normal outflow integral reduces to one over the surface of the strip. If we label the value of ψ on the side of the strip nearest the x-z plane by the symbol $\psi_-(\xi)$ and the value of ψ on the side away from the x-z plane is called $\psi_+(\xi)$, then Eq. (7.2.7) finally reduces, in this case, to

$$\psi(\mathbf{r}) = \frac{1}{4\pi}\int_0^{l_z} dz_0 \int_a^a [\psi_+(\xi) - \psi_-(\xi)]\left[\frac{\partial}{\partial\eta}G(\mathbf{r}|x_1 + \xi, y_1 + \eta|\omega)\right]_{\eta=0} d\xi \tag{11.3.13}$$

Therefore, if we obtain a good approximation to the value of ψ at the surface of the strip, we can get a good approximation to ψ anywhere in the room. As pointed out above, the value of ψ at the strip surface should not be very sensitive to the presence or absence of the room walls, as long as the strip is small and not close to a wall. We can thus use for $\psi(\mathbf{r}_0^s)$ its value for free waves, without the presence of walls and use Eq. (11.3.13) to take the effect of the walls into account.

From Eq. (11.2.103) we can see that the value at the strip surface of the plane-plus-scattered wave for an incident wave at angle u with respect to the x axis is

$$\psi(\xi,u) = \sqrt{8\pi}\sum_m i^m \left\{\frac{Se_m(h,\cos u)Se_m(h,\cos\vartheta)}{M_m^e} Je_m(h,1) \right.$$
$$\left. + \frac{So_m(h,\cos u)So_m(h,\cos\vartheta)}{M_m^0}\left[Jo_m(h,1) - ie^{-i\delta_m^0}\sin\delta_m^0\, Ho_m(h,1)\right]\right\}$$

But the series over the even functions is just the part of a plane wave, which is even about $\vartheta = 0, \pi$. From Eq. (11.2.94) we can see that

$$\sqrt{8\pi}\sum i^m \frac{Se_m(h,\cos u)Se_m(h,\cos\vartheta)}{M_m^e} Je_m(h,\cosh\mu)$$
$$= e^{(2ih/a)\xi\cos u}\cos\left[\left(\frac{2h}{a}\right)\eta\sin u\right]$$

The term in brackets in the second sum is proportional to the Wronskian $\Delta(Jo_m,Ho_m) = i$. Substituting in the approximate expressions (11.2.101), we finally obtain, to the second approximation in h,

$$\psi(\xi,u) \simeq e^{ih\cos\vartheta\cos u} + ih\sin u\sin\vartheta - \tfrac{1}{8}h^2\sin(2u)\sin(2\vartheta)$$

where $\xi = \frac{1}{2}a\cos\vartheta$ and where h is given below (11.3.12).

Referring to Eq. (11.3.12), we see that the value of ψ at the surface of the strip, when we add up the effects of the wave in the four directions, u, $-u$, $\pi - u$, and $\pi + u$, is

$$\psi \simeq \cos\left[\frac{\pi\nu(x_1+\xi)}{l_x}\right]\cos\left(\frac{\pi\sigma y_1}{l_y}\right)\cos\left(\frac{\pi\tau z_0}{l_z}\right)$$
$$- \left(\frac{\pi\sigma a}{2l_y}\right)\cos\left(\frac{\pi\nu x_1}{l_x}\right)\sin\left(\frac{\pi\sigma y_1}{l_y}\right)\cos\left(\frac{\pi\tau z_0}{l_z}\right)\sin\vartheta + \cdots$$

where we have not computed out the h^2 term, since it will not be needed. The quantity $\psi_+(\xi)$ is the value of ψ for $0 < \vartheta < \pi$, where $\sin\vartheta = \sqrt{1-(2\xi/a)^2}$, and $\psi_-(\xi)$ is the value of ψ for $\pi < \vartheta < 2\pi$, where $\sin\vartheta = -\sqrt{1-(2\xi/a)^2}$. Therefore,

$$[\psi_+(\xi) - \psi_-(\xi)] \simeq -\left(\frac{\pi\sigma a}{l_y}\right)\cos\left(\frac{\pi\nu x_1}{l_x}\right)\sin\left(\frac{\pi\sigma y_1}{l_y}\right)\cos\left(\frac{\pi\tau z_0}{l_z}\right)\sqrt{1-(2\xi/a)^2} \qquad (11.3.14)$$

to the second order in the small quantity h (h is small when a is small compared to the wavelength).

To compute (11.3.13) we next need an expression for the gradient of G. Using the second expression in (11.3.10) with y and z interchanged, we have

$$\left[\frac{\partial}{\partial\eta}G\right]_{\eta=0} = \frac{4\pi}{l_xl_z}\sum_{m,n}\epsilon_m\epsilon_m\left[\frac{\cos(\pi m x_0/l_x)\cos(\pi n z_0/l_z)}{\sin(k_{mn}l_y)}\right]\cdot$$
$$\cdot\cos\left(\frac{\pi m x}{l_x}\right)\cos\left(\frac{\pi n z}{l_z}\right)\begin{cases}\sin(k_{mn}y_1)\cos[k_{mn}(l_y-y)]; & y>y_1\\ -\cos(k_{mn}y)\sin[k_{mn}(l_y-y_1)]; & y<y_1\end{cases}$$

where $x_0 = x_1 + \xi$ and $k_{mn} = \sqrt{(\omega/c)^2 - (\pi m/l_x)^2 - (\pi n/l_z)^2}$. The integral of (11.3.13) over z_0 reduces all the terms in the sum over n to zero except for those terms with $n = \tau$, because of the orthogonality of the factors $\cos(\pi\tau z_0/l_z)$ from ψ and $\cos(\pi n z_0/l_z)$ from G. In the integral over ξ, the factor $\cos(\pi m x_0/l_x)$ may be taken to be $\cos(\pi m x_1/l_x)$ as long as the strip width is small compared to a wavelength. Consequently,

$$\int_{-\frac{1}{2}a}^{\frac{1}{2}a}\cos\left[\frac{\pi m}{l_x}(x_1+\xi)\right]\sqrt{1-(2\xi/a)^2}\,d\xi \simeq \tfrac{1}{4}\pi a\cos(\pi m x_1/l_x)$$

Therefore, the approximate expression for the solution of the Helmholtz equation which has zero normal gradient both on the room walls and on the strip, is

$$\psi(\mathbf{r}) \simeq \frac{\pi^2\sigma a^2}{4l_xl_y}\sum_{m=0}^{\infty}\epsilon_m\left[\frac{\cos(\pi m x_1/l_x)\cos(\pi\nu x_1/l_x)}{\sin(k_{m\tau}l_y)}\right]\sin\left(\frac{\pi\sigma y_1}{l_y}\right)\cdot$$
$$\cdot\cos\left(\frac{\pi m x}{l_x}\right)\cos\left(\frac{\pi\tau z}{l_z}\right)\begin{cases}-\sin(k_{m\tau}y_1)\cos[k_{m\tau}(l_y-y)]; & y>y_1\\ \cos(k_{m\tau}y)\sin[k_{m\tau}(l_y-y_1)]; & y<y_1\end{cases} \qquad (11.3.15)$$

where $k_{m\tau} = \sqrt{k^2 - (\pi m/l_x)^2 - (\pi\tau/l_z)^2}$ and $k = \omega/c$, ω being the allowed angular frequency for the distorted eigenfunction.

Computing the Eigenvalue. We have not yet completed our task, for the allowed value of k has not been obtained and the expression we have just written down for $\psi(\mathbf{r})$ is not correct unless we put in the correct value of k. To do this we resort again to Green's theorem. We take the equation for $\psi_0 = \cos(\pi\nu x/l_x)\cos(\pi\sigma y/l_y)\cos(\pi\tau z/l_z)$, multiplied by ψ, the correct solution, approximated by Eq. (11.3.15) and subtract from it the equation for ψ multiplied by ψ_0,

$$\psi\nabla^2\psi_0 - \psi_0\nabla^2\psi = (k^2 - k_0^2)\psi\psi_0$$

where k_0 is given in Eq. (11.3.11) and k is the quantity we wish to compute.

Integrating both sides over all the volume inside the room and outside the strip allows us to change the left-hand integral to a surface integral, by (7.2.2). This is zero except for ψ grad ψ_0 along the surface of the strip. We finally have, in the notation of (11.3.13),

$$k^2 = k_0^2 - \int_0^{l_z} dz \int_{-\frac{1}{2}a}^{\frac{1}{2}a} [\psi_+(\xi) - \psi_-(\xi)] \left[\frac{\partial\psi_0}{\partial y}\right]_{y=y_1} d\xi \left[\iiint \psi\psi_0\, dv\right]^{-1}$$

But we can obtain an approximate value for the difference $[\psi_+ - \psi_-]$ from Eq. (11.3.14), and the calculation of $\partial\psi_C/\partial y$ is easy. We expect ψ to be approximately equal to ψ_0 over most of the volume of the room, so that the volume integral should be approximately equal to $l_x l_y l_z/\epsilon_x\epsilon_y\epsilon_z$.

Therefore we have, to the second order in a/l_y,

$$k^2 \simeq \left(\frac{\pi\nu}{l_x}\right)^2 + \left(\frac{\pi\tau}{l_z}\right)^2 + \left(\frac{\pi\sigma}{l_y}\right)^2 \left[1 - \left(\frac{\pi\epsilon_\nu\epsilon_\sigma}{4}\right)\left(\frac{a}{l_y}\right)^2 \cos^2\!\left(\frac{\pi\nu x_1}{l_x}\right) \sin^2\!\left(\frac{\pi\sigma y_1}{l_y}\right)\right] \quad (11.3.16)$$

where the allowed frequency is, of course, $ck/2\pi$.

We see, in the first place, that the allowed frequencies are reduced in value (if they are changed at all). This seems reasonable, for the presence of a strip should slow the air down in a room and thus lower all resonance frequencies. Next, we see that the amount of reduction is proportional to the square of the y component of the fluid velocity, in the unperturbed wave, at the position of the strip. For instance, wherever the velocity is parallel to the $x - z$ plane [*i.e.*, where $\sin(\pi\sigma y_1/l_y) = 0$] k^2 does not differ from k_0^2. This also seems reasonable, because the "slowing down" of the oscillations is caused by the flow around the strip, and the amount of reduction depends on how much flow there is perpendicular to the strip surface. Flow parallel to the surface is not distorted by the plane. It is perhaps more accurate to say that the flow distortion around the strip increases the effective mass of the fluid in comparison to its stiffness, so that the resonance frequencies decrease, as long as a/l_y is small.

We can now return to Eq. (11.3.15) and insert the value of k into the terms to obtain an approximate solution for ψ. Since the difference between k and k_0 is small, it is not important to include it in the expressions for $k_{m\tau}$, except for $m = \nu$. Therefore, we can set $k_{m\tau} = k_m \simeq \sqrt{(\pi/l_x)^2(\nu^2 - m^2) + (\pi\sigma/l_y)^2}$, $m \neq \nu$. On the other hand,

$$k_{\nu\tau} = k_\nu \simeq \left(\frac{\pi\sigma}{l_y}\right) - \left(\frac{\epsilon_\nu}{4}\right)\left(\frac{\pi^2\sigma a^2}{l_x l_y{}^2}\right) \cos^2\!\left(\frac{\pi\nu x_1}{l_x}\right) \sin^2\!\left(\frac{\pi\sigma y_1}{l_y}\right)$$

Consequently $\sin(k_\nu l_y)$ turns out to be quite small, much smaller than $\sin(k_m l_y)$ for $m \neq \nu$. This one large term, for $m = \nu$, can be taken out of the sum, and since

$$\sin(k_\nu l_y) \simeq -(-1)^\sigma \left(\frac{\pi^2\sigma a^2}{l_x l_y}\right)\left(\frac{\epsilon_\nu}{4}\right)\cos^2\left(\frac{\pi\nu x_1}{l_x}\right)\sin^2\left(\frac{\pi\sigma y_1}{l_y}\right)$$

we have finally

$$\psi(\mathbf{r}) \simeq \psi_0 + \left(\frac{\pi^2\sigma a^2}{4l_x l_y}\right)\sum_{m\neq\nu}\epsilon_m\left[\frac{\cos(\pi m x_1/l_x)\cos(\pi\nu x_1/l_x)\sin(\pi\sigma y_1/l_y)}{\sin(k_m l_y)}\right]\cdot$$
$$\cdot\cos\left(\frac{\pi m x}{l_x}\right)\cos\left(\frac{\pi\tau z}{l_z}\right)\begin{cases} -\sin(k_m y_1)\cos[k_m(l_y - y)]; & y > y_1 \\ \cos(k_m y_1)\sin[k_m(l_y - y_1)]; & y < y_1\end{cases} \qquad (11.3.17)$$

In addition to the unperturbed part, ψ_0, there is a small term representing the scattering from the strip, expressed in terms of the other standing waves for different m. We can say that the strip "couples" some of the standing waves to others. As long as the strip axis, from floor to ceiling, is parallel to the z axis, then no wave of a given value of τ (z quantum number) can be coupled to another for a different value of τ, though the different waves for different x and y quantum numbers are, most of them, coupled by this strip.

Transmission through Ducts. A series of three-dimensional problems, which are closely related to some of the two-dimensional ones dealt with in the previous section, are those concerned with the transmission of waves through ducts. Suppose the duct is a cylinder, with a boundary having a cross-sectional shape which allows separable solutions of the wave equation, into the coordinates ξ_1 and ξ_2, for instance. Then the whole wave function may be separated into a product of a function ϕ of ξ_1 and ξ_2 and a function F of z, the distance along the axis of the cylinder,

$$\psi = \phi(\xi_1,\xi_2)F(z);\quad \nabla^2\phi + k_t^2\phi = 0$$
$$(d^2F/dz^2) + k_z^2F = 0;\quad k^2 = (\omega/c)^2 = k_t^2 + k_z^2$$

The boundary conditions on ϕ at the boundary ($\xi_1 = K$, for instance) fix the allowed values of k_t, the transverse wave number. The value of k is fixed by fixing the frequency of the wave transmitted, thus fixing ω. Consequently the longitudinal wave number k_z is fixed by fixing ω and choosing a particular transverse standing wave to fit the boundary conditions, which fixes k_t. The solution for F may be a traveling wave $e^{ik_z z - i\omega t}$ or it may be a combination of waves in opposite directions. If it is a simple traveling wave, we see that its phase velocity *ω/k_z is never smaller than c, the wave velocity in free space.* For k_z is never larger than $k = \omega/c$ and is frequently much smaller, if k_t is large. In fact, for the higher modes of transverse oscillation k_t may be larger than k, in which case k_z is imaginary and there is no true wave motion. In this case the wave is so busy oscillating back and forth across the duct that it forgets to travel along the duct, so to speak.

For the boundary condition of zero normal gradient at the inner surface of the cylindrical duct, the lowest transverse mode is a constant for ϕ, with $k_t = 0$. In this case $k_z = k = \omega/c$, the phase velocity equals the free-space velocity c and there is no dispersion. For all higher transverse modes $k_z < k$, the phase velocity is larger than c and there is dispersion. In the case of zero value on the surface, even the lowest transverse mode has a k_t larger than zero, and all k_z's are less than k. In this case there is a certain frequency, proportional to this lowest value of k_t, called the *cutoff frequency*, below which no true wave motion is possible down the duct.

The transmission of waves along cylindrical ducts is quite a simple matter unless one brings in the effects at one end or the other or unless one deals with a change in the duct cross section at some point. These cases are, of course, the ones which make a study of duct transmission have practical importance, so it is useful to treat a few typical examples. The first ones will assume an infinitely long tube and will be concerned entirely with the generation of the wave at the input end. In this case, if there is wave motion along the duct, it will be away from the input end.

Acoustic Transients in a Rectangular Duct. If one knows the longitudinal velocity of the fluid at the input end of an acoustic duct, one can compute the wave motion in the duct. The boundary conditions are that the fluid velocity is tangential to the duct surface, at the surface of rigid boundaries. So, for a rectangular duct, the general expression for a wave traveling away from the input end ($z = 0$), for the velocity potential, is

$$\psi = \sum_{m,n} A_{mn} \cos\left(\frac{\pi m x}{l_x}\right) \cos\left(\frac{\pi n y}{l_y}\right) e^{i\omega[(\tau/c)z - t]} \tag{11.3.18}$$

where

$$\tau = \tau_{mn} = \sqrt{1 - (\pi mc/\omega l_x)^2 - (\pi nc/\omega l_y)^2}; \quad m, n = 0, 1, 2, \ldots$$

for a rectangular duct of sides l_x and l_y for simple harmonic waves. For any given value of ω, τ is imaginary for large enough values of m and n. We then take τ to be positive imaginary, so that in this case

$$\tau = i(c\kappa_{mn}/\omega); \quad \kappa_{mn} = \sqrt{(\pi m/l_x)^2 + (\pi n/l_y)^2 - (\omega/c)^2} \tag{11.3.19}$$

where κ is the *attenuation constant* for the higher mode, since the z factor becomes $e^{-\kappa z}$, indicating an attenuation along the duct, rather than wave motion.

If the pressure is specified at the input end of the duct, $p_0(x,y)e^{-i\omega t} = i\omega\rho\psi_{z=0}$, then the series coefficients are

$$A_{mn} = i\left(\frac{\epsilon_m \epsilon_n}{\rho l_x l_y \omega}\right) F_{mn}; \quad F_{mn} = \int_0^{l_y} \int_0^{l_y} p_0 \cos\left(\frac{\pi m x}{l_x}\right) \cos\left(\frac{\pi n y}{l_y}\right) dx\, dy$$

where F_{mn} is the (m,n)th component of the force required to drive the wave along the duct. Ordinarily, however, the longitudinal air velocity at the $z = 0$ end is specified, and the pressure is then computed. If this velocity $(v_z)_{z=0} = v_0(x,y)e^{-i\omega t} = (\partial\psi/\partial z)_{z=0}$ then

$$A_{mn} = -\frac{(ic\epsilon_m\epsilon_n/l_xl_y)}{\sqrt{\omega^2 - \omega_{mn}^2}} I_{mn}; \quad I_{mn} = \int_0^{l_x}\int_0^{l_y} v_0 \cos\left(\frac{\pi mx}{l_x}\right)\cos\left(\frac{\pi ny}{l_y}\right) dx\, dy$$

where $\omega_{mn} = \pi c\sqrt{(m/l_x)^2 + (n/l_y)^2}$ will be called the *cutoff frequency* of the (m,n)th wave, for reasons which will be apparent shortly.

The ratio F_{mn}/I_{mn} may be called the *impedance* of the (m,n)th wave

$$Z_{mn} = \rho c \frac{\omega}{\sqrt{\omega^2 - \omega_{mn}^2}} = \frac{\rho c}{\tau_{mn}} \tag{11.3.20}$$

When the driving frequency ω is larger than the cutoff frequency of the (m,n)th wave, Z_{mn} and τ_{mn} are real and the wave is propagated along the tube with a velocity c/τ_{mn}. As ω is made smaller, this velocity increases until, at $\omega = \omega_{mn}$, it is infinite, τ_{mn} is zero and Z_{mn} is infinite. Below this cutoff frequency, the (m,n)th wave does not propagate but attenuates, and the impedance is imaginary

$$Z_{mn} = -i(\rho\omega/\tau_{mn}) = -i\omega(\rho c/\sqrt{\omega_{mn}^2 - \omega^2})$$

corresponding to an effective mass $M_{mn} = \rho c/\sqrt{\omega_{mn}^2 - \omega^2}$ per unit area of duct cross section. Consequently, for any given driving frequency ω only those waves having cutoff frequencies ω_{mn} less than ω are really propagated and contribute any real part to the input impedance; all higher waves contribute reactive parts only and do not extend very far away from the source at $z = 0$.

If the generator at $z = 0$ is an impulsive one, the resultant motion may be computed by use of Laplace transforms. Suppose that the initial velocity is impulsive, $(v_z)_{z=0} = v_0(x,y)\delta(t)$. Then according to (11.1.16), the impulsive wave is the one for which the wave of Eq. (11.3.18) is the Laplace transform. Use of the tables at the end of this chapter shows that the Laplace transform of $u(t - b)J_0(a\sqrt{t^2 - b^2})$ is $[\exp(-b\sqrt{p^2 + a^2})/\sqrt{p^2 + a^2}]$. The (m,n)th term of the series (11.3.18) is

$$-\left(\frac{c\epsilon_m\epsilon_n}{l_xl_y}\right) I_{mn}\cos\left(\frac{\pi mx}{l_x}\right)\cos\left(\frac{\pi ny}{l_y}\right)\frac{\exp[-(z/c)\sqrt{p^2 + \omega_{mn}^2}]}{\sqrt{p^2 + \omega_{mn}^2}}; \quad \omega = ip$$

which corresponds to the expression found in the table. Therefore, the impulsive wave has the velocity potential

$$\psi = -u\left(t - \frac{z}{c}\right)\left(\frac{c}{l_xl_y}\right)\sum_{m,n}\epsilon_m\epsilon_n \cos\left(\frac{\pi mx}{l_x}\right)\cos\left(\frac{\pi ny}{l_y}\right)J_0\left[\omega_{mn}\sqrt{t^2 - \left(\frac{z}{c}\right)^2}\right] \tag{11.3.21}$$

For the lowest partial wave, $\omega_{00} = 0$, the time-dependent factor is simply $J_0(0) = 1$, and this wave consists of a pulse traveling along the tube (pressure is $-\rho\partial\psi/\partial t$ which changes the step function u into a delta function, a true pulse). The higher the partial wave, the larger ω_{mn} is, and the more rapidly the Bessel function oscillates. The lowest wave has a phase velocity equal to c, there is no dispersion of the wave, it starts as a pulse and continues as a pulse along the duct. The higher modes are dispersive, however, their phase velocities (c/τ_{mn}) do vary with frequency, so that the pulse is spread out as it traverses the tube, leaving an oscillatory wake as it moves along (see Fig. 2.7).

Corresponding analyses of electromagnetic waves inside ducts (wave guides) will be given in Chap. 13.

Constriction in Rectangular Duct. Obstructions in a duct produce partial reflections of a wave traveling in the duct. If the obstruction be

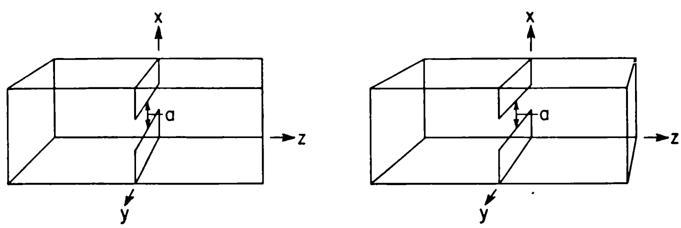

Fig. 11.4 Reflection and transmission of wave in duct from slit in plane barrier.

small enough to produce only slight rearrangement in the flow pattern (such as would be the case for a narrow strip run across the tube, for instance), then methods using Eq. (11.3.13) will be useful to obtain approximate formulas. If the obstruction is a major one, however, we must use other methods. An example is that of a rectangular duct, of sides l_x and l_y, having a rigid barrier set up across it at $z = 0$, with the barrier pierced by a slit through its center, of width a, parallel to the y axis [in other words, the slit is between $x = \frac{1}{2}(l_x - a)$ and $x = \frac{1}{2}(l_x + a)$ for $z = 0$ and $0 \le y \le l_y$]. A wave coming from the left ($z < 0$) strikes this partial barrier, part is reflected to the left and part goes through the slit to continue on to the right.

Here, as we have discovered several times before, it is possible to consider the flow near and in the slit as being in phase, as long as the wavelength of the wave is considerably longer than the slit width. All we need to know is the general shape of the velocity potential and its gradient, within the slit, corresponding to uniform flow through the slit. We then fit this to the wave solutions on either side of the barrier, thereby fixing the amplitude of the flow through the slit.

There is no particular need to make this calculation for the most general wave propagated down the duct. The approximation we are to make is best for long wavelengths, and the results of the reflection are simple only at low frequencies. For as long as the frequency is low enough, only the $m = n = 0$ mode wave can be transmitted along the tube, and the reflected wave will be the same sort as the incident wave, far enough from the barrier. The effect of the barrier may then be expressed simply by giving the amplitude and phase of the reflected wave for an incident wave of unit amplitude and standard phase. At frequencies higher than the next lowest cutoff frequency, ω_{10} or ω_{01}, at least two kinds of waves are reflected, and more than two quantities are needed to describe the reflection.

For a plane wave incident on the barrier, therefore, we can assume that the velocity potential is

$$\psi(x,z) = \begin{cases} e^{ikz} + (1 - A_0)e^{-ikz} + \sum_{m=1}^{\infty} A_m \cos\left(\frac{\pi m x}{l_x}\right) e^{+\kappa_m z}; & z < 0 \\ A_0 e^{ikz} - \sum_{m=1}^{\infty} A_m \cos\left(\frac{\pi m x}{l_x}\right) e^{-\kappa_m z}; & z > 0 \end{cases} \tag{11.3.22}$$

where $k = \omega/c$, $\kappa_m = \sqrt{(\pi m/l_x)^2 - k^2}$ $(k < \pi/l_x)$, and where we have set up the series so as to ensure that the gradient $\partial\psi/\partial z$ is continuous at $z = 0$. We should now solve the problem by adjusting the coefficients A so that ψ, given by both series, is continuous in the slit, its slope already being continuous there. This would require the solving of an infinite set of simultaneous equations, however, so we try to use the solutions for steady flow conditions to help us make a start toward obtaining a successive approximations solution.

Referring to Eqs. (10.1.24) *et seq.*, we see that, for steady flow, the velocity potential in the slit is a constant and the z component of the gradient of ψ in the slit is a constant divided by $\sin\vartheta$, where $x = \frac{1}{2}(l_x - a\cos\vartheta)$ for $z = 0$. Consequently, as long as $k \ll 2\pi/a$, we have

$$\left(\frac{\partial\psi}{\partial z}\right)_{z=0} \simeq \begin{cases} 0; & 0 \le x < \frac{1}{2}l_x - \frac{1}{2}a \\ (2Q/\pi a l_y)\csc\vartheta; & \frac{1}{2}l_x - \frac{1}{2}a < x < \frac{1}{2}l_x + \frac{1}{2}a \\ 0; & \frac{1}{2}l_x + \frac{1}{2}a < x \le l_x \end{cases}$$

where Q is the total flow through the slit (the amplitude of flow in this case). But the series expansion for this gradient is, from (11.3.22), supposed to be $ikA_0 + \sum \kappa_m A_m \cos\left(\frac{\pi m x}{l_x}\right)$ so that the coefficients A are

$$A_0 \simeq \frac{2Q}{i\pi k a l_x l_y}\int \csc\vartheta\, dx = -i\left(\frac{Q}{k l_x l_y}\right)$$

$$A_m \simeq \frac{4Q}{\pi a \kappa_m l_x l_y}\int \cos\left(\frac{\pi m x}{l_x}\right)\csc\vartheta\, dx = \left(\frac{2Q}{\kappa_m l_x l_y}\right)\cos(\tfrac{1}{2}\pi m) J_0\left(\frac{\pi m a}{2 l_x}\right)$$

where we have used Eq. (5.3.65) to obtain the last expression. As before, we have reduced our unknowns down to one quantity, Q.

In order to compute Q we must compute the value of ψ at the center of the slit, and adjust Q so that ψ is continuous across the slit. Approaching from the $z > 0$ side, the value of ψ at the center is

$$\psi(\tfrac{1}{2}l_x,0) \simeq -i\left(\frac{Q}{kl_xl_y}\right) - \left(\frac{2Q}{l_xl_y}\right)\sum_{n=1}\left(\frac{1}{\kappa_{2n}}\right)J_0\left(\frac{\pi na}{l_x}\right)$$
$$\simeq -i\left(\frac{Q}{kl_xl_y}\right) - \left(\frac{2Q}{\pi l_xl_y}\right)\int_0^\pi\left\{\sum_{n=1}^{\infty}\cos\left[\pi n\left(1-\frac{a}{l_x}\cos\vartheta\right)\right]\frac{d\vartheta}{\kappa_{2n}}\cos(\pi n)\right\}$$

The series does not converge well, so it would be desirable to find a series similar to it, which can be summed, to subtract and add. For this reason we have written out the second form, which just redefines the expression for the coefficients as integrals. The series inside the integral sign, however, is nearly simple enough to be summable, and if we replace κ_{2n} by $2\pi n/l_x$, which it approaches for large values of n, we can sum it:

$$\sum_{n=1}^{\infty}\left(\frac{l_x}{2\pi n}\right)\cos\left(\frac{\pi na}{l_x}\cos\vartheta\right) = -\operatorname{Re}\left\{\frac{l_x}{2\pi}\ln[1-e^{(\pi ia/l_x)\cos\vartheta}]\right\}$$
$$= -\left(\frac{l_x}{4\pi}\right)\ln\left[2-2\cos\left(\frac{\pi a}{l_x}\cos\vartheta\right)\right] = -\frac{l_x}{4\pi}\ln\left[4\sin^2\left(\frac{\pi a}{2l_x}\cos\vartheta\right)\right]$$

Since $\pi a/2l_x \ll 1$, we can consider the quantity in the brackets proportional to $\cos^2\vartheta$ and can finally obtain for

$$\int_0^\pi\sum_{n=1}^{\infty}\left(\frac{l_x}{2\pi n}\right)\cos\left(\frac{\pi na}{l_x}\cos\vartheta\right)d\vartheta \simeq -\tfrac{1}{2}l_x\ln\left(\frac{\pi a}{2l_x}\right)$$

Consequently, we designate the series part of $\psi(\tfrac{1}{2}l_x,0)$ as $(Q/\pi l_y)W(a/l_x)$, where the function

$$W\left(\frac{a}{l_x}\right) = \sum_{n=1}^{\infty}\left(\frac{2}{l_x\kappa_{2n}}\right)J_0\left(\frac{\pi na}{l_x}\right);\quad \kappa_{2n} = \sqrt{\left(\frac{2\pi n}{l_x}\right)^2 - k^2}$$
$$\simeq \ln\left(\frac{2l_x}{\pi a}\right) - \sum_{n=1}^{\infty}\left[\frac{1}{\pi n}-\frac{2}{l_x\kappa_{2n}}\right]J_0\left(\frac{\pi na}{l_x}\right) \tag{11.3.23}$$

is real and positive, being large when a/l_x is small. The series in the second line is quite rapidly convergent.

Returning again to the equations for continuity of ψ in the slit, we see that the equation determining Q is the one requiring the two expressions for ψ in (11.3.22) to be equal at $z = 0$, $x = \frac{1}{2}l_x$:

$$2 + i\left(\frac{Q}{kl_xl_y}\right) + \left(\frac{Q}{l_y}\right)W = -i\left(\frac{Q}{kl_xl_y}\right) - \left(\frac{Q}{l_y}\right)W$$

or, for a unit incident plane wave, the flow through the slit, in phase and amplitude, is

$$Q \simeq \frac{ikl_xl_y}{1 - ikl_xW(a/l_x)} \tag{11.3.24}$$

and the velocity potential is

$$\psi \simeq \begin{cases} e^{ikz} - \dfrac{ikl_xW}{1 - ikl_xW}e^{-ikz} \\ \qquad + \dfrac{2ik}{1 - ikl_xW}\displaystyle\sum_{n=1}^{\infty}(-1)^nJ_0\left(\frac{\pi na}{l_x}\right)\cos\left(\frac{2\pi nx}{l_x}\right)\frac{e^{\kappa_{2n}z}}{\kappa_{2n}}; & z < 0 \\ \dfrac{e^{ikz}}{1 - ikl_xW} - \dfrac{2ik}{1 - ikl_xW}\displaystyle\sum_{n=1}^{\infty}(-1)^nJ_0\left(\frac{\pi na}{l_x}\right)\cos\left(\frac{2\pi nx}{l_x}\right)\frac{e^{-\kappa_{2n}z}}{\kappa_{2n}}; \\ & z > 0 \end{cases}$$

Both of these series converge quite well as long as z is not zero.

Let us now review our findings. For a/l_x small, W is large, though $kl_xW(a/l_x)$ may not be large if (kl_x) is small compared with unity. Nevertheless, for any value of (kl_x), we can make a small enough compared to l_x so that $kl_xW(a/l_x)$ is considerably larger than unity. In this case the amplitude of the reflected wave is nearly equal to the amplitude of the incident wave, and the phase change on reflection is nearly zero. The series terms represent the distortion of the wave near the slit. They become negligible for $|z|$ larger than a few wavelengths because of the exponentials. Beyond this region of distortion, there are only plane waves (as long as $k < 2\pi/l_x$), an incident plus reflected wave on the left, a transmitted wave on the right. If we made no measurements near the slit, the waves would appear to us as if

$$\psi \simeq \begin{cases} e^{ikz} - \left[\dfrac{ikl_xW}{1 - ikl_xW}\right]e^{-ikz}; & z < 0 \\ (1 - ikl_xW)^{-1}e^{ikz} & z > 0 \end{cases}$$

In fact, if we made measurements on the plane waves some distance away from the slit, it would appear as though there were a membrane or other obstruction at $z = 0$, across which a pressure drop

$$ik\rho c[\psi_- - \psi_+] \simeq -2ik\rho c\left[\frac{ikl_xW}{1 - ikl_xW}\right]$$

occurred. There appears also to be a net velocity of this membrane equal to Q/l_xl_y, so that the *effective impedance* of the barrier (whatever it is) would appear to be

$$\frac{\text{Pressure drop}}{\text{Mean velocity}} \simeq \rho c[-2ikl_xW] = -i\omega(2\rho l_xW) = Z_s \quad (11.3.25)$$

where the quantity in the parentheses, multiplied by l_xl_y (to get it back to a per duct rather than a per unit area quantity), may be called the *effective mass* of the fluid in the slit. This effective mass, which has approximately the value $(2\rho l_x^2l_y)\ln(2l_x/\pi a)$ when $k \ll 2\pi/l_x$ (or $\lambda \gg l_x$), is the effective mass which is driven back and forth through the slit by the incident wave, thereby producing the reflected and transmitted wave. To the incident wave a combined impedance, of obstacle and free duct beyond, is presented.

According to Eqs. (2.1.13) and (2.1.15), when a wave e^{ikz} (without dispersion) strikes a termination of impedance Z_0 at $z = 0$, the reflected wave is related in amplitude and phase to the incident one by the factor $-(A_-/A_+) = -e^{-2\pi(\alpha-i\beta)}$ which is related to Z_0 by the equations

$$e^{-2\pi(\alpha-i\beta)} = \left(\frac{A_-}{A_+}\right) = \frac{\rho c - Z_0}{\rho c + Z_0}; \quad Z_0 = \rho c\,\frac{1-(A_-/A_+)}{1+(A_-/A_+)} = \rho c\tanh[\pi(\alpha - i\beta)]$$

In the present case $A_+ = 1$ and $A_- = ikl_xW/(1 - ikl_xW)$ so that the impedance presented to the incident wave is

$$Z_0 \simeq \rho c(1 - 2ikl_xW) = \rho c + Z_s$$

where Z_s is the effective impedance of the barrier with slit. Since ρc is the impedance of an open duct, we have that Z_0 is given by the impedance Z_s of the constriction, in series with the duct on the other side of the slit.

When the driving frequency ω is made larger than the cutoff frequency ω_{10}, then the reflected and transmitted waves are not simple plane waves any longer, and it requires more than a single complex impedance to describe them in terms of the incident wave.

Our solution is not exact, of course; we have arranged to have ψ continuous only at the center of the slit and there will be discontinuities elsewhere. These could be corrected for by an additional series, which would introduce a slight discontinuity in gradient, and so on. Ordinarily, this first approximation is sufficiently accurate for our needs, however.

Wave Transmission around Corner. It is sometimes useful to be able to compute the transmission of a wave around an elbow bend in a duct. As an example of this type of calculation we consider a right angle bend in a duct of cross section l_x by l_y, with the axis of the duct in the x-z plane. We studied the steady flow around such a bend in Sec. 10.2. The velocity potential given in Eq. (10.2.57) and the transformation shown in Fig. 10.23 will be applicable here, if we let $a = b = l_x$, so that $\alpha = 1$. The velocity potential ψ is related to w by the equation $w = e^{-\pi F l_y/Q}$, where $F = \psi + i\chi$. We have here chosen Q to be the total flow around the corner, so that Q/l_y is the former Q, the flow per unit thickness perpendicular to the figure.

In these terms the relation between $z = x + iy$ and F is

$$\begin{aligned} z &= \frac{2l_x}{\pi}\left[\tanh^{-1}\sqrt{\tanh\left(\frac{\pi F l_y}{2Q}\right)} - \tan^{-1}\sqrt{\tanh\left(\frac{\pi F l_y}{2Q}\right)}\right] \\ |v| &= \left|\frac{dF}{dz}\right| = \left|\frac{(Q/l_x l_y)}{\sqrt{\tanh(\pi F l_y/2Q)}}\right| \end{aligned} \tag{11.3.26}$$

where $|v|$ is the magnitude of the steady-state velocity at the point F (or z). What we do is to consider the flow in the square region $BP'DP$ in Fig. 11.5 to be that of an incompressible fluid, whereas the flow from A to $P'D$ and from DP to C is wave motion for a compressible fluid. As long as the wavelength is longer than l_x, the neglect of compression in the corner is not serious, and it enables us to use Eq. (11.3.26) to fit the wave solutions at the boundaries $P'D$ and DP.

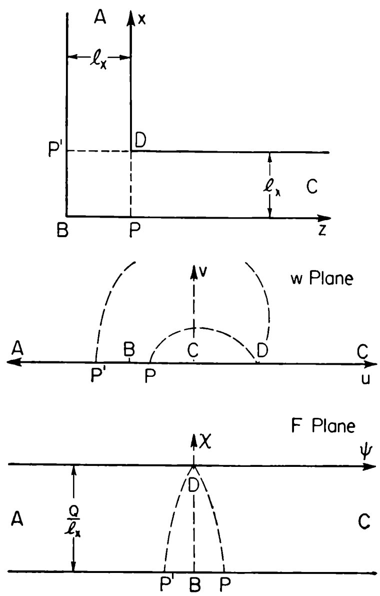

Fig. 11.5 Reflection and transmission of wave around corner in duct. Figures show conformal transforms to calculate steady flow across DP' and DP.

As far as the wave solutions go, we can consider the two arms to be connected by some sort of transformer which insists on a certain distribution of velocity across the line $P'D$ and the same distribution across PD. We can consider the arm from A to $P'D$ separate from the arm from PD to C and, for the wave part, take the z axis along both arms, the positive range being from PD to C, the negative range from A to $P'D$. Except for the elbow region $BP'DP$, the wave along the tube, when $k < \pi/l_x$ has the usual form,

$$\psi(x,z) = \begin{cases} e^{ikz} + (1 - A_0)e^{-ikz} + \displaystyle\sum_{m=1}^{\infty} A_m \cos\left(\frac{\pi m x}{l_x}\right) e^{\kappa_m z}; & z < 0 \\ A_0 e^{ikz} - \displaystyle\sum_{m=1}^{\infty} A_m \cos\left(\frac{\pi m x}{l_x}\right) e^{-\kappa_m z}; & z > 0 \end{cases} \tag{11.3.27}$$

where $\kappa_m = \sqrt{(\pi m/l_x)^2 - k^2}$ and where the (x,z) origin for $z < 0$ is at P' and that for $z > 0$ is at P. To find the initial dependence on x of the normal gradient at $z = 0$, we use the steady-state solution indicated in Eq. (11.3.26), computing the value of v along the lines DP and DP'.

First we must find the equivalent of the lines DP and DP' on the F plane. The point D, of course, is the origin $F = 0$, but the point P ($-il_x$ on the z plane) is not quite so easily found. It must correspond, on the F plane, to a positive real part and an imaginary part $-i(Q/l_y)$. But $\tanh[(\pi\psi l_y/2Q) - \frac{1}{2}i\pi] = \coth(\pi\psi l_y/2Q)$, which is larger than unity. Therefore along the line BC

$$z = -il_x + (2l_x/\pi)[\tanh^{-1}\sqrt{\tanh(\pi\psi l_y/2Q)} - \tan^{-1}\sqrt{\coth(\pi\psi l_y/2Q)}]$$

and for z to equal $-il_x$, we must choose $\tanh(\pi\psi_0 l_y/2Q) = \gamma^2$ so that $\tanh^{-1}\gamma = \tan^{-1}(1/\gamma)$. The solution of this is $\gamma = 0.7341$, $\tanh^{-1}\gamma = 0.9376$, $\tanh(\pi\psi_0 l_y/2Q) = 0.5389$, $\pi\psi_0 l_y/2Q = 0.6026$, so that the point P in the F plane is $F = (Q/l_y)[0.3836 - i]$ and the point P' is $F = (Q/l_y)[-0.3836 - i]$. At both points in the z plane, the fluid velocity is parallel to the boundary in the z plane and has the value

$$v_0 = \left|\frac{(Q/l_x l_y)}{\sqrt{\tanh[(\pi\psi_0 l_y/2Q) - \frac{1}{2}i\pi]}}\right| = 0.7341(Q/l_x l_y)$$

at the point $x = 0$, $z = 0$ on the first sketch of Fig. 11.5. From the symmetry of the solution (we can add another elbow, in the other direction, corresponding to the flow through a T-shaped duct), we can see that $dv_0/dx = 0$ at point P.

As we go along the dotted lines PD in either of the three plots of Fig. 11.5, the z component of the velocity, $v_0(x)$, changes in value, increasing steadily to infinity at D. If we know the exact course of the dotted line in the F plane, we could obtain the exact behavior of v_0 as a function of x, but it would undoubtedly be such a complicated function of x that we could do nothing further with it. However, since we know the value of v_0 at $x = 0$ and the fact that $\int v_0\, dx = \frac{Q}{l_y}$, if we can determine the behavior of v_0 close to the point $D(x = l_x)$, we can perhaps choose a simpler function of x which will satisfy the requirements sufficiently accurately.

Near $F = 0$, Eq. (11.3.26) shows that a series expansion for z in terms of $\tanh(\pi F l_y/2Q) = \delta^2$ gives us

$$z = \frac{2l_x}{\pi}[\delta + \tfrac{1}{3}\delta^3 + \cdots - \delta + \tfrac{1}{3}\delta^3 - \cdots] \simeq \left(\frac{4l_x}{3\pi}\right)\delta^3$$

or $$\delta = \sqrt{\tanh(\pi F l_y/2Q)} \simeq (3\pi z/4l_x)^{\frac{1}{3}}$$

In order that z follow along the dotted line from D to P, z must be equal to $-i\zeta = \zeta e^{-\frac{1}{2}i\pi}$, where $\zeta = (l_x - x)$ in the coordinates of the first sketch of Fig. 11.5. This means that F is approximately equal to $(2Q/\pi l_y)e^{-\frac{1}{3}i\pi} \cdot$ $\cdot\,(3\pi\zeta/4l_x)^{\frac{2}{3}}$, the dotted line starting off from D in the F plane at an angle of $-60°$ to the ψ axis. The velocity v_0, in magnitude and direction, is the complex conjugate of dF/dz, which is, along DP near the point D, approximately $(Q/l_xl_y)(4l_x/3\pi\zeta)^{\frac{1}{3}}e^{-\frac{1}{6}i\pi}$, pointed down at an angle of 30° from the horizontal. Therefore along DP, near $D(x = l_x)$, the horizontal component of velocity is

$$v_0(x) \to \left(\frac{Q}{l_xl_y}\right)\cos\left(\frac{\pi}{6}\right)\left[\frac{(8/3\pi)}{2-2(x/l_x)}\right]^{\frac{1}{3}} = 0.821\,\frac{(Q/l_xl_y)}{\sqrt[3]{2[1-(x/l_x)]}}$$

We now have all the limiting conditions for $v_0(x)$, and we must find a simple form which will approximate its behavior. The simplest form which has the right dependence on x, both near $x = 0$ (point P) and $x = l_x$ (point D), is $v_0 = C/\sqrt[3]{1-(x/l_x)^2}$. If it were the exact solution, C would be $0.734(Q/l_xl_y)$ at $x = 0$ and $0.821(Q/l_xl_y)$ at $x = l_x$, so this form cannot be exact, but if C has some intermediate value, the approximation will be quite good. Let us see what value C must have in order that the integral of v_0 over the duct equal Q, the total flow around the bend. We have

$$Q = l_yC\int_0^{l_x}\frac{dx}{\sqrt[3]{1-(x/l_x)^2}} = Cl_xl_y\int_0^{\pi}\sin^{\frac{1}{3}}\phi\,d\phi = \frac{\sqrt{\pi}\,\Gamma(2/3)}{2\Gamma(7/6)}\,Cl_xl_y$$

or

$$C = \frac{2\Gamma(7/6)}{\sqrt{\pi}\,\Gamma(2/3)}\left(\frac{Q}{l_xl_y}\right) = 0.7714\left(\frac{Q}{l_xl_y}\right)$$

which is indeed intermediate between the limiting values quoted above.

Consequently, we shall assume that along the line PD (and also along the line $P'D$) the z component of the gradient of ψ is approximately $C[1-(x/l_x)^2]^{-\frac{1}{3}}$, with C given above. And now it is possible to go back to Eq. (11.3.27) and fit the Fourier series for the gradient of ψ to this form. The typical equation for the coefficients is, according to Eq. (5.3.63),

$$\begin{aligned}\tfrac{1}{2}l_xA_m\kappa_m &= C\int_0^{l_x}\frac{\cos(\pi mx/l_x)}{\sqrt[3]{1-(x/l_x)^2}}\,dx = Cl_x\int_0^1\frac{\cos(\pi mu)}{(1-u^2)^{\frac{1}{3}}}\,du\\ &= Cl_x\left[\frac{\sqrt{\pi}\,\Gamma(2/3)}{2(\frac{1}{2}\pi m)^{\frac{1}{6}}}\right]J_{\frac{1}{6}}(\pi m)\end{aligned}$$

or

$$A_m = \frac{2\Gamma(7/6)}{\kappa_m}\left[\frac{J_{\frac{1}{6}}(\pi m)}{(\frac{1}{2}\pi m)^{\frac{1}{6}}}\right]\left(\frac{Q}{l_xl_y}\right);\quad A_0 = \left(\frac{Q}{ikl_xl_y}\right)$$

To calculate the value of Q and thus to solve the problem (at least to the approximation we are considering here), we must somehow relate the value of ψ on one side of the elbow to the value on the other side. Of course, if we could fit the value for every value of x across the duct, we would have solved the problem exactly; but we are not this ambitious and we know that, with the approximations used, as long as k is smaller than π/l_x, arranging to fit ψ at one point will provide a reasonably good fit at all other points. The two easiest points to make the fit are at $x = l_x$ (at point D) where the requirement is that $\psi_- = \psi_+$, or at $x = 0$ (at points P' and P) where the requirement is that $\psi_+ = \psi_- + 0.7672(Q/l_y)$, as the earlier discussion shows. The results will differ somewhat, of course.

We chose the fit at point D, as being somewhat simpler and also because this is the point of maximum flow. At this point the series for the higher terms is $(Q/l_y)G(kl_x)$, where

$$G(kl_x) = 2\Gamma(7/6) \sum_{m=1}^{\infty} \frac{(-1)^m}{l_x\kappa_m} \left[J_{\frac{1}{6}}(\pi m)/(\tfrac{1}{2}\pi m)^{\frac{1}{6}}\right] \tag{11.3.28}$$

so that the joining equation is $2 = 2(Q/l_y)[(1/ikl_x) - G]$, or

$$Q = 2ikl_xl_y/[1 - ikl_xG(kl_x)]$$

Once this is computed, as a function of kl_x (for kl_x less than π), all the other expressions may be obtained, just as they were for the case of the barrier with slit. In particular, we see that the impedance of the elbow, plus the duct beyond, to a plane wave incident on it, is

$$Z_0 \simeq \rho c + Z_e; \quad Z_e = -\rho c(2ikl_xG) = -i\omega(2\rho l_xG) \tag{11.3.29}$$

where Z_e is the effective impedance of the elbow alone, which appears in series with the impedance ρc of the rest of the duct.

At low frequencies Z_e acts like a mass reactance, with effective mass (for the whole tube) of $[2\rho l_x^2 l_y G(kl_x)]$. For very small values of (kl_x), the function G approaches the constant value

$$G_0 = \Gamma(7/6) \sum_{m=1}^{\infty} (-1)^m(\tfrac{1}{2}\pi m)^{-\frac{7}{6}} J_{\frac{1}{6}}(\pi m) = 0.2518$$

For somewhat larger values of $(kl_x) < 1$, we can expand $G(kl_x)$ in a power series

$$G(kl_x) = G_0 + \tfrac{1}{2}(kl_x)^2G_1 + \tfrac{3}{8}(kl_x)^4G_2 + \tfrac{5}{16}(kl_x)^6G_3 + \cdots$$

where
$$G_n = 2^{\frac{7}{6}}\Gamma(7/6) \sum_{m=1}^{\infty} \left[\frac{(-1)^m}{(\pi m)^{2n+\frac{7}{6}}}\right] J_{\frac{1}{6}}(\pi m)$$
$$G_1 = 0.01306; \quad G_2 = 0.001217, \ . \ . \ .$$

Parenthetically, the series for the G_n's converge absolutely, but computation time may be saved by the following procedure. For m large, the Bessel function takes on its asymptotic form

$$J_{\frac{1}{6}}(\pi m) \to \sqrt{\frac{2}{\pi^2 m}} \cos\left(\pi m - \frac{\pi}{3}\right) = \frac{1}{2\pi} \sqrt{\frac{2}{m}} (-1)^m$$

Therefore the series for G_n may be rewritten

$$G_n = \frac{2^{\frac{2}{3}}\Gamma(7/6)}{\pi^{2(n+1)+\frac{1}{6}}} \left\{ \zeta(2n + \tfrac{5}{3}) - \sum_{m=1}^{\infty} \frac{1}{m^{2n+\frac{5}{3}}} [1 - (-1)^m \pi \sqrt{2m}\, J_{\frac{1}{6}}(\pi m)] \right\}$$

where $\zeta(2n + \frac{5}{3})$ is the Riemann zeta function

$$\zeta(z) = \sum_{m=1}^{\infty} \left(\frac{1}{m^z}\right) = \prod_p (1 - p^{-z})^{-1}$$

where the product is over all positive prime numbers ($p = 2, 3, 5, 7, 11, 13$). The series to add to the zeta function, to obtain G_n, converges quite rapidly. In fact, for $n > 2$, the formula

$$G_n \simeq \frac{0.12343}{\pi^{2n}} \zeta(2n + \tfrac{5}{3}) - \frac{0.00764}{\pi^{2n}}$$

will give accuracy better than four significant figures.

Membrane in Circular Tube. A further example of calculations involving transmission in ducts involves the reflection from and transmission through an elastic membrane stretched across a circular tube of radial cross section a. If the waves are started in a manner symmetrical to the duct axis, all the motion will be symmetrical and the permissible modes of vibration will have the form

$$J_0(\pi\alpha_n r/a)e^{+ik\tau_n z - i\omega t}; \quad \tau_n = \sqrt{1 - (\pi\alpha_n/ka)^2}$$

and where, if the boundary condition corresponds to acoustic waves (Neumann conditions), the constant α_n is the nth root of the equation $[dJ_0(\pi\alpha)/d\alpha] = -J_1(\pi\alpha) = 0$. It is one of the roots labeled α_{0n} in the table at the end of this chapter. Of course r is the radial coordinate, z the distance along the tube, and $k = \omega/c$, where c is the velocity of acoustic waves in air.

On the other hand, if the membrane is elastic, of density M per unit area under a tension T per unit length, and if the membrane is held rigidly at its edges ($r = a$), the modes of free vibration for the symmetrical cases are

$$\zeta = J_0(\pi\beta_n r/a)e^{-i\omega_n t}; \quad \omega_n = \pi C\beta_n/a$$

where ζ is the displacement of the point (r,ϕ) of the membrane from equilibrium in the z direction, $C = \sqrt{T/M}$ is the velocity of transverse waves in the membrane, $\omega_n/2\pi$ is the resonance frequency of the nth mode, and β_n is the nth root of the equation $J_0(\pi\beta) = 0$ (labeled β_{0n} in the table at the end of this chapter).

Now suppose a plane wave ($n = 0$) of frequency less than $\alpha_1/2a$ is started to the right from the left-hand end of the tube, some distance from the membrane, which is at $z = 0$. The wave will strike the membrane, part will be reflected and part will be transmitted, moving the membrane in the process. We can assume that the acoustic wave is given by the velocity potential

$$\psi = \begin{cases} e^{ikz} + (1 - A_0)e^{-ikz} + \sum_{n=1}^{\infty} A_n J_0\left(\frac{\pi\alpha_n r}{a}\right) e^{\kappa_n z}; & z < 0 \\ A_0 e^{ikz} - \sum_{n=1}^{\infty} A_n J_0\left(\frac{\pi\alpha_n r}{a}\right) e^{-\kappa_n z}; & z > 0 \end{cases} \tag{11.3.30}$$

where $\kappa_n = \sqrt{(\pi\alpha_n/a)^2 - k^2}$. The combinations of the exponentials and the choice of coefficients A are such that the z component of the velocity at $z = 0$ is the same on both sides of the membrane (and, of course, is equal to that of the membrane) and so that at large distances from the membrane ($|z|$ large) there is an outgoing wave at the right and an incoming plus outgoing one at the left.

The values of the coefficients A are determined by fitting these expressions into the equation of motion of the membrane. For motions symmetric about the center of the membrane, the equation for the displacement ζ is

$$M\frac{\partial^2\zeta}{\partial t^2} = \frac{T}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\zeta}{\partial r}\right) + p(r) \tag{11.3.31}$$

where, by Eq. (11.3.30), the driving pressure is

$$p(r) = 2i\omega\rho\left[1 - A_0 + \sum_{n=1}^{\infty} A_n J_0(k_n r)\right] e^{-i\omega t}$$

where ρ is the density of air in the duct (M is the membrane mass per unit area) and where $k_n = \pi\alpha_n/a$ is *not* equal to ω_n/c.

We can solve Eq. (11.3.31) in two ways (though, of course, the results must be the same no matter which way is used), one by expanding ζ as a sum of the eigenfunctions $J_0(\pi\beta_m r/a)$ which go to zero at $r = a$ and the other by treating (11.3.31) as an ordinary inhomogeneous equation to be solved by the use of Eq. (5.3.19). We shall have enough series, anyway, so we choose the latter method, since it does not immediately lead to a double series. The independent solutions of the homogeneous equation are $J_0(Kr)$ and $N_0(Kr)$, where $K = \omega/C = \omega\sqrt{M/T}$; with Wronskian $\Delta(J_0,N_0) = J_0(Kr)N_0'(Kr) - N_0(Kr)J_0'(Kr) = 2/\pi Kr$, where J_0', N_0' denote derivatives with respect to the arguments Kr.

The solution of (11.3.31) is thus the application of Eq. (5.2.19), setting the limits of integration and the constants so that ζ is zero at $r = a$ and is not infinite at $r = 0$, and using the formulas for integration of Bessel function products given at the end of Chap. 10. The results are

$$\zeta = \frac{-\pi}{2T}\left\{J_0(Kr)\left[\int_r^a N_0(Kx)p(x)x\,dx - \frac{N_0(Ka)}{J_0(Ka)}\int_0^a J_0(Kx)p(x)x\,dx\right] + N_0(Kr)\int_0^r J_0(kx)p(x)x\,dx\right\}$$

$$= -i\left(\frac{2K\rho}{MC}\right)e^{-i\omega t}\left\{\frac{1-A_0}{K^2}\left[1 - \frac{J_0(Kr)}{J_0(Ka)}\right] + \sum_{n=1}^{\infty}\frac{A_n}{K^2-k_n^2}\left[J_0(k_nr) - \frac{J_0(Kr)}{J_0(Ka)}J_0(k_na)\right]\right\} \qquad (11.3.32)$$

which does go to zero at $r = a$.

The values of the coefficients A, and therefore the shape of the membrane and of the sound wave, may be determined by requiring that the velocity of motion of the membrane, $-i\omega\zeta$, is equal to the air velocity in the z direction at $z = 0$:

$$-i\omega\zeta = ikA_0 + \sum_{n=1}^{\infty}\kappa_nA_nJ_0(k_nr); \quad \kappa_n = \sqrt{k_n^2 - k^2}$$

This leads to an infinite number of simultaneous equations in the infinite number of unknowns A_n. In order to obtain a solution, it is best to proceed by successive approximations. Usually the ratio ρ/M between membrane mass per unit area and air density is small; if this is so, all the A's are small with respect to unity (unless the frequency is such as to produce membrane resonance).

When there is no membrane resonance, when $\rho/M \ll 1$ and when the frequency is less than the first cutoff frequency of the tube ($k < k_1$) then, to the zeroth approximation, the membrane velocity is

$$ikA_0 + \sum_{n=1}^{\infty}\kappa_nA_nJ_0(k_nr) \simeq -2\left(\frac{\rho}{M}\right)\left[1 - \frac{J_0(Kr)}{J_0(Ka)}\right](1 - A_0)$$

Values of the A's, to the first approximation in ρ/M, may be obtained by multiplying both sides by $J_0(k_nr)r\,dr = J_0(\pi\alpha_nr/a)r\,dr$ and integrating over r from zero to a.

$$A_0 \simeq \frac{2\rho J_2(Ka)}{ikMJ_0(Ka) + 2\rho J_2(Ka)} \to -i\omega\left(\frac{\rho ca^2}{4T}\right); \quad Ka \to 0$$

$$A_n \simeq \frac{-2(\rho/M)}{\kappa_nJ_0^2(k_na)}\left(\frac{K^2}{K^2-k_n^2}\right)\left[\frac{J_0(Ka)J_2(k_na) - J_0(k_na)J_2(Ka)}{J_0(Ka) + (2\rho/ikM)J_2(Ka)}\right]$$

$$\to -\frac{2\rho}{M\kappa_n}\left[\frac{J_2(k_na)}{J_0^2(k_na)}\right]; \quad K \to k_n \qquad (11.3.33)$$

As a matter of fact, by using the factor $(1 - A_0)$, instead of 1, we have arranged so that formulas (11.3.33) are correct to the first approximation in ρ/M even when there is membrane resonance. Such resonance occurs when K is such that $J_0(Ka) = 0$, in which case $A_0 \to 1$ and $A_n \to -i(k/\kappa_n)[K^2/(K^2 - k_n^2)][1/J_0(k_n a)]$, which are values very much larger than the values for most values of K but which are not infinite. Likewise none of the A's go to infinity when K goes to k_n [as a matter of fact, there is not even a resonance at $K \to k_n$, $\omega = C\pi\alpha_n/a$; the resonances occur at $\omega = C\pi\beta_n/a$, where $J_0(Ka) = 0$].

Referring to the discussion of Eqs. (11.3.25) *et seq.*, we see that the ratio of reflected to incident wave amplitudes is $A_-/A_+ = -(1 - A_0)$, so that the effective impedance of the membrane plus tube beyond to a plane wave striking the left-hand side of the membrane is

$$Z_0 = \rho c \frac{1 - (A_-/A_+)}{1 + (A_-/A_+)} = i\omega M \left[\frac{J_0(Ka)}{J_2(Ka)}\right] + \rho c$$

the resistive term ρc being the acoustic resistance of the tube beyond the membrane, to which the membrane adds a series reactance. At very low frequencies, this membrane reactance becomes

$$X_0 \to 8T/-i\omega a^2$$

which is a stiffness reactance, caused by the tension T in the membrane. At membrane resonance $J_0(Ka) = 0$ and the impedance of the membrane goes to zero, the impedance of the system being just that of the tube, ρc. When $J_2(Ka) = 0$, the membrane oscillates so that as much air is pushed in one direction as in the other, and the net impedance is infinite. For frequencies larger than $k_1 c/a\pi = \alpha_1 c/2a$, higher mode waves can be transmitted along the tube, and the phenomenon becomes more complicated than can be described in terms of a single impedance.

Radiation from Tube Termination. Another calculation which is of use in acoustics problems is that of the projection of a wave out of the end of a circular tube. The nature of the radiation and its reaction back on the wave inside the tube depends on the sort of termination chosen. The simplest, as far as computation goes, is the flanged termination, where the end of the tube is a hole in a flat wall, large compared to the wavelength. The case of no flange will be treated in Sec. 11.4. Here we consider the case of a circular tube of radius a, with axis the negative z axis, ending at $z = 0$, with the $x - y$ plane a rigid wall, except for the circular open end of the tube.

If a plane wave is transmitted to the right along the tube, it will reach the open end, part will radiate out into the open, to the right of the wall, and part will be reflected back into the tube. Below the first cutoff frequency ($k < k_1 = \pi\alpha_1/a$), the form of this wave is

$$\psi = e^{ikz} - A_0 e^{-ikz} + \sum_{n=1}^{\infty} A_n J_0(k_n r) e^{\kappa_n z}; \quad z < 0 \tag{11.3.34}$$

as before, where $\kappa_n = \sqrt{k_n^2 - k^2}$. To the zeroth approximation in the (presumably) small quantities A_n, the velocity of air at the open end of the tube is

$$v_0 \simeq \begin{cases} ik(1 + A_0); & r < a \\ 0; & r > a \end{cases}$$

To compute the radiation to the right, out of the open end, we can use the Green's function method. The Green's function useful here is the one having zero normal gradient at $z = 0$;

$$G_k(\mathbf{r}|\mathbf{r}_0) = (e^{ikR}/R) + (e^{ikR'}/R')$$

where $k = \omega/c$, $(R)^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$, and $(R')^2 = (x - x_0)^2 + (y - y_0)^2 + (z + z_0)^2$, corresponding to a unit source at

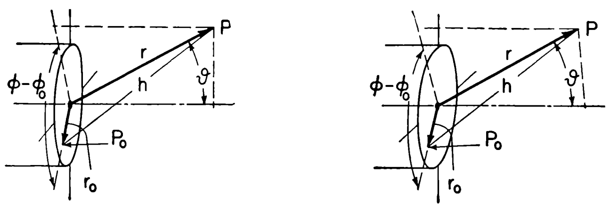

Fig. 11.6 Angles and distances for radiation of waves from open end of cylinder.

(x_0,y_0,z_0) and its image at $(x_0,y_0,-z_0)$. Using the expression for the velocity at $z = 0$ (which equals minus the normal gradient) and Eq. (7.2.10), we have for the radiated wave,

$$\psi(\mathbf{r}) = -\frac{1}{2\pi}\int_0^{2\pi} d\phi_0 \int_0^a r_0\, dr_0 v_0(r_0)\, \frac{e^{ikh}}{h} \tag{11.3.35}$$

where $h^2 = r^2 + r_0^2 - 2rr_0 \sin\vartheta \cos(\phi - \phi_0)$, the point P_0 on the $z = 0$ plane being defined by the polar coordinates r_0, ϕ_0, the point P, at the end of the vector $\mathbf{r}$, being defined by the spherical coordinates r, ϑ, ϕ (with the reference axis the extension of the tube axis and origin the center of the open hole), and h being the distance between P_0 and P.

To determine the coefficients A_n and thus the distribution of pressure and velocity at the opening, we have to match expressions for pressure at $z = 0$, at least approximately. Using the integral expression, the pressure at the point (r,ϕ) on the plane $z = 0$ is

$$p(\mathbf{r}) = -\frac{i\omega\rho}{2\pi}\int_0^{2\pi} d\phi_0 \int_0^a r_0\, dr_0 v_0(r_0)\, \frac{\exp(ik\sqrt{r^2 + r_0^2 - 2rr_0\cos\phi_0})}{\sqrt{r^2 + r_0^2 - 2rr_0\cos\phi_0}}$$

But, as the table at the end of Chap. 10 shows, we have that

$$\left[\frac{e^{ikh}}{h}\right] = \int_0^\infty J_0(xh)\,\frac{x\,dx}{\sqrt{x^2 - k^2}}$$

where $\sqrt{\lambda^2 - k^2} = -i\sqrt{k^2 - \lambda^2}$ when $\lambda < k$ and both λ and k are real (in other words, the path of integration for λ goes just *below* the point $\lambda = k$). We also note that

$$J_0(x\sqrt{r^2 + r_0^2 - 2rr_0\cos\phi_0}) = \sum_{m=0}^\infty \epsilon_m \cos(m\phi_0) J_m(xr_0) J_m(xr)$$

These substitutions seem to be going backward, adding more integrations to the original two, but it does enable us to get ahead eventually, for we can then integrate over ϕ_0:

$$\begin{aligned} p(\mathbf{r}) &= -\frac{i\omega\rho}{2\pi}\int_0^{2\pi} d\phi_0 \int_0^a r_0\,dr_0 v_0(r_0)\,\cdot \\ &\qquad\qquad \cdot \sum_{m=0}^\infty \epsilon_m \cos(m\phi_0) \int_0^\infty \frac{J_m(xr_0)J_m(xr)}{\sqrt{x^2 - k^2}}\,x\,dx \\ &= -ik\rho c \int_0^\infty x\,dx \int_0^a v_0(r_0)\,\frac{J_0(xr_0)J_0(xr)}{\sqrt{x^2 - k^2}}\,r_0\,dr_0 \end{aligned}$$

and we are ready to insert a value of $v_0(r_0)$.

If we assume that the coefficients A_n, in the series expression for ψ inside the tube, are small, we can insert the approximate expression $v_0 \simeq ik(1 + A_0)$ in the integral for p and then equate this to the series expression obtained from Eq. (11.3.34),

$$\begin{aligned} ik\rho c\left[(1 - A_0) + \sum_{n=1}^\infty A_n J_0(k_n r)\right] & \\ \simeq \rho c k^2 a(1 + A_0) \int_0^\infty J_0(xr)J_1(xa)\,\frac{dx}{\sqrt{x^2 - k^2}} & \end{aligned}$$

from which we can obtain approximate expressions for the A's. For instance, by multiplying both sides by $r\,dr$ and integrating from zero to a, all the terms on the left-hand side vanish except the one with the factor $(1 - A_0)$, because of the orthogonality of the eigenfunctions $J_0(k_n a)$. The result is

$$\begin{aligned} \frac{1 - A_0}{1 + A_0} &\simeq -2ik \int_0^\infty [J_1(xa)]^2\,\frac{dx}{x\sqrt{x^2 - k^2}} \\ &= 2\mu \int_0^\mu \frac{[J_1(u)]^2\,du}{u\sqrt{\mu^2 - u^2}} - 2i\mu \int_\mu^\infty \frac{[J_1(u)]^2\,du}{u\sqrt{u^2 - \mu^2}} \end{aligned}$$

where $u = xa$ and $\mu = ka = \omega a/c = 2\pi a/\lambda$.

Both of these integrals are given in the tables on page 1324. The first turns out to be just $1 - (1/\mu)J_1(2\mu)$, and the second is just $1/\mu$ times the Struve function $S_1(2\mu)$, defined on page 1324. We note that, to this approximation, the effective acoustic impedance of the opening, to a plane wave coming along the tube from the left, is just ρc times $(1 - A_0)/(1 + A_0)$,

$$Z_0 \simeq \rho c\left[1 - \frac{1}{ka}J_1(2ka) - \frac{i}{ka}S_1(2ka)\right] \to \tfrac{1}{2}\rho c(ka)^2 - i\omega(8\rho a/3\pi); \quad ka \ll 1$$

as long as $k < \pi\alpha_1/a$, so that only the lowest mode can be transmitted along the tube. The limiting values show that, at low frequencies, the open end has less impedance than a continuation of the same tube to infinity.

We can now express the other A's in terms of A_0 by multiplying the equation for the pressure by $J_0(k_n r)r\,dr$ and integrating from zero to a $(k_n = \pi\alpha_{0n}/a)$. The integral

$$\int_0^a J_0(xr)J_0(k_n r)r\,dr = \frac{xa}{x^2 - k_n^2}\left[J_0(k_n a)J_1(xa) - \frac{k_n}{x}J_0(xa)J_1(k_n a)\right]$$

where the second term in the brackets is zero since $J_1(k_n a) = J_1(\pi\alpha_{0n}) = 0$. Integration of both sides finally gives

$$A_n \simeq \frac{1 + A_0}{J_0(\pi\alpha_{0n})}[\Theta_n(\mu) - i\chi_n(\mu)]; \quad \mu = ka = 2\pi a/\lambda$$
$$\Theta_n = 2\mu\int_0^\mu \frac{J_1^2(u)}{u^2 - u_n^2}\frac{u\,du}{\sqrt{\mu^2 - u^2}}; \quad \chi_n = 2\mu\int_\mu^\infty \frac{J_1^2(u)}{u^2 - u_n^2}\frac{u\,du}{\sqrt{u^2 - \mu^2}} \tag{11.3.36}$$

where $u_n = k_n a = \pi\alpha_{0n}$. For $\mu < u_1$, these quantities diminish quite rapidly with increasing n, thereby justifying our method of approximating the answer. Tables of these functions are available.[1] Their limiting values for low frequencies are:

$$\Theta_n \to -(\mu^4/3\pi^2\alpha_{0n}^2); \quad \chi_n \to -g_n\mu; \quad \mu \to 0,\ n > 0$$
$$g_1 = 0.0920; \quad g_2 = 0.0356; \quad g_3 = 0.0194, \ldots$$

The functions for $n = 0$ were given earlier:

$$\Theta_0 = 1 - \left(\frac{1}{\mu}\right)J_1(2\mu) \to \tfrac{1}{2}\mu^2; \quad \chi_0 = \left(\frac{1}{\mu}\right)S_1(2\mu) \to \left(\frac{8\mu}{3\pi}\right); \quad \mu \to 0$$

We can now return to Eq. (11.3.35) to compute the radiation out of the open end of the tube, to the first approximation in the small quanti-

[1] See, for instance, Morse, "Vibration and Sound," 2d ed., McGraw-Hill, New York, 1948.

ties A_n. For large distances (*i.e.*, many wavelengths) from the opening, we can set $h \simeq r - r_0 \sin\vartheta \cos(\phi - \phi_0)$ and, using the expression

$$v_0 \simeq \frac{2}{1 + \Theta_0 - i\chi_0}\left\{ik + \sum_{n=1}^{\infty} \kappa_n(\Theta_n - i\chi_n)\frac{J_0(k_n r)}{J_0(k_n a)}\right\}$$

where $k_n = \pi\alpha_{0n}/a$ and $\kappa_n = \sqrt{k_n^2 - k^2}$, we obtain

$$\psi \simeq \frac{-(1/\pi)}{1 + \Theta_0 - i\chi_0}\left[\frac{e^{ikr}}{r}\right]\left\{\int_0^a r_0\,dr_0\left[ik + \sum \kappa_n(\Theta_n - i\chi_n)\frac{J_0(k_n r_0)}{J_0(k_n a)}\right]\int_0^{2\pi} e^{-ikr_0 \sin\vartheta\cos(\phi_0 - \phi)}\,d\phi_0\right\}$$

The integral over ϕ_0 produces $2\pi\, J_0(kr_0 \sin\vartheta)$, where ϑ is the angle between the vector $\mathbf{r}$ and the axis of the tube, extended beyond the opening in the flat wall at $z = 0$ (*i.e.*, it is the spherical angle between r and the z axis). The integral over r_0 can then be carried out to obtain

$$\psi \simeq \frac{-i\mu a}{1 + \Theta_0 - i\chi_0}\left[\frac{e^{ikr}}{r}\right]\left\{\Phi_0(\mu\sin\vartheta) - i\sum_{n=1}^{\infty}\left(\frac{\kappa_n}{k}\right)(\Theta_n - i\chi_n)\Phi_n(\mu\sin\vartheta)\right\} \tag{11.3.37}$$

where

$$\mu = ka \quad\text{and}\quad \Phi_n(s) = \frac{2sJ_1(s)}{s^2 - (\pi\alpha_{0n})^2} \xrightarrow[s\to 0]{} \begin{cases} 1; & n = 0 \\ -(s/\pi\alpha_{0n})^2; & n > 0 \end{cases}$$

From this we can obtain the pressure and intensity of the radiated wave at large distances from the opening, for a unit amplitude plane wave sent along the tube toward the opening. This holds, in the form given, only for frequencies such that $k < k_1$, *i.e.*, below the first cutoff frequency for the tube.

We could, of course, discuss waves outside and inside other cylinders, elliptic or parabolic, but such cases are rare in practice and nothing new would be learned of importance. A few cases will be met in the problems. Instead we take up a different sort of problem, before going on to noncylindrical coordinates.

Transmission in Elastic Tubes. There are cases where the variations in pressure of the sound wave traveling down a circular tube produce appreciable stretching of the walls. This is particularly true when the tube contains a comparatively incompressible fluid (such as water) and the outside of the tube is in air so that no additional external support is given the tube. The response of the tube walls to unsymmetric modes (for $m > 0$) will be quite different from the response for the symmetric ones ($m = 0$); the former will depend on the resistance of the wall to bending, whereas the latter will depend only on the simple Young's modulus for stretching. The symmetric case is thus more straightforward; it is also more often encountered.

Suppose that at one cross section (one value of z) the excess pressure, over and above the equilibrium pressure, is $p(z,t)$. We should expect the radius a of the circular cross section to be somewhat larger than at equilibrium, by an amount proportional to p. An increase in internal pressure of p must be balanced by an increase in tensile stress in the pipe wall of ap/h, where h is the thickness of the pipe wall. The radius of the pipe will therefore be increased by a fractional amount ap/hE_t, where E_t is Young's modulus for the material of the tube wall. Therefore, for slow variations of pressure the radius of the tube will be changed from the equilibrium value a to the value $a + p(a^2/hE_t)$, as long as the tube walls have little stiffness, so that the stretching at one point is not affected by stretching at nearby points.

If the pressure changes are rapid, however, we must also take into account the effective mass of the tube wall. Calling the increase in radius δ, the equation relating δ to the excess internal pressure will be

$$\rho_t h \frac{d^2\delta}{dt^2} + \delta(hE_t/a^2) = p$$

where ρ_t is the density of the material in the tube wall. Consequently, if we can neglect the internal friction of the tube wall, the radial velocity of the tube wall v_r (which is also the radial velocity of the fluid at the inner surface of the wall) for an oscillating pressure $p(z)e^{-i\omega t}$ is

$$v_r = [p(z)/Z_t]e^{-i\omega t}; \quad Z_t = -i\omega(\rho_t h) - (hE_t/i\omega a^2)$$

where Z_t may be called the *transverse mechanical impedance* of the tube wall. We are assuming that the air outside the tube has negligible reaction on the tube and that the tube walls are flexible (like rubber) rather than stiff (like steel).

A symmetrical sound wave along the inside of the circular tube is given by the velocity potential

$$\psi = A \exp(ikz\sqrt{1-\sigma^2})\, J_0(k\sigma r)e^{-i\omega t}; \quad k = \omega/c \qquad (11.3.38)$$

where the value of the parameter σ is determined by the impedance Z_t. The excess pressure at the tube wall at the point (z,a) is then

$$p = -\rho(\partial\psi/\partial t) = i\omega\rho\psi = i\omega\rho A \exp(ikz\sqrt{1-\sigma^2})\, J_0(k\sigma a)e^{-i\omega t}$$

where ρ is the density of the fluid. The radial velocity of the fluid at the wall at the same point is

$$v_r = \frac{\partial\psi}{\partial r} = -k\sigma A \exp(ikz\sqrt{1-\sigma^2})\, J_1(k\sigma a)e^{-i\omega t}$$

and σ is determined by the equation

$$\frac{p}{v_r} = -i\omega\rho_t h - \frac{hE_t}{i\omega a^2} = -\frac{i\omega\rho}{k\sigma}\frac{J_0(k\sigma a)}{J_1(k\sigma a)}$$

For low frequencies, such that $\omega^2 \ll E_t/\rho_t a^2$, it turns out that the lowest root of σ is imaginary, so we should have used the hyperbolic Bessel function $I_0(k\tau r) = J_0(ik\tau r)$ $(\sigma = i\tau)$ for ψ. Likewise, in practice, hE_t/a^2 is large enough so that τ is considerably smaller than $1/ka$ and we can get along with only the first few terms of the series expansion for I_0. Consequently, we can write the velocity potential

$$\psi \simeq A[1 + \tfrac{1}{4}(k\tau r)^2]\exp(ikz\sqrt{1+\tau^2} - i\omega t); \quad 2\rho\omega^2 a^3 \ll hE_t$$

and the equation for τ becomes

$$-i\omega\rho_t h + \frac{ihE_t}{\omega a^2} \simeq \frac{2i\omega\rho}{k^2\tau^2 a} = i\,\frac{2\rho c^2}{\omega\tau^2 a}$$

or

$$\tau^2 \simeq \frac{(2\rho c^2 a/h)}{E_t - \omega^2 a^2\rho_t} \simeq \left(\frac{2\rho c^2 a}{hE_t}\right)\left[1 + \frac{\omega^2 a^2 \rho_t}{E_t}\right]$$

Therefore, the velocity, to this degree of accuracy, is

$$\psi \simeq A\left[1 + \left(\frac{\rho\omega^2 a r^2}{2hE_t}\right) + \left(\frac{\rho\rho_t\omega^4 a^3 r^2}{2hE_t^2}\right)\right] e^{i\omega[(z/c_t)-t]}$$

where c_t is the velocity of this wave in its transmission along the tube

$$c_t \simeq c\left[1 - \frac{\rho c^2 a}{hE_t} - \frac{\rho\rho_t\omega^2 c^2 a^3}{hE_t^2}\right]$$

which is smaller than the velocity c of sound in an infinite amount of the fluid which is in the tube. For stiff tubes ($hE_t \gg \rho c^2 a$) the velocity is not much reduced. The tube is dispersive for waves, since c_t depends on ω.

The higher symmetrical modes may be computed in terms of the angle

$$\alpha_0(z) = \tan^{-1}\left[\frac{-z}{J_0(z)}\frac{dJ_0(z)}{dz}\right] = \tan^{-1}\left[\frac{zJ_1(z)}{J_0(z)}\right]$$

which is tabulated at the end of this chapter and at the end of the book. The limiting values are:

$$\begin{aligned} \alpha_0(z) &\to \tfrac{1}{2}z^2; && z \to 0 \\ &\to \tan^{-1}[z\tan(z - \tfrac{1}{2}m\pi - \tfrac{1}{4}\pi)]; && z \to \infty \\ &\to -\tan^{-1} y; && z = iy \to i\infty \end{aligned}$$

The equation for σ is then

$$\cot[\alpha_0(k\sigma a)] = -\frac{hE_t/\rho a}{(kca)^2} + \left(\frac{\rho_t h}{\rho a}\right)$$

The tables for α_0 at the back of the book may be used to calculate the values of σ for the higher modes. For these higher modes σ is real and the velocity of propagation $c_t = c/\sqrt{1 - \sigma^2}$ is larger than c.

For the lowest mode, for very elastic tubes, or for higher frequencies $k\sigma a$ is large and imaginary, whence

$$\tau \simeq \frac{\rho c \omega a^2/h}{E_t - \rho_t \omega^2 a^2}; \quad \rho\omega^2 a^3/h > E_t > \rho_t \omega^2 a^2$$

and the velocity of propagation is

$$c_t = c/\sqrt{1 + \tau^2}$$

which may be quite slow when τ is large. Finally, when $\rho_t\omega^2 a^2 > E_t$, σ is real even for the lowest mode and c_t is larger than c for all modes. In this limiting case the walls of the tube are "mass controlled" and their motion is out of phase with the pressure, which pushes the waves along faster than they would go in free space. The example we have worked out here is of course a very simplified case; in actual practice we would usually have to consider the reaction of the medium outside the tube and the stiffness of the tube. The formulas are reasonably satisfactory for water inside a thin rubber tube, however, or for the behavior of arterial blood.

Spherical Coordinates. Spherical coordinates r, ϑ, ϕ have been discussed in connection with the Laplace equation [see Eqs. (10.3.25) *et seq.*]. Because of the nature of the scale factors ($h_r = 1$, h_ϑ and h_ϕ depend on r) only the r factor depends on the value of k in the Helmholtz equation and, therefore, the ϑ and ϕ factors are the same for Laplace and Helmholtz equations. Separating the Helmholtz equation yields $\psi = R(r)\Theta(\vartheta)\Phi(\phi)$, with

$$(d^2\Phi/d\phi^2) + m^2\Phi = 0$$

$$\frac{1}{\sin\vartheta}\frac{d}{d\vartheta}\left(\sin\vartheta\,\frac{d\Theta}{d\vartheta}\right) + \left[n(n+1) - \frac{m^2}{\sin^2\vartheta}\right]\Theta = 0$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left[k^2 - \frac{n(n+1)}{r^2}\right]R = 0$$

The separation with regard to the separation constants, m, n, k is nearly complete: Θ and Φ are independent of k, Φ is independent of n, and R is independent of m.

When ϕ is free to vary from zero to 2π without bounds, the factor Φ must be periodic in ϕ and m must be an integer or zero. The eigenfunction factors are $\sin(m\phi)$ and $\cos(m\phi)$ or $e^{\pm im\phi}$. When ϑ can range from zero to π, the constant n must also be an integer and the Θ factor is a spherical harmonic, defined in Eqs. (5.3.36) and (10.3.25). To refresh our memory, we write down the definitions for the angle functions, which are the same we used in Chap. 10:

$$\Theta = P_n^m(\cos\vartheta) = \sin^m\vartheta\, T_{n-m}^m(\cos\vartheta)$$
$$= \frac{(n+m)!}{2^n n!(n-m)!}\sin^m\vartheta\, F(m-n, m+n+1|m+1|\sin^2\tfrac{1}{2}\vartheta)$$

The properties of this function were fully discussed in Sec. 10.3 and were tabulated at the end of Chap. 10. The complete angular functions for spherical coordinates are called *spherical harmonics:*

$$\begin{aligned} Y_{emn}(\vartheta,\phi) &= \cos(m\phi)P_n^m(\cos\vartheta); \quad n = 0, 1, 2, 3, \ldots \\ Y_{0mn}(\vartheta,\phi) &= \sin(m\phi)P_n^m(\cos\vartheta); \quad m = 0, 1, 2, \ldots, n-1, n \\ Y_{e0n}(\vartheta,\phi) &= Y_n(\vartheta) = P_n(\cos\vartheta) \end{aligned}$$

The functions for $m = 0$ are called *zonal harmonics*, those for $m = n$ are called *sectoral harmonics*, and the rest are called *tesseral harmonics*. We saw, on page 1280, how they were related to the derivatives, with respect to x, y, and z, of $1/\sqrt{x^2+y^2+z^2}$. We also have given, on page 1270, the integral representations of these functions.

We shall also take up here, what could have been discussed in Chap. 10 but was withheld to lend variety to the present chapter, some of the symmetry properties of the spherical harmonics. We have several times pointed out that the operator $\mathbf{r} \times \nabla$ is a rotational operator. We meant by this statement that the effect on a function $\psi(\mathbf{r})$ of a clockwise rotation by an angle $|d\omega|$ about an axis, through the origin, pointed in the direction of $d\omega$ is $d\omega \cdot \mathbf{r} \times \nabla\psi(\mathbf{r})$. The displacement of the point $\mathbf{r}$ by such a rotation is $d\omega \times \mathbf{r}$ and the difference between ψ at $\mathbf{r} + d\omega \times \mathbf{r}$ and ψ at $\mathbf{r}$ is

$$(d\omega \times \mathbf{r}) \cdot \nabla\psi = d\omega \cdot (\mathbf{r} \times \nabla\psi)$$

The interesting point about spherical harmonics, which we wish to make here, is that the rotational operator $\mathbf{r} \times \nabla$ has a remarkably simple effect on them.

We can most easily show this by using the complex spherical harmonic

$$X_n^m(\vartheta,\phi) = Y_{emn} + iY_{0mn} = e^{im\phi}P_n^m(\cos\vartheta)$$

The vector $\mathbf{r} \times \nabla\psi = \mathbf{R}(\psi)$ in spherical coordinates is

$$\mathbf{R}(\psi) = \mathbf{a}_\varphi(\partial\psi/\partial\vartheta) - (\mathbf{a}_\vartheta/\sin\vartheta)(\partial\psi/\partial\phi) \tag{11.3.39}$$

The z component of this, corresponding to the effect of rotation about the spherical axis, is

$$R_z(\psi) = \partial\psi/\partial\phi$$

as, of course, it should be. Instead of considering the x and y components of R, we shall take the complex quantity

$$R_x + iR_y = ie^{i\phi}\left(\frac{\partial\psi}{\partial\phi} + i\cot\vartheta\,\frac{\partial\psi}{\partial\phi}\right)$$

which will display the results more easily.

Use of the recursion formulas for the spherical harmonics, given on page 1326, will show that

$$\begin{aligned} R_z(X_n^m) &= (\mathbf{r} \times \nabla X_n^m)_z = imX_n^m \\ R_x(X_n^m) + iR_y(X_n^m) &= -iX_n^{m+1} \\ R_x(X_n^m) - iR_y(X_n^m) &= -i(n-m+1)(n+m)X_n^{m-1} \end{aligned} \tag{11.3.40}$$

Consequently rotation of X_n^m about the z axis simply changes the phase of X_n^m, a rather obvious statement since rotation about the z axis is the same as a change of value of ϕ. A less obvious result, however, is that a rotation about x or y of X_n^m brings in only two neighboring functions, X_n^{m+1} and X_n^{m-1}, rather than all of the others. The translation of these rules to real operators R_x and R_y and real harmonics Y, is easy. Incidentally, the connection between these results and Eqs. (1.6.40) *et seq.*, concerned with angular momentum, is fairly close.

If we wish to consider the operators R_x, R_y, R_z as components of a vector rotation operator, we might inquire what the square of the magnitude of the vector does, when operating on a spherical harmonics. It is

$$\begin{aligned} R^2X_n^m &= [R_z^2 + \tfrac{1}{2}(R_x + iR_y)(R_x - iR_y) + \tfrac{1}{2}(R_x - iR_y)(R_x + iR_y)]X_n^m \\ &= n(n+1)X_n^m \end{aligned}$$

so that the spherical harmonic is also an eigenfunction for the operator R^2. This does not have much significance in classical physics, but in wave mechanics, where the vector operator R is proportional to the angular momentum of the field about the origin, the square of R is just the square of the total angular momentum.

In other words, spherical harmonics are eigenfunctions for rotation about the spherical (z) axis, which display very simple recursion formulas for rotation about x and y. Since a spherical boundary has invariance to rotation about any axis through its center, it is, perhaps, not astonishing that solutions in spherical coordinates, suitable for spherical boundaries, have simple properties under such rotation (such as measured by the operator R^2, for example).

The set of spherical harmonics is a complete mutually orthogonal set of eigenfunctions for the coordinates, ϑ, ϕ. The normalizing factor is

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\vartheta\, d\vartheta [Y_{emn}]^2 = \frac{(4\pi/\epsilon_m)}{2n+1}\frac{(n+m)!}{(n-m)!} = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\vartheta\, d\vartheta [Y_{0mn}]^2$$

so that, for any piecewise continuous function F of ϑ, ϕ in the range $0 < \vartheta < \pi$, $0 < \phi < 2\pi$,

$$\begin{aligned} F(\vartheta,\phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{\epsilon_m}{4\pi}(2n+1)\frac{(n-m)!}{(n+m)!}\int_0^{2\pi} dv \int_0^{\pi} \sin u\, du\, F(u,v)\, \cdot \\ \cdot\, \{Y_{emn}(\vartheta,\phi)Y_{emn}(u,v) + Y_{0mn}(\vartheta,\phi)Y_{0mn}(u,v)\} \end{aligned} \tag{11.3.41}$$

The radial factors are related to the Bessel functions discussed in Secs. 5.3 and 10.3 and tabulated at the end of Chap. 10. Letting $R = J(r)/\sqrt{r}$, we have

$$\frac{d^2J}{dr^2} + \frac{1}{r}\frac{dJ}{dr} + \left[k^2 - \frac{(n+\frac{1}{2})^2}{r^2}\right]J = 0$$

which is the equation satisfied by the functions

$$J_{n+\frac{1}{2}}(kr), \quad N_{n+\frac{1}{2}}(kr), \quad H_{n+\frac{1}{2}}(kr)$$

In order to make for simplicity in the asymptotic expansion, we shall define our radial solutions as follows:

$$\begin{aligned}
j_n(kr) &= \sqrt{\frac{\pi}{2kr}}\,J_{n+\frac{1}{2}}(kr) \to \frac{1}{kr}\sin[kr - \tfrac{1}{2}\pi n]; \qquad kr \to \infty \\
n_n(kr) &= \sqrt{\frac{\pi}{2kr}}\,N_{n+\frac{1}{2}}(kr) \to \frac{-1}{kr}\cos[kr - \tfrac{1}{2}\pi n]; \quad kr \to \infty \qquad (11.3.42) \\
h_n(kr) &= \sqrt{\frac{\pi}{2kr}}\,[J + iN] \to \frac{i^{-n}}{ikr}\,e^{ikr}; \qquad kr \to \infty
\end{aligned}$$

which will be called *spherical Bessel functions* of the first, second, and third kind, respectively [see Eq. (5.3.67)].

Spherical Bessel Functions. The unusual thing about these Bessel functions is that their series expansion about $z = \infty$ is not an asymptotic one; it is exact. The reason for this is that the series remaining, after removing the factor with the essential singularity, is not an infinite series (which diverges) but is a polynomial. It is easiest to show this with h_n; the other two functions j_n and n_n may then be obtained by taking the real and imaginary parts of $h_n(kr)$.

The equation for h_n indicates that it has a pole of order n at $r = 0$ and that it has an essential singularity at $r = \infty$. To take out the essential singularity, we first set $r = u/ik$ in (11.3.41), then set $J = e^u S(u)/u$, and finally in the equation for S, set $u = 1/w$, in order to study the behavior of S at $r \to \infty$. We obtain

$$\frac{d^2S}{dw^2} + \left(\frac{2}{w} - \frac{2}{w^2}\right)\frac{dS}{dw} - \frac{n(n+1)}{w^2}S = 0$$

A series solution of this equation is possible,

$$S = 1 - \tfrac{1}{2}n(n+1)w + \tfrac{1}{8}(n-1)n(n+1)(n+2)w^2 - \cdots$$
$$= \sum_{m=0}^{n} \frac{(n+m)!}{m!(n-m)!}(-\tfrac{1}{2}w)^m$$

which terminates with the nth power of w. Consequently our spherical Hankel function, in order to satisfy the required limiting values at

$r \to \infty$, must be

$$h_n(z) = \frac{e^{iz}}{iz} i^{-n} \sum_{m=0}^{n} \frac{(n+m)!}{m!(n-m)!} \left(\frac{i}{2z}\right)^m$$
$$= -ie^{iz} \left[\frac{(2n)!}{2^n n! z^{n+1}}\right] F(-n|-2n|-2iz); \quad z = kr \qquad (11.3.43)$$

which expression is valid for all values of $z > 0$. The real part of it is $j_n(z)$ and the imaginary part is $n_n(z)$. Other properties of these functions are given in the table at the end of the present chapter. For instance, the roots of $j_n(\pi\beta) = 0$ and $dj_n(\pi\alpha)/d\alpha = 0$ are tabulated, quantities which are useful in calculating solutions inside spheres, with Dirichlet or Neumann conditions at the spherical surface.

Green's Function and Plane Wave Expansion. By the methods of Eq. (7.2.63) or from the properties of the Bessel functions given at the end of Chap. 10, we find that the Green's function for unbounded space, for an outgoing wave of frequency $\omega/2\pi$ is

$$G(r|r_0|\omega) = e^{ikR}/R = ikh_0(kR)$$
$$= ik \sum_{n=0}^{\infty} (2n+1) \sum_{m=0}^{n} \epsilon_m \frac{(n-m)!}{(n+m)!} \cos[m(\phi - \phi_0)] P_n^m(\cos\vartheta_0) \cdot$$
$$\cdot P_n^m(\cos\vartheta) \begin{cases} j_n(kr_0)h_n(kr); & r > r_0 \\ j_n(kr)h_n(kr_0); & r < r_0 \end{cases} \qquad (11.3.44)$$

By letting the source go to $\vartheta_0 = \pi$, $r_0 \to \infty$, we obtain the expansion for the plane wave e^{ikz}, as with Eq. (7.2.52),

$$e^{ikr\cos\vartheta} = \sum_{n=0}^{\infty} (2n+1) i^n P_n(\cos\vartheta) j_n(kr) \qquad \mathbf{(11.3.45)}$$

or, for a plane wave in the general direction u, v,

$$e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_{n=0}^{\infty} (2n+1) i^n \sum_{m=0}^{n} \epsilon_m \frac{(n-m)!}{(n+m)!} \cos[m(\phi - v)] \cdot$$
$$\cdot P_n^m(\cos u) P_n^m(\cos\vartheta) j_n(kr) \qquad (11.3.46)$$

where the vector $\mathbf{r}$ is of length r and has the spherical angles ϑ, ϕ and the vector $\mathbf{k}$ is of length k and has the spherical angles u, v.

This last expansion enables us to calculate an integral expression for the separated solutions in spherical coordinates. We multiply both sides by Y_{emn} or Y_{0mn} of u, v and integrate over u, v obtaining

$$
\begin{aligned}
Y_{emn}(\vartheta,\phi)j_n(kr) &= \frac{1}{4\pi i^n}\int_0^{2\pi} dv \int_0^{\pi} e^{i\mathbf{k}\cdot\mathbf{r}}\, Y_{emn}(u,v)\sin u\, du \\
Y_{0mn}(\vartheta,\phi)j_n(kr) &= \frac{1}{4\pi i^n}\int_0^{2\pi} dv \int_0^{\pi} e^{i\mathbf{k}\cdot\mathbf{r}}\, Y_{0mn}(u,v)\sin u\, du
\end{aligned}
\tag{11.3.47}
$$

which are related to the integral representations of Eq. (5.3.67). Similar integral expressions for Bessel functions of the second or third kinds can also be devised, by modifying the contour of integration for u. For instance using contour B of Fig. 5.10 (going from $i\infty - \epsilon$ to $-i\infty + \epsilon$, $0 < \epsilon < \frac{1}{2}\pi$) and by using integration by parts n times, we have

$$
\begin{aligned}
2\pi \int_B e^{ikr\cos u} P_n(\cos u)\sin u\, du &= \frac{4\pi}{2^n n!}\int_{i\infty}^{1} e^{ikrz}\left[\frac{d^n}{dz^n}(z^2-1)^n\right] dz \\
= \frac{4\pi}{n!}\left(\frac{ikr}{2}\right)^n \int_{i\infty}^{1} e^{ikrz}(1-z^2)^n\, dz &= \frac{4\pi}{in!}\left(\frac{ikr}{2}\right)^n \int_{-\infty}^{i} e^{krt}(1+t^2)^n\, dt \\
&= 4\pi i^n \sqrt{\pi/2kr}\, H_{n+\frac{1}{2}}(kr) = 4\pi i^n h_n(kr)
\end{aligned}
$$

and from this we can obtain

$$
P_n(\cos\vartheta)h_n(kr) = \frac{1}{4\pi i^n}\int_0^{2\pi} dv \int_B e^{i\mathbf{k}\cdot\mathbf{r}} P_n(\cos u)\sin u\, du \tag{11.3.48}
$$

and so on.

These representations will be put to many uses in succeeding pages. There they will be used to obtain the relationship between wave solutions in circular cylinder coordinates, $\eta = \sqrt{x^2 + y^2}$, $\phi = \tan^{-1}(y/x)$, z, and solutions in spherical coordinates $r = \sqrt{x^2 + y^2 + z^2}$, $\vartheta = \cos^{-1}(z/r)$, $\phi = \tan^{-1}(y/x)$. With the use of the relations $\eta = r\sin\vartheta$, $z = r\cos\vartheta$, of

$$
\begin{aligned}
i\mathbf{k}\cdot\mathbf{r} &= ikr[\cos\vartheta\cos u + \sin\vartheta\sin u\cos(\phi - v)] \\
&= ik[z\cos u + \eta\sin u\cos(\phi - v)]
\end{aligned}
$$

and of Eq. (11.2.21), for the corresponding integral representation of $\cos(m\phi)J_m(k\rho)$, Eq. (11.3.47) gives us

$$
P_n^m(\cos\vartheta)j_n(kr) = \tfrac{1}{2}i^{m-n}\int_0^{\pi} e^{ikz\cos u} J_m(k\eta\sin u)P_n^m(\cos u)\sin u\, du \tag{11.3.49}
$$

Similarly the solution $\cos(m\phi)J_m(k\eta)$ should be expressible in terms of a series $\Sigma B_n P_n^m(\cos\vartheta)$. The coefficients B_n should be the coefficients of an expansion in spherical harmonics,

$$
\begin{aligned}
B_n &= \frac{2n+1}{2}\frac{(n-m)!}{(n+m)!}\int_0^{\pi}\cos(m\phi)J_n(kr\sin u)P_n^m(\cos u)\sin u\, du \\
&= \frac{2n+1}{4\pi i^m}\frac{(n-m)!}{(n+m)!}\int_0^{2\pi} dv \int_0^{\pi} e^{ikr\sin u\cos(\phi - v)}\cos(mv)P_n^m(\cos u)\sin u\, du \\
&= i^{n-m}(2n+1)\frac{(n-m)!}{(n+m)!}\cos(m\phi)P_n^m(0)j_n(kr)
\end{aligned}
$$

where $P_n^m(0) = 0$ when $n - m$ turns out to be an odd integer, and is

$(-1)^l[(2m+2l)!/2^{m+2l}l!(m+l)!]$ when $n - m = 2l$ is even. Consequently,

$$J_m(k\eta) = \sum_{l=0}^{\infty} \frac{(2m+4l+1)(2l)!}{2^{m+2l}l!(m+l)!} P_{m+2l}^m(\cos\vartheta)j_{m+2l}(kr) \qquad (11.3.50)$$

and, in particular

$$\begin{aligned} J_0(k\eta) &= \sum_{n=0}^{\infty} (4n+1)\left[\frac{\Gamma(n+\frac{1}{2})}{n!\Gamma(\frac{1}{2})}\right] P_{2n}(\cos\vartheta)j_{2n}(kr) \\ H_0(k\eta) &= \sum_{n=0}^{\infty} (4n+1)\left[\frac{\Gamma(n+\frac{1}{2})}{n!\Gamma(\frac{1}{2})}\right] P_{2n}(\cos\vartheta)h_{2n}(kr) \end{aligned} \qquad (11.3.51)$$

where the second formula has been obtained by methods analogous to those used for Eq. (11.3.48).

One might wonder whether the series given above converge properly. For instance series (11.3.45) does not look promising. However an examination of the series expansion for $j_n(kr)$ (which converges absolutely for all finite values of kr) shows that, when $n \gg \frac{1}{2}kr$, $j_n(kr) \to (kr/2)^n[\Gamma(\frac{3}{2})/\Gamma(n+\frac{3}{2})]$. Consequently the rest of the series, for n larger than $N \gg \frac{1}{2}kr$, is like $\sum \left(\frac{ikr}{2}\right)^n \left[\frac{\Gamma(\frac{3}{2})}{\Gamma(n+\frac{3}{2})}\right] P_n(\cos\vartheta)$, which converges absolutely. Of course when kr is large, many terms in the series must be included before their size begins to diminish, but eventually they do diminish and eventually we can neglect the rest of the series.

Waves inside a Sphere. The resonance frequencies for waves inside a sphere are determined by the roots of the function $j_n(kr)$. If, for instance, the boundary conditions are suitable for sound waves inside a sphere of radius a, then we must use the function of the first kind, $j_n(\pi\alpha_{ns}r/a)$, which is finite at $r = 0$, with the values of α such that $dj_n(\pi\alpha)/d\alpha = 0$. The quantity α_{ns} is the sth root of this equation; values for the first few of these roots are given in the table at the end of this chapter. The eigenfunctions for the spherical interior, for Neumann boundary conditions are therefore

$$\psi_{mns}^e(\mathbf{r}) = \cos(m\phi)P_n^m(\cos\vartheta)j_n(\pi\alpha_{ns}r/a) \qquad (11.3.52)$$

plus another set, having $\sin(m\phi)$ instead of $\cos(m\phi)$, labeled with superscript o (odd). These are mutually orthogonal eigenfunctions, with normalizing constants

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\vartheta\, d\vartheta \int_0^a [\psi_{mns}^e]^2 r^2\, dr = \frac{(2\pi a^3/\epsilon_m)(n+m)!}{(2n+1)(n-m)!}\left[\frac{(\pi\alpha_{ns})^2 - n(n+1)}{(\pi\alpha_{ns})^2}\right] j_n^2(\pi\alpha_{ns}) = \Lambda_{mns}^2$$

and equal ones for the odd functions. All these waves for a given pair of values of n and s for all allowed values of m (0 to n) and for both even [$\cos(m\phi)$] and odd [$\sin(m\phi)$], have the same resonance frequency, $\omega_{ns}/2\pi$, where $\omega_{ns} = \pi c \alpha_{ns}/a$.

The standing waves for m and n smaller than s correspond to those waves reflecting almost normally from the spherical surface and, therefore, focusing strongly at the center of the sphere. The first maximum comes a distance approximately $ma/\pi\alpha_{ns} \simeq ma/\pi s$ away from the center of the sphere, from there on out to the outer boundary the amplitude diminishes, roughly, inversely proportional to the distance from the center of the sphere. On the other hand the waves for which m or n is much larger than s avoid the center of the sphere and are large only for r larger than $[a(n - s - 1)/n]$, approximately. We note that for these cases ($s \ll n$), $(\pi\alpha_{ns}) \simeq n + \frac{1}{2}\pi s$, so factor $[(\pi\alpha_{ns})^2 - n(n+1)]/(\pi\alpha_{ns})^2$ is quite a bit smaller than unity, indicating that, over most of the interior of the sphere, the wave amplitude is quite small for such modes.

The Green's function for this spherical enclosure is

$$G_k(\mathbf{r}|\mathbf{r}_0) = \sum_{m,n,s} \frac{4\pi}{\Lambda^2_{mns}(k^2_{ns} - k^2)} [\psi^e_{mns}(\mathbf{r}_0)\psi^e_{mns}(\mathbf{r}) + \psi^0_{mns}(\mathbf{r}_0)\psi^0_{mns}(\mathbf{r})] \quad (11.3.53)$$

where $k_{ns} = \pi\alpha_{ns}/a$. It is, of course, not equal to the Green's function of Eq. (11.3.44), for the present function satisfies Neumann conditions at $r = a$, whereas the former satisfies outgoing wave requirements at $r \to \infty$.

Two examples of wave motion inside a sphere will be worked out, not that one often encounters sound waves inside spherical enclosures, but to bring out a few more techniques which will be useful in other cases of more practical interest.

Vibrations of a Hollow, Flexible Sphere. A problem of some interest in several connections concerns the free vibrations of a spherical, airtight membrane, kept at a uniform tension T (dynes per centimeter) by excess air pressure inside the sphere. At equilibrium the membrane is a hollow sphere of radius a, and the equilibrium excess pressure inside is $(2T/a)(2\pi aT = \pi a^2 P)$. In the first analysis we shall neglect the reaction of the air outside the sphere; the radiation load will be discussed later in this section.

It is particularly important to separate these vibrations up into various modes with differing spherical harmonic dependence on angle over the surface of the sphere, for each order of the spherical harmonic will have a different natural frequency. In addition, the restoring force, bringing the membrane back to its equilibrium shape, is different for the zero-order case than it is for the higher order cases.

We consider first the zero-order case, where the vibrations are spherically symmetric. Here the air inside the sphere moves in accordance with the velocity potential $\psi = Aj_0(kr)e^{-i\omega t}$, where $\omega = kc$ is to be determined. The excess pressure p just inside the membrane and the radial velocity of the membrane are then

$$p_a = i\omega\rho Aj_0(ka)e^{-i\omega t};\quad v_r = -kAj_1(ka)e^{-i\omega t}$$

and the ratio of these must equal the acoustical impedance of the membrane for symmetrical vibration. If the membrane density is ρ_s and its thickness is h, the mass of the membrane per unit area can be written $h\rho_s$ so that the mass load is $-i\omega h\rho_s$. If the radius of the sphere is increased by an amount η over the equilibrium value a, the tension in the membrane is increased over the equilibrium value by an amount $2\eta hE/a$, where E is the modulus of elasticity for stretching of the membrane. The excess of pressure over equilibrium just inside the membrane must then be $4\eta hE/a^2$, so that the stiffness impedance is $i(4Eh/\omega a^2)$.

Consequently for free, symmetric vibrations we must have

$$\frac{p_a}{v_r} = -i\rho c\frac{j_0(ka)}{j_1(ka)} = -i\omega h\rho_s + i(4Eh/\omega a^2)$$

or

$$[j_0(ka)/j_1(ka)] = (\rho_s h/\rho a)(ka) - (4Eh/\rho c^2ka^2) \tag{11.3.54}$$

The various values $\pi\gamma_{0s}$ of ka which satisfy this equation fix the allowed frequencies of symmetric vibration, $\omega_{0s} = \pi\gamma_{0s}c/a$. If the membrane mass is the controlling factor, the lowest natural frequency will be

$$\pi\gamma_{01} \simeq \sqrt{\frac{3\rho a}{\rho_s h}}\left[1 + \left(\frac{2Eh}{3\rho c^2a}\right)\right];\quad \rho_s h \gg 3\rho a;\quad 3\rho c^2a \gg 2Eh \tag{11.3.55}$$

The higher frequencies for this symmetry are obtained from tables of values for the spherical Bessel functions.

To obtain the resonance frequencies for the other angular symmetries, we must return to Sec. 1.3 to discuss curvature of surfaces. The restoring force of the membrane for these higher modes is only the change in balance between pressure and tension caused by a change in curvature of the membrane. Reference to Eqs. (1.3.6) and a consideration of our discussions of string and membrane curvature at the beginning of Sec. 2.1 indicate that the curvature (or "bulginess") of a surface ξ_1 = constant, from which the restoring force of the membrane can be computed, is $C = -(1/h_1h_2h_3)(\partial h_2h_3/\partial\xi_1)$. The force per unit area exerted by a membrane, under tension T, curved to coincide with the surface ξ_1 = constant is then TC, and the pressure required to hold the membrane in this shape is $-TC$. For the sphere at equilibrium ($h_r = 1$, $h_\vartheta = r$, $h_\phi = r\sin\vartheta$) $C = 2/a$ and the pressure is thus $2T/a$ as we mentioned before.

But as we are interested in the curvature and net force when the membrane is not in equilibrium, we have to go further. This is not difficult to do if the displacement is small. Suppose the actual surface is

$\xi_1 = \eta(\xi_2,\xi_3) +$ constant, where the magnitude of the displacement η is small enough so that the angle between the normal to the ξ_1 surface and the normal to the actual surface is small. Then the unit vector normal to the displaced surface is $\mathbf{e} = \mathbf{a}_1 - (\mathbf{a}_2/h_2)(\partial\eta/\partial\xi_2) - (\mathbf{a}_3/h_3)(\partial\eta/\partial\xi_3)$, to a good approximation. To obtain the curvature of the displaced surface we compute the component of $(\partial\mathbf{e}/\partial\xi_2)$ along ξ_2 and the component of $(\partial\mathbf{e}/\partial\xi_3)$ along ξ_3 and add.

$$C_\eta \simeq \left[\frac{-1}{h_1h_2h_3}\frac{\partial}{\partial\xi_1}(h_2h_3)\right] + \frac{1}{h_2h_3}\left[\frac{\partial}{\partial\xi_2}\left(\frac{h_3}{h_2}\frac{\partial\eta}{\partial\xi_2}\right) + \frac{\partial}{\partial\xi_3}\left(\frac{h_2}{h_3}\frac{\partial\eta}{\partial\xi_3}\right)\right] \quad (11.3.56)$$

where the value of the first term is taken at $\xi_1 + \eta$.

Therefore when the originally spherical membrane coincides with the surface $r = a + \eta(\xi_2,\xi_3)$, being under tension T, the reaction force per unit area of membrane is TC_η and the excess reaction over and above the equilibrium value is

$$\begin{aligned}TC_\eta + \left(\frac{2T}{a}\right) &\simeq -\left(\frac{2T}{a+\eta}\right) + \left(\frac{2T}{a}\right) + \frac{T}{a^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left[\sin\vartheta\,\frac{\partial\eta}{\partial\vartheta}\right] \\ &\quad + \frac{T}{a^2\sin^2\vartheta}\frac{\partial^2\eta}{\partial\phi^2} \simeq \frac{T}{a^2\sin\vartheta}\left[\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\,\frac{\partial\eta}{\partial\vartheta}\right) + \frac{1}{\sin\vartheta}\frac{\partial^2\eta}{\partial\phi^2}\right] + \frac{2T\eta}{a^2}\end{aligned}$$

and, if the displacement is $\eta = \eta_0 Y_{emn}(\vartheta,\phi)e^{-i\omega t}$, then the excess reaction force per unit area is

$$-p = -(n-1)(n+2)(T/a^2)\eta_0 Y_{emn}(\vartheta,\phi)e^{-i\omega t} \quad (11.3.57)$$

This is an interesting result; the angle functions Y turn out to be eigenfunctions for the operator measuring curvature. The acoustic impedance of the membrane for this type of motion is thus $[-i\omega\rho_s h + i(n-1)(n+2)(T/\omega a^2)]$. The same result occurs with the odd functions Y_{omn}. We note that the membrane has no restoring force for the $n = 1$ type displacements (such as $Y = \cos\vartheta$); this is quite in order, for such motions represent a displacement of the sphere as a whole to one side or another, which makes no change in curvature.

If the displacement η of the membrane is to be proportional to one specific spherical harmonic, Y_{emn} or Y_{omn}, the wave motion inside the sphere must have the same dependence on angle;

$$\psi = A \,{}^{\cos}_{\sin}(m\phi)P_n^m(\cos\vartheta)j_m(kr)e^{-i\omega t}$$

The ratio between pressure and radial velocity at $r = a$, for this type of motion, is

$$z(a) = \left[\frac{i\omega\rho j_n(ka)}{kj_n'(ka)}\right] = -i\omega\rho a\cot[\alpha_n(ka)]$$

where $\tan(\alpha_n) = -[zj'_n(z)/j_n(z)]$, as shown in the tables at the end of this chapter.

The allowed frequencies of vibration of the membrane plus enclosed air are then obtained by requiring that the impedance of the membrane match the impedance of the air just inside the membrane,

$$\cot[\alpha_n(ka)] = \left(\frac{\rho_s h}{\rho a}\right) - \frac{(n-1)(n+2)}{(ka)^2}\left(\frac{T}{\rho a c^2}\right);\quad n > 1 \qquad (11.3.58)$$

The allowed values of ka obtained by solving this equation, which can be called $\pi\gamma_{ns}$, may be used to obtain the resonance frequencies for the system, $\omega_{ns} = \pi\gamma_{ns}c/a$ for $n > 1$. The symmetrical case, $n = 0$, was given in Eq. (11.3.54) and the $n = 1$ case has no restoring force, as was noted just above.

Vibrating String in Sphere. For the second example we shall take the case of a flexible string stretched along a diameter of a hollow sphere, which diameter can be made coincident with the spherical axis ($\vartheta = 0$ and $\vartheta = \pi$). This string has a cross-sectional radius η_1, which is much smaller than the radius a of the inner surface of the spherical enclosure, and has a tension T and mass per unit length, ρ_s, such that the velocity of transverse waves along it is c_s (the velocity of acoustic waves in the air in the sphere is c). Suppose, by some means, this string is set into vibration in its fundamental mode, so the displacement of the string is in the x plane; this will couple with the air in the enclosure and the problem is to calculate the resulting acoustic field inside the sphere.

When we deal with the sound waves near the outer walls, it is best to use the spherical coordinates r, ϑ, ϕ but when we are to deal with events near the surface of the string, we can use the cylindrical coordinates $\eta = r \sin \vartheta$, $z = r \cos \vartheta$, and ϕ, which is the same as the ϕ for the spherical coordinates. The combined system is string plus sound. The string vibrates, "causing" the sound wave, which reacts back on the string, "causing" it to vibrate. The coupling is at the surface of the string.

Since we cannot solve the problem exactly, we have to start the cycle somewhere, assume a motion or driving force, solve for the consequent motions and forces in turn until we come back to the assumed one, and see how well it corresponds. The place we start will be with the assumed force on the string, which we assume to be independent of position along the string, of an amount $F_0e^{-i\omega t}$ per unit length of string. In this case the displacement of the part of the string at distance z from the center is

$$\xi = \left(\frac{F_0}{k_s^2 T}\right)\left[\frac{\cos(k_s z)}{\cos(k_s a)} - 1\right]e^{-i\omega t};\quad k_s = \frac{\omega}{c_s};\quad c_s^2 = T/\rho_s$$

If the motion is in the x, z plane, the x component of the velocity of the part of the string at z is then $-i\omega\xi$ and the radial component of the air velocity at the surface ($\eta = \eta_1$) of the string is

$$v_\rho = \left(\frac{F_0}{i\omega\rho_s}\right)\cos\phi\left[\frac{\cos(k_s z)}{\cos(k_s a)} - 1\right]e^{-i\omega t}$$

If there were no sphere present and the string were infinite in extent, the velocity potential for the sound wave radiated from the string would be

$$\left(\frac{F_0}{i\omega\rho_s}\right)\cos\phi\left[\frac{\cos(k_s z)}{\cos(k_s a)}\frac{1}{\sqrt{k^2-k_s^2}}\frac{Z_1(\eta\sqrt{k^2-k_s^2})}{Z_1'(\eta_1\sqrt{k^2-k_s^2})} - \frac{Z_1(k\eta)}{kZ_1'(k\eta_1)}\right]e^{-i\omega t}$$

where $k = \omega/c$, and Z_1 is some linear combination of J_1 and N_1 which satisfies the boundary conditions at infinity, and Z_1' is its derivative. If η_1, the radius of the string, is much smaller than the wavelength $2\pi/k$, then the largest part of Z will be its Neumann function part (except for the very unlikely case where the boundary conditions demand no Neumann function at all). Therefore near the string surface $Z_1(x) \to A/x$ and $Z_1'(x) \to -(A/x^2)$ and thus

$$\psi_a \simeq \left(\frac{F_0}{i\omega\rho_s}\right)\cos\phi\left[1 - \frac{\cos(k_s z)}{\cos(k_s a)}\right]\frac{\eta_1^2}{\eta};\quad \eta_1 < \eta \ll \frac{2\pi}{k} \tag{11.3.59}$$

We next assume that the radius η_1 of the string is enough smaller than the radius a of the spherical enclosure, so that the actual velocity potential near the string is hardly altered by the presence of the sphere, though it is profoundly affected near the spherical wall. Consequently, in the Green's equation

$$\psi(\mathbf{r}) = \frac{1}{4\pi}\oint[\psi(\mathbf{r}_0)\,\mathrm{grad}_0\,G - G\,\mathrm{grad}_0\,\psi(\mathbf{r}_0)]\cdot d\mathbf{S}_0$$

we could use the G of Eq. (11.3.53) and its derivative along the string surface, with very little error. Naturally we must use a Green's function having zero normal gradient at $r_0 = a$ [as given in (11.3.52)]. Then a solution satisfying Neumann boundary conditions at $r = a$ would be given in terms of an integral over the surface of the string,

$$\psi(\mathbf{r}) = \frac{\eta_1}{4\pi}\int_0^{2\pi}d\phi_0\int_{-a}^{a}dz_0\left[G\frac{\partial}{\partial\eta_0}\psi_a - \psi_a\frac{\partial}{\partial\eta_0}G\right]_{\eta_0=\eta_1}$$

where G is given in Eq. (11.3.53) and ψ_a in (11.3.59).

The integral over ϕ_0 cancels all but the $m = 1$, even terms in the sum for G. Collecting all the terms we have

$$\psi(\mathbf{r}) = \left(\frac{F_0\eta_1^2}{2\omega\rho_s a^2}\right)\sum_{n,s}\frac{i^{-n}(\pi\alpha_{ns})^3[(2n+1)/n(n+1)]}{[(\pi\alpha_{ns})^2 - n(n+1)][(\pi\alpha_{ns})^2 - k^2a^2]}\frac{1}{j_n^2(\pi\alpha_{ns})}\cdot$$
$$\cdot\cos\phi\,P_n^1(\cos\vartheta)j_n(\pi\alpha_{ns}r/a)\int_{-a}^{a}dz_0\left[\frac{\cos(k_s z_0)}{\cos(k_s a)} - 1\right]\cdot$$
$$\cdot\int_0^{2\pi}e^{i(\pi\alpha_{ns}/a)z_0\cos u}P_n^1(\cos u)\sin^2 u\,du$$

where we have used Eq. (11.3.49) to change $P_n^1(\cos\vartheta_0)j_n(\pi\alpha_{ns}r_0/a)$ into a function of η_0 and z_0, which may then be differentiated by η_0. In the integral expression (11.3.49) we have also set $J_1(\pi\alpha_{ns}\eta_1\sin u/a)$ equal to $(\pi\alpha_{ns}\eta_1\sin u/2a)$ since η_1 is much smaller than a (this is not a good approximation for n or s large, but these terms are small in any case). We next use the table at the end of Chap. 10 to obtain an expression for P_n^1 in terms of derivatives, and integrate over u by parts (changing to $x = \cos u$) obtaining

$$\frac{1}{2^n n!}\int_{-1}^{1} e^{igzx}(1-x^2)\left[\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^n\right]dx$$
$$= 2i^{n+1}\frac{(\frac{1}{2}gz)^{n+1}}{n!}\int_{-1}^{1} e^{igzx}\left\{-(1-x^2)^{n+1}\right.$$
$$\left.+(1-x^2)^n\left[\frac{2(n+1)x}{igz}+\frac{n(n+1)}{(igz)^2}\right]\right\}dx$$
$$= i^{n+1}\sqrt{\pi}\,\frac{n+1}{\sqrt{\frac{1}{2}gz}}\left\{-J_{n+\frac{3}{2}}(gz)-\frac{n}{gz}J_{n+\frac{1}{2}}(gz)+J_{n-\frac{1}{2}}(gz)\right\}$$
$$= 2i^{n+1}(n+1)\left\{\frac{n+1}{gz}j_n(gz)-2j_{n+1}(gz)\right\};\quad g=\frac{\pi\alpha_{ns}}{a};\quad n>0$$

where we have used one of the integral representations of the Bessel functions to obtain the answer.

This quantity must then be integrated over z_0 to obtain the coefficients in the expansion for ψ. Since we are investigating only forces on the string (and thus motions of the string) which are symmetric with respect to the center of the string and sphere, only sound waves of this symmetry should enter and, sure enough, all even values of n disappear. The integrals for n odd cannot be expressed in finite form and usually must be computed numerically. We can set

$$\int_{-a}^{a} dz_0\left[\frac{\cos(k_sz_0)}{\cos(k_sa)}-1\right]\int_0^{\pi} e^{igz_0\cos u}P_n^1(\cos u)\sin^2 u\,du = 0;\qquad n=0,2,4,\ldots$$
$$= \frac{2i^{n+1}(n+1)a}{\cos(k_sa)}A_n(k_sa;\pi\alpha_{ns});\quad n=1,3,5,\ldots$$

where

$$A_n(\alpha;\beta) = \int_{-1}^{1}[\cos(\alpha w)-\cos(\alpha)]\left[\left(\frac{n+1}{\beta w}\right)j_n(\beta w)-2j_{n+1}(\beta w)\right]dw$$

is a dimensionless function of the parameters α, β, which is finite for finite, real values of α, β.

We can now write

$$\psi(r,\vartheta,\phi)\simeq i\left(\frac{F_0\eta_1^2}{\omega\rho_sa}\right)\sum_{n,s}\frac{(\pi\alpha_{ns})^2[(2n+1)/n]A_n(k_sa;\pi\alpha_{ns})}{[(\pi\alpha_{ns})^2-n(n+1)][(\pi\alpha_{ns})^2-k^2a^2]}\cdot$$
$$\cdot\frac{\cos\phi\,P_n^1(\cos\vartheta)j_n(\pi\alpha_{ns}r/a)}{\cos(k_sa)j_n^2(\pi\alpha_{ns})}\qquad(11.3.60)$$

where we sum over odd values of n. The force on the string, caused by the sound waves, per unit length of string, in the positive x direction is the integral of the x component of the pressure at $\eta = \eta_1$. For instance, this force at $z = 0$, the center of the string, is

$$F_s = -i\omega\rho\eta_1 \int_0^{2\pi} \psi(\eta_1,\tfrac{1}{2}\pi,\phi) \cos\phi \, d\phi$$

where the parameter k, entering into the terms ka and $k_sa = kca/c_s$ is as yet unfixed.

This force F_s is, in general, not equal to the originally assumed force F_0. In the first place, it is not independent of z, as F_0 was assumed to be. In the second place, it is usually considerably smaller than F_0, because of the dimensionless factor $\rho\eta_1^3/\rho_s a$, which is quite small since η_1 is considerably smaller than a and the air density ρ is much smaller than ρ_s/η_1^2. However, there are two terms in the denominator of each term, $[(\pi\alpha_{ns})^2 - (ka)^2]$ and $\cos(k_sa) = \cos(kac/c_s)$; if either of these becomes quite small, then F_s can be made to equal F_0 at any point along the string.

Of course, if F_s equaled F_0 at all points, we would have the exact solution for one of the free vibrations of the string-air system. To obtain this exact solution we would have had to use F_s instead of F_0 in our equation and would have had the expression

$$\cos(k_sz_0)\left[\tan(k_sa)\int_0^{k_sa} F_s\left(\frac{\xi}{a}\right)\cos(\xi)\,d\xi - \int_{k_sz_0}^{k_sa} F_s\left(\frac{\xi}{a}\right)\sin(\xi)\,d\xi\right] - \sin(k_sz_0)\int_0^{k_sz_0} F_s\left(\frac{\xi}{a}\right)\cos(\xi)\,d\xi$$

instead of $F_0\left[\dfrac{\cos(k_sa_0)}{\cos(k_sa)} - 1\right]$ in the expression for F_0A_n. This would have resulted in an integral equation for F_s which would have been too complicated to solve. What we have done, instead, is to assume that F_s, for some free vibrations, is nearly constant and equal to F_0, independent of z_0. For some of the lower resonances this is nearly true, and we can expect to obtain an approximate solution by setting F_s at $z = 0$ equal to F_0.

Resonance Frequencies of the System. Resonance of the combined system occurs only when the string is near one of its resonances, *i.e.*, when $\cos(kac/c_s) \to 0$, or when the air in the spherical enclosure is near resonance, *i.e.*, when $ka \to (\pi\alpha_{ns})$. In the former case the string has most of the energy and "drives" the air in the sphere; in the latter case the standing sound wave in the sphere has the energy and "drives" the string. In either case, because $\rho\eta_1^3/\rho_s a$ is small, the coupling between air and string is small, and the resonance frequencies of the system are close either to the resonance frequencies of the string or else to those for the air in the sphere. For the first few resonances of each, which are symmetrical about $z = 0$, F_s should be nearly independent of z, and F_s, as obtained from Eq. (11.3.60), should be approximately equal to F_0.

For instance, near the lowest sound resonance of the appropriate symmetry, we have $k = (\pi\alpha_{11}/a) - \epsilon$, where ϵ is small. To the first approximation in ϵ we need only consider the $n = 1$, $s = 1$ term in series (11.3.60), and we finally obtain

$$\epsilon_{11} \simeq \frac{(\pi\rho\eta_1^4/2\rho_s a^3)(\pi\alpha_{11})^2 A_1(\pi\alpha_{11}c/c_s;\pi\alpha_{11})}{[(\pi\alpha_{11})^2 - 2]j_i^2(\pi\alpha_{11})\cos(\pi\alpha_{11}c/c_s)} \tag{11.3.61}$$

where we have set $P_1^1(0) = 1$ and $j_1(\pi\alpha_{11}\eta_1/a) \simeq \frac{1}{3}(\pi\alpha_{11}\eta_1/a)$. This expression will serve to obtain the resonance frequency by use of the equation $\nu_{11} \simeq (c\alpha_{11}/2a) - (c\epsilon_{11}/2\pi)$ and, by substitution of $k_{11}a = \pi\alpha_{11} - \epsilon a$ for ka and also $k_{11}ac/c_s$ for $(k_s a)$ in Eq. (11.3.60), serves to compute ψ for this lowest resonance of the air. To the first approximation, only the $n = 1$, $s = 1$ term needs to be considered and

$$\psi_{11} \simeq F_0(3a/\pi\alpha_{11}\eta_1)\cos\phi\, P_1^1(\cos\vartheta)j_1(\pi\alpha_{11}r/a)$$

where F_0 is now just an adjustable constant, which is fitted to the prescribed amplitude of motion of the string or of the air. We note that if ν_{11} is below the lowest string resonance, then ν_{11} is less than the cavity resonance without the string. Below its lowest frequency the string acts as an additional mass load on the air oscillations.

For the lowest resonance having the string carrying most of the energy, we set $k = (\pi c_s/2ac) - \delta$ and obtain

$$\delta \simeq \left(\frac{\pi\rho\eta_1^4}{3\rho_s a^3}\right)\sum_{n,s}\frac{(\pi\alpha_{ns})^3[(2n+1)/n]A_n(\frac{1}{2}\pi;\pi\alpha_{ns})}{[(\pi\alpha_{ns})^2 - n(n+1)][(\pi\alpha_{11})^2 - (\pi c_s/2c)^2]j_n^2(\pi\alpha_{ns})} \tag{11.3.62}$$

from which we can compute the string and air motion.

The case where the cavity resonance is close to a string resonance is the case of *degeneracy* and must be solved separately. When the lowest cavity resonance frequency $c\alpha_{11}/2a$ differs from the lowest string resonance $c_s/4a$ by the small amount $\Delta\nu$, that is, when $\pi c_s/2c = (\pi\alpha_{11}) + (2\pi a\Delta\nu/c)$ and $2\pi\Delta\nu/c \ll 1$, for instance, we set $ka = (\pi\alpha_{11}) - \gamma$ and find that the two allowed values for ka, from which the two resonance frequencies may be computed, are

$$(ka)_{\text{res}} \simeq (\pi\alpha_{11}) + \left(\frac{\pi a\Delta\nu}{c}\right) \pm \sqrt{(\pi a\Delta\nu/c)^2 + K} \tag{11.3.63}$$

$$K \simeq \left(\frac{\pi\rho\eta_1^4 c_s}{2\rho_s a^2 c}\right)\frac{(\pi\alpha_{11})^2 A_1(\pi\alpha_{11}c/c_s;\pi\alpha_{11})}{[(\pi\alpha_{11})^2 - 2]j_1^2(\pi\alpha_{11})}$$

As shown in Fig. 11.7, when $|\Delta\nu|$ is larger than $(c/\pi a)\sqrt{K}$, the two allowed frequencies are some distance apart, the one approaching the horizontal line corresponding to the solution of Eq. (11.3.61) and the other approaching that of Eq. (11.3.62). As the resonance frequency of the string alone approaches that of the resonator alone ($\Delta\nu \to 0$), the two resonance frequencies do not merge but are still a distance apart proportional to $\sqrt{K}$ at $\Delta\nu = 0$. Then, as $\Delta\nu$ changes sign, the mode which had concentrated the energy in the string (diagonal line) changes into the mode which concentrates the energy in the sound waves (horizontal line) and vice versa. The parallelism between this and the results for simple coupled oscillators is obvious.

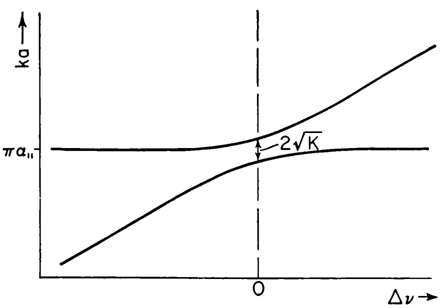

Fig. 11.7 Behavior of resonance frequencies passing through condition of degeneracy ($\Delta v = 0$) according to Eq. (11.3.63). Constant K is coupling constant.

Radiation from Sphere. For external problems we use the phase angles analogous to those used for radiation and scattering from a cylinder (see page 1373 and tables at end of this chapter),

$$\begin{aligned} j_n(z) &= D_n(z)\sin[\delta_n(z)]; & (d/dz)j_n(z) &= -D'_n(z)\sin[\delta'_n(z)] \\ n_n(z) &= -D_n\cos\delta_n; & (d/dz)n_n(z) &= D'_n\cos\delta'_n \\ h_n(z) &= -iD_ne^{i\delta_n}; & (d/dz)h_n(z) &= iD'_ne^{i\delta_n'} \end{aligned} \tag{11.3.64}$$

Therefore the velocity potential for sound waves outside a sphere of radius a, which has a radial velocity at its surface of $v_0(\vartheta,\phi)e^{-i\omega t}$, is

$$\psi(r,\vartheta,\phi) = \frac{e^{-i\omega t}}{4\pi ik}\sum_{m,n}\epsilon_m(2n+1)\frac{(n-m)!}{(n+m)!}\frac{e^{-i\delta_n'(ka)}}{D'_n(ka)}h_n(kr)P_n^m(\cos\vartheta)\cdot$$
$$\cdot\int_0^{2\pi}d\phi_0\int_0^{\pi}v_0(\vartheta_0,\phi_0)\cos[m(\phi-\phi_0)]\,P_n^m(\cos\vartheta_0)\sin\vartheta_0\,d\vartheta_0 \tag{11.3.65}$$

The velocity potential, radial velocity, and pressure at large distances from the sphere are

$$\psi \to \left(\frac{ca}{ikr}\right) e^{ik(r-ct)} f(\vartheta,\phi); \quad v_r \to \left(\frac{ca}{r}\right) e^{ik(r-ct)} f(\vartheta,\phi)$$

$$p \to \left(\frac{\rho c^2 a}{r}\right) e^{ik(r-ct)} f(\vartheta,\phi)$$

$$f(\vartheta,\phi) = \frac{-i}{cka} \sum_{n,m} \frac{i^{-n}\epsilon_m}{4\pi} (2n+1) \frac{(n-m)!}{(n+m)!} \frac{e^{-i\delta_n'}}{D_n'} P_n^m(\cos\vartheta) \cdot \tag{11.3.66}$$

$$\cdot \int_0^{2\pi} d\phi_0 \int_0^{\pi} v_0(\vartheta_0,\phi_0) \cos[m(\phi - \phi_0)] P_n^m(\cos\vartheta_0) \sin\vartheta_0 \, d\vartheta_0$$

The average intensity S of the radiated sound at the point (r,ϑ,ϕ) $(r \gg a)$ is just $\frac{1}{2}|pv_r|$ so that

$$S = \left(\frac{\rho c^3 a^2}{2r^2}\right) |f(\vartheta,\phi)|^2$$

and the total power radiated by the vibrating sphere

$$P = \frac{\rho c^3}{8\pi k^2} \sum_{n=0}^{\infty} (2n+1) \int_0^{2\pi} d\phi_0 \int_0^{2\pi} d\phi_1 \int_0^{\pi} \sin\vartheta_0 \, d\vartheta_0 \int_0^{\pi} \sin\vartheta_1 \, d\vartheta_1 \cdot$$

$$\cdot \left[\frac{v_0(\vartheta_0,\phi_0)}{cD_n'(ka)}\right] \left[\frac{v_0(\vartheta_1,\phi_1)}{cD_n'(ka)}\right] P_n[\cos\vartheta_0 \cos\vartheta_1 + \sin\vartheta_0 \sin\vartheta_1 \cos(\phi_0 - \phi_1)] \tag{11.3.67}$$

For very long wavelengths ($ka \ll 1$), the formulas simplify, as in the cylindrical case. In this case the approximate formulas show that $[1/D_0'(ka)] \simeq (ka)^2$ and $1/D_n'$ for $n > 0$ diminishes with still higher powers of (ka); $\delta_0'(ka) \simeq \frac{1}{3}(ka)^3$ and δ_n' for $n > 0$ is also a higher order term. Therefore, unless the integral of $v_0(\vartheta,\phi)$ over ϕ and ϑ is zero, we have

$$f(\vartheta,\phi) \to \frac{-ika}{4\pi c} \int_0^{2\pi} d\phi_0 \int_0^{\pi} v_0(\vartheta_0,\phi_0) \sin\vartheta_0 \, d\vartheta_0; \quad ka \to 0$$

The radiated wave is spherically symmetric, as though the sphere were moving outward and inward uniformly over the whole surface. The total amplitude of inward-outward flow

$$Q = a^2 \int_0^{2\pi} d\phi_0 \int_0^{\pi} v_0(\vartheta_0,\phi_0) \sin\vartheta_0 \, d\vartheta_0; \quad f(\vartheta,\phi) \to -i(kQ/4\pi ac)$$

is called the *equivalent strength* of the vibrating sphere for radiation. The intensity of radiation a distance r from the center of the sphere is then, in the long-wave limit,

$$S \to (\rho c k^2 Q^2/32\pi^2 r^2) = (\rho\omega^2 Q^2/32\pi^2 c r^2); \quad ka \to 0$$

and the limiting expression for the total power radiated from the sphere is $4\pi S r^2$

$$P \to 4\pi S r^2 \to (\rho\omega^2 Q^2/8\pi c); \quad ka \to 0$$

For the very short-wavelength limit the wave tends to propagate radially outward from each point on the sphere, so that the angle dependence of the wave at any distance is the same as the angle dependence of the motion of the sphere's surface. As the tables at the end of this chapter show, when $ka \gg 1$, the phase angles $\delta'_n(ka)$ for $n < N = \sqrt{ka}$, are approximately equal to $ka - \frac{1}{2}\pi(n + 1)$ and the amplitudes $D'_n(ka)$ are equal to $1/ka$, whereas for $n > N$ the amplitudes D'_n increase without limit as $n \to \infty$. Therefore, the angle-distribution function of Eq. (11.3.66) becomes

$$f(\vartheta,\phi) \to \frac{1}{c} \sum_{m,n} \frac{\epsilon_m}{4\pi} (2n + 1) \frac{(n - m)!}{(n + m)!} \int_0^{2\pi} d\phi_0 \int_0^{\pi} v_0(\vartheta_0,\phi_0) \cdot$$

$$\cdot P_n^m(\cos \vartheta_0)P_n^m(\cos \vartheta) \cos[m(\phi - \phi_0)] \sin \vartheta_0 \, d\vartheta_0 = [v_0(\vartheta,\phi)/c]; \quad ka \to \infty$$

so that for very short wavelengths $v_r \to [av_0(\vartheta,\phi)/r]e^{ik(r-ct)}$ and $p = \rho c v_r$, so that $S \to (\rho c a^2/r^2)|v_0(\vartheta,\phi)|^2$ for $ka \to \infty$. This result also corresponds to the findings for polar coordinates discussed on page 1376.

Dipole Source. Next to the simple source, where the sphere expands and contracts uniformly over its surface, the case where the sphere vibrates as a whole backward and forward along the z axis is the simplest. If the displacement along the z axis is $Ae^{-i\omega t}$, the radial velocity of the surface of the sphere at point (a,ϑ,ϕ) is $-i\omega A \cos \vartheta e^{-i\omega t}$, and consequently the angle distribution factor f is

$$f(\vartheta,\phi) = i(A/a)[e^{-i\delta_1'(ka)}/D_1'(ka)] \cos \vartheta$$

and the power radiated is

$$P = \frac{2\pi\rho c^3|A|^2}{3[D_1'(ka)]^2} \to \begin{cases} \frac{1}{6}\pi\rho c^3 k^6 a^6 |A|^2; & ka \to 0 \\ \frac{2}{3}\pi\rho c^3 k^2 a^2 |A|^2; & ka \to \infty \end{cases}$$

The excess pressure on the surface of the sphere, from the wave produced by the motion, is

$$p = \rho c \omega A \frac{D_1(ka)}{D_1'(ka)} e^{i(\delta_1 - \delta_1') - i\omega t} \cos \vartheta$$

where $(-i\omega A)$ is, of course, the velocity amplitude of the sphere's motion. The net reaction force, opposing the motion from the radiation field, is then the integral of the z component of p over the sphere. Leaving out the time factor, $e^{-i\omega t}$, this is

$$F = \tfrac{4}{3}\pi a^2 \rho c \omega A \left(\frac{D_1}{D_1'}\right) e^{i(\delta_1 - \delta_1')} = -\tfrac{4}{3}\pi a^2 \rho c \omega A \frac{h_1(ka)}{h_1'(ka)}$$

$$= -\tfrac{4}{3} i(\pi a^2 \rho c) \frac{\omega^2(\omega + i\omega_0)}{[\omega + \omega_0(i + 1)][\omega + \omega_0(i - 1)]} A; \quad \omega_0 = c/a$$

when we insert the expressions for h_1 and h_1^i, in terms of $ka = \omega/\omega_0$, found in the tables at the end of this chapter. The radiation impedance of the sphere to sidewise vibrational motion is, therefore,

$$Z_r = \frac{F}{-i\omega A} = M_a \frac{\omega\omega_0(\omega + i\omega_0)}{\omega^2 + 2i\omega_0\omega - 2\omega_0^2}$$

where $M_a = \frac{4}{3}\pi a^3\rho$ is the mass of a volume of air equal to the volume of the sphere. At low frequencies ($\omega \ll \omega_0 = c/a$), this is a mass reactance, $-\frac{1}{2}i\omega M_a$, equivalent to a mass load of $\frac{1}{2}M_a$; at high frequencies ($\omega \gg \omega_0$), it is $\frac{4}{3}\pi a^2\rho c$, a pure resistance.

Next we suppose the sphere to have mass M_0. The combined mechanical impedance of mass and radiation load is then $Z_r - i\omega M_0$ and, if a force of $F_\omega e^{-i\omega t}$ is applied to the sphere in the z direction, the velocity of the sphere and the velocity potential of the sound wave at large distances from the sphere are:

$$U_\omega = \frac{(F_\omega M_0/-i\omega)[\omega + \omega_0(i+1)][\omega + \omega_0(i-1)]e^{-i\omega t}}{[M_0\omega + i\omega_0(M_0 + \frac{1}{2}M_a) + \omega_0\sqrt{M_0^2 - \frac{1}{4}M_a^2}] \cdot [M_0\omega + i\omega_0(M_0 + \frac{1}{2}M_a) - \omega_0\sqrt{M_0^2 - \frac{1}{4}M_a^2}]}$$

$$\psi_\omega \to \frac{(F_\omega M_0 ac/r)\exp\left\{i\omega\left[\left(\frac{r-a}{c}\right) - t\right]\right\}\cos\vartheta}{[M_0\omega + i\omega_0(M_0 + \frac{1}{2}M_a) + \omega_0\sqrt{M_0^2 - \frac{1}{4}M_a^2}] \cdot [M_0\omega + i\omega_0(M_0 + \frac{1}{2}M_a) - \omega_0\sqrt{M_0^2 - \frac{1}{4}M_a^2}]}$$

This can now be processed by Fourier or Laplace transform techniques to obtain the transient behavior of the sphere plus radiation when acted on by a force on the sphere in the z direction. If, for instance, we strike the sphere a blow of unit impulse at $t = 0$, the velocity potential at time t at the point (r,ϑ,ϕ) is

$$\begin{aligned} g(\mathbf{r}|t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\psi_\omega}{F_\omega}\right)d\omega = 0; \quad ct > r - a \\ &= \frac{a^2\cos\vartheta}{r\sqrt{M_0^2 - \frac{1}{4}M_a^2}}\exp\left[\left(1 + \frac{M_a}{2M_0}\right)\left(\frac{r-a}{a} - \frac{ct}{a}\right)\right] \cdot \\ &\qquad \cdot \sin\left[\sqrt{1 - \left(\frac{M_a}{2M_0}\right)^2}\left(\frac{ct}{a} - \frac{r-a}{a}\right)\right]; \quad ct > r - a \end{aligned}$$

The motion is considerably damped, only the first half wavelength being of appreciable size. The wavelength of the radiation is, incidentally, equal to $2\pi a$ when the mass of the sphere is considerably larger than the mass of an equal volume of air.

The radiation from any transient motion of the sphere, caused by any arbitrary force $f(t)$ applied to the sphere, is then given by Eq. (11.1.15).

Radiation from a Collection of Sources. If the radiation is caused by a collection of sources inside a sphere of radius a, instead of by the vibration of the surface of the sphere, we use the Green's function of Eq. (11.3.44). Suppose the source strength per unit volume is $q(r,\vartheta,\phi)e^{-i\omega t}$, which is zero for $r > a$. Then the velocity potential for $r \gg a$ is $\psi \to (e^{ikr-i\omega t}/r)f(\vartheta,\phi)$, where

$$\begin{aligned}f(\vartheta,\phi) &= \iiint dV_0\, q(x_0,y_0,z_0)e^{-i(k/r)(xx_0+yy_0+zz_0)}\\ &= \sum_{m,n} \epsilon_m i^{-n}(2n+1)\frac{(n-m)!}{(n+m)!}P_n^m(\cos\vartheta)\cdot\\ &\quad\cdot\int_0^{2\pi}\cos[m(\phi-\phi_0)]\,d\phi_0\int_0^{\pi}P_n^m(\cos\vartheta_0)\sin\vartheta_0\,d\vartheta_0\cdot\\ &\qquad\cdot\int_0^a q(r_0,\vartheta_0,\phi_0)j_n(kr_0)r_0^2\,dr_0 \qquad (11.3.68)\end{aligned}$$

These formulas should be compared with the analogous ones for the static case, given in (10.3.42) and (10.3.44). The expansion in spherical harmonics does not appear much different from (10.3.42), in fact the integrals are not much different for $n > ka$. Of course the different spherical harmonic terms are here all multiplied by the same function of r, namely e^{ikr}/r, instead of by the different powers $1/r^{n+1}$. The radiation field, for all higher orders of the spherical harmonics, diminishes much more slowly with distance than does the corresponding static field. The integral multiplying the spherical harmonic Y_{emn} or Y_{0mn} is the (m,n)th component of the source distribution.

The integral in x, y, z is more apparently different from its static counterpart. In the first place the dependence on x_0, y_0, z_0 is through an exponential instead of an algebraic function.

If (ka) is larger than unity, an expansion of the exponential in powers of x, y, z does not converge well and a direct integration of the exponential as it stands returns us to the second formula, using spherical harmonics.

If, however, $ka < 1$, the series expansion of the exponential is easily convergent. The zeroth order term is just the total *source strength* $\iiint q\,dV$. The first-order terms of $f(\vartheta,\phi)$ are $-ik\dfrac{x}{r}\displaystyle\iiint x_0q(x_0,y_0,z_0)\,dV_0$ and others for y and z. The integral is called the x component of the *dipole moment* of the distribution of sources (see page 1278). The second-order terms have integrals called *quadrupole moments* and so on. The relation between these multipole moments and the integrals over spherical harmonics is the same as for the static case, as discussed on page 1281.

Radiation from Piston in Sphere. If the portion of the sphere of radius a from $\vartheta = 0$ to $\vartheta = \vartheta_0$ is a piston, moving with radial velocity $Ue^{-i\omega t}$, and the rest of the sphere is rigid, the expression for the velocity potential outside the sphere is

$$\psi = \frac{Ue^{-i\omega t}}{2ik} \sum_{n=0}^{\infty} \frac{e^{-i\delta_n'(ka)}}{D_n'(ka)} [P_{n-1}(\cos\vartheta_0) - P_{n+1}(\cos\vartheta_0)]P_n(\cos\vartheta)h_n(kr) \tag{11.3.69}$$

after performing the integrations of Eq. (11.3.65). The coefficient in the brackets, for $n = 0$, is $[1 - \cos\vartheta_0]$.

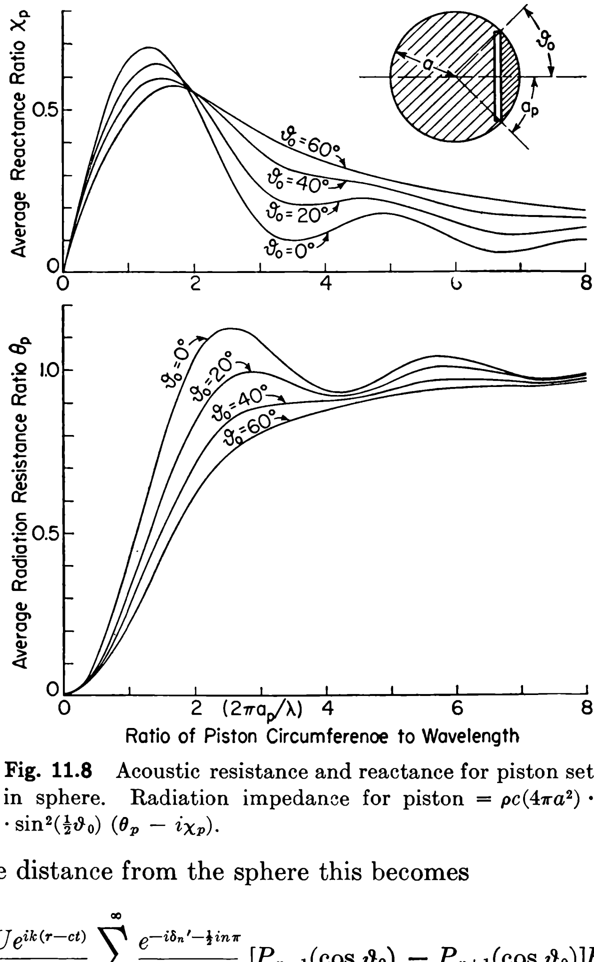

Fig. 11.8 Acoustic resistance and reactance for piston set in sphere. Radiation impedance for piston $= \rho c(4\pi a^2) \cdot \sin^2(\frac{1}{2}\vartheta_0)\ (\theta_p - i\chi_p)$.

At large distance from the sphere this becomes

$$\psi \to \frac{-Ue^{ik(r-ct)}}{2k^2r} \sum_{n=0}^{\infty} \frac{e^{-i\delta_n' - \frac{1}{2}in\pi}}{D_n'(ka)} [P_{n-1}(\cos\vartheta_0) - P_{n+1}(\cos\vartheta_0)]P_n(\cos\vartheta)$$

At the surface of the sphere the pressure is $(ik\rho c\psi)_{r=a}$, and the integral of this over the surface of the piston $(0 < \vartheta < \vartheta_0)$ is

$$\begin{aligned} F &= \left(\frac{i\rho c}{2}\right) Ue^{-i\omega t} \sum_{n=0}^{\infty} \left[\frac{h_n(ka)}{h_n'(ka)}\right] \frac{2\pi a^2}{2n+1} [P_{n-1}(\cos\vartheta_0) - P_{n+1}(\cos\vartheta_0)]^2 \\ &= 4\pi a^2 \rho c U e^{-i\omega t} \sin^2(\tfrac{1}{2}\vartheta_0)[\theta_p - i\chi_p] \end{aligned}$$

where

$$\theta_p = \frac{1}{4\sin^2(\frac{1}{2}\vartheta_0)} \sum_{n=0}^{\infty} \frac{[P_{n-1}(\cos\vartheta_0) - P_{n+1}(\cos\vartheta_0)]^2}{(2n+1)(ka)^2[D'_n(ka)]^2} \tag{11.3.70}$$

$$\chi_p = \frac{1}{4\sin^2(\frac{1}{2}\vartheta_0)} \sum_{n=0}^{\infty} \frac{[P_{n-1}(\cos\vartheta_0) - P_{n+1}(\cos\vartheta_0)]^2}{(2n+1)} \left[\frac{D_n(ka)}{D'_n(ka)}\right] \cdot \cdot \cos[\delta_n(ka) - \delta'_n(ka)]$$

Curves of values of θ_p and χ_p are shown in Fig. 11.8, plotted as function of $2\pi a_p/\lambda = 2ka\sin(\frac{1}{2}\vartheta_0)$, the ratio of the equivalent circumference of the piston to the wavelength. We notice that at long wavelengths (compared to $2\pi a_p$) the resistive part is smaller than the reactive part, which is a mass reactance, proportional to k. For short wavelengths $\theta_p \to 1$ and $\chi_p \to 0$, so that the impedance is real and equal to ρc times the area of the piston, $\pi a_p^2 = 4\pi a^2 \sin^2(\frac{1}{2}\vartheta_0)$.

Scattering of Plane Wave from Sphere. The general technique and the principles involved in the calculation of the scattering of plane waves have been discussed, in connection with the circular cylinder, on pages 1376 to 1382. For scattering from spheres, we start with the expansion of a plane wave, given in Eq. (11.3.45). If the boundary condition is that ψ is to be zero at $r = a$, the solution must be

$$\begin{aligned}\psi &= \sum_{n=0}^{\infty} (2n+1)i^n P_n(\cos\vartheta) \left[j_n(kr) - \frac{j_n(ka)}{h_n(ka)} h_n(kr)\right] e^{-i\omega t} \\ &= \sum_{n=0}^{\infty} (2n+1)i^n P_n(\cos\vartheta) e^{-i\delta_n(ka)}[\cos(\delta_n) j_n(kr) + \sin(\delta_n) n_n(kr)] e^{-i\omega t} \\ &= e^{ik(z-ct)} + \psi_s e^{-i\omega t}\end{aligned}$$

$$\psi_s = -\sum_n (2n+1) i^{n+1} e^{-i\delta_n(ka)} \sin[\delta_n(ka)] P_n(\cos\vartheta) h_n(kr) \tag{11.3.71}$$

The scattered amplitude at large distances from the sphere, the ratio S between scattered intensity and incident intensity, and the ratio Q between total power scattered and incident intensity (the effective cross section for scattering from the sphere) are

$$\psi_s \to -\frac{e^{ikr}}{kr} \sum_n (2n+1) e^{-i\delta_n} \sin(\delta_n)\, P_n(\cos\vartheta)$$

$$S = \frac{1}{k^2r^2} \sum_{m,n} (2m+1)(2n+1)\cos(\delta_m - \delta_n) \cdot \cdot \sin\delta_m \sin\delta_n\, P_m(\cos\vartheta)\, P_n(\cos\vartheta)$$

$$Q = 4\pi a^2 \sum_{n=0}^{\infty} \frac{2n+1}{(ka)^2} \sin^2[\delta_n(ka)] \tag{11.3.72}$$

The effective cross section Q may be obtained in another way, which will be useful later, when we consider absorption by the sphere. We compute the work done on the surface of the sphere by the incident pressure and this, divided by the incident intensity $(\frac{1}{2}\rho ck^2)$, is equal to Q. The incident pressure $i\omega\rho\psi$ at the surface of the sphere is

$$p_0 = i\rho ck\Sigma i^n(2n + 1)P_n(\cos\vartheta)j_n(ka)$$

and the actual velocity (we use the *inward* velocity) at $r = a$ is

$$\begin{aligned}-v_r &= -k\sum i^n(2n + 1)P_n(\cos\vartheta)\frac{e^{-i\delta_n}}{D_n}[-n_n(ka)j'_n(ka) + n'_n(ka)j_n(ka)] \\ &= -\frac{k}{(ka)^2}\sum i^n(2n + 1)P_n(\cos\vartheta)\left(\frac{e^{-i\delta_n}}{D_n}\right)\end{aligned}$$

using the expression for the Wronskian, $\Delta(j_n,n_n) = 1/(ka)^2$. The power taken from the incident wave at the surface of the sphere is the integral of $\frac{1}{2}\operatorname{Re}(-\bar{p}_0v_r)$ over the sphere's surface;

$$\begin{aligned}\int\tfrac{1}{2}\operatorname{Re}[-\bar{p}_0v_r]\,dA &= (\tfrac{1}{2}\rho ck^2)\frac{4\pi a^2}{(ka)^2}\operatorname{Re}\left\{\sum(2n + 1)ie^{-i\delta_n}\left[\frac{j_n(ka)}{D_n(ka)}\right]\right\} \\ &= (\tfrac{1}{2}\rho ck^2)\frac{4\pi a^2}{(ka)^2}\sum(2n + 1)\sin^2\delta_n\end{aligned}$$

When this is divided by the incident intensity $(\frac{1}{2}\rho ck^2)$, the quotient is just Q. The power delivered to the sphere by the incident wave, in this case, all goes into the scattered wave. For the case of Neumann boundary conditions, where δ'_n is substituted in the formulas of Eq. (11.3.72) instead of δ_n, the value of v_r is zero at $r = 0$; what is needed in this case is the product of the actual pressure with the radial velocity of the incident wave, which can be written $\frac{1}{2}\operatorname{Re}[-p\bar{v}_{0r}]$.

For long wavelengths $(ka \ll 1)$, $\delta_0 \to ka$, $\delta_1 \to \frac{1}{3}(ka)^3$, etc., so to the first approximation, for Dirichlet conditions,

$$S \to (a/r)^2;\quad Q \to 4\pi a^2;\quad ka \to 0$$

The scattering is spherically symmetrical in this limit, and the limiting cross section Q is larger than the geometrical cross section πa^2. If Neumann conditions had been imposed, the expression for Q would have δ'_n instead of δ_n and the limiting value of Q for $ka \to 0$ would have been $\frac{4}{9}\pi a^2(ka)^4$, which is much smaller than πa^2 for $ka \ll 1$. As mentioned with the cylinder, Dirichlet conditions seem to affect the wave more at long wavelengths than do Neumann conditions.

For very short wavelengths, the scattered amplitude breaks up into two parts: the shadow-forming wave, directed in the direction of the plane wave, and the reflected wave, spread over all angles. For ka large, $\delta_n(ka)$ is large for $n < ka$ and is small for $n > ka$. Consequently an approximate expression for the cross section is

$$Q \to \frac{2\pi}{k^2} \sum_{n=0}^{ka} (2n + 1) = 2\pi a^2; \quad ka \gg 1$$

where we have substituted for $\sin^2 \delta_n$ its average value $\frac{1}{2}$ for the values of n less than ka, and have neglected the rapidly converging series for $n > ka$. This cross section is twice the geometrical cross section πa^2,

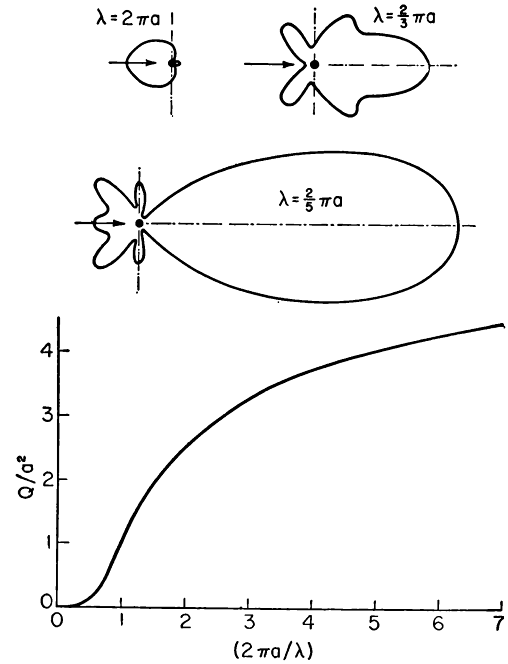

Fig. 11.9 Scattered intensity and scattering cross section Q for plane wave of wavelength λ scattered from sphere of radius a $[v_r(S) = 0]$.

since it includes the shadow-forming part as well as the reflected wave. Either, by itself, would equal πa^2.

Later in this chapter we shall show that, when ka is very large, the scattered intensity becomes

$$S \to \frac{a^2}{4r^2} + \left(\frac{a^2}{4r^2}\right) \cot^2(\tfrac{1}{2}\vartheta) J_1^2(ka \sin \vartheta)$$

The first term of this is the reflected wave, with uniform distribution in all directions and with total integral equal to πa^2. The second term corresponds to the shadow-forming wave, which should really be combined with the plane wave before calculating the intensity. This shadow-forming wave has a peak in the forward direction ($\vartheta = 0$) a factor $\frac{1}{2}ka$

larger than for other directions; away from the forward direction it soon becomes smaller than the first term. The behavior of the scattered intensity for the Neumann boundary condition ($v_r = 0$ at $r = a$), as a function of frequency, is shown in Fig. 11.9. In this case the scattering pattern is distorted even for long wavelengths, but the behavior for short wavelengths, with the formation of the forward peak, is similar to the Dirichlet case.

There are times when the boundary conditions at $r = a$ cannot be simulated by simple Dirichlet or Neumann conditions. In some cases the waves penetrate the sphere, but at a different speed than in the air and with, perhaps, absorption. Two somewhat different boundary conditions arise: one resulting in a generalized homogeneous boundary condition at $r = a$ and the other resulting in an effective index of refraction for $r < a$. The first case is more appropriate when air does not penetrate deeply into the sphere, the surface of the sphere yields a little to the pressure, but there is no true wave motion inside the sphere. In this case we can assume, with reasonable accuracy, that there is an acoustic impedance of the surface

$$Z = -(p/v_r)_a = -[i\omega\rho\psi/(\partial\psi/\partial r)]_a = R - iX$$

which is more or less independent of the angle of incidence of the wave though it will vary with ω.

The second case is more appropriate when the wave motion really penetrates throughout the sphere and when the substance within $r = a$ is more or less uniform, though its properties can be quite different from those of air. In this case we can include the wave motion inside $r = a$ and fit the two solutions at $r = a$. Inside the sphere, the wave equation is $\nabla^2\psi - [(n + iq)/c]^2(\partial^2\psi/\partial t^2) = 0$ where $(n + iq)$ is the complex index of refraction of the material. In both cases energy is absorbed by the sphere (unless R_a or q is zero) in contrast to the cases for simple Dirichlet or Neumann conditions, where all the incoming energy was sent out again, either as undistorted plane wave or as scattered wave.

Scattering from a Sphere with Complex Index of Refraction. We shall discuss here the second case, where the wave motion penetrates throughout the sphere, but the medium has an index of refraction $(n + iq)$ and an effective density ρ_s, differing from ρ, the air density. A plane wave in this medium will have a velocity potential, pressure, and velocity

$$\psi = Ae^{-kqx+ik(nx-ct)};\quad p = i\omega\rho_s\psi;\quad v_x = (ikn - kq)\psi$$

where $k = \omega/c$, as before. The average intensity at the point x is then $\frac{1}{2}\rho_s cnk^2|A|^2e^{-2kqx}$, indicating that the wave attenuates in the medium.

For such an index of refraction, the possible solutions inside the sphere, which are finite for $r < a$ and which can be used to fit boundary conditions at $r = a$, are the set

$$\begin{matrix}\cos\\ \sin\end{matrix}(m\phi)P_l^m(\cos\vartheta)\, j_l\,[k(n+iq)r]$$

To fit the boundary conditions we must compute the ratio between radial gradient and value of the potential at $r = a$. We accordingly define the angles, also given in the tables at the end of this chapter,

$$\tan\alpha_n(z) = -[zj'_n(z)/j_n(z)]; \quad \tan\beta_n(z) = -[zn'_n(z)/n_n(z)]$$

For the interior functions we need only the angles α_n, for the complex argument $k(n + iq)a$. We can call this angle $\alpha_n[k(n + iq)a] = \alpha_n^i$ to simplify the notation.

Next we set up the velocity potential, inside and out, which is to fit together at $r = a$ and be a plane-plus-scattered wave outside the sphere,

$$\psi = \begin{cases} \displaystyle\sum_{m=0}^{\infty} (2m+1)i^m A_m P_m(\cos\vartheta)j_m[k(n+iq)r]e^{-i\omega t}; & r < a \\ \displaystyle\sum_{m=0}^{\infty} (2m+1)i^m P_m(\cos\vartheta)e^{-i\eta_m}[\cos(\eta_m)j_m(kr) \\ \qquad\qquad + \sin(\eta_m)n_m(kr)]e^{-i\omega t}; & r > a \end{cases}$$

where the coefficients A_m and the phase angles η_m are to be adjusted so that pressure and radial velocity are continuous at $r = a$. The two equations for the nth terms are

$$A_m i\omega\rho_s j_m[k(n+iq)a] = i\omega\rho e^{-i\eta_m}[\cos(\eta_m)j_m(ka) + \sin(\eta_m)n_m(ka)]$$
$$A_m k(n+iq)aj'_m[k(n+iq)a] = kae^{-i\eta_m}[\cos(\eta_m)j'_m(ka) + \sin(\eta_m)n'_m(ka)]$$

Dividing one by the other, we obtain

$$\frac{k(n+iq)a}{j_m[k(n+iq)a]}\,j'_m[k(n+iq)a] = -\tan\alpha_m^i = ka\,\frac{\rho_s}{\rho}\,\frac{j'_m(ka) + \tan\eta_m n'_m(ka)}{j_m(ka) + \tan\eta_m n_m(ka)}$$

or
$$\tan\eta_m = \tan[\delta_m(ka)]\,\frac{\rho\tan\alpha_m^i - \rho_s\tan\alpha_m^0}{\rho\tan\alpha_m^i - \rho_s\tan\beta_m^0} \tag{11.3.73}$$

where $\alpha_m^i = \alpha_m[k(n + iq)a]$, $\alpha_m^0 = \alpha_m^0(ka)$ and $\beta_m^0 = \beta_m(ka)$. This gives us the two limiting cases as well as the intermediate ones. For instance when $\rho \gg \rho_s$ then $\eta_m \to \delta_m(ka)$, but when $\rho \ll \rho_s$ then $\eta_m \to \delta'_m(ka)$. If $q > 0$, the angles η_m are complex; we can write them

$$\eta_m = \chi_m - i\kappa_m$$

The values of χ and κ may be obtained from Eq. (11.3.73).

The scattered wave is then

$$\psi_s = -\sum_{n=0}^{\infty} i^{n+1}(2n+1)P_n(\cos\vartheta)e^{-i\eta_n}\sin\eta_n\, h_n(kr)$$
$$\to -\frac{e^{ikr}}{kr}\sum_{n=0}^{\infty}(2n+1)P_n(\cos\vartheta)e^{-i\eta_n}\sin\eta_n; \quad r\to\infty$$

the intensity of the scattered wave per unit incident intensity is

$$S = \frac{1}{4k^2r^2} \sum_{m,n} (2m + 1)(2n + 1)P_m(\cos \vartheta)P_n(\cos \vartheta) \cdot$$
$$\cdot [1 + \cos 2(\chi_m - \chi_n)\, e^{-2(\kappa_m+\kappa_n)} - e^{-2\kappa_m} \cos(2\chi_m) - e^{-2\kappa_n} \cos(2\chi_n)]$$

and, finally, the scattering cross section, the total power scattered per unit incident intensity, is

$$Q_s = \frac{4\pi a^2}{(ka)^2} \sum_{n=0}^{\infty} (2n + 1)e^{-2\kappa_n}[\sinh^2 \kappa_n + \sin^2 \chi_n] \qquad (11.3.74)$$

This is not the only cross section, however, when there is absorption. A certain amount of power is lost inside the sphere, without being reradiated at all. The plane-plus-scattered wave may be rewritten as

$$e^{ikz} + \psi_s \to \frac{i}{2kr} \sum_{n=0}^{\infty} (2n + 1)P_n(\cos \vartheta)[(-1)^n e^{-ikr} - e^{ikr-2i\eta_n}]; \quad r \to \infty$$

The first terms represent the incoming plane wave; there is no absorption or phase change in the incoming wave. On the other hand the second terms, representing the outgoing waves, are modified by the scattering angle η_n. As long as η_n is real (as long as $\kappa_n = 0$), there is as much power going outward from the sphere as there is coming in. But if $\eta_n = \chi_n - i\kappa_n$ and $\kappa_n > 0$, then less is outward bound than is sent in. The part of the wave which never gets out, which is absorbed inside the sphere, is the part measured by the difference in total outward radiation. We take the difference between the amount radiated if there were no absorption ($\kappa_n = 0$), $\left(\frac{\pi}{k^2}\right) \sum (2n + 1)$, and the amount radiated with absorption, $\left(\frac{\pi}{k^2}\right) \sum (2n + 1)e^{-4\kappa_n}$. This difference, the *absorption cross section*, is the power absorbed by the sphere per unit incident intensity,

$$Q_a = \frac{2\pi a^2}{(ka)^2} \sum_{n=0}^{\infty} (2n + 1)e^{-2\kappa_n} \sinh(2\kappa_n) \qquad (11.3.75)$$

The sum of these two cross sections is the *extinction cross section*, the fraction of energy lost by the plane wave, either through scattering or by absorption:

$$Q_e = Q_s + Q_a = \frac{4\pi a^2}{(ka)^2} \sum_{n=0}^{\infty} (2n + 1)e^{-\kappa_n}[\cosh \kappa_n \sin^2 \chi_n + \sinh \kappa_n \cos^2 \chi_n]$$

This last, total cross section can also be obtained by computing the power lost to the incident plane wave at the surface of the sphere. We multiply the pressure of the plane wave by the inward velocity of the sphere's surface and add to it the inward velocity of the plane wave times the actual pressure at the surface, and obtain

$$\int \tfrac{1}{2}\,\mathrm{Re}[-\bar{p}_0 v_r - p\bar{v}_{0r}]\,dA = (\tfrac{1}{2}\rho c k^2)4\pi a^2 \sum_{n=0}^{\infty} (2n+1)D_n(ka)D'_n(ka)e^{-\kappa_n} \cdot$$
$$\cdot\,\{\cosh\kappa_n \sin\chi_n[\sin\delta_n \sin(\chi_n - \delta'_n) + \sin\delta'_n \sin(\delta_n - \chi_n)] + \sinh\kappa_n \cos\chi_n[\sin\delta_n\cos(\chi_n - \delta'_n) - \sin\delta'_n\cos(\delta_n - \chi_n)]\}$$

which, after some juggling of terms, reduces to the formula of (11.3.75).

Somewhat similar formulas are obtained if we start with an acoustic impedance of the material. When this impedance is $Z = R - iX$, the boundary condition is

$$\left(\frac{p}{-v_r}\right)_a = \frac{-i\omega\rho\psi}{(\partial\psi/\partial r)_{r=a}} = Z = R - iX$$

The expression for ψ outside the sphere is the same as before, and the equations for scattering, absorption, and total cross section are the same as before. The only difference is the equation for the phase angle $\eta_n = \chi_n - i\kappa_n$, which now is

$$\tan\eta_n = \tan\delta'_n\left[\frac{iZ + \rho cka\cot\alpha_n}{iZ + \rho cka\cot\beta_n}\right]$$

When $Z \to \infty$ the requirement is to have zero radial velocity at $r = a$ and $\eta_n \to \delta'_n$; whereas when $Z \to 0$, the angle $\eta_n \to \delta_n$.

It sometimes is easier, from a computational point of view, to use the *reflection coefficient* $R_m = e^{-2i\eta_m}$ instead of the phase angle η_m, in terms of which to express our results. The total wave function and the scattered wave at large distances from the sphere, are

$$\psi \to \frac{i}{2kr}\sum_{m=0}^{\infty}(2m+1)P_m(\cos\vartheta)[(-1)^m e^{-ikr} - R_m e^{ikr}]$$

$$\psi_s \to \frac{ie^{ikr}}{2kr}\sum_{m=0}^{\infty}(2m+1)P_m(\cos\vartheta)(1 - R_m)$$

The scattering cross section, the absorption and total cross sections then become much simpler in form:

$$Q_s = \frac{\pi a^2}{k^2a^2}\sum(2m+1)|1 - R_m|^2;\quad Q_a = \frac{\pi a^2}{k^2a^2}\sum(2m+1)[1 - |R_m|^2]$$

$$Q_t = \frac{\pi a^2}{k^2a^2}\sum(2m+1)[2 - R_m - \bar{R}_m] \qquad (11.3.76)$$

The formulas for the reflection coefficients R_m, in terms of the internal wave or the surface impedance, are

$$-R_m = \frac{ka\rho_s \bar{h}'_m(ka) + \rho \tan(\alpha^i_m)\bar{h}_m(ka)}{ka\rho_s h'_m(ka) + \rho \tan(\alpha^i_m) h_m(ka)} = \frac{\rho c \bar{h}_m(ka) - iZ\bar{h}'_m(ka)}{\rho c h_m(ka) - iZh'_m(ka)} \tag{11.3.77}$$

We note that, when iZ or $\tan(\alpha^i_m)$ are real, $|R_m| = 1$ and $Q_a = 0$. Also, since

$$R_m = e^{-2i\delta_m} \frac{[1 + (iZD'_m/\rho c D_m)\cos(\delta_m - \delta'_m)] + (Z/\rho c D^2_m k^2 a^2)}{[1 + (iZD'_m/\rho c D_m)\cos(\delta_m - \delta'_m)] - (Z/\rho c D^2_m k^2 a^2)}$$

the particular values of ka for which the bracket is zero result in $R_m = -1$, for which the total cross section is largest. Such resonance peaks are encountered in both acoustical and neutron scattering.

Scattering from a Helmholtz Resonator. Another problem of sufficient interest to give in detail is that of sound wave diffraction around a hollow sphere pierced with a circular opening of diameter small compared to that of the sphere. Such a sphere with opening is the simplest possible form of a *Helmholtz resonator*, and its behavior in a sound field is of considerable interest. We let the radius of the sphere be a and the center of the hole be on the z axis, so that the opening is between $\vartheta = 0$ and $\vartheta = \vartheta_1$ (the radius of the hole being $a \sin \vartheta_1$) and the solid spherical shell is between $\vartheta = \vartheta_1$ and $\vartheta = \pi$.

Any excess pressure across this hole will cause a flow of air in and out through it. If the hole radius is smaller than a wavelength, the flow pattern near the hole will be similar to the steady-state flow given in Eq. (10.3.58) and discussed in the pages following that equation. The only fact we need to know here is that the pressure is uniform across the hole, and the velocity normal to the plane of the hole is $C/\sqrt{a_h^2 - \eta^2}$, where a_h is the radius of the hole ($a \sin \vartheta_1$ in this case) and η is the distance, in the plane, from the center of the hole. In the case discussed here, with a_h quite a bit smaller than a, a good approximation to this behavior is obtained by specifying that over the opening p is constant and

$$v_r \simeq \frac{V}{\sqrt{\cos\vartheta - \cos\vartheta_1}}; \quad 0 \le \vartheta < \vartheta_1$$

with $v_r = 0$ for $\vartheta_1 < \vartheta \le \pi$. The relationship between the constant V and the pressure will have to be worked out.

Let us assume that, across the surface of the hole, the average excess pressure from the incident sound wave, whatever it is, is $p_0 e^{-i\omega t}$. This acts as a driving force on the air in the hole, moving it in and out and causing wave motion inside and outside the sphere. The velocity potential of this forced motion may be written

$$\psi_f = \begin{cases} a \sum_{n=0}^{\infty} A_n P_n(\cos\vartheta) \left[\dfrac{j_n(kr)}{kaj_n'(ka)}\right] e^{-i\omega t}; & r < a \\ a \sum_{n=0}^{\infty} A_n P_n(\cos\vartheta) \left[\dfrac{h_n(kr)}{kah_n'(ka)}\right] e^{-i\omega t}; & r > a \end{cases} \tag{11.3.78}$$

which is arranged to have the radial velocity at $r = a$ be

$$v_r = \sum_{n=0}^{\infty} A_n P_n(\cos\vartheta) e^{-i\omega t}$$

An exact solution would involve solution for all the infinite set of coefficients A_n. As on page 1388, we obtain an approximate solution by assuming a ratio between the A's and solving for the single remaining unknown, V.

This radial velocity, if it is to approximate the steady-state distribution through a hole, should equal (approximately) $[V/\sqrt{\cos\vartheta - \cos\vartheta_1}]\, e^{-i\omega t}$ for $0 \le \vartheta < \vartheta_1$ and equal zero for $\vartheta_1 < \vartheta \le \pi$. The tables at the end of Chap. 10 have just such a series. We find that

$$V\sqrt{2} \sum_{n=0}^{\infty} \sin[(n + \tfrac{1}{2})\vartheta_1] P_n(\cos\vartheta) = \begin{cases} V/\sqrt{\cos\vartheta - \cos\vartheta_1}; & 0 \le \vartheta < \vartheta_1 \\ 0; & \vartheta_1 < \vartheta \le \pi \end{cases}$$

so that $A_n \simeq \sqrt{2}\, V \sin[(n + \frac{1}{2})\vartheta_1]$, expressing the A's in terms of the single unknown V.

To the approximation we are expecting, the pressure should be uniform over the hole. But the pressure computed from the series for $r < a$ should not equal that computed from the series for $r > a$; the difference should equal the driving pressure p_0, and this relation should fix the value of V. The series for the pressure at $\vartheta = 0$, $r = a$ do not converge, but the difference between the series does (just barely!). The pressure $p_0 e^{-i\omega t}$ should equal the pressure computed from the $r < a$ series minus that computed from the $r > a$ series, so that the equation for V is

$$\begin{aligned} p_0 &= i\rho c \sqrt{2}\, V \sum_{n=0}^{\infty} \sin[(n + \tfrac{1}{2})\vartheta_1] \left[\frac{j_n(ka)}{j_n'(ka)} - \frac{j_n(ka) + in_n(ka)}{j_n'(ka) + in_n'(ka)}\right] \\ &= -i\sqrt{2}\, V \frac{\rho c}{k^2a^2} \sum_{n=0}^{\infty} \sin[(n + \tfrac{1}{2})\vartheta_1] \left[\frac{e^{-i\delta_n'}}{(D_n')^2 \sin(\delta_n')}\right] \end{aligned}$$

where the argument for δ_n' and D_n' is (ka).

Convergence can be improved by adding and subtracting a series which approaches the behavior of this series for large values of n. For

$n \gg ka$ we have

$$\frac{-e^{-i\delta_n'}}{(ka)^2(D_n')^2\sin(\delta_n')} \to ka\,\frac{2n+1}{n(n+1)} = \frac{ka}{n} + \frac{ka}{n+1}$$

so that the comparison series is

$$\sqrt{2}\,ka\left\{\sum_{n=1}^{\infty}\frac{1}{n}\sin[(n+\tfrac{1}{2})\vartheta_1] + \sum_{m=1}^{\infty}\frac{1}{m}\sin[(m-\tfrac{1}{2})\vartheta_1]\right\};\quad m = n+1$$

$$= \sqrt{8}\,ka\cos(\tfrac{1}{2}\vartheta_1)\,\mathrm{Im}\left[\sum_{n=1}^{\infty}\frac{1}{n}e^{in\vartheta_1}\right] = -\sqrt{8}\,ka\cos(\tfrac{1}{2}\vartheta_1)\,\mathrm{Im}[\ln(1-e^{i\vartheta_1})]$$

$$= \sqrt{2}\,ka(\pi - \vartheta_1)\cos(\tfrac{1}{2}\vartheta_1)$$

We therefore have for the driving pressure

$$p_0 = -[\sqrt{2}\,\rho c V/\sin(\tfrac{1}{2}\vartheta_1)]\zeta(ka,\vartheta_1)$$

where

$$\zeta(ka,\vartheta_1) = -ika\sin(\tfrac{1}{2}\vartheta_1)\left\{(\pi-\vartheta_1)\cos(\tfrac{1}{2}\vartheta_1)\right.$$

$$\left. - \sum_{n=0}^{\infty}\sin[(n+\tfrac{1}{2})\vartheta_1]\left[\frac{e^{-i\vartheta_n'}}{(ka)^3(D_n')^2\sin\delta_n'} + \frac{1-\delta_{0n}}{n} + \frac{1}{n+1}\right]\right\} \tag{11.3.79}$$

may be called the *specific impedance* of the hole, for reasons shortly to become apparent. The quantity V is not the average velocity through the opening. The total flow amplitude is

$$Q = 2\pi\int_0^{a_h}\frac{Vy\,dy}{\sqrt{\cos\vartheta - \cos\vartheta_1}} = 2\pi Va^2\int_{\cos\vartheta_1}^{1}\frac{dx}{\sqrt{x-\cos\vartheta_1}}$$
$$= 4\pi a^2 V\sqrt{2}\sin(\tfrac{1}{2}\vartheta_1)$$

But this should equal the average velocity amplitude, v_0, times the area of the opening, $4\pi a^2\sin^2(\frac{1}{2}\vartheta_1)$. Therefore the relation between V and the average velocity amplitude v_0 is $V = \frac{1}{2}\sqrt{2}\sin(\frac{1}{2}\vartheta_1)v_0$. Consequently, $p_0 = \rho c v_0\zeta(ka,\vartheta_1)$, and $\zeta(ka,\vartheta_1)$ is the acoustic impedance of the opening (including the internal and external waves) in units of ρc.

For long wavelengths ($ka \ll 1$), this impedance becomes

$$\zeta \to -ika[(\pi-\vartheta_1)\cos(\tfrac{1}{2}\vartheta_1) - \sin(\tfrac{1}{2}\vartheta_1)]\sin(\tfrac{1}{2}\vartheta_1)$$
$$+ i(3/ka)\sin^2(\tfrac{1}{2}\vartheta_1) + (ka)^2\sin^2(\tfrac{1}{2}\vartheta_1);\quad ka \to 0$$
$$\to -i(\tfrac{1}{2}\pi ka\vartheta_1) + i(3\vartheta_1^2/4ka) + (\tfrac{1}{2}ka\vartheta_1)^2;\quad \vartheta_1 \to 0$$

For small values of ϑ_1 and of ka, the mechanical impedance of the air in the hole, the ratio between $\pi(a\vartheta_1)^2p_0$ and v_0, is

$$(F/v_0) \to -i\omega(\tfrac{1}{2}\pi^2a_h^3\rho) + \frac{i}{\omega}(\pi a_h^2)^2\left(\frac{\rho c^2}{\frac{4}{3}\pi a^3}\right) + \omega^2(\pi a_h^2)^2\left(\frac{\rho}{4\pi c}\right)$$

where $a_h \simeq a\vartheta_1$. The first term is the mass reactance of the air in the hole, given in Eq. (10.3.60). The second term is the stiffness reactance of the air inside the sphere, recognizable by the ratio between the specific stiffness of the air, ρc^2, and the volume of the sphere. The last term is the radiation resistance, coming from the waves sent out to infinity. Resonance occurs at $\omega = \omega_0 \simeq \sqrt{3\vartheta_1 c^2/2\pi a^2}$; also for higher frequencies near where $\sin \delta_n'$ goes to zero. If we were taking viscosity into account, we would have had to add a fourth term, also resistive, representing the friction of motion through the hole.

Next we have to couple this forced motion to an incident wave. For instance, p_0 may be produced by a plane wave in the direction $\vartheta = \alpha$, $\phi = 0$ with respect to the axis of the hole. The plane-plus-scattered wave, taking the boundary condition to be $v_r = 0$ at $r = a$ (to the zeroth approximation the hole is not present), turns out to be

$$\psi = \sum_{m,n} \epsilon_m(2n+1)i^n \frac{(n-m)!}{(n+m)!} \cos(m\phi)P_n^m(\cos\alpha)P_n^m(\cos\vartheta) \cdot \cdot \left[j_n(kr) - \frac{j_n'(ka)}{h_n'(ka)} h_n(kr) \right] e^{-i\omega t}$$
$$= e^{ik[z\cos\alpha + x\sin\alpha) - i\omega t} - i \sum_n (2n+1)i^n P_n[\cos\alpha\cos\vartheta + \sin\alpha\sin\vartheta\cos\phi] \cdot \cdot e^{-i\delta'_n(ka)} \sin[\delta_n'(ka)]h_n(kr)e^{-i\omega t} \qquad (11.3.80)$$

The average pressure over the area of the hole turns out to be

$$p_0 = \frac{ik\rho c}{2\sin^2(\frac{1}{2}\vartheta_1)} \sum_{n=0}^{\infty} i^n P_n(\cos\alpha)[P_{n-1}(\cos\vartheta_1) - P_{n+1}(\cos\vartheta_1)] \frac{e^{-i\delta_n'(ka)}}{(ka)^2 D_n'(ka)}$$
$$\to ik\rho c[1 + \tfrac{3}{2}ika\cos\alpha\cos^2(\tfrac{1}{2}\vartheta_1)]; \quad ka \to 0$$

which is to be put into the expression for the coefficients of series (11.3.78)

$$A_n = -\frac{p_0}{\rho c\zeta(ka,\vartheta_1)} \sin(\tfrac{1}{2}\vartheta_1)\sin[(n+\tfrac{1}{2})\vartheta_1]$$

to obtain the part of the wave produced by motion of air in the hole.

The complete expression for the scattered wave, including the wave from motion in the hole, at large distances from the resonator, is

$$\psi_s \simeq -\frac{e^{ikr-i\omega t}}{kr} \sum_{m,n} \left\{ \epsilon_m(2n+1)\frac{(n-m)!}{(n+m)!} P_n^m(\cos\alpha)e^{-i\delta_n'}\sin\delta_n' - i^{-n}\frac{\delta_{0m}p_0}{k\rho c\zeta(ka,\vartheta_1)} \sin(\tfrac{1}{2}\vartheta_1)\sin[(n+\tfrac{1}{2})\vartheta_1]\frac{e^{-i\delta_n'}}{D_n'} \right\} \cos(m\phi)P_n^m(\cos\vartheta) \qquad (11.3.81)$$
$$\to -\frac{ae^{ikr-i\omega t}}{r}(ka)^2 \left\{ \tfrac{1}{3} - [\cos\alpha\cos\vartheta + \sin\alpha\sin\vartheta\cos\phi] - \tfrac{1}{3}\left[\frac{(1+\frac{3}{2}ika\cos\alpha)(1-\frac{3}{2}ika\cos\vartheta)}{1-(2\pi k^2a^2/3\vartheta_1) - \frac{1}{3}i(ka)^3} \right] \right\}; \quad ka \to 0; \quad \vartheta_1 \to 0$$

From this can be obtained the expressions for scattered intensity and effective cross section.

For very long wavelengths the part of the wave coming from the hole cancels the spherically symmetric part of the wave scattered by the spherical shell. The presence of the hole "short-circuits" the bulk of the sphere as far as the spherically symmetric part, having only the "dipole" scattering. The only term in ϑ_1 (to this approximation) is the mass reactance term in the impedance ζ. The smaller the hole, the lower the resonance frequency for the resonator, $(ka)_0 \simeq \sqrt{3\vartheta_1/2\pi}$. At resonance the part of the wave coming from the hole is very much larger than the part scattered from the shell, approximately by the ratio $(2\pi/3\vartheta_1)^{\frac{1}{2}}$ to 1.

$$\psi_s \to i\,\frac{ae^{ikr-i\omega t}}{r}\sqrt{\frac{2\pi}{3\vartheta_1}};\quad \omega = \omega_0,\ \vartheta_1 \ll 1$$

To the first approximation in (ka) and in ϑ_1, the scattering cross section is

$$Q \simeq 4\pi a^2(ka)^4 \left\{\frac{(2\pi k^2a^2/3\vartheta_1)^2 + (\frac{1}{3}k^3a^3)^2}{[1 - (2\pi k^2a^2/3\vartheta_1)]^2 + (\frac{1}{3}k^3a^3)^2} + 3\right\} \quad (11.3.82)$$

which is very much smaller than the sphere's area $(4\pi a^2)$ (as long as $ka < 1$) *except* at the resonance frequency of the resonator, when

$$Q \simeq (24\pi/\vartheta_1)(\pi a^2);\quad ka = \sqrt{3\vartheta_1/2\pi}$$

At this frequency the effective cross section for scattering is considerably larger than the geometrical cross section (πa^2).

The two expressions for ψ_s and Q at resonance would be profoundly modified if we had included the effect of viscosity. The chief effect of viscosity is to add another resistive term to the impedance, principally arising from the flow through the hole. If the amplitude of motion is small enough so that vortices are not present, the effect is to add a real term to the impedance ζ, which results in an additional term in the denominator of the last line of Eq. (11.3.81), making it $1 - (2\pi k^2a^2/3\vartheta_1) - \frac{1}{3}i(ka)^3 - i\delta$ and making the denominator of Eq. (11.3.82) $[1 - (2\pi k^2a^2/3\vartheta_1)]^2 + [\frac{1}{3}(ka)^3 + \delta]^2$. The term δ has a different dependence on frequency than $\frac{1}{3}(ka)^3$ and is larger than it, though usually smaller than unity, over the range of frequency for which the formulas are valid Therefore the expressions for ψ_s, S, and Q are not changed appreciably except near the resonance, where $ka \simeq \sqrt{3\vartheta_1/2\pi}$. When viscosity is included, the resonance peak is not so high as before. For instance Q at resonance is approximately $(\pi a^2)(3\vartheta_1/\pi\delta)^2$ instead of $(\pi a^2)(24\pi/\vartheta_1)$ and, since $(24\pi/\vartheta_1) > (3\vartheta_1/\pi\delta)^2 > 1$ in most cases, the peak is still quite high, though not so high as when viscosity is omitted.

Scattering from an Ensemble of Scatterers. The subject of the diffraction of waves by an assemblage of scatterers is a big one, covering the range of items from the blue of the sky to electric conductivity in metals, from reverberation in sea water to X-ray diffraction. It would carry us too far afield to cover this whole subject in detail, indeed it would require at least an additional chapter to do it justice. Instead we shall only touch on a few of the simpler concepts and examples.

The degree of complication involved in the analysis depends very markedly on the strength of the scattering. If the scattering cross section is a small fraction of the area irradiated by the incident wave, as it is in the case of X rays or sound waves in "bubbly" water, it is a good approximation to consider that the scattering from any one center does not depend on the presence of other scatterers (multiple scattering can

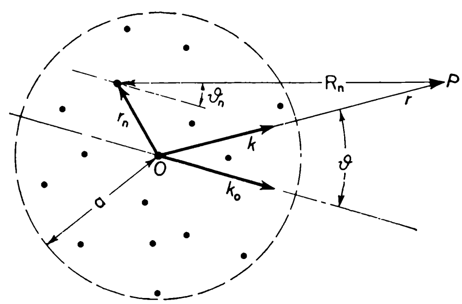

Fig. 11.10 Angles, position and propagation vectors for scattering of plane wave by ensemble of scatterers.

be neglected). On the other hand for electron waves in metals and for some acoustical cases, the wave about each scatterer is profoundly influenced by the presence of the other scatterers and the classification into singly and multiply scattered waves has lost its meaning. We start with the simple, weak-scattering case.

Referring to Fig. 11.10, we assume that there are N scatterers all distributed inside a sphere of radius a, the nth scatterer being located with respect to the center of the sphere by the vector $\mathbf{r}_n$. The incident plane wave is $\psi_i = e^{ikr\cos\vartheta}$ (we have defined the wave potential so that $|\psi|^2$ is the intensity) with unit intensity. If the total scattering is small, each scatterer in the ensemble behaves as if it alone were in this incident wave, and each sends a scattered wave radially outward, having a distribution-in-angle function $f(\vartheta)$. Consequently the scattered wave at point P is

$$\psi_s = \sum_{n=1}^{N} \frac{f(\vartheta_n)}{R_n} \exp[i(\mathbf{k}_0 \cdot \mathbf{r}_n + kR_n)]$$

where we have taken the phase to be zero at the origin. When $r \gg a$, this becomes, to the first approximation in a/r,

$$\psi_s \simeq \frac{e^{ikr}}{r} f(\vartheta) \sum_{n=1}^{N} \exp(i\boldsymbol{\mu} \cdot \mathbf{r}_n) \tag{11.3.83}$$

where $\boldsymbol{\mu} = \mathbf{k}_0 - \mathbf{k}$, $\mu = 2k \sin(\frac{1}{2}\vartheta)$.

Each component of the wave at P has its own phase; in some cases all, or nearly all, of the waves are in phase and the scattered beam is intense: in other cases none, or few, of the components are in phase and the scattered wave is considerably weaker. In the former case the scattering is said to be *coherent;* in the latter, *incoherent.* The relative amounts of coherent and incoherent waves in the scattered wave depend to some extent on the degree of regularity of positioning of the scatterers.

To take one extreme first, let us begin by assuming that the scatterers happen all to be arranged in regularly spaced planes, perpendicular to the vector $\boldsymbol{\mu} = \mathbf{k} - \mathbf{k}_0$. Suppose there are L scatterers in each plane and M planes, each a distance d apart, so that $LM = N$ and $d < 2a$. We then set the x axis parallel to $\boldsymbol{\mu}$; the x coordinate of the nth plane of scatterers is then $x = x_1 + (n - 1)d$. Equation (11.3.83) then becomes

$$\begin{aligned} \psi_s &= \left(\frac{e^{ikr}}{r}\right) f(\vartheta) L e^{i\mu x_1} \sum_{n=1}^{M} e^{i\mu(n-1)d} \\ &= N \left(\frac{e^{ikr}}{r}\right) f(\vartheta) e^{i\mu(x_1 + \frac{1}{2}Nd)} \left[\frac{\sin(kMd \sin \frac{1}{2}\vartheta)}{M \sin(kd \sin \frac{1}{2}\vartheta)}\right] \end{aligned} \tag{11.3.84}$$

For $\vartheta \to 0$, in the direction of the incident wave, the scattered wave $\psi_s = Nf(\vartheta)(e^{ikr}/r)$ interferes with the incident wave to cause a shadow. If $Mkd = 2\pi Md/\lambda$ is considerably smaller than unity, this limiting expression is valid for all values of ϑ and the total scattered wave is just N times the wave scattered from one scatterer. In this case, where $2\pi a$ is smaller than the wavelength, all of the scattered wave is coherent and the ensemble of scatterers act like a single scatterer of N times the original "strength."

If, however, $Mkd > 1 \gg (kd)$ $(4\pi a > \lambda \gg 2\pi d)$, then the simple form for ψ_s is multiplied by $\{\sin[Mkd \sin(\frac{1}{2}\vartheta)]/Mkd \sin(\frac{1}{2}\vartheta)\}$. If we arrange things so that the x axis (perpendicular to the planes of scatterers) stays parallel to $\boldsymbol{\mu}$ as we change ϑ, then the amplitude of the scattered wave decreases rapidly as ϑ increases and, except for the sharp forward, shadow-causing beam, the scattered wave is considerably smaller than N times the corresponding amplitude for a single scatterer.

Finally, when the wavelength λ is smaller than $2\pi d$, we must use the exact form of (11.3.84). In this case, when $kd \sin(\frac{1}{2}\vartheta) = (2\pi d/\lambda) \sin(\frac{1}{2}\vartheta)$ is equal to an integer, m, times π, the denominator goes to zero again, and a strong beam is produced. In other words, whenever the angle ϑ is such that $2d \sin(\frac{1}{2}\vartheta) = m\lambda$ (and when $\boldsymbol{\mu}$ is kept perpendicular to the planes of scatterers), a strong beam is produced. These angles are just the Bragg angles for diffraction of X rays from a crystal lattice.

When $\boldsymbol{\mu}$ is not perpendicular to regularly spaced planes of scatterers, or when the scatterers are not arranged in any regular array, it is easier to see what is happening by computing the scattered intensity $|\psi_s|^2$ (per unit incident intensity) than it is to work with the amplitude of ψ_s. (As a matter of fact $|\psi_s|^2$ is usually easier to measure experimentally than is ψ_s.) From Eq. (11.3.83) we have

$$S = \frac{|f(\vartheta)|^2}{r^2} \sum_{m,n=1}^{N} \exp(i\boldsymbol{\mu} \cdot \mathbf{R}_{mn})$$

where $\mu = 2k \sin(\frac{1}{2}\vartheta)$, $\boldsymbol{\mu} = \mathbf{k}_0 - \mathbf{k}$ and $\mathbf{R}_{mn} = \mathbf{r}_m - \mathbf{r}_n$ is the vector from the nth to the mth scatterer. We first separate off the $m = n$ terms:

$$S = \frac{|f(\vartheta)|^2}{r^2}\left[N + \sum_{n\neq m} \exp(i\boldsymbol{\mu} \cdot \mathbf{R}_{mn})\right]$$

The first term is the intensity which one would compute if the N scatterers sent out their waves independently of each other so that we add their intensities, not their amplitudes. This is the *incoherent* part of the scattering. The second term is the *coherent* part, which takes into account the phase relations between the various scattered waves. If the scatterers are far enough apart on the average, or $\mu = 2k\sin(\frac{1}{2}\vartheta)$ is large enough, so that $\boldsymbol{\mu} \cdot \mathbf{R}_{mn} > \frac{1}{2}\pi$ for most m's and n's, and if the scatterers are so distributed that the values of $\boldsymbol{\mu} \cdot \mathbf{R}_{mn}$ come at random (*i.e.*, if $\boldsymbol{\mu}$ is not perpendicular to regularly spaced planes as discussed earlier), then the exponentials in the double sum will cancel out and leave only the incoherent term.

For μ small enough, however, there is still a coherent part, which will predominate in the forward direction. For instance, if the scatterers are distributed in a random manner over the interior of the sphere of radius a, so that the *average* density is uniform and equal to $(3N/4\pi a^3)$, this sum will equal, approximately,

$$\begin{aligned}\frac{3N(N-1)}{4\pi a^3}\int_0^{2\pi} d\phi \int_0^1 dx \int_0^a e^{i\mu r x}\, r^2\, dr &= \frac{3N(N-1)}{a^3}\int_0^a \frac{\sin(\mu r)}{\mu r}\, r^2\, dr \\ = 3N(N-1)\,\frac{\sin(\mu a) - \mu a\cos(\mu a)}{\mu^3 a^3} &\to N(N-1)[1 - \tfrac{1}{10}\mu^2 a^2];\quad \mu a \to 0\end{aligned}$$

where the factor $N(N - 1)$ arises because, though we sum over m from 1 to N, when we sum over n we leave out the $n = m$ term.

This formula shows that for $\mu a \ll 1$ $(2\pi a \sin\frac{1}{2}\vartheta \ll \lambda)$ the total scattered intensity has a strong peak,

$$S \to N^2 \frac{|f(\vartheta)|^2}{r^2}\left[1 - \frac{2}{5}\frac{N-1}{N} k^2a^2 \sin^2(\tfrac{1}{2}\vartheta) + \cdots\right]$$

proportional to the *square* of the number of scatterers. Here the amplitudes of the scattered waves add, and the coherent scattering predominates. On the other hand when $4a \sin(\frac{1}{2}\vartheta) \gg \lambda$, the coherent scattering becomes relatively small (particularly if N is large) and the scattered intensity is predominantly the incoherent scattering.

$$S \to N|f(\vartheta)|^2/r^2$$

proportional to the first power of the number of scatterers.

Scattering of Sound from Air Bubbles in Water. In the previous discussion we have assumed that the scattering was "weak" enough to allow us to neglect the effect of the scattered waves on the wave incident on another scatterer. To illustrate the effects of multiple scattering, we shall investigate the case of the scattering of sound waves in water by a distribution of small air bubbles. In order to carry the computations through, we assume that the radius of the bubbles is small compared to the wavelength, so that the scattering from individual bubbles is spherically symmetric. First we should see what the scattering from a single bubble is like.

Since the bubble is considerably smaller than a wavelength, the excess pressure p from the incident sound wave (over and above the average pressure P) is practically uniform over the surface of the bubble. Let $p = p_0e^{-i\omega t}$. The relation between excess pressure p and change of bubble volume, for adiabatic expansion of the air in the bubble, is $p/P = -\gamma(dV/V)$ or, since $\gamma P = \rho_a c_a^2$ for air,

$$-(\partial p/\partial t) = i\omega p = (3\rho_a c_a^2/4\pi a^3)(\partial V/\partial t) = (3\rho_a c_a^2/a)v_r$$

where γ is the ratio of specific heats (1.4 for air) and v_r is the radial velocity of the surface of the bubble.

The induced motion of the bubble surface sets up a radially outgoing wave in the water

$$\psi_s = (A/r)e^{ikr-i\omega t}; \quad \omega = kc_w$$

where ρ_w, c_w are the density and sound velocity of water, ρ_a, c_a the corresponding values for air, at mean pressure P. The excess pressure of incident and scattered wave just outside the bubble must equal the excess pressure inside; also the radial velocity must match at $r = a$:

$$p_0 + \left(\frac{i\omega\rho_w}{a}\right)A = p = -i\left(\frac{3\rho_a c_a^2}{\omega a}\right)v_r = i\left(\frac{3\rho_a c_a^2}{\omega a^3}\right)A + \left(\frac{3\rho_a c_a^2}{a^2 c_w}\right)A$$

where we have assumed that $ka \ll 1$ so that $e^{ika} \simeq 1$. Finally the velocity potential of the scattered wave is, to this approximation,

$$\psi_s \simeq \frac{(ap_0/i\omega\rho_w)}{(\omega_0/\omega)^2 - 1 - i(B/\omega)} \frac{e^{ikr-i\omega t}}{r}$$

where $\omega_0^2 = 3\rho_a c_a^2/a^2\rho_w$ is the resonance frequency of the air bubble and $B = 3\rho_a c_a^2/a\rho_w c_w$ is the radiation resistance load on the spherical surface. In practice there are other resistive losses, viscosity effects, and other losses because the changes of pressure in small bubbles are not strictly adiabatic and, therefore, not exactly reversible, so the last term in the denominator should be replaced by the somewhat larger $-i\delta(\omega)$, which depends on frequency in a more complicated way than does $-i(B/\omega)$. Function δ is, in general, somewhat larger than unity at $\omega = \omega_0$.

Therefore, if the velocity potential incident on the bubble is $\psi_i e^{-i\omega t}$, the scattered velocity potential "produced" by the incident wave is

$$\psi_s \simeq \frac{a\psi_i}{(\omega_0/\omega)^2 - 1 - i\delta(\omega)} \frac{e^{ikr-i\omega t}}{r} = \left(\frac{i\psi_i}{2k}\right)(1 - R_0)\frac{e^{ikr-i\omega t}}{r}$$

where R_0 is the reflection coefficient of Eq. (11.3.76). The scattering, absorption, and total cross sections of the bubble are then

$$Q_t \simeq \frac{4\pi a^2[1 + (\delta/ka)]}{\left[\left(\frac{\omega_0}{\omega}\right)^2 - 1\right]^2 + \delta^2}; \quad Q_s = \frac{4\pi a^2}{\left[\left(\frac{\omega_0}{\omega}\right)^2 - 1\right]^2 + \delta^2};$$

$$Q_a = \frac{4\pi a^2(\delta/ka)}{\left[\left(\frac{\omega_0}{\omega}\right)^2 - 1\right]^2 + \delta^2}$$

All of these exhibit the resonance at $\omega = \omega_0$ between the stiffness of the air in the bubble and the effective mass of the water just outside the bubble. Both ω_0 and δ are functions of bubble radius a.

Now let us consider the effect of a number of bubbles. Suppose we have N bubbles all inside a sphere of radius P. They vary in size and are random in space distribution, but we can assume that there is a density function $n(\mathbf{r},a)\,da$ which gives the average density, at the point specified by vector $\mathbf{r}$, of bubbles having radius between a and $a + da$, so that the total number of bubbles is

$$N = \int_0^{a_m} da \iiint n(\mathbf{r},a)\,dV \quad \text{and} \quad n(\mathbf{r}) = \int_0^{a_m} n(\mathbf{r},a)\,da$$

is the average density of bubbles of all size at the point $\mathbf{r}$. We assume that there is a maximum radius of bubble a_m, and that the volume integration, which is over the sphere of radius P, includes all the bubbles of any size. In order that the previous scattering calculations be valid, we must assume that the wavelength of sound in free water is considerably longer than $2\pi a_m$.

If $c_w/\omega \gg a_m$, we can write the velocity potential at some point

$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ as the sum of the incident wave ψ_i and the spherically symmetric scattered waves from the N bubbles, the nth bubble being at $\mathbf{r}_n$;

$$\psi(\mathbf{r}) = \psi_i(\mathbf{r}) + \sum_{n=1}^{N} \left(\frac{A_n}{r}\right) e^{ikR_n}$$

where $k = \omega/c_w$ and R_n is $|\mathbf{r} - \mathbf{r}_n|$ the distance from the nth bubble to the point P. The magnitude A_n is related to the velocity potential at the position of the nth bubble. It is not the actual potential ψ there, for ψ includes the effect of the scattered wave from this same nth bubble, and the "driving force" should be the incident wave plus the effects of the other bubbles. In other words

$$A_n = \frac{a\psi_n}{(\omega_0/\omega)^2 - 1 - i\delta}; \quad \text{where } \psi_n = \psi_i(\mathbf{R}_n) + \sum_{m \neq n} \left(\frac{A_m}{R}\right) e^{ikR_{mn}}$$

where $R_{mn} = |\mathbf{r}_m - \mathbf{r}_n|$. Here we have included the effect on bubble n of the scattering from all the other bubbles.

The equations requisite for further progress may then be written

$$\psi(\mathbf{r}) = \psi_i(\mathbf{r}) + \sum_{n=1}^{N} \psi_n \left(\frac{g_n}{R_n}\right) e^{ikR_n}$$

$$\psi_n = \psi(\mathbf{r}_n) - \left(\frac{g_n}{R_{nn}}\right) \psi_n e^{ikR_{mn}} = \psi_i(\mathbf{r}_n) + \sum_{m \neq n} \left(\frac{g_m}{R_{mn}}\right) \psi_m e^{ikR_{mn}}$$

$$g_n = \frac{a}{[\omega_0(a)/\omega]^2 - 1 - i\delta(\omega,a)}; \quad Q_{sn} = 4\pi|g_n|^2$$

The fields for specific configurations of bubbles are not particularly interesting, even if they could be obtained. What is more interesting is the average field, obtained by averaging over all configurations of bubbles inside the sphere of radius P. Such an average is indicated by the brackets $\langle \quad \rangle$. Averaging ψ_i produces no change, since it is the incident field, independent of the configuration of bubbles; consequently, $\langle\psi_i(\mathbf{r})\rangle = \psi_i(\mathbf{r})$. The configurational average for ψ is then

$$\langle\psi(\mathbf{r})\rangle = \psi_i(\mathbf{r}) + \iiint G(\mathbf{r}_n)\, \langle\psi_n(\mathbf{r}_n)\rangle \frac{e^{ikR_n}}{R_n}\, dV_n$$

where $\langle\psi_n(\mathbf{r})\rangle$ is the average external field acting on the nth particle, due to the incident field plus the scattering of all other bubbles except the nth. The quantity $G(\mathbf{r})$ is the average scattering density of the bubbles at r, averaged over all sizes there;

$$G(\mathbf{r}) = \int_0^P n(\mathbf{r},a) g(a)\, da$$

It is the average scattering strength times the average density of bubbles at $\mathbf{r}$.

The average of ψ_n, the external field acting on the nth particle, should be approximately equal to the average of ψ, the total field, at the same point. For only one scatterer, it should equal ψ_i; for a large number of bubbles, it should differ from $\langle\psi\rangle$ by terms of the order of $1/N$. Consequently, for N large we have

$$\langle\psi(\mathbf{r})\rangle \simeq \psi_i(\mathbf{r}) + \iiint G(\mathbf{r}_n)\,\langle\psi(\mathbf{r}_n)\rangle\,\frac{e^{ikR_n}}{R_n}\,dV_n$$

which is an integral equation for $\langle\psi(\mathbf{r})\rangle$ equivalent to the partial differential equation

$$\nabla^2\langle\psi\rangle + k^2\langle\psi\rangle = -4\pi G\langle\psi\rangle \quad \text{or} \quad \nabla^2\langle\psi\rangle + k_s^2(r)\langle\psi\rangle = 0$$

$$\text{where}\quad k_s(r) = k\left[1 + \left(\frac{4\pi G}{k^2}\right)\right] \simeq k + \frac{2\pi}{k}\int_0^P \frac{n(r,a)a\,da}{(\omega_0/\omega)^2 - 1 - i\delta(a)}$$

Thus the configurational average of the velocity potential is a solution of the wave equation for an index of refraction $1 + (4\pi G/k^2)$ which varies with the density of bubbles. Because of the δ term, this index of refraction is complex, producing attenuation as the wave traverses the bubble-filled region. The Green's function for this wave function is the solution of the equation

$$\nabla_r^2 K(\mathbf{r}|\mathbf{r}_0) + k_s^2(r)K(\mathbf{r}|\mathbf{r}_0) = -4\pi\delta(\mathbf{r} - \mathbf{r}_0)$$

which goes as $1/R$ when $R = |\mathbf{r} - \mathbf{r}_0|$ goes to zero and represents an outgoing wave for R large.

A similar configurational average of the square of the velocity potential, $|\psi(\mathbf{r})|^2$, can also be carried out. The integral equation corresponding to that for $\langle\psi(\mathbf{r})\rangle$ turns out to be

$$\langle|\psi(\mathbf{r})|^2\rangle \simeq |\langle\psi(\mathbf{r})\rangle|^2 + \frac{1}{4\pi}\iiint S(\mathbf{r}_0)\langle|\psi(\mathbf{r}_0)|^2\rangle|K(\mathbf{r}|\mathbf{r}_0)|^2\,dV_0$$

$$\text{where}\qquad S(\mathbf{r}) = 4\pi\int_0^P n(\mathbf{r},a)|g(a)|^2\,da = \int_0^P n(\mathbf{r},a)Q_s(a)\,da$$

is the total scattering cross section of bubbles per unit volume at the point specified by vector $\mathbf{r}$. This equation shows that the coherent part of the scattered wave is given by the square of the configurational average of $\psi(\mathbf{r})$, which is affected by the bubbles through their effect on the index of refraction $\sqrt{1 - 4\pi G(\mathbf{r})}$. In addition to the coherent wave $|\langle\psi(\mathbf{r})\rangle|^2$, there is also the incoherent scattering (given by the integral) proportional to the density S of scattering cross section, with the radiation from each element of volume proportional to $\langle|\psi(\mathbf{r})|^2\rangle$ and attenuated in its passage through the bubbly region, according to the square of the Green's function $K(\mathbf{r}|\mathbf{r}_0)$.

Spheroidal Coordinates. The prolate spheroidal coordinates were defined in Eq. (10.3.46). They may also be expressed in terms of the angle variables $\mu = \cosh^{-1} \xi$, $\vartheta = \cos^{-1} \eta$,

$$z = \tfrac{1}{2}a \cosh \mu \cos \vartheta; \quad \begin{matrix} x \\ y \end{matrix} = \tfrac{1}{2}a \sinh \mu \sin \vartheta \begin{matrix} \cos \\ \sin \end{matrix} \phi$$

$$h_\mu = h_\vartheta = \tfrac{1}{2}a \sqrt{\cosh^2 \mu - \cos^2 \vartheta} = \frac{a}{\sqrt{8}} \sqrt{\cosh 2\mu - \cos 2\vartheta};$$

$$h_\phi = \tfrac{1}{2}a \sinh \mu \sin \vartheta \tag{11.3.85}$$

$$\nabla^2\psi = \frac{(4/a^2)}{\cosh^2 \mu - \cos^2 \vartheta}\left[\frac{1}{\sinh \mu}\frac{\partial}{\partial \mu}\left(\sinh \mu \frac{\partial \psi}{\partial \mu}\right) + \frac{1}{\sin \vartheta}\frac{\partial}{\partial \vartheta}\left(\sin \vartheta \frac{\partial \psi}{\partial \vartheta}\right) + \frac{\sinh^2 \mu + \sin^2 \vartheta}{\sinh^2 \mu \sin^2 \vartheta}\frac{\partial^2 \psi}{\partial \phi^2}\right]$$

from which equations for the separated solutions may be found. The oblate coordinates are given in Eq. (10.3.55) and may be formally obtained from the prolate case by changing a into $-ia$ and, simultaneously ξ into $i\xi$, or else changing $\cosh \mu$ into $i \sinh \mu$ in Eqs. (11.3.85). Consequently, when solutions for the prolate case have been obtained, the oblate solutions involve a fairly simple extension to imaginary values of parameters and coordinates.

The factored solutions of the Helmholtz equation $(\nabla^2\psi + k^2\psi) = 0$ in prolate spheroidal coordinates are trigonometric functions of the axial coordinate ϕ [which are $\begin{matrix} \cos \\ \sin \end{matrix}(m\phi)$, $m = 0, 1, 2, \ldots$ when all of ϕ from 0 to 2π is allowed] and solutions of the following equations (in the coordinates ξ and η)

$$\begin{aligned} \frac{d}{d\xi}\left[(\xi^2 - 1)\frac{dJ}{d\xi}\right] - \left[A - h^2\xi^2 + \frac{m^2}{\xi^2 - 1}\right] J &= 0 \\ \frac{d}{d\eta}\left[(1 - \eta^2)\frac{dS}{d\eta}\right] + \left[A - h^2\eta^2 - \frac{m^2}{1 - \eta^2}\right] S &= 0 \end{aligned} \tag{11.3.86}$$

where $h = \frac{1}{2}ak = \frac{1}{2}(\omega a/c) = \pi a/\lambda$, where $\psi = \begin{matrix} \cos \\ \sin \end{matrix}(m\phi)S(\eta)J(\xi)$ and where A is the separation constant. We notice that the two equations are really the same, the one for S involving the behavior of the solution between the singular points -1, $+1$, and the one for J for the range from $+1$ to ∞. We can, therefore, study the solutions of the one equation for z, where η comprises the range -1 to $+1$, and ξ the range from $+1$ to ∞.

These solutions were touched on in Chap. 5 [see Eq. (5.3.95)]. We found that the regular singular points are at ± 1, with indices $\pm\frac{1}{2}m$, and that there is an irregular singular point at infinity. There we also showed that the angle function $S(\eta)$ could be expressed in terms of a series of associated Legendre functions

$$S(\eta) = \sum_n d_n P^m_{m+n}(\eta) = (1 - \eta^2)^{\frac{1}{2}m} \sum_n d_n T^m_n(\eta)$$

where the recursion formula relating successive coefficients is

$$\frac{n(n-1)h^2}{(2n+2m-1)(2n+2m-3)} d_{n-2} + h^2 \frac{(n+2m+1)(n+2m+2)}{(2n+2m+3)(2n+2m+5)} d_{n+2} + \left[h^2 \frac{2(n+m)(n+m+1) - m^2 - 1}{(2n+2m+3)(2n+2m-1)} + (n+m)(n+m+1) - A\right] d_n = 0$$

For most values of A, the solution S which is finite at $\eta = 1$ is infinite at $\eta = -1$. But the recursion equations may be solved by the continued-fraction techniques discussed in Sec. 5.2 (in connection with Mathieu equation), and a convergent series may be obtained for a discrete set of values of the separation constant A. There are two sets of finite solutions, one for even values of n, the other for odd values; we arrange each in order of increasing value of A and can finally write,

$$\begin{aligned} S_{ml}(h,\eta) &= (1 - \eta^2)^{\frac{1}{2}m} \sum_{n=0}^{\infty} d_{2n}(h|m,l) T^m_{2n}(\eta) \\ &\qquad\qquad l = m,\ m+2,\ m+4,\ \ldots \\ &= (1 - \eta^2)^{\frac{1}{2}m} \sum_{n=0}^{\infty} d_{2n+1}(h|m,l) T^m_{2n+1}(\eta) \\ &\qquad\qquad l = m+1,\ m+3,\ \ldots \end{aligned} \tag{11.3.87}$$

where $A_{m,l}(h) < A_{m,l+1}$. For a given m, the lowest value of A is labeled A_{mm} (for $l = m$), the next for $l = m + 1$, and so on. The corresponding functions S_{ml} are eigenfunctions, the set for a given m and different l being mutually orthogonal.

When $h \to 0$, the equation for S reduces to that for a single spherical harmonic, $P^m_l(\eta) = (1 - \eta^2)^{\frac{1}{2}m} T^m_{l-m}(\eta)$ and $A_{ml} \to l(l+1)$. We can normalize our function S so that its behavior near $\eta = 1$ is close to that of P, no matter what value h has. In other words, since we have $T^m_{l-m}(1) = [(l+m)!/2^m m!(l-m)!]$, we require that

$$\sum_n{}' \frac{(n+2m)!}{n!} d_n(h|m,l) = \frac{(l+m)!}{(l-m)!} \tag{11.3.88}$$

where the prime over the summation sign, indicating that only even values of n are included if $(l - m)$ is even and only odd values of n are included if $(l - m)$ is odd, enables us to include both even and odd functions in one formula.

Power series expansions for the d's and other constants for h small may also be obtained. For instance, setting

$$S_{00} = d_0P_0(\eta) + d_2P_2(\eta) + d_4P_4(\eta) + \cdots$$

and

$$A_{00} = a_2h^2 + a_4h^4 + \cdots$$

we have

$$\begin{aligned}&[\tfrac{1}{3}h^2d_0 + \tfrac{2}{15}h^2d_2 - Ad_0]P_0(\eta)\\ &\qquad + [6d_2 + \tfrac{2}{3}h^2d_0 + \tfrac{11}{21}h^2d_2 + \tfrac{4}{21}h^2d_4 - Ad_2]P_2(\eta)\\ &\qquad + [20d_4 + \tfrac{12}{35}h^2d_2 + \tfrac{39}{77}h^2d_4 + \tfrac{30}{143}h^2d_6 - Ad_4]P_4(\eta) + \cdots = 0\end{aligned}$$

Setting $d_2 = [h^2\alpha_2 + h^4\alpha_4 + \cdots]d_0$ and $d_4 = [h^4\beta_4 + \cdots]d_0$, and equating the coefficients of P_0, P_2, . . . to zero separately, we obtain

$$a_2 = \tfrac{1}{3};\quad a_4 = \tfrac{2}{15}\alpha_2 \quad\text{and}\quad \alpha_2 = -\tfrac{1}{9};\quad -\alpha_2a_2 + 6\alpha_4 + \tfrac{11}{21}\alpha_2 = 0$$
$$\text{or}\quad a_4 = -\tfrac{2}{135};\quad \alpha_4 = \tfrac{2}{567} \quad\text{and}\quad 20\beta_4 + \tfrac{12}{35}\alpha_2 = 0 \quad\text{or}\quad \beta_4 = \tfrac{1}{525}$$

Consequently

$$S_{00} \simeq d_0P_0 - d_0[\tfrac{1}{9}h^2 - \tfrac{2}{567}h^4]P_2 + (h^4/525)d_0P_4 + \cdots$$

Since this is to equal unity at $\eta = 1$, we must have

$$d_0\left[1 - \tfrac{1}{9}h^2 + \frac{11}{2025}h^4 - \cdots\right] = 1$$
$$\text{or}\quad d_0 = 1 + \tfrac{1}{9}h^2 + \frac{14}{2025}h^4 + \cdots$$

By this means approximate formulas can be built up for small values of h:

$$\begin{aligned}
S_{00} &\simeq \left(1 + \tfrac{1}{9}h^2 + \frac{14}{2025}h^4\right)P_0(\eta) - (\tfrac{1}{9}h^2 + \tfrac{5}{567}h^4)P_2(\eta) + \frac{h^4}{525}P_4(\eta)\\
A_{00} &\simeq \tfrac{1}{3}h^2 - \tfrac{2}{135}h^4;\quad \Lambda_{00} \simeq 2\left[1 + \tfrac{2}{9}h^2 + \frac{116}{2025}h^4\right]\\
S_{01} &\simeq \left(1 + \tfrac{1}{25}h^2 + \frac{144}{55125}h^4\right)P_1(\eta)\\
&\qquad - \left(\tfrac{1}{25}h^2 + \frac{29}{6525}h^4\right)P_3(\eta) + \frac{h^4}{2205}P_5(\eta)\\
A_{01} &\simeq 2 + \tfrac{3}{5}h^2 - \tfrac{6}{875}h^4;\quad \Lambda_{01} \simeq \tfrac{2}{3} + \tfrac{4}{75}h^2 + \frac{92}{18375}h^4\\
S_{11} &\simeq \left(1 + \tfrac{2}{25}h^2 + \frac{1193}{275625}h^4\right)P_1^1(\eta)\\
&\qquad - \left(\tfrac{1}{75}h^2 + \frac{16}{16875}h^4\right)P_3^1(\eta) + \frac{h^4}{11025}P_5^1(\eta)\\
A_{11} &\simeq 2 + \tfrac{1}{5}h^2 - \tfrac{4}{875}h^4;\quad \Lambda_{11} \simeq \tfrac{4}{3} + \tfrac{16}{75}h^2 + \frac{17104}{826875}h^4 \qquad (11.3.89)
\end{aligned}$$

where the quantities Λ_{ml} are the normalization constants;

$$\Lambda_{ml}(h) = {\sum_n}' [d_n(h|ml)]^2 \left(\frac{2}{2n+2m+1}\right) \frac{(n+2m)!}{n!} = \int_{-1}^{1} [S_{ml}]^2 \, d\eta$$

Second solutions may be obtained in terms of series of the Legendre functions of the second kind, with the series extended over Q_n^m's for negative n; however, second solutions are seldom needed for the angle functions.

The Radial Functions. The function

$$F_{ml}(\eta) = (1 - \eta^2)^{-\frac{1}{2}m} S_{ml}(h,\eta) = {\sum_n}' d_n T_n^m(\eta)$$

(where the prime on the summation sign still indicates inclusion of even n's when $l - m$ is even, odd n's when $l - m$ is odd) is a solution of the differential equation ($\eta = z$)

$$L_z(f) = (z^2 - 1)f'' + 2(m + 1)zf' + [m(m + 1) - A + h^2z^2]f = 0 \tag{11.3.90}$$

which has been discussed in Chap. 5 [see Eq. (5.3.95)]. We showed there that a solution of this equation was the integral representation

$$f(z) = \int_{-1}^{1} e^{ihzt}(1 - t^2)^m F(t) \, dt; \quad L_z(f) = 0$$

where F is a solution of $L_t(F) = 0$. Using this as an integral transformation, we can set F_{ml} inside the integral and come out with a solution of Eq. (11.3.90) which turns out to be useful for the ξ coordinate. Using one of the relations between T_n^m and the spherical Bessel functions j_n given at the end of this chapter, we have

$$\int_{-1}^{1} e^{ihzt}(1 - t^2)^{\frac{1}{2}m} S_{ml}(h,t) \, dt = \frac{2}{(hz)^m} {\sum_n}' i^n d_n \frac{(n+2m)!}{n!} j_{n+m}(hz)$$

and, consequently, the radial function of the first kind,

$$\begin{aligned} je_{ml}(h,z) &= \tfrac{1}{2} i^{m-l} h^m \frac{(l-m)!}{(l+m)!} (z^2 - 1)^{\frac{1}{2}m} \int_{-1}^{1} e^{ihzt}(1 - t^2)^{\frac{1}{2}m} S_{ml}(h,t) \, dt \\ &= \frac{(l-m)!}{(l+m)!} \left(\frac{z^2-1}{z^2}\right)^{\frac{1}{2}m} {\sum_n}' i^{n+m-l} d_n(h|ml) \frac{(n+2m)!}{n!} j_{n+m}(hz) \\ &\to (1/hz) \cos[hz - \tfrac{1}{2}\pi(l+1)]; \quad hz \to \infty \end{aligned} \tag{11.3.91}$$

is a solution of the first of Eqs. (11.3.86). We have normalized so that the function has the same asymptotic behavior as does the limiting solution for $h \to 0$ ($je_{ml} \to j_l$).

It is not obvious that je_{ml} has the same behavior at $z = 1$ as does $S_{ml}(h,z)$. We mentioned a few pages back that the indices for both singular points, ± 1, are $\pm\frac{1}{2}m$ and we see that the S functions are chosen (the value of A is picked) so that they contain *only* the $+\frac{1}{2}m$ index at *both* $+1$ and -1 (*i.e.* they are finite at both points). At $z = \pm 1$ the Bessel functions $j_{n+m}(hz)$ are analytic and, since the series converges, the series is analytic at $z = \pm 1$. The index at $z = \pm 1$ is thus given by the factor $(z^2 - 1)^{\frac{1}{2}m}$, which indicates an index $+\frac{1}{2}m$ as with the S function. Consequently $je_{ml}(h,z)$ must be proportional to $S_{ml}(h,z)$. Of course, the presence of the factor z^{-m} in the formula for je_{ml} seems to indicate a pole at $z = 0$, but it turns out that the functions $j_{n+m}(hz)$ go to zero at $z = 0$ as $(hz)^{n+m}$, so that for nonnegative values of n, j_{n+m}/z^m is analytic at $z = 0$.

The factor of proportionality between je_{ml} and S_{ml} may be found by relating them at $z = 0$. For even values of $(l - m)$ there is a term $i^{m-l}d_0(2m)!j_m(hz)$ in the primed sum for je; this term does not vanish at $z = 0$.

$$je_{ml} \to \frac{(l-m)!}{(l+m)!} i^{-l} d_0(2m)! \frac{h^m}{1 \cdot 3 \cdot 5 \cdots (2m+1)}; \qquad z \to 0,\ l - m = 0, 2, 4, 6, \ldots$$

But for $l - m$ even

$$S_{ml}(h,0) = \sum_{n=0}^{\infty} d_{2n} T_{2n}^m(0) = \sum_n d_{2n}(-1)^n \frac{1 \cdot 1 \cdot 3 \cdot 5 \cdots (2m+2n-1)}{2^n(n)!}$$

For odd values of $l - m$, only odd values of n appear in the summation and

$$je_{ml}(h,z) \to z \frac{(l-m)!}{(l+m)!} i^{-l+1} d_1(2m+1)! \frac{h^{m+1}}{1 \cdot 3 \cdot 5 \cdots (2m+3)}; \quad z \to 0$$

All the T_n^m's go to zero at $z = 0$, for odd values of n, consequently we have to match slopes. Since $dT_n^m/dz = T_{n-1}^{m+1}$, we have

$$S_{ml}(h,z) \to z \sum_{n=0}^{\infty} d_{2n+1} T_{2n}^{m+1}(0)$$

$$= z \sum_n d_{2n+1}(-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n+2m+1)}{2^n(n)!}$$

Consequently, we can set $S_{ml}(h,z) = \lambda_{ml}(h)je_{ml}(h,z)$, where

$$\begin{aligned}
\lambda_{ml}(h) &= \frac{i^l}{d_0h^m}\frac{2m+1}{2^m m!}\frac{(l+m)!}{(l-m)!}S_{ml}(h,0); \quad l = m, m+2, m+4, \ldots \\
&= \frac{i^l}{\pi h^m}\frac{(l+m)!}{(l-m)!}\frac{2^{2m+1}}{(2m)!}\Gamma(m+\tfrac{3}{2}) \cdot \\
&\quad \cdot \sum_n (-1)^n \frac{d_{2n}}{d_0}\frac{\Gamma(n+m+\frac{1}{2})}{(n)!}; \quad l-m = 0, 2, \ldots \\
&= \frac{i^{l-1}}{\pi h^{m+1}}\frac{(l+m)!}{(l-m)!}\frac{2^{2m+3}}{(2m+1)!}\Gamma(m+\tfrac{5}{2}) \cdot \\
&\quad \cdot \sum_n (-1)^n \frac{d_{2n+1}}{d_1}\frac{\Gamma(n+m+\frac{3}{2})}{(n)!}; \quad l-m = 1, 3, \ldots \\
&= \frac{i^{l-1}}{d_1h^{m+1}}\frac{(2m+2)(2m+3)}{2^m m!}\frac{(l+m)!}{(l-m)!}S'_{ml}(h,0) \\
&\qquad l = m+1, m+3, \ldots \qquad (11.3.92)
\end{aligned}$$

A second solution is needed for the radial coordinate ξ. This solution will have a singularity at $\xi = \pm 1$; a logarithmic singularity plus a term of the sort $(\xi^2 - 1)^{-\frac{1}{2}m}$. We may as well normalize it so that its asymptotic behavior is simply related to je. One way is to substitute for the $j_{n+m}(hz)$ functions in the series of (11.3.91) the corresponding spherical Neumann functions $n_{n+m}(hz)$, so that

$$\begin{aligned}
ne_{ml}(h,z) &= \frac{(l-m)!}{(l+m)!}\left(\frac{z^2-1}{z^2}\right)^{\frac{1}{2}m} {\sum_n}' i^{n+m-l}d_n(h|ml)\frac{(n+2m)!}{n!}n_{n+m}(hz) \\
&\to (1/hz)\sin[hz - \tfrac{1}{2}\pi(l+1)]; \quad hz \to \infty \qquad (11.3.93)
\end{aligned}$$

Unfortunately this series does not converge well for hz small; in fact, it is an asymptotic series, not being absolutely convergent for any finite values of hz. Other expansions will be discussed later. From the asymptotic behavior, however, we see that the Wronskian

$$\Delta(je_{ml},ne_{ml}) = 1/h(z^2 - 1) \qquad (11.3.94)$$

Green's Function and Other Expansions. By methods which should be habitual by now, we obtain the Green's function expansion in prolate spheroidal coordinates

$$\begin{aligned}
\frac{e^{ikR}}{R} &= 2ik \sum_{m,l} \frac{\epsilon_m}{\Lambda_{ml}} S_{ml}(h, \cos\vartheta_0)S_{ml}(h, \cos\vartheta)\cos[m(\phi - \phi_0)] \cdot \\
&\quad \cdot \begin{cases} je_{ml}(h, \cosh\mu_0)he_{ml}(h, \cosh\mu); & \mu > \mu_0 \\ je_{ml}(h, \cosh\mu)he_{ml}(h, \cosh\mu_0); & \mu < \mu_0 \end{cases} \qquad (11.3.95)
\end{aligned}$$

where $he_{ml} = je_{ml} + ine_{ml}$ is the radial function which reduces to the

Hankel function when $h \to 0$, and where

$$R^2 = \tfrac{1}{4}a^2[\cosh^2 \mu + \cosh^2 \mu_0 - \sin^2 \vartheta - \sin^2 \vartheta_0 \\ - 2 \cosh \mu \cosh \mu_0 \cos \vartheta \cos \vartheta_0 - 2 \sinh \mu \sinh \mu_0 \sin \vartheta \sin \vartheta_0 \cos(\phi - \phi_0)]$$

Likewise the plane wave expansion is

$$e^{i\mathbf{k}\cdot\mathbf{r}} = 2 \sum_{m,l} \frac{\epsilon_m i^l}{\Lambda_{ml}(h)} S_{ml}(h, \cos \vartheta_0) \cos[m(\phi - \phi_0)] \cdot \\ \cdot S_{ml}(h, \cos \vartheta) je_{ml}(h, \cosh \mu) \quad (11.3.96)$$

where
$$\mathbf{k} \cdot \mathbf{r} = k[z \cos \vartheta_0 + x \sin \vartheta_0 \cos \phi_0 + y \sin \vartheta_0 \sin \phi_0] \\ = h[\cosh \mu \cos \vartheta \cos \vartheta_0 + \sinh \mu \sin \vartheta \sin \vartheta_0 \cos(\phi - \phi_0)]$$

These expansions may be used, among other things, to compute various integral relations between the various functions. For instance, multiplying both sides of Eq. (11.3.96) by $\cos[m(\phi - \phi_0)]Se_{ml}(h, \cos \vartheta)$ and integrating over ϕ and $\cos \vartheta$ produces the following:

$$\int_0^{2\pi} \cos[m(\phi - \phi_0)]\, d\phi \int_0^{\pi} e^{i\mathbf{k}\cdot\mathbf{r}} S_{ml}(h, \cos \vartheta) \sin \vartheta\, d\vartheta \\ = 4\pi i^l S_m(h, \cos \vartheta_0) je_{ml}(h, \cosh \mu)$$

But expanding the factor $e^{ih \sinh \mu \sin \vartheta \sin \vartheta_0 \cos(\phi - \phi_0)}$ in $e^{i\mathbf{k}\cdot\mathbf{r}}$ gives us

$$4\pi i^l S_{ml}(h, \cos \vartheta_0) je_{ml}(h, \cosh \mu) \\ = 2\pi i^m \int_0^{\pi} e^{ih \cosh \mu \cos \vartheta \cos \vartheta_0} J_m(h \sinh \mu \sin \vartheta \sin \vartheta_0) S_{ml}(h, \cos \vartheta) \sin \vartheta\, d\vartheta$$

and, letting $\vartheta_0 \to 0$ and dividing out $\sin^m \vartheta_0$ on both sides, we finally have

$$je_{ml}(h, \cosh \mu) = \tfrac{1}{2} i^{m-l} h^m \sinh^m \mu \frac{(l + m)!}{(l - m)!} \int_0^{\pi} e^{ih \cosh \mu \cos \vartheta} \cdot \\ \cdot S_{ml}(h, \cos \vartheta) \sin^{m+1} \vartheta\, d\vartheta$$

which is to be compared with Eq. (11.3.91).

Another integral representation is obtained by multiplying both sides of Eq. (11.3.95) by $\cos[m(\phi - \phi_0)]S_{ml}(h, \cos \vartheta)$ and integrating over ϕ and ϑ:

$$S_{ml}(h, \cos \vartheta_0) je_{ml}(h, \cosh \mu_0) he_{ml}(h, \cosh \mu) \\ = \frac{1}{4\pi} \int_0^{2\pi} \cos[m(\phi - \phi_0)]\, d\phi \int_0^{\pi} h_0(kR) S_{ml}(h, \cos \vartheta) \sin \vartheta\, d\vartheta \quad (11.3.97)$$

If we let $\vartheta_0 \to \tfrac{1}{2}\pi$ and $\mu_0 \to 0$, the left-hand side of this equation approaches either

$$\sinh^m \mu_0 [i^{m-l} d_0 h^m/(2m + 1)] he_{ml}(h, \cosh \mu); \quad \text{if } l = m, m + 2, m + 4, \ldots$$

or

$$\sinh^m \mu_0 [i^{m+1-l} d_1 h^{m+1}/(2m + 2)(2m + 3)] \cos \vartheta_0 he_{ml}(h, \cosh \mu)$$

if $l = m + 1, m + 3, \ldots$.

Next we look at the right-hand side of this equation as $\vartheta_0 \to \frac{1}{2}\pi$ and $\mu_0 \to 0$. Taking first the case of $l = m, m + 2, m + 4, \ldots$, we can first let $\vartheta_0 \to \frac{1}{2}\pi$ and μ_0 be small, in which case, to the first order in the small quantity $\sinh \mu_0$,

$$kR \simeq h\sqrt{\rho^2 + \eta^2 - 2\rho\eta\cos(\phi - \phi_0)};\quad \text{where } \rho^2 = \cosh^2\mu - \sin^2\vartheta$$
and $$\eta^2 = [\sinh^2\mu \sin^2\vartheta/\cosh^2\mu - \sin^2\vartheta]\sinh^2\mu_0$$

Using the expansion formula on page 1323, we can expand the function $h_0(kR)$ into

$$\begin{aligned}
h_0(kR) &\simeq \frac{\sqrt{\pi/2h}}{[\rho^2 + \eta^2 - 2\rho\eta\cos(\phi - \phi_0)]^{\frac{1}{4}}} H_{\frac{1}{2}}[h\sqrt{\rho^2 + \eta^2 - 2\rho\eta\cos(\phi - \phi_0)}] \\
&\simeq \frac{e^{-\frac{1}{2}i\psi}}{1 - \frac{1}{2}(\eta/\rho)\cos(\phi - \phi_0)}\sqrt{\frac{\pi}{2h\rho}} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi_0)} J_m(h\eta) H_{m+\frac{1}{2}}(h\rho) \\
&\simeq \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi_0)} J_m(h\eta) h_m(h\rho)\left[1 + \frac{1}{2}\frac{\eta}{\rho} e^{-i(\phi-\phi_0)}\right]
\end{aligned}$$

When this expression is inserted in Eq. (11.3.97) and integration over ϕ is carried out, we can then let $\mu_0 \to 0$, dividing out by $\sinh^m \mu_0$ on both sides, finally obtaining a formula for he_{ml}, for $l = m, m + 2, m + 4, \ldots$,

$$\begin{aligned}
he_{ml}(h, \cosh\mu) = i^{l-m}(2m + 1)\left[\frac{\sinh^m\mu}{2^{m+2}m!d_0}\right] \cdot \\
\cdot \int_0^\pi \frac{1}{\rho^m}\left[h_m(h\rho) + (-1)^m h_{-m}(h\rho)\right. \\
\left. - (-1)^m\left(\frac{m}{h\rho}\right) h_{m+1}(h\rho)\right] S_{ml}(h, \cos\vartheta)\sin^{m+1}\vartheta\, d\vartheta
\end{aligned} \tag{11.3.98}$$

where

$$\rho = \cosh^2\mu - \sin^2\vartheta = \tfrac{1}{2}\cosh 2\mu + \tfrac{1}{2}\cos 2\vartheta = u^2 + v^2 + 2uv\cos 2\vartheta;$$
$$u = \tfrac{1}{2}e^{\mu};\quad v = \tfrac{1}{2}e^{-\mu}$$

From this, by one more transformation, we can obtain a series expansion for he. For instance, for $m = 0$ the quantity in brackets inside the integral becomes (see page 1574)

$$2h_0(h\rho) = \sum_{n=0}^{\infty} (-1)^n(2n + 1)P_n(\cos 2\vartheta) j_n(hv) h_n(hu)$$

so that, for $l = 0, 2, 4, \ldots$,

$$\begin{aligned}
he_{0l}(h, \cosh\mu) = \left(\frac{i^l}{2d_0}\right)\sum_{n=0}^{\infty}(2n + 1) j_n(\tfrac{1}{2}he^{-\mu}) h_n(\tfrac{1}{2}he^{\mu}) \cdot \\
\cdot \int_0^\pi P_n(\cos 2\vartheta) S_{0l}(h, \cos\vartheta)\sin\vartheta\, d\vartheta
\end{aligned}$$

from which, by use of Eq. (11.3.87), we can obtain a new expansion for he which converges satisfactorily at $\mu = 0$. Since

$$\begin{aligned}P_0(\cos 2\vartheta) &= P_0(\cos\vartheta)\\ P_1(\cos 2\vartheta) &= \tfrac{4}{3}P_2(\cos\vartheta) - \tfrac{1}{3}P_0(\cos\vartheta)\\ P_2(\cos 2\vartheta) &= \tfrac{48}{35}P_4(\cos\vartheta) - \tfrac{4}{7}P_2(\cos\vartheta) + \tfrac{1}{5}P_0(\cos\vartheta)\end{aligned}$$

and so on, we have

$$\begin{aligned}he_{0l}(h,\cosh\mu) = i^l \Bigg\{ & j_0(\tfrac{1}{2}he^{-\mu})h_0(\tfrac{1}{2}he^{\mu}) - \left[\tfrac{4}{5}\frac{d_2(h|0l)}{d_0(h|0l)} - 1\right] j_1(\tfrac{1}{2}he^{-\mu})h_1(\tfrac{1}{2}he^{\mu})\\ & + \left[\tfrac{16}{21}\frac{d_4(h|0l)}{d_0(h|0l)} - \tfrac{4}{7}\frac{d_2(h|0l)}{d_0(h|0l)} + 1\right] j_2(\tfrac{1}{2}he^{-\mu})h_2(\tfrac{1}{2}he^{\mu}) + \cdots \Bigg\}\end{aligned}$$

for $l = 0, 2, 4, \ldots$.

This series just barely converges for h small. For instance, for $he^{\mu} \ll 1$, by use of Eqs. (11.3.89), by using the series expansions for j and h and by use of the formula

$$e^{-\mu} + \tfrac{1}{3}e^{-1\mu} + \tfrac{1}{5}e^{-5\mu} + \cdots = \tanh^{-1}(e^{-\mu})$$

we obtain, for $he^{\mu} \ll 1$,

$$ne_{00}(h,\cosh\mu) \simeq -\frac{2}{h}\{\tfrac{1}{3}h^2\cosh\mu + (1 + \tfrac{11}{72}h^2 - \tfrac{1}{24}h^2\cosh 2\mu)\tanh^{-1}e^{-\mu}\}$$

For the case $m = 1$, the quantity in brackets in the integral of Eq. (11.3.98) is $2h_1(h\rho)$, and we finally obtain

$$\begin{aligned}he_{1l}(h,\cosh\mu) &= \frac{3i^{l-1}\sinh\mu}{hd_0}\sum_n (-1)^n(2n+3)j_{n+1}(\tfrac{1}{2}he^{-\mu})\,\cdot\\ &\qquad\cdot\, h_{n+1}(\tfrac{1}{2}he^{\mu})\int_0^\pi T_n^1(\cos 2\vartheta)S_{1l}(h,\cos\vartheta)\sin^2\vartheta\,d\vartheta\\ &= \frac{12}{h}i^{l-1}\sinh\mu\,\Bigg\{j_1(\tfrac{1}{2}he^{-\mu})h_1(\tfrac{1}{2}he^{\mu})\\ &\qquad - \left[\tfrac{24}{5}\frac{d_2}{d_0} - 3\right] j_2(\tfrac{1}{2}he^{-\mu})h_2(\tfrac{1}{2}he^{\mu})\\ &\qquad + \left[\tfrac{80}{11}\frac{d_4}{d_0} - 8\frac{d_2}{d_0} + 6\right] j_3(\tfrac{1}{2}he^{-\mu})h_3(\tfrac{1}{2}he^{\mu}) + \cdots\Bigg\}\end{aligned}$$

Carrying on in the same way as above we find, for $he^{\mu} \ll 1$,

$$\begin{aligned}ne_{11}(h,\cosh\mu) \simeq -\frac{3}{2h^2}\frac{1}{\sinh\mu}\{&[(1 - \tfrac{19}{75}h^2)\cosh\mu - \tfrac{1}{75}h^2\cosh^2\mu]\\ &- 2\sinh^2\mu[(1 + \tfrac{3}{50}h^2) + \tfrac{1}{10}h^2\cosh^2\mu]\tanh^{-1}e^{-\mu}\}\end{aligned}$$

Returning now to the case of $l = m + 1, m + 3, \ldots$, we find we must take the derivative of the right-hand side of (11.3.97) with respect to $\cos\vartheta_0$ before setting $\vartheta_0 = \frac{1}{2}\pi$. This means taking the derivative of $h_0(kR)$:

$$\left[\frac{dh_0(kR)}{d\cos\vartheta_0}\right]_{\vartheta_0=\frac{1}{2}\pi}$$

$$= \frac{h\cosh\mu\cosh\mu_0\cos\vartheta}{\sqrt{\rho^2+\eta^2-2\rho\eta\cos(\phi-\phi_0)}}\,h_1[h\sqrt{\rho^2+\eta^2-2\rho\eta\cos(\phi-\phi_0)}]$$

$$= h\cosh\mu\cosh\mu_0\cos\vartheta\sum_{m=-\infty}^{\infty}\frac{e^{im(\phi-\phi_0)}}{\rho}J_m(h\eta)h_{m+1}(h\rho)\left[1+\frac{3}{2}\frac{\eta}{\rho}e^{-i(\phi-\phi_0)}\right]$$

so that, setting $\sinh\mu_0\to 0$ and expanding J_m and h_{m+1}, we finally have, for $l = m+1, m+3, \ldots$

$$he_{ml}(h,\cosh\mu) = i^{l-m-1}\frac{(2m+2)(2m+3)}{2^{m+2}m!d_1(h|ml)}\cosh\mu\sinh^m\mu\cdot$$
$$\cdot\int_0^\pi\frac{1}{\rho^{m+1}}\left[h_{m+1}(h\rho)+(-1)^m h_{-m+1}(h\rho)\right.$$
$$\left.-(-1)^m\frac{3m}{\rho}h_{-m+2}(h\rho)\right]S_{ml}(h,\cos\vartheta)\cos\vartheta\sin^{m+1}\vartheta\,d\vartheta$$

For $m = 0$, the quantity in brackets is $2h_1(h\rho)$, so that

$$he_{0l}(h,\cosh\mu) = i^{l-1}\frac{6\cosh\mu}{hd_1}\sum_{n=0}^{\infty}(-1)^n(2n+3)j_{n+1}(\tfrac{1}{2}he^{-\mu})h_{n+1}(\tfrac{1}{2}he^{\mu})\cdot$$
$$\cdot\int_0^\pi T_n^1(\cos 2\vartheta)S_{01}(h,\cos\vartheta)\cos\vartheta\sin\vartheta\,d\vartheta$$
$$= i^{l-1}\frac{12\cosh\mu}{h}\left\{j_1(\tfrac{1}{2}he^{-\mu})h_1(\tfrac{1}{2}he^{\mu})-\left[\tfrac{12}{7}\frac{d^3}{d_1}+1\right]j_2(\tfrac{1}{2}he^{-\mu})h_2(\tfrac{1}{2}he^{\mu})\right.$$
$$\left.+\left[\tfrac{80}{33}\frac{d_5}{d_1}+\frac{d_3}{d_1}+2\right]j_3(\tfrac{1}{2}he^{-\mu})h_3(\tfrac{1}{2}he^{\mu})-\cdots\right\}$$

and, for $he^\mu \ll 1$, we have

$$ne_{0l}(h,\cosh\mu) \simeq -\frac{6}{h^2}\{[(1+\tfrac{17}{200}h^2)\cosh\mu-\tfrac{1}{40}h^2\cosh 3\mu]\tanh^{-1}e^{-\mu}$$
$$-\tfrac{1}{2}[(1+\tfrac{1}{12}h^2)-\tfrac{1}{20}h^2\cosh 2\mu]\}$$

These approximate formulas, together with the corresponding ones for the solutions of the first kind,

$$\begin{aligned} je_{00}(h,\cosh\mu) &\simeq 1-\tfrac{1}{18}h^2-\tfrac{1}{6}h^2\sinh^2\mu \\ je_{0l}(h,\cosh\mu) &\simeq (\tfrac{1}{3}h+\tfrac{1}{75}h^3)\cosh\mu \\ je_{0l}(h,\cosh\mu) &\simeq (\tfrac{1}{3}h+\tfrac{2}{75}h^3)\sinh\mu \end{aligned} \qquad (11.3.99)$$

for $he^\mu \ll 1$, suffice to calculate radiation and scattering at the long-wavelength limit.

Oblate Spheroidal Coordinates. The coordinates given in Eqs. (10.3.55) are, in terms of $\rho = \sinh^{-1}\xi$ and $\vartheta = \cos^{-1}\eta$,

$$
\begin{aligned}
&z = b\sinh\rho\cos\vartheta; \quad \begin{matrix} x \\ y \end{matrix} = b\cosh\rho\sin\vartheta \begin{matrix}\cos \\ \sin\end{matrix}\phi \\
&h_\rho = h_\vartheta = b\sqrt{\sinh^2\rho + \cos^2\vartheta} = (b/\sqrt{2})\sqrt{\cosh 2\rho + \cos 2\vartheta} \qquad (11.3.100) \\
&h_\phi = b\cosh\rho\sin\vartheta
\end{aligned}
$$

where $\rho = 0$ is a disk perpendicular to the z axis, of radius b.

These formulas can be obtained from the prolate spheroidal coordinate system of Eq. (11.3.81) if the prolate coordinate μ is changed into $\rho - \frac{1}{2}i\pi$ and if, at the same time, the parameter a is changed into $2ib$. Consequently all the formulas, obtained for the prolate wave functions, can be changed into the corresponding formulas for the oblate case, by changing μ into $\rho - \frac{1}{2}\pi i$ and $h = ka/2$ into $ig = ikb$ (with $k = \omega/c$ as usual).

For instance, the radiation from a vibrating disk is given in terms of these functions. Suppose the disk, of radius b, to be moving bodily back and forth, perpendicular to its plane (which can be taken parallel to the $x - y$ plane for convenience). In terms of the oblate spheroidal coordinates, the normal velocity at the disk surface $\rho = 0$ is $v_0e^{-i\omega t}$ for $0 \le \vartheta \le \frac{1}{2}\pi$ and is $-v_0e^{-i\omega t}$ for $\frac{1}{2}\pi \le \vartheta \le \pi$. The wave from it may be expressed in terms of the series

$$\psi = \sum_l A_l S_{0l}(ig, \cos\vartheta) he_{0l}(ig, -i\sinh\rho)$$

where

$$
\begin{aligned}
A_l\Lambda_{0l}G_l &= bv_0e^{-i\omega t}\int_{-1}^{1} S_{0l}(ig,\eta)\eta\, d\eta \\
&= \begin{cases} 0; & l = 0, 2, 4, \ldots \\ \frac{2}{3}bv_0e^{-i\omega t}d_1(ig|0l); & l = 1, 3, 5, \ldots \end{cases} \\
G_l &= [(d/d\rho)he_{0l}(ig, -i\sinh\rho)]_{\rho=0}
\end{aligned}
$$

The pressure back on the disk is then

$$p = \tfrac{2}{3}i\rho cgv_0e^{-i\omega t}\sum_l{}' \left[\frac{he_{0l}(ig, -i\sinh\mu)}{\dfrac{d}{d\rho}he_{0l}(ig, -i\sinh\rho)}\right]_{\rho=0} \cdot \left[\frac{d_1(ig|0l)}{\Lambda_{0l}}\right] S_{0l}(ig, \cos\vartheta)$$

(where the prime indicates that only odd values of l are included in the summation), and the ratio of the total force to the disk velocity, which is the acoustic impedance of the free disk, is

$$
\begin{aligned}
Z &= \tfrac{8}{9}\pi i\rho cgb^2 \sum_1{}' \frac{d_1(ig|0l)}{\Lambda_{0l}}\left[\frac{he_{0l}(ig,0)}{\dfrac{d}{d\rho}he_{0l}(ig, -ig\sinh\rho)}\right]_{\rho=0} \\
&\simeq -\tfrac{8}{3}i\rho cgb^2 + \frac{16}{27\pi}\rho cg^4b^2; \quad g = 2\pi h/\lambda \ll 1 \\
&= -i\omega(8b^3\rho/3) + (16\rho\omega^4b^6/27\pi c^2)
\end{aligned}
$$

which gives the effective mass of the air load, $8b^3\rho/3$, and the radiation resistance for long wavelengths λ.

Many other calculations can be carried out by use of these solutions, some of which will be mentioned in the problem section.

The other coordinate systems for which the wave equation separates, conical, parabolic, and so on, have not been investigated thoroughly, nor does there seem to be any physical application, using these coordinates, of sufficient importance to merit giving them space here. The general methods of calculation of the solutions, the plane wave expansions, the Green's functions, and the scattering formulas, would all be worked out by methods parallel to our work so far.

11.4 *Integral and Variational Techniques*

It must have long since become evident that the problems involving simple boundary conditions on simple, separable boundaries are both straightforward and, usually, not particularly interesting. For spheres and circular cylinders the solutions are well known, and even the scattering problem has answers which must suffer the contempt bred by familiarity. Of course, in the case of the more exotic separable surfaces, such as strips or disks or flattened cones, the fundamental eigenfunction solutions have elements of novelty which will merit continued interest, at least until adequate numerical tables have been published, but they also will eventually be "more of the same." Since few cases in "real life" are as regular as these, they are also less useful in practical cases.

The problems where the bounding surface is "not quite separable," on the other hand, are more difficult, more interesting and more nearly conform to actuality. The scattering of waves from an object which is not quite a sphere, the propagation of waves in a duct which is not everywhere uniform in cross section or boundary properties, are problems we have discussed in Sec. 9.3 and have already encountered in this chapter (see pages 1448 and 1490). They cannot usually be solved exactly, but it is important that approximate solutions be found which are both compact in form and have a minimum error.

For such problems the use of Green's functions, of integral equation techniques, and of variational principles offer methods of utility and power. It is our purpose, in this section, to discuss a few typical problems of this sort and to illustrate how the methods discussed in Chaps. 7, 8, and 9 may be applied to them. Some of these examples involve the penetration of waves through holes in an otherwise regular surface, others involve irregularities in boundary conditions, in only a few of them can an exact solution be obtained in usable form.

A number of such problems have already been discussed, as examples in Secs. 9.3 and 9.4 and also earlier in this chapter. For example, we treated the case of a pierced diaphragm in an otherwise regular duct. Here the procedure was to set up solutions on each side of the duct which satisfy the conditions far from the diaphragm (outgoing transmitted wave on one side, incident and reflected wave on the other) and then to require that the two solutions join smoothly in value and gradient over the opening in the diaphragm. This gave rise to an infinite set of equations in an infinite set of unknowns, the coefficients of the eigenfunction expansions on each side of the diaphragm, which could not be solved exactly. What was done was to *assume an approximate form* of the solution right in the opening (in most cases the solution of the Laplace equation for the corresponding boundary) which provided approximate relations between the unknown expansion coefficients and, to the approximation assumed, reduced the number of unknowns from infinity to one or two. The values of these could then be determined by joining the solution at one or two points in the opening; if the chosen approximate form is nearly correct, the solutions will nearly join over the rest of the opening.

But this technique, though conceptually simple and fairly easy to apply, has several drawbacks. There is no simple way to improve the solution nor, indeed, to find out how good it is. Let us, therefore, take up a problem of this sort again, to see what contribution the techniques of Chaps. 7, 8, and 9 can make.

Iris Diaphragm in Pipe. The case we shall discuss first is that of the transmission of sound in a rigid pipe, of circular cross section of radius a, partially closed by a rigid diaphragm at $z = 0$, having a concentric hole in it of radius b. We have already treated a similar case (see page 1443) of a slit in a diaphragm in a rectangular pipe. We set the solution for $z > 0$ to be waves in the positive direction (for the upper modes, the z factor was a decreasing exponential) and the solution for $z < 0$ to be an incident plus a reflected wave (plus the higher modes, diminishing as $z \to -\infty$). The two solutions were then adjusted to have zero gradient at the rigid part of the diaphragm and to have equal gradients in the open hole or slit. For an exact solution we would then have had to adjust things so the value of the velocity potential was also continuous at the hole, but this was too difficult to do exactly, so an approximate form was assumed for the z gradient in the hole, having a single adjustable constant, which was then set so as to ensure continuity in value of ψ along the axis of the tube. If our choice of the gradient function in the hole is a good one, the discontinuity in ψ over the rest of the hole should be small; but it is rather difficult to check back to see how small this discontinuity actually is.

Let us restate our procedure in terms of Green's functions; the results will help us see how improvements can be made. The Green's function which has zero normal gradient at the surface of the tube and also along the diaphragm surface $z = 0$, and which has outgoing waves for $z \to \infty$, will be called $G_k^+(\mathbf{r}|\mathbf{r}_0)$ and the one suitable for $z < 0$ will be called $G_k^-(\mathbf{r}|\mathbf{r}_0)$. For the circular tube the usual procedures show that

$$G_k^+(r,\phi,z|r_0,\phi_0,z_0) = \frac{4i}{a^2}\sum_{m,n} \epsilon_m \cos[m(\phi - \phi_0)] \cdot$$
$$\cdot \frac{J_m(\pi\alpha_{mn}r/a)J_m(\pi\alpha_{mn}r_0/a)}{k_{mn}[1 - (m/\pi\alpha_{mn})^2]J_m^2(\pi\alpha_{mn})} \begin{cases} \cos(k_{mn}z_0)e^{ik_{mn}z}; & z > z_0 \\ \cos(k_{mn}z)e^{ik_{mn}z_0}; & z < z_0 \end{cases} \tag{11.4.1}$$

where α_{mn} is the nth root of the equation $[dJ_m(\pi\alpha)/d\alpha] = 0$ and $k_{mn}^2 = k^2 - (\pi\alpha_{mn}/a)^2$. For m, n large, $k_{mn} = i\sqrt{(\pi\alpha_{mn}/a)^2 - k^2} \to i\pi n/a$. The function G_k^- is obtained by reversing the signs of z and z_0 in the above.

It is not difficult to see that the whole problem is solved once we know the distribution of air flow through the opening at $z = 0$, $r < b$, which we may call $u_z^0(r)$. For then the normal gradient of the velocity potential ψ is just $-u_z^0$ in the open area, zero elsewhere on the boundary, and from Green's theorem [see Eq. (7.2.10)] we have that the velocity potential inside the tube is (omitting the time factor $e^{-i\omega t}$)

$$\psi = \begin{cases} \dfrac{1}{4\pi}\displaystyle\int_0^{2\pi} d\phi_0 \int_0^b r_0\,dr_0\, u_z^0(r_0)G_k^+(r,\phi,z|r_0,\phi_0,0); & z > 0 \\ 2A\cos(kz) - \dfrac{1}{4\pi}\displaystyle\int_0^{2\pi} d\phi_0 \int_0^b r_0\,dr_0\, u_z^0(r_0)G_k^-(r,\phi,z|r_0,\phi_0,z_0); & z < 0 \end{cases} \tag{11.4.2}$$

where $k = \omega/c = 2\pi/\lambda$ as usual. On the side $z < 0$, there is an incident plane wave of amplitude A, combined with a similar reflected wave adjusted in phase so that, for this part, the gradient at $z = 0$ is zero. This would be the solution if there were no hole in the diaphragm; the modification of the reflected wave produced by the hole comes in the integral of $G^-u_z^0$, which is also a wave going to $-\infty$. On the $z > 0$ side, there is only a transmitted wave, the integral of $G^+u_z^0$, a wave going to $+\infty$.

When the wave vector k is smaller than the cutoff value $\pi\alpha_{01}/a$, the only wave reaching $z \to \infty$ is the plane wave for $m = n = 0$ in the Green's function, and the limiting form for the transmitted wave is

$$\psi \to (i\tau A/\pi k a^2)e^{ikz}; \quad z \to \infty; \quad k < \pi\alpha_{01}/a$$

where
$$\tau = \frac{2\pi}{A}\int_0^b u_z^0(r_0)r_0\,dr_0 \tag{11.4.3}$$

is the amplitude of flow of air through the hole in the diaphragm, relative to the incident amplitude. The ratio of the square of the amplitude of ψ for $z \to \infty$ to the square of the amplitude A of the incident wave, $T = |\tau/\pi k a^2|^2$, is the *transmission factor* for the opening, the ratio of transmitted to incident intensity. This transmission (and reflection) can be computed easily from u_z^0.

In Sec. 11.3 we assumed that the form of u_z^0 was approximately equal to that for steady flow through the opening (the solution for $k = 0$) and made our calculations accordingly, adjusting the amplitude of u_z^0 so as to get approximate continuity of ψ across $z = 0$. Here we wish to discover an integral equation for u_z^0, which may be solved by successive approximations and from which we can estimate the degree of accuracy of the approximations. Actually, of course, for many problems, it is not necessary to know the exact details of u_z^0 or of ψ near the diaphragm, if we can obtain a reasonably accurate value for τ, defined in Eq. (11.4.3); without computing u_z^0 directly, we can compute the transmission and reflection factors

$$T = |\tau/\pi k a^2|^2; \quad R = 1 - T \tag{11.4.4}$$

giving the fraction of the power incident on the diaphragm which is transmitted through the hole and the fraction reflected, respectively. These quantities are often all that are needed.

The integral equation for u_z^0 is obtained by requiring that there be continuity of ψ across the hole; in other words, that the ψ obtained from the $z > 0$ expression of Eq. (11.4.2) be equal to that obtained from the $z < 0$ expression at $z = 0$. Since $G^+ \to G^-$ as $|z| \to 0$, we have for our integral equation

$$A = \frac{1}{4\pi} \int_0^{2\pi} d\phi_0 \int_0^b r_0\, dr_0\, u_z^0(r_0) G_k(r,\phi,0|r_0,\phi_0,0); \quad r < b$$

If this could be solved exactly we would, as we have shown, have the whole problem solved.

A Variational Principle. The equation is not too easy to solve directly, but it is in a suitable form for obtaining a variational principle for u_z^0 and τ (see Sec. 9.4). Multiplying both sides by $u_z^0(r)$ and integrating over ϕ and r, we have

$$\frac{1}{4\pi} \iiiint u_z^0(r) u_z^0(r_0) G_k\, d\phi\, d\phi_0\, r r_0\, dr\, dr_0 = A \iint u_z^0\, d\phi\, r\, dr = A^2 \tau$$

or $$A^2\tau = 2A \iint u_z^0\, d\phi\, r\, dr - \frac{1}{4\pi} \iiiint u_z^0(r) u_z^0(r_0) G_k\, d\phi\, d\phi_0\, r r_0\, dr\, dr_0 \tag{11.4.5}$$

If now we vary u_z^0 by small amounts δu away from its correct form, we have

$$2 \iint d\phi\, r\, dr\, \delta u \left[A - \frac{1}{4\pi} \iint u_z^0(r_0) G_k\, d\phi\, r_0\, dr_0 \right] = A^2 \delta\tau$$

which is just equivalent to the equation for A if $\delta\tau = 0$. Consequently the correct form of u_z^0 is the one which gives a stationary value to the integral of Eq. (11.4.5). We first vary simply the amplitude of u_z^0, setting $u_z^0 = B\chi(r)$, where B is to be varied, to find an extremum;

$$\delta\left[2AB\iint \chi\,d\phi\,r\,dr - \frac{B^2}{4\pi}\iiiint \chi(r)\chi(r_0)G_k\,d\phi\,d\varphi_0\,rr_0\,dr\,dr_0\right] = 0$$

The best value of B turns out to be

$$B = \frac{4\pi A\iint\chi\,d\phi\,r\,dr}{\iiiint\chi(r)\chi(r_0)G_k\,d\phi\,d\phi_0\,rr_0\,dr\,dr_0}$$

so that A, the amplitude of the incident wave, can be canceled out, leaving a variational principle relating the flow amplitude τ to the flow distribution χ for unit incident wave amplitude [see Eq. (9.4.8)],

$$[\tau] = \frac{4\pi\iint\chi(r)\,d\phi\,r\,dr\iint\chi(r_0)\,d\phi_0\,r_0\,dr_0}{\iiiint\chi(r)\chi(r_0)G_k(r,\phi,0|r_0,\phi_0,0)\,d\phi_0\,d\phi\,rr_0\,dr_0\,dr} \qquad (11.4.6)$$

The expression on the right has an extremal for the correct shape of χ, the extremal value is τ. Put another way; even if χ is not exactly the correct form, the computed ratio of integrals will differ from the correct value of τ to the second order in the small variation of χ. Furthermore Eq. (11.4.2) may be manipulated, as per Chap. 9, to obtain an iterated expression for χ, which must be more nearly equal to the correct form than the first try. We note that both numerator and denominator involve two χ's, consequently the amplitude of the trial function used for χ need not be fixed till later; it will not affect the value of τ. If we wish to compute only the transmission and reflection parameters, we never need to fix the amplitude of χ.

Calculating the Transmission Factor. Suppose we try the form used in Sec. 11.3 for the iris diaphragm in a circular pipe,

$$\chi \simeq C[1 - (r/b)^2]^{-\frac{1}{2}}$$

which is a fairly good approximation for $b < \frac{1}{2}a$. To compute the integrals of Eq. (11.4.6), we need the value of the general integral

$$\begin{aligned}\int_0^b \left[1 - \left(\frac{r}{b}\right)^2\right]^{n-\frac{1}{2}} J_0(\beta r)r\,dr &= b^2\int_0^{\frac{1}{2}\pi}\sin\vartheta\cos^{2n}\vartheta J_0(\beta b\sin\vartheta)\,d\vartheta \\ &= \frac{2^{n-\frac{1}{2}}b^2\Gamma(n+\frac{1}{2})}{(\beta b)^{n+\frac{1}{2}}}J_{n+\frac{1}{2}}(\beta b) = \frac{1\cdot 1\cdot 3\cdot 5\cdots(2n-1)}{(\beta b)^n}b^2 j_n(\beta b)\end{aligned}$$

Using this result for various values of n and β, we have

$$\iint\chi\,d\phi\,r\,dr = 2\pi Cb^2 j_0(0) = 2\pi b^2 C$$

$$\iiiint \chi(r)\chi(r_0)G_k \, d\phi \, d\phi_0 \, rr_0 \, dr \, dr_0$$

$$= \left(\frac{16\pi^2 C^2 b^4}{ka^2}\right) \left\{ i + \sum_{n=1}^{\infty} \frac{k}{K_n} \left[\frac{j_0(\pi\alpha_{0n}b/a)}{J_0(\pi\alpha_{0n})} \right]^2 \right\}$$

where $K_n^2 = (\pi\alpha_{0n}/a)^2 - k^2$ ($k < \pi\alpha_{01}/a$ and $b < \frac{1}{2}a$). Therefore an approximate value of the transmission factor is

$$T \simeq \frac{1}{1 + (\Sigma)^2}; \quad (\Sigma) = \sum_{n=1}^{\infty} \frac{(ka^3/b^2)}{\sqrt{1 - (ka/\pi\alpha_{0n})^2}} \frac{\sin^2(\pi\alpha_{0n}b/a)}{(\pi\alpha_{0n})^3 J_0^2(\pi\alpha_{0n})} \tag{11.4.7}$$

The series may be improved in convergence by remembering that $J_0^2(\pi\alpha_{0n}) \to 2/\pi^2\alpha_{0n}$ as $n \to \infty$, so that the comparison series to add and subtract is

$$S = \left(\frac{ka^3}{2\pi b^2}\right) \sum_{n=1}^{\infty} \left(\frac{1}{n^2}\right) \sin^2\left(\frac{\pi n b}{a}\right) = \left(\frac{ka^3}{4\pi b^2}\right) \sum_{n=1}^{\infty} \frac{1}{n^2} \left[1 - \cos\left(\frac{2\pi n b}{a}\right)\right]$$

$$= \left(\frac{ka^3}{4\pi b^2}\right) \mathrm{Re}[I(e^{2\pi i b/a})]$$

where
$$I(x) = -\int_x^1 \frac{\ln(1-u)}{u} \, du = \ln x \ln(1-x) - \int_1^x \frac{\ln t}{t-1} \, dt$$
$$= \ln x \ln(1-x) - \ln x - \tfrac{1}{4}(\ln x)^2$$
$$- [\text{a series in odd powers of } \ln x]; \quad |\ln x| < 2\pi$$

where we have obtained the last few relations by changing variables of integration and integrating by parts. Consequently,

$$S = \pi\left(\frac{ka^2}{4b}\right)\left[1 - \left(\frac{b}{a}\right)\right]$$

and the sum (Σ) in Eq. (11.4.7) may be written

$$(\Sigma) = (\pi ka^2/4b) - \tfrac{1}{4}\pi ka$$
$$+ \left(\frac{ka^3}{b^2}\right) \sum_{n=1}^{\infty} \frac{2\pi n^2 - (\pi\alpha_{0n})^3 \sqrt{1 - (ka/\pi\alpha_{0n})^2} \, J_0^2(\pi\alpha_{0n})}{2\pi n^2 (\pi\alpha_{0n})^3 \sqrt{1 - (ka/\pi\alpha_{0n})^2} \, J_0^2(\pi\alpha_{0n})} \sin^2\left(\pi\alpha_{0n}\frac{b}{a}\right) \tag{11.4.8}$$

where the modified sum converges quite rapidly and may be considered as a correction to the first two terms.

When b/a is small (iris aperture small compared to tube cross section), the first term in the expression for (Σ) is the largest, and the transmission factor

$$T \to \frac{1}{1 + (\pi ka^2/4b)^2} \to 16(ka)^{-2}(b^2/\pi^2 a^2); \quad b \ll ka^2$$

The fraction of the incident wave which gets through a very small hole is proportional to the ratio of area of hole to area of tube cross section, but inversely proportional to the square of the frequency (unless the frequency is too small). On the other hand, if the frequency goes to zero ($ka \ll b/a$), then T approaches unity; if the frequency is low enough (*i.e.*, if there is time enough in each cycle), the sound will get through the aperture with practically no hindrance.

We cannot ask for the high-frequency limit of this formula, for we assumed the frequency to be less than the first cross mode, so that only the lowest mode is transmitted. Nor should we expect that the formula holds very well for b nearly equal to a (aperture radius nearly equal pipe radius), for the trial function we have used for χ is not a good approximation for this case.

If we had used the techniques of Sec. 11.3, we would have obtained a very similar expression for T, differing only by the fact that, with the earlier technique, we required continuity just along the cylinder axis, whereas in the present case we have required continuity on the average over the hole, so to speak.

A better approximation, one which should give satisfactory results even for $b \to a$ (though not for $ka \to \infty$), would be obtained by setting

$$\chi \simeq \frac{C}{\sqrt{1 - (r/b)^2}} + \gamma C \sqrt{1 - \left(\frac{r}{b}\right)^2}$$

where we vary γ to obtain a stationary value of the expression of Eq. (11.4.6).

Setting $(d/d\gamma)(1/\tau) = 0$ gives the simplest formula; the value of γ for stationary τ is

$$\gamma = -\frac{b}{a}\left\{\sum_{n=1}^{\infty} \frac{j_0(\pi\alpha_{0n}b/a)j_2(\pi\alpha_{0n}b/a)}{(\pi\alpha_{0n})\sqrt{1 - (ka/\pi\alpha_{0n})^2}\,J_0^2(\pi\alpha_{0n})}\right\} \cdot \left\{\sum_{n=1}^{\infty} \frac{j_1(\pi\alpha_{0n}b/a)j_2(\pi\alpha_{0n}b/a)}{(\pi\alpha_{0n})^2\sqrt{1 - (ka/\pi\alpha_{0n})^2}\,J_0^2(\pi\alpha_{0n})}\right\}^{-1} \tag{11.4.9}$$

and the best value of the transmission constant is

$$T \simeq \left\{1 + \frac{(ka)^2}{(1 + \frac{1}{3}\gamma)^4}\left[\sum_{n=1}^{\infty} \frac{j_0(\pi\alpha_{0n}b/a) + (\gamma a/\pi\alpha_{0n}b)j_1(\pi\alpha_{0n}b/a)}{(\pi\alpha_{0n})\sqrt{1 - (ka/\pi\alpha_{0n})^2}\,J_0^2(\pi\alpha_{0n})}\right]^2\right\}^{-1} \tag{11.4.10}$$

where γ is given by Eq. (11.4.9). These series may also be summed by methods similar to those resulting in Eq. (11.4.8).

Hole in Infinite Plane. The same general method may be applied to the limiting case when a goes to infinity, when the series expression for G becomes the integral representation,

$$G_k^+ = 2 \sum_{m=0}^{\infty} \epsilon_m \cos[m(\phi - \phi_0)] \int_0^\infty \frac{J_m(ur)J_m(ur_0)}{\sqrt{u^2 - k^2}} \cdot$$

$$\cdot\, u\, du \begin{cases} \cosh(z_0 \sqrt{u^2 - k^2}) \exp(-z \sqrt{u^2 - k^2}); & z > z_0 \\ \cosh(z \sqrt{u^2 - k^2}) \exp(-z_0 \sqrt{u^2 - k^2}); & z < z_0 \end{cases} \quad (11.4.11)$$

Setting $\chi = C/\sqrt{1 - (r/b)^2}$ we have $\iint \chi\, d\phi\, r\, dr = 2\pi b^2 C$ as before, but

$$\iiiint \chi(r)\chi(r_0)G_k\, d\phi\, d\phi_0\, rr_0\, dr\, dr_0 = 8\pi^2 i b^4 C^2 \int_0^\infty \frac{u\, du}{\sqrt{k^2 - u^2}} j_0^2(ub)$$
$$= \frac{8\pi^2 b^2 C^2}{k} \left\{ i \int_0^1 \frac{\sin^2(kbv)}{v\sqrt{1 - v^2}} dv + \int_1^\infty \frac{\sin^2(kbv)}{v\sqrt{v^2 - 1}} dv \right\}$$

These two integrals may be evaluated numerically for any value of kb. For kb small, we can expand them

$$\int_0^1 \frac{\sin^2(kbv)}{v\sqrt{1 - v^2}} dv \simeq k^2 b^2 \int_0^1 \frac{v\, dv}{\sqrt{1 - v^2}} - \tfrac{1}{3}k^4 b^4 \int_0^1 \frac{v^3\, dv}{\sqrt{1 - v^2}}$$
$$= k^2 b^2 - \tfrac{2}{9} k^4 b^4$$

$$\int_1^\infty \frac{\sin^2(kbv)}{v\sqrt{v^2 - 1}} dv \simeq \int_1^\infty \frac{\sin^2(kbv)}{v^2} dv + (kb)^2 \int_1^\infty \left[\frac{v}{\sqrt{v^2 - 1}} - 1 \right] dv$$
$$\simeq \tfrac{1}{2}\pi kb - k^2 b^2 + k^2 b^2 = \tfrac{1}{2}\pi kb$$

and obtain an approximate value for the net flow through the hole,

$$\tau \simeq 2\pi k b^2 \{ i(k^2 b^2 - \tfrac{2}{9} k^4 b^4) + \tfrac{1}{2}\pi kb \}^{-1}$$

Instead of the transmission factor T, what is needed here is the *effective area* of the opening, the ratio between the total power passing through the hole to the intensity of the incident wave. This may be obtained in terms of τ by the following useful trick. The intensity of the incident wave $Ae^{ikz - i\omega t} = \psi_i$ is

$$I_i = -\operatorname{Re}\left\{ \tfrac{1}{2}\rho \frac{\partial \bar{\psi}_i}{\partial t} \frac{\partial \psi_i}{\partial z} \right\} = \tfrac{1}{2}\rho c k^2 |A|^2$$

whereas the total power transmitted through the hole is the integral of the intensity for the full solution for $z > 0$ over the area of the hole,

$$P_t = -\operatorname{Re}\left\{ \tfrac{1}{2} i\rho c k \iint \bar{\psi}^+ \left(\frac{\partial \psi^+}{\partial z} \right) d\phi_0\, r_0\, dr_0 \right\}$$

The *effective area* of the hole for transmitting the wave is, therefore,

$$Q_t = \operatorname{Re}\left\{ \frac{i}{k|A|^2} \iint \bar{\psi}^+(r_0) u_z^0(r_0)\, d\varphi_0\, r_0\, dr_0 \right\}$$

But from Eq. (11.4.2) or (11.4.5) we realize that, for continuity, the value of ψ^+ in the hole is just equal to the value of the incident wave there, that is, $\psi^+(r_0) = A$. Therefore,

$$Q_t = \operatorname{Im}\left\{\frac{-1}{kA}\iint u_z^0(r_0)\,d\phi_0\,r_0\,dr_0\right\} = \operatorname{Im}\left[-\frac{\tau}{k}\right] \tag{11.4.12}$$

where τ is the flow integral defined in Eq. (11.4.3) (since τ has dimensions of length, Q has dimensions of area).

Using the expression for τ obtained from the variational principle, we find the effective area for transmission of a normally incident plane wave through a circular hole in an infinite plane is

$$Q_t \simeq \pi b^2\left[\frac{8}{\pi^2} - \frac{16}{9\pi^2}k^2b^2\right]; \quad \text{hole, Neumann conditions} \tag{11.4.13}$$

an effective opening, for long wavelengths, a little less than the actual area of opening. The first term of this may be checked by using previous calculations. We can consider the air in the circular opening as an effective mass $\frac{1}{2}\pi^2b^3\rho$ [see Eq. (10.3.60)], acted on by an incident force $2\pi b^2P_0$, where P_0 is the pressure amplitude of the incident wave and $P_0^2/\rho c$ its intensity. The velocity amplitude of the air is then $(4P_0/\pi kb\rho c)$ and, since the radiation resistance for radiating a wave to the right is approximately $\frac{1}{2}\rho ck^2b^2$ [see discussion prior to Eq. (11.3.36)], the power penetrating through the hole is $\frac{1}{2}\rho ck^2b^2(4P_0/\pi kb\rho c)^2(\pi b^2) = (8\pi b^2P_0^2/\pi^2\rho c)$ and the ratio between this power and the incident intensity is $8\pi b^2/\pi^2$, which is the first term in Eq. (11.4.13). A more accurate expression can be obtained by using two terms in the variational expression for u_z^0. Incidentally $2Q_t$ is also the scattering cross section for waves incident normally on a disk of radius b, where ψ goes to zero on the disk, according to Babinet's principle (see page 1431).

The calculation of the penetration of waves through a round hole in a diaphragm, with homogeneous Dirichlet conditions on the boundary, can be calculated in the same manner. The Green's function is modified so it goes to zero at $z = 0$, and the variational expression for τ is

$$[\tau] = \frac{4\pi k^2[\iint\psi_0(r_0)\,d\phi_0\,r_0\,dr_0]^2}{\iiiint \psi_0(r_0)\psi_0(r)\left[\dfrac{\partial^2}{\partial z\,\partial z_0}G_k(\mathbf{r}|\mathbf{r}_0)\right]_{z=0} d\phi\,d\phi_0\,rr_0\,dr\,dr_0} \tag{11.4.14}$$

where now the value ψ_0 of the velocity potential in the opening, instead of its slope, is used and is balanced by the use of the gradients of the Green's function. One assumes a form $C\sqrt{1-(r/b)^2}$ for ψ, and adds a term in $[1-(r/b)^2]^{\frac{3}{2}}$ for better accuracy. Using the single term for ψ, one computes a cross section

$$\begin{aligned} Q_t \simeq \pi b^2(8/27\pi^2)(kb)^4[1 &+ 0.32k^2b^2 \\ &+ 0.049k^4b^4 + \cdots]; \quad \text{hole, Dirichlet conditions} \end{aligned} \tag{11.4.15}$$

which is a quite accurate result for $kb < 1$. Indeed a numerical calculation of the integrals in Eq. (11.4.14) shows that the expression for Q computed using a single term for ψ_0 yields reasonably accurate results out to $kb \simeq 3$ and the two-term form for ψ_0 gives a good value for Q out to $kb \simeq 5$. These results, by Babinet's principle, also hold for the scattering of sound waves normally incident on a rigid disk (disk, Neumann conditions).

Reflection in Lined Duct. We turn now to other wave problems soluble by integral techniques. One such is the transmission of sound inside ducts lined with sound-absorbing material. Suppose that the duct is rectangular in cross section, of sides a and b (axis along z axis) and that the inner surface for $z < 0$ is perfectly rigid. For $z > 0$ the sides $x = 0$, $y = 0$, $y = b$ are also rigid but the side $x = a$ is covered with material which allows a certain amount of air motion normal to the surface. We assume that the correct boundary condition for $x = a$, $z > 0$, is that the normal gradient of the velocity potential at the surface is a constant times the value of ψ there

$$\partial\psi/\partial x = ik\eta\psi; \quad \text{at } x = a, z > 0 \qquad (11.4.16)$$

where η is called the *specific acoustic admittance* of the surface.

Since $i\rho kc\psi$ is the pressure and $\partial\psi/\partial x$ the velocity in the x direction, we could also express our boundary condition by saying that the normal specific acoustic impedance, the ratio of pressure at the surface to velocity of air normal to the surface, at the surface, $z_n = ik\rho c\psi/(\partial\psi/\partial x)$ or

$$\eta = \rho c/z_n = \gamma + i\sigma$$

where γ is the specific acoustic conductance (always positive) and σ is the specific acoustic susceptance of the surface (σ can be positive, masslike, or negative, springlike).

The solutions of the wave equation inside the duct will have a factor $\cos(\pi ny/b)$ which is common for positive and negative values of z, since no coupling will exist between waves for different n. We thus need not bother about the y factor, n might as well be set equal to zero. The factor for the x direction for $z > 0$ is somewhat more complicated. We choose a factor $\cos(\pi\mu_m x/a)$ for $z > 0$, where μ_m is fixed in value by the boundary condition at $x = a$ (for $z > 0$)

$$(\pi\mu_m)\tan(\pi\mu_m) = -ika\eta(k) = ka(\sigma - i\gamma) \qquad (11.4.17)$$

This equation was discussed in Sec. 4.7; its roots are complex: for $(ka\eta)$ smaller than $m + 1$, approximate values of the roots are

$$\mu_0 \simeq \frac{1}{\pi}\sqrt{-ika\eta}\,[1 + \tfrac{1}{6}ika\eta]$$

$$\mu_m = m - \left(\frac{ika\eta}{\pi^2 m}\right) + \left(\frac{k^2a^2\eta^2}{\pi^4 m^3}\right) + \cdots; \quad m > 0$$

For larger values of $(ka\eta)$, tables of the roots are available.

Consequently, the possible modes of wave transmission along the part of the tube for $z > 0$ have the form

$$\varphi_{mn}e^{ik_{mn}z} = \cos(\pi\mu_m x/a)\cos(\pi n y/b)e^{ik_{mn}z-ikct}$$

where $k_{mn}^2 = k^2 - (\pi\mu_m/a)^2 - (\pi n/b)^2$. Since μ is complex, k_{mn} is complex and the wave attenuates as it travels in the z direction. When the real part of $(\pi\mu_m/a)^2$ plus $(\pi n/b)^2$ becomes larger than k^2, then the imaginary part of k_{mn} becomes quite large and the corresponding modes are sharply damped; for smaller values of m and n, there is still damping (unless η is purely imaginary) but it is much less marked. Thus we have no sharp difference in kind between modes below and above their cutoff, as we did for the nonabsorbing boundary, only a difference in degree. In most cases however this difference in degree is marked enough to constitute a real separation; below some value of m (for $n = 0$), the attenuation is quite small compared to the attenuation above this same m, for a given value of k. For small values of k, only k_{00} has a relatively small imaginary part; the higher modes damp out much more quickly.

But to solve the present problem we use a Green's function fitting the rigid boundary conditions for $z < 0$ and then by its use set up an appropriate integral equation for the modified boundary for $z > 0$. The Green's function for $n = 0$ (we need not bring in dependence on the y axis, hence may consider the problem a two-dimensional one) is

$$G_k(x,z|x_0,z_0) = \left(\frac{2\pi i}{ka}\right)e^{ik|z-z_0|} + \sum_{m=1}^{\infty}\left(\frac{4\pi}{aK_m}\right)\cos\left(\frac{\pi m x_0}{a}\right)\cos\left(\frac{\pi m x}{a}\right)e^{-K_m|z-z_0|} \tag{11.4.18}$$

where $K_m^2 = (\pi m/a)^2 - k^2$. (We assume that the frequency is less than the lowest cutoff.)

The result of using Green's theorem is that $\psi(x,z)$ is equal to an integral over the two sides of the duct ($x = 0$, $x = a$) of G times the gradient $\partial\psi/\partial x$ plus integrals over the ends at $\pm\infty$ of $G(\partial\psi/\partial z) - \psi(\partial G/\partial z)$. The integral over the end at $+\infty$ vanishes because $\psi \to 0$ as $z \to \infty$ (because of the absorption of energy by the wall). The integral over the end at $-\infty$ is not zero, however, and this makes the further analysis rather more involved. There is no reason why we have to stick by the simple Green's function of Eq. (11.4.18), however. We can always add to it some solution of the homogeneous equation which fits the same boundary condition on the sides, and the result will be another Green's function. Since only the first term of G_k extends to $-\infty$, we insert a term which cancels this at $z_0 = -\infty$, obtaining

$$\gamma_k(x,z|x_0,z_0) = \left(\frac{2\pi i}{ka}\right)[e^{ik|z-z_0|} - e^{ik(z-z_0)}] + \sum_{m=1}^{\infty}\left(\frac{4\pi}{aK_m}\right)\cos\left(\frac{\pi m x_0}{a}\right)\cos\left(\frac{\pi m x}{a}\right)e^{-K_m|z-z_0|} \quad (11.4.19)$$

The integral of $\gamma(\partial\psi/\partial z) - \psi(\partial\gamma/\partial z)$ across both ends, at $\pm\infty$, then vanishes and the Green's integral reduces to an integral of $(\partial\psi/\partial x_0)_{x_0=a}$ times $\gamma_k(x,z|a,z_0)$ over z_0 from 0 to ∞. But this surface ($x_0 = a$, $z_0 > 0$) is covered with material which relates ψ and $\partial\psi/\partial x_0$ by Eq. (11.4.16), so we finally arrive at the equation

$$\psi(x,z) = \frac{ik\eta}{4\pi}\int_0^\infty \psi(a,z_0)\gamma_k(x,z|a,z_0)\,dz_0 \quad (11.4.20)$$

Setting $x = a$ results in a simple integral equation for $\psi(a,z)$; the solution of this may then be reinserted in (11.4.20) to obtain $\psi(x,z)$.

This integral equation is of the Wiener-Hopf type (see page 978) since γ is a function of $(z - z_0)$; we shall, therefore, use the procedures of the Fourier transform to calculate $\psi(a,z)$. Actually the function $\psi(a,z)$ (or, rather, its Fourier transform) is all that is needed to calculate the relative amplitude of the wave reflected back to $-\infty$ by the absorbing material. For the form of ψ as $z \to -\infty$ must be

$$\psi \to \alpha e^{ikz} + \beta e^{-ikz}; \quad z \to -\infty$$

and from (11.4.20), using the asymptotic form for γ, we have

$$\psi \to \frac{\eta}{2a}\left\{e^{ikz}\int_0^\infty \psi(a,z_0)e^{-ikz_0}\,dz_0 - e^{-ikz}\int_0^\infty \psi(a,z_0)e^{ikz_0}\,dz\right\}$$

Comparing the two, we see that the ratio between the reflected and incident amplitudes is proportional to the ratio between two Fourier transforms of $\psi(a,z_0)$. If

$$\Phi_+(\omega) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \psi(a,z_0)e^{i\omega z_0}\,dz_0$$

then the ratio between reflected and incident amplitudes is

$$r(k) = \beta/\alpha = -[\Phi_+(k)/\Phi_+(-k)] \quad (11.4.21)$$

Fourier Transform of the Integral Equation. Referring to Sec. 4.8, we see that we split $\psi(a,z)$ into two functions: $\phi_+(z)$ which is equal to $\psi(a,z)$ for $z > 0$ and $\phi_-(z)$ which equals $\psi(a,z)$ for $z < 0$;

$$\phi_+(z) = \begin{cases}\psi(a,z); & z > 0\\ 0; & z < 0\end{cases}; \quad \phi_-(z) = \begin{cases}0; & z > 0\\ \psi(a,z); & z < 0\end{cases}$$

We can find the Fourier transforms of these,

$$\Phi_+(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \psi(a,z_0)e^{i\omega z_0}\,dz_0; \quad \Phi_-(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 \psi(a,z_0)e^{i\omega z_0}\,dz_0$$

Because of the asymptotic behavior of ψ, we see that the integral for Φ_+ converges for $\operatorname{Im}\omega > -\operatorname{Im} k_0$ [where $k_0^2 = k^2 - (\pi\mu_0/a)^2$; μ_0 is defined in Eq. (11.4.17); since $\operatorname{Re}\eta > 0$ we have $\operatorname{Im} k_0 > 0$] and that the integral for Φ_- converges for $\operatorname{Im}\omega < 0$, so the two functions overlap in the band

$$-\operatorname{Im} k_0 < \operatorname{Im}\omega < 0; \quad k_0^2 = k^2 - (\pi\mu_0/a)^2 \simeq k^2 + (ik\eta/a)$$

According to Eq. (8.5.6), the double transform of a function of $(z - z_0)$ is $\delta(\omega - \omega_0)$ times a function of ω which is the Fourier transform with respect to $(z - z_0)$. In other words, since $\gamma_k = \gamma(z - z_0)$, the double Fourier transform is $\Gamma\delta(\omega - \omega_0)$, where

$$\Gamma(\omega|x,x_0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \gamma(\zeta)e^{i\omega\zeta}\,d\zeta; \quad \zeta = z - z_0$$

if the integral converges. We now see another reason (or is it the same reason?) for changing from G to γ by adding the term in $e^{ik(z-z_0)}$. If this were not added, the integral over the part of the Green's function for $\zeta = (z - z_0) > 0$ would converge only for $\operatorname{Im}\omega > 0$, the integral for $\zeta < 0$ would converge only for $\operatorname{Im}\omega < 0$, and nowhere would there be a common region of analyticity. By adding the extra term, we have arranged that the asymptotic behavior of γ for $\zeta \to \infty$ is as $e^{-K_1\zeta}$, where $K_1 = \sqrt{(\pi/a^2) - k^2} > 0$. The behavior as $\zeta \to -\infty$ is still as before, but we can now adjust ω so the whole integral converges, by keeping the imaginary part of ω between 0 and $-K_1$.

The result is (for $0 > \operatorname{Im}\omega > -K_1$)

$$\Gamma(\omega|x,x_0) = \left(\frac{2}{a}\right)\sqrt{2\pi}\left\{\frac{-1}{k^2 - \omega^2} + 2\sum_{m=1}^\infty \frac{\cos(\pi m x_0/a)\cos(\pi m x/a)}{(\pi m/a)^2 - (k^2 - \omega^2)}\right\}$$
$$\to -(2/\sigma)\sqrt{2\pi}\cot(\sigma a); \quad x,\, x_0 \to a$$

where $\sigma^2 = k^2 - \omega^2$, and where we have used a procedure similar to that for Eq. (4.3.7) to sum the series.

But there is a much neater way of obtaining Γ, by solving the Fourier transform of the equation for γ. This equation (which is the same as the equation for G) is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial\zeta^2} + k^2\right)\gamma = -4\pi\delta(x - x_0)\delta(\zeta)$$

so the equation for Γ is

$$\left(\frac{\partial^2}{\partial x^2} + k^2 - \omega^2\right)\Gamma = -2\sqrt{2\pi}\,\delta(x - x_0)$$

The solution of this equation, having zero normal gradient at $x = 0$ and $x = a$, is

$$\begin{aligned}\Gamma(\omega|x,x_0) &= \frac{-2\sqrt{2\pi}}{\sigma\sin(\sigma a)}\begin{cases}\cos(\sigma x_0)\cos\sigma(x - a); & x > x_0\\ \cos(\sigma x)\cos\sigma(x_0 - a); & x < x_0\end{cases}\\ &\to -(2/\sigma)\sqrt{2\pi}\cot(\sigma a); \qquad x, x_0 \to a \qquad (11.4.22)\end{aligned}$$

where $\sigma = \sqrt{k^2 - \omega^2}$. Of course we may wonder whether this is the transform of γ or of G or of some other function with some other solution of the homogeneous equation added to G. The answer is that it represents any of these (which give a convergent Fourier transform) only within certain ranges of the imaginary part of ω. It represents γ, as we have seen, for $0 > \operatorname{Im}\omega > -K_1$. If we had added $-e^{ik(z_0-z)}$ to G, so that the "free wave" were present only for $z > 0$ instead of for $z < 0$, the resulting convergent Fourier transform would be the Γ of (11.4.22) within the range $0 < \operatorname{Im}\omega < K_1$, and so on.

However we have chosen to use γ, and its Fourier transform is Γ within the range $0 > \operatorname{Im}\omega > -K_1$, which overlaps the regions of analyticity of Φ_+ and of Φ_-, the transforms of ψ for $z > 0$ and $z < 0$, respectively. Hence we can go on to the next step of the Wiener-Hopf process. The Fourier transform of integral equation

$$\psi(a,z) = \frac{ik\eta}{4\pi}\int_0^\infty \psi(a,z_0)\gamma_k(a,z|a,z_0)\,dz_0$$

is, by Eqs. (8.5.5) and (8.5.51),

$$\Phi_+(\omega) + \Phi_-(\omega) = \left(\frac{ik\eta}{2\sqrt{2\pi}}\right)\Phi_+(\omega)\Gamma(\omega|a,a) = -ik\eta\left[\frac{\cot(\sigma a)}{\sigma}\right]\Phi_+(\omega)$$

or

$$\Phi_+(\omega) = \left[\frac{-\sigma\sin(\sigma a)}{ik\eta\cos(\sigma a) + \sigma\sin(\sigma a)}\right]\Phi_-(\omega) \qquad (11.4.23)$$

Factoring the Transformed Equation. If now the ratio between Φ_+ and Φ_- can be factored, to obtain two functions Υ_+ and Υ_- [see Eq. (8.5.52)] such that $\Upsilon_+(\sigma)$ is regular (*i.e.*, has no zeros or poles) for $\operatorname{Im}\sigma > -\operatorname{Im}\mu_0$, and $\Upsilon_-(\sigma)$ is regular for $\operatorname{Im}\sigma < 0$, then we can obtain both Φ_+ and Φ_- and solve our problem exactly. The zeros of the quantity in brackets are for $\sigma = n\pi/a$ ($n = \ldots, -1, 0, 1, 2, \ldots$). Its poles are at the roots of the equation $\sigma\tan(\sigma a) = -ik\eta$; in other words, the roots of σ are related to the roots of Eq. (11.4.17);

$$\sigma_m = \frac{\pi\mu_m}{a} \xrightarrow[\eta\to 0]{} \begin{cases}\sqrt{-ik\eta/a}; & m = 0\\ (\pi m/a) - (ik\eta/\pi m); & m \neq 0\end{cases} \qquad (11.4.24)$$

where m can be a negative as well as a positive integer, the root for $-m$ being the negative of the root for $+m$. We can now rewrite the ratio to be factored in terms of infinite products [see Eq. (4.3.9)]

$$\frac{\sigma\sin(\sigma a)}{ik\eta\cos(\sigma a)+\sigma\sin(\sigma a)} = \frac{a(k^2-\omega^2)\prod_{n=1}^{\infty}\left[1-\left(\frac{a}{n\pi}\right)^2(k^2-\omega^2)\right]}{ik\eta\prod_{m=0}^{\infty}\left[1-\left(\frac{a}{\pi\mu_m}\right)^2(k^2-\omega^2)\right]}$$

$$= a(k+\omega)(k-\omega)\cdot$$

$$\cdot\frac{\prod_{n=1}^{\infty}\left[\sqrt{1-\left(\frac{ka}{\pi n}\right)^2}+i\left(\frac{\omega a}{\pi n}\right)\right]\prod_{n=1}^{\infty}\left[\sqrt{1-\left(\frac{ka}{\pi n}\right)^2}-i\left(\frac{\omega a}{\pi n}\right)\right]}{ik\eta\prod_{m=0}^{\infty}\left[\sqrt{1-\left(\frac{ka}{\pi\mu_m}\right)^2}+i\left(\frac{\omega a}{\pi\mu_m}\right)\right]\prod_{m=0}^{\infty}\left[\sqrt{1-\left(\frac{ka}{\pi\mu_m}\right)^2}-i\left(\frac{\omega a}{\pi\mu_m}\right)\right]}$$

where we can, if we wish, multiply and divide terms by $e^{i\omega a/n\pi}$ to ensure convergence. To improve the symmetry, we can set $\sqrt{(\pi\mu_0/a)^2-k^2} = -ik_0 \simeq -i\sqrt{k^2+i(k\eta/a)}$ and make the first factors of the two lower infinite products be $-(a/\pi\mu_m)^2(k_0+\omega)(k_0-\omega)$. The factors $(k+\omega)(k-\omega)$ times the first infinite product in the numerator are all regular for $\operatorname{Im}\omega < 0$, as is also the first product in the denominator, whereas the second products, above and below the line, are regular for $\operatorname{Im}\omega > -\operatorname{Im}k_0 < 0$. Consequently, in the equation

$$\Phi_+(\omega)\,\frac{\left(\frac{ik\eta}{\pi^2\mu_0^2}\right)(\omega+k_0)\prod_{m=1}^{\infty}\left[\sqrt{1-\left(\frac{ka}{\pi\mu_m}\right)^2}-i\left(\frac{\omega a}{\pi\mu_m}\right)\right]e^{i\omega a/\pi m}}{\prod_{n=1}^{\infty}\left[\sqrt{1-\left(\frac{ka}{\pi n}\right)^2}-i\left(\frac{\omega a}{\pi n}\right)\right]e^{i\omega a/\pi n}}$$

$$= \Phi_-(\omega)\,\frac{(\omega+k)(\omega-k)\prod_{n=1}^{\infty}\left[\sqrt{1-\left(\frac{ka}{\pi n}\right)^2}+i\left(\frac{\omega a}{\pi n}\right)\right]e^{-i\omega a/\pi n}}{(\omega-k_0)\prod_{m=1}^{\infty}\left[\sqrt{1-\left(\frac{ka}{\pi\mu_m}\right)^2}+i\left(\frac{\omega a}{\pi\mu_m}\right)\right]e^{-i\omega a/\pi m}}$$

the left side is regular in the upper half plane for ω (plus a strip from $\operatorname{Im}\omega = 0$ to $\operatorname{Im}\omega > -\operatorname{Im}k_0$), and the right side is regular in the lower half plane, so there is enough overlap to ensure that one side is the analytic continuation of the other. Consequently, each side is an integral function (see page 382) over the finite part of the ω plane. Investigation of the asymptotic dependence of each indicates that it has no singularity at infinity; it has no singularity elsewhere, therefore, *each must be a constant C*.

We have thus obtained a solution for the Fourier transform of ψ for $z > 0$ and for $z < 0$. The first is the more important function; we shall write it out

$$\Phi_+(\omega) = C \frac{\prod_{n=1}^{\infty} \left[\sqrt{1 - \left(\frac{ka}{\pi n}\right)^2} - i\left(\frac{\omega a}{\pi n}\right)\right] e^{i\omega a/\pi n}}{(\omega + k_0) \prod_{m=1}^{\infty} \left[\sqrt{1 - \left(\frac{ka}{\pi \mu_m}\right)^2} - i\left(\frac{\omega a}{\pi \mu_m}\right)\right] e^{i\omega a/\pi m}} \tag{11.4.25}$$

in the appropriate region of regularity. From it, by the inverse Fourier transform, the value of $\psi(a,z)$ for $z > 0$ can be found and then, by use of Eq. (11.4.20), $\psi(x,z)$ can be computed. But such painful tasks need not confront us if we wish to calculate only the amount of wave reflected back to $-\infty$ by the discontinuity in duct lining at $z = 0$. For, by Eq. (11.4.21), this can be obtained directly from the Fourier transforms we have just evaluated;

$$r(k) = -\frac{\Phi_+(k)}{\Phi_+(-k)} = \frac{(k - k_0)}{(k + k_0)} \prod_{n=1}^{\infty} \frac{\left[\sqrt{1 - \left(\frac{ka}{\pi n}\right)^2} - i\left(\frac{ka}{\pi n}\right)\right]}{\left[\sqrt{1 - \left(\frac{ka}{\pi n}\right)^2} + i\left(\frac{ka}{\pi n}\right)\right]} e^{2iak/\pi n} \cdot$$

$$\cdot \prod_{m=1}^{\infty} \frac{\left[\sqrt{1 - \left(\frac{ka}{\pi \mu_m}\right)^2} + i\left(\frac{ka}{\pi \mu_m}\right)\right]}{\left[\sqrt{1 - \left(\frac{ka}{\pi \mu_m}\right)^2} - i\left(\frac{ka}{\pi \mu_m}\right)\right]} e^{-2iak/\pi m}$$

$$= \frac{(k - k_0)}{(k + k_0)} \prod_{m=1}^{\infty} \left[\frac{\sqrt{1 - \left(\frac{ka}{\pi \mu_m}\right)^2} + i\left(\frac{ka}{\pi \mu_m}\right)}{\sqrt{1 - \left(\frac{ka}{\pi m}\right)^2} + i\left(\frac{ka}{\pi m}\right)}\right]^2; \quad ka < \pi \tag{11.4.26}$$

where the second form is obtained by multiplying and dividing by extra infinite products and eventually using the formula

$$\lim_{\omega \to k} \prod_{m=1}^{\infty} \left[\frac{1 - \left(\frac{a}{\pi m}\right)^2 (k^2 - \omega^2)}{1 - \left(\frac{a}{\pi \mu_m}\right)^2 (k^2 - \omega^2)}\right] = 1$$

Equation (11.4.26) gives an expression for the ratio between the reflected and incident waves. The first factor $(k - k_0)/(k + k_0)$ [where

$k_0^2 = k^2 - (\pi\mu_0/a)^2 \simeq k^2 + i(k\eta/a)]$ is the result one would obtain from a first-order calculation, using the average rate of loss of energy to compute the attenuation. The infinite product term is the correction to this. Only a few terms in the product sequence need be computed to obtain satisfactory accuracy, for the factors approach unity quite rapidly as m increases. We note also that $r \to 0$ as $\eta \to 0$ or as $a \to \infty$. We note that the solutions for microwave transmission (see Chap. 13) along a rectangular wave guide having a change in conductivity of one surface of the wave guide at $z = 0$ may be obtained in a quite similar way.

Radiation from End of Circular Pipe. As another example of the use of the Wiener-Hopf technique, we consider the case of a rigid pipe of circular cross section of radius a, with axis along the z axis, extending to $z \to -\infty$ but terminating at $z = 0$. In other words the only rigid surface is at $\rho = a$, $z < 0$, in terms of the cylindrical coordinate system ρ, φ, z. Here the Green's function to use will be the simplest possible one

$$g(\mathbf{r}|\mathbf{r}_0) = (1/R)e^{ikR}$$

where R is the distance from the observation point ρ, φ, z (or r, ϑ, φ in spherical coordinates, with the center of the pipe opening as origin) to the source point ρ_0, φ_0, z_0 (or r_0, ϑ_0, φ_0). This Green's function does not satisfy the boundary conditions on the surface of the pipe, but it is the solution of the inhomogeneous equation

$$(\nabla^2 + k^2)g = -4\pi\delta(\mathbf{r} - \mathbf{r}_0)$$

having outgoing waves at $r \to \infty$.

Two possible solutions are of physical interest; one representing a wave starting inside the pipe from $z = -\infty$ and striking the open end of the pipe, a part of it being reflected back down the pipe to $z \to -\infty$ and a part getting out into the open, arriving at $r \to \infty$ with an amplitude $[f(\vartheta)/r]$, where ϑ is the angle between the radius vector of the observation point r, ϑ, φ and the extended axis of the pipe;

$$\psi \to \begin{cases} Ae^{ikz} + Be^{-ikz}; & \text{inside pipe, } z \to -\infty \\ [f(\vartheta)/r]e^{ikr}; & \text{outside pipe, } r \to \infty \end{cases} \tag{11.4.27}$$

We note that the angle ϑ may go from zero to a value as close to π as is desired, if we make r large enough. The other situation of interest represents the reciprocal of this physical situation, a source outside the pipe, at a point r_0, ϑ_0, φ_0 ($r_0 \to \infty$) sending a wave toward the pipe opening; some of the incident wave will penetrate the pipe, sending a wave down it to $z = -\infty$, the rest will be reflected from the pipe opening, back to $r = \infty$;

$$\psi_r \to \begin{cases} Ce^{-ikz}; \quad \text{inside pipe, } z \to -\infty \\ \exp\{-ikr[\cos\vartheta\cos\vartheta_0 + \sin\vartheta\sin\vartheta_0\cos(\varphi - \varphi_0)]\} \\ \qquad\qquad + [f_r(\vartheta)/r]e^{ikr}; \quad \text{outside pipe, } r \to \infty \end{cases} \tag{11.4.28}$$

But before we discuss the relations between these functions and their parameters A, B, C, f, f_r, we shall show how one obtains an integral equation for ψ, for example, which will be solved to give these useful relations. The usual Green's equation (7.27) is

$$\psi(\mathbf{r}) = \frac{1}{4\pi} \oint [g(\mathbf{r}|\mathbf{r}_0^s) \operatorname{grad}_0 \psi(\mathbf{r}_0^s) - \psi(\mathbf{r}_0^s) \operatorname{grad}_0 g(\mathbf{r}|\mathbf{r}_0^s)] \cdot d\mathbf{A}_0 \quad (11.4.29)$$

where the surface to be integrated over is A_1 the sphere of radius L ($L \to \infty$) outside the pipe; A_2 the plane circular area of radius a inside the pipe at $z_0 \to -\infty$; A_3 the inner and outer surface of the pipe itself. Since both g and ψ go to zero as $1/r$ on surface A_1, the product of one by the radial gradient of the other, goes to zero as $1/r^3$, so the integral of this over A_1 is zero. Likewise, since g also goes as $1/r$ inside the pipe as $z \to -\infty$ (though ψ does not), the integral over A_2 likewise vanishes. Since $\operatorname{grad}_\rho \psi$ is zero at the inner and outer surface of the pipe, part of the integral over A_3 is zero and we have left, for the case of Eq. (11.4.28),

$$\psi(\mathbf{r}) = \frac{-1}{4\pi} \int_{A_3} \psi(\mathbf{r}_0^s)[\operatorname{grad}_0 g(\mathbf{r}|\mathbf{r}_0^s)] \cdot d\mathbf{A}_0$$

If we specify that our solution ψ is axially symmetric (*i.e.*, the simple plane wave mode is started down the pipe from $z = -\infty$), then the integration over φ will simply give 2π and the whole integration over A_3 will be the integral over z (for $\rho = a$) from $-\infty$ to 0 for the inside surface and from 0 to $-\infty$ for the outside surface. The gradient of g for the inside surface is $\partial g/\partial \rho_0$, for the outside surface is $-(\partial g/\partial \rho_0)$. Consequently, the integral equation for ψ is

$$\psi(\rho,z) = \tfrac{1}{2}a \int_{-\infty}^{0} [\psi^+(a,z_0) - \psi^-(a,z_0)] \left[\frac{\partial}{\partial a} g_s(\rho,z|a,z_0)\right] dz_0 \quad (11.4.30)$$

where ψ^+ is the value of ψ just outside the pipe, ψ^- is the value just inside, and g_s is the part of g which is independent of $\varphi - \varphi_0$. From this equation (or rather from a modification of it) we later obtain a solution for $[\psi^+ - \psi^-]$.

Formulas for Power Radiated and Reflected. Before we solve the integral equation, it would be well to find out what are the quantities of physical interest and how they are related to $[\psi^+ - \psi^-]$. Referring to Eq. (11.4.27) for the wave started in the pipe at $z = -\infty$, we would like to know the magnitude of the reflection coefficient $R = B/A$, the ratio between the amplitude B of the wave reflected back down the tube from the open end to the amplitude of the primary wave coming from $-\infty$; we would also like to compute the distribution in angle $f(\vartheta)$ of the wave radiated out into the open from the end of the pipe. These can be obtained by another application of Green's theorem, or else by using Eq.

(11.4.29) in the special case of r large enough that $\psi(r)$ may be given its asymptotic form $[f(\vartheta)/r]e^{ikr}$ and that g may be given its asymptotic form $(e^{ikr}/r)e^{-ikr_0\cos\theta}$ where θ is the angle between r and r_0.

The outer spherical surface A_1 is made enough larger than r that this integral still vanishes, but the integral over the disk at $z = -L$, $\rho < a$ cannot now be neglected. It is

$$\frac{e^{ikr}}{4\pi r}\int_0^{2\pi} d\varphi_0 \int_0^a \biggl\{\exp[-ikz_0\cos\vartheta - ik\rho_0\sin\vartheta\cos(\varphi_0-\varphi)]\frac{\partial}{\partial z_0}(Ae^{ikz_0}+Be^{-ikz_0}) - (Ae^{ikz_0}+Be^{-ikz_0})\frac{\partial}{\partial z_0}\exp[-ikz_0\cos\vartheta - ik\rho_0\sin\vartheta\cos(\varphi_0-\varphi)]\biggr\}_{z_0=-L}\rho_0\,d\rho_0$$

$$= ik\frac{e^{ikr}}{2r}\int_0^a \rho_0\,d\rho_0 J_0(k\rho_0\sin\vartheta)[A(1+\cos\vartheta)e^{-ikL(1-\cos\vartheta)} - B(1-\cos\vartheta)e^{ikL(1+\cos\vartheta)}]$$

$$= \frac{e^{ikr}}{r}\left(\frac{ia}{2}\right)\frac{J_1(ka\sin\vartheta)}{\sin\vartheta}[A(1+\cos\vartheta)e^{-ikL(1-\cos\vartheta)} - B(1-\cos\vartheta)e^{ikL(1+\cos\vartheta)}]$$

where we have used the integral representation for the Bessel function.

The integral over both surfaces of the pipe is modified by the use of the asymptotic form for g,

$$\frac{e^{ikr}}{4\pi r}\int_0^{2\pi} d\varphi_0\int_{-L}^0 dz_0\left\{\chi(z_0)\frac{\partial}{\partial a}\exp[-ikz_0\cos\vartheta - ika\sin\vartheta\cos(\varphi_0-\varphi)]\right\} = \left(\frac{e^{ikr}}{r}\right)\tfrac{1}{2}ka\sin\vartheta J_1(ka\sin\vartheta)\int_{-L}^0 \chi(z_0)e^{-ikz_0\cos\vartheta}\,dz_0$$

where $\chi(z_0) = [\psi^+(a,z_0) - \psi^-(a,z_0)]$, the difference in values of ψ just outside and inside the pipe, which enters the integral equation (11.4.30). Collecting all these, we obtain an expression for the distribution in angle of the wave coming out of the end of the pipe,

$$f(\vartheta) = \lim_{L\to\infty}\left\{\tfrac{1}{2}a\sqrt{k^2-\zeta^2}\,J_1(a\sqrt{k^2-\zeta^2})\int_{-L}^0 \chi(z_0)e^{-iz_0\zeta}\,dz_0 + \tfrac{1}{2}ia\frac{J_1(a\sqrt{k^2-\zeta^2})}{\sqrt{k^2-\zeta^2}}[A(k+\zeta)e^{-iL(k-\zeta)} - B(k-\zeta)e^{iL(k+\zeta)}]\right\} \tag{11.4.31}$$

where $\zeta = k\cos\vartheta$.

Viewed as a function of the complex variable ζ, the term containing A and B vanishes for $\operatorname{Im}\zeta > \operatorname{Im}k$ (we assume, for the time being, that k has a small, positive, imaginary part, which can eventually be set zero).

On the other hand, the integral involving χ, which is the Fourier transform of the surface discontinuity function $\chi(z_0)$,

$$\Xi(\zeta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \chi(z_0) e^{-iz_0\zeta}\, dz_0$$
$$\chi(z) = \begin{cases} 0; & z > 0 \\ \psi^+(a,z) - \psi^-(a,z); & z < 0 \end{cases} \qquad (11.4.32)$$

is valid only for Im $\zeta >$ Im k, because of the asymptotic form of $\psi^-(a,z)$ [see Eq. (11.4.27)]. Consequently, we infer that Eq. (11.4.31) really means that $f(\vartheta)$ is equal to the first term (involving Ξ) for Im $\zeta >$ Im k and equals the second term (involving A and B) for Im $\zeta <$ Im k. The two terms must be equal at $\zeta = \pm k$. Since $J_1(z) \to \frac{1}{2}z$ $(z \to 0)$, we have

$$-(2i/ka^2)f(0) = A = i\Omega^+; \quad (2i/ka^2)f(\pi) = B = i\Omega^-$$
$$\Omega^+ = \sqrt{2\pi} \lim_{\zeta\to k}[(\zeta - k)\Xi(\zeta)]; \quad \Omega^- = \sqrt{2\pi} \lim_{\zeta\to -k}[(\zeta + k)\Xi(k)] \qquad (11.4.33)$$

The quantities Ω^+ and Ω^- are the residues of the transform $\Xi(\zeta)$ at its poles at $\pm k$.

Consequently, the *reflection amplitude*, the ratio between reflected and primary wave inside the pipe, is given in terms of the behavior of the Fourier transform of the discontinuity of ψ at the pipe surface near its poles at $\zeta = \pm k$.

$$B/A = \Omega^-/\Omega^+ = -\sqrt{R}\, e^{2ikl}; \quad R = |B/A|^2 \qquad (11.4.34)$$

where l is the "end correction" for the pipe, the apparent place at which the waves are reflected. The square of $|B/A|$ is the reflection factor. The intensity of the primary wave is proportional to $|A|^2$ (call the proportionality factor D, it will cancel out); the power radiated out of the open end is

$$P_{\text{rad}} = \pi a^2 D(|A|^2 - |B|^2) = \pi a^2 D|A|^2(1 - R^2)$$

If this power were distributed evenly in all directions, the intensity at large distances from the open end would be $(a^2D/4r^2)|A|^2(1 - R^2)$; instead it is $D|f(\vartheta)|^2/r^2$. The ratio of these two might be called the *angle-distribution factor*

$$\alpha(\vartheta) = \frac{4|f(\vartheta)|^2/a^2}{|A|^2(1 - R^2)} = \frac{(ka)^2|f(\vartheta)|^2}{(1 - R^2)|f(0)|^2} \qquad (11.4.35)$$

where we may formally express $f(\vartheta)$ in terms of the Fourier transform of the surface discontinuity function [according to our discussion of Eq. (11.4.31)];

$$f(\vartheta) = \tfrac{1}{2}ka \sin\vartheta\, J_1(ka \sin\vartheta)\Xi(k\cos\vartheta) \qquad (11.4.36)$$

where we mean by $\Xi(k \cos \vartheta)$ the analytic continuation of $\Xi(\zeta)$ onto the real axis from its region of validity Im $\zeta >$ Im k.

Further utilization of the relations already written down results in a relation between the amplitude of the reflection coefficient and the angle-distribution factor,

$$|R|^2 = |\Omega^-/\Omega^+|^2 = [\alpha(\pi)/\alpha(0)] \tag{11.4.37}$$

and an explicit expression for $\alpha(\vartheta)$ in terms of Ξ and the Ω's,

$$\alpha(\vartheta) = [k \sin\vartheta\, J_1(ka \sin\vartheta)]^2 \frac{|\Xi(k\cos\vartheta)|^2}{|\Omega^+|^2 - |\Omega^-|^2} \tag{11.4.38}$$

Finally we can relate the reciprocal solution (11.4.28) for a plane wave coming in from $r = \infty$, striking the end of the tube, part being reflected and part sent into the tube, by applying Green's theorem (7.2.2) to ψ and ψ_r. We obtain, for the ratio between the intensity of the wave penetrating the tube to that of the incident wave,

$$|C|^2 = [\alpha(\vartheta_0)/\alpha(0)] \tag{11.4.39}$$

and for the effective cross section of the tube end for absorbing sound, the ratio of the total power entering the tube to the incident intensity,

$$\sigma_a(\vartheta) = \pi a^2 |C|^2 = \pi a^2 [\alpha(\vartheta)/\alpha(0)] \tag{11.4.40}$$

where $\alpha(\vartheta)$ is given by Eq. (11.4.38). (We have omitted the subscript 0 for ϑ, since it is no longer needed.) Thus the fraction of an incident plane wave, at an angle ϑ to the tube axis, which is absorbed by the pipe, is proportional to the fraction of the wave sent along the inside of the pipe, which gets out the end and travels outward at an angle ϑ to the pipe axis (which is, of course, the principle of reciprocity for this problem).

The Fourier Transform of the Integral Equation. But none of these equations will produce answers unless we compute the Fourier transform $\Xi(\zeta)$, defined in Eq. (11.4.32). We should obtain this by solving the Fourier transform of the integral equation (11.4.30). However, that equation is not yet in form appropriate for application of the Wiener-Hopf machinery. In the first place $(\partial/\partial a)g_s(\rho,z|a,z_0)$ is not a symmetric function of r and r_0, because of the partial derivative. Also $\psi(a,z)$ is not simply related to $\chi(z) = \psi^+(a,z) - \psi^-(a,z)$.

If we take the derivative of both sides of (11.4.30) with respect to ρ and then set $\rho = a$, the resulting Green's function will be symmetric in r, r_0 and the derivative of ψ will be zero for $z < 0$, by the boundary conditions. Consequently, the integral equation to work with is

$$\int_{-\infty}^{\infty} \chi(z_0)\gamma(z - z_0)\, dz_0 = \begin{cases} v_\rho(z); & z > 0 \\ 0; & z < 0 \end{cases} \tag{11.4.41}$$

where χ is defined in Eq. (11.4.32), $v_\rho(z) = (2/a)[\partial\psi/\partial\rho]_{\rho=a}$ and

$$\gamma(z - z_0) = \int_0^{2\pi} \left[\frac{\partial^2}{\partial\rho\,\partial\rho_0} g(\rho,z,\varphi|\rho_0,z_0,\varphi_0)\right]_{\rho,\rho_0=a} d\varphi_0$$

where we have integrated over φ_0 to obtain 2π times the part of g which is independent of $(\varphi - \varphi_0)$.

We next take the Fourier transforms of these quantities. The transform $\Xi(\zeta)$ of $\chi(z_0)$ has already been defined in Eq. (11.4.32); $(k^2 - \zeta^2)\Xi(\zeta)$ is regular for $\operatorname{Im} \zeta > -\operatorname{Im} k$. The Fourier transform of $v_\rho(z)$,

$$V(\zeta) = \frac{1}{\sqrt{2\pi}} \int_0^\infty v_\rho(z) e^{i\zeta z}\, dz$$

may be shown to be regular in the region $\operatorname{Im} \zeta < \operatorname{Im} k$, by using the asymptotic form for ψ and differentiating it with respect to ρ.

The double Fourier transform of $\gamma(z - z_0)$ is [see Eq. (8.5.6)] equal to $\Gamma(\zeta)\delta(\zeta - \zeta_0)$ where $\Gamma(\zeta)$ is the Fourier transform of $\gamma(z)$. This function Γ may be obtained directly by applying a Fourier transform to the equation for g [see Eq. (11.4.22)], to obtain its transform G with respect to $(z - z_0)$,

$$\left[\frac{\partial^2}{\partial z^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2} + k^2\right] g = -\frac{4\pi}{\rho}\delta(z - z_0)\delta(\rho - \rho_0)\delta(\varphi - \varphi_0)$$

$$\left[\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2} + (k^2 - \zeta^2)\right] G = -\frac{2\sqrt{2\pi}}{\rho}\delta(\rho - \rho_0)\delta(\varphi - \varphi_0)$$

Expanding G in terms of a Fourier series in $(\varphi - \varphi_0)$ and solving the radial equations, we find that the Fourier transform of g with respect to $z - z_0$ is

$$G(\rho,\varphi|\rho_0,\varphi_0|\zeta) = \frac{i\pi}{\sqrt{2\pi}} \sum_{m=0}^{\infty} \epsilon_m \cos[m(\varphi - \varphi_0)] \begin{cases} H_m(\rho_0 \sqrt{k^2 - \zeta^2}) J_m(\rho \sqrt{k^2 - \zeta^2}); & \rho < \rho_0 \\ J_m(\rho_0 \sqrt{k^2 - \zeta^2}) H_m(\rho \sqrt{k^2 - \zeta^2}); & \rho > \rho_0 \end{cases}$$

where H_m is the Hankel function of the first kind, $H_m^{(1)}$. Taking the derivative of this with respect to ρ and ρ_0, setting both ρ and ρ_0 equal to a, and then integrating over φ results in the Fourier transform we desire:

$$\begin{aligned} \Gamma(\zeta) &= i\pi \sqrt{2\pi} \frac{\partial}{\partial a}[H_0(a\sqrt{k^2 - \zeta^2})] \frac{\partial}{\partial a}[J_0(a\sqrt{k^2 - \zeta^2})] \\ &= i\pi\sqrt{2\pi}\,(k^2 - \zeta^2) H_1(a\sqrt{k^2 - \zeta^2}) J_1(a\sqrt{k^2 - \zeta^2}) \end{aligned} \tag{11.4.42}$$

which is regular in the range $|\operatorname{Im} \zeta| < \operatorname{Im} k$.

Therefore [see Eq. (8.5.5)], the Fourier transform of integral equation (11.4.41) is

$$2\pi^2 i(k^2 - \zeta^2) H_1(a\sqrt{k^2 - \zeta^2}) J_1(a\sqrt{k^2 - \zeta^2}) \Xi(\zeta) = V(\zeta) \tag{11.4.43}$$

where all the terms are regular within the strip $|\operatorname{Im} \zeta| < \operatorname{Im} k$. This is now ready to factorize.

Factoring the Transformed Equation. The quantity $2\pi(k^2 - \zeta^2)\Xi(\zeta)$ is regular in the "upper" half plane $\operatorname{Im}\zeta > -\operatorname{Im}k$, whereas $V(\zeta)$ is regular for the "lower" half plane $\operatorname{Im}\zeta < \operatorname{Im}k$ (we assume $\operatorname{Im}k > 0$ for our analysis). If we can factor the remaining part,

$$\pi i H_1(a\sqrt{k^2 - \zeta^2})J_1(a\sqrt{k^2 - \zeta^2})$$

into Υ_+/Υ_-, where Υ_+ is regular in the upper half plane and Υ_- in the lower half plane, we shall have obtained our answer. Here the logarithmic singularity in H_1 defeats attempts at separating H_1 into an infinite product expression, although J_1 can be so expressed. We are, therefore, forced back to the use of contour integrals as discussed at the end of Chap. 8.

The function $\ln[\pi i H_1(a\sqrt{k^2 - \zeta^2})J_1(a\sqrt{k^2 - \zeta^2})]$ is regular in the strip $|\operatorname{Im}\zeta| < \operatorname{Im}k$, consequently it may be represented by a Cauchy integral, with contour deformed to fit just inside the strip of regularity;

$$\ln\Upsilon_+ - \ln\Upsilon_- = \frac{1}{2\pi i}\int_{-\infty - i\epsilon}^{\infty - i\epsilon} \frac{\ln[\pi i H_1(a\sqrt{k^2 - t^2})J_1(a\sqrt{k^2 - t^2})]}{t - \zeta}\,dt - \frac{1}{2\pi i}\int_{-\infty + i\epsilon}^{\infty + i\epsilon} \frac{\ln[\pi i H_1(a\sqrt{k^2 - t^2})J_1(a\sqrt{k^2 - t^2})]}{t - \zeta}\,dt$$

Since the first integral is regular for $\operatorname{Im}\zeta > -\epsilon$ (where $0 < \epsilon < \operatorname{Im}k$), it can be taken for $\ln\Upsilon_+$ and, since the second integral is regular for $\operatorname{Im}\zeta < +\epsilon$, it can be taken for $\ln\Upsilon_-$. When k is made real ($\epsilon \to 0$), the integral for $\ln\Upsilon_-$ differs from that of $\ln\Upsilon_+$ only by the way in which the contour for t avoids the point $t = \zeta$; for Υ_- it goes above $t = \zeta$, for Υ_+ it goes below.

We, therefore, have

$$\Upsilon_+(\zeta) = \exp\left\{\frac{1}{2\pi i}\int_{-k}^{k} \frac{\ln[\pi i H_1(a\sqrt{k^2 - t^2})J_1(a\sqrt{k^2 - t^2})]\,dt}{t - \zeta} + \frac{\zeta}{\pi i}\int_k^\infty \frac{\ln[2K_1(a\sqrt{t^2 - k^2})I_1(a\sqrt{t^2 - k^2})]\,dt}{(t^2 - \zeta^2)}\right\} \tag{11.4.44}$$

where K_1 and I_1 are the hyperbolic Bessel functions defined at the end of Chap. 10 and $\Upsilon_-(\zeta) = 1/\Upsilon_+(-\zeta)$. If ζ is on the real axis and $|\zeta| < k$, the first integral will be singular. Its value will be πi times the residue of the integrand at $t = \zeta$ plus the principal value of the integral [see Eq. (4.2.9)] (for Υ_- one takes $-\pi i$ times the residue plus the principal value);

$$\Upsilon_+(k\cos\vartheta) = \sqrt{\pi i H_1(ka\sin\vartheta)J_1(ka\sin\vartheta)}\cdot \exp\left\{i\,\frac{ka\cos\vartheta}{\pi}\,\mathcal{P}\int_0^{ka} \frac{x\ln[\pi i H_1(x)J_1(x)]\,dx}{[x^2 - (ka\sin\vartheta)^2]\sqrt{(ka)^2 - x^2}} + \frac{ika\cos\vartheta}{\pi}\int_0^\infty \frac{x\ln[1/2I_1(x)K_1(x)]\,dx}{[x^2 + (ka\sin\vartheta)^2]\sqrt{x^2 + (ka)^2}}\right\} \tag{11.4.45}$$

where $\mathcal{P}$ is the symbol for the principal value of the integral. When $\zeta = \pm k$, there is no singularity and

$$\Upsilon_+(k) = \frac{1}{\Upsilon_+(-k)} = \exp\left\{\frac{ika}{\pi}\int_0^{ka}\frac{\ln[\pi i H_1(x)J_1(x)]}{x\sqrt{(ka)^2 - x^2}}\,dx + \frac{ika}{\pi}\int_0^\infty \frac{\ln[1/2I_1(x)K_1(x)]}{x\sqrt{x^2+(ka)^2}}\,dx\right\}$$

Returning to the transformed equation,

$$(k^2 - \zeta^2)\Xi(\zeta)\Upsilon_+(\zeta) = V(\zeta)\Upsilon_-(\zeta)/2\pi$$

the left-hand side of which is regular in the half plane $\operatorname{Im}\zeta > -\epsilon$ $(0 < \epsilon < \operatorname{Im} k)$ and the right side of which is regular in the half plane

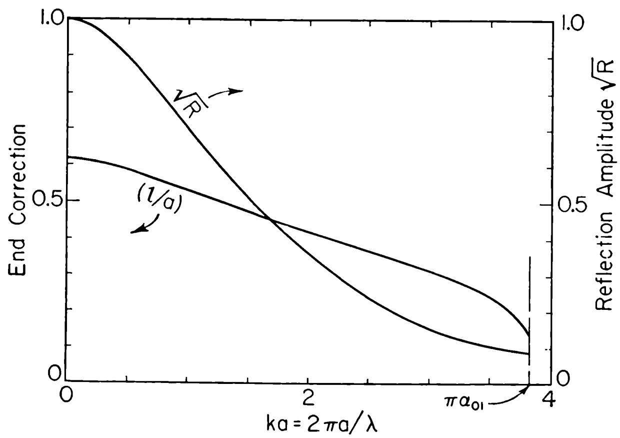

Fig. 11.11 Values of reflection coefficient R and end correction l for plane wave issuing from open end of unflanged pipe.

$\operatorname{Im}\zeta < \epsilon$. We can show that the left side (therefore also the right) stays finite as $\operatorname{Re}\zeta \to \infty$ within the strip $|\operatorname{Im}\zeta| < \epsilon$. Consequently, one side is the analytic continuation of the other and both represent a constant C and therefore,

$$\begin{aligned}\Xi(\zeta) &= [C/(k^2 - \zeta^2)\Upsilon_+(\zeta)]\\ \Omega^+ &= \lim_{\zeta\to k}[(\zeta - k)\Xi(\zeta)] = -C/2k\Upsilon_+(k)\\ \Omega^- &= C/2k\Upsilon_+(-k) = C\Upsilon_+(k)/2k\end{aligned} \tag{11.4.46}$$

From these quantities we can compute the quantities of physical interest in our solution, in terms of the function $\Upsilon_+(\zeta)$. Function Υ_+ is not, of course, given in terms of known, tabulated functions, but it is given in terms of a pair of integrals, which may be evaluated numerically as accurately as we wish. The specific formula for the reflection amplitude and reflection factor R, for example, is

$$B/A = -\Upsilon_+(k)/\Upsilon_+(-k) = [\Upsilon_+(k)]^2 = -\sqrt{R}\,e^{2\pi ikl}$$
$$\sqrt{R} = \exp\left\{-\frac{2ka}{\pi}\int_0^{ka}\frac{\tan^{-1}[-J_1(x)/N_1(x)]}{x\sqrt{(ka)^2-x^2}}\,dx\right\}$$
$$\frac{l}{a} = \frac{1}{\pi}\int_0^{ka}\frac{\ln[\pi J_1(x)\sqrt{J_1^2(x)+N_1^2(x)}]}{x\sqrt{(ka)^2-x^2}}\,dx + \frac{1}{\pi}\int_0^{\infty}\frac{\ln[1/2I_1(x)K_1(x)]}{x\sqrt{x^2+(ka)^2}}\,dx \tag{11.4.47}$$

Curves for the square root of the reflection factor, $\sqrt{R}$, and the end correction l/a are shown in Fig. 11.11. Practically all the primary wave is reflected back into the pipe ($|R| \to 1$) when ka is small; practically all gets out into the open ($|R| \to 0$) when ka is large. Finally the angle-distribution function,

$$\alpha(\vartheta) = \frac{4}{\pi\sin^2\vartheta}\,\frac{J_1(ka\sin\vartheta)}{[J_1^2(ka\sin\vartheta)+N_1^2(ka\sin\vartheta)]}\,\frac{|R|}{1-|R|^2}\cdot$$
$$\cdot\exp\left\{\frac{2ka\cos\vartheta}{\pi}\,\mathcal{P}\int_0^{ka}\frac{x\tan^{-1}[-J_1(x)/N_1(x)]\,dx}{(x^2-k^2a^2\sin^2\vartheta)\sqrt{k^2a^2-x^2}}\right\}$$

is given in terms of the principal value of an integral. This function is plotted as a function of angle ϑ for different values of ka in Fig. 11.12.

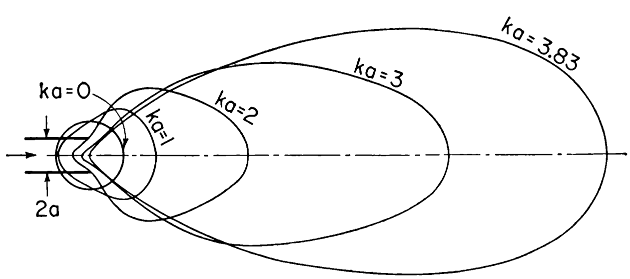

Fig. 11.12 Polar diagram of distribution in angle $\alpha(\vartheta)$ of intensity of wave radiated from open end of pipe.

We see that, as ka increases, the wave coming out of the pipe is concentrated more and more in the direction of the axis of the pipe.

None of these solutions are valid for values of ka large enough so that a higher mode can be transmitted along the pipe, but when $ka < \pi\alpha_{01} = 3.832$, the results are exact and as accurate as one has energy to compute the integrals involved.

Radiation from Vibrating Source. The next problem to consider is that of radiation out to infinity from an object of finite size enclosing the origin, where we specify either value or normal gradient at the surface A of the object. This closed surface may then be described by giving its equation in spherical coordinates about the origin

$$r = a(\vartheta,\varphi); \quad \text{surface } A$$

where, since the origin is inside, we have a finite value of the function a for every pair of values of ϑ, φ in the range $0 < \varphi < 2\pi$, $0 < \vartheta < \pi$. It will not restrict our study overmuch if we rule out all surfaces for which a has multiple values for some ranges of ϑ, φ; in fact we shall restrict things still further, for simplicity in later discussion, and say that the surface A is everywhere convex outward, there being no plane which is tangent to A at two separate points.

If we apply Green's theorem to the radiated wave ψ and to the free-space wave function $g_k(\mathbf{r}|\mathbf{r}_0) = e^{ikR}/R$ [see Eq. (7.2.7)], we obtain

$$\psi(\mathbf{r}) = \frac{-1}{4\pi} \oint \left[g_k(\mathbf{r}|\mathbf{r}_0^s) \frac{\partial}{\partial n_0} \psi(\mathbf{r}_0^s) - \psi(\mathbf{r}_0^s) \frac{\partial}{\partial n_0} g_k(\mathbf{r}|\mathbf{r}_0^s) \right] dA \quad (11.4.48)$$

where $\partial/\partial n_0$ is the outward-pointing, normal gradient to the surface A in the r_0 coordinates. [The minus sign takes into account that the region in which ψ is given is *outside* the surface A, so the gradient given in Eq. (7.2.7) is $-\partial/\partial n_0$.] Next we insert the Fourier transform of g_k [see Eqs. (11.3.6) *et seq.*],

$$g_k(\mathbf{r}|\mathbf{r}_0) = \frac{1}{2\pi^2} \iiint \frac{e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}_0)}\, dp_x\, dp_y\, dp_z}{p^2 - k^2}$$

where the wave vector $\mathbf{p}$ has components p_x, p_y, p_z, magnitude p and direction given by the polar angles u and v. The integration is over all the p space, and to obtain outgoing waves we avoid the pole at $p = -k$ by going above it in the p complex plane and going below the pole at $p = k$.

Inserting this into Eq. (11.4.48), we obtain

$$\psi(\mathbf{r}) = \frac{-1}{8\pi^3} \iiint \left\{ \oint \left[i(\mathbf{n}_0 \cdot \mathbf{p})\psi(\mathbf{r}_0^s) + \frac{\partial}{\partial n_0} \psi(\mathbf{r}_0^s) \right] e^{-i\mathbf{p}\cdot\mathbf{r}_0^s}\, dA_0 \right\} \cdot \frac{e^{i\mathbf{p}\cdot\mathbf{r}}}{p^2 - k^2}\, dp_x\, dp_y\, dp_z$$

where $\mathbf{n}_0$ is a unit, outward-pointing vector normal to the surface A_0. The quantity in the braces is a function of $\mathbf{p}$ but not of $\mathbf{r}_0$. Next we integrate over p_z, going above (on the complex p_z plane) the pole at $p_z = -\sqrt{k^2 - p_x^2 - p_y^2}$ and below the pole at $p_z = +\sqrt{k^2 - p_x^2 - p_y^2}$. When z is greater than any z on the surface A_0, the integral reduces to $2\pi i$ times the residue of the integrand at $p_z = \sqrt{k^2 - p_x^2 - p_y^2}$.

We now transform from rectangular coordinates p_x, p_y in p space to spherical coordinates p, u, v by setting $p_x = k \sin u \cos v$ and $p_y = k \sin u \sin v$ (u must be complex in order that p_x and p_y range to infinity). The area element $dp_x\, dp_y$ becomes $k^2 \cos u \sin u\, du\, dv$ and the expression for ψ, after integration over p_z, becomes

$$\psi(\mathbf{r}) = \frac{ik}{4\pi}\int_0^{2\pi} dv \int_{-i\infty}^{i\infty} \sin u \, du \, F(u,v,k) \cdot \\ \cdot \exp\{ikr[\cos u \cos \vartheta + \sin u \sin \vartheta \cos(v - \varphi)]\} \quad (11.4.49)$$

where ϑ and φ are the spherical angles for $\mathbf{r}$ and where

$$F(u,v,k) = \frac{-1}{2\pi}\oint\left[i(\mathbf{n}_0 \cdot \mathbf{k})\psi(\mathbf{r}_0^s) + \frac{\partial}{\partial n_0}\psi(\mathbf{r}_0^s)\right] e^{-i\mathbf{k}\cdot\mathbf{r}_0^s}\, dA_0$$

with $\mathbf{k} = \mathbf{a}_x k \sin u \cos v + \mathbf{a}_y k \sin u \sin v + \mathbf{a}_z k \cos u$. The integration over u is along a contour which ensures convergence of the integral. The expression, of course, is valid only if $\mathbf{r}$ is outside surface A.

We have thus obtained a relationship between the Green's function formulation and the expansion of the problem in plane waves, as discussed at the beginning of Sec. 11.3 [see Eq. (11.3.3)]. We do not propose to evaluate the integral given above for F. We shall develop other procedures for determining F in terms of the boundary conditions; what was attained here was to show that the expression given in Eq. (11.4.49) represents an outgoing wave radiated from some source of finite size.

Angle Distribution of Radiated Wave. Function F has a simple and useful physical meaning, which can be demonstrated by letting r go to infinity in (11.4.49) and obtaining the asymptotic expression for ψ by means of the method of steepest descent (see Sec. 4.6). We first change our variables of integration from angle u with respect to the z axis and angle v with respect to the x axis to angle θ with respect to the vector $\mathbf{r}$ and angle ϕ about $\mathbf{r}$, where

$$\begin{aligned} \sin u \, du \, dv &= \sin \theta \, d\theta \, d\phi \\ \cos \theta &= \cos u \cos \vartheta + \sin u \sin \vartheta \cos(v - \varphi) \\ \cos u &= \cos \theta \cos \vartheta + \sin \theta \sin \vartheta \cos \phi \\ \sin \theta \sin \phi &= \sin u \sin(v - \varphi) \end{aligned}$$

If the limits for u are $\pm i\infty$, these are also the limits of θ; therefore,

$$\psi(\mathbf{r}) = \frac{ik}{4\pi}\int_0^{2\pi} d\phi \int_{-i\infty+\frac{1}{2}i\pi}^{i\infty-\frac{1}{2}i\pi} \sin \theta \, d\theta \, F(u,v,k) e^{ikr\cos\theta}$$

where we have adjusted our contour for θ so that the imaginary part of $\cos \theta$ is positive when $\theta \to \pm i\infty$, so that the integrand vanishes at the two limits.

The saddle point of the exponential is at $\theta = 0$ ($u = \vartheta$, $v = \varphi$) so the phase of the exponential will not change and its magnitude will go to zero as one goes away from θ along either of the lines defined by

$$\cos \theta = 1 + i\zeta; \quad \theta = \xi + i\eta$$
$$\cos \xi = \operatorname{sech} \eta; \quad \sinh \eta \tanh \eta = \zeta; \quad \eta \text{ positive or negative}$$

The integral then becomes

$$\psi(\mathbf{r}) = ke^{-ikr}\int_0^\infty F(u,v,k)e^{-kr\zeta}\,d\zeta$$

When $kr \to \infty$, the only part of the integrand which contributes is for ζ very near zero, *i.e.*, for u very near ϑ and v very near φ, so that the asymptotic form for the solution ψ is

$$\psi(\mathbf{r}) \to F(\vartheta,\varphi,k)(e^{ikr}/r); \quad kr \to \infty \tag{11.4.50}$$

Consequently, the amplitude of the plane wave factor in the integral representation of ψ in terms of plane waves is proportional to the angle-distribution factor $F(\vartheta,\varphi,k)$ for the wave at large distances from the source. (This is not true unless the integral over u runs from $-i\infty$ to $+i\infty$, thus eliminating all incoming waves.) The relationship is not surprising, but it is well to demonstrate it and to calculate the factor of proportionality $ik/4\pi$.

Applying the Boundary Conditions. But we still have to determine the angle-distribution factor F in terms of the specifications at the boundary of the radiator $r = a(\vartheta,\varphi)$. Suppose we specify the value of ψ on the boundary surface, $\psi_0(\vartheta,\varphi)$ on $r = a(\vartheta,\varphi)$. This value is related to the angle-distribution factor, F, given in Eqs. (11.4.49) and (11.4.50), by the integral equation

$$\psi_0(\vartheta,\varphi) = \frac{ik}{4\pi}\int_0^{2\pi} dv \int_{-i\infty}^{i\infty} \sin u\, du\, F(u,v)\cdot \\ \cdot \exp\{ika(\vartheta,\varphi)[\cos u\cos\vartheta + \sin u \sin\vartheta\cos(v-\varphi)]\} \tag{11.4.51}$$

for the unknown F in terms of the known functions ψ_0 and a.

This equation is not easy to solve in general; in fact, if the surface $a(\vartheta,\varphi)$ belongs to a separable system of coordinates and if ka is not too large, it is best to use the expansions in eigenfunctions already discussed in Sec. 11.3. But if ka is large (larger than 10, roughly), we can use the method of steepest descent (see Sec. 4.6) to relate F and ψ_0 approximately. We first separate the function F into its magnitude and phase

$$F(u,v) = f(u,v)e^{-i\beta(u,v)} \tag{11.4.52}$$

and then study the behavior of the phase of the integrand of Eq. (11.4.51).

We have chosen our contour for u so that the predominant part of the integral comes from the range of u near its real axis; in fact we saw in the preceding pages that the contour can be arranged so that the largest contribution comes from the region near $u = \vartheta$, $v = \varphi$. This result is somewhat modified by the fact that a is a function of ϑ and φ. The total exponential in the integral of Eq. (11.4.51) is

$$\begin{aligned} t(u,v|\vartheta,\varphi) &= ika(\vartheta,\varphi)\cos\theta - i\beta(u,v) \\ \cos\theta &= \cos u\cos\vartheta + \sin u\sin\vartheta\cos(v-\varphi) \end{aligned} \tag{11.4.53}$$

If ka is large compared to unity (> 10), this exponential variation will be much more rapid than the variation of the amplitude f, so we need consider only t in deciding just where the integrand has its largest value, and can choose β to make t zero at this point, as well as to make $\partial t/\partial u$ and $\partial t/\partial v$ both zero there.

The quantity $a(\vartheta,\varphi) \cos \theta$ is the component of $\mathbf{a}(\vartheta,\varphi)$ in the (u,v) direction and, if we did not have β to include, the maximum value of $a \cos \theta$ would be at $\theta = 0$ (the unit vector $\mathbf{u}$ of Fig. 11.13, in the direction u, v in the same direction as the vector $\mathbf{a}$ in the direction ϑ, φ, where

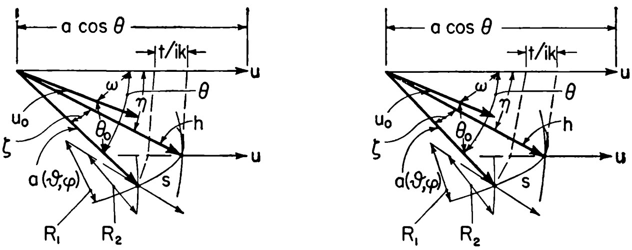

Fig. 11.13 Angles and directions involved in calculation of radiation from a surface $a(\vartheta,\varphi)$.

the boundary value ψ_0 is given). But if we then choose $\beta = ka$, this will make β a function of ϑ and φ, not u and v. The correct way to choose the direction $\mathbf{u}_0$ for the saddle point of t so that β can be a function of u, v and have t stationary with respect to small motions of $\mathbf{u}$ about $\mathbf{u}_0$ is as follows.

For each direction (u,v) of vector $\mathbf{u}$ we find the point on the radiating surface which has its normal parallel to $\mathbf{u}$ (we have restricted our surface shapes so that there is only one such point). Call the vector to this point $\mathbf{h}$ and its spherical angles σ, τ (these, of course, are functions of u, v). The projection of $\mathbf{h}$ on $\mathbf{u}$ is the distance from the origin to the intersection with $\mathbf{u}$ of the tangent plane to the surface which is perpendicular to $\mathbf{u}$. In other words, if we hold u, v fixed and vary ϑ, φ, *the values of* ϑ, φ *which make* $\mathbf{a} \cdot \mathbf{u} = a(\vartheta,\varphi) \cos \theta$ *maximum* are σ, τ and thus $\mathbf{a}(\sigma,\tau) = \mathbf{h}$ (which is, of course, a function of u, v).

We now choose β to be $k\mathbf{h}(u,v) \cdot \mathbf{u} = ka(\sigma,\tau) \cos \eta$, where $\cos \eta = \cos u \cos \sigma + \sin u \sin \sigma \cos(v - \tau)$. It is then obvious that the exponent

$$t = ik[\mathbf{a}(\vartheta,\varphi) - \mathbf{h}(u,v)] \cdot \mathbf{u}$$

is *both zero and stationary* when $\mathbf{u}$ is pointed so that $\mathbf{h}(u,v)$ coincides with $\mathbf{a}(\vartheta,\varphi)$. Thus we have arranged to have β a function just of u and v and still have t both zero and stationary for some (u,v) for all values of ϑ, φ.

For the chosen ϑ, φ we now choose our axes for integrating over u, v with respect to the direction $\mathbf{u}_0(\vartheta,\varphi)$ (the direction of the normal to the surface at ϑ, φ, at angles u_0, v_0). As we move $\mathbf{u}$ away from the direction of $\mathbf{u}_0$, the vector $\mathbf{h}$ moves away from the direction $\mathbf{a}(\vartheta,\varphi)$. The relation between the corresponding angles ω (equals the angle $\mathbf{u}$, $\mathbf{u}_0$) and ζ (the angle $\mathbf{h}$, $\mathbf{a}$) depends on the curvature of the surface at ϑ, φ. Close to ϑ, φ the surface is nearly the shape of a tangent ellipsoid, with two principal radii of curvature, R_1 and R_2, as shown in Fig. 11.13. If we base our second angle of integration α (to accompany η in orienting $\mathbf{u}$ with respect to $\mathbf{u}_0$) on the plane of the principal axis R_1, we can readily see that the relation between the angle of rotation of $\mathbf{h}$ and the angle of rotation of $\mathbf{u}$ is the ratio between $a(\vartheta,\varphi)$ and the radius of curvature for the direction α,

$$\left(\frac{\cos^2\alpha}{R_1} + \frac{\sin^2\alpha}{R_2}\right)\zeta = \frac{\omega}{a(\vartheta,\varphi)}\sqrt{1 - \sin^2\theta_0\cos^2(\alpha - \alpha_0)};\quad \zeta, \omega \ll 1 \tag{11.4.54}$$

$$\cos\zeta = \cos\vartheta\cos\sigma + \sin\vartheta\sin\sigma\cos(\varphi - \tau)$$
$$\cos\omega = \cos u\cos u_0 + \sin u\sin u_0\cos(v - v_0)$$

where α_0 is the angle between the plane of $\mathbf{a}$ and $\mathbf{u}_0$ and the plane of the principal axis R_1, which is the reference plane for the rotational angle α. [The radical comes in because the surface at the point ϑ, φ is normal to $\mathbf{u}_0$, not to $\mathbf{a}$, so that rotation of $\mathbf{a}$ by an angle ζ produces a different linear displacement, depending on the orientation angle $(\alpha - \alpha_0)$.]

When ζ is small, we find that $t \simeq \frac{1}{2}ikS^2[(1/R_1)\cos^2\alpha + (1/R_2)\sin^2\alpha]$ and that $S \simeq \zeta a/\sqrt{1 - \sin^2\theta_0\cos^2(\alpha - \alpha_0)}$. Therefore, for small values of angle ω,

$$t \simeq \tfrac{1}{2}ik\,\frac{R_1R_2}{R_2\cos^2\alpha + R_1\sin^2\alpha}\,\omega^2 = -x$$

We can adjust the contour for u (or, rather, ω) from $-i\infty$ to $+i\infty$ so that x stays real, going from zero to infinity in both directions, from u_0 to $+i\infty$ and from u_0 to $-i\infty$. Consequently, our integral representation becomes (since $\sin\omega\,d\omega \simeq \omega\,d\omega$ for ω small)

$$\psi_0(\vartheta,\varphi) = \frac{ik}{2\pi}\int_0^{2\pi} d\alpha \int_0^\infty \left[\frac{R_2\cos^2\alpha + R_1\sin^2\alpha}{ikR_1R_2}\right] e^{-x} f(u,v)\,dx$$

where we now have to express the amplitude function $f(u,v)$ in such a way as to relate it to the specified value ψ_0 on the surface. The best way is to expand f as a Taylor series about u_0, v_0. Expressing displacements from u_0 in terms of angular displacement component ω_x in the plane of the principal radius R_1 (more precisely those angular displacements of $\mathbf{u}$ which produce an angular displacement of $\mathbf{h}$ in the plane of R_1) and component ω_y in the plane of R_2,

$$f(u,v) = f(u_0,v_0) + f_x\omega_x + f_y\omega_y + \tfrac{1}{2}f_{xx}\omega_x^2 + \tfrac{1}{2}f_{yy}\omega_y^2 + f_{xy}\omega_x\omega_y + \cdots$$

where $\omega_x = \omega\cos\alpha$, $\omega_y = \omega\sin\alpha$, and the subscripts on f indicate differentials of f with respect to ω_x, ω_y, evaluated at $\omega_x = \omega_y = 0$ (*i.e.*, at u_0, v_0).

Performing the integrations we obtain

$$\psi_0(\vartheta,\varphi) \simeq \left(\frac{R_1 + R_2}{2R_1R_2}\right) f(u_0,v_0) + \left(\frac{\frac{1}{16}R_1^2 + \frac{1}{8}R_1R_2 + \frac{5}{16}R_2^2}{ikR_1^2R_2^2}\right) f_{xx}(u_0,v_0) + \left(\frac{\frac{5}{16}R_1^2 + \frac{1}{8}R_1R_2 + \frac{1}{16}R_2^2}{ikR_1^2R_2^2}\right) f_{yy}(u_0,v_0) + \cdots$$

which series converges well as long as $kR_1R_2/(R_1 + R_2)$ is large enough. In order to obtain an expression for f the asymptotic amplitude of the radiated wave, in terms of the amplitude of f on the source, we must invert the series. The lowest approximation, good when $kR \to \infty$ (*i.e.*, good for the "geometrical optics" case) is

$$f(u,v) = \left(\frac{2R_1R_2}{R_1 + R_2}\right) \psi_0(\sigma,\tau) \tag{11.4.55}$$

where σ, τ is the point on the source surface for which the direction u, v is normal. In other words, for the limit of very short wavelengths, the waves are radiated normally from each point of the surface, the relative amplitude being proportional to the harmonic mean radius of curvature $[2R_1R_2/(R_1 + R_2)]$ at the point in question.

The next approximation is obtained by substituting the lowest approximation back into the series equation, remembering the relationship between ω and ζ. The final equation may be expressed as follows: We find the point σ, τ on the surface for which u, v is normal and determine the planes of the principal radii of curvature of the surface at this point. We call ϑ_1 the angle displacement from σ, τ in the plane of R_1, that in the orthogonal direction ϑ_2. Then the angle-distribution factor $f(u,v)$, giving the amplitude of the asymptotic wave in the u, v direction, is

$$\begin{aligned} f(u,v) \simeq \left(\frac{2R_1R_2}{R_1 + R_2}\right) \psi_0(\sigma,\tau) &- \left(\frac{\frac{1}{8}R_1^2 + \frac{1}{4}R_1R_2 + \frac{5}{8}R_2^2}{ikaR_2(R_1 + R_2)}\right)\left[(1 - \sin^2\theta_0 \cos^2\alpha_0) \frac{\partial}{\partial\vartheta_1}\left(\frac{R_1}{a}\frac{\partial\psi_0}{\partial\vartheta_1}\right)\right]_{\sigma,\tau} \\ &- \left(\frac{\frac{5}{8}R_1^2 + \frac{1}{4}R_1R_2 + \frac{1}{8}R_2^2}{ikaR_1(R_1 + R_2)}\right)\left[(1 - \sin^2\theta_0 \sin^2\alpha_0) \frac{\partial}{\partial\vartheta_2}\left(\frac{R_2}{a}\frac{\partial\psi_0}{\partial\vartheta_2}\right)\right]_{\sigma,\tau} \end{aligned} \tag{11.4.56}$$

where $a = a(\sigma,\tau)$, θ_0 is the angle between the direction $\mathbf{u}(u,v)$ and the direction $\mathbf{h}(\sigma,\tau)$ from the origin to the point on the surface having a normal in the direction $\mathbf{u}$, α_0 is the angle between the plane containing $\mathbf{u}$ and $\mathbf{h}$ and the plane of the principal radius R_1, and where the values of the derivatives are taken at the point σ, τ.

This expression shows how, to the first order in the small quantity $1/ka$, the finite wavelength "blurs out" the sharp geometrical-optics pattern given by Eq. (11.4.56). The expression is a reasonably good approximation when the boundary value $\psi(\vartheta,\varphi)$ does not change rapidly with ϑ_1 or ϑ_2. A detailed study of the case where ψ_0 changes discontinuously shows that diffraction bands of the usual Fresnel type are produced in f where the discontinuity occurs in the geometrical-optics case of Eq. (11.4.55).

The same analysis, with the same general results, will obtain if we specify normal gradient of ψ, instead of value of ψ, on the surface $r = a(\vartheta,\varphi)$. The higher approximations in the series, of which Eq. (11.4.56) constitutes the first two terms, may also be obtained, by considering the higher terms in the expansion of t and f. The algebraic difficulties are considerable, however, and will not be gone into here. The two-dimensional case, which is of course much simpler, has been developed to all orders, and the case where there is a sharp discontinuity in ψ is also worked out (see bibliography at the end of this chapter, paper by Lax and Feshbach).

Scattering of Waves, Variational Principle. Scattering problems, as well as radiation problems, can be solved by the methods of Green's function—integral equation—variational principle. The case of the scattering of a plane wave from an object of finite dimensions (surface area, volume), having a specified boundary condition (Dirichlet, Neumann, or mixed) at its surface, is a useful one to demonstrate. We use the simple Green's function

$$g_k(\mathbf{r}|\mathbf{r}_0) = (1/R)e^{ikR}$$

for free space for outgoing waves, and we ask for the solution which has zero gradient (for example) on the scatterer's surface and which corresponds to an incident plane wave $Ce^{i\mathbf{k}_i\cdot\mathbf{r}}$, in the z direction with wave number k and wave vector $\mathbf{k}_i = k\mathbf{a}_z$. By our usual methods we see that such a solution will satisfy the integral equation

$$\psi(\mathbf{r}) = Ce^{i\mathbf{k}_i\cdot\mathbf{r}} + \frac{1}{4\pi}\oint \psi(\mathbf{r}_0^s)\,\frac{\partial}{\partial n_0}\,g_k(\mathbf{r}|\mathbf{r}_0^s)\,dA_0 \tag{11.4.57}$$

where the integral is over the surface of the scatterer and $\mathbf{n}_0$ is a normal vector pointing *away* from the excluded inside of the scatterer (therefore, in the opposite sense to the direction of Chap. 7).

Constant C is the amplitude of the incident wave; the incident intensity is some constant D times $|C|^2$. The asymptotic form of the scattered wave may be obtained by using the asymptotic form $(e^{ikr}/r)e^{-i\mathbf{k}_s\cdot\mathbf{r}_0}$ of g_k, where $\mathbf{k}_s = k\mathbf{a}_r$ is the scattered wave vector, in the direction $\mathbf{r}$ of the point of observation at infinity. This scattered wave is, of course, proportional to the incident amplitude,

$$\psi_s \to Cf(\mathbf{k}_i|\mathbf{k}_s)\,\frac{e^{ikr}}{r}$$

$$f(\mathbf{k}_i|\mathbf{k}_s) = \frac{-i}{4\pi C}\oint (\mathbf{k}_s\cdot\mathbf{n}_0)\psi(\mathbf{r}_0^s)e^{-i\mathbf{k}_s\cdot\mathbf{r}_0^s}\,dA_0 \tag{11.4.58}$$

where the function f is the scattered amplitude per incident amplitude. Its square gives the intensity of the scattered wave at an angle ϑ, φ to the incident wave and the integral of $|f|^2$ over ϑ and φ is the *cross section* of the scatterer for scattering a plane wave of incident wave vector $\mathbf{k}_i$ (that is, of wavelength $2\pi/k$ and direction $\mathbf{a}_z$).

For the purpose of setting up a variational principle it is necessary to have the angle-distribution function $f(\mathbf{k}_i|\mathbf{k}_s)$ appear explicitly in integral equation (11.4.57). This may be done by multiplying and dividing the first term on the right-hand side by f, a procedure which also eliminates the arbitrary incident amplitude C;

$$\psi(\mathbf{r}) = \frac{-ie^{i\mathbf{k}_i\cdot\mathbf{r}}}{4\pi f(\mathbf{k}_i|\mathbf{k}_s)} \oint (\mathbf{k}_s\cdot\mathbf{n}_0)\psi(\mathbf{r}_0^s)e^{-i\mathbf{k}_s\cdot\mathbf{r}_0^s}\,dA_0 + \frac{1}{4\pi}\oint \psi(\mathbf{r}_0^s)\,\frac{\partial}{\partial n_0}\,g_k(\mathbf{r}|\mathbf{r}_0^s)\,dA_0 \tag{11.4.59}$$

from which we can obtain $\psi(\mathbf{r})$ if we know $\psi(\mathbf{r}_0^s)$ on the surface of the scatterer. To obtain an integral equation for $\psi(\mathbf{r}_0^s)$ alone we could set $\mathbf{r}$ on the surface in this equation; but this would result in a three-term integral equation and would involve the nonsymmetrical function $\partial g_k/\partial n_0$. A much simpler form of the equation is obtained by taking the normal gradient of both sides of Eq. (11.4.59) at the surface. The left-hand side is then zero, for the normal gradient of ψ on the surface is zero, and we have

$$\frac{(\mathbf{k}_i\cdot\mathbf{n})e^{i\mathbf{k}_i\cdot\mathbf{r}^s}}{f(\mathbf{k}_i|\mathbf{k}_s)} \oint (\mathbf{k}_s\cdot\mathbf{n}_0)\psi(\mathbf{r}_0^s)e^{-i\mathbf{k}_s\cdot\mathbf{r}_0^s}\,dA_0 = -\oint \psi(\mathbf{r}_0^s)\left[\frac{\partial^2}{\partial n\,\partial n_0}\,g_k(\mathbf{r}^s|\mathbf{r}_0^s)\right]dA_0 \tag{11.4.60}$$

A variational principle can now be constructed by multiplying both sides by $\tilde{\psi}(\mathbf{r}^s)$ and integrating over the surface of the scatterer in the $\mathbf{r}^s$ coordinates;

$$-[f] = \frac{\oint (\mathbf{k}_i\cdot\mathbf{n}_1)\tilde{\psi}(\mathbf{r}_1^s)e^{i\mathbf{k}_i\cdot\mathbf{r}_1^s}\,dA_1 \oint (\mathbf{k}_s\cdot\mathbf{n}_0)\psi(\mathbf{r}_0^s)e^{-i\mathbf{k}_s\cdot\mathbf{r}_0^s}\,dA_0}{\oint dA \oint dA_0\tilde{\psi}(\mathbf{r}^s)\psi(\mathbf{r}_0^s)\left[\dfrac{\partial^2}{\partial n\,\partial n_0}\,g_k(\mathbf{r}^s|\mathbf{r}_0^s)\right]} \tag{11.4.61}$$

The correct forms for ψ and $\tilde{\psi}$ give a stationary value for $[f]$ which value is equal to the angle-distribution function $f(\mathbf{k}_i|\mathbf{k}_s)$. That this is true may be seen by varying $\tilde{\psi}$ and equating the coefficient of $\delta\tilde{\psi}$ to zero. The equation also involves $\tilde{\psi}$, however, and we cannot use it until we have discovered what kind of function it is. This is determined by setting the coefficient of $\delta\psi$ equal to zero, which results in the equation

$$\frac{(\mathbf{k}_s\cdot\mathbf{n}_0)e^{-i\mathbf{k}_s\cdot\mathbf{r}_0^s}}{f(\mathbf{k}_i|\mathbf{k}_s)} \oint (\mathbf{k}_i\cdot\mathbf{n})\tilde{\psi}(\mathbf{r}^s)e^{i\mathbf{k}_i\cdot\mathbf{r}^s}\,dA = -\oint \tilde{\psi}(\mathbf{r}^s)\left[\frac{\partial^2}{\partial n\,\partial n_0}\,g_k(\mathbf{r}^s|\mathbf{r}_0^s)\right]dA_0$$

Comparison between this and Eq. (11.4.60) shows that $\tilde{\psi}$ *is the solution of the reciprocal problem,* where a plane wave is sent in with wave vector $-\mathbf{k}_s$, from the point r, ϑ, φ which was the observation point for ψ and which returns from the obstacle a scattered wave with wave vector $-\mathbf{k}_i$ to the point on the z axis at $-\infty$, which was the source point for ψ. The fact that the angle-distribution function f is the same for this reciprocal problem as it is for the original one is only another way of stating the principle of reciprocity. As indicated on page 873, the adjoint solution $\tilde{\psi}$ is related to the process of interchanging source and observer.

Angle-distribution Function and Total Cross Section. We can now state our variational principle for scattering as follows: We assume a general form for the value of ψ on the surface, for an incoming plane wave in the direction $\mathbf{k}_i$ and a corresponding value $\tilde{\psi}$ for the incoming wave in the direction $-\mathbf{k}_s$, and compute the function f given in Eq. (11.4.61), a function of the parameters involved in ψ and $\tilde{\psi}$. The form of ψ (and the corresponding form for $\tilde{\psi}$) which gives zero value to all the first derivatives of f with respect to the parameters of the form of ψ (and thus of $\tilde{\psi}$) is the best form, and the resulting function f is nearest to the correct angle-distribution function $f(\mathbf{k}_i|\mathbf{k}_s)$. In the limit of complete flexibility for the forms of ψ and $\tilde{\psi}$, the forms giving the stationary value of f are the correct forms for ψ and $\tilde{\psi}$ on the boundary surface and the corresponding value of f is the correct $f(\mathbf{k}_i|\mathbf{k}_s)$.

If we wish the form of ψ away from the boundary, we can insert the best value of ψ on the surface into Eq. (11.4.59). But we seldom need to do this, for we usually wish only the angle distribution of scattering and the total cross section for scattering

$$Q(\mathbf{k}_i) = \int^{2\pi} d\varphi \int_0^\pi |f(\mathbf{k}_i|\mathbf{k}_s)|^2 \sin\vartheta \, d\vartheta \qquad (11.4.62)$$

which can be obtained from f by integration over the angles φ, ϑ locating the direction of $\mathbf{k}_s$ with respect to $\mathbf{k}_i$. As a matter of fact, we need not go through this integration but can obtain Q directly from f, according to the following argument.

The scattered intensity in the direction $\mathbf{k}_s$, per unit incident intensity, is

$$|f(\mathbf{k}_i|\mathbf{k}_s)|^2 \left(\frac{1}{r}\right)^2 = -\left(\frac{1}{k|C|^2}\right) \mathrm{Im}\left[\psi_s \frac{\partial\tilde{\psi}_s}{\partial r}\right]_{r\to\infty}$$

where ψ_s is the scattered wave [this is, of course, the second term on the right of Eq. (11.4.57) which goes to $(Cf/r)e^{ikr}$ for $r \to \infty$]. The integral of this over the sphere at infinity gives the cross section Q. But we do not need to integrate over an infinitely large sphere; any integral of $(1/k|C|^2)\ \mathrm{Im}[\psi_s(\partial\tilde{\psi}_s/\partial n)]$ over any closed surface outside the scatterer will give the same result. In particular we could integrate over a surface an infinitesimal distance outside the scattering surface,

$$Q(\mathbf{k}_i) = \frac{-1}{k|C|^2} \operatorname{Im}\left\{\oint \psi_s(\mathbf{r}^s) \left[\frac{\partial}{\partial n} \bar{\psi}_s(\mathbf{r}^s)\right] dA\right\} \tag{11.4.63}$$

But, at any point outside the scatterer, the total ψ is the sum of the incident plane wave $Ce^{i\mathbf{k}_i\cdot\mathbf{r}}$ and the scattered wave ψ_s; since the normal gradient of ψ at the surface is zero, the normal gradient of ψ_s at the surface is equal to minus the normal gradient of the plane wave there,

$$(\partial\psi_s/\partial n) = -iC(\mathbf{k}_i\cdot\mathbf{n})e^{i\mathbf{k}_i\cdot\mathbf{r}^s}; \quad (\partial\bar{\psi}_s/\partial n) = i\bar{C}(\mathbf{k}_i\cdot\mathbf{n})e^{-i\mathbf{k}_i\cdot\mathbf{r}^s}$$

Consequently,

$$Q(\mathbf{k}_i) = \left(\frac{1}{k|C|^2}\right) \operatorname{Im}\left\{i\bar{C} \oint (\mathbf{k}_i\cdot\mathbf{n})\psi_s(\mathbf{r}^s)e^{-i\mathbf{k}_i\cdot\mathbf{r}^s}\, dA\right\}$$
$$= \left(\frac{1}{k|C|^2}\right) \operatorname{Im}\left\{i\bar{C} \oint (\mathbf{k}_i\cdot\mathbf{n})[\psi(\mathbf{r}^s) - Ce^{i\mathbf{k}_i\cdot\mathbf{r}^s}]e^{-i\mathbf{k}_i\cdot\mathbf{r}^s}\, dA\right\}$$

The integral of the second term in the brackets is zero, because the normal outflow integral of a constant vector $\mathbf{k}_i$ is zero. The integral of the first term, according to Eq. (11.4.58), is proportional to the limiting value of f when $\mathbf{k}_s$ is made equal to $\mathbf{k}_i$. Therefore,

$$Q(\mathbf{k}_i) = (4\pi/k) \operatorname{Im}[f(\mathbf{k}_i|\mathbf{k}_i)] \tag{11.4.64}$$

Instead of integrating $|f|^2$ over all directions of $\mathbf{k}_s$, we obtain Q from the imaginary part of the limiting value of f itself, for $\mathbf{k}_s \to \mathbf{k}_i$ [see Eq. (9.3.23)].

If the surface is a sphere of radius a, the Green's function may be expanded in spherical harmonics [see Eq. (11.3.44)]. When we take the gradients (which are now derivatives with respect to r and r_0), we remember that r_0 is put on the surface first, consequently the factor suitable for $r > r_0$ is used.

$$\frac{\partial^2}{\partial r\, \partial r_0} g_k = ik^3 \sum_{n=0}^{\infty} (2n+1) \sum_{m=0}^{n} \epsilon_m \frac{(n-m)!}{(n+m)!} \cos[m(\varphi - \varphi_0)] \cdot$$
$$\cdot P_n^m(\cos\vartheta_0) P_n^m(\cos\vartheta) j_n'(kr_0) h_n'(kr) \tag{11.4.65}$$

with r_0 and r then set equal to a.

Scattering from Spheres. If the scattering surface is a sphere and if ψ and $\bar{\psi}$ on the surface are expanded in spherical harmonics, the results of the present technique are just those analogous to Eqs. (11.3.72) for $\partial\psi/\partial n = 0$ (as, of course, they must be). If we set $\mathbf{k}_i$ along the spherical axis, then ψ is independent of φ and

$$\psi(\mathbf{r}^s) = \sum_{n=0}^{\infty} A_n P_n(\cos\vartheta)$$

where A_n are the arbitrary parameters to vary; then, if ϑ_s, φ_s are the angles relating $\mathbf{k}_s$ to $\mathbf{k}_i$, we have that

$$\tilde{\psi}(\mathbf{r}^s) = \sum_{n=0}^{\infty} (-1)^n A_n \sum_{m=0}^{n} \epsilon_m \frac{(n-m)!}{(n+m)!} \cdot$$
$$\cdot P_n^m(\cos\vartheta_s) P_n^m(\cos\vartheta) \cos[m(\varphi - \varphi_s)]$$

Likewise the plane wave expansions multiplied by $(\mathbf{k}\cdot\mathbf{n})$ are

$$(\mathbf{k}_i\cdot\mathbf{n})e^{i\mathbf{k}_i\cdot\mathbf{r}} = k\sum_{n=0}^{\infty} i^n[nP_{n-1}(\cos\vartheta) + (n+1)P_{n+1}(\cos\vartheta)]j_n(ka)$$

$$(\mathbf{k}_s\cdot\mathbf{n})e^{-i\mathbf{k}_s\cdot\mathbf{r}} = k\sum_{n=0}^{\infty} (-i)^n j_n(ka) \cdot$$
$$\cdot \sum_m \left[n\frac{(n-m-1)!}{(n+m-1)!} P_{n-1}^m(\cos\vartheta_s)P_{n-1}^m(\cos\vartheta) \right.$$
$$\left. - (n+1)\frac{(n-m+1)!}{(n+m+1)!}P_{n+1}^m(\cos\vartheta_s)P_{n+1}^m(\cos\vartheta)\right]\cos[m(\varphi-\varphi_s)]$$

Combining all these and performing the integrations of Eq. (11.4.61), we have

$$-[f] = \frac{N^2}{ikM}; \quad N = \sum_{n=0}^{\infty}(-1)^n A_n j_n'(ka)P_n(\cos\vartheta_s)$$

$$M = \sum_{n=0}^{\infty} A_n^2 \left[\frac{P_n(\cos\vartheta_s)}{2n+1}\right] j_n'(ka)h_n'(ka)$$

where

$$j_n'(ka) = \frac{1}{k}\frac{d}{da}j_n(ka)$$
$$= \left(\frac{n}{2n+1}\right)j_{n-1}(ka) - \left(\frac{n+1}{2n+1}\right)j_{n+1}(ka) = -D_n'\sin\delta_n'$$
$$h_n'(ka) = \frac{1}{k}\frac{d}{da}h_n(ka) = iD_n'e^{i\delta_n'}$$

Setting the partial derivative of $[f]$ with respect to each A_n equal to zero gives rise to the infinite sequence of equations

$$A_n = (-1)^n\frac{2n+1}{h_n'(ka)}\left(\frac{M}{N}\right)$$

which seems, at first sight, to be a rather complicated set of simultaneous equations, for M and N contain all the A's. However, we notice that, if the equations are solved, then $M = N$, so that we have the simple set of relations

$$A_n = [(-1)^n(2n + 1)/h'_n(ka)] = [(-1)^n(2n + 1)e^{-i\delta_n'}/iD'_n]$$

and

$$\begin{aligned} f(\mathbf{k}_i|\mathbf{k}_s) &= -\frac{1}{k}\sum_{n=0}^{\infty}(2n + 1)e^{-i\delta_n'}\sin\delta'_n P_n(\cos\vartheta_s) \\ Q(\mathbf{k}_i) &= \frac{4\pi}{k^2}\sum_{n=0}^{\infty}(2n + 1)\sin^2\delta'_n \end{aligned} \tag{11.4.66}$$

[using Eq. (11.4.64) for Q] which are just those analogous to Eqs. (11.3.72) for Neumann conditions.

Thus our variational principle gives us the same exact solution, if we expand in spherical harmonics as if we had carried out the procedures of Sec. 11.3. This is not surprising, of course; the two results must be the same. But the present technique is somewhat better designed to obtain the quantities of physical interest with a somewhat smaller amount of work. For example, since we have determined A_n, the value of ψ on the surface of the sphere is immediately given (except for a factor of proportionality),

$$\psi(\mathbf{r}^s) = -iA\sum_{n=0}^{\infty}(-1)^n\left[\frac{2n + 1}{D'_n(ka)}\right]e^{-i\delta_n'(ka)}P_n(\cos\vartheta) \tag{11.4.67}$$

We also see that, if we use a trial function ψ with a finite number of spherical harmonics, the best results which can be obtained are the same series for f, Q, and $\psi(\mathbf{r}^s)$, with the sums over a finite number of n's. But, except for an investigation of the limiting behavior for short wavelengths, which we shall take up later, the use of the present technique for a spherical scattering surface is just duplicating what can be done by the use of eigenfunctions. The method comes into its own when the scattering surface is of more complex shape, such that eigenfunction solutions are not fully tabulated or else are not possible, because the surface is not a part of a separable coordinate system.

Scattering from a Strip. Although the Mathieu function solutions for the scattering of waves from a strip are known, the variational calculation of this case is of some advantage, for the results, though not exact, are in a more compact form than Eqs. (11.2.102) *et seq.* We use the steady-state flow approximation for ψ on the surface, as we have done many times before, and then apply Eq. (11.4.61) to obtain a fairly good approximation for the scattering.

The integrals of Eq. (11.4.61) are over both sides of the strip, which reduce to integrals of the difference in value of ψ on the two sides of the strip, integrated over one side. Suppose the strip to be in the y, z plane, with its axis along the z axis and to have width a (strip is in region $x = 0$, $|y| < \frac{1}{2}a$). Suppose the direction of the incident wave is in the x, y plane, at an angle φ_i to the x axis and we wish to measure the amount scattered at angle φ_s to the x axis. Equation (10.11.28) indicates that the difference between the value of ψ at point $(0^+,y)$ and that at point $(0^-,y)$ (for $|y| < \frac{1}{2}a$) is $B\sqrt{1-(2y/a)^2}\cos\varphi_i$, for a steady flow of fluid past a strip at incident angle φ_i with respect to the normal to the strip. Here B is a constant which will cancel out in the variational calculation, so it does not need to be evaluated. Similarly, the difference for a flow in the $-\varphi_s$ direction would be $-B\sqrt{1-(2y/a)^2}\cos\varphi_s$.

Inserting these assumed forms in Eq. (11.4.61), we obtain for one of the integrals of the numerator

$$\begin{aligned}
\oint (\mathbf{k}_i \cdot \mathbf{n})\tilde{\psi}(\mathbf{r}^s)e^{i\mathbf{k}_i\cdot\mathbf{r}^s}\,dA
&= -kB\cos\varphi_i\cos\varphi_s\int_{-\frac{1}{2}a}^{\frac{1}{2}a}\sqrt{1-\left(\frac{2y}{a}\right)^2}\,e^{iky\sin\varphi_i}\,dy \\
&= -\tfrac{1}{2}kaB\cos\varphi_i\cos\varphi_s\int_0^\pi (1-\cos^2 u)e^{i(ka/2)\sin\varphi_i\cos u}\,du \\
&= -\tfrac{1}{2}\pi kaB\cos\varphi_i\cos\varphi_s[J_0(\tfrac{1}{2}ka\sin\varphi_i)+J_2(\tfrac{1}{2}ka\sin\varphi_i)] \\
&= -(2\pi B/\sin\varphi_i)\cos\varphi_i\cos\varphi_s J_1(\tfrac{1}{2}ka\sin\varphi_i)
\end{aligned}$$

The other integral in the numerator is

$$(2\pi B/\sin\varphi_s)\cos\varphi_i\cos\varphi_s J_1(\tfrac{1}{2}ka\sin\varphi_s)$$

To compute the integral in the denominator we need a convenient form for g. This turns out to be the Fourier integral representation

$$g_k(x,y|x_0y_0) = i\int_{-\infty}^{\infty}\exp[i\omega(y-y_0)+i|x-x_0|\sqrt{k^2-\omega^2}]\,\frac{d\omega}{\sqrt{k^2-\omega^2}}$$

where the radical is positive imaginary when $\omega > k$. We thus have

$$\left[\frac{\partial^2}{\partial x\,\partial x_0}g_k\right]_{x,x_0\to 0} = i\int_{-\infty}^{\infty}\sqrt{k^2-\omega^2}\,e^{i\omega(y-y_0)}\,d\omega$$

so the integral in the denominator becomes

$$\begin{aligned}
\oint dA\oint dA_0\tilde{\psi}(\mathbf{r}^s)\psi(\mathbf{r}_0^s)\left[\frac{\partial^2}{\partial n\,\partial n_0}g_k(\mathbf{r}^s|\mathbf{r}_0^s)\right]
&= -iB^2\cos\varphi_i\cos\varphi_s\int_{-\infty}^{\infty}\sqrt{k^2-\omega^2}\,d\omega\,\cdot \\
&\quad\cdot\int_{-\frac{1}{2}a}^{\frac{1}{2}a}\sqrt{1-\left(\frac{2y}{a}\right)^2}\sqrt{1-\left(\frac{2y_0}{a}\right)^2}\,e^{i\omega(y-y_0)}\,dy\,dy_0 \\
&= -8\pi^2B^2i\cos\varphi_i\cos\varphi_s\int_0^\infty\sqrt{k^2-\omega^2}\,[J_1(\tfrac{1}{2}a\omega)]^2\,\frac{d\omega}{\omega^2} \\
&= 8\pi^2B^2\cos\varphi_i\cos\varphi_s(R-iX)
\end{aligned}$$

where

$$R = \int_0^1 \sqrt{1 - x^2}\, J_1^2(\tfrac{1}{2}kax) \left(\frac{dx}{x^2}\right) = \int_0^{\frac{1}{2}\pi} \cot^2 u\, J_1^2(\tfrac{1}{2}ka \sin u)\, du$$
$$X = \int_1^\infty \sqrt{x^2 - 1}\, J_1^2(\tfrac{1}{2}kax) \left(\frac{dx}{x^2}\right) = \int_0^{\frac{1}{2}\pi} \sin u \tan u\, J_1^2(\tfrac{1}{2}ka \sec u)\, du$$

These integrals may be evaluated numerically. They are finite for all real values of ka.

The best value of the angle-distribution function f and of the effective width Q can then be written down

$$f(\mathbf{k}_i|\mathbf{k}_s) \simeq \frac{-\cot \varphi_i \cot \varphi_s}{2(R - iX)} J_1(\tfrac{1}{2}ka \sin \varphi_i)\, J_1(\tfrac{1}{2}ka \sin \varphi_s)$$
$$Q(\mathbf{k}_i) \simeq \frac{2\pi}{k} \left(\frac{X}{R^2 + X^2}\right) \cot^2 \varphi_i\, J_1^2(\tfrac{1}{2}ka \sin \varphi_i) \tag{11.4.68}$$

The scattered wave is zero if either the incident or the scattered wave is in the plane of the strip (φ_i or $\varphi_s = \frac{1}{2}\pi$); if the incident wave strikes the strip normally ($\varphi_i = 0$), the effective width is $\frac{1}{8}\pi k a^2 X/(R^2 + X^2)$. A quite similar (though very much more difficult) calculation has been carried out for the disk (see bibliography under Levine and Schwinger; actually the calculations were done for a round hole in a plane, but this is the same, by Babinet's principle).

Scattering of Short Waves. As a last example of the use of the integral methods, we treat the scattering of a plane wave by a rigid sphere in the limit that the wavelength $2\pi/k$ is very small compared to the radius of the sphere a. In this case the most convenient form of Green's equation is the one relating the scattered wave ψ_s and the Green's function $g_k = (1/R)e^{ikR}$ for the infinite domain:

$$\psi_s(\mathbf{r}) = \frac{1}{4\pi} \oint \left[\psi_s(\mathbf{r}_0^s) \frac{\partial}{\partial n_0} g_k(\mathbf{r}|\mathbf{r}_0^s) - g_k(\mathbf{r}|\mathbf{r}_0^s) \frac{\partial}{\partial n_0} \psi_s(\mathbf{r}_0^s)\right] dA_0$$

where we have again reversed the sign of $\mathbf{n}_0$, so it is now pointed outward, away from the excluded inside of the scatterer.

Since the normal gradient of the whole solution ψ is to be zero at the surface of the scatterer, the normal gradient of ψ_s must equal minus the normal gradient of the incident plane wave ψ_i there, so

$$\psi_s(\mathbf{r}) = \frac{1}{4\pi} \oint \left[\psi_s(\mathbf{r}_0^s) \frac{\partial}{\partial n_0} g_k(\mathbf{r}|\mathbf{r}_0^s) + g_k(\mathbf{r}|\mathbf{r}_0^s) \frac{\partial}{\partial n_0} \psi_i(\mathbf{r}_0^s)\right] dA_0 \tag{11.4.69}$$

where $\psi = \psi_i + \psi_s$ and ψ_i is the plane wave $Ce^{i\mathbf{k}_i\cdot\mathbf{r}}$.

For very short waves we know that the part of the surface for which $\mathbf{n} \cdot \mathbf{k}_i$ is positive (if the surface is everywhere convex) is in the "shadow," whereas those parts for which $\mathbf{n} \cdot \mathbf{k}_i$ is negative are "illuminated" by the incident wave. What we mean by "shadow" is that ψ_s almost completely cancels ψ_i, in value as well as in normal gradient, there. In the "illuminated" part, on the other hand, while $\partial\psi_s/\partial n$ cancels $\partial\psi_i/\partial n$, ψ_s is approximately equal to ψ_i at the surface. Therefore, our scattered wave can be represented approximately by integrals of the product of the Green's function and the incident wave, different combinations for the two portions of the surface;

$$\psi_s(\mathbf{r}) \simeq \frac{1}{4\pi} \iint\limits_I \left[\psi_i(\mathbf{r}_0^s) \frac{\partial}{\partial n_0} g_k(\mathbf{r}|\mathbf{r}_0^s) + g_k(\mathbf{r}|\mathbf{r}_0^s) \frac{\partial}{\partial n_0} \psi_i(\mathbf{r}_0^s) \right] dA_0$$
$$+ \frac{1}{4\pi} \iint\limits_S \left[g_k(\mathbf{r}|\mathbf{r}_0^s) \frac{\partial}{\partial n_0} \psi_i(\mathbf{r}_0^s) - \psi_i(\mathbf{r}_0^s) \frac{\partial}{\partial n_0} g_k(\mathbf{r}|\mathbf{r}_0^s) \right] dA_0 \quad (11.4.70)$$

where the subscript I indicates the integration is to be over the illuminated portion and the subscript S is to indicate the part in shadow.

These two integrands have very different behavior. The first represents the "reflected wave," discussed on page 1380; the second is the "shadow-forming" wave, which is needed to cancel the incident wave. If the second integral were over a closed surface, it would exactly reproduce (and exactly cancel) the incident wave; since it is over only a part of a surface, it cancels only the incident wave behind the object.

A transformation, called the *Maggi transformation*, allows the integration to be considerably simplified for the shadow-forming wave. We note that the vector

$$\mathbf{A} = (g_k \operatorname{grad}_0 \psi_i - \psi_i \operatorname{grad}_0 g_k)$$

considered as a function of $\mathbf{r}_0^s$, has zero divergence (except when $\mathbf{r}$ is on the surface also). For

$$\operatorname{div} \mathbf{A} = g_k \nabla_0^2 \psi_i - \psi_i \nabla_0^2 g_k = 4\pi \psi_i(\mathbf{r}_0^s) \delta(\mathbf{r} - \mathbf{r}_s^0)$$

which is zero as long as $\mathbf{r}$ is not on the surface. Therefore, $\mathbf{A}$ can be considered to be the curl of a vector (see page 53) $\mathbf{B}$ and, using Stokes' theorem (page 43), we have

$$\frac{1}{4\pi} \iint\limits_S \operatorname{curl} \mathbf{B} \cdot d\mathbf{A}_0 = \frac{1}{4\pi} \oint \mathbf{B} \cdot d\mathbf{s} \quad (11.4.71)$$

where the line integral of $\mathbf{B}$ *is along the line which separates the shadow from the illuminated part of the surface* (call it the *shadow line*). The Maggi transformation gives an expression for $\mathbf{B}$ in terms of g_k and of ψ_i. We substitute our surface integral by an equivalent line source along the shadow line.

But actually, we do not need to compute the vector **B**, beyond noting the obvious fact that it depends only on the Green's function and the incident wave and thus does not depend on the shape of the scattering surface (only on the shape of the shadow line). Therefore, we have shown that the *shadow-forming wave is the same for all surfaces which have the same shadow line* [at least to the degree of approximation represented by Eq. (11.4.70)]. In particular it is the same as the one for an opaque film, the edge of which coincides with the shadow line of the original scatterer.

To show how this goes, we return to the sphere of radius a. The equivalent surface, as far as the shadow-forming wave goes, is a disk of radius a, concentric with the sphere and normal to the direction of the incident wave. The shadow-forming wave is thus the negative of the wave radiated by a disk vibrating with the uniform phase and amplitude of the incident plane wave. Close to the disk this wave has the familiar Fresnel diffraction rings around its edge; far from the origin the usual Frauenhofer pattern appears. Since we have been calculating the asymptotic form of ψ_s, we shall compute the Frauenhofer case.

The integral is thus over φ_0 and r_0, polar coordinates on the disk, ψ_i is Ce^{ikz}, and the asymptotic form for g_k is $(e^{ikr}/r)e^{-i\mathbf{k}_s\cdot\mathbf{r}_0}$ where $\mathbf{k}_s$ is a vector in the direction ϑ, φ, pointing to the point of observation. The shadow-forming wave is thus, approximately,

$$
\begin{aligned}
-ikC(1 + \cos\vartheta)\frac{e^{ikr}}{4\pi r}\int_0^{2\pi} d\varphi_0 \int_0^a r_0\,dr_0\, e^{-ikr_0\sin\vartheta\cos(\varphi_0-\varphi)}
&= -\tfrac{1}{2}ikC(1+\cos\vartheta)\frac{e^{ikr}}{r}\int_0^a J_0(kr_0\sin\vartheta)r_0\,dr_0 \\
&= -\tfrac{1}{2}ika^2C\left[\left(\frac{1+\cos\vartheta}{ka\sin\vartheta}\right)J_1(ka\sin\vartheta)\right]\frac{e^{ikr}}{r}
\end{aligned}
$$

The quantity in brackets is unity at $\vartheta = 0$, straight beyond the disk, in the center of the shadow; it drops to zero at about $\sin\vartheta = 3.8/ka$ (which is a small angle when $|ka| \gg 1$) and remains small for larger ϑ. This term is thus important only directly behind the object (which is not surprising since it is the shadow-forming term).

The first integral in Eq. (11.4.70), representing the reflected wave, cannot be manipulated in quite so cavalier a manner. Because of the plus sign in place of the minus, it cannot be represented by a line integral and, for short wavelengths, it must be calculated by the method of steepest descent. The calculation is very similar to that given on page 1539. We shall consider only the case of the sphere here. We let the unit vector normal to the surface at ϑ_0, φ_0 be $\mathbf{n}_0$, vector $\mathbf{k}_i = k\mathbf{a}_z$ be the incident wave vector, and $\mathbf{k}_s = k\mathbf{a}_r$ be in the direction of the observation point; then the first integral of Eq. (11.4.70) becomes

$$\left(\frac{ia^2C}{4\pi}\right)\int_0^{2\pi} d\varphi_0 \int_{\frac{1}{2}\pi}^{\pi} (\mathbf{k}_i - \mathbf{k}_s)\cdot \mathbf{n}_0 e^{i(\mathbf{k}_i-\mathbf{k}_s)\cdot\mathbf{n}_0 a} \sin\vartheta_0\, d\vartheta_0 \left(\frac{e^{ikr}}{r}\right) = \left(\frac{a^2C}{4\pi}\right)\left(\frac{\partial F}{\partial a}\right)\frac{e^{ikr}}{r}$$

where
$$F = \int_0^{2\pi} d\varphi_0 \int_{\frac{1}{2}\pi}^{\pi} e^{i(\mathbf{k}_i-\mathbf{k}_s)\cdot\mathbf{n}_0 a} \sin\vartheta_0\, d\vartheta_0$$

The saddle point for ϑ_0 in the integral F is the point on the sphere where vector $\mathbf{n}_0$ is antiparallel to vector $(\mathbf{k}_i - \mathbf{k}_s)$. This is a vector pointed "halfway between" $\mathbf{k}_i$ and $\mathbf{k}_s$. It corresponds to the point on the sphere at which the incident wave is specularly reflected in the direction $\mathbf{k}_s$. Since vector $(\mathbf{k}_i - \mathbf{k}_s)$ has magnitude $2k \sin(\frac{1}{2}\vartheta)$ and is pointed in the direction $\frac{1}{2}\vartheta$, φ, we take the unit vector $\mathbf{n}_s$, in this direction, as the axis for integration over the sphere; instead of ϑ_0, φ_0 we use the corresponding angles u, v. Our integral is then

$$F = \int dv \int e^{2ika\sin(\vartheta/2)\cos u} \sin u\, du$$

where the limits are rather complicated because they correspond to $\frac{1}{2}\pi \le \vartheta_0 \le \pi$, which is not simple in terms of u.

However we are expecting to find an asymptotic expression for F, so we will integrate over u in the complex plane, arranging so that, near $u = \pi$, the exponential will have constant phase and will have its maximum at $u = \pi$. When this is done, the exact nature of the limits on u and v will not be important except when $a\mathbf{n}_s$ is very close to the shadow line, *i.e.*, when $\vartheta \to 0$. Consequently,

$$\begin{aligned} F &\simeq 2\pi \int_\pi^{i\infty} e^{2ika\sin(\vartheta/2)\cos u} \sin u\, du \\ &= -2\pi i e^{-2ika\sin(\vartheta/2)} \int_0^\infty e^{-2ka\sin(\vartheta/2)x}\, dx \\ &= -[\pi i e^{-2ika\sin(\vartheta/2)}/ka \sin(\tfrac{1}{2}\vartheta)] \end{aligned}$$

so that the first integral of Eq. (11.4.20) is, approximately,

$$-(aC/2r)\exp\{ik[r - 2a\sin(\tfrac{1}{2}\vartheta)]\}$$

the phase of the reflected wave being delayed by the amount $2ak \sin(\frac{1}{2}\vartheta)$, as "geometrical optics" would require.

To the order of approximation inherent in our assumptions as to the nature of ψ_s on the spherical surface, the asymptotic form for ψ_s is thus

$$\psi_s \to Cf(\vartheta)(e^{ikr}/r); \quad r/a \to \infty$$

where

$$f(\vartheta) \simeq \tfrac{1}{2}ia\left\{ie^{-2ika\sin(\vartheta/2)} - ka\left[\left(\frac{1+\cos\vartheta}{ka\sin\vartheta}\right)J_1(ka\sin\vartheta)\right]\right\} \tag{11.4.72}$$

as $ka \to \infty$. The reflected wave has amplitude independent of ϑ, to this approximation. The shadow-forming wave has a sharp peak in the direction $\vartheta = 0$; it is negligible in other directions.

The expression for the reflected wave is valid except when $\vartheta \to 0$; we thus cannot use Eq. (11.4.64) for this part; to obtain Q we integrate $|f|^2$ over all directions. But since the reflected part of f is independent of ϑ, we can obtain this part of the cross section by multiplying the square of its magnitude by 4π, obtaining πa^2, for the power reflected per unit incident intensity. Also integrating $|f|^2$ for the shadow-forming part, we find its cross section is also πa^2. Consequently, the total cross section for *scattering* is $2\pi a^2$, twice the effective cross section of the sphere, for very short wavelengths, as stated on pages 1381 and 1485. One half of this is the reflected part, the other half the shadow-forming part.

Problems for Chapter 11

11.1 A string of variable mass density $\rho = \rho_0[1 + b(x)]$ gm per cm length is stretched with tension T between rigid supports a distance l apart (the supports are at $x = 0$ and $x = l$). Use the perturbation technique of Chap. 9 to obtain first-order expressions for the shapes and frequencies of the various normal modes of vibration of the string.

11.2 A uniform string of density ρ and tension T extends from $x = 0$ to $x = l$. Transverse motion is opposed by a frictional force $R(\partial y/\partial t)$ dynes per cm. Show that if a transverse force $F_0e^{-i\omega t}$ is applied to the $x = 0$ end, the displacement will be

$$y = (F_0/\rho c\alpha)e^{-(\alpha x/c)-i\omega t}; \quad \alpha^2 = -\omega^2 - i\omega(R/\rho)$$

Show (by use of the tables of Laplace transforms at the end of this chapter) that, if the transverse force at $x = 0$ is a delta function impulse at $t = 0$, then the displacement of the string will be

$$y = u\left(t - \frac{x}{c}\right)e^{-Rt/2\rho}\,J_0\left[\left(\frac{iR}{2\rho}\right)\sqrt{t^2 - \left(\frac{x}{c}\right)^2}\right]$$

11.3 A rectangular enclosure, with inner boundaries the planes $x = 0, x = a, y = 0, y = b, z = 0, z = l$, is an acoustical resonator, being filled with air at standard pressure and temperature. The strip of the wall $z = 0$ between $x = x_0 - \frac{1}{2}\Delta$ and $x = x_0 + \frac{1}{2}\Delta$ vibrates with velocity $v_z = Ue^{-i\omega t}$; the rest of the boundary surface is rigid. Show that the mean acoustic impedance at the surface of the driving strip, the ratio between mean pressure and velocity for the strip, is equal to

$$Z_{\text{av}} = i\rho c\left\{\frac{\Delta}{\pi\tan(kl)} - \frac{8a^2k}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin^2(\pi n\Delta/2a)\cos^2(\pi n x_0/a)}{n^3\tau_n\tanh(\pi n l\tau_n/a)}\right\}$$

where $k = \omega/c$, $\tau_n^2 = 1 - (ak/\pi n)^2$ and where we have assumed that $\omega < (\pi c/a)$. Find the limiting value of Z as $\omega \to 0$. Show that it behaves as though the strip had a stiffness and a mass load and find the limiting values of effective mass and stiffness.

11.4 A rectangular duct, with rigid side walls at $x = 0, x = a, y = 0, y = b$, extends from $z = 0$ to $z = -\infty$. The acoustic impedance of the end $z = 0$, $[i\omega\rho\psi/(\partial\psi/\partial z)]_{z=0} = Z(x)$ is a function of x only [see discussion preceding Eq. (11.2.14)]. Show that the integral equation for the velocity potential inside the duct, when an incident simple-harmonic plane wave is sent along the duct from $-\infty$, is

$$\psi(x,z) = \left\{2A\cos(kz) + \left(\frac{ik\rho c}{4\pi}\right)\int_0^a \frac{\psi(x_0,0)}{Z(x_0)} G_k(x,z|x_0,0)\,dx_0\right\} e^{-i\omega t}$$

where $\omega = kc$, A is the incident amplitude. The Green's function is

$$G_k = \left(\frac{4\pi i}{ak}\right)e^{-ikz} + \sum_{n=1}^{\infty}\left(\frac{8\pi}{a\kappa_n}\right)\cos\left(\frac{\pi n x_0}{a}\right)\cos\left(\frac{\pi n x}{a}\right)e^{\kappa_n z};\quad z \le 0$$

where $\kappa_n^2 = (\pi n/a)^2 - k^2$ (if we assume that $\omega < \pi c/a$). Show that, to the first order in the quantity $\eta(\xi) = [\rho c/Z(a\xi)]$ (assumed small) the wave function at $z = 0$ is

$$\psi(x,0) \simeq 2A\left\{1 - \int_0^1 \eta(\xi)\,d\xi + \sum_{n=1}^{\infty}\left(\frac{2ik}{\kappa_n}\right)\cos\left(\frac{\pi n x}{a}\right)\int_0^1 \eta(\xi)\cos(\pi n\xi)\,d\xi\right\}$$

Show that the reflected wave for $z \ll 0$ has the form $A(1 - T)e^{-ikz}$ where T might be called the absorption factor (if T is real, it equals the fraction of incident energy absorbed by the barrier at $z = 0$). Calculate the expression for T to the second order in η. Is it real if η is real? If not, why not?

11.5 Set up a variational principle for Prob. 11.4. Show that, if the shape of the trial function $\varphi(\xi)$ is adjusted so that

$$[T] = \frac{2\left[\int_0^1 \eta(\xi)\varphi(\xi)\,d\xi\right]^2}{\int_0^1 \eta(\xi)\varphi^2(\xi)\,d\xi - \left(\frac{ika}{4\pi}\right)\int_0^1 d\lambda \int_0^1 d\xi\,\eta(\lambda)\varphi(\lambda)G_k(a\lambda,0|a\xi,0)\eta(\xi)\varphi(\xi)}$$

is stationary, then φ is proportional to $\psi(x,0)$ (compute the proportionality factor) and the stationary value of $[T]$ is the absorption factor T of the previous problem.

11.6 The impedance of the barrier at $z = 0$ in Probs. 11.4 and 11.5 is $10\rho c$ for $0 \le x \le \frac{1}{2}a$ and ∞ for $\frac{1}{2}a \le x \le a$. Calculate and plot the first-order approximation of $\psi(x,0)$ for $k = \pi/\sqrt{2a}$. Calculate T to the second order, using the formula of Prob. 11.4, for the same value of k.

Compute T, using the formula of Prob. 11.5, using $\varphi = 1$, and compare the two values.

11.7 A circular membrane of mass σ per unit area, under tension T, is rigidly clamped along its edge, a circle of radius a. The membrane closes one side of an airtight vessel (as does the diaphragm of a kettle-drum). The speed of sound in air is enough greater than that of waves on the membrane so that we can say that the reaction pressure of the air in the vessel is proportional to the mean displacement of the membrane from equilibrium, $p = -\rho c_a^2 \pi a^2 \bar{\eta}/V$, where ρ is the density of air, c_a the sound velocity in air, V the volume of the vessel, and $\bar{\eta}$ the mean displacement of the membrane. Show that the equation of motion of the membrane is

$$\nabla^2\eta - (1/c^2)(\partial^2\eta/\partial t^2) = (\rho c_a^2/VT) \int_0^a \int_0^{2\pi} \eta r \, dr \, d\varphi$$

where $c^2 = T/\sigma$. Show that the allowed frequencies of free vibration of the membrane-vessel system is obtained from the roots of the equation

$$(\omega a/c)^2 J_0(\omega a/c) + (\pi \rho c_a^2 a^4/\sigma c^2 V) J_2(\omega a/c) = 0$$

for the vibrations independent of φ and from the solutions of $J_m(\omega a/c) = 0$ for the angle-dependent motion.

11.8 A circular membrane of radius a, density σ, tension T, held rigidly at its edges, is started into motion by a pressure pulse $\delta(t)$, spread equally over one face. Show that, if all other effects of the air load are neglected, the subsequent motion of the diaphragm is

$$\psi = \frac{2a}{\sigma c} \sum_m \frac{J_0(\pi\beta_m r/a)}{(\pi\beta_m)^2 J_1(\pi\beta_m)} \sin(\pi\beta_m ct/a)$$

where β_m is the mth root of $J_0(\pi\beta_m) = 0$.

11.9 What is the effect on the natural frequencies of vibration of a circular membrane of a small additional mass attached to its center?

11.10 A cylindrical wave, $Ae^{-i\omega t}H_0^{(1)}(kR)(kc = \omega)$, is sent out from the line $\varphi' = 0$, $r' = b$, where $R^2 = r^2 + b^2 - 2rb\cos\varphi$. This wave is scattered from a cylinder of radius a $(a < b)$ whose axis is at the point $r = 0$. Show that, if the boundary condition is that the velocity potential is zero at $r = a$, then the form of the wave at large distances is

$$\psi \to A\sqrt{\frac{2}{\pi kr}}\, e^{ik(r-ct)} \sum_{m=0}^{\infty} (2 - \delta_{0m}) \cos(m\varphi) i^{-m} e^{-i\pi/4} J_m(kb) \left[\frac{H_m^{(1)}(kb)}{H_m^{(1)}(ka)}\right]$$

where $k = \omega/c$. What is the expression for the distribution in angle of the intensity?

11.11 A cylinder of radius a vibrates to and fro so that the radial velocity at its surface is $U \cos\varphi e^{-i\omega t}$. Compute the radiated velocity potential, the asymptotic expression for the intensity, and the total power radiated per unit length of cylinder. Show that the net reaction force back on the cylinder, because of its motion through the air, is

$$F = -ia\pi\rho cU\left[\frac{C_1(ka)}{C_1'(ka)}\right]e^{i(\delta_1-\delta_1')-i\omega t} \to \begin{cases} -i\omega(\pi a^2\rho)Ue^{-i\omega t}; & ka \ll 1 \\ \pi a\rho cUe^{-i\omega t}; & ka \gg 1 \end{cases}$$

11.12 Obtain formulas similar to Eqs. (11.3.76) for the scattering and absorption of sound from a long cylinder of radius a, covered with sound-absorbing material such that the ratio of pressure to radial velocity at $r = a$ is $-2\rho c$. Plot scattering and absorption cross sections as functions of ka for the range $0 \le ka \le 2$. What is the angular distribution of the scattered wave at $ka = 2$?

11.13 Compute the separation constants be_2 and be_4 and the Fourier series for $Se_2(h, \cos\vartheta)$ and $Se_4(h, \cos\vartheta)$, in powers of h, out to the fourth power of h, by the method of page 1412 and also by the use of Eqs. (11.2.87) and (11.2.88) and compare.

11.14 A flexible membrane of mass σ per unit area, under tension T, has its outer edge held stationary along an elliptic boundary of major axis $\frac{1}{2}a\cosh\mu_0$ and minor axis $\frac{1}{2}a\sinh\mu_0$. One side of this membrane is subjected to a periodic overpressure $P_0e^{-i\omega t}$, the amplitude being the same at each point of the membrane. Show that the expression for the displacement ψ of the membrane at the point μ, ϑ (elliptic coordinates) is

$$\psi = \frac{P_0e^{-i\omega t}}{\sigma\omega^2}\left\{1 - \sum_{m=0}^{\infty}\frac{2\pi B_0^e(h,2m)}{M_{2m}^e(h)Je_{2m}(h, \cosh\mu_0)}Se_{2m}(h, \cos\vartheta)\cdot\right.$$
$$\left.\cdot\, Je_{2m}(h, \cosh\mu)\right\};\quad 0 \le \mu \le \mu_0$$

where $h = a\omega/2c$ and $c^2 = T/\sigma$. Using Eqs. (11.2.85) and (11.2.88), compute and plot the amplitude of ψ as a function of μ for $\vartheta = 0$, $\vartheta = \frac{1}{2}\pi$ for $h = 1, 2$. (Use Table XVI, Appendix, where appropriate.)

11.15 Express the function $Se_1(h, \cos\vartheta)$, for h large, in terms of the derivative of the theta function ϑ_2 or ϑ_4 with respect to ϑ, as indicated on page 1418. From this result calculate the allowed frequency of vibration of an elliptic membrane of great eccentricity, for the mode corresponding to Se_1.

11.16 A hollow duct has an elliptic cylinder boundary, with major axis of its cross section A and minor axis B. Discuss the transmission of acoustic waves along this duct for the modes corresponding to Se_0, So_1, and Se_1. Calculate and plot in suitable units the cutoff frequencies (see page 1442) for these modes as function of B over the range $0 < B < A$, for A constant.

11.17 An electromagnetic wave of wavelength $2\pi/k$ is incident on a long conducting strip of width $a = 4/k$, the direction of the wave being perpendicular to the axis of the strip and at an angle 30° to the plane of the strip. Compute and plot the distribution in angle of the scattered wave for each direction of polarization of the wave, as a function of the angle of scattering.

11.18 Calculate and plot the acoustical radiation resistance and reactance of a vibrating strip of width a [see Eq. (11.2.97)] as a function of $h = \frac{1}{2}ak$ for the range $(0 \leq h \leq 3)$. Values of $B_1^o(h,m)$ are

$h^2 =$	1	2	3	4	5	7	9
$m = 1$	1.0981	1.2051	1.3215	1.4480	1.5850	1.8929	2.2504
$m = 3$	0.0103	0.0202	0.0299	0.0394	0.0486	0.0668	0.0855

11.19 Calculate and plot the electrical conductance and susceptance of a strip of width a carrying alternating current [see Eq. (11.2.99)] as a function of $h = \frac{1}{2}ak$ for $0 \leq h \leq 3$. Values of $B_0^e(h,m)$ are

$h^2 =$	1	2	3	4	5	7	9
$m = 0$	1.1393	1.3090	1.5117	1.7488	2.0219	2.6818	3.5061
$m = 2$	0.0596	0.1128	0.1594	0.1998	0.2349	0.2938	0.3428
$m = 4$	0.0003	0.0013	0.0029	0.0050	0.0078	0.0150	0.0243

11.20 A sound wave of wavelength $2\pi/k$ and pressure amplitude P_0 is incident on a long rigid strip of width a, the direction of the wave being perpendicular to the axis of the strip and at an angle u to the plane of the strip. Show that the net force, per unit length, on the strip because of the sound wave is

$$F = iP_0\pi a\sqrt{2\pi}\sum_{n=0}^{\infty}(-1)^m\frac{B_1^o(h,\,2m+1)e^{i\delta_{2m+1}^o}}{M_{2m+1}^0 C_{2m+1}^0}\,So_{2m+1}(h,\,\cos u)$$

Find the expression for F, valid for small values of $h = \frac{1}{2}ak$, to the second order in h. Compute and plot the magnitude and phase angle of F/aP for $u = 90°$ for $(0 \leq h \leq 3)$. Use the tables at the end of the book and the values given in Prob. 11.18.

11.21 A membrane of infinite extent, of mass σ per unit area, under tension T, is in contact on both sides with a medium of infinite extent of density ρ and speed of sound c_0. Show that the equation of motion for the transverse motion of the membrane, if it is vibrating with simple harmonic motion of frequency $\omega/2\pi$, is

$$\nabla^2\eta(x,y) + K_m^2\eta(x,y) + \left(\frac{K_m^2\rho}{\pi\sigma}\right)\int_0^{2\pi} du \int_0^\infty \eta(x_0,y_0)e^{iK_0R}\,dR = \left[\frac{f(x,y)}{T}\right]$$

where $K_0^2 = (\omega/c_0)^2$; $K_m^2 = (\omega/c_m)^2 = (\sigma\omega^2/T)$, where $\eta(x,y)$ is the transverse displacement of the point (x,y) on the membrane, where $R^2 = (x - x_0)^2 + (y - y_0)^2$, and where $f(x,y)e^{-i\omega t}$ is the transverse driving force on the membrane, per unit area. Show that free plane waves can travel along the membrane, of the form $e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}$, whenever the magnitude k of the wave vector $\mathbf{k}$ satisfies the equation

$$k^2 - K_m^2\left[1 + \frac{2(\rho/\sigma)}{\sqrt{k^2 - K_0^2}}\right] = 0$$

Discuss the physical significance of the two types of waves, one for k near K_0 and the other for k near K_m (if ρ/σ is small). Show that, if $c_0 < c_m$, the latter type is attenuated but that, if $c_0 > c_m$, neither type is attenuated. What is the physical reason for this?

11.22 The membrane of Prob. 11.21 is being driven by a force $\delta(x - x_1)e^{-i\omega t}$, applied along the line $x = x_1$. Show that the displacement of the membrane is

$$\eta = \frac{-1}{4\pi^2\sigma c_m^2}\int_{-\infty}^{\infty} e^{ik(x-x_1)}\left[k^2 - K_m^2 - \frac{(2K_m^2\rho/\sigma)}{\sqrt{k^2 - K_0^2}}\right]^{-1} dk$$

Compute approximate expressions for η when $c_0 \gg c_m$ and when $c_0 \ll c_m$, assuming in both cases that $\rho/\sigma \ll 1$.

11.23 A duct of circular cross section of radius a, of infinite length along the z axis, has an iris diaphragm at $z = 0$, with a circular hole in it, such that the part from $r = 0$ to $r = b$ is open, the part from $r = b$ to $r = a$ is a rigid metal plate. Show that the velocity potential for sound waves in the duct is

$$\psi(r,\varphi,z) = \begin{cases} \dfrac{1}{4\pi}\displaystyle\int_0^{2\pi} d\varphi_0 \int_0^b u_0(r_0)G_k^+(r,\varphi,z|r_0,\varphi_0,0)r_0\,dr_0; & z > 0 \\ 2A\cos(kz) - \dfrac{1}{4\pi}\displaystyle\int_0^{2\pi} d\varphi_0 \int_0^b u_0(r_0)G_k^-(r,\varphi,z|r_0,\varphi_0,0)r_0\,dr_0; & z < 0 \end{cases}$$

for the case where a plane wave Ae^{ikz} from $-\infty$ is incident on the diaphragm. The function u_0 is the velocity amplitude through the circular hole, G^+ is the Green's function for $z > 0$ which has zero normal gradient at $r = a$ and at $z = 0$, and G^- is the corresponding Green's function for $z < 0$. Obtain an expansion in terms of Bessel functions for the G's and show that the integral equation for u_0 is

$$2A = \int_0^b u_0(r_0)G(r|r_0)r_0\,dr_0$$

where

$$G(r|r_0) = \frac{4i}{a^2k} + \sum_{n=1}^{\infty} \frac{4J_0(\pi\alpha_{0n}r/a)J_0(\pi\alpha_{0n}r_0/a)}{a^2J_0^2(\pi\alpha_{0n})\sqrt{(\pi\alpha_{0n}/a)^2 - k^2}}$$

when $k < \pi\alpha_{01}/a$. Show that a fraction $|Q/\pi a^2k|^2$ of the incident intensity is transmitted through the hole to $z \to \infty$, where $\iint u_0\, d\varphi_0 r_0\, dr_0 = AQ$. Show that, by assuming $u_0 = [AQ/2\pi b^2 \sqrt{1 - (r_0/b)^2}]$, setting $r = 0$ in G, and using the equation

$$\int_0^{\frac{1}{2}\pi} J_0(z \sin w) \sin w\, dw = j_0(z)$$

the approximate solution

$$Q = \pi a^2 k \left[i + \sum_{n=1}^{\infty} \frac{kj_0(\pi\alpha_{0n}b/a)}{J_0^2(\pi\alpha_{0n})\sqrt{(\pi\alpha_{0n}/a)^2 - k^2}} \right]^{-1}$$

can be obtained.

11.24 Show that a variational principle for solving the integral equation of Prob. 11.23 is

$$[Q] = \frac{4\pi \left[\int_0^b \varphi(r_0) r_0\, dr_0 \right]^2}{\int_0^b r_1\, dr_1 \int_0^b \varphi(r_1) G(r_1|r_0) \varphi(r_0) r_0\, dr_0}$$

Using the trial function $\varphi = [1 - (r/b)^2]^{-\frac{1}{2}}$, compute Q and compare with the results of Prob. 11.23.

11.25 A rectangular duct of infinite extent along the z axis has constant width b in the y direction but has a sudden increase in width in the x direction, from a_- to a_+, at $z = 0$ (in other words, the bounding surfaces are $y = \pm\frac{1}{2}b$, $x = \pm\frac{1}{2}a_-$ for $z < 0$, $x = \pm\frac{1}{2}a_+$ for $z > 0$, $a_+ > a_-$, and the parts of the $z = 0$ plane between $|x| = \frac{1}{2}a_-$ and $\frac{1}{2}a_+$). Assuming that the velocity across the plane $z = 0$, $|x| < \frac{1}{2}a_-$, $|y| < \frac{1}{2}b$ is approximately $V_0[1 - (2x/a_-)^2]^{-\frac{1}{2}}$, calculate the reflection and transmission when a plane wave is sent along the duct from $z = -\infty$.

11.26 Solve the problem of the acoustic radiation out of a circular tube, set in a plane wall, discussed in Eqs. (11.3.34) *et seq.*, by the methods of Prob. 11.22. Suppose the tube is of radius a, that it extends to $z = -\infty$ and terminates at $z = 0$, the wall $z = 0$, $r > a$ being rigid and the region $z > 0$ being completely open. A wave of amplitude A is started from $z = -\infty$, part of it radiating out the end of the tube and part reflected back to $-\infty$. Show that the appropriate Green's function for $z > 0$ is

$$G_k^+ = 2i \sum_{m=0}^{\infty} \epsilon_m \cos[m(\varphi - \varphi_0)] \int_0^\infty J_m(kr) \cdot$$

$$\cdot J_m(kr_0) \frac{\lambda d\lambda}{K} \begin{cases} \cos(Kz)e^{ikz_0}; & z < z_0 \\ \cos(Kz_0)e^{ikz}; & z > z_0 \end{cases}$$

where $K = \sqrt{k^2 - \lambda^2} = i\sqrt{\lambda^2 - k^2}$. Assume that the z component of the air velocity at the opening $z = 0$ is $v_z = U[1 - (r/a)^2]^{-\frac{1}{3}}$ and solve for U by equating the value of ψ for $r = 0$, computed by Green's function for $z > 0$ with that for $z < 0$ as $z \to 0$. Use the formula

$$\int_0^{\frac{1}{2}\pi} J_0(z \sin w) \sin w \cos^{\frac{1}{3}} w \, dw = \frac{(\Gamma\frac{2}{3})}{(2z^2)^{\frac{1}{3}}} J_{\frac{2}{3}}(z) \to \frac{1}{2}\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{5}{3})}; \quad z \to 0$$

to show that an approximate value of U, when $k < \pi\alpha_{01}/a$, is

$$U \simeq \left[\frac{2Ak\Gamma(5/3)}{\Gamma(2/3)}\right] \left\{ i + 2^{\frac{1}{3}} i a^2 k \Gamma(5/3) \int_0^\infty J_{\frac{2}{3}}(\lambda a) \frac{\lambda \, d\lambda}{(2\lambda a)^{\frac{2}{3}} K} + 2^{\frac{1}{3}} k \Gamma(2/3) \sum_{n=1}^{\infty} \frac{J_{\frac{2}{3}}(\pi\alpha_{0n})}{(2\pi\alpha_{0n})^{\frac{2}{3}} K_n J_0^2(\pi\alpha_{0n})} \right\}^{-1}$$

where $K_n^2 = (\pi\alpha_{0n}/a)^2 - k^2$. Compute the total power radiated out of this hole per unit incident intensity in terms of U/A.

11.27 A sphere of radius a and mass M is vibrating to and fro inside a hollow sphere of inner radius $b = 2a$. The spheres are concentric when the inner one is at equilibrium. Calculate the mechanical impedance load of the air between the spheres on the motion of the inner sphere. What is the motion of the inner sphere when it is struck an impulsive blow at $t = 0$?

11.28 A sphere of radius a has a surface with acoustic impedance $Z = 2\rho c$ (resistive, independent of ω). Plot the scattering and absorption cross section [see Eq. (11.3.76)] of the sphere as a function of ka. Compute and plot the magnitude of the ratio between the net force on the sphere and the pressure of the incident wave, as a function of ka.

11.29 Set up the variational principle appropriate for the Helmholtz resonator discussed in Eqs. (11.3.78) *et seq.*

11.30 Compare Eqs. (11.2.103) and (11.4.68) to the third order in the quantity $h = \frac{1}{2}ka$.

11.31 A disk of radius a, placed with its center at the origin and its surface in the x, y plane, scatters a plane sound wave which is sent in a direction in the x, z plane, at an angle ϑ_i to the z axis. The scattered wave is measured at some distance from the disk, at angle ϑ_s with respect to the z axis, the plane containing its direction and the z axis making an angle ϕ_s with the (x,y) plane. Use Eq. (11.4.61) to compute the scattering formula, with

$$g_k = i \sum_{m=0}^{\infty} \epsilon_m \cos[m(\phi - \phi_0)] \int_0^\infty J_m(\lambda r) J_m(\lambda r_0) \frac{\lambda\, d\lambda}{\sqrt{k^2 - \lambda^2}}$$

and using for trial functions $\varphi^+ - \varphi^- = [1 + \alpha r \sin \vartheta_i \cos \phi_i]$; $\tilde{\varphi}^+ - \tilde{\varphi}^- = [1 + \alpha r \sin \vartheta_s \cos(\phi - \phi_s)]$. Vary the parameter α to obtain best expressions for φ and $\tilde{\varphi}$. Show that the best function ψ, to use in the integral of Eq. (11.4.57) to obtain an improved form of ψ, is

$$\psi_1(\mathbf{r}) = -4\pi i C \varphi(\mathbf{r}) \frac{\oint (\mathbf{k}_i \cdot \mathbf{n}) \tilde{\varphi}(\mathbf{r}^s) e^{i\mathbf{k}_i \cdot \mathbf{r}^s}\, dA}{\oint\oint dA\, dA_0\, \tilde{\varphi}(\mathbf{r}) \varphi(\mathbf{r}_0) \left[\dfrac{\partial^2}{\partial n\, \partial n_0} g_k(\mathbf{r}^s|\mathbf{r}_0^s) \right]}$$

11.32 An incident plane wave $Ce^{i\mathbf{k}_i \cdot \mathbf{r}}$ is scattered by an object with index of refraction n compared to the rest of space. Show that a variational principle for the angle distribution factor $f(\mathbf{k}_i|\mathbf{k}_s)$ is

$$[f] = \frac{(n^2 - 1)k^2 \int \tilde{\varphi}(\mathbf{r}_0) e^{i\mathbf{k}_i \cdot \mathbf{r}_0}\, dv_0 \int \varphi(\mathbf{r}) e^{-i\mathbf{k}_s \cdot \mathbf{r}}\, dv}{\int \tilde{\varphi}(\mathbf{r}) \varphi(\mathbf{r})\, dv - (n^2 - 1)k^2 \int dv \int dv_0\, \tilde{\varphi}(\mathbf{r}) g_k(\mathbf{r}|\mathbf{r}_0) \varphi(\mathbf{r}_0)}$$

where each integration is over the volume of the scattering object and where $g_k = (e^{ikR}/R)$ $(R = |\mathbf{r} - \mathbf{r}_0|)$. Compute f when the object is a sphere of radius a and when the trial functions are $\varphi = e^{i\mathbf{k}_i \cdot \mathbf{r}}$ and $\tilde{\varphi} = e^{-i\mathbf{k}_s \cdot \mathbf{r}}$. Use the expansions of g_k and of $e^{i\mathbf{k} \cdot \mathbf{r}}$ in spherical harmonics.

Cylindrical Bessel Functions

[See Eq. (5.3.63) and tables of Chap. 10]

$$J_m(z) \xrightarrow[z \to 0]{} \frac{1}{m!} \left(\frac{z}{2}\right)^m \xrightarrow[z \to \infty]{} \sqrt{\frac{2}{\pi z}} \cos[z - \tfrac{1}{2}\pi(m + \tfrac{1}{2})]$$

$$\left.\begin{aligned} N_0(z) &\xrightarrow[z \to 0]{} \frac{2}{\pi} [\ln z - 0.11593] \\ N_m(z) &\xrightarrow[z \to 0]{} -\frac{(m-1)!}{\pi} \left(\frac{2}{z}\right)^m \quad (m > 0) \end{aligned}\right\} \xrightarrow[z \to \infty]{} \sqrt{\frac{2}{\pi z}} \sin[z - \tfrac{1}{2}\pi(m + \tfrac{1}{2})]$$

$$H_m(z) = J_m(z) + iN_m(z) \xrightarrow[z \to \infty]{} \sqrt{\frac{2}{\pi z}}\, e^{iz - \frac{1}{2}i\pi(m + \frac{1}{2})}$$

Amplitudes and Phase Angles ($m = 0, 1, 2, \ldots$) (see Table XIV, Appendix).

$$J_m(z) = C_m(z)\sin[\delta_m(z)]; \quad [dJ_m(z)/dz] = -C'_m(z)\sin[\delta'_m(z)]$$
$$N_m(z) = -C_m(z)\cos[\delta_m(z)]; \quad [dN_m(z)/dz] = C'_m(z)\cos[\delta'_m(z)]$$
$$H_m(z) = -iC_m(z)e^{i\delta_m(z)}; \quad [dH_m(z)/dz] = iC'_m(z)e^{i\delta_m'(z)}$$
$$\tan[\alpha_m(z)] = \frac{-z}{J_m(z)}\left[\frac{d}{dz}J_m(z)\right]; \quad \tan[\beta_m(z)] = \frac{-z}{N_m(z)}\left[\frac{d}{dz}N_m(z)\right]$$
$$\tan[\delta'_m(z)] = \tan[\delta_m(z)]\tan[\alpha_m(z)]\cot[\beta_m(z)]$$
$$\tan[\gamma_m(z)] = \tan\delta_m\left[\frac{\cos\beta_m}{\cos\alpha_m}\right] = \tan\delta'_m\left[\frac{\sin\beta_m}{\sin\alpha_m}\right]$$
$$\Delta(J_m,N_m) = (2/\pi z) = C_m(z)C'_m(z)\sin[\delta_m(z) - \delta'_m(z)]$$

Asymptotic Values:

For $z \gg m$ and $z \gg 1$; $C_m \simeq \sqrt{\dfrac{2}{\pi z}} \simeq C'_m$; $\quad \delta_m \simeq z - \frac{1}{2}\pi(m - \frac{1}{2})$

$$\delta'_m \simeq z - \tfrac{1}{2}\pi(m + \tfrac{1}{2}); \quad \tan\alpha_m \simeq z\tan\delta'_m; \quad \tan\beta_m \simeq z\tan\delta_m$$

For $z \ll 2m + 1$, we have $C_0 \simeq \sqrt{1 + \left(\dfrac{2}{\pi}\ln z\right)^2}$, $C'_0 \simeq \dfrac{2}{\pi z}$:

$$\alpha_0 \simeq \tfrac{1}{2}z^2; \quad \delta'_0 \simeq \tfrac{1}{4}\pi z^2; \quad \tan\beta_0 \simeq \frac{2}{\pi}\cdot\tan\delta_0 \simeq -(1/\ln z)$$
$$C_m \simeq \frac{(m-1)!}{\pi}\left(\frac{2}{z}\right)^m; \quad C'_m \simeq \frac{m!}{2\pi}\left(\frac{2}{z}\right)^{m+1}; \quad m = 1, 2, 3, \ldots$$
$$\delta_m \simeq \frac{\pi m}{(m!)^2}\left(\frac{z}{2}\right)^{2m} \simeq -\delta'_m; \quad -\tan\alpha_m \simeq m \simeq \tan\beta_m$$

For $m > 1$; $z - m = 0$; $C_m = C^0_m \simeq 0.8946m^{-\frac{1}{3}} + 0.0059m^{-\frac{5}{3}} + \cdots$:

$$C'_m = C^{0\prime}_m \simeq 0.8217m^{-\frac{2}{3}} + 0.0895m^{-\frac{4}{3}} + \cdots$$
$$\delta_m = \delta^0_m \simeq \tfrac{1}{6}\pi - 0.0114m^{-\frac{4}{3}} + \cdots$$
$$\delta'_m = \delta^{0\prime}_m \simeq -\tfrac{1}{6}\pi + 0.1892m^{-\frac{2}{3}} + \cdots$$
$$\tan\alpha_m = \tan\alpha^0_m \simeq -0.9185m^{\frac{2}{3}} + 0.2000 + \cdots$$
$$\tan\beta_m = \tan\beta^0_m \simeq 0.9185m^{\frac{2}{3}} + 0.2000 + \cdots$$

For $m > 1$; $|z - m| \ll m^{\frac{1}{3}}$; to the first order in $(z - m)$:

$$C_m \simeq C^0_m - \tfrac{1}{2}C^{0\prime}_m(z - m); \quad C'_m \simeq C^{0\prime}_m - \frac{1}{m}C^{0\prime}_m(z - m)$$
$$\delta_m \simeq \delta^0_m + 0.7954m^{-\frac{1}{3}}(z - m); \quad \delta'_m \simeq \delta^{0\prime}_m$$
$$\tan\alpha_m \simeq \tan\alpha^0_m\left[1 + \left(\frac{z-m}{m}\right)\tan\alpha^0_m\right]$$
$$\tan\beta_m \simeq \tan\beta^0_m\left[1 + \left(\frac{z-m}{m}\right)\tan\beta^0_m\right]$$

For $z < m$; $p = \sqrt{1 - (z/m)^2} \gg 0.6m^{-\frac{1}{3}}$:

$$\tan \alpha_m \simeq -mp \simeq -\tan \beta_m$$

$$\tan \delta_m \simeq \tfrac{1}{2}\left[\left(\frac{1-p}{1+p}\right)e^{2p}\right]^m \simeq -\tan \delta'_m$$

$$C_m \simeq [\pi mp \tan \delta_m]^{-\frac{1}{2}};\quad C'_m \simeq \sqrt{mp}\,[\pi z^2 \tan \delta_m]^{-\frac{1}{2}}$$

For $z > m$; $q = \sqrt{(z/m)^2 - 1} \gg 0.6m^{-\frac{1}{3}}$:

$$\delta_m \simeq m[q - \tan^{-1} q] + \tfrac{1}{4}\pi;\quad \delta'_m \simeq \delta_m - \tfrac{1}{2}\pi + \left(\frac{1+q^2}{2mq^3}\right)$$

$$\tan \alpha_m \simeq -mq \cot \delta_m;\quad \tan \beta_m \simeq mq \tan \delta_m$$

$$C_m \simeq \sqrt{2/\pi mq};\quad C'_m \simeq \sqrt{2mq/\pi z^2}$$

Roots. β_{mn} is the nth root of $J_m(\pi\beta) = 0$.

β_{mn}	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$m = 0$	0.7655	1.7571	2.7546	3.7535	4.7527
$m = 1$	1.2197	2.2331	3.2383	4.2411	5.2429
$m = 2$	1.6348	2.6792	3.6988	4.7097	5.7168
$m = 3$	2.0308	3.1070	4.1428	5.1639	6.1781
$m = 4$	2.4153	3.5221	4.5748	5.6073	6.6294

$$\beta_{m1} \simeq (m/\pi) + 0.5907m^{\frac{1}{3}};\quad m \gg 1$$

$$\beta_{mn} \simeq n + \tfrac{1}{2}m - \tfrac{1}{4} - \tfrac{1}{8}[(4m^2 - 1)/(n + \tfrac{1}{2}m - \tfrac{1}{4})];\quad n \gg 1; n > m$$

α_{mn} is the nth root of $[dJ_m(\pi\alpha)/d\alpha] = 0$.

α_{mn}	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$m = 0$	0.0000	1.2197	2.2331	3.2383	4.2411
$m = 1$	0.5861	1.6970	2.7140	3.7261	4.7312
$m = 2$	0.9722	2.1346	3.1734	4.1923	5.2036
$m = 3$	1.3373	2.5513	3.6115	4.6428	5.6624
$m = 4$	1.6926	2.9547	4.0368	5.0815	6.1103

$$\alpha_{m1} \simeq (m/\pi) + 0.2574m^{\frac{1}{3}};\quad m \gg 1$$

$$\alpha_{mn} \simeq n + \tfrac{1}{2}m - \tfrac{3}{4} - \tfrac{1}{8}[(4m^2 - 1)/(n + \tfrac{1}{2}m - \tfrac{3}{4})];\quad n \gg 1; n > m$$

Weber Functions

[Parabolic wave functions. See Eqs. (11.2.63) *et seq.*]

$$D_m(z) = 2^{\frac{1}{2}m}e^{-\frac{1}{4}z^2+\frac{1}{2}i\pi m}U_2(-\tfrac{1}{2}m|\tfrac{1}{2}|\tfrac{1}{2}z^2) \xrightarrow[z\to\infty]{} z^m e^{-\frac{1}{4}z^2} \quad \text{Re } z > 0$$

$$= 2^{\frac{1}{2}m}e^{-\frac{1}{4}z^2}\left[\frac{\sqrt{\pi}}{\Gamma(\frac{1}{2} - \frac{1}{2}m)}F(-\tfrac{1}{2}m|\tfrac{1}{2}|\tfrac{1}{2}z^2) - \sqrt{\frac{\pi}{2}}\frac{2z}{\Gamma(-\frac{1}{2}m)}F(\tfrac{1}{2} - \tfrac{1}{2}m|\tfrac{3}{2}|\tfrac{1}{2}z^2)\right]$$

where U and F are the confluent hypergeometric functions defined in the tables of Chap. 5. The functions $D_m(z)$, $D_m(-z)$ and $D_{-m-1}(iz)$ satisfy the equation

$$\frac{d^2}{dz^2}D_m(z) + [(m+\tfrac{1}{2}) - \tfrac{1}{4}z^2]D_m(z) = 0$$

where $D_m(z) = e^{-i\pi m}D_m(-z) + \dfrac{\sqrt{2\pi}}{\Gamma(-m)}e^{-\frac{1}{2}i\pi(m+1)}D_{-m-1}(iz)$

$$D_{m+1}(z) - zD_m(z) + mD_{m-1}(z) = 0$$

$$\frac{d}{dz}D_m(z) + \tfrac{1}{2}zD_m(z) - mD_{m-1}(z) = 0$$

$$\frac{d}{dz}D_m(z) - \tfrac{1}{2}zD_m(z) + D_{m+1}(z) = 0$$

$$\int_0^\infty e^{-\frac{1}{4}u^2}u^{m-1}D_m(u)\,du = 2^{-\frac{1}{2}m}\Gamma(m)\cos(\tfrac{1}{4}m\pi);\quad \operatorname{Re} m > 0$$

$$D_m(ax+by) = e^{\frac{1}{4}(bx-ay)^2}\left[\frac{a}{\sqrt{a^2+b^2}}\right]^m \cdot \sum_{n=0}^{\infty}\frac{\Gamma(m+1)}{\Gamma(m-n+1)n!}D_{m-n}(\sqrt{a^2+b^2}\,x)D_n(\sqrt{a^2+b^2}\,y)\left(\frac{b}{a}\right)^n$$

For m an integer,

$$D_m(z) = (-1)^m e^{\frac{1}{4}z^2}\left[\frac{d^m}{dz^m}e^{-\frac{1}{2}z^2}\right] = 2^{-\frac{1}{2}m}e^{-\frac{1}{4}z^2}H_m(z\sqrt{\tfrac{1}{2}})$$

$$D_{-m}(z) = e^{\frac{1}{4}z^2}\int_z^\infty du_1\int_{u_1}^\infty du_2\cdots\int_{u_{m-1}}^\infty du_m\,e^{-\frac{1}{2}u_m^2}$$

$$D_0(z) = e^{-\frac{1}{4}z^2};\quad D_{-1}(z) = e^{\frac{1}{4}z^2}\int_z^\infty e^{-\frac{1}{2}u^2}\,du$$

The separated equation for one factor in the solution of the Helmholtz equation in parabolic coordinates is

$$(d^2y/dx^2) + [(2m+1)i + x^2]y = 0;\quad x = \lambda\sqrt{k}$$

having solutions $D_m(x\sqrt{2i})$ and $D_{-m-1}(z\sqrt{-2i})$. The equation for the other factor is

$$(d^2y/dx^2) + [-(2m+1)i + x^2]y = 0;\quad x = \mu\sqrt{k}$$

having solutions $D_m(x\sqrt{-2i})$ and $D_{-m-1}(x\sqrt{2i})$. These solutions are related by the following relations.

$$D_m(x\sqrt{2i}) = \frac{\Gamma(m+1)}{\sqrt{2\pi}}\left[e^{\frac{1}{2}im\pi}D_{-m-1}(-x\sqrt{-2i}) + e^{-\frac{1}{2}im\pi}D_{-m-1}(x\sqrt{-2i})\right]$$
$$= e^{-i\pi m}D_m(-x\sqrt{2i}) + \frac{\sqrt{2\pi}}{\Gamma(-m)}e^{-\frac{1}{2}i\pi(m+1)}D_{-m-1}(-x\sqrt{-2i})$$
$$= e^{i\pi m}D_m(-x\sqrt{2i}) + \frac{\sqrt{2\pi}}{\Gamma(-m)}e^{\frac{1}{2}i\pi(m+1)}D_{-m-1}(x\sqrt{-2i})$$

Wronskian $\Delta[D_m(x\sqrt{2i}),\, D_{-m-1}(x\sqrt{-2i})] = i^{-m}/\sqrt{2i}$

For exterior problems m is an integer, when

$$D_m(-x\sqrt{2i}) = (-1)^m D_m(-x\sqrt{2i});$$
$$D_m(-x\sqrt{-2i}) = (-1)^m D_m(x\sqrt{-2i})$$
$$D_{-m-1}(-x\sqrt{2i}) = -(-1)^m D_{-m-1}(x\sqrt{2i}) + \sqrt{2\pi}\,(i^m/m!)D_m(x\sqrt{-2i})$$
$$D_{-m-1}(-x\sqrt{-2i}) = -(-1)^m D_{-m-1}(x\sqrt{-2i}) + \sqrt{2\pi}\,(1/i^m m!)D_m(x\sqrt{2i})$$

Addition Theorems:

$$i\pi H_0^{(1)}(\Omega) = \frac{\sqrt{8\pi}}{i}\sum_{m=0}^{\infty}\frac{i^m}{m!}D_m(u\sqrt{-2i})\cdot$$
$$\cdot D_m(v\sqrt{-2i})\begin{cases} D_m(x\sqrt{2i})\,D_{-m-1}(y\sqrt{-2i}); & y > x \\ D_m(y\sqrt{2i})\,D_{-m-1}(x\sqrt{-2i}); & x > y\end{cases}$$
$$4\Omega^2 = (u^2+v^2+x^2+y^2)^2 - 4(uv+xy)^2$$
$$= (u^2-v^2-x^2+y^2)^2 - 4(ux-vy)^2$$

$$e^{i(\lambda\cos\phi+\mu\sin\phi)^2} = e^{\frac{1}{2}i(\lambda^2+\mu^2)}\sec\phi\sum_{n=0}^{\infty}\frac{i^n\tan^n\phi}{n!}D_n(\lambda\sqrt{-2i})D_n(\mu\sqrt{2i})$$

For interior problems, m is complex, and we use the solutions

$$H_e(a,x) = e^{-\frac{1}{2}ix^2}F(\tfrac{1}{4}+\tfrac{1}{4}ia|\tfrac{1}{2}|ix^2)$$
$$= 1 - \tfrac{1}{2}ax^2 + \tfrac{1}{24}(a^2-2)x^4 - \tfrac{1}{720}(a^3-14a)x^6 + \cdots$$
$$= \Gamma(\tfrac{3}{4})\sqrt{\tfrac{1}{2}x}\,[J_{-\frac{1}{4}}(\tfrac{1}{2}x^2) - \tfrac{3}{2}aJ_{\frac{3}{4}}(\tfrac{1}{2}x^2) + \tfrac{7}{8}a^2J_{\frac{7}{4}}(\tfrac{1}{2}x^2) + \cdots]$$
$$= \frac{2^{\frac{3}{4}}\pi^{\frac{1}{2}}e^{-\frac{1}{4}\pi a}}{|\Gamma(\frac{1}{4}+\frac{1}{4}ia)|}\mathrm{Re}[e^{\frac{1}{8}i\pi+i\sigma(a)+\frac{1}{4}ia\ln 2}D_{-\frac{1}{2}-\frac{1}{2}ia}(x\sqrt{2i})]$$

where $\Gamma(\frac{1}{4}+\frac{1}{4}ia) = |\Gamma(\frac{1}{4}+\frac{1}{4}ia)|e^{i\sigma(a)}$, and

$$H_0(a,x) = xe^{-\frac{1}{2}ix^2}F(\tfrac{3}{4}+\tfrac{1}{4}ia|\tfrac{3}{2}|ix^2)$$
$$= x - \tfrac{1}{6}ax^3 + \tfrac{1}{120}(a^2-6)x^5 - \tfrac{1}{5040}(a^3-26a)x^7 + \cdots$$
$$= \Gamma(\tfrac{5}{4})\sqrt{2x}\,[J_{\frac{1}{4}}(\tfrac{1}{2}x^2) - \tfrac{5}{6}aJ_{\frac{5}{4}}(\tfrac{1}{2}x^2) + \tfrac{3}{8}a^2J_{\frac{9}{4}}(\tfrac{1}{2}x^2) + \cdots]$$
$$= -\frac{\pi^{\frac{1}{2}}e^{-\frac{1}{4}\pi a}}{2^{\frac{3}{4}}|\Gamma(\frac{3}{4}+\frac{1}{4}ia)|}\mathrm{Re}[e^{i\tau(a)+\frac{1}{4}ia\ln 2}D_{-\frac{1}{2}-\frac{1}{2}ia}(x\sqrt{2i})]$$

where $\Gamma(\frac{3}{4}+\frac{1}{4}ia) = |\Gamma(\frac{3}{4}+\frac{1}{4}ia)|e^{i\tau(a)}$

These functions satisfy the equation $(d^2y/dx^2) + (a + x^2)y = 0$. Their asymptotic expansions, for x and a real, $x \to \infty$, are

$$H_e(a,x) \to \frac{2\Gamma(\frac{1}{2})e^{-\frac{1}{4}\pi a}}{|\Gamma(\frac{1}{4} + \frac{1}{4}ia)|\sqrt{x}} \cos[\tfrac{1}{2}x^2 + \tfrac{1}{2}a \ln x - \tfrac{1}{8}\pi - \sigma(a)]$$

$$H_0(a,x) \to \frac{2\Gamma(\frac{3}{2})e^{-\frac{1}{4}\pi a}}{|\Gamma(\frac{3}{4} + \frac{1}{4}ia)|\sqrt{x}} \cos[\tfrac{1}{2}x^2 + \tfrac{1}{2}a \ln x - \tfrac{3}{8}\pi - \tau(a)]$$

Mathieu Functions

(See pages 562 and 1408; also Tables XVI, XVII of Appendix)

Solutions of Mathieu's equation:

$$\frac{d^2y}{d\phi^2} + [b - h^2 \cos^2 \phi]y = 0$$

or $$(z^2 - 1)\left(\frac{d^2y}{dz^2}\right) + z\left(\frac{dy}{dz}\right) + (h^2z^2 - b)y = 0; \quad \text{for } z = \cos\phi$$

Eigenfunction Solutions. Periodic in ϕ, suitable for ϕ real. Even solutions about $\phi = 0$; $b = be_{2m}$ or be_{2m+1}.

$$Se_{2m}(h, \cos\phi) = \sum_{n=0}^{\infty} B_{2n} \cos(2n\phi); \quad \sum_n B_{2n} = 1$$

$$Se_{2m+1}(h, \cos\phi) = \sum_{n=0}^{\infty} B_{2n+1} \cos[(2n + 1)\phi]; \quad \sum_n B_{2n+1} = 1$$

When it is necessary to indicate the particular series, the coefficients are written in full, $B_n^e(h,m)$.

$\int_0^{2\pi} [Se_m]^2\, d\phi = M_m^e$; normalization constant

$Se_m(h,1) = 1$; $[dSe_m/d\phi]_{\phi=0} = 0$; $Se_m(0, \cos\phi) = \cos(m\phi)$

$Se_{2m}(h,0) = \sum_n (-1)^n B_{2n}$; $Se'_{2m}(h,0) = Se_{2m+1}(h,0) = 0$

$$\left[\frac{d}{d\phi} Se_{2m+1}(h, \cos\phi)\right]_{\phi=\frac{1}{2}\pi} = Se'_{2m+1}(h,0) = \sum_n (-1)^{n-1}(2n + 1)B_{2n+1}$$

Odd solutions about $\phi = 0$; $b = bo_{2m}$ or bo_{2m+1}.

$$So_{2m}(h, \cos\phi) = \sum_{n=1}^{\infty} B_{2n} \sin(2n\phi); \quad \sum_n (2n)B_{2n} = 1$$

$$So_{2m+1}(h, \cos\phi) = \sum_{n=0}^{\infty} B_{2n+1} \sin[(2n + 1)\phi]; \quad \sum_n (2n + 1)B_{2n+1} = 1$$

When it is necessary to indicate the particular series, the coefficients are written $B_n^e(h,m)$.

$$\int_0^{2\pi} [So_m]^2\, d\phi = M_m^e; \quad \text{normalization constant}$$

$$So_m(h,1) = 0; \quad [dSo_m/d\phi]_{\phi=0} = 1; \quad So_m(0, \cos\phi) = \sin(m\phi)$$

$$So'_{2m}(h,0) = \sum_n (-1)^n (2n) B_{2n}; \quad So_{2m}(h,0) = 0$$

$$So_{2m+1}(h,0) = \sum_n (-1)^n B_{2n+1}; \quad So'_{2m+1}(h,0) = 0$$

Corresponding Radial Solutions. For $\mu = i\phi$ and for values of the B's and of b corresponding to the angular functions Se, So.

Even functions:

$$\begin{aligned}
Je_{2m}(h, \cosh\mu) &= \sqrt{\tfrac{1}{2}\pi} \sum_{n=0}^{\infty} (-1)^{n-m} B_{2n} J_{2n}(h\cosh\mu) \\
&= \frac{(-1)^m B_0 \sqrt{\frac{1}{2}\pi}}{Se_{2m}(h,0)}\, Se_{2m}(h, \cosh\mu) \\
&= \frac{(-1)^m}{Se_{2m}(h,0)} \sqrt{\tfrac{1}{2}\pi} \sum_{n=0}^{\infty} B_{2n} J_{2n}(h\sinh\mu) \\
&= (\sqrt{\tfrac{1}{2}\pi}/B_0) \sum_{n=0}^{\infty} (-1)^{n-m} B_{2n} J_n(\tfrac{1}{2}he^{\mu}) J_n(\tfrac{1}{2}he^{-\mu}) \\
&\xrightarrow[\mu\to\infty]{} \frac{1}{\sqrt{h\cosh\mu}} \cos[h\cosh\mu - \tfrac{1}{2}\pi(2m + \tfrac{1}{2})]
\end{aligned}$$

$$\begin{aligned}
Je_{2m+1}(h, \cosh\mu) &= \sqrt{\tfrac{1}{2}\pi} \sum_{n=0}^{\infty} (-1)^{n-m} B_{2n+1} J_{2n+1}(h\cosh\mu) \\
&= \frac{(-1)^{m+1} h B_1}{Se'_{2m+1}(h,0)} \sqrt{\tfrac{1}{8}\pi}\, Se_{2m+1}(h, \cosh\mu) \\
&= \frac{(-1)^m \sqrt{\frac{1}{2}\pi}}{Se'_{2m+1}(h,0)} \sum_{n=0}^{\infty} (2n+1) B_{2n+1} J_{2n+1}(h\sinh\mu) \\
&= (\sqrt{\tfrac{1}{2}\pi}/B_1) \sum_{n=0}^{\infty} (-1)^{n-m} B_{2n+1} [J_n(\tfrac{1}{2}he^{-\mu}) J_{n+1}(\tfrac{1}{2}he^{\mu}) \\
&\qquad + J_{n+1}(\tfrac{1}{2}he^{-\mu}) J_n(\tfrac{1}{2}he^{\mu})] \\
&\xrightarrow[\mu\to\infty]{} \frac{1}{\sqrt{h\cosh\mu}} \cos[h\cosh\mu - \tfrac{1}{2}\pi(2m + \tfrac{3}{2})]
\end{aligned}$$

Values and slopes at $\mu = 0$, $Je_{2m}(h,1) = \sqrt{\tfrac{1}{2}\pi}\,[(-1)^m B_0/Se_{2m}(h,0)]$:

$$Je_{2m+1}(h,1) = \tfrac{1}{2}h\sqrt{\tfrac{1}{2}\pi}\,[(-1)^m B_1/Se'_{2m+1}(h,0)];\quad [dJe_m/d\mu]_{\mu=0} = 0$$

Odd functions:

$$\begin{aligned}
Jo_{2m}(h, \cosh\mu) &= \sqrt{\tfrac{1}{2}\pi}\tanh\mu \sum_{n=1}^{\infty} (-1)^{n-m}(2n)B_{2n}J_{2n}(h\cosh\mu)\\
&= \frac{-i(-1)^m(h^2B_2/4)\sqrt{\tfrac{1}{2}\pi}}{So'_{2m}(h,0)}\,So_{2m}(h,\cosh\mu)\\
&= \frac{(-1)^m\coth\mu}{So'_{2m}(h,0)}\sqrt{\frac{\pi}{2}}\sum_{n=1}^{\infty}(2n)B_{2n}J_{2n}(h\sinh\mu)\\
&= (\sqrt{\tfrac{1}{2}\pi}/B_2)\sum_{n=1}^{\infty}(-1)^{n-m}B_{2n}[J_{n-1}(\tfrac{1}{2}he^{-\mu})J_{n+1}(\tfrac{1}{2}he^{\mu})\\
&\qquad - J_{n+1}(\tfrac{1}{2}he^{-\mu})J_{n-1}(\tfrac{1}{2}he^{\mu})]\\
&\xrightarrow[\mu\to\infty]{} \frac{1}{\sqrt{h\cosh\mu}}\cos[h\cosh\mu - \tfrac{1}{2}\pi(2m+\tfrac{1}{2})]
\end{aligned}$$

$$\begin{aligned}
Jo_{2m+1}(h, \cosh\mu) &= \sqrt{\tfrac{1}{2}\pi}\tanh\mu \sum_{n=0}^{\infty} (-1)^{n-m}(2n+1)B_{2n+1}J_{2n+1}(h\cosh\mu)\\
&= \frac{-i(-1)^m B_1 h}{So_{2m+1}(h,0)}\sqrt{\frac{\pi}{8}}\,So_{2m+1}(h,\cosh\mu)\\
&= \frac{(-1)^m}{So_{2m+1}(h,0)}\sqrt{\frac{\pi}{2}}\sum_{n=0}^{\infty}B_{2n+1}J_{2n+1}(h\sinh\mu)\\
&= (\sqrt{\tfrac{1}{2}\pi}/B_1)\sum_{n=0}^{\infty}(-1)^{n-m}B_{2n+1}[J_n(\tfrac{1}{2}he^{-\mu})J_{n+1}(\tfrac{1}{2}he^{\mu})\\
&\qquad - J_{n+1}(\tfrac{1}{2}he^{-\mu})J_n(\tfrac{1}{2}he^{\mu})]\\
&\xrightarrow[\mu\to\infty]{} \frac{1}{\sqrt{h\cosh\mu}}\cos[h\cosh\mu - \tfrac{1}{2}\pi(2m+\tfrac{3}{2})]
\end{aligned}$$

where the prime in Se' and So' denotes differentiation with respect to ϕ.
Values and slopes at $\mu = 0$; $Jo_m(h,1) = 0$:

$$[dJo_{2m}/d\mu]_{\mu=0} = \tfrac{1}{4}h^2(-1)^m\sqrt{\frac{\pi}{2}}\,[B_2/So'_{2m}(h,0)]$$

$$[dJo_{2m+1}/d\mu]_{\mu=0} = \tfrac{1}{2}h(-1)^m\sqrt{\frac{\pi}{2}}\,[B_1/So_{2m+1}(h,0)]$$

Second Solutions. For same values of b. Second solutions for angular functions may be obtained by setting $\mu = i\phi$ in the following, if the convergence of the series allows it.

$$Ne_{2m}(h, \cosh \mu) = \sqrt{\tfrac{1}{2}\pi} \sum_{n=0}^{\infty} (-1)^{n-m} B_{2n} N_{2n}(h \cosh \mu)$$

$$= (\sqrt{\tfrac{1}{2}\pi}/B_0) \sum_{n=0}^{\infty} (-1)^{n-m} B_{2n} N_n(\tfrac{1}{2}he^{\mu}) J_n(\tfrac{1}{2}he^{-\mu})$$

$$\xrightarrow[\mu\to\infty]{} \frac{1}{\sqrt{h \cosh \mu}} \sin[h \cosh \mu - \tfrac{1}{2}\pi(2m + \tfrac{1}{2})]$$

$$Ne_{2m+1}(h, \cosh \mu) = \sqrt{\tfrac{1}{2}\pi} \sum_{n=0}^{\infty} (-1)^{n-m} B_{2n+1} N_{2n+1}(h \cosh \mu)$$

$$= \frac{1}{B_1}\sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} (-1)^{n-m} B_{2n+1}[J_n(\tfrac{1}{2}he^{-\mu}) N_{n+1}(\tfrac{1}{2}he^{\mu}) + J_{n+1}(\tfrac{1}{2}he^{-\mu}) N_n(\tfrac{1}{2}he^{\mu})]$$

$$\xrightarrow[\mu\to\infty]{} \frac{1}{\sqrt{h \cosh \mu}} \sin[h \cosh \mu - \tfrac{1}{2}\pi(2m + \tfrac{3}{2})]$$

Wronskian, $\Delta(Je_m, Ne_m) = 1$

Slope at $\mu = 0$; $\left[\dfrac{d}{d\mu} Ne_m\right]_{\mu=0} = [1/Je_m(h,1)]$

Value at $\mu = 0$; $Ne_{2m}(h,1) = \dfrac{1}{B_0}\sqrt{\dfrac{\pi}{2}} \displaystyle\sum_{n=0}^{\infty} (-1)^{n-m} J_n(\tfrac{1}{2}h) N_n(\tfrac{1}{2}h) B_{2n}$

$$Ne_{2m+1}(h,1) = \frac{1}{B_1}\sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} (-1)^{n-m} B_{2n+1}[J_n(\tfrac{1}{2}h) N_{n+1}(\tfrac{1}{2}h) + J_{n+1}(\tfrac{1}{2}h) N_n(\tfrac{1}{2}h)]$$

$$No_{2m}(h, \cosh \mu) = \sqrt{\tfrac{1}{2}\pi} \tanh \mu \sum_{n=0}^{\infty} (-1)^{n-m} (2n) B_{2n} N_{2n}(h \cosh \mu)$$

$$= (\sqrt{\tfrac{1}{2}\pi}/B_2) \sum_{n=1}^{\infty} (-1)^{n-m} B_{2n}[J_{n-1}(\tfrac{1}{2}he^{-\mu}) N_{n+1}(\tfrac{1}{2}he^{\mu}) - J_{n+1}(\tfrac{1}{2}he^{-\mu}) N_{n-1}(\tfrac{1}{2}he^{\mu})]$$

$$\xrightarrow[\mu\to\infty]{} \frac{1}{\sqrt{h \cosh \mu}} \sin[h \cosh \mu - \tfrac{1}{2}\pi(2m + \tfrac{1}{2})]$$

$$No_{2m+1}(h, \cosh \mu)$$
$$= \sqrt{\tfrac{1}{2}\pi} \tanh \mu \sum_{n=0}^{\infty} (-1)^{n-m}(2n+1)B_{2n+1}N_{2n+1}(h \cosh \mu)$$
$$= (\sqrt{\tfrac{1}{2}\pi}/B_1) \sum_{n=0}^{\infty} (-1)^{n-m}B_{2n+1}[J_n(\tfrac{1}{2}he^{-\mu})N_{n+1}(\tfrac{1}{2}he^{\mu}) - J_{n+1}(\tfrac{1}{2}he^{-\mu})N_n(\tfrac{1}{2}he^{\mu})]$$
$$\xrightarrow[\mu\to\infty]{} \frac{1}{\sqrt{h \cosh \mu}} \sin[h \cosh \mu - \tfrac{1}{2}\pi(2m + \tfrac{3}{2})]$$

Wronskian, $\Delta(Jo_m,No_m) = 1$
Value at $\mu = 0$; $No_m(h,1) = -1/[dJo_m/d\mu]_{\mu=0}$
Slope at $\mu = 0$;

$$No'_{2m}(h,1) = (\sqrt{\tfrac{1}{2}\pi}/B_2) \sum_{n=1}^{\infty} (-1)^{n-m}(2n)B_{2n}[2J_n(\tfrac{1}{2}h)N_n(\tfrac{1}{2}h) - J_{n-1}(\tfrac{1}{2}h)N_{n+1}(\tfrac{1}{2}h) - J_{n+1}(\tfrac{1}{2}h)N_{n-1}(\tfrac{1}{2}h)]$$

$$No'_{2m+1}(h,1) = (\sqrt{\tfrac{1}{2}\pi}/B_1) \sum_{n=0}^{\infty} (-1)^{n-m}B_{2n+1}\{-(2n+1)[J_n(\tfrac{1}{2}h)N_{n+1}(\tfrac{1}{2}h) + J_{n+1}(\tfrac{1}{2}h)N_n(\tfrac{1}{2}h)] + h[J_n(\tfrac{1}{2}h)N_n(\tfrac{1}{2}h) + J_{n+1}(\tfrac{1}{2}h)N_{n+1}(\tfrac{1}{2}h)]\}$$

Series Expansions [see Eq. (11.2.93)]:

$$i\pi H_0^{(1)}(kR) = 4\pi i \Bigg[\sum_{m=0}^{\infty} \left(\frac{1}{M_m^e}\right) Se_m(h, \cos \vartheta_0)Se_m(h, \cos \vartheta) \cdot \cdot \begin{cases} Je_m(h, \cosh \mu_0)He_m(h, \cosh \mu); & \mu > \mu_0 \\ Je_m(h, \cosh \mu)He_m(h, \cosh \mu_0); & \mu < \mu_0 \end{cases}$$
$$+ \sum_{m=1}^{\infty} \left(\frac{1}{M_m^0}\right) So_m(h, \cos \vartheta_0)So_m(h, \cos \vartheta) \cdot \cdot \begin{cases} Jo_m(h, \cosh \mu_0)Ho_m(h, \cosh \mu); & \mu > \mu_0 \\ Jo_m(h, \cosh \mu)Ho_m(h, \cosh \mu_0); & \mu < \mu_0 \end{cases} \Bigg]$$

where

$$R^2 = (\tfrac{1}{2}a)^2[\cosh^2 \mu + \cosh^2 \mu_0 - \sin^2 \vartheta - \sin^2 \vartheta_0 - 2 \cosh \mu \cosh \mu_0 \cos \vartheta \cos \vartheta_0 - 2 \sinh \mu \sinh \mu_0 \sin \vartheta \sin \vartheta_0]; \quad h = \tfrac{1}{2}ak$$

$$\exp[ih(\cosh \mu \cos \vartheta \cos u + \sinh \mu \sin \vartheta \sin u)]$$
$$= \sqrt{8\pi} \sum_m i^m \left[\left(\frac{1}{M_m^e}\right) Se_m(h, \cos u)Se_m(h, \cos \vartheta)Je_m(h, \cosh \mu) + \left(\frac{1}{M_m^0}\right) So_m(h, \cos u)So_m(h, \cos \vartheta)Jo_m(h, \cosh \mu) \right]$$

Amplitudes and Phase Angles [see Eq. (11.2.100)]:

$$Je_m(h,1) = C^e_m \sin\delta^e_m;\quad Ne_m(h,1) = -C^e_m \cos\delta^e_m;\quad He_m(h,1) = -iC^e_m e^{i\delta_m{}^e}$$
$$Je'_m(h,1) = 0;\quad Ne'_m(h,1) = 1/Je_m(h,1);\quad He'_m(h,1) = i/Je_m(h,1)$$
$$Jo_m(h,1) = 0;\quad No_m(h,1) = -1/Jo'_m(h,1);\quad Ho_m(h,1) = -i/Jo'_m(h,1)$$
$$Jo'_m(h,1) = -C^0_m \sin\delta^0_m;\quad No'_m(h,1) = C^0_m\cos\delta^0_m;\quad Ho'_m(h,1) = iC^0_m e^{i\delta_m{}^0}$$

where the prime denotes differentiation by μ and where

$$He_m = Je_m + iNe_m,\ Ho_m = Jo_m + iNo_m$$

Spherical Bessel Functions

(See pages 1465 *et seq.;* also Tables XII and XV, Appendix)

$$j_n(z) = \sqrt{\frac{\pi}{2z}}\, J_{n+\frac{1}{2}}(z) \xrightarrow[z\to 0]{} \frac{z^n}{1\cdot 3\cdot 5\cdots(2n+1)}$$
$$\xrightarrow[z\to\infty]{} \frac{1}{z}\cos[z - \tfrac{1}{2}\pi(n+1)];\quad n \text{ an integer}$$
$$n_n(z) = \sqrt{\frac{\pi}{2z}}\, N_{n+\frac{1}{2}}(z) \xrightarrow[z\to 0]{} -\frac{1\cdot 1\cdot 3\cdot 5\cdots(2n-1)}{z^{n+1}}$$
$$\xrightarrow[z\to\infty]{} \frac{1}{z}\sin[z - \tfrac{1}{2}\pi(n+1)];\quad n_n(z) = (-1)^{n+1}j_{-n-1}(z)$$
$$h_n(z) = j_n(z) + in_n(z) \xrightarrow[z\to\infty]{} \frac{1}{z}\, i^{-n-1}e^{iz};\quad h_{-n}(z) = i(-1)^{n-1}h_{n-1}(z)$$
$$j_n(z)n_{n-1}(z) - j_{n-1}(z)n_n(z) = \Delta(j_n,n_n) = (1/z^2)$$
$$j_0(z) = \frac{1}{z}\sin z;\quad n_0(z) = -\frac{1}{z}\cos z;\quad h_0(z) = -\frac{i}{z}e^{iz}$$
$$j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z};\quad n_1(z) = -\frac{\cos z}{z^2} - \frac{\sin z}{z};\quad h_1(z) = -\left(\frac{z+i}{z^2}\right)e^{iz}$$
$$j_2(z) = \left(\frac{3}{z^3} - \frac{1}{z}\right)\sin z - \frac{3}{z^2}\cos z;\quad n_2(z) = -\left(\frac{3}{z^3} - \frac{1}{z}\right)\cos z - \frac{3}{z^2}\sin z;$$
$$h_2(z) = -\left(\frac{-iz^2 + 3z + 3i}{z^3}\right)e^{iz};\quad \text{etc.}$$
$$j_n(z) = (-1)^n z^n \left(\frac{d}{z\,dz}\right)^n\left(\frac{\sin z}{z}\right);\quad n_n(z) = -(-1)^n z^n\left(\frac{d}{z\,dz}\right)^n\left(\frac{\cos z}{z}\right)$$
$$h_n(z) = -i(-1)^n\left(\frac{d}{z\,dz}\right)^n\left(\frac{e^{iz}}{z}\right)$$

If $f_n(z)$ is a linear combination of $j_n(z)$ and $n_n(z)$, with coefficients independent of n or z, we have

$$\frac{1}{z^2}\frac{d}{dz}\left(z^2\frac{df_n}{dz}\right) + \left[1 - \frac{n(n+1)}{z^2}\right]f_n = 0$$

$$\left(\frac{2n+1}{z}\right)f_n(z) = f_{n-1}(z) + f_{n+1}(z)$$

$$(2n+1)\frac{d}{dz}f_n(z) = nf_{n-1}(z) - (n+1)f_{n+1}(z)$$

$$\int z^{n+2}f_n(z)\,dz = z^{n+2}f_{n+1}(z)$$

$$\int z^{1-n}f_n(z)\,dz = -z^{1-n}f_{n-1}(z)$$

$$\int [f_n(z)]^2z^2\,dz = \tfrac{1}{2}z^3[f_n^2(z) - f_{n-1}(z)f_{n+1}(z)]$$

$$\int f_n(z)g_m(z)\,dz = z^2\frac{f_{n-1}(z)g_m(z) - f_n(z)g_{m-1}(z)}{(n-m)(n+m+1)} - z\frac{f_n(z)g_m(z)}{(n+m+1)}$$

where $g_m(z)$ is another linear combination of $j_m(z)$ and $n_m(z)$.

$$\int f_n(\alpha x)g_n(\beta x)x^2\,dx = \left(\frac{x^2}{\alpha^2 - \beta^2}\right)[\beta f_n(\alpha x)g_{n-1}(\beta x) - \alpha f_{n-1}(\alpha x)g_n(\beta x)]$$

Series Expansions:

$$\frac{f_m(k\sqrt{r^2 + r_0^2 - 2rr_0\cos\vartheta})}{(r^2 + r_0^2 - 2rr_0\cos\vartheta)^{\frac{1}{2}m}}$$
$$= \frac{1}{(krr_0)^m}\sum_{n=0}^{\infty}(2n + 2m + 1)T_n^m(\cos\vartheta)j_{n+m}(kr)f_{n+m}(kr_0); \quad r_0 > r > 0$$

Since $h_0(kR) = \dfrac{1}{ikR}e^{ikR}$ $(R^2 = r^2 + r_0^2 - 2rr_0\cos\vartheta)$, then

$$\frac{e^{ikR}}{R} = ik\sum_{n=0}^{\infty}(2n+1)P_n(\cos\vartheta)j_n(kr)h_n(kr_0); \quad r_0 > r > 0$$

and, letting r_0 go to infinity and using the asymptotic forms of $h_{n+m}(kr_0)$,

$$e^{ikr\cos\vartheta} = \sum_{n=0}^{\infty}(2n+1)i^nP_n(\cos\vartheta)j_n(kr)$$
$$= \frac{1}{(kr)^m}\sum_{n=0}^{\infty}(2n+2m+1)i^mT_n^m(\cos\vartheta)j_{n+m}(kr)$$

Definite Integrals:

$$j_n(z) = \frac{1}{2i^n}\int_{-1}^{1} e^{izt}P_n(t)\,dt = \int_0^1 J_n(zt)\,\frac{t^{n+1}\,dt}{\sqrt{1-t^2}}$$

$$\int_0^\infty e^{-at}j_n(bt)t^m\,dt = \frac{\Gamma(\frac{1}{2})\Gamma(n+m+1)}{2^{n+1}\Gamma(n+\frac{1}{2})}\,\frac{b^n}{(a^2+b^2)^{\frac{1}{2}(m+n+1)}}\cdot$$
$$\cdot F\left(\frac{n+m+1}{2};\frac{n+1-m}{2}\left|\frac{2n+3}{2}\right|\frac{b^2}{a^2+b^2}\right)$$

$$\int_0^\pi e^{iz\cos\vartheta\cos u}J_m(z\sin\vartheta\sin u)P_n^m(\cos u)\sin u\,du = 2i^{n-m}P_n^m(\cos\vartheta)j_n(z)$$

$$\int_0^\infty e^{-at}j_n(bt)j_n(ct)t\,dt = \left(\frac{1}{2bc}\right)Q_n\left(\frac{a^2+b^2+c^2}{2bc}\right)$$

Others can be obtained by expressing j_n in terms of $J_{n+\frac{1}{2}}$ and then using the formulas on page 1324.

Amplitudes and Phase Angles:

$$j_n(z) = D_n\sin\delta_n;\quad n_n(z) = -D_n\cos\delta_n;\quad h_n(z) = -iD_n(z)e^{i\delta_n(z)}$$
$$dj_n/dz = -D_n'\sin\delta_n';\quad dh_n/dz = iD_n'(z)e^{i\delta_n'(z)};\quad \text{etc.}$$

$$\tan\alpha_n = -\frac{z}{j_n}\frac{d}{dz}(j_n);\quad \tan\beta_n = -\frac{z}{n_n}\frac{d}{dz}(n_n)$$

$$\tan\delta_n = -(j_n/n_n);\quad \tan\delta_n' = -(j_n'/n_n') = \tan\delta_n\left[\frac{\tan\alpha_n}{\tan\beta_n}\right]$$

$$\tan\gamma_n = \tan\delta_n\left[\frac{\cos\beta_n}{\cos\alpha_n}\right] = \tan\delta_n'\left[\frac{\sin\beta_n}{\sin\alpha_n}\right]$$

$$\frac{1}{z}\frac{d}{dz}(zj_n) = -E_n\sin\epsilon_n;\quad \frac{1}{z}\frac{d}{dz}(zh_n) = iE_ne^{i\epsilon_n}$$

$$D_0(z) = 1/z;\quad \delta_0(z) = z;\quad E_0(z) = 1/z;\quad \epsilon_0(z) = z - \tfrac{1}{2}\pi$$
$$D_0'(z) = D_1(z);\quad \delta_0'(z) = \delta_1(z);\quad \sin(\delta_n - \delta_n') = 1/z^2D_nD_n'$$

Asymptotic Values:

For $z \gg n$ and $z \gg 1$, $D_n(z) \simeq \left(\frac{1}{z}\right) \simeq D_n'(z)$:

$$\delta_n(z) \simeq z - \tfrac{1}{2}n\pi;\quad \delta_n'(z) \simeq z - \tfrac{1}{2}\pi(n+1)$$
$$\tan\alpha_n \simeq z\tan[z - \tfrac{1}{2}\pi(n+1)];\quad \tan\beta_n \simeq -z\cot[z - \tfrac{1}{2}\pi(n+1)]$$

For $z \ll |2n-1|$:

$$D_n \simeq \frac{1\cdot1\cdot3\cdot5\cdots(2n-1)}{z^{n+1}};\quad D_n' \simeq \frac{n+1}{z}D_n;\quad \delta_0 \simeq z;\quad \delta_0' \simeq \tfrac{1}{3}z^3$$

$$\delta_n \simeq \frac{z^{2n+1}}{1\cdot3\cdot5\cdots(2n+1)\cdot1\cdot1\cdot3\cdot5\cdots(2n-1)};$$
$$\delta_n' \simeq -\frac{n}{n+1}\delta_n;\quad n>0$$

$$\alpha_0 \simeq \tfrac{1}{3}z^2;\quad \tan\beta_0 \simeq 1+z^2;\quad \epsilon_n \simeq -\frac{n+1}{n}\delta_n;\quad n>0$$

$$\tan\alpha_n \simeq -n;\quad \tan\beta_n \simeq (n+1);\quad E_n \simeq (n/z)D_n;\quad n>0$$

Roots. β_{ln} is the nth root of $j_l(\pi\beta) = 0$:

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$l = 0$	1.0000	2.0000	3.0000	4.0000	5.0000
$l = 1$	1.4303	2.4590	3.4709	4.4775	5.4816
$l = 2$	1.8346	2.8950	3.9226	4.9385	5.9489
$l = 3$	2.2243	3.3159	4.3602	5.3870	6.4050
$l = 4$	2.6046	3.7258	4.7873	5.8255	6.8518

α_{ln} is the nth root of $[dj_l(\pi\alpha)/d\alpha] = 0$:

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$l = 0$	0	1.4303	2.4590	3.4709	4.4775
$l = 1$	0.6626	1.8909	2.9303	3.9485	4.9591
$l = 2$	1.0638	2.3205	3.3785	4.4074	5.4250
$l = 3$	1.4369	2.7323	3.8111	4.8525	5.8786
$l = 4$	1.7974	3.1323	4.2321	5.2869	6.3224

γ_{ln} is the nth root of $(d/d\gamma)[\pi\gamma j_l(\pi\gamma)] = 0$:

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$l = 0$	0.5000	1.5000	2.5000	3.5000	4.5000
$l = 1$	0.8734	1.9470	2.9656	3.9744	4.9796
$l = 2$	1.2319	2.3692	3.4101	4.4311	5.4440
$l = 3$	1.5831	2.7762	3.8400	4.8745	5.8965
$l = 4$	1.9296	3.1728	4.2591	5.3076	6.3393

Spheroidal Functions

(See page 1502 *et seq.*)

Angle Functions:

$$S_{ml}(h,z) = \sum_n{}' d_n(h|m,l)P^m_{n+m}(z) = (1 - z^2)^{\frac{1}{2}m} \sum_n{}' d_n T^m_n(z)$$

[where the prime on the summation sign indicates that only even values of n are included if $(l - m)$ is even, only odd values of n if $(l - m)$ is odd] is a solution of

$$\frac{d}{dz}\left[(1 - z^2)\frac{dS}{dz}\right] + \left[A_{ml}(h) - h^2z^2 - \frac{m^2}{1 - z^2}\right]S = 0$$

which is finite and continuous over the range $-1 \leq z \leq 1$, for different allowed values of the separation constant $A_{ml}(l = m, m+1, m+2, \ldots)$ where $A_{ml} < A_{m,l+1}$. We require that, as $z \to 0$, $(1 - z^2)^{-\frac{1}{2}m} S_{ml}(h,z) \to [(l+m)!/(l-m)!2^m m!] = T^m_{l-m}(1)$, so that

$$\sum_n{}' \frac{(n+2m)!}{n!} d_n(h|m,l) = \frac{(l+m)!}{(l-m)!}$$

The functions S are a set of orthogonal eigenfunctions. The normalizing constant is

$$\int_{-1}^{1} |S_{ml}|^2 \, dz = \sum_n{}' [d_n(h|m,l)]^2 \left(\frac{2}{2n+2m+1}\right) \frac{(n+2m)!}{n!} = \Lambda_{ml}(h)$$

Values for special values of z and h:

$$S_{ml}(h,z) \to \frac{(l+m)!}{2^m m!(l-m)!} (1 - z^2)^{\frac{1}{2}m}; \quad \text{as } z \to 1$$

$$S_{ml}(h,0) = \sum_{n=0}^{\infty} (-1)^n \frac{1 \cdot 1 \cdot 3 \cdot 5 \cdots (2m+2n-1)}{2^n(n)!} d_{2n}; \quad l - m \text{ even}$$

$$\left[\frac{d}{dz} S_{ml}(h,z)\right]_{z=0}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2m+2n+1)}{2^n(n)!} d_{2n+1}; \quad l - m \text{ odd}$$

when $h \to 0$, $S_{ml}(h,z) \to P^m_l(z)$, $A_{ml} \to l(l+1)$.

Radial Functions. Solutions of the first kind, finite at $z = \pm 1$:

$$je_{ml}(h,z) = \frac{(l-m)!}{(l+m)!} \left(\frac{z^2-1}{z^2}\right)^{\frac{1}{2}m} \sum_n{}' i^{n+m-l} \frac{(n+2m)!}{n!} d_n(h|ml) j_{n+m}(hz)$$

$$\to \frac{1}{hz} \cos[hz - \tfrac{1}{2}\pi(l+1)]; \quad hz \to \infty$$

$$= [1/\lambda_{ml}(h)] S_{ml}(h,z)$$

where the d's are the same coefficients as for $S_{ml}(h,z)$, where

$$\lambda_{ml}(h) = \frac{i^l}{d_0(h|ml)} \frac{2m+1}{(2h)^m m!} \frac{(l+m)!}{(l-m)!} S_{ml}(h,0); \quad l = m, m+2, m+4, \ldots$$

$$= \frac{2i^{l-1}}{d_1(h|ml)} \frac{(2m+2)(2m+3)}{(2h)^{m+1} m!} \frac{(l+m)!}{(l-m)!} \left[\frac{d}{dz} S_{ml}(h,z)\right]_{z=0};$$

$$l = m+1, m+3, \ldots$$

and where $j_m(hz)$ is the spherical Bessel function of the first kind.

Solutions of the second kind:

$$ne_{ml}(h,z) = \frac{(l-m)!}{(l+m)!}\left(\frac{z^2-1}{z^2}\right)^{\frac{1}{2}m} \sum_n{}' i^{n+m-l}\,\frac{(n+2m)!}{n!}\,d_n(h|ml)n_{n+m}(hz)$$

$$\to \frac{1}{hz}\sin[hz - \tfrac{1}{2}\pi(l+1)];\quad hz \to \infty$$

$$\Delta(je,ne) = je_{ml}\frac{d}{dz}ne_{ml} - ne_{ml}\frac{d}{dz}je_m = \frac{1}{h(z^2-1)}$$

Solutions of the third kind:

$$he_{ml}(h,z) = je_{ml}(h,z) + ine_{ml}(h,z) \to \frac{i^{-l-1}}{hz}e^{ihz};\quad hz \to \infty$$

$$he_{0l}(h,\cosh\mu) = i^l\left[j_0(u)h_0(v) - \left(\tfrac{4}{5}\frac{d_2}{d_0} - 1\right)j_1(u)h_1(v) + \left(\tfrac{16}{21}\frac{d_4}{d_0} - \tfrac{4}{7}\frac{d_2}{d_0} + 1\right)j_2(u)h_2(v) - \cdots\right];\quad l = 0, 2, \ldots$$

$$= i^{l-1}\,\frac{12\cosh\mu}{h}\left[j_1(u)h_1(v) - \left(\tfrac{12}{7}\frac{d_3}{d_1} + 1\right)j_2(u)h_2(v) + \left(\tfrac{80}{33}\frac{d_5}{d_1} + \tfrac{4}{3}\frac{d_3}{d_1} + 2\right)j_3(u)h_3(v) - \cdots\right];\quad l = 1, 3, \ldots$$

$$he_{1l}(h,\cosh\mu) = i^{l-1}\,\frac{12\sinh\mu}{h}\left[j_1(u)h_1(v) - \left(\tfrac{24}{5}\frac{d_2}{d_0} - 3\right)j_2(u)h_2(v) + \left(\tfrac{80}{11}\frac{d_4}{d_0} - 8\frac{d_2}{d_0} + 6\right)j_3(u)h_3(v) - \cdots\right];\quad l = 1, 3, \ldots$$

where $u = \frac{1}{2}he^{-\mu}$, $v = \frac{1}{2}he^{\mu}$.

Addition Theorems:

$$\frac{e^{ikR}}{R} = 2ik\sum_{m,l}\frac{\epsilon_m}{\Lambda_{ml}(h)}S_{ml}(h,\cos\vartheta_0)S_{ml}(h,\cos\vartheta)\cos[m(\phi-\phi_0)]\cdot \cdot\begin{cases} je_{ml}(h,\cosh\mu_0)he_{ml}(h,\cosh\mu); & \mu > \mu_0 \\ je_{ml}(h,\cosh\mu)he_{ml}(h,\cosh\mu_0); & \mu < \mu_0\end{cases}$$

where

$$kR = h^2[\cosh^2\mu + \cosh^2\mu_0 - \sin^2\vartheta - \sin^2\vartheta_0 - 2\cosh\mu\cosh\mu_0\cos\vartheta\cos\vartheta_0 - 2\sinh\mu\sinh\mu_0\sin\vartheta\sin\vartheta_0\cos(\phi-\phi_0)]$$

$$e^{i\mathbf{k}\cdot\mathbf{r}} = 2\sum_{m,l}\frac{\epsilon_m i^l}{\Lambda_{ml}(h)}S_{ml}(h,\cos\vartheta_0)S_{ml}(h,\cos\vartheta)\cos[m(\phi-\phi_0)]je_{ml}(h,\cosh\mu)$$

where

$$\mathbf{k}\cdot\mathbf{r} = h[\cosh\mu\cos\vartheta\cos\vartheta_0 + \sinh\mu\sin\vartheta\sin\vartheta_0\cos(\phi-\phi_0)]$$

For prolate spheroidal coordinates, where the surface $\mu = 0$ is the line between $z = \pm\frac{1}{2}a$, $x = y = 0$, we set $h = (\frac{1}{2}ka)$ and $z = \cos\vartheta$ or $\cosh\mu$ to obtain the angle and radial solutions, respectively. For oblate spheroidal coordinates, where the surface $\rho = 0$ is the circular disk of radius b, in the x, y plane, with center at the origin, we set $h = ikb$ and $z = \cos\vartheta$ for the angle functions, but $z = -i\sinh\rho$ $(\mu = \rho - \frac{1}{2}i\pi)$ for the radial functions.

Short Table of Laplace Transforms

The functions $f(t)$ and $F(\omega)$ $(\omega = ip)$, satisfying the convergence conditions given on page 1340, are related by the equations

$$F(\omega) = \int_0^\infty e^{-pt}f(t)\,dt;\quad f(t) = \frac{1}{2\pi}\int_{-\infty+i\epsilon}^{\infty+i\epsilon} e^{-i\omega t}F(\omega)\,d\omega;\quad \epsilon \text{ real and } > 0$$

Original function $f(t)$	Laplace transform $F(p)$ $(p = -i\omega)$
$Af(t)$	$AF(p)$
$f(at)$	$\frac{1}{a}F(p/a)$
df/dt	$pF(p) - f(0)$
d^2f/dt^2	$p^2F(p) - pf(0) - f'(0)$
$\int_0^t f(t)\,dt$	$(1/p)F(p)$
$t^nf(t)$	$(-1)^n(d^nF/dp^n)$
$(1/t)f(t)$	$\int_p^\infty F(q)\,dq$
$\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau$	$F_1(p)F_2(p)$
$\begin{cases} f(t-a); & t \geq a \\ 0; & t < a \end{cases}$	$e^{-ap}F(p)$
$e^{at}f(t)$	$F(p-a)$
$\frac{1}{\sqrt{\pi t}}\int_0^\infty e^{-u^2/4t}f(u)\,du$	$\frac{1}{\sqrt{p}}F(\sqrt{p})$
$t^{\frac{1}{2}\nu}\int_0^\infty u^{-\frac{1}{2}\nu}J_\nu(2\sqrt{ut})\,f(u)\,du$	$p^{-\nu-1}F(1/p)$; Re $\nu > -\frac{1}{2}$
$\int_0^\infty \frac{t^uf(u)}{\Gamma(u+1)}\,du$	$(1/p)F(\ln p)$

If $f(t)$ is periodic with period a, $f(t+a) = f(t)$, for $t > 0$, and if $F_0(p) = \int_0^a f(t)e^{-pt}\,dt$, then

$f(t)$	$\frac{F_0(p)}{1 - e^{-ap}}$

Laplace Transforms of Specific Functions. When $f(t) = 0$ for $t < 0$ and is given below for $t \geq 0$ then the Laplace transform $F(p)$ $(p = -i\omega)$ is

$f(t)$	$F(p)$
$t^\nu;\quad \operatorname{Re}\nu > -1$	$\Gamma(\nu+1)/p^{\nu+1}$
$\sin(at)$	$a/(p^2+a^2)$
$\cos(at)$	$p/(p^2+a^2)$
$e^{-bt}\sin(at+\phi)$	$\dfrac{a\cos\phi + (p+b)\sin\phi}{(p+b)^2+a^2}$
$(1/t)\sin(at)$	$\tan^{-1}(a/p)$
$t^\nu \sin(at)$	$\dfrac{1}{2i}\dfrac{\Gamma(\nu+1)}{(p^2+a^2)^{\nu+1}}[(p+ia)^{\nu+1} - (p-ia)^{\nu+1}]$
$t^\nu \cos(at)$	$\dfrac{1}{2}\dfrac{\Gamma(\nu+1)}{(p^2+a^2)^{\nu+1}}[(p+ia)^{\nu+1} + (p-ia)^{\nu+1}]$
$e^{-bt}[A\sinh(at) + B\cosh(at)]$	$\left[\dfrac{aA}{(p+b)^2-a^2}\right] + \left[\dfrac{B(p+b)}{(p+b)^2-a^2}\right]$
$t^\nu e^{at};\quad \operatorname{Re}\nu > -1$	$\Gamma(\nu+1)/(p-a)^{\nu+1}$
$F(\alpha\|\gamma\|t)$ (confluent hypergeometric function)	$(1/p)\,F(\alpha,1\|\gamma\|1/p)$ (hypergeometric function)
$\begin{cases}(t^2-a^2)^\nu; & t>a\\ 0; & t<a\end{cases}\quad \operatorname{Re}\nu > -\frac{1}{2}$	$i\sqrt{\dfrac{\pi a}{2p}}\,\dfrac{2^\nu\Gamma(\nu+1)}{(ap)^\nu}\,e^{\frac{1}{2}\pi i(\nu+\frac{1}{2})}H^{(1)}_{\nu+\frac{1}{2}}(iap)$
$\begin{cases}(2at-t^2)^{n-\frac{1}{2}}; & (0<t<2a)\\ 0; & t>2a\end{cases}\quad n>0$	$\sqrt{\pi}\,\Gamma(n+\frac{1}{2})2^n(a/p)^n e^{-ap-\frac{1}{2}\pi in}J_n(iap)$
$e^{-bt}J_\nu(at);\quad \operatorname{Re}\nu > -1$	$\dfrac{a^\nu}{\sqrt{(p+b)^2+a^2}}\cdot[p+b+\sqrt{(p+b)^2+a^2}]^{-\nu}$
$\begin{cases}J_0(a\sqrt{t^2-b^2}); & t>b\\ 0; & t<b\end{cases}$	$e^{-b\sqrt{p^2+a^2}}/\sqrt{p^2+a^2}$
$u(t-a) = \begin{Bmatrix}0; & t<a\\ 1; & t>a\end{Bmatrix} = \int_{-\infty}^{t-a}\delta(t)\,dt$	$(1/p)e^{-ap}$
$\delta(t-a) = \lim\limits_{\Delta\to 0}\left\{\dfrac{1}{\Delta}[u(t-a) - u(t-a-\Delta)]\right\}$	e^{-ap}
$\frac{1}{2}[e^b\delta(t+a) - e^{-b}\delta(t-a)]$	$\sinh(ap+b)$

$f(t)$	$F(p)$
$\sum_{n=0}^{\infty} \delta(t - na)$	$1/(1 - e^{-ap})$
$\sum_{n=0}^{\infty} (-1)^n \delta(t - na)$	$1/(1 + e^{-ap})$
$\sum_{n=0}^{\infty} (-1)^n u(t - na)$	$1/p(1 + e^{-ap})$
$2 \sum_{n=0}^{\infty} \delta(t - a - 2na)$	$1/\sinh(ap)$
$2 \sum_{n=0}^{\infty} (-1)^n \delta(t - a - 2na)$	$1/\cosh(ap)$
$\delta(t) + 2 \sum_{n=1}^{\infty} (-1)^n \delta(t - 2na)$	$\tanh(ap)$
$u(t) + 2 \sum_{n=1}^{\infty} (-1)^n u(t - 2na)$	$(1/p) \tanh(ap)$
$2 \sum_{n=0}^{\infty} f(t - 2na - a) \cdot u(t - a - 2na)$	$F(p)/\sinh(ap)$
$2 \sum_{n=0}^{\infty} (-1)^n f(t - 2na - a) \cdot u(t - a - 2na)$	$F(p)/\cosh(ap)$
$f(t) + 2 \sum_{n=1}^{\infty} (-1)^n f(t - 2na) \cdot u(t - 2na)$	$F(p) \tanh(ap)$
$2 \sum_{n=0}^{\infty} e^{-(2n+1)b} \delta(t - a - 2na)$	$1/\sinh(ap + b)$
$2 \sum_{n=0}^{\infty} e^{-(2n+1)b} \delta(t - c - a - 2na)$	$e^{-cp}/\sinh(ap + b)$
$\sum_{n=0}^{\infty} e^{-(2n+1)b}[e^d \delta(t + c - a - 2na) - e^{-d} \delta(t - c - a - 2na)]$	$\sinh(cp + d)/\sinh(ap + b)$

$f(t)$	$F(p)$
For $q = \sqrt{p/a^2}$ and $erfc(z) = (2/\sqrt{\pi}) \int_z^\infty e^{-w^2}\,dw \to \begin{cases} 1 - (2z/\sqrt{\pi}); & z \to 0 \\ (e^{-z^2}/z\sqrt{\pi}); & z \to \infty \end{cases}$	
$(x/2a\sqrt{\pi t^3})\, e^{-x^2/4a^2t}$	e^{-qx}
$(a/\sqrt{\pi t})\, e^{-x^2/4a^2t}$	$(1/q)e^{-qx}$
$erfc(x/2a\sqrt{t})$	$(1/p)e^{-qx}$
$\left(t + \dfrac{x^2}{2a^2}\right) erfc\left(\dfrac{x}{2a\sqrt{t}}\right) - \dfrac{x}{a}\sqrt{\dfrac{t}{\pi}}\, e^{-x^2/4a^2t}$	$(1/p^2)e^{-qx}$
$(a/\sqrt{\pi t})\, e^{-x^2/4a^2t} - ha^2 e^{hx+a^2h^2t} \cdot erfc\left(\dfrac{x}{2a\sqrt{t}} + ah\sqrt{t}\right)$	$\dfrac{e^{-qx}}{q+h}$
$a^2 e^{hx+a^2h^2t} erfc\left(\dfrac{x}{2a\sqrt{t}} + ah\sqrt{t}\right)$	$\dfrac{e^{-qx}}{q(q+h)}$
$(1/h) erfc(x/2a\sqrt{t}) - \dfrac{1}{h} e^{hx+a^2h^2t} erfc\left(\dfrac{x}{2a\sqrt{t}} + ah\sqrt{t}\right)$	$\dfrac{e^{-qx}}{p(q+h)}$

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CHAPTER 12

Diffusion, Wave Mechanics

We offer a mixed bag in this chapter: a discussion of more of the aspects of diffusion and of kinetic theory followed by a sketch of the techniques of solving the simple Schroedinger equation derived in Sec. 2.6. These two topics have no close logical connection; they are the two topics left to be dealt with before we go on to a discussion of vector fields.

We have treated some steady-state, diffusion problems already in Chap. 10, since they were solutions of the Laplace or Poisson equation. In the first section of this chapter we shall discuss cases which vary with time, where the differences between solutions of the parabolic partial differential equation (see page 691) and the elliptic (wave) equation become apparent. In a number of cases of interest in the study of kinetic theory problems the diffusion equation is not a good approximation; we are forced to compute the distribution function, which measures the probability of presence of a particle in a given element of phase space. The second section of this chapter discusses a few typical computations for distribution functions.

The final section deals with typical solutions of the nonrelativistic wave equation for an elementary particle, such as an electron or a proton. There will be no attempt to make this an exhaustive treatise on wave mechanics; other texts are available. Only enough examples will be given to illustrate the more usual techniques of calculation: the computation of the allowed energies of bound states, the steady-state scattering of particles by a field of force, and a few cases of time-dependent solutions.

12.1 *Solutions of the Diffusion Equation*

In Sec. 2.4 we pointed out that solutions of the diffusion equation

$$\nabla^2\psi = (1/a^2)(\partial\psi/\partial t) \tag{12.1.1}$$

(where $a^2 = \kappa/\rho C$, κ being the heat conductivity, ρ the density, and C the specific heat of the medium) differ from solutions of the wave equation because the first time derivative produces irreversible motion, in contrast to the reversible motion corresponding to the second time derivative of the wave equation.

For example, a string of length l, having wave velocity c, when held at rest in a configuration $\psi_0(x)$ until $t = 0$ and then released, has a subsequent motion given by

$$\psi(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{\pi n x}{l}\right) \cos\left(\frac{\pi n c t}{l}\right); \quad A_n = \frac{2}{l} \int_0^l \psi_0(x) \sin\left(\frac{\pi n x}{l}\right) dx$$

On the other hand, a slab of heat-conducting material of thickness l, having a temperature distribution $\psi_0(x)$ (x is the distance from one side of the slab, normal to the surface; $x = l$ corresponds to the other surface) at $t = 0$ and held at zero temperature at the two surfaces, has a subsequent distribution of temperature given by

$$\psi(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{\pi n x}{l}\right) \exp\left[-\left(\frac{\pi n a}{l}\right)^2 t\right] \tag{12.1.2}$$
$$A_n = \frac{2}{l} \int_0^l \psi_0(x) \sin\left(\frac{\pi n x}{l}\right) dx$$

The space-dependent factors in the series are the same; the time-dependent factor here is an exponential, whereas for the wave solution the factor is a trigonometric function. In the wave solution, time may be reversed without altering its nature (in fact, in the example given the solution is not altered at all); in the diffusion solution, a reversal of time profoundly alters the solution (in fact, the solution diverges for negative values of time).

Transient Surface Heating of a Slab. A few simple cases will further emphasize the difference between diffusion and wave solutions. Suppose the slab of thickness l has its face at $x = 0$ held at zero temperature [since a constant temperature may be added to the solution of (12.1.1), "zero temperature" is any convenient, reference temperature, independent of time] and the surface $x = l$ is alternately raised and lowered in temperature:

$$\psi_{x=l} = e^{-i\omega t}$$

The time dependence of the steady state ψ is thus $e^{-i\omega t}$ (the real part of ψ is the actual temperature) and the space part must satisfy the equation

$$\frac{d^2\psi}{dx^2} + \left(\frac{i\omega}{a^2}\right)\psi = 0$$

with the boundary conditions that $\psi = 0$ at $x = 0$ and $\psi = e^{-i\omega t}$ at $x = l$.

The solution is

$$\psi(\omega,x) = \frac{\sin[(x/a)\sqrt{i\omega}]}{\sin[(l/a)\sqrt{i\omega}]} e^{-i\omega t} \tag{12.1.3}$$

If ψ is the temperature, $-\kappa \operatorname{grad} \psi = \mathbf{J}$ is the heat energy flow per unit area, so that the heat flow across the face $x = 0$ is

$$\begin{aligned} J(\omega,0) &= -\sqrt{i\rho C\kappa\omega} \csc[(l/a)\sqrt{i\omega}]\, e^{-i\omega t} \\ &= -\frac{\sqrt{\rho C\kappa\omega}}{\sqrt{\sin^2(\sqrt{\frac{1}{2}\omega}\, l/a) + \sinh^2(\sqrt{\frac{1}{2}\omega}\, l/a)}} e^{-i\omega t + \frac{1}{4}i\pi - i\alpha} \end{aligned} \tag{12.1.4}$$

where $\tan\alpha = \cot(\sqrt{\frac{1}{2}\omega}\, l/a) \tanh(\sqrt{\frac{1}{2}\omega}\, l/a)$. When $\sqrt{\frac{1}{2}\omega}\,(l/a)$ is very small, the heat flow is large and in phase with the temperature fluctuation. At low frequencies the heat flow has a chance to stabilize throughout the slab, and the temperature distribution is the steady-state flow times a slowly varying time factor. For high frequencies $[\sqrt{\frac{1}{2}\omega}\,(l/a)$ large], the heat flow does not have time to "settle down"; the surface temperature has reversed itself before the last "surge" has penetrated very deeply into the slab, and a sort of "heat wave" travels into and through the slab. For $\sqrt{\frac{1}{2}\omega}\,(l/a)$ large, the amplitude of the heat flow out of the other side of the slab is $2\sqrt{\rho C\kappa\omega} \exp[-\sqrt{\frac{1}{2}\omega}\,(l/a)]$ times the amplitude of temperature fluctuation of the "driven" side, and the phase lag is $\sqrt{\frac{1}{2}\omega}\,(l/a) - \frac{1}{4}\pi$ radians behind the "driving" temperature oscillation. There is, therefore, transmission of heat energy oscillations across the slab, though the attenuation is great and the dispersion is large [the phase lag is proportional to l and the argument of the exponential may be written $i(\sqrt{\frac{1}{2}\omega}/a)(l - \sqrt{2\omega}\, at) - \frac{1}{4}i\pi$ so that the phase velocity is $a\sqrt{2\omega} = \sqrt{2\omega\kappa/\rho C}$, which depends on frequency].

We are now ready to use the Laplace transform to compute the flow of heat out from the surface $x = 0$ if a temperature pulse $\psi(l) = \delta(t)$ is applied to the surface $x = l$. According to Eq. (11.1.16), this heat flow should be

$$J_\delta(t,0) = \frac{1}{2\pi} \int_{-\infty+i\epsilon}^{\infty+i\epsilon} J(\omega,0)\, d\omega$$

To evaluate this integral, we must investigate the residues at the poles of $J(\omega,0)$. There is no pole at $\omega = 0$. When $\omega = -i(\pi n a/l)^2$, with n a positive integer, the $\sin[(l/a)\sqrt{i\omega}]$ term in the denominator vanishes; further investigation reveals no other singularities in the finite part of the ω plane. To find the residues we set $\omega = -i(\pi n a/l)^2 + \epsilon$, whence $\sin[(l/a)\sqrt{i\omega}] \to \frac{1}{2}i\epsilon(-1)^n(l^2/\pi n a^2)$ so that the residue at the nth pole is $-(2\kappa^2\pi^2 n^2/i\rho C l^3)(-1)^n e^{-(\pi n a/l)^2 t}$. Since all these poles are on or below the real axis of ω, the heat flow for t negative is zero (which is not surprising) and we have, finally, that the function giving the pulse of heat, coming out the $x = 0$ side is

$$J_\delta(t,0) = \frac{2\pi^2\kappa^2}{\rho C l^3} \sum_{n=1}^{\infty} (-1)^n n^2 e^{-(\pi n a/l)^2 t} u(t)$$

a function which converges for $t > 0$, but which is not a very well-behaved series at $t = 0$. The series may be integrated, however, to give the heat flow out of $x = 0$ for an impressed temperature $T(t)u(t)$ at $x = l$;

$$J(t) = u(t)\left(\frac{2\pi^2\kappa^2}{\rho C l^3}\right) \sum_{n=1}^{\infty} (-1)^n n^2 \int_0^t T(\tau) e^{-(\pi n a/t)^2(t-\tau)}\, d\tau \qquad (12.1.5)$$

which is reasonably convergent for $t > 0$. We shall see later what to do for t near zero.

We could, of course, have computed the leftward flow of heat, or the temperature distribution for any point x, inside the slab, rather than at $x = 0$. The temperature distribution, for example, is

$$\begin{aligned}\psi_\delta(t,x) &= -\left(\frac{2\pi a^2}{l^2}\right) \sum_n (-1)^n n \sin\left(\frac{\pi n x}{l}\right) e^{-(\pi n a/l)^2 t} \\ &= \left(\frac{a^2}{l}\right) \frac{d}{dx} \vartheta_3\left(\frac{\pi x}{2l} - \tfrac{1}{2}\pi,\, e^{-(\pi a/l)^2 t}\right) \qquad (12.1.6)\end{aligned}$$

where the function ϑ_3 is one of the theta functions defined in Eq. (4.5.70). This relation enables us to utilize some of the many pseudoperiodic properties of these functions. For example, we can make use of the Poisson sum rule to obtain [see also Eqs. (11.2.90) *et seq.*]

$$\vartheta_3(u,q) = \sqrt{\frac{-\pi}{\ln q}} \sum_{n=-\infty}^{\infty} \exp\left[\frac{1}{\ln q}(u - \pi n)^2\right]$$

to obtain the useful formulas

$$\psi_\delta(t,x) = \frac{a}{\sqrt{\pi t}} \frac{d}{dx} \sum_{n=-\infty}^{\infty} \exp\left[-\frac{(x - 2nl - l)^2}{4a^2 t}\right] \qquad (12.1.7)$$

$$J_\delta(t,x) = -\left(\frac{\kappa a^2}{l}\right) \frac{d^2}{dx^2} \vartheta_3 = \frac{-\kappa}{\pi a} \frac{d}{dt} \left\{\sqrt{\frac{\pi}{t}} \sum_{n=-\infty}^{\infty} \exp\left[-\frac{(x - 2nl - l)^2}{4a^2 t}\right]\right\}$$

which tells us further interesting properties of ψ and J.

Green's Functions and Image Sources. In the first place the formulas show clearly that the temperature distribution corresponds to a number of diffusion Green's functions (for dipoles, because we pair source and image at each face; the mathematical counterpart of the dipole is the derivative) of the sort defined in Eq. (7.4.10), placed at $x = l + 2nl$. But this sequence of image source functions is just the solution we would have obtained if we had gone about solving our problem by the Green's function technique. We can view our excursion as a roundabout method of getting the Green's function solution or as a roundabout method of proving a useful series expansion for the theta function, whichever seems more fundamental, the mathematics or the physics.

The series of Eqs. (12.1.7) (expressing the distribution in terms of source and image) are useful in seeing what happens to the pulse of heat, particularly as $t \to 0$, where the series of Eq. (12.1.6) do not converge well. For small values of the time, the temperature is extremely small except close to $x = l$. Since we have not here taken into account the finite velocity of heat discussed on page 865, some heat will start to come out of the face $x = 0$ for t as small (positive) as we wish. Since the amount is quite small when t is small, it is hardly worth while to add the complication of finite velocity to the solutions. For more general cases, where the face $x = l$ is maintained at temperature $T_0(t)u(t)$, it is usually easier to use the series of Eq. (12.1.6) in the formulas

$$\begin{aligned} T(t,x) &= \int_0^t T_0(t-\tau)\psi_\delta(\tau,x)\,d\tau \\ J(t,x) &= -\kappa \int_0^t T_0(t-\tau)\,\frac{d}{dx}\,\psi_\delta(\tau,x)\,d\tau \end{aligned} \tag{12.1.8}$$

although the series of Eq. (12.1.7) may be used if, for any reason, it is more appropriate. It should be pointed out again that, if T_0 is a simple function of t (a step function, for instance), the resulting answer could have been obtained more easily by modification of the simple series (12.1.2); only when T_0 is not simple is the Laplace transform technique necessary.

Radiative Heating. Another boundary condition at $x = l$ is for the heat to be transmitted to the slab by radiation, the heat flow into the slab being proportional to the difference between the temperature of the slab at $x = l$ and the temperature $T_r(t)$ of the radiator. If the radiator temperature is $e^{-i\omega t}$, above and below some "zero temperature," the boundary condition is that

$$-J = \kappa h(T_r - \psi) \quad \text{or} \quad (d\psi/dx) + h\psi = he^{-i\omega t}; \quad \text{at } x = l$$

The corresponding solution for $\psi = 0$ at $x = 0$ is

$$\psi(\omega,x) = \frac{ah\sin[(x/a)\sqrt{i\omega}]\,e^{-i\omega t}}{ha\sin[(l/a)\sqrt{i\omega}] + \sqrt{i\omega}\cos[(l/a)\sqrt{i\omega}]} \tag{12.1.9}$$

which exhibits a phase lag of the temperature cycle, which is more than that for Eq. (12.1.3) and which is greater, the higher the frequency.

To convert this into a transient distribution corresponding to a delta function dependence of T_r on time, we must first find the zeros of the denominator and then compute the residues of the integral of $\psi(\omega,x)$ over ω. The zeros are related to the roots ν_n of the equation $\tan(\pi\nu) + (\pi\nu/hl) = 0$, which has the following limiting values:

$$\begin{aligned} \nu_n &\simeq n[1 - (\pi/hl)]; & hl &\ll 1/n \\ &\simeq n + \tfrac{1}{2} + [hl/\pi^2(n + \tfrac{1}{2})]; & n &\gg 1/hl \end{aligned}$$

Intermediate values must be computed, as functions of hl. The poles of the integrand are at $\omega = -i(\pi\nu_n a/l)^2$ and, computing residues, we finally have

$$\psi_\delta(t,x) = -2\pi a^2 h^2 \sum_{n=1}^{\infty} \frac{\nu_n}{h^2l^2 + hl + \pi^2\nu_n^2} \left[\frac{\sin(\pi\nu_n x/l)}{\cos(\pi\nu_n)}\right] e^{-(\pi\nu_n a/l)^2 t} u(t) \tag{12.1.10}$$

which has a behavior similar to that of Eq. (12.1.6).

Of course if the slab were of infinite thickness, the solutions would be simpler. If the conductive medium is in the region $x > 0$, then the basic solution, for simple harmonic boundary conditions, is

$$\psi(\omega,x) = \exp[-(x/a)\sqrt{p} + pt]; \quad p = -i\omega$$

We can now devise a Laplace transform procedure to calculate the distribution of temperature inside the region $x > 0$ for any reasonable sort of boundary conditions at $x = 0$, $t > 0$.

For example, suppose we require that the temperature of the face $x = 0$ be $\varphi_0(t)u(t)$. The Laplace transform of this is

$$\Phi_0(p) = \int_0^\infty e^{-pt}\varphi_0(t)\,dt \tag{12.1.11}$$

and the Laplace transform of the temperature distribution $\psi(x,t)$ in the medium is

$$\Psi(x,p) = \Phi_0(p)e^{-qx}; \quad q = \sqrt{p/a^2}$$

The corresponding transforms $\psi(x,t)$ may be obtained from the table at the end of Chap. 11, for some of the simpler forms of $\Phi_0(p)$. If, for instance, the temperature at $x = 0$ is zero until $t = 0$ and then rises linearly as At, the Laplace transform is $\Phi_0(p) = A/p^2$ and the corresponding temperature distribution in the medium will be

$$\psi(x,t) = A\left(t + \frac{x^2}{2a^2}\right) erfc\left(\frac{x}{2a\sqrt{t}}\right) - A\,\frac{x}{a}\sqrt{\frac{t}{\pi}}\,e^{-x^2/4a^2t}$$

where $erfc(z) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_z^\infty e^{-w^2}\,dw$. The temperature distribution corresponding to a delta function surface temperature is the one for $\Phi_0 = 1$ and consequently, according to Eqs. (11.1.16) and (12.1.8),

$$\psi(x,t) = \frac{x}{2a^2} \int_0^t \frac{1}{\sqrt{\pi\tau^3}} \varphi_0(t - \tau) e^{-x^2/4a^2\tau} \, d\tau \tag{12.1.12}$$

On the other hand, if the face at $x = 0$ is heated by radiation, so that $-(d\psi/dx) + h\psi = h\varphi_0(t)$, the corresponding boundary condition on the transform results in

$$\Psi(x,p) = [h\Phi_0(p)/(q + h)]e^{-qx}$$

from which the temperature distribution $\psi(x,t)$ may be determined. Both of these results, of course, may also be obtained from Green's function considerations, using the Green's functions of Sec. 7.4.

Transient Internal Heating of a Slab. To compute the temperature distribution in a slab of thickness l (between $x = 0$ and $x = l$), which is being heated internally by a source distribution $P(x,t)$ cal per sec per unit volume, we use the inhomogeneous equation for the temperature $\psi(x,t)$,

$$\partial\psi/\partial t = a^2\nabla^2\psi + 4\pi a^2 s(x,t) \tag{12.1.13}$$

where $a^2 = \kappa/\rho c$ and $s = P/4\pi\kappa$. A simple technique for computing ψ for nearly any sort of P is to compute ψ for a simple harmonic P and use the Laplace transform. Accordingly, we first set down the solution of Eq. (12.1.13) for $s = \delta(x - x_0)e^{-pt}$. This is a solution of $a^2\nabla^2\psi + p\psi = 0$, except at $x = x_0$, where it has a discontinuity of slope of -4π. If ψ is to be zero at the two slab surfaces, we have $\psi = G(p|x|x_0)$, where

$$G(p|x|x_0) = \frac{4\pi a}{\sqrt{p}\,\sin\left(\frac{l\sqrt{p}}{a}\right)} \begin{cases} \sin\left(\frac{x}{a}\sqrt{p}\right)\sin\left(\frac{l - x_0}{a}\sqrt{p}\right); & x < x_0 \\ \sin\left(\frac{x_0}{a}\sqrt{p}\right)\sin\left(\frac{l - x}{a}\sqrt{p}\right); & x > x_0 \end{cases} \tag{12.1.14}$$

and the temperature distribution for $s = (1/4\pi\kappa)P(x,p)e^{-pt}$ is, then

$$\psi(x,p) = \frac{1}{4\pi\kappa} \int_0^l P(x_0,p)G(p|x|x_0)\, dx_0 \, e^{-pt}$$

The special case of uniform input, where the input power per unit volume is the real part of P_0e^{-pt}, with P_0 independent of x, gives a temperature distribution

$$\psi(x,p) = \mathrm{Re}\left\{\frac{(a^2/\kappa)P_0e^{-pt}}{p\cos(l\sqrt{p}/2a)}\left[\cos\left(\frac{l - 2x}{2a}\sqrt{p}\right) - \cos\left(\frac{l}{2a}\sqrt{p}\right)\right]\right\} \tag{12.1.15}$$

An interesting application of this formula occurs in the case of a strip of metal, immersed in a medium (such as liquid helium) of high heat conductivity, the strip being heated by an alternating current of frequency $\omega/2\pi$. Since the heat input to the strip is proportional to the square of the current at any instant, the power P per unit volume is $P_0 - \mathrm{Re}(P_0 e^{-2i\omega t})$, where P_0 is the mean power input density, the fluctuation in input power having twice the frequency of the heating current. In this case the temperature distribution across the strip, which is supposed to have thickness l and width much greater than l, is

$$\psi = \frac{P_0}{2\kappa}\, x(l - x) - \mathrm{Re}\left\{\frac{a^2 P_0 e^{-2i\omega t}}{2i\omega\kappa \cos(l\sqrt{2i\omega}/2a)} \cdot \left[\cos\left(\frac{l-2x}{2a}\sqrt{2i\omega}\right) - \cos\left(\frac{l}{2a}\sqrt{2i\omega}\right)\right]\right\}$$

and the flow of heat out of the surface $x = 0$, per unit area per second, is

$$-J_0 = \tfrac{1}{2} l P_0 \left\{1 - \mathrm{Re}\left[\frac{2a}{l\sqrt{2i\omega}} \tan\left(\frac{l\sqrt{2i\omega}}{2a}\right) e^{-2i\omega t}\right]\right\}$$
$$\to \begin{cases} lP_0 \sin^2(\omega t); & l\sqrt{2\omega}/2a \ll 1 \\ \tfrac{1}{2} l P_0 \left[1 - \dfrac{2a}{l\sqrt{2\omega}} \cos(2\omega t - \tfrac{1}{4}\pi)\right]; & l\sqrt{2\omega}/2a \gg 1 \end{cases} \tag{12.1.16}$$

when the power input is $P_0 \sin^2(\omega t)$. For low frequencies, the outflow of energy is in phase with the current; it drops to zero or nearly so every half cycle. For high frequencies, the outflow does not fluctuate so markedly and what fluctuation there is, is not in phase with the heat generation.

If the heat input varies in a nonoscillatory way, the procedure of the Laplace transform may be invoked to calculate the resulting temperature. For example, if a pulse of heat $\delta(t)$ is impressed on each unit volume, the temperature is

$$\begin{aligned} \psi_\delta(x,t) &= \frac{a^2}{2\pi i\kappa} \int_{-\infty}^{\infty} \frac{e^{-pt}}{p\cos(l\sqrt{p}/2a)} \left[\cos\left(\left(\frac{l-2x}{2a}\right)\sqrt{p}\right) - \cos\left(\frac{l}{2a}\sqrt{p}\right)\right] dp \\ &= \frac{4a^2}{\pi\kappa} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin\left[\frac{\pi x}{l}(2n+1)\right] \exp\left[-\left(\frac{\pi a}{l}\right)^2 (2n+1)^2 t\right] u(t) \\ J_\delta(x,t) &= -\frac{2a^2}{l}\, \vartheta_2\left(\frac{x}{l}; \frac{4\pi i a^2 t}{l^2}\right) u(t) \\ &= -\sqrt{\frac{\kappa}{\pi\rho c t}} \sum_{n=-\infty}^{\infty} (-1)^n \exp\left[-\frac{(x - nl)^2}{4a^2 t}\right] u(t) \end{aligned} \tag{12.1.17}$$

where we have again used the properties of the theta functions, as given in the table at the end of Chap. 4. The temperature distribution and heat flow out of the face $x = 0$, for a power input $P(t)$ per unit volume throughout the slab, are as before,

$$T(x,t) = \int_0^t P(t-\tau)\psi_\delta(x,\tau)\,d\tau; \quad U_0(t) = -\int_0^t P(t-\tau)J_\delta(0,\tau)\,d\tau$$

Diffusion and Absorption of Particles. The diffusion equation is approximately obeyed by diffusing particles, such as electrons in a gas or neutrons in matter, as long as the distances involved are large compared to the mean free path of the particles and as the times involved are large compared to the mean free times between collisions. We saw, in Eq. (2.4.42), that the diffusion constant a^2 is equal to one-third the mean velocity v_a of the particles, times the mean free path λ_a and that, in addition to the time- and space-derivative terms there are terms representing the absorption and production of particles. If all the particles in the region are near enough alike in speed and heterogeneous enough in direction of motion so that they all behave not too differently from the mean behavior, then the equation giving their density ρ as a function of position and time is

$$\partial\rho/\partial t = a^2\nabla^2\rho - \chi\rho + q \tag{12.1.18}$$

where $a^2 = \frac{1}{3}v_a\lambda_a$ (v_a the average velocity and λ_a the mean free path of the particles), χ is the mean rate of absorption of the particles in the medium, and q is the number of particles "created" in the medium per unit volume, as defined in the discussion preceding Eq. (2.4.42).

The properties of the particles and the medium through which they pass, which are of importance here, are therefore: the mean velocity v_a and mean free path λ_a of the particles; the fraction κ of the collisions in which the particle is absorbed; also q, the rate at which particles come into the distribution, either by "creation" in the medium (as by fission or radioactivity) or by the initial scattering from some incident, uniform beam, some particles of which penetrate well into the medium before they suffer their first collision. (We, of course, take their place of "creation" to be the position at which they are introduced into ρ by their first collision.) In terms of these physical quantities, the diffusion constant $a^2 = \frac{1}{3}\lambda_a v_a$, as already mentioned, v_a/λ_a is the rate at which an average particle suffers collisions, and $\kappa v_a/\lambda_a$, being the rate at which the average particle suffers an absorbing collision, must be equal to χ, the absorption rate.

An example which will illustrate some of the behavior of the solutions of this equation is as follows: A sudden shower of particles falls on the surface $x = 0$ of a semi-infinite mass of material in the region $x > 0$. They may be considered to be "created" in the material at the place they are first scattered. But if N cross the surface $x = 0$, a number $(N/\lambda_i)e^{-x/\lambda_i}\,dx$ will be scattered into the distribution between x and $x + dx$ beneath the surface, so that for a delta function shower the source function q will equal $(1/\lambda_i)e^{-x/\lambda_i}\delta(t)$. We are to solve Eq. (12.1.18) for this q, to find the subsequent behavior of ρ. Of course, in the case of an actual pulse of particles, the first collisions deep in the slab will occur later than those near the surface because of the finite velocity v_i of the particles in the incident beam. Consequently, a more correct form for q would be $(1/\lambda_i)e^{-x/\lambda_i}\delta(t - x/v_i)$. But this effect is negligible if v_i is large compared with the diffusion rate v_a or if the changes in density with time are slow compared to the collision rate v_i/λ_i. We here neglect the term x/v_i in δ, though it could be included, if need be. Constant λ_i is, of course, the mean free path for the incident particles.

The boundary condition on ρ at $x = 0$ is, according to Eq. (2.4.34), that $\rho = \gamma(\partial\rho/\partial x)$ at $x = 0$ (where $\gamma = 0.71\lambda_a$) if the region $x < 0$ is free space, such that all particles leaving the material never return. However, to start with, let us solve the problem with the much simpler boundary condition $\rho = 0$ at $x = 0$. From Eq. (2.4.43) we can obtain the Green's function for the one-dimensional case considered here, for $\rho = 0$ at $x = 0$:

$$G(t|x|x_0) = \frac{e^{-\chi t}}{2a\sqrt{\pi t}}\left[e^{-(x-x_0)^2/4a^2t} - e^{-(x+x_0)^2/4a^2t}\right]u(t) \qquad (12.1.19)$$

The particle density for a sudden burst of particles at $t = 0$ is then (for $t > 0$)

$$\begin{aligned}\varphi_\delta(x,t) &= \frac{1}{\lambda_i}\int_0^\infty e^{-x_0/\lambda_i}G(t|x|x_0)\,dx_0 \\ &= \frac{1}{2\lambda_i}e^{-\chi t+(a^2t/\lambda_i^2)}\left[e^{-x/\lambda_i}\,erfc\left(\frac{a\sqrt{t}}{\lambda_i} - \frac{x}{2a\sqrt{t}}\right)\right. \\ &\qquad\left. - e^{x/\lambda_i}\,erfc\left(\frac{a\sqrt{t}}{\lambda_i} + \frac{x}{2a\sqrt{t}}\right)\right]u(t) \qquad (12.1.20)\end{aligned}$$

where

$$erfc(w) = \frac{2}{\sqrt{\pi}}\int_w^\infty e^{-u^2}\,du \rightarrow \begin{cases} 2 + (1/w\sqrt{\pi})\,e^{-w^2}; & w \rightarrow -\infty \\ 1 - (2w/\sqrt{\pi}); & w \rightarrow 0 \\ e^{-w^2}(1/w\sqrt{\pi})[1 - (1/2w^2)]; & w \rightarrow \infty \end{cases}$$

The factor $e^{-\chi t}$ corresponds to the loss by absorption, the factor $e^{(a^2t/\lambda_i^2)}$ to the rise in density because of flow in to x from higher densities at earlier times.

For t small compared to $(x/a)^2$, the first term in the brackets in Eq. (12.1.20) is much larger than the second and the whole expression just reduces to $(1/\lambda_i)e^{-x/\lambda_i}$, except for x very near zero. For t large compared to $(\lambda_i/a)^2$, both terms in the bracket nearly cancel each other and leave

$$\varphi_\delta \to x\left(\frac{\frac{1}{2}\lambda_i}{a^3t^{\frac{3}{2}}\sqrt{\pi}}\right)e^{-\chi t - x^2/4a^2t}; \quad t \gg \lambda_i^2/a^2$$

At $t = 0$, $\varphi_\delta = (1/\lambda_i)e^{-x/\lambda_i}$; shortly thereafter the part near $x = 0$ "melts away" to zero; as time goes on, more of the function, deeper below the surface, is reduced until finally there is only a low maximum far inside the material.

If, instead of a burst of particles at $t = 0$, a steady stream had been maintained, things would have eventually settled down to a steady state, where as many particles were absorbed or left the surface $x = 0$ as were shot at it per second. The equation to be satisfied,

$$a^2(d^2\rho/dx^2) - \chi\rho = -(1/\lambda_i)e^{-x/\lambda_i} \tag{12.1.21}$$

has a steady-state solution of

$$\varphi_s(x) = \frac{\lambda_i}{a^2 - \lambda_i^2\chi}\left[e^{-x\sqrt{\chi}/a} - e^{-x/\lambda_i}\right]$$

which is zero at $x = 0$ and, for large values of x, behaves exponentially, following the exponent x/λ_i or $x\sqrt{\chi}/a$, whichever is the smaller.

However, as mentioned earlier, $(\chi/a^2) = (\kappa v_a/\lambda_a)(3/\lambda_a v_a) = (3\kappa/\lambda_a^2)$, where κ is the fraction of collisions for which the particle is absorbed, usually a quite small quantity. Therefore, $\sqrt{\chi}/a$ is usually smaller than $1/\lambda_i$, unless the mean free path for the incident beam is very much larger than the mean free path λ_a for the distribution ρ, and therefore the deep penetration of the particles (the behavior of ρ for large x) is chiefly determined by the absorption-diffusion process (the first exponential) rather than by the penetration of the incident beam (second exponential).

The transient solution, corresponding to a unit stream of particles started at $t = 0$ and continuing, is then

$$\begin{aligned}\varphi &= \varphi_s(x) - \int_0^\infty \varphi_s(x_0)G(t|x|x_0)\,dx_0 \\ &= \varphi_s(x) - \frac{\lambda_i^2}{\lambda_i^2\chi - a^2}\left\{\varphi_\delta(x,t) - \frac{1}{2\lambda_i}\left[e^{-x\sqrt{\chi}/a}erfc\left(\sqrt{\chi t} - \frac{x}{2a\sqrt{t}}\right)\right.\right. \\ &\qquad \left.\left. - e^{x\sqrt{\chi}/a}erfc\left(\sqrt{\chi t} + \frac{x}{2a\sqrt{t}}\right)\right]\right\}\end{aligned} \tag{12.1.22}$$

These and other solutions have been obtained for the boundary condition $\rho = 0$ at $x = 0$. However, as we mentioned earlier, the more correct boundary condition for the particle density ρ is that $\rho = \gamma(\partial\rho/\partial x)$ at $x = 0$, where $\gamma \simeq 0.71\lambda_a$. We can convert our solutions φ into solutions for ρ, satisfying the more correct boundary conditions, however, by setting

$$\rho - \gamma(\partial\rho/\partial x) = \varphi(x,t)$$

or

$$\rho(x,t) = -(e^{x/\gamma}/\gamma)\int_\infty^x e^{-u/\gamma}\varphi(u,t)\,du$$
$$= \frac{1}{\gamma}\int_0^\infty e^{-w/\gamma}\,\varphi(x+w,\,t)\,dw \qquad (12.1.23)$$

where, if ρ satisfies Eq. (12.1.18), φ satisfies

$$\partial\varphi/\partial t = a^2\nabla^2\varphi - \chi\varphi + q - \gamma(\partial q/\partial x)$$

If $\varphi = 0$ at $x = 0$, then $\rho = \gamma(\partial\rho/\partial x)$ at $x = 0$.

For example, the steady-state solution for ρ, satisfying Eq. (12.1.21), will correspond to φ satisfying (assuming $\lambda_a = \lambda_i$)

$$a^2(\partial^2\varphi/\partial x^2) - \chi\varphi = -(1.71/\lambda_i)e^{-x/\lambda_i}$$

which corresponds to $\varphi = 1.71\varphi_s(x)$, with φ_s given above. Performing the integration for ρ, we have

$$\rho_s(x) = \frac{1.71\lambda_i}{a^2 - \lambda_i^2\chi}\left[\frac{ae^{-x\sqrt{\chi}/a}}{a+\gamma\sqrt{\chi}} - \frac{\lambda_i e^{-x/\lambda_i}}{\lambda_i+\gamma}\right] \qquad (12.1.24)$$

which satisfies the proper boundary conditions for ρ. The other, transient solutions may be calculated, if need be, using Eq. (12.1.23).

Of course, the boundary condition with $\gamma = 0.71\lambda_a$ is a very good approximation only if no sources are present within a distance λ_a of the boundary, which is certainly not the case here. Consequently, the constant 1.71 in Eq. (12.1.24) should be somewhat different in value.

Fission and Diffusion. When the material through which neutrons are diffusing is, in part, made up of fissionable material, neutrons are "created" in the material as well as "destroyed." These neutrons have two properties which profoundly affect the behavior of the chain reaction. In the first place, not all the neutrons are given off immediately, some are delayed an appreciable amount before being emitted; in the second place, the neutrons "created" by the fission process usually have energies much higher than the average neutron energy and must be slowed down before they can produce more fission. To see how these properties affect the phenomenon, we shall separate them artificially, taking into account first the delay and then the slowing-down.

In the first place, let us neglect both delay and slowing-down and assume that a certain percentage of the neutrons in a given volume produce fissions each second and that the fissions immediately produce more neutrons of the same average energy. In this case the quantity q in Eq. (12.1.18) is simply proportional to ρ; the proportionality factor, R, is a constant. The only change this makes in the equation we have been solving, therefore, is that, instead of the absorption factor χ, we should now use $(\chi - R)$, the net loss rate, in all our formulas. In this case, consequently, if the *reproduction rate* R turns out to be enough larger than the absorption rate χ, then the neutron density continually increases in magnitude and the chain reaction is self-supporting. Just how much larger than χ must R be depends on the configuration of the material, as will shortly be demonstrated.

Of course, in terms of average velocity v_a of the neutrons and their mean free path λ_a, it was already pointed out that $\chi = \kappa v_a/\lambda_a$ where κ is the fraction of collisions which result in absorption of the colliding neutron. Similarly, we can define ζ to be the fraction of collisions which cause fission and ν to be the mean number of neutrons "created" by a fission; then

$$R = \nu\zeta v_a/\lambda_a$$

Our previous remark therefore is that a chain reaction is not possible, no matter what the configuration, unless $\nu\zeta > \kappa$, that is, unless the fraction of fission neutrons created per collision is greater than the fraction of absorptions per collision.

When the neutrons produced by the fission are, to some extent, delayed before they are emitted, then the term q is not just proportional to $\rho(x,t)$. If we neglect the effect of slowing-down, q turns out to be related to ρ at the same point x, but at different times τ, earlier than t. Supposing that the neutrons from the fission process are emitted in a single radioactive process, their rate of emission is an exponential time term $e^{-\alpha t}$ (a fairly good approximation even when several processes are taking place) and the equation which ρ is to satisfy is

$$\frac{\partial}{\partial t}\rho(x,t) = a^2\frac{\partial^2}{\partial x^2}\rho(x,t) - \chi\rho(x,t) + \alpha R\int_0^\infty e^{-\alpha\tau}\rho(x, t-\tau)\,d\tau + q \qquad (12.1.25)$$

the integral term of which takes delay into account (approximately) but not the effect of slowing-down. The quantity $1/\alpha$ is the mean delay of the neutrons before emission. The last term, q, corresponds to the extra neutrons introduced into the region from outside. We note that when ρ is independent of time, the integral term reduces to $R\rho(x)$. The effects of delay in neutron emission are, of course, noticeable only in nonstationary motions of the system.

Laplace Transform Solution. A solution of this equation by Laplace transform methods is indicated. We consider the case of a slab of thickness l, between $x = 0$ and $x = l$ and, for the time being, set $\rho = 0$ at $x = 0$ and $x = l$. We analyze the transient q into its simple-harmonic components, and first solve the problem for $q = \delta(x - x_0)e^{-i\omega t}$. We assume a steady-state solution of the form

$$\varphi(x|x_0|\omega) = \sum_{m=1}^{\infty} A_m \sin\left(\frac{\pi m x}{l}\right) e^{-i\omega t}$$

substituting this into Eq. (12.1.25), multiplying by $\sin(\pi n x/l)$ and integrating over x, we obtain an equation for the coefficient A_n,

$$\tfrac{1}{2}l\left[-i\omega + \beta_n + \chi - \frac{\alpha R}{\alpha - i\omega}\right] A_n = \sin\left(\frac{\pi n x_0}{l}\right)$$

where $\beta_n = (\pi n a/l)^2$ has the dimensions of a frequency, as does χ and R. The solution will be given in terms of the roots of the equation $\omega^2 + i\omega(\alpha + \chi + \beta_n) + \alpha(R - \beta_n - \chi) = 0$, which may be written as $-iw_n^+$ and $-iw_n^-$, where

$$\begin{aligned} w_n^+ &= \tfrac{1}{2}(\alpha + \chi + \beta_n) + \tfrac{1}{2}\sqrt{(\beta_n + \chi - \alpha)^2 + 4\alpha R} \\ &\quad \to \beta_n + \chi + (\alpha R/\beta_n); \quad \beta_n \gg \chi,\ R,\ \alpha \\ w_n^- &= \tfrac{1}{2}(\alpha + \chi + \beta_n) - \tfrac{1}{2}\sqrt{(\beta_n + \chi - \alpha)^2 + 4\alpha R} \\ &\quad \to \alpha - (\alpha R/\beta_n); \quad \beta \gg \chi,\ R,\ \alpha \end{aligned}$$

It is useful to express these roots directly in terms of the physical constants mentioned above:

$$\left.\begin{matrix} w_n^+ \\ w_n^- \end{matrix}\right\} = \frac{v_a}{2\lambda_a}\left\{\tfrac{1}{3}\lambda_a^2\left(\frac{\pi n}{l}\right)^2 + \kappa + \delta \pm \sqrt{\left[\tfrac{1}{3}\left(\frac{\pi n \lambda_a}{l}\right)^2 + \kappa - \delta\right]^2 + 4\nu\zeta\delta}\right\}$$

where κ, as we said before, is the probability of absorption per collision, $1/\delta$ is the mean delay of the neutrons in units of mean free time ($\alpha = \delta v_a/\lambda_a$ or $1/\delta = v_a/\lambda_a\alpha$), and $\nu\zeta$ is the mean number of neutrons produced per collision (ν the mean number per fission and ζ the probability of fission at a collision).

In terms of these quantities, we have

$$\varphi(x|x_0|\omega) = \frac{2i\omega - 2\alpha}{l} \sum_{n=1}^{\infty} \frac{\sin(\pi n x_0/l)\sin(\pi n x/l)}{(\omega + iw_n^+)(\omega + iw_n^-)}\, e^{-i\omega t}$$

and the transient solution for particles entering at x_0 at time t, corresponding to $q = \delta(x - x_0)\delta(t)$ is, as before,

$$\begin{aligned} \varphi_\delta(x|x_0|t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(x|x_0|\omega)\, d\omega \\ &= \frac{2}{l}\sum_{n=1}^{\infty} \frac{\sin(\pi n x_0/l)\sin(\pi n x/l)}{\sqrt{[\frac{1}{3}(\pi n \lambda_a/l)^2 + \kappa - \delta]^2 + 4\nu\zeta\delta}} \left\{\left[\left(\frac{\lambda_a w_n^+}{v_a}\right) - \delta\right]\exp(-w_n^+ t)\right. \\ &\qquad \left. + \left[\delta - \left(\frac{\lambda_a w_n^-}{v_a}\right)\right]\exp(-w_n^- t)\right\} u(t) \qquad (12.1.26) \end{aligned}$$

The density of particles resulting from $q(x,t)$ neutrons per second per unit volume introduced into the slab at time t and position x inside the slab (inserted from outside, not by fission, for the results of fission are included in the solution) to the approximation mentioned after Eq. (12.1.25) is then

$$\rho(x,t) = \int_0^t d\tau \int_0^l dx_0 q(x_0,\tau)\varphi_\delta(x|x_0|t - \tau)$$

If we wish to take into account the more correct boundary conditions that $\rho = \pm 0.71\lambda(\partial\rho/\partial x)$ at the two faces of the slab, this can be done approximately (if $l \gg \lambda$) by setting the faces of the actual slab at $x = 0.71\lambda$ and at $x = l - 0.71\lambda$ so that the length l in Eq. (12.1.26) is the actual thickness of the slab plus 1.42λ.

Several interesting things may be discovered by an examination of Eq. (12.1.26). In the first place, if $R = 0$ ($\zeta = 0$), the second exponential term disappears and the solution becomes

$$\rho = \frac{2}{l}\sum_{n=1}^{\infty} \sin\left(\frac{\pi n x_0}{l}\right) \sin\left(\frac{\pi n x}{l}\right) \exp\left\{-\left(\frac{tv_a}{\lambda_a}\right)\left[\kappa + \tfrac{1}{3}\left(\frac{\pi n \lambda_a}{l}\right)^2\right]\right\}$$

where κ is the mean rate of loss of neutrons per collision, by absorption (it is the ratio of the absorption cross section to the total cross section). It can be seen that the term $\frac{1}{3}(\pi n\lambda_a/l)^2$ is the corresponding loss, from the nth partial mode, because of diffusion out of the slab. If we want to conserve neutrons, both terms must be kept small; the material of the medium must be such that the great majority of the collisions are elastic and the slab must be several wavelengths thick so that the loss of neutrons per mean free time, $\frac{1}{3}(\pi n\lambda_a/l)^2$, by diffusion out of the slab is also quite small. This limiting expression for ρ, for $R = 0$, is, of course, a solution of Eq. (12.1.18).

If now R (or $\nu\zeta$) is increased slowly, the second exponential term in (12.1.26) appears smaller in magnitude than the first term at $t = 0$. As long as $\nu\zeta$ is small, the rate of decay of this second term is determined by α ($= \delta v_a/\lambda_a$) the rate of production of the delayed neutrons; if they are delayed only a short time, α will be large and the second term will be negligible in a short time. If α is small, however, if the neutrons coming from the fission come slowly, then the second term will decay slowly with time so that after a sufficiently long time only the second terms will remain and the remaining density will correspond to the remaining delayed neutrons, not to the diffusion constant a^2.

If R is made large enough, w_1^- will become negative and the chain reaction will become unstable, the density increasing without limit. This will occur when

$$\sqrt{(\beta_1 + \chi - \alpha)^2 + 4\alpha R} > (\alpha + \chi + \beta_1)$$

or when

$$R > \chi + \beta_1 = \chi + (\pi a/l)^2$$

or when

$$\nu\zeta > \kappa + \tfrac{1}{3}(\pi\lambda_a/l)^2 \tag{12.1.27}$$

This is an obvious requirement: To have a chain reaction, the average number of fission neutrons created per collision must be more than the average number of neutrons absorbed per collision plus the corresponding number lost by diffusion out of the slab. Since neutrons are lost either by absorption or by leaving the slab, the reproduction factor R must be larger than the absorption factor χ by the amount $\frac{1}{3}(\pi a/l)^2$ before the chain reaction is self-supporting. The thinner the slab or the larger the diffusion constant a^2, the more "enriched" must be the material before it will sustain a chain reaction. [We note that if we had a perfect reflector for neutrons at the two boundaries, the eigenfunctions would be cosines rather than sines, the term $n = 0$ would be included in the series, and the density would become unstable when $R > \chi$ (when $\nu\zeta > \kappa$), and loss by diffusion would not occur.]

It is also to be noticed that, when the neutrons are all "prompt" (when α goes to infinity), the solution turns out to be

$$\rho = \frac{2}{l} \sum_{n=1}^{\infty} \sin\left(\frac{\pi n x_0}{l}\right) \sin\left(\frac{\pi n x}{l}\right) \exp\left\{-t\left[\chi - R + \left(\frac{\pi n a}{l}\right)^2\right]\right\}$$

which is the solution of Eq. (12.1.18) with $q = R\rho$. In this limiting case, when R is greater than the critical amount, ρ begins immediately to increase. On the other hand, if δ is small (long delay), Eq. (12.1.26) shows that ρ may initially decrease even if w_1^- is negative. For the first term in the brackets is an exponentially decaying term, which is larger than the second term for t small if δ is small enough. The density ρ at first decreases until the increasing second term catches up, until the delayed neutrons put in enough appearance to build up the reaction.

The Slowing-down of Particles. To take into account the fact that fission neutrons usually have to slow down after they are "created," in order to cause more fissions, we must consider solutions of the diffusion equation given in Eq. (2.4.54). This equation resulted from the following considerations: We introduce high-speed particles into a medium which tends to slow them down, each collision producing a fractional decrease in energy. Thus we can use the average number of collisions suffered by the particle since it was introduced, instead of time, to measure the rate at which these particles diffuse outward. If, for example, the high-energy particles were inserted at a given point in the medium, slower particles would be found at other places; the slower the particle, the more widespread would be its distribution. Equation (2.4.54) expresses just this behavior; it will be discussed again in Sec. 12.2.

The dimensionless parameter, called the *age* of the particle,

$$\tau = (MQ/mQ_m)\ln(v_i/v) = (1/\eta)\ln(v_i/v); \quad v = v_i e^{-\eta\tau}$$

(where $mQ_m/MQ = \eta$ is the mean fractional loss of speed per collision) is the mean number of collisions suffered by a particle of speed v, since it was introduced into the medium with speed v_i. [We assume MQ/mQ_m to be independent of v in this case, for clarity in exposition, though Eq. (2.4.53) shows that the analysis may be carried on even though this is not so.] The quantity which corresponds to density here, $\psi = n_tQ_mp^4\rho/M = [p^3v\rho(v)\eta/\lambda]$ is the total number of particles per unit volume which have a speed greater than v (an age less than τ or a momentum greater than $p = mv$) at the beginning of a given second and which have speed less than v (age greater than τ) at the end of the same second. The space distribution of the introduced, high-speed particles corresponds, therefore, to an initial condition on ψ. As we shall see shortly, τ is proportional to time when λ is constant ($\lambda = 1/n_tQ$).

The equation for ψ in terms of space distribution and the *age parameter* τ is, according to Eq. (2.4.54),

$$\partial\psi/\partial\tau = \tfrac{1}{3}\lambda^2\nabla^2\psi - \kappa_i\psi \tag{12.1.28}$$

where λ is the mean free path for the higher speed particles (here assumed constant but it need not be to have the equation hold) and the dimensionless constant κ_i is equal to the fraction of particles absorbed by the medium per collision. Solutions of this equation are, of course, quite similar to those we obtained earlier in this section; the interpretation is now somewhat different. For example, if particles are "created" at depth x_i inside a slab of thickness l, so that the number introduced per second per unit volume with speed v_i is $\delta(x - x_i)$, then the value of ψ for a speed v is

$$\psi(x|x_i|\tau) = \frac{2}{l}\sum_{n=1}^{\infty} \sin\left(\frac{\pi n x_i}{l}\right)\sin\left(\frac{\pi n x}{l}\right)\exp\left\{-\tau\left[\tfrac{1}{3}\left(\frac{\pi n\lambda}{l}\right)^2 + \kappa_i\right]\right\}$$

and the density of particles having speeds between v and $v + dv$ is then $mp^2\rho(v)\,dv = P(v)\,dv$, where

$$P(v) = \left(\frac{\lambda}{v^2\eta}\right)\psi = \frac{2\lambda}{\eta l v^2}\sum_{n=1}^{\infty}\sin\left(\frac{\pi n x_i}{l}\right)\sin\left(\frac{\pi n x}{l}\right)\left(\frac{v}{v_i}\right)^{[(\pi n\lambda/l)^2+3\kappa_i](1/3\eta)} \tag{12.1.29}$$

If $2\eta > \frac{1}{3}(\pi\lambda/l)^2 + \kappa_i$, then there will be an infinite number of particles at zero velocity, according to this solution; in this case, the particles are slowed down so fast they do not diffuse out of the slab rapidly enough. Actually this never occurs, for various reasons: first, because the conditions for validity of Eq. (12.1.28) are not met in the limit of vanishing v; also, η is usually small enough so it would not occur anyway.

Fission, Diffusion, and Slowing-down. To return to our slab with fissionable material, if $\rho(x,t)$ particles per unit volume, having low enough speed to cause fission, are present at x, t, then $R\rho(x,t)$ fission neutrons will be produced per second per unit volume, according to our definition of R. These neutrons must be slowed down before they reach a velocity at which they can produce more fissions, and in the slowing-down some are diffused, absorbed, or lost from the slab. Thus we restrict ρ to be a measure of the density of only those neutrons which are capable of producing fission. In general, only the slow ones will be included, the ones slower than a certain limiting velocity v_ρ. (The criterion actually is not so sharp as this, but to the first approximation we shall assume that all neutrons of speed less than v_ρ can produce fission and all neutrons of greater speed are not included in ρ.) From the discussion of slowing-down, given above, we see that if $R\rho(x_i,t_i)$ high-speed neutrons (of mean speed v_i) are produced per second per cubic centimeter by fission at x_i, t_i, then

$$R\int_0^l \rho(x_i,t_i)\psi(x|x_i|\tau_i)\,dx_i = R\sum_{n=1}^{\infty}\left\{\frac{2}{l}\int_0^l \rho(x_i,t_i)\sin\left(\frac{\pi n x_i}{l}\right)dx_i\right\}\cdot \cdot\sin\left(\frac{\pi n x}{l}\right)\left(\frac{v_\rho}{v_i}\right)^{[(\pi n\lambda/l)^2+3\kappa_i](1/3\eta)}$$

neutrons per second enter the distribution ρ at x, that is, come to have speeds less than v_ρ.

The average length of time for the neutrons to slow down from v_i to v_ρ is obtained by using the expression λ/v for the mean free time, the average time between collisions. An infinitesimal length of time dt is related to $d\tau$, the fractional number of collisions, by the relation

$$dt = (\lambda/v)\,d\tau = -(\lambda/\eta v^2)\,dv$$

or the time delay is

$$t - t_i = (\lambda/\eta)[(1/v_\rho) - (1/v_i)]; \quad \eta = mQ_m/MQ$$

on the average (as long as λ is fairly independent of v). If $v_i \gg v_\rho$, the approximate value of this mean delay, $\lambda/\eta v_\rho$, is considerably longer than the mean free time λ/v_ρ of the slowed-down neutrons in ρ, but it is usually shorter than the mean delay of the delayed neutrons. If we take the mean delay to be $1/\alpha$ (which is to be at least as large as $\lambda/\eta v_\rho$), then an approximate expression for the neutrons entering the slowed-down distribution at x, t is [see Eq. (12.1.25)]

$$\sum_{n=1}^{\infty}\left\{\alpha R_n\int_0^\infty e^{-\alpha u}\,du\,\frac{2}{l}\int_0^l \rho(x_i, t-u)\sin\left(\frac{\pi n x_i}{l}\right)dx_i\right\}\sin\left(\frac{\pi n x}{l}\right) \tag{12.1.30}$$

where $R_n = R(v_\rho/v_i)^{(1/3\eta)[(\pi n\lambda/l)^2+3\kappa_i]}$ and $\alpha = (\delta v_\rho/\lambda) < (\eta v_\rho/\lambda)$. The quantity R_n is smaller than R because some of the neutrons have been absorbed or have diffused out of the slab before they were slowed down enough to enter the distribution ρ.

If now we invoke our Laplace transform technique, setting the expression above into Eq. (12.1.25) in place of the next to the last term given there and setting $q = \delta(x - x_0)e^{-pt}$, we can obtain a solution to (12.1.25) by setting

$$\rho = \sum_{m=1}^{\infty} A_m \sin\left(\frac{\pi m x}{l}\right) e^{-pt}$$

as before, obtaining for A_n,

$$\left[-p + \beta_n + \chi - \frac{\alpha R_n}{\alpha - p}\right] A_n = \frac{2}{l} \sin\left(\frac{\pi n x_0}{l}\right)$$

differing only from the earlier expression by having R_n instead of R. Consequently, we can use the whole of the analysis leading to Eq. (12.1.26), except that we now insert R_n instead of R in the formulas (R_1 for the first term of the series, R_2 for the second term, etc.).

The solution for a pulse of neutrons introduced at x_0 and $t = 0$, taking into account the effect of slowing-down (approximately) is thus the series of Eq. (12.1.26) with R_n inserted in the nth term (also in w_n^+ and w_n^-) instead of R. The discussion of the behavior of the solution follows the same general line as before. A self-sustaining chain reaction will not start unless $R_1 > \chi + \beta_1$, or

$$\nu > (1/\zeta)[\kappa_\rho + \tfrac{1}{3}(\pi\lambda/l)^2](v_i/v_\rho)^{(1/3\eta)[(\pi n\lambda/l)^2+3\kappa_i]}$$

where we have set $R = (\nu\zeta v_\rho/\lambda)$, $a^2 = \frac{1}{3}v_\rho\lambda$, and $\chi = \kappa v_\rho/\lambda$, and where we have allowed for the fact that κ may be different for the slow neutrons ($v < v_\rho$) than for the fast ($v > v_\rho$). Thus the fission neutrons which are lost before they have a chance to cause more fission are lost in two ways: they are lost by absorption or diffusion out of the slab while they are slowing down (represented by the v_i/v_ρ factor, which is the dominant factor if $v_i \gg v_\rho$), and they are lost after they have slowed down for, even when they are slow enough to cause fissions, only a small fraction ζ of the collisions produce fission [represented by the factor $(\kappa_\rho/\zeta) + (\frac{1}{3}\zeta)(\pi\lambda/l)^2$ in the inequality]. Since v_i is usually much larger than v_ρ, κ_i/η must be quite small and $l\sqrt{\eta}$ must be quite a bit larger than λ or else ν will need to be impossibly large before a chain reaction starts. Since η, the fraction of energy lost per collision, is approximately proportional to the ratio of masses of the neutron and the target atom, the material of the slab should have a high proportion of light atoms, but the atoms chosen must be ones which do not absorb neutrons (κ small). In any case, the thickness of the slab l must be considerably larger than a mean free path λ in order that a chain reaction start for a ν of reasonable size.

General Case, Diffusion Approximation. We can now generalize our discussion by calculating a chain reaction in a piece of homogeneous material of any size or shape. We solve the Helmholtz equation for the boundary condition that ρ be zero on a surface just 0.71 of a mean free path outside the boundary of the material, obtaining the sequences k_n (corresponding to $\pi n/l$ for the slab) of eigenvalues and of $\psi_n(r)[\sin(\pi n x/l)$ for the slab], the eigenfunctions and their normalizing constants Λ_n. The properties of the material of importance (to recapitulate our definitions) are its mean free path for neutrons λ (assumed independent of v), the mean speed v_i of the neutrons emitted from the fission process, the mean speed v_ρ below which neutrons will cause fission, the fractional loss in speed η per collision during the slowing down process ($\eta \simeq m/M$), the fraction κ of these collisions at which the neutron is absorbed without fission (κ_i for the fast neutrons and κ_ρ for those of speed less than v_ρ), the mean delay $1/\alpha = \lambda/v_\rho\delta$ between the fission process and the appearance of the resulting slowed-down neutrons in the distribution $\rho(\delta < \eta)$ and, finally, the rate R at which a neutron, of speed less than v_ρ, can cause a fission, which is equal to v_ρ/λ times the fraction ζ of collisions which produce fissions times the mean number ν of neutrons given off per fission ($R = \nu\zeta v_\rho/\lambda$).

The density of fission-producing neutrons at the point r at time $t > 0$, caused by the introduction of one slow neutron at r_0 at $t = 0$, is then

$$\rho_\delta(r|r_0|t) = \sum_n \frac{[\psi_n(\mathbf{r}_0)\psi_n(\mathbf{r})/\Lambda_n]}{\sqrt{(\frac{1}{3}\lambda^2 k_n^2 + \kappa_\rho - \delta)^2 + 4\nu_n\zeta\delta}} \left[\left(\frac{\lambda}{v_\rho} w_n^+ - \delta\right) e^{-w_n^+ t} + \left(\delta - \frac{\lambda}{v_\rho} w_n^-\right) e^{-w_n^- t}\right] \qquad (12.1.31)$$

which takes into account (approximately) the effect of diffusion, absorption, fission, delayed emission, and slowing-down. The rates of decay are

$$\left.\begin{matrix} w_n^+ \\ w_n^- \end{matrix}\right\} = \frac{v_\rho}{2\lambda} \left[\tfrac{1}{3}\lambda^2 k_n^2 + \kappa_\rho + \delta \pm \sqrt{(\tfrac{1}{3}\lambda^2 k_n^2 + \kappa_\rho - \delta)^2 + 4\nu_n\zeta\delta}\right]$$

where $\delta = \alpha\lambda/v_\rho$ is the ratio between the mean free time and the mean delay time of the fission neutrons ($\delta < \eta$) (see above) and

$$\nu_n = \nu(v_\rho/v_i)^{(1/\eta)[\frac{1}{3}\lambda^2 k_n{}^2 + \kappa_i]}$$

is the average number of neutrons emitted per fission, which avoid absorption or loss by diffusion out of the reactor while slowing down, and survive to join the distribution of slow neutrons, for the nth partial wave.

The distribution eventually builds up, and the chain reaction is self-sustaining, if w_1^-, the second decay rate for the lowest mode $\psi_1(k_1 < k_n, n > 1)$, is negative. This happens when

$$\nu > (1/\zeta)(\tfrac{1}{3}\lambda^2 k_1^2 + \kappa_\rho)(v_i/v_\rho)^{(1/\eta)(\frac{1}{3}\lambda^2 k_1{}^2+\kappa_i)} \tag{12.1.32}$$

If w_1^- is kept at exactly zero by adjusting κ, the density ρ soon has entirely the shape ψ_1 of the lowest mode, since all the higher modes attenuate with time. Since k_1 is inversely proportional to the shortest diameter of the reactor, the reactor dimensions must be considerably larger than a mean free path λ in order that Eq. (12.1.32) may be satisfied.

Heating of a Sphere. The calculation of diffusion problems, for configurations other than parallel-sided slabs, is carried out in a way which should be familiar by now. We shall consider only one other case here: that of a sphere of radius R. Consider, for example, the case of a heat-conducting sphere turning slowly in front of a source of radiant heat (such as the earth and the sun). In this case the boundary condition is that the rate of intake of energy is specified at the surface: if the axis of rotation is the z axis ($z = r \cos \vartheta$) and the radiation comes from a point a large distance in the direction of the positive x axis ($x = r \sin \vartheta \cos \varphi$) then the rate of inflow is $I_0 \sin \vartheta \cos \varphi$. However, if we take axes fixed with respect to the sphere, which turns about the z axis with an angular velocity ω, then the inflow of heat at the surface $r = a$ is $I_0 \sin \vartheta \cos(\omega t - \varphi)$ and the distribution of temperature inside the sphere is the solution of Eq. (12.1.1) which has κ times its radial gradient at $r = R$ equal to the inflow specified. This solution is the real part of

$$\psi(r,\omega) = \left(\frac{I_0}{\sqrt{i\kappa\rho C\omega}}\right)\frac{j_1(\sqrt{i\omega}\, r/a)}{j_1'(\sqrt{i\omega}\, R/a)} \sin \vartheta\, e^{i(\varphi-\omega t)} \tag{12.1.33}$$

where j_1 is the spherical Bessel function, j_1' is its derivative with respect to its argument, and C is the specific heat of the material ($a^2 = \kappa/\rho C$). This solution, of course, assumes that what is absorbed on the heated side is reradiated on the other; any net energy intake or outflow would require a nonperiodic term, to be discussed later.

When ω is small compared to $R/a = R\sqrt{\rho C/\kappa}$, then the argument of j_1 is small, for $r \le R$, so that $j_1(z) \simeq \frac{1}{3}z$ in the range and $j_1'(z) \simeq \frac{1}{3}$. Consequently, for $\omega a \ll R$,

$$\mathrm{Re}[\psi(r,\omega)] \simeq (I_0/\kappa) r \sin \vartheta \cos(\varphi - \omega t) = (I_0/\kappa)[x \cos \omega t + y \sin \omega t]$$

the temperature gradient being uniform across the sphere, pointed in the direction of the source. On the other hand, when ω, the angular velocity, is large compared to (R/a), $j_1(z) \to -(1/z) \cos z \simeq -(1/2z)e^{-iz}$, for $z = \sqrt{i\omega}\,(r/a)$ and r near the surface, and $j_1'(z) \simeq (i/2z)e^{-iz}$. Consequently, when $\omega a \gg R$ and $R - r \ll R$, the temperature distribution is

$$\mathrm{Re}[\psi(r,\omega)] = \frac{I}{\sqrt{\kappa\rho C\omega}} e^{-\sqrt{\frac{1}{2}\omega}(R-r)/a} \cos\left[\omega t - \sqrt{\frac{\rho C\omega}{2\kappa}}(R - r) - \varphi - \tfrac{1}{4}\pi\right]$$

corresponding to a wave of wavelength $\sqrt{8\pi^2\kappa/\rho C\omega}$ and of wave velocity $\sqrt{2\kappa\omega/\rho C}$, which is rapidly attenuated as it moves below the surface, reducing by a factor $e^{-2\pi}$ every further wavelength it penetrates. The wave front rotates with φ as the sphere rotates, being 45° behind in phase.

If, on the other hand, the whole sphere is being irradiated uniformly, the inflow of energy being proportional to the difference between the surroundings and the surface temperature of the sphere, then we can use the spherically symmetric solution of Eq. (12.1.1). If the boundary condition is that $\partial\psi/\partial r = h(e^{-i\omega t} - \psi)$ at $r = R$, then the temperature inside the sphere is the real part of

$$\psi(r,\omega) = \left(\frac{R^2h}{r}\right) \frac{\sin(\sqrt{i\omega}\, r/a)/\sin(\sqrt{i\omega}\, R/a)}{(\sqrt{i\omega}\, R/a)\cot(\sqrt{i\omega}\, R/a) - 1 + Rh} e^{-i\omega t} \qquad (12.1.34)$$

and if the radiation is turned on at $t = 0$, the sphere being at zero temperature before this and the heater being at unit temperature thereafter, the subsequent temperature at r, t is

$$T = \frac{-1}{2\pi i}\int_{-\infty}^{\infty} \psi(r,\omega)\frac{d\omega}{\omega}; \quad t > 0$$

which may be evaluated by the usual method of residues.

Finally if, by some means, the spherical shell at $r = r_0$ is being heated periodically so that the power input density is $\delta(r - r_0)e^{-i\omega t}$, the corresponding Green's function,

$$\varphi(r|r_0|\omega) = \left(\frac{ar_0}{\kappa r}\right)\frac{[e^{-i\omega t}/\sqrt{i\omega}]}{\sin(\sqrt{i\omega}\, R/a)}\begin{cases} \sin\left[\dfrac{\sqrt{i\omega}}{a} r\right]\sin\left[\dfrac{\sqrt{i\omega}}{a}(R - r_0)\right]; & r < r_0 \\ \sin\left[\dfrac{\sqrt{i\omega}}{a} r_0\right]\sin\left[\dfrac{\sqrt{i\omega}}{a}(R - r)\right]; & r > r_0 \end{cases} \qquad (12.1.35)$$

may be used to compute the transient temperature, caused by fluctuations of internal heating, when the boundary condition is that $\psi = 0$ at $r = R$. By methods which are by now, we hope, quite familiar, we obtain for the transient temperature resulting from a power input $\delta(r - r_0)\delta(t)$, a pulse at $t = 0$ in a shell at $r = r_0$,

$$\varphi_\delta(r|r_0|t) = \left(\frac{2r_0a^2}{\kappa rR}\right)\sum_{n=1}^{\infty}\sin\left(\frac{\pi nr_0}{a}\right)\sin\left(\frac{\pi nr}{a}\right)e^{-(\pi na/R)^2t};\quad t > 0 \quad (12.1.36)$$

The temperature at r, t caused by a power input density $P(r,t)u(t)$ inside the sphere, for temperature zero at $t = 0$ for all r and at $r = 0$ for all t, is then

$$T(r,t) = \int_0^t d\tau \int_0^R dr_0 P(r_0,\tau)\varphi_\delta(r|r_0|t - \tau)$$

From this may be obtained the temperature distribution of the earth, for example, for any assumed radial and time distribution of radioactive heating.

12.2 *Distribution Functions for Diffusion Problems*

In a number of cases the crude approximations represented by the diffusion equation are not good enough; for some reason we may desire to probe more deeply into the atomic details underlying the diffusion picture than is possible by the methods of the preceding section. We must then turn to the *distribution function* $f(\mathbf{r},\mathbf{p},t)$, the density of particles in the element of phase space at the position $\mathbf{r}$, momentum $\mathbf{p}$, and time t, a field in six-dimensional phase space (plus time). The general properties of this function were discussed in Sec. 2.4. We shall investigate a few of the techniques which have been found useful in computing f and indicate the solution of a few typical problems which have turned out to be of practical interest.

We confine our attention, at first, to problems which depend on only one space dimension, corresponding to the "slab" problems of the previous section. Many examples of importance are not too different from such a case; in addition it thus avoids a superfluity of subscripts and complexities of algebra which would only obscure the main points. We also leave out the exchange of energy during collision and the effects of a force, such as electric or magnetic forces, on the particles. Consequently, at the beginning we can consider the particles as having all the same speed v, the effect of collisions being simply to change their direction of motion.

With all these restrictions, the resulting distribution function is a function of the one coordinate x, of time t, and of the angle ϑ between the direction of motion of the particle and the x axis, but of none of the other possible variables. In other words, the quantity $f(x,\mu,t)\,dx\,d\mu$ ($\mu = \cos\vartheta$) is the number of particles, at time t, in the volume element consisting of the slice of a prism, of unit cross-sectional area, between the planes x and $x + dx$, having the cosine of the inclination ϑ of their motion with respect to the x axis between μ and $\mu + d\mu$. Referring to Eqs. (2.4.16) and (2.4.35) and to Fig. 12.1, the equation for f is

$$\frac{\partial}{\partial t} f(x,\mu,t) + \mu v \frac{\partial}{\partial x} f(x,\mu,t) = -n_t Q_t v f(x,\mu,t) + n_t v \iint d\Omega' \sigma(\theta') f(x,\mu',t) + S(x,\mu,t) \quad (12.2.1)$$

where n_t is the mean number of target atoms (or molecules or nuclei) per cubic centimeter, where S is the mean density of particles created per second per cubic centimeter with directions between μ and $\mu + d\mu$, where v is the mean particle speed, and where $\mu = \cos \vartheta$. The quantity σ gives the angle distribution of the particles scattered elastically from a solid angle at $\vartheta' = \cos^{-1} \mu'$ into a solid angle at $\vartheta = \cos^{-1} \mu$, such that

$$Q_e = 2\pi \int_0^\pi \sigma(\vartheta) \sin \vartheta \, d\vartheta$$

is the cross section for elastic scattering for the particles at velocity v from the target atoms. The quantity $Q_t = Q_e/\kappa$ is the total cross section,

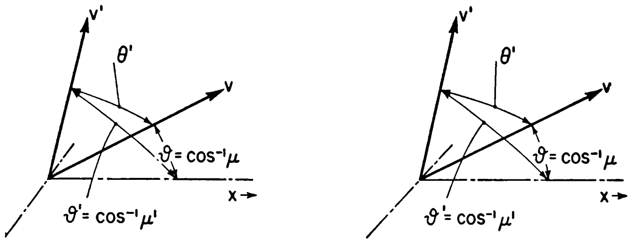

Fig. 12.1 Angles and velocity vectors involved in transport equation.

including a possible absorption cross section, so that $\kappa \leq 1$ is the mean fraction of particles returned to the distribution and $1 - \kappa$ the fraction absorbed per collision. Factor κ is sometimes called the *albedo factor*, since it determines the reflectivity (called the albedo) of a slab of the material.

We can now express our lengths in terms of the mean free path $1/n_tQ_t$ [see Eq. (2.4.12)] as unit and time in units of mean free time $1/n_tQ_tv$. We set $x = n_tQ_t\xi$ and $t = n_tQ_tv\tau$ and obtain

$$\frac{\partial}{\partial \tau} f(\xi,\mu,\tau) + \mu \frac{\partial}{\partial \xi} f(\xi,\mu,\tau) + f(\xi,\mu,\tau) = \kappa \iint d\Omega' \alpha(\theta') f(\xi,\mu',\tau) + s(\xi,\mu,\tau) \quad (12.2.2)$$

where $s = S/n_tQ_tv$ and where $\alpha(\theta') = \sigma(\theta')/Q_e$ is the angle-distribution factor, normalized to unity, so that

$$2\pi \int_0^\pi \alpha(\vartheta) \sin \vartheta \, d\vartheta = 1$$

We now wish to calculate the f's resulting from various assumed behaviors of α and source function s, for various boundary conditions.

Uniform Space Distribution. First we assume that the region under consideration is so far away from the boundary planes that their effects are negligible and that the source function s is independent of x (uniform space distribution of sources). Then f is also independent of x. If, in addition, the newly introduced particles are isotropic, so that s is independent of μ, then f will be independent of μ and the problem becomes trivial. The transport equation (12.2.2) reduces to

$$(df/d\tau) + (1 - \kappa)f = s(\tau)$$

which has the solution

$$f(t) = e^{(\kappa-1)\tau}\left[f(0) + \int_0^\tau e^{(1-\kappa)y}s(y)\,dy\right] \tag{12.2.3}$$

where the origin of τ is set so that $s(\tau) = 0$ for $\tau \leq 0$.

The nature of the solution depends critically on the value of $(1 - \kappa)$, the fraction of particles absorbed per collision. If this is greater than zero, then f will remain bounded for all values of τ (unless s goes to infinity). As long as some particles are absorbed, there will always be a finite value of f large enough so that the *number* of particles absorbed per unit time, $(1 - \kappa)f$, can counterbalance s, the number of particles introduced per unit time. If, on the other hand, no particles are absorbed (albedo factor $\kappa = 1$), then

$$f(\tau) = f(0) + \int_0^\tau s(y)\,dy$$

and since s is never negative, f tends toward infinity. No particles are absorbed, so that all particles introduced stay on indefinitely. If particles are continually introduced, the density increases without limit.

If s is not isotropic in the direction of the introduced particles, we have to take into account the dependence on angle of the particles scattered at a collision (the variation of α with ϑ). The most obvious method of solution would then be to expand all dependence on direction of particle motion into series of spherical harmonics. For example, if the source function is symmetric about some axis which can be called the x axis, then this is also the axis of symmetry for the distribution f. Consequently, we can write

$$f(\mu,\tau) = \sum_{n=0}^{\infty} F_n(\tau)P_n(\mu); \quad s(\mu,\tau) = \sum_{n=0}^{\infty} S_n(\tau)P_n(\mu)$$

where $\mu = \cos\vartheta$ (see Fig. 12.1), where the coefficients $S_n(\tau)$ are known but the functions $F_n(\tau)$ are to be determined. The scattering function α

may also be expanded in spherical harmonics of θ', the angle between $\mathbf{v}'$ and $\mathbf{v}$ (see Fig. 12.1),

$$\alpha(\theta') = \frac{1}{4\pi}\sum_{n=0}^{\infty}(2n+1)A_nP_n(\cos\theta'); \quad A_n = 2\pi\int_{-1}^{1}\alpha(\cos^{-1}\mu)\,P_n(\mu)\,d\mu$$
$$= \frac{1}{4\pi}\sum_{n=0}^{\infty}(2n+1)A_n\sum_{m=0}^{n}\epsilon_m\frac{(n-m)!}{(n+m)!}P_n^m(\cos\vartheta')P_n^m(\cos\vartheta)\cdot$$
$$\cdot\cos[m(\varphi-\varphi')] \qquad (12.2.4)$$

where A_0 must equal unity in order that the normalization requirement $2\pi\int\alpha\,d\mu = 1$ be satisfied.

When the series are inserted in Eq. (12.2.2) (when the $\partial/\partial\xi$ term can be neglected), there results a set of simultaneous equations

$$(d/d\tau)F_0 + (1-\kappa)F_0 = S_0(\tau)$$
$$(d/d\tau)F_n + (1-\kappa A_n)F_n = S_n(\tau); \quad n > 0$$

These may be solved analogously with Eq. (12.2.3). For example, for the case where both f and s are zero for $\tau \leq 0$, we have

$$f(\mu,\tau) = e^{-\tau}\int_0^\tau e^y\sum_{n=0}^{\infty}e^{\kappa A_n(\tau-y)}S_n(y)P_n(\mu)\,dy = e^{-\tau}\int_0^\tau e^y s(\mu,y)\,dy$$
$$+ e^{-\tau}\int_0^\tau e^y\sum_{n=0}^{\infty}[e^{\kappa A_n(\tau-y)}-1]S_n(y)P_n(\mu)\,dy \qquad (12.2.5)$$

When κ is small compared to unity (when most of the particles are absorbed at each collision), the second form of the solution is the more useful, for then the second term is small and

$$f(\mu,\tau) \simeq e^{-\tau}\int_0^\tau e^y s(\mu,y)\,dy$$
$$+ \kappa e^{-\tau}\int_0^\tau e^y\sum_{n=0}^{\infty}(\tau-y)A_nS_n(y)P_n(\mu)\,dy; \quad A_0 = 1$$

In this case, when only a few particles are elastically scattered, the particle distribution f at any time is a sort of average of the source function s, in magnitude and direction, averaged over the previous mean free time (unit interval in τ), plus a correction term depending on the angular distribution of the elastic scattering (*i.e.*, on the A_n's). The mean free time is thus a sort of relaxation time for the distribution function.

If the albedo factor κ is not small, but if the angle-distribution function $\alpha(\theta')$ is not very markedly dependent on the scattering angle θ', then all the A_n's except the first will be small and a good approximation to the distribution function is

$$f(\mu,\tau) \simeq e^{(\kappa-1)\tau} \int_0^\tau e^{(1-\kappa)y} S_0(y)\, dy + e^{-\tau} \int_0^\tau e^y [s(\mu,y) - S_0(y)]\, dy$$

where $S_0(\tau) = \frac{1}{2} \int_{-1}^{1} s(\mu,\tau)\, d\mu$ is the average value of s over all angles for the time τ. The relaxation time for the angle-dependent part is one mean free time. The angle-dependent part comes to equilibrium with the source distribution much more rapidly than the isotropic part, in this case.

When $s(\mu,\tau)$ has some asymptotic form which is independent of τ after $\tau = T$, then the distribution function reaches a steady state after a time somewhat longer than $T + (1 - \kappa)^{-1}$. This steady-state distribution can be written

$$\begin{aligned} f(\mu,\tau) &\to \sum_{n=0}^{\infty} \frac{S_n(T)}{1 - \kappa A_n} P_n(\mu); \quad \tau - T - \frac{1}{1-\kappa} \gg 1 \\ &\simeq s(\mu,T) + \kappa \sum_{n=0}^{\infty} A_n S_n(T) P_n(\mu); \quad \kappa \ll 1 \\ &\simeq \left(\frac{\kappa}{1-\kappa}\right) S_0(T) + s(\mu,T); \quad A_n \ll 1,\ n > 0 \end{aligned}$$

The second form shows that, if most of the particles are absorbed per collision, the asymptotic form for f is nearly the same as that of the particles introduced, s. On the other hand, if the absorption is small $(1 - \kappa \ll 1)$, the third form shows that when the scattering is nearly isotropic the steady-state distribution is nearly isotropic. We note that, if there is no absorption ($\kappa = 1$), there cannot be a steady state, for the number of particles is continually increasing as particles are introduced.

Approximations for Forward Scattering. In the next section [see Eqs. (12.3.60) *et seq.*] we shall find that at high speeds the angular distribution of charged particles scattered from atoms is approximately proportional to $[1 + \frac{1}{2}(v/u)^2 \sin^2 (\frac{1}{2}\vartheta)]^{-2}$, where v is the speed of the incident particle, ϑ is its angle of elastic scattering, and u is a constant determined by the properties of the particle and of the target atom. When v is much larger than u, nearly all the scattering is forward, for $\vartheta < \frac{1}{2}\pi$. In such cases it is useful to concentrate our attention on the region near $\vartheta = 0$ and to do our calculations on the plane tangential to the unit sphere at $\vartheta = 0$. In other words, instead of integrating over ϑ' and φ', we integrate over ρ' and φ', where $\rho' = 2 \tan(\frac{1}{2}\vartheta')$ is the distance from the origin of the point on the plane which corresponds to

ϑ', φ'. The error of such a representation will be quite large in the region $\rho > 2$, but if there is only a small amount of back scattering, the error will not be important. The criterion for validity will be that f for $\rho > 2(\vartheta > \frac{1}{2}\pi)$ must be negligibly small compared to f for $\rho \ll 2$ (ϑ near zero).

When this is possible, we can use rectangular coordinates $\beta = \rho \cos \varphi$ and $\eta = \rho \sin \varphi$ on the tangent plane and can then expand our functions in terms of Hermite polynomials of β and η. For example, a fairly good approximation to the angle-scattering function for high speeds is

$$\alpha(\vartheta) \simeq \left(\frac{v^2}{\pi u^2}\right) e^{-(v/u)^2\rho^2} \simeq \left(\frac{v^2}{\pi u^2}\right) e^{-(x^2+y^2)}$$

where $x = (2v/u) \tan(\frac{1}{2}\vartheta) \cos \varphi$ and $y = (2v/u) \tan(\frac{1}{2}\vartheta) \sin \varphi$. The element of area on the tangent plane corresponding to $d\Omega = \sin \vartheta \, d\vartheta \, d\varphi$ on the unit sphere is approximately $(u/v)^2 \, dx \, dy$ so that the integral of α, times $(u/v)^2 \, dx \, dy$, over the tangent plane is unity.

To carry through the integration of Eq. (12.2.2), we express the coordinates ϑ, φ and ϑ', φ' in terms of the rectangular coordinates x, y and x', y'; then the relative coordinates corresponding to θ' will be $(x - x')$ and $(y - y')$. We next express both f and s in terms of Hermite polynomials,

$$f = \sum_{m,n} e^{-(x^2+y^2)} F_{mn}(\tau,\xi) H_{2m}(x) H_{2n}(y)$$
$$s = \sum_{mn} e^{-(x^2+y^2)} S_{mn}(\tau,\xi) H_{2m}(x) H_{2n}(y)$$

where we have used only the even polynomials because f and s are supposed to be symmetric about the origin (for the same reason $F_{mn} = F_{nm}$ and $S_{mn} = S_{nm}$).

If now we can express the function $\alpha(\theta')$ in the integral in terms of Hermite polynomials of x', we shall be able to set up a sequence of equations relating the coefficients F_{mn} with the coefficients S_{mn}. The function $\alpha(\theta')$, in terms of the coordinates (x,y) of the final direction of the particle and of (x',y') for the initial direction, is

$$\alpha(\theta') = (v^2/\pi u^2) e^{-(x-x')^2-(y-y')^2}$$

and the relationship needed comes from the generating function of the Hermite polynomials

$$e^{-x^2+2xx'-x'^2} = e^{-x^2} \sum_m \frac{(x')^m}{m!} H_m(x)$$
$$= e^{-x^2} \sum_{m,s} \frac{1}{2^m s!(m - 2s)!} H_{m-2s}(x') H_m(x)$$

Inserting this and a similar relationship for (y,y') in the integral, integration over (x',y') leaves only even values of m in the above sum. Equating coefficients of Hermite polynomials, we obtain finally the (approximate) equations

$$\frac{\partial}{\partial\tau}F_{mn} + \frac{\partial}{\partial\xi}F_{mn} + (1-\kappa)F_{mn} \simeq \kappa \sum_{s,t} \frac{F_{m-s,n-t}}{2^{2s+2t}s!t!} + S_{mn} \qquad (12.2.6)$$

where the summation is over all values of s less than $m+1$, all values of t less than $n+1$, with the exception of the term $s = t = 0$, which has been taken over on the left side.

We have assumed here that v is enough larger than u so that $\mu = \cos\vartheta \simeq 1 - \frac{1}{2}(u/v)^2(x^2+y^2)$ may be considered to be unity over the region of the (x,y) plane where the F's are appreciable in value. Thus for distributions where the scattering involves only small angles of deflection, distance into the slab and time are on equal terms.

As an example, we consider the case of a steady-state beam of particles falling on the surface $\xi = 0$ of the slab, penetrating it, and being scattered as they penetrate. In this case, f is independent of τ and dependent only on ξ, x, and y. To obviate complicated boundary conditions, we can consider the particles as entering the distribution only after their first collision. If the incident beam is I_0 particles per second per unit area (in mean free paths squared), all traveling in the ξ direction (normal to the slab face), then the intensity of undeviated particles ξ mean free paths in from the surface is $I_0e^{-\xi}$ and the distribution of particles just after their first scattering (these are the "created" particles, according to our convention) is

$$s(\xi,x,y) = I_0e^{-\xi}\alpha(\vartheta) = (v^2/\pi u^2)e^{-\xi-x^2-y^2}$$

or

$$S_{mn}(\xi) = (v^2/\pi u^2)\delta_{m0}\delta_{n0}e^{-\xi}$$

The solution for F_{00} is then

$$F_{00} = \left(\frac{v^2}{\pi u^2}\right)e^{-(1-\kappa)\xi}\int_0^\xi e^{-\kappa y}\,dy$$

which rises linearly from zero at $\xi = 0$, passes through a maximum, and then drops to zero proportionally to the exponential $e^{-(1-\kappa)\xi}$, which is slowly decaying if the capture fraction $1-\kappa$ is small.

The other terms in the series for f may then be computed. The next several are

$$F_{01} = F_{10} = \left(\frac{v^2}{\pi u^2}\right)\tfrac{1}{4}\kappa e^{-(1-\kappa)\xi}\int_0^\xi dy_1\int_0^{y_1} e^{-\kappa y_2}\,dy_2$$

$$F_{11} = \left(\frac{v^2}{\pi u^2}\right)e^{-(1-\kappa)\xi}\int_0^\xi dy_1\int_0^{y_1}\left[\tfrac{1}{16}\kappa e^{-\kappa y_2} + \tfrac{1}{8}\kappa^2\int_0^{y_2} e^{-\kappa y_3}\,dy_3\right]dy_2$$
$$= 2F_{02} = 2F_{20}$$

$$F_{21} = F_{12} = 3\left(\frac{v^2}{\pi u^2}\right) e^{-(1-\kappa)\xi} \int_0^\xi dy_1$$
$$\int_0^{y_1} \left[\frac{\kappa}{384} e^{-\kappa y_2} + \int_0^{y_2} \left(\frac{3\kappa^2}{128} e^{-\kappa y_3} + \frac{\kappa^3}{32} \int_0^{y_3} e^{-\kappa y_4}\, dy_4\right) dy_3\right] dy_2$$
$$= 3F_{03} = 3F_{30};\quad \text{etc.}$$

All the F_{mn}'s having the same value of $(m + n)$ are proportional to each other (this simply corresponds to the requirement that f be a function of $x^2 + y^2$ and not of x^2 or y^2 separately). All the F's except F_{00} increase proportional to ξ^2 for ξ small, and F_{mn} is proportional to $\xi^{m+n}e^{-(1-\kappa)\xi}$ for ξ large.

If there is no absorption ($\kappa = 1$), this series eventually diverges, each term increasing without limit as ξ increases. In this case each particle keeps on colliding, without being absorbed, until the probability of its motion being at a large angle ϑ is large and our original assumption (on which we based the approximation, that most particles were moving nearly parallel to the z axis) is no longer true. If, however, there is appreciable absorption ($\kappa < 1$), then the series does not diverge, for most particles are absorbed before they are scattered appreciably away from z direction. Alternative solutions are in Probs. 12.14 to 12.16.

The same kind of analysis can be used for the time dependent case, when particles are introduced, at $t = 0$, throughout the slab, initially moving in the positive z direction. The distribution "spreads out," the mean deviation from the z direction increasing without limit if $\kappa = 1$ (no absorption) but remaining within bounds if $\kappa < 1$. But we have spent enough time on extremely restricted cases; it is time to look at the more general behavior of distribution functions.

General Considerations, Steady-state Case. The steady-state equation for the distribution function for the case where f depends on only one space coordinate and where there is no source of particles in the material,

$$\mu(\partial/\partial\xi)f(\xi,\mu,\varphi) + f(\xi,\mu,\varphi) = R(\xi,\mu,\varphi) \qquad (12.2.7)$$

is obtained from Eq. (12.2.2). Distance ξ is measured in terms of mean free path $1/n_tQ_t$, $\vartheta = \cos^{-1}\mu$ is the angle between the velocity and the x (that is, ξ) axis, and φ is the angle between the (v,x) plane and the (x,y) plane. The function

$$R(\xi,\mu,\varphi) = \kappa\int\int \alpha(\theta')f(\xi,\mu',\varphi')\, d\varphi'\, d\mu'$$

is the number of particles entering the distribution at (ξ,μ,φ) per mean free time, because of collisions, and might be called the *collision recovery function.* Constant $\kappa = Q_s/Q_t$ is the albedo factor and α is the angular distribution of the elastically scattered particles, normalized so that

$$2\pi \int_0^\pi \alpha(\vartheta) \sin\vartheta\, d\vartheta = 1$$

Angle θ' is the angle between the velocity before collision (at ϑ',φ') and that after collision (at ϑ,φ);

$$\cos\theta' = \cos\vartheta' \cos\vartheta + \sin\vartheta' \sin\vartheta \cos(\varphi - \varphi')$$

When α is given in terms of spherical harmonics as in Eq. (12.2.4),

$$\alpha(\theta') = \frac{1}{4\pi}\sum_{n=0}^{\infty}(2n+1)A_nP_n(\cos\theta')$$

the coefficient A_0 must equal unity for α to be normalized. For isotropic scattering, all the A's are zero except $A_0 = 1$.

The integral of f over all directions, $\rho(\xi) = \int\int f(\xi,\mu,\varphi)\,d\varphi\,d\mu$ is the particle density at depth ξ. The mean flux of particles in the ξ direction at any point is proportional to

$$J(\xi) = \int\int \mu f(\xi,\mu,\varphi)\,d\varphi\,d\mu$$

Integrating Eq. (12.2.7) over μ and φ, we obtain [use Eq. (12.2.4) and integrate over μ, φ first]

$$\begin{aligned}(d/d\xi)J(\xi) &= -\rho(\xi) + \kappa\int\int d\mu\,d\varphi\int\int\alpha(\theta')f(\xi,\mu',\varphi')\,d\mu'\,d\varphi' \\ &= -(1-\kappa A_0)\rho(\xi) = -(1-\kappa)\rho(\xi)\end{aligned} \tag{12.2.8}$$

which is the *equation of continuity* for particles. If $(1-\kappa) = 0$, no particles are absorbed and the flux J must be independent of ξ. If particles are absorbed ($\kappa < 1$), then (in the absence of sources) flux J must decrease with ξ.

Another integral invariant of the particle motion is the *second moment* of the distribution

$$K(\xi) = \int\int \mu^2 f(\xi,\mu,\varphi)\,d\varphi\,d\mu$$

If the mean flux J is analogous to the dipole moment of the distribution, then K is analogous to its quadrupole moment. If all the particles are moving at right angles to the x axis, then K is zero; if all of them are moving parallel to the x axis (either in the positive or negative direction), then K has its maximum value for a given ρ.

We obtain the equation for K by multiplying Eq. (12.2.7) by μ and integrating:

$$(d/d\xi)K(\xi) = -J(\xi) + \kappa\int\int\mu\,d\mu\,d\varphi\int\int\alpha(\theta')f(\xi,\mu',\varphi')\,d\mu'\,d\varphi'$$

Expanding $\alpha(\theta')$ in a series of spherical harmonics [see Eq. (12.2.4)] and integrating first over μ and φ, the integral becomes

$$\kappa A_1\int\int\mu' f(\xi,\mu',\varphi')\,d\mu'\,d\varphi' = \kappa A_1 J(\xi)$$

so that the equation for the variation of the second moment of the distribution becomes

$$(d/d\xi)K(\xi) = -(1-\kappa A_1)J(\xi) \tag{12.2.9}$$

This sequence of equations for the higher moments can be continued, if need be, each higher moment being expressed in terms of the preceding moment, the albedo factor κ, and a coefficient of the expansion of the distribution in angle $\alpha(\theta)$. In general (though not always), the coefficients A_n are less than unity, so none of the higher moments is likely to be independent of ξ.

One could imagine a technique of solution involving this procedure. We expand f in terms of spherical harmonics

$$f(\xi,\vartheta,\varphi) = \sum_{mn} [F_{mn}\cos(m\varphi) + F^0_{mn}\sin(m\varphi)]P^m_n(\cos\vartheta)$$

$$\rho(\xi) = 4\pi F_{00};\quad J(\xi) = \tfrac{4}{3}\pi F_{01};\quad K(\xi) = \tfrac{4}{3}\pi F_{00} + \tfrac{8}{15}\pi F_{02};\quad \text{etc.}$$

Inserting this series into Eq. (12.2.7) and utilizing the expansion of α given in Eq. (12.2.4) and the equation

$$\mu P^m_n(\mu) = \left(\frac{n-m+1}{2n+1}\right)P^m_{n+1}(\mu) + \left(\frac{n+m}{2n+1}\right)P^m_{n-1}(\mu)$$

we eventually arrive at the following set of simultaneous equations for the coefficients $F_{mn}(\xi)$:

$$\frac{d}{d\xi}F_{m,n+1} + \left(\frac{2n+3}{n+m+1}\right)(1-\kappa A_n)F_{mn} + \left(\frac{2n+3}{2n-1}\right)\left(\frac{n-m}{n+m+1}\right)\frac{d}{d\xi}F_{m,n-1} = 0 \quad (12.2.10)$$

with a similar set for the coefficients F^0_{mn}.

One might try to compute f by computing the F's in sequence. Unfortunately it is not usually easy to express our boundary conditions at the surface of the slab in terms of the F's. For example, suppose the slab is semi-infinite, the medium filling all space for $\xi > 0$, and suppose that an incident beam of particles, all moving in the positive ξ direction, falls on the surface $\xi = 0$. The distribution f at $\xi = 0$ has the following complicated form: for $\mu > 0$ it is $I_0\delta(1-\mu)$ (this represents the distribution of the incident particles) and for $\mu < 0$ it is not zero but is unknown (this represents the particles which have suffered one or more collisions in the medium and which rebound back out of the material). In fact, in many problems, the thing we wish to compute is just the behavior of $f(0,\mu,\varphi)$ for $\mu < 0$, for it gives the distribution in angle of the diffuse reflection of particles from the surface of the medium. Nevertheless the solutions given here will be useful in describing limiting conditions at large distances from boundaries. See also Probs. 12.8 to 12.13.

Letting the dependence of the F's on ξ be through the exponential $e^{-k\xi}$ reduces Eqs. (12.2.10) to a set of simultaneous algebraic equations from which one could, in principle, solve for allowed values of k and corresponding relative values of the F's. However, the recursion formulas for the F's do not result in very convergent forms to be solved for k, so this trail will not be followed any further.

We might note, however, that a substitution of the expression $f = e^{-k\xi}g(\mu)$ in the integrodifferential equation (12.2.7) gives some possible solutions in the case of isotropic scattering ($\alpha = 1/4\pi$). For then $R(\xi,\mu,\varphi)$ would be independent of μ and

$$(1 - \mu k)g(\mu) = G; \quad R(\xi,\mu,\varphi) = e^{-k\xi}\,G$$

Substituting $g = G/(1 - \mu k)$ back in the integral for R gives

$$G = \frac{\kappa G}{4\pi}\iint \frac{d\mu'\,d\varphi'}{1 - k\mu'}; \quad 1 = \tfrac{1}{2}\kappa \int_{-1}^{1} \frac{d\mu'}{1 - k\mu'} = \left(\frac{\kappa}{k}\right)\tanh^{-1} k$$

so that the equation for k is

$$\frac{\tanh(k/\kappa)}{k/\kappa} = \kappa \quad \text{or} \quad \kappa = \frac{k}{\tanh^{-1} k} \tag{12.2.11}$$

This equation is the same as Eq. (4.4.4), which was discussed for its general behavior for all complex values of k/κ. From the figures in Chap. 4, we can see the general nature of the roots k and their dependence on κ. For example, for $\kappa = 1$, k equals zero, or $\pm 4.493i$, or $\pm 7.725i$, or $\pm 10.904i$, etc.; for $\kappa = 0.5$, k equals ± 0.9575, or $\pm 2.138i$, or $\pm 3.798i$, or $\pm 5.406i$, etc., whereas for $\kappa = 0$, k equals unity. Consequently a solution of Eq. (12.2.7) for isotropic scattering, for $\kappa = 0.5$, would be

$$f(\xi,\mu) = \frac{C_1 e^{-0.96\xi}}{1 - 0.96\mu} + \frac{C_2 e^{+0.96\xi}}{1 + 0.96\mu} + B_1 \frac{\cos(2.14\xi) + 2.14\mu \sin(2.14\xi)}{1 + (2.14\mu)^2} + \cdots$$

if this could be made to fit the boundary conditions at $\xi = 0$.

For $\kappa = 1$ (no absorption), another simple solution is possible, even' when α is not independent of θ. We can insert $f = G[\mu - k\xi]$ in Eq. (12.2.7) and show that k must equal $(1 - A_1)$. Expressing G in terms of the net flux J (which is independent of ξ when $\kappa = 1$), we have

$$f = (3/4\pi)J[\mu - (1 - A_1)\xi] \tag{12.2.12}$$

which is also a solution that does not fit simply onto usual boundary conditions. We will see, however, on page 1623, that it is appropriate as an asymptotic form of the solution.

Integral Relations between the Variables. It is, of course, possible to obtain a formal solution of Eq. (12.2.7) by considering R to be a known function of ξ, μ, and φ and treating the equation as an inhomogeneous equation in ξ. The solution has, as a part, the solution of the homogeneous part of the equation, $I_0(\mu,\varphi)e^{-\xi/\mu}$. This part, for $\mu > 0$, represents those incident particles coming into the medium from outside the left-hand face; for $\mu < 0$, it represents those particles coming in through the right-hand face. For example, for a slab of material between the planes $\xi = 0$ and $\xi = a$, the formal solution of Eq. (12.2.7) is

$$\begin{aligned} f(\xi,\mu,\varphi) &= f(0,\mu,\varphi)e^{-\xi/\mu} + \frac{1}{\mu}\int_0^\xi R(\eta,\mu,\varphi)e^{-(\xi-\eta)/\mu}\,d\eta \\ f(\xi,-\mu,\varphi) &= f(a,-\mu,\varphi)e^{(\xi-a)/\mu} \\ &\qquad + \frac{1}{\mu}\int^a R(\eta,-\mu,\varphi)e^{(\xi-\eta)/\mu}\,d\eta; \quad 0 < \mu \le 1 \end{aligned} \tag{12.2.13}$$

We have, of course, had to split the solution into two parts, one part for the particles with positive ξ component of velocity, the other with negative ξ component. This is, of course, what is needed, for we can specify only the incoming distribution at the surface, not the outgoing. Here we specify, for instance, $f(0,\mu,\varphi)$ and solve to find $f(0,-\mu,\varphi)$.

This pair of Eqs. (12.2.13) is not a solution, however, for R itself depends on f. It is an integral equation for f, which can be changed into a still simpler integral equation for the mean density ρ. For isotropic scattering ($\alpha = 1/4\pi$), for example, R is just equal to $\kappa\rho/4\pi$ so, in the first place, f can be determined from

$$f(\xi,\mu,\varphi) = \begin{cases} f(0,\mu,\varphi)e^{-\xi/\mu} + \dfrac{\kappa}{4\pi\mu}\displaystyle\int_0^\xi \rho(\eta)e^{(\eta-\xi)/\mu}\,d\eta; & 0 < \mu \le 1 \\ f(a,\mu,\varphi)e^{(a-\xi)/\mu} - \dfrac{\kappa}{4\pi\mu}\displaystyle\int_\xi^a \rho(\eta)e^{(\eta-\xi)/\mu}\,d\eta; & -1 \le \mu < 0 \end{cases}$$

which are similar to Eqs. (2.4.19). As mentioned in Sec. 2.4, the first term in the topmost expression is the remnant of the incident beam which has traversed ξ/μ mean free paths in penetrating at an angle $\cos^{-1}\mu$ from the surface $\xi = 0$ without collision. The second term, the integral, is the contribution to the distribution function of these particles which had their last collision at a depth η (since they are traveling to the right, $\mu > 0$, η must be less than ξ) and reach the depth ξ without further collision.

Finally, having seen that (at least for isotropic scattering) the distribution function f may be computed once the density is known, we may set up an integral equation for ρ. This has been done in Sec. 2.4 [see Eq. (2.4.20)]. All that is needed is integration of Eq. (12.2.13) over μ. Since $\rho(\xi) = \iint f(\xi,\mu,\varphi)\,d\mu\,d\varphi$, we have

$$\rho(\xi) = \Phi(\xi) + \tfrac{1}{2}\kappa\int_0^a \rho(\eta)E_1(|\eta - \xi|)\,d\eta \tag{12.2.14}$$

where $$E_n(x) = \int_1^\infty e^{-xy}\left(\frac{dy}{y^n}\right)$$

and $$\Phi(\xi) = \int_0^{2\pi} d\varphi \int_1^\infty \left[f\left(0, \frac{1}{y}, \varphi\right)e^{-\xi y} + f\left(a, -\frac{1}{y}, \varphi\right)e^{-(a-\xi)y}\right] dy$$

This is a Fredholm equation of the second kind (see Chap. 8); when $\Phi = 0$, it is homogeneous and, when $a \to \infty$, the resulting Eq. (2.4.20) is called the *Milne equation.* We defer solving this equation for a few pages and discuss alternate ways of solving Eq. (12.2.7) first.

Two general cases represent the majority of the diffusion problems [of the sort represented by Eq. (12.2.7)] which we wish to solve. One is the case where the slab is quite thick ($a \to \infty$), the particles are introduced from the far side or are created in the slab some distance from the surface $\xi = 0$, and no particles are in the region $\xi < 0$ except those which come out from the slab. This is the case for light diffusing out from the interior of a star, reaching the surface and radiating away; it is also the case for neutrons penetrating a shield, coming from the far side, some of them reaching the outer surface and getting away. In fact it is approximately the case for every free surface of a material containing diffusing particles as long as the surface is nearly plane and the density of particles is dependent chiefly on the depth below the surface (at least in a region of the dimensions of several mean free paths), and as long as there are no particles incident on this part of the surface, coming into the slab from outside. Such a problem can be called the *problem of diffuse emission.*

We see that, for this case, we set $a \to \infty$ and, since there are no particles incident on the face $\xi = 0$, the function $f(0,\mu,\varphi)(\mu > 0)$ in Eq. (12.2.13) is zero. Everything is symmetric about the ξ axis so that f and R are independent of φ. We are not usually interested, in this case, in the absorption of the particles inside the slab; we are primarily interested in the distribution in angle of the particles which leave the surface $\xi = 0$. Consequently we set $\kappa = 1$ and Eqs. (12.2.13) become

$$f(\xi,\mu) = \begin{cases} \dfrac{1}{\mu}\displaystyle\int_0^{\xi} R(\eta,\mu)e^{-(\xi-\eta)/\mu}\,d\eta; & 0 < \mu \leq 1 \\ -\dfrac{1}{\mu}\displaystyle\int_{\xi}^{\infty} R(\eta,\mu)e^{-(\xi-\eta)/\mu}\,d\eta; & 0 > \mu \geq -1 \end{cases} \tag{12.2.15}$$

and, when the scattering is isotropic, $R(\eta,\mu) = \rho(\eta)/4\pi$, the distribution function f is simply related to the density $\rho(\xi)$, with the integral equation for ρ the standard Milne equation

$$\rho(\xi) = \tfrac{1}{2}\int_0^{\infty} \rho(\eta)E_1(|\xi - \eta|)\,d\eta \tag{12.2.16}$$

The boundary condition on f is that $f(0,\mu) = 0$ for $(0 < \mu \leq 1)$. We wish to compute values of $f(0,\mu)$ for $(0 > \mu \geq -1)$, giving the distribution in angle of the emergent radiation.

The other general case of interest is the one where the slab is thick enough or absorbent enough so that we can neglect particles coming through from the other side (or those created in the slab) and can concentrate on the effect of particles incident on the face at $\xi = 0$. We take a unidirectional beam at an angle φ_0, $\cos^{-1} \mu_0$, falling uniformly on the face $\xi = 0$, so that

$$f(0,\mu,\varphi) = I_0\delta(\mu - \mu_0)\delta(\varphi - \varphi_0); \quad 0 < \mu \leq 1$$

We wish to find the distribution in angle, $f(0,\mu,\varphi)$ $(0 > \mu \geq -1)$, of the particles diffusely reflected back out of the slab. This might be called the *problem of diffuse reflection.*

In this case, in general, the diffusely reflected radiation is dependent on φ as well as on μ, since the incident beam depends on φ. But in the case of isotropic scattering, the particle "forgets" its initial direction of motion after its first collision, so we can divide up the distribution function into the as yet undeflected part of the incident beam $I_0\delta(\mu - \mu_0)\delta(\varphi - \varphi_0)e^{-\xi/\mu_0}$, and the rest of the distribution, $f_d(\xi,\mu)$, accounting for the particles which have collided at least once, which is independent of φ. In this case, Eq. (12.2.13) for f_d is

$$f_d(\xi,\mu) = \begin{cases} \dfrac{\kappa}{4\pi\mu} \displaystyle\int_0^\xi \rho(\eta)e^{(\eta-\xi)/\mu}\, d\eta; & 0 < \mu \leq 1 \\ \dfrac{-\kappa}{4\pi\mu} \displaystyle\int_\xi^\infty \rho(\eta)e^{(\eta-\xi)/\mu}\, d\eta; & 0 > \mu \geq -1 \end{cases} \tag{12.2.17}$$

where we note that the only part of f which depends on φ is the inhomogeneous part representing those incident particles which have not yet suffered their first collision.

Incidentally, the solution of the distribution in angle of diffusely reflected particles enables us to obtain one further integral relation which will be of use later. Suppose we say that the diffusely reflected distribution, when the incident beam is at angle $\cos^{-1} \mu_0$ with respect to the ξ axis, is $(I_0/\mu)S(\mu,\mu_0) = f(0,-\mu)$. The function $S(\mu,\mu_0)$ is the ξ component of the intensity reflected back at angle $\cos^{-1}(-\mu)$ for an incident beam of unit intensity at incident angle $\cos^{-1} \mu_0$; it is defined only for μ, $\mu_0 \geq 0$. If we deal with isotropic scattering, S will be independent of $\varphi - \varphi_0$, otherwise we must write $S(\mu,\varphi;\mu_0,\varphi_0)$. It is not difficult to show that $S(\mu,\mu_0) = S(\mu_0,\mu)$.

The function $S(\mu,\mu_0)$ is, of course, just the function we need to solve the problem of diffuse reflection. A useful integral relationship involving S may be obtained by realizing that at depth ξ below the surface the inward-bound particles may be considered to be "incident" on the surface $\xi = \xi$, penetrating still further and being diffusely reflected back outward again. At depth ξ, the particles inward bound are, first, the remnant of the incident particles which have not yet collided, $I_0\delta(\mu - \mu_0)\delta(\varphi - \varphi_0)e^{-\xi/\mu_0}$, plus the rest of the distribution, $f_d(\xi,\mu)$ for $\mu > 0$. For each direction of the inward-bound particles, there is a scattered distribution $S(\mu,\mu_0)/\mu$ returning toward $\xi = 0$ from the material to the right of ξ. We can, therefore, calculate the distribution of outward-bound particles in terms of the scattering function S and the inward-bound distribution:

$$f(\xi,-\mu) = \frac{1}{\mu} I_0 e^{-\xi/\mu_0} S(\mu,\mu_0) + \frac{2\pi}{\mu} \int_0^1 S(\mu,\mu') f_d(\xi,\mu')\, d\mu' \quad (12.2.18)$$

for isotropic scattering. This will turn out to be a useful integral relation.

Likewise, for the case of diffuse emission, the only reason $f(\xi,-\mu)$ is any different from $f(0,-\mu)$, the emitted radiation, is because there is a region to the left of ξ, sending particles, inward bound, past the plane $\xi = \xi$, which are reflected back out according to the function S:

$$f(\xi,-\mu) = f(0,-\mu) + \frac{2\pi}{\mu} \int_0^1 S(\mu,\mu') f(\xi,\mu')\, d\mu' \quad (12.2.19)$$

This also will be useful.

Calculating the Diffuse Scattering. The function $S(\mu,\mu_0)$, the ξ component of the intensity of the radiation diffusely reflected at angle $\cos^{-1}(-\mu)$ because of a unit beam incident at angle $\cos^{-1}\mu_0$, is a sort of Green's function for the distribution function, in terms of which the problems of diffuse emission and diffuse reflection may be solved. A considerable amount of juggling of the equations already written is involved before we reach the end result.

We start with the differentiation of Eq. (12.2.18) with respect to ξ, then setting $\xi = 0$;

$$\mu f'(0,-\mu) = -\frac{1}{\mu_0} I_0 S(\mu,\mu_0) + 2\pi \int_0^1 S(\mu,\mu') f'_d(0,\mu')\, d\mu'$$

where $f'(\xi,\mu) = [\partial f(\xi,\mu)/\partial\xi]$. We can now use Eq. (12.2.7) (for isotropic scattering)

$$\mu f'_d(0,\mu) = -f_d(0,\mu) + R(0); \quad \mu f'(0,-\mu) = f(0,\mu) - R(0)$$

But the quantity $R(0)$ is the total number of particles scattered in any direction by the target atoms near the surface $\xi = 0$; the number of the incident plus diffusing particles scattered there:

$$R(0) = \left(\frac{\kappa}{4\pi}\right) I_0 \left[1 + 2\pi \int_0^1 S(\mu'',\mu_0)(d\mu''/\mu'')\right] = [\kappa\rho(0)/4\pi]$$

We also need to use the fact that $f(0,-\mu) = (I_0/\mu)S(\mu,\mu_0)$ and $f_d(0,\mu) = 0$ $(\mu > 0)$ and the symmetry relation $S(\mu',\mu_0) = S(\mu_0,\mu')$.

Substituting this in the equation for $\mu f'(0,-\mu)$ we obtain the simple-looking equation for the reflection function S,

$$\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)S(\mu,\mu_0)$$
$$= \frac{\kappa}{4\pi}\left[1+2\pi\int_0^1 S(\mu'',\mu_0)\frac{d\mu''}{\mu''}\right]\left[1+2\pi\int_0^1 S(\mu,\mu')\frac{du'}{\mu'}\right]$$

or
$$S(\mu,\mu_0) = \frac{\kappa}{4\pi}\left(\frac{\mu\mu_0}{\mu+\mu_0}\right)H(\mu_0)H(\mu) \tag{12.2.20}$$

where
$$H(\mu) = 1+2\pi\int_0^1 S(\mu,\mu')\frac{d\mu'}{\mu'}$$

The function H is a standard distribution in angle, in terms of which the scattering function S may be computed. To compute H, we obtain its integral equation, by substituting the expression for S into the definition of H;

$$H(\mu) = 1+\tfrac{1}{2}\kappa\mu H(\mu)\int_0^1 \frac{H(\mu')}{\mu+\mu'}\,d\mu' \tag{12.2.21}$$

It is not difficult to show that $H(\mu)$ is normalized so that

$$\int_0^1 H(\mu)\,d\mu = \left(\frac{2}{\kappa}\right)[1-\sqrt{1-\kappa}] \tag{12.2.22}$$

and that an alternative integral equation for H is

$$\frac{1}{H(\mu)} = \sqrt{1-\kappa}+\tfrac{1}{2}\kappa\int_0^1 H(\mu')\left(\frac{\mu'\,d\mu'}{\mu+\mu'}\right) \tag{12.2.23}$$

This last equation may be solved by successive iteration, to obtain quite accurate values for H. The results can be tested by use of Eq. (12.2.22). A few sample values of H are given in the accompanying table, for different values of κ and μ. The table is completed by noting that $H(\mu) = 1$ for $\kappa = 0$ and that $H(0) = 1$.

Table of Values of $H(\mu)$

$\mu =$	0.2	0.4	0.6	0.8	1.0
$\kappa = 0.2$	1.0389	1.0555	1.0659	1.0732	1.0786
$\kappa = 0.4$	1.0858	1.1252	1.1509	1.1694	1.1834
$\kappa = 0.6$	1.1452	1.2186	1.2689	1.3063	1.3354
$\kappa = 0.8$	1.2286	1.3611	1.4590	1.5358	1.5982
$\kappa = 1.0$	1.4503	1.8293	2.1941	2.5527	2.9078

By use of the functions $H(\mu)$, we can obtain the distribution in angle of the particles at the surface $\xi = 0$ for an incident beam of particles all traveling at angle $\cos^{-1}\mu_0$ to the ξ axis.

$$f(0,\mu) = \begin{cases} I_0\delta(\mu - \mu_0)\delta(\varphi - \varphi_0); & 0 < \mu \leq 1 \\ (I_0/4\pi)[\kappa\mu_0/(\mu_0 - \mu)]H(\mu_0)H(-\mu); & 0 > \mu \geq -1 \end{cases} \tag{12.2.24}$$

We note that the effect of the albedo constant κ comes in directly, by changing the total amount of reflection, and indirectly, by changing the distribution in angle of the diffuse reflection; the nearer κ is to zero, the more uniform is the distribution.

The solution for a more "spread-out" distribution of incident particles may be obtained from this by integration over μ_0. If this distribution is axially symmetric, so that the number of particles (all at the same velocity) having the cosine of their angle of incidence between μ_0 and $\mu_0 + d\mu_0$ is $2\pi f_0(\mu_0)\, d\mu_0$, then the distribution in angle at the surface $\xi = 0$ is

$$f(0,\mu) = \begin{cases} f_0(\mu); & 0 < \mu \leq 1 \\ \frac{1}{2}\kappa H(-\mu) \displaystyle\int_0^1 \frac{\mu_0 f(\mu_0)H(\mu_0)}{\mu_0 - \mu}\, d\mu_0; & 0 > \mu \geq -1 \end{cases} \tag{12.2.25}$$

Once the distribution in angle at $\xi = 0$ is obtained, Eq. (12.2.7) may be solved directly to find the distribution function for $\xi > 0$. For example, the function f may be expanded in spherical harmonics as indicated in Eq. (12.2.10), and the dependence of the coefficients as functions of ξ may be calculated once their values at $\xi = 0$ are known. Or else the Laplace transform of the distribution function is used, as will be considered later.

Calculating the Diffuse Emission. When there is no absorption ($\kappa = 1$) and when there are no particles incident on the surface $\xi = 0$, then the particles in the semi-infinite slab will gradually diffuse out of the surface $\xi = 0$. Since, when $\kappa = 1$, the net flux J is everywhere the same [see Eq. (12.2.8)], we would expect this flux to be negative, corresponding to the net flow out of the face $\xi = 0$. Here the equation relating the distribution in angle of the emergent particles $f(0,-\mu)$ and the scattering function $S(\mu,\mu_0)$ is given in Eq. (12.2.19); from it we can calculate $f(0,-\mu)$ in terms of the function H.

As before, we differentiate Eq. (12.2.19) and set $\xi = 0$;

$$\mu f'(0,-\mu) = 2\pi \int_0^1 S(\mu,\mu')f'(0,\mu')\, d\mu'$$

But $\mu' f'(0,\mu') = R(0)$ and $\mu f'(0,-\mu) = f(0,-\mu) - R(0)$, where, for isotropic scattering and no absorption, $R(\xi) = \rho(\xi)/4\pi$. Consequently,

$$f(0,-\mu) = \frac{1}{4\pi}\rho(0)\left[1 + 2\pi \int S(\mu,\mu')\frac{d\mu'}{\mu'}\right] = \frac{1}{4\pi}\rho(0)H(\mu) \tag{12.2.26}$$

and we have obtained the distribution of emitted particles in terms of the same function $H(\mu)$ which produced the expressions for diffuse reflection.

In the case of no absorption, the asymptotic dependence of f on ξ, μ, and the net flux is given in Eq. (12.2.12). When J is negative and when the scattering is isotropic, we can say that

$$f(\xi,\mu) = -(3/4\pi)(\xi - \mu)J + f_1(\xi,\mu)$$

where $f_1 \to 0$ as $\xi \to \infty$. One can also obtain a relation between f_1 and the scattering function $S(\mu,\mu')$. Since $(3/4\pi)(\mu - \xi)J$ is an exact solution of Eq. (12.2.7), the scattering of the correction term f_1 must just produce f_1 again; in other words,

$$f(\xi,-\mu) = -\frac{3}{4\pi}(\xi - \mu)J + \frac{2\pi}{\mu}\int_0^1 S(\mu,\mu')f_1(\xi,\mu')\,d\mu'; \quad 0 < \mu \leq 1$$

However, since $f(0,\mu') = 0$ for $\mu > 0$, we must have $f_1(0,\mu') = (3/4\pi)\mu'$ and we finally obtain

$$f(0,-\mu) = -\left(\frac{3}{4\pi}\right)J\left[\mu + \frac{2\pi}{\mu}\int_0^1 S(\mu,\mu')\mu'\,d\mu'\right] \tag{12.2.27}$$

giving the distribution of emergent radiation in terms of J and the scattering function S.

Integrating $2\pi\mu$ times Eq. (12.2.26) over μ from -1 to 0 ($f = 0$ for $\mu > 0$) and integrating 2π times Eq. (12.2.27) over the same range of μ, we find that

$$-J = \tfrac{1}{4}\rho(0)\left[1 + 4\pi\int_0^1 d\mu \int_0^1 d\mu'\,S(\mu,\mu')\left(\frac{\mu'}{\mu}\right)\right]$$

and

$$\rho(0) = -\tfrac{3}{4}J\left[1 + 4\pi\int_0^1 d\mu \int_0^1 d\mu'\,S(\mu,\mu')\left(\frac{\mu'}{\mu}\right)\right]$$

so that we finally obtain an interesting interrelation between mean flux and particle density at the surface $\xi = 0$;

$$\rho(0) = -\sqrt{3}\,J \tag{12.2.28}$$

which is valid for the conservative case ($\kappa = 1$) with isotropic scattering. In this case the diffuse emission is

$$f(0,-\mu) = -(\sqrt{3}/4\pi)JH(\mu) \tag{12.2.29}$$

where J is a negative quantity.

The same sort of analysis can be applied when the albedo constant κ is not unity. In this case [see Eq. (12.2.11)] the asymptotic form is $Ge^{k\xi}/(1 + \mu k)$, where k is the positive, real root of $k = \tanh(k/\kappa)$. Since no extra particles come in across the face $\xi = 0$ (in fact, particles leave there) and since there is absorption ($\kappa < 1$), the distribution function $f(\xi,\mu)$ and the density $\rho(\xi)$ will have to increase exponentially with increasing ξ in order to take care of the absorption and still have enough particles to come out of the face $\xi = 0$. Thus we shall set

$$f(\xi,\mu) = [Ge^{k\xi}/(1 + \mu k)] + f_1(\xi,\mu); \quad f_1 \xrightarrow[\xi \to \infty]{} 0$$

and obtain

$$f(\xi,-\mu) = \frac{Ge^{k\xi}}{1 - \mu k} + \frac{2\pi}{\mu} \int_0^1 S(\mu,\mu')f_1(\xi,\mu)\, d\mu$$

or since $f_1(0,\mu') = -G/(1 + \mu' k)$, we have

$$\begin{aligned} f(0,-\mu) &= G\left\{\frac{1}{1 - k\mu} - \frac{2\pi}{\mu}\int_0^1 \frac{S(\mu,\mu')}{1 + k\mu'}\, d\mu'\right\} \\ &= G\left\{\frac{1}{1 - k\mu} - \tfrac{1}{2}\kappa H(\mu)\int_0^1 \frac{\mu' H(\mu')\, d\mu'}{(\mu + \mu')(1 + k\mu')}\right\} \end{aligned}$$

After further juggling we finally obtain, for the case of $\kappa < 1$, no radiation incident on the face $\xi = 0$, for the diffuse radiation emitted by the face $\xi = 0$,

$$f(0,-\mu) = \frac{G}{H(1/k)}\left[\frac{H(\mu)}{1 - k\mu}\right] \tag{12.2.30}$$

A great deal more detail must be gone into in order to calculate more complicated problems, such as the penetration through slabs of finite thickness or the effect of nonisotropic scattering on the diffuse reflection or emission. These other problems, however, may be solved by methods similar to the ones we have outlined, though the details are necessarily more tedious. They can be found in books and articles specializing in this subject.

Solution by Laplace Transforms. Another way of arriving at the solution of the problem of diffuse emission is by means of the Laplace transform. This, of course, should have been evident from inspection of Eqs. (12.2.15) and (12.2.16), the equation relating the distribution function (for the case of no absorption, isotropic scattering, and no incident beam) with the particle density $\rho(\xi)$ and the integral equation for ρ. For the kernel of the integral equation for ρ is $E_1(|\xi - \eta|)$ [E_1 defined in Eq. (12.2.14)] a function of the difference $|\xi - \eta|$; in Sec. 8.4 we saw that such kernels were particularly amenable to Laplace or Fourier transform techniques. Other advantages of the method will become apparent as we proceed.

The Laplace transforms of the distribution function and of the particle density are defined as

$$\begin{aligned} F(p,\mu) &= \int_0^\infty f(\xi,\mu)e^{-p\xi}\, d\xi; \quad \text{Re}\, p > 0 \\ P(p) &= \int_0^\infty \rho(\xi)e^{-p\xi}\, d\xi = 2\pi \int_{-1}^1 F(p,\mu)\, d\mu \end{aligned} \tag{12.2.31}$$

We should also notice another useful item, that the Laplace transform of the density is related to the angular distribution of the emergent radiation. For Eqs. (12.2.15) for f in terms of ρ (for isotropic scattering, no absorption, $R = \rho/4\pi$) shows that this emergent distribution is

$$f(0,-\mu) = \frac{1}{4\pi\mu}\int_0^\infty \rho(\eta)e^{-\eta/\mu}\,d\eta = \left(\frac{1}{4\pi\mu}\right)P\left(\frac{1}{\mu}\right);\quad (0 < \mu \le 1) \tag{12.2.32}$$

This is one of the additional "dividends" of the Laplace transform solution; if we wish only the distribution in angle of the emergent radiation, we do not need to make the inverse transformation to find ρ from P; we can use P itself. This result corresponds indirectly to the result we have obtained in our previous analysis, that it is much easier to obtain $f(0,-\mu)$ than it is to obtain $f(\xi,\mu)$. Of course, we could stop here, if we wished, for we have already obtained $f(0,-\mu)$ [see Eq. (12.2.29)], and we could use Eqs. (12.2.32) and the inversion transforms to obtain $\rho(\xi)$ and thence $f(\xi,\mu)$.

However, there are still other "dividends" to consider, so we shall pursue the method further. For example, the Laplace transform of the transport equation for this case,

$$\mu f'(\xi,\mu) + f(\xi,\mu) = \rho(\xi)/4\pi$$

is

$$F(p,\mu) = \left(\frac{1}{1+p\mu}\right)\left[\frac{1}{4\pi}P(p) + \mu f(0,\mu)\right] \tag{12.2.33}$$

which is a simple algebraic relation between the transforms for f and ρ and the distribution function at the surface. By multiplying by $2\pi\,d\mu$ and integrating over μ we can finally obtain an equation for the transform of the density,

$$P(p)\left[1 - \frac{1}{p}\tanh^{-1} p\right] = 2\pi\int_{-1}^{1}\left[\frac{\mu f(0,\mu)}{1+p\mu}\right]d\mu \tag{12.2.34}$$

We now proceed to "haul ourselves up by our bootstraps" using the method of Wiener and Hopf. This method was discussed in Sec. 8.4 in connection with this same problem; we review the results here simply to maintain the thread of our discussion. There are three factors in Eq. (12.2.34); $P(p)$, $[1 - (1/p)\tanh^{-1} p]$, and

$$j(p) = 2\pi\int_{-1}^{0}\left[\frac{\mu f(0,\mu)}{1+p\mu}\right]d\mu$$

The function $P(p)$ is analytic over the whole region $\mathrm{Re}\, p > 0$ in order that its inverse transform be zero for $\xi < 0$. Also $[1 - (1/p)\tanh^{-1} p]$ turns out to be analytic in the strip $-1 < \mathrm{Re}\, p < 1$. The function $j(p)$ is analytic over the whole region $\mathrm{Re}\, p < 1$, because $\mu f(0,\mu)$ is bounded over the range $0 > \mu \ge -1$ and is zero in the range $\mu > 0$. (If f were not zero for $\mu > 0$, the integral could not be restricted to $0 > \mu \geq -1$, the integral would have poles in the region Re $p < 1$, and the method would fail.) But when there is no radiation incident on the surface $\xi = 0$, so that $f(0,\mu)$ is zero for $\mu > 0$, we have $j(p)$ analytic throughout the region Re $p < 1$ and going to zero proportional to $1/p$ as $p \to \infty$.

If now we can decompose $[1 - (1/p) \tanh^{-1} p]$ into the ratio of two functions, one of which is analytic in the region Re $p > 0$ and the other in the region Re $p < 1$, we can multiply up by one function to obtain an equation, one side of which is analytic in the region Re $p > 0$ and the other in the region Re $p < 1$. They are equal to each other in the band $0 <$ Re $p < 1$ and, by analytic continuation, they are equal over the plane, representing a function analytic everywhere (which must be a constant).

As in Sec. 8.4, we take the logarithm of $[1 - (1/p) \tanh^{-1} p]$ and try to set it equal to the difference of two functions, analytic in the right places. But we cannot use just $[1 - (1/p) \tanh^{-1} p]$, for it is zero at $p = 0$ (going as p^2) and the logarithm would have a singularity there. If we take $(1/p^2)[1 - (1/p) \tanh^{-1} p]$, the logarithm will go to infinity when $p \to \infty$, but if we let

$$[(p^2 - 1)/p^2][1 - (1/p) \tanh^{-1} p] = [\tau_+(p)/\tau_-(p)]$$

the logarithm of both sides will still be analytic in the strip $1 >$ Re $p > -1$ and will go to zero as $p \to \infty$. To find the τ's which satisfy this equation and which are analytic in the required region, we use the Cauchy integral formula (4.2.8) and with the contour along the edge of the strip within which $[1 - (1/p) \tanh^{-1} p]$ is analytic:

$$\ln\left[\left(1 - \frac{1}{p^2}\right)\left(1 - \frac{1}{p}\tanh^{-1} p\right)\right] = \ln \tau_+(p) - \ln \tau_-(p) \qquad (12.2.35)$$

$$\ln \tau_+ = \frac{1}{2\pi i}\int_{\beta - i\infty}^{\beta + i\infty} \ln\left[\left(1 - \frac{1}{s^2}\right)\left(1 - \frac{1}{s}\tanh^{-1} s\right)\right]\frac{ds}{s - p}; \quad 1 > \beta > 0$$

$$\ln \tau_- = \frac{1}{2\pi i}\int_{-\beta - i\infty}^{-\beta + i\infty} \ln\left[\left(1 - \frac{1}{s^2}\right)\left(1 - \frac{1}{s}\tanh^{-1} s\right)\right]\frac{ds}{s - p}$$

The first integral (therefore, τ_+) is analytic for Re $p < \beta$, and the second (therefore, τ_-) is analytic for Re $p > -\beta$. Consequently, readjusting Eq. (12.2.34),

$$P(p)\left[\frac{p^2}{(p + 1)\tau_-(p)}\right] = \left[\frac{p - 1}{\tau_+(p)}\right] 2\pi \int_{-1}^{1}\left[\frac{\mu f(0,\mu)}{1 + p\mu}\right] d\mu$$

we have that the left-hand side is analytic in the whole region Re $p > 0$ and the right-hand side is analytic in the region Re $p < \beta$. Furthermore, since τ_+ approaches a constant divided by p for large p, the quantity on the right (consequently, the quantity on the left) approaches a constant as $p \to \infty$.

Since both sides are (by analytic continuation) analytic everywhere and bounded at infinity, both sides equal a constant C everywhere. Therefore,

$$P(p) = (C/p^2)(p + 1)\tau_-(p) \tag{12.2.36}$$

To compute the value of the constant C, we find the limiting value of both sides as $p \to 0$. When $p = 0$, the function $j(p)$ in Eq. (12.2.34) is just equal to J, the constant net flux of particles drifting in the negative direction toward and out of the surface $\xi = 0$. When $p \to 0$, the function $1 - (1/p) \tanh^{-1} p \to -\frac{1}{3}p^2$; consequently ($q_\infty$ a constant),

$$P(p) \to -3J\left[\frac{1}{p^2} + \frac{q_\infty}{p} + \cdots\right]; \quad p \to 0 \tag{12.2.37}$$

Incidentally, we can thus show that the transform of P, the density ρ, has an asymptotic form $\rho \to -3J(q_\infty + \xi + \cdots)$ as $\xi \to \infty$, which corresponds to the asymptotic form of Eq. (12.2.12).

Returning to Eq. (12.2.35), we can show, by a bit of contour stretching, that $\tau_- \to \sqrt{3}$ as $p \to 0$. Finally we obtain that $C = -\sqrt{3}\,J$ where J is the net flux in the direction of positive ξ (and, therefore, a negative quantity in this problem) and thus

$$P(p) = -\sqrt{3}\,J\left(\frac{p+1}{p^2}\right)\cdot$$
$$\cdot \exp\left\{\frac{1}{2\pi i}\int_{-\beta-i\infty}^{-\beta+i\infty} \ln\left[\left(1 - \frac{1}{s^2}\right)\left(1 - \frac{1}{s}\tanh^{-1} s\right)\right]\frac{ds}{s-p}\right\} \tag{12.2.38}$$

(for $1 > \beta > 0$) with asymptotic behavior given in Eq. (12.2.37). The constant q_∞ appearing there may be computed by expanding the integral of Eq. (12.2.38) in powers of p; it is

$$q_\infty = 0.7104461 \cdots \tag{12.2.39}$$

Consideration of the relation between Eqs. (12.2.9), (12.2.32), and (12.2.37) shows that $K = 2\pi\int \mu^2 f\,d\mu = -J(q_\infty + \xi)$.

By comparing Eq. (12.2.38) with Eqs. (12.2.32) and (12.2.29), we see that

$$H(\mu) = (1 + \mu)\cdot$$
$$\cdot \exp\left\{\frac{\mu}{2\pi i}\int_{-\beta-i\infty}^{-\beta+i\infty} \ln\left[\left(1 - \frac{1}{s^2}\right)\left(1 - \frac{1}{s}\tanh^{-1} s\right)\right]\frac{ds}{\mu s - 1}\right\} \tag{12.2.40}$$

which enables us to compute diffuse reflection as well as emission, though this formula is good only for $\kappa = 1$, whereas the earlier equation for H is valid $0 < \kappa < 1$. The angular distribution of particles coming out of the surface may thus be obtained by evaluation of the integral of Eq. (12.2.40) or by successive iterations of the integral equation (12.2.23).

The results, of course, are the same. But now we can go on to compute the density of particles as a function of ξ and thus, by use of Eq. (12.2.15), the distribution function for any value of ξ, if this is needed.

The Variational Calculation of Density. The density function

$$\rho(\xi) = 2\pi \int_{-1}^{1} f(\xi,\mu)\, d\mu$$

may be computed from Eq. (12.2.38) by the Laplace inversion formula

$$\rho(\xi) = \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} P(p) e^{p\xi}\, dp \tag{12.2.41}$$

though the procedure is quite tedious. The result was shown in Fig. 2.20, where there was a discussion of the relationship between this exact solution and the diffusion-equation approximation. We saw there that, more than a mean free path deep in the slab, the density is quite closely equal to its asymptotic form

$$\rho(\xi) \to -3J(0.710446 + \xi + \cdots); \quad \xi \gg 1 \tag{12.2.42}$$

[see Eqs. (2.4.31) and (2.4.34)]. We see from Eqs. (12.2.15) that the asymptotic form for the distribution function is

$$f(\xi,\mu) \to -\frac{3}{4\pi} J(0.710446 + \xi - \mu + \cdots); \quad \xi \gg 1 \tag{12.2.43}$$

In this region f is so nearly spherically symmetric ($\xi \gg \mu$) that the diffusion approximation (see Sec. 2.4) is satisfactory. For $\xi < 1$, near the surface, however, the diffusion approximation is not good enough for detailed calculations although the "$\frac{2}{3}$ approximation" given in Eq. (2.4.31) is not bad, as shown in Fig. 2.21.

The density function may also be obtained by a variational technique. The integral equation to be solved is

$$\rho(\xi) = \tfrac{1}{2} \int_0^\infty \rho(\eta) E_1(|\xi - \eta|)\, d\eta$$

[see Eq. (12.2.16)]. But, from the asymptotic form for ρ, we can set

$$\rho(\xi) = -3J[\xi + q(\xi)] \tag{12.2.44}$$

where
$$q(\xi) \xrightarrow[\xi \to \infty]{} q_\infty = -\frac{K(0)}{J} = -\frac{2\pi}{J} \int_{-1}^{0} f(0,\mu)\mu^2\, d\mu$$
$$= \frac{1}{2J} \int_0^\infty d\eta \int_{-1}^{0} \rho(\eta) e^{\eta/\mu} \mu\, d\mu$$
$$= -\frac{1}{2J} \int_0^\infty \rho(\eta) E_3(\eta)\, d\eta$$

which is obtained by using Eq. (12.2.15) and the definition

$$E_n(\eta) = \int_1^\infty e^{-\eta y} \frac{dy}{y^n}$$

Expressing ρ in terms of $q(\xi)$, we finally have

$$q(\xi) \xrightarrow[\xi\to\infty]{} q_\infty = \tfrac{3}{2}\left[\tfrac{1}{4} + \int_0^\infty q(\eta)E_3(\eta)\,d\eta\right] \tag{12.2.45}$$

On the other hand, expressing ρ in terms of q on both sides of the integral equation for ρ (integrating again by parts for part of the integral), we obtain an integral equation for q,

$$q(\xi) = \tfrac{1}{2}E_3(\xi) + \tfrac{1}{2}\int_0^\infty q(\eta)E_1(|\xi - \eta|)\,d\eta \tag{12.2.46}$$

To obtain a variational equation to minimize, we set

$$D = \frac{\int_0^\infty q^*(\xi)\left[q^*(\xi) - \tfrac{1}{2}\int_0^\infty q^*(\eta)E_1(|\xi-\eta|)\,d\eta\right]d\xi}{\left[\int_0^\infty q^*(\eta)E_3(\eta)\,d\eta\right]^2} \tag{12.2.47}$$

One can show (see Chap. 9) that the minimum value of D is for $q^*(\xi) = q(\xi)$ and that this minimum value is (see Eq. 12.2.45)

$$D_{\min} = 1/(\tfrac{4}{3}q_\infty - \tfrac{1}{2})$$

Consequently a choice of q^* as a reasonable function (going to zero as $\xi \to \infty$), having parameters which may be varied to minimize D, will at the same time give a best form for q and a value of q_∞ which may be compared with the correct value, given in Eq. (12.2.39), to check the closeness of fit.

For example, letting $q = \alpha$, a constant, D comes out to be $\frac{9}{4}$ and thus $q_\infty = \alpha = \frac{17}{24} = 0.7083$, which is within less than 1 per cent of the correct value. On the other hand, assuming that

$$q^*(\xi) = q_\infty[1 - AE_2(\xi) + BE_3(\xi)] \tag{12.2.48}$$

results in a minimum value for D of 2.235831 for $A = 0.3428949$ and $B = 0.3158704$. The resulting value for q_∞ is then 0.7104457, which corresponds to the correct value out to the sixth decimal place, so the corresponding expression for ρ, using Eq. (12.2.44), is quite close to the correct one indeed. From it (if we wish) we can compute $H(\mu)$ and $f(\xi,\mu)$.

This set of calculations completes our discussion of the classical Milne problem, for the diffusion of particles (or light) through a scattering medium when the scattering is isotropic and when the particles do not lose energy at each collision. This last assumption is the most crucial one, for we have several times indicated how we could extend our calculations to cases of nonisotropic scattering, as we have also indicated how we could include annihilation and creation of particles in the medium. We can also carry on calculations for other space symmetry than the one-dimensional case considered. For example, the transport equation for radiation spreading out spherically in all directions from an origin is

$$\frac{\partial}{\partial\tau} f(r,\mu,\tau) + \mu \frac{\partial}{\partial r} f(r,\mu,\tau) + \left(\frac{1-\mu^2}{r}\right) \frac{\partial}{\partial\mu} f(r,\mu,\tau) + f(r,\mu,\tau) = \kappa \iint d\Omega' \, \alpha(\theta') f(r,\mu',\tau) + s(r,\mu,\tau) \quad (12.2.49)$$

where the angle between the direction of motion and the radius vector corresponding to r is $\cos^{-1}\mu$. Comparison with Eq. (12.2.2) shows that the effect of the spherical symmetry is to add a term in $\partial f/\partial\mu$, which makes solution more difficult, but it can be carried out approximately (by the variational method if need be).

But, when we must include the fact that collision slows down the particles, as well as changing their direction of motion, then we must bring a new "dimension" into the problem which alters it profoundly. If the target atoms are much heavier than the diffusing particles (usually the case for electrons but not usually the case for neutrons), the particles can have many collisions before their velocity is appreciably reduced; in other words the slowing-down length is much longer than the collision length. And as long as we deal with conditions within a slowing-down length of the boundary, we can use the solutions discussed up till now in this section, because the particles in this outer layer do not lose much speed before they escape from the medium. For heavy target atoms, this outer layer is a thick one, sometimes including the whole of the material; but for light target atoms, the layer is thin and we must consider energy loss in calculating the distribution function deep below the surface.

Loss of Energy on Collision. The effects of the loss of energy on collision were discussed in Sec. 2.4; we take up only a few examples here to show how some of the techniques, discussed in later chapters, will help obtain answers. We shall confine our remarks to cases of elastic collision, where the energy lost by the particle is taken up as kinetic energy of the target atom and to cases where the scattering of the particle by the target atom (in coordinates moving with the center of mass of particle target) is isotropic.

In most cases there is a maximum energy of the particles, either the energy of the particles incident on the material or the initial energy of the particles created in the material. It is usual to express the kinetic energy E of the particle (and thus its velocity) in terms of this initial energy E_0; the most convenient is a logarithmic scale, $u = \ln(E_0/E)$. The energy parameter u then varies from zero (maximum energy, $E = E_0$) to infinity (minimum energy, $E = 0$). The distribution function f is thus a function of position, represented by the vector $\mathbf{r}$; of kinetic energy, represented by u; of the direction of the particle velocity, given by the unit vector $\mathbf{a}_u$; and of time. This u is proportional to the "age" variable used in Eqs. (2.4.54) and (12.1.28).

The scattering function α is a function of the angle between the incident direction $\mathbf{a}'_u$ and the final direction $\mathbf{a}_u$ of the particle. Because the particle loses energy, the initial and final energies differ and, because the target atom moves after the collision, the distribution in angle of the scattered particles, in coordinates at rest with the majority of the targets, is not uniform even if the scattering is isotropic in the center-of-mass system. Moreover, there is a relationship between the angle of scattering θ' and the ratio between initial kinetic energy E' and the final energy E of the particle. If Θ is the angle of scattering in the center-of-mass system and if the ratio between the mass of the target atom and that of the particle is M, then the relations between θ', Θ, and initial and final energy parameters u' and u are

$$\begin{aligned}(M+1)e^{-\frac{1}{2}w}\cos\theta' &= (M\cos\Theta + 1)\\ \cos\Theta &= 1 - [(M+1)^2/2M](1 - e^{-w})\\ \cos\theta' &= \tfrac{1}{2}(M-1)e^{\frac{1}{2}w} - \tfrac{1}{2}(M+1)e^{-\frac{1}{2}w}\end{aligned} \tag{12.2.50}$$

where $w = u - u'$. We also notice that the largest possible energy loss is for $\theta' = \pi$, where $w_\pi = 2\ln[(M+1)/(M-1)]$ except for $M = 1$ (target same mass as particle) in which case $w_\pi = \infty$ and $\theta'_{\max} = \frac{1}{2}\pi$.

The integral R, for the total number of particles being scattered into the region of phase space under consideration, must be an integral over u' as well as over the angle element $d\Omega'$. But, because of the relationship between angle and energy, the scattering function α has a delta-function factor (for isotropic scattering in the center-of-mass system):

$$\alpha(\theta',w) = \begin{cases} \dfrac{(M+1)^2}{8\pi M}\, e^{-w}\delta[\cos\theta' - \tfrac{1}{2}(M-1)e^{\frac{1}{2}w} + \tfrac{1}{2}(M+1)e^{-\frac{1}{2}w}]; & w \le w_\pi \\ 0; & w > w_\pi \end{cases}$$

which is normalized so that

$$\int du' \iint \alpha(\theta', u - u')\, d\Omega' = \frac{(M+1)^2}{4M}\int_0^{w_\pi} e^{-w}\, dw = 1$$

The average energy loss per collision (on a logarithmic scale) and the average cosine of the angle of deflection are then

$$
\begin{aligned}
w_{\text{av}} &= [\ln(E_0/E)]_{\text{av}} = 1 - \frac{(M-1)^2}{2M}\ln\left(\frac{M+1}{M-1}\right) \\
(\cos\theta')_{\text{av}} &= \int du' \iint \cos\theta'\, \alpha(\theta', u-u')\, d\Omega' = \frac{2}{3M}
\end{aligned}
\tag{12.2.51}
$$

Before we write down the equation of transfer, which is to take the place of Eq. (12.2.1), we make one more change to simplify the equation. We can no longer make $n_tQ_tx = \xi$, for (n_tQ_t), the number of collisions per unit distance traveled by a particle at speed v, varies with v and we no longer can consider v constant. We can, however, achieve some simplification by changing the dependent variable to

$$\psi(\mathbf{r},u,\mathbf{a}_u) = n_tQ_tvf(\mathbf{r},u,\mathbf{a}_u)$$

where $\psi\, du\, d\Omega$ is the *number of collisions* per second per unit volume made by particles in the energy range du with velocities directed in the solid angle $d\Omega$. We also set $\lambda(u) = 1/n_tQ_t$, the mean free path for particles of energy corresponding to $u = \ln(v_0^2/v^2)$, and set the ratio of Q_s, the cross section for scattering without capture to Q_t, the total cross section equal to $\kappa(u)$, the albedo factor for energy corresponding to u.

With all these changes, the transport equation becomes

$$
\begin{aligned}
&\frac{\lambda(u)}{v}\frac{\partial}{\partial t}\psi(\mathbf{r},u,\mathbf{a}_u) + \lambda(u)\mathbf{a}_u\cdot\operatorname{grad}\psi(\mathbf{r},u,\mathbf{a}_u) + \psi(r,u,\mathbf{a}_u) \\
&\qquad = \int_0^u \kappa(u')\,du' \iint d\Omega'\,\alpha(\theta', u-u')\psi(\mathbf{r},u',\mathbf{a}'_u) + S(\mathbf{r},u,\mathbf{a}_u)
\end{aligned}
\tag{12.2.52}
$$

where S is (as before) the number of particles created per second per unit volume in the energy range denoted by u, with direction of motion given by $\mathbf{a}_u$.

Uniform Space Distribution. In the interior of the diffusing medium, where we do not have to worry about the effects of boundaries (which effects have already been discussed), it is often the case that both ψ and S are more or less independent of $\mathbf{r}$. In this case, we can drop the troublesome term in grad ψ and can also disregard the direction of motion of the particles by integrating Eq. (12.2.52) over all directions. As a matter of fact, even if ψ does depend on $\mathbf{r}$ and $\mathbf{a}_u$, if the diffusing medium occupies enough space so that only a negligible fraction of the particles are lost by leaving the surface each second, we can integrate Eq. (12.2.52) over space and over all directions $\mathbf{a}_u$, obtaining ($\eta = \int \cdot \cdot \cdot \int \psi\, dV\, d\Omega$).

$$\frac{\lambda(u)}{v}\frac{\partial}{\partial t}\eta(u,t) + \eta(u,t) = q(u,t) + \int_0^u du'\,\kappa(u')a(u-u')\eta(u',t) \tag{12.2.53}$$

where $\eta\, du$ is the average number of collisions suffered, per unit time, by a particle in the energy range given by du, $q\,du$ is the total number of particles created per second in the same range, and

$$a(u - u') = \begin{cases} \displaystyle\iint d\Omega'\, \alpha(\theta',w) = \left[\frac{(M+1)^2}{4M}\right] e^{u'-u}; & u - u' \le w_\pi \\ 0; & u - u' > w_\pi \end{cases} \qquad (12.2.54)$$

is the corresponding scattering function.

We first consider the steady-state situation, when q and, therefore, η are independent of time. It is sufficient to solve the case where a unit number of particles, all at the same energy E_0, is introduced each second, so that $q = \delta(u)$. Other rates and distributions in energy may be calculated by manipulating this solution. The result of dropping the time dependence is to reduce Eq. (12.2.53) into a simple Volterra integral equation, which may be changed into a differential equation as long as u is smaller than $w_\pi = 2 \ln[(M + 1)/(M - 1)]$.

For example, when $M = 1$ (neutrons in hydrogen), $w_\pi \to \infty$, and the differential equation for $\eta(u)$ is

$$d\eta/du = -[1 - \kappa(u)]\eta$$

The solution, omitting those particles which have not yet collided after creation, is

$$\eta(u) = \eta(0) \exp\left\{-\int_0^u [1 - \kappa(y)]\, dy\right\}; \quad u > 0 \qquad (12.2.55)$$

which is to be compared with Eq. (2.4.55). The factor $[1 - \kappa(y)]$ is the fraction of collisions which result in absorption of a particle. For no absorption, κ will be unity for all values of u and $\eta(u) = 1$ for $u > 0$. The distribution function f, in that case, will be proportional to $1/vn_tQ_t$, an ever-increasing number as we look at slower and slower particles (unless $n_tQ_t \to Av^{-\gamma}$ with $\gamma \ge 1$). When we attempt to reach a steady state with no absorption, we must have the source on for a very long time, and thus we collect very large amounts of slow particles.

The reason the $M = 1$ case is simpler than those for $M > 1$ is that, for $M = 1$, particles of all possible energies from E_0 to 0 can come from a single collision of a particle with energy E_0 with a target atom. When $M > 1$, there is a certain energy $E_{\min} = E_0 e^{-w_\pi} = E_0[(M - 1)/(M + 1)]^2$, below which it is impossible to reach, by means of a single collision, from E_0. As might be expected, there is a discontinuity in the function η at this value of u, which makes the process of solving for η more complicated.

Since Eq. (12.2.53) has a kernel which is a function of $u - u'$, we can try using the Laplace transform if there is no capture, $\kappa(u') = 1$. Then the integral term in Eq. (12.2.53) may be transformed by means of the faltung theorem: the Laplace transform of

$$\int^u \eta(u')a(u-u')\,du' \quad \text{is} \quad H(p)A(p)$$

where H and A are the Laplace transforms of $\eta(u)$ and $a(u)$. The Laplace transform of a is

$$A(p) = \frac{(M+1)^2}{4M}\int_0^{w_\pi} e^{-u(p+1)}\,du = \frac{(M+1)^2}{4M(p+1)}\left[1-\left(\frac{M-1}{M+1}\right)^{2(p+1)}\right] \tag{12.2.56}$$

Consequently, the Laplace transform of the equation

$$\eta(u) = \int_0^u a(u-u')\eta(u')\,du' + q(u)$$

is

$$H(p) = H(p)A(p) + Q(p)$$

where $Q(p)$ is the Laplace transform of $q(u)$; if $q(u) = \delta(u)$, then $Q(p) = 1$.

The equation for the Laplace transform of $\eta(u)$ is thus

$$H(p) = \frac{4M}{(M+1)^2}\,\frac{(p+1)}{[4M(p+1)/(M+1)^2] - 1 + [(M-1)/(M+1)]^{2(p+1)}} \tag{12.2.57}$$

for the steady-state case, with no absorption, for a source function $q(u) = \delta(u)$. This may then be inverted to obtain the average number of collisions per second, η, as a function of u for the no-capture case:

$$\eta(u) = \frac{4M}{2\pi i(M+1)^2}\cdot$$

$$\cdot\int_{\beta-i\infty}^{\beta+i\infty}\frac{(p+1)e^{pu}\,dp}{[4M(p+1)/(M+1)^2] - 1 + (M-1/M+1)^{2(p+1)}} \tag{12.2.58}$$

where $\beta > 0$. Most of the poles of the integrand are to the left of the imaginary axis of p, but there is one at $p = 0$, with residue equal to $[(M+1)^2/4Mw_{\text{av}}]$ where w_{av}, given in Eq. (12.2.51), is the mean energy loss in one collision. Therefore, for u large (low energy particles), the asymptotic value for the collision function η is

$$\eta \to (1/w_{\text{av}});\quad u \to \infty \tag{12.2.59}$$

for the no-capture, steady-state case, a generalization of (12.2.55) for $\kappa = 1$.

The solution for smaller values of u may be obtained by finding the residues at the other poles of the integrand of Eq. (12.2.58) and adding. This is done in several articles devoted to this subject and need not be duplicated here. Likewise the case where $\kappa(u) < 1$, when capture occurs, complicates the case and does not warrant discussing here.

The time-dependent case, for no capture, may also be solved by the Laplace transform technique, though the most general case is not simple. In the case of $M = 1$ (target atom same mass as particle), Eq. (12.2.53) simplifies to

$$\frac{\lambda(u)}{v}\frac{\partial}{\partial t}\eta(u,t) + \eta(u,t) = \int_0^u e^{u'-u}\eta(u',t)\,du' + \delta(u)\,\delta(t) \qquad (12.2.60)$$

We take the Laplace transform with respect to time, obtaining

$$\left[1 + \frac{s\lambda(u)}{v}\right] Z(u,s) = \int_0^u e^{u'-u}Z(u',s)\,du' + \frac{e^{-u}}{1 - \left(\frac{s\lambda_0}{v_0}\right)}$$

where
$$H(u,s) = \int_0^\infty \eta(u,t)e^{-st}\,dt = \frac{\delta(u)}{1 + \left(\frac{s\lambda_0}{v_0}\right)} + Z(u,s)$$

Differentiating this with respect to u leads to a differential equation, which can be solved for Z, yielding finally, for the Laplace transform of η,

$$H(u,s) = \frac{\delta(u)}{1 + (s\lambda_0/v_0)} + \frac{\exp\left\{-s\int_0^u du'/[s + (v'/\lambda(u'))]\right\}}{[1 + (s\lambda_0/v_0)][1 + (s\lambda(u)/v)]} \qquad (12.2.61)$$

which may be inverted to obtain η, once we have determined the nature of the dependence of the mean free path $\lambda(u)$ on u. For example, if $\lambda(u)$ were proportional to the velocity v, the inversion would be simple. If λ is constant, on the other hand, it is possible to solve Eq. (12.2.53) for the case $M > 1$. See also Probs. 12.12 and 12.16.

Age Theory. When the space variation of the distribution function f and, therefore, of the collision rate function $\psi = n_tQ_tvf$, cannot be neglected, we must return to Eq. (12.2.52). As indicated in Secs. 2.4 and 12.1, when M is large, so that only a small amount of energy is lost per collision and when a steady state is reached, the "age theory" approximation is appropriate, except where the particle density varies markedly in one mean free path (when the techniques of the beginning of this section are needed). We have already [see Eqs. (2.4.56) and (12.1.29)] worked out some of the solutions for this approximation; it will be necessary here only to rederive the fundamental equation, in order to show the relationship between the results obtained earlier and the material we have discussed in the present section.

If we can assume that the space rate of change is small, we can assume that it is nearly uniform in angle (nearly independent of $\mathbf{a}_u$) and that its small asymmetry is directed along the direction of greatest space change of ψ. In other words, we can assume that

$$\psi(\mathbf{r},u,\mathbf{a}_u) \simeq \frac{1}{4\pi}\left[\psi_0(\mathbf{r},u) + g(u)\mathbf{a}_u \cdot \operatorname{grad} \psi_0(\mathbf{r},u)\right] \tag{12.2.62}$$

where
$$\psi_0(\mathbf{r},u) = \iint \psi(\mathbf{r},u,\mathbf{a}_u)\, d\Omega$$

and where $g(u)$ is supposed to be small. If now we substitute this in Eq. (12.2.52) (leaving out time dependence) and compute the average of this over all directions $\mathbf{a}_u$ (that is, integrate over Ω), we get an equation [see the discussion just preceding Eq. (2.4.41)]

$$\tfrac{1}{3}\lambda(u)g(u)\nabla^2\psi_0(\mathbf{r},u) + \psi_0(\mathbf{r},u) = \int_0^u a(u - u')\psi_0(\mathbf{r},u')\kappa(u')\, du' + S_0(\mathbf{r},u) \tag{12.2.63}$$

where $S_0(\mathbf{r},u) = \iint S(\mathbf{r},u,\mathbf{a}_u)\, d\Omega$ is the "zero moment" of the source function and $a(u - u')$ is the function given in Eq. (12.2.54).

In the case of M large, w_π is small and the range of u' over which the integrand is not zero is small. If ψ_0 varies smoothly and relatively slowly with u, then $\psi_0(\mathbf{r}, u - w_\pi)$ will not differ much from $\psi_0(\mathbf{r},u)$ and we can expand $\psi_0(\mathbf{r},u')$ in a Taylor's series:

$$\kappa(u')\psi_0(\mathbf{r},u') = \kappa(u)\psi_0(\mathbf{r},u) + (u' - u)(\partial/\partial u)[\kappa(u)\psi_0(\mathbf{r},u)] + \cdots$$

Inserting this in Eq. (12.2.63) and utilizing the relationships

$$\int_{u-w_\pi}^{u} a(u - u')\, du' = 1$$

$$\int_{u-w_\pi}^{u} (u - u')a(u - u')\, du' = w_{\text{av}} = 1 - \frac{(M-1)^2}{2M}\ln\left(\frac{M+1}{M-1}\right)$$

we finally obtain (for $u > w_\pi$)

$$\tfrac{1}{3}\lambda(u)g(u)\nabla^2\psi_0(\mathbf{r},u) = -w_{\text{av}}\kappa(u)(\partial/\partial u)\psi_0(\mathbf{r},u) - [1 - \kappa(u)]\psi_0(\mathbf{r},u) + S_0(\mathbf{r},u)$$

where $g(u)$ is yet to be determined.

To fix $g(u)$ and also to estimate the accuracy with which Eq. (12.2.62) represents ψ, we multiply Eq. (12.2.52) by the cosine of the angle ϑ between $\mathbf{a}_u$ and grad ψ_0 and integrate over all directions of $\mathbf{a}_u$. We obtain

$$\begin{aligned}
&\tfrac{1}{3}\lambda(u)\operatorname{grad}\psi_0 + \tfrac{1}{3}g(u)\operatorname{grad}\psi_0 - S_1(\mathbf{r},u)\\
&\quad = \tfrac{1}{3}\kappa(u)g(u)\operatorname{grad}\psi_0 \iint \cos\vartheta\, d\Omega \cdot \\
&\qquad\qquad \cdot \int_{u-w_\pi}^{u} du' \iint \cos\vartheta'\, \alpha(\theta', u - u')\, d\Omega' + \cdots\\
&\quad = (2/9M)\kappa(u)g(u)\operatorname{grad}\psi_0 + \cdots
\end{aligned}$$

where $S_1(\mathbf{r},u) = \iint \cos\vartheta\, S(\mathbf{r},u,a_u)\, d\Omega$ is the first moment of the source distribution. From this, to our assumed order of approximation and if S_1 can be neglected, we obtain

$$g(u) = -\lambda(u)/[1 - (2/3M)\kappa(u)]$$

and, finally, the "age equation"

$$\nabla^2\psi_0(\mathbf{r},T) - (\partial/\partial T)\psi_0(\mathbf{r},T) - \epsilon(T)\psi_0(\mathbf{r},T) = -\sigma(r,T) \quad (12.2.64)$$

where the "age" of the particles

$$T = \tfrac{1}{3}\int_0^u \frac{\lambda^2(u')\,du'}{w_{\mathrm{av}}\kappa(u')[1 - (2/3M)\kappa(u')]} \quad (12.2.65)$$

and $$\epsilon = \frac{3(1-\kappa)[1 - (2\kappa/3M)]}{\lambda^2};\quad \sigma = \frac{3[1 - (2\kappa/3M)]}{\lambda^2}S_0$$

The "age" T has the dimensions of an area and is proportional to the square of the mean distance a particle has drifted by the time it reaches the energy $E = E_0e^{-u}$. We remember that the distribution function is $f(\mathbf{r},u,\mathbf{a}_u) = [\lambda(u)/v]\psi(\mathbf{r},u,\mathbf{a}_u)$ and that ψ is given in terms of ψ_0 by Eq. (12.2.62).

This approximate form for f and the corresponding equation for ψ_0 is valid if and where $\lambda|\text{grad}\,\psi_0|$ is considerably smaller than ψ_0. It is not a good approximation for particle energies near E_0 (when u is less than w_π); nor is it valid if $\lambda(u)$ or $\kappa(u)$ changes appreciably in a range w_π of u.

Comparison with Eqs. (2.4.54) shows that we have chosen here a different scale for the age T of the particles than we did previously, to achieve a simpler form for the equation. The present variable T is roughly proportional to a mean square of the mean free path λ times the earlier τ, which was the average number of collisions required to reduce the particle from its original energy E_0 to its final energy $E = E_0e^{-u}$. The former variable, τ, had the advantage of an obvious physical meaning; the latter variable, T, has the advantage of making the equations easier to solve when $\lambda(u)$ varies much with u. Comparison with Eq. (12.1.28) shows the same difference, with the additional difference that the present $\kappa(u)$ is the fraction of particles *not* absorbed per collision, whereas the former κ_i was the fraction absorbed.

Since, in this section, we are showing how to obtain solutions when the diffusion equation *cannot* be used and since we have already, in Secs. 2.4 and 12.1, given examples of the method of solving the age-theory variety of diffusion equation, we shall not pursue the age-theory approximation here.

Indeed we have reached a convenient stopping place for this section. We have shown how to solve the diffusion problem in two important situations where the ordinary diffusion equation or the age-theory form is not adequate. Near the boundaries of the diffusing medium we were able to improve on the ordinary diffusion approximation, in regard to the distribution in angle of the particles emerging from the surface into free space and in regard to the particle density within a mean free path of the surface as long as we could neglect slowing-down effects in this region near the surface. In the interior of the material we were able to obtain improvements on the results of simple age-theory calculations for cases where a fairly large fraction of energy is lost per collision (light target atoms) as long as we could assume a negligible space variation of the particle density and as long as the fraction $(1 - \kappa)$ absorbed per collision is small. In other cases the usual diffusion or age theory, discussed in Sec. 12.1, is usually adequate.

We must now go on to study a quite different sort of equation.

12.3 *Solutions of Schroedinger's Equation*

The motion of a particle of mass M in a potential $V(\mathbf{r},\mathbf{p})$, where $\mathbf{r}$ and $\mathbf{p}$ are its position and momentum, respectively, is determined by the Schroedinger equation,

$$\mathcal{H}(\mathbf{r},\mathbf{p})\Psi = i\hbar(\partial\Psi/\partial t) \qquad (12.3.1)$$

where $\mathcal{H}$ is the Hamiltonian operator and may be written as the sum of the kinetic and potential energy of the particle:

$$\mathcal{H} = (p^2/2M) + V(\mathbf{r},\mathbf{p})$$

Equation (12.3.1), of course, neglects relativistic and spin effects. We have already discussed the origin and physical interpretation of the Schroedinger equation in some detail in Sec. 2.6 [see Eq. (2.6.38)] so that we shall content ourselves here with a summary of those items which are pertinent to this section.

Definitions. Consider first the equations satisfied by Ψ. It will be recalled that in order to convert (12.3.1) into a differential equation, it is necessary to express either $\mathbf{r}$ or $\mathbf{p}$ as a differential operator. In the latter case

$$\mathbf{p} = (\hbar/i)\nabla$$

so that Eq. (12.3.1) may be written:

$$[-(\hbar^2/2M)\nabla^2 + V(\mathbf{r},(\hbar/i)\nabla)]\Psi = i\hbar(\partial\Psi/\partial t)$$

or

$$[-\nabla^2 + U]\Psi = (i/\alpha^2)(\partial\Psi/\partial t)$$

where

$$U = (2M/\hbar^2)V; \quad \alpha^2 = (\hbar/2M)$$

In these equations the wave function Ψ is a function of $\mathbf{r}$ and t.

An entirely equivalent equation, the Schroedinger equation in momentum space, may be obtained by replacing $\mathbf{r}$ by the operator

$$\mathbf{r} = -(\hbar/i)\nabla_p$$

where the x component of ∇_p is $\partial/\partial p_x$. Calling the corresponding wave function $\Phi(p,t)$, that is, replacing Ψ by Φ in (12.3.1), one obtains the Schroedinger equation in momentum space:

$$[(p^2/2M) + V((-\hbar/i)\nabla_p,\mathbf{p})]\Phi(\mathbf{p},t) = i\hbar(\partial\Phi/\partial t)$$

The functions Φ and Ψ are Fourier transforms of each other:

$$\Phi(\mathbf{p},t) = \left(\frac{1}{2\pi\hbar}\right)^{\frac{3}{2}} \int_{-\infty}^{\infty} e^{i(\mathbf{p}\cdot\mathbf{r})/\hbar}\Psi(\mathbf{r},t)\, dv$$

Let us turn now to the physical interpretation of the wave function Ψ. (To obtain an interpretation for Φ it is only necessary to paraphrase the discussion below in terms of momentum space.) The key to the significance of Ψ is in the statement that $|\Psi|^2$ *dv is the relative probability of finding a particle in a volume element dv at* $\mathbf{r}$ *at a time t.* It now becomes possible to evaluate the average value of any dynamical quantity, F. (It should be recalled that such a quantity does not usually have a specific value in quantum mechanics.) The average value of $F(\mathbf{r},\mathbf{p})$ is:

$$F_{\text{av}}(t) = \frac{\iiint \bar{\Psi}(\mathbf{r})F\left[\mathbf{r}, \left(\frac{\hbar}{i}\right)\nabla\right]\Psi(\mathbf{r})\, dv}{\iiint|\Psi|^2\, dv} \tag{12.3.2}$$

If we consider our system to be made up of a number of noninteracting identical particles,[1] then $|\Psi|^2$ gives the density of particles at the point $\mathbf{r}$. Corresponding to this density there is a current density $\mathbf{J}$ satisfying the continuity equation:

$$\operatorname{div}\mathbf{J} + (\partial\rho/\partial t) = 0$$

From this and the Schroedinger equations, $\mathbf{J}$ is

$$\mathbf{J} = -\frac{i\hbar}{2M}[\bar{\Psi}\operatorname{grad}\Psi - \Psi\operatorname{grad}\bar{\Psi}];\quad \mathbf{J} = 2\alpha^2\operatorname{Im}[\bar{\Psi}\operatorname{grad}\Psi] \tag{12.3.3}$$

In this section we shall be principally interested in states which have a definite energy E; that is, satisfy the eigenvalue equation

$$\mathcal{H}\Psi = E\Psi$$

Comparing with the Schroedinger equation, we see that the time dependence of Ψ for such states is given by

$$i\hbar(\partial\Psi/\partial t) = E\Psi;\quad |\Psi|^2 \text{ independent of } t$$

States which have a definite energy are, therefore, *stationary states.* Solving for $\Psi(t)$ yields:

$$\Psi(t) = e^{-(iEt/\hbar)}\psi;\quad \psi = \Psi(0)$$

The function ψ also satisfies $\mathcal{H}\psi = E\psi$, or more definitely:

$$\nabla^2\psi + [\epsilon - U]\psi = 0 \tag{12.3.4}$$

where $\quad \epsilon = (2M/\hbar^2)E;\quad U = (2M/\hbar^2)V$

[1] We assume that the exclusion principle has little effect here.

As a single example of this review, consider the plane wave solution

$$\Psi = e^{i(\mathbf{p}\cdot\mathbf{r}-Et)/\hbar}$$

The density $|\Psi|^2 = 1$ which means that the relative probability of finding the particle in a volume element dv is independent of its position, *i.e.* it is equally likely to be anywhere. The current density $\mathbf{J} = \mathbf{p}/M$, that is, just the velocity of the particle. For the plane wave, the momentum is definite and equal to $\mathbf{p}$ since

$$\mathfrak{p}\psi = \mathbf{p}\psi$$

This definiteness of $\mathfrak{p}$ has its concomitant complete indefiniteness of position as we have shown above and as is demanded by the uncertainty principle.

On the other hand, the wave function $e^{-\beta r}$ (β = constant), describing the ground state of the electron in a hydrogen atom, will have zero current since it is real; it will not have a definite momentum. Its momentum wave function is

$$\Phi(\mathbf{p}) = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} \iiint e^{i(\mathbf{p}\cdot\mathbf{r})/\hbar} e^{-\beta r}\, dv$$

This integral may be easily evaluated:

$$\Phi(\mathbf{p}) = \frac{4\pi}{(2\pi\hbar)^{\frac{3}{2}}} \int_0^\infty \frac{\sin(pr/\hbar)}{(pr/\hbar)} e^{-\beta r} r^2\, dr = \frac{8\pi}{(2\pi\hbar)^{\frac{3}{2}}} \frac{\beta}{[\beta^2 + (p/\hbar)^2]^2}$$

The square of this quantity $|\Phi(\mathbf{p})|^2\, dp_x\, dp_y\, dp_z$ gives the relative probability of finding the electron with a momentum between $\mathbf{p}$ and $d\mathbf{p}$.

In this section we shall take up a few examples of the technique of solution of Eqs. (12.3.1) and (12.3.4). As with the rest of this work, the intent is primarily to elucidate methods of solution, so there will be no attempt to give a connected account of atomic structure, for example. We shall illustrate many of the special techniques by using one-dimensional examples; a few problems, which are by their nature multidimensional ones, will also be discussed.

The Harmonic Oscillator. One of the simplest and most often encountered dynamical problems is that of a particle in an elastic field of force, corresponding to the one-dimensional potential function $\frac{1}{2}Kx^2$. In classical physics the particle may be at rest with zero energy at the equilibrium position $x = 0$ or it may be in simple-harmonic motion of frequency $\omega/2\pi = \sqrt{K/4\pi^2M}$ and of amplitude $\sqrt{2E/K}$, where E is its energy. Since the probability of presence of the classical particle is in inverse proportion to its velocity, this probability is

$$P = \begin{cases} \dfrac{(1/\pi)}{\sqrt{(2E/K) - x^2}}; & x^2 < 2E/K \\ 0; & x^2 > 2E/K \end{cases}$$

a quantity which will be of interest to compare with the corresponding $|\Psi|^2$.

For a stationary state in wave mechanics, the wave function Ψ must have an exponential time dependence, $\Psi = \psi(x)e^{-iEt/\hbar} = \psi(x)e^{-i\epsilon\alpha^2 t}$, where E is the stationary energy of the particle and $\epsilon = (2M/\hbar^2)E$. In this state every measurement of the energy of the particle will give the value E. In the present case the equation for the space-dependent factor ψ is [see Eq. (2.6.28)]

$$(d^2\psi/dx^2) + (\epsilon - \beta^2x^2)\psi = 0 \tag{12.3.5}$$

where $\beta = M\omega/\hbar$ and $\omega^2 = K/M$. This is an equation with an irregular singular point (of species higher than the lowest) at infinity. Change of the independent variable to $z = \beta x^2$ shows that ψ may be expressed in terms of the confluent hypergeometric function of z. In fact ψ must be just proportional to the *Weber function*

$$D_n(x\sqrt{2\beta}) = 2^{\frac{1}{2}n}e^{-\frac{1}{2}\beta x^2 + i\pi n}U_2(-\tfrac{1}{2}n|\tfrac{1}{2}|\beta x^2); \quad \mathrm{Re}\, x > 0$$

defined in Eq. (11.2.63), with properties given in the table at the end of Chap. 11. In the present case $n = (\epsilon/2\beta) - \frac{1}{2} = (E/\hbar\omega) - \frac{1}{2}$.

The function D_n is chosen to vanish as $x \to +\infty$, but it does not usually vanish as $x \to -\infty$. Using the relation between $D_n(x)$, $D_n(-x)$ and $D_{-n-1}(ix)$, we find that

$$D_n(x\sqrt{2\beta}) \to \begin{cases} (x\sqrt{2\beta})^n e^{-\frac{1}{2}\beta x^2}; & x \to \infty\,. \\ \dfrac{\sqrt{2\pi}}{\Gamma(-n)}\,\dfrac{e^{+\frac{1}{2}\beta x^2}}{(-x\sqrt{2\beta})^{n+1}}; & x \to -\infty \end{cases}$$

unless n is zero or a positive integer, in which case $D_n(z) = (-1)^n D_n(-z)$. If we wish to have a wave function which is quadratically integrable we must have $n = 0, 1, 2, \ldots$, which limits the energy of the particle to discrete allowed values

$$\epsilon_n = 2\beta n + \beta \quad \text{or} \quad E_n = \hbar\omega(n + \tfrac{1}{2}) \tag{12.3.6}$$

In order that the state represented by Ψ be a stationary state, for a definite energy, the space factor ψ must be a standing wave, which requirement restricts the allowed energy in the same way that the frequency of free vibration of a string is restricted. The particle is not allowed to have zero energy [this is related to the uncertainty principle, Eq. (2.6.2), for it is not allowed to have a particle with exactly zero momentum definitely at $x = 0$]. The allowed energy levels are evenly spaced, the spacing being $\hbar\omega$. This result is to be compared with Eq. (2.6.31), where the harmonic oscillator was computed by operator methods.

When n is an integer, the Weber functions become proportional to the Hermite polynomials discussed in the table at the end of Chap. 6.

Using the integral relations to calculate the normalizing factors, we find that the allowed wave function corresponding to the energy $\hbar\omega(n + \frac{1}{2})$ is the nth normalized eigenfunction

$$\varphi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{\beta}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\beta x^2} H_n(x\sqrt{\beta}) \tag{12.3.7}$$

where $$H_n(z) = (-1)^n e^{z^2}(d^n/dz^n)e^{-z^2}; \quad \beta = M\omega/\hbar$$

is a polynomial of nth degree in z, with either all even powers of z up to n (if n is even) or all odd powers up to n (if n is odd), the coefficients of successive terms alternating in sign. The result of this is that the n roots of $H_n(z) = 0$ are all real and are symmetrically spaced about $z = 0$.

The average behavior of the harmonic oscillator may be studied by evaluating the average of the displacement and of the square of the displacement. Through Eq. (12.3.2) and the recursion formula given in the table at the end of Chap. 6, we see that

$$\begin{aligned} x\sqrt{\beta}\,\varphi_n(x) &= \sqrt{\tfrac{1}{2}n}\,\varphi_{n-1}(x) + \sqrt{\tfrac{1}{2}(n+1)}\,\varphi_{n+1}(x) \\ \beta x^2\varphi_n(x) &= \tfrac{1}{2}\sqrt{n(n-1)}\,\varphi_{n-2} + (n+\tfrac{1}{2})\varphi_n \\ &\qquad + \tfrac{1}{2}\sqrt{(n+1)(n+2)}\,\varphi_{n+2} \end{aligned}$$

The mean displacement of the particle in the nth state is

$$\int_{-\infty}^{\infty} x\varphi_n^2(x)\,dx = 0; \quad \text{since the } \varphi\text{'s are orthogonal}$$

indicating that the oscillations are symmetrical about the equilibrium point $x = 0$. The mean square displacement

$$\int_{-\infty}^{\infty} x^2\varphi_n^2(x) = \frac{n+\frac{1}{2}}{\beta} = \frac{E_n}{K}$$

is just half of the square of the maximum displacement the classical particle would have if it had energy $E_n = h\omega(n + \frac{1}{2})$. As far as average behavior goes, therefore, the quantized oscillator corresponds to the classical oscillator, except that only the discrete values E_n of the energy are allowed.

As an exercise in the application of the method of steepest descent and as a comparison with the classical results, let us compute our asymptotic value of ψ for n large. We have, setting $A_n^2 = (\beta/2^n n!\sqrt{\pi})$, and using Eq. (5.3.53),

$$\varphi_n = A_n 2^n \frac{\Gamma(\frac{1}{2}n+1)}{2\pi i} e^{-\frac{1}{2}z^2} \oint e^{z^2t-(\frac{1}{2}n+1)\ln t+(\frac{1}{2}n-\frac{1}{2})\ln(1-t)}\,dt$$

where the contour is shown in Fig. 5.12, going from minus infinity, about $t = 0$ in a positive direction and back to $-\infty$. The saddle point for the exponent in the integral is where $f(t) = z^2t - (\frac{1}{2}n + 1)\ln t + (\frac{1}{2}n - \frac{1}{2})\ln(1 - t)$ has zero derivative, which is at the roots of the equation,

$$f'(t) = z^2 - \frac{\frac{1}{2}n + 1 - \frac{3}{2}t}{t(1 - t)} = 0$$

When $z^2 < 2n + 1$, the two saddle points are symmetrically placed above and below the real t axis; the resulting integral has a cosine factor. When $z^2 > 2n + 1$, the roots are on the real t axis and only the one nearest $t = 0$ needs to be traversed. Proceeding as in Sec. 4.6, we eventually find, for $n \gg 1$,

$$\varphi_n \simeq \begin{cases} \dfrac{\beta^{\frac{1}{4}}\sqrt{2/\pi}}{(2n - z^2)^{\frac{1}{4}}}\cos[\frac{1}{2}z\sqrt{2n - z^2} - \frac{1}{2}\pi n + n\sin^{-1}(z/\sqrt{2n})]; & (z^2 < 2n) \\ \sqrt{n!}(\beta/\pi)^{\frac{1}{4}}z^n e^{-\frac{1}{2}z^2} & (z^2 > 2n) \end{cases}$$

where $z^2 = \beta x^2$ and $\beta = M\omega/\hbar$.

The square of the first expression may be written in simpler form;

$$|\varphi_n|^2 \simeq \frac{(2/\pi)}{\sqrt{a_n^2 - x^2}}\cos^2\left[\frac{1}{2}\beta x\sqrt{a_n^2 - x^2} + \frac{1}{2}\beta a_n^2 \sin\left(\frac{x}{a_n}\right) - \frac{1}{2}\pi n\right]; \quad x < a_n$$

where $a_n^2 = (2n + 1)/\beta = 2E_n/M\omega^2 = 2E_n/K$, with $K = M\omega^2$. The average value (the average value of the $\cos^2$ term is $\frac{1}{2}$) of $|\varphi_n|^2$ for $n \gg 1$ is thus on the average equal to the classical probability expression, P, given at the beginning of this subsection, over most of the range of x which is allowed classically.

The difference between the quantum mechanics and the classical mechanics thus lies both in the spectra of allowed energies and in the probability density for the presence of the particle. In the quantum mechanical case only the energy values $E_n = \hbar\omega(n + \frac{1}{2})$ are allowed, whereas in the classical case all positive values are allowed. When $\hbar\omega$ is small compared to the range of energy values under consideration, this difference is not particularly noticeable; but when $\hbar\omega$ is not small, the difference is pronounced. The probability of presence of the particle $|\Psi_n|^2$, in the quantum mechanical case, has a series of nodes a distance approximately $2\pi/\beta a_n$ apart near the origin for a state having just the energy E_n. For small values of n, $|\Psi_n|^2$ differs considerably from the value of classical P, given in Eq. (12.3.4); for large values of n, the average of $|\Psi_n|^2$ over several wavelengths approaches P for $x < a_n$. If we are not certain of the energy, so that Ψ is a mixture of states for neighboring energies, the nodes for the different states will not coincide and the expression for $|\Psi|^2$ will be fairly close to P for E large compared to $\hbar\omega$. Whether $\hbar\omega$ is large or small compared to E, however, $|\Psi|^2$ does not go to infinity at $x = a_n$ (whereas P does) and it is not zero for $x > a_n$ (whereas

P is). In quantum mechanics there is a nonzero (though usually small) probability that the particle be found in a region where the potential energy is greater than the total energy. This follows from the wavelike properties of the function Ψ.

Short-range Forces. Another simple potential function, useful in the study of nuclear forces, is the limiting case of an attractive force which acts over only a very short range of x. The corresponding potential energy is zero over most of the range of x, but near zero has a deep, narrow depression, often called a "potential well." The limiting case is V equal to $-V_0\delta(x)$ and the corresponding Schroedinger equation for a constant-energy state is

$$(d^2\psi/dx^2) + [\epsilon + 2\alpha\delta(x)]\psi = 0 \qquad (12.3.8)$$

where $\alpha = MV_0/\hbar^2$, $\epsilon = 2ME/\hbar^2$. By integrating the equation over a small interval of x about zero, we see that there must be a discontinuity in slope of ψ at $x = 0$ equal to $-2\alpha\psi(0)$;

$$(\partial\psi/\partial x)_{0^-} - (\partial\psi/\partial x)_{0^+} = 2\alpha\psi(0)$$

There is just one "bound" state (with negative energy) for this potential. It is

$$\psi_0(x) = \sqrt{\alpha/2}\, e^{-\alpha|x|}; \quad \epsilon = -\alpha^2; \quad E = MV_0^2/2\hbar^2$$

having a maximum at $x = 0$ and becoming vanishingly small for $|x|$ large.

The positive energy states correspond to free particles, traveling with constant velocity v except when they "hit" the short range force at $x = 0$. We use the free-wave functions $e^{\pm ikx}(\epsilon = k^2;\ k = Mv/h)$ and join them at $x = 0$ to fit the requirements of discontinuous slope there. For example, a possible solution is

$$\psi = \begin{cases} e^{ikx} - [\alpha/(\alpha + ik)]e^{-ikx}; & x < 0 \\ [ik/(\alpha + ik)]e^{ikx}; & x > 0 \end{cases}$$

Reference to Eq. (12.3.3) indicates that this represents a stream of particles coming from $-\infty$ and striking the potential well at $x = 0$, some being reflected back to $-\infty$ and some continuing on to $+\infty$. The earlier discussion shows that the squares of the coefficients of the various terms here obtained correspond to the ratio of reflected to incident current density R and of transmitted to incident current density T. These turn out to be

$$R = \alpha^2/(\alpha^2 + k^2); \quad T = 1 - R = k^2/(\alpha^2 + k^2)$$

The deeper the well, the greater the reflection; the larger the particle velocity ($k = Mv/\hbar$), the smaller the reflection. Further discussion of the physical inferences to be drawn from such formulas will await the exegesis of Eqs. (12.3.29) and (12.3.31).

The Effect of a Perturbation. A problem often encountered in quantum mechanics is the following: A system in a definite, steady state has applied to it at $t = 0$ an additional, perturbing force; to find the subsequent behavior of the system. One of the simplest such problems can be solved exactly for the harmonic oscillator and will provide a model for other more complicated cases. Suppose, for $t < 0$ the system described by Eq. (12.3.5) was in the nth state, with energy $E_n = \hbar\omega(n + \frac{1}{2})$ and wave function $\varphi_n e^{-i\omega(n+\frac{1}{2})t}$, where φ_n is given in Eq. (12.3.7); at $t = 0$ the perturbing potential $+ Fxu(t)$ is turned on. Subsequent to $t = 0$, therefore, the equation for the space part of the wave function is

$$(d^2\psi/dx^2) + [\epsilon - \beta^2 x^2 - (2M/\hbar^2)Fx]\psi = 0$$

or

$$(d^2\psi/dx^2) + [\epsilon + (F/\omega\hbar)^2 - \beta^2(x + b)^2]\psi = 0$$

where $b = F/M\omega^2 = F/K$ is the displacement of the equilibrium caused by the perturbing force F.

The new solution which is finite everywhere is, of course, some linear combination of the displaced eigenfunctions.

$$\Psi = \sum_{m=0}^{\infty} A_m \varphi_m(x + b) \exp\left[-i\omega(m + \tfrac{1}{2})t + \left(\frac{i}{\hbar}\right) E_b t \right] \tag{12.3.9}$$

where $E_b = F^2/2M\omega^2 = \frac{1}{2}(F^2/K)$ is the depression of the potential energy minimum caused by the perturbing force. To obtain the complete solution, we must compute the particular linear combination of $\varphi_m(x + b)$'s which equals $\varphi_n(x)$ at $t = 0$.

To obtain this result we use the relation

$$\begin{aligned}\varphi_n(x - b) = &\sum_{m=0}^{n} \varphi_m(x) e^{\frac{1}{4}\gamma^2} \frac{(-\gamma/\sqrt{2})^{n-m}}{(n - m)!} \sqrt{\frac{n!}{m!}} F(n + 1|n - m + 1| - \tfrac{1}{2}\gamma^2) \\ &+ \sum_{m=n+1}^{\infty} \varphi_m(x) e^{\frac{1}{4}\gamma^2} \frac{(\gamma/\sqrt{2})^{m-n}}{(m - n)!} \sqrt{\frac{m!}{n!}} F(m + 1|m - n + 1| - \tfrac{1}{2}\gamma^2)\end{aligned} \tag{12.3.10}$$

(where $\gamma^2 = \beta b^2 = M\omega F^2/\hbar K^2$), derived from the formulas at the end of Chap. 6. From it we obtain, finally, the solution

$$\Psi(x,t) = \begin{cases} \varphi_n(x) e^{-i\omega(n+\frac{1}{2})t} & t < 0 \\ \displaystyle\sum_{m=0}^{\infty} A_m^n \varphi_m(x + b) e^{-i\omega(m+\frac{1}{2})t+(i/\hbar)E_b t}; & t > 0 \end{cases} \tag{12.3.11}$$

where $E_b = \frac{1}{2}(F^2/K) = \frac{1}{2}M\omega^2 b^2 = \frac{1}{2}\omega\hbar\gamma^2$ and where

$$A^n_m = \begin{cases} e^{\frac{1}{4}\gamma^2} \dfrac{(-\gamma/\sqrt{2})^{n-m}}{(n-m)!} \sqrt{\dfrac{n!}{m!}} F(n+1|n-m+1|-\frac{1}{2}\gamma^2); & m \le n \\ e^{\frac{1}{4}\gamma^2} \dfrac{(\gamma/\sqrt{2})^{m-n}}{(m-n)!} \sqrt{\dfrac{m!}{n!}} F(m+1|m-n+1|-\frac{1}{2}\gamma^2); & m \ge n \end{cases}$$

When the perturbing force is small compared to $K\sqrt{\beta}$, then γ is a small quantity and, to the first approximation in γ, the solution is

$$\Psi \simeq \begin{cases} \varphi_n(x)e^{-i\omega(n+\frac{1}{2})t}; \quad t < 0 \\ \varphi_n(x+b) + \gamma[\sqrt{\frac{1}{2}(n+1)}\,\varphi_{n+1}e^{-i\omega t} - \sqrt{\frac{1}{2}n}\,\varphi_{n-1}e^{i\omega t}] \cdot \\ \qquad \cdot\, e^{-i\omega(n+\frac{1}{2})t+(i/\hbar)E_b t}; \quad t > 0 \end{cases}$$

The particle density, to this approximation, is then

$$|\Psi|^2 \simeq \begin{cases} [\varphi_n(x)]^2; \quad t < 0 \\ [\varphi_n(x+b)]^2 + 2\gamma[\sqrt{\frac{1}{2}(n+1)}\,\varphi_n\varphi_{n+1} \\ \qquad - \sqrt{\frac{1}{2}n}\,\varphi_n\varphi_{n-1}]\cos(\omega t); \quad t > 0 \end{cases} \tag{12.3.12}$$

The particle density, therefore, before $t = 0$, is in a steady state characterized by the eigenfunction φ_n; after the perturbation is turned on at $t = 0$, the state becomes a combination of several eigenfunctions and the density oscillates with time. Since, for the eigenfunctions given in Eq. (12.3.7),

$$\frac{1}{\sqrt{\beta}}\frac{d}{dx}\varphi_n(x) = -\sqrt{\tfrac{1}{2}(n+1)}\,\varphi_{n+1}(x) + \sqrt{\tfrac{1}{2}n}\,\varphi_{n-1}(x)$$

the approximate expression for the particle density after $t = 0$ may also be written

$$\begin{aligned} |\Psi|^2 &\simeq [\varphi_n(x+b)]^2 - 2b[\varphi_n(x+b)(d/dx)\varphi_n(x+b)]\cos(\omega t); \quad t > 0 \\ &\simeq [\varphi_n(x+b-b\cos\omega t)]^2 \end{aligned}$$

This shows that the perturbation has disturbed the distribution $(\varphi_n)^2$, which is symmetric about $x = 0$, producing a distribution which is on the average symmetrical about the new equilibrium position $x = -b$, but which has a small antisymmetric oscillatory part. At $t = 0$ the center of the distribution is still at $x = 0$, but at time $t = \pi/\omega$ later, it is at $x = -2b$, at time $t = 2\pi/\omega$ it is back at $x = 0$, and so on, the center of the distribution oscillating back and forth with frequency $\omega/2\pi$ and amplitude b. This sort of oscillation is, of course, the sort of motion which a classical particle would execute if it were acted on by a similar perturbation so that, in this simple case at least, there is a vague correspondence between the behavior of the probability density $|\Psi|^2$ and the related classical particle.

Approximate Formulas for General Perturbation. The techniques discussed in Chap. 9 may be applied here to compute the stationary states of systems which differ only slightly from the harmonic oscillator, or to calculate the behavior of harmonic oscillators perturbed by some small additional force. Suppose the perturbing potential is $V'(x)$ so that the resulting Schroedinger equation for a steady state [corresponding to Eq. (12.3.5)] is

$$(d^2\psi/dx^2) + (\epsilon - \beta^2x^2)\psi = (2M/\hbar^2)V'(x)\psi \qquad (12.3.13)$$

where $\epsilon = 2ME/\hbar^2$. This may be changed into an integral equation for ψ and ϵ by the methods discussed in Chap. 9:

$$\psi(x) = \left(\frac{2M}{\hbar^2}\right)\int_{-\infty}^{\infty} G_\epsilon(x|x_0)V'(x_0)\psi(x_0)\,dx_0 \qquad (12.3.14)$$

Kernel G is the Green's function for the harmonic oscillator, satisfying the equation

$$(d^2G/dx^2) + (\epsilon - \beta^2x^2)G = \delta(x - x_0)$$

where $\epsilon = 2ME/\hbar^2$ and $\beta = M\omega/\hbar$ and the boundary condition that G goes to zero at $|z| \to \infty$.

In the present case the most useful expression for the Green's function is the eigenfunction series

$$G_\epsilon(x|x_0) = \sum_{n=0}^{\infty} \frac{\varphi_n(x)\bar{\varphi}_n(x_0)}{\epsilon - \epsilon_n} \qquad (12.3.15)$$

where the φ's are the eigenfunctions for the unperturbed harmonic oscillator given in (12.3.7) and ϵ_n are the corresponding values of $2ME/\hbar^2$, $\epsilon_n = 2\beta(n + \frac{1}{2})$. Introducing this expansion into (12.3.14), we obtain

$$\psi(x) = \frac{2M}{\hbar^2}\sum_{m=0}^{\infty}\frac{\varphi_m(x)}{\epsilon - \epsilon_m}\int_{-\infty}^{\infty}\bar{\varphi}_m(x_0)V'(x_0)\psi(x_0)\,dx_0$$

We shall want our function to approach a particular solution, say φ_n when V' goes to zero. Let us, therefore, adjust the normalization of ψ so that the coefficient of φ_n is unity. Therefore,

$$\psi = \varphi_n(x) + \sum_{m\neq n}\frac{\varphi_m(x)}{E - E_n}\int_{-\infty}^{\infty}\bar{\varphi}_m(x_0)V'(x_0)\psi(x_0)\,dx_0$$

and $$E = E_n + \int_{-\infty}^{\infty}\bar{\varphi}_n(x_0)V'(x_0)\psi(x_0)\,dx_0 \qquad (12.3.16)$$

where $E_n = \hbar\omega(n + \frac{1}{2})$. This is an exact pair of equations which may be solved, as most integral equations are solved, by a set of iterations.

We first place $\psi = \varphi_n$ in the right-hand side of (12.3.16), yielding a new approximation for ψ which is again inserted in the right-hand side, and so on. The corresponding value of the energy may then be evaluated by inserting ψ from the first equation into the second (see Sec. 9.1 for further details).

The first approximation is $\epsilon = \epsilon_n + (2M/\hbar^2)V'_{nn}$ so that

$$E^1 = \hbar\omega(n + \tfrac{1}{2}) + V'_{nn}; \quad \psi^1 = \varphi_n(x) + \sum_{m \neq n} \frac{V'_{mn}}{E - E_m} \varphi_m(x) \qquad (12.3.17)$$

where
$$V'_{mn} = \int_{-\infty}^{\infty} \bar{\varphi}_m(x_0) V'(x_0) \varphi_n(x_0)\, dx_0$$

is called the *matrix element* of V' with respect to the unperturbed states m and n. In the expression for ψ^1, we may substitute E^1 for E. The second approximation is then obtained by substituting ψ^1 into the right-hand side of Eq. (12.3.16):

$$E^2 = \hbar\omega(n + \tfrac{1}{2}) + V'_{nn} + \sum_{m \neq n} \frac{V'_{nm}V'_{mn}}{E - E_m}$$

$$\psi^2 = \varphi_n(x) + \sum_{m \neq n} \frac{V'_{mn}}{E - E_m} \varphi_m(x) + \sum_{\substack{m \neq n \\ p \neq n}} \frac{V'_{mp}V'_{pn}}{(E - E_p)(E - E_m)} \varphi_m(x)$$

Here E in the sum in the first expression may be replaced by E_n to obtain a formula for the energy accurate to second order. In the expression for ψ^2, the E in the first sum must be replaced by E^1, while in the second sum E_n is sufficient for second-order accuracy. It is now possible to indicate the complete expansion:

$$\begin{aligned} E &= \hbar\omega(n + \tfrac{1}{2}) + V'_{nn} + \sum_{m \neq n} \frac{V'_{nm}V'_{mn}}{E - E_m} + \sum_{\substack{m \neq n \\ p \neq n}} \frac{V'_{nm}V'_{mp}V'_{pn}}{(E - E_p)(E - E_m)} \\ &\quad + \sum_{\substack{m \neq n \\ p \neq n \\ q \neq n}} \frac{V'_{nm}V'_{mp}V'_{pq}V'_{qn}}{(E - E_p)(E - E_q)(E - E_m)} + \cdots \\ \psi &= \varphi_n(x) + \sum_{m \neq n} \frac{V'_{mn}}{E - E_m} \varphi_m(x) + \sum_{\substack{m \neq n \\ p \neq n}} \frac{V'_{mp}V'_{pn}}{(E - E_p)(E - E_m)} \varphi_m(x) \\ &\quad + \sum_{\substack{m \neq n \\ p \neq n \\ q \neq n}} \frac{V'_{mq}V'_{qp}V'_{pn}}{(E - E_q)(E - E_p)(E - E_m)} \varphi_m + \cdots \end{aligned} \qquad (12.3.18)$$

For a given order of approximation (say the νth), E in (12.3.18) must be replaced by $E^{(\nu-2)}$, $E^{(\nu-3)}$, etc., in the successive summations which appear in the expression for E, while in the expression for ψ, E must be replaced by $E^{(\nu-1)}$, $E^{(\nu-2)}$ in these successive sums. For example,

$$E^2 = \hbar\omega(n + \tfrac{1}{2}) + V'_{nn} + \sum_{m \neq n} \frac{V'_{nm}V'_{mn}}{\hbar\omega(n - m)}$$

$$\psi^2 = \varphi_n(x) + \sum_{m \neq n} \frac{V'_{mn}}{\hbar\omega(n - m) + V'_{nn}} \varphi_m(x) + \sum_{\substack{m \neq n \\ p \neq n}} \frac{V'_{mp}V'_{pn}}{[\hbar\omega(n - p)][\hbar\omega(n - m)]} \varphi_m(x)$$

The conditions under which these series converge were discussed in Chap. 9. Other perturbation formulas, useful when these series are poorly convergent, are also given in Chap. 9.

As an illustration of the perturbation method, we can compute energies and wave functions for the linear perturbation Fx, for which an exact solution was obtained in Eq. (12.3.11). From the relation

$$x\sqrt{\beta}\,\varphi_n(x) = \sqrt{\tfrac{1}{2}(n + 1)}\,\varphi_{n+1}(x) + \sqrt{\tfrac{1}{2}n}\,\varphi_{n-1}(x)$$

we obtain expressions for the matrix elements V'_{sn};

$$V'_{sn} = \left(F\sqrt{\frac{\hbar}{M\omega}}\right) \begin{cases} \sqrt{\tfrac{1}{2}(n + 1)}; & s = n + 1 \\ \sqrt{\tfrac{1}{2}n}; & s = n - 1 \\ 0; & s \text{ all other values} \end{cases}$$

To the second order, therefore, Eq. (12.3.18) indicates that the perturbed energies are

$$E \simeq \hbar\omega(n + \tfrac{1}{2}) + \frac{F^2\hbar}{2M\omega}\left(\frac{n - n - 1}{\hbar\omega}\right) = E_n - E_b$$

and, to the first approximation, the perturbed wave functions are

$$\begin{aligned} \psi &\simeq \varphi_n(x) - \gamma[\sqrt{\tfrac{1}{2}(n + 1)}\,\varphi_{n+1}(x) - \sqrt{\tfrac{1}{2}n}\,\varphi_{n-1}(x)] \\ &\simeq \varphi_n(x) + b\frac{d}{dx}\varphi_n(x) \simeq \varphi_n(x + b) \end{aligned}$$

which correspond to the solutions of Eq. (12.3.9).

Momentum Wave Functions. Reference to Sec. 2.6 [see the discussion after Eq. (2.6.28)] indicates that if the energy operator $\mathcal{H}$ for a stationary state is expressed in terms of the momentum p and position q of the particle, the equation for the space-dependent wave function is obtained by changing each p in H into $-\hbar i(\partial/\partial q)$ and operating by $\mathcal{H}$ on ψ. This type of wave function, for the harmonic oscillator, is given in Eq. (12.3.7). The corresponding momentum wave function, the square of which gives the probability that the particle has a given momentum, is given by changing x into $(\hbar i)(\partial/\partial p)$ in $\mathcal{H}$ and using the resulting operator on the wave function. For the harmonic oscillator the resulting equation for the function, with the time term $e^{-iEt/\hbar}$ divided off, is

$$\frac{d^2\chi}{dp^2} + \left[\frac{2E}{M\hbar^2\omega^2} - \frac{p^2}{M^2\omega^2\hbar^2}\right]\chi = 0 \tag{12.3.19}$$

This is exactly the same form as Eq. (12.3.5), and must have the same kind of solutions. The allowed values of the energy, for which χ is finite for all values of p, are

$$E_n = \hbar\omega(n + \tfrac{1}{2})$$

the same as Eq. (12.3.6) (as, of course, is required because we are talking about the same stationary states, asking only for their representation in terms of momentum instead of position). The corresponding momentum wave functions are

$$\chi_n(p) = \frac{1}{\sqrt{2^n n!}}\left(\frac{1}{M\omega\hbar}\right)^{\frac{1}{4}} e^{-p^2/2M\omega\hbar} H_n\left(\frac{p}{\sqrt{M\omega\hbar}}\right) \tag{12.3.20}$$

which are again normalized to unity, since the integral of χ^2 over all values of p must correspond to certainty that the particle momentum is in the range $-\infty < p < +\infty$.

The mean momentum is zero and, by use of the recurrence relations of the Hermite polynomials H_n, we can show that the mean square value of the momentum of the particle in the nth state is

$$\int_{-\infty}^{\infty} p^2\chi_n^2(p)\,dp = \hbar\omega M(n + \tfrac{1}{2}) = ME_n$$

which is equal to the mean square momentum of a classical particle having energy E_n.

The momentum wave functions are useful in finding the momentum behavior of the quantized oscillator, or in computing the perturbation caused by a term dependent on p, for example. According to Eq. (2.6.24), the space and momentum wave functions are related by being Fourier transforms of each other:

$$\chi_n(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{-ipx/\hbar}\,\varphi_n(x)\,dx \tag{12.3.21}$$

which may be verified by going back to the integral representation of the Weber functions (or which may be considered to provide, ready-made, an interesting integral equation satisfied by the Hermite polynomials).

Bound and Free States. When the potential energy has a minimum but goes asymptotically to a finite value at $x = +\infty$ or $-\infty$, then some of the allowed energies will be discrete values, corresponding to states for which the particle is bound in the potential valley. For other ranges of energy, higher than the asymptotic value, all energies will be allowed, the particle being free to travel to infinity. As an example, we compute the wave functions and allowed energies for a particle of mass m in a potential field

$$V(x) = V_0 \cosh^2 \mu \left\{\tanh[(x - \mu d)/d] + \tanh(\mu)\right\}^2$$

shown in Fig. 12.2. For V_0 positive, this potential field has its minimum value ($V = 0$) at $x = 0$. As x is increased positive, the potential rises to an asymptotic value $V_0 e^{2\mu}$ for $x \to +\infty$; as x is made negative, V also rises to an asymptotic value $V_0 e^{-2\mu}$, for $x \to -\infty$.

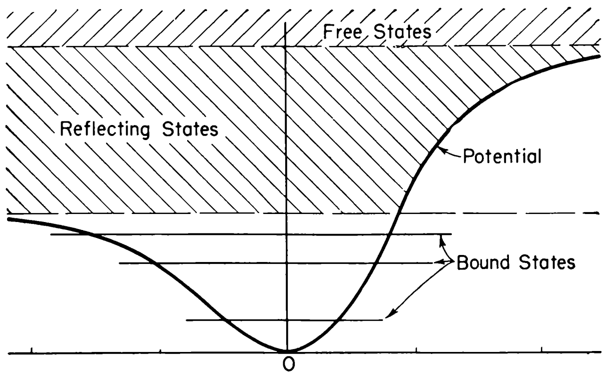

Fig. 12.2 One-dimensional potential energy exhibiting bound, reflecting, and free states.

Classically, the particle could not have a negative energy; for energies between zero and $V_0 e^{-2\mu}$ ($\mu > 0$), the particle would oscillate back and forth in the potential valley; for energies between $V_0 e^{-2\mu}$ and $V_0 e^{2\mu}$, the particle could come from $-\infty$, be reflected by the potential rise to the right of the minimum, and go back to $-\infty$; and for energies greater than $V_0 e^{2\mu}$, the particle could move from $-\infty$ to $+\infty$ or from $+\infty$ to $-\infty$.

To find the corresponding quantum mechanical behavior, we express distances in terms of the scale factor d and energies in terms of the unit energy $\hbar^2/2md^2$. For a state with energy E, the equation for the space factor ($\Psi = \psi e^{-iEt/}$) is

$$(d^2\psi/dz^2) + [\epsilon - v \cosh 2\mu - v \sinh 2\mu \tanh z + v \cosh^2 \mu \operatorname{sech}^2 z]\psi = 0 \quad (12.3.22)$$

where $z = [(x - \mu d)/d]$, $\epsilon = (2md^2/\hbar^2)E$, $v = (2md^2/\hbar^2)V_0$.

Surprisingly enough, this equation is related to the hypergeometric equation. We first set

$$\psi = e^{-az} \operatorname{sech}^b z\, F(z)$$

Then, if the parameters a and b are given by the equations

$$\begin{aligned} a^2 + b^2 &= -\epsilon + v \cosh 2\mu; \quad 2ab = v \sinh 2\mu \\ \text{or} \qquad a &= \tfrac{1}{2}\sqrt{ve^{2\mu} - \epsilon} - \tfrac{1}{2}\sqrt{ve^{-2\mu} - \epsilon} = \tfrac{1}{2}\kappa_+ - \tfrac{1}{2}\kappa_- \\ b &= \tfrac{1}{2}\sqrt{ve^{2\mu} - \epsilon} + \tfrac{1}{2}\sqrt{ve^{-2\mu} - \epsilon} = \tfrac{1}{2}\kappa_+ + \tfrac{1}{2}\kappa_- \end{aligned} \tag{12.3.23}$$

(κ_+^2 is the difference between particle and potential energy at $x = +\infty$ and κ_-^2 is the same difference at $x = -\infty$) the equation for F is

$$F'' - 2[a + b \tanh z]F' + [v \cosh^2 \mu - b(b + 1)] \operatorname{sech}^2 z\, F = 0$$

which is turned into a hypergeometric equation by changing the independent variable, setting $u = \frac{1}{2}[1 - \tanh(z)] = e^{-z}/(e^z + e^{-z})$;

$$u(1 - u)\frac{d^2F}{du^2} + [a + b + 1 - 2(b + 1)u]\frac{dF}{du} + [v \cosh^2 \mu - b(b + 1)]F = 0$$

Comparison with Eq. (5.2.42) indicates that a solution which is finite at $u = 0$ ($x \to +\infty$) is the hypergeometric function

$$F = F\left(b + \tfrac{1}{2} - \sqrt{v \cosh^2 \mu + \tfrac{1}{4}},\, b + \tfrac{1}{2} + \sqrt{v \cosh^2 \mu + \tfrac{1}{4}}\,\middle|\, a + b + 1 \,\middle|\, u\right)$$

Since both b and a are real when $\epsilon < ve^{+2\mu}$, the function ψ vanishes at $x \to \infty$ for this range of energy. A second solution is u^{-a-b} times another F, resulting in a ψ which behaves at $x = \infty$ as $e^{(a+b)z}$, going to infinity there, when $\epsilon < ve^{2\mu}$. Consequently, as long as ϵ (the particle energy in units of $\hbar^2/2md^2$) is smaller than $ve^{2\mu}$ (the asymptotic values of the potential energy at $z \to \infty$), this second solution is ruled out by the requirement of quadratic integrability.

The solution of the Schroedinger equation which is finite at $x = \infty$ is, therefore,

$$\psi = \frac{Ne^{-az}}{(e^z + e^{-z})^b} F\left(b + \tfrac{1}{2} - \sqrt{v \cosh^2 \alpha + \tfrac{1}{4}},\; b + \tfrac{1}{2} + \sqrt{v \cosh^2 \alpha + \tfrac{1}{4}}\,\middle|\, a + b + 1 \,\middle|\, \frac{e^{-z}}{e^z + e^{-z}}\right) \tag{12.3.24}$$

where $z = (x/d) - \mu$, a and b are given in Eq. (12.3.23), and N is the normalizing factor.

According to Eq. (5.2.49), this solution has the following limiting behavior near $z = -\infty$:

$$\psi \to \frac{\Gamma(a + b + 1)\Gamma(b - a)e^{(a-b)z}}{\Gamma(b + \frac{1}{2} - \sqrt{v \cosh^2 \mu + \frac{1}{4}})\, \Gamma(b + \frac{1}{2} + \sqrt{v \cosh^2 \mu + \frac{1}{4}})} + \frac{\Gamma(a + b + 1)\Gamma(a - b)e^{(b-a)z}}{\Gamma(a + \frac{1}{2} - \sqrt{v \cosh^2 \mu + \frac{1}{4}})\, \Gamma(a + \frac{1}{2} + \sqrt{v \cosh^2 \mu + \frac{1}{4}})}$$

which goes to infinity whether $a > b$ or $b > a$, unless one of the terms is zero. From the symmetry of the terms in a and b and from the symmetry of the definitions of a and b [see Eq. (12.3.23)], we shall obtain all the cases if we choose $b > a$; choosing $a > b$ would simply duplicate the results. If $b > a$, in order that ψ be finite as $z \to -\infty$, we must have the argument of one of the gamma functions in the denominator of the second term a negative integer, so the second term will vanish. This means that the only allowed energies in the range $\epsilon < ve^{-2\mu}$ are those for which

$$b = b_n = \sqrt{v \cosh^2 \mu + \tfrac{1}{4}} - (n + \tfrac{1}{2}); \quad a_n = \frac{\tfrac{1}{2}v \sinh 2\mu}{\sqrt{v \cosh^2 \mu + \tfrac{1}{4}} - (n + \tfrac{1}{2})}$$

where n is zero or any positive integer for which $b_n > a_n$. In other words, n must be less than the quantity $\sqrt{v \cosh^2 \mu + \tfrac{1}{4}} - \tfrac{1}{2} - \sqrt{\tfrac{1}{2}v \sinh 2\mu}$.

We note that no discrete, bound levels are allowed if $\sqrt{\tfrac{1}{2}v \sinh 2\mu}$ is larger than $\sqrt{v \cosh^2 \mu + \tfrac{1}{4}} - \tfrac{1}{2}$; in other words, unless

$$v > e^{2\mu} \tanh \mu; \quad \mu > 0$$

there are no bound levels at all. If the potential energy becomes too "skewed" (μ too large), there is not enough minimum to "support" a bound state. Within these limits on v and μ, however, one or more states are possible, having wave functions which vanish at both $x = +\infty$ and $x = -\infty$, which correspond to the classical, bound states, where the particle oscillates with finite amplitude about the potential minimum with periodic (but not simple-harmonic) motion. In the classical case the frequency of oscillation would depend on the amplitude of motion, being $\sqrt{V_0/2\pi^2 md^2 \cosh^2 \mu}$ for small amplitudes, but diminishing for larger amplitudes. The sign of the non-simple-harmonic behavior in the wave-mechanical case is that the allowed energy levels are not equally spaced, as they are for a simple, parabolic potential.

Existence of Bound States. We have just shown that, within certain limits of μ and v, the allowed energy levels in the range $\epsilon < ve^{-2\mu}$ are the discrete values

$$\epsilon_n = v \cosh 2\mu - [\sqrt{v \cosh^2 \mu + \tfrac{1}{4}} - (n + \tfrac{1}{2})]^2 - \frac{v^2 \sinh^2 2\mu}{[\sqrt{v \cosh^2 \mu + \tfrac{1}{4}} - (n + \tfrac{1}{2})]^2} \tag{12.3.25}$$

where $n = 0, 1, 2, \ldots, < [\sqrt{v \cosh^2 \mu + \tfrac{1}{4}} - \tfrac{1}{2} - \sqrt{\tfrac{1}{2}v \sinh 2\mu}]$

The corresponding wave function is

$$\psi_n = \frac{Ne^{-a_n z}}{(e^z + e^{-z})^{b_n}} F\left(-n, 2\sqrt{v \cosh^2 \mu + \tfrac{1}{4}} - n \middle| a_n + b_n + 1 \middle| \frac{e^{-z}}{e^z + e^{-z}}\right) \tag{12.3.26}$$

where a_n and b_n are given in Eq. (12.3.23).

When v is small, there may only be a single bound level. For instance, for the case $\mu = 0$ (potential valley symmetric about $x = 0$) and for $v \ll \frac{1}{4}$, the only bound state is for $n = 0$. The allowed energy, to the second order in the small quantity v, is

$$\epsilon \simeq v - v^2; \quad E \simeq V_0 - (2md^2/\hbar^2)V_0^2$$

This level is only slightly below the asymptotic value V_0 of the potential energy, being "just barely bound."

One can show, in general, that, if V is zero from $-\infty$ to $+\infty$ except for a small range of x, where it has a very shallow "valley," there will always be at least one bound level. For if we consider the case for E just less than zero and greater than the minimum of V in the valley, then in the region where $E > V$ the wave function will curve downward and we can adjust things so that it will slope toward the axis at both ends of the range of x for which $E > V$. In the region where $V = 0$, E will be less than V, and here the solution will be $e^{-\alpha x}$ for $x \to \infty$ and $e^{\alpha x}$ for $x \to -\infty$, in order that ψ be finite, where $\alpha^2 = -(2mE/\hbar^2)$ $(E < 0)$. No matter how shallow the dip in V and no matter how small the related curve downward of ψ in this region, we can always make α small enough (by making $-E$ smaller, by making the state "just barely bound") so that the two exponential solutions for x large will join on smoothly.

Therefore, at least one bound level is always possible in this case, no matter how shallow the potential valley. The same conclusion is not valid, however, if the asymptotic value of V as $x \to \infty$ differs from its asymptotic value when $x \to -\infty$. For then we cannot adjust E so that the difference in slopes of the two exponential terms, representing ψ for $|x|$ large, is as small as possible. Both of these general conclusions are illustrated in the present example, where we see that when $\mu = 0$ and $V(+\infty) = V(-\infty)$, one bound level is always present, no matter how small v is, whereas when $\mu \neq 0$ and $V(+\infty) \neq V(-\infty)$ no bound level is present when v is smaller than $e^{2\mu} \tanh \mu$.

We should note, in passing, that these conclusions are valid only for one-dimensional problems. For two and three dimensions, when V is zero everywhere except for a small region near the origin, where V has a shallow minimum, it is possible to make this depression too small in extent or too shallow (or both) for it to have *any* bound level.

On the other hand, when $v \cosh^2 \mu$ is large, there are several bound levels. Near the potential minimum at $x = 0$, the potential energy is parabolic,

$$V(x) \simeq (x/d)^2 V_0 \operatorname{sech}^2 \mu + \cdots; \quad x \ll d$$

so that the classical frequency of small vibrations about equilibrium is $\omega/2\pi$ where $\omega = \sqrt{2V_0/md^2 \cosh^2 \mu}$. Consequently, the constant v is related to ω by the equation

$$v = (m^2d^4/\hbar^2)\omega^2 \cosh^2 \mu$$

When $v \cosh^2 \mu$ is considerably larger than $\frac{1}{4}$, Eq. (12.3.25) for the energy may be expanded for n small, yielding

$$E_n \simeq \hbar\omega(n + \tfrac{1}{2}) - \left(\frac{\hbar^2}{2md^2}\right)(1 + \tfrac{3}{2}\tanh^2 \mu)(n + \tfrac{1}{2})^2 + \cdots$$

for the lower levels. Thus the first few bound levels correspond to the harmonic oscillator formula (12.3.6) with a correction term corresponding to the difference between the present potential energy and the simple parabolic form.

Reflection and Transmission. When the energy ϵ is greater than $ve^{-2\mu}$ but less than $ve^{2\mu}$, both solutions are finite at $x \to -\infty$ but only one at $x = +\infty$. The function given in Eq. (12.3.24) is finite everywhere for every energy in the range $ve^{-2\mu} < \epsilon < ve^{2\mu}$. In this range the quantity $\kappa_+ = \sqrt{ve^{2\mu} - \epsilon}$ is real but the quantity $\sqrt{ve^{-2\mu} - \epsilon} = -ik_-$ is imaginary, k_-^2 being the asymptotic value of the particle's kinetic energy (in units of $\hbar^2/2md^2$) at $x = -\infty$. The correct wave function, for $ve^{2\mu} > \epsilon > ve^{-2\mu}$, is then

$$\psi = \frac{Ne^{-\frac{1}{2}(\kappa_+ + ik_-)z}}{(e^z + e^{-z})^{\frac{1}{2}\kappa_+ - \frac{1}{2}ik_-}} F\left(b + \tfrac{1}{2} - \gamma, b + \tfrac{1}{2} + \gamma \middle| \kappa_+ + 1 \middle| \frac{e^{-z}}{e^z + e^{-z}}\right) \tag{12.3.27}$$

where $b = \frac{1}{2}\kappa_+ - \frac{1}{2}ik_-$ and $\gamma = \sqrt{v \cosh^2 \mu + \frac{1}{4}}$. Using Eq. (5.2.49), we find the asymptotic behavior of ψ at $x \to -\infty$ to be

$$\psi \to N\Gamma(\kappa_+ + 1) \left\{ \frac{\Gamma(-ik_-)e^{ik_-z}}{\Gamma(\frac{1}{2}\kappa_+ + \frac{1}{2} - \gamma - \frac{1}{2}ik_-)\Gamma(\frac{1}{2}\kappa_+ + \frac{1}{2} + \gamma - \frac{1}{2}ik_-)} + \frac{\Gamma(ik_-)e^{-ik_-z}}{\Gamma(\frac{1}{2}\kappa_+ + \frac{1}{2} - \gamma + \frac{1}{2}ik_-)\Gamma(\frac{1}{2}\kappa_+ + \frac{1}{2} + \gamma + \frac{1}{2}ik_-)} \right\}$$

The first term corresponds to a steady stream of particles of momentum $p_- = \sqrt{2m(E - V_0e^{-2\mu})}$ traveling in the positive direction. As Eq. (12.3.3) indicates, this is represented wave mechanically by a simple-harmonic wave

$$\exp(ip_-x/\hbar) = \exp[i(z + \mu)\sqrt{\epsilon - ve^{-2\mu}}] = \exp[ik_-(z + \mu)]$$

the square of the magnitude of the coefficient of this exponential being proportional to the particle current density in the positive direction. The second term, of course, corresponds to a wave (and consequently a stream of particles) in the negative direction. Since the coefficient of this second term is the complex conjugate of the coefficient of the first term, the two currents are equal in magnitude and the net current is zero.

As with classical particles, the stream projected to the right cannot reach $+\infty$, for the potential energy there, $V_0e^{2\mu}$, is larger than the particle energy, so all of them are reflected back to $-\infty$.

When the energy ϵ is larger than $ve^{2\mu}$ (in units of $\hbar^2/2md^2$), then the quantity $\sqrt{ve^{2\mu} - \epsilon} = -ik_+$ is also imaginary, and all solutions of Eq. (12.3.22) are finite at both $x = \infty$ and $x = -\infty$. The solution of Eq. (12.3.24) in this range ($\epsilon > ve^{2\mu}$),

$$\begin{aligned}
\psi &= Ne^{\frac{1}{2}i(k_+-k_-)z}(e^z + e^{-z})^{\frac{1}{2}i(k_++k_-)} \cdot \\
&\cdot F\left(-\tfrac{1}{2}ik_+ - \tfrac{1}{2}ik_- + \tfrac{1}{2} - \gamma,\ -\tfrac{1}{2}ik_+ - \tfrac{1}{2}ik_- + \tfrac{1}{2} + \gamma \middle| 1 - ik_+ \middle| \frac{e^{-z}}{e^z + e^{-z}}\right) \\
&\to Ne^{ik_+z};\quad z \to +\infty \qquad\qquad (12.3.28)\\
&\to N\Gamma(1 - ik_+)\left\{\frac{\Gamma(-ik_-)e^{ik_-z}}{\Gamma(\frac{1}{2} + \gamma - \frac{1}{2}ik_+ - \frac{1}{2}ik_-)\Gamma(\frac{1}{2} - \gamma - \frac{1}{2}ik_+ - \frac{1}{2}ik_-)}\right. \\
&\qquad \left. + \frac{\Gamma(ik_-)e^{-ik_-z}}{\Gamma(\frac{1}{2} + \gamma - \frac{1}{2}ik_+ + \frac{1}{2}ik_-)\Gamma(\frac{1}{2} - \gamma - \frac{1}{2}ik_+ + \frac{1}{2}ik_-)}\right\};\quad z \to -\infty
\end{aligned}$$

where $\gamma = \sqrt{v\cosh^2\mu + \frac{1}{4}}$ and $k_- = \sqrt{\epsilon - ve^{-2\mu}}$, represents a stream of particles moving, some to the right and some to the left at $z \to -\infty$, but only to the right at $z \to +\infty$. This is not in accord with classical mechanics, for if a stream of particles were started at $-\infty$ with energy greater than the potential energy everywhere in the range $-\infty < x < +\infty$, then all the particles would continue to move to the right and none would be turned back. It is in accord, however, with wave motion through a medium of refractive index proportional to $(E - V)$; if the index changes with x in the range $-\infty < x < +\infty$, then part of the wave, which starts to the right at $-\infty$ will be reflected back to $-\infty$ and only part of the wave will be transmitted on to $+\infty$.

The relative magnitudes of the transmitted and reflected waves are obtained by using the formula $\Gamma(z)\Gamma(1 - z) = (\pi/\sin\pi z)$. For example in the square of the magnitude of the e^{ik_-z} term is the factor $\Gamma(1 - ik_+)\Gamma(1 + ik_+) = (\pi k_+/\sinh\pi k_+)$ and also the factor $\Gamma(ik_-)\Gamma(-ik_-) = (\pi/k_-\sinh\pi k_-)$. Dealing similarly with the gamma functions in the denominator and remembering that the current density is proportional to k times the square of the amplitude of the exponential, we find that the incident current (to the right at $x = -\infty$), reflected current (to the left at $x = -\infty$), and transmitted current (to the right at $x = +\infty$) are proportional to I_i, I_r, and I_t, respectively, where

$$\begin{aligned}
I_i &= \tfrac{1}{2}k_+|N|^2\,\frac{\cosh[\pi(k_+ + k_-)] + \cos(2\pi\gamma)}{\sinh(\pi k_+)\sinh(\pi k_-)} \\
I_r &= \tfrac{1}{2}k_+|N|^2\,\frac{\cosh[\pi(k_- - k_+)] + \cos(2\pi\gamma)}{\sinh(\pi k_+)\sinh(\pi k_-)} \\
I_t &= k_+|N|^2
\end{aligned}$$

It is not difficult to see that $I_i - I_r = I_t$ as indeed it must to satisfy the equation of continuity.

If, then, a unit current density is incident from $x = -\infty$, the reflected and transmitted current will be R and T, where

$$\begin{aligned} R &= \frac{\cosh[\pi(k_- - k_+)] + \cos(2\pi\gamma)}{\cosh[\pi(k_- + k_+)] + \cos(2\pi\gamma)} \\ T &= \frac{2\sinh(\pi k_+)\sinh(\pi k_-)}{\cosh[\pi(k_+ + k_-)] + \cos(2\pi\gamma)} = 1 - R \end{aligned} \tag{12.3.29}$$

where $\gamma = \sqrt{v\cosh^2\mu + \frac{1}{4}}$, $k_+ = \sqrt{\epsilon - ve^{2\mu}}$, and $k_- = \sqrt{\epsilon - ve^{-2\mu}}$. As the energy ϵ approaches the asymptotic potential $ve^{2\mu}$, k_+ approaches zero, T approaches zero, and R approaches unity. As v approaches zero, so that the potential becomes independent of x, R approaches zero and T approaches unity (for positive ϵ).

Another interesting phenomenon may be best displayed by considering the symmetric case, $\mu = 0$. Here the allowed bound levels are

$$\epsilon_n = v - [\sqrt{v + \tfrac{1}{4}} - (n + \tfrac{1}{2})]^2; \quad n = 0, 1, \ldots, < \sqrt{v + \tfrac{1}{4}} - \tfrac{1}{2}$$

and the "reflection coefficient" of the potential valley for incident particles of energy $\epsilon > 0$ is

$$R = \frac{1 + \cos(2\pi\sqrt{v + \frac{1}{4}})}{\cosh(2\pi\sqrt{\epsilon - v}) + \cos(2\pi\sqrt{v + \frac{1}{4}})} = 1 - T$$

We note that this coefficient varies periodically with $\sqrt{v + \frac{1}{4}}$, being maximum when $\sqrt{v + \frac{1}{4}}$ is an integer and being zero when $\sqrt{v + \frac{1}{4}} = \frac{1}{2}, \frac{3}{2}, \ldots, N + \frac{1}{2}, \ldots$. In this latter case the wavelets reflected from various parts of the potential valley interfere to destroy the reflected wave and allow all of it to be transmitted. This is completely analogous to the case of the quarter-wave film put on lens surfaces to minimize the light reflected from these surfaces.

But, as with all quantum phenomena, there is also a particle "explanation" of the phenomenon. When $\sqrt{v + \frac{1}{4}}$ is just a little larger than $N + \frac{1}{2}$, then there is a bound level ϵ_N just below the top of the valley wall and, as v is decreased through $(N + \frac{1}{2})^2 - \frac{1}{4}$ to a value slightly less than this, this top bound level is "squeezed out" into the continuum of free levels above $\epsilon = v$. When v is nearly equal to $(N + \frac{1}{2})^2 - \frac{1}{4}$, it turns out that the wave function (for ϵ just larger than v) is very much larger in the potential valley than it is at $|x| = \infty$, whereas for $v \simeq N^2 - \frac{1}{4}$, its amplitude in the valley is not much different from its amplitude as $|x| \to \infty$. A slow particle coming in to the valley comes in at an energy very close to an allowed bound energy; even though it is in a free state, it stays a relatively long time in the valley and evidently "forgets" to be reflected. Even when v is slightly less than $(N + \frac{1}{2})^2 - \frac{1}{4}$ and the bound level has been "squeezed out," it seems to exist as a "virtual level" which traps slow incoming particles for a time and sends few of them back to $-\infty$.

This effect of "virtual levels" on the reflecting power of potential minima will also be encountered in two- and three-dimensional problems.

We have now discussed the behavior of the wave function, for the potential given in Eq. (12.3.22), for all values of the energy. We have seen that no states are allowed for energy less than the minimum potential energy, which is in accord with classical dynamics. For energies between the potential minimum and the lowest asymptotic value V_-, the allowed states are bound, the wave function vanishing at $x = \pm\infty$, which accords with classical results; but only a discrete set of energies in

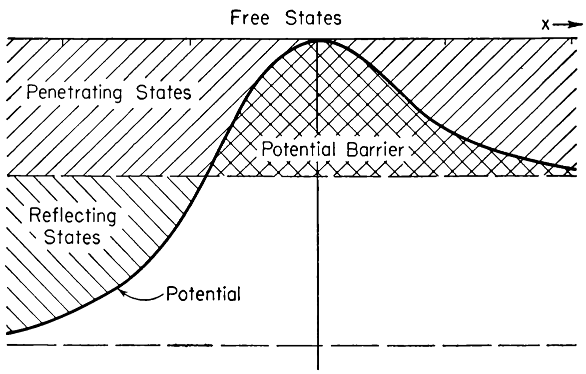

Fig. 12.3 Potential function allowing penetration of waves through potential barrier.

this range is allowed, which differs from the classical case, where all energies above the minimum would be allowed. For particle energies greater than V_- but smaller than the asymptotic value V_+ of the potential at $x = +\infty$, all energies are allowed, the wave function vanishes at $x = +\infty$ but extends out to $-\infty$, where it corresponds to an incident wave (in the positive x direction) and a reflected wave of equal amplitude, which is in accord with the classical picture of a stream of particles being reflected from the side of the potential plateau. Finally, for particle energies larger than V_+, all energies are allowed, in accord with the classical case, but here reflection of the wave occurs from the valley sides in contradiction to a classical particle, which would never be reflected if it had this much energy. The useful wave functions in this range are the ones given in Eq. (12.3.28), corresponding to an incident beam from $-\infty$, a transmitted beam to $+\infty$, and a reflected beam back to $-\infty$, and the converse one corresponding to an incident beam from $+\infty$, a transmitted one to $-\infty$, and a reflected beam back to $+\infty$. Both of these wave functions are finite over the whole range $-\infty < x < +\infty$ but do not tend to zero at either limit.

Penetration through a Potential Barrier. The same potential energy may be turned upside down by reversing the signs of V_0 and μ to give a field with asymptotic value $-ve^{2\mu}$ (in units of $\hbar^2/2md^2$) at $x \to -\infty$, of value zero at $x = 0$, and of asymptotic value $-ve^{-2\mu}$ at $x \to \infty$. Classically, particles with energy larger than the top of the mountain ($\epsilon > 0$), would all pass over the peak, going from $-\infty$ to $+\infty$, particles coming from $-\infty$ with energy less than the top value would all be reflected back to $-\infty$. From our previous discussion we would expect, in quantum mechanics, a certain amount of reflection when $\epsilon > 0$; it will be of interest to see whether we get transmission through the potential peak for particles with energy less than zero.

The equation for the wave function, for the potential energy function $-v \cosh^2 \mu(\tanh z - \tanh \mu)^2$ and for $z = (x/d) + \mu$, is

$$(d^2\psi/dz^2) + [\epsilon + v \cosh 2\mu - v \sinh 2\mu \tanh z - v \cosh^2 \mu \operatorname{sech}^2 z]\psi = 0$$

which is to be compared with Eq. (12.3.22). By methods similar to those used heretofore, we obtain the solution, for $\epsilon > -ve^{-2\mu}$, which represents a transmitted beam at $x = +\infty$ and an incident and reflected beam at $x = -\infty$;

$$\begin{aligned}
\psi &= Ne^{\frac{1}{2}i(k_++k_-)z}(e^z + e^{-z})^{\frac{1}{2}i(k_+-k_-)z} \cdot \\
&\quad \cdot F\left(\tfrac{1}{2} + i\frac{k_- - k_+ + \beta}{2}, \tfrac{1}{2} + i\frac{k_- - k_+ - \beta}{2} \Big| 1 - ik_+ \Big| \frac{e^{-z}}{e^z + e^{-z}}\right) \\
&\to Ne^{ik_+z}; \quad z \to \infty \\
&\to \frac{N\Gamma(1 - ik_+)\Gamma(-ik_-)e^{ik_-z}}{\Gamma\left(\frac{1}{2} - i\frac{k_+ + k_- + \beta}{2}\right)\Gamma\left(\frac{1}{2} - i\frac{k_+ + k_- - \beta}{2}\right)} \\
&\qquad + \frac{N\Gamma(1 - ik_+)\Gamma(ik_-)e^{-ik_-z}}{\Gamma\left(\frac{1}{2} + i\frac{k_- - k_+ + \beta}{2}\right)\Gamma\left(\frac{1}{2} + i\frac{k_- - k_+ - \beta}{2}\right)}; \quad z \to -\infty
\end{aligned} \tag{12.3.30}$$

where $\beta = \sqrt{4v \cosh^2 \mu - 1}$, $k_+ = \sqrt{\epsilon + ve^{-2\mu}}$, and $k_- = \sqrt{\epsilon + ve^{2\mu}}$. The quantity k_+^2 is the kinetic energy of the particle when it is at $x = +\infty$, k_-^2 is its kinetic energy at $x = -\infty$. Both of these are positive (the k's are real) as long as ϵ is larger than $-ve^{-2\mu}$. This solution is finite everywhere as long as the k's are real.

Manipulating the gamma functions as before, we see that, if the solution is normalized to represent an incident beam of particles of unit intensity, coming from $-\infty$, then the current T, transmitted to $+\infty$ and the current R, reflected back to $-\infty$, for all values of ϵ for which k_+ and k_- are real, are

$$T = \frac{2\sinh(\pi k_+)\sinh(\pi k_-)}{\cosh[\pi(k_- + k_+)] + \cosh(\pi\beta)};$$
$$R = \frac{\cosh[\pi(k_- - k_+)] + \cosh(\pi\beta)}{\cosh[\pi(k_- + k_+)] + \cosh(\pi\beta)} = 1 - T \qquad (12.3.31)$$

These formulas show that for every value of ϵ larger than $-ve^{-2\mu}$ there is some current transmitted to $+\infty$ and some reflected back to $-\infty$. This is in contrast to the results of classical mechanics, where there would be no reflection for $\epsilon > 0$ and no transmission for $\epsilon < 0$, when the particle would have to "penetrate" the potential barrier at $x = 0$ to get to $x = \infty$. In the limiting cases, the quantum mechanical results approach the classical ones, for when $\epsilon \gg v$, k_+ and k_- are large and nearly equal, in which case T is nearly unity and R is nearly zero. On the other hand when v and/or μ is large and ϵ is less than zero (less than the top of the peak) then k_+, though real, is considerably smaller than k_- or β; in which case T is small and R is nearly unity. The higher and "thicker" the potential hill is, the less is transmitted through it. Nevertheless the results show that a potential barrier of finite height and width does not prevent a small fraction of the particles hitting it from penetrating and appearing on the other side! This behavior is essential in explaining the facts of radioactivity, for example. A plot of the transmitted fraction, T, as function of the particle energy ϵ, is given in Fig. 12.4 to show the general results.

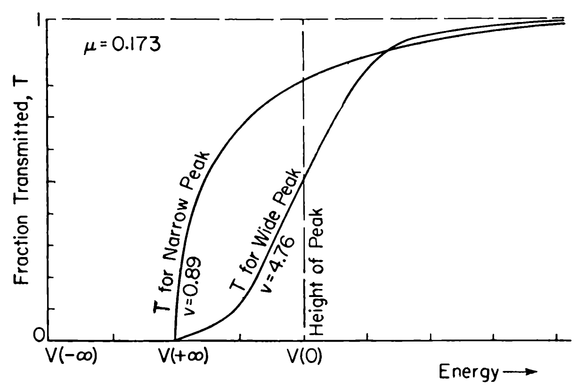

Fig. 12.4 Transmission of particles through potential barrier of Fig. 12.3.

For $\epsilon < -ve^{-2\mu}$ the quantity k_+ is imaginary, and we can call it $i\kappa_+$. In this case ψ goes to zero at $x = +\infty$, and the coefficients of incident and reflected waves at $x = -\infty$ have the same magnitude, indicating perfect reflection. The potential barrier, for this range of energy, extends to $+\infty$, so the reflection is complete. Finally, for $\epsilon < -ve^{2\mu}$, no solution can be found which is finite everywhere, so these energies are not allowed.

Central Force Fields, Angular Momentum. When a particle of mass M is in a potential field $V(r)$ which is centrally symmetric about some point (which may be taken to be the origin) the resulting Schroedinger equation for constant energy E is

$$\nabla^2\psi + [\epsilon - v(r)]\psi = 0$$

where $\epsilon = 2ME/\hbar^2$ and $v = 2MV/\hbar^2$, may be separated in spherical coordinates r, ϑ, φ. The angle dependence factor is the usual spherical harmonic

$$\begin{gathered} \psi = (1/r)X_l^m(\vartheta,\varphi)R(r); \quad X_l^m = e^{im\varphi}P_l^m(\cos\vartheta) \\ m = -l, -l+1, \ldots, l-1, l; \quad l = 0, 1, 2, \ldots \end{gathered} \tag{12.3.32}$$

where the eigenfunctions P_l^m are tabulated in the table at the end of Chap. 10 and the equation for the radial factor is

$$\frac{d^2R}{dr^2} + \left[\epsilon - \frac{l(l+1)}{r^2} - v(r)\right]R = 0 \tag{12.3.33}$$

where, for ψ to be finite everywhere, R must go to zero as $r \to 0$ and must stay finite as $r \to \infty$.

We see that the energy ϵ cannot depend on the quantum number m, since it does not come into the equation for R and ϵ. This degeneracy corresponds to the symmetry of the potential field, the various combinations of wave functions for different m corresponding to different orientations of the wave with respect to the polar axis [see, for example, the discussion of Eq. (10.3.38)]. The classical manifestation of this same symmetry is, of course, the constancy of the angular momentum of the particle about the center of force and the constancy of the direction of the angular momentum. This constancy can also be translated into quantum language.

The angular momentum $\mathbf{r} \times \mathbf{p}$ of the particle about the origin corresponds to an operator [see Eqs. (1.6.90) and (2.6.17) and page 1464].

$$\begin{aligned} \mathfrak{M} &= \frac{\hbar}{i}\mathbf{r} \times \text{grad} = \frac{\hbar}{i}\left[\mathbf{a}_\varphi \frac{\partial}{\partial\vartheta} - \frac{\mathbf{a}_\vartheta}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right] \\ &= \frac{\hbar}{i}\left[\mathbf{a}_z\frac{\partial}{\partial\varphi} - \mathbf{a}_x\left(\sin\varphi\frac{\partial}{\partial\vartheta} + \cos\varphi\cot\vartheta\frac{\partial}{\partial\varphi}\right)\right. \\ &\qquad\qquad \left. + \mathbf{a}_y\left(\cos\varphi\frac{\partial}{\partial\vartheta} - \sin\varphi\cot\vartheta\frac{\partial}{\partial\varphi}\right)\right] \end{aligned}$$

The component along the polar axis, $\mathfrak{M}_z = (\hbar/i)(\partial/\partial\varphi)$, is one for which ψ is an eigenfunction, for

$$\mathfrak{M}_z X_l^m = m\hbar X_l^m$$

if the φ factor of ψ is $e^{im\varphi}$. The wave function is not an eigenfunction for

$\mathfrak{M}_x$ or $\mathfrak{M}_y$, for neither of these operators produces a constant times ψ when they operate on ψ. As we have shown in Eq. (1.6.41) and can demonstrate by using the properties of the spherical harmonics, the operators

$$(\mathfrak{M}_x + i\mathfrak{M}_y)X_l^m = \hbar X_l^{m+1}; \quad (\mathfrak{M}_x - i\mathfrak{M}_y)X_l^m = \hbar(l+m)(l-m+1)X_l^{m-1} \tag{12.3.34}$$

raise or lower the quantum number m, changing the wave function by their operation.

Finally, the operator corresponding to the square of the total angular momentum is another operator for which ψ is an eigenfunction:

$$\begin{aligned}(\mathfrak{M}_x^2 + \mathfrak{M}_y^2 + \mathfrak{M}_z^2)X_l^m &= -\hbar^2\left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\,\frac{\partial}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\right]X_l^m \\ &= [\mathfrak{M}_z^2 + \tfrac{1}{2}(\mathfrak{M}_x + i\mathfrak{M}_y)(\mathfrak{M}_x - i\mathfrak{M}_y) + \tfrac{1}{2}(\mathfrak{M}_x - i\mathfrak{M}_y)(\mathfrak{M}_x + i\mathfrak{M}_y)]X_l^m \\ &= \hbar^2 l(l+1)X_l^m\end{aligned}$$

Consequently, we can say that a choice of spherical coordinates in terms of which to express the wave function, and a choice of X_l^m for the angle dependence corresponds to a decision to have our wave functions be eigenfunctions for the total angular momentum and for the component of the angular momentum along the polar axis, in addition to being eigenfunctions for the total energy operator. We could, of course, use other combinations of spherical harmonics, such as the ones

$$Y_{ml}^e = \cos(m\varphi)P_l^m \quad \text{or} \quad Y_{ml}^0 = \sin(m\varphi)P_l^m$$

but these would not be eigenfunctions for the z component of $\mathfrak{M}$, so we prefer the complex X functions.

Central Force Fields, the Radial Equation. Since a great many quantum mechanical problems involve a central force field or a field which is not very different from a central field, it is useful to work out some of the cases for which we can obtain an exact solution, so that we can use them as a basis for perturbation calculations for other problems. One potential for which an exact solution may be found is the harmonic oscillator, $V = \frac{1}{2}M\omega^2r^2$, where $\omega/2\pi$ is the classical frequency of oscillation. The corresponding equation for R turns out to be proportional to a confluent hypergeometric function of r^2, which is infinite at $r \to \infty$ unless one of the indices is a negative integer, in which case the solution is one of the *Laguerre polynomials* tabulated at the end of Chap. 6.

$$R(r) = Nr^{l+1}e^{-\frac{1}{2}\beta r^2}L_k^{l+\frac{1}{2}}(\beta r^2); \quad k = 0, 1, 2, \ldots$$

where β, as before, equals $M\omega/\hbar$ and the Laguerre polynomial $L_n^a(z)$ has the following properties:

$$\frac{d^2}{dz^2} L_n^a + \left(\frac{a+1}{z} - 1\right)\frac{d}{dz} L_n^a + \left(\frac{n}{z}\right) L_n^a = 0 \qquad (12.3.35)$$

$$L_n^a(z) = \frac{[\Gamma(n+a+1)]^2}{n!\Gamma(a+1)} F(-n|a+1|z) \to \frac{\Gamma(n+a+1)}{n!}(-z)^n; \quad z \to \infty$$

$$\int_0^\infty z^a e^{-z} L_m^a(z) L_n^a(z)\, dz = \delta_{mn} \frac{[\Gamma(n+a+1)]^3}{n!}$$

$$\int_0^\infty z^{a+1} e^{-z} L_n^a(z) L_n^a(z)\, dz = [\Gamma(n+a+1)]^3 \left[\frac{2n+a+1}{n!}\right]$$

The only values of n for which the confluent hypergeometric series terminates with a finite number of terms, and for which R is finite at $r \to \infty$ is for $n = 0, 1, 2, \ldots$. This limits the allowed values of the energy $E = (\hbar^2/2M)\epsilon$ to the following discrete values

$$E_n = \hbar\omega(2k + l + \tfrac{3}{2}) = \hbar\omega(n + \tfrac{1}{2}); \quad n = 2k + l + 1 = 1, 2, 3, \ldots \qquad (12.3.36)$$

The spacing of these levels is $\hbar\omega$, as with the linear harmonic oscillator given in Eq. (12.3.6), but in the present case the lowest level is $\frac{3}{2}\hbar\omega$ above the potential minimum, whereas in the one-dimensional case the lowest level is $\frac{1}{2}\hbar\omega$ above the minimum (for a two-dimensional harmonic oscillator it would be $\hbar\omega$ above the minimum).

The corresponding, normalized eigenfunction for the harmonic potential is

$$\psi_{mln}(r,\vartheta,\varphi) = \sqrt{\beta^3 \frac{2l+1}{4\pi(2n+1)} \frac{(l-m)!}{(l+m)!} \frac{(\frac{1}{2}n - \frac{1}{2}l - \frac{1}{2})!}{[\Gamma(\frac{1}{2}n + \frac{1}{2}l + 1)]^3}}\, e^{im\varphi} P_l^m(\cos\vartheta) \cdot$$
$$\cdot (\beta r^2)^{\frac{1}{2}l} e^{-\frac{1}{2}\beta r^2} L^{l+\frac{1}{2}}_{\frac{1}{2}n-\frac{1}{2}l-\frac{1}{2}}(\beta r^2) \qquad (12.3.37)$$
$$l = n-1, n-3, n-5, \ldots, 1 \text{ or } 0$$
$$m = 0, 1, 2, \ldots, l; \quad \beta = M\omega/\hbar$$

or its complex conjugate. We see that each level is degenerate, there being $2l + 1$ different values of m, each with the same energy for each value of l. In addition, the set of states for a given value of n, for different allowed values of l (all odd values if n is even, all even if n is odd) have the same energy. If n is odd, there are $\frac{1}{2}n - \frac{1}{2}$ different values of l; if n is even, there are $\frac{1}{2}n$. Consequently, the number of states having energy $\hbar\omega(n + \frac{1}{2})$ is 1 when $n = 1$, 3 when $n = 2$, 6 when $n = 3$, 10 when $n = 4$, and so on, the order of the degeneracy equaling $\frac{1}{2}n(n + 1)$ in general.

Coulomb Potential. Another potential for which an exact solution may be found is the coulomb potential $V = -(\eta^2/r)$. As indicated in Eq. (5.2.55) and preceding, the equation for the radial factor R for this potential is

$$\frac{d^2R}{dr^2} + \left[-\kappa^2 + \frac{2M\eta^2}{\hbar^2 r} - \frac{l(l+1)}{r^2}\right] R = 0; \quad \kappa^2 = -\epsilon$$

On the other hand the equation for the function $z^b e^{-\frac{1}{2}z} F(-c|a+1|z) = f(z)$ is

$$\frac{d^2f}{dz^2} + \left[\frac{a+1-2b}{z}\right]\frac{df}{dz} + \left[-\tfrac{1}{4} + \frac{2c+a+1}{2z} + \frac{b(b-a)}{z^2}\right]f = 0 \tag{12.3.38}$$

A correspondence between the two equations is obtained by setting $z = 2\kappa r$, $a = 2l+1$, $b = l+1$, and $\kappa = M\eta^2/\hbar^2(c+l+1)$. Consequently, the solution of the radial equation which is finite at $r = 0$ is

$$\begin{aligned} R &= N(2\kappa r)^{l+1}e^{-\kappa r}F\left(l+1-\frac{M\eta^2}{\hbar^2\kappa}\,|2l+2|2\kappa r\right) \\ &\to N(2l+1)!\left\{\frac{\exp[\kappa r - (M\eta^2/\hbar^2\kappa)\ln(2\kappa r)]}{\Gamma[l+1-(M\eta^2/\hbar^2\kappa)]}\right. \\ &\qquad \left. -(-1)^l\frac{\exp[-\kappa r + (M\eta^2/\hbar^2\kappa)\ln(2\kappa r) + i\pi]}{\Gamma[l+1+(M\eta^2/\hbar^2\kappa)]}\right\}; \quad r\to\infty \end{aligned}$$

This function is finite at $r \to \infty$ for all positive values of the particle energy ($\kappa = ik$, $k^2 = \epsilon$), but for negative energies (κ real) it is finite only when the first term is zero, *i.e.*, when $M\eta^2/\hbar^2\kappa$ is an integer equal to or greater than $l+1$. In other words the allowed negative energies are those for which

$$\epsilon_n = -(M\eta^2/\hbar^2 n)^2; \quad E_n = -\frac{M\eta^4}{2n^2\hbar^2}; \quad n = l+1, l+2, \ldots \tag{12.3.39}$$

These values are, of course, the hydrogen-atomic, bound levels, giving the Balmer series of energy differences. The normalized wave function, corresponding to the quantum numbers n, l, m, for the coulomb potential, is then

$$\begin{aligned} \psi_{mln}(r,\vartheta,\varphi) &= \sqrt{\left(\frac{2M\eta^2}{\hbar^2 n}\right)^3 \frac{2l+1}{8\pi n}\frac{(l-m)!}{(l+m)!}\frac{(n-l-1)!}{[(n+l)!]^3}}\,\cdot \\ &\quad \cdot e^{im\varphi}P_l^m(\cos\vartheta)\left(\frac{2M\eta^2}{\hbar^2 n}r\right)^l e^{-(M\eta^2/\hbar^2 n)r}L_{n-l-1}^{2l+1}\left(\frac{2M\eta^2}{\hbar^2 n}r\right) \end{aligned} \tag{12.3.40}$$

and its complex conjugate.

This system is also multiply degenerate, all states for the same value of n having the same energy. Since there are $2l+1$ different values of m for each value of l and since all values of l from zero to $n-1$ are allowed for each value of n, there are n^2 different states which have the allowed energy E_n. The state for $l = 0$ is called the *s state* (1*s* state for $n = 1$, 2*s* for $n = 2$, etc.), the three states (for a given n) for $l = 1$ are *p states*, the five for $l = 2$ are *d states*, and so on, the sequence traditionally used in atomic problems being *s, p, d, f, g, h*, . . . , with a number preceding the letter to indicate the value of n. The value of m, as was pointed out before, simply indicates the orientation of the angular momentum in space and is of interest in the case where the electron moves in a uniform magnetic field as well as in the electric field of the nucleus.

When E is positive, the wave function, for a given value of m and l, which is finite everywhere is

$$\psi(k,l,m|r,\vartheta,\varphi) = \frac{e^{(\pi M\eta^2/2\hbar^2 k)}}{(2l+1)!}\left|\Gamma\left(l+1+\frac{iM\eta^2}{\hbar^2 k}\right)\right|(2kr)^l e^{ikr}\cdot$$
$$\cdot e^{im\varphi}P_l^m(\cos\vartheta)F\left(l+1+\frac{iM\eta^2}{\hbar^2 k}\,|2l+2|2ikr\right) \quad (12.3.41)$$
$$\rightarrow\left(\frac{1}{kr}\right)\sin\left[kr+\left(\frac{M\eta^2}{\hbar^2 k}\right)\ln(2kr)-\tfrac{1}{2}\pi l-\Omega_{kl}\right];\quad kr\rightarrow\infty$$

where $k^2 = \epsilon = 2ME/\hbar^2$ and

$$\Gamma\left(l+1+\frac{iM\eta^2}{\hbar^2 k}\right) = \left|\Gamma\left(l+1+\frac{iM\eta^2}{\hbar^2 k}\right)\right| e^{i\Omega_{kl}}$$
$$\left|\Gamma\left(l+1+\frac{iM\eta^2}{\hbar^2 k}\right)\right|^2 = \left[l^2+\left(\frac{M\eta^2}{\hbar^2 k}\right)^2\right]\left[(l-1)^2+\left(\frac{M\eta^2}{\hbar^2 k}\right)^2\right]\cdots$$
$$\cdots\left[1+\left(\frac{M\eta^2}{\hbar^2 k}\right)^2\right]\frac{(\pi M\eta^2/\hbar^2 k)}{\sinh(\pi M\eta^2/\hbar^2 k)}$$

The fact that the coulomb potential η^2/r still has its effect on the particle even at very large values of r is evidenced by the logarithmic term in the argument of the sine in the asymptotic form for ψ. Not even at very large distances does the wavelength settle down to being just equal to $2\pi/k$.

Inverse Cube Force. In classical dynamics, the inverse cube and inverse fifth-power central forces ($V \propto 1/r^2$ or $1/r^4$) also have solutions representable in elementary functions. This is not true in quantum mechanics. For example, the radial equation for the attractive inverse cube force is

$$\frac{d^2R}{dr^2}+\left[\epsilon+\frac{\gamma^2-l(l+1)}{r^2}\right]R = 0$$

where $l(l+1)$ must be less than the force constant γ^2 in order that the particle be bound when it has the angular momentum corresponding to quantum number l. We see that the cases for $l > 0$ are the same as that for $l = 0$ if we substitute $\gamma^2 - l(l+1)$ for γ^2, so we need only consider the $l = 0$ case. The equation for a possible bound state ($\epsilon = -\kappa^2$),

$$\frac{d^2R}{dr^2}+\left[-\kappa^2+\frac{\gamma^2}{r^2}\right]R = 0$$

has, for a solution which is finite at $r\rightarrow\infty$, the spherical Hankel function of imaginary argument and complex order

$$\frac{1}{r}R = h_{ip-\frac{1}{2}}(i\kappa r) = \sqrt{\frac{\pi}{2i\kappa r}}\left[\frac{e^{\pi p}}{\sinh(\pi p)}J_{ip}(i\kappa r) - \frac{1}{\sinh(\pi p)}J_{-ip}(i\kappa r)\right]$$
$$\to \frac{1}{i\kappa r}e^{-\kappa r - \frac{1}{2}i\pi(ip+\frac{1}{2})};\quad \kappa r \to \infty$$
$$\to \sqrt{\frac{2i\pi}{\kappa r}}\frac{e^{\frac{1}{2}\pi p}}{|\Gamma(1+ip)|\sinh(\pi p)}\sin[p\ln(\tfrac{1}{2}\kappa r) - \Phi_p];\quad \kappa r \to 0$$

where $p = \sqrt{\gamma^2 - \frac{1}{4}}$ and $\Gamma(1 + ip) = |\Gamma(1 + ip)|e^{i\Phi_p}$. We note that, for $r \to 0$ when p is real ($\gamma > \frac{1}{2}$), the function does not go to zero or infinity, but oscillates with ever increasing frequency, according to the sine of the logarithm of r.

When the potential strength is not large ($\gamma < \frac{1}{2}$) though it is attractive (γ real), then p is imaginary and the order of the Bessel functions is real. When this is true, no wave function can be found, for *any* negative energy, which is finite at both $r = 0$ and $r = \infty$. The Hankel function, given above (for ip real) goes to zero at $r \to \infty$ but is infinite at $r = 0$; the Bessel function $J_{ip}(i\kappa r)$ is zero at $r \to 0$ but is infinite at $r \to \infty$. Hence for a weak attractive inverse cube force there is no bound level, though there are finite wave functions for positive energies. The Hankel function, though it goes to infinity, is quadratically integrable at $r \to 0$; thus if the boundary condition is integrability, not finiteness, *all* negative energies would be allowed.

For $\gamma > \frac{1}{2}$ the order of the Bessel functions, ip, is imaginary, and the function written down is finite over the whole range of r, *no matter what negative value of energy we choose.* This seems to be quite a confusing result. Perhaps the singularity in the force field is great enough to destroy the discreteness of bound states. Another point of view, however, is that the requirements for stationary states may not be just finiteness (or integrability). Let us take the orthogonality integral of two of the functions given above for two different values of κ. Multiplying the equation for one by the other and subtracting the reverse combination and integrating over r we obtain [as with Eqs. (6.3.16) *et seq.*]

$$(\kappa_1^2 - \kappa_2^2)\int_0^\infty R_1R_2\,dr = \left[R_2\frac{dR_1}{dr} - R_1\frac{dR_2}{dr}\right]_0$$
$$= \frac{2i\pi p}{\sqrt{\kappa_1\kappa_2}}\left[\frac{e^{\pi p}}{|\Gamma(1+ip)|^2\sinh^2(\pi p)}\right]\sin\left[p\ln\left(\frac{\kappa_2}{\kappa_1}\right)\right]$$

Consequently, the two functions are not orthogonal unless $p\ln(\kappa_2/\kappa_1)$ is π times an integer, positive or negative.

Thus a requirement that the state functions for bound states, for $\gamma > \frac{1}{2}$, be a mutually orthogonal set imposes a quantization of energy. But it is a strange quantization, for it does not uniquely fix the levels, it just fixes levels relative to each other. If we say the energy $E_0 = -\kappa_0^2$ is allowed, then the wave functions for the following infinite set of energies $E_n = -\kappa_n^2$ of bound states are all mutually orthogonal;

$$E_n = -\kappa_0^2 e^{2\pi n/p}; \quad n = \ldots, -2, -1, 0, 1, 2, 3, \ldots$$

If these are the "allowed" bound levels, they extend to $-\infty$ and have an accumulation point at zero energy.

For positive energies (κ imaginary) all solutions are finite for $\gamma > \frac{1}{2}$, so there is *no* required phase angle for the radial solution as $r \to \infty$ and, consequently, no scattering *unless* we require that the solutions be mutually orthogonal, which will fix their phase at $r = 0$ and, correspondingly, their phase at $r \to \infty$.

Central fields, with even greater concentration than $1/r^2$ at the origin, may be solved, formally. But the difficulties attending the inverse cube case, just discussed, appear in an exaggerated form in these cases. The radial equation has an irregular singular point at $r = 0$ as well as at $r \to \infty$, so that asymptotic series must be used at both ends of the range of r. As with the $1/r^2$ case, both solutions for small r are integrable but oscillate with ever-increasing frequency as $r \to 0$. The solutions for large r have the asymptotic form $e^{-\kappa r}$ or $e^{+\kappa r}$, so one solution is finite (or integrable) over the whole range of r for any value of κ. Unless we apply the phase condition for orthogonality, mentioned above, there is no quantization of the bound states.

But such potential fields are of little interest in physics, if any. We should pass on to other, more useful, aspects.

Coulomb Field in Parabolic Coordinates. Both the soluble central force problems produce degenerate bound levels, a number of states having the same allowed energy. A related property of the solution is that the Schroedinger equation may be separated in several different coordinate systems. The equation for the harmonic potential $V = \frac{1}{2}M\omega^2 r^2 = \frac{1}{2}M\omega^2(x^2 + y^2 + z^2)$ may be separated in rectangular coordinates x, y, z and circular cylinder coordinates ρ, θ, z as well as spherical coordinates r, ϑ, φ. The equation for the coulomb potential $V = \eta^2/r$ may be separated in parabolic and in prolate spheroidal coordinates as well as in spherical coordinates. Solutions in these other coordinates correspond to the same set of allowed energies; in fact the new solutions must be linear combinations of the spherical solutions for the degenerate states corresponding to a given level.

To illustrate these points, we consider the case of the coulomb potential in the parabolic coordinates (see page 1296) defined by

$$x = \sqrt{\lambda\mu}\cos\varphi; \quad y = \sqrt{\lambda\mu}\sin\varphi; \quad z = \tfrac{1}{2}(\lambda - \mu); \quad r = \tfrac{1}{2}(\lambda + \mu)$$
$$h_\lambda = \tfrac{1}{2}\sqrt{(\mu/\lambda) + 1}; \quad h_\mu = \tfrac{1}{2}\sqrt{(\lambda/\mu) + 1}; \quad h_\varphi = \sqrt{\lambda\mu}; \quad \lambda, \mu \geq 0$$

which separates the Schroedinger equation (for constant energy E) into

$$\psi = L(\lambda)M(\mu)C(\varphi);\quad C'' + m^2C = 0$$
$$L'' + (1/\lambda)L' + [-\tfrac{1}{4}\kappa^2 + (\kappa\sigma/\lambda) - (m^2/4\lambda^2)]L = 0$$
$$M'' + (1/\mu)M' + [-\tfrac{1}{4}\kappa^2 + (\kappa\tau/\mu) - (m^2/4\mu^2)]M = 0$$

where $\kappa^2 = -\epsilon = 2ME/\hbar^2$ and $\sigma + \tau = M\eta^2/\hbar^2\kappa$ are the separation constants. Comparison with Eq. (12.3.38) shows that the solution which is finite for $\lambda = 0$ is

$$L = (\kappa\lambda)^{\frac{1}{2}m}e^{-\frac{1}{2}\kappa\lambda}F(\tfrac{1}{2}m + \tfrac{1}{2} - \sigma|m + 1|\kappa\lambda)$$

A similar expression, with λ changed to μ and σ changed to τ, is the correct form for the factor M. The factor C is, of course, a trigonometric function or imaginary exponential; m must be an integer in order that C be periodic in φ.

As before, when κ is real (E negative), ψ will be finite only at infinity if $\frac{1}{2}m + \frac{1}{2} - \sigma$ and $\frac{1}{2}m + \frac{1}{2} - \tau$ are zero or negative integers.

$$\sigma = \tfrac{1}{2}m + \tfrac{1}{2} + s;\quad \tau = \tfrac{1}{2}m + \tfrac{1}{2} + t;\quad s, t = 0, 1, 2, \ldots$$
$$\sigma + \tau = n = s + t + m + 1 = 1, 2, 3, \ldots = M\eta^2/\hbar^2\kappa$$

which coincides with Eq. (12.3.39) for the allowed values of energy for the bound states (as it should). Expressing the confluent hypergeometric functions in terms of the Laguerre polynomials $L_s^m(\kappa\lambda)$ and $L_t^m(\kappa\mu)$ appropriate for the bound states, we are ready to compute the normalization constant. This involves an integration of $|\psi|^2$ times the volume element, $\frac{1}{4}(\lambda + \mu)\,d\lambda\,d\mu\,d\varphi$. which involves a bit more algebra than before to reach the end result. We finally have, for the bound state n, s, m, the normalized wave function

$$\psi_{msn}(\lambda,\mu,\varphi) = \sqrt{\frac{1}{\pi}\left(\frac{M\eta^2}{\hbar^2 n}\right)^3 \frac{s!(n-s-m-1)!}{n[(s+m)!(n-s-1)!]^3}}\, e^{im\varphi}\,\cdot$$
$$\cdot\left[\frac{M\eta^2}{\hbar^2 n}\sqrt{\lambda\mu}\right]^m e^{-(M\eta^2/2\hbar^2 n)(\lambda+\mu)} L_s^m\left(\frac{M\eta^2\lambda}{\hbar^2 n}\right) L_{n-s-m-1}^m\left(\frac{M\eta^2\mu}{\hbar^2 n}\right) \quad (12.3.42)$$

Comparing this set of wave functions with those given in Eq. (12.3.40) for the same potential in spherical coordinates, we see that

$$\psi_{001}(r,\vartheta,\varphi) = \psi_{001}(\lambda,\mu,\varphi)$$
$$\psi_{002}(r,\vartheta,\varphi) = \sqrt{\tfrac{1}{2}}\,[\psi_{002}(\lambda,\mu,\varphi) + \psi_{012}(\lambda,\mu,\varphi)]$$
$$\psi_{012}(r,\vartheta,\varphi) = \sqrt{\tfrac{1}{2}}\,[\psi_{002}(\lambda,\mu,\varphi) - \psi_{012}(\lambda,\mu,\varphi)]$$
$$\psi_{112}(r,\vartheta,\varphi) = \psi_{102}(\lambda,\mu,\varphi);\quad \text{etc.}$$

In general, one of the spherical functions, for a given m and n, is equal to a linear combination of parabolic functions, all for the same m and n but for different values of s. Vice versa (of course) each of the parabolic functions can be built out of spherical functions, for the same m and n but different l values.

Rutherford Scattering. Solutions in parabolic coordinates can also be used to represent solutions for positive energies, where $\kappa = ik$, $k^2 = \epsilon = (Mv/\hbar)^2$ where M is the mass of the particle and v its speed at $r \to \infty$. We have

$$\psi = Ne^{im\varphi}(\lambda\mu)^{\frac{1}{2}m}e^{-\frac{1}{2}ik(\lambda+\mu)}F(\tfrac{1}{2}m + \tfrac{1}{2} - \sigma|m + 1|ik\lambda)F(\tfrac{1}{2}m + \tfrac{1}{2} - \tau|m + 1|ik\mu)$$

where $\sigma + \tau = -i(M\eta^2/\hbar^2k)$. It is possible to choose σ so that the resulting solution, for $m = 0$, corresponds to a plane wave, coming from the left along the z axis, striking the coulomb field concentrated near the origin, and, in part, being scattered away from the origin in various directions. The wave coming from the left should have a factor $e^{ikz} = e^{\frac{1}{2}ik(\lambda-\mu)}$, so that we should choose $F(\frac{1}{2}m + \frac{1}{2} - \sigma|1|ik\lambda)$ to be equal to $e^{ik\lambda}$. To do this, we use the $m = 0$ case (solution symmetrical about the z axis). We also need to set $\sigma = -\frac{1}{2}$, because $F(1|1|ik\lambda) = e^{ik\lambda}$. Then $\frac{1}{2} - \tau = i(M\eta^2/\hbar^2k) = i(\eta^2/\hbar v)$ and the solution, for this case, turns out to be

$$\psi = \Gamma\left(1 - \frac{i\eta^2}{\hbar v}\right) e^{(\pi\eta^2/2\hbar v)}e^{\frac{1}{2}ik(\lambda-\mu)}F\left(\frac{i\eta^2}{\hbar v}\,|1|ik\mu\right)$$
$$\to e^{ikz-i(\eta^2/\hbar v)\ln[k(r-z)]} + \left[\frac{\eta^2 e^{i(\eta^2/\hbar v)\ln(1-\cos\vartheta)-2i\delta}}{2Mv^2\sin^2(\frac{1}{2}\vartheta)r}\right] e^{ikr+i(\eta^2/\hbar v)\ln(kr)} \qquad (12.3.43)$$

where $k = Mv/\hbar$, $\Gamma[1 + (i\eta^2/\hbar v)] = |\Gamma[1 + (i\eta^2/\hbar v)]|e^{i\delta}$ and $\vartheta = \cos^{-1}(z/r)$ is the angle of scattering. The asymptotic form is valid for $(r - z) \gg \hbar/Mv$. The constant η^2, the magnitude of the coulomb field, is positive for an attractive field, is negative for a repulsive field. (In the latter case there would be no bound states, only free states, with positive energy.)

We see that, well away from the positive z axis (the line, in the direction of the incident beam, going through the center of the field), the wave function breaks up into two parts: the first term being the incident wave going (in general) in the direction of the z axis, though with some distortion (the logarithmic term in the exponent) because a coulomb field "warps" a plane wave even at very large distances; the second term consisting of a radially outgoing, "scattered" wave with an amplitude which depends on the angle of scattering ϑ according to the factor $(\eta^2/2Mv^2r)\csc^2(\frac{1}{2}\vartheta)$. The square of this amplitude gives the scattered *intensity* at the angle ϑ; this corresponds to the *Rutherford formula* for scattering, the classical expression for scattering from a coulomb field. The scattering is greater, the smaller the angle of scattering ϑ, the smaller the velocity v of the incoming particle, or the larger the potential parameter η.

When $r = z$ (that is, along the positive z axis) the coordinate μ is zero, and

$$\psi \to \sqrt{\frac{(\pi\eta^2/\hbar v)e^{\pi\eta^2/\hbar v}}{\sinh(\pi\eta^2/\hbar v)}}\, e^{ikz-i\delta}; \quad \vartheta \to 0; \quad z \to r$$

We notice that when the coulomb force is weak or the particle velocity large ($\pi\eta^2/\hbar v$ small), the amplitude of the wave near the center of force is approximately equal to the amplitude of the incident wave (unity). For attractive fields (η^2 positive) when the coulomb force is strong or the particle velocity small, the amplitude near the origin is $\sqrt{2\pi\eta^2/\hbar v}$ times the incident amplitude. On the other hand, for repulsive fields (η^2 negative), the amplitude near the origin is the small quantity $e^{\pi\eta^2/\hbar v}\sqrt{-2\pi\eta^2/\hbar v}$ times the incident amplitude when $(-\pi\eta^2/\hbar v)$ is large. In other words, fast particles are as likely to be near the origin as anywhere else; slow particles tend to concentrate near the center if the force is attractive, tend to shun the center if the force is repulsive, not an unexpected result.

Other Soluble Central Force Systems. Equation (12.3.33), for the radial factor in the wave function, is not soluble in terms of known functions for useful potentials which depend on powers of r other than the square [see Eq. (12.3.37)] or the inverse first [see Eq. (12.3.40)]. A solution may be obtained for $V = \beta r$, but this is of no particular interest. Likewise a solution can be obtained for a repulsive inverse cube force ($V = +\gamma^2/r^2$) but, as indicated on page 1666, the solution for an attractive inverse cube force is not a satisfactory one.

Radial fields with higher positive powers than 2 introduce worse irregularity in the singular point at infinity than can be handled by the confluent hypergeometric function; fields with powers of r less than -2 introduce an additional irregular singular point at $r = 0$ which has not been completely investigated (the fields are of no great practical interest, in addition).

Exact solutions for potentials which are more complicated in form than just powers of r may sometimes be obtained for the $l = 0$ cases. What is done is to transform the independent variable from r to z, which is some function of r, in which form the modified equation for R may come out in simple form. For example, the radial equation for the attractive exponential potential, $V = -Be^{-r/d}$,

$$(d^2R/dr^2) + [-\kappa^2 + b^2e^{-r/d}]R = 0; \quad \epsilon = -\kappa^2; \quad b^2 = 2MB/\hbar^2$$

transforms, when z is set equal to $2bde^{-r/2d}$, to the Bessel equation

$$\frac{d^2R}{dz^2} + \frac{1}{z}\frac{dR}{dz} + \left(1 - \frac{4d^2\kappa^2}{z^2}\right)R = 0$$

Consequently, the radial function which goes to zero at $r \to \infty$ $(z \to 0)$ is $J_{2d\kappa}\,(2bde^{-r/2d})$. To have this same function go to zero at $r = 0$, we must adjust κ and thus the order of J, so that

$$J_{2d\kappa}(2bd) = 0$$

The *largest* value of κ for which this can be done will give the lowest allowed energy $-\kappa^2$, so it will be labeled $\kappa_1(d,b)$, and so on. When $2bd$ is large compared to unity, we can use the asymptotic expression for the first zero of a Bessel function of large order, given on page 1564. We obtain $2bd \simeq 2d\kappa + 1.8558(2d\kappa)^{\frac{1}{3}}$, or

$$\kappa_1(d,b) \simeq b - 1.8558(b/4d^2)^{\frac{1}{3}}$$

or
$$E_1 \simeq -B + 3.7116(\hbar^2B^2/8d^2M)^{\frac{1}{3}}; \quad 2bd \gg 1 \qquad (12.3.44)$$

The lowest level, when the potential well is deep, is thus above the bottom ($E = -B$) by an amount which varies inversely as the two-thirds power of the "range" d of the potential well.

This solution is valid only for the states with zero angular momentum ($l = 0$); for $l > 0$ the "centrifugal force term" $-l(l + 1)/r^2$ is present; as it cannot be transformed into a simple power of z, the resulting equation in z is not soluble in terms of known functions. If d or b is small enough, the term $-2/r^2$ may be larger than $b^2e^{-r/d}$ for all values of r, in which case there are no bound states for this potential, for $l > 0$.

To find other soluble cases for $l = 0$, we can take the known eigenfunction polynomials and transform their equations, seeking a case which gives terms corresponding to those of Eq. (12.3.33); a second derivative term, a factor multiplying R which has a constant term (the allowed energy) which depends on the quantum number, and a variable term (the potential energy) which must be independent of the quantum numbers. When we find such a case, the corresponding wave function is thus already at hand.

For example, if $R = Nz^{\frac{1}{2}a+\frac{1}{2}}e^{-\frac{1}{2}z}L_n^a(z)/\sqrt{z'}$, where $z = z(r)$ and $z' = dz/dr$, then the equation for R in terms of r is

$$\frac{d^2R}{dr^2} + \left[-\tfrac{1}{4}(z')^2 + (n + \tfrac{1}{2}a + \tfrac{1}{2})\,\frac{(z')^2}{z} + \frac{a^2 - 1}{4}\left(\frac{z'}{z}\right)^2 - \tfrac{3}{4}\left(\frac{z''}{z'}\right)^2 + \tfrac{1}{2}\left(\frac{z'''}{z'}\right)\right] R = 0 \quad (12.3.45)$$

If $z = br^\mu$, the equation becomes

$$R'' + \left[-\tfrac{1}{4}\mu^2b^2r^{2\mu-2} + (n + \tfrac{1}{2}a + \tfrac{1}{2})\mu^2br^{\mu-2} - \frac{a^2\mu^2 - 1}{4r^2}\right] R = 0$$

which has no constant term unless $\mu = 1$ or 2 [which cases have already been discussed, see Eqs. (12.3.37) and (12.3.40)]. On the other hand for $z = be^{-r/d}$, the equation for ψ in terms of r,

$$R'' + [-(b^2/4d^2)e^{-2r/d} + (n + \tfrac{1}{2}a + \tfrac{1}{2})(b/d^2)e^{-r/d} - (a^2/4d^2)]R = 0$$

has a constant term, which may be identified with the energy. At first sight it appears that one of the potential energy terms depends on the quantum number n, but we soon see that a may be adjusted so that $(n + \frac{1}{2}a + \frac{1}{2})$ will be independent of n and a.

Consequently, the radial equation (12.3.33) for a potential energy

$$V = De^{2(r_0-r)/d} - 2De^{(r_0-r)/d}; \quad v = 2MV/\hbar^2$$

will correspond to the equation above if b were made equal to

$$\left(\frac{d}{\hbar}\right)\sqrt{8MD}e^{r_0/d}$$

and a were set equal to $(d/\hbar)\sqrt{8MD} - 2(n + \frac{1}{2})$. The potential energy written down here has a minimum at $r = r_0$ of value $-D$ and curvature such that the classical frequency (or, rather, the angular velocity $\omega = 2\pi\nu$) of small oscillations about this minimum would be $\omega = \sqrt{2D/d^2M}$. It goes to zero as $r \to \infty$ and, if D and r_0 are both large, it becomes very large and positive at $r = 0$. The function R is adjusted to be zero at $z = 0$ $(r = \infty)$ and at $z = \infty$ $(r = -\infty)$, whereas the function we wish to be the radial factor of the wave function should go to zero at $r = 0$ $[z_0 = (d/h)\sqrt{8MD}\,e^{r_0/d}]$. If z_0 is large, however, the function R is quite small anyway, because of the exponential term $e^{-\frac{1}{2}z_0}$, so that it would take only a trivial change in energy ϵ to make R go to zero at $z = z_0$ rather than at $z = \infty$, and this negligibly small energy change would produce a negligibly small change in the value of R for $z \ll z_0$, where R is large. Consequently, as long as $e^{r_0/d}$ is considerably larger than unity, the allowed bound energies and wave functions are, approximately,

$$\begin{aligned} E_n &\simeq -D + \frac{\hbar}{d}\sqrt{\frac{2D}{M}}\,(n + \tfrac{1}{2}) - \frac{\hbar^2}{2Md^2}\,(n + \tfrac{1}{2})^2 \\ &= -D + \hbar\omega(n + \tfrac{1}{2}) - (\hbar^2\omega^2/4D)(n + \tfrac{1}{2})^2 \\ \psi_n &\simeq Nz^{\alpha-n}e^{-\frac{1}{2}z+\frac{1}{2}(r/d)}L_n^{2\alpha-2n-1}(z) \end{aligned}$$

where $\omega = \sqrt{2D/d^2M}$, $z = 2\alpha e^{(r_0-r)/d}$ and $\alpha = (d/\hbar)\sqrt{2MD}$. We see that n can be zero or any positive integer less than $\alpha - \frac{1}{2}$. To obtain the normalization constant (or for any other integral involving ψ_n^2) it is allowable, within the limits of the approximation ($e^{r_0/d}$ large) to integrate from $z = 0$ to $z = \infty$.

Another possible potential function, for which an exact solution for $l = 0$ is possible, is the form

$$V(r) = V_0 \tanh^2(r/d)$$

which is the symmetric case of the potential considered on pages 1651, *et seq.* Solutions for this potential, which vanish at $x = \pm\infty$, were given in Eq. (12.3.26). In the present case, however, we need to have $\psi = 0$ at $r = 0$. This restricts the allowed states to those for states having the quantum number an odd integer, for such wave functions are antisymmetric about the origin, and if the potential has been made symmetric, every other wave function will vanish at $r = 0$. Consequently, the wave function and desired energy levels are:

$$\begin{aligned} R_{00n}(r) &= N[\cosh z]^{-\gamma+2n+\frac{3}{2}} F\left(-2n-1, 2\gamma-2n-1 \middle| \gamma-2n-\tfrac{1}{2} \middle| \frac{e^{-z}}{e^z+e^{-z}}\right) \\ &= N'[\cosh z]^{-\gamma+2n+\frac{3}{2}} T_{2n+1}^{\gamma-2n-\frac{3}{2}}(\tanh z) \\ \epsilon_n &= 2\gamma(2n+\tfrac{3}{2}) - (2n+1)(2n+2) - \tfrac{1}{2}; \quad \gamma = \sqrt{v+\tfrac{1}{4}}; \quad z = r/d \\ E_n &= \sqrt{\frac{2}{M}}\left(\frac{\hbar}{d}\right)\sqrt{V_0 + \left(\frac{\hbar^2}{8Md^2}\right)}\,(2n+\tfrac{3}{2}) \\ &\qquad - \left(\frac{\hbar^2}{2Md^2}\right)[(2n+1)(2n+2)+\tfrac{1}{2}] \end{aligned}$$

where we have picked out every other level of the set given in Eq. (12.3.25) (by using $2n + 1$ instead of n) since only the odd eigenfunctions go to zero at $r = 0$. An exact solution for the more general potential in Eq. (12.3.22) can also be found for the radial coordinate, if we adjust our energy so that the solution, which is zero at $r \to \infty$, is also zero at $r = 0$ (instead of $r \to -\infty$).

This is about as far as we can go in getting exact solutions for central force problems in terms of known functions. Other solutions and allowed energies, for other potential fields, must be obtained by the approximation methods discussed in Chap. 9. The method of factorization [see Eq. (6.3.18) and tables at the end of Chap. 6] does not produce any others, though it may simplify the calculation of R.

Perturbations of Degenerate Systems. The technique of calculation of the energies and eigenfunctions, if the potential does not differ much from one for which an exact solution is known, follows that given in Eqs. (12.3.15) *et seq.* There is one additional complication, however, which is present because of the symmetry in space of the solutions for the central force problem. As we have already seen, this produces degeneracies, several states having the same allowed energy. Consequently, when we set up a perturbed wave function as an unperturbed function φ_n plus a correction sum involving φ_m's for m not equal to n, we are faced with an initial problem: Which of the several φ's having a given unperturbed energy do we choose to be φ_n? Presumably we should choose for our unperturbed wave function a linear combination of all φ's for the chosen energy and then let the perturbation pick the suitable combination.

To show how this goes, we consider the case of a Schroedinger equation

$$\mathcal{H}\psi = [V - (\hbar^2/2M)\nabla^2]\psi = E\psi$$

If now the potential energy V can be split into a potential V_0 for which we have exact solutions, plus a perturbing potential V_1, which is small compared to V_0, we can then start our iterative procedure. We set

$\mathcal{H} = \mathcal{H}_0 + V_1$, where $\mathcal{H}_0 = [V_0 - (h^2/2M)\nabla^2]$ and where

$$\mathcal{H}_0\chi_{mn} = E_n^0\chi_{mn}$$

can be solved exactly for the allowed energies E_n^0 and eigenfunctions χ_{mn} for the unperturbed system. We have taken into account, in our notation, the possibility of degeneracy, for we have used two subscripts on χ, only one on E^0. The set χ_{mn}, for the different allowed values of m (one or more), are all for the same energy E_n. The Green's function for the unperturbed system is a solution of

$$(E - \mathcal{H}_0)G = \delta(\mathbf{r} - \mathbf{r}_0)$$

and is

$$G_E(\mathbf{r}|\mathbf{r}_0) = \sum_{m,n} \frac{\bar{\chi}_{mn}(\mathbf{r}_0)\chi_{mn}(\mathbf{r})}{E - E_n^0}; \quad m = 0, 1, 2, \ldots, M_n; \quad n = 0, 1, 2, \ldots$$

The corresponding integral equation, corresponding to the equation $(\mathcal{H}_0 + V_1)\psi = E\psi$, is

$$\psi(\mathbf{r}) = \sum_{m,n} \frac{\chi_{mn}(\mathbf{r})}{E - E_n^0} \iiint \bar{\chi}_{mn}(\mathbf{r}_0) V_1(\mathbf{r}_0)\psi(\mathbf{r}_0)\, dv_0 \qquad (12.3.46)$$

So far this has been essentially the same as the procedure on page 1647. Now, however, we assume that ψ does not necessarily reduce to a particular χ_{mn} when V_1 goes to zero, but to some linear combination of all the χ's for the particular energy E_n^0. We set

$$\psi = \sum_r C_r\chi_{rn} + \sum_{s,p\neq n} A_{sp}\chi_{sp}$$

Substituting this in Eq. (12.3.46), we obtain, first, for the coefficients C_m,

$$(E - E_n^0)C_m = \sum_r C_r u_{mn,rn} + \sum_{s,p\neq n} A_{sp}u_{mn,sp}$$

where
$$u_{mn,sp} = \iiint \bar{\chi}_{mn} V_1 \chi_{sp}\, dv$$

For this to correspond to the first of Eqs. (12.3.16), we should have the first series reduce to a single term. In other words we should so choose the coefficients C_m of the linear combination that

$$\sum_r C_r u_{mn,rn} = E_n^1 C_m$$

For this to be true, the determinant of the coefficients C_m must be zero:

$$\Delta_n = \begin{vmatrix} (u_{1n,1n} - E_n^1) & u_{1n,2n} & \cdots & u_{1n,Mn} \\ u_{2n,1n} & (u_{2n,2n} - E_n^1) & \cdots & u_{2n,Mn} \\ \cdots & \cdots & \cdots & \cdots \\ u_{Mn,1n} & u_{Mn,2n} & \cdots & (u_{Mn,Mn} - E_n^1) \end{vmatrix} = 0 \qquad (12.3.47)$$

which is an M_nth order equation for E_n^1 in terms of the matrix components $u_{rn,mn}$. There are M_n roots, which can be ordered and labeled $E_{\sigma n}^1$. (Some of these roots may be equal; the perturbation may not have removed all the degeneracy, of course.) Corresponding to the root $E_{\sigma n}^1$ are the coefficients $C_{\sigma m}$ (which may be normalized so that $\sum_m |C_{\sigma m}|^2 = 1$) and the linear combinations of wave functions

$$\varphi_{\sigma n} = \sum_m C_{\sigma m}\chi_{mn}; \quad \iiint |\varphi_{\sigma n}|^2 \, dv = 1$$

There are M_n of these linear combinations φ for each allowed value E_n^0, solutions of the unperturbed equation $\mathcal{H}_0\varphi_{\sigma n} = E_n^0\varphi_{\sigma n}$ for the same energy E_n^0. They are mutually orthogonal (see page 60) and are normalized. In fact they are just as good wave functions for the unperturbed wave equation as are the functions χ. They are actually better than the functions χ, for the problem at hand, for they have the property (from the very way they were formed) of possessing a matrix for V_1 which is diagonal for a given n,

$$\iiint \bar{\varphi}_{\sigma n} V_1 \varphi_{\tau n} \, dv = \delta_{\sigma\tau} E_{\sigma n}^1$$

although the matrix components for different n,

$$\iiint \bar{\varphi}_{\sigma p} V_1 \varphi_{\tau n} \, dv = U_{\sigma p,\tau n}; \quad p \neq n$$

may not be zero, even if $\sigma \neq \tau$.

If now we start our perturbation calculation over again with the eigenfunctions φ, preformed to "fit" the perturbation V_1, the process goes almost as easily as if we had no degenerate states. Since the perturbation does not "mix up" the φ's, we need to have only one for a given E_n for a leading term for

$$\psi = \varphi_{\sigma n} + \sum_{\tau,p\neq n} A_{\tau p}\varphi_{\tau p}$$

Substituting the unperturbed set $\varphi_{\sigma n}$ for χ_{mn} in Eq. (12.3.46), we can now obtain a set of equations identical with Eqs. (12.3.16), from which we can compute a successive approximation solution

$$\begin{aligned}
\psi^{-1}(\mathbf{r}) &= 0; \quad \psi^0(\mathbf{r}) = \varphi_{\sigma n}(\mathbf{r}) \\
\psi^\nu(\mathbf{r}) &= \varphi_{\sigma n}(\mathbf{r}) + \sum_{\tau,m\neq n} \frac{\iiint \bar{\varphi}_{\tau m} V_1 \psi^{\nu-1} \, dv_0}{E_n^0 - E_m^0 + \iiint \bar{\varphi}_{\tau n} V_1 \psi^{\nu-2} \, dv_0} \\
E^\nu &= E_n^0 + \iiint \bar{\varphi}_{\sigma n}(\mathbf{r}_0) V_1(\mathbf{r}_0) \psi^{\nu-1}(\mathbf{r}_0) \, dv_0
\end{aligned} \tag{12.3.48}$$

from which, by successive approximations, we can calculate the allowed energies and wave functions for the perturbed state, in a manner completely analogous to those given in Eqs. (12.3.17) and (12.3.18). The functions ψ^ν and the energies E^ν approach the correct solutions as $\nu \to \infty$, within the radius of convergence given in Sec. 9.1.

Recapitulating, to compute the effect of a perturbation V_1 on a degenerate system, we first find that particular set of eigenfunctions $\varphi_{\sigma n}$ for which the matrix of V_1 is diagonal for all states having the same unperturbed energy E_n^0. Using this "preformed" set of functions, the calculations for the effect of V_1 on the system may be carried out as though the system were not degenerate.

The Stark Effect. As an example of this sort of calculation, we shall compute the effect on various central force systems caused by a uniform force field (a uniform electric field, for example, if the particle is charged). The perturbing potential is $Fz = Fr \cos \vartheta = \frac{1}{2}F(\lambda - \mu)$. If the wave function is separated in spherical coordinates (the wave functions being eigenfunctions for the z component and square of the angular momentum as well as for the energy), then the angle factors are spherical harmonics, as indicated in Eqs. (12.3.33) *et supra*. For both harmonic oscillator and coulomb potential states, the energy is independent of l as well as of m, so these states are degenerate. To see whether the spherical states, characterized by the quantum numbers l and m, are the appropriate ones for a uniform field perturbation, we need only to see whether the matrix components of Fz are diagonal in the quantum numbers m and l. But these quantum numbers involve the integration over the angle factor X_{ml}, so that we can make this investigation before we decide which radial force state is to be the unperturbed state.

Since the perturbation potential, $V_1 = Fr \cos \vartheta$, is independent of the axial angle φ, we can be sure that the matrix elements U will be diagonal in the quantum number m. In other words

$$\begin{aligned} U_{mln,m'l'n'} &= NN' \int_0^{2\pi} e^{im\varphi - im'\varphi}\, d\varphi \int_0^\pi \cos \vartheta\, P_l^m P_{l'}^{m'} \sin \vartheta\, d\vartheta \int_0^\infty r R_{ln} R_{l'n'} r^2\, dr \\ &= \delta_{mm'} NN' \int_0^\pi \cos \vartheta\, P_l^m P_{l'}^{m'} \sin \vartheta\, d\vartheta \int_0^\infty r R_{ln} R_{l'n'} r^2\, dr \end{aligned}$$

We note, first, that, if the polar axis of the spherical coordinates were not pointed along the direction of the perturbing force, the expression for V_1 *would* contain φ and thus U would *not* be diagonal for m. We have, by our choice of orientation, diagonalized part of the matrix.

The matrix is *not* diagonal in the quantum number l, however, for the integral over ϑ is zero when $m = m'$, $l = l'$ (the additional cosine term makes the integrand antisymmetric about $\vartheta = \frac{1}{2}\pi$). Since

$$\cos \vartheta\, P_l^m(\cos \vartheta) = \frac{l - m + 1}{2l + 1} P_{l+1}^m(\cos \vartheta) + \frac{l + m}{2l + 1} P_{l-1}^m(\cos \vartheta)$$

we see that the only components of U differing from zero are those for which $l' = l \pm 1$. There is nothing further to be done about this, moreover; we have fixed the orientation of the spherical coordinate system by requiring that the matrix U be diagonal for m; to diagonalize it for l, we must choose wave functions and quantum numbers suitable to another coordinate system.

The coordinate system to use, for the perturbation of the three-dimensional harmonic oscillator by a uniform field, is the rectangular system, x, y, z. The wave function, built out of three factors of the sort given in Eq. (12.3.7),

$$\varphi_{ksn}(x,y,z) = \frac{(\beta/\pi)^{\frac{3}{4}}}{\sqrt{2^n k!s!(n-s-k-1)!}} e^{-\frac{1}{2}\beta r^2} \cdot \\ \cdot H_k(x\sqrt{\beta})\, H_s(y\sqrt{\beta})\, H_{n-k-s-1}(z\sqrt{\beta})$$

is a solution of the Schroedinger equation for the central force $V_0 = \frac{1}{2}M\omega^2r^2 = \frac{1}{2}M\omega^2(x^2 + y^2 + z^2)$ and is a linear combination of the solutions given in Eq. (12.3.37), for different m and l, but for the same n. The energy for all these degenerate states is, of course, $\hbar\omega(n + \frac{1}{2})$. The solutions φ above, however, are the useful ones for the uniform force perturbation, because the matrix components of Fz are diagonal in both k and s, because of the orthogonality of the polynomials H_k and H_s. As a matter of fact, in this case, we know from Eq. (12.3.9) that the perturbed wave function is $\varphi_{ksn}(x,y,z + b)$, where $b = F/M\omega^2$ (the x and y factors not being perturbed at all) and that the perturbed energy levels are

$$E = \hbar\omega(n + \tfrac{1}{2}) - (F^2/2M\omega^2)$$

The coulomb field wave functions cannot be separated in rectangular coordinates, but here the wave functions in parabolic coordinates are appropriate, those given in Eq. (12.3.42). The perturbation $\frac{1}{2}F(\lambda - \mu)$ is independent of φ (if we have chosen the parabolic axis parallel to the perturbing force) so that U is diagonal with regard to m. The rest of the integral involves

$$\iint_0^\infty (\lambda\mu)^m e^{-\gamma(\lambda+\mu)} L_s^m(\gamma\lambda) L_{s'}^m(\gamma\lambda) L_{n-s-m-1}^m(\gamma\mu) L_{n-s'-m-1}^m(\gamma\mu)(\lambda^2 - \mu^2)\, d\lambda\, d\mu$$

which is zero unless $s' = s$ (though, if we had written down the integral for n' and n, it would have been seen that not all nondiagonal terms $U_{n's'm',nsm}$ are zero; when $n' \neq n$ some of the terms with $n' + s' \neq n + s$ are not zero). Consequently, the parabolic wave functions are the proper ones to use for this perturbation. The first-order correction to the energy results in (see page 785)

$$E \simeq -\left(\frac{M\eta^4}{2n^2\hbar^2}\right) - \tfrac{3}{2}F\left(\frac{\hbar^2}{M\eta^2}\right) n(n - 2s - m - 1) \quad (12.3.49)$$

for the parabolic state with quantum numbers n, s, and m, for the system having potential energy $-(\eta^2/r) + Fz$. It should be noted that, if the uniform force extends to infinity, there is some large negative value of z for which the potential energy is equal to E, beyond which the particle can move freely to $-\infty$. As we have seen on page 1660, a particle in wave mechanics can penetrate a potential barrier eventually, though it takes a very long time if the barrier is high or thick; a uniform field provides a barrier through which the particle can leak from the region near the origin to the region near $z = -\infty$. Therefore, strictly speaking, *all* negative energy states are allowed and the quantized states near the origin are not permanently stationary states, since the particle will eventually find itself outside the influence of the center of force. However, if F is relatively small, the potential barrier is very thick, it will take an extremely long time for the particle to leak out, and the energy states given in Eq. (12.3.49) will be stationary states to all intents and purposes.

Momentum Eigenfunctions. Sometimes, for example in the treatment of the scattering of X rays by atoms, it is of interest to compute the probability density in momentum space rather than in configuration space. As shown in Sec. 2.6, the momentum wave function is a solution of a related Schroedinger equation, where $i\hbar(\partial/\partial p_x)$ is substituted for x, and p_x is substituted for $-i\hbar(\partial/\partial x)$ in the equation for the usual eigenfunction ψ_m. But, rather than solve a new equation, we can take advantage of the fact that the two wave functions are Fourier transforms of each other. The momentum wave function, χ, the square of which gives the probability density for momentum, which satisfies the equation

$$\frac{1}{2M}|\mathbf{p}|^2\chi + V(i\hbar\,\mathrm{grad})\chi = E\chi \qquad (12.3.50)$$

(where $i\hbar$ grad has been substituted for $\mathbf{r}$ in the potential function V) is related to the position eigenfunction, ψ, which satisfies the usual equation

$$-(\hbar^2/2M)\nabla^2\psi + V(r)\psi = E\psi$$

by the relation (where N is the number of dimensions involved)

$$\begin{aligned}\chi(\mathbf{p}) &= \left[\frac{1}{\sqrt{2\pi\hbar}}\right]^N \iiint e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar}\psi(\mathbf{r})\,dv_r \\ \psi(\mathbf{r}) &= \left[\frac{1}{\sqrt{2\pi\hbar}}\right]^N \iiint e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}\chi(\mathbf{p})\,dv_p\end{aligned} \qquad (12.3.51)$$

This integral relationship also holds for the more general, time-dependent wave functions Ψ and Ξ, where the terms $E\psi$ and $E\chi$ in their respective equations are replaced by $i\hbar(\partial\Psi/\partial t)$ and $i\hbar(\partial\Xi/\partial t)$.

A simple example of this is the one-dimensional harmonic oscillator.

The wave equation for χ was given in Eq. (12.3.19) and the eigenfunction solutions, as given in Eq. (12.3.20), are

$$\chi_n(\mathbf{p}) = \frac{i^{-n}}{\sqrt{2^n n!\hbar\sqrt{\pi\beta}}} e^{-\frac{1}{2}(p^2/\hbar^2\beta)} H_n\left(\frac{p}{\hbar\sqrt{\beta}}\right)$$

which is to be compared with Eq. (12.3.7). To show that this solution also satisfies Eqs. (12.3.51), we use the generating function for the Hermite polynomials, given in the tables at the end of Chap. 6;

$$\frac{1}{\sqrt{2\pi\beta\hbar}} \sum_{n=0}^{\infty} t^n \sqrt{\frac{2^n}{n!}} \int_{-\infty}^{\infty} e^{-i\rho z}\varphi_n(z)\, dz = \frac{1}{\sqrt{2\pi\hbar\sqrt{\pi\beta}}} \int_{-\infty}^{\infty} e^{-t^2+2tz-\frac{1}{2}z^2-i\rho z}\, dz$$

$$= \frac{1}{\sqrt{\hbar\sqrt{\pi\beta}}} e^{t^2-2it\rho-\frac{1}{2}\rho^2} = \sum_{n=0}^{\infty} t^n \sqrt{\frac{2^n}{n!}}\, \chi_n(\rho) \quad (12.3.52)$$

where $z = x\sqrt{\beta}$ and $\rho = (p/\hbar\sqrt{\beta})$. Equating coefficients of t^n, we see that φ_n and χ_n satisfy the relationship (12.3.51) and (12.3.21).

We can, of course, build up the wave function for the three-dimensional harmonic oscillator, with potential function $\frac{1}{2}M\omega^2r^2$. We substitute $p/\hbar\sqrt{\beta}$ in Eq. (12.3.37) and make the necessary changes in the normalization constant, obtaining

$$\chi_{mln}(p,\theta,\phi) = \sqrt{\frac{2l+1}{8\pi\hbar^6\beta^3}\frac{(l-m)!}{(l+m)!}\frac{(\frac{1}{2}n-\frac{1}{2}l-\frac{1}{2})!}{[\Gamma(\frac{1}{2}n+\frac{1}{2}l+1)]^3}}\, e^{im\phi}P_l^m(\cos\theta)\,\cdot$$
$$\cdot\,(p^2/\beta\hbar^2)^{\frac{1}{2}l}e^{-\frac{1}{2}p^2/\beta\hbar^2}L^{l+\frac{1}{2}}_{\frac{1}{2}n-\frac{1}{2}l-\frac{1}{2}}(p^2/\beta\hbar^2) \quad (12.3.53)$$

which (with the exception of a power of i that can be omitted) is the Fourier transform of ψ_{mln} given in Eq. (12.3.37). When n is a positive integer, when $n - l$ is a positive, odd integer, and when m is zero or an integer not greater than l, then χ or its complex conjugate represents a stationary state for an allowed energy $\hbar\omega(n + \frac{1}{2})$. Here we have separated our wave function into the *spherical momentum coordinates* p (the magnitude of $\mathbf{p}$), θ the angle between $\mathbf{p}$ and the z axis, and ϕ the angle between the zp plane and the xz plane.

The calculation of integral (12.3.51) for the momentum eigenfunction for the attractive coulomb potential $-(\eta^2/r)$, corresponding to the position function given in Eq. (12.3.40), is not quite so easy. The exponent of the Fourier integral factor, $i\mathbf{p}\cdot\mathbf{r}/\hbar$, when expressed in spherical space and momentum coordinates, is

$$i(\mathbf{p}\cdot\mathbf{r}/\hbar) = i(pr/\hbar)[\sin\vartheta\sin\theta\cos(\varphi-\phi) + \cos\vartheta\cos\theta]$$

The part of the integral over the φ coordinate may be evaluated by use of the integral expression for the Bessel function. We have

$$\int_0^{2\pi} e^{-i(pr/\hbar)\sin\theta\sin\vartheta\cos(\varphi-\phi)+im\varphi}\,d\varphi = 2\pi i^m e^{im\phi} J_m\left(\frac{pr}{\hbar}\sin\vartheta\sin\theta\right)$$

The next integral, over ϑ, may also be expressed in terms of Bessel functions;

$$\int_0^{\pi} e^{-i(pr/\hbar)\cos\vartheta\cos\theta} J_m\left(\frac{pr}{\hbar}\sin\vartheta\sin\theta\right) P_l^m(\cos\vartheta)\sin\vartheta\,d\vartheta$$
$$= -(-i)^{l+m}\sqrt{\frac{2\pi\hbar}{pr}}\,P_l^m(\cos\theta)J_{l+\frac{1}{2}}\left(\frac{pr}{\hbar}\right)$$

The first integral of Eqs. (12.3.51) is then

$$\chi = -(-i)^l\sqrt{\frac{\hbar^2 n^2}{8M^2\eta^4}\,\frac{2l+1}{4\pi np}\,\frac{(l-m)!}{(l+m)!}\,\frac{(n-l-1)!}{[(n+l)!]^3}}\,e^{im\phi}P_l^m(\cos\theta)\cdot$$
$$\cdot\int_0^\infty u^{l+\frac{3}{2}}e^{-\frac{1}{2}u}J_{l+\frac{1}{2}}(\tfrac{1}{2}uw)L_{n-l-1}^{2l+1}(u)\,du$$

where $u = 2M\eta^2 r/\hbar^2 n$ and $w = \hbar np/M\eta^2$. Using the integral relation between Laguerre and Gegenbauer polynomials given in the table at the end of Chap. 6, we have, finally (we neglect factors like i^l),

$$\chi_{nlm}(p,\theta,\phi) = \sqrt{\left(\frac{2\hbar n}{M\eta^2}\right)^3\frac{2l+1}{2\pi}\,\frac{(l-m)!}{(l+m)!}\,\frac{n(n-l-1)!}{(n+l)!}}\,\cdot$$
$$\cdot\, e^{im\phi}P_l^m(\cos\theta)\left(\frac{1-z}{2}\right)^2(1-z^2)^{\frac{1}{2}l}T_{n-l-1}^{l+\frac{1}{2}}(z) \quad (12.3.54)$$

where $z = [(\hbar np/M\eta^2)^2 - 1]/[(\hbar np/M\eta^2)^2 + 1]$. The variable z goes from -1 to $+1$ as p goes from zero to infinity. The allowed energies for bound states are those given in Eq. (12.3.39).

The momentum wave functions for a few of the lowest states are

$$\chi_{100} = \sqrt{\frac{1}{2\pi^2}\left(\frac{2\hbar}{M\eta^2}\right)^3}\left[\left(\frac{\hbar p}{M\eta^2}\right)^2+1\right]^{-2}$$
$$\chi_{210} = \sqrt{\frac{8}{\pi^2}\left(\frac{4\hbar}{M\eta^2}\right)^3}\left(\frac{2\hbar p}{M\eta^2}\right)\left[\left(\frac{2\hbar p}{M\eta^2}\right)^2+1\right]^{-3}\cos\theta$$
$$\chi_{200} = \sqrt{\frac{2}{\pi^2}\left(\frac{4\hbar}{M\eta^2}\right)^3}\left[\left(\frac{2\hbar p}{M\eta^2}\right)^2-1\right]\left[\left(\frac{2\hbar p}{M\eta^2}\right)^2+1\right]^{-3}$$

functions which diminish for large values of p. The second has a node in the $x - y$ plane, and the third has a spherical node at $p = M\eta^2/2\hbar$.

From these wave functions we can find the mean square of the particle momentum when the system is in the state characterized by the quantum numbers n, l, m. This is

$$\iiint [\chi_{nlm}]^2 p^2\,dv_p = \left(\frac{M\eta^2}{\hbar n}\right)^2$$

which is largest for the lowest bound state ($n = 1$) and which decreases as n increases, being independent of the quantum numbers l and m.

The angular momentum of the particle may be computed by methods symmetric to those used in deriving Eqs. (12.3.34):

$$\mathfrak{M} = \mathbf{r} \times \mathbf{p} = i\hbar\, \mathbf{p} \times \mathrm{grad}_p = i\hbar \left[\mathbf{a}_z \frac{\partial}{\partial \phi} - \mathbf{a}_x \left(\sin\phi \frac{\partial}{\partial \theta} + \cos\phi \cot\theta \frac{\partial}{\partial \phi} \right) + \mathbf{a}_y \left(\cos\phi \frac{\partial}{\partial \theta} - \sin\phi \cot\theta \frac{\partial}{\partial \phi} \right) \right]$$

As before, we find that the function χ_{mln} is an eigenfunction for $\mathfrak{M}_z$ with eigenvalue $m\hbar$ (or $-m\hbar$ for its complex conjugate) and is also an eigenfunction for $|\mathfrak{M}|^2$ with eigenvalue $\hbar^2 l(l + 1)$.

Scattering from Central Fields. The case of importance for unbound states, with positive energies, is the one representing the particle coming in from a certain direction, striking the central field and being scattered by it. At very large distances the wave function for the incident particle is a plane wave (in the positive z direction, for example, proportional to $e^{ipz/\hbar}$) corresponding to a known, initial momentum p in the z direction. The scattered wave, at large distances, will be a radially outgoing wave $e^{ipr/\hbar}/r$ with amplitude depending on the angle of scattering ϑ. If the incident plane wave has unit amplitude, representing a stream of particles of unit incident current density, then the square of the amplitude of the scattered wave represents the density of the current scattered at the corresponding angle; the integral of this over all angles is the total scattered current per unit incident current density. This integral has the dimensions of an area; it represents the effective area of the force field for scattering particles and is called the elastic scattering *cross section* of the field for the incoming particles.

We discussed the scattering of waves in Sec. 9.3 and computed some examples in Chap. 11; the procedure here is the same. We solve the radial equation (12.3.33) for positive energy $\epsilon = k^2$, for the solution R which is finite at $r = 0$ and which goes to $(1/k)\sin(kr - \eta_l - \frac{1}{2}\pi l)$ for large enough values of r to make $v(r)$ negligible (we assume v goes to zero as $r \to \infty$). The distribution in angle of the scattered wave and the total cross section for scattering are then given in terms of the phase angles η_l, as shown in Eqs. (11.3.72). (We use η_l here instead of the δ_l used there.)

Not all potential fields can be computed in this simple manner. For example, Eq. (12.3.41) shows that, for a coulomb field, the phase angles do not reach an asymptotic value but continue to change no matter how large r is, because of the logarithmic term in the asymptotic form. The coulomb field does not approach zero fast enough to allow the radial functions to settle down to a form as simple as $(1/k)\sin(kr - \eta_l - \frac{1}{2}\pi l)$

as $r \to \infty$. Of course we have been able to calculate the scattering from a coulomb field by the use of parabolic coordinates; the result shows why the series must be handled carefully, the cross section is infinite, the coulomb field affects the motion of particles even out to infinity.

For potential fields which go to zero more rapidly than $1/r$, however, the solution has an asymptotic value of the phase angle, from which the scattering can be computed as indicated by Eqs. (9.3.18) and (11.3.71). As an example of the possible calculations, we shall compute the scattering of an electron of energy $E = \hbar^2k^2/2M$ by a field

$$V(r) = \begin{cases} (Ze^2/a) - (Ze^2/r); & r < a \\ 0; & r > a \end{cases}$$

which corresponds to a nucleus of charge Ze at the origin plus a negative neutralizing spherical shell at $r = a$. Inside the sphere, the factor obtained by solving

$$\frac{d^2R_l}{dr^2} + \left[-K^2 + \frac{2MZe^2}{\hbar^2 r} - \frac{l(l+1)}{r^2}\right]R_l = 0; \quad K^2 = \left(\frac{2MZe^2}{\hbar^2 a}\right) - \left(\frac{2ME}{\hbar^2}\right)$$

is

$$\frac{1}{r}R_l(r) = N(2Kr)^l e^{-Kr} F\left(l + 1 - \frac{MZe^2}{\hbar^2 K}\middle| 2l + 2\middle| 2Kr\right); \quad r < a$$

The important property of these interior functions is the ratio of the gradient of R_l to its value at $r = a$. Using the properties of the confluent hypergeometric functions, we obtain

$$\tan\Phi_l = \left[\frac{r}{R_l}\frac{dR_l}{dr}\right]_{r=a} = \left(\frac{MZe^2}{\hbar^2K}\right) - 1 - (Ka) + \left(l + 1 - \frac{MZe^2}{\hbar^2K}\right)\frac{F\left(l + 2 - \frac{MZe^2}{\hbar^2K}\middle| 2l + 2\middle| 2Ka\right)}{F\left(l + 1 - \frac{MZe^2}{\hbar^2K}\middle| 2l + 2\middle| 2Ka\right)}$$

Outside $r = a$ we use a combination $\cos\eta_l j_l(kr) + \sin\eta_l n_l(kr)$ with η_l so adjusted that its ratio of slope to value is the same as that for the interior function at $r = a$. Using the phase angles defined at the end of Chap. 11, we have[1]

$$\tan\Phi_l = ka\frac{j_l'\cos\eta_l + n_l'\sin\eta_l}{j_l\cos\eta_l + n_l\sin\eta_l}$$

or

$$\tan\eta_l = \tan\delta_l\frac{\tan\Phi_l + \tan\alpha_l}{\tan\Phi_l + \tan\beta_l} = \tan\gamma_l\frac{\sin(\Phi_l + \alpha_l)}{\sin(\Phi_l + \beta_l)}$$

where the argument of the angles α, β, δ, γ is ka, with $k^2 = \epsilon = 2ME/\hbar^2$.

[1] The definition of phase η_l employed here corresponds to the asymptotic behavior of $R_l = \sin(kr - \frac{1}{2}l\pi - \eta_l)$. In quantum mechanics, where the potentials are commonly attractive and the waves are "pulled in" rather than out as by a barrier in acoustics, it is more usual to employ $(-\eta_l)$ as the phase shift.

We use a combination of the exterior-interior functions for different l's such that the incoming wave, for large r, corresponds to the plane wave [see Eq. (9.3.15)]

$$e^{ikz} = \sum_{l=0}^{\infty} i^l\,(2l+1)P_l\,(\cos\vartheta)j_n(kr)$$

Consequently for $r > a$, the incident plus scattered wave is

$$\sum_{l=0}^{\infty} (2l+1)i^l P_l(\cos\vartheta)e^{-i\eta_l}[\cos(\eta_l)j_l(kr) + \sin(\eta_l)n_l(kr)]$$

as in Eq. (11.3.71). The scattered wave, scattered amplitude, and scattering cross section for $kr \gg 1$ are then

$$\begin{aligned}
\psi_s &\to -\frac{e^{ikr}}{kr}\sum_{l=0}^{\infty}(2l+1)e^{-i\eta_l}\sin(\eta_l)P_l(\cos\vartheta)\\
S &= \frac{1}{k^2r^2}\sum_{l,n}(2l+1)(2n+1)\cos(\eta_l-\eta_n)\sin\eta_l\sin\eta_n\,\cdot\\
&\qquad\qquad\cdot\, P_l(\cos\vartheta)P_n(\cos\vartheta)\\
Q &= \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\eta_l
\end{aligned} \tag{12.3.55}$$

We might note that there is a relationship between the total cross section Q and the asymptotic amplitude of the scattered wave, analogous to Eq. (11.4.64) and mentioned on page 1069. Since Eqs. (12.3.55) are the correct forms for any reasonable sort of central field, where the phase angles η_l are determined by the field, then the expansion of the angle-distribution factor $f(\vartheta)$, defined by the equation

$$\psi \to e^{ikr\cos\vartheta} + (\epsilon^{ikr}/r)f(\vartheta); \quad r \to \infty$$

in terms of the phase angles η_l,

$$f(\vartheta) = -\frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{-i\eta_l}\sin(\eta_l)P_l(\cos\vartheta)$$

and the limiting relation between f and Q, obtained by inspection of the two series,

$$Q = (4\pi/k)\,\mathrm{Im}[f(0)] \tag{12.3.56}$$

are correct in general. In particular, this last relation should hold whether f and Q are given in terms of a series of spherical harmonics or are given in closed form.

It was shown in Chap. 9, page 1069, that Eq. (12.3.56) is nothing more than an expression of the conservation of incident particles. The primary beam beyond the scattering center ($\vartheta = 0$) must be reduced in intensity by an amount sufficient to account for the total number scattered; the scattered wave for $\vartheta = 0$ is just the shadow-forming wave (see page 1381), so the total cross section must be proportional to the part of $f(0)$ which is out of phase with the incident wave.

Ramsauer and Other Resonance Effects. For the example considered here, the phase angles depend on two dimensionless parameters, $ka = (a/\hbar)\sqrt{2ME} = Mva/\hbar$, proportional to the velocity of the incident particle, and $\beta = \sqrt{MaZe^2/2\hbar^2}$, which is a measure of the "strength" of

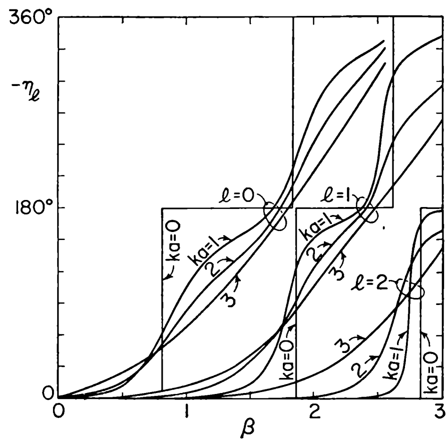

Fig. 12.5 Phase angles η_l for scattering from screened coulomb field, as functions of field-strength parameter β, for different values of velocity parameter ka.

the attractive field, being proportional to the square root of the "charge factor" Z and the radius a of the boundary beyond which $V = 0$. It turns out that the phase angles η, for $k = 0$, all start at zero for β small, then jump discontinuously to $-\pi$, -2π, etc., as β increases. For $k > 0$, the angles $-\eta$ also rise with β, but not so discontinuously, as shown in Fig. 12.5. The explanation of the discontinuity for $k = 0$ lies in the behavior of the bound levels for the potential V. For β very small, no bound level exists; the first bound level (for $l = 0$) appears very close to the top of the potential hole, for β just above the value for which η_0 jumps discontinuously from 0 to $-\pi$ for $k = 0$; each new level appears, for a given l, for β just above a similar jump of η_l.

Viewed as a function of ka, for different values of β, the phase angles behave as shown in Fig. 12.6. The angle $-\eta_0$ varies linearly with ka for small values of ka; for other l's the angle $-\eta_l(ka) \simeq n\pi + \mu_l(ka)^{l+1}$ for $ka \ll 1$, where the constant μ_l approaches $\pm\infty$ as β approaches one of the values corresponding to a discontinuous jump of $\eta_l(0)$. Consequently, only the $l = 0$ term in Q stays constant as $ka \to 0$, all other terms approach zero, except when β has just the value for $\mu_l \to \infty$, in which case the cross section approaches infinity as k goes to zero. On the other hand, for β a little below this (in other words, when the level has been "squeezed out" a little) $|\sin \eta_l|$ first rises and then falls as ka is decreased and the cross section also first rises and then becomes quite small as $ka \to 0$ (see the curves of Fig. 12.6 for $\beta = 1.75$).

To put these statements in a more general form, when the ratio of slope to value of the interior function R_l is equal to the ratio of slope

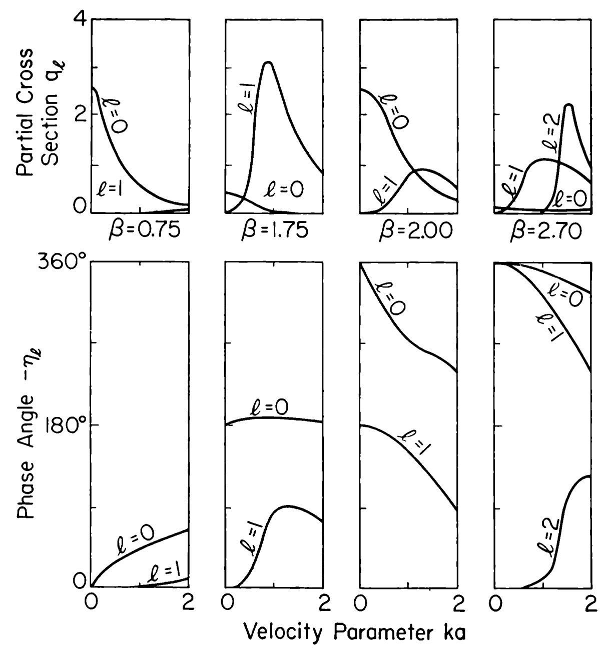

Fig. 12.6 Phase angles $-\eta_l$ and partial cross sections $q_l = (2l + 1) \sin^2(\eta_l)/k^2a^2$ as function of ka for different values of l and β, to illustrate the effect of "virtual levels."

to value of j_l at $r = a$, then $\tan \Phi_l$ equals $-\tan \alpha_l$ and $\eta_l = 0$. This is nearly true for energies which would correspond to an allowed bound state if the coulomb field were not cut off at $r = a$ but were to continue to infinity. When this happens for ϵ small (*i.e.*, when the "virtual level" is just above zero), the total cross section drops to very low values as $\epsilon \to 0$. This phenomenon, in the case of electron scattering from atoms, is called the *Ramsauer effect.*

Maxima (called *resonance peaks*) in the cross section may occur when $\eta_l = \frac{1}{2}\pi$, that is, whenever $\Phi_l = -\beta_l$. At the maxima, the *partial cross section* Q_l

$$Q_l = (4\pi/k^2)(2l + 1)\sin^2(\eta_l) \xrightarrow[\eta_l \to \frac{1}{2}\pi]{} (4\pi/k^2)(2l + 1)$$

is the maximum possible cross section. Suppose this occurs at $\epsilon = \epsilon_r$. In the neighborhood of this peak

$$\cot \eta_l \simeq \cot \delta_l \frac{(\epsilon - \epsilon_r) \sec^2 \beta_l (d\Phi_l/d\epsilon)_{\epsilon=\epsilon_r}}{\tan \alpha_l - \tan \beta_l}$$

where the angles interrelating the Bessel functions, α_l and β_l, are given on page 1575.

Employing the Wronskian relation

$$j_l' n_l - n_l' j_l = 1/ka$$

yields
$$\tan \delta_l (\tan \alpha_l - \tan \beta_l) = [1/n_l^2(ka)]$$
and
$$\cot \eta_l \simeq n_l^2 (\epsilon - \epsilon_r) \sec^2 \beta_l (d\Phi_l/d\epsilon)_{\epsilon=\epsilon_r}$$

The partial cross section now takes on the appearance of a typical resonance curve:

$$Q_l = \left(\frac{4\pi}{k^2}\right)(2l+1)\frac{1}{1+\cot^2 \eta_l}$$
$$\simeq \left(\frac{\pi}{k^2}\right)(2l+1)\frac{4\cos^4 \beta_l/[n_l^4(d\Phi_l/d\epsilon)_{\epsilon=\epsilon_r}]}{(\epsilon-\epsilon_r)^2 + \cos^4 \beta_l/[n_l^4(d\Phi_l/d\epsilon)^2_{\epsilon=\epsilon_r}]}$$

the width of the resonance at half maximum, Γ, is

$$\Gamma = \frac{2\cos^2 \beta_l}{n_l^2 (d\Phi_l/d\epsilon)_{\epsilon=\epsilon_r}}$$

The resonance will be particularly narrow and very noticeable at low energies, for there $n_l(ka)$ is particularly large. This conclusion is of course conditioned by the energy sensitivity of $(d\Phi_l/d\epsilon)_{\epsilon=\epsilon_r}$ and is valid only if this derivative does not approach zero rapidly as $\epsilon \to 0$. Since Φ_l describes the behavior in the interior region $r \leq a$, Φ_l will be insensitive to change in ϵ for small ϵ only if the potential energy in the interior region is much larger than ϵ (that is, only if the kinetic energy is large for $r < a$, $\epsilon \to 0$). This condition is not met in the problem being considered but is the situation for the scattering of slow neutrons by nuclei and manifests itself in the phenomena of resonance scattering. We note that, as l increases, (ka) being kept constant, the widths decrease rapidly and the resonances become much sharper while, for constant l as the energy increases, the widths increase so that eventually the strong fluctuation in Q which results at a resonance will disappear at a sufficiently high energy. Some of these phenomena are apparent in Fig. 12.6; they are more noticeable in nuclear scattering calculations.

Approximate Calculations for Slow Incident Particles. When the energy of the incident particle is quite small (when $ka \ll 1$), then $\eta_l \to 0$ for $l \to 0$ and the majority of the scattered wave corresponds to the $l = 0$, spherically symmetric part. This is the case for slow neutrons (less than a few thousand electron-volts energy), and it is likewise the case for very slow electrons (less than an electron-volt for scattering from atoms). In this energy range, we need compute only the radial function for $l = 0$,

$$\frac{d^2R_0}{dr^2} + [k^2 - v(r)]R_0 = 0; \quad R_0 \to \begin{cases} 0; & r \to 0 \\ (1/k)\sin(kr - \eta_l); & r \to \infty \end{cases}$$

which is a simpler equation to deal with than are the ones for $l > 0$. (In both cases, we neglect exchange effects, which complicate both calculations but which may be neglected for heavier atoms and nuclei.)

For example, we can solve exactly several cases, for potential functions which cannot be solved when the $l(l+1)/r^2$ term is present. In the case given on page 1670, for example, for the potential $V = -Be^{-r/d}$, the solution for $l = 0$ for positive energy $E = (\hbar^2/2M)k^2$ is the linear combination

$$R_0 = N\left[\frac{J_{2ikd}(2bde^{-r/2d})}{J_{2ikd}(2bd)} - \frac{J_{-2ikd}(2bde^{-r/2d})}{J_{-2ikd}(2bde^{-r/2d})}\right]; \quad b^2 = 2MB/\hbar^2$$

which goes to zero at $r = 0$, as required. The constant N is adjusted so that R_0 has the required asymptotic behavior.

$$J_{2ikd}(2bde^{-r/2d}) \xrightarrow[r\to\infty]{} \frac{1}{\Gamma(1 + 2ikd)}\, e^{2ikd\ln(bd) - ikr}$$

Since $\quad \Gamma(1 + 2ikd) = \sqrt{(2\pi kd)/\sinh(2\pi kd)}\; e^{i\Omega(2kd)}$

and $\quad J_{2ikd}(2bd) = |J|e^{i\Phi(kd,bd)}$

then N can be written as $(i/2k)\sqrt{|J|^2(2\pi kd)/\sinh(2\pi kd)}$ and the asymptotic form for R_0 is

$$R_0 \to (1/k)\sin[kr - 2kd\ln(bd) + \Omega + \Phi]; \quad r \to \infty$$

which determines the phase angle η_0. For sufficiently small values of the energy, the $l = 0$ term is the only one differing appreciably from the Bessel functions $j_l(kr)$ of the unperturbed plane wave, the scattered intensity

$$S \simeq \frac{1}{k^2r^2}\sin^2[\Omega + \Phi - 2kd\ln(bd)]$$

is independent of angle, and the scattering cross section

$$Q \simeq (4\pi/k^2)\sin^2[\Omega + \Phi - 2kd\ln(bd)]$$

is just $4\pi r^2 S$.

When the energy is small enough so that $kd \ll 1$, then both the gamma and Bessel functions may be simplified. For example, since

$$[(d/dz)\Gamma(z)]_{z=1} = -\gamma = -0.5772$$

and since

$$[(d/d\nu)J_\nu(a)]_{\nu=0} = \tfrac{1}{2}\pi N_0(a)$$

we have that

$$\Gamma(1 + 2ikd) \simeq 1 - 1.1544ikd; \quad \Omega \simeq -1.1544kd$$
$$J_{2ikd}(2bd) \simeq J_0(2bd) + \pi ikdN_0(2bd); \quad \Phi \simeq -\pi kd \cot[\delta_0(2bd)]$$

where $\tan[\delta_0(x)] = -[J_0(x)/N_0(x)]$. Consequently, the $l = 0$ phase angle, in the limit of small energies, becomes

$$\eta_0 \simeq kd\{\ln(b^2d^2) + \pi \cot[\delta_0(2bd)] + 1.1544\}$$

and the total cross section becomes

$$Q \simeq 4\pi d^2\{1.1544 + \ln(b^2d^2) + \pi \cot[\delta_0(2bd)]\}^2 \qquad (12.3.57)$$

Finally, when $2bd$ is much smaller than unity (the potential well is shallow or narrow or both) then $\cot \delta_0 \simeq (-2/\pi)[\ln bd + \gamma - (bd)^2]$. The phase shift η_0 is then

$$\eta_0 \simeq -2kd(bd)^2; \quad 2bd \ll 1$$

and

$$Q \simeq 16\pi d^2(bd)^4; \quad 2bd \ll 1$$

This result agrees, of course, with the value for η_0 obtained by perturbation techniques in which the effect of the interaction is assumed to be small. The perturbation result is called the *Born approximation* and is discussed immediately below.

The Born Approximation. For fields small enough or particle velocities high enough, it is possible to consider the whole scattering field as a perturbation, modifying an incident plane wave, by setting

$$\nabla^2\psi + k^2\psi = (2M/\hbar^2)V(r)\psi; \quad k = \sqrt{2ME/\hbar^2} = Mv/\hbar$$

considering the right-hand term as an inhomogeneous part and "solving" by the use of the Green's function. If we wish the incident wave to be a plane wave of unit amplitude, e^{ikz}, the resulting integral equation for ψ is

$$\psi(r) = e^{ikz} - \frac{M}{2\pi\hbar^2} \iiint \frac{e^{ikR}}{R} V(\mathbf{r}_0)\psi(\mathbf{r}_0)\, dv_0 \qquad (12.3.58)$$

where $R = \sqrt{r^2 + r_0^2 - 2rr_0 \cos \Omega}$ is the distance between r and r_0 and Ω is the angle between the two vectors. We can, of course, replace e^{ikz} by any other solution of the homogeneous Helmholtz equation which we wish, to correspond to other possible incident waves.

This integral equation may be solved by successive approximations, provided the integral converges. If the integral converges absolutely, *i.e.*, if $\int|V|\,dv$ is finite, this series of iterations may converge (see page 1074), but the first few terms in the sequence sometimes turn out to be remarkably good approximations even when this is not true. A detailed examination of the integral shows that the criterion for rapidity of convergence of the iteration may be expressed in terms of the De Broglie wavelength $\lambda = \hbar/Mv = 2\pi/k$ and the distance r_c of closest approach of the particle in the repulsive field $|V(r)|$ (r_c is the root of the equation $|V(r)| = E$). It turns out that the first approximation to the solution of Eq. (13.3.58) is certainly adequate whenever $\lambda \ll r_c$; it is often adequate even when this criterion is not satisfied.

An examination of the value of ψ at $r = 0$ will often suffice to indicate whether the first few approximations converge. The first approximation to ψ is, of course, obtained by setting e^{ikz} for ψ in the integral of Eq. (12.3.58). For $r = 0$ the first approximation is

$$\begin{aligned}\psi_1(0) &= \frac{-M}{2\pi\hbar^2} \iiint \frac{e^{ikr(1+\cos\vartheta)}}{r} V(r) r^2 \, dr \sin\vartheta \, d\vartheta \, d\varphi \\ &= -\frac{2M}{k\hbar^2} \int_0^\infty V(r) e^{ikr} \sin(kr) \, dr\end{aligned}$$

and if $\psi_1(0)$ is small compared to the unit amplitude of the plane incident wave, $\psi_1(r)$ will presumably be small for $r > 0$. If now $V(r) \to A/r^\alpha$ as $r \to 0$ and $V \to B/r^\beta$ as $r \to \infty$, then α must be less than 2 (the force near the origin cannot be inverse cube) for this integral to converge, for any value of k, and β must be greater than 1 (the force at great distances must fall off faster than inverse square) for the integral to stay finite as $k \to 0$.

If these limiting requirements are met, then the first approximation will be small for all values of k (and the higher approximations correspondingly smaller) if $(2M/\hbar^2)\int V(r) r \, dr$ is small compared to unity. As k increases, the integral decreases in magnitude because the integrand oscillates faster and faster because of the factor $e^{ikr} \sin kr$. If V goes to zero monotonically as r increases, $\psi_1(0)$ will be, in general, smaller than the integral over r from zero to the first zero of $\sin kr$;

$$|\psi_1(0)| \leq \frac{2M}{\hbar^2} \int_0^{2/k} V(r) r \, dr$$

If $V(r) \to A/r$ as $r \to 0$ and $V \leq A/r^\beta (\beta > 1)$ as $r \to \infty$, then $\psi_1(0) \leq 4MA/k\hbar^2 = 4A/v\hbar$. But the distance of closest approach r_c, mentioned in the previous page, is approximately determined by equating the kinetic energy, $\frac{1}{2}Mv^2$, to the value of $|V|$ at r_c. For the higher energies, $|V| \to A/r$ so that $r_c \simeq 2A/Mv^2$ or $A \simeq \frac{1}{2}Mv^2 r_c$ so that the criterion that $\psi_1(0)$ be small is that $2Mvr_c/\hbar = 4\pi r_c/\lambda$ be small, where $\lambda = 2\pi\hbar/Mv$ is the De Broglie wavelength of the incoming particle. Thus, for this example, we arrive at our requirement that the iteration approximation converges when r_c is smaller than $\lambda/2\pi$ and that the first approximation is sufficient when r_c is considerably smaller than $\lambda/2\pi$ (see discussion on page 1092).

When the first approximation is adequate, we need not calculate ψ close in to the center of force; only the asymptotic form is needed to compute the scattering. When r is much larger than r_0,

$$\frac{e^{ikR}}{R} \to \frac{e^{ikr}}{r} e^{-ikr_0 \cos \Omega}; \quad r \gg r_0$$

If $\mathbf{k}_s$ is the vector of magnitude k and direction parallel to $\mathbf{r}$, the direction of motion of the part of the scattered beam which we are measuring, then $kr_0 \cos \Omega = \mathbf{k}_s \cdot \mathbf{r}_0$. Also the incident plane wave may be written $e^{i\mathbf{k}_i \cdot \mathbf{r}}$ where $\mathbf{k}_i$ is the vector representing the incident beam. Then the asymptotic form for the first approximation to ψ is

$$\psi(\mathbf{r}) \to e^{i\mathbf{k}_i \cdot \mathbf{r}} - \frac{Me^{ikr}}{2\pi\hbar^2 r} \iiint e^{i(\mathbf{k}_i - \mathbf{k}_s)\cdot \mathbf{r}_0} V(r_0)\, dv_0 \qquad (12.3.58)$$

This formula is known as the *Born approximation*. Within its range of validity it is a remarkably simple and useful formula. It states that the scattered wave, at very large distances from the scatterer, has an amplitude equal to $(\sqrt{2\pi}\, M/\hbar^2 r)$ times *the Fourier transform of the scattering potential* V with respect to the vector difference $\mathbf{K} = \mathbf{k}_i - \mathbf{k}_s$. Since the magnitude of either $\mathbf{k}_i$ or $\mathbf{k}_s$ is $k = Mv/\hbar$, we see that the magnitude of $\mathbf{K}$ is $\mu = 2k \sin(\frac{1}{2}\vartheta)$ where ϑ is the *angle of scattering*, the angle between $\mathbf{k}_s$ and $\mathbf{k}_i$. We also see that $\mathbf{K}$ is $1/\hbar$ times the momentum given to the center of force by the particle during the scattering process.

Since $V(r)$ is spherically symmetric, we can integrate over the angle part of dv_0, giving

$$\begin{gathered}\psi(\mathbf{r}) \to e^{i\mathbf{k}_i \cdot \mathbf{r}} + \frac{e^{ikr}}{r} f(\mu); \quad \mu = (2Mv/\hbar) \sin(\tfrac{1}{2}\vartheta) \\ f(\mu) \simeq -\frac{2M}{\hbar^2} \int_0^\infty V(r_0) \frac{\sin(\mu r_0)}{(\mu r_0)} r_0^2\, dr_0\end{gathered} \qquad (12.3.59)$$

The function f is the relative amplitude of that part of the beam which is scattered at the angle ϑ. The integral of its square over all angles is the total cross section Q. We note that, to this approximation, the dependence of f (and thus of $|f|^2 = S$, the angle-distribution function) on incident particle velocity v and on angle of scattering ϑ is through the quantity $\mu = (2Mv/\hbar) \sin(\frac{1}{2}\vartheta)$. As indicated above, this formula for f is valid if $\mathbf{K}$ is large enough. For example, for the potential on page 1682, we find that the results obtained from Eq. (12.3.59) correspond quite closely to the exact solution, obtained on page 1683, as long as μ is larger than about $2Z$.

In many cases the potential field causing the scattering comes from a nucleus of charge Ze, surrounded by a spherically symmetric distribution $\rho(r)$ of electrons, enough to make up a neutral atom. In this case, the potential function is

$$V(r) = \frac{4\pi}{r} \int_r^\infty \rho(r)r^2\,dr - 4\pi \int_r^\infty \rho(r)r\,dr; \quad 4\pi \int_0^\infty \rho r^2\,dr = Ze^2$$

Inserting this in the expression for the angle-distribution function and integrating by parts, we have

$$f(\mu) \simeq \frac{2MZe^2}{\mu^2\hbar^2}\,[F(0) - F(\mu)]; \quad F(\mu) = \frac{4\pi}{Ze^2} \int_0^\infty \rho(r)\,\frac{\sin(\mu r)}{\mu r}\,r^2\,dr \qquad (12.3.60)$$

where $F(0) = 1$, by definition of the electron charge for the neutral atom. Function $F(\mu)$ is called the *structure factor* of the charge distribution.

When μ is large enough so that $1/\mu$ is small compared to the "radius" of the atom (the value of r within which about three-fourths of the electronic charge lies), then $F(\mu)$ is small compared to $F(0) = 1$. For this range the angle-distribution factor is

$$f(\mu) \to f_R(\mu) = \frac{2MZe^2}{\hbar^2\mu^2} = \frac{Ze^2}{2Mv^2\sin^2(\frac{1}{2}\vartheta)}; \quad F(\mu) \ll 1$$

the square of which is the angular distribution of scattering, according to the Rutherford formula, for charged particles from the coulomb field of a nucleus of charge Ze. To this first approximation, whenever the incident particles are fast enough (v large) and come close enough to be scattered by an appreciable angle (ϑ large enough), then the particles pay practically no attention to the distributed electronic charge ρ but are scattered primarily by the central nucleus.

On the other hand, when μ times the atomic "radius" is small compared to unity (for slow incident particles or for glancing collisions), we can expand $[\sin(\mu r)/\mu r]$ in the integral for $F(\mu)$, to obtain

$$F(\mu) \to 1 - \mu^2\left(\frac{2\pi}{3Ze^2}\right) \int_0^\infty \rho(r)r^4\,dr; \quad \mu \to 0$$

and the limiting values of $f(\mu)$ and of the cross section for scattering Q ($\to 4\pi|f|^2$ for μ small) are

$$f(\mu) \to \tfrac{1}{3}Z(r_a^2/a_0); \quad Q \to (4\pi/9)(Z^2r_a^4/a_0^2)$$

where

$$r_a^2 = (4\pi/Ze^2) \int_0^\infty \rho(r)r^4\,dr$$

is the mean-square distance of the electronic charge from the nucleus and where $a_0 = \hbar^2/Me^2 = 0.532 \times 10^{-8}$ cm is the "first Bohr orbit radius" for the hydrogen atom. The limiting cross section for very slow incident particles is thus proportional to the square of the nuclear charge and to the fourth power of the "rms radius of the atom" r_a, if the Born approximation is valid in this limiting case. We note that, for this approximation, the scattering is the same for repulsive fields as for equal and opposite attractive fields.

The quantity $1 - F(\mu) = [f(\mu)/f_R(\mu)]$ is, consequently, the ratio of the actual scattering amplitude to the Rutherford amplitude for the same scattering angle and nuclear charge. When the Born approximation is valid, $1 - F$ is a function only of $\mu = (2Mv/\hbar)\sin(\frac{1}{2}\vartheta)$ and of the radial distribution of electronic charge. It increases as μ^2 when μ is small and rises fairly smoothly up to an asymptotic value unity when μ is quite large. This function may be measured experimentally by measuring the scattering of high-speed electrons (or protons) from a given atom. And since F is a Fourier transform of the electronic charge density, we can invert the integral to obtain ρ or V from an experimentally determined $F(\mu)$. We have

$$\begin{aligned}\rho(r) &= \frac{Ze^2}{2\pi^2 r}\int_0^\infty F(\mu)\sin(\mu r)\mu\,d\mu;\quad F(\mu) = 1 - \left[\frac{f(\mu)}{f_R(\mu)}\right] \\ V(r) &= Ze^2 + \frac{\hbar^2}{\pi m}\int_0^\infty \sin(\mu r)[f(\mu) - f_R(\mu)]\mu\,d\mu\end{aligned} \tag{12.3.61}$$

where $f(\mu)$ is the amplitude of the scattered wave, as defined in Eq. (12.3.59) and $f_R(\mu) = 2MZe^2/\hbar^2\mu^2$ is the Rutherford scattering amplitude. Consequently, if we know the scattering amplitude f for all values of μ and if we also know Z, we can compute both ρ and V (to the approximation inherent in the Born approximation).

Phase Angles by Born Approximation. In some cases the potential function V is too large to be amenable to treatment by the first-order Born approximation, yet the $l = 0$ phase angle η_0 is the only large term in the exact solution of Eq. (12.3.56). This is often the case in nuclear scattering problems, where the potential well is quite narrow and deep. In this case, we solve the radial equations (12.3.33), for each l, as accurately as we can. For example, for $l = 0$ we could start with an unperturbed form which includes one of the potentials discussed on pages 1670 *et seq.*, for which we have exact solutions; whereas for the other values of l we could take the spherical Bessel function. We can set $V(r) = V_0(r) + V_1(r)$ where we know the solutions for V_0 for the radial function for $l = 0$,

$$\frac{d^2R}{dr^2} + \left[k^2 - \frac{2M}{\hbar^2}V_0(r)\right]R = 0$$

and where $V_1(r)$ is considerably smaller than $V_0(r)$. Suppose we call $y_{01}(r)$ the solution of this equation which goes to zero at $r = 0$ and which has the asymptotic form

$$y_{01}(r) \to \sin(kr - \Phi_0);\quad r \to \infty$$

and we call y_{02} the independent solution which has the asymptotic form

$$y_{02}(r) \rightarrow -\cos(kr - \Phi_0); \quad r \rightarrow \infty$$

and which, usually, diverges at $r = 0$. We note that the Wronskian of these solutions is k.

The equation which we wish to solve, for $l = 0$, is, however,

$$\frac{d^2R_0}{dr^2} + \left[k^2 - \frac{2M}{\hbar^2} V_0(r)\right] R_0 = \frac{2M}{\hbar^2} V_1(r)R_0$$

If this is treated as an inhomogeneous equation, Eq. (5.2.19) shows that the integral equation for R_0 is

$$R_0(r) = a_0 y_{01}(r) - \int_0^r K_0(r|r_0)R_0(r_0)\,dr_0$$

where

$$K_0(r|r_0) = \frac{2M}{k\hbar^2}\,[y_{01}(r)V_1(r_0)y_{02}(r_0) - y_{02}(r)V_1(r_0)y_{01}(r_0)]$$

Solving this by successive iterations, the "zeroth" approximation is $a_0Y_{00} = a_0y_{01}$, the next is $a_0Y_{01} = -a_0\int K_0Y_{00}\,dr_0$ and, in general,

$$R_0(r) = a_0 \sum_{m=0}^{\infty} (-1)^m Y_{0m}(r); \quad Y_{00}(r) = y_{01}(r)$$

$$Y_{0m}(r) = \int_0^r K_0(r|r_0)Y_{0,m-1}(r_0)\,dv_0$$

or

$$R_0(r) = a_0\frac{2M}{k\hbar^2}\left[y_{01}(r)\sum_{m=0}^{\infty} F_{m1}(r) - y_{02}(r)\sum_{n=1}^{\infty} F_{n2}(r)\right]$$

$$F_{01} = \frac{k\hbar^2}{2M}; \quad F_{m1}(r) = (-1)^m \int_0^r V_1(r_0)y_{02}(r_0)Y_{0,m-1}(r_0)\,dr_0$$

$$F_{n2}(r) = (-1)^n \int_0^r V_1(r_0)y_{01}(r_0)Y_{0,n-1}(r_0)\,dr_0$$

When $r \rightarrow \infty$, $V_1(r_0) \rightarrow 0$ fast enough so that the F's all reach constant values asymptotically (or else the successive approximation does not converge at all) and for $r \rightarrow \infty$

$$R_0(r) \rightarrow b_0 \sin[kr - \eta_0]$$

where $\quad b_0 = a_0(2M/k\hbar^2)\{[\Sigma F_{m1}(\infty)]^2 + [\Sigma F_{n2}(\infty)]^2\}^{\frac{1}{2}}$

and $\quad \eta_0 = \Phi_0 - \tan^{-1}\{[\Sigma F_{n2}(\infty)]/[\Sigma F_{m1}(\infty)]\}$

The angle η_0* is the phase angle to insert in the first terms of Eqs. (12.3.56). If V_1 is sufficiently small compared to V_0, only $F_{12}(\infty)$ needs to be calculated, the higher order terms being negligible; in which case

* See footnote on p. 1682.

the equations for R_0 and η_0 become

$$R_0(r) \to a_0 \sin(kr - \eta_0); \quad r \to \infty$$
$$\eta_0 \simeq \Phi_0 + \tan^{-1}\left\{\left(\frac{2M}{k\hbar^2}\right) \int_0^\infty V_1(r_0)[y_{01}(r_0)]^2 \, dr_0\right\} \tag{12.3.62}$$

which can be computed once we know Φ_0 and the first solution, $y_{01}(r)$.

The terms in Eqs. (12.3.56) for $l > 0$ may also be calculated by a similar procedure. In many cases, however, the higher phase angles may be computed, to sufficient accuracy, by using all of V as a perturbation. In other words, we set $V_0 = 0$; in which case the two solutions of the radial equation for $l > 0$ are

$$y_{l1}(r) = (kr)j_l(kr) \to \sin(kr - \tfrac{1}{2}l\pi)$$
$$y_{l2}(r) = (kr)n_l(kr) \to -\cos(kr - \tfrac{1}{2}l\pi)$$

In this case a calculation, similar to that given in Eqs. (12.3.62), obtains

$$R_l(r) \to a_l \sin(kr - \tfrac{1}{2}l\pi - \eta_l); \quad r \to \infty$$
$$\eta_l \simeq \tan^{-1}\left\{\left(\frac{2Mk}{\hbar^2}\right) \int_0^\infty V(r_0)[j_l(kr_0)]^2 r_0^2 \, dr_0\right\} \tag{12.3.63}$$

to the first order in the successive approximation solution of the integral equation for R_l.

When the magnitude of the potential energy $V(r)$ is considerably smaller than k^2 for $r > r_v$ then for all values of l larger than kr_v the spherical Bessel functions in the integral for η_l may be replaced by the first terms in their series expansion

$$j_l(kr_0) \simeq \frac{(kr_0)^l}{1 \cdot 3 \cdot 5 \cdots (2l + 1)}$$

over the whole range of integration where $V(r_0)$ is large. In this case an even simpler formula is

$$\eta_l \simeq \frac{(2M/\hbar^2)k^{2l+1}}{[1 \cdot 3 \cdot 5 \cdots (2l + 1)]^2} \int_0^{r_v} V(r_0) r_0^{2l+2} \, dr_0$$

For the forms of V usually encountered, this expression diminishes in value quite rapidly as l is increased.

We may show that the Born approximation (12.3.59) and the angular distribution obtained from formula (12.3.63) for η_l are identical. For this purpose we require the expansion of $(\sin \mu r)/\mu r$ in Legendre polynomials of the angle ϑ between $\mathbf{k}_i$ and $\mathbf{k}_s$, which expansion may be obtained by taking the imaginary part of the expansion of $e^{i\mu r}/\mu r$ (see table at end of Chap. 11);

$$\frac{\sin \mu r}{\mu r} = \sum_l (2l + 1) P_l(\cos \vartheta) j_l^2(kr)$$

Substituting in Eq. (12.3.59) yields

$$\psi_s = -\frac{2M}{\hbar^2}\frac{e^{ikr}}{r}\sum_l (2l+1)P_l(\cos\vartheta)\int_0^\infty j_l^2(kr_0)V(r_0)r_0^2\,dr_0$$

Compare this with Eq. (12.3.56) in which we make the approximation $\eta_l \ll 1$;

$$\psi_s = -\frac{e^{ikr}}{kr}\sum_l (2l+1)\eta_l P_l(\cos\vartheta)$$

Upon substituting Eq. (12.3.63) for η_l in this equation, assuming $\eta_l \ll 1$, we obtain an expression exactly equal to the result obtained from an expansion of the Born approximation in Legendre functions as given above.

As an example, consider the scattering potential

$$V(r) = -Be^{-r/d}$$

We need to employ the Born approximation (12.3.63) here only for $l > 0$ since for $l = 0$ the exact solution was obtained on page 1687:

$$\begin{aligned}\eta_0 &= 2kd\ln(bd) - \Omega - \Phi; && b^2 = 2MB/\hbar^2\\ &\simeq 2kd\{\ln(bd) + 0.5772 + \tfrac{1}{2}\pi\cot[\delta_0(2bd)]\}; && kd \ll 1\\ &\simeq -0.2319kd; && kd \ll 1;\quad bd \ll 1\end{aligned}$$

For values of $l > 0$, we use Eqs. (12.3.63) to obtain approximate formulas for the phase angles (see page 1575);

$$\begin{aligned}\eta_l &\simeq -b^2k\int_0^\infty e^{-r/d}[j_l(kr)]^2r^2\,dr = \left(\frac{b^2}{2k}\right)\frac{d}{d(1/d)}\,Q_l\left(\frac{1+2k^2d^2}{2k^2d^2}\right)\\ &= \frac{-(ld^3b^2/k)}{1+4k^2b^2}\left\{Q_{l-1}\left(1+\frac{1}{2k^2d^2}\right) - \left(1+\frac{1}{2k^2d^2}\right)Q_l\left(1+\frac{1}{2k^2d^2}\right)\right\}\\ &\qquad\to -b^2d^2(kd)^{2l+1}\left[\frac{2\cdot4\cdot6\cdots(2l+2)}{1\cdot3\cdot5\cdots(2l+1)}\right];\quad kd \ll l\end{aligned}$$

The last formula ensures satisfactory convergence of the scattering and cross-section series of Eqs. (12.3.56) as long as $kd < \frac{1}{2}$. When kd is much larger than unity, the scattering may be computed by using the Born approximation formulas of Eqs. (12.3.59). For intermediate ranges of kd, the exact expression for η_0 and the formulas giving η_l in terms of the Legendre functions of the second kind result in fairly accurate series for S and Q. Since application of Eq. (12.3.63) to all η_l's is just the Born approximation (12.3.59), the cross section may be written as $Q_B + (4\pi/k^2)\,(\sin^2\eta_0 - \sin^2\eta_0^B)$ where Q_B is the Born approximation cross section and η_0^B is given by Eq. (12.3.63) with $l = 0$.

The Variational Method. The method of variation of parameters, discussed in Chap. 9, may be used to compute wave functions and energies for bound states, or the scattering of free particles, as accurately as we please, if we are willing to choose a trial function with enough parameters. As an example of the method, we shall carry out the variational computations on an attractive central force of the form $V(r) = -Be^{-r/d}$, for which we have already made some calculations.

For the bound states we usually minimize the energy. We assume a normalized wave function $\phi(\alpha,\beta, \ldots ,\mathbf{r})$ having the required general properties (normalized, finite, symmetric about the origin for the lowest state, etc.) and having the details of its shape determined by the parameters α, β, . . . within the limits imposed by the general requirements. We then compute the integral

$$J(\alpha,\beta, \ldots) = \iiint \left[(\text{grad } \phi)^2 + \left(\frac{2M}{\hbar^2}\right) V\phi^2 \right] dv$$
$$= \iiint \left[-\phi\nabla^2\phi + \left(\frac{2M}{\hbar^2}\right) V\phi^2 \right] dv \quad (12.3.64)$$

which is a function of the parameters α, β, ... The minimum value of J then gives the best value of $2M/\hbar^2$ times the allowed energy $(2ME/\hbar^2 = \epsilon)$ of the bound state which can be obtained with the assumed form of ϕ. This best value is always larger than the correct value. As the functional form ϕ is made more flexible by the addition of more variational parameters, the approximate value of the energy as obtained by the variational method approaches the correct value from above. Also the values of the parameters for which J is minimum make ϕ as close to the exact solution ψ of the Schroedinger equation

$$\nabla^2\psi + [\epsilon - (2M/\hbar^2)V]\psi = 0$$

as is possible within the limitations of the choice of form of ϕ.

A number of considerations should enter in making a choice of form of ϕ. An important one is symmetry; if V is a central force field, the ϕ for a lowest state is symmetric about the origin, ϕ's for higher states must be orthogonal to this, etc. The function should also have an asymptotic behavior fairly close to that of ψ; if V is negligibly small compared to $|E| = |\hbar^2\epsilon/2M|$ for $r > r_v$, then $\psi \to A \exp(-\sqrt{-\epsilon}r)$ for $r > r_v$. Finally (but not the least important) ϕ should be an analytic form which allows the calculations to be carried out without too great computational difficulty.

The simplest form for the ground-state wave function is the simple exponential

$$\phi(\alpha,r) = \sqrt{\frac{\alpha^3}{\pi}}\, e^{-\alpha r}; \quad \iiint \phi^2\, dv = 1$$

In this case we sacrifice correctness of asymptotic behavior to simplicity of form. If $V = -Be^{-r/d}$, the variational integral

$$J(\alpha) = \alpha^2 - [b^2\alpha^3/(\alpha + \tfrac{1}{2}\delta)^3]; \quad b^2 = (2MB/\hbar^2); \quad \delta = 1/d$$

The equation $dJ/d\alpha = 0$ then reduces to finding the real, positive roots of the biquadratic equation

$$(\alpha + \tfrac{1}{2}\delta)^4 = \tfrac{3}{4}b^2\delta\alpha$$

which may be solved graphically, algebraically, or by approximate means. For example, if $b \gg \delta$ (deep, narrow potential hole), then

$$\alpha \simeq (\tfrac{3}{4}b^2\delta)^{\frac{1}{3}} - \tfrac{2}{3}\delta; \quad J \simeq -b^2 + 3(\tfrac{3}{4}b^2\delta)^{\frac{2}{3}}$$

or

$$E \simeq -B + 3.931(\hbar^2B/8d^2M)^{\frac{1}{3}}$$

which is to be compared with the approximation to the exact solution given in Eq. (12.3.44). The energy computed by the variational method, using this simple trial function, gives a lowest state which is too high by about 5 per cent of its height above the bottom of the hole $(-B)$.

Another form which is a bit more flexible is

$$\phi(\alpha,\beta;r) = \sqrt{\frac{\alpha\beta(\alpha+\beta)}{2\pi(\beta-\alpha)^2}}\left(\frac{1}{r}\right)[e^{-\alpha r} - e^{-\beta r}]$$

where we lose no generality if we assume that $\beta > \alpha$. The corresponding variational integral is

$$J(\alpha,\beta) = \alpha\beta - \frac{4b^2\alpha\beta(\alpha+\beta)}{(2\alpha+\delta)(\alpha+\beta+\delta)(2\beta+\delta)}$$

which could, of course, be minimized for both α and β, by setting the two partials of J, with respect to α and with respect to β, equal to zero and solving the two resulting equations simultaneously for α and β, then substituting back in for J, to obtain the allowed energy. This requires the solution of the simultaneous equations

$$\begin{aligned}(2\alpha+\delta)^2(\alpha+\beta+\delta)^2(2\beta+\delta)^2 &= 4b^2\delta(2\alpha+\delta)[3\beta^2 + 2\beta(\alpha+\delta) \\ &\qquad + \alpha(\alpha+\delta)] \\ &= 4b^2\delta(2\beta+\delta)[3\alpha^2 + 2\alpha(\beta+\delta) \\ &\qquad + \beta(\beta+\delta)]\end{aligned}$$

which may be solved numerically, but are rather difficult for algebraic solution.

Another possibility involves using only parameter β for variation, using parameter α to ensure correct asymptotic behavior. In other words, we require α to equal $\sqrt{-\epsilon}$, or to the first approximation, we require α to equal the square root of minus the minimum value of J. We thus again have two simultaneous equations to solve to determine α and β. The first is obtained by requiring that J be a minimum for β $(\partial J/\partial\beta = 0)$;

$$(2\alpha+\delta)(\alpha+\beta+\delta)^2(2\beta+\delta)^2 = 4b^2\delta[3\beta^2 + 2\beta(\alpha+\delta) + \alpha(\alpha+\delta)]$$

and the other by requiring that the J, for which this first equation holds, be equal to $-\alpha^2$;

$$(2\alpha + \delta)(\alpha + \beta + \delta)(2\beta + \delta) = 4b^2\beta$$

This set of simultaneous equations is somewhat easier to solve than the first set; it gives results nearly as accurate. Moreover, when we have finished calculating α and β, the best value of J is just $-\alpha^2$.

The solution is $\beta = \sqrt{\frac{1}{2}\delta(\alpha + \delta)}$. The corresponding relation between the binding energy and depth is, recalling that $\delta = 1/d$,

$$(2bd)^2 = (2\alpha d + 1)[1 + \sqrt{2(1 + \alpha d)}]^2; \quad \alpha = \sqrt{-\epsilon}$$

Comparing this with the exact relationship $J_{2\alpha d}(2bd) = 0$, we find that it gives excellent results for $\alpha d < 1$, being in error at $\alpha d = 4$ by 3 per cent and at $\alpha d = 0$ by $\frac{2}{3}$ per cent. For greater values of αd, expression (12.3.44) is more appropriate.

How may this result be improved? One may of course add on additional terms to the form $(1/r)[e^{-\alpha r} - e^{-\beta r}]$ as for example one may employ $(1/r)\{[e^{-\alpha r} - e^{-\beta r}] + C[e^{-\alpha r} - e^{-\gamma r}]\}$ where now C and γ as well as β are variational parameters. This, however, becomes unwieldy very rapidly; the addition of many such terms involves the solution of secular determinants for various values of a group of nonlinear parameters β and γ and others which would occur in the additional terms. Another procedure, the *variation-iteration* method to be given below, is more economical and has the added advantage of yielding a lower bound to $(bd)^2$ as well as the upper bounds which the variational method normally gives.

Variation-Iteration Method. We shall review the method which is described in Chap. 9, page 1138, specializing the notation to the Schroedinger equation for a bound state in the field of an attractive central potential. For $l = 0$ let $\psi = R/r$, where R satisfies

$$(d^2R/dr^2) + [-\kappa^2 + b^2u(r)]R = 0; \quad \kappa^2 = -\epsilon$$

The first step is the conversion of this equation into an integral equation through the application of the Green's function:

$$(d^2/dr^2)G_\kappa(r|r') - \kappa^2 G_\kappa(r|r') = -\delta(r - r')$$
$$G_\kappa = \frac{1}{\kappa}\begin{cases} \sinh(\kappa r)e^{-\kappa r'}; & r < r' \\ \sinh(\kappa r')e^{-\kappa r}; & r > r' \end{cases}$$

The integral equation satisfied by R is

$$R(r) = b^2 \int_0^\infty G_\kappa(r|r')u(r')R(r')\,dr'$$

We may regard this integral equation as an eigenvalue problem with b^2 as the eigenvalue. It here should be recalled that b^2 is the potential depth whereas κ^2 is proportional to the energy. Thus determining b^2, as we shall now do, corresponds to finding the potential strength for a given potential form u which will give a level with a given energy. This arrangement corresponds nicely to the situation in nuclear physics where the energy is known from experiment, and it is desired to find the corresponding potential energy. Of course if the calculation determining b^2 can be carried out analytically, then one arrives at a relation between b^2 and κ^2 which may be inverted to give κ^2 in terms of b^2.

We may now state the iteration part of the method. Let R_0 be an initial guess for R, which may have been obtained in many ways (for example, by the variation method with a simple trial function). Then the first iterate R_1 is given by

$$R_1(r) = \int_0^\infty G_\kappa(r|r')u(r')R_0(r')\,dr'$$

and generally

$$R_n(r) = \int_0^\infty G_\kappa(r|r')u(r')R_{n-1}(r')\,dr'$$

Note that the eigenvalue b^2 need not appear in these recurrence relations since only the form of the wave functions is relevant, not their normalization. These wave functions R_n are now inserted into the variational expressions (12.3.65) and into one which may be obtained directly from the integral equation satisfied by R [see Eq. (9.4.42)]. Here the quantity J to be varied is

$$J = \frac{\int_0^\infty dr\,[u(r)R^2(r)]}{\int_0^\infty dr \int_0^\infty dr'\,[R(r)u(r)G_\kappa(r|r')u(r')R(r')]} \tag{12.3.65}$$

The approximations to the eigenvalue b^2 are then labeled $(b^2)^{(n)}$ where (n) may be either integer or half integer and $(b^2)^{(0)}$ is obtained by the insertion of R_0/r for variational principle (12.3.65). The other approximations are

$$\begin{aligned}
(b^2)^{(\frac{1}{2})} &= (0,0)/(0,1) \\
(b^2)^{(1)} &= (0,1)/(1,1) \\
&\cdots\cdots\cdots\cdots \\
(b^2)^{(n+\frac{1}{2})} &= (n,n)/(n,\,n+1) \\
(b^2)^{(n+1)} &= (n,\,n+1)/(n+1,\,n+1)
\end{aligned}$$

When the element (p,q) is defined by

$$(p,q) = \int_0^\infty R_p u R_q\,dr$$

it may be shown (see Chap. 9, page 1139) that

$$(b^2)^{(0)} \geq (b^2)^{(\frac{1}{2})} \geq (b^2)^{(1)} \geq \cdots \geq (b^2)^{(n)} \geq (b^2)^{(n+\frac{1}{2})} \geq b^2$$

So that the successive approximations to b^2 form a monotonic sequence approaching b^2 from above.

From the discussion in Chap. 9, page 1144, we may also obtain a lower bound for b^2:

$$(b^2) \geq (b^2)^{(n+\frac{1}{2})}\left[1 - \frac{(b^2)^{(n)} - (b^2)^{(n+\frac{1}{2})}}{b_1^2 - (b^2)^{(n+\frac{1}{2})}}\right]$$

Here b_1^2 is the second eigenvalue of the integral equation for R. This corresponds to a potential depth for which $-\kappa^2$ is the second rather than the lowest eigenvalue. It is not necessary to know b_1^2 exactly; inserting a lower bound for b_1^2 in the above inequality will maintain it.

Such a lower bound may be obtained from the expansion [see Chap. 9, Eq. (9.4.113)]:

$$S = \int_0^\infty dr \int_0^\infty dr' \, [u(r)G_\kappa(r|r')G_\kappa(r'|r)u(r)] = \sum_{n=0}^\infty \frac{1}{(b_n^4)}$$

where (b_0^2) is the first eigenvalue discussed above, b_1^2 the next, etc. We see that

$$\frac{1}{b_1^4} = S - \frac{1}{b_0^4} - \sum_{n=2}^\infty \frac{1}{b_n^4}$$

or

$$\frac{1}{b_1^4} < \left(S - \frac{1}{b_0^4}\right)$$

or

$$\frac{1}{b_1^4} < \left\{S - \left[\frac{1}{(b^2)^{(n+\frac{1}{2})}}\right]^2\right\}$$

since $(b^2)^{(n+\frac{1}{2})} > b_0^2$. We thus have a lower bound for b_1^2 which may be obtained by direct integration of S and by use of the approximate value of b^2 for the smallest eigenvalue. Inserting the expression for G_κ given above into the definition of S, we obtain

$$S = \frac{2}{\kappa^2}\int_0^\infty dr \, u(r)e^{-2\kappa r}\int_r^\infty dr' \, u(r')\sinh^2 \kappa r'$$

which may readily be evaluated for any given potential form $u(r)$.

As an example, we return to the exponential potential $u = e^{-\delta r}$, using the trial function $\phi(\alpha,\beta;r)$ defined on page 1697 and setting $\alpha^2 = -\epsilon = \kappa^2$ as before. This function, with $\beta = \sqrt{\frac{1}{2}\delta(\alpha + \delta)}$, is consequently the initial function R_0; to obtain the iterates, we use G_κ with κ set equal to α. Thus

$$R_1 = \frac{2}{\delta(\delta + 2\alpha)}\left[e^{-\alpha r} - e^{-(\alpha+\delta)r}\right] - \frac{2}{(\delta + \beta)^2 - \alpha^2}\left[e^{-\alpha r} - e^{-(\beta+\delta)r}\right]$$

From this we may determine $(b^2)^{(\frac{1}{2})}$

$$(b^2)^{(\frac{1}{2})}d^2 = \frac{4[(\alpha+\beta)d+2][(\alpha+\beta)d+1][\beta d+1][\alpha d+1][2\alpha d+1]}{(2\beta d+1)[(10\alpha^2d^2+14\alpha d+5)+(7\alpha d+5)(\beta-\alpha)d+(\beta-\alpha)^2d^2]}$$

while
$$S = d^4/2(1+\alpha d)(1+2\alpha d)^2$$

It is also possible to obtain $(b^2)^{(1)}$. For $(\alpha d) = 0$, $(b^2)^{(0)}d^2 = 1.4571$, $(b^2)^{(\frac{1}{2})}d^2 = 1.4466$. The exact value is just one-quarter of the square of the first root of the Bessel function, equaling 1.4458. We see that as a result of one iteration the error was reduced from $\frac{2}{3}$ to $\frac{1}{20}$ per cent. Calculation of $(b^2)^{(1)}$ (which may be evaluated since R_1 is available) would be even closer to the exact answer. The value $(b^2)^{(\frac{1}{2})}$ given above is an upper bound. To obtain the lower bound we first show that $(b_1^2d^2) > (6.78)$ so that

$$(b^2d^2) > (1.4466)[1 - (0.0105/5.33)]$$

Hence
$$1.4466 > b^2d^2 > 1.4437; \quad \text{correct value, } 1.4458$$

We see that the correct value of $(bd)^2$ is determined to better than 0.3 per cent. As usual, the lower bound is not as close to the proper value of $(bd)^2$ as the upper bound.

It is not necessary to start with as good an initial trial function R_0 as was done above. Of course, it will take more iterations to achieve the accuracy obtained above after one iteration. However, in many cases, particularly when numerical work is required, it may be more economical not to carry out a complicated variational procedure but to rely on the economy achieved as a consequence of the repetitive nature of the iterative process.

Higher bound states (if they exist) may be computed, using a trial function which is orthogonal to the ϕ obtained for the lowest state. For example, an approximate lowest state for angular momentum equal to $\hbar$ is obtained by using

$$\phi(\gamma;r) = N\cos\vartheta\, re^{-\gamma r}$$

or

$$\phi(\alpha,\gamma;r) = N'\cos\vartheta(e^{-\alpha r} - e^{-\gamma r})$$

where N and N' are normalization constants making $\iiint\phi^2\,dv = 1$.

Variational Principles for Scattering. The scattering of particles by a field of force may also be treated by the variational method. Here we find the stationary value of $k\cot\eta_l$, with η_l the phase angle (see page 1681), rather than minimizing the energy. It is characteristic of the various forms of the variational principle to be discussed below that the form of the wave function is important only in the region where the force field exists.

We shall limit the discussion to slow incident particles where only the $l = 0$ functions differ very much from the corresponding Bessel functions, so that only the $l = 0$ phase angle η_0 is required for the computation of the scattering. The radial equation to be solved is (for $\psi = R/r$)

$$(d^2R/dr^2) + [k^2 - U(r)]R = 0; \quad k^2 = 2ME/\hbar^2; \quad U(r) = 2MV(r)/\hbar^2$$

For large r, we know that $R \to S \equiv C \sin(kr - \eta_0)/\sin \eta_0$ (C an arbitrary constant) so that S satisfies

$$(d^2S/dr^2) + k^2S = 0$$

The variational principles are expressed in terms of R and S, or in terms of $S - R = w$. The function w differs substantially from zero only in the region where $U \neq 0$ as may be seen from the equation it satisfies:

$$(d^2w/dr^2) + [k^2 - U]w + US = 0$$

Following the results of Chap. 9, we may obtain two variational principles which are quite analogous to Eq. (12.3.65). In the first we are required to obtain the stationary value of

$$[K] = \int_0^\infty \left[\left(\frac{dS}{dr}\right)^2 - \left(\frac{dR}{dr}\right)^2 - U(r)R^2 + k^2(S^2 - R^2)\right] dr \qquad (12.3.66)$$

When S is normalized so that $S(0) = 1$ (that is, $C = -1$), then $K = [k \cot \eta_0]$ (see page 1108 for discussion of the meaning of the brackets).

This variational principle can be employed to establish the dependence of $k \cot \eta_0$ on k^2. Let the exact solution at $k^2 = k_0^2$ be R_0 and its corresponding asymptotic dependence S_0. These may be inserted as variational wave functions in (12.3.66). Then

$$\left[\frac{d(k \cot \eta_0)}{d(k^2)}\right]_{k=k_0} = \int_0^\infty (S_0^2 - R_0^2)\, dr$$

which yields immediately the slope of $k \cot \eta_0$ at $k^2 = k_0^2$ as a function of k^2, that is, of the energy. It is immediately clear that only the behavior of R_0, in the region where U is important (where R_0 does not have its asymptotic value S_0) enters into the slope. If in this region $|U| \gg k^2$, that is, the magnitude of the potential energy is much larger than k^2, the variation of the wave function with k^2 will be small and we may expect that the above expression for the slope will be accurate for a range of values of k_0. Or in other words, for small k^2, low-energy particles, the function $k \cot \eta_0$ shows a straight-line dependence on k^2; this straight-line dependence holds as long as $k^2 \ll |U|$, that is, $E \ll |V|$.

Because the slope given above does not depend upon asymptotic values of R_0, these being canceled out by S_0 we may insert quite crude approximations to R_0 and still obtain accurate results. For example, in the case of the exponential potential considered in the preceding section, it was found that $R_0 = e^{-\alpha r} - e^{-\beta r}$ and hence $S_0 = e^{-\alpha r}$, $\beta d = \sqrt{\frac{1}{2}(1 + \alpha d)}$

was a fairly good wave function (a better one being given by R_1 on page 1700). If $\hbar^2\alpha^2/2M$, the binding energy of the system, is much smaller than V, the value of the integral for $d(k \cot \eta_0)/d(k^2)$ will not change much in going from an energy $-(\hbar^2\alpha^2/2M)$ to $+E$, so that we may insert this wave function for R_0 in the integral, even though its behavior for $r \to \infty$ is far from correct. Then

$$\frac{d[k \cot \eta_0]}{d(k^2)} \simeq d\left[\frac{3\beta d - \alpha d}{2\beta d(\alpha d + \beta d)}\right]$$

We, however, require more than just the slope; we need its value at k_0. For this purpose we may introduce a trial function R with one or more parameters and a corresponding S into (12.3.66) and vary the parameters to determine the best possible u and the corresponding $k \cot \eta_0$. Variational principle (12.3.66) is not the most convenient for this purpose, however, since S depends explicitly on the unknown phase shift η. If we insist that this be chosen as the exact phase shift rather than treating it as a variational parameter, one obtains a second variational principle [see Eq. (9.4.57)] which does not involve S. The quantity to be varied so as to obtain a stationary value is

$$L = (B + \sqrt{B^2 - 4AC})/2A$$

where
$$A = \int_0^\infty U\left(\frac{\sin^2 kr}{k^2}\right) dr$$
$$B = 1 + \int_0^\infty \frac{U \sin(2kr)}{k} dk - 2\int_0^\infty \frac{U \sin(kr)}{k} w\, dr$$
$$C = \int_0^\infty \left[\left(\frac{dw}{dr}\right)^2 - (k^2 - U)\right] dr - 2\int_0^\infty Uw \cos(kr)\, dr + \int_0^\infty U \cos^2 kr\, dr$$

where again $w = S - R$. If $w(0)$ is placed equal to 1, then L for the correct w gives just $k \cot \eta_0$. Again we emphasize the fact that w, being the difference between R and its asymptotic dependence S, will differ from 0 only in the region where U is appreciable compared to k^2.

The Born approximation (12.3.63) may be obtained from this variational principle by inserting $w = \cos kr$ as a trial function. This choice corresponds to taking R proportional to $\sin kr$, the undistorted incident plane wave. Then $C = 0$, $B = 1$, and $L = 1/A$ or

$$\tan \eta_0 \simeq \frac{1}{k}\int_0^\infty U \sin^2 kr\, dr$$

which is the Born approximation when η_0 is small.

A second useful type of approximate form for R is the one we already have employed to evaluate the slope. Again we say that for $U \gg k^2$, changing the energy from $\hbar^2k^2/2M$ to $-(\hbar^2\alpha^2/2M)$, $U \gg \alpha^2$ will not reflect itself strongly in a change in w so that we may take $w = e^{-\beta r}$ in L and so obtain the constant value of $[k \cot \eta_0]$, the slope as a function of k^2 having been discussed earlier. Since we are dealing with small k, we shall place $k = 0$ and obtain $k \cot \eta_0$ for that value. The values of A, B, and $B^2 - 4AC$ are

$$\begin{aligned} A &= -2b^2d^3 \\ B &= 1 - [2(b^2d^2)(\beta d)(2 + \beta d)/(\beta d + 1)^2] \\ B^2 - 4AC &= 1 + 4b^2d^2(\beta d) - [8(b^4d^4)(\beta^4d^4)(2 + \beta d)/(1 + \beta d)^4(1 + 2\beta d)] \end{aligned}$$

We may eliminate b^2d^2, if we wish, by employing the result obtained for $(2bd)^2$ on page 1698 which may be written

$$(2bd)^2 = (2\beta d - 1)(2\beta d + 1)^3$$

so that the final expression for $k \cot \eta_0$ at $k = 0$ depends only on the combination βd (and, therefore, αd) except for a proportionality factor of d.

In the energy region between the regions of validity of the Born approximation and the above (which may be termed the strong field approximation), one must actually carry out the variational procedure. For the case of slow incident particles, $w = e^{-\gamma r}$ is a useful trial function where γ is the variational parameter.

Variation-Iteration Method for Scattering. As in the case of bound states, we may set up an integral equation for the solution of the scattering problem. The iteration method may then be applied to the integral equation; the integral equation also provides a variational principle for $[k \cot \eta_l]$.

We again limit the discussion to the case $l = 0$. The integral equation is, written in a form most convenient for iteration:

$$R = \int_0^\infty \Gamma_k(r|r_0)U(r_0)R(r_0)\,dr_0$$

where

$$\begin{aligned} \Gamma_k(r|r_0) &= k \cot \eta_0 \frac{\sin(kr)\sin(kr_0)}{k^2} - \frac{1}{k}\begin{cases} \sin(kr)\cos(kr_0); & r < r_0 \\ \cos(kr)\sin(kr_0); & r > r_0 \end{cases} \\ &= k \cot \eta_0\,[\sin(kr)\sin(kr_0)/k^2] + G_k(r|r_0) \end{aligned}$$

The iterative scheme is given by the recursion relation between the nth iterate and the $(n + 1)$ iterate:

$$R_{n+1} = \int_0^\infty \Gamma_k(r|r_0)U(r_0)R_n(r_0)\,dr_0$$

Evaluating R_{n+1} involves substituting an approximate value for $k \cot \eta_0$ which may be obtained from the variational principle given below in terms of a quantity J which is to be varied, the best possible solution for a given form of R occurring at the stationary value of J. When the exact wave functions are introduced into J, it equals $k \cot \eta_0$ so that we have a variational principle for $k \cot \eta_0$. $[J]$ is

$$[J] = \frac{\int_0^\infty R^2 U\, dr - \int_0^\infty \int_0^\infty R(r)U(r)G_k(r|r_0)U(r_0)R(r_0)\, dr\, dr_0}{\left[\frac{1}{k}\int_0^\infty UR \sin(kr)\, dr\right]^2} \tag{12.3.67}$$

The iteration scheme is combined with the variational principle, by inserting R_n in Eq. (12.3.67) which yields the nth approximation for $k \cot \eta_0$, which we shall label $J^{(n)}$

$$J^{(n)} = J^{(n-1)} + \frac{\int_0^\infty R_n U[R_n - R_{n+1}]\, dr}{\left[\frac{1}{k}\int_0^\infty UR_n \sin(kr)\, dr\right]^2}$$

It should be emphasized that the theory of convergence of the iterative process for this type of problem has not been developed so completely as for the iterative process for the bound-state problem discussed earlier. Successful applications of this scheme due to Schwinger may be found in the literature where it is seen that the method is excellent for cases in which $U \gg k^2$, that is, the low-energy situation, the only case where it has been applied extensively.

As a final comment, we note that, by introducing the trial function $R = \sin(kr)$ in Eq. (12.3.67), an improved Born approximation is obtained:

$$k \cot \eta_0 \simeq k \cot \eta_0^{(B)} - \cot^2 \eta_0^{(B)} \int_0^\infty \int_0^\infty \sin(kr)U(r)G_k(r|r_0)U(r_0)\sin(kr_0)\, dr\, dr_0$$

or

$$k \cot \eta_0 \simeq k \cot \eta_0^{(B)} + 2\left[\frac{\cot^2 \eta_0^{(B)}}{k}\right] \int_0^\infty dr\, U \sin(kr) \int_r^\infty U \sin(kr_0)\cos(kr_0)\, dr_0$$

where $\eta_0^{(B)}$ is the Born approximation phase shift, given in Eq. (12.3.64).

Variational Principle for the Angle Distribution. A variational principle analogous to that obtained from Eq. (11.4.57) may be obtained by use of the three-dimensional Green's function, which does for the angle-distribution function $f(\vartheta)$ what Eq. (12.3.67) does for the phase angles. We change the Schroedinger equation

$$\nabla^2\psi + k^2\psi = U(r)\psi; \quad k^2 = (2ME/\hbar^2); \quad U = (2MV/\hbar^2)$$

into the integral equation for ψ,

$$\psi(\mathbf{r}) = Ce^{i\mathbf{k}_i\cdot\mathbf{r}} - \frac{1}{4\pi}\int \frac{e^{ikR}}{R} U(r_0)\psi(\mathbf{r}_0)\, dv_0$$

where $\mathbf{k}_i$ is the vector of amplitude k pointed in the direction of the incident beam of particles. The asymptotic form of ψ is then

$$\psi(\mathbf{r}) \to Ce^{i\mathbf{k}_i\cdot\mathbf{r}} + Cf(\vartheta)(e^{ikr}/r); \quad r \to \infty$$
$$f(\vartheta) = -\frac{1}{4\pi C}\int e^{-i\mathbf{k}_s\cdot\mathbf{r}_0}U(r_0)\psi(\mathbf{r}_0)\,dv_0$$

where $f(\vartheta)$ is the angle-distribution factor for the scattered wave, $|f(\vartheta)|^2$ is the differential cross section (see page 1066), and its integral over all directions is the total cross section Q for elastic scattering from the potential field $V(r)$. We have multiplied and divided by C, the amplitude of the incident beam, in order that our results come out per unit incident intensity.

The vector $\mathbf{k}_s$ has magnitude k and is pointed in the direction of the observer, at an angle ϑ to $\mathbf{k}_i$. We now substitute back in the integral equation for C, making it homogeneous,

$$\psi(\mathbf{r}) = -\frac{e^{i\mathbf{k}_i\cdot\mathbf{r}}}{4\pi f(\vartheta)}\int e^{-i\mathbf{k}_s\cdot\mathbf{r}_0}U(r_0)\psi(\mathbf{r}_0)\,dv_0 - \frac{1}{4\pi}\int\left(\frac{e^{ikR}}{R}\right)U(r_0)\psi(\mathbf{r}_0)\,dv_0$$

To obtain a variational principle for $f(\vartheta)$, we carry on as has been done in Secs. 9.4 and 11.4, multiplying by $\tilde{\psi}(r)U(r)$, where $\tilde{\psi}$ is the solution for an incident wave coming in the direction $-\mathbf{k}_s$ and being scattered in the direction $-\mathbf{k}_i$ (the reciprocal case, where source and observer are reversed) and integrating over dv. The final variational principle is

$$[f(\vartheta)] = \frac{-\int e^{i\mathbf{k}_i\cdot\mathbf{r}}U(r)\tilde{\psi}(\mathbf{r})\,dv\int e^{-i\mathbf{k}_s\cdot\mathbf{r}_0}U(r_0)\psi(\mathbf{r}_0)\,dv_0}{4\pi\int\tilde{\psi}(\mathbf{r})U(r)\psi(\mathbf{r})\,dv + \int\int\tilde{\psi}(\mathbf{r})U(r)(e^{ikR}/R)U(r_0)\psi(\mathbf{r}_0)\,dv\,dv_0} \tag{12.3.68}$$

The quantity $[f]$ is stationary when the correct expressions for ψ and $\tilde{\psi}$ are used, in which case $[f(\vartheta)]$ equals $f(\vartheta)$; the values of the parameters for an assumed ψ (and corresponding $\tilde{\psi}$) which make J minimum correspond to the best fit for the assumed forms of ψ and $\tilde{\psi}$, and the resulting minimum value of $[f]$ is the best possible value of $f(\vartheta)$ for the forms used. This is also an iteration scheme, for the "best form" for ψ may then be inserted in the right-hand side of the integral equation for ψ to obtain a still better form, which may then be inserted in (12.3.68) to improve f. The relation between this procedure and that discussed in the preceding subsection should be obvious; the present formulation is simpler in principle but usually more difficult in execution because three-dimensional integrals are involved instead of one-dimensional ones.

If the form chosen for ψ is a plane wave (which is not a bad approximation for large values of k), the resulting $[f]$ is closely related to the Born approximation for elastic scattering, given in Eqs. (12.3.58) and (12.3.59). The three-dimensional Fourier transform for the potential function is

$$u(\mathbf{K}) = (2\pi)^{-\frac{3}{2}}\int e^{i\mathbf{K}\cdot\mathbf{r}}U(r)\,dv$$

and the Fourier transform of the Green's function $e^{ikR}/4\pi R$ is

$$g_k(\mathbf{K}|\mathbf{r}_0) = (2\pi)^{-\frac{3}{2}}\int \frac{e^{ikR}}{4\pi R}\,e^{i\mathbf{K}\cdot\mathbf{r}}\,dv = (2\pi)^{-\frac{3}{2}}\,\frac{e^{-i\mathbf{K}\cdot\mathbf{r}_0}}{K^2 - k^2}$$

[see Eq. (11.3.6)]. If we now insert the trial functions $\psi \simeq e^{i\mathbf{k}_i\cdot\mathbf{r}}$ and $\tilde{\psi} \simeq e^{-i\mathbf{k}_s\cdot\mathbf{r}}$ into Eq. (12.3.67) and use the faltung theorem (4.8.25) in the double integral, we finally obtain

$$f(\vartheta) \simeq -\sqrt{\tfrac{1}{2}\pi}\,\frac{[u(\mathbf{k}_i - \mathbf{k}_s)]^2}{u(\mathbf{k}_i - \mathbf{k}_s) + \dfrac{1}{\sqrt{8\pi^3}}\displaystyle\int \frac{u(\mathbf{k}_i - \mathbf{K})u(\mathbf{K} - \mathbf{k}_s)}{K^2 - k^2}\,dv_K}$$

where the integration is over the whole three-dimensional volume of "K space."

If we neglect the integral, we obtain the Born approximation,

$$f_B(\vartheta) = -\sqrt{\tfrac{1}{2}\pi}\,u(\mathbf{k}_i - \mathbf{k}_s) = -\frac{1}{4\pi}\int e^{i(\mathbf{k}_i-\mathbf{k}_s)\cdot\mathbf{r}}U(r)\,dv$$

The higher Born approximations (see Sec. 9.3) are obtained by expanding the denominator of Eq. (12.3.67) and using iteration to obtain successively better approximations for ψ. We see, however, that, if the integral over K space above is *not* small compared to $u(\mathbf{k}_i - \mathbf{k}_s)$, then the Born series will not converge very well and that it would be better to use the formula above without expanding the denominator.

To see how the calculation goes, we try the potential $U(r) = -(\gamma e^{-\lambda r}/r)$ or $V(r) = -(\eta^2 e^{-\lambda r}/r)$ $(\eta^2 = \hbar^2\gamma/2M)$, a potential which is coulomb in form near the center of force but which falls off more rapidly than coulomb for large r. If this were the potential for an electron of charge $-e$, it would correspond to the field of a point charge η^2/e at $r = 0$, surrounded by a distribution of charge of density $-(\eta^2\lambda^2 e^{-\lambda r}/4\pi e r)$, having total charge $-\eta^2/e$ which just cancels the effect of the central charge at large r. (It has also been suggested by Yukawa as a possible field between nucleons.) The Fourier transform of the potential is

$$u(\mathbf{K}) = -\gamma/\sqrt{\tfrac{1}{2}\pi}\,(K^2 + \lambda^2)$$

so that the Born approximation is

$$f(\vartheta) \simeq f_B(\vartheta) = \gamma/[|\mathbf{k}_i - \mathbf{k}_s|^2 + \lambda^2] = 2M\eta^2/\hbar^2[\lambda^2 + 4k^2\sin^2(\tfrac{1}{2}\vartheta)]$$

which is to be compared with the Rutherford formula on page 1669 $(k = Mv/\hbar,\ \eta^2 = Ze^2)$.

The faltung integral, corresponding to the double integral in the denominator of (12.3.67), is then evaluated by a method given in Sec. 9.3 [see Eq. (9.3.55)],

$$\frac{\gamma^2}{\sqrt{2\pi^5}}\int\frac{dv_K}{(K^2-k^2)[|\mathbf{K}-\mathbf{k}_i|^2+\lambda^2][|K-k_s|^2+\lambda^2]}=\sqrt{\frac{2}{\pi}}\,\gamma^2H(\vartheta)$$

$$H(\vartheta)=\frac{1}{2k\sin(\frac{1}{2}\vartheta)\sqrt{\lambda^4+4k^2[\lambda^2+k^2\sin^2(\frac{1}{2}\vartheta)]}}\cdot$$
$$\cdot\left\{\tan^{-1}\frac{\lambda k\sin(\frac{1}{2}\vartheta)}{\sqrt{\lambda^4+4k^2[\lambda^2+k^2\sin^2(\frac{1}{2}\vartheta)]}}\right.$$
$$\left.+\tfrac{1}{2}i\ln\frac{\sqrt{\lambda^4+4k^2[\lambda^2+k^2\sin^2(\frac{1}{2}\vartheta)]}+2k^2\sin(\frac{1}{2}\vartheta)}{\sqrt{\lambda^4+4k^2[\lambda^2+k^2\sin^2(\frac{1}{2}\vartheta)]}-2k^2\sin(\frac{1}{2}\vartheta)}\right\}$$

The imaginary part of H results from the fact that we must go around the poles of the integrand at $K = \pm k$, above the pole at $-k$, and below that at $+k$, corresponding to the requirement that the Fourier transform of $e^{ikR}/4\pi R$ shall represent an outgoing wave [see discussion of Eq. (7.2.31)]. The formula for the angle-distribution function $f(\vartheta)$, for the assumption of plane waves as trial functions for ψ and $\tilde{\psi}$ in (12.3.67), is then

$$f(\vartheta)\simeq\frac{\{\gamma/[\lambda^2+4k^2\sin^2(\frac{1}{2}\vartheta)]\}^2}{\{\gamma/[\lambda^2+4k^2\sin^2(\frac{1}{2}\vartheta)]\}-\gamma^2H(\vartheta)}$$
$$=f_B(\vartheta)\left\{\frac{1}{1-\gamma[\lambda^2+4k^2\sin^2(\frac{1}{2}\vartheta)]H(\vartheta)}\right\}$$

which differs from the Born approximation f_B by the factor in braces. If γ is small, this factor produces only a small correction; if γ is large, the modification may be considerable. In general, the modified angle-distribution factor is a better fit to the correct solution than is the simple Born approximation.

The limiting values of the correction term for small values of the variables are

$$[\lambda^2+4k^2\sin^2(\tfrac{1}{2}\vartheta)]H(\vartheta)\to\frac{\lambda}{2k^2}+\frac{i}{k}\ln\left[\frac{2k}{\lambda}\sin(\tfrac{1}{2}\vartheta)\right];\quad\lambda\ll k$$
$$\to[1/2(\lambda-2ik)];\qquad\lambda\gg k\sin\tfrac{1}{2}\vartheta$$

which reveals several interesting results. In the first place, we note that in the limit $\lambda\to 0$ the Born approximation $f_B\to[\lambda/4k^2\sin^2(\frac{1}{2}\vartheta)]$ is exactly correct as far as amplitude of f goes [see, for example, Eq. (12.3.43)]. In this limit the correction term in the braces is imaginary and is, indeed, related to the second term in the expansion of the imaginary exponential occurring in the correct expression for f, given in Eq. (12.3.43). As well as a single, additive correction term can do it, the correction term H tends to modify the phase of f, not its magnitude, in this limiting case.

One might try using Eq. (12.3.56) to compute the total cross section from f. This formula, exactly correct for the exact solution, fails completely for the Born approximation, since f_B is real. It is possible in the present case, however, to integrate $|f_B|^2$ over all angles of scattering to obtain the "Born cross section,"

$$Q_B = 2\pi \int_0^\pi |f_B|^2 \sin\vartheta \, d\vartheta = \frac{4\pi\gamma^2}{\lambda^2(\lambda^2 + 4k^2)}$$

There is an imaginary part of H, however, so that use of Eq. (12.3.56) gives a nonzero result in this next approximation,

$$\begin{aligned} Q &= (4\pi/k)\,\mathrm{Im}[f(0)] \simeq f_B(0)\,\mathrm{Im}\{1/[1 - \lambda^2 H(0)]\} \\ &\simeq \{4\pi\gamma^2/\lambda^2[(\lambda - \tfrac{1}{2}\gamma)^2 + 4k^2]\} \end{aligned}$$

which is fairly close to the Born cross section, particularly if γ is smaller than λ. Thus what is effectively a second-order expression for $f(\vartheta)$ can be used to obtain a first-order expression for Q, which bears out the discussion on page 1073. A second-order expression for Q may be obtained by integrating the $f(\vartheta)$ of the preceding page, including the H correction, over all angles (though this integration is not a simple affair).

Returning to the original integral equation for ψ, we can now obtain an improved form for ψ by iteration,

$$\begin{aligned} \psi^{(2)}(r) &= e^{i\mathbf{k}_i\cdot\mathbf{r}} + \int \frac{e^{ikR-\lambda r_0}}{4\pi R r_0} e^{i\mathbf{k}_i\cdot\mathbf{r}_0}\, dv_0 \\ &= e^{i\mathbf{k}_i\cdot\mathbf{r}} + \frac{\gamma}{2\pi^2}\int \frac{e^{-i\mathbf{K}\cdot\mathbf{r}}\, dv_K}{(K^2 - k^2)[|K - k_i|^2 + \lambda^2]} \end{aligned}$$

Either of these forms may be computed by expansion in spherical harmonics.

Two Particles, One-dimensional Case. The problem of the interaction of two similar particles with a fixed potential field and with each other is essentially equivalent to the classical three-body problem. The fixed potential field, acting alike on both particles, can be considered to be the field of a third body, heavy enough so the center of gravity of the system is coincident with its position (the origin). The classical three-body problem is essentially nonseparable, in part because of the possible collisions of the particles. The quantum mechanical problem is also essentially nonseparable and for very much the same reason.

To see just what the difficulties are, without being confused with extraneous mathematical complexities, we choose the extremely simplified case of two one-dimensional particles, each acted on by a very simplified potential field. The displacement of particle 1 from the center of the force field will be called x and the displacement of particle 2 will be y. To begin with we assume that the field acting on each particle is a very deep, very narrow potential well at the origin, a delta function of the sort discussed on page 1644. In this case the field acts on each particle, so the equation for the combined wave function, when we neglect the interparticle field, is

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \epsilon + 2a\delta(x) + 2a\delta(y)\right]\psi = 0$$

for a steady state having energy $\hbar^2\epsilon/2M$. The constant M is the mass of each particle, and the constant $a = MV_0/\hbar^2$ is the parameter of the potential function, V_0 being the "strength" of the delta function well.

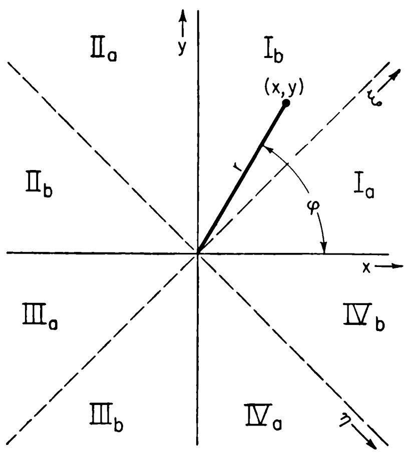

Fig. 12.7 Coordinates and quadrant designations for system of two one-dimensional particles.

The most important fact which this equation displays is that the wave function for the two-particle system is a two-dimensional function and that the equation and system can be best pictured in a plane, as shown in Fig. 12.7. The potential center produces deep, narrow valleys just along the x and y axes; elsewhere the potential is zero. From another point of view, this equation shows that ψ is a solution of the two-dimensional Helmholtz equation, with $\epsilon = k^2$, except along the coordinate axes. Along $x = 0$ and $y = 0$ are semireflecting mirrors, which impose a discontinuity in slope in the wave, the discontinuity in slope normal to the mirror being $-2a$ times the magnitude of the wave at the mirror. If the two particles are free, both starting from $+\infty$, with velocity v_1 and v_2, respectively, the wave function will consist of an incident plane wave

$$\psi_i = e^{-ik_1x-ik_2y}; \quad k_1 = Mv_1/\hbar; \quad k_2 = Mv_2/\hbar$$

plus all the reflections of this wave from the "mirrors" along the x and y axes.

The reflections vary with angle of incidence; for instance, for the wave ψ_i striking the $x = 0$ mirror the incident, reflected, and transmitted waves are

$$\psi = \begin{cases} e^{-ik(x\cos u + y\sin u)} - \left(\dfrac{a}{a + ik\cos u}\right) e^{ik(x\cos u - y\sin u)}; & x > 0 \\ \left(\dfrac{ik\cos u}{a + ik\cos u}\right) e^{-ik(x\cos u + y\sin u)}; & x < 0 \end{cases}$$

where $k^2 = k_1^2 + k_2^2 = \epsilon$; $k_1 = k\cos u$; $k_2 = k\sin u$. The incident wave has angle of incidence with respect to the y axis of $u = \tan^{-1}(k_2/k_1)$; the reflected and transmitted wave amplitudes depend on these angles of incidence. The incident wave ψ_i has reflections from both mirrors, and the combined reflected, transmitted, and incident wave in each of the quadrants labeled in Fig. 12.7 is

$$\psi = \begin{cases} e^{-ikr\cos(\varphi-u)} + \dfrac{a^2 e^{ikr\cos(\varphi-u)}}{(a+ik\cos u)(a+ik\sin u)} \\ \qquad - \dfrac{ae^{ikr\cos(\varphi+u)}}{(a+ik\cos u)} - \dfrac{ae^{-ikr\cos(\varphi+u)}}{(a+ik\sin u)}; & \text{quadrant I} \\ \dfrac{ik\cos ue^{-ikr\cos(\varphi-u)}}{(a+ik\cos u)} - \dfrac{iak\cos ue^{-ikr\cos(\varphi+u)}}{(a+ik\cos u)(a+ik\sin u)}; & \text{quadrant II} \\ \dfrac{-k^2\sin u\cos ue^{-ikr\cos(\varphi-u)}}{(a+ik\cos u)(a+ik\sin u)}; & \text{quadrant III} \\ \dfrac{ik\sin ue^{-ikr\cos(\varphi-u)}}{(a+ik\sin u)} - \dfrac{iak\sin ue^{ikr\cos(\varphi+u)}}{(a+ik\cos u)(a+ik\sin u)}; & \text{quadrant IV} \end{cases} \tag{12.3.69}$$

where $r = \sqrt{x^2+y^2}$, and $\varphi = \tan^{-1}(y/x)$ are polar coordinates in the x, y plane. The two particles coming in from the right are partly reflected by the potential valleys and partly transmitted. Each particle maintains its kinetic energy, but it has a chance of reversing the direction of its motion. The relative intensities of the reflected and transmitted beams are given by the squares of the magnitudes of the coefficients.

The Green's Function. The Green's function for this system can be obtained from Eq. (11.2.22). We multiply ψ by $e^{ik(x_0\cos u + y_0\sin u)}$ and integrate over u from $-i\infty$ to $+i\infty$. Since we have satisfied the joining conditions (which are homogeneous) for each value of u, the whole integral will also satisfy the conditions. The first term in quadrant I is then just $\pi i H_0^{(1)}(kR)$, where $R^2 = (x-x_0)^2 + (y-y_0)^2$; it is the wave radiated from the point source at x_0, y_0 in the first quadrant. The third term corresponds to a wave which seems to come from the image point $-x_0$, y_0; it is not a symmetric wave, for it has all the higher order Hankel functions in it. By use of the generating function for the trigonometric functions given in the table at the end of Chap. 10, we can expand

$$\frac{a}{a+ik\cos u} = \frac{a}{\sqrt{a^2+k^2}} \sum_{n=0}^{\infty} \epsilon_n i^{-n} \left[\frac{\sqrt{a^2+k^2}-a}{k}\right]^n \cos(nu)$$

and, consequently, the third term in the expression for the first quadrant becomes

$$-\frac{\pi i a}{\sqrt{a^2+k^2}} \sum_{m=0}^{\infty} \epsilon_m \left[\frac{\sqrt{a^2+k^2}-a}{k}\right]^m H_m^{(1)}(kR_2)\cos(m\varphi_2)$$

$$x + x_0 = R_2\cos\varphi_2; \quad y - y_0 = R_2\sin\varphi_2$$

a wave reflected from the "mirror" at $x = 0$, seeming to come from the image in quadrant II and having the necessary angular dependence to help fit the joining conditions at $x = 0$ and $y = 0$. All the other integrals can be similarly expanded in terms of angle-dependent waves arising from the source or from one of its three images. The number of terms needed depends on the quadrant in which ψ is measured; for instance, for quadrant III only the wave from the true source (x_0,y_0) is needed, but its angle dependence is no longer uniform, it has been distorted by transmission through the "mirrors."

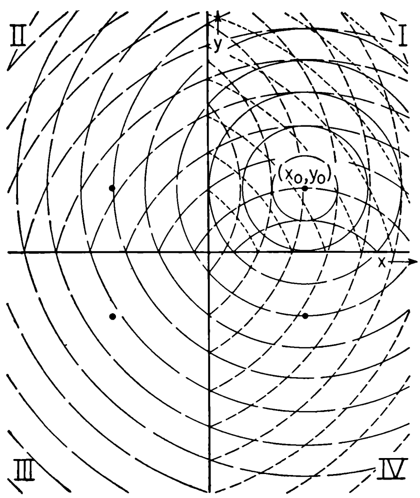

Fig. 12.8 Sketch of wave fronts for Green's function from source at (x_0,y_0). Solid-line wave fronts in quadrant I are of uniform amplitude; amplitude of dashed wave fronts varies with angle.

The Green's function is symbolically represented in Fig. 12.8. There is a symmetric wave arising from the source (x_0,y_0) (solid-line wave fronts) plus transmitted or reflected waves which have amplitude depending on angle (dashed-line wave fronts). The expression pictured here, obtained by integrating Eq. (12.3.69) over u, is valid almost everywhere over the x, y plane. For example if the source (x_0,y_0) is in any other quadrant, we simply rotate the figure, relabeling the quadrants. But the expression we have set up and represented *will not be valid when the source is exactly on the x or y axis.* Here symmetry would seem to require a symmetric wave from the source (at x_0, 0, for example) plus a single reflected wave from the image (at $-x_0$, 0). But such a wave will not satisfy the requirements of discontinuity of slope at the x axis; it is *too* symmetric.

What we have forgotten is that the present problem, viewed from the wave-reflection point of view, is one which gives rise to surface waves; viewed from the particle point of view, it has bound states. We must look at these surface waves to see what occurs.

The only bound state of the system (without interaction between particles) may easily be expressed in terms of simple exponentials, as was the case for Eq. (12.3.8). We have

$$\psi = \begin{cases} Ne^{-ax-ay} = Ne^{-\alpha\xi}; & \text{quadrant I} \\ Ne^{ax-ay} = Ne^{\alpha\eta}; & \text{quadrant II} \\ Ne^{ax+ay} = Ne^{\alpha\xi}; & \text{quadrant III} \\ Ne^{-ax+ay} = Ne^{-\alpha\eta}; & \text{quadrant IV} \end{cases} \tag{12.3.70}$$

where $x = (1/\sqrt{2})(\xi + \eta)$, $y = (1/\sqrt{2})(\xi - \eta)$, and $\alpha = \sqrt{2}\,a$. The wave function resembles a "pagoda roof," coming to a highest point at $r = 0$ and sloping down in all directions, with the slope gradually leveling off at large r. The contour lines for ψ (one might continue the metaphor and call them the "shingle lines") are squares with the sides parallel to the ξ, η axes; the roof ridges are along the x, y axes, where the potential valleys are. The allowed energy for the system is, of course,

$$E = -(MV_0^2/\hbar^2); \quad \epsilon = -2a^2$$

Incidentally, this bound wave function may be expressed in terms of r and φ, in terms of the semicylindrical functions defined in the tables at the end of this chapter. We see that the series

$$\psi = \sum_{m=0}^{\infty} \epsilon_m(-1)^m \cos(4m\varphi) J_{4m}^{(4)}(i\sqrt{2}\,ar) \tag{12.3.71}$$

expresses ψ for the whole range of φ from 0 to 2π, and that it takes into account the discontinuities in gradient of ψ at $\varphi = 0, \frac{1}{2}\pi, \pi$, and $\frac{3}{2}\pi$. In fact, by use of the properties of the functions $J_{4m}^{(4)}(z)$, we can show directly that

$$\begin{aligned}
\nabla^2\psi &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\varphi^2} \\
&= -\left(\frac{4a}{\pi r}\right)e^{-ar}\sum_m \epsilon_m \cos(4m\varphi) + 2a^2\psi \\
&= -\frac{2a}{r}\left[\delta(\varphi) + \delta(\varphi - \tfrac{1}{2}\pi) + \delta(\varphi - \pi) + \delta(\varphi - \tfrac{3}{2}\pi)\right]\psi + 2a^2\psi \\
&= -2a[\delta(x) + \delta(y)]\psi + 2a^2\psi
\end{aligned}$$

thus demonstrating that the series of (12.3.71) is a solution of the equation with delta function potentials.

Surface Waves. There are also partly bound and partly free states, as for instance when particle 2 is in the bound state and particle 1 comes in from the right with speed v_1. The wave function is then

$$\psi = \begin{cases} e^{-a|y| - ik_a x} - [a/(a + ik_a)]e^{-a|y| + ik_a x}; & x > 0 \\ [ik_a/(a + ik_a)]e^{-a|y| - ik_a x}; & x < 0 \end{cases} \tag{12.3.72}$$

where $k_a = Mv_1/\hbar$ and $\epsilon = k_a^2 - a^2$. Although particle 1 may be reflected from the valley at $x = 0$, as long as there is no particle interaction, the particles cannot change places. To have particle 1 bound, we must start with particle 1 bound and particle 2 free.

These are the surface waves we mentioned above, which become prominent in the Green's function whenever the source is on one of the coordinate axes. In fact a detailed analysis of the nature of this Green's function (which need not be detailed here; we are not trying to compute this problem, we only want to get its over-all pattern, to understand its general peculiarities, so we can be helped in understanding the similar peculiarities of other multiparticle problems) shows that it has a circular outgoing wave, which is angle dependent, going to zero along the axis (being the limit of the primary wave and its reflection as the two merge) and its angle-dependent reflection, coming from $-x_0$, 0; but in addition there is a surface wave, of the sort given in Eq. (12.3.72), traveling away from the source and being reflected at $x = 0$. It requires both surface and area waves to satisfy the conditions along the x axis, if the source is on this axis. We note that the wavelength of the surface wave, $2\pi/k_a$, is shorter than that for the free waves, $2\pi/k$, for if $\epsilon = k^2$, then $k_a^2 = k^2 + a^2$.

This behavior would also be apparent if we should expand the Green's function in terms of a series and integral of products of source and observer eigenfunctions of the homogeneous equation [see Eqs. (7.2.39), (7.2.41), and (7.2.42)]. In the present case this consists of a double integral of the free-wave parts over all values of the wave numbers k_1 and k_2, plus single integrals of the surface waves over all values of their k's, plus a single, separate term for the bound states. When either source or observer is far from either x or y axis, the surface-wave and bound-state terms are negligible, near the axes the surface-wave terms become important, near the origin the bound-state term is important. All these terms enter when we integrate Eq. (12.3.74).

Consequently, if we have a distribution of sources in the plane, the solution of the inhomogeneous equation will involve the integration of the usual Green's function over the area distribution, plus an integration of the surface-wave part of the Green's function over the amounts of source which are on the coordinate axes.

The Effects of Interaction between the Particles. Now let us see what happens when we do include an interaction between the two particles. Suppose it is a repulsive short-range force, a delta function of $(x - y)$, along the ξ axis, diagonal to the x, y axes of Fig. 12.7. Here the wave function, instead of having a ridge, as it does for the attractive fields along x and y, has a "trough," the discontinuity in normal gradient being positive. Insertion of this interaction potential at once complicates the problem considerably. In the first place it destroys some of the symmetry of the system; there is a potential ridge or "mirror" along the ξ axis but none along the η axis. This mirror acts to interchange the energies of the particle so that now, when we send in a wave corresponding to particle 1 having speed $v_1 = \hbar k_1/M$ and particle 2 having $v_2 = \hbar k_2/M$, a part of the reflected wave will have x and y reversed and thus have particle 1 leaving with speed v_2 and particle 2 with speed v_1.

Even in the case where both particles are free, the additional mirror introduces a complication at the origin which cannot be satisfied by a simple sum of plane waves. One can try, for instance, sending in particle 1 from the right with speed v_1 (wave number k_1) and sending in particle 2 from the left with smaller velocity v_2 (wave number $-k_2$). The angle of the incident wave with the x axis is, therefore, less than $\frac{1}{4}\pi$ and the incident wave is coming in from infinity in quadrant IV. Some of this wave will be reflected from the x axis (particle 2 reflected from the origin) and some from the y axis (particle 1 reflected from the origin), but some will penetrate these "mirrors" and strike the new mirror along the ξ axis, where it will be reflected. This corresponds to a reflection of particle 1 from particle 2 and an exchange of kinetic energies; the direction of the reflected wave is now $\frac{1}{2}\pi - u$, corresponding to particle 1 having wave number k_2 and particle 2 wave number $-k_1$.

So far everything is clear; but if we continue in this manner, adding the various waves reflected and transmitted at each surface, for each octant, requiring that the discontinuity in normal slope at $x = 0$ or $y = 0$ be $-2a$ times ψ, as before, and that the discontinuity in normal slope at $\eta = 0$ be $+2b$ times ψ, we then find we cannot make the problem "close," that we have either not enough component waves or too many joining conditions. If $b = 0$, several of these joining conditions vanish and the problem *can* be solved by combinations of plane waves, but this is the case considered before, for no interaction between the particles. As soon as the interaction is included, a finite sum of plane waves is not sufficient to satisfy all the joining conditions and also the requirement of periodicity in the polar angle φ (that is, the wave in the second quadrant obtained by going from quadrant IV via quadrant I must be the same as that obtained by going via quadrant III).

Evidently the origin, where all the mirrors meet, does some reflecting on its own account and sends out a wave, which includes some surface waves as well as "free" waves. The new mirror at $\eta = 0$ does not allow surface waves along itself, the sign of b precludes this, but its presence transforms part of any wave into surface waves. In "particle language," the interaction force makes an interchange of energy possible between the particles; in particular, if both particles start out free, one particle can give up energy and drop into the bound state, the other particle going off with the rest of the energy.

Some of these matters can be more clearly seen if we change the equation of motion of the pair of particles,

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + [\epsilon + 2a\delta(x) + 2a\delta(y)]\psi = 2b\delta(x - y)\psi \quad (12.3.73)$$

into the integral equation

$$\psi(x,y) = \frac{-b}{2\pi} \iint \delta(x_0 - y_0)\psi(x_0,y_0)G_k(x,y|x_0,y_0)\, dx_0\, dy_0$$
$$= -\frac{b}{2\pi} \int_{-\infty}^{\infty} \psi(x_0,x_0)G_k(x,y|x_0,x_0)\, dx_0 \qquad (12.3.74)$$

where G_k is the Green's function obtained by integrating Eq. (12.3.69) over u, for $\epsilon = k^2$, for a source at x_0, y_0. This expresses the solution ψ at any point x, y in terms of the solution along the ξ axis ($x = y$), where the perturbing "ridge" of potential is. The solution is formed by considering the value of ψ at each point on the ξ axis, times $-(b/2\pi)$, to be a source, producing waves in the rest of the space. If the correct values of ψ along the ξ axis are put in, the correct solution ψ will be reproduced; even if ψ is not known exactly, an approximate form for ψ, put in the integral, will produce a fairly good approximation for ψ over the rest of the plane.

An integral equation for $\psi(x,x)$ along the ξ axis is obtained by setting $y = x = \xi/\sqrt{2}$. If $\phi_0(\xi) = \psi(\xi/\sqrt{2},\xi/\sqrt{2})$ is the value of ψ at the point $(\xi,0)$ on the ξ axis and if $G_k(\xi|\xi_0) = G_k(\xi/\sqrt{2},\xi/\sqrt{2}|\xi_0/\sqrt{2},\xi_0/\sqrt{2})$ is the value of the Green's function at $(\xi,0)$ when the unit source is at $(\xi_0,0)$, then the integral equation is

$$\phi_0(\xi) = -\frac{b}{\sqrt{8\pi^2}} \int_{-\infty}^{\infty} \phi_0(\xi_0)G_k(\xi|\xi_0)\, d\xi_0 \qquad (12.3.75)$$

When this equation has been solved, it can be put back into Eq. (12.3.74) to obtain the general solution ψ for energy $\epsilon = k^2$.

The kernel $G_k(\xi|\xi_0)$, for k real, is a combination of outgoing waves in ξ, for $\xi > \xi_0$; is a combination of waves in both directions for $0 < \xi < \xi_0$; and is a combination of waves going to $\xi \to -\infty$ for $\xi < 0$; in addition, at $\xi_0 = 0$ it is a singular function, which on integration over ξ_0 gives the value of ϕ_0 at $\xi = 0$ times a function corresponding to the surface waves, a combination of real and imaginary exponential functions. We do not intend here to compute ϕ_0 or even $G_k(\xi|\xi_0)$. We can see, without computation, what sort of form $\psi(x,y)$ will have, from the general properties of the Green's function $G_k(x,y|x_0,y_0)$, which we have already outlined and from the form of Eq. (12.3.74).

We take first the case of both particles free. The exact solution of Eq. (12.3.73), as we have seen, cannot be expressed in terms of an incident plane wave plus reflected plane waves. Nevertheless the value of ψ along the ξ axis is not too different from the value of the ψ of Eq. (12.3.69) (the solution without interaction) along the ξ axis. If this is put into the integral of Eq. (12.3.74), the result of the integration should be a fairly good approximation to the solution of (12.3.73).

The integration over most of the range of ξ_0 $(= x_0 \sqrt{2})$ corresponds to the additional plane wave reflections of solution (12.3.69) from the additional "mirror" along the ξ axis. In addition to these reflected waves, however, is the term from the singular part of G which gives rise to surface waves traveling out from the origin along both the x and y axes, plus a certain amount of circular wave going radially out from the origin, the amplitude of these waves being proportional to the value of ψ at the origin.

Meaning of the Results. Let us now translate this result back into particle language. We send in the two particles from infinity, one with speed v_1, the other with speed v_2. There is a possibility that both particles continue on past the potential valley at the origin with their original speeds (the transmitted wave); there is also a possibility that one particle is reflected from the origin and returns from whence it came with its original speed, the other continuing past the origin with its original speed (the waves reflected from the x and y "mirrors"); and there is a possibility that both particles are reflected from the origin but retain their original speeds (the wave reflected from both x and y "mirrors"). This much results without the effect of the interaction force, the ξ "mirror."

But in addition there is a possibility that both particles interchange their speeds, leaving the $x = y$ line in either direction (plane waves reflected from the ξ "mirror"); there is also the possibility that, at the origin, where all three potentials act together, one particle drops into the bound state, the other particle leaving the origin in either direction with a speed corresponding to the energy thus given up (the surface waves); lastly, there is a (relatively small) probability that at the origin one particle gives up *part* of its kinetic energy to the other and the two leave with speeds different from their original values, but such as to make the sum of their kinetic energies equal to the original sum (circular wave going out radially from the origin). All the possibilities mentioned in this last sentence are produced by the particle interaction term, which makes the problem nonseparable and which introduces the multiferous complexities (and possibilities) in many-particle systems.

If the interaction term were attractive instead of repulsive, as we have here assumed it, there would be an additional outgoing surface wave along the ξ axis, corresponding to the particles leaving the origin bound together, carrying the leftover energy as mutual kinetic energy.

One last consideration, before we turn to problems more closely related to reality: we need to indicate how the interaction force modifies the wave function for the state where both particles are bound. We could compute this from Eq. (12.3.74) by using a Green's function composed of the exponentially diminishing Hankel functions of imaginary argument, but it will suffice here to indicate pictorially how the function is modified from the "pagoda roof" shape, discussed on page 1713. If the interaction force is repulsive, the binding energy of the particles will be reduced and ψ at large distances will go to zero slower than the ψ of Eq. (12.3.70). Presumably the "roof ridges" along x and y drop down from the peak at the origin roughly exponentially with an exponential factor smaller than the a of Eq. (12.3.70). The "roof angle" at these ridges must be the same as before, since this is determined by the potential valleys along x and y.

However, in addition to the ridges, the repulsive interaction force requires a V-shaped "trough" in the roof along the ξ axis. This can be done by making the roof in the quadrants II and IV "bulge out" a bit, the ψ contours (the "shingle lines") being curves concave towards the origin. In quadrants I and III the roof then slopes down more steeply away from the ridges along x and y axes, with a little additional downward curvature, enabling the two sections to meet along the axis with a V-shaped "trough." If the repulsive interaction is too large, the roof cannot be fitted to so sharp a V without "wrinkling" it elsewhere; which is equivalent to saying that if the repulsive interaction force is too great there can be no bound state, one particle or both must be free. If the interaction force is attractive, there is another "ridge" along the ξ axis and the roof must "sag" a bit between ridges (instead of bulging) in order to fit together. The sagging or bulging is an additional indication that simple exponential functions are not sufficient to represent the solution; thus that the solution is essentially nonseparable.

Coupled Harmonic Oscillators. The only separable case of interacting particles is that of coupled harmonic oscillators, a case which does not demonstrate many of the points brought out in the last subsection because there are no free states. If the frequency of oscillation of each particle by itself in the force field is $\omega/2\pi$, and if the two particles attract each other with a force $C(x - y)$, the Schroedinger equation, analogous to (12.3.5), for the system is

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + [\epsilon - \beta^2x^2 - \beta^2y^2 - \gamma^2(x - y)^2]\psi = 0$$

where $\epsilon = 2ME/\hbar^2$, $\beta = M\omega/\hbar = (1/\hbar)\sqrt{KM}$, and $\gamma = (1/\hbar)\sqrt{CM}$. This is better expressed in terms of the normal coordinates ξ, η, where

$$x = \frac{1}{\sqrt{2}}(\xi + \eta); \quad y = \frac{1}{\sqrt{2}}(\xi - \eta); \quad \xi = \frac{1}{\sqrt{2}}(x + y);$$
$$\eta = \frac{1}{\sqrt{2}}(x - y)$$

The equation then takes on the separable form

$$\frac{\partial^2\psi}{\partial \xi^2} + \frac{\partial^2\psi}{\partial \eta^2} + [\epsilon - \beta^2\xi^2 - \beta_+^2\eta^2]\psi = 0 \qquad (12.3.76)$$

where $\beta_+^2 = \beta^2 + \gamma^2 = (M/\hbar^2)(K + C) = (M\omega_+/h)^2; \omega_+^2 = \omega^2 + (C/M)$. The solution is, following Eq. (12.3.7),

$$\psi_{mn}(\xi,\eta) = \frac{(\beta\beta_+)^{\frac{1}{4}}}{\sqrt{\pi 2^{m+n} m! n!}} e^{-\frac{1}{2}\beta\xi^2 - \frac{1}{2}\beta_+\eta^2} H_m(\xi\sqrt{\beta}) H_n(\eta\sqrt{\beta_+}) \tag{12.3.77}$$

with corresponding energies

$$E_{mn} = \hbar\omega(m + \tfrac{1}{2}) + \hbar\omega_+(n + \tfrac{1}{2})$$

We note that we cannot say that particle 1 has a particular energy and particle 2 has some particular energy, as we might if the interaction C were zero. Here the quantum numbers m and n *both* apply to *both particles.*

All the levels are raised by the interaction, by the difference between $\hbar\omega_+(n + \frac{1}{2})$ and $\hbar\omega(n + \frac{1}{2})$ (which would be the value if there were no interactive attraction). The states that are least changed are those for $n = 0$, which correspond to a small amplitude of motion in the η direction, the particles moving more or less together as they oscillate. The interaction has removed the degeneracy in the levels. For $C = 0$, the nth level has n fold degeneracy; for C small but not zero, the levels are not spread out much but are equally spaced from the lowest one, $\hbar\omega(n + \frac{1}{2}) + \frac{1}{2}\hbar\omega_+$ to the highest, $\frac{1}{2}\hbar\omega + \hbar\omega_+(n + \frac{1}{2})$.

We note, as of interest later, that the states can be distinguished by their symmetry with respect to interchange of particles. States with *even* values of n are unchanged when x and y are interchanged (η changes sign) whereas the ψ for n an *odd* integer changes sign when x and y are interchanged. The former states are said to have *even symmetry,* the latter *odd symmetry.* We also note that the interaction does not affect the function or energies for the ξ coordinate, only those concerned with the question of symmetry.

Central Force Fields, Several Particles, Angular Momentum. The behavior of several identical particles in a central force field, with interaction forces between the particles, is much the most interesting problem of this sort, because it is a very good approximation to the state of affairs in a multielectron atom and is a useful approximation (though not very close) to the state of affairs in a multiparticle nucleus. Without the interaction potentials, the Schroedinger equation is

$$\left[\sum_{n=1}^{N} \nabla_n^2 + \epsilon - \sum_{n=1}^{N} v(r_n)\right]\psi = 0 \tag{12.3.78}$$

When there is no interaction, it is possible to specify a particular energy for a particular particle, the total energy being the sum of the one-particle energies and the wave function the product of the one-particle ψ's.

But this is not necessary, the same total energy is obtained if we interchange the energies and ψ's of any pair of particles; the system has an $(N!)$-fold degeneracy with respect to interchange of states among particles. But, as with any degenerate state (see page 1673), we need not choose these factored functions as eigenfunctions, any linear combination of the $N!$ functions for the same ϵ will also be an eigenfunction; we wish to choose those combinations which are "suitable" for the interaction forces which will eventually be applied. When we include interaction, of course, we can no longer say that particle n has a particular energy; all we can say is that the system as a whole has energy ϵ. (Though we are not going into the question in this treatise, it should be noted that for some particles the Pauli principle drastically reduces this degeneracy by adding the further requirement that ψ change sign on interchange of particles.)

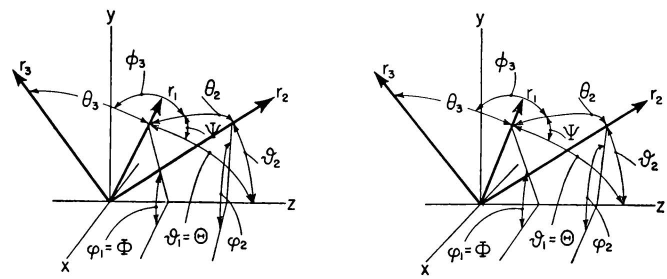

Fig. 12.9 Angles involved in calculation of total angular momentum of system.

The same remarks can be made concerning angular momentum. The factored solutions of Eq. (12.3.78) correspond to each particle having a definite total angular momentum, labeled by the quantum number l_i (see page 1661) and a definite component of this angular momentum along the polar axis, corresponding to the quantum number m_i. But actually, when interaction is taken into account, the individual particles have no specified angular momentum; all that can be said is that the N particles as a whole have total angular momentum given by the quantum number l and that the component of this momentum along the polar axis is $\hbar m$. It is of interest to find what combinations of the factored ψ's correspond to a given l and m. Or, what is equivalent (see page 1677), it is of interest to see in what coordinate systems we can separate off the total angular momentum operator.

One possible choice of coordinates, not so symmetric as might be desired but satisfactory for the present, is to refer the radius vector for one particle to the z axis and the radius vectors of all the other particles to the chosen $\mathbf{r}$ instead of to the z axis. The angles to be used are shown in Fig. 12.9. Radius vector $\mathbf{r}_1$ for the first particle is referred to the z

axis and to the $x - z$ plane, by the angles $\Theta = \vartheta_1$ and $\Phi = \varphi_1$; $\mathbf{r}_1$ is thus the new polar axis for all the other $\mathbf{r}$'s. The orientation of the other $\mathbf{r}$'s about $\mathbf{r}_1$ is given with reference to the (r_1,r_2) plane; consequently, the angle between this plane and the (r_1,z) plane, Ψ, is the third angle specifying the general orientation. These three angles are Euler angles [see Eq. (1.3.8)] for the system.

The relative orientation of radius vector $\mathbf{r}_n$ is then given by the angle Θ_n between $\mathbf{r}_n$ and $\mathbf{r}_1$ and the angle ϕ_n between the (r_2,r_1) plane and the (r_nr_1) plane. The relations between these angles and the usual spherical angles ϑ_n, φ_n, giving the orientation of $\mathbf{r}_n$ with respect to z and the x, z plane, are

$$\begin{aligned}
\vartheta_1 &= \Theta; \quad \varphi_1 = \Phi \\
\cos\vartheta_2 &= \cos\Theta\cos\Theta_2 + \sin\Theta\sin\Theta_2\cos\Psi \\
\sin\vartheta_2\sin(\varphi_2 - \Phi) &= \sin\Theta_2\sin\Psi \\
\cos\vartheta_n &= \cos\Theta\cos\Theta_n + \sin\Theta\sin\Theta_n\cos(\Psi + \phi_n) \\
\sin\vartheta_n\sin(\varphi_n - \Phi) &= \sin\Theta_n\sin(\Psi + \phi_n); \quad n > 2
\end{aligned} \tag{12.3.79}$$

After a considerable struggle, all the coordinates r_n, ϑ_n, φ_n may be expressed in terms of the new coordinates $r_1, r_2, \ldots, r_N, \Phi, \Theta, \Psi, \Theta_2, \Theta_3, \phi_3, \ldots, \Theta_N, \phi_N$, and the wave equation (12.3.78) may be written in their terms. Unfortunately the new coordinates are not mutually orthogonal, so the resulting wave equation is a rather complicated affair. The important point, however, is that the part involving the Euler angles Θ, Φ, Ψ *does* separate off and may be solved by itself.

A somewhat easier way to set up the equation for the Euler-angle factor in the wave function, however, is to express the total angular momentum components as operators in the new coordinates (see page 1661). The final result is:

$$\begin{aligned}
\mathfrak{M}_x &= \frac{\hbar}{i}\left[-\cos\Phi\cot\Theta\frac{\partial}{\partial\Phi} - \sin\Phi\frac{\partial}{\partial\Theta} + \frac{\cos\Phi}{\sin\Theta}\frac{\partial}{\partial\Psi}\right] \\
\mathfrak{M}_y &= \frac{\hbar}{i}\left[-\sin\Phi\cot\Theta\frac{\partial}{\partial\Phi} + \cos\Phi\frac{\partial}{\partial\Theta} + \frac{\sin\Phi}{\sin\Theta}\frac{\partial}{\partial\Psi}\right] \\
\mathfrak{M}_z &= \frac{\hbar}{i}\frac{\partial}{\partial\Phi}
\end{aligned}$$

The equation for the eigenvalues B and eigenfunctions Υ for $\mathfrak{M}^2/\hbar^2$ is then

$$\begin{aligned}
(1 + \cot^2\Theta)\frac{\partial^2\Upsilon}{\partial\Phi^2} + \frac{\partial^2\Upsilon}{\partial\Theta^2} + \frac{1}{\sin^2\Theta}\frac{\partial^2\Upsilon}{\partial\Psi^2} \\
- \frac{2\cot\Theta}{\sin\Theta}\frac{\partial^2\Upsilon}{\partial\Phi\,\partial\Psi} + \cot\Theta\frac{\partial\Upsilon}{\partial\Theta} + B\Upsilon = 0
\end{aligned} \tag{12.3.80}$$

The angles Φ and Ψ are cyclical, and the factors for them will be trigonometric functions of an integer times Φ or Ψ. As usual in quantum mechanics, we prefer the complex exponential, so we set

$$\Upsilon = e^{im\Phi+ik\Psi}H(\Theta);\quad m,\ k \text{ any integers, positive or negative}$$

The remaining equation for H may be turned into that for the hypergeometric equation by setting

$$H = \sin^d(\tfrac{1}{2}\Theta)\cos^s(\tfrac{1}{2}\Theta)F(\Theta)$$

where d is the magnitude of the difference, $|m - k|$, and s is the magnitude of the sum, $|m + k|$, of the quantum numbers. Then, if we set $z = \frac{1}{2}(1 - \cos\Theta) = \sin^2(\frac{1}{2}\Theta)$, the equation for F becomes

$$z(1 - z)\frac{d^2F}{dz^2} + [c - (a + b + 1)z]\frac{dF}{dz} - abF = 0$$

where

$$c = 1 + d;\quad a + b = a + d + s$$
$$ab = (d + s)(d + s + 2) - B$$
$$d = |m - k|;\quad s = |m + k|$$

This is the familiar hypergeometric equation. The solution, which is finite at $\Theta = 0$, is the series $F(a, b|1 + d|z)$; the second solution is infinite there. However, this series is infinite at $\Theta = \pi$ ($z = 1$) unless either a or b is a negative integer $-\nu$; consequently, the only finite solutions of (12.3.80) are

$$\Upsilon = e^{im\Phi+ik\Psi}\sin^d(\tfrac{1}{2}\Theta)\cos^s(\tfrac{1}{2}\Theta)F(-\nu, 1 + d + s + \nu|1 + d|\sin^2\tfrac{1}{2}\Theta)$$

with the corresponding eigenvalue for B, the eigenvalue for $\mathfrak{M}^2/\hbar^2$,

$$B = \left(\nu + \frac{d + s}{2}\right)\left(\nu + \frac{d + s}{2} + 1\right) = l(l + 1);$$
$$l = \nu + \tfrac{1}{2}(d + s) = 0, 1, 2, \ldots$$

which proves that the square of the total angular momentum of all the particles in the central force (plus interaction, if this is present) is $\hbar^2 l(l + 1)$, and the projection of this on the z axis is $\hbar m$. The quantity $\frac{1}{2}(d + s)$ is the larger of the two integers $|m|$ or $|k|$.

To see more clearly the properties of these eigenfunctions, let us express them more specifically in terms of l. In the first place we consider m to be positive; for negative m's we can take the complex conjugates of these. There will be $2l + 1$ different eigenfunctions for the given values of m and of l, for k can go from $+l$ to $-l$. For $|k| \leq m$, the functions have the form

$$\Upsilon_{lmk} = e^{im\Phi+ik\Psi}\sin^m(\Theta)\cot^k(\tfrac{1}{2}\Theta)F(m - l, l + m + 1|m - k + 1|\sin^2\tfrac{1}{2}\Theta)$$

for $l \geq k > m$, they are

$$\Upsilon_{lmk} = e^{im\Phi+ik\Psi}\sin^k(\Theta)\cot^m(\tfrac{1}{2}\Theta)F(k - l, l + k + 1|k - m + 1|\sin^2\tfrac{1}{2}\Theta) \tag{12.3.81}$$

whereas for $-l \leq k < -m$, they are

$$\Upsilon_{lmk} = e^{im\Phi+ik\Psi} \sin^{-k}(\Theta) \tan^m(\tfrac{1}{2}\Theta)F(-k-l, l-k+1|m-k+1| \sin^2 \tfrac{1}{2}\Theta)$$

The complete solution of Eq. (12.3.78) is obtained by using a linear combination of products of these functions of Φ, Θ, Ψ with functions of $r_1, r_2, \ldots, r_N, \Theta_2, \Theta_3, \ldots, \Theta_n, \phi_3, \phi_4, \ldots \phi_N$. Even when the interaction potentials are added to Eq. (12.3.78), to give the complete Schroedinger equation for the system, these interactions depend only on the r's and on the relative position angles $\Theta_2, \ldots, \phi_N$ and not on Φ, Θ, or Ψ, so the factors Υ will be the same though the other factors will be altered by the interaction. The factors Υ are still not quite the correct factors which turn up in the final solutions when the interaction is "turned on," however.

The correct factors turn out to be linear combinations of the Υ's which are classified according to (1) their behavior under an inversion of the coordinate system and (2) their symmetry or antisymmetry on the interchange of any pair of particles. Since the interaction potentials are invariant against an inversion and are symmetric on interchange of any pair of particles, the final wave function ψ can always be expressed in terms of eigenfunctions which are either odd or even with respect to inversion and are either symmetric or antisymmetric with respect to particle interchange.

Inversion and Parity. *Inversion* is defined as the substitution for each coordinate of every particle by its negative, *i.e.*,

$$\mathbf{r}_i \to -\mathbf{r}_i$$

The process of inversion is equivalent to changing from the usual right-hand coordinate system to a left-hand coordinate system. Interactions which depend only upon the distance between particles are clearly invariant against such an inversion so that, if $\psi(\mathbf{r}_1,\mathbf{r}_2, \ldots)$ is a solution of the Schroedinger equation, $\psi(-\mathbf{r}_1,-\mathbf{r}_2, \ldots)$ is also a solution with the same energy and the same angular momentum quantum numbers. Since there is only one such solution, the two ψ's must be proportional to each other

$$\psi(\mathbf{r}_1,\mathbf{r}_2, \ldots) = \alpha\psi(-\mathbf{r}_1,-\mathbf{r}_2, \ldots)$$

By performing an inversion on both sides of this equation, it is clear that $\alpha^2 = 1$, so that $\alpha = \pm 1$. If

$$\psi(\mathbf{r}_1,\mathbf{r}_2, \ldots) = -\psi(-\mathbf{r}_1,-\mathbf{r}_2, \ldots)$$

the wave function is said to be of *odd parity* while, if there is no change of sign on inversion, the wave function is said to be of *even parity*.

The wave function ψ consists of sums of products of the Υ's and functions of the interparticle distances. Hence the parities are fixed by taking the correct linear combinations of the Υ's. To determine these, the behavior of the Eulerian angles under inversion is required:

$$\Theta \to \pi - \Theta; \quad \Phi \to \pi + \Phi; \quad \Psi \to \pi - \Psi$$

Examples of the proper linear combinations for the $l = 1$ case will be given below.

Symmetrizing for Two-particle Systems. The process of symmetrizing is usually rather tedious. How it works out may be illustrated by considering the case of two particles. Here the exchange of particles 1 and 2 replaces Θ by ϑ_2 and Φ by φ_2 where r_2, ϑ_2, and φ_2 are the coordinates of particle 2 in a spherical coordinate system. The solution of

$$[\nabla_1^2 + \nabla_2^2 + \epsilon - v(r_1) - v(r_2) - w(r_{12})]\psi = 0$$

must be expressible as a linear combination of products of the Υ's and functions f of r_1, r_2 and $\Theta_2 = \vartheta_{12}$. Since $r_{12}^2 = r_1^2 + r_2^2 - 2r_1r_2 \cos \vartheta_{12}$ we see that the whole potential energy is completely independent of Φ, Θ, and Ψ. We need to compute the differential equation for the factors f, for each of the proper combinations of the Υ's.

The easiest way to do this is to consider what happens when w goes to zero and compare it with the form of the Υ's. In this case the solutions must be linear combinations of one-particle functions, of the general form

$$u = e^{im_1\varphi_1 + im_2\varphi_2} P_{l_1}^{m_1}(\cos \vartheta_1) P_{l_2}^{m_2}(\cos \vartheta_2) R_{n_1l_1}(r_1) R_{n_2l_2}(r_2)$$

of the sort discussed on pages 1720 *et seq.* We consider first the case for $l = 0$ (zero total angular momentum). Here the function Υ is a constant, independent of Φ, Θ, or Ψ, so the total solution, which is a linear combination of the u's when $w = 0$, must be a function of $\Theta_2 = \vartheta_{12}$, r_1 and r_2 alone. Such a function is

$$\begin{aligned}\psi^0 &= f^0_{l_1n_1n_2}(\Theta_2, r_1, r_2) = P_{l_1}(\cos \vartheta_{12}) R_{n_1l_1}(r_1) R_{n_2l_1}(r_2) \\ &= \left\{ \sum_{m_1=0}^{l_1} \epsilon_{m_1} \cos[m_1(\varphi_1 - \varphi_2)] \frac{(l_1 - m_1)!}{(l_1 + m_1)!} P_{l_1}^{m_1}(\cos \vartheta_1) P_{l_1}^{m_1}(\cos \vartheta_2) \right\} \cdot \\ &\qquad \cdot R_{n_1l_1}(r_1) R_{n_2l_1}(r_2); \quad w = 0 \qquad (12.3.82)\end{aligned}$$

corresponding to the two particles having equal total angular momenta ($l_2 = l_1$) oppositely oriented ($m_2 = -m_1$) so that the net angular momentum is just zero. Of course, we could use a combination of such functions for different values of l_1, but each different value of l_1 corresponds to a different energy even for the no-interaction case.

Consequently, the function given in Eq. (12.3.82) is the only one, for a given unperturbed energy $\epsilon_{n_1l_1n_2l_1}$, which has zero total angular momentum; it occurs only for those unperturbed energies corresponding to

$l_2 = l_1$. Working backward from the form of f^0, we can see that the differential equation for f for the case when $l = m = 0$ (Υ = constant, $\psi = f^0$) is

$$\frac{1}{r_1^2}\frac{\partial}{\partial r_1}\left(r_1\frac{\partial f^0}{\partial r_1}\right) + \frac{1}{r_2^2}\frac{\partial}{\partial r_2}\left(r_2\frac{\partial f^0}{\partial r_2}\right) + \left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right)\frac{1}{\sin\vartheta_{12}}\frac{\partial}{\partial\vartheta_{12}}\left(\sin\vartheta_{12}\frac{\partial f^0}{\partial\vartheta_{12}}\right) + [\epsilon - v(r_1) - v(r_2) - w(r_{12})]f^0 = 0 \quad (12.3.83)$$

which holds even when w is not zero. When $w \neq 0$, of course, f^0 is no longer proportional to $P_{l_1}(\cos\vartheta_{12})$; the dependence on ϑ_{12} is much more complicated than that because $w(r_{12})$ depends on ϑ_{12}. Incidentally ψ^0 is symmetric to interchange of particles 1 and 2 for all these states; it is independent of the angle $(\varphi_2 - \varphi_1)$, reversal of which reverses the positions of the pair.

Turning now to the case of $l = 1$, there are nine possible factors Υ_{1mk};

	$(k = 1)$	$(k = 0)$	$(k = -1)$
$(m = 1)$	$e^{i\Phi+i\Psi}(1 + \cos\Theta)$;	$e^{i\Phi}\sin\Theta$;	$e^{i\Phi-i\Psi}(1 - \cos\Theta)$
$(m = 0)$	$e^{i\Psi}\sin\Theta$;	$\cos\Theta$;	$e^{-i\Psi}\sin\Theta$
$(m = -1)$	$e^{-i\Phi+i\Psi}(1 - \cos\Theta)$;	$e^{-i\Phi}\sin\Theta$;	$e^{-i\Phi-i\Psi}(1 + \cos\Theta)$

Instead of classifying the solutions according to k, we shall divide them in accordance with their parity and the absolute value of k. These combinations are:

	$(\lvert k\rvert = 1$, even$)$	$(\lvert k\rvert = 1$, odd$)$	$(k = 0$, odd$)$
$(m = 1)$	$e^{i\Phi}[\cos\Psi + i\sin\Psi\cos\Theta]$	$e^{i\Phi}[i\sin\Psi + \cos\Psi\cos\Theta]$	$e^{i\Phi}\sin\Theta$
$(m = 0)$	$\sin\Psi\sin\Theta$	$\cos\Psi\sin\Theta$	$\cos\Theta$
$(m = -1)$	$e^{-i\Phi}[\cos\Psi - i\sin\Psi\cos\Theta]$	$e^{-i\Phi}[i\sin\Psi - \cos\Psi\cos\Theta]$	$e^{-i\Phi}\sin\Theta$

For a given m, the final expression for the symmetric and antisymmetric wave functions can involve only linear combinations of wave functions having the same parity. Note also that, since there are nine independent states, we shall have just nine independent wave functions after symmetrization. Returning to the vector picture, we can say that these nine states (or linear combinations of them) correspond to the two electrons' angular momenta being relatively oriented so that the vector sum is $l = 1$. This can occur in three ways: the individual l's are equal and not antiparallel; or else $l_1 = l_2 + 1$ and the two are antiparallel; or else $l_1 = l_2 - 1$ and the two are antiparallel. For each of the three cases there are three orientations of the net angular momentum with respect to the z axis.

Starting with the $m = 0$ trio, we take first the wave function having even parity, $\sin \Psi \sin \Theta$. We see whether we can find a solution of the sort $\psi = \sin \Psi \sin \Theta\, f(\Theta_2, r_1, r_2)$ which can be expanded (for $w = 0$) in factors of the u type. The proper combination is

$$\begin{aligned}\psi^0 &= \sin \Psi \sin \Theta\, P^1_{l_1}(\cos \Theta_2) R_{n_1 l_1}(r_1) R_{n_2 l_1}(r_2)\\ &= \sin \vartheta_1 \sin \vartheta_2 \sin(\varphi_2 - \varphi_1) T^1_{l_1-1}(\cos \Theta_2) R_{n_1 l_1}(r_1) R_{n_2 l_1}(r_1)\end{aligned}$$

But according to the addition formula for the Gegenbauer polynomials we have

$$\begin{aligned}\sin(\varphi_2 - \varphi_1) T^1_{l_1-1}(\cos \Theta_2) = 2 \sum_{m_1=1}^{l_1} \frac{m_1(l_1 - m_1)!}{(l_1 + m_1)!} [\sin \vartheta_1 \sin \vartheta_2]^{m_1-1} \cdot \\ \cdot T^{m_1}_{l_1-m_1}(\cos \vartheta_1) T^{m_1}_{l_1-m_1}(\cos \vartheta_2) \sin[m_1(\varphi_2 - \varphi_1)]\end{aligned}$$

Consequently, the proper combination for the case for $w = 0$ is

$$\begin{aligned}\sin \Psi \sin \Theta f^0 &= \sin \Psi \sin \Theta P^1_{l_1}(\cos \vartheta_{12}) R_{n_1 l_1}(r_1) R_{n_2 l_1}(r_2)\\ &= 2 \sum_{m_1=1}^{l_1} \frac{m_1(l_1 - m_1)!}{(l_1 + m_1)!} P^{m_1}_{l_1}(\cos \vartheta_1) P^{m_1}_{l_1}(\cos \vartheta_2) \sin[m(\varphi_2 - \varphi_1)] \cdot \\ &\qquad \cdot R_{n_1 l_1}(r_1) R_{n_2 l_1}(r_2)\end{aligned}$$

and the corresponding equation for $f(\vartheta_{12}, r_1, r_2)$ (even for $w \neq 0$) is

$$\begin{aligned}&\frac{1}{r_1^2} \frac{\partial}{\partial r_1}\left(r_1^2 \frac{\partial f}{\partial r_1}\right) + \frac{1}{r_2^2} \frac{\partial}{\partial r_2}\left(r_2^2 \frac{\partial f}{\partial r_2}\right)\\ &\quad + \left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right)\left[\frac{1}{\sin \vartheta_{12}} \frac{\partial}{\partial \vartheta_{12}}\left(\sin \vartheta_{12} \frac{\partial f}{\partial \vartheta_{12}}\right) - \frac{f}{\sin^2 \vartheta_{12}}\right]\\ &\quad + [\epsilon - v(r_1) - v_2(r_2) - w(r_{12})]f = 0 \qquad (12.3.84)\end{aligned}$$

differing from Eq. (12.3.83), for $l = 0$, by the term in $\sin^{-2} \vartheta_{12}$. These functions correspond to the case of both particles of the same l, with vector sum equal to unity with zero component along z. Since the function changes sign with reversal of $(\varphi_2 - \varphi_1)$, it is antisymmetric with respect to interchange of particles.

The other pair of functions for $m = 0$ are more complicated, in that we have specifically to consider the interchanged function $\tilde{f} = f(\vartheta_{12}, r_2, r_1)$. To shorten an already long discussion, we state results without detailed proof. We take the combination having odd parity

$$\psi = \cos \Theta[f(\vartheta_{12}, r_1, r_2) + \cos \vartheta_{12} f(\vartheta_{12}, r_2, r_1)] + \sin \Theta \cos \Psi \sin \vartheta_{12} f(\vartheta_{12}, r_2, r_1)$$

In the case $w = 0$, we can set $f^0(\vartheta_{12}, r_1, r_2) = P_{l_1}(\cos \vartheta_{12}) R_1(r_1) R_2(r_2)$, in which case, after a little juggling, we see that

$$\begin{aligned}\psi^0 &= \cos\vartheta_1 P_{l_2}(\cos\vartheta_{12})R_1(r_1)R_2(r_2) + \cos\vartheta_2 P_{l_2}(\cos\vartheta_{12})R_2(r_1)R_1(r_2) \\ &= F(1,2) + F(2,1)\end{aligned}$$

where $F(1,2) = \sum_{m_1=0}^{l_2} \frac{\epsilon_{m_1}}{(2l_2+1)} \cos[m_1(\varphi_1 - \varphi_2)] P_{l_2}^{m_1}(\cos\vartheta_2) \cdot$

$$\cdot \left[\frac{(l_2 - m_1 + 1)!}{(l_2 + m_1)!} P_{l_2+1}^{m_1}(\cos\vartheta_1) + \frac{(l_2 - m_1)!}{(l_2 + m_1 - 1)!} P_{l_2-1}^{m_1}(\cos\vartheta_1) \right] R_1(r_1)R_2(r_2)$$

and $F(2,1)$ is obtained from this by interchanging ϑ_1, φ_1, r_1 with ϑ_2, φ_2, r_2. This obviously is a combination of states of antiparallel particle angular momentum, where one particle or the other has an l one greater or one less than the other, the requirements of symmetry mixing up the two possibilities. It is also obvious that these functions are unchanged when the particles are interchanged. The equation for $f(\vartheta_{12},r_1,r_2)$ in this case includes a term with the inverted function $\tilde{f} = f(\vartheta_{12},r_2,r_1)$;

$$\begin{aligned}\frac{1}{r_1^2}\frac{\partial}{\partial r_1}\left(r_1^2 \frac{\partial f}{\partial r_1}\right) + \frac{1}{r_2^2}\frac{\partial}{\partial r_2}\left(r_2^2 \frac{\partial f}{\partial r_2}\right) + \left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right)\frac{1}{\sin\vartheta_{12}}\frac{\partial}{\partial\vartheta_{12}}\left(\sin\vartheta_{12}\frac{\partial f}{\partial\vartheta_{12}}\right) \\ + \frac{2}{r_1^2}\left(\cot\vartheta_{12}\frac{\partial f}{\partial\vartheta_{12}} - f\right) - \frac{2}{r_2^2\sin\vartheta_{12}}\frac{\partial\tilde{f}}{\partial\vartheta_{12}} + [\epsilon - v(r_1) - v(r_2) - w]f = 0\end{aligned} \tag{12.3.85}$$

The third combination, antisymmetric for interchange of particles and again of odd parity, is

$$\psi = \cos\Theta\,[f - \cos\vartheta_{12}\tilde{f}] - \sin\Theta\cos\Psi\sin\vartheta_{12}\tilde{f}$$

which, when $w = 0$, reduces to

$$\cos\vartheta_1 P_{l_2}(\cos\vartheta_{12})R_1(r_1)R_2(r_2) - \cos\vartheta_2 P_{l_2}(\cos\vartheta_{12})R_2(r_1)R_1(r_2)$$

The equation for f (for $w \neq 0$) is now

$$\begin{aligned}\frac{1}{r_1^2}\frac{\partial}{\partial r_1}\left(r_1^2 \frac{\partial f}{\partial r_1}\right) + \frac{1}{r_2^2}\frac{\partial}{\partial r_2}\left(r_2^2 \frac{\partial f}{\partial r_2}\right) + \left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right)\frac{1}{\sin\vartheta_{12}}\frac{\partial}{\partial\vartheta_{12}}\left(\sin\vartheta_{12}\frac{\partial f}{\partial\vartheta_{12}}\right) \\ + \frac{2}{r_1^2}\left(\cot\vartheta_{12}\frac{\partial f}{\partial\vartheta_{12}} - f\right) + \frac{2}{r_2^2\sin\vartheta_{12}}\frac{\partial\tilde{f}}{\partial\vartheta_{12}} + [\epsilon - v(r_1) - v(r_2) - w]f = 0\end{aligned} \tag{12.3.86}$$

which again is a state corresponding to one particle having angular momentum either one greater or one less than the other, the projection of the resultant on the z axis being zero. The equation for f differs from (12.3.85) only in the sign of the $(\partial\tilde{f}/\partial\vartheta_{12})$ term.

The other six states may be combined in similar ways; we write down the summary of results:

$$
\begin{aligned}
\psi_{11s} &= e^{i\Phi}[f_{11s} \sin\Theta + \tilde{f}_{11s}(\sin\Theta \cos\vartheta_{12} \\
&\qquad + i \sin\Psi \sin\vartheta_{12} + \cos\Theta \cos\Psi \sin\vartheta_{12})] \\
&= e^{i\varphi_1} \sin\vartheta_1 f_{11s} + e^{i\varphi_2} \sin\vartheta_2 \tilde{f}_{11s} \\
\psi_{11a} &= e^{i\varphi_1} \sin\vartheta_1 f_{11a} - e^{i\varphi_2} \sin\vartheta_2 \tilde{f}_{11a} \\
\psi_{110} &= e^{i\Phi}(\cos\Psi - i \cos\Theta \sin\Psi) f_{110} \\
&= -(e^{i\varphi_1} \sin\vartheta_1 \cos\vartheta_2 - e^{i\varphi_2} \cos\vartheta_1 \sin\vartheta_2) \csc\vartheta_{12} f_{110} \qquad (12.3.87) \\
\psi_{10s} &= f_{10s} \cos\Theta + \tilde{f}_{10s}(\cos\Theta \cos\vartheta_{12} + \sin\Theta \sin\vartheta_{12} \cos\Psi) \\
&= \cos\vartheta_1 f_{10s} + \cos\vartheta_2 \tilde{f}_{10s} \\
\psi_{10a} &= \cos\vartheta_1 f_{10a} - \cos\vartheta_2 \tilde{f}_{10a} \\
\psi_{100} &= \sin\Psi \sin\Theta f_{100} \\
\psi_{1,-1s} &= \bar{\psi}_{1,1s};\quad \psi_{1,-1a} = \bar{\psi}_{1,1a};\quad \psi_{1,-10} = \bar{\psi}_{1,10}
\end{aligned}
$$

where the first subscript on ψ gives the value of total angular momentum l, the second its z component m, and the last its symmetry properties. The factors f having last subscript 0 are solutions of Eq. (12.3.84) and those with subscript a satisfy Eq. (12.3.86); the resulting ψ's are antisymmetric with respect to interchange of particles. Those factors f having last subscript s satisfy Eq. (12.3.85), and the resulting ψ's are then symmetric for particle exchange. These functions are still further modified if the particles have spins, which also interact.

The analysis for $l > 1$ goes very much the same but is still more tedious. There are 25 different functions for $l = 2$, corresponding to the vector sum of the one-particle angular momenta being 2, with five different orientations of this resultant with respect to the z axis. Such a resultant can result from particle 2 having an l_2 ranging from $l_1 - 2$ to $l_1 + 2$ giving 5×5 states. The actual solutions are various linear combinations of these, and the equations for the corresponding $f(\vartheta_{12}, r_1, r_2)$ are all nonseparable when the interaction $w(r_{12}) \neq 0$. The corresponding analysis for more than two particles is still more tedious.

Bound, Free, and "Surface" States. In most of the cases of interest the potential energy terms $v(r_n)$ and $w(r_{mn})$ go to zero for large values of the radial argument. The situation is similar to that considered on pages 1710 *et seq.*, except that the configuration space appropriate for ψ is $3N$-dimensional. For positive energy all values are allowed and most states correspond to all particles being free, coming in from infinity, being reflected by the potential "well" near the origin, and going back out to infinity. Possible states correspond to a "plane" wave $\exp(\Sigma i \mathbf{k}_n \cdot \mathbf{r}_n)$ for the incident part, for the energy $\epsilon = \Sigma k_n^2$. As long as the interaction terms w are zero, each particle is affected only by its own potential $v(\mathbf{r}_n)$, which corresponds to a reflection of the plane wave at each of the $(3N - 3)$ hypersurfaces corresponding to each of the particles in turn being at the origin. These hypersurfaces take the place of the "mirrors" in the earlier example.

In addition to the "oblique" waves, where every particle is free, there are "surface" waves, corresponding to one or more of the particles in a bound state, these waves being confined to the neighborhood of one or the other of the hypersurface "mirrors" mentioned in the previous paragraph. The energy ϵ for these surface waves may be positive or it may be negative, when the energy of binding of the bound particles (which is negative) more than cancels the sum of the kinetic energies of the ones which are free. Consequently, for part of its negative range also, the energy has continuous allowed values. These surface wave states are always degenerate, for any one of the particles can be bound and the others free. As long as interaction is neglected, however, these states do not "mix up"; if one of the particles starts out bound, it stays bound. Finally there are the strictly discrete energy levels corresponding to states where all particles are bound. [Sometimes these states are not present as, for example, for a coulomb field (Ze^2/r) with several more electrons than the Z of the field.] These bound states have wave functions which are large only near the origin and have negative energy values, though some of these discrete levels (for several particles in higher bound states) may be higher than the possible continuous levels for some of the surface waves. As long as there is no interaction, none of these states "mix up."

Introduction of the interaction terms w adds extra "mirrors" along those hypersurfaces corresponding to two particles coinciding in position. As with the simple problem discussed before, these extra terms at once "mix up" the previous states by adding reflections from the extra mirrors (one particle exchanges states with another), by allowing "oblique" waves to change into "surface" waves (one particle sinks into a bound state, giving its extra energy to another) or vice versa and, finally, by producing an extra "hyperspherically outgoing" wave coming from the $3N$-dimensional origin [all particles redistribute their kinetic energies, coming back out with new wave numbers k'_n, the only requirement being that $\Sigma(k'_n)^2 = \Sigma(k_n)^2 = \epsilon$].

This characteristic behavior can be demonstrated by use of the two-particle model discussed in the previous two subsections. The part of ψ depending on the Euler angles can be separated off, since it determines the orientation of the system in space and not the relative orientation of the two radius vectors. This γ factor is a linear combination of the functions given in Eq. (12.3.82). The factor f, depending on the radial distances r_1 and r_2 and the angle between them, ϑ_{12}, is the one modified when interparticle forces are "turned on." We notice that, when the interaction term w is neglected, this factor turns out to be a spherical harmonic of $\cos \vartheta_{12}$ times a product of radial functions of r_1 and of r_2.

We can picture the wave function somewhat better if we change to hyperspherical coordinates R and α, such that $r_1 = R \cos \alpha$, $r_2 = R \sin \alpha$. The factor f is then a function of R, 2α, and ϑ_{12}. The geometry is symbolized in Fig. 12.10, where 2α goes from 0 to π, as does ϑ_{12}, so that only four octants are needed. The construction to obtain r_1 and r_2 is shown; if one of these is placed along the line OP and the other along OQ, the distance between the two ends is r_{12}. In this geometry the bound states have factors f which are small except near the origin; the "surface" waves are concentrated near the polar axes $2\alpha = 0$, $2\alpha = \pi$; and the interaction forces are large near the line $2\alpha = \frac{1}{2}\pi$, $\vartheta_{12} = 0$ (the line OP). The geometry is only symbolic, however, for the scale factors for the coordinates do not correspond to the actual scale factors.

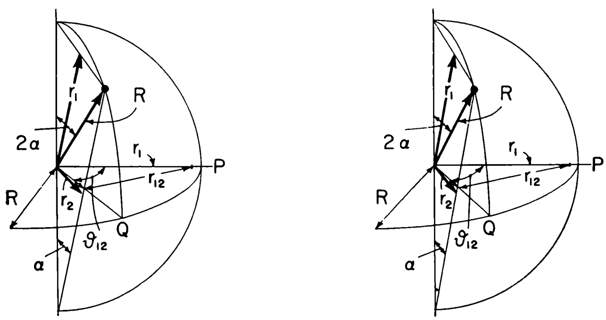

Fig. 12.10 Geometric construction relating R and α with r_1, r_2, and ϑ_{12}.

Green's Function for the Two-particle System. In order to continue the examination of the behavior of a system of two particles in a central force field plus interaction, it will be necessary to obtain a Green's function for the two particles without central force or interaction. One needs the solution of the six-dimensional inhomogeneous Helmholtz equation

$$[\nabla_1^2 + \nabla_2^2 + k^2]G = -\delta(\mathbf{r}_1 - \mathbf{r}_1')\delta(\mathbf{r}_2 - \mathbf{r}_2')$$

The most "physical" coordinate system in which to expand G would be the one with the Euler angles, Φ, Θ, Ψ, ϑ_{12}, α, and R, which we have been discussing for the past several pages. But these are not orthogonal coordinates; it would be best to expand G in orthogonal coordinates first and then change the series over, after expansion, if needed.

The most useful orthogonal set, not too different from the Euler-angle set and also not too different from the set used when the particles are considered separately, is the *two-particle hyperspherical system,* defined as follows:

$$\begin{aligned}
&x_1 = r\cos\alpha\sin\vartheta_1\cos\varphi_1; \quad x_2 = r\sin\alpha\sin\vartheta_2\cos\varphi_2 \\
&y_1 = r\cos\alpha\sin\vartheta_1\sin\varphi_1; \quad y_2 = r\sin\alpha\sin\vartheta_2\sin\varphi_2 \\
&z_1 = r\cos\alpha\cos\vartheta_1; \quad z_2 = r\sin\alpha\cos\vartheta_2 \qquad\qquad (12.3.88) \\
&h_r = 1; \quad h_\alpha = r; \quad h_{\vartheta_1} = r\cos\alpha; \quad h_{\varphi_1} = r\cos\alpha\sin\vartheta_1 \\
&h_{\vartheta_2} = r\sin\alpha; \quad h_{\varphi_2} = r\sin\alpha\sin\vartheta_2; \quad r_1 = r\cos\alpha; \quad r_2 = r\sin\alpha \\
&r_{12} = r\sqrt{1 - \sin 2\alpha[\cos\vartheta_1\cos\vartheta_2 + \sin\vartheta_1\sin\vartheta_2\cos(\varphi_1 - \varphi_2)]}
\end{aligned}$$

The volume element in these coordinates is $r^5 \sin^2\alpha \cos^2\alpha \sin\vartheta_1 \sin\vartheta_2$ $d\vartheta_1\, d\vartheta_2\, d\varphi_1\, d\varphi_2\, d\alpha\, dr$, the surface area of a hypersphere of radius r is $\pi^3 r^5$, and the Helmholtz equation is

$$\begin{aligned}(\nabla_1^2 + \nabla_2^2 + k^2)\psi &= \frac{1}{r^5}\frac{\partial}{\partial r}\left(r^5 \frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2 \sin^2\alpha \cos^2\alpha}\frac{\partial}{\partial\alpha}\left(\sin^2\alpha \cos^2\alpha \frac{\partial\psi}{\partial\alpha}\right) \\ &\quad + \frac{1}{r^2\cos^2\alpha}\left[\frac{1}{\sin\vartheta_1}\frac{\partial}{\partial\vartheta_1}\left(\sin\vartheta_1 \frac{\partial\psi}{\partial\vartheta_1}\right) + \frac{1}{\sin^2\vartheta_1}\frac{\partial^2\psi}{\partial\varphi_1^2}\right] \\ &\quad + \frac{1}{r^2\sin^2\alpha}\left[\frac{1}{\sin\vartheta_2}\frac{\partial}{\partial\vartheta_2}\left(\sin\vartheta_2 \frac{\partial\psi}{\partial\vartheta_2}\right) + \frac{1}{\sin^2\vartheta_2}\frac{\partial^2\psi}{\partial\varphi_2^2}\right] + k^2\psi = 0 \qquad (12.3.89)\end{aligned}$$

The dependence of ψ on the φ's and ϑ's is well known; the corresponding factors are spherical harmonics with quantum numbers l_1, m_1, l_2, m_2, and the results of the operations in the brackets are $l_1(l_1+1)\psi$ and $l_2(l_2+1)\psi$, respectively. The equation for the α factor is then

$$\frac{1}{\sin^2\alpha\cos^2\alpha}\frac{d}{d\alpha}\left(\sin^2\alpha\cos^2\alpha\frac{dA}{d\alpha}\right) - \left[\frac{l_1(l_1+1)}{\cos^2\alpha} + \frac{l_2(l_2+1)}{\sin^2\alpha}\right]A = -BA$$

which may be transformed into that for a hypergeometric function of $\sin^2\alpha$. The solution which is finite at $\alpha = 0$ and the corresponding value of the separation constant B is

$$\begin{aligned} A &= \cos^{l_1}\alpha \sin^{l_2}\alpha\, F(-n, l_1 + l_2 + n + 2|l_2 + \tfrac{3}{2}|\sin^2\alpha) \\ B &= (l_1 + l_2 + 2n)(l_1 + l_2 + 2n + 4)\end{aligned} \qquad (12.3.90)$$

where n must be zero or a positive integer for A to be finite at $\alpha = \frac{1}{2}\pi$; in which case F is a finite polynomial (*a Jacobi polynomial*) of $n + 1$ terms.

The final equation, for the radial part of ψ, is

$$\frac{1}{r^5}\frac{d}{dr}\left(r^5\frac{dR}{dr}\right) + \left[k^2 - \frac{(l_1 + l_2 + 2n + 2)^2 - 4}{r^2}\right]R = 0$$

which has, as solutions, the Bessel functions $r^{-2}J_{l_1+l_2+2n+2}(kr)$ or $r^{-2}N_{l_1+l_2+2n+2}(kr)$. Consequently, the complete solution of Eq. (12.3.89) which is finite at $r = 0$ is

$$\begin{aligned}\psi = {}& {\cos \atop \sin}(m_1\varphi_1) P_{l_1}^{m_1}(\cos\vartheta_1)\, {\cos \atop \sin}(m_2\varphi_2) P_{l_2}^{m_2}(\cos\vartheta_2)\cos^{l_1}\alpha\sin^{l_2}\alpha \cdot \\ & \cdot F(-n, l_1 + l_2 + n + 2|l_2 + \tfrac{3}{2}|\sin^2\alpha)(1/r^2)J_{l_1+l_2+2n+2}(kr)\end{aligned}$$

For outgoing waves, we use the Hankel function $H^{(1)}_{l_1+l_2+2n+2}(kr)$ instead of J; in which case, of course, ψ is infinite at $r = 0$.

The Green's function may be expressed in terms of the six-dimensional source-observer distance R, defined as

$$\begin{aligned} R^2 = (r)^2 + (r')^2 & \\ & - 2rr'\cos\alpha\cos\alpha'[\cos\vartheta_1\cos\vartheta_1' + \sin\vartheta_1\sin\vartheta_1'\cos(\varphi_1 - \varphi_1')] \\ & - 2rr'\sin\alpha\sin\alpha'[\cos\vartheta_2\cos\vartheta_2' + \sin\vartheta_2\sin\vartheta_2'\cos(\varphi_2 - \varphi_2)]\end{aligned}$$

It is, simply,

$$G_k(\mathbf{r}|\mathbf{r}') = \left(\frac{ik^2}{16\pi^2R^2}\right)H_2^{(1)}(kR) \simeq \begin{cases} (1/4\pi^3R^4); & kR \ll 1 \\ [\frac{1}{2}ik^4/(2\pi ikR)^{\frac{5}{2}}]e^{ikR}; & kR \gg 1 \end{cases} \quad (12.3.91)$$

Of course, we also wish the expansion of G in terms of the coordinates $r, \alpha, \ldots, \varphi_2$ of the observer and $r', \alpha', \ldots, \varphi_2'$ of the source, with respect to some arbitrary origin and polar axis. This expansion may be obtained by the methods outlined in Chap. 7. It is

$$G_k(\mathbf{r}|\mathbf{r}') = \frac{i}{16\pi(rr')^2} \cdot$$

$$\cdot \sum_{l_1l_2n}(2l_1+1)(2l_2+1)\frac{(l_1+l_2+2n+2)(l_1+l_2+n+1)!\Gamma(l_2+n+\frac{3}{2})}{n![\Gamma(l_2+\frac{3}{2})]^2\Gamma(l_1+n+\frac{3}{2})} \cdot$$

$$\cdot \cos^{l_1}\alpha \sin^{l_2}\alpha\, F(-n, l_1+l_2+n+2|l_2+\tfrac{3}{2}|\sin^2\alpha) \cdot$$

$$\cdot \cos^{l_1}\alpha' \sin^{l_2}\alpha'\, F(-n, l_1+l_2+n+2|l_2+\tfrac{3}{2}|\sin^2\alpha') \cdot$$

$$\cdot \left\{\sum_{m_1m_2} \epsilon_{m_1}\epsilon_{m_2}\frac{(l_1-m_1)!(l_2-m_2)!}{(l_1+m_1)!(l_2+m_2)!}\cos[m_1(\varphi_1-\varphi_1')]\cos[m_2(\varphi_2-\varphi_2')] \cdot \right.$$

$$\left. \cdot P_{l_1}^{m_1}(\cos\vartheta_1)P_{l_1}^{m_1}(\cos\vartheta_1')P_{l_2}^{m_2}(\cos\vartheta_2)P_{l_2}^{m_2}(\cos\vartheta_2')\right\} \cdot$$

$$\cdot \begin{cases} J_{l_1+l_2+2n+2}(kr)H_{l_1+l_2+2n+2}(kr'); & r < r' \\ J_{l_1+l_2+2n+2}(kr')H_{l_1+l_2+2n+2}(kr); & r > r' \end{cases} \quad (12.3.92)$$

From this one can obtain the expansion for a "plane wave"; particle 1, for example, coming in along the z_1 axis from $-\infty$ with wave number $k_1 = Mv_1/\hbar = k\cos\beta$ and particle 2 with wave number $k_2 = Mv_2/\hbar = k\sin\beta$ in a direction Θ, along the x_2, z_2 plane. One sets the source at $r' \to \infty$, $\alpha' = \beta$, $\vartheta_1' = -\pi$, $\vartheta_2' = -\Theta$, $\varphi_2' = \pi$ and equates Eqs. (12.3.91) and (12.3.92)

$$\exp[ikr\cos\beta\cos\alpha\cos\vartheta_1 + ikr\sin\beta\sin\alpha(\cos\Theta\cos\vartheta_2$$

$$+ \sin\Theta\sin\vartheta_2\cos\varphi_2)] = \frac{\pi}{k^2r^2}\sum_{l_1l_2n}(-1)^n i^{l_1+l_2}(2l_1+1)(2l_2+1) \cdot$$

$$\cdot \frac{(l_1+l_2+2n+2)(l_1+l_2+n+1)!\,\Gamma(l_2+n+\frac{3}{2})}{n![\Gamma(l_2+\frac{3}{2})]^2\Gamma(l_1+n+\frac{3}{2})} \cdot$$

$$\cdot \cos^{l_1}\alpha' \sin^{l_2}\alpha' F(-n, l_1+l_2+n+2|l_2+\tfrac{3}{2}|\sin^2\beta)\cos^{l_1}\alpha\sin^{l_2}\alpha \cdot$$

$$\cdot F(-n, l_1+l_2+n+2|l_2+\tfrac{3}{2}|\sin^2\alpha)P_{l_1}(\cos\vartheta_1)J_{l_1+l_2+2n+2}(kr) \cdot$$

$$\cdot \left\{\sum_{m_2=0}^{l_2}\epsilon_{m_2}\frac{(l_2-m_2)!}{(l_2+m_2)!}P_{l_2}^{m_2}(\cos\Theta)P_{l_2}^{m_2}(\cos\vartheta_2)\cos(m\varphi_2)\right\} \quad (12.3.93)$$

When we are using the Green's function in an integral equation, it often occurs that we are dealing only with solutions of a given symmetry or orientation, in which case we need use only that part of G which has the corresponding symmetry or orientation. As all the eigenfunctions in Eq. (12.3.92) are members of orthogonal sets, the other terms do not intrude. This property of the series is particularly useful when we are dealing with functions of a given total angular momentum; we can pick out that part of the series corresponding to a given l and m and use that part alone.

For example, we say in Eq. (12.3.83) that a function having zero total angular momentum has the form $P_{l_1}(\cos\vartheta_{12})R_1(r_1)R_2(r_2)$. Therefore the part of G having $l = 0$ corresponds to the part of the sum for $l_2 = l_1$, $m_2 = m_1$, which is symmetric in $(\varphi_1 - \varphi_2)$;

$$G_{l=0}(\mathbf{r}|\mathbf{r}') = \frac{i}{8\pi(rr')^2}\sum_{nl_1}(2l_1+1)\frac{(l_1+n+1)(2l_1+n+1)!}{n![\Gamma(l_1+\frac{3}{2})]^2}\cdot$$
$$\cdot(\sin\alpha'\cos\alpha')^{l_1}F(-n,\,2l_1+n+2|l_1+\tfrac{3}{2}|\sin^2\alpha')(\sin\alpha\cos\alpha)^{l_1}$$
$$\cdot F(-n,\,2l_1+n+2|l_1+\tfrac{3}{2}|\sin^2\alpha)P_{l_1}(\cos\vartheta'_{12})P_{l_1}(\cos\vartheta_{12})$$
$$\cdot\begin{cases} J_{2l_1+2n+2}(kr)H_{2l_1+2n+2}(kr'); & r<r' \\ J_{2l_1+2n+2}(kr')H_{2l_1+2n+2}(kr); & r>r'\end{cases}$$

where the partial sum is independent of the Euler angles Φ, Θ, and Ψ. We note that

$$F(-n,\,n+2l+2|l+\tfrac{3}{2}|\sin^2\alpha) = \frac{2^{l+\frac{1}{2}}n!\Gamma(l+\frac{3}{2})}{(n+2l+1)!}T_n^{l+\frac{1}{2}}(\cos 2\alpha)$$

Similar partial sums can be made for other values of l and m, by using the combinations given in Eq. (12.3.87), for example. These partial sums are each combinations of functions of Φ, Θ, Ψ times functions of α, α', ϑ_{12}, ϑ'_{12}, r, and r'. In some cases we need to use functions with particles interchanged. This is done by interchanging ϑ_1, φ_1 with ϑ_2, φ_2 and changing α into $\frac{1}{2}\pi - \alpha$, which interchanges r_1 and r_2. In such cases we may use the relation

$$F(-n,\,l_1+l_2+n+2|l_2+\tfrac{3}{2}|\sin^2\alpha)$$
$$= (-1)^{l_1+l_2+n}[\Gamma(l_2+\tfrac{3}{2})/\Gamma(l_1+\tfrac{3}{2})]F(-n,\,l_1+l_2+n+2|l_1-\tfrac{3}{2}|\cos^2\alpha)$$

to find the right combination.

Still another representation for the Green's function, which is of use at times, is the Fourier integral representation. By methods which are familiar by now, we find that

$$G_k(\mathbf{r}|\mathbf{r}') = \frac{1}{64\pi^6}\int\cdots\int\frac{e^{i\xi_1(x_1-x_1')+\cdots+i\zeta_2(z_2-z_2')}}{\xi_1^2+\cdots+\zeta_2^2-k^2}\,d\xi_1\cdots d\zeta_2 \qquad (12.3.94)$$

where the sixfold integration is over the components ξ_1, η_1, ζ_1, ξ_2, η_2, ζ_2 of the wave number along the rectangular axes. This integral can, if necessary, be transformed into hyperspherical coordinates by use of

Eqs. (12.3.88); in this case it is usually best also to transform the wave-number space into hyperspherical coordinates, with a "radius" κ and five angles.

Bound States. For the bound states we need to find both allowed energy and corresponding wave function. To compute energy we can use the variational technique; we can then improve the form of the wave function by using the results of the variational calculation in the integral equation for ψ, using the Green's function (see Chap. 9). The integral for two particles to be minimized is

$$H = \int \cdot \cdot \cdot \int \bar{\phi}[-\nabla_1^2 - \nabla_2^2 + v(r_1) + v(r_2) + w(r_{12})]\phi \, dv_1 \, dv_2$$

The minimum value of H for a normalized wave function ϕ is the allowed value of the energy.

As a simple example, we can consider the atom of nuclear charge Ze, having two bound electrons in their lowest state. In this case $v(r) = -(Zq^2/r)$ and $w(r_{12}) = q^2/r_{12}$ where $q^2 = 2Me^2/\hbar^2$, e being the electronic charge. A very simple choice of wave function for this state is

$$\phi = (\mu^3/\pi)e^{-\mu(r_1+r_2)}$$

which is already normalized. The kinetic energy part of the integral is, since ϕ is symmetric,

$$T = 2\int \cdot \cdot \cdot \int \bar{\phi}\nabla_1^2\phi \, dv_1 \, dv_1 = 2\mu^2$$

The average nuclear potential is

$$V = -2Zq^2\int \cdot \cdot \cdot \int (1/r_1)\phi^2 \, dv_1 \, dv_2 = -2\mu Z q^2$$

The integral for the interaction potential is obtained by expanding $1/r_{12}$ in spherical harmonics:

$$\frac{1}{r_{12}} = \sum_{n=0}^{\infty} P_n(\cos\vartheta_{12}) \begin{cases} (r_1^n/r_2^{n+1}); & r_2 > r_1 \\ (r_2^n/r_1^{n+1}); & r_1 > r_2 \end{cases}$$

Since the wave function is independent of ϑ_{12}, only the first term of this series contributes to the integral, so we have

$$\begin{aligned} W &= q^2 \int \cdot \cdot \cdot \int \phi^2 \left(\frac{1}{r_{12}}\right) dv_1 \, dv_2 \\ &= 8\mu^6 q^2 \int_0^\infty e^{-2\mu r_1} r_1^2 \, dr_1 \left[\frac{1}{r_1}\int_0^{r_1} e^{-2\mu r_2} r_2^2 \, dr_2 + \int_{r_1}^\infty e^{-2\mu r_2} r_2 \, dr_2\right] \\ &= \tfrac{5}{8}\mu q^2 \end{aligned}$$

Consequently $H = 2\mu^2 - 2\mu q^2(Z - \frac{5}{16})$; the best value of μ is the root of $dH/d\mu = 0$, which is $\mu = \frac{1}{2}q^2(Z - \frac{5}{16})$, and the best value of the energy of the lowest bound state which can be obtained from this simple form of trial function is

$$\epsilon \simeq -\tfrac{1}{2}(Z - \tfrac{5}{16})^2 q^4 \quad \text{or} \quad E \simeq -(Z - \tfrac{5}{16})^2 (Me^4/\hbar^2) \qquad (12.3.95)$$

This is actually a fairly good approximation to the energy, in spite of the acknowledged poorness of the trial function. The measured values of the energy of the lowest states of the two-electron systems for $Z = 2, 3, 4, 5, 6 (He, Li^+, Be^{++}, B^{+++}, C^{++++})$, in units of $-(Me^4/\hbar^2)$ are

$$-2\epsilon_{\text{exp}} = 5.807,\ 14.560,\ 27.313,\ 44.065,\ 64.813$$

whereas the corresponding values from Eq. (12.3.95) are

$$-2\epsilon \simeq 5.695,\ 14.445,\ 27.195,\ 43.945,\ 64.695$$

an almost constant difference, corresponding to an error of less than 2 per cent for $Z = 2$ and less than 0.2 per cent for $Z = 6$. In fact an extremely good value for ϵ would be $-\frac{1}{2}(Z - \frac{5}{16})^2 - 0.120$.

We could now insert this approximate form for ψ into the right-hand side of the integral equation which is formed by using the Green's function obtained in the previous subsection:

$$\psi(\mathbf{r}) = -\int \cdot \cdot \cdot \int [v(r_1') + v(r_2') + w(r_{12}')]\psi(\mathbf{r}')G_{\sqrt{\epsilon}}(\mathbf{r}|\mathbf{r}')\, dv_1'\, dv_2'$$

where the vectors $\mathbf{r}$, $\mathbf{r}'$ are six-dimensional vectors in the configuration space of the two particles. We have set the k^2 of the Green's function to correspond to the bound-state energy ϵ (negative, so that k is imaginary), assuming that we have determined the energy fairly accurately [as, for example, in Eq. (12.3.95)]. Inserting the approximate form of ϕ in the integral and performing the integration should yield an improved form for ψ.

The trial wave function is a function of r and α [see Eqs. (12.3.88)] only, being independent of ϑ_1, ϑ_2, φ_1, and φ_2. Since the functions v are likewise dependent only on r and α, the first two terms of the integral involve only the term in series (12.3.92) for $l_1 = l_2 = m_1 = m_2 = 0$. Since

$$w(r_{12}) = \frac{q^2}{r_{12}} = \left(\frac{1}{r}\right) \sum_{l=0}^{\infty} P_l(\cos \vartheta_{12}) \begin{cases} \sec \alpha \tan^l \alpha; & 0 \le \alpha \le \frac{1}{4}\pi \\ \csc \alpha \cot^l \alpha; & \frac{1}{4}\pi \le \alpha \le \frac{1}{2}\pi \end{cases}$$

the third term in the integral will involve only those terms for which $l_1 = l_2$, $m_1 = m_2 = 0$, in fact, the terms written out for $G_{l=0}$.

The terms in the integral independent of ϑ_{12} are fairly easily obtained. The part of G which is used here is

$$G_0 = \frac{i}{8\pi^2(rr')^2} \sum_{n=0}^{\infty} \frac{\sin[2(n+1)\alpha] \sin[2(n+1)\alpha']}{\sin \alpha \cos \alpha \sin \alpha' \cos \alpha'} \cdot$$

$$\cdot \begin{cases} J_{2n+2}(i\sqrt{2}\,\mu r') H_{2n+2}(i\sqrt{2}\,\mu r); & r > r' \\ H_{2n+2}(i\sqrt{2}\,\mu r') J_{2n+2}(i\sqrt{2}\,\mu r); & r < r' \end{cases}$$

where we have set $k^2 = \epsilon = -2\mu^2 = -\frac{1}{2}q^4(Z - \frac{5}{16})^2$, as per Eq. (12.3.95). The functions $v(r_1') + v(r_2')$ become $-(q^2Z/r')(\csc\alpha' + \sec\alpha')$ and the approximate wave function is

$$(\mu^3/\pi)e^{-\mu r'(\sin\alpha' + \cos\alpha')} = (\mu^3/\pi)e^{-\sqrt{2}\mu r'\cos\beta}$$

where $\beta = \alpha' - \frac{1}{4}\pi$. The first term in the series for $w(r_{12})$, which also is independent of ϑ_{12}, is $(q^2/r)\begin{matrix}\sec\\ \csc\end{matrix}\alpha$, where the secant is used when $0 < \alpha < \frac{1}{4}\pi$ and the cosecant when $\frac{1}{4}\pi < \alpha < \frac{1}{2}\pi$.

Putting all these factors together and using the semicylindrical functions defined at the end of this chapter, we obtain for the part of ψ independent of ϑ_{12} (*i.e.*, using only the first term of w),

$$\begin{aligned}
&\frac{\mu^3q^2}{2\pi r^2}\sum_n \frac{\sin[2(n+1)\alpha]}{\sin(2\alpha)}\int_0^\infty \{\ \}(r')^2\,dr'\,\sqrt{2}\cdot \\
&\quad\cdot\left\{Z\int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi}\cos[2(n+1)\beta + \tfrac{1}{2}n\pi]\cos\beta e^{-\sqrt{2}\mu r\cos\beta}\,d\beta\right. \\
&\qquad\left. - \int_0^{\frac{1}{4}\pi}\cos[2(n+1)\beta + \tfrac{1}{2}n\pi](\cos\beta + \sin\beta)e^{-\sqrt{2}\mu r\cos\beta}\,d\beta\right\} \\
&= \frac{q^2}{2r^2}\sum_m (-1)^m\frac{\sin[2(2m+1)\alpha]}{\sin(2\alpha)}\left\{H_{4n+2}(iz)\int_0^z J_{4n+2}(iz')\cdot\right. \\
&\quad\cdot\left[(Z - \tfrac{1}{2})\frac{d}{dz'}J_{4n+2}^{(4)}(iz') - \tfrac{1}{2}\frac{d}{dz'}E_{4n+2}^{(4)}(iz')\right](z')^2\,dz' \\
&\qquad + J_{4n+2}(iz)\int_z^\infty H_{4n+2}(iz')\left[(Z - \tfrac{1}{2})\frac{d}{dz'}J_{4n+2}^{(4)}(iz')\right. \\
&\qquad\qquad\left.\left. - \tfrac{1}{2}\frac{d}{dz'}E_{4n+2}^{(4)}(iz')\right](z')^2\,dz'\right\}
\end{aligned}$$

where $\{\ \}$ in the first integral is the combination of H_{2n+2} and J_{2n+2} which occurs in G and where $z = \sqrt{2}\,\mu r$; $z' = \sqrt{2}\,\mu r'$. This part of the integral is similar to the series expression (12.3.71); it does not depend on the angle ϑ_{12}; consequently, it is not much better than the trial function we started from.

The next few terms in the expansion of w do make an improvement, however. The integrals are similar to those carried out for the first term, and the integrals over r may be carried out explicitly, if need be, by using the integral representations of the Bessel functions.

Variational Calculation. But, in most cases, it is better to use the variational technique to improve the wave function. If one wishes to have a lowest state wave function which takes the interaction between particles into account, one could try a function which is small for r_{12} small but is independent of r_{12} when r_{12} is large, such as

$$\phi(\mu,\lambda) = e^{-\mu(r_1+r_2)}[1 - \lambda e^{-\mu r_{12}}]$$

with two parameters, λ and μ. This function is not normalized, so that the integral for H would have to be divided by the integral of the square of ϕ.

Even before we carry out the detailed calculations, we can work out a few general properties of the variational integrals. For example, by noting that μ is an over-all scale factor, we can see that

$$\int \cdots \int \phi^2 \, dv_1 \, dv_2 = \mu^{-6} N(\lambda)$$

$$\int \cdots \int \phi[\nabla_1^2 + \nabla_2^2]\phi \, dv_1 \, dv_2 = -2\mu^{-4} T(\lambda)$$

$$\int \cdots \int \left[\frac{1}{r_1} + \frac{1}{r_2}\right] \phi^2 \, dv_1 \, dv_2 = 2\mu^{-5} V(\lambda)$$

$$\int \cdots \int \left(\frac{1}{r_{12}}\right) \phi^2 \, dv_1 \, dv_2 = \mu^{-5} W(\lambda)$$

where the quantities N, T, V, and W are independent of μ. We can then, formally, minimize the energy for μ;

$$H = 2\mu^2(T/N) - 2\mu Z q^2(V/N) + \mu q^2(W/N)$$

for

$$\partial H/\partial\mu = 0; \quad \mu = \tfrac{1}{2}q^2[Z(V/T) - \tfrac{1}{2}(W/T)];$$
$$\epsilon(\lambda) = -\tfrac{1}{2}q^4[Z(V/T) - \tfrac{1}{2}(W/T)]^2 \qquad (12.3.96)$$

an obvious generalization of Eq. (12.3.95). The best value of λ and the correspondingly best value of the energy are then obtained by minimizing $\epsilon(\lambda)$. (It should be noted, however, that this simple formulation of the variation is possible only when the potential energies are all coulomb potentials, inversely proportional to a distance.)

The most convenient coordinates in which to compute the integrals just defined are the Hylleraas coordinates, $u = r_{12}$, $s = r_1 + r_2$, $t = r_1 - r_2$. The usual volume element is $dv_1 \, dv_2 = d\varphi_1 \sin\vartheta_1 \, d\vartheta_1 \, r_1^2 \, dr_1 \, d\varphi_2 \sin\vartheta_2 \, d\vartheta_2 \, r_2^2 \, dr_2$ which can be modified by referring the direction of $\mathbf{r}_2$ to $\mathbf{r}_1$ (see Fig. 12.9).

$$dv_1 \, dv_2 = d\varphi_1 \sin\vartheta_1 \, d\vartheta_1 \, d\Psi \sin\vartheta_{12} \, d\vartheta_{12} \, r_1^2 \, dr_1 \, r_2^2 \, dr_2$$

If the functions involved depend only on r_1, r_2, and r_{12} the angles φ_1, ϑ_1, and Ψ may be integrated over, producing a factor $8\pi^2$, and the complete integral would be

$$\int \cdots \int F(r_1,r_2,r_{12}) \, dv_1 \, dv_2 = \int\int\int F \, dv$$

where $dv = 8\pi^2 \sin\vartheta_{12} \, d\vartheta_{12} \, r_1^2 \, dr_1 \, r_2^2 \, dr_2$.

Since $r_{12}^2 = r_1^2 + r_2^2 - 2r_1r_2\cos\vartheta_{12}$, we have that $r_1r_2\sin\vartheta_{12}\,d\vartheta_{12} = r_{12}\,dr_{12}$; consequently,

$$dv = 8\pi^2 r_{12} r_1 r_2 \, dr_{12} \, dr_1 \, dr_2 = \pi^2 u(s^2 - t^2) \, du \, ds \, dt$$

where $u = r_{12}$, $s = r_1 + r_2$, $t = r_1 - r_2$, and $-u \le t \le u$, $0 \le u \le s < \infty$. If the integrand F is symmetric in t, the total integral can be written as

$$\int \cdots \int F\, dv_1\, dv_2 = 2\pi^2 \int_0^\infty ds \int_0^s u\, du \int_0^u (s^2 - t^2) F\, dt$$

and, if we set $\sigma = \mu s$, $\tau = \mu t$, and $\eta = \mu u$, the potential integrals for the trial function $e^{-\sigma}(1 - \lambda e^{-\eta})$ are

$$W(\lambda) = 2\pi^2 \int_0^\infty d\sigma\, e^{-2\sigma} \int_0^\sigma d\eta (1 - 2\lambda e^{-\eta} + \lambda^2 e^{-2\eta}) \int_0^\eta (\sigma^2 - \tau^2)\, d\tau$$

$$V(\lambda) = 4\pi^2 \int_0^\infty \sigma\, d\sigma\, e^{-2\sigma} \int_0^\sigma \eta\, d\eta (1 - 2\lambda e^{-\eta} + \lambda^2 e^{-2\eta}) \int_0^\eta d\tau$$

The integral of the kinetic energy is more complicated. We use the integral of the square of the gradients of ϕ rather than the integral of the Laplacians (see below). By using the formulas

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial r_1}\left(\frac{x_1}{r_1}\right) + \frac{\partial f}{\partial r_{12}}\left(\frac{x_1 - x_2}{r_{12}}\right); \quad \frac{\partial f}{\partial r_1} = \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t}; \quad \text{etc.}$$

one can show that

$$[\nabla_1\phi]^2 + [\nabla_2\phi]^2 = 2\left(\frac{\partial\phi}{\partial s}\right)^2 + 2\left(\frac{\partial\phi}{\partial t}\right)^2 + 2\left(\frac{\partial\phi}{\partial u}\right)^2 + \frac{4}{u(s^2 - t^2)}\left(\frac{\partial\phi}{\partial u}\right)\left[s(u^2 - t^2)\left(\frac{\partial\phi}{\partial s}\right) + t(s^2 - u^2)\left(\frac{\partial\phi}{\partial t}\right)\right]$$

if ϕ depends only on r_1, r_2, and r_{12}. Therefore, for the trial function chosen,

$$\begin{aligned} T &= -\tfrac{1}{2}\mu^4 \int \cdots \int \phi[\nabla_1^2 + \nabla_2^2]\phi\, dv_1\, dv_2 \\ &= \tfrac{1}{2}\mu^4 \int \cdots \int [(\nabla_1\phi)^2 + (\nabla_2\phi)^2]\, dv_1\, dv_2 \\ &= 2\pi^2 \int_0^\infty d\sigma\, e^{-2\sigma} \int_0^\sigma \eta\, d\eta \int_0^\eta (\sigma^2 - \tau^2)\, d\tau \bigg[(1 - \lambda e^{-\eta})^2 \\ &\qquad + \lambda^2 e^{-2\eta} + \frac{2\lambda\sigma}{\eta} \frac{\eta^2 - \tau^2}{\sigma^2 - \tau^2} (1 - \lambda e^{-\eta}) e^{-\eta} \bigg] \end{aligned}$$

From these integrals, inserted in Eq. (12.3.96), one may compute a more accurate value of ϵ. By using still more flexible trial functions with more parameters, one may come as close as one pleases to the correct result.

Scattering of Electron from Hydrogen Atom. As an example of another use of the multidimensional Green's function, we shall indicate the procedure for computing the results of a collision between an electron and a hydrogen atom ($Z = 1$, one bound electron, one free; we neglect the electronic spin and symmetry effects as we have been doing so far). The integral equation for the system is

$$\psi(\mathbf{r}) = q^2 \int \cdots \int \left[\frac{1}{r_1'} + \frac{1}{r_2'} - \frac{1}{r_{12}'}\right] \psi(\mathbf{r}')G_k(\mathbf{r}|\mathbf{r}')\, dv_1'\, dv_2'$$

where the form of the Green's function we use here is the Fourier integral representation of Eq. (12.3.94) and where $q^2 = 2Me^2/\hbar^2$ as before.

The state to be studied is where particle 1 is initially free, coming toward the origin with kinetic energy k_1^2, and where particle 2 is bound in the lowest state, with energy $-\frac{1}{4}q^4$. The corresponding wave function is, in part, a "surface wave" (see page 1713) clinging closely to the hypersurface $r_2 = 0$. There will be, however, "reflected" waves, induced by the interaction potential; some of these will also be surface waves, one or the other particle being left behind in a bound state; but if $(k_1^2 - \frac{1}{4}q^4)$ is positive, there will also be a certain amount of "free" wave, corresponding to both particles leaving the attractive center.

The first approximation to the solution of the integral equation may be obtained by inserting, for ψ, in the integral the "surface" wave corresponding to the initial state. [This is equivalent to the Born approximation for a single particle, see Eqs. (12.3.58) *et seq.*] We try

$$\psi^i \simeq e^{ik_1z_1}(q^3/\sqrt{8\pi})e^{-\frac{1}{2}q^2r_2} = e^{ik_1z_1}\phi_{001}(r_2)$$

[where ϕ_{001} is the lowest one of the eigenfunction solutions ψ_{mln} given in Eq. (12.3.40)] and use the integral equation to compute the next approximation. One useful item emerges immediately: if $1/r_2$ were the only term in the brackets in the integral equation, then ψ^i would be the *exact* solution of the integral equation. Consequently, the integral of ψ^iG times q^2/r_2 must just equal ψ^i, and

$$\psi \simeq \psi^i + q^2 \int \cdots \int \left[\frac{1}{r_1'} - \frac{1}{r_{12}'}\right] e^{ik_1z_1'}\phi_{001}(\mathbf{r}_2')G_k(\mathbf{r}|\mathbf{r}')\, dv_1'\, dv_2' \quad (12.3.97)$$

where $k^2 = k_1^2 - \frac{1}{4}q^4$. The Green's function G has nonzero amplitude for both r_1 large and for r_2 large, corresponding to the fact that either particle 1 or particle 2 can come away from the origin after the collision. In order to see in more detail what is happening, we shall have to organize this equation into more recognizable forms. For example, we may be interested in the behavior of particle 1 when it rebounds from the center of force, leaving particle 2 in a bound state (not necessarily the lowest). In this case it is appropriate to expand the integral into a series of the eigenfunctions for particle 2, times functions of r_1

$$\psi \simeq \psi^i + \sum_{nlm} \phi_{mln}(\mathbf{r}_2) f_{mln}(\mathbf{r}_1) \tag{12.3.98}$$

$$f_{mln}(\mathbf{r}_1) = q^2 \iiint dv_2\, \bar{\phi}_{mln}(\mathbf{r}_2) \int \cdots \int \left[\frac{1}{r_1'} - \frac{1}{r_{12}'}\right] \cdot e^{ik_1z_1'} \phi_{001}(\mathbf{r}_2') G_k(\mathbf{r}|\mathbf{r}')\, dv_1'\, dv_2'$$

The function f_{mln} therefore gives the behavior of particle 1 when we are sure that particle 2 is left in the state mln. This sum, of course, must include the free-state wave functions for particle 2 in order to have the complete set of ϕ's and, for some of these, particle 1 would not be free since energy is conserved. This need not disturb us, for we have set up the expansion in this form specifically to compute the behavior of 1 when 2 is left bound, so we shall concentrate on the coefficients of the lower terms of the series and rely on another type of expansion to clarify the situation.

Inserting the Fourier integral representation of G in the equation for f, we have

$$f_{mln}(\mathbf{r}_1) = \frac{q^2}{64\pi^6} \int \cdots \int \frac{e^{i\xi_1(x_1 - x_1') + \cdots + i\zeta_2(z_2 - z_2')}}{\xi_1^2 + \cdots + \zeta_2^2 - k_1^2 + \frac{1}{4}q^4} \bar{\phi}_{mln}(\mathbf{r}_2) \phi_{001}(\mathbf{r}_2') \cdot \cdot e^{ik_1z_1'} \left(\frac{1}{r_1'} - \frac{1}{r_{12}'}\right) dv_2\, dv_1'\, dv_2'\, d\xi_1 \cdots d\zeta_2$$

The integral over dv_2 can be carried out without much trouble. The result is the Fourier transform of $\bar{\phi}_{mln}$, the function $\bar{\chi}_{mln}$ defined in Eq. (12.3.54). It is a function of the components ξ_2, η_2, ζ_2 of the wave number vector κ_2, with poles at $\kappa_2 = \pm i(q^2/2n)$, corresponding to the energy $\epsilon = -(q^4/4n^2)$ of the nth bound states for particle 2.

The integration over ξ_2, η_2, ζ_2 is not quite so easy. If the denominator $(\xi_1^2 + \cdots + \zeta_2^2 - k_1^2 + \frac{1}{4}q^4)$ were not present, the result would be just to give us back $\bar{\phi}_{mln}(\mathbf{r}_2')$, multiplied by $8\pi^3$, since the integrations over dv_2 and $d\xi_2\, d\eta_2\, d\zeta_2$ would simply be the Fourier integral theorem. Put another way, the integral over κ_2 space can be reduced to contour integrals about the poles of $\bar{\chi}$ and of the factor $[\kappa_1^2 + \kappa_2^2 - k_1^2 + \frac{1}{4}q^4]^{-1}$. The residues at the poles of $\bar{\chi}$ are proportional to the Fourier transform of χ, $\bar{\phi}_{mln}$. The residues about the poles of the other factor correspond to the distortion of the particle 2 wave function because of the presence of particle 1, and can be made small by making r_1 large. For the time being, we disregard these terms.

Elastic and Inelastic Scattering. Considering just the residues about the poles of $\bar{\chi}$, we have, using Eq. (11.2.5),

$$f_{mln}(\mathbf{r}_1) \simeq \frac{q^2}{8\pi^3} \int \cdots \int \frac{e^{i\xi_1(x_1-x_1')+i\eta_1(y_1-y_1')+i\zeta_1(z_1-z_1')}}{\xi_1^2+\eta_1^2+\zeta_1^2-k_1^2+\frac{1}{4}q^4[1-(1/n^2)]} e^{ikz_1'} \cdot \bar{\phi}_{mln}(\mathbf{r}_2')\phi_{001}(\mathbf{r}_2')\left(\frac{1}{r_1'}-\frac{1}{r_{12}'}\right) dv_1'\, dv_2'\, d\xi_1\, d\eta_1\, d\zeta_1$$

$$\simeq q^2 \int \cdots \int \frac{e^{ik_nR_1}}{4\pi R_1} \bar{\phi}_{mln}(\mathbf{r}_2')\phi_{001}(\mathbf{r}_2')\left(\frac{1}{r_1'}-\frac{1}{r_{12}'}\right) e^{ik_1z_1'}\, dv_1'\, dv_2'$$

$$\xrightarrow[r_1\to\infty]{} \left(\frac{q^2}{4\pi r_1}\right) e^{ik_nr_1} \int \cdots \int e^{i(\mathbf{k}_i-\mathbf{k}_n)\cdot\mathbf{r}_1'} \bar{\phi}_{mln}(\mathbf{r}_2')\phi_{001}(\mathbf{r}_2')\left(\frac{1}{r_1'}-\frac{1}{r_{12}'}\right) dv_1'\, dv_2'$$

where $\mathbf{k}_i = k_1\mathbf{a}_z$ is a vector equal to $M/\hbar$ times the incident velocity of particle 1 and $\mathbf{k}_n$ is proportional to the velocity of particle 1 after leaving the atom in state n. The direction of $\mathbf{k}_n$ is at the scattering angle ϑ_1 to the z_1 axis, and the magnitude of $\mathbf{k}_n$ is related to k_1 and the energies of initial and final bound states $-\frac{1}{4}q^4$ and $-\frac{1}{4}(q^4/n^2)$ by the equation

$$k_n^2 = k_1^2 - \tfrac{1}{4}q^4[1-(1/n^2)]$$

We note that vector $\mathbf{k}_1$, corresponding to the term for particle 1 leaving the atom unchanged (elastic scattering), has the same magnitude k_1 as $\mathbf{k}_i$ but is in a different direction. We note also that, for all states except the initial state, the $1/r_1'$ integral is zero because of the orthogonality of the eigenfunctions ϕ.

The integral over dv_2' results in a function of $\mathbf{r}_1'$ which is just the field of the nucleus minus that of the charge density $|\phi_{001}|^2$ (for the lowest f) or the field of the "exchange charge density" $\bar{\phi}_{mln}\phi_{001}$ for f_{mln}. Consequently, we can say that the probability that particle 1 is scattered elastically from the system particle 2 plus nucleus is equal to the Fourier transform of the potential field of the system for particle 1, for the wave number $(\mathbf{k}_1 - \mathbf{k}_i)$ corresponding to the change in momentum of particle 1 caused by the collision. Similarly the probability of inelastic scattering, leaving particle 2 in state (mln), is the Fourier transform of the potential of particle 1 caused by the exchange charge density $\bar{\phi}_{mln}\phi_{001}$, for the wave number $(\mathbf{k}_n - \mathbf{k}_i)$ proportional to the change in momentum produced by the inelastic collision. We can call the magnitude of this wave number μ_n, where

$$\mu_n^2 = |\mathbf{k}_n - \mathbf{k}_i|^2 = k_1^2 + k_n^2 - 2k_1k_n\cos\vartheta_1 \to [2k_1\sin(\tfrac{1}{2}\vartheta_1)]^2;\quad k_1^2 \gg \tfrac{1}{4}q^4;\quad \vartheta_1 \gg 1-(k_n/k_1)$$

One can now use the machinery of the Born approximation [see Eqs. (12.3.59) and (12.3.60)] to compute the elastic and inelastic scattering of particle 1. For the elastic scattering, the potential causing the scattering is that of the nucleus minus that of the charge density $\rho(r) = |\phi_{001}(r)|^2$; for the inelastic scattering (for the state mln) the potential is that of the exchange charge density $\bar{\phi}_{mln}\phi_{001}$ [here Eqs. (12.3.59) and (12.3.60) must be modified because the ρ, and therefore V, is not spherically symmetric].

When k_1^2 is very large compared to the binding energy $\frac{1}{4}q^4$, we can obtain a somewhat simplified formula by integrating over dv_1' first and using the formula

$$\frac{1}{4\pi}\iiint e^{i\mathbf{k}\cdot\mathbf{r}_1'}\left(\frac{1}{r_{12}'}\right)dv_1' = \left(\frac{1}{k^2}\right)e^{i\mathbf{k}\cdot\mathbf{r}_2'}$$

obtained by Fourier transform. We have

$$f_{mln} \xrightarrow[k_1\to\infty]{} \left(\frac{q^2}{\mu^2 r_1}\right)e^{i\mu r_1}\iiint [1 - e^{-i\mathbf{k}\cdot\mathbf{r}_2'}]\bar{\phi}_{mln}(\mathbf{r}_2')\phi_{001}(\mathbf{r}_2')\,dv_2'$$

where $\mu = |\mathbf{k}| \simeq 2k_1\sin(\frac{1}{2}\vartheta_1)$, which is valid except for the scattering angle ϑ_1 very small. Since, for $k_1 \gg \frac{1}{4}q^2$, the momentum of the inelastically scattered particles is not much smaller than the momentum of the elastically scattered particles, we are interested here in the *total* scattered current for particle 1, both elastic and inelastic. This is obtained by summing the squares of the absolute magnitudes of all the terms in Eq. (12.3.98):

$$S(\vartheta_1) \simeq r_1^2\sum_{nlm}|f_{mln}|^2$$

where S is the total scattered current of particle 1, per unit solid angle, at scattering angle ϑ_1, per unit incident current density.

From the completeness of eigenfunctions, we can show that, for a sum over all possible states,

$$\sum_{mln}\left|\iiint \bar{\phi}_{mln}G\phi_{001}\,dv\right|^2 = \iiint \bar{\phi}_{001}|G|^2\phi_{001}\,dv$$

which is called the *matrix sum rule*. Consequently,

$$\begin{aligned} S(\vartheta_1) &\simeq 2\left(\frac{q^4}{\mu^4}\right)\iiint \bar{\phi}_{001}[1 - \cos(\mathbf{k}\cdot\mathbf{r}_2')]\phi_{001}\,dv \\ &\simeq 2(q^4/\mu^4)[F(0) - F(\mu)] \end{aligned}$$

where F is the *structure factor* for the state (001), defined in Eq. (12.3.60). Comparing that equation with the present one, we see that the current of *elastically* scattered particles is equal to $(q^4/\mu^4)[F(0) - F(\mu)]^2$, whereas the *total* scattered current (elastic and inelastic) is $2(q^4/\mu^4)[F(0) - F(\mu)]$ when the incident momentum is large enough. (For a target atom with more than one particle, this last formula is modified somewhat.)

Exchange of Particles. Having discussed the possibilities for particle 1 to rebound from the atom, we must now consider the probability that particle exchange takes place, particle 2 coming out and particle 1 remaining behind. (This corresponds to reflection from the ξ axis in the simple example of page 1710.) Here it is more convenient to expand in terms of eigenfunctions for particle 1. Instead of Eq. (12.3.98), we write

$$\psi \simeq \psi^i + \sum_{mln} \phi_{mln}(\mathbf{r}_1) g_{mln}(\mathbf{r}_2) \tag{12.3.99}$$

$$g_{mln} = q^2 \iiint dv_1 \bar{\phi}_{mln}(\mathbf{r}_1) \int \cdot \cdot \cdot \int \left(\frac{1}{r_1'} - \frac{1}{r_{12}'}\right) e^{ik_1z_1'} \cdot \\ \cdot \phi_{001}(\mathbf{r}_2') G_k(\mathbf{r}|\mathbf{r}')\, dv_1'\, dv_2'$$

This series is a complete duplication of series (12.3.98), and we run a danger of duplicating some behavior if we count all the g's as well as all the f's. However, if we have considered only those f's for which $\phi_{mln}(\mathbf{r}_1)$ is a bound state, then the first set corresponds to particle 2 bound and the second to particle 1 bound and there is no duplication. (There may *in addition* be some states with neither particle bound; we consider these later.)

Applying the same approximations to this integral as to f_{mln}, we find that

$$g_{mln}(r_1) \xrightarrow[r_2 \to \infty]{} \left(\frac{q^2}{4\pi r_2}\right) e^{ik_nr_2} \int \cdot \cdot \cdot \int e^{i\mathbf{k}_1\cdot\mathbf{r}_1 - i\mathbf{k}_n\cdot\mathbf{r}_2} \left(\frac{1}{r_1'} - \frac{1}{r_{12}'}\right) \cdot \\ \cdot \bar{\phi}_{mln}(\mathbf{r}_1')\, \phi_{001}(\mathbf{r}_2')\, dv_1'\, dv_2'$$

which may be manipulated, in various special cases, in a manner similar to the f's.

If the kinetic energy k_1^2 of the incident particle is less than the binding energy $\frac{1}{4}q^4$ of the initial state, then it will be impossible for both particles to be free at the same time and, if we are measuring only those particles which leave the atom, we can write

$$\psi \to \psi^i + \Sigma \phi_{mln}(\mathbf{r}_2) f_{mln}(\mathbf{r}_1) + \Sigma \phi_{mln}(\mathbf{r}_1) g_{mln}(\mathbf{r}_2)$$

for r_1 or r_2 large (r large). Both sums are over only those states for which $k_n^2 = k_1^2 - \frac{1}{4}q^4[1 - (1/n^2)]$ is positive. We are here neglecting possible states where *both* particles are bound (negative ion). These states are included (approximately) in the integral for ψ but vanish when r is made large.

Finally, if k_1^2 is larger than $\frac{1}{4}q^4$, both particles can be free. To compute the probability of this, we go back to Eq. (12.3.97) and investigate the behavior of the integral for r large and α not 0 or $\frac{1}{2}\pi$ (both r_1 and r_2 large). To make this investigation it will be easiest to use the form of Green's function given in Eq. (12.3.91). For the case of r large and r' small, we can write

$$R \simeq r^2 - r_1' \cos\alpha \cos\Theta_1 - r_2' \sin\alpha \cos\Theta_2$$
$$\cos\Theta_j = [\cos\vartheta_j \cos\vartheta_j' + \sin\vartheta_j \sin\vartheta_j' \cos(\varphi_j - \varphi_j')]$$

If we set the wave-number vector of particle 1 as $\mathbf{K}_1$, having magnitude $k\cos\alpha$ ($k^2 = k_1^2 - \frac{1}{4}q^4$) and direction at angle ϑ_1 with respect to the z axis (angle Θ_1 with respect to r_1'), and set the wave-number vector of particle 2 as $\mathbf{K}_2$, with magnitude $k \sin \alpha$ and angles ϑ_2 and Θ_2 with respect to z and to r'_2, then

$$G_k(\mathbf{r}|\mathbf{r}') \to [\tfrac{1}{2}ik^4/(2\pi ikr)^{\frac{5}{2}}]e^{ikr - i\mathbf{K}_1\cdot\mathbf{r}_1' - i\mathbf{K}_2\cdot\mathbf{r}_2'}$$

and the part of the wave function, for both r_1 and r_2 large, is

$$\psi \to \psi^i + r^{-\frac{5}{2}}e^{ikr}\Omega(k,\alpha,\vartheta_1,\vartheta_2) \tag{12.3.100}$$

$$\Omega = \sqrt{\frac{k^3q^4}{2^7\pi^5 i^3}} \int \cdots \int \left(\frac{1}{r'_1} - \frac{1}{r'_{12}}\right) e^{i(\mathbf{k}_i - \mathbf{K}_1)\cdot\mathbf{r}_1' - i\mathbf{K}_2\cdot\mathbf{r}_2'}\phi_{001}(\mathbf{r}'_2)\, dv'_1\, dv'_2$$
$$= \left(\frac{k}{2\pi i}\right)^{\frac{3}{2}}\left(\frac{q^2}{\nu^2}\right) \iiint [1 - e^{i(\mathbf{k}_1 - \mathbf{K}_1)\cdot\mathbf{r}_2'}]e^{-i\mathbf{K}_2\cdot\mathbf{r}_2'}\phi_{001}(\mathbf{r}'_2)\, dv'_2$$

where $\quad \nu = |\mathbf{k}_1 - \mathbf{K}_1| = \sqrt{k_1^2 + k^2\cos^2\alpha - 2k_1k\cos\alpha\cos\vartheta_1}$
and $\quad |\mathbf{K}_1| = k\cos\alpha; \quad |\mathbf{K}_2| = k\sin\alpha; \quad k^2 = k_1^2 - \frac{1}{4}q^4$

This part of the wave function is expressed in hyperspherical coordinates and, to obtain the total intensity $|\Omega|^2$, must be integrated over the surface element $r^5 \sin^2\alpha \cos^2\alpha\, d\alpha \sin\vartheta_1\, d\vartheta_1\, d\varphi_1 \sin\vartheta_2\, d\vartheta_2\, d\varphi_2$. The function gives the amplitude of the wave in the directions ϑ_1 and ϑ_2 for the division of energy given by the angle α. If k_1^2 is less than $\frac{1}{4}q^4$, k is imaginary and there is no wave motion of this type.

The final result is quite analogous to the two-dimensional case discussed on page 1717. We send a surface wave ψ^i in and a number of surface waves come out, clinging to the $r_2 = 0$ surface (particle 2 bound) with amplitudes $|f_{mln}|$; in addition there are other surface waves clinging to the $r_1 = 0$ surface (particle 1 bound) with amplitudes $|g_{mln}|$; and finally, if k_1^2 is large enough, there is a free wave (both particles free) with a continuous distribution of the excess energy between the two particles (angle α), with corresponding amplitude $|\Omega|$. (We note that this amplitude is zero for $\alpha = 0$ or $\frac{1}{2}\pi$; waves in these directions are the surface waves with wave number k_n rather than k.) Collecting terms, we see that the part of ψ which stays finite for r very large (r_1 or r_2 or both large) is

$$\psi \to e^{i\mathbf{k}_i\cdot\mathbf{r}_1}\phi_{001}(\mathbf{r}_2) + \sum_{mln} \phi_{mln}(\mathbf{r}_2)f_{mln}(\mathbf{r}_1)$$
$$+ \sum_{mln} \phi_{mln}(\mathbf{r}_1)g_{mln}(\mathbf{r}_2) + r^{-\frac{5}{2}}e^{ikr}\Omega(k,\alpha,\vartheta_1,\vartheta_2) \tag{12.3.101}$$

where the sum over mln is for all bound states for which $k_1^2 - \frac{1}{4}q^4[1 - (1/n^2)]$ is positive and where the term in Ω is present only when $k_1^2 > \frac{1}{4}q^4$ (in which case all *bound* states are included in the sums).

This formula is, of course, only approximate. The correct solution should have the same general asymptotic form as the one given, but the values of the f's, g's and Ω would be somewhat different. The range of validity of the approximation may be determined in the same manner as was discussed on page 1689. In general the formulas are valid for high incident kinetic energies and become less accurate for slow incident particles. There is no satisfactory solution for inelastic collisions (and exchange) for very low incident speeds (just above the excitation energy, for example).

Recapitulation. We must now terminate this section on wave mechanics. No mention can be made here of the important subjects of particle spin, or the Pauli exclusion principle, or many other important matters. Many other volumes, specifically concerned with quantum theory, discuss such matters; this book is concerned primarily with analytic techniques of solution of field problems and not primarily with the physics underlying the equations. And we have already touched on most of the techniques utilized in quantum mechanics: the determination of bound and free states, the use of the perturbation, and the variational methods and the employment of the integral equation in connection with the Born approximation. A few of the complications arising in the many-particle problem have been discussed, enough to show their difficulty and some of the means used to circumvent the difficulties. Any further discussion will lead us too far from our main task. We must go on to our last major subject, the calculation of purely vector fields.

Problems for Chapter 12

12.1 A wire of radius R is placed in a medium where the temperature is oscillating, $T = T_0 + \Delta T e^{-i\omega t}$. The resistivity of the wire material, for temperatures near T_0 is $\rho = \rho_0 + \rho_1(T - T_0)$. Calculate the resistance of the wire per unit length as a function of time. Calculate the resistance as a function of time when the temperature at $r = R$ is $T_0 + \Delta T u(t)$.

12.2 Calculate the series expressing the flow of heat into the bar of Prob. 10.1, as a function of time when the heater is turned on at $t = 0$, its temperature being $T_0 u(t)$.

12.3 When an accelerated ion impinges on a metallic surface, it gives up its kinetic energy eV_0 to a small region at the surface of the metal. The temperature of the metal at the point (x,y,z), at a time t after the collision, is approximately given by the Green's function

$$T = (A/t^{\frac{3}{2}}) \exp[-a^2(x^2 + y^2 + z^2)/4t]u(t)u(x)$$

where a is the diffusion constant of the metal, $x = y = z = 0$ is the point of impact of the ion, the region $x \geq 0$ is occupied by metal, and the region $x \leq 0$ is free space, and where A is so adjusted that the integral of the additional heat energy, ρCT, over the metal, is equal to eV_0. The rate of evaporation of metal from its surface $x = 0$, in terms of the surface temperature T_s is

$$J_e = B \exp(-E_0/kT_s)$$

where E_0 is the average energy required to remove a metal atom from the surface. Compute the total amount of metal evaporated from the surface (sputtered off) by the impact of one ion, as a function of B, E_0, V_0 and the thermal properties of the metal.

12.4 In liquid helium II the relation between heat flow $\mathbf{J}$ cal per sec per cm^2 and temperature differential τ is *not* that $\mathbf{J}$ is proportional to grad τ but that $\partial\mathbf{J}/\partial t$ is proportional to grad τ;

$$\partial J/\partial t = -\rho_2 C c_2^2 \operatorname{grad} \tau$$

where the actual temperature is $T_0 + \tau$, where $\rho_2 C_2$ is the specific heat per cubic centimeter of the helium and where c_2 is a quantity of the dimensions of velocity. Show that the temperature obeys a true wave equation, with c_2 the wave velocity (called the velocity of *second sound* or of thermal waves). A slab of conducting material of ordinary thermal and electrical properties, of thickness l, is immersed in a bath of liquid helium II, large enough to be considered as extending to infinity. Show that the boundary conditions at both surfaces of the slab (take them at $x = \pm\frac{1}{2}l$) are that the ratio between the heat flow J into the helium and the temperature differential τ is $J = \rho_2 C_2 c_2 \tau$ for the time-varying parts of J and τ. Suppose that the electrical conductivity of the slab is σ, its density ρ, specific heat C, and thermal conductivity κ and that an electrical current density $I \cos(\omega t/2)$ traverses each part of the slab. Show that the temperature distribution in the slab ($x < \frac{1}{2}l$) and in the helium ($x > \frac{1}{2}l$) is given by the real parts of the following expressions:

$$\begin{aligned} T &= T_0 + (I^2R/4\sigma\kappa)(\tfrac{1}{4}l^2 - x^2) \\ &+ \left(\frac{I^2R}{2\sigma\gamma^2\kappa^2}\right)\left[\frac{\cos(\gamma x) - \cos(\gamma l/2) + (\gamma\kappa/\rho_2 C_2 c_2)\sin(\gamma l/2)}{\cos(\gamma l/2) - \gamma(\kappa/\rho_2 C_2 c_2)\sin(\gamma l/2)}\right] e^{-i\omega t}; \qquad x < \tfrac{1}{2}l \\ &= -\left(\frac{I^2R}{2\sigma\gamma^2\kappa^2}\right)\left[1 - \left(\frac{\rho_2 C_2 c_2}{\gamma\kappa}\right)\cot\left(\frac{\gamma l}{2}\right)\right]^{-1} e^{i(\omega/c_2)(x - ct)}; \qquad x > \tfrac{1}{2}l \end{aligned}$$

where $\gamma^2 = i\omega\rho C/\kappa$ and where R is the factor to convert joules into calories.

12.5 A neutron shield consists of material, between the planes $x = 0$ and $x = l$, having mean free path for neutrons equal to λ_a ($\ll l$) and such that a fraction κ of the collisions results in absorption of the neutron. At $t = 0$, a "pulse" of neutrons of density $\delta(t)$ impinges on the surface $x = l$. Show that approximate expressions for the neutron density in the slab (use the approximate boundary condition that $\rho = 0$ at $x = 0, l$) are ($0 \le x \le l$, $0 < t$)

$$\rho_\delta(x,t) = \frac{e^{-\chi t}}{2a\sqrt{\pi}\,t^{\frac{3}{2}}} \sum_{n=-\infty}^{\infty} [(2n+1)l - x] \exp\left\{\left(\frac{-1}{4a^2t}\right)[x - (2n+1)l]^2\right\}$$
$$= -\left(\frac{2\pi a^2}{l^2}\right) \sum_{m=l}^{\infty} (-1)^m m \sin\left(\frac{\pi m x}{l}\right) \exp\left\{-\left[\chi + \left(\frac{\pi m a}{l}\right)^2\right] t\right\}$$

where $a^2 = \lambda_a v_a/3$ and $\chi = \kappa v_a/\lambda_a$. Derive expressions for the neutron flux into the slab at $x = l$ and out of the slab at $x = 0$; discuss the dependence of these quantities on (l/λ_a), κ, and t. What are the corresponding expressions for the case when $\rho(l) = u(t)$?

12.6 A beam of I electrons per second of initial velocity v_i is introduced into a hollow cylindrical container, the boundaries of which are $z = 0$, $z = l$, $r = a$. The beam is introduced through a small hole at the origin and is directed along the axis of the cylinder. The mean free path of electrons in the gas inside the container is λ ($\ll l$), and the fraction of kinetic energy lost per collision with a gas atom is η. Assume that there are no inelastic collisions and that all electrons which strike the container walls are removed from the distribution. Show that, in this case, the steady-state charge density of electrons inside the container is

$$\rho = \left(\frac{6eI\lambda^2}{l^2a^2v_i}\right) \sum_{m,n} \left[\frac{m}{J_1^2(\pi\beta_{0n})}\right] \frac{J_0(\pi\beta_{0n}r/a)\sin(\pi m z/l)}{\left[1 + \left(\frac{\pi m\lambda}{l}\right)^2\right]\left[\left(\frac{\pi m\lambda}{l}\right)^2 + \left(\frac{\pi\lambda\beta_{0n}}{a}\right)^2 - \left(\frac{3}{\eta}\right)\right]}$$

as long as $3/\eta$ is smaller than $(\pi\lambda/l)^2 + (\pi\lambda\beta_{01}/a)^2$. [The numbers β_{0n} are the roots of the equation $J_0(\pi\beta) = 0$.]

12.7 Compute the initial behavior of a homogeneous, spherical reactor, using Eqs. (12.1.31) and assuming reasonable values for the constants involved. Take the effective radius of the sphere, a, to be 10 times the mean free path λ; assume $\delta = \frac{1}{10}$ and $\kappa_i = \kappa_\rho = \frac{1}{5}$; let $\eta = \frac{1}{3}$ and $v_i/v_\rho = 10$ and adjust $(\nu\zeta)$ so that w_1^-, for the lowest mode, is exactly zero. Plot the neutron density at $r = 0$ as a function of $t\lambda/v_\rho$ when a single neutron is inserted at the origin at $t = 0$. Plot also the flux of neutrons out from the surface as a function of $t\lambda/v_\rho$.

12.8 A medium of infinite extent is composed of atoms which scatter intruding particles uniformly in all directions [*i.e.*, $\alpha = 1/4\pi$ in Eq. (12.2.2)] with negligible loss of energy per collision. Compute the steady-state distribution function $f(\mathbf{r},\mathbf{a}_v)$ [where $\mathbf{a}_v$ is the unit vector in the direction of the particle velocity $\mathbf{v}$, and $\mathbf{r}$ is the position vector in the distance units of Eq. (12.2.2)] arising from a source distribution $s(\mathbf{r},\mathbf{a}_v) = (1/4\pi)s(\mathbf{r})$ of particles (each elementary source sends out new particles equally in all directions) by means of the Fourier transform. Set

$$F(\mathbf{k},\mathbf{a}_v) = (2\pi)^{-\frac{3}{2}}\int e^{i\mathbf{k}\cdot\mathbf{r}} f(\mathbf{r},\mathbf{u}_v)\,dv \quad\text{and}\quad S(\mathbf{k}) = (2\pi)^{-\frac{3}{2}}\int e^{i\mathbf{k}\cdot\mathbf{r}} s(\mathbf{r})\,dv$$

and show that Eq. (12.2.2) becomes (for steady state)

$$(i\mathbf{k}\cdot\mathbf{a}_v + 1)F(\mathbf{k},\mathbf{a}_v) = \frac{1}{4\pi}\left[S(\mathbf{k}) + \kappa\int F(\mathbf{k},\mathbf{a}'_v)\,d\Omega'\right]$$

Utilize Eq. (6.3.44) to obtain an expansion of F in spherical harmonics of $\mathbf{a}_k\cdot\mathbf{a}_v$ ($\mathbf{a}_k$ is the unit vector in the direction of $\mathbf{k}$, so that $\mathbf{k} = \mathbf{k}\mathbf{a}_k$)

$$F(\mathbf{k},\mathbf{a}_v) = \frac{(i/4\pi)S(\mathbf{k})}{k - i\kappa Q_0(i/k)}\sum_{n=0}^{\infty}(2n+1)P_n(\mathbf{a}_k\cdot\mathbf{a}_v)Q_n\left(\frac{i}{k}\right)$$

12.9 Solve for the distribution function of Prob. 12.8 for a point source of particles at the origin, $S = (2\pi)^{-\frac{3}{2}}$. Evaluate the inverse Fourier integral for f by expanding the exponential $e^{-i\mathbf{k}\cdot\mathbf{r}}$ in the integral into spherical harmonics of $(\mathbf{a}_k\cdot\mathbf{a}_r)$ and, by using the formula $iQ_0(i/k) = \tan^{-1}k$, obtain

$$f(\mathbf{r},\mathbf{a}_v) = \frac{i}{16\pi^3}\sum_{n=0}^{\infty}(-i)^nP_n(\mathbf{a}_r\cdot\mathbf{a}_v)\int_{-\infty}^{\infty}\frac{Q_n(i/k)h_n(kr)}{k - \kappa\tan^{-1}k}\,k^2\,dk$$

Show that this may formally be evaluated as a contour integral around the pole at $k = i\sigma$, where σ is the positive root of $\sigma = \tanh(\sigma/\kappa)$, giving

$$f(\mathbf{r},\mathbf{a}_v) = \frac{\sigma^2}{8\pi^3}\left(\frac{1-\sigma^2}{1-\kappa-\sigma^2}\right)\sum_{n=0}^{\infty}(-i)^nP_n(\mathbf{a}_r\cdot\mathbf{a}_v)Q_n\left(\frac{1}{\sigma}\right)h_n(i\sigma r)$$

Discuss the convergence of this series for κ small and for $(1-\kappa)$ small.

12.10 Solve for the steady-state distribution function as a function of ξ and μ [see Eq. (12.2.2)] when the source is a steady current density I_0, pointed in the positive ξ direction, applied at $\xi = 0$, when the scattering function $\alpha(\theta')$ has a pronounced forward peak; $\alpha(\theta' \ll \frac{1}{2}\pi) \gg \alpha(\theta' > \frac{1}{2}\pi)$. Show that in this case a reasonable approximation to the distribution function is the solution of the modified equation, for $1-\mu$ small,

$$\frac{\partial}{\partial\xi}f(\xi,\mu) + f(\xi,\mu) - \tfrac{1}{2}\kappa\sum_{n=0}^{\infty}(2n+1)A_nP_n(\mu)\int_{-1}^{1}P_n(\lambda)f(\xi,\lambda)\,d\lambda$$

where the A's are defined in Eq. (12.2.4). Show that a solution of this equation, satisfying the appropriate boundary condition that $f(0,\mu) = (I_0/2\pi)\delta(1-\mu)$, is

$$f(\xi,\mu) = \frac{I_0}{4\pi}\sum_{n=0}^{\infty}(2n+1)P_n(\mu)\exp[(\kappa A_n - 1)\xi]$$

Discuss the details of the solution when $\alpha(\theta) = [(N+1)/4\pi]\cos^{2N}(\frac{1}{2}\theta)$ where $N \gg 1$. What is the result for $N \to \infty$? Why?

12.11 Suppose that, at each collision, the incident particle (neutron, for example) is absorbed by the target nucleus and then re-emitted equally in all directions with a reduced kinetic energy, such that the fraction of particles re-emitted in the energy range between E and $E + dE$ is $\alpha(E/E')^\alpha(dE/E)$ where E' is the kinetic energy of the incident particle ($E' \geq E$). Set up the equation analogous to Eq. (12.2.2) for the distribution function $f(\xi,\mu,\epsilon)$ for the number of particles in a unit volume a distance ξ mean free paths in from the surface of the scattering medium, having direction of motion an angle $\vartheta = \cos^{-1}\mu$ to the ξ axis and with kinetic energy given by the parameter $\epsilon = \ln(E_0/E)$. Show that, for distributions independent of ξ, the equation for f is

$$f(\mu,\epsilon) = \frac{\alpha}{4\pi}\int_0^\epsilon e^{\alpha(\epsilon'-\epsilon)}R(\epsilon')\,d\epsilon' + s(\mu)\delta(\epsilon)$$

where $R(\epsilon) = 2\pi\int f\,d\mu$ and where s is the source function, feeding in particles at the initial energy E_0. Show, by the use of the Laplace transform in ϵ, that

$$f(\mu,\epsilon) = \tfrac{1}{2}\alpha\epsilon\int_{-1}^{1} s(\mu)\,d\mu + s(\mu)\delta(\epsilon)$$

What is the general solution, for any source function $s(\mu,\epsilon)$ ($\epsilon \geq 0$)?

12.12 Suppose the fraction of particles re-emitted in the range between ϵ and $\epsilon + d\epsilon$ ($\epsilon = \ln E_0/E$) in Prob. 12.11 is $w(\epsilon)$ ($\epsilon \geq 0$) where $\int_0^\infty w(\epsilon)\,d\epsilon \leq 1$. Show that the equation for the Laplace transform ($\epsilon \to \eta$) for $f \to \varphi$ and the solution for the transform for $R \to \rho$ are

$$\varphi(\mu,\eta) = (1/4\pi)\omega(\eta)\rho(\eta) + s(\mu);\quad \omega(\eta) = \int_0^\infty w(\epsilon)e^{-\eta\epsilon}\,d\epsilon$$
$$\rho(\eta) = \sigma/[1 - \omega(\eta)];\quad \sigma = 2\pi\int_{-1}^{1} s(\mu)\,d\mu$$

Show that when $w = [\kappa\delta(\epsilon) + \alpha\kappa' u(\epsilon - \epsilon_0)e^{-\alpha(\epsilon-\epsilon_0)}]$ ($\kappa + \kappa' \leq 1$) then

$$f(\mu,\epsilon) = \frac{(\sigma/4\pi)}{1-\kappa}\left\{\kappa\delta(\epsilon) + \sum_{n=1}^{\infty}\left(\frac{\kappa'\alpha}{1-\kappa}\right)^n \frac{(\epsilon - n\epsilon_0)^{n-1}}{(n-1)!}\,e^{-\alpha(\epsilon-n\epsilon_0)}u(\epsilon - n\epsilon_0)\right\} + s(\mu)\delta(\epsilon)$$

Discuss the physical significance of the sequence of step functions u in the answer.

12.13 Apply the inverse Laplace transform in ξ to Eq. (12.2.2) for the case of a semi-infinite medium ($x \geq 0$) for uniform scattering of particles with negligible loss of energy per collision. Show that when $s(\xi,\mu) = (I/2\pi)\delta(\xi)\delta(1 - \mu)$ (corresponding to an incident beam falling normally on the surface $\xi = 0$), the equation for the Laplace transform ($f \to F$, $\xi \to p$) of f is given by the equation

$$(1 + \mu p)F(p,\mu) = (\kappa/4\pi)G(p) + (I/2\pi)\delta(1 - \mu);\quad G(p) = 2\pi\int_{-1}^{1} F\,d\mu$$

By integration over μ and rearrangement of terms, show that

$$G(p) = \frac{Ip}{(p+1)(p - \kappa \tanh^{-1} p)}; \quad G(p) = \int_0^\infty R(\xi) e^{-p\xi}\, d\xi$$

Discuss the calculation of $R(\xi) = 2\pi \int f\, d\mu$ and thence of f.

12.14 Suppose the scattering is predominantly small angle, so that we can neglect back scattering. We can then expand $f(\mu')$ by Taylor's series in the integral of Eq. (12.2.2), by setting $\mu' \simeq \mu(1 - \frac{1}{2}\theta'^2) + \theta' \sqrt{1 - \mu^2} \cos \beta$, where $d\Omega' \simeq \theta'\, d\theta'\, d\beta$. Show that, to the second order in ϑ, Eq. (12.2.2) becomes

$$\frac{\partial f}{\partial \xi} + (1 - \kappa) f - \frac{\Delta^2 \kappa}{2\vartheta} \frac{\partial}{\partial \vartheta} \left(\vartheta \frac{\partial f}{\partial \vartheta} \right) \simeq s(\vartheta, \xi)$$

where $\mu = \cos \vartheta \simeq 1 - \frac{1}{2}\vartheta^2$ and $\Delta^2 = 2\pi \int_0^\pi \alpha(\theta')(\theta')^3\, d\theta' \ll 1$ (constant Δ might be called the mean scattering angle). Show that when a beam $s(\vartheta)$ is incident on the plane $x = 0$, the solution of this approximate equation is

$$\begin{aligned} f(\vartheta, \xi) &\simeq \int_0^\infty J_0(\lambda\vartheta) \exp[-(1 - \kappa + \tfrac{1}{2}\Delta^2 \kappa\, \lambda^2)\xi] \lambda\, d\lambda \int_0^\infty s(\theta) J_0(\lambda\theta)\theta\, d\theta \\ &= \frac{e^{(\kappa-1)\xi}}{\Delta^2 \kappa \xi} \int_0^\infty I_0\left(\frac{\vartheta\theta}{\kappa\, \Delta^2 \xi}\right) \exp\left(\frac{-\vartheta^2 - \theta^2}{2\kappa\, \Delta^2 \xi}\right) s(\theta)\theta\, d\theta \end{aligned}$$

12.15 Combine the techniques suggested by Prob. 12.14 and the first part of Prob. 12.12. Assume that $\Delta = \Delta_0(1 - e^{-\gamma\epsilon})$ but that $w(\epsilon)$ is independent of ϑ. Compute $f(\vartheta, \xi, \epsilon)$.

12.16 Suppose that a particle scattered through an angle θ' loses a fraction $(2m/M)(1 - \cos \theta')$ of its kinetic energy [see Eq. (2.4.45)]. Using the approximations discussed in Prob. 12.14 for small-angle scattering, show that the equation for f is

$$\left(\frac{\partial f}{\partial \xi}\right) + (1 - \kappa) f - \kappa \Delta^2 \frac{m}{M} \left(\frac{\partial f}{\partial \epsilon}\right) + \frac{\kappa \Delta^2}{2\vartheta} \frac{\partial}{\partial \vartheta} \left(\vartheta \frac{\partial f}{\partial \vartheta} \right) = s(\vartheta, \xi) \delta(\epsilon)$$

where $\epsilon = \ln(E_0/E)$ and Δ^2 is defined in Prob. 12.14. Show that, if $s(\vartheta, \xi) = e^{-\xi}\delta(1 - \mu)$ (normally incident beam, attenuated as it penetrates), the distribution function f is

$$f \simeq \left(\frac{1}{\kappa\, \Delta^2 \epsilon}\right) \exp\left[-\xi + \frac{M\epsilon}{m\Delta^2} - \frac{m\vartheta^2}{2M\epsilon} \right] \left[1 - u\left(\epsilon - \frac{\kappa\, \Delta^2 m}{M}\, \xi \right) \right]$$

(*Hint:* Use the Laplace transform for ξ and ϵ and the Hankel transform for ϑ.)

12.17 The potential energy of a particle of mass M is ∞ for $x < 0$, is $U_0 \hbar^2 / 2M$ (a constant) for $a < x < b$, and is zero elsewhere. Show that the solution of the Schroedinger equation for the energy $k^2\hbar^2/2M$ is

$$\psi(k,x) = \begin{cases} (1/A)\sin(kx); & 0 < x < a \\ (1/A)[\sin(ka)\cosh\kappa(k - a) & \\ \qquad + (k/\kappa)\cos(ka)\sinh\kappa(x - a)]; & a < x < b \\ \sin[k(x - b) + \Omega]; & b < x \end{cases}$$

where

$$A^2 = \csc^2\varphi\{\sin^2(ka + \varphi)e^{2\kappa(b-a)} + \sin^2(ka - \varphi)e^{-2\kappa(b-a)} - 2\cos(2\varphi)\sin(ka + \varphi)\sin(ka - \varphi)\}$$

$$\sin\varphi = (k/\sqrt{U_0});\quad \kappa = \sqrt{U_0}\cos\varphi$$

and $$\Omega = \tan^{-1}\left\{\frac{\sin(ka + \varphi)e^{\kappa(b-a)} + \sin(ka - \varphi)e^{-\kappa(b-a)}}{\sin(ka + \varphi)e^{\kappa(b-a)} - \sin(ka - \varphi)e^{-\kappa(b-a)}}\tan\varphi\right\}$$

Show that, when $k^2 < U_0$ and $\kappa(b - a)$ is large, ψ is large in the region $(0 < x < a)$ for only narrow regions of energy. Find the value of these "virtual energy levels" and discuss their physical significance.

12.18 For Prob. 12.17, if the initial value of ψ is $\psi_0(x)$, show that the subsequent solution of the Schroedinger time-dependent equation is

$$\Psi(x,t) = \frac{2}{\pi}\int_0^\infty e^{-ik^2\tau}\psi(k,x)\,dk \int_0^\infty \psi_0(\xi)\psi(k,\xi)\,d\xi$$

where $\tau = \hbar t/2M$ is proportional to time. Suppose that, initially, the particle is certainly inside the "valley region" $0 < x < a$, so that $\psi = \sqrt{2/a}\sin(\pi nx/a)u(a - x)$; $x > 0$, for example. Show that, when $(\pi n/a)^2 \ll U_0$ and $\exp[\sqrt{U_0}(b - a)] \gg 1$, a good approximation to Ψ in the "valley" is

$$\sqrt{2a}\left(\frac{(n\pi)^2}{\pi a^4 U_0}\right)\exp[-2\sqrt{U_0}(b - a)]\int_{-\infty}^{\infty}\frac{\sin[(k_n + \lambda)x]}{(\lambda^2 + \delta_n^2)}\exp[-ik_n^2 - 2ik_n\lambda\tau]\,d\lambda;\quad 0 < x < a;\quad \tau > 0$$

where

$$k_n = (n\pi/a) - (n\pi/a^2\sqrt{U_0})$$

and $$\delta_n = [(2\pi n)^2/a^3U_0]\exp[-2\sqrt{U_0}(b - a)]$$

Show that this is approximately equal to

$$\sqrt{2/a}\sin(k_nx)\exp\left\{-\left[ik_n^2 + \frac{(2\pi n)^3}{a^4U_0}\exp[-2\sqrt{U_0}(b - a)]\right]\tau\right\};\quad 0 < x < a;\quad \tau > 0$$

What is the physical significance of the real part of the exponential?

12.19 The one-dimensional potential energy for a particle of mass M is $-(U_0\hbar^2/2M)$ for $-a < x < a$ and is zero for $|x| > a$. What are the bound-state energies and wave functions? How many bound states are there for U_0 very small? The three-dimensional potential energy for the same particle is $-(U_0\hbar^2/2M)$ for the radius $r < a$ and is zero for $r > a$. What are the bound-state energies and wave functions? How many bound states are there for U_0 very small?

12.20 A system has a Hamiltonian operator $\mathfrak{H}_0$ and is in a stationary state characterized by φ_n, solution of $\mathfrak{H}_0\varphi_n = E_n\varphi_n$, up to $t = 0$. At $t = 0$ a perturbing energy $\mathfrak{H}_1$ is suddenly applied. By formal application of the Laplace transform, show that the solution of the time-dependent Schroedinger equation subsequent to $t = 0$ is

$$\begin{aligned}\Psi(t) &= \frac{i}{2\pi}\int_{-\infty+i\epsilon}^{\infty+i\epsilon} \frac{\exp(-iEt/\hbar)}{(E - \mathfrak{H}_0 - \mathfrak{H}_1)}\,\varphi_n\,dE \\ &= \exp[-i(\mathfrak{H}_0 + \mathfrak{H}_1)t/\hbar]\varphi_n \\ &= \frac{i}{2\pi}\int_{-\infty+i\epsilon}^{\infty+i\epsilon} \exp\left(-\frac{iEt}{\hbar}\right)\sum_{n=0}^{\infty}\frac{(1 - \mathcal{P}_n)\mathfrak{H}_1^n}{[E - \mathfrak{H}_0 - \mathcal{P}_n\mathfrak{H}_1]^{n+1}}\,\varphi_n\,dE\end{aligned}$$

where operator $\mathcal{P}_n$ is defined in the discussion preceding Eq. (9.1.43). From the last formulation, compute a series expansion for $\Psi(t)$, in terms of the eigenfunctions φ_m, analogous to the perturbation series (9.1.12). By rearrangement, obtain series analogous to the Feenberg and the Fredholm series of Sec. 9.1.

12.21 Show that the Schroedinger equation for the hydrogen atom ($V = -e^2/r$) can be separated in prolate spheroidal coordinates, with the nucleus at one of the foci. Compute the energies and wave functions for the lowest four bound states.

12.22 Compute the angle-distribution function for scattering from a potential field $U = A\exp(-\alpha^2r^2)$, by use of the variational principle of Eq. (12.3.68), taking the trial function to be the incident wave $\exp(ikz)$. (Hint: Perform the double integration in the denominator in configuration space, not momentum or K space. Over what range of k does this result differ by less than 10 per cent from the Born approximation?

12.23 A particle of mass M is in a potential field

$$V = M\omega^2x^2 + (b^2/x^2)$$

Show that its allowed energies are

$$W_n = 2\hbar\omega(n + \tfrac{1}{2} + \tfrac{1}{2}\alpha);\quad n = 0, 1, 2, \ldots$$

where $\alpha^2 = (2Mb^2/\hbar^2) + \frac{1}{4}$, and that the corresponding wave functions are

$$\psi_n = N_n e^{-\frac{1}{2}z^2} z^{\alpha+\frac{1}{2}} L_{n+\alpha}^{\alpha}(z^2)$$

where $z = x\sqrt{M\omega/\hbar}$. Express the normalizing constant N_n in terms of n, a, M, ω, and $\hbar$.

12.24 A particle of mass M moves in a field

$$V = \left(\frac{\hbar^2}{2M}\right)\frac{b(b+1)}{r^2}\sec^2\vartheta - \frac{Ze^2}{r}$$

Show that its allowed bound-state energies are

$$W_n = -(Me^4Z^2/\hbar^2\sigma^2); \quad \sigma = b+1, b+2, \ldots, b+n, \ldots$$

and that the corresponding wave function is

$$\psi = Ne^{im\varphi}\cos^{b+1}\vartheta \sin^m \vartheta F(m - l + 1, \tfrac{1}{2}l + \tfrac{1}{2}m + b + 1|b + \tfrac{3}{2}|\tfrac{1}{2}\cos^2\vartheta) \cdot \\ \cdot x^{l+b}e^{-\frac{1}{2}x} L_{\sigma+l+b}^{2l+2b+1}(x)$$

What values must l have for ψ to be finite?

12.25 A harmonic oscillator is perturbed by a potential bx^3, where b is small compared to $\frac{1}{2}M\omega^2$. Calculate the perturbation in the energy levels, to the second order in b.

12.26 A particle is acted on by a central force of potential

$$-(\hbar^2b^2/2M)e^{-r/d}.$$

Compute the lowest allowed energy as a function of $1/bd$ for $0 < 1/bd < 2$, by using tables of Bessel functions (see page 1670). Compute this energy also by the variational technique, using the trial wave function $Ne^{-\alpha r}$, and compare the results.

12.27 Compute the scattering phase angle η_0, for the potential field of Prob. 12.26, for $bd = \frac{1}{2}$, as function of kd, for $0 < kd < 1$ (see page 1687). Compute η_0 also by means of the Born approximation and by the variational method (see pages 1692 and 1703) and compare.

12.28 Obtain the Green's function of page 1711 for the source point on the x axis. What is it when the source point is at the origin?

12.29 Work out the details of the derivation of Eq. (12.3.85).

12.30 Compute a next approximation to the helium ground-state wave function by using the variational wave function of page 1734 and the Green's function of Eq. (12.3.94) in the equation on page 1735.

12.31 A system, with Hamiltonian operator $\mathcal{H}_0$, before $t = 0$, has a doubly degenerate ground state; $\mathcal{H}_0\varphi_1 = E\varphi_1$ and $\mathcal{H}_0\varphi_2 = E\varphi_2$ (where φ_1 is orthogonal to φ_2). After $t = 0$, an additional term $\mathcal{H}_1$ is present in the Hamiltonian having the following properties:

$$\mathcal{H}_1\varphi_1 = \hbar(\alpha\varphi_1 + \beta\varphi_2); \quad \mathcal{H}_1\varphi_2 = h(\beta\varphi_1 - \alpha\varphi_2)$$

Show that, if the system had been in state 1 before $t = 0$, the wave function for $t > 0$ will be

$$\Psi = \left\{\left[\cos(\gamma t) - \frac{i\alpha}{\gamma}\sin(\gamma t)\right]\varphi_1 - \frac{i\beta}{\gamma}\sin(\gamma t)\varphi_2\right\} e^{-iEt/\hbar}$$

where $\gamma^2 = \alpha^2 + \beta^2$. Discuss the physical significance of this result.

12.32 Two systems, one with two discrete energy levels

$$\mathcal{H}_1\varphi_n = E_n\varphi_n(n = 0, 1; E_0 = 0; E_1 = E)$$

and the other with one discrete level (set at zero energy) and a continuum of levels (ejected particle)

$$\mathcal{H}_2\chi_0 = 0; \quad \mathcal{H}_2\chi_k = \epsilon(k)\chi_k; \quad \chi_k = e^{i\mathbf{k}\cdot\mathbf{r}_2}$$

Beginning at $t = 0$, the two systems are coupled by an energy term $\mathcal{H}_{12}$, having the properties

$$\mathcal{H}_{12}\varphi_0\chi_k = M(k)\varphi_1\chi_0 e^{i\mathbf{k}\cdot\mathbf{r}_0}; \quad \mathcal{H}_{12}\varphi_1\chi_0 = \int \bar{M}(k)\varphi_0\chi_k e^{-i\mathbf{k}\cdot\mathbf{r}_0}\,dv_k$$

where dv_k is the volume element in three-dimensional "k space" and where $\mathbf{r}_0$ can be said to locate the center of gravity of system 1. By setting

$$\Psi = a_0(t)\varphi_1\chi_0 e^{-iEt/\hbar} + \int a_k(t)\varphi_0\chi_k e^{-i\epsilon(k)t/\hbar}\,dv_k$$

in the time-dependent Schroedinger equation and solving for the a's, show that, if ϵ depends only on the magnitude of $\mathbf{k}$ and if M is small compared to E, we have

$$a_0 \simeq e^{-\gamma t}; \quad a_k \simeq \bar{M}(k)e^{-i\mathbf{k}\cdot\mathbf{r}_0}[(e^{-\gamma t+i(\epsilon-E)t/\hbar} - 1)/(E - \epsilon(k) - i\hbar\gamma)]$$

where $\gamma = [\pi|M(k_1)|^2k_1^2/\hbar\epsilon'(k_1)]$, k_1 being the root of the equation $\epsilon(k) = E$ and ϵ' being the derivative of ϵ with respect to k. Hence show that the wave function for $t > 0$ is

$$\Psi \simeq \varphi_1\chi_0 e^{-\gamma t - iEt/\hbar} + \frac{4\pi\bar{M}(k_1)}{r_2'}\varphi_0 e^{ik_1r_2' - iEt/\hbar}u\left[t - \frac{\hbar r_2'}{\epsilon'(k_1)}\right]\exp\left\{-\gamma\left[t - \frac{\hbar r_2'}{\epsilon'(k_1)}\right]\right\}$$

where $r_2' = |\mathbf{r}_2 - \mathbf{r}_0|$. Assuming that φ represents an atom or nucleus and that χ_0, χ_k represents a particle (photon, positron, etc.) in its "latent" or emitted state, respectively, discuss the physical significance of these results.

Jacobi Polynomials

The polynomial $F(-n, n + a|c|z)$ is called the *Jacobi polynomial* of nth order for the parameters a and c ($n = 0, 1, 2, \ldots$; $c \neq 0, 1, 2, \ldots, n - 1$). These polynomials are orthogonal in the range $0 \leq z \leq 1$ for the density function $z^{c-1}(1 - z)^{a-c}$:

$$\int_0^1 z^{c-1}(1-z)^{a-c}F(-m, m+a|c|z)F(-n, n+a|c|z)\,dz$$
$$= \delta_{mn}\frac{n![\Gamma(c)]^2\Gamma(n+a-c+1)}{(a+2n)\Gamma(a+n)\Gamma(c+n)}; \quad \operatorname{Re} c > 0; \quad \operatorname{Re} a - c > -1$$

General Relationships:

$$F(-n, n+a|c|z) = \frac{z^{1-c}(1-z)^{c-a}}{c(c+1)\cdots(c+n-1)}\frac{d^n}{dz^n}[z^{c+n-1}(1-z)^{a+n-c}]$$
$$\frac{d}{dz}F(-n, n+a|c|z) = -\frac{n(n+a)}{c}F(-n+1, n+a+1|c+1|z)$$
$$zF(-n, n+a|c|z) = \frac{c-1}{2n+a}[F(-n, n+a-1|c-1|z) - F(-n-1, n+a|c-1|z)]$$
$$F(-n, n+a|c|z) = \frac{(a+n)(c+n)}{c(2n+a)}F(-n, n+a+1|c+1|z) - \frac{n(n+a+c)}{c(2n+a)}F(-n+1, n+a|c+1|z)$$
$$= \frac{\Gamma(c)\Gamma(c-a)}{\Gamma(n+c)\Gamma(c-a-n)}F(-n, n+a|a-c+1|1-z)$$
$$= \frac{\Gamma(c)\Gamma(2n+a)}{\Gamma(n+a)\Gamma(n+c)}(-z)^nF\left(-n, 1-c-n|1-2n-a|\frac{1}{z}\right)$$
$$= \frac{\Gamma(c)z^{1-c}}{\Gamma(n+a)\Gamma(c-n-a)}\int_0^z (z-t)^{c-n-a-1}t^{n+a-1}(1-t)^n\,dt$$

Special Cases:

$$F(-n, n+2\beta+1|\beta+1|z) = \frac{2^\beta n!\Gamma(\beta+1)}{\Gamma(n+2\beta+1)}T_n^\beta(1-2z)$$
$$F(-n, n|\tfrac{1}{2}|z) = \cos[2n\sin^{-1}\sqrt{z}]$$
$$F(-n, n+2|\tfrac{3}{2}|z) = \frac{\sin[2(n+1)\sin^{-1}\sqrt{z}]}{2(n+1)\sqrt{z(1-z)}}$$
$$F(0,a|c|z) = 1; \quad F(-1, a+1|c|z) = 1 - \left(\frac{a+1}{c}\right)z$$
$$F(-2, a+2|c|z) = 1 - \left[\frac{2(a+2)}{c}\right]z + \left[\frac{(a+2)(a+3)}{c(c+1)}\right]z^2; \cdots$$
$$F(-n, n+a|c|z) = \sum_{s=0}^{n}(-1)^s\left[\frac{n!\Gamma(c)\Gamma(a+n+s)}{(n-s)!s!\Gamma(a+n)\Gamma(c+s)}\right]z^s$$
$$z^m = \sum_{n=0}^{m}(-1)^n\left[\frac{(a+2n)m!\Gamma(a+n)\Gamma(c+m)}{n!(m-n)!\Gamma(a+n+m+1)\Gamma(c)}\right]F(-n, n+a|c|z)$$

$$F(-n, n + 2m|m + \tfrac{1}{2}|\sin^2\alpha) = \frac{(n+1)\cdots(n+m)(n+m)\cdots(n+2m-1)}{(-1)^m\frac{1}{2}\cdot\frac{3}{2}\cdots(m-\frac{1}{2})}\cdot \left[\frac{1}{\sin 2\alpha}\frac{d}{d\alpha}\right]^m \cos[2(n+m)\alpha]$$

$$F(-n, n + l_1 + l_2 + 2|l_2 + \tfrac{3}{2}|\sin^2\alpha) = \sum_{m=0}^{n} \epsilon_m(-1)^m \cdot \left\{\sum_{s=m}^{n} \frac{(-1)^s n!(2s)!(n+l_1+l_2+s+1)!\Gamma(l_2+\frac{3}{2})}{2^{2s}s!(n-s)!(s+m)!(s-m)!(n+l_1+l_2+1)!\Gamma(s+l_2+\frac{3}{2})}\right\}\cdot \cos(2m\alpha)$$

Semicylindrical Functions

$$J_m^{(2)}(z) = \frac{2}{\pi i^m}\int_0^{\frac{1}{2}\pi} e^{iz\cos u}\cos(mu)\,du;\quad m = 0, 1, 2, 3, \ldots$$

$$= J_m(z) - \begin{cases} \dfrac{2i}{\pi}\left[\dfrac{z}{m^2-1} + \dfrac{z^3}{(m^2-1)(m^2-9)} + \cdots\right]; & m = 0, 2, 4, \ldots \\ \dfrac{2i}{m\pi}\left[1 + \dfrac{z^2}{m^2-4} + \dfrac{z^4}{(m^2-4)(m^2-16)} + \cdots\right]; & m = 1, 3, 5, \ldots \end{cases}$$

$$\simeq H_m^{(1)}(z) + \begin{cases} 2i/\pi z; & m = 0, 2, 4, \ldots \\ 2mi/\pi z^2; & m = 1, 3, 5, \ldots \end{cases} \quad z \gg m$$

$$\frac{1}{z}\frac{d}{dz}\left(z\frac{dJ_m^{(2)}}{dz}\right) + \left(1 - \frac{m^2}{z^2}\right)J_m^{(2)} = \begin{cases} 2i/\pi z; & m = 0, 2, 4, \ldots \\ 2mi/\pi z^2; & m = 1, 3, 5, \ldots \end{cases}$$

$$e^{iz|\cos\phi|} = \sum_{n=0}^{\infty} (-1)^n \cos(2n\phi)J_{2n}^{(2)}(z)$$

$$\frac{d}{dz}J_0^{(2)}(z) = -J_1^{(2)}(z);\quad \frac{d}{dz}J_m^{(2)}(z) = \tfrac{1}{2}J_{m-1}^{(2)}(z) - \tfrac{1}{2}J_{m+1}^{(2)}(z);\quad m = 1, 2, \ldots$$

$$\left(\frac{m}{z}\right)J_m^{(2)}(z) = \tfrac{1}{2}J_{m-1}^{(2)}(z) + \tfrac{1}{2}J_{m+1}^{(2)}(z) + \begin{cases} 0; & m = 2, 4, 6, \ldots \\ 2i/\pi z; & m = 1, 3, 5, \ldots \end{cases}$$

Likewise, for $n > 2$, we can define a function $J_m^{(n)}(z)$ such that, over the range $-(\pi/n) \le \phi \le \pi/n$, the series

$$e^{iz\cos\phi} = \sum_{m=0}^{\infty} \epsilon_m i^{nm}\cos(nm\phi)J_{nm}^{(n)}(z)$$

holds, repeating itself n times in the full range of ϕ. In other words,

$$\sum_{m=0}^{\infty} \epsilon_m i^{nm} \cos(nm\phi) J_{nm}^{(n)}(z) = \begin{cases} e^{iz\cos\phi}; & -\pi/n \le \phi \le \pi/n \\ e^{iz\cos(\phi - 2\pi/n)}; & \pi/n \le \phi \le 3\pi/n \\ \cdots \end{cases}$$

The general properties of these functions are:

$$\begin{aligned} J_m^{(n)}(z) &= \frac{n}{\pi i^m} \int_0^{\pi/n} e^{iz\cos u} \cos(mu)\, du \\ &= \frac{n}{2\pi i^m} \int_{-\pi/n}^{\pi/n} e^{iz\cos u + imu}\, du \\ &= \sum_{s=0}^{\infty} \frac{n i^{s-m}}{\pi s!} z^s \int_0^{\pi/n} \cos^s u \cos(mu)\, du; \quad m = 0, 1, 2, 3, \ldots \\ &\to \frac{n}{2}\sqrt{\frac{2}{\pi z}}\, e^{iz - \frac{1}{2}i\pi(m+\frac{1}{2})} + \frac{n \cos(m\pi/n)}{\pi i^{m-1} \sin(\pi/n)} \left(\frac{1}{z}\right) e^{iz\cos(\pi/n)}; \quad z \to \infty \end{aligned}$$

$$\frac{d}{dz} J_0^{(n)}(z) = -J_1^{(n)}(z); \quad \frac{d}{dz} J_m^{(n)}(z) = \tfrac{1}{2}J_{m-1}^{(n)}(z) - \tfrac{1}{2}J_{m+1}^{(n)}(z); \quad m > 0$$

$$\left(\frac{m}{z}\right) J_m^{(n)}(z) = \tfrac{1}{2}J_{m-1}^{(n)}(z) + \tfrac{1}{2}J_{m+1}^{(n)}(z) - \left(\frac{n i^{-m}}{\pi z}\right) e^{iz\cos(\pi/n)} \sin(m\pi/n)$$

$$\begin{aligned} &\frac{1}{z}\frac{d}{dz}\left(z \frac{dJ_m^{(n)}}{dz}\right) + \left(1 - \frac{m^2}{z^2}\right) J_m^{(n)}(z) \\ &\qquad = \left(\frac{n}{\pi i^m}\right)\left[\left(\frac{i}{z}\right) \sin\left(\frac{\pi}{n}\right) \cos\left(\frac{m}{n}\pi\right) - \left(\frac{m}{z^2}\right) \sin\left(\frac{m}{n}\pi\right)\right] e^{iz\cos(\pi/n)} \end{aligned}$$

A closely related function is

$$E_m^{(n)}(z) = \frac{n}{\pi i^m} \int_0^{\pi/n} e^{iz\cos u} \sin(mu)\, du \to \frac{in}{\pi i^m z} [e^{iz\cos(\pi/n)} - e^{iz}]; \quad z \to \infty$$

$$\frac{d}{dz} E_m^{(n)}(z) = \tfrac{1}{2}E_{m-1}^{(n)}(z) - \tfrac{1}{2}E_{m+1}^{(n)}(z)$$

$$\left(\frac{m}{2}\right) E_m^{(n)}(z) = \tfrac{1}{2}E_{m-1}^{(n)}(z) + \tfrac{1}{2}E_{m+1}^{(n)}(z) + \left(\frac{n i^{-m}}{\pi z}\right)\left[\cos\left(\frac{\pi m}{n}\right) - 1\right]$$

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CHAPTER 13

Vector Fields

The fields we have been discussing in detail in the past three chapters have been scalar fields; only a single number has been needed to be assigned to each point in space to specify the physical situation. In most cases, of course, the quantity of physical interest was actually a vector, the electric intensity, or the fluid velocity, for example. But the situation was such that the vector could be obtained directly from the scalar field, usually by taking the gradient. It usually requires considerable simplification of the problem in order to arrive at so convenient a result: requiring that the electric charges be at rest or neglecting the viscosity of the moving fluid or the like. But the relative convenience of the use of scalar fields is so great that we use them whenever there is any possible excuse for it.

Naturally the calculation of vector fields, which cannot be expressed in terms of the gradient of a scalar, is a more arduous task than it is for scalar fields, since three numbers must be calculated for each point in space, rather than one. These three numbers can be expressed in several ways, depending on the particular way we choose to express the vector. The most obvious way, of course, would be to specify the x, y, and z components of the vector at each point, in which case we would, in effect, have three scalar fields to compute so that the boundary conditions are satisfied. Another way is to specify the components of the vector field along each of the three directions of those curvilinear coordinates which happen to be appropriate for the problem at hand; this again reduces the problem to that of calculating three scalar fields, the magnitudes of these three components.

A procedure usually more in keeping with the physics of the situation, however, is to obtain the vector field from three scalar fields by more general vector operations. As we have seen in Chap. 1 [see Eq. (1.5.15)], any vector field may be expressed as the sum of the gradient of a scalar potential and the curl of a vector potential. One of the scalars specifying the field may, therefore, be the scalar potential; the other two will serve to specify uniquely the vector potential. Only two values at each point are needed to specify the vector potential, rather than three, because the auxiliary condition, that the vector potential have zero divergence, reduces the independent quantities needed for complete specification from three to two.

We can see why this latter procedure is preferable by discussing how we fit boundary conditions by vector fields. The "basic" boundary conditions for scalar fields are the homogeneous Dirichlet (that the field go to zero at the boundary) or homogeneous Neumann (that the gradient of the field normal to the boundary be zero). In the vector case we must somehow specify the values or normal gradients, at the boundary, of all three of the basic scalar fields mentioned in the last paragraph. If we are clumsy in our choice of scalar fields by which to represent our vector field, we shall find the imposition of boundary conditions an extremely complicated task; if we are clever in our choice, we shall find the task fairly simple.

A review of the various vector fields discussed in Chaps. 2 and 3, which cannot be expressed as the gradient of a scalar (the electromagnetic field, the velocity of a viscous fluid, or the displacement of an elastic solid, for instance) indicates that the "natural" boundary conditions would relate to the values or gradients of the components of the vector fields normal or tangential to the boundary surface, at this surface. If F_n and $\mathbf{F}_t$ are these components, respectively, one possible boundary condition would be that $F_n = 0$, another would be $\mathbf{F}_t = 0$, and so on. Actually, of course, $\mathbf{F}_t$ is the amplitude of the two-dimensional vector formed by projecting $\mathbf{F}$ on the boundary surface; when we have inhomogeneous boundary conditions, we shall need to specify both magnitude and direction of this projection. If $\mathbf{n}$ is the unit vector normal to the boundary surface, then the tangential projection is contained in $\mathbf{n} \times \mathbf{F}$, so that Dirichlet conditions can be specified by specifying the scalar $F_n = \mathbf{n} \cdot \mathbf{F}$ and the two-dimensional vector $\mathbf{n} \times \mathbf{F}$.

In other cases the curl of the vector potential $\mathbf{F}$ may be the vector of physical interest, in which case we have the generalizations of the Neumann conditions, involving the specifying of values of $\mathbf{n} \cdot (\text{curl } \mathbf{F})$ and of $\mathbf{n} \times (\text{curl } \mathbf{F})$.

To show how important is the proper choice of three scalars with which to express the vector field, let us consider for a moment the fitting of boundary conditions on a spherical surface. If we use F_x, F_y, and F_z, the rectangular components of the vector, to be the three scalars, the equations for these scalars will be much the same as those for the usual scalar field. For instance, if the equation which $\mathbf{F}$ is to satisfy is the vector Helmholtz equation, $\nabla^2\mathbf{F} + k^2\mathbf{F} = 0$, then the equations for all three rectangular components would be scalar Helmholtz equations, which we learned how to solve in Chap. 11. But, having obtained the solutions, we would still be in trouble in applying the boundary conditions, because we have expressed our vector in rectangular components, which have no simple relation to the spherical boundary surface. Even if we calculated F_x, F_y, and F_z as solutions of the Helmholtz equation in the spherical coordinates r, ϑ, φ, we still would have to "mix up" the three solutions in a complicated way in order to arrange that the vector **F** be tangential to the spherical surface everywhere, for instance. The problem could, in principle, be solved, but we would run into the same sort of practical difficulties which were discussed at the beginning of Chap. 5 and which led us there to look for solutions for scalar fields which separated in coordinates appropriate for the boundary. Here we find a need to express our vector field in terms of component scalar fields which bear a simple relationship to the boundary surface.

The solution which leaps to mind, of course, is to express our vector in terms of its components along the unit axes of the coordinate system appropriate for the boundary. In the example mentioned, the suggestion would be to use F_r, F_ϑ, and F_φ, the three components along $\mathbf{a}_r$, $\mathbf{a}_\vartheta$, and $\mathbf{a}_\varphi$, to be the three scalars in terms of which to express our vector field. But here we run into another difficulty, which may quickly be seen by referring to the expression for $\nabla^2\mathbf{F}$ in terms of F_r, F_ϑ, F_φ, given at the end of Chap. 1. We see there that the three component equations for the vector Helmholtz equation do not separate into an equation for F_r alone, another for F_ϑ alone, and a third for F_φ alone. We are faced with a set of three simultaneous equations, each equation involving *all three components*. It is not impossible to solve them but their solution involves, in general, the solution of a sixth-order differential equation, which we should seek earnestly to avoid. The same dilemma is reached for all curvilinear coordinate systems. Even if the equation for a scalar field could be separated in these coordinates, the equations for the three components of **F** along the three unit vectors for these coordinates may not separate and, even if they do separate, they "mix up" the components so that each component enters into all three equations.

But why do we need to go at this problem blindly, without taking suggestions from its physical counterparts? In all the cases treated in Chap. 2, we found that there is good reason to split the vector field into two parts: one part obtained from a scalar potential by taking the gradient and the other derived from a vector potential by taking the curl. The first of these fields is usually called the *longitudinal* (or lamellar) component, since a gradient points in the direction of greatest rate of change of the scalar potential. (It points in the direction of propagation of a plane wave, for instance.) The second is called the *transverse* (or solenoidal) component, since the curl of a vector is usually transverse to the direction of greatest change. (It points at right angles to the direction of propagation of a plane wave, for instance.) In many cases the two components differ somewhat in behavior; the velocity of transverse waves in an elastic medium is different from that of longitudinal waves, for example. In many other cases, even if the two satisfy the same equation (as they do in the electromagnetic field), there is no bar to making the separation this way.

This separation into longitudinal and transverse components, $\mathbf{F} = \mathbf{F}_l + \mathbf{F}_t$, has several advantages. In the first case, the longitudinal component is, as we have said, the gradient of a scalar potential, and consequently all the techniques of the previous three chapters are at once available to provide the solution. We know how to obtain separated solutions, for example, and we know how to fit boundary conditions for scalar potentials. Consequently our task in this chapter should be concentrated on the transverse component, obtained from a vector potential by taking the curl. This field may always be derived from a *pair* of scalar fields; although any vector field requires three scalars to define it, the requirement that the field be transverse (*i.e.*, that it be a curl, *i.e.*, that its divergence be zero) imposes one linear relationship between the three scalars, which reduces the number of independent scalars to two.

In the first section of this chapter we shall show how this division of the vector field into longitudinal and transverse parts enables us to simplify the procedure of applying boundary conditions. We shall investigate the requirements on the various coordinate systems in order that the transverse solution have relations with coordinate surfaces analogous to those enjoyed by separated scalar potentials. (In other words, we shall generalize the concept of *separability* to include transverse vector fields and their equations.) We shall then recapitulate the discussions of Chaps. 6 and 7 to discuss the general properties of transverse vector eigenfunctions and vector Green's functions and their uses in fitting boundary conditions. In Secs. 13.2 and 13.3 we shall go on to discuss solutions of the vector Laplace, Helmholtz, and wave equations for different coordinate systems and boundary conditions.

13.1 *Vector Boundary Conditions, Eigenfunctions, and Green's Functions*

Our first task in this section is to verify some of the statements made in the introduction to this chapter. The equation we shall use to exemplify our results will be the vector Helmholtz equation:

$$\nabla^2\mathbf{F} + k^2\mathbf{F} = \operatorname{grad}(\operatorname{div}\mathbf{F}) - \operatorname{curl}(\operatorname{curl}\mathbf{F}) + k^2\mathbf{F} = 0 \qquad (13.1.1)$$

The vector Laplace equation may be obtained from it by setting $k = 0$, and the solutions of the vector wave equation may be obtained from the solutions of (13.1.1) by the Laplace transform, as discussed in Sec. 11.1.

The corresponding inhomogeneous equation may be obtained by adding a term to the right-hand side of (13.1.1).

As we noted in Sec. 1.5 [see Eq. (1.5.15)], the vector solution $\mathbf{F}$ may always be separated into a longitudinal and a transverse part

$$\begin{aligned} \mathbf{F} &= \mathbf{F}_l + \mathbf{F}_t; \quad \mathbf{F}_l = \text{grad } \varphi \\ \mathbf{F}_t &= \text{curl } \mathbf{A}; \quad \text{curl } \mathbf{F}_l = 0; \quad \text{div } \mathbf{F}_t = 0 \end{aligned} \tag{13.1.2}$$

where φ is the *scalar potential* and $\mathbf{A}$ the *vector potential*. The longitudinal part (or, rather, the corresponding scalar potential) has been treated in previous chapters and need not be detailed again; we carry it along, when necessary, for completeness.

The vector potential $\mathbf{A}$ also has a longitudinal and a transverse part, but the longitudinal part has no physical significance, since the procedure of taking the curl of $\mathbf{A}$, to obtain $\mathbf{F}$, wipes out the longitudinal part. Consequently, *we usually assume that the vector potential* $\mathbf{A}$ *has zero divergence.* A few exceptions will be made, particularly in the case of the vector Laplace equation.

This statement deserves some further comment in the case of the electromagnetic field, for it corresponds to a choice of gauge which is not relativistically invariant, as was indicated on page 212. The equations for the electromagnetic potentials in this gauge are

$$\begin{aligned} \nabla^2\varphi &= -(4\pi\rho/\epsilon) \\ \text{curl}(\text{curl } \mathbf{A}) + \frac{\mu\epsilon}{c^2}\left(\frac{\partial^2\mathbf{A}}{\partial t^2}\right) &= \frac{4\pi\mu}{c}\mathbf{J} - \frac{\epsilon\mu}{c}\,\text{grad}\left(\frac{\partial\varphi}{\partial t}\right) \\ \mathbf{B} = \text{curl } \mathbf{A}; \quad \mathbf{E} &= -\text{ grad } \varphi - (1/c)(\partial\mathbf{A}/\partial t) \end{aligned} \tag{13.1.3}$$

In a few cases it is advisable to add an arbitrary gradient to this vector potential, arranging it so that div $\mathbf{A}$, instead of being zero, is equal to some arbitrary scalar field U. The equations for the modified potentials $\mathbf{A}_u$ and φ_u are then

$$\begin{aligned} \nabla^2\varphi_u &= -(4\pi\rho/\epsilon) - (1/c\epsilon)(\partial U/\partial t) \\ \nabla^2\mathbf{A}_u - \frac{\mu\epsilon}{c^2}\left(\frac{\partial^2\mathbf{A}_u}{\partial t^2}\right) &= -\frac{4\pi\mu}{c}\mathbf{J} + \text{grad } U + \frac{\epsilon\mu}{c}\,\text{grad}\left(\frac{\partial\varphi_u}{\partial t}\right) \end{aligned}$$

In this case, in order that the electric and magnetic fields be not changed by this gauge transformation, we must have that

$$\mathbf{A}_u = \mathbf{A} - \text{grad } \chi; \quad \varphi_u = \varphi + (1/c)(\partial\chi/\partial t); \quad \text{where } \nabla^2\chi = -U$$

In particular, if we wish to transform to the gauge where $\mathbf{A}$ and φ form the components of a four-vector, then we must set div $\mathbf{A} = -(\epsilon\mu/c)(\partial\varphi/\partial t)$, that is, $\mathbf{A}_r = \mathbf{A} - \text{grad } \chi_r$; $\varphi_r = \varphi + (1/c)(\partial\chi_r/\partial t)$, where $\nabla^2\chi_r = (\epsilon\mu/c)\cdot$ $\cdot\,(\partial\varphi_r/\partial t)$ and where $\mathbf{A}_r$ and φ_r satisfy Eqs. (2.5.15). Consequently, we can perform our original calculations for $\mathbf{A}$ and φ in the gauge of Eqs.

(13.1.3) and later transform to another gauge by the transformations indicated here. Another gauge of occasional utility is given in Eqs. (3.4.20).

The Transverse Field in Curvilinear Coordinates. Suppose the curvilinear coordinates ξ_1, ξ_2, ξ_3 with scale factors h_1, h_2, h_3, are appropriate for the boundary surface under consideration, the boundary being, for example, the surface $\xi_1 = C$. The longitudinal part of the field is grad φ and, if the Helmholtz equation, for example, is separable in these coordinates, the boundary conditions for this part are applied in the manner outlined in previous chapters. The transverse part is, as we pointed out a few pages ago, dependent on two scalar fields. It would be advantageous to choose these scalars, and the mode of deriving the transverse field from them, so that the part of the field derived from one scalar would be tangential to the surface $\xi_1 = C$ and the other would be normal to it.

A vector normal to the ξ_1 surfaces is $\mathbf{a}_1 f$, where f is some scalar function of ξ_1, ξ_2, ξ_3 which is to be determined. This is not, however, always a transverse field, for the divergence of this vector is $(1/h_1h_2h_3)[\partial(h_2h_3f)/\partial\xi_1]$, which is seldom zero even if f is independent of ξ_1.

The vector

$$\mathbf{M} = \operatorname{curl}(\mathbf{a}_1 f) = \frac{\mathbf{a}_2}{h_1h_3}\frac{\partial}{\partial\xi_3}(h_1f) - \frac{\mathbf{a}_3}{h_1h_2}\frac{\partial}{\partial\xi_2}(h_1f)$$

is tangential to the surface $\xi_1 = C$, however, and since its divergence is zero, it could be one solution, if f is such a scalar field as to make $\mathbf{M}$ satisfy the prescribed equation. If the equation is the vector Helmholtz equation, which is

$$-\nabla^2\mathbf{M} = -\operatorname{grad}\operatorname{div}\mathbf{M} + \operatorname{curl}\operatorname{curl}\mathbf{M} = \operatorname{curl}\operatorname{curl}\mathbf{M} = k^2\mathbf{M}$$

(since div $\mathbf{M} = 0$), we need only to arrange things so that curl $\mathbf{M} =$ curl curl $(\mathbf{a}_1 f)$ is equal to $k^2\mathbf{a}_1 f$ plus the gradient of some scalar, for then the curl of this will just be $k^2\mathbf{M}$. But

$$\operatorname{curl}\mathbf{M} = -\frac{\mathbf{a}_1}{h_2h_3}\left[\frac{\partial}{\partial\xi_2}\left(\frac{h_3}{h_1h_2}\frac{\partial}{\partial\xi_2}h_1f\right) + \frac{\partial}{\partial\xi_3}\left(\frac{h_2}{h_1h_3}\frac{\partial}{\partial\xi_3}h_1f\right)\right] + \frac{\mathbf{a}_2}{h_1h_3}\frac{\partial}{\partial\xi_1}\left(\frac{h_3}{h_1h_2}\frac{\partial}{\partial\xi_2}h_1f\right) + \frac{\mathbf{a}_3}{h_1h_2}\frac{\partial}{\partial\xi_1}\left(\frac{h_2}{h_1h_3}\frac{\partial}{\partial\xi_3}h_1f\right)$$

This certainly does not look to be much like $k^2\mathbf{a}_1 f$ + grad u, but a second look shows that the ξ_2 and ξ_3 components can be made to look like the corresponding components of a gradient if, and only if, h_3/h_1h_2 and h_2/h_1h_3 *are both independent* of ξ_1, that is, if both h_1 and h_2/h_3 are independent of ξ_1. For then we can reverse the order of differentiation and can also rearrange the ξ_1 component, by adding and subtracting a term to finish off the gradient, obtaining finally

$$\operatorname{curl} \mathbf{M} = \frac{1}{h_1^2} \operatorname{grad}\left(\frac{\partial}{\partial \xi_1} h_1 f\right) - \mathbf{a}_1 \left\{\frac{1}{h_1^3}\left(\frac{\partial^2}{\partial \xi_1^2} h_1 f\right) + \frac{1}{h_2 h_3} \frac{\partial}{\partial \xi_2}\left(\frac{h_3}{h_1 h_2} \frac{\partial}{\partial \xi_2} h_1 f\right) + \frac{1}{h_2 h_3} \frac{\partial}{\partial \xi_3}\left(\frac{h_2}{h_1 h_3} \frac{\partial}{\partial \xi_3} h_1 f\right)\right\}$$

The first term will not have a zero curl unless h_1 is independent of ξ_2 and ξ_3 as well; in other words, h_1 must be constant and may thus be unity. The last two terms then become equivalent to two terms in the expression for $\nabla^2 f$ for, since $h_1 = 1$, h_3/h_2 is equal to $h_1h_2h_3/h_2^2 = g_2(\xi_1,\xi_3)f_2(\xi_2)$ and h_2/h_3 is equal to $h_1h_2h_3/h_3^2 = g_3(\xi_1,\xi_2)f_3(\xi_3)$ where the functions g_n and f_n are defined on page 510. The first term in the braces is the only one which prevents it from being $\nabla^2 f$. However it can be adjusted so the expression in braces is $w(\xi_1)\nabla^2\psi$, where $f = w(\xi_1)\psi$, if the first term, which may then be written as $[\partial^2(w\psi)/\partial\xi_1^2]$, is equal to $(w/f_1)[\partial(f_1\partial\psi/\partial\xi_1)/\partial\xi_1]$, where $h_2h_3 = g_1(\xi_2,\xi_3)f_1(\xi_1)$. Writing this equation out, we finally obtain

$$2\left(\frac{dw}{d\xi_1}\right)\left(\frac{\partial\psi}{\partial\xi_1}\right) + \psi\left(\frac{d^2w}{d\xi_1^2}\right) = \frac{w}{f_1}\left(\frac{df_1}{d\xi_1}\right)\left(\frac{\partial\psi}{\partial\xi_1}\right)$$

which must hold for any of the forms of ψ which we may wish to use.

This equation can be satisfied, for any ψ, if $d^2w/d\xi_1^2 = 0$ and if $2 \ln w = \ln f_1$. The first equation is solved if w is either 1 or ξ_1 and the second equation if f_1 is either 1 or ξ_1^2. When all this occurs, we finally have

$$\operatorname{curl} \mathbf{M} = \operatorname{grad}(\partial w\psi/\partial\xi_1) - \mathbf{a}_1 w \nabla^2\psi$$

and, if ψ is a solution of the scalar Helmholtz equation, $\nabla^2\psi = -k^2\psi$, we have

$$\operatorname{curl} \mathbf{M} = \operatorname{grad}(\partial f/\partial\xi_1) + k^2\mathbf{a}_1 f; \quad f = w\psi$$

and thus
$$\operatorname{curl}(\operatorname{curl} \mathbf{M}) = k^2\mathbf{M}$$

so that $\mathbf{M} = \operatorname{curl}(\mathbf{a}_1 w\psi)$ has zero divergence and is a solution of the vector Helmholtz equation.

A scrutiny of the various separable coordinates given in the table at the end of Chap. 5 shows that there are only six of the eleven separable coordinate systems in which one of the scale factors is unity, and the ratio of the other two scale factors is independent of the coordinate corresponding to the unity scale factor: rectangular coordinates, in which x, y, or z may be ξ_1; the three cylindrical coordinates, in which z corresponds to ξ_1 in the sense used here; and, finally, spherical and conical coordinates, in which the preferred coordinate is the radius.

In the first four cases, the function f_1 is 1 so the function w is also 1; in the last two cases, $f_1 = r^2$ so the function $w = r$.

We have just shown that for six of the eleven coordinate systems which allow separation of the scalar Helmholtz equation, it is possible to set up a transverse solution of the vector Helmholtz equation, which is tangential to one of the coordinate surfaces, of the general form

$$\mathbf{M} = \operatorname{curl}[\mathbf{a}_1 w(\xi_1)\psi] = \operatorname{grad}(w\psi) \times \mathbf{a}_1 \tag{13.1.4}$$

where ψ is a solution of the scalar Helmholtz equation, ξ_1 is z and $w = 1$ for the cylindrical coordinates (including cartesian) and where $\xi_1 = r$, $w = r$ for the spherical and conical coordinates.

Thus we have obtained one of the two scalar fields which are to generate our transverse solution and have indicated how it can generate its part of the transverse field. We need one other, which should, if possible, generate a field normal to a coordinate surface or else the curl of which is tangential to the surface.

We have seen that $\mathbf{a}_1 f$ is not, in general, a transverse solution (for many coordinate systems it is not even a solution). However,

$$\mathbf{N} = \frac{1}{k}\operatorname{curl}\operatorname{curl}[\mathbf{a}_1 w(\xi_1)\chi] = k\mathbf{a}_1 w\chi + \frac{1}{k}\operatorname{grad}(\partial w\chi/\partial\xi_1) \tag{13.1.5}$$

where χ is a solution of the scalar Helmholtz equation [and w is as in Eq. (13.1.4)], is a transverse solution of the vector Helmholtz equation for the same six coordinate systems as before, which has a curl,

$$\operatorname{curl}\mathbf{N} = k\operatorname{grad}(w\chi) \times \mathbf{a}_1$$

tangential to the ξ_1 surfaces. $\mathbf{N}$ is not identical with $\mathbf{M}$ even if $\chi = \psi$. In fact $\mathbf{N}$ is often perpendicular to $\mathbf{M}$ when $\chi = \psi$.

We have thus defined three scalar fields, φ, ψ, and χ, all solutions of the *scalar* Helmholtz equation, which can be the basis for the most general solution of the *vector* Helmholtz equation and which are in a form that allows of fairly simple application of boundary conditions:

$$\begin{aligned} \mathbf{L} &= \operatorname{grad}\varphi; \quad \mathbf{M} = \operatorname{curl}(\mathbf{a}_1 w\psi) = \operatorname{grad}(w\psi) \times \mathbf{a}_1 \\ \mathbf{N} &= \frac{1}{k}\operatorname{curl}\operatorname{curl}(\mathbf{a}_1 w\chi) = k\mathbf{a}_1 w\chi + \frac{1}{k}\operatorname{grad}\left[\frac{\partial(w\chi)}{\partial\xi_1}\right] \end{aligned} \tag{13.1.6}$$

The first of these is the longitudinal part of the solution, and the other two are the transverse parts. We have put in the factor $1/k$ in N so that the dimensions of φ, $w\chi$, and $w\psi$ are all the same.

For the other five coordinate systems, we can obtain solutions by taking three solutions of the scalar Helmholtz equation as the three rectangular components of the vector, but then the fitting of boundary conditions is well-nigh impossible of attainment. In fact we are justified in defining *separation* for the vector solution as the process of breaking the solution into the components **L**, **M**, and **N** as defined in (13.1.6), with the scalar generating fields separated into factors as discussed in Sec. 5.1. In the sense of this definition, therefore, only six coordinate systems allow separation of the solution of the vector Helmholtz equation. For other vector equations, separation, in the sense of this paragraph, is even more restricted. For static elasticity, for instance, the equation separates only in spherical and two-dimensional polar coordinates; not even rectangular coordinates allow simple fitting of boundary conditions.

Vector Green's Theorems. In order to show the relationship between boundary conditions and solutions and in order to obtain vector Green's functions for longitudinal and transverse vector fields, we need to develop the vector analogues to Green's theorem (7.2.2). This theorem relates the behavior of scalar fields on the boundary surface to their average behavior throughout the volume inside the boundary, gives us an indication of the nature of the boundary conditions required to specify fields inside the boundary, and is the basic relationship upon which we have developed the Green's function technique of solving boundary value problems. Its ultimate source is, of course, Gauss' theorem, Eq. (1.4.7), relating divergence to normal outflow integral for a vector field.

Here we are relating two vectors **E** and **F**, one of which will be identified with a Green's function later and the other will be related to the required solution, whereas earlier we were relating two scalars. Two properties, instead of one, will need to be specified on the surface, one related to the curl and the other to the divergence. We start with the statement of Gauss' theorem for two vectors formed from **E** and **F**,

$$\begin{aligned}\iiint \operatorname{div}(\mathbf{E} \operatorname{div} \mathbf{F})\, dv &= \oint (\mathbf{E}\cdot\mathbf{n}) \operatorname{div} \mathbf{F}\, dA \\ \iiint \operatorname{div}(\mathbf{E}\times \operatorname{curl} \mathbf{F})\, dv &= \oint (\mathbf{E}\times \operatorname{curl}\mathbf{F})\cdot \mathbf{n}\, dA \\ &= \oint (\mathbf{n}\times\mathbf{E})\cdot \operatorname{curl}\mathbf{F}\, dA = \oint \mathbf{E}\cdot(\operatorname{curl}\mathbf{F}\times\mathbf{n})\, dA\end{aligned}$$

where, for ease in manipulation, we have set the axial vector $d\mathbf{A}$ equal to the outward-pointing, unit vector **n**, normal to the surface, times the scalar magnitude dA of the surface element.

Next we look up in our table of vector operations, given at the end of Chap. 1, to see that

$$\begin{aligned}\operatorname{div}(\mathbf{E} \operatorname{div} \mathbf{F}) &= \mathbf{E}\cdot \operatorname{grad}(\operatorname{div}\mathbf{F}) + (\operatorname{div}\mathbf{F})(\operatorname{div}\mathbf{E}) \\ \operatorname{div}(\mathbf{E}\times\operatorname{curl}\mathbf{F}) &= (\operatorname{curl}\mathbf{E})\cdot(\operatorname{curl}\mathbf{F}) - \mathbf{E}\cdot\operatorname{curl}(\operatorname{curl}\mathbf{F})\end{aligned}$$

Reversing **E** for **F** and subtracting, we finally obtain the equivalent Green's theorems for vectors

$$\iiint [\mathbf{E} \cdot \text{grad div } \mathbf{F} - \mathbf{F} \cdot \text{grad div } \mathbf{E}]\, dv = \oint [(\text{div } \mathbf{F})\mathbf{E} - (\text{div } \mathbf{E})\mathbf{F}] \cdot \mathbf{n}\, dA$$

$$-\iiint [\mathbf{E} \cdot \text{curl curl } \mathbf{F} - \mathbf{F} \cdot \text{curl curl } \mathbf{E}]\, dv = \oint [(\mathbf{n} \times \mathbf{E}) \cdot \text{curl } \mathbf{F} + (\mathbf{n} \times \text{curl } \mathbf{E}) \cdot \mathbf{F}]\, dA$$

and finally

$$\begin{aligned}\iiint [\mathbf{E} \cdot \nabla^2\mathbf{F} - \mathbf{F} \cdot \nabla^2\mathbf{E}]\, dv &= \oint [\mathbf{E} \times \text{curl } \mathbf{F} + \mathbf{E}(\text{div } \mathbf{F}) - \mathbf{F} \times \text{curl } \mathbf{E} - \mathbf{F}(\text{div } \mathbf{E})] \cdot \mathbf{n}\, dA \\ &= \oint \{[\mathbf{E} \text{ div } \mathbf{F} - \mathbf{F} \text{ div } \mathbf{E}] \cdot \mathbf{n} \\ &\qquad - [\mathbf{E} \cdot (\mathbf{n} \times \text{curl } \mathbf{F}) + \text{curl } \mathbf{E} \cdot (\mathbf{n} \times \mathbf{F})]\}\, dA \qquad (13.1.7)\end{aligned}$$

where, of course, $\nabla^2\mathbf{F} = \text{grad div } \mathbf{F} - \text{curl curl } \mathbf{F}$. Vector $\mathbf{E}$ will eventually become the vector field to be calculated in terms of the boundary conditions, and vector $\mathbf{F}$ is to be the Green's function.

We see that the boundary conditions break up into ones for the longitudinal field and ones for the transverse field. If the divergence of the solution is zero at the boundary, then we can use a transverse Green's function and the solution will be transverse everywhere within the boundary (with a few unusual exceptions mentioned later). Similarly, if the curl of the solution is zero at the boundary, we can use a longitudinal Green's function and the solution will be longitudinal, in which case we can return to use of a scalar potential; the first of Eqs. (13.1.7) is equivalent to Eq. (7.2.2).

The kinds of boundary conditions which may be applied to vector fields are already foreshadowed in these equations. Two of them, $(\mathbf{E} \cdot \mathbf{n})$ and $(\mathbf{n} \times \mathbf{E})$, correspond to fixing the normal and tangential components of the field itself at the surface; if these are the boundary conditions to be applied, we choose a Green's function $\mathbf{F}$ which is zero at the boundary. For the case where the field represents the displacement of an elastic medium, for instance, these boundary conditions would be used when the surface of the medium is to be distorted from its equilibrium state. In the case of the vector electromagnetic potential say, in the gauge of Eq. (3.4.20)], the normal component of $\mathbf{A}$ is determined by the time integral of the charge on the boundary surface, and the tangential component of $\mathbf{A}$ is determined by the surface current.

The alternate set of boundary conditions corresponds somehow to fixing the normal gradients of the various components of $\mathbf{E}$, div $\mathbf{E}$, and $(\mathbf{n} \times \text{curl } \mathbf{E})$; and if these boundary conditions are to be applied, both divergence and curl of the Green's function must be zero at the surface. In the case of the elastic solid, these boundary conditions on $\mathbf{E}$ correspond to a specification of pressure and tangential stress at the surface. The corresponding case for the electromagnetic vector potential would correspond to a specification of electric and magnetic fields at the surface (curl $\mathbf{A}$ is proportional to $\mathbf{H}$).

Furthermore, if the physics of the situation, either through the boundary conditions or the equations, specifies that the field be either longitudinal or transverse everywhere inside the boundary, we can take this into account by specializing the Green's function, and the corresponding boundary conditions simplify. For instance, if the field is everywhere transverse, only the projection of **E** on the surface, $(\mathbf{n} \times \mathbf{E})$, or else the projection of curl **E**, need be specified on the boundary. The corresponding Green's function has zero divergence, and in addition either it or its curl has zero tangential component on the boundary. On the other hand, if the field is everywhere longitudinal, only the normal component of **E** or else its divergence need be specified at the boundary and the corresponding longitudinal Green's function is required to have its normal component or its divergence zero.

The Green's Function a Dyadic. In the last part of Chap. 7 we indicated that the Green's function, in general, was the kernel of an integral operator which served to transform the boundary conditions and/or the source density into the solution. When the solution is to be a scalar, this kernel could also be a scalar, but when the source function and the boundary conditions are vectors, the kernel must be a vector operator, a dyadic of the type discussed in Sec. 1.6. This result is, one might say, a natural extension of the function-space concept of eigenfunctions and Green's functions which was discussed in Chaps. 6 and 7. There we indicated that the boundary conditions could be thought of as a vector in abstract vector space, with unit vectors $\mathbf{e}(x^s)$, and that the Green's function is a vector operator transforming the boundary values into values of the solution. As long as the boundary conditions and source function are scalars, G is an operator in vector space but a scalar in "actual" space; if the boundary conditions and source densities are actual vectors, the Green's function must be a vector operator in actual space as well as in function space. It must transform the vector boundary conditions and source densities into the vector solution. In other words, the Green's function for the vector case must be a *dyadic* function of source and observer position.

This dyadic Green's function $\mathfrak{G}(\mathbf{r}|\mathbf{r}_0)$ must have the same general properties which were enumerated on page 804 for the scalar Green's function: (1) it must satisfy a reciprocity relationship, (2) it will serve to generate the solution from both boundary conditions and source functions, and (3) the resulting solutions will have discontinuities just outside the boundaries. For use with the Helmholtz equation, it must satisfy the inhomogeneous dyadic equation

$$\nabla^2\mathfrak{G}(\mathbf{r}|\mathbf{r}_0) + k^2\mathfrak{G}(\mathbf{r}|\mathbf{r}_0) = -4\pi\mathfrak{J}\delta(\mathbf{r} - \mathbf{r}_0) \tag{13.1.8}$$

where $\mathfrak{J}$ is the idemfactor, the dyadic analogue of unity ($\mathfrak{J} \cdot \mathbf{F} = \mathbf{F}$ for any **F**). The operator ∇^2 operates on the **r** dependence of $\mathfrak{G}$ (the operator

∇_0^2 operates on the $\mathbf{r}_0$ dependence) and acts on all nine of its components. The nine cartesian components of the dyadic $\nabla^2\mathfrak{G}$ are the Laplacians of the nine cartesian components of $\mathfrak{G}$. When expressed in components along curvilinear coordinates, it is a more complex function, but this can be worked out by the methods of Sec. 1.5. A few examples will be given later.

We can then show, by methods analogous to those of page 808, that $\mathfrak{G}$ satisfies the reciprocity relationship

$$\mathfrak{G}(\mathbf{r}|\mathbf{r}_0) = \mathfrak{G}(\mathbf{r}_0|\mathbf{r})$$

We can also show that $\mathfrak{G}$ is a symmetric dyadic; in other words, that $\mathfrak{G} \cdot \mathbf{F}$ is the same vector as $\mathbf{F} \cdot \mathfrak{G}$, for any $\mathbf{F}$.

We may then use this dyadic Green's function to find the solution of the inhomogeneous vector equation, such as the Helmholtz equation

$$\nabla^2\mathbf{F} + k^2\mathbf{F} = -4\pi\mathbf{Q}(\mathbf{r}) \tag{13.1.9}$$

within some boundary surface S, subject to boundary conditions on S. As we have done many times before, we multiply Eq. (13.1.8) on the left by $\mathbf{F}$ and (13.1.9) on the right by $\mathfrak{G}$, subtract, and integrate over the volume inside S, obtaining the *vector* equation

$$\textstyle\iiint[(\nabla^2\mathbf{F}) \cdot \mathfrak{G} - \mathbf{F} \cdot (\nabla^2\mathfrak{G})]\,dv = 4\pi\iiint[\mathbf{F}\,\delta(\mathbf{r} - \mathbf{r}_0) - \mathbf{Q} \cdot \mathfrak{G}]\,dv$$

The equation may also be written with $\mathbf{F}$ and $\mathfrak{G}$ in reverse order, since $\mathfrak{G}$ is a symmetric dyadic.

By reason of Eq. (13.1.7) and by exchanging $\mathbf{r}$ and $\mathbf{r}_0$, we can, therefore, show that, inside and on the boundary surface S,

$$\begin{aligned}\mathbf{F}(\mathbf{r}) = \iiint \mathfrak{G}(\mathbf{r}|\mathbf{r}_0) \cdot \mathbf{Q}(\mathbf{r}_0)\,dv_0 \\ + \frac{1}{4\pi}\oint \{[\nabla_0 \cdot \mathbf{F}(\mathbf{r}_0^s)][\mathfrak{G}(\mathbf{r}|\mathbf{r}_0^s) \cdot \mathbf{n}] - (\nabla_0 \cdot \mathfrak{G})[\mathbf{F}(\mathbf{r}_0^s) \cdot \mathbf{n}] \\ - \mathfrak{G}(\mathbf{r}|\mathbf{r}_0^s) \cdot [\mathbf{n} \times (\nabla_0 \times \mathbf{F})] - (\nabla_0 \times \mathfrak{G}) \cdot [\mathbf{n} \times \mathbf{F}(\mathbf{r}_0^s)]\}\,dA_0\end{aligned} \tag{13.1.10}$$

where the meaning of the various vector operations on the vectors and dyadics, indicated in the surface integral, may be worked out from the rules discussed in Chap. 1, though they will be illustrated in detail for a few cases shortly. The term $(\nabla_0 \cdot \mathbf{F})$ is a scalar whereas $(\mathfrak{G} \cdot \mathbf{n})$ is a vector; the term $(\nabla_0 \cdot \mathfrak{G})$ is a vector whereas $(\mathbf{F} \cdot \mathbf{n})$ is a scalar; the term $(\mathbf{n} \times \text{curl}\,\mathbf{F})$ is a vector and $\mathfrak{G}$ is a dyadic; the term $(\mathbf{n} \times \mathbf{F})$ is a vector and $(\nabla \times \mathfrak{G})$ a dyadic; so that the whole of the quantity inside the braces is a vector. The volume integral is over the whole volume enclosed by S, and the surface integral is over the whole of S (in the $\mathbf{r}_0$ coordinates). This is the equivalent of Eq. (7.2.7) and is the starting point for the various Green's function techniques for the vector solutions.

For example, if the boundary conditions are that the values of both tangential and normal parts of $\mathbf{F}$ ($\mathbf{F}_t$ and F_n) are specified on the boundary, then the Green's function dyadic is adjusted so that it goes to zero on the boundary ($\mathfrak{G}$ is zero when $\mathbf{r}$ is on S and $\mathbf{r}_0$ is not or when $\mathbf{r}_0$ is on S and $\mathbf{r}$ is not) and

$$\mathbf{F}(\mathbf{r}) = \iiint \mathfrak{G}(\mathbf{r}|\mathbf{r}_0) \cdot \mathbf{Q}(\mathbf{r}_0)\, dv_0 \qquad (13.1.11)$$
$$- \frac{1}{4\pi} \oint \{[\boldsymbol{\nabla}_0 \cdot \mathfrak{G}(\mathbf{r}|\mathbf{r}_0^s)]F_n(\mathbf{r}_0^s) + [\boldsymbol{\nabla}_0 \times \mathfrak{G}(\mathbf{r}|\mathbf{r}_0^s)] \cdot [\mathbf{n} \times \mathbf{F}_t(\mathbf{r}_0^s)]\}\, dA_0$$

where $\mathfrak{G}(\mathbf{r}^s|\mathbf{r}_0) = \mathfrak{G}(\mathbf{r}|\mathbf{r}_0^s) = 0$. We see here that the component of $\mathbf{F}$ normal to the surface (a scalar) is dealt with separately from the tangential components, which come in as the tangential vector $\mathbf{F}_t$. ($\mathbf{F}_t$ is the tangential vector obtained by projecting $\mathbf{F}$ on the plane tangent to the surface; $\mathbf{n} \times \mathbf{F} = \mathbf{n} \times \mathbf{F}_t$ is the tangential vector formed by rotating $\mathbf{F}_t$ 90° about $\mathbf{n}$ as an axis, the rotation being clockwise when looking in the direction of positive $\mathbf{n}$.) If it turns out that $\mathbf{F}$ is to be a transverse solution (which could occur only if div $\mathbf{Q} = 0$) then $\mathfrak{G}$ may be taken transverse (that is, $\boldsymbol{\nabla} \cdot \mathfrak{G} = -\boldsymbol{\nabla}_0 \cdot \mathfrak{G} = 0$) and the normal component of $\mathbf{F}$ at the surface need not be specified, only the transverse. This is not surprising, for requiring that div $\mathbf{F}$ be zero everywhere establishes a relation between the three scalar fields generating $\mathbf{F}$ and only two sets of boundary conditions are needed to specify all three. Vice versa, if $\mathbf{F}$ is longitudinal, $\mathfrak{G}$ may be taken longitudinal ($\boldsymbol{\nabla} \times \mathfrak{G} = -\boldsymbol{\nabla}_0 \times \mathfrak{G} = 0$), and only the normal component of $\mathbf{F}$ at the boundary need be specified.

On the other hand, if the boundary conditions are that the divergence and the tangential component of the curl of $\mathbf{F}$ be specified on S, then $\mathfrak{G}$ should be adjusted so that its divergence is zero and its tangential curl is zero at S, in which case

$$\mathbf{F}(\mathbf{r}) = \iiint \mathfrak{G}(\mathbf{r}|\mathbf{r}_0) \cdot \mathbf{Q}(\mathbf{r}_0)\, dv_0$$
$$+ \frac{1}{4\pi} \oint \{[\mathfrak{G} \cdot \mathbf{n}](\mathrm{div}_0\, \mathbf{F}) - \mathfrak{G} \cdot [\mathbf{n} \times \mathrm{curl}_0\, \mathbf{F}]\}\, dA_0 \qquad (13.1.12)$$

where $\mathrm{div}[\mathfrak{G}(\mathbf{r}^s|\mathbf{r}_0)] = \mathrm{div}_0[\mathfrak{G}(\mathbf{r}|\mathbf{r}_0^s)] = 0$ and likewise $\mathbf{n} \times \mathrm{curl}[\mathfrak{G}(\mathbf{r}^s|\mathbf{r}_0)] = \mathbf{n} \times \mathrm{curl}_0[\mathfrak{G}(\mathbf{r}|\mathbf{r}_0^s)] = 0$. In this case divergence of $\mathbf{F}$ is dealt with separately from the tangential components of its curl. If $\mathbf{F}$ is to be transverse (which means that $\mathbf{Q}$ is transverse and div $\mathbf{F}$ is zero on S), then the only boundary conditions to be specified are the two tangential components of the curl of $\mathbf{F}$.

A general inhomogeneous boundary condition for vector solutions would be that some linear combination of the normal component of $\mathbf{F}$ and its divergence has a specified value on the boundary:

$$[\mathrm{div}\, \mathbf{F} + \alpha(\mathbf{n} \cdot \mathbf{F})]_{\mathbf{r}=\mathbf{r}^s} = H(\mathbf{r}^s)$$

and a linear combination of tangential component and the tangential component of its curl is equal to some tangential vector on the boundary:

$$[(\mathbf{n} \times \text{curl}\, \mathbf{F}) + \beta(\mathbf{n} \times \mathbf{F})]_{\mathbf{r}=\mathbf{r}^s} = \mathbf{K}(\mathbf{r}^s) \times \mathbf{n}$$

where both α and β may be functions of $\mathbf{r}^s$. This case is analogous to Eq. (7.2.11). The proper gambit here is to take

$$[\nabla \cdot \mathfrak{G} + \alpha(\mathbf{n} \cdot \mathfrak{G})] = 0 \text{ and } \mathbf{n} \times [\nabla \times \mathfrak{G} - \beta\mathfrak{G}] = 0$$

on the boundary, in which case

$$\mathbf{F}(r) = \iiint \mathfrak{G}(\mathbf{r}|\mathbf{r}_0) \cdot \mathbf{Q}(\mathbf{r}_0)\, dv_0 + \frac{1}{4\pi} \oint \{H(\mathbf{r}_0^s)[\mathfrak{G}(\mathbf{r}|\mathbf{r}_0^s) \cdot \mathbf{n}] + \mathfrak{G}(\mathbf{r}|\mathbf{r}_0^s) \cdot [\mathbf{n} \times \mathbf{K}(\mathbf{r}_0^s)]\}\, dA_0 \quad (13.1.13)$$

which is the vector analogue of Eq. (7.2.12).

Vector Eigenfunctions. Before we go on to calculate the dyadic Green's functions for various cases, it would be well to say a few words about the eigenfunction solutions of the vector Helmholtz equation (as an example of vector equations). These are obtained from eigenfunction solutions of the corresponding scalar equation by the methods discussed earlier in this chapter. They can be most easily set up in two ways:

$$\begin{aligned} &\mathbf{F} = \mathbf{i}U + \mathbf{j}V + \mathbf{k}W = \mathbf{L} + \mathbf{M} + \mathbf{N} \\ &\mathbf{L} = \text{grad}\,\varphi; \quad \mathbf{M} = \text{curl}(\mathbf{a}_1 w\psi); \quad \mathbf{N} = (1/k)\,\text{curl}\,\text{curl}(\mathbf{a}_1 w\chi) \end{aligned} \quad (13.1.14)$$

where U, V, W or φ, ψ, χ are eigenfunction solutions of the scalar equation $\nabla^2\varphi + k^2\varphi = 0$, etc. The problem here is to adjust k and the relative magnitudes of U, V, and W or φ, ψ, and χ so that the boundary conditions are satisfied all over the boundary.

It can be seen that this adjustment is comparatively easy if it can be arranged that one scalar, such as U or ψ, will satisfy the boundary conditions all by itself, without requiring assistance from V, W or φ, χ. If this simplification is not possible, the determination of the allowed values of k is quite difficult. A little reflection will serve to show that this remark is only another way of pointing out the additional requirements which our concept of separability imposes on coordinates for vector solutions. Examples of the way this works out in practice will be given shortly.

The usual homogeneous boundary conditions imposed on vector eigenfunctions are either the Dirichlet condition $\mathbf{F} = 0$ or else the Neumann condition $\mathbf{n} \times \text{curl}\,\mathbf{F} = 0$ and $\text{div}\,\mathbf{F} = 0$ on the boundary. We can also require that $\mathbf{F}$ be transverse ($\text{div}\,\mathbf{F} = 0$ on boundary) and at the same time satisfy Dirichlet conditions on the boundary, in which case the correct Dirichlet condition is that $\mathbf{n} \times \mathbf{F} = 0$ there (specifying F_n also would be overdoing it, if we have already imposed the condition $\operatorname{div} \mathbf{F} = 0$). Incidentally, specifying that $\operatorname{div} \mathbf{F} = 0$ on the boundary is usually equivalent to requiring that $\mathbf{F}$ be transverse everywhere within the boundary. For, taking the divergence of $\nabla^2\mathbf{F} + k^2\mathbf{F} = 0$, we see that the scalar $\operatorname{div} \mathbf{F}$ is a solution of the scalar Helmholtz equation. Only for a set of discrete values of k can there be a scalar solution of $\nabla^2 f + k^2 f = 0$ within a closed boundary when it is zero everywhere on the boundary. Consequently, unless the eigenvalues of the vector Helmholtz equation are identical with those of the scalar one with $f = 0$ on the boundary, when $\operatorname{div} \mathbf{F} = 0$ on the boundary, it will be zero throughout the interior.

Let us now work out a few simple vector eigenfunctions so that we may see, more concretely, some of the complications we encounter and how they may be overcome. The simplest case is for a rectangular enclosure of sides l_x, l_y, l_z with origin at one corner. If Dirichlet conditions are imposed, the eigenfunctions can be written

$$\begin{aligned} \mathbf{F} &= (\mathbf{i} \text{ or } \mathbf{j} \text{ or } \mathbf{k}) \left[\sin\left(\frac{\pi n_x x}{l_x}\right) \sin\left(\frac{\pi n_y y}{l_y}\right) \sin\left(\frac{\pi n_z z}{l_z}\right)\right] \\ k^2 &= \pi^2 \left[\left(\frac{n_x}{l_x}\right)^2 + \left(\frac{n_y}{l_y}\right)^2 + \left(\frac{n_z}{l_z}\right)^2\right] \\ &n_x,\, n_y,\, n_z, = 1, 2, 3, \ldots\,; \quad \mathbf{F} = 0 \text{ on boundary} \end{aligned} \tag{13.1.15}$$

there being three orthogonal vectors for each trio of values of n_x, n_y, and n_z. None of these vectors is either longitudinal or transverse, nor can we rearrange things to obtain a longitudinal eigenfunction and two transverse ones for each trio of solutions of this sort. As indicated earlier, solutions satisfying boundary conditions which specify all three components of the solution at the boundary are, in general, neither longitudinal nor transverse (*i.e.*, neither divergence nor curl is zero). We might point out what is obvious here, that the set (13.1.5) is a complete set of vector eigenfunctions, in terms of which any piecewise continuous vector field may be represented inside the boundary.

Now let us relax our requirements a little, specifying only that the tangential components of $\mathbf{F}$ be zero at the boundaries. This requirement is met, of course, by the set (13.1.15), but also by any linear combination of this and the set $\mathbf{F} = \mathbf{U}$ or $\mathbf{V}$ or $\mathbf{W}$, where

$$\begin{aligned} \mathbf{U} &= \mathbf{i} \cos(\pi n_x x/l_x) \sin(\pi n_y y/l_y) \sin(\pi n_z z/l_z) \\ \mathbf{V} &= \mathbf{j} \sin(\pi n_x x/l_x) \cos(\pi n_y y/l_y) \sin(\pi n_z z/l_z) \\ \mathbf{W} &= \mathbf{k} \sin(\pi n_x x/l_x) \sin(\pi n_y y/l_y) \cos(\pi n_z z/l_z) \end{aligned} \tag{13.1.16}$$

which has the same eigenvalues for k as does set (13.1.15). This also is a complete set of eigenfunctions. (Each scalar amplitude is a complete set for scalar functions inside the boundary and there is one set for each component of a vector field.) Consequently, the two sets, (13.1.15) and (13.1.16), are too many; we need one more boundary condition to limit our eigenfunctions. This could be the requirement that the normal component of **F** be zero at the boundary, in which case we have only set (13.1.15). Or else it could be that the divergence of **F** be zero at the boundaries, in which case we use the set (13.1.16) alone. This set, however, is not transverse, for with these boundaries the eigenvalues for a scalar wave, div **F**, with zero values at the boundaries, are the same as the eigenvalues for the vector waves. We have inadvertently included, in each wave of the threefold set, some of the longitudinal wave.

In the present case, however, with boundary conditions of zero tangential **F** and zero divergence, we can do what we could not do with set (13.1.15), choose linear combinations so that one set is longitudinal and two are transverse with zero divergence everywhere inside the boundary. This set is

$$\begin{aligned}
\mathbf{L} &= (\pi n_x/l_x)\mathbf{U} + (\pi n_y/l_y)\mathbf{V} + (\pi n_z/l_z)\mathbf{W} \\
&= \operatorname{grad}[\sin(\pi n_x x/l_x)\sin(\pi n_y y/l_y)\sin(\pi n_z z/l_z)] \\
\mathbf{M} &= -(\pi n_z/l_z)\mathbf{V} + (\pi n_y/l_y)\mathbf{W} \\
&= \operatorname{curl}[\mathbf{i}\sin(\pi n_x x/l_x)\cos(\pi n_y y/l_y)\cos(\pi n_z z/l_z)] \\
&= -\mathbf{i}\times\operatorname{grad}[\sin(\pi n_x x/l_x)\cos(\pi n_y y/l_y)\cos(\pi n_z z/l_z)] \qquad (13.1.17) \\
\mathbf{N} &= \frac{\pi^2}{k}\left[\left(\frac{n_y}{l_y}\right)^2 + \left(\frac{n_z}{l_z}\right)^2\right]\mathbf{U} - \frac{\pi^2}{k}\left(\frac{n_x n_y}{l_x l_y}\right)\mathbf{V} - \frac{\pi^2}{k}\left(\frac{n_x n_z}{l_x l_z}\right)\mathbf{W} \\
&= (1/k)\operatorname{curl}\operatorname{curl}[\mathbf{i}\cos(\pi n_x x/l_x)\sin(\pi n_y y/l_y)\sin(\pi n_z z/l_z)] \\
\operatorname{div}\mathbf{F} &= 0; \quad (\mathbf{n}\times\mathbf{F}) = 0 \quad \text{on boundary}
\end{aligned}$$

where **L** is the longitudinal vector and **M** and **N** are the transverse ones.

We note several items of interest in this result. In the first place, the functions φ, ψ, χ, defined in Eq. (13.1.14), which generate **L**, **M**, **N**, are all different and satisfy different scalar boundary conditions. Function φ is zero for the boundary, $\mathbf{i}\psi$ has zero normal component on the boundary, and $\mathbf{i}\chi$ has zero tangential component there. They each are a complete set of scalar eigenfunctions, however, and they each possess an identical set of eigenvalues. Finally we see that, for a given trio of quantum numbers, n_x, n_y, n_z, the vector fields **L**, **M**, **N** are mutually orthogonal everywhere. It will turn out that this is nearly always the case with vector fields **L**, **M**, **N** as defined in Eq. (13.1.14) when they are subject to "reasonable" boundary conditions.

Next we look for solutions with tangential curls which are zero on the boundary. There are three of these having the general form $\mathbf{i}U$, $\mathbf{j}V$, and $\mathbf{k}W$:

$$\begin{aligned} \mathbf{u} &= \mathbf{i}\sin(\pi n_x x/l_x)\cos(\pi n_y y/l_y)\cos(\pi n_z z/l_z) \\ \mathbf{v} &= \mathbf{j}\cos(\pi n_x x/l_x)\sin(\pi n_y y/l_y)\cos(\pi n_z z/l_z) \\ \mathbf{w} &= \mathbf{k}\cos(\pi n_x x/l_x)\cos(\pi n_y y/l_y)\sin(\pi n_z z/l_z) \end{aligned} \tag{13.1.18}$$

In addition, the function $\mathbf{L}$ of Eqs. (13.1.17) (or, for that matter the gradient of any solution of the scalar Helmholtz equation) has zero curl at the surface, so we have more than we need. But this should be expected, since the complete set of Neumann boundary conditions should be that the divergence should be zero on the boundary, in addition to the zero requirement for the tangential curl. Some further juggling with various components shows that the required set is

$$\begin{aligned} \mathbf{L} &= \operatorname{grad}[\sin(\pi n_x x/l_x)\sin(\pi n_y y/l_y)\sin(\pi n_z z/l_z)] \\ \mathbf{M} &= (\pi n_z/l_z)\,\mathbf{v} - (\pi n_y/l_y)\mathbf{w} \\ &= \operatorname{curl}[\mathbf{i}\cos(\pi n_x x/l_x)\sin(\pi n_y y/l_y)\sin(\pi n_z z/l_z)] \\ \mathbf{N} &= \frac{\pi^2}{k}\left[\left(\frac{n_y}{l_y}\right)^2 + \left(\frac{n_z}{l_z}\right)^2\right]\mathbf{u} - \frac{\pi^2}{k}\frac{n_x n_y}{l_x l_y}\mathbf{v} - \frac{\pi^2}{k}\frac{n_x n_z}{l_x l_z}\mathbf{w} \\ &= (1/k)\operatorname{curl}\operatorname{curl}[\mathbf{i}\sin(\pi n_x x/l_x)\cos(\pi n_y y/l_y)\cos(\pi n_z z/l_z)] \\ \operatorname{div}\mathbf{F} &= 0;\quad (\mathbf{n}\times\operatorname{curl}\mathbf{F}) = 0 \quad \text{on boundary} \end{aligned} \tag{13.1.19}$$

As before, $\mathbf{L}$ is longitudinal and $\mathbf{M}$ and $\mathbf{N}$ are transverse. Again the corresponding φ, ψ, and χ are different scalar functions, but each comprises a complete set of scalar eigenfunctions.

Finally we can set up a trio of eigenvectors with zero normal component and zero tangential curl (the fourth possible combination of homogeneous boundary conditions). In fact the set of Eq. (13.1.18) is just this set. They may be rearranged into one longitudinal and two transverse sets, if we wish:

$$\begin{aligned} \mathbf{L} &= (\pi n_x/l_x)\mathbf{u} + (\pi n_y/l_y)\mathbf{v} + (\pi n_z/l_z)\mathbf{w} \\ &= -\operatorname{grad}[\cos(\pi n_x x/l_x)\cos(\pi n_y y/l_y)\cos(\pi n_z z/l_z)] \end{aligned}$$

plus $\mathbf{M}$ and $\mathbf{N}$ as given in Eqs. (13.1.19) (13.1.20)

$$(\mathbf{n}\cdot\mathbf{F}) = 0;\quad (\mathbf{n}\times\operatorname{curl}\mathbf{F}) = 0 \quad \text{on boundary}$$

Thus we have demonstrated the possibility of obtaining triple sets of eigenvector solutions which satisfy any of the four possible combinations of homogeneous boundary conditions. It is not difficult to see how still more general boundary conditions, as utilized in Eq. (13.1.13), may also be satisfied.

Other sets of eigenfunctions, for other coordinates, will be worked out later in this chapter. What we have seen thus far suffices to indicate the general trend of affairs, however. If the boundary conditions are that $\mathbf{F}$ be zero on the surface, then the easiest solution is to analyze $\mathbf{F}$ into its three rectangular components U, V, W, which are solutions of the scalar equation with homogeneous Dirichlet conditions. Naturally the eigenvalues for U, V, and W are identical, so that for each natural frequency there is a threefold freedom of vibration, corresponding to the threefold degeneracy of this case.

If the boundary condition is any of the other three homogeneous conditions, or a combination of them, it is usually best to express $\mathbf{F}$ in terms of the longitudinal eigenvectors $\mathbf{L}$ and the two sets of transverse vectors $\mathbf{M}$ and $\mathbf{N}$. These also are orthogonal to each other. In the case of the simple Helmholtz equation and rectangular boundaries, the eigenvalues for each set are the same as for the other two sets, giving again threefold degeneracy. In some other cases, this will not be true, the allowed frequencies of different types of waves being different.

An example is the equation for the vibrations of an isotropic elastic medium,

$$(\lambda + 2\mu) \operatorname{grad} \operatorname{div} \mathbf{F} - \mu \operatorname{curl} \operatorname{curl} \mathbf{F} + \rho\omega^2\mathbf{F} = 0$$

When the boundary conditions are such that the motion separates into a longitudinal or a transverse part [such as for Eqs. (13.1.17), (13.1.19) and (13.1.20)], then the equation separates and the eigenfunctions are as given in the equations, though the natural frequencies of the longitudinal waves differ from those of the transverse waves, by the ratio $\sqrt{\mu/(\lambda + 2\mu)}$, as can be seen from the above equation. On the other hand, for the boundary condition $\mathbf{F} = 0$, where the solutions will not break up into longitudinal and transverse parts, the equation does not separate, even for rectangular coordinates. The difficulty is that, for this boundary condition, longitudinal waves produce transverse waves on reflection, and vice versa. Since they travel at different speeds, a combination which results in periodic motion is not a simple one and cannot be represented by means of a few products of sines and cosines.

Whenever the boundary conditions do allow separation, however, the eigenvectors $\mathbf{F}_n(r)$ may be labeled by a quartet of numbers, symbolized here by the subscript n. The first of these numbers indicates which of the three families $\mathbf{L}$, $\mathbf{M}$, or $\mathbf{N}$ (or else, if it is more convenient, $\mathbf{i}U$, $\mathbf{j}V$, or $\mathbf{k}W$) is meant and the other three are the quantum numbers of the underlying scalar eigenfunctions, φ, ψ, or χ (or else, U, V, or W). We can show that the set $\mathbf{F}_n$, for all allowed values of the quartet n, is a complete set of mutually orthogonal functions, so that

$$\iiint \bar{\mathbf{F}}_n \cdot \mathbf{F}_m \, dv = \begin{cases} 0; & m \neq n \\ \Lambda_n; & m = n \end{cases} \tag{13.1.21}$$

where the integration is over the volume enclosed by the boundary. Therefore any piecewise continuous vector function $\mathbf{E}$ satisfying the same conditions on the boundary may be represented (inside the boundary) by the series

$$\mathbf{E} = \sum_n a_n \mathbf{F}_n; \quad a_n = \frac{1}{\Lambda_n} \iiint \bar{\mathbf{F}}_n \cdot \mathbf{E}\, dv \tag{13.1.22}$$

The method of proving these statements is completely analogous to that used in Sec. 6.3 for scalar eigenfunctions. In Chap. 3 we saw that each vector equation we are investigating is the Lagrange-Euler equation corresponding to a Lagrange density within the boundary, such that the solutions $\mathbf{F}_n$ each represent a stationary value of the integral of the Lagrange density throughout the enclosed volume. From this we can show the orthogonality of the functions and their completeness in the manner already outlined in Sec. 6.3.

Green's Function for the Vector Helmholtz Equation. Continuing our discussion parallel to the earlier treatment of scalar eigenfunctions, it is not difficult to show that the Green's function dyadics for the interior of a closed boundary may be expanded in terms of eigenvectors. For the Helmholtz equation, the Green's function must satisfy Eq. (13.1.8). It is not difficult to show that

$$\mathfrak{G}(\mathbf{r}|\mathbf{r}_0) = 4\pi \sum_n \frac{\bar{\mathbf{F}}_n(\mathbf{r})\mathbf{F}_n(\mathbf{r}_0)}{\Lambda_n(k_n^2 - k^2)} \tag{13.1.23}$$

where $\mathbf{F}_n$ is an eigenfunction solution of $\nabla^2\mathbf{F}_n + k_n^2\mathbf{F}_n = 0$ satisfying the same homogeneous conditions as does $\mathfrak{G}$, and k_n^2 is its corresponding eigenvalue. If $\mathbf{F}_n$ is a complex vector, then its conjugate is also present, so that $\mathfrak{G}$ is a Hermitian dyadic. It is also obvious that $\mathfrak{G}$ is a symmetric dyadic, which is also symmetric with respect to $\mathbf{r}$, $\mathbf{r}_0$. The juxtaposition of the two vectors $\bar{\mathbf{F}}_n$ and $\mathbf{F}_n$ (not a scalar or vector product) makes the result a dyadic [see Eq. (1.6.7)].

We point out that, viewed as a function of k, $\mathfrak{G}$ has poles at each eigenvalue k_n of k, with residue $-2\pi\bar{\mathbf{F}}_n(r)\mathbf{F}_n(r_0)/k_n$ at the nth pole, corresponding to the infinite response to a driving force at a resonant frequency (unless damping is present). We note, in this case, that $\mathfrak{G}$ can be made transverse simply by omitting the longitudinal eigenvectors $\mathbf{L}$ from the sum, whereas the sum over the $\mathbf{L}$'s alone gives the longitudinal Green's function.

For exterior problems, when the enclosed volume is infinite, the eigenvalues form a continuous set. The corresponding Green's function may be obtained by changing the series of Eq. (13.1.23) into an integral and by choosing the path of integration so as to obtain outgoing waves, as was done for Eq. (7.2.42). Or else the series may be separated into a series over the "angle coordinates," multiplied by a discontinuous function of the "radial coordinate," as was done for the scalar case in Eqs. (7.2.51) and (7.2.63). We shall do this, later in the chapter, for several coordinate systems.

The Green's function for the infinite domain can be obtained also in closed form, as was the case with the scalar problem. We are here in the anomalous position of trying to find a *dyadic* solution to an inhomogeneous equation in order to find the vector solutions; we shall have to be careful in using our vector-dyadic symbolism, particularly with respect to the delta function. However, as we want only the simplest radial dyadic solution, we go ahead, expressing our results in rectangular components as we go along in order to make the vector-dyadic symbolism as simple as possible. In order to shorten the writing of formulas wherever possible, we shall, *for this section only*, change our symbolism as follows: $x = x_1$, $y = x_2$, $z = x_3$; $\mathbf{i} = \mathbf{e}_1$, $\mathbf{j} = \mathbf{e}_2$, $\mathbf{k} = \mathbf{e}_3$.

It turns out, also, that the calculations for the vector Helmholtz equation are simpler than those for the vector Laplace equation, so we go through the former first.

The equation which the Green's function for the Helmholtz equation must satisfy is

$$\sum_{n,m=1}^{3} \mathbf{e}_m\mathbf{e}_n \left\{ \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] G_{mn} + k^2 G_{mn} \right\} = -4\pi\delta(\mathbf{r} - \mathbf{r}_0) \sum_{n=1}^{3} \mathbf{e}_n\mathbf{e}_n \tag{13.1.24}$$

where $\mathfrak{G} = \sum_{m,n} \mathbf{e}_m\mathbf{e}_n G_{mn}(\mathbf{r}|\mathbf{r}_0|k)$; $(\omega = kc)$, G_{mn} being the components of $\mathfrak{G}$ along the rectangular axes. This gives us nine scalar equations for the nine components, and reference to Eq. (7.2.17) shows that, for the infinite domain with specification of outgoing waves at infinity, we obtain

$$\mathfrak{G}(\mathbf{r}|\mathbf{r}_0|k) = \sum_{n=1}^{3} \mathbf{e}_n\mathbf{e}_n \left(\frac{e^{ikR}}{R} \right) = \left(\frac{e^{ikR}}{R} \right) \mathfrak{J} \tag{13.1.25}$$

The corresponding function for two dimensions (if it is needed) is

$$\mathfrak{G} = \sum_{n=1}^{2} \mathbf{e}_n\mathbf{e}_n i\pi H_0(kR)$$

The function for one dimension is just the scalar one; vectors and dyadics do not differ from scalars in one dimension.

If all equations to be encountered were simple Helmholtz equations, there would be little else to say, except to determine expansions of $\mathfrak{G}$ in various coordinates. However, most vector solutions are not quite so simple as this. In many physical applications the divergence of the resulting vector field must be zero: this is true for the velocity field of a viscous, incompressible fluid; it is also true for the electromagnetic vector potential, in the gauge making $\varphi = 0$ (see page 332), wherever there is no free charge ρ. The equation for the simple-harmonic vibrations of an isotropic elastic medium is

$$(\lambda + 2\mu) \text{ grad div } \mathbf{s} - \mu \text{ curl curl } \mathbf{s} + \rho\omega^2\mathbf{s} = 0 \qquad (13.1.26)$$

which indicates that the wave velocity of the longitudinal part of the wave differs from that of the transverse part, as was pointed out in Chap. 2. Here again the transverse part of $\mathbf{s}$ must be separated from its longitudinal part; each part must be solved separately and then put together, with coefficients dependent on λ and μ, to obtain the complete solution. In these more complicated situations, we must separate the Green's dyadic of Eq. (13.1.25) into its longitudinal and transverse parts.

Longitudinal and Transverse Green's Dyadics. The problem is to find a dyadic field with zero curl and another with zero divergence such that the sum of the two just equals the dyadic defined in Eq. (13.1.24). Presumably this could be solved by straightforward processes, but as so often happens, it can more easily be found by guessing the answer and proving that it is the answer. Of course such a procedure does not prove that the solution, when guessed, is the only possible answer, but we have already proved in Chap. 1 that the separation of a vector into its longitudinal and transverse parts is a unique process.

One would expect that the longitudinal part would be the gradient of some function of $\mathbf{r}$ and $\mathbf{r}_0$. The symmetry required of Green's functions by the reciprocity condition indicates that, if it is a gradient in the $\mathbf{r}$ coordinates, it must also be the gradient in the $\mathbf{r}_0$ coordinates (which we shall at times call the $\mathbf{r}'$ coordinates) so the simplest form of this function is

$$\mathfrak{L}(\mathbf{r}|\mathbf{r}'|k) = \frac{1}{k^2}\sum \mathbf{e}_m\mathbf{e}_n \frac{\partial}{\partial x_m}\frac{\partial}{\partial x'_n} g(\mathbf{r}|\mathbf{r}'|k) = \frac{1}{k^2}\nabla g(\mathbf{r}|\mathbf{r}'|k)\nabla' \qquad (13.1.27)$$

where $g(\mathbf{r}|\mathbf{r}'|k)$ is the usual scalar Green's function e^{ikR}/R for the infinite domain and $R^2 = (x_1 - x'_1)^2 + (x_2 - x'_2)^2 + (x_3 - x'_3)^2$. Incidentally, we see that $\partial g/\partial x_m = -(\partial g/\partial x'_m)$. The factor $1/k^2$ is included in order to keep the dimensionality of $\mathfrak{L}$ the same as the dimensionality of g, for example, a reciprocal length.

As the symbolic notation $\nabla g \nabla'$ for this function indicates, both curl $\mathfrak{L}$ and curl' $\mathfrak{L}$ are zero, so this is an operator which will produce longitudinal vectors out of any kind of vectors. To make clear what is meant by this statement, we note that to obtain a solution we operate on some vector function $\mathbf{F}$ of $\mathbf{r}'$ by the Green's dyadic and then integrate over $\mathbf{r}'$. The result, if we use $\nabla g \nabla'$, is the integral of

$$F_x(\mathbf{r}') \text{ grad}(\partial g/\partial x') + F_y(\mathbf{r}') \text{ grad}(\partial g/\partial y') + F_z(\mathbf{r}') \text{ grad}(\partial g/\partial z')$$

over $\mathbf{r}'$. The curl of this in the unprimed coordinates is zero, no matter what function the components F_x, F_y, F_z are of $\mathbf{r}'$.

We might guess that a corresponding dyadic for the transverse waves would be related to the curl of some vector function of the scalar Green's function g. Here requirements of symmetry make the function which can be symbolized by $\nabla \times (\mathfrak{J}g) \times \nabla'$ the simplest of the type. Consequently, we define

$$\begin{aligned}\mathfrak{T}(\mathbf{r}|\mathbf{r}'|k) &= -(1/k^2)\nabla \times [\mathfrak{J}g(\mathbf{r}|\mathbf{r}'|k)] \times \nabla' \\ &= \frac{1}{k^2}\nabla \times \left[\mathbf{i}\frac{\partial g}{\partial z'}\mathbf{j} - \mathbf{i}\frac{\partial g}{\partial y'}\mathbf{k} - \mathbf{j}\frac{\partial g}{\partial z'}\mathbf{i} + \mathbf{j}\frac{\partial g}{\partial x'}\mathbf{k} + \mathbf{k}\frac{\partial g}{\partial y'}\mathbf{i} - \mathbf{k}\frac{\partial g}{\partial x'}\mathbf{j}\right] \\ &= \frac{1}{k^2}\left\{\mathbf{ii}\left[\frac{\partial^2 g}{\partial y\,\partial y'} + \frac{\partial^2 g}{\partial z\,\partial z'}\right] - \mathbf{ij}\frac{\partial^2 g}{\partial y\,\partial x'} - \mathbf{ik}\frac{\partial^2 g}{\partial z\,\partial x'} + \cdots\right\}\end{aligned} \tag{13.1.28}$$

But since $\partial g/\partial x' = -(\partial g/\partial x)$, etc., the first term in this last expression can be changed into

$$-\mathbf{ii}[(\partial^2 g/\partial y^2) + (\partial^2 g/\partial z^2)](1/k^2) = -(\mathbf{ii}/k^2)[\nabla^2 g + (\partial^2 g/\partial x\,\partial x')]$$

and the second term can be changed to $-(\mathbf{ij}/k^2)(\partial^2 g/\partial x\,\partial y')$, etc. Consequently, we have that

$$\mathfrak{T}(\mathbf{r}|\mathbf{r}'|k) = -\frac{\mathfrak{J}}{k^2}\nabla^2 g - \frac{1}{k^2}\sum_{m,n}\mathbf{e}_m\mathbf{e}_n\frac{\partial^2 g}{\partial x_m\,\partial x'_n}$$

But $\nabla^2 g = -k^2 g - 4\pi\delta(\mathbf{r} - \mathbf{r}')$, so that, finally, we have that

$$\mathfrak{J}g(\mathbf{r}|\mathbf{r}'|k) = \mathfrak{J}\frac{e^{ikR}}{R} = \mathfrak{L}(\mathbf{r}|\mathbf{r}'|k) + \mathfrak{T}(\mathbf{r}|\mathbf{r}'|k) - \frac{4\pi}{k^2}\mathfrak{J}\delta(\mathbf{r} - \mathbf{r}') \tag{13.1.29}$$

But this is not the most convenient method of separating our general Green's function, because of the additional delta function which has both a longitudinal and a transverse part. While we have separated $\mathfrak{J}(e^{ikR}/R)$ into longitudinal and transverse parts for all of space except for $\mathbf{r} = \mathbf{r}'$, at the origin we still have a delta function left over. Moreover we know, from Eq. (13.1.23), that $\mathfrak{J}(e^{ikR}/R)$ may be expanded into a convergent series of eigenfunctions for the infinite domain, whereas the expansion of a delta function

$$\mathfrak{J}\delta(\mathbf{r} - \mathbf{r}') = \sum_n \bar{\mathbf{F}}_n(\mathbf{r})\mathbf{F}_n(\mathbf{r}')/\Lambda_n$$

is very poorly convergent; consequently, the expansions of $\mathfrak{L}$ and $\mathfrak{T}$ will not converge well. Obviously what is needed is to combine the "longitudinal part" of the delta function with $\mathfrak{L}$ and the "transverse part" with $\mathfrak{T}$ so as to express $\mathfrak{J}g(\mathbf{r}|\mathbf{r}'|k)$ as the simple sum of one longitudinal dyadic and one transverse dyadic.

It is not simple to set out closed expressions for the longitudinal and the transverse parts of the discontinuous dyadic $\mathfrak{J}\delta(\mathbf{r} - \mathbf{r}')$, but it is easy enough to define their integral properties and to determine their eigenvector expansions. We define the dyadic function $\mathfrak{D}_l(\mathbf{r} - \mathbf{r}')$ as being that function which, when operating on any vector field $\mathbf{F}(\mathbf{r}')$ yields, on integrating over the $\mathbf{r}'$ coordinates, just the *longitudinal part* of $\mathbf{F}(\mathbf{r})$. Similarly the function $\mathfrak{D}_t(\mathbf{r} - \mathbf{r}')$ operating on $\mathbf{F}(r')$, integrated over $\mathbf{r}'$, yields the *transverse* part of $\mathbf{F}$. If, for a given region and equation, we have obtained our set of longitudinal eigenvectors $\mathbf{L}_n(\mathbf{r})$ [see Eq. (13.1.14)] and our two sets of transverse eigenfunctions $\mathbf{M}(\mathbf{r})$ and $\mathbf{N}(\mathbf{r})$, then

$$\begin{aligned} \mathfrak{D}_l(\mathbf{r} - \mathbf{r}') &= \sum_n \frac{1}{\Lambda_n} \bar{\mathbf{L}}_n(\mathbf{r})\mathbf{L}_n(\mathbf{r}') \\ \text{and} \qquad \mathfrak{D}_t(\mathbf{r} - \mathbf{r}') &= \sum_n \left[\frac{1}{\Lambda'_n} \bar{\mathbf{M}}_n(\mathbf{r})\mathbf{M}_n(\mathbf{r}') + \frac{1}{\Lambda''_n} \bar{\mathbf{N}}_n(\mathbf{r})\mathbf{N}_n(\mathbf{r}')\right] \end{aligned} \tag{13.1.30}$$

where n now stands for the trio of quantum numbers defining the scalar eigenfunctions φ_n, ψ_n, and χ_n, and the different normalizing factors for $\mathbf{L}_n$, $\mathbf{M}_n$, and $\mathbf{N}_n$ are distinguished by the prime and double prime on the Λ's.

We can therefore define two Green's functions, a longitudinal $\mathfrak{G}_l$ and a transverse $\mathfrak{G}_t$, for the infinite domain, such that

$$\begin{aligned} \mathfrak{G}_l(\mathbf{r}|\mathbf{r}'|k) &= (1/k^2)[\nabla g(\mathbf{r}|\mathbf{r}'|k)\nabla' - 4\pi\mathfrak{D}_l(\mathbf{r} - \mathbf{r}')] \\ \mathfrak{G}_t(\mathbf{r}|\mathbf{r}'|k) &= (1/k^2)[-\nabla \times \mathfrak{J}g(\mathbf{r}|\mathbf{r}'|k) \times \nabla' \\ &\qquad - 4\pi\mathfrak{D}_t(\mathbf{r} - \mathbf{r}')] \\ \mathfrak{J}g(\mathbf{r}|\mathbf{r}'|k) &= \mathfrak{G}_l(\mathbf{r}|\mathbf{r}'|k) + \mathfrak{G}_t(\mathbf{r}|\mathbf{r}'|k) \\ \mathfrak{J}\delta(\mathbf{r} - \mathbf{r}') &= \mathfrak{D}_l(\mathbf{r} - \mathbf{r}') + \mathfrak{D}_t(\mathbf{r} - \mathbf{r}') \end{aligned} \tag{13.1.31}$$

where $g(\mathbf{r}|\mathbf{r}'|k)$ is the scalar Green's function for the infinite domain, e^{ikR}/R for the three-dimensional Helmholtz equation. Except at the point $\mathbf{r} = \mathbf{r}'$, $\mathfrak{G}_l$ is just equal to $\mathfrak{L}$ and $\mathfrak{G}_t$ is equal to $\mathfrak{T}$, and so they may be computed in closed form. For instance, for $g = e^{ikR}/R$

$$\begin{aligned} \mathfrak{L} &= \left[\mathfrak{J}\left(\frac{1 - ikR}{k^2R^2}\right) - \frac{\mathbf{RR}}{R^2}\left(\frac{3 - ikR - k^2R^2}{k^2R^2}\right)\right]\frac{e^{ikR}}{R} \\ \mathfrak{T} - \frac{4\pi}{k^2}\mathfrak{J}\delta(\mathbf{r} - \mathbf{r}') &= \left[-\mathfrak{J}\left(\frac{1 - ikR - k^2R^2}{k^2R^2}\right)\right. \\ &\qquad \left. + \frac{\mathbf{RR}}{R^2}\left(\frac{3 - ikR - k^2R^2}{k^2R^2}\right)\right]\frac{e^{ikR}}{R} \end{aligned} \tag{13.1.32}$$

But the expansions for $\mathfrak{G}_l$ and $\mathfrak{G}_t$ are more convergent than are those for $\mathfrak{L}$ and $\mathfrak{T}$ because the discontinuity at $\mathbf{R} = 0$, for the $\mathfrak{G}$'s, is much simpler than is the case for $\mathfrak{L}$ and $\mathfrak{T}$. For instance, for the eigenvector solutions $\mathbf{L}_n$, $\mathbf{M}_n$, and $\mathbf{N}_n$ for the infinite domain

$$\mathfrak{G}_l(\mathbf{r}|\mathbf{r}'|k) = 4\pi \sum_n \frac{\bar{\mathbf{L}}_n(\mathbf{r})\mathbf{L}_n(\mathbf{r}')}{\Lambda_n(k_n^2 - k^2)}$$
$$\mathfrak{G}_t(\mathbf{r}|\mathbf{r}'|k) = 4\pi \sum_n \left\{ \frac{\bar{\mathbf{M}}_n(\mathbf{r})\mathbf{M}_n(\mathbf{r}')}{\Lambda_n'(k_n^2 - k^2)} + \frac{\bar{\mathbf{N}}_n(\mathbf{r})\mathbf{N}_n(\mathbf{r}')}{\Lambda_n''(k_n^2 - k^2)} \right\} \tag{13.1.33}$$

Thus we have completely separated the Green's function into its longitudinal and transverse parts, each of which has a reasonably convergent expansion. Expansions of the type of Eqs. (7.2.51) and (7.2.63), for external problems, may also be obtained, as will be shown later.

It is important to calculate what sort of inhomogeneous equations these two functions satisfy, so we can utilize Green's theorem to calculate longitudinal and transverse wave solutions. Since curl $\mathfrak{G}_l = 0$, we have

$$(\text{grad div} + k^2)\mathfrak{G}_l(\mathbf{r}|\mathbf{r}'|k) = \frac{1}{k^2} \sum_{m,n} \mathbf{e}_m\mathbf{e}_n \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_n'} (\nabla^2 + k^2)g$$
$$- \frac{4\pi}{k^2} \text{grad div}[\mathfrak{J}\delta(\mathbf{r} - \mathbf{r}')] - 4\pi\mathfrak{D}_l(\mathbf{r} - \mathbf{r}') = -4\pi\mathfrak{D}_l(\mathbf{r} - \mathbf{r}')$$

and, by subtraction from the equation for $\mathfrak{G}$, (13.1.24), since div $\mathfrak{G}_t = 0$,

$$(-\text{ curl curl} + k^2)\mathfrak{G}_t(\mathbf{r}|\mathbf{r}'|k) = -4\pi\mathfrak{D}_t(\mathbf{r} - \mathbf{r}') \tag{13.1.34}$$

Consequently, a solution $\mathbf{F}_l$ of the equation

$$\text{grad div } \mathbf{F}_l + k^2\mathbf{F}_l = -4\pi\mathbf{Q}_l$$

where $\mathbf{Q}_l$ is a longitudinal field, will be longitudinal. From Eq. (13.1.7), it is

$$\mathbf{F}_l(\mathbf{r}) = \iiint \mathfrak{G}_l(\mathbf{r}|\mathbf{r}'|k) \cdot \mathbf{Q}_l(\mathbf{r}')\, dv'$$
$$+ (1/4\pi) \oint [(\text{div}' \mathbf{F})(\mathfrak{G}_l \cdot \mathbf{n}) - (\text{div}' \mathfrak{G}_l)(\mathbf{F} \cdot \mathbf{n})]\, dA' \tag{13.1.35}$$

Contrariwise, a solution $\mathbf{F}_t$ of the equation

$$-\text{ curl curl } \mathbf{F}_t + k^2\mathbf{F}_t = -4\pi\mathbf{Q}_t$$

where $\mathbf{Q}_t$ has zero divergence, is a transverse vector. It is

$$\mathbf{F}_t(\mathbf{r}) = \iiint \mathfrak{G}_t(\mathbf{r}|\mathbf{r}'|k) \cdot \mathbf{Q}_t(\mathbf{r}')\, dv'$$
$$- (1/4\pi) \oint [\mathfrak{G}_t \cdot (\mathbf{n} \times \text{curl}' \mathbf{F}) + (\text{curl}' \mathfrak{G}_t) \cdot (\mathbf{n} \times \mathbf{F})]\, dA' \tag{13.1.36}$$

Both of these formulas hold both within and on the boundary S.

If $\mathbf{Q}$ is *not* transverse in the formula

$$-\text{ curl curl } \mathbf{F} + k^2\mathbf{F} = -4\pi\mathbf{Q}$$

then $\mathbf{F}$ is not all transverse. We use as a Green's function

$$\mathfrak{G}_c = [\mathfrak{G}_t - (4\pi/k^2)\mathfrak{D}_l] = \mathfrak{J}g(\mathbf{r}|\mathbf{r}'|k) - \mathfrak{L}(\mathbf{r}|\mathbf{r}'|k)$$

in which case we can quickly show that the transverse part of the function $\mathbf{F}$ is given by Eq. (13.1.36), using $\mathfrak{G}_c$ instead of $\mathfrak{G}_t$ and using $\mathbf{Q}_t$, the transverse part of $\mathbf{Q}$. The longitudinal part of $\mathbf{F}$ is simply

$$\mathbf{F}_l = -(4\pi/k^2)\mathbf{Q}_l \tag{13.1.37}$$

where $\mathbf{Q}_l$ is the longitudinal part of $\mathbf{Q}$. This last may be checked by direct substitution in the equation for $\mathbf{F}$. In the present case, the formula for $\mathbf{F}_t$ in terms of $\mathfrak{G}_c$ is valid within the boundary but not on it, because of the additional delta function in the surface integral.

Green's Dyadic for Elastic Waves. Finally, the equation for elastic waves in an isotropic medium may be written

$$c_l^2 \operatorname{grad} \operatorname{div} \mathbf{F} - c_t^2 \operatorname{curl} \operatorname{curl} \mathbf{F} + \omega^2\mathbf{F} = -4\pi\mathbf{Q} \tag{13.1.38}$$

where $c_l = \sqrt{(\lambda + 2\mu)/\rho}$ is the velocity of longitudinal waves, $c_t = \sqrt{\mu/\rho}$ the velocity of transverse waves, ρ is the density of the medium, $\mathbf{Q} = \mathbf{f}/\rho$ where $\mathbf{f}$ is the body force per unit volume, and $\omega/2\pi$ is the driving frequency. The Green's function here is

$$\mathfrak{G}_e(\mathbf{r}|\mathbf{r}'|\omega) = \frac{1}{c_l^2}\,\mathfrak{G}_l\left(\mathbf{r}|\mathbf{r}'|\,\frac{\omega}{c_l}\right) + \frac{1}{c_t^2}\,\mathfrak{G}_t\left(\mathbf{r}|\mathbf{r}'|\,\frac{\omega}{c_t}\right) \tag{13.1.39}$$

where $\mathfrak{G}_l$ and $\mathfrak{G}_t$ are the dyadics defined in Eq. (13.1.31). We note that the two parts of the solution have different basic scalar Green's functions g, corresponding to the different velocities of propagation of the two types of wave. They add together at $R \to 0$ but become out of phase for R large. A solution of Eq. (13.1.38) is then

$$\begin{aligned}\mathbf{F}(\mathbf{r}) = \iiint &\mathfrak{G}_e(\mathbf{r}|\mathbf{r}'\omega) \cdot \mathbf{Q}(\mathbf{r}')\,dv' \\ &+ (1/4\pi)c_l^2 \oint [(\mathfrak{G}_e \cdot \mathbf{n})(\operatorname{div}' \mathbf{F}) - (\operatorname{div}' \mathfrak{G}_e)(\mathbf{F} \cdot \mathbf{n})]\,dA' \\ - (1/4\pi)c_t^2 &\oint [\mathfrak{G}_e \cdot (\mathbf{n} \times \operatorname{curl}' \mathbf{F}) + (\operatorname{curl}' \mathfrak{G}_e) \cdot (\mathbf{n} \times \mathbf{F})]\,dA'\end{aligned} \tag{13.1.40}$$

where, of course, $\operatorname{div} \mathfrak{G}_e = (1/c_l^2) \operatorname{div} \mathfrak{G}_l$ and $\operatorname{curl} \mathfrak{G}_e = (1/c_t^2) \operatorname{curl} \mathfrak{G}_t$, the values being taken for $\mathbf{r}'$ on the boundary S.

It should be emphasized again that solutions of the inhomogeneous Helmholtz equation do not need the complications of this subsection to arrive at a solution; the simple Green's dyadic, such as the one given in Eq. (13.1.25), is sufficient. If the inhomogeneous source field $\mathbf{Q}$ is transverse, the solution [given in Eqs. (13.1.10), (13.1.11), (13.1.12), or (13.1.13)] will automatically be transverse; if $\mathbf{Q}$ is longitudinal, the solution will automatically be longitudinal. It is only when we have to solve the equation

$$a \operatorname{grad} \operatorname{div} \mathbf{F} - b \operatorname{curl} \operatorname{curl} \mathbf{F} + k^2\mathbf{F} = -4\pi\mathbf{Q}$$

with $a \neq b$, that we have to separate the Green's function into longitudinal and transverse parts, as worked out in the present subsection.

Finally, it is instructive to note the behavior of the dyadics $\mathfrak{G}_l$ and $\mathfrak{G}_t$ for kR large and kR small but not zero. We display this by noting the form of the vector field resulting when each $\mathfrak{G}$ operates on the unit vector $\mathbf{i} = \mathbf{e}_1$ (the results for operating on $\mathbf{j}$ or $\mathbf{k}$ are quite similar, since the dyadics have symmetry about $R = 0$). For comparison we also include the results for the simple dyadic $\mathfrak{G} = \mathfrak{I}g$. For $kR \to \infty$, we have

$$\begin{aligned} &\mathfrak{G} \cdot \mathbf{i} \to \mathbf{i}(e^{ikR}/R); \quad \mathfrak{G}_l \cdot \mathbf{i} \to \mathbf{a}_R \cos \vartheta (e^{ikR}/R) \\ &\mathfrak{G}_t \cdot \mathbf{i} \to (\mathbf{i} - \mathbf{a}_R \cos \vartheta)(e^{ikR}/R) = \mathbf{a}_\vartheta \sin \vartheta (e^{ikR}/R) \end{aligned} \tag{13.1.41}$$

where $\mathbf{a}_R$ is the unit vector pointing from the source point $\mathbf{r}'$ to the observation point $\mathbf{r}$, ϑ is the angle between $\mathbf{a}_R$ and the x axis ($\cos \vartheta = \mathbf{i} \cdot \mathbf{a}_R$), and $\mathbf{a}_\vartheta$ is a unit vector perpendicular to $\mathbf{a}_R$, coplanar with $\mathbf{i}$ and $\mathbf{a}_R$, which points toward the x axis rather than away from it. These formulas display the fact that $\mathfrak{G} = \mathfrak{G}_l + \mathfrak{G}_t$ and also show why $\mathfrak{G}_l$ is called longitudinal and $\mathfrak{G}_t$ is called transverse.

For $1 \gg kR > 0$, we have

$$\begin{aligned} &\mathfrak{G} \cdot \mathbf{i} \simeq \mathbf{i}/R \\ &\mathfrak{G}_l \cdot \mathbf{i} \simeq -\mathfrak{G}_t \cdot \mathbf{i} \simeq (\mathbf{a}_\vartheta \sin \vartheta - 2\mathbf{a}_R \cos \vartheta)(1/k^2R^3) \\ &\qquad = -[2\mathbf{i}P_2(\cos \vartheta) - P_2^1(\cos \vartheta)(\mathbf{j} \cos \varphi + \mathbf{k} \sin \varphi)](1/k^2R^3) \end{aligned}$$

where φ is the angle between the plane of $(\mathbf{i},\mathbf{a}_R)$ and the (x,y) plane. We note that for this range of values of kR, $\mathfrak{G}_l$ and $\mathfrak{G}_t$ are very much larger than $\mathfrak{G}$ but are opposite in sign, so they nearly cancel out, leaving only $\mathfrak{G}$. The first form for $\mathfrak{G}_l \cdot \mathbf{i}$ shows that for $\mathbf{a}_R$ along the x axis the vector points in the negative x direction, whereas for $\mathbf{a}_R$ perpendicular to the x axis it points in the positive x direction; for intermediate directions it is always in the $(\mathbf{i},\mathbf{a}_R)$ plane, at an angle $[\pi + \cot^{-1}(\frac{1}{3} \csc 2\vartheta + \cot 2\vartheta)]$ to the x axis. The second form for $\mathfrak{G}_l \cdot \mathbf{i}$ shows that these very large values of the field, for kR small, integrate to zero when integrated over all directions (multiply by $\sin \vartheta \, d\vartheta \, d\varphi$ and integrate over ϑ and φ, keeping kR constant). The motion of the medium near $\mathbf{r} = \mathbf{r}'$ is quite similar to a "smoke ring" or ring vortex, with the axis of the ring the x axis (or whatever the direction of the constant vector on which $\mathfrak{G}$ operates).

Solutions of the Vector Laplace Equation. In the limit $k \to 0$, some properties of the solutions diverge or go to zero. The solutions $\mathbf{i}U + \mathbf{j}V + \mathbf{k}W$ are still proper solutions and, if boundary conditions allow their use, the solution of the vector equation becomes just the solution of three scalar equations of the sort discussed in Chap. 10. But when we try to separate our solutions into a longitudinal set and two transverse sets [as in Eq. (13.1.6)], we find ourselves in trouble. The difficulty seems to be in the longitudinal solution, though at first sight the second transverse solution $\mathbf{N}$ seems to be the culprit because of the $1/k$ factor. But if we multiply $\mathbf{N}$ by k, to keep it finite when we let $k \to 0$, we then see that $k\mathbf{N}$ approaches a *longitudinal* vector field as k vanishes, for it becomes $\operatorname{grad}[\partial(w\chi)/\partial\xi_1]$, where χ is a solution of the Laplace equation.

Further examination shows that, in the limit $k \to 0$, the set of solutions $\mathbf{N}$ become linear combinations of the set of solutions $\mathbf{L}$, of Eq. (13.1.6). They are no longer an independent set, and our separation into longitudinal and transverse solutions has lost us one needed set of solutions. Looking at the solutions $\mathbf{L}$ we see that, in the limit $k \to 0$, the set $\mathbf{L}$ (and thus $\mathbf{N}$, for the two sets become the same) is *both* transverse *and* longitudinal or *neither* transverse *nor* longitudinal, depending on the point of view. For, in the limit, φ is a solution of the scalar Laplace equation, so that $\operatorname{div}(\operatorname{grad} \varphi) = \operatorname{div} \mathbf{L} = 0$; $\mathbf{L}$ has *both* zero divergence *and* zero curl (and so does $\mathbf{N}$).

We, therefore, have two sets of solutions, $\mathbf{N}$ (or $\mathbf{L}$) and $\mathbf{M}$, one of which, $\mathbf{M}$, has zero divergence and nonzero curl and the other of which, $\mathbf{N}$, has zero divergence and zero curl. If we are searching for transverse solutions only, for use as vector potentials, for example, this will be satisfactory; only the set $\mathbf{M}$ will be involved in the final answer when we take the curl of the vector potential, but the set $\mathbf{N}$ may be needed to satisfy the boundary conditions.

Our difficulties enter when we wish to find a true longitudinal solution of the *vector* Laplace equation, *i.e.*, a vector field $\mathbf{F}$ having zero curl but *nonzero* divergence, which satisfies the equation $\operatorname{grad} \operatorname{div} \mathbf{F} = 0$. If it is to have zero curl, $\mathbf{F}$ should be the gradient of some scalar, χ, and $\operatorname{grad} \operatorname{div} \operatorname{grad} \chi = \operatorname{grad}(\nabla^2\chi) = 0$. The dilemma now becomes more clear: if we take χ to be the solution of $\nabla^2\chi = 0$, then $\mathbf{F} = \operatorname{grad} \chi$ will have zero divergence; one solution is possible, $\nabla^2\chi = \text{constant}$, but this does not serve to generate a complete set of longitudinal fields.

However, the vector field, $\mathbf{a}_1\chi$, where χ is a solution of $\nabla^2\chi = 0$ and where $\mathbf{a}_1$ is the unit vector along the x, y, or z coordinate, is a solution of the vector Laplace equation which, in general, does have a nonzero divergence. This solution is not a longitudinal one for, in general, its curl is not zero either. We might expect to separate off the longitudinal part of $\mathbf{a}_1\chi$ by the use of Eq. (1.5.16),

$$\mathbf{F}_l = \operatorname{grad}\left\{ \iiint \frac{(\partial\chi/\partial\xi_1)}{R}\, dv' \right\}$$

An examination of various possible solutions χ of the scalar Laplace equation shows that, whenever the volume integral inside the braces is not zero or a constant, it *diverges*. Therefore, we cannot find a purely longitudinal solution of the vector Laplace equation having a nonzero divergence. If we must consider solutions with nonzero divergence, we must use either the set $\mathbf{i}U + \mathbf{j}V + \mathbf{k}W$ or the mixed set $\mathbf{a}_1\chi + \mathbf{M} + \mathbf{N}$.

The problem is still more embarrassing in the case of the static displacement of an isotropic elastic medium, for here the form $\mathbf{a}_1\chi$ is not usually a possible solution. In addition the transverse parts of the solution, **M** and **N**, are not particularly important in the static elastic case compared to the longitudinal part; they represent the unavoidable twisting of the medium in order to have the displacements required by the boundary conditions, but it is the longitudinal part which plays the major role with respect to internal strains and body forces, such as gravity.

The equation to be solved is

$$\nabla \cdot \mathfrak{T} = (\lambda + 2\mu) \text{ grad div } \mathbf{s} - \mu \text{ curl curl } \mathbf{s} \\ = (\lambda + \mu) \text{ grad div } \mathbf{s} + \mu \nabla^2 \mathbf{s} = -\mathbf{F}$$

where **s** is the displacement of the medium, $\mathfrak{T}$ is the stress dyadic, and **F** is the body force. What this states, in the absence of a $k^2\mathbf{s}$ term, is that the divergence of $\mathfrak{T}$, if it is not zero, is a longitudinal vector, for in cases of practical interest **F**, being the gradient of a potential energy ($\mathbf{F} = -\text{ grad } V$), is longitudinal. Consequently, the contribution of the transverse parts of **s** to the divergence of $\mathfrak{T}$ and thus to the balancing of the body forces is precisely zero. We must have curl curl of the **M** and **N** parts of **s** zero no matter what **F** is (as long as **F** is a conservative field).

However, the proper equation for the longitudinal part of **s** may be obtained from the equation above. We set $\mathbf{s} = \mathbf{L} = \text{grad } \varphi$ in, to obtain

$$(\lambda + \mu) \text{ grad}(\nabla^2\varphi) + \mu\nabla^2(\text{grad } \varphi) = (\lambda + 2\mu) \text{ grad}(\nabla^2\varphi) = \text{grad } V \tag{13.1.42}$$

and, if we wish to make sure that the transverse solutions do not obtrude, we take the divergence of this, obtaining

$$\nabla^4\varphi = \text{div grad(div grad } \varphi) = [1/(\lambda + 2\mu)]\nabla^2 V \tag{13.1.43}$$

where, if the body-force potential is a solution of Laplace's equation, we have that φ must be a solution of the *biharmonic equation* $\nabla^4\varphi = 0$. This, then, produces the longitudinal part of the displacement. We have, of course, to adjust our solution of Eq. (13.1.43) to ensure that Eq. (13.1.42) is satisfied, and then enough of the transverse solutions is added to satisfy boundary conditions.

Actually, in many cases the displacement is not the most useful function to solve for first. Because of the presence of the relatively unimportant parts **M** and **N**, the solution gets cluttered up with nonessentials. In addition many boundary conditions of practical interest are given in terms of surface stresses, rather than in terms of displacements. Consequently, it would often be advisable, in the static case, to consider stress to be the primary field, rather than strain. Since the stress dyadic $\mathfrak{T}$ is symmetric, it can be expressed in terms of symmetric differential operations on a scalar *stress function* Ω;

$$\mathfrak{T} = \nabla\Omega\nabla; \quad \nabla^4\Omega = \nabla^2 V$$

with Ω adjusted so that $\nabla \cdot \mathfrak{T} = \text{grad}(\nabla^2\Omega) = \text{grad } V$. (The three equations for the three components of this equation are often called the *equations of compatibility*.) The stress function Ω is thus a solution of the same equation as for φ. The scalar Ω and the displacement $\mathbf{s}$ are related by the equation

$$\nabla\Omega\nabla = \lambda\mathfrak{J} \text{ div } \mathbf{s} + \mu(\nabla\mathbf{s} + \mathbf{s}\nabla)$$

If Ω is known, then $\mathbf{s}$ may be determined by integrating this equation.

The same difficulties enter when we wish to use Green's function techniques to calculate our solution. As long as we do not have to separate out longitudinal and transverse parts, there is little trouble. We can use Eq. (13.1.10), with

$$\mathfrak{G}(\mathbf{r}|\mathbf{r}_0) = \mathfrak{J}/R \tag{13.1.44}$$

for the Green's dyadic for infinite space and no infinities obtrude. But when we have to separate the effects of longitudinal and transverse parts, as we do for example in the elastic solid case, we have troubles. The equations for $\mathfrak{G}$ are

$$\begin{aligned} \text{grad div } \mathfrak{G} &= -(\mathfrak{J}/R^3) + 3(\mathbf{RR}/R^5) \\ \text{curl curl } \mathfrak{G} &= 4\pi\mathfrak{J}\delta + \text{grad div } \mathfrak{G} \end{aligned}$$

so that $\mathfrak{G}$ is not a solution of the static elastic equation

$$(\lambda + 2\mu) \text{ grad div } \mathfrak{G}_e - \text{curl curl } \mathfrak{G}_e = -4\pi\mathfrak{J}\delta(\mathbf{r} - \mathbf{r}') \tag{13.1.45}$$

Although we cannot always break a vector solution of Laplace's equation into finite longitudinal and transverse parts, it turns out we can break up the dyadic $\mathfrak{G}$. We use the formula

$$\nabla \times \mathfrak{J}f(R) \times \nabla' = -\mathfrak{J}\nabla^2 f - \nabla f \nabla'$$

and adjust f so that $\nabla^2 f = 1/R$. For the infinite domain, this requires that $f(R) = \frac{1}{2}R$. Consequently, we define

$$\begin{aligned} \mathfrak{E} = \tfrac{1}{2}\nabla R\nabla' &= -\mathfrak{G}(\mathbf{r}|\mathbf{r}') - \tfrac{1}{2}\nabla \times \mathfrak{J}R \times \nabla' \\ &= -\tfrac{1}{2}\mathfrak{G} + \tfrac{1}{2}(\mathbf{RR}/R^3) \end{aligned} \tag{13.1.46}$$
$$\text{grad div } \mathfrak{E} = -\text{ grad div } \mathfrak{G}$$

We can then see that the correct solution of Eq. (13.1.45) is a linear combination of $\mathfrak{G}$ and $\mathfrak{E}$. Inserting the combination into Eq. (13.1.45) and using Eq. (13.1.46) to determine the coefficients, we find that, for the infinite domain,

$$\mathfrak{G}_e(\mathbf{r}|\mathbf{r}') = \frac{1}{\mu}\mathfrak{G} + \frac{\lambda + \mu}{\mu(\lambda + 2\mu)}\mathfrak{E} = \frac{\lambda + 3\mu}{2\mu(\lambda + 2\mu)}\frac{\mathfrak{J}}{R} + \frac{\lambda + \mu}{2\mu(\lambda + 2\mu)}\frac{\mathbf{RR}}{R^3} \tag{13.1.47}$$

and the solution of the equation

$$(\lambda + 2\mu) \text{ grad div } \mathbf{F} - \mu \text{ curl curl } \mathbf{F} = -4\pi\mathbf{Q}$$

is

$$\begin{aligned}\mathbf{F}(\mathbf{r}) = &\iiint \mathfrak{G}_e(\mathbf{r}|\mathbf{r}') \cdot \mathbf{Q}(\mathbf{r}')\, dv' \\ &+ (1/4\pi)(\lambda + 2\mu) \oint [(\text{div}' \mathbf{F})(\mathfrak{G}_e \cdot \mathbf{n}) - (\text{div}' \mathfrak{G}_e)(\mathbf{F} \cdot \mathbf{n})]\, dA' \\ &- (1/4\pi)\mu \oint [\mathfrak{G}_e \cdot (\mathbf{n} \times \text{curl}' \mathbf{F}) + (\text{curl}' \mathfrak{G}_e) \cdot (\mathbf{n} \times \mathbf{F})]\, dA' \end{aligned} \quad (13.1.48)$$

within and on the surface integrated over in the last two integrals. For finite boundaries, the function $\mathfrak{G}_e$ is modified by using a solution of the homogeneous equation, which is analytic inside the boundary, in order to have the modified Green's dyadic fit one of the four homogeneous boundary conditions discussed earlier in this section. The technique was discussed in Chap. 7 and will be illustrated again, later in this chapter. As we shall see later, it will sometimes be better to calculate the stress field rather than the displacement.

One final case of interest is the solution of the inhomogeneous equation

$$\text{curl curl } \mathbf{F} = 4\pi\mathbf{Q} \quad (13.1.49)$$

where $\mathbf{Q}$ is a transverse vector. This is the case with the electromagnetic field, for instance, when there is no free charge q and the current density $\mathbf{J}$ does not change with time (in order for this to hold div $\mathbf{J} = 0$). Then if we set $\mathbf{Q} = \mathbf{J}/c$ and $\mathbf{F} = \mathbf{A}$, the vector potential, the magnetic intensity $\mathbf{H} = (1/\mu)$ curl $\mathbf{A}$, then $\mathbf{A}$ is related to $\mathbf{J}$ as given in Eq. (13.1.49).

Since we are here interested only in the transverse parts of the vectors $\mathbf{A}$ and $\mathbf{J}$ and, in particular, with the curl of $\mathbf{A}$, it would seem that we might as well calculate the vector (curl $\mathbf{F}$) directly; this will remove the troublesome longitudinal parts and simplify the calculation. We use the Green's function $\mathfrak{C}(\mathbf{r}|\mathbf{r}') = \text{curl } \mathfrak{G}(\mathbf{r}|\mathbf{r}')$ for our Green's function, rather than $\mathfrak{G}$ itself. This has most of the properties of a Green's function, except that it changes sign when $\mathbf{r}$ is interchanged with $\mathbf{r}'$. It must satisfy the equation

$$\text{curl curl } \mathfrak{C} = 4\pi \text{ curl}[\mathfrak{J}\delta(\mathbf{r} - \mathbf{r}')] \quad (13.1.50)$$

Combining this with Eq. (13.1.49), we have

$$\begin{aligned}\iiint [(\text{curl}' \text{ curl}' \mathfrak{C}) \cdot \mathbf{F} - \mathfrak{C} \cdot \text{curl}' \text{ curl}' \mathbf{F}]\, dv' \\ = -4\pi \iiint \mathbf{F} \cdot \text{curl}'[\mathfrak{J}\delta]\, dv' - 4\pi \iiint \mathfrak{C} \cdot \mathbf{Q}\, dv' \\ = \oint [\mathfrak{C} \cdot (\mathbf{n} \times \text{curl}' \mathbf{F}) + \text{curl}' \mathfrak{C} \cdot (\mathbf{n} \times \mathbf{F})]\, dA'\end{aligned}$$

But, since

$$\begin{aligned}\iiint \text{div}'(\mathbf{F} \times \mathfrak{J}\delta)\, dv' &= \oint (\mathbf{F} \times \mathbf{n})\delta(\mathbf{r} - \mathbf{r}')\, dA' \\ &= \iiint [\delta\mathfrak{J} \cdot (\text{curl}' \mathbf{F}) - \mathbf{F} \cdot \text{curl}'(\mathfrak{J}\delta)]\, dv'\end{aligned}$$

or $$-\iiint \mathbf{F} \cdot \text{curl}'(\mathfrak{J}\delta)\, dv' = -\text{ curl } \mathbf{F}(\mathbf{r}) + \begin{cases} 0; & \mathbf{r} \text{ inside } S \\ (\mathbf{F} \times \mathbf{n}); & \mathbf{r} \text{ on } S \end{cases}$$

so that, finally,

$$\operatorname{curl}\mathbf{F}(\mathbf{r}) = -\iiint \mathfrak{C}\cdot\mathbf{Q}\,dv' - \frac{1}{4\pi}\oint [\mathfrak{C}\cdot(\mathbf{n}\times\operatorname{curl}'\mathbf{F}) + \operatorname{curl}'\mathfrak{C}\cdot(\mathbf{n}\times\mathbf{F})]\,dA' \tag{13.1.51}$$

However, since $\mathfrak{C}$ is the curl of the usual Green's dyadic $\mathfrak{G}$, solution of $\nabla^2\mathfrak{G} = -4\pi\mathfrak{I}\delta(\mathbf{r}-\mathbf{r}')$, we can integrate Eq. (13.1.51) to obtain an expression for the vector potential in terms of the usual dyadic $\mathfrak{G}$. The result may have in it an arbitrary amount of a longitudinal field, but this field has no effect when we take the curl of $\mathbf{F}$ to find the magnetic intensity (or the fluid velocity, when we are computing viscous flow in an incompressible fluid). Thus

$$\mathbf{F} = \iiint \mathfrak{G}\cdot\mathbf{Q}\,dv' - \frac{1}{4\pi}\oint [\mathfrak{G}\cdot(\mathbf{n}\times\operatorname{curl}'\mathbf{F}) + (\operatorname{curl}'\mathfrak{G})\cdot(\mathbf{n}\times\mathbf{F})]\,dA' \tag{13.1.52}$$

is a satisfactory expression for computing a vector potential from which we take only the curl.

Green's Functions for the Wave Equation. For the solutions of the wave equation, we can either solve directly, by the methods used to compute Eqs. (7.3.5) and (7.3.8), or we can use the methods of the Laplace transform. For instance, to transform from the solutions of the Helmholtz equation, for a given frequency $\omega = ip$, to the solutions for an impulsive force at $t = t_0$, we multiply by $e^{-i\omega(t-t_0)}(d\omega/2\pi)$ and integrate from $-\infty + i\epsilon$ to $+\infty + i\epsilon$. Or else we look up, in the table of Laplace transforms, at the end of Chap. 11, the function $f(t)$ which, when multiplied by e^{-pt} and integrated over t from 0 to ∞, gives the correct Helmholtz solution, as function of $p = -i\omega$.

Suppose we wish the Green's dyadic for a unit pulse of force at $\mathbf{r} = \mathbf{r}_0$. For infinite space, the dyadic must be a solution of

$$\nabla^2\mathfrak{G} - (\partial^2/\partial t^2)\mathfrak{G} = -4\pi\delta(\mathbf{r}-\mathbf{r}_0)\delta(t-t_0)\mathfrak{I}$$

Since the right-hand side of this equation is the Laplace transform of $-4\pi\mathfrak{I}\delta(\mathbf{r}-\mathbf{r}_0)$, the Green's dyadic required must be the Laplace transform of the simple dyadic given in Eq. (13.1.25), for the infinite domain,

$$\mathfrak{G}_0(\mathbf{r},t|\mathbf{r}_0,t_0) = (\mathfrak{I}/R)\delta[t - t_0 - (R/c)] \tag{13.1.53}$$

If boundaries are at finite distances and the Green's function for the Helmholtz equation is $\mathfrak{G}_0(\mathbf{r}|\mathbf{r}_0|ip/c) + \mathfrak{F}$, where $\mathfrak{F}$ is some dyadic which fits the boundary conditions and is analytic inside the boundary, then the corresponding time-dependent Green's function is

$$\mathfrak{G}(\mathbf{r},t|\mathbf{r}_0,t_0) = \mathfrak{G}_0(\mathbf{r},t|\mathbf{r}_0,t_0) + \mathfrak{F}_0(\mathbf{r},t|\mathbf{r}_0,t_0)$$

where
$$\int_0^\infty e^{-pt}\,\mathfrak{F}_0(\mathbf{r},t|\mathbf{r}_0,0)\,dt = \mathfrak{F}\left(\mathbf{r}|\mathbf{r}_0|\,\frac{ip}{c}\right)$$

$\mathfrak{F}$ being the Laplace transform of $\mathfrak{F}_0$.

These functions are sufficient for computing the corresponding formulas to Eqs. (13.1.10) to (13.1.12). The expression, analogous to Eq. (7.3.5), is

$$\begin{aligned}
\mathbf{F}(\mathbf{r},t) = \int_0^{t+} dt_0 \iiint dv_0\, \mathfrak{G}(\mathbf{r},t|\mathbf{r}_0,t_0)\cdot\mathbf{Q}(r_0t_0) \\
+ \frac{1}{4\pi}\int_0^{t+} dt_0 \oint [(\mathrm{div}_0\,\mathbf{F})(\mathfrak{G}\cdot\mathbf{n}) - (\mathrm{div}_0\,\mathfrak{G})(\mathbf{F}\cdot\mathbf{n}) \\
- \mathfrak{G}\cdot(\mathbf{n}\times\mathrm{curl}_0\,\mathbf{F}) - (\mathrm{curl}_0\,\mathfrak{G})\cdot(\mathbf{n}\times F)]\,dA_0 \\
- \frac{1}{c^2}\iiint \left[\left(\frac{\partial}{\partial t_0}\mathfrak{G}\right)\cdot\mathbf{F} - \mathfrak{G}\cdot\left(\frac{\partial}{\partial t_0}\mathbf{F}\right)\right]_{t_0=0} dv_0
\end{aligned} \tag{13.1.54}$$

where the last integral is one over the initial values and initial rates of change of $\mathbf{F}$ inside the boundary.

When we have to deal with longitudinal or transverse waves, we have to take the Laplace transform of the two Green's functions defined in Eq. (13.1.31):

$$\begin{aligned}
\mathfrak{G}_l(\mathbf{r},t|\mathbf{r}_0,t_0|c) &= -\frac{\mathfrak{I}}{R}\left[\left(\frac{c}{R}\right)^2 v(z) + \frac{c}{R}u(z)\right] \\
&\quad + \frac{\mathbf{RR}}{R^3}\left[3\frac{c^2}{R}v(z) + \frac{c}{R}u(z) + \delta(z)\right] - 4\pi\mathfrak{D}_l(\mathbf{r}-\mathbf{r}_0)v(z) \\
\mathfrak{G}_t(\mathbf{r},t|\mathbf{r}_0,t_0|c) &= \frac{\mathfrak{I}}{R}\left[\left(\frac{c}{R}\right)^2 v(z) + \frac{c}{R}u(z) + \delta(z)\right] \\
&\quad - \frac{\mathbf{RR}}{R^3}\left[3\left(\frac{c}{R}\right)^2 v(z) + \frac{c}{R}u(z) + \delta(z)\right] + 4\pi\mathfrak{D}_t(\mathbf{r}-\mathbf{r}_0)v(z)
\end{aligned} \tag{13.1.55}$$

where $z = t - t_0 - (R/c)$ and where $u(z)$ and $v(z)$ are successive integrals of the delta function

$$u(z) = \int_{-\infty}^{z}\delta(x)\,dx = \begin{cases}0; & z<0\\ 1; & z>0\end{cases};\quad v(z) = \int_{-\infty}^{z}u(x)\,dx = \begin{cases}0; & z<0\\ z; & z>0\end{cases}$$

At large distances only the pulse survives, the transverse and longitudinal waves being at right angles to each other, the sum of the two equaling $(\mathfrak{I}/R)\delta(z)$. At short distances ($R \ll ct$) the ring vortex mentioned on page 1784 builds up steadily after $t = 0$, according to the function $v[t - t_0 - (R/c)]$.

Using these pulsed waves, transverse and longitudinal, one can extend formulas (13.1.35), (13.1.36) and (13.1.40) to the time variable case, by integrating the volume and surface integrals of these formulas over t_0 from 0 to $t + \epsilon$ and by adding an additional volume integral of initial value and time rate of change of $\mathbf{F}$, analogous to the third integral in Eq. (13.1.54). The dyadic for the isotropic elastic medium, for example, is a combination of a longitudinal dyadic with wave velocity $c_l = \sqrt{(\lambda + 2\mu)/\rho}$ and a transverse dyadic with wave velocity $c_t = \sqrt{\mu/\rho}$, analogous to the combination of Eq. (13.1.39). Applications of many of these formulas will be given in succeeding sections.

13.2 *Static and Steady-state Solutions*

Static and steady-state vector fields occur in three cases of physical interest: the magnetic field caused by steady currents, the static strains in an elastic medium, and the steady-state flow of a viscous fluid. The solutions and boundary conditions are somewhat different for each of the three cases. Other cases of physical interest, such as the static electric field or the flow of a nonviscous fluid, may be obtained from a scalar field by use of the techniques discussed in Chap. 10, so it would be redundant to discuss them again here. Our main interest in this chapter will be in vector fields which *cannot* be expressed in terms of a single scalar potential.

In the case of the magnetic field $\mathbf{H}$, we compute the vector potential $\mathbf{A}$, with $\mathbf{H} = \operatorname{curl} \mathbf{A}$. In the presence of a current density $\mathbf{J}$ we have

$$\operatorname{curl}\operatorname{curl} \mathbf{A} = 4\pi\mu\mathbf{J}/c \tag{13.2.1}$$

with the permeability μ usually unity except for ferromagnetic material. To the first approximation iron may be considered to have infinite permeability, in which case $\mathbf{H}$ would have to be normal to the surface of the iron. Consequently, we must have the normal component of $\mathbf{A}$ be zero at the surface of the iron and the normal gradient of the tangential component of $\mathbf{A}$ must also be zero there.

Since, for a steady state, $\operatorname{div} \mathbf{J}$ must be zero, we can just as well let $\mathbf{A}$ be the solution of the vector Poisson equation

$$\nabla^2\mathbf{A} = -(4\pi\mu\mathbf{J}/c) \tag{13.2.2}$$

where it is occasionally useful to include a longitudinal part for $\mathbf{A}$ for ease of computation. The presence or absence of this longitudinal part will not affect the distribution of the magnetic field, which depends only on the transverse part of $\mathbf{A}$.

The displacement $\mathbf{s}$ of an isotropic elastic medium under the action of a body force $4\pi\mathbf{F}$ dynes per cc is

$$\begin{aligned}(\lambda + 2\mu) \operatorname{grad}\operatorname{div} \mathbf{s} - \mu \operatorname{curl}\operatorname{curl} \mathbf{s} \\ = (\lambda + \mu) \operatorname{grad}\operatorname{div} \mathbf{s} + \mu\nabla^2\mathbf{s} = -4\pi\mathbf{F}\end{aligned} \tag{13.2.3}$$

As mentioned earlier, only in two-dimensional polar coordinates and in spherical coordinates can separation of this equation be achieved, so that boundary conditions can be satisfied exactly; in other systems only approximate solutions have been obtained. Here also, whenever possible, it is more convenient to divide up the field into a longitudinal and transverse part, though this is not always advisable. The boundary conditions either specify the surface displacement or the tractions applied to the surface. If the boundary surface is perpendicular to the z axis, the traction $\mathbf{T}$ will depend on the rates of change of $\mathbf{s}$ near the surface as follows:

$$\mathbf{T} = \mathfrak{T} \cdot \mathbf{k} = \lambda\mathbf{k} \operatorname{div} \mathbf{s} + \mu \operatorname{grad} s_z + \mu(\partial\mathbf{s}/\partial z) \tag{13.2.4}$$

How these boundary conditions are to be applied will be taken up shortly. As we have seen [Eq. (13.1.42)], the solution for $\mathbf{s}$ may usually be written as $\mathbf{s} = \operatorname{grad} \varphi + \mathbf{M} + \mathbf{N}$, where $\mathbf{M}$ and $\mathbf{N}$ are transverse solutions of the vector Laplace equation and where φ is a scalar solution of $\nabla^4\varphi = 0$, subject to the equations of compatibility.

Finally, the relation between pressure and velocity in the steady-state flow of an incompressible fluid is [see Eq. (2.3.14)]

$$\operatorname{grad} P + \eta \operatorname{curl} \operatorname{curl} \mathbf{v} = \rho\mathbf{v} \times \operatorname{curl} \mathbf{v}$$

where $\mathbf{v}$ is the fluid velocity, ρ its density, η its coefficient of viscosity, and P is the "dynamic pressure" corresponding to Bernouilli's principle

$$P = p + \tfrac{1}{2}\rho v^2 + V$$

where p is the actual pressure and V is the potential corresponding to the body forces, such as gravity. (The force per unit volume is minus the gradient of V.) The boundary conditions are usually that $\mathbf{v} = 0$ at the boundary and, of course, that $\operatorname{div} \mathbf{v} = 0$ throughout.

This equation is not linear and so is not amenable to the techniques of this chapter. Whenever $\mathbf{v}$ is small enough, however (we shall see later how small this must be), we neglect the nonlinear term $\rho\mathbf{v} \times \operatorname{curl} \mathbf{v}$ and obtain the linear equation

$$-\operatorname{curl} \operatorname{curl} \mathbf{v} = \nabla^2\mathbf{v} = -4\pi\mathbf{Q}; \quad \operatorname{div} \mathbf{v} = 0 \tag{13.2.5}$$

where $\mathbf{Q} = -(1/4\pi\eta) \operatorname{grad} P$ is a longitudinal vector and where, to the first approximation in $\mathbf{v}$, $P \simeq p + V$. If, on the other hand, the fluid velocity is nearly equal to a constant vector U, so that $\mathbf{v} = \mathbf{U} + \mathbf{u}$, where $\mathbf{u}$ is small enough, we can use an alternate form of the steady flow equations

$$\operatorname{grad}(p + V) + \rho\mathbf{v} \cdot (\nabla\mathbf{v}) + \eta \operatorname{curl} \operatorname{curl} \mathbf{v} = 0$$

If the constant vector $\mathbf{U}$ is along the z axis, then to the first order in the small quantity $\mathbf{u}$ we have

$$-\eta \text{ curl curl } \mathbf{u} = \eta\nabla^2\mathbf{u} = \text{grad}(p + V) + \rho U(\partial\mathbf{u}/\partial z) \qquad (13.2.6)$$

which is, again, a linearized equation.

Our task, in this section, is to indicate how these equations are to be solved in various coordinate systems and for various boundary conditions.

Two-dimensional Solutions. Two-dimensional cases do not need all of the machinery we have developed in this section; in fact we have already worked out some of the cases in Chap. 10. The solutions are of two types: a vector pointing in the z direction, with amplitude which depends only on x and y; and a vector parallel to the x, y plane, with direction and amplitude independent of z. We go over a few, to remind

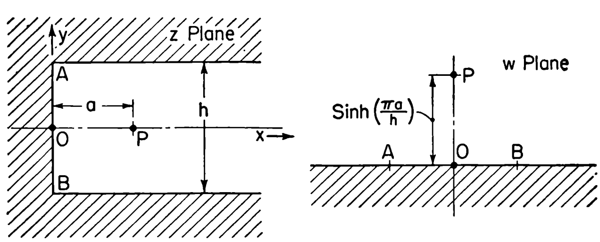

Fig. 13.1 Conformal transform for magnetic field about wire in slot.

ourselves of some of the techniques and in preparation for more difficult cases.

In the case of the magnetic field, when the current responsible for the field is all in the z direction, the vector potential $\mathbf{A}$ is all in the z direction and its magnitude is a solution of the Poisson equation in x and y, with the current density a two-dimensional source density. For instance, in the situation shown in Fig. 13.1 a wire carrying current I passes through the point $x = a$, $y = 0$ and is inside a slot in a mass of iron, the air space being the region $x > 0$, $|y| < \frac{1}{2}h$. Here all that is needed is to modify the two-dimensional Green's function $-2 \ln R$ sufficiently so that the normal gradient of A_z is zero at the surface of the iron. This may be done by use of a Fourier series in y, using the expansion of the Green's function, or it may be computed by use of the conformal transform given in Eq. (4.7.7).

The conformal transform to the w plane is

$$w = -\cosh(\pi x/h)\sin(\pi y/h) + i\sinh(\pi x/h)\cos(\pi y/h)$$

The point P ($x = a$, $y = 0$) corresponds to $w = i\sinh(\pi a/h)$. A source at this point, with the corresponding boundary condition along the real w

axis is

$$A_z = -2I \ln |[w - i \sinh(\pi a/h)][w + i \sinh(\pi a/h)]|$$

and, transformed back to the x, y plane, it becomes

$$A_z = -I \ln\left\{\left[\cosh\left(\frac{2\pi a}{h}\right) - \cosh\left(\frac{2\pi x}{h}\right)\cos\left(\frac{2\pi y}{h}\right)\right]^2 + \sinh^2\left(\frac{2\pi x}{h}\right)\sin^2\left(\frac{2\pi y}{h}\right)\right\} + 2I \ln 2$$

Consequently, the magnetic field is

$$\begin{aligned} H_x &= -\frac{4\pi I}{h}\frac{(\cosh\alpha\cosh\xi - \cos\eta)\sin\eta}{(\cosh\alpha - \cosh\xi\cos\eta)^2 + \sinh^2\xi\sin^2\eta} \\ H_y &= -\frac{4\pi I}{h}\frac{(\cosh\alpha\cos\eta - \cosh\xi)\sinh\xi}{(\cosh\alpha - \cosh\xi\cos\eta)^2 + \sinh^2\xi\sin^2\eta} \end{aligned} \tag{13.2.7}$$

where $\alpha = 2\pi a/h$, $\xi = 2\pi x/h$, $\eta = 2\pi y/h$. This, of course, may be expanded in a Fourier series in y. For ξ considerably larger than η,

$$\begin{aligned} H_x &\to -(4\pi I/h)(\cosh\alpha/\cosh\xi) \to 0 \\ H_y &\to +(4\pi I/h)\tanh\xi \to (4\pi I/h) \end{aligned}$$

a uniform magnetic field between the upper and lower surfaces of the slot.

For viscous flow we can take, for a simple example, the flow through a rectangular duct with axis parallel to the z axis, of width l_x and height l_y. We can assume that the pressure is uniformly distributed along the tube,

$$P = -Fz$$

where F is the pressure drop per unit distance along the duct. Using Eq. (13.2.5), we have

$$\nabla^2\mathbf{v} = -(F/\eta)\mathbf{k}$$

with the boundary conditions that $\mathbf{v} = 0$ at $x = 0$, l_x, and $y = 0$, l_y. Obviously the fluid velocity is in the z direction, so that the amplitude v_z must satisfy a Poisson equation with a constant density F/η. The solution is the series

$$\mathbf{v} = \frac{16F}{\pi^4\eta}\mathbf{k}\sum_{m,n}\frac{\sin[(\pi x/l_x)(2m+1)]\sin[(\pi y/l_y)(2n+1)]}{\{[(2m+1)/l_x]^2 + [(2n+1)/l_y]^2\}(2m+1)(2n+1)} \tag{13.2.8}$$

with a total flow along the duct

$$Q = \frac{64Fl_xl_y}{\pi^6\eta}\sum_{m,n}\frac{1}{\{[(2m+1)/l_x]^2 + [(2n+1)/l_y]^2\}(2m+1)^2(2n+1)^2}$$

This formula holds as long as the dimensionless ratio $v_z\rho\sqrt{l_xl_y}/\eta$ is less than a universal constant, called the *Reynolds number*. If the pressure drop is increased beyond the amount required for the flow to reach this limiting value, a discontinuous change takes place in the flow; turbulence sets in, the pressure drop for a given flow increases considerably, and the velocity distribution is quite different from that given in Eq. (13.2.8). Obviously what has happened is that the nonlinear terms have become large enough to count and the result is turbulent flow. We note that the smaller η is, the *smaller* is the limiting velocity above which turbulence sets in.

Another example is given on page 1197, of flow through a slit of width a in the x, z plane. Expressed in terms of the elliptic coordinates μ and ϑ, the velocity, pressure, and vorticity $\mathbf{w} = \frac{1}{2}$ curl $\mathbf{v}$ are

$$\begin{aligned}\mathbf{v} &= \mathbf{a}_\mu\left(\frac{4Q}{\pi a}\right)\frac{\sin^2\vartheta}{\sqrt{\sinh^2\mu + \sin^2\vartheta}} \\ p &= \frac{16\eta Q}{\pi a^2}\,\frac{\sin^2\vartheta - e^{-\mu}\sinh\mu}{\sinh^2\mu + \sin^2\vartheta} \\ w &= \frac{4Q}{\pi a^2}\,\frac{\sin 2\vartheta}{\sinh^2\mu + \sin^2\vartheta}\end{aligned} \tag{13.2.9}$$

where Q is the volume flow through the slit per unit length of slit in the z direction.

An analogous elastic problem to that of Eq. (13.2.8) is that of a vertical rectangular duct filled with rubber (for instance) which is sagging because of the force of gravity $\rho g\mathbf{k}$. Here the displacement is in the z direction and is a pure shear strain. We have to solve the equation

$$\mu\nabla^2 s_z = -\rho g$$

subject to the requirements that s_z be zero at $x = 0, l_x$; $y = 0, l_y$. The formula for $\mathbf{s}$ is the same as that for $\mathbf{v}$ in Eq. (13.2.8), except that $\rho g/\mu$ is substituted for F/η.

Polar Coordinates. A simple example of a two-dimensional solution in the coordinates r, θ, z is that of the flow of a viscous fluid through a pipe of circular cross section of radius a. Here again the pressure drop is to be F, so that $P = -Fz$ and, from Eq. (13.2.5) we have

$$\nabla^2\mathbf{v} = -\left(\frac{F}{\eta}\right)\mathbf{k};\quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = -\left(\frac{F}{\eta}\right)$$

if we require that $\mathbf{v}$ is in the z direction and that its magnitude be dependent only on r. The solution which goes to zero at $r = a$ is

$$\mathbf{v} = \mathbf{k}(F/4\eta)(a^2 - r^2) \tag{13.2.10}$$

and the total flow is $\pi F a^4/8\eta$, which relates the total flow to the pressure drop, the radius, and the coefficient of viscosity, for flow slow enough so that turbulence is not present.

As an example, in elasticity we consider a long tube of inner radius a and outer radius b, with a pressure difference between inner and outer faces; we set $\mathbf{s} = \operatorname{grad} \varphi$ with φ a solution of $\nabla^4\varphi$ for φ dependent on r alone,

$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\varphi}{dr}\right)\right]\right\} = 0 \tag{13.2.11}$$

Consequently,

$$\varphi = C_1r^2 + C_2 \ln r + C_3 + C_4r^2 \ln r$$

with the constants to be adjusted to fit the boundary conditions and equations of compatibility. As a matter of fact, we can immediately set $C_4 = 0$, for the $r^2 \ln r$ term gives a nonzero term in the expression $\operatorname{grad}(\nabla^2\varphi)$, which is part of the compatibility equation $\nabla \cdot \mathfrak{T} = 0$. Without C_4, the displacement vector $\mathbf{s}$ and stress tensor $\mathfrak{T}$ are

$$\begin{aligned}
\mathbf{s} &= \mathbf{a}_r(d\varphi/dr) = \mathbf{a}_r[2C_1r + (C_2/r)] \\
\mathfrak{T} &= \lambda\mathfrak{J} \operatorname{div} \mathbf{s} + \mu(\nabla\mathbf{s} + \mathbf{s}\nabla) \\
&= \mathbf{a}_r\mathbf{a}_r[4C_1(\lambda + \mu) - 2\mu(C_2/r^2)] + \mathbf{a}_\theta\mathbf{a}_\theta[4C_1(\lambda + \mu) + 2\mu(C_2/r^2)]
\end{aligned}$$

If the pressure inside the cylinder is P and that outside is zero, we must have $\mathfrak{T} \cdot \mathbf{a}_r = -\mathbf{a}_rP$ at $r = a$ and $= 0$ at $r = b$. Consequently, we have

$$\begin{aligned}
\mathbf{s} &= \mathbf{a}_r \frac{Pa^2}{2(b^2 - a^2)} \frac{b}{(\lambda + \mu)} \left[\frac{r}{b} + \frac{(\lambda + \mu)b}{\mu r}\right] \\
\mathfrak{T} &= \frac{Pa^2}{b^2 - a^2}\left[\mathbf{a}_r\mathbf{a}_r\left(1 - \frac{b^2}{r^2}\right) + \mathbf{a}_\theta\mathbf{a}_\theta\left(1 + \frac{b^2}{r^2}\right)\right]
\end{aligned} \tag{13.2.12}$$

The radial stress is thus compressive (negative) throughout the range $a < r < b$, and the tangential stress is tensile with its greatest value $[P(b^2 + a^2)/(b^2 - a^2)]$ at $r = a$. No matter how large b is compared to a, this tensile stress is never smaller than P, so that if P is larger than the tensile strength of the material of the cylinder, it will break no matter how thick the wall. Note that the expansive stress $|\mathfrak{T}| = [2Pa^2/(b^2 - a^2)]$ is independent of r and also that the radial expansion of the outer surface of the cylinder is $[Pa^2b(\lambda + 2\mu)/2(b^2 - a^2)(\lambda + \mu)\mu]$, which is larger, the smaller the shear modulus μ is compared to the modulus λ. The displacement vector, in this case, has no transverse terms, since all motion is radial and there is no rotation of elements of the medium.

Circular Cylinder Coordinates. The three sets of solutions of the vector Laplace equation in circular cylinder coordinates are:

$$\mathbf{M}_{skm}(r,\theta,z) = \frac{1}{k}\,\mathrm{curl}\left[\mathbf{a}_z \begin{matrix}\cos\\ \sin\end{matrix}(m\theta)e^{kz}J_m(kr)\right]$$
$$= \tfrac{1}{2}\mathbf{a}_r \begin{matrix}-\sin\\ \cos\end{matrix}(m\theta)e^{kz}[J_{m-1}(kr) + J_{m+1}(kr)]$$
$$- \tfrac{1}{2}\mathbf{a}_\theta \begin{matrix}\cos\\ \sin\end{matrix}(m\theta)e^{kz}[J_{m-1}(kr) - J_{m+1}(kr)]$$
$$\mathbf{N}_{skm}(r,\theta,z) = \frac{1}{k}\,\mathrm{grad}\left[\begin{matrix}\cos\\ \sin\end{matrix}(m\theta)e^{kz}J_m(kr)\right] \tag{13.2.13}$$
$$= \tfrac{1}{2}\mathbf{a}_r \begin{matrix}\cos\\ \sin\end{matrix}(m\theta)e^{kz}[J_{m-1}(kr) - J_{m+1}(kr)] + \mathbf{a}_z \begin{matrix}\cos\\ \sin\end{matrix}(m\theta)e^{kz}J_m(kr)$$
$$+ \tfrac{1}{2}\mathbf{a}_\theta \begin{matrix}-\sin\\ \cos\end{matrix}(m\theta)e^{kz}[J_{m-1}(kr) + J_{m+1}(kr)]$$
$$\mathbf{G}_{skm}(r,\theta,z) = 2\mathbf{a}_z \begin{matrix}\cos\\ \sin\end{matrix}(m\theta)e^{kz}J_m(kr) - \mathbf{N}_{skm}(r,\theta,z)$$

where we have subtracted a certain amount of vector $\mathbf{N}$, in the last expression, so that $\iint \mathbf{N}\cdot\mathbf{G}r\,dr\,d\theta$ is zero. The subscript s becomes e when we use the upper choice of the trigonometric functions shown in the formulas, becomes o when we use the lower choice. The functions $\mathbf{M}$ and $\mathbf{N}$ have zero divergence, and function $\mathbf{N}$ also has zero curl; $\mathbf{G}$ has nonzero divergence and curl. The differential relations between the functions are

$$\mathrm{curl}\,\mathbf{G} = 2k\mathbf{M};\quad \mathrm{curl}\,\mathbf{M} = k\mathbf{N};\quad \mathrm{curl}\,\mathbf{N} = 0$$
$$\mathrm{curl}\,\mathrm{curl}\,\mathbf{G} = 2k^2\mathbf{N} = \mathrm{grad}\,\mathrm{div}\,\mathbf{G};\quad \mathrm{div}\,\mathbf{M} = \mathrm{div}\,\mathbf{N} = 0$$

for the same subscripts.

Referring to Eq. (10.3.22), we see that these solutions may be expressed in terms of two-dimensional vector solutions in a plane at an angle u

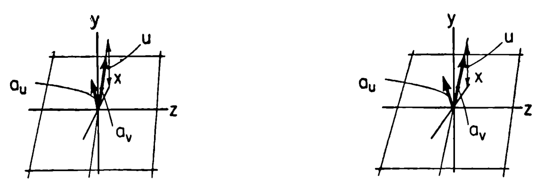

Fig. 13.2 Angles and coordinates for rotating complex plane about z axis, by angle u.

with respect to the x, z plane, integrated over u. Referring to Fig. 13.2, we define the unit vector

$$\mathbf{a}_u = -\mathbf{i}\sin u + \mathbf{j}\cos u$$

normal to the u plane and the orthogonal unit vectors

$$\mathbf{a}_v = \mathbf{i}\cos u + \mathbf{j}\sin u;\quad \mathbf{a}_z = \mathbf{k}$$

in the u plane. In terms of these unit vectors, therefore, our vector solutions are

$$\begin{aligned}
\mathbf{M}_{skm} &= \frac{1}{2\pi i^{m+1}} \int_0^{2\pi} \mathbf{a}_u e^{kX} \frac{\cos}{\sin}(mu)\,du \\
\mathbf{N}_{skm} &= \frac{1}{2\pi i^{m+1}} \int_0^{2\pi} (i\mathbf{a}_z - \mathbf{a}_v) e^{kX} \frac{\cos}{\sin}(mu)\,du \\
\mathbf{G}_{skm} &= \frac{1}{2\pi i^{m+1}} \int_0^{2\pi} (i\mathbf{a}_z + \mathbf{a}_v) e^{kX} \frac{\cos}{\sin}(mu)\,du
\end{aligned} \tag{13.2.14}$$

where $X = z + i(x\cos u + y\sin u) = z + ir\cos(\theta - u)$.

The Green's dyadic for this case, obtained from Eq. (10.3.23), may be written

$$\mathfrak{G} = \frac{\mathfrak{J}}{R} = \mathfrak{J} \sum_{m=0}^{\infty} \epsilon_m \cos[m(\theta - \theta_0)] \int_0^\infty J_m(kr) J_m(kr_0) e^{-k|z-z_0|}\,dk$$

which may also be written in terms of integrals of $\mathbf{M}$, $\mathbf{N}$, $\mathbf{G}$ over k. The vector potential from current I, flowing through a single loop of narrow wire in the circle $r = a$, $z = 0$, $0 < \theta < 2\pi$, is then given by the integral of $(I\mathfrak{G} \cdot \mathbf{a}_{\theta_0})\delta(r_0 - a)\delta(z_0)$ over r_0, θ_0, z_0, where

$$\mathbf{a}_{\theta_0} = \mathbf{a}_r \sin(\theta - \theta_0) + \mathbf{a}_\theta \cos(\theta - \theta_0)$$

Consequently, the vector potential from a single loop of current is

$$\begin{aligned}
\mathbf{A} &= 2\pi I a \mathbf{a}_\theta \int_0^\infty e^{-k|z|} J_1(kr) J_1(ka)\,dk \\
&= Ia \int_0^\infty dk \int_0^\infty \mathbf{M}_{ek0}(r,\theta,z)[\mathbf{a}_{\theta_0} \cdot \mathbf{M}_{ek0}(a,\theta_0,0)]\,d\theta_0; \quad z < 0
\end{aligned}$$

where integration over the whole circuit yields a combination of only transverse solutions $\mathbf{M}$ (see page 1791). The resulting magnetic field is

$$\begin{aligned}
\mathbf{H} = \operatorname{curl} \mathbf{A} = 2\pi I a \Big[\mathbf{a}_r \int_0^\infty e^{-k|z|} J_1(kr) J_1(ka) k\,dk \\
+ \mathbf{a}_z \int_0^\infty e^{-k|z|} J_0(kr) J_1(ka) k\,dk \Big]
\end{aligned} \tag{13.2.15}$$

Juggling of contour integral relations shows that

$$\mathbf{A} = \mathbf{a}_\theta \frac{\pi I a^2 r}{(a^2 + r^2 + z^2)^{\frac{3}{2}}} F\left(\tfrac{3}{4}, \tfrac{5}{4} \,\middle|\, 2 \,\middle|\, \frac{4a^2 r^2}{(a^2 + r^2 + z^2)^2}\right)$$

For $(a^2 + r^2 + z^2) \gg 2ar$, the hypergeometric series F becomes unity, and the formulas become simple. For instance, for r and $z \gg a$,

$$\mathbf{A} \simeq \mathbf{a}_\theta \pi I a^2 \left[\frac{r}{(r^2 + z^2)^{\frac{3}{2}}}\right]$$

$$\mathbf{H} \simeq \frac{\pi I a^2}{(a^2 + z^2)^{\frac{5}{2}}} [3rz\mathbf{a}_r + (2z^2 - r^2)\mathbf{a}_z] = -\pi I a^2 \operatorname{grad}\left[\frac{z}{(r^2 + z^2)^{\frac{3}{2}}}\right]$$

which is the same as the magnetic field caused by a magnetic dipole of strength $(\pi a^2 I)$ pointed in the z direction at the origin.

Spherical Coordinates. The one set of coordinates which allows fairly general solutions for all three major vector Laplace-type equations (magnetic field, viscous flow, and elastic displacement) is the spherical set r, ϑ, φ. Our solutions may be expressed in terms of constant vectors ($\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ or a combination) times a spherical harmonic times a function of r (r^n or r^{-n-1} for the Laplace equation). But this, as we have shown, does not allow easy satisfaction of boundary conditions on spherical boundaries, and we make use of the techniques already discussed in this chapter. For the vector Laplace equation, for instance, we take the curl of $\mathbf{r}$ times the solution of the scalar Laplace equation, $r^n Y_n^m(\vartheta,\varphi)$ (where Y is a spherical harmonic of the nth order) or $r^{-n-1}Y_n^m(\vartheta,\varphi)$. This gives us the tangential solution $\mathbf{M}$. For the solution with zero curl and divergence $\mathbf{N}$, we use the gradient of $r^n Y_n^m$ or $r^{-n-1}Y_n^m$.

These solutions can be expressed in terms of the mutually orthogonal vector spherical harmonics, $\mathbf{P}$, $\mathbf{B}$, and $\mathbf{C}$ discussed in Sec. 13.3 and tabulated at the end of this chapter. All that is necessary to know about them here is that the trio, for a given value of m, n, and s (odd or even), are mutually perpendicular for each value of ϑ and φ, and that their integral properties are those given in the table.

The detailed expression of $\mathbf{M}$ and $\mathbf{N}$ in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ may be easiest obtained by using Eqs. (10.3.34). We may also save intermediate computations by using the complex form for the spherical harmonic:

$$X_n^m(\vartheta,\varphi) = e^{im\varphi}P_n^m(\cos\vartheta) = Y_{mn}^e(\vartheta,\varphi) + iY_{mn}^o(\vartheta,\varphi) \qquad (13.2.16)$$

where $m > 0$. We also use the convention for the complex conjugate that

$$\overline{X}_n^m(\vartheta,\varphi) = e^{-im\varphi}P_n^m(\cos\vartheta) = X_n^{-m}(\vartheta,\varphi)$$

The formulas are then:

$$\begin{aligned}
\mathbf{M}_{mn}^1(r,\vartheta,\varphi) &= \operatorname{curl}[\mathbf{r}r^n X_n^m(\vartheta,\varphi)] = -\mathbf{r}\times\operatorname{grad}[r^n X_n^m(\vartheta,\varphi)] \\
&= r^n\{\tfrac{1}{2}\mathbf{i}[(1+\delta_{0m})iX_n^{m+1} + (n+m)(n-m+1)(1-\delta_{0m})iX_n^{m-1}] \\
&\quad + \tfrac{1}{2}\mathbf{j}[(1+\delta_{0m})X_n^{m+1} - (n+m)(n-m+1)(1-\delta_{0m})X_n^{m-1}] - \mathbf{k}miX_n^m\} \\
\mathbf{N}_{mn}^1(r,\vartheta,\varphi) &= \operatorname{grad}[r^{n+1}X_{n+1}^m(\vartheta,\varphi)] \qquad (13.2.17)\\
&= r^n\{\tfrac{1}{2}\mathbf{i}[-(1+\delta_{0m})X_n^{m+1} + (n+m)(n+m+1)(1-\delta_{0m})X_n^{m-1}] \\
&\quad + \tfrac{1}{2}\mathbf{j}[(1+\delta_{0m})iX_n^{m+1} + (n+m)(n+m+1)(1-\delta_{0m})iX_n^{m-1}] \\
&\quad + \mathbf{k}(n+m+1)X_n^m\}
\end{aligned}$$

where, for $m = 0$, we use only the real parts of $\mathbf{M}$ and $\mathbf{N}$. The real part of $\mathbf{M}_{mn}^1$ is $\mathbf{M}_{emn}^1$ and its imaginary part is $\mathbf{M}_{omn}^1$. We note that, for $\mathbf{M}^1$, m ranges from zero to n; for $\mathbf{N}^1$, m ranges from zero to $n+1$.

Reference to the tables at the end of this chapter shows that $\mathbf{M}^1_{smn} = \sqrt{n(n+1)}\, r^n \mathbf{C}^s_{mn}(\vartheta,\varphi)$ where s may be e (even) or o (odd). There are also solutions which vanish at infinity but diverge at $r \to 0$:

$$\begin{aligned}\mathbf{M}^2_{smn}(r,\vartheta,\varphi) &= -\mathbf{r} \times \operatorname{grad}[r^{-n-1} Y^s_{mn}(\vartheta,\varphi)] \\ &= \sqrt{n(n+1)}\, r^{-n-1}\mathbf{C}^s_{mn}(\vartheta,\varphi)\end{aligned} \tag{13.2.18}$$

The corresponding zero-curl–zero-divergence solutions are the real and imaginary parts of

$$\begin{aligned}\mathbf{N}^2_{mn}(r,\vartheta,\varphi) &= \operatorname{grad}[r^{-n} X^m_{n-1}(\vartheta,\varphi)] \\ &= r^{-n-1}\{\tfrac{1}{2}\mathbf{i}[-(1+\delta_{0m})X^{m+1}_n + (n-m)(n-m+1)(1-\delta_{0m})X^{m-1}_n] \\ &\quad + \tfrac{1}{2}\mathbf{j}[(1+\delta_{0m})iX^{m+1}_n + (n-m)(n-m+1)(1-\delta_{0m})iX^{m-1}_n] \\ &\quad - \mathbf{k}(n-m)X^m_n\}\end{aligned} \tag{13.2.19}$$

Again, for $\mathbf{M}^2$, m goes from zero to n, but for $\mathbf{N}^2$, m goes from zero to $n-1$; also n cannot be zero, and we use only the real part for $m = 0$.

We can express the angular dependence of $\mathbf{N}^1$ and $\mathbf{N}^2$ in terms of the vector harmonics $\mathbf{P}$ and $\mathbf{B}$ given at the end of the chapter. We have

$$\begin{aligned}\mathbf{N}^1_{smn} &= \sqrt{n+1}\, r^n[\sqrt{n+2}\,\mathbf{B}^s_{m,\,n+1}(\vartheta,\varphi) + \sqrt{n+1}\,\mathbf{P}^s_{m,\,n+1}(\vartheta,\varphi)] \\ \mathbf{N}^2_{smn} &= \sqrt{n}\, r^{-n-1}[\sqrt{n-1}\,\mathbf{B}^s_{m,\,n-1}(\vartheta,\varphi) - \sqrt{n}\,\mathbf{P}^s_{m,\,n-1}(\vartheta,\varphi)]\end{aligned} \tag{13.2.20}$$

which shows their orthogonality to the set $\mathbf{M}^1$, $\mathbf{M}^2$.

We need another set of vector solutions, orthogonal to $\mathbf{M}^1$ and $\mathbf{N}^1$ or to $\mathbf{M}^2$ and $\mathbf{N}^2$. These may be obtained by reversing the angle dependence of $\mathbf{N}^1$ and $\mathbf{N}^2$:

$$\begin{aligned}\mathbf{G}^1_{smn}(r,\vartheta,\varphi) &= \sqrt{n}\, r^n[\sqrt{n-1}\,\mathbf{B}^s_{m,\,n-1}(\vartheta,\varphi) - \sqrt{n}\,\mathbf{P}^s_{m,\,n-1}(\vartheta,\varphi)] \\ \mathbf{G}^1_{mn} &= r^n\{\tfrac{1}{2}\mathbf{i}[-(1+\delta_{0m})X^{m+1}_n + (n-m)(n-m+1)(1-\delta_{0m})X^{m-1}_n] \\ &\quad + \tfrac{1}{2}\mathbf{j}[(1+\delta_{0m})iX^{m+1}_n + (n-m)(n-m+1)(1-\delta_{0m})iX^{m-1}_n] \\ &\quad - \mathbf{k}(n-m)X^m_n\} \\ &\qquad n = 1, 2, 3, \ldots;\quad m = 0, 1, 2, \ldots, n-1;\quad s = e, o \qquad (13.2.21) \\ \mathbf{G}^2_{smn}(r,\vartheta,\varphi) &= \sqrt{n+1}\, r^{-n-1}[\sqrt{n+2}\,\mathbf{B}^s_{m,\,n+1}(\vartheta,\varphi) \\ &\quad + \sqrt{n+1}\,\mathbf{P}^s_{m,\,n+1}(\vartheta,\varphi)] \\ \mathbf{G}^2_{mn} &= r^{-n-1}\{\tfrac{1}{2}\mathbf{i}[-(1+\delta_{0m})X^{m+1}_n + (n+m)(n+m+1)(1-\delta_{0m})X^{m-1}_n] \\ &\quad + \tfrac{1}{2}\mathbf{j}[(1+\delta_{0m})iX^{m+1}_n + (n+m)(n+m+1)(1-\delta_{0m})iX^{m-1}_n] \\ &\quad + \mathbf{k}(n+m+1)X^m_n\} \\ &\qquad n = 0, 1, 2, \ldots;\quad m = 0, 1, 2 \ldots, n+1;\quad s = e, o\end{aligned}$$

This set of solutions has neither zero divergence nor zero curl. The differential relations between all three functions are:

$$\begin{aligned}&\operatorname{curl}\mathbf{N}^1_{smn} = \operatorname{curl}\mathbf{N}^2_{smn} = 0 = \operatorname{div}\mathbf{N}^1_{smn} = \operatorname{div}\mathbf{N}^2_{smn} \\ &\operatorname{div}\mathbf{M}^1_{smn} = \operatorname{div}\mathbf{M}^2_{smn} = 0;\quad \operatorname{curl}\mathbf{M}^1_{smn} = (n+1)\mathbf{N}^1_{sm,n-1} \\ &\operatorname{curl}\mathbf{M}^2_{smn} = n\mathbf{N}^2_{sm,\,n+1};\quad \operatorname{div}\mathbf{G}^1_{smn} = -n(2n+1)r^{n-1}Y^s_{m,\,n-1}(\vartheta,\varphi) \\ &\operatorname{div}\mathbf{G}^2_{smn} = -(n+1)(2n+1)r^{-n-2}Y^s_{m,\,n+1}(\vartheta,\varphi) \\ &\operatorname{curl}\mathbf{G}^1_{smn} = -(2n+1)\mathbf{M}^1_{sm,\,n-1};\quad \operatorname{curl}\mathbf{G}^2_{smn} = (2n+1)\mathbf{M}^2_{sm,\,n+1}\end{aligned} \tag{13.2.22}$$

The functions for $m = 0$ take on simpler forms, only the even forms ($s = e$) occur, and

$$\begin{aligned}
\mathbf{M}^1_{e0n} &= \mathbf{M}^1_n = r^n[-\mathbf{i}\sin\varphi + \mathbf{j}\cos\varphi]P^1_n(\cos\vartheta) = \mathbf{a}_\varphi r^n P^1_n \\
\mathbf{M}^2_{e0n} &= \mathbf{M}^2_n = \mathbf{a}_\varphi r^{-n-1}P^1_n(\cos\vartheta) \\
\mathbf{N}^1_{e0n} &= \mathbf{N}^1_n = r^n[-(\mathbf{i}\cos\varphi + \mathbf{j}\sin\varphi)P^1_n + \mathbf{k}(n+1)P_n] \\
&= r^n[(n+1)\mathbf{a}_r P_{n+1} - \mathbf{a}_\vartheta P^1_{n+1}] = -r^n(\mathbf{a}_r T^1_{n-1} - \mathbf{k}T^1_n) \\
\mathbf{N}^2_{e0n} &= \mathbf{N}^2_n = r^{-n-1}[-(\mathbf{i}\cos\varphi + \mathbf{j}\sin\varphi)P^1_n - \mathbf{k}nP_n] \\
&= -r^{-n-1}[n\mathbf{a}_r P_{n-1} + \mathbf{a}_\vartheta P^1_{n-1}] = -r^{-n-1}(\mathbf{a}_r T^1_{n-1} - \mathbf{k}T^1_{n-2}) \\
\mathbf{G}^1_{e0n} &= \mathbf{G}^1_n = -r^n[n\mathbf{a}_r P_{n-1} + \mathbf{a}_\vartheta P^1_{n-1}] = -r^n(\mathbf{a}_r T^1_{n-1} - \mathbf{k}T^1_{n-2}) \\
\mathbf{G}^2_{e0n} &= \mathbf{G}^2_n = r^{-n-1}[(n+1)\mathbf{a}_r P_{n+1} - \mathbf{a}_\vartheta P^1_{n+1}] = -r^{-n-1}(\mathbf{a}_r T^1_{n-1} - \mathbf{k}T^1_n)
\end{aligned} \tag{13.2.23}$$

Finally, for $n = 0$, $\mathbf{M}^1$, $\mathbf{M}^2$, $\mathbf{N}^2$, and $\mathbf{G}^1$ are zero, and the pair of three elementary solutions are

$$\begin{gathered}
\mathbf{N}^1_{e00} = \mathbf{k};\quad \mathbf{N}^1_{e10} = \mathbf{i};\quad \mathbf{N}^1_{010} = \mathbf{j} \\
\mathbf{G}^2_{e00} = \mathbf{k}/r;\quad \mathbf{G}^2_{e10} = \mathbf{i}/r;\quad \mathbf{G}^2_{010} = \mathbf{j}/r
\end{gathered}$$

By use of the formulas at the end of this chapter, we can express the usual spherical harmonic functions, times constant vectors, in terms of the $\mathbf{M}$, $\mathbf{N}$, and $\mathbf{G}$ functions. For instance, for $m = 0$,

$$\begin{aligned}
&\mathbf{k}[r^nP_n(\cos\vartheta)] = \frac{1}{2n+1}[\mathbf{N}^1_n - \mathbf{G}^1_n];\quad \mathbf{k}\left[\frac{P_n}{r^{n+1}}\right] = \frac{-1}{2n+1}[\mathbf{N}^2_n - \mathbf{G}^2_n] \\
&\mathbf{i}[r^nP_n] = \frac{1}{n(n+1)(2n+1)}[(2n+1)\mathbf{M}^1_{o1n} + n\mathbf{N}^1_{e1n} + (n+1)\mathbf{G}^1_{e1n}] \\
&\mathbf{i}\left[\frac{P_n}{r^{n+1}}\right] = \frac{1}{n(n+1)(2n+1)}[(2n+1)\mathbf{M}^2_{o1n} + (n+1)\mathbf{N}^2_{e1n} + n\mathbf{G}^2_{e1n}] \\
&\mathbf{j}[r^nP_n] = \frac{1}{n(n+1)(2n+1)}[-(2n+1)\mathbf{M}^1_{e1n} + n\mathbf{N}^1_{o1n} + (n+1)\mathbf{G}^1_{o1n}] \\
&\mathbf{j}\left[\frac{P_n}{r^{n+1}}\right] = \frac{1}{n(n+1)(2n+1)}[-(2n+1)\mathbf{M}^2_{e1n} + (n+1)\mathbf{N}^2_{o1n} + n\mathbf{G}^2_{o1n}]
\end{aligned} \tag{13.2.24}$$

Green's Dyadic for Vector Laplace Equation. The dyadic Green's function, which is a solution of the equation

$$\nabla^2\mathfrak{G} = -\left(\frac{4\pi}{r^2\sin\vartheta}\right)\delta(r - r_0)\delta(\vartheta - \vartheta_0)\delta(\varphi - \varphi_0)\mathfrak{J} \tag{13.2.25}$$

may be expressed in terms of a series of the functions $\mathbf{M}$, $\mathbf{N}$, and $\mathbf{G}$. The angle functions

$$\begin{aligned}
\mathbf{D}^1_{mn}(\vartheta,\varphi) &= \sqrt{n(n+1)}\,\mathbf{C}^e_{mn}(\vartheta,\varphi);\quad \mathbf{D}^2_{mn} = \sqrt{n(n+1)}\,\mathbf{C}^o_{mn} \\
\mathbf{D}^3_{mn} &= \sqrt{n+1}\,[\sqrt{n+2}\,\mathbf{B}^e_{m,n+1} + \sqrt{n+1}\,\mathbf{P}^e_{m,n+1}] \\
\mathbf{D}^4_{mn} &= \sqrt{n+1}\,[\sqrt{n+2}\,\mathbf{B}^o_{m,n+1} + \sqrt{n+1}\,\mathbf{P}^o_{m,n+1}] \\
\mathbf{D}^5_{mn} &= \sqrt{n}\,[\sqrt{n-1}\,\mathbf{B}^e_{m,n-1} - \sqrt{n}\,\mathbf{P}^e_{m,n-1}] \\
\mathbf{D}^6_{mn} &= \sqrt{n}\,[\sqrt{n-1}\,\mathbf{B}^o_{m,n-1} - \sqrt{n}\,\mathbf{P}^o_{m,n-1}]
\end{aligned} \tag{13.2.26}$$

constitute a complete, orthogonal set of vector functions of the angles ϑ and φ. In terms of them we can expand the dyadic $\mathfrak{G}$,

$$\mathfrak{G} = \sum_{\sigma\mu\nu} \mathbf{F}^\sigma_{\mu\nu}(r;r_0,\vartheta_0,\varphi_0)\mathbf{D}^\sigma_{\mu\nu}(\vartheta,\varphi) \tag{13.2.27}$$

where the functions $\mathbf{F}$ may be determined by the method outlined in obtaining Eq. (7.2.63). Application of the vector Laplace operator yields

$$\nabla^2\mathfrak{G} = \sum_{\sigma\mu\nu} \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\mathbf{F}^\sigma_{\mu\nu}\right) - \frac{n(n+1)}{r^2}\mathbf{F}^\sigma_{\mu\nu}\right]\mathbf{D}^\sigma_{\mu\nu}(\vartheta,\varphi)$$

Substituting this in Eq. (13.2.25), multiplying both sides by $\mathbf{D}^1_{mn}(\vartheta,\varphi)\cdot$ $\cdot \sin\vartheta\, d\vartheta\, d\varphi$ and integrating over ϑ and φ gives us an equation for $\mathbf{F}^1_{mn}$,

$$n(n+1)\frac{4\pi/\epsilon_m}{2n+1}\frac{(n+m)!}{(n-m)!}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\mathbf{F}^1_{mn}\right) - \frac{n(n+1)}{r^2}\mathbf{F}^1_{mn}\right] = -\frac{4\pi}{r^2}\mathbf{D}^1_{mn}\delta(r-r_0)$$

The solution of this equation which is finite at $r = 0$ and $r = \infty$ is

$$\mathbf{F}^1_{mn} = \frac{\epsilon_m}{n(n+1)}\frac{(n-m)!}{(n+m)!}\mathbf{D}^1_{mn}(\vartheta_0,\varphi_0)\begin{cases} r^n/r_0^{n+1}; & r < r_0 \\ r_0^n/r^{n+1}; & r > r_0 \end{cases}$$

Similarly we can obtain solutions for the F's for other values of the superscript s. For instance, if we had multiplied by $D^3_{mn}(\vartheta,\varphi)$ and integrated both sides over ϑ and φ, we would have obtained

$$\mathbf{F}^3_{mn} = \frac{\epsilon_m}{(n+1)(2n+1)}\frac{(n-m+1)!}{(n+m+1)!}\mathbf{D}^3_{mn}\begin{cases} r^n/r_0^{n+1}; & r < r_0 \\ r_0^n/r^{n+1}; & r > r_0 \end{cases}$$

Finally, taking into account the relation between the $\mathbf{D}$'s and the functions $\mathbf{M}$, $\mathbf{N}$, and $\mathbf{G}$ gives us the expansion of the Green's dyadic:

$$\begin{aligned}\mathfrak{G} = \frac{\mathfrak{J}}{R} = \sum_{smn} \epsilon_m \frac{(n-m)!}{(n+m)!}\Bigg\{&\frac{1}{n(n+1)}\mathbf{M}^1_{smn}(r,\vartheta,\varphi)\mathbf{M}^2_{smn}(r_0,\vartheta_0,\varphi_0) \\ &+ \frac{1}{(n+1)(2n+1)}\left(\frac{n-m+1}{n+m+1}\right)\mathbf{N}^1_{smn}(r,\vartheta,\varphi)\mathbf{G}^2_{smn}(r_0,\vartheta_0,\varphi_0) \\ &+ \frac{1}{n(2n+1)}\left(\frac{n+m}{n-m}\right)\mathbf{G}^1_{smn}(r,\vartheta,\varphi)\mathbf{N}^2_{smn}(r_0,\vartheta_0,\varphi_0)\Bigg\}\end{aligned} \tag{13.2.28}$$

where $s = e, o; m = 0, 1, 2, \ldots, n; n = 0, 1, 2, \ldots$; and $r < r_0$. For $r > r_0$, we interchange r, ϑ, φ and r_0, ϑ_0, φ_0.

This expansion has several interesting characteristics. It, of course, satisfies the requirements of reciprocity, $\mathfrak{G}(\mathbf{r}|\mathbf{r}_0) = \mathfrak{G}(\mathbf{r}_0|\mathbf{r})$. It automatically separates off the transverse parts, $\mathbf{M}^1\mathbf{M}^2$, from the rest. But it also exhibits a curious cross-product behavior in the functions $\mathbf{N}$ and $\mathbf{G}$; the last two terms might be expected to be $\mathbf{N}^1\mathbf{N}^2$ and $\mathbf{G}^1\mathbf{G}^2$, but they are not. This last property is closely related to the singular behavior of the Green's function for the Laplace equation, as compared to the Green's function for the vector Helmholtz equation (*i.e.*, as $k \to 0$) which was discussed on page 1785. The functions $\mathbf{M}$ and $\mathbf{N}$ are limits of corresponding solutions for $k \neq 0$, but the function $\mathbf{G}$ is not. The result of the cross-product arrangement is that the part of $\mathfrak{G}$ which is transverse in $\mathbf{r}$ is not transverse in $\mathbf{r}_0$, which is presumably a concomitant of the singular behavior mentioned above.

Magnetic Field Solutions. When we use the Green's dyadic of Eq. (13.2.28) for the magnetic field from a current distribution, we find that if the current distribution is steady state, with zero divergence, then the vector potential $\mathbf{A}$ will be transverse, as was indicated on page 1788. A first, quite simple, example will be that of the field caused by the rotation of a sphere of radius a, charged uniformly with a charge Q, about the z axis with angular velocity ω. The surface current density is then

$$\mathbf{J} = \left(\frac{\omega Q}{4\pi a}\right)\mathbf{a}_\varphi \sin\vartheta = \left(\frac{\omega Q}{4\pi a}\right)\mathbf{a}_\varphi P_1^1(\cos\vartheta)$$

and the vector potential arising from it is (for $\mu = 1$)

$$\begin{aligned}\mathbf{A} &= \sqrt{n(n+1)}\left(\frac{\omega Q a}{4\pi c}\right)\int_0^{2\pi} d\varphi_0 \int_0^{\pi} \mathfrak{G}\cdot\mathbf{C}_{01}(\vartheta_0,\varphi_0)\sin\vartheta_0\, d\vartheta_0 \\ &= \left(\frac{\omega Q a}{3c}\right)\begin{cases}(1/a^2)\mathbf{M}_1^1; & r<a\\ a\mathbf{M}_1^2; & r>a\end{cases} = \left(\frac{\omega Q}{3c}\right)\mathbf{a}_\varphi \sin\vartheta \begin{cases}(r/a); & r<a\\ (a^2/r^2); & r>a\end{cases}\end{aligned} \tag{13.2.29}$$

The magnetic field is obtained from this by the use of Eqs. (13.2.22):

$$\begin{aligned}\mathbf{H} = \operatorname{curl}\mathbf{A} &= \left(\frac{\omega Q}{3c}\right)\begin{cases}(2/a)\mathbf{N}_0^1; & r<a\\ -a^2\mathbf{N}_2^2; & r>a\end{cases}\\ &= \left(\frac{\omega Q}{3c}\right)\begin{cases}(2/a)\mathbf{k}; & r<a\\ (a^2/r^3)(3\mathbf{a}_r\cos\vartheta - \mathbf{k}); & r>a\end{cases}\end{aligned} \tag{13.2.30}$$

This field is uniform inside the sphere and that of a dipole outside.

Another, somewhat more complex, example is that of the field from a wire loop at $r = a$, $\vartheta = \frac{1}{2}\pi$, carrying current J. In this case the current density is

$$\mathbf{J} = (J/a)\delta(r-a)\delta(\vartheta - \tfrac{1}{2}\pi)\mathbf{a}_\varphi$$

and the vector potential is, for $\mu = 1$,

$$\begin{aligned}\mathbf{A} &= \left(\frac{\pi J}{c}\right)\mathbf{a}_\varphi \sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{2^{2n}n!(n+1)!}P_{2n+1}^1(\cos\vartheta)\begin{cases}(a/r)^{2n+2}; & r>a\\ (r/a)^{2n+1}; & r<a\end{cases}\\ &= \frac{\pi J}{c}\sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{2^{2n}n!(n+1)!}\begin{cases}a^{2n+2}\mathbf{M}_{2n+2}^2; & r>a\\ a^{-2n-1}\mathbf{M}_{2n+1}^1; & r<a\end{cases}\end{aligned}$$

and the magnetic field is

$$\mathbf{H} = \frac{\pi J}{ac} \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n} n!(n+1)!} \cdot$$
$$\cdot \begin{cases} (2n+2)(r/a)^{2n}[\mathbf{k}T_{2n}^1 - \mathbf{a}_r T_{2n-1}^1]; & r < a \\ (2n+1)(a/r)^{2n+3}[\mathbf{a}_r T_{2n+1}^1 - \mathbf{k}T_{2n}^1]; & r > a \end{cases} \qquad (13.2.31)$$

where T_{2n}^1 is the Gegenbauer polynomial of $\cos\vartheta$, as defined in the table at the end of Chap. 6. This series converges except at $r = a$.

If, inside the loop of wire and concentric with the loop, a sphere of iron of radius $b < a$ is placed, the field is modified so that the normal gradient of tangential $\mathbf{A}$ is zero at $r = b$. Since all $\mathbf{M}$'s are tangential, we need only add a field which vanishes at $r = \infty$, having normal gradient equal to minus that of the $\mathbf{A}$ above at $r = b$. This is

$$\mathbf{A} = \left(\frac{\pi J}{c}\right) \mathbf{a}_\varphi \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n} n!(n+1)!} P_{2n+1}^1(\cos\vartheta) \cdot$$
$$\cdot \begin{cases} \left[\left(\dfrac{r}{a}\right)^{2n+1} + \dfrac{2n+1}{2n+2}\left(\dfrac{b^2}{ar}\right)^{2n+2}\left(\dfrac{a}{b}\right)\right]; & r < a \\ \left(\dfrac{a}{r}\right)^{2n+2}\left[1 + \dfrac{2n+1}{2n+2}\left(\dfrac{a}{b}\right)^{4n+3}\right]; & r > a \end{cases}$$

A related expression is obtained for $\mathbf{H}$. The sphere of iron increases the magnitude of the nth term given in Eq. (13.2.31) for $r > a$ by a factor

$$\left[1 + \frac{(2n+1)}{(2n+2)}\left(\frac{b}{a}\right)^{4n+3}\right]$$

At large distances from the ring the field reduces to a dipole field

$$\mathbf{H} \to \frac{\pi J}{ac}\left(\frac{a}{r}\right)^3\left[1 + \tfrac{1}{2}\left(\frac{b}{a}\right)^3\right][3\mathbf{a}_r \cos\vartheta - \mathbf{k}]; \quad r \gg a$$

Here the sphere of iron has increased the field strength by a factor $[1 + \frac{1}{2}(b/a)^3]$.

Viscous Flow about a Sphere. The calculation of the flow of an incompressible viscous fluid about spherical boundaries may also involve the Green's dyadic. As long as there is no pressure gradient, Eq. (13.2.5) indicates that the equation for $\mathbf{v}$ is homogeneous and the fluid velocity is that combination of the zero-divergence solutions $\mathbf{M}_{smn}^1$, $\mathbf{M}_{smn}^2$, $\mathbf{N}_{smn}^1$, and $\mathbf{N}_{smn}^2$ which satisfies the boundary conditions. For instance, for fluid between two concentric spheres of radii a and b ($a < b$), the inner one being at rest and the outer rotating with angular velocity ω about the z axis, we choose the linear combination of $\mathbf{M}_1^1$ and $\mathbf{M}_1^2$ which has zero velocity at $r = a$ and velocity $\mathbf{a}_\varphi b\omega \sin\vartheta$ at $r = b$. This is

$$\mathbf{v} = \frac{b^3\omega}{b^3 - a^3}[\mathbf{M}_1^1 - a^3\mathbf{M}_1^2] = \frac{b^3\omega}{r^2}\frac{r^3 - a^3}{b^3 - a^3}\mathbf{a}_\varphi \sin\vartheta \qquad (13.2.32)$$

From Eq. (2.3.10) we see that the stress dyadic in the fluid, for zero pressure, is

$$\mathfrak{T} = \eta(\nabla\mathbf{v} + \mathbf{v}\nabla) = \frac{3\eta b^3\omega}{b^3 - a^3}\left(\frac{a^3}{r^3}\right)(\mathbf{a}_r\mathbf{a}_\varphi + \mathbf{a}_\varphi\mathbf{a}_r)\sin\vartheta$$

using the vector operator formulas at the end of Chap. 1. Finally the torque on the inner sphere is

$$T = 2\pi a^3 \int_0^\pi |\mathfrak{T}\cdot\mathbf{a}_\varphi| \sin^2\vartheta\, d\vartheta = 8\pi\frac{\eta a^3 b^3\omega}{b^3 - a^3}.$$

On the other hand when the fluid motion causes pressure differences in the fluid, the equation for the velocity is an inhomogeneous one, (13.2.5). The vector $\mathbf{Q} = -(1/4\pi\eta)\operatorname{grad} P$ has zero curl and, if the liquid is incompressible, it must also have zero divergence; consequently, it should be a linear combination of the solutions $\mathbf{N}$. As an example of such solutions, let us assume a pressure distribution $p_n = A_n r^{-n} P_{n-1}(\cos\vartheta)$, with gradient

$$\operatorname{grad} p_n = A_n\mathbf{N}_n^2(r,\vartheta,\varphi) = A_n r^{-n-1}[\mathbf{k}T_{n-2}^1(\cos\vartheta) - \mathbf{a}_r T_{n-1}^1(\cos\vartheta)]$$

outside a sphere of radius $r = a$. The velocity distribution is then a solution of the inhomogeneous equation (13.2.5), with enough solution of the homogeneous equation to satisfy the boundary conditions.

The particular integral, for this distribution of pressure, is then

$$\begin{aligned}
\mathbf{v}_i &= -\frac{1}{4\pi\eta}\iiint \mathfrak{G}\cdot\operatorname{grad}(p_n)\, dv \\
&= -\frac{A_n}{4\pi\eta n(2n+1)}\mathbf{D}_{0n}^5(\vartheta,\varphi)\iint [\mathbf{D}_{0n}^5(\vartheta_0,\varphi_0)]^2 \sin\vartheta_0\, d\vartheta_0\, d\varphi_0 \cdot \\
&\qquad\qquad \cdot\left\{r^{-n-1}\int_a^r r\, dr + r^n\int_r^\infty r^{-2n}\, dr\right\} \\
&= \frac{A_n}{\eta(2n-1)(2n+1)}[(n - \tfrac{1}{2})a^2 - (n + \tfrac{1}{2})r^2]\mathbf{N}_n^2(r,\vartheta,\varphi) \\
&= \frac{A_n}{2\eta}\left[\frac{a^2 r^{-n-1}}{(2n+1)} - \frac{r^{-n+1}}{2n-1}\right]\sqrt{n}\,[\sqrt{n-1}\,\mathbf{B}_{0,n-1} - \sqrt{n}\,\mathbf{P}_{0,n-1}]
\end{aligned}$$

But this particular integral does not have zero divergence; in fact, its divergence is $[nA_n/r^n\eta(2n - 1)]P_{n-1}(\cos\vartheta)$, as can be shown by use of the tables at the end of this chapter. We must therefore add a solution of the homogeneous equation which has a divergence to cancel this. The only solution having a divergence of the right sort is, according to Eq. (13.2.22), the function $\mathbf{G}^2_{n-2}$; therefore we see that the function $[nA_n/\eta(n - 1)(2n - 1)(2n - 3)]\mathbf{G}^2_{n-2}$ has a divergence which just cancels that of $\mathbf{v}_i$. Consequently, the fluid velocity outside the sphere is

$$\mathbf{v} = \frac{A_n}{2\eta}\left\{\left[\frac{a^2}{2n+1} - \frac{r^2}{2n-1}\right]\mathbf{N}^2_n(r,\vartheta,\varphi) + \frac{2n}{(n-1)(2n-1)(2n-3)}\mathbf{G}^2_{n-2}(r,\vartheta,\varphi)\right\} + \mathbf{C}(r,\vartheta,\varphi)$$

where $\mathbf{C}$ is the combination of solutions $\mathbf{M}$ and $\mathbf{N}$ which will allow the boundary conditions to be fulfilled.

If the boundary conditions are that $\mathbf{v}$ is zero at $r = a$ and is equal to $B_{n-2}\mathbf{N}^1_{n-2}(r,\vartheta,\varphi)$ for r large compared to a, then we adjust A_n so the $\mathbf{G}^2$ term is canceled by the $\mathbf{N}^1$ term at $r = a$ and we add enough $\mathbf{N}^2$ so that $\mathbf{v} = 0$ at $r = a$. The final result is

$$\mathbf{v} = B_n\left\{[\mathbf{N}^1_n - a^{2n+1}\mathbf{G}^2_n] + \tfrac{1}{2}a^{2n+1}\frac{(n+1)(2n+1)}{(n+2)}(r^2 - a^2)\mathbf{N}^2_{n+2}\right\} \tag{13.2.33}$$

where we have changed from n to $n + 2$. The pressure is equal to $-\eta[B_n/(n + 2)](n + 1)(2n + 1)(2n + 3)(a^{2n+1}/r^{n+2})P_{n+1}(\cos\vartheta)$. If the behavior of $\mathbf{v}$ for $r \gg a$ is given by series $\sum_n B_n\mathbf{N}^1_n$, then the solution $\mathbf{v}$ which satisfies the boundary condition $\mathbf{v} = 0$ at $r = a$ is a series of the solutions given in (13.2.33).

The case of greatest interest is the one for steady flow of a fluid past a stationary sphere of radius a, where the velocity vector for $r \gg a$ is just $\mathbf{k}U = U\mathbf{N}^1_0$, so that $n = 0$ and $B_0 = U$. Consequently, the expression for the velocity, correct for small values of U, where Eq. (13.2.5) is valid, is

$$\begin{aligned}\mathbf{v} &= U[(\mathbf{N}^1_0 - a\mathbf{G}^2_0) + \tfrac{1}{4}a(r^2 - a^2)\mathbf{N}^2_2] \\ &= U\left[\mathbf{k}\left(1 - \frac{a}{r}\right) + \tfrac{1}{4}\left(\frac{a^3}{r^3} - \frac{a}{r}\right)(2\mathbf{a}_r\cos\vartheta + \mathbf{a}_\vartheta\sin\vartheta)\right]\end{aligned} \tag{13.2.34}$$

and the excess pressure is

$$p = -\tfrac{3}{2}(U\eta a/r^2)\cos\vartheta$$

The stress dyadic is, according to Eq. (2.3.10),

$$\begin{aligned}\mathfrak{T} &= \tfrac{3}{2}(U\eta a/r^2)\mathfrak{I} \cos\vartheta + \eta(\nabla\mathbf{v} + \mathbf{v}\nabla) \\ &= \tfrac{3}{2}U\eta(a^3/r^4)\mathfrak{I}\cos\vartheta + \tfrac{9}{2}U\eta\mathbf{a}_r\mathbf{a}_r\left(\frac{a}{r^2} - \frac{a^3}{r^4}\right)\cos\vartheta \\ &\qquad - \tfrac{3}{2}U\eta(\mathbf{a}_r\mathbf{a}_\vartheta + \mathbf{a}_\vartheta\mathbf{a}_r)(a^3/r^4)\sin\vartheta\end{aligned}$$

By symmetry, the net force on the sphere ($r = a$) is in the z direction, so that the net force is

$$\begin{aligned}F &= a^2\iint \mathbf{k}\cdot\mathfrak{T}\cdot\mathbf{a}_r \sin\vartheta\, d\vartheta\, d\varphi; \quad r = a \\ &= \tfrac{3}{2}aU\eta\iint \sin\vartheta\, d\vartheta\, d\varphi = 6\pi aU\eta \end{aligned} \tag{13.2.35}$$

which is *Stokes' law* for the viscous drag on a slowly moving sphere.

We note that the z component of the drag on the sphere is the same on each element of its surface. We note that to this approximation the drag is proportional to the relative velocity of sphere and liquid and also to the radius of the sphere. Finally, we note that the results given in Eqs. (13.2.34) and (13.2.35) are for the steady-state condition, after the relative motion has been going on for a long time. We see that the distortion of the velocity field extends far out from the sphere, because of the term $\mathbf{k}(aU/r)$ particularly. If the relative motion is begun at $t = 0$, it will take quite a while before this field settles down to its steady state.

Elastic Distortion of a Sphere. The homogeneous equation for the steady-state distortion of a sphere,

$$(\lambda + 2\mu)\,\text{grad div } \mathbf{s} - \mu\,\text{curl curl } \mathbf{s} = 0$$

has, for the two transverse sets of solutions, just the sets $\mathbf{M}$ and $\mathbf{N}$ already defined in Eqs. (13.2.17) to (13.2.20). The longitudinal set $\mathbf{G}$ defined in Eqs. (13.2.21), however, is not a solution and we must seek another combination of powers of r and angle functions $\mathbf{B}$ and $\mathbf{P}$ which will satisfy the new equation.

This may be done, using the equations given at the end of this chapter. For instance, to find a solution analogous to $\mathbf{G}_{mn}$, we set

$$\mathbf{E}^1_{mn} = Ar^n\mathbf{B}_{m,n-1} + Br^n\mathbf{P}_{m,n}$$

and operate on this by the differential operators of the equation:

$$\begin{aligned}\text{grad div } \mathbf{E} &= [(n+2)B - \sqrt{n(n-1)}\,A]\mathbf{N}^1_{m,n-2} \\ \text{curl curl } \mathbf{E} &= \sqrt{\frac{n}{n-1}}\,[\sqrt{n(n-1)}\,B - (n+1)A]\mathbf{N}^1_{m,n-2}\end{aligned}$$

from which we can compute the ratio between A and B to satisfy the elastic equation. Likewise a combination $\mathbf{E}^2_{mn}$ can be formed of $r^{-n-1}\mathbf{B}_{m,n+1}$

and $r^{-n-1}\mathbf{P}_{m,n+1}$. These are

$$\mathbf{E}^1_{smn} = r^n\left[\sqrt{n(n-1)}\,\mathbf{E}^s_{m,n-1} + \frac{\lambda(n+2)+\mu(n+4)}{\lambda(n-1)+\mu(n-3)}\,n\mathbf{P}^s_{m,n-1}\right]$$
$$\mathbf{E}^2_{smn} = r^{-n-1}\left[\sqrt{(n+1)(n+2)}\,\mathbf{B}^s_{m,n+1} - \frac{\lambda(n+2)+\mu(n+4)}{\lambda(n-1)+\mu(n-3)}\,(n+1)\mathbf{P}^s_{m,n+1}\right] \tag{13.2.36}$$

They reduce to the functions $\mathbf{G}^1$ and $\mathbf{G}^2$ of Eq. (13.2.21) when $\lambda = -\mu$ (as does the elastic equation reduce to the vector Laplace equation in this case). When $\lambda \neq -\mu$, however, the angular dependence of the $\mathbf{E}$'s is not orthogonal to the angle dependence of the functions $\mathbf{M}$ and $\mathbf{N}$. This makes it more difficult to set up an expression for the Green's dyadic of a form similar to that of Eq. (13.2.28); another form will be found more amenable.

In addition we shall need the expression for the traction on the surface of the sphere, $(\mathfrak{T}\cdot\mathbf{a}_r)_{r=a}$. To obtain this we need to compute the radial traction

$$\mathfrak{T}\cdot\mathbf{a}_r = \mathbf{a}_r\left[\lambda\,\mathrm{div}\,\mathbf{s} + 2\mu\frac{\partial s_r}{\partial r}\right] + \mu\mathbf{a}_\vartheta\left[\frac{1}{r}\frac{\partial s_r}{\partial\vartheta} + r\frac{\partial}{\partial r}\left(\frac{s_\vartheta}{r}\right)\right] + \mu\mathbf{a}_\varphi\left[\frac{1}{r\sin\vartheta}\frac{\partial s_r}{\partial\varphi} + r\frac{\partial}{\partial r}\left(\frac{s_\varphi}{r}\right)\right] \tag{13.2.37}$$

for each of the functions $\mathbf{M}$, $\mathbf{N}$, $\mathbf{E}$, for $r = a$. From the formulas at the end of this chapter, we can obtain the following useful formulas, for the axially symmetric cases, $m = 0$:

$$\begin{aligned}
&\text{For } \mathbf{s} = r^n\mathbf{C}_{0n}; && \mathfrak{T}\cdot\mathbf{a}_r = \mu(n-1)r^{n-1}\mathbf{C}_{0n}\\
&\text{For } \mathbf{s} = r^n\mathbf{B}_{0,n+1}; && \mathfrak{T}\cdot\mathbf{a}_r = -\lambda\sqrt{(n+1)(n+2)}\,r^{n-1}\mathbf{P}_{0,n+1} + \mu(n-1)r^{n-1}\mathbf{B}_{0,n+1}\\
&\text{For } \mathbf{s} = r^n\mathbf{B}_{0,n-1}; && \mathfrak{T}\cdot\mathbf{a}_r = -\lambda\sqrt{n(n-1)}\,r^{n-1}\mathbf{P}_{0,n-1} + \mu(n-1)r^{n-1}\mathbf{B}_{0,n-1}\\
&\text{For } \mathbf{s} = r^n\mathbf{P}_{0,n+1}; && \mathfrak{T}\cdot\mathbf{a}_r = [\lambda(n+2)+2\mu n]r^{n-1}\mathbf{P}_{0,n+1} + \mu\sqrt{(n+1)(n+2)}\,r^{n-1}\mathbf{B}_{0,n+1}\\
&\text{For } \mathbf{s} = r^n\mathbf{P}_{0,n-1}; && \mathfrak{T}\cdot\mathbf{a}_r = [\lambda(n+2)+2\mu n]r^{n-1}\mathbf{P}_{0,n-1} + \mu\sqrt{n(n-1)}\,r^{n-1}\mathbf{B}_{0,n-1}\\
&\text{For } \mathbf{s} = \mathbf{M}^1_n; && \mathfrak{T}\cdot\mathbf{a}_r = \mu(n-1)(1/r)\mathbf{M}^1_n\\
&\text{For } \mathbf{s} = \mathbf{N}^1_n; && \mathfrak{T}\cdot\mathbf{a}_r = 2\mu n(1/r)\mathbf{N}^1_n
\end{aligned} \tag{13.2.38}$$

When the boundary conditions on the elastic sphere are the displacements of its surface, we employ a set of solutions which are mutually orthogonal functions of the angle variables ϑ, φ at the surface $r = a$. Two of the needed three sets are

$$\mathbf{M}^1_{mn} = \sqrt{n(n+1)}\, r^n \mathbf{C}_{mn}(\vartheta,\varphi)$$
$$\mathbf{N}^1_{mn} = \sqrt{(n+1)(n+2)}\, r^n \mathbf{B}_{m,\,n+1}(\vartheta,\varphi) + (n+1) r^n \mathbf{P}_{m,\,n+1}(\vartheta,\varphi)$$

The third set must be a combination of $\mathbf{E}_{mn}$ and $\mathbf{N}_{m,\,n-2}$ which is orthogonal to $\mathbf{N}$ at $r = a$. Such a combination would be

$$\sqrt{n(n-1)}\, a^n \mathbf{B}_{m,\,n-1} - n a^n \mathbf{P}_{m,\,n-1}; \quad \text{at } r = a$$

and this combination is

$$\mathbf{H}^1_{mn} = \frac{1}{\lambda(4n-1) + \mu(8n-3)} \{n(2n+1)(\lambda+\mu)a^2\mathbf{N}^1_{m,\,n-2} - (2n-1)[\lambda(n-1) + \mu(n-3)]\mathbf{E}^1_{mn}\} \tag{13.2.39}$$

With these three sets we can compute solutions corresponding to any displacement of the boundary surface. As an example, suppose we flatten one portion of the spherical surface. The displacement of the surface is then

$$\mathbf{s}_{r=a} = \begin{cases} -a\mathbf{k}(\cos\vartheta - \cos\vartheta_0); & 0 \le \vartheta \le \vartheta_0 \\ 0; & \vartheta_0 \le \vartheta \le \pi \end{cases} \tag{13.2.40}$$

But, as is shown in Eq. (13.2.24), we can set up the function

$$\begin{aligned} \frac{1}{2n+1}[\mathbf{N}^1_n - \mathbf{H}^1_n] &= \frac{1}{2n+1}[\sqrt{(n+1)(n+2)}\,\mathbf{B}_{n+1} + (n+1)\mathbf{P}_{n+1}] \\ &\quad + \frac{(r^n/2n+1)\sqrt{n(n-1)}}{\lambda(4n-1)+\mu(8n-3)}\Big\{(2n-1)[\lambda(n-1)+\mu(n-3)] \\ &\qquad - n(2n+1)(\lambda+\mu)(a/r)^2\Big\}\mathbf{B}_{n-1} \\ &\quad + \frac{nr^n/(2n+1)}{\lambda(4n-1)+\mu(3n-3)}\Big\{(2n-1)[\lambda(n+2)+\mu(n+4)] \\ &\qquad - n(2n+1)(n-1)(\lambda+\mu)(a/r)^2\Big\}\mathbf{P}_{n-1} \\ &\xrightarrow[r\to a]{} \mathbf{k}a^nP_n(\cos\vartheta) \end{aligned} \tag{13.2.41}$$

and from the tables at the end of Chap. 10, we can show that

$$\begin{aligned} F(\vartheta) &= \begin{cases} -a(\cos\vartheta - \cos\vartheta_0); & 0 \le \vartheta \le \vartheta_0 \\ 0; & \vartheta_0 \le \vartheta \le \pi \end{cases} \\ &= -\tfrac{1}{2}a\sum_{n=0}^{\infty}\left[\frac{P_{n+2}(\cos\vartheta_0)}{(2n+3)} - \frac{2(2n+1)}{(2n+3)(2n-1)}P_n(\cos\vartheta_0) \right. \\ &\qquad \left. + \frac{P_{n-2}(\cos\vartheta_0)}{(2n-1)}\right]P_n(\cos\vartheta) \end{aligned}$$

where we stipulate that $P_{-1} = 1$ and $P_{-2}(\cos\vartheta) = P_1(\cos\vartheta)$. From these two items we construct a displacement function which satisfies the boundary condition of Eq. (13.2.40) and is finite everywhere within the sphere

$$\mathbf{s} = -\tfrac{1}{2}\sum_{n=0}^{\infty} a^{1-n}\left[\frac{P_{n+2}(\cos\vartheta_0)}{(2n+1)(2n+3)} - \frac{2P_n(\cos\vartheta_0)}{(2n+3)(2n-1)} + \frac{P_{n-2}(\cos\vartheta_0)}{(2n+1)(2n-1)}\right][\mathbf{N}_n^1(r,\vartheta,\varphi) - \mathbf{H}_n^1(r,\vartheta,\varphi)] \quad (13.2.42)$$

$$\begin{aligned}\mathbf{s} = -\tfrac{1}{2}a\{&[\tfrac{2}{3} - \cos\vartheta_0 + \tfrac{1}{3}P_2(\cos\vartheta_0)]\mathbf{k} \\ &+ [1 - \tfrac{6}{5}\cos\vartheta_0 + \tfrac{1}{5}P_3(\cos\vartheta_0)]\left(\frac{r}{a}\right)\mathbf{k}P_1(\cos\vartheta) \\ &+ [\tfrac{1}{3} - \tfrac{10}{21}P_2(\cos\vartheta_0) + \tfrac{1}{7}P_4(\cos\vartheta_0)]\left[\left(\frac{r}{a}\right)^2 P_2(\cos\vartheta) - \frac{2(\lambda+\mu)}{7\lambda+13\mu}\left(\frac{a^2-r^2}{a^2}\right)\right]\mathbf{k} + \cdots\end{aligned}$$

which is the required solution. As the terms for $n > 2$ are not parallel to the z axis, the displacement $\mathbf{s}$ is not entirely along $\mathbf{k}$.

By using Eqs. (13.2.42) we can compute the stress in the sphere

$$\begin{aligned}\mathfrak{T} = &-[1 - \tfrac{6}{5}P_1(\cos\vartheta_0) + \tfrac{1}{5}P_3(\cos\vartheta_0)][\tfrac{1}{2}\lambda\mathfrak{J} + \mu\mathbf{kk}] \\ &- \frac{1}{a}[\tfrac{1}{3} - \tfrac{10}{21}P_2(\cos\vartheta_0) + \tfrac{1}{7}P_4(\cos\vartheta_0)]\left\{3\lambda z\frac{3\lambda+5\mu}{7\lambda+13\mu}\mathfrak{J}\right. \\ &\left.+ 6\mu z\frac{3\lambda+5\mu}{7\lambda+13\mu}\mathbf{kk} - \tfrac{3}{2}\mu\frac{\lambda+3\mu}{7\lambda+13\mu}[x(\mathbf{ik}+\mathbf{ki}) + y(\mathbf{jk}+\mathbf{kj})]\right\} \\ &- \cdots \qquad (13.2.43)\end{aligned}$$

which converges absolutely for $r < a$.

Tractions Specified on Surface. There are times when the traction is specified on the surface of the sphere, rather than the displacement, the displacement being the quantity to be determined. For instance, suppose the traction at $r = a$ is

$$[\mathfrak{T}\cdot\mathbf{a}_r]_{r=a} = \begin{cases}\alpha\mathbf{a}_\varphi\sin\vartheta; & 0 \le \vartheta < \vartheta_0 \\ 0; & \vartheta_0 < \vartheta < \pi - \vartheta_0 \\ -\alpha\mathbf{a}_\varphi\sin\vartheta; & \pi - \vartheta_0 < \vartheta \le \pi\end{cases}$$

which corresponds to the polar regions of the sphere (out to latitude ϑ_0) being twisted in opposite directions by being fastened to rods of great rigidity, which are rotated in opposite directions with torque T. The constant α is related to this torque by the equation

$$T = \tfrac{2}{3}\pi a^3\alpha(1 - \cos\vartheta_0)^2(2 + \cos\vartheta_0)$$

In this case we need only the solutions $\mathbf{M}_n^1(r,\vartheta,\varphi) = \mathbf{a}_\varphi r^n P_n^1(\cos\vartheta)$ and, as a matter of fact, we need only these functions for even values of $n > 0$, for we wish a solution which changes sign when ϑ is changed to $\pi - \vartheta$. We, therefore, set

$$\mathbf{s} = \sum_{n=0}^{\infty} A_n \mathbf{M}^1_{2n+2}; \quad (\mathfrak{T} \cdot \mathbf{a}_r) = \mathbf{a}_\varphi \mu \sum_{n=0}^{\infty} (2n+1) A_n r^{2n+1} P^1_{2n+2}(\cos\vartheta)$$

where P^1_{2n+2} is one of the associated Legendre functions discussed in Chap. 10. To determine the constants A_n, we multiply both sides of the equation for $(\mathfrak{T} \cdot \mathbf{a}_r)_{r=a}$ by $P^1_{2m+2} \sin\vartheta\, d\vartheta$ and integrate over ϑ

$$\begin{aligned}
\mu(2n+1)A_n a^{2n+1} \int_{-1}^{1} [P^1_{2n+2}(x)]^2\, dx &= 2\mu \frac{2n+1}{4n+5} A_n a^{2n+1}(2n+2)(2n+3) \\
&= 2\alpha \int_{\cos\vartheta_0}^{1} \sqrt{1-x^2}\, P^1_{2n+2}(x)\, dx \\
&= \frac{2\alpha}{(2n+1)(2n+4)} \sin^2\vartheta_0 P^2_{2n+2}(\cos\vartheta_0)
\end{aligned}$$

$$A_n = \frac{(3T/2\pi\mu)a^{-2n-4}}{(1-\cos\vartheta_0)^2(2+\cos\vartheta_0)} \left[\frac{(4n+5)(2n)!}{(2n+1)(2n+4)!}\right] \sin^2\vartheta_0 P^2_{2n+2}(\cos\vartheta_0)$$

Consequently, the displacement $\mathbf{s}$ of the part of the sphere at point (r, ϑ, φ) is

$$\begin{aligned}
\mathbf{s} = \mathbf{a}_\varphi &\frac{(3T/2\pi\mu a^2)}{(1-\cos^2\vartheta_0)^2(2+\cos\vartheta_0)} \cdot \\
&\cdot \sum_{n=0}^{\infty} \frac{(4n+5)(2n)!}{(2n+1)(2n+4)!} [\sin^2\vartheta_0 P^2_{2n+2}(\cos\vartheta_0)] \left(\frac{r}{a}\right)^{2n+2} P^1_{2n+2}(\cos\vartheta) \\
&\to \mathbf{a}_\varphi \left(\frac{T}{4\pi\mu a^2}\right) \sum_{n=0}^{\infty} \frac{4n+5}{2n+1} \left(\frac{r}{a}\right)^{2n+2} P^1_{2n+2}(\cos\vartheta); \quad \vartheta_0 \to 0 \qquad (13.2.44)
\end{aligned}$$

which indicates the large amount of twisting near the region of application of the torque and the reduction of this farther from the areas of application. A use of the tables at the end of Chap. 10 will show that

$$\sum_{n=0}^{\infty} \left(\frac{4n+5}{2n+1}\right) x^{2n+2} P^1_{2n+2}(z) = 3x\sqrt{1-z^2} \int_0^x \frac{1-x^2}{x} \left[\frac{1}{(1+x^2-2xz)^{\frac{3}{2}}} - \frac{1}{(1+x^2+2xz)^{\frac{3}{2}}}\right] dx$$

so that a closed analytic expression for $\mathbf{s}$ may be obtained for $\vartheta_0 \to 0$.

Many other examples could be worked out for the behavior of an elastic sphere, but no new principles would exhibit themselves. A Green's function can be arrived at by a limiting process from the elastic wave solutions, but this will be discussed in the next section.

13.3 *Vector Wave Solutions*

As we have seen in Sec. 13.1, the solutions of the vector Helmholtz equation, $\nabla^2\mathbf{A} + k^2\mathbf{A} = 0$, are more regular in their behavior than are the solutions of the vector Laplace equation, for $k = 0$, discussed in Sec. 13.2. Many of the wave solutions are quite similar to the scalar wave solutions discussed in Chap. 11. We shall attempt here to point out these similarities but chiefly to emphasize the points of difference.

The fields we spend most of our time on will be the electromagnetic field, the elastic medium, and the motions of a viscous fluid, though there are several other manifestations of vector wave fields in nature. The equations for these fields, and the boundary conditions which they satisfy, have already been outlined in Chap. 2 and again in Sec. 13.1. They will be taken up again as needed.

The only additional comment needed here concerns the time-dependent equation for viscous flow. When the fluid velocity is small enough (see pages 162 and 1792), Eq. (2.3.14) becomes

$$\nabla^2\mathbf{v} - (\rho/\eta)(\partial\mathbf{v}/\partial t) = (1/\eta)\,\mathrm{grad}\,P \tag{13.3.1}$$

where $\mathrm{div}\,\mathbf{v} = 0$ and $P = p + V$. This is not the inhomogeneous vector wave equation, it is the vector diffusion equation; consequently, the solutions are not vector waves. We discuss the solutions here because of their vector properties and also because separating off the time dependence (in an exponential factor) leaves the equation for the spatial factor a vector Helmholtz equation, which is similar to the equations dealt with for electromagnetic and elastic waves. The techniques for solution are thus sufficiently similar to make it advantageous to take them up here rather than elsewhere.

Reflection of Plane Waves from a Plane. A plane electromagnetic wave in vacuum of frequency $\omega/2\pi = kc/2\pi$ and of direction of propagation parallel to the unit vector $\mathbf{a}_i = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta$ is

$$\begin{aligned}\mathbf{A} &= (c/i\omega)E_0\mathbf{a}_e e^{ik(\mathbf{a}_i\cdot\mathbf{r}-ct)};\quad \varphi = 0\\ \mathbf{E} &= -(\partial\mathbf{A}/c\,\partial t) = E_0\mathbf{a}_e e^{ik(\mathbf{a}_i\cdot\mathbf{r}-ct)}\\ \mathbf{H} &= \mathbf{a}_i\times\mathbf{E} = E_0\mathbf{a}_h e^{ik(\mathbf{a}_i\cdot\mathbf{r}-ct)}\end{aligned} \tag{13.3.2}$$

where* $\mathbf{a}_i$, $\mathbf{a}_e = -\mathbf{a}_x\sin\theta\sin\psi + \mathbf{a}_y\cos\theta\sin\psi + \mathbf{a}_z\cos\psi = \mathbf{a}_h\times\mathbf{a}_i$ and $\mathbf{a}_h = \mathbf{a}_i\times\mathbf{a}_e = \mathbf{a}_x\sin\theta\cos\psi - \mathbf{a}_y\cos\theta\cos\psi + \mathbf{a}_z\sin\psi$ constitute a right-hand, orthogonal set of unit vectors at the Euler angles θ and ψ to the x, y, z axes. In terms of a reflecting surface at the y, z plane, the angle θ is the angle of incidence and (since the direction of polarization

* In this section we again use $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ instead of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ to obviate confusion with $i = \sqrt{-1}$ and $k = \omega/c$.

of the wave is along **H**) the angle ψ measures the angle between the direction of polarization $\mathbf{a}_h$, and the x, y, plane, which is called the *plane of incidence*. This is shown in Fig. 13.3.

If the boundary surface at $x = 0$ is effectively a perfect conductor, we must have that $\mathbf{E} \times \mathbf{a}_x = 0$ at $x = 0$ and, therefore, that $\mathbf{A} \times \mathbf{a}_x = 0$. Since div **A** is also zero, this tangential boundary condition on both components of tangential **A** is sufficient to fix the whole solution. To the incident wave must be added a reflected wave, with direction of propagating along $\mathbf{a}'_i = -\mathbf{i}\cos\theta + \mathbf{j}\sin\theta$, of such an angle χ between the direction of polarization and the plane of incidence (and reflection) that its

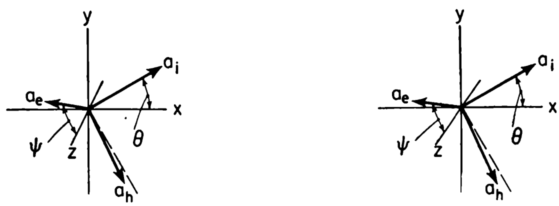

Fig. 13.3 Vectors used in describing reflection of transverse wave from y-z plane.

tangential electric field will cancel that of the incident wave. For a perfect conductor at $x = 0$, this obtains when $\chi = \psi$ and the incident plus reflected waves are (for $x < 0$),

$$\begin{aligned}
\mathbf{E} &= E_0[\mathbf{a}_e e^{ikx\cos\theta} - \mathbf{a}'_e e^{-ikx\cos\theta}]e^{iky\sin\theta - i\omega t} \\
&= 2E_0[-\mathbf{a}_x \sin\theta \sin\psi \cos(kx\cos\theta) \\
&\qquad + (\mathbf{a}_y \cos\theta \sin\psi + \mathbf{a}_z \cos\psi)i\sin(kx\cos\theta)]e^{iky\sin\theta - i\omega t} \\
\mathbf{H} &= 2E_0[\mathbf{a}_x \sin\theta \cos\psi\, i\sin(kx\cos\theta) \\
&\qquad - (\mathbf{a}_y \cos\theta\cos\psi - \mathbf{a}_z\sin\psi)\cos(kx\cos\theta)]e^{iky\sin\theta - i\omega t}
\end{aligned} \tag{13.3.3}$$

where $\mathbf{a}'_e = \mathbf{a}_x \sin\theta\sin\psi + \mathbf{a}_y\cos\theta\sin\psi + \mathbf{a}_z\cos\psi$ and $\mathbf{a}'_h = \mathbf{a}'_i \times \mathbf{a}'_e$. Vector **H** is still everywhere perpendicular to **E**.

We defer consideration of less simple boundary conditions for electromagnetic waves until we take up the simple reflection of elastic waves. In the case of elastic waves we can have both longitudinal (compressional) and transverse (shear) waves

$$\begin{aligned}
\mathbf{s}_c &= S_c\mathbf{a}_i e^{ik_c\mathbf{a}_i\cdot\mathbf{r} - i\omega t}; \quad \mathbf{s}_s = S_s\mathbf{a}_h e^{ik_s\mathbf{a}_i\cdot\mathbf{r} - i\omega t} \\
k_c &= \omega/c_c; \quad k_s = \omega/c_s
\end{aligned} \tag{13.3.4}$$

where the two different wave velocities c_c and c_s are given in Eqs. (2.2.2) and (2.2.3). The corresponding stress dyadics are given in Eqs. (2.2.14) and (2.2.16):

$$\begin{aligned}
\mathfrak{T}_c &= ik_c[\lambda\mathfrak{J} + 2\mu\mathbf{a}_i\mathbf{a}_i]S_c e^{ik_c\mathbf{a}_i\cdot\mathbf{r} - i\omega t} \\
\mathfrak{T}_s &= ik_s\mu[\mathbf{a}_i\mathbf{a}_h + \mathbf{a}_h\mathbf{a}_i]S_s e^{ik_s\mathbf{a}_i\cdot\mathbf{r} - i\omega t}
\end{aligned}$$

The boundary condition at the surface $x = 0$ would be that $\mathfrak{T} \cdot \mathbf{a}_x = 0$ if the surface is a free surface, with no medium for positive x. In order to satisfy this, we must add reflected waves; even if the incident wave is purely compressional, a certain amount of shear wave is reflected unless the angle of incidence is zero. For an incident compressional wave, for example, the combination is

$$\mathbf{s} = S_i \mathbf{a}_i e^{ik_c \mathbf{a}_i \cdot \mathbf{r} - i\omega t} + S' \mathbf{a}_i' e^{ik_c \mathbf{a}_i' \cdot \mathbf{r} - i\omega t} + S'' \mathbf{a}_i'' \times \mathbf{a}_z e^{ik_s \mathbf{a}_i'' \cdot \mathbf{r} - i\omega t}$$

where the direction of the reflected shear wave is not equal to the direction of the reflected compressional wave because $k_c \neq k_s$ and we must have $k_c \sin\theta = k_s \sin\theta'$ in order that the dependence on y of all the waves match on the surface $x = 0$. Also we have set $\psi' = 0$ because of the symmetry of the problem, so that $\mathbf{a}_h'' = \mathbf{a}_i'' \times \mathbf{a}_z$ in which $\mathbf{a}_i'' = -\mathbf{a}_x \cos\theta' + \mathbf{a}_y \sin\theta'$. The traction across $x = 0$ is then

$$(\mathfrak{T} \cdot \mathbf{a}_x)_0 = i\{k_c[\lambda \mathbf{a}_x + 2\mu \mathbf{a}_i \cos\theta] S_i + k_c[\lambda \mathbf{a}_x - 2\mu \mathbf{a}_i' \cos\theta] S' \\ + k_s \mu [\mathbf{a}_i'' \sin\theta' - (\mathbf{a}_i'' \times \mathbf{a}_z)\cos\theta'] S''\} e^{ik_c y \sin\theta - i\omega_t}$$

which must be set zero by suitable adjustment of S' and S''. This has the following solution: $\sin\theta' = (c_s/c_c)\sin\theta$, and

$$S' = -S_i \frac{\lambda + 2\mu\cos^2\theta - \mu \sin 2\theta \tan 2\theta'}{\lambda + 2\mu\cos^2\theta + \mu\sin 2\theta \tan 2\theta'}$$

$$S'' = 2S_i \frac{\lambda + 2\mu\cos^2\theta}{\lambda + 2\mu\cos^2\theta + \mu\sin 2\theta\tan 2\theta'} \frac{\sin 2\theta}{\cos 2\theta'}$$

for the amplitudes of reflected compressional and shear waves. The angle θ' of the reflected shear wave is less than $\frac{1}{4}\pi$ for $0 \leq \theta \leq \frac{1}{2}\pi$ because $(c_s/c_c)^2 = \mu/(\lambda + 2\mu)$ which is always less than $\frac{1}{2}$. Consequently, $\tan(2\theta')$ never reaches infinity. For normal incidence $S'' = 0$ and the reflected wave is purely compressional.

Boundary Impedance. When the material of the reflecting plane is not a perfect reflector, the boundary conditions are somewhat more arduous a task. We utilize the ideas of impedance, developed in Chap. 3, to put them in as tractable a form as possible.

Starting with the electromagnetic field, we note first that, since div $\mathbf{A}$ is supposed to be zero in the gauge under consideration, we need to impose only two boundary conditions to specify the field. We choose these to be the relation between tangential components of $\mathbf{E}$ and $\mathbf{H}$, because these components are continuous across a boundary, whereas the normal components have a discontinuity. The fact that the divergences of $\mathbf{E}$ and $\mathbf{H}$ are specified on both sides of the boundary assures that the normal components will automatically satisfy their boundary conditions if tangential $\mathbf{E}$ and $\mathbf{H}$ are made continuous across the boundary.

But we can narrow the requirements still further if we remember, from Eq. (3.4.23) that, for each point in space for a simple-harmonic wave, we can set up a dyadic "ratio" between $c\mathbf{H}$ and $-4\pi\mathbf{E}$ which may be called the impedance dyadic of the wave at that point. Here we are not concerned with the dyadic changing the full vector **H** into **E**, but with the two-dimensional dyadic transforming the tangential components of **H** into the tangential components of **E**. Such a dyadic is not a proper three-dimensional dyadic, in that it will not survive a rotation of axes in three space, but it is a perfectly good dyadic in the two space of the boundary surface (which is all we need here).

We define the dyadic, $\mathfrak{Z}_t$, the *normal impedance* for the boundary under consideration, by the equation

$$4\pi(\mathbf{n} \times \mathbf{E}) = c\mathfrak{Z}_t \cdot [\mathbf{H} - \mathbf{n}(\mathbf{n} \cdot \mathbf{H})] \tag{13.3.5}$$

where **n** is the unit vector normal to the surface, pointing into the surface [$\mathbf{n} = \mathbf{a}_x$ in the example of Eq. (13.3.3)] and the quantity in brackets on the right is just a fancy way to write the tangential part of **H**. The two-dimensional dyadic $\mathfrak{Z}_t$ is not simply expressed in terms of the three-dimensional field impedance dyadic $\mathfrak{Z}$ of Eq. (3.4.23), but in any particular case it can be worked out, as we shall shortly see.

If now we can work out the components of the normal impedance $\mathfrak{Z}_t$ just inside the surface ($x = +\delta$), then we must adjust the ratio of tangential **E** and **H** just outside the surface ($x = -\delta$) so that $\mathfrak{Z}_t$ *is continuous across the surface.* If we are not interested in the magnitude of the field inside the surface, this requirement of continuity of normal impedance dyadic is the *sole boundary condition required.*

To see how this works out, let us work out the normal impedance dyadic, just inside the surface $x = 0$, for a plane wave traveling away from the surface (x increasing), which would correspond to a transmitted wave arising from an incident wave striking the surface from outside ($x < 0$). The angle between the direction of transmission and the normal to the surface **i** is taken to be θ', which is to be related to the angle of incidence θ later. We assume the medium to have any value of dielectric constant ϵ, permeability μ, and conductivity σ and use Eq. (2.5.19) and that following it as the equations of motion for **A**. We call the complex quantity n, where

$$n^2 = \mu\epsilon + (4\pi i\mu\sigma/\omega)$$

the *index of refraction* of the medium. Then, taking different amplitudes for the case when the magnetic vector is parallel to the plane of incidence (x,y), and the case when **H** is normal to it, we have, for $x > 0$,

$$\mathbf{A} = (1/ik)[E'_\perp(-\mathbf{a}_x \sin\theta' \sin\psi' + \mathbf{a}_y \cos\theta' \sin\psi') + E'_\parallel \mathbf{a}_z \cos\psi']e^{ikn(x\cos\theta' + y\sin\theta') - i\omega t}$$

and

$$\begin{aligned} 4\pi\mathbf{a}_x \times \mathbf{E} &= 4\pi(-\mathbf{a}_y E'_{\parallel} + \mathbf{a}_z E'_{\perp} \cos\theta') e^{ikn(x\cos\theta' + y\sin\theta') - i\omega t} \\ c[\mathbf{H} - \mathbf{a}_x(\mathbf{a}_x \cdot \mathbf{H})] &= (cn/\mu)(-\mathbf{a}_y E'_{\parallel}\cos\theta' + \mathbf{a}_z E'_{\perp}) e^{ikn(x\cos\theta' + y\sin\theta') - i\omega t} \\ k &= \omega/c \end{aligned}$$

These vectors are not parallel, but the normal impedance dyadic has a quite simple form:

$$\mathfrak{Z}_t^+ = (4\pi\mu/cn)[\mathbf{a}_y\mathbf{a}_y \sec\theta' + \mathbf{a}_z\mathbf{a}_z\cos\theta']; \quad x > 0 \tag{13.3.6}$$

which is independent of the values of E'_n and $E'_{\perp}$ and is constant throughout the extent of the simple plane wave, and thus certainly constant over the area of the surface illuminated by the incident wave. All we need now is to adjust the values of incident and reflected wave so that the corresponding value of $\mathfrak{Z}$ just outside the surface ($x < 0$) is equal to this dyadic at $x = 0$.

We note that only the simple plane wave produces an impedance as simple as this. Incidentally, we also note that, when the index of refraction n is large (either because $\mu\epsilon$ is large or because $\mu\sigma/\omega$ is large), the normal impedance dyadic of the surface is small.

The field for a plane wave of unit amplitude, at an angle of incidence θ and angle of polarization ψ, plus a reflected wave with unknown amplitudes $R_{\parallel}$ and $R_{\perp}$ for the two polarizations, is, for $x < 0$,

$$\mathbf{A} = \frac{1}{ik}\{[-\mathbf{a}_x\sin\theta\sin\psi + \mathbf{a}_y\cos\theta\sin\psi + \mathbf{a}_z\cos\psi]e^{ikx\cos\theta} \\ + [R_{\perp}(\mathbf{a}_x\sin\theta + a_y\cos\theta) + R_{\parallel}\mathbf{a}_z]e^{-ikx\cos\theta}\}e^{iky\sin\theta - i\omega t}$$

and, at $x = 0$,

$$\begin{aligned} 4\pi\mathbf{a}_x \times \mathbf{E} &= 4\pi[-\mathbf{a}_y(\cos\psi + R_{\parallel}) + \mathbf{a}_z\cos\theta(\sin\psi + R_{\perp})]e^{iky\sin\theta - i\omega t} \\ c[\mathbf{H} - \mathbf{a}_x(\mathbf{a}_x\cdot\mathbf{H})] &= c[-\mathbf{a}_y\cos\theta(\cos\psi - R_{\parallel}) + \mathbf{a}_z(\sin\psi - R_{\perp})]e^{iky\sin\theta - i\omega t} \end{aligned}$$

so that the impedance at $x = 0$ is

$$\mathfrak{Z}_t^- = \frac{4\pi}{c}\left[\mathbf{a}_y\mathbf{a}_y\sec\theta\,\frac{\cos\psi + R_{\parallel}}{\cos\psi - R_{\parallel}} + \mathbf{a}_z\mathbf{a}_z\cos\theta\,\frac{\sin\psi + R_{\perp}}{\sin\psi - R_{\perp}}\right]$$

We note that this impedance, for incident and reflected waves, depends on x (the formula for $x \neq 0$ is obtained by multiplying each R in the formula for $\mathfrak{Z}$ for $x = 0$ by $e^{-2ikx\cos\theta}$), whereas the impedance for a single plane wave is independent of x, y, or z.

First we need to relate the angle of transmission θ' to the angle of incidence (and reflection) θ. We do this by noting that the tangential fields are continuous across the boundary and, therefore, that the dependence on y (and z and t) is the same on both sides of the boundary. We have equated the dependence on t, there is no dependence on z, and the dependence on y will match if the angle of incidence θ and the angle of refraction θ' satisfy *Snell's law*

$$n \sin \theta' = \sin \theta; \quad \cos \theta' = \sqrt{1 - (1/n)^2 \sin^2 \theta}$$

Next, in order that $\mathfrak{Z}_t^-$ at $x = 0$ may be equal to $\mathfrak{Z}_t^+$, we must adjust the reflected amplitude so that

$$\begin{aligned} R_\parallel &= -\cos \psi \left[\frac{1 - (\mu/n)(\cos \theta/\cos \theta')}{1 + (\mu/n)(\cos \theta/\cos \theta')} \right] \\ R_\perp &= -\sin \psi \left[\frac{1 - (\mu/n)(\cos \theta'/\cos \theta)}{1 + (\mu/n)(\cos \theta'/\cos \theta)} \right] \end{aligned} \tag{13.3.7}$$

which give the reflected amplitudes for the two polarizations. We note that by this method we need not compute the transmitted intensity in order to get the R's, only the impedance is needed. These formulas give the well-known equations for the reflected intensities of polarized beams.

If n is large, either because $\epsilon\mu$ or $\mu\sigma/\omega$ is large, then the angle of refraction θ' is always small. Since the conductivity σ of most metals is of the order of 10^{11} esu, the quantity $\mu\sigma/\omega$ is large even for frequencies of the order of megacycles and the values of $\cos \theta'$ and $\sec \theta'$ in Eq. (13.3.6) differ from unity by less than 0.01 per cent. Consequently the normal impedance dyadic for a metal surface perpendicular to the x axis is $\mathfrak{Z}_t \simeq 4\pi\mu/cn(\mathbf{a}_y\mathbf{a}_y + \mathbf{a}_z\mathbf{a}_z)$. Using the approximate form

$$n \simeq \sqrt{2\pi\mu\sigma/\omega}\,(1 + i); \quad \mu\sigma \gg \omega$$

and generalizing the form of the dyadic for a unit normal $\mathbf{n}$ to the surface in any direction, we have, finally, for the normal impedance dyadic of a metal surface,

$$\mathfrak{Z}_t \cong -(4\pi/c)\sqrt{\mu\omega/4\pi\sigma}\,(\mathbf{n} \times \mathfrak{J} \times \mathbf{n})e^{-\frac{1}{4}i\pi} \tag{13.3.8}$$

where $\mathbf{n}$ is the normal into the surface (making an acute angle with the incident direction). This expression will be of considerable use later.

It is particularly useful when we can separate our field solutions into ones of the type $\mathbf{M}$ and $\mathbf{N}$ of Sec. 13.1, one tangential to the surface and the other with its curl tangential to the surface. The $\mathbf{M}$ type of solution corresponds to $\mathbf{A}$ and $\mathbf{E}$ tangential to the surface and $\mathbf{H} = \text{curl}\,\mathbf{A}$; the $\mathbf{N}$ type corresponds to $\mathbf{H}$ tangential. For the $\mathbf{M}$ type, $\mathbf{E}$ tangential, we can write $\mathbf{E} = (i\omega/c)\mathbf{A} = \mathbf{a}_z E(x,y)$ where we have chosen the x direction in the direction of $\mathbf{n}$ and the z direction that in the direction of $\mathbf{E}$, which is perpendicular to $\mathbf{a}_x$. Since $\text{div}\,\mathbf{E} = 0$, the derivative of $\mathbf{E}$ with respect to z must be zero at the point in question. From Eqs. (2.5.18), for a simple harmonic field, in vacuum, $\mathbf{H} = (c/i\omega)[(\partial E/\partial y)\mathbf{a}_x - (\partial E/\partial x)\mathbf{a}_y]$ so the normal impedance is $\mathfrak{Z}_t = (4\pi i\omega/c^2)[E/(\partial E/\partial x)](\mathbf{a}_y\mathbf{a}_y + \mathbf{a}_z\mathbf{a}_z)$.

Comparing this with Eq. (13.3.8) we can say that the boundary conditions at the surface of a metal are such that when the vector potential **A** is tangential to the surface the relation between its magnitude at the surface and the normal gradient of its magnitude at the surface (into the surface) is

$$A = \frac{1}{ik}\sqrt{\frac{\mu\omega}{4\pi\sigma}}\, e^{-\frac{1}{4}i\pi}\left(\frac{\partial A}{\partial n}\right); \quad k = \omega/c \tag{13.3.9}$$

for **M** type solutions. The values of μ and σ are, of course, for the material beyond the boundary surface ($x > 0$). The corresponding relation for the **N** type solution (curl $\mathbf{A} = \mathbf{H}$ tangential) is

$$H = \frac{1}{ik}\sqrt{\frac{4\pi\sigma}{\mu\omega}}\, e^{\frac{1}{4}i\pi}\left(\frac{\partial H}{\partial n}\right)$$

When σ is very large (as it is with metals), **A** must be very small compared to $\partial\mathbf{A}/\partial n$ if **A** and **E** are parallel to the surface, but $\partial\mathbf{H}/\partial n$ must be very small compared to **H** if $\mathbf{H} =$ curl **A** is parallel to the surface.

Elastic Wave Reflection. In the case of electromagnetic waves, the divergence condition (div $\mathbf{A} = 0$) is supposed to hold not only at the boundary but throughout space: consequently, only two boundary conditions are needed to be specified. We picked the tangential components and thus were able to express our boundary condition in terms of the two-dimensional dyadic $\mathfrak{Z}_t$. There is no similar over-all requirement for the elastic displacement, so three boundary conditions must be satisfied at the surfaces $x = 0$. Part of the requirements may be expressed in terms of an impedance, of the sort given on page 1816, but it represents no simplification of procedure in this case, so we may as well match displacements and tractions at the surface.

For an example, take the case of the reflection of elastic plane waves at the interface $x = 0$ between two media, one ($x < 0$) having elastic constants λ_- and μ_- and the other ($x > 0$) having constants λ_+ and μ_+. If the direction of the incident wave (taken to be pure compressional) is parallel to $\mathbf{a}_i = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta$ coming in from the left, then the angle of reflection of the compressional wave is also θ but the angle of reflection of the shear wave, θ', and the corresponding angles θ_+ and θ'_+ for the refracted waves in the second medium are related by the requirement that the y dependence of all waves correspond at $x = 0$, giving the generalized Snell's laws:

$$(1/c_c^-)\sin\theta = (1/c_s^-)\sin\theta' = (1/c_c^+)\sin\theta_+ = (1/c_s^+)\sin\theta'_+$$
$$(c_c^-)^2 = [(\lambda_- + 2\mu_-)/\rho_-]; \quad (c_s^+)^2 = \mu_+/\rho_+$$

The expressions for the displacement and stress are

$$\mathbf{s} = \begin{cases} [\mathbf{a}_i e^{ik_c^- \mathbf{r}\cdot\mathbf{a}_i} + \mathbf{a}_i' C e^{ik_c^- \mathbf{r}\cdot\mathbf{a}_i'} + (\mathbf{a}_i'' \times \mathbf{a}_z) D e^{ik_s^- \mathbf{r}\cdot\mathbf{a}_i''}] e^{-i\omega t}; & x < 0 \\ [A\mathbf{a}_r e^{ik_c^+ \mathbf{r}\cdot\mathbf{a}_r} + B(\mathbf{a}_r'' \times \mathbf{a}_z) e^{ik_s^+ \mathbf{r}\cdot\mathbf{a}_r''}] e^{-i\omega t}; & x > 0 \end{cases} \tag{13.3.10}$$

$$\mathfrak{T} = \begin{cases} ik_c^- \{(\lambda_- \mathfrak{J} + 2\mu_- \mathbf{a}_i\mathbf{a}_i) e^{ik_c^- \mathbf{r}\cdot\mathbf{a}_i} + C(\lambda_- \mathfrak{J} + 2\mu_- \mathbf{a}_i'\mathbf{a}_i') e^{ik_c^- \mathbf{r}\cdot\mathbf{a}_i'} \\ \qquad + D\mu_-(c_c^-/c_s^-)[\mathbf{a}_i''(\mathbf{a}_i'' \times \mathbf{a}_z) + (\mathbf{a}_i'' \times \mathbf{a}_z)\mathbf{a}_i''] e^{ik_s^- \mathbf{r}\cdot\mathbf{a}_i''}\}; & x < 0 \\ ik_c^+ \{A(\lambda_+ \mathfrak{J} + 2\mu_+ \mathbf{a}_r\mathbf{a}_r) e^{ik_c^+ \mathbf{r}\cdot\mathbf{a}_r} \\ \qquad + B\mu_+(c_c^+/c_s^+)[\mathbf{a}_r''(\mathbf{a}_r'' \times \mathbf{a}_z) + (\mathbf{a}_r'' \times \mathbf{a}_z)\mathbf{a}_r''] e^{ik_s^+ \mathbf{r}\cdot\mathbf{a}_r''}\}; & x > 0 \end{cases}$$

for an incident compressional wave of unit amplitude, where

$$k_c^- c_c^- = \omega, \text{ etc.}; \quad \mathbf{a}_i' = -\mathbf{i}\cos\theta + \mathbf{j}\sin\theta; \quad \mathbf{a}_i'' = -\mathbf{i}\cos\theta' + \mathbf{j}\sin\theta'$$
$$\mathbf{a}_r = \mathbf{i}\cos\theta_+ + \mathbf{j}\sin\theta_+; \quad \mathbf{a}_r'' = \mathbf{i}\cos\theta_+' + \mathbf{j}\sin\theta_+'$$

By equating the pairs of expressions for the x and y components of $\mathbf{s}$ and the x and y components of $\mathbf{a}_x \cdot \mathfrak{T}$, the traction at the surface,

$$(1 - C)\cos\theta + D\sin\theta' = A\cos\theta_+ + B\sin\theta_+'; \quad \text{etc.}$$

we obtain four simultaneous equations which may be solved for A, B, C, D, the amplitudes of the two reflected and two refracted waves compared to the unit-amplitude incident wave.

Waves in a Duct. The next arrangement of boundaries in order of difficulty is that of the inside of a duct or wave guide of rectangular cross section of width a (in the x direction) and height b (in the y direction). Assuming, first, that the duct extends indefinitely in the z direction, we shall investigate a wave propagated in the positive z direction.

If the wave is an electromagnetic one and the interior of the duct is equivalent to vacuum as far as propagation goes, then we can use the boundary conditions of Eqs. (13.3.9) *et seq.*, assuming that the walls are metallic. We start with the case of perfect conductivity, where, if $\mathbf{E}$ is tangential, $\mathbf{A}_t$ is zero at the surface and, if $\mathbf{H}$ is tangential, the normal gradient of $\mathbf{H}$ is zero at the surface. The function $\mathbf{M}$ cannot be tangential to the whole of the duct surface, but it can be perpendicular to the z axis, tangential to one pair of duct walls, and normal to the other, which is quite satisfactory.

We consequently call the $\mathbf{M}$ type solution the *transverse electric* wave, setting

$$\mathbf{A} = \frac{1}{ik}\mathbf{E} = \frac{1}{ik}\operatorname{curl}[\mathbf{a}_z\psi(x,y)\, e^{ik_z z - i\omega t}]$$

where ψ is a solution of the two-dimensional, scalar Helmholtz equation $\nabla^2\psi + k_t^2\psi = 0$ and where $c^2(k_z^2 + k_t^2) = c^2k^2 = \omega^2$. The boundary conditions on ψ are those which make tangential $\mathbf{A}$ and $\mathbf{E}$ zero on the surface. This turns out to make the gradient $\partial\psi/\partial n$ go to zero on the surface everywhere; in other words, for the *transverse electric waves*

$$\mathbf{A} = \frac{1}{ik}\mathbf{E} = \mathbf{M}_{mn} = \frac{1}{ik}\,\mathrm{curl}\left[\mathbf{a}_z \cos\left(\frac{\pi m x}{a}\right)\cos\left(\frac{\pi n y}{b}\right) e^{ik_z z - i\omega t}\right]$$
$$= \left[i\mathbf{a}_x\left(\frac{\pi n}{bk}\right)\cos\left(\frac{\pi m x}{a}\right)\sin\left(\frac{\pi n y}{b}\right) - i\mathbf{a}_y\left(\frac{\pi m}{ak}\right)\sin\left(\frac{\pi m x}{a}\right)\cos\left(\frac{\pi n y}{b}\right)\right] e^{ik_z z - i\omega t}$$
$$k_z^2 = k^2 - k_{mn}^2;\quad k_{mn}^2 = (\pi m/a)^2 + (\pi n/b)^2;\quad k = \omega/c \tag{13.3.11}$$
$$\mathbf{H} = \mathrm{curl}(\mathbf{M}_{mn}) = \left\{-k_z\left[\mathbf{a}_x\left(\frac{\pi m}{ak}\right)\sin\left(\frac{\pi m x}{a}\right)\cos\left(\frac{\pi n y}{b}\right) + \mathbf{a}_y\left(\frac{\pi n}{bk}\right)\cos\left(\frac{\pi m x}{a}\right)\sin\left(\frac{\pi n y}{b}\right)\right] - ik\mathbf{a}_z\left[\left(\frac{\pi m}{ak}\right)^2 + \left(\frac{\pi n}{bk}\right)^2\right]\cos\left(\frac{\pi m x}{a}\right)\cos\left(\frac{\pi m y}{b}\right)\right\} e^{ik_z z - i\omega t}$$

For waves in the opposite direction along z, we simply reverse the sign of k_z.

These waves are dispersive, having a phase velocity $c_z = \omega/k_z$ greater than c; in fact, if m and n are large enough, c_z is imaginary and the corresponding waves are attenuated along z. Comparison with Eq. (11.3.18) shows that for electromagnetic waves the boundary conditions rule out the purely longitudinal wave ($m,n = 0$) which, for the scalar (acoustic) case, corresponds to transmission tangential to the duct walls. A purely longitudinal wave could not have the tangential electric intensity

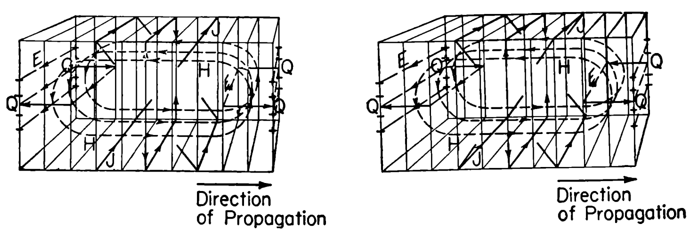

Fig. 13.4 Fields, surface currents, and charges for the lowest transverse electric wave along a rectangular wave guide.

zero at all duct walls, so all waves must involve a certain amount of to-and-fro reflection from the side walls. This produces the dispersion. All waves have a cutoff frequency $\omega_{mn} = ck_{mn}$, below which the wave is attenuated, rather than transmitted.

We also note that, although $\mathbf{E}$ is perpendicular to the duct axis, $\mathbf{H}$ is not. This does not mean that the wave is not a transverse one, however, for its divergence is zero and $\mathbf{H}$ is perpendicular to $\mathbf{E}$ everywhere; the wave is a combination of plane waves at various angles to the z axis, and there is no reason to expect *both* $\mathbf{E}$ and $\mathbf{H}$ to be perpendicular to $\mathbf{a}_z$.

An examination of the field and the charge-current-density distribution along the duct walls for the lowest wave, for $m = 0$, $n = 1$ if $b > a$ (note that *either* m or n can be zero, but not both) shows how the wave propagates. The fields are [we take the real parts of (13.3.11)]:

$$\mathbf{E} = -\mathbf{a}_x(\pi/b)\sin(\pi y/b)\cos(k_z z - \omega t);\quad k_z^2 = (\omega/c)^2 - (\pi/b)^2$$
$$\mathbf{H} = -\mathbf{a}_y\left(\frac{\pi k_z}{kb}\right)\sin\left(\frac{\pi y}{b}\right)\cos(k_z z - \omega t) + \mathbf{a}_z\left(\frac{\pi^2}{kb^2}\right)\cos\left(\frac{\pi y}{b}\right)\sin(k_z z - \omega t)$$

for $\omega > \omega_{01} = \pi c/b$. These are shown in Fig. 13.4 for the range $0 \le (k_z z - \omega t) \le \frac{1}{2}\pi$, the electric lines of force being dashed lines and the magnetic lines dot-dashed.

The charge density Q on the $x = 0$ wall is $(1/4\pi)\mathbf{a}_x \cdot \mathbf{E}_{x=0}$, with similar expressions for the other three walls. According to the discussion preceding Eq. (3.4.24), the surface current density on the wall $x = 0$ is $(c/4\pi)\mathbf{a}_x \times \mathbf{H}_{x=0}$. Consequently, the surface charge and current densities are, for $x = 0$,

$$Q = -(1/4b)\sin(\pi y/b)\cos(k_z z - \omega t)$$
$$\mathbf{J} = -\mathbf{a}_z(ck_z/4bk)\sin(\pi y/b)\cos(k_z z - \omega t) - \mathbf{a}_y(c\pi^2/4\pi kb^2)\cos(\pi y/b)\sin(k_z z - \omega t)$$

For $x = a$, the signs of Q and $\mathbf{J}$ are reversed. For the walls $y = 0$ and $y = b$, $Q = 0$ and

$$\mathbf{J} = \mathbf{a}_x(c\pi^2/4kb^2)\sin(k_z z - \omega t)$$

At $(k_z z - \omega t) = 0$, the two vertical walls ($x = 0,a$) have their maximum charge and the electric field between them has its maximum value; the only current being on these vertical walls, in the direction of propagation (z axis) so that $\mathbf{H}$ may be vertical and $\mathbf{S} = (c/4\pi)\mathbf{E} \times \mathbf{H}$, which equals $\frac{1}{4}\pi\mathbf{a}_z(k_z c/kb^2)\sin^2(\pi y/b)$, has maximum value, twice its average value, pointed in the z direction. At $(k_z z - \omega t) = \frac{1}{2}\pi$ (one-fourth wavelength ahead of zero phase, in space, one-fourth period before it, in time) the charge is zero and the current across the horizontal walls is a maximum; the electric field is zero and the magnetic field is in the z direction. This wave consists, essentially, of the motion of charge from one vertical wall to the other twice a cycle, with the concomitant currents and fields caused by (or causing, it is much the same) the charge flow. The higher transverse electric modes correspond to more complicated oscillations of surface charges.

The current and electric field expressions make it possible to calculate an equivalent impedance for this lowest wave. The total current going to and fro in the z direction along the face $x = 0$ is, at $z = 0$,

$$I = -(ck_z/2\pi k)e^{-i\omega t}$$

and the average potential difference between the walls $x = 0$ and $x = l$ is

$$V = -2ae^{-i\omega t}$$

The ratio might be called the impedance of this lowest mode and $1/a$ of the ratio would be the impedance per unit width of duct

$$Z = 4\pi k/ck_z = [4\pi\omega/c\sqrt{\omega^2 - (c/b)^2}]$$

The impedance aZ would be that measured if we tried to drive the duct by feeding the current as specified into the faces $x = 0$ and $x = l$ at $z = 0$.

Incidentally this value of the impedance, considered simply as ratio of potential drop (per unit width) to total current, turns out to be just equal to the impedance of the field in the z direction, as defined in Eq. (3.4.23). The vector $-c\mathbf{a}_z \times \mathbf{H}$ is $-\mathbf{a}_x(\pi ck_z/kb)\sin(\pi y/b)e^{-i\omega t}$, which is the equivalent of current, and the vector $4\pi\mathbf{E}$ is the equivalent of voltage. The ratio is just that given above. As a matter of fact, by Eq. (3.4.23) the impedance of the (m,n)th wave, given in Eq. (13.3.11), is just

$$Z_{mn} = (4\pi k/ck_z) = [4\pi\omega/c\sqrt{\omega^2 - (\pi cm/a)^2 - (\pi cn/b)^2}] \qquad (13.3.12)$$

If $\omega > \omega_{mn}$, this is resistive; if $\omega < \omega_{mn}$, Z is imaginary, corresponding to an inductive reactance.

The $\mathbf{N}$ type modes are (using the definitions of k_{mn} and k_z as before),

$$\begin{aligned}\mathbf{A} = \frac{\mathbf{E}}{ik} = \mathbf{N}_{mn} &= \frac{1}{k^2}\operatorname{curl}\operatorname{curl}\left[\mathbf{a}_z \sin\left(\frac{\pi mx}{a}\right)\sin\left(\frac{\pi ny}{b}\right)e^{ik_zz-i\omega t}\right] \\ &= \left\{\frac{ik_z}{k^2}\left[\left(\frac{\pi m}{a}\right)\mathbf{a}_x\cos\left(\frac{\pi mx}{a}\right)\sin\left(\frac{\pi ny}{b}\right) + \left(\frac{\pi n}{b}\right)\mathbf{a}_y\sin\left(\frac{\pi mx}{a}\right)\cos\left(\frac{\pi ny}{b}\right)\right]\right. \\ &\qquad \left. + \left(\frac{k_{mn}^2}{k^2}\right)\mathbf{a}_z\sin\left(\frac{\pi mx}{a}\right)\sin\left(\frac{\pi ny}{b}\right)\right\}e^{ik_zz-i\omega t} \qquad (13.3.13) \\ \mathbf{H} &= \left[\mathbf{a}_x\left(\frac{\pi n}{b}\right)\sin\left(\frac{\pi mx}{a}\right)\cos\left(\frac{\pi ny}{b}\right)\right. \\ &\qquad \left. - \mathbf{a}_y\left(\frac{\pi m}{a}\right)\cos\left(\frac{\pi mx}{a}\right)\sin\left(\frac{\pi ny}{b}\right)\right]e^{ik_z-zi\omega t}\end{aligned}$$

and are called *transverse magnetic waves.* (Here also, to reverse the direction of the wave, we reverse the sign of k_z.) The lowest of these waves is for $n = m = 1$ (we note that *neither* m nor n can be zero for this type) as depicted in Fig. 13.5. Current and charge oscillate longitudinally, with $\mathbf{E}$ going from a ring of plus charge to one of minus and $\mathbf{H}$ making transverse loops inside the region of greatest surface current. We note, incidentally, that no function of the set $\mathbf{M}$ corresponds to a function of set $\mathbf{N}$. Moreover, the longitudinal impedance of these waves, the ratio between $4\pi c\mathbf{a}_z \times \mathbf{E}$ and $-\mathbf{H}$, is just equal to that given in Eq. (13.3.12).

The Green's Function. In order to solve many problems of engineering interest in wave guide design, we need the Green's dyadic in simple form. We start by noting that we can write the sets **M** and **N** in the following simple notation:

$$\begin{aligned}\mathbf{M}_{mn} &= (1/ik)\,\mathrm{curl}[\mathbf{a}_z\psi_{mn}\exp(iz\sqrt{k^2-k_{mn}^2}-ikct)]\\ &= \mathbf{B}_{mn}(x,y)\exp(iz\sqrt{k^2-k_{mn}^2}-ikct)\\ \mathbf{N}_{mn} &= (1/k^2)\,\mathrm{curl\,curl}[(\mathbf{a}_z\chi_{mn})\exp(iz\sqrt{k^2-k_{mn}^2}-ikct)]\\ &= \mathbf{C}_{mn}(x,y)\exp[iz\sqrt{k^2-k_{mn}^2}-ikct]\end{aligned}$$

where

$$\begin{aligned}\mathbf{B}_{mn} &= (i/k)[\mathbf{a}_z \times \mathrm{grad}\,\psi_{mn}]\\ \mathbf{C}_{mn} &= (k_{mn}^2/k^2)\mathbf{a}_z\chi_{mn} + (i\sqrt{k^2-k_{mn}^2/k^2})\,\mathrm{grad}\,\chi_{mn}\end{aligned}$$

and where the functions ψ_{mn} and χ_{mn} are mutually orthogonal, scalar eigenfunctions, solutions of the two-dimensional equation

$$(\partial^2\psi/\partial x^2) + (\partial^2\psi/\partial y^2) + (k_{mn}^e)^2\psi = 0$$

(The eigenvalue for χ is k_{mn}^h which equals k_{mn}^e for the rectangular duct

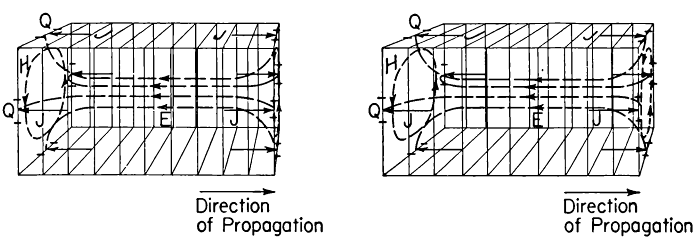

Fig. 13.5 Fields, surface currents, and charges for the lowest transverse magnetic wave along a rectangular wave guide.

but may not in other cases.) For a rectangular duct, $\psi_{mn} = \cos(\pi mx/a)\cdot\cos(\pi ny/b)$ and $\chi_{mn} = \sin(\pi mx/a)\sin(\pi ny/b)$. (At conductor boundary $\partial\psi/\partial n = \chi = 0$).

There is, in addition, the longitudinal set

$$\begin{aligned}\mathbf{L}_{mn} &= (1/ik)\,\mathrm{grad}[\chi_{mn}\exp(iz\sqrt{k^2-k_{mn}^2}-ikct)]\\ &= \mathbf{D}_{mn}(x,y)\exp[iz\sqrt{k^2-k_{mn}^2}-ikct]\\ \mathbf{D}_{mn} &= \frac{1}{k}\sqrt{k^2-k_{mn}^2}\,\mathbf{a}_z\chi_{mn} - \frac{i}{k}\,\mathrm{grad}\,\chi_{mn}\end{aligned}$$

which is orthogonal to the sets **M** and **N**. We note that the space parts of **L** and of **N** contain different linear combinations of the two terms $\mathbf{a}_z\chi_{mn}$ and grad χ_{mn}. Of course, if we do not wish to separate into transverse and longitudinal sets, we could use $\mathbf{a}_z\chi_{mn}$ and $(1/ik)$ grad χ_{mn}, which are both orthogonal to the set $\mathbf{M}_{mn}$ and to each other.

Now suppose we wish to solve the equation

$$\nabla^2\mathfrak{G} + k^2\mathfrak{G} = -4\pi\delta(x - x_0)\delta(y - y_0)\delta(z - z_0)e^{-ikct}\mathfrak{J}$$

subject to appropriate boundary conditions on the duct surfaces and at the far ends of the duct. We can assume first that the full expansion of $\mathfrak{G}$ (including the longitudinal part) has the form

$$\mathfrak{G}(\mathbf{r}|\mathbf{r}_0|k) = \sum_{mn} \{[\mathbf{a}_z \times \operatorname{grad} \psi_{mn}]\mathbf{F}_{mn}(\mathbf{r}_0,z) + \mathbf{a}_z\chi_{mn}\mathbf{G}_{mn}(\mathbf{r}_0,z) \\ + \operatorname{grad} \chi_{mn}\mathbf{H}_{mn}(\mathbf{r}_0,z)\} e^{-ikct}$$

Inserting this into the equation for $\mathfrak{G}$, multiplying through by $\mathbf{a}_z \times \operatorname{grad} \psi_{mn}$, $\mathbf{a}_z\chi_{mn}$, and $\operatorname{grad} \chi_{mn}$ in turn, and integrating over x and y, we find expressions for $\mathbf{F}$, $\mathbf{G}$, and $\mathbf{H}$. We set $\iint \chi_{mn}^2\, dx\, dy = \Lambda_{mn}^h$ (for the rectangular duct it is $ab/\epsilon_m\epsilon_n$) and we have, by Green's theorem,

$$\iint |\mathbf{a}_z \times \operatorname{grad} \psi_{mn}|^2\, dx\, dy = \iint (\operatorname{grad} \psi_{mn}) \cdot (\operatorname{grad} \psi_{mn})\, dx\, dy \\ = \oint \psi \operatorname{grad}_n \psi\, dA - \iint \psi\nabla^2\psi\, dx\, dy = k_{mn}^2\Lambda_{mn}^e$$

where $\Lambda_{mn}^e = \iint \psi_{mn}^2\, dx\, dy$ (for the rectangular duct this is also $ab/\epsilon_m\epsilon_n$).

We find, for instance, that

$$\mathbf{F}_{mn} = \frac{4\pi}{k_{mn}^2\Lambda_{mn}^e} [\mathbf{a}_z \times \operatorname{grad} \psi_{mn}(x_0,y_0)] f_{mn}(z,z_0)e^{-ikct}$$

where f satisfies the equation

$$(\partial^2 f/\partial z^2) + (k^2 - k_{mn}^2)f = -\delta(z - z_0)$$

and is subject to the appropriate boundary conditions. In the present case, with outgoing waves in both directions, away from the source,

$$f_{mn} = \frac{-1}{2i\sqrt{k^2 - k_{mn}^2}} \exp(i|z - z_0|\sqrt{k^2 - k_{mn}^2})$$

Similarly,

$$\mathbf{G}_{mn} = \frac{4\pi}{\Lambda_{mn}^h} [\mathbf{a}_z\chi_{mn}(x_0,y_0)] g_{mn}(z,z_0)e^{-ikct}$$

where, for outgoing waves, g is the same as f. (If the tube were terminated at a given pair of values of z, g would differ from f because $\mathbf{G}$ points along the duct whereas $\mathbf{F}$ and $\mathbf{H}$ point across the duct.)

Eventually, we find that

$$\mathfrak{G}(\mathbf{r}_0|\mathbf{r}|k) = 4\pi \sum_{m,n} \left\{ \frac{\mathbf{a}_z\mathbf{a}_z}{\Lambda_{mn}^h} \chi_{mn}(x_0,y_0)\chi_{mn}(x,y) g_{mn} \right. \\ + \frac{1}{k_{mn}^2\Lambda_{mn}^e} [\mathbf{a}_z \times \operatorname{grad} \psi_{mn}(x_0,y_0)][\mathbf{a}_z \times \operatorname{grad} \psi_{mn}(x,y)] f_{mn} \\ \left. + \frac{1}{k_{mn}^2\Lambda_{mn}^h} [\operatorname{grad} \chi_{mn}(x_0,y_0)][\operatorname{grad} \chi_{mn}(x,y)] f_{mn} \right\} \quad (13.3.14)$$

This Green's function includes both longitudinal and transverse parts, which may be used to obtain transverse solutions, as indicated on page 1783. However, if we prefer to have $\mathfrak{G}$ include only the transverse part, we should transfer back to the vectors **B** and **C** and leave out the **D** terms. To expand $\mathfrak{G}$ in terms of **B**, **C**, and **D**, multiply both sides by $\bar{\mathbf{B}}_{mn}$, for example, and integrate over x and y, which determines the appropriate coefficient of the new expansion. Eventually we can write

$$\begin{aligned}\mathfrak{G}(\mathbf{r}|\mathbf{r}_0|k) = 4\pi \sum_{m,n} \Bigg\{ & \frac{(k/k^e_{mn})^2}{\Lambda^e_{mn}} \bar{\mathbf{B}}_{mn}(x_0,y_0)\mathbf{B}_{mn}(x,y)f_{mn} \\ & + \frac{(k/k^h_{mn})^2}{\Lambda^h_{mn}} \left[\left(\frac{k^h_{mn}}{k}\right)^2 \mathbf{a}_z\chi_{mn}(x_0,y_0)g_{mn} \right. \\ & \left. - (i/k^2)\sqrt{k^2 - (k^h_{mn})^2}\,\mathrm{grad}_0\,\chi_{mn}(x_0,y_0)f_{mn} \right] \mathbf{C}_{mn}(x,y) \\ & + \frac{1}{\Lambda^h_{mn}} \left[\frac{1}{k}\sqrt{k^2 - (k^h_{mn})^2}\,\mathbf{a}_z\chi_{mn}(x_0,y_0)g_{mn} \right. \\ & \left. + \frac{1}{k}\,\mathrm{grad}_0\,\chi_{mn}(x_0,y_0)f_{mn} \right] \mathbf{D}_{mn}(x,y) \Bigg\} e^{-ikct}\end{aligned}$$

It should be emphasized that this type of expansion is satisfactory only when $f_{mn} = g_{mn}$.

Specializing to the case of the rectangular duct of infinite extent, where $g_{mn} = f_{mn} = (i/2\sqrt{k^2 - k^2_{mn}})\exp(i|z - z_0|\sqrt{k^2 - k^2_{mn}})$, $(k^e_{mn})^2 = (k^h_{mn})^2 = k^2_{mn} = (\pi m/a)^2 + (\pi n/b)^2$, and $\Lambda^e_{mn} = \Lambda^\epsilon_{mn} = ab/\epsilon_m\epsilon_n$, and writing down only the transverse part, we have, finally,

$$\begin{aligned}\mathfrak{G}_t(\mathbf{r}|\mathbf{r}_0|k) = \left(\frac{2\pi ik}{ab}\right) \Bigg\{ & \sum_{k_{mn}<k} \left(\frac{\epsilon_m\epsilon_m}{k^2_{mn}\tau_{mn}}\right) [\bar{\mathbf{B}}_{mn}(x_0,y_0)\mathbf{B}_{mn}(x,y) \\ & + \bar{\mathbf{C}}_{mn}(x_0,y_0)\mathbf{C}_{mn}(x,y)]e^{ik\tau_{mn}|z-z_0|} \\ & - ik \sum_{k_{mn}>k} \left(\frac{\epsilon_m\epsilon_n}{k^2_{mn}\kappa_{mn}}\right) [\bar{\mathbf{B}}_{mn}(x_0,y_0)\mathbf{B}_{mn}(x,y) \\ & + \bar{\mathbf{C}}_{mn}(x_0,y_0)\mathbf{C}_{mn}(x,y)]e^{-\kappa_{mn}|z-z_0|} \Bigg\} e^{-ikct}\end{aligned} \tag{13.3.15}$$

where $\tau^2_{mn} = 1 - (k_{mn}/k)^2$ and $\kappa^2_{mn} = k^2_{mn} - k^2$ and where the vectors **B** and **C** are the coefficients of $\exp(ik_z z - i\omega t)$ in the expressions for **A** in Eqs. (13.3.11) and (13.3.13), respectively. The first series in this expression contains all the terms which give rise to genuine wave motion along the duct; the second contains all the waves which attenuate rapidly because they are excited below their cutoff frequency. As we shall see later, the first series gives rise to a resistive term in the driving impedance; the second series, to a reactive term. Incidentally, since $\mathfrak{G}_t$ is a symmetric dyadic, we can reverse the order of $\bar{\mathbf{B}}$ and $\mathbf{B}$, etc. For other boundary conditions at the ends of the duct, it is usually better to use Eq. (13.3.14).

Generation of Waves by Wire. To see more clearly how this Green's function is used to solve problems of practical interest, we compute the field produced in a rectangular wave guide of infinite length, of inner dimensions a in the x direction, and b in the y direction, by a wire, carrying current I placed along the line $z = 0$, $y = \frac{1}{2}b$. We use Eq. (13.1.52) where, since we want only the transverse part of the field, we can use $\mathfrak{G}_t$ instead of the full dyadic $\mathfrak{G}$, which includes the longitudinal part. The inhomogeneous vector $\mathbf{Q}$ is, in the present case (with $\mu = \epsilon = 1$), $\mathbf{Q} = (1/c)\mathbf{a}_x I\delta(y - \frac{1}{2}b)\delta(z)$, where we have divided out the time factor $e^{-i\omega t}$ since we are solving the Helmholtz equation for the space part.

The only complicated part of the calculation is the integration of the various vector functions $\bar{\mathbf{B}}_{mn}(x_0,y_0)$, $\bar{\mathbf{C}}_{mn}(x_0,y_0)$ times the function $\mathbf{Q}$ over x_0 and y_0,

$$\int_0^a dx \int_0^b dy\, \bar{\mathbf{B}}_{mn} \cdot \mathbf{Q} = \begin{cases} 0; & m > 0 \text{ or } n = 2, 4, 6, \ldots \\ -i(\pi n/bkc)I(-1)^{\frac{1}{2}n-\frac{1}{2}}; & m = 0 \text{ and } n = 1, 3, 5, \ldots \end{cases}$$

$$\int_0^a dx \int_0^b dy\, \bar{\mathbf{C}}_{mn} \cdot \mathbf{Q} = 0 \text{ for all allowed values of } m \text{ and } n.$$

If, however, we insist that the current be carried by an infinitesimal wire, the electric field at the wire will be infinite (at least that part producing the reactance). So we "spread out" the delta function over y making the current pass through a strip of width Δ centrally located in the duct. The only difference this will make in the integrals will be that the ones for $\mathbf{B}$, for $m = 0$ and $n = 1, 3, 5, \ldots$, will then become $-i(2I/\Delta kc)(-1)^{\frac{1}{2}n-\frac{1}{2}}\sin(\pi n\Delta/2b)$, reducing to $-i(\pi nI/bkc)(-1)^{\frac{1}{2}n-\frac{1}{2}}$ for n small, but differing from this for n sufficiently large, enough so the series converges.

We can then compute the integral of $\mathfrak{G}_t \cdot \mathbf{Q}$ [the surface integral of Eq. (13.1.52) is zero because of the boundary conditions]. For the case where the driving frequency is above the lowest cutoff frequency ($n = 1$) but below the next one ($n = 3$), this is, for $z > 0$,

$$\mathbf{A} = i\mathbf{a}_x\left(\frac{8\pi I}{abc}\right)\frac{\sin(\pi y/b)}{\sqrt{k^2 - (\pi/b)^2}}\exp\left[i\sqrt{k^2 - \left(\frac{\pi}{b}\right)^2}\, z\right]$$
$$+ \mathbf{a}_x\left(\frac{16I}{ac}\right)\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)}\frac{\sin[\pi(2n+1)y/b]}{\sqrt{(\pi/b)^2(2n+1)^2 - k^2}}\sin\left[\frac{\pi\Delta}{2b}(2n+1)\right]\cdot$$
$$\cdot\exp[-z\sqrt{(\pi/b)^2(2n+1)^2 - k^2}]$$

where we have included the effect of the finite width Δ of the strip in the sum $(n > 0)$ but not in the first term. The electric intensity at $y = \frac{1}{2}b$, $z = 0$, the center of the strip carrying current, is then

$$\mathbf{E} \simeq -\mathbf{a}_x \frac{(8\pi I\omega/abc)}{\sqrt{\omega^2 - (\pi c/b)^2}} + \mathbf{a}_x i\omega\left(\frac{16Ib}{\pi a\Delta c^2}\right) \sum_{n=1}^{\infty} \frac{\sin[(\pi\Delta/2b)(2n+1)]}{(2n+1)\sqrt{(2n+1)^2 - (kb/\pi)^2}}$$

The second series does not converge well, so it is advisable to subtract and add a series nearly like it, which can be summed. By analytic continuation (first making ζ complex, in the first quadrant, and then letting it become real and small) we find that

$$\sum_{n=1}^{\infty} \frac{\sin(2n+1)\zeta}{(2n+1)^2} = -\sin\zeta + \mathrm{Re}\left[\int_0^\zeta \sum_{n=0}^{\infty} e^{i\eta(2n+1)} \frac{d\eta}{2n+1}\right]$$
$$= -\sin\zeta - \tfrac{1}{2}\,\mathrm{Re}\left[\int_0^\zeta \ln\left(\frac{1-e^{i\eta}}{1+e^{i\eta}}\right) d\eta\right] \simeq \tfrac{1}{2}\zeta\ln(2/e^2\zeta); \quad \zeta \ll 1$$

Consequently, the potential drop along the wire, the voltage which must be applied to produce the current which generates the field, is

$$V = -aE_x \simeq \frac{(8\pi I\omega/bc)}{\sqrt{\omega^2 - (\pi c/b)^2}} - i\omega\left(\frac{4I}{c^2}\right)[\ln(4b/\pi e^2\Delta) + 2\Sigma] \qquad (11.3.16)$$

where

$$\Sigma = \sum_{n=1}^{\infty}\left[\frac{1}{\sqrt{(2n+1)^2 - (kb/\pi)^2}} - \frac{1}{2n+1}\right] \simeq 0.0266(\omega b/\pi c)^2; \quad kb \ll 3\pi$$

is a small correction term to the closed-formula, logarithmic approximation to the series.

The ratio V/I is the input impedance of the wire in the wave guide, for waves going to infinity in both directions, for frequencies between ω_{01} and ω_{03}. If one or both ends of the duct are closed, so as to reflect back the wave, the first term of Eq. (11.3.16) will be considerably changed but the reactive term will be very little modified, unless the closure is very close to the wire, for the "waves" causing this term do not extend far from the wire. The first term, divided by I, is, of course, the resistance and the second term, over I, is the reactance, which is inductive as long as the quantity in brackets is positive (*i.e.*, if Δ is small enough). Near $\omega = \omega_{01} = \pi c/b$, the resistive term is large, going to infinity as $\omega \to \omega_{01}$. For ω less than ω_{01}, the impedance is entirely reactive and no true wave goes along the duct in either direction. We note also that the resistive part of V/I is equal to $2/b$ times the impedance Z_{01}, given in Eq. (13.3.12) for the (0,1) mode. From the first term of the impedance, the resistive term, R, we can compute the power radiated away from the wire, $P = \frac{1}{2}RI^2$.

Losses along the Duct Walls. When the duct walls are not perfectly conducting, the tangential electric intensity at the walls is not exactly zero at the walls and the longitudinal wave number k_z becomes complex, corresponding to attenuation of the waves because of energy absorption at the walls. According to Eqs. (13.3.5) and (13.3.8), the vector $\mathbf{n} \times \mathbf{E}$ should equal $-\sqrt{\mu\omega/4\pi\sigma}\, e^{-\frac{1}{4}i\pi}$ (a small factor) times the tangential part of $\mathbf{H}$. Since we have adjusted the factors $\pi m/a$ and $\pi n/b$ in $\mathbf{M}_{mn}$ and $\mathbf{N}_{mn}$ to make $\mathbf{n} \times \mathbf{E}$ equal to zero at the boundary, we must modify them so that $\mathbf{n} \times \mathbf{E}$ is small but not zero there. We can do this, for instance for the $\mathbf{M}$ set, by changing $\pi mx/a$ into $(\pi x/a)[m - (2\varphi_{mx}/\pi)] + \varphi_{mx} = \Omega_{mx}$ where φ_x is small compared to $\frac{1}{2}\pi$. In this case the factor $\sin(\pi mx/a)$ occurring in the x component of $\mathbf{M}_{mn}$ becomes $\sin \varphi_{mx}$ at $x = 0$ and $(-1)^{m-1} \sin \varphi_{mx}$ at $x = a$. (We are tacitly assuming that the conductivity of all walls is equal so that the boundary conditions are similar at both parallel walls.)

The tangential electric vector at the $x = 0$ wall is then (remembering that $\mathbf{n}$ is *into* the wall and thus equals $-\mathbf{a}_x$ in this case)

$$(\mathbf{n} \times \mathbf{E})_{x=0} = -\mathbf{a}_z(\pi/a)[m - (2\varphi_{mx}/\pi)] \sin \varphi_{mx} \cos \Omega_{ny}$$

for the transverse electric field. But the tangential magnetic field for $\mathbf{M}_{mn}$ has a y component as well as an x component, so the transverse electric wave cannot, by itself, satisfy the new boundary conditions, we must add a small amount of the corresponding $\mathbf{N}_{mn}$ wave. Consequently, we set $\mathbf{A} = \mathbf{M}_{mn} + \alpha_{mn}\mathbf{N}_{mn}$ and proceed to compute α, φ_{mx}, and φ_{ny} by fitting boundary conditions at the duct walls, $x = 0$ and $y = 0$. For instance, for the $x = 0$ wall

$$-\mathbf{a}_x \times \mathbf{E} \simeq -\mathbf{a}_z \left[\frac{\pi m}{a} - \alpha\left(\frac{\pi n}{b}\right)\frac{k_z}{k}\right] \varphi_{mx} \cos \Omega_{ny} + \mathbf{a}_y \alpha (ik_{mn}^2/k) \varphi_{mx} \sin \Omega_{ny}$$
$$= -iZ\mathbf{H}_{\text{tan}} = -\mathbf{a}_z(k_{mn}^2/k) Z \cos \Omega_{ny} + \mathbf{a}_y i \left[\left(\frac{\pi n}{b}\right)\frac{k_z}{k} + \alpha\left(\frac{\pi m}{a}\right)\right] Z \sin \Omega_{ny}$$

where $Z = \sqrt{\mu\omega/4\pi\sigma}\, e^{\frac{1}{4}\pi i}$ and where we have assumed that φ and Z are both small enough so we can neglect terms in φ^2, φZ, etc.

There is a similar set of equations for the $y = 0$ wall. Between them we can calculate the correction factors φ_{xm}, φ_{yn}, and α. What is then apparent is that the fields for Z small are not greatly different from those for $Z = 0$, except that there is a small, residual tangential $\mathbf{E}$ on the surface, proportional to $\mathbf{H}$, owing to the fact that the nodes of the functions $\sin \Omega_{mx}$, etc., do not now exactly coincide with the duct walls. The only place where there is a noticeable difference from the $Z = 0$ functions is at the boundaries, for the **E** functions. When we finish the computations, we find that the φ's are proportional to the small quantity $Z = \sqrt{\mu\omega/8\pi\sigma}\,(1 + i)$ and that the transmission wave number k_z is now

$$k_z = \left[k^2 - \left(\frac{\pi}{a}\right)^2\left(m - \frac{2}{\pi}\varphi_{mx}\right)^2 - \left(\frac{\pi}{b}\right)^2\left(n - \frac{2}{\pi}\varphi_{ny}\right)^2\right]^{\frac{1}{2}}$$
$$\simeq \sqrt{k^2 - (\pi m/a)^2 - (\pi n/b)^2} + i\,(\text{constant})\,\sqrt{\mu\omega/8\pi\sigma}$$

where the (constant) is obtained from the solution of the equations for the φ's. The factor in **E** and **H** which depends on Z is then $e^{ik_z{}^0z-\kappa z}$ where $(k_z^0)^2 = k^2 - k_{mn}^2$ and κ is a small quantity which is a measure of the attenuation of the wave because of loss of energy at the walls, being proportional to $\sqrt{\mu\omega/4\pi\sigma}$.

But a simpler method can be found to obtain a first-order expression for κ, rather than the rather tedious solution of the equations for α and the φ's. It lies in the use of Eqs. (13.3.5) and (13.3.8), together with the fact that, if energy is lost in the walls, the energy flow down the tube must be attenuated. The total energy flow down the tube is, of course, the integral of $\mathbf{S} = (c/4\pi)(\bar{\mathbf{E}} \times \mathbf{H})$ over the tube cross section, averaged over time. If energy is lost at the walls, both $\bar{\mathbf{E}}$ and $\mathbf{H}$ will have attenuation factors $e^{-\kappa z}$ so that $S_{\text{av}} = S_0e^{-2\kappa z}$ and the power lost to the walls per unit length of duct is $2\kappa\iint S_{\text{av}}\,dA$.

The power lost can be computed, to the first order of the small quantity $\sqrt{\mu\omega/4\pi\sigma}$, as follows. We assume that the values of **H** near the walls are essentially unchanged by the introduction of small energy loss at the walls (the formulas for $-iZ\mathbf{H}_{\text{tan}}$, given above, bear this out) and thus that we can use the expressions for **H** obtained for infinite conductivity, such as the ones given in Eqs. (13.3.11) and (13.3.13). The surface current producing (or produced by) **H** is then, from the discussion preceding Eq. (3.4.23), equal to $-(c/4\pi)\mathbf{n} \times \mathbf{H}$, where **n** is the normal into the surface and **H** is the magnetic field at the surface. If the tangential electric field at the surface is either zero or is out of phase with $\mathbf{n} \times \mathbf{H}$, no power is lost to the walls, but if $-(c/4\pi)(\mathbf{n} \times \mathbf{H}) \cdot \bar{\mathbf{E}}$ has a real part, this real part, integrated over the perimeter of the duct cross section, must equal the energy lost. Since $-(c/4\pi)(\mathbf{n} \times \mathbf{H}) \cdot \bar{\mathbf{E}} = \mathbf{n} \cdot [(c/4\pi)(\bar{\mathbf{E}} \times \mathbf{H})]$, we see that this does equal the component of energy flow into the wall. Therefore

$$2\kappa \iint S_{\text{av}}\,dA = \text{Re} \oint ds\left[\left(\frac{c}{4\pi}\right)\mathbf{n} \cdot (\bar{\mathbf{E}} \times \mathbf{H})\right]$$
$$= \left(\frac{c}{4\pi}\right)\text{Re} \oint ds\,[(\mathbf{n} \times \bar{\mathbf{E}}) \cdot \mathbf{H}_{\text{tan}}]$$

where the integration $\oint ds$ is around the perimeter of the cross section and the integration $\iint dA$ is over the whole area of the cross section. The fields used in computing S_{av} and the field $\mathbf{H}_{tan}$ may be the ones for infinite conductance, but the field used in $(\mathbf{n} \times \mathbf{E})$ must be the one for finite conductivity. However, Eqs. (13.3.5) and (13.3.8) enable us to compute this in terms of $\mathbf{H}_{tan}$. We have that $(\mathbf{n} \times \mathbf{E}) = \sqrt{\mu\omega/4\pi\sigma}\, e^{-\frac{1}{4}i\pi}\mathbf{H}_{tan}$ at the boundary and, consequently,

$$\kappa = \tfrac{1}{2}\sqrt{\frac{\mu\omega}{8\pi\sigma}}\,\frac{\oint ds|\mathbf{H}_{tan}|^2}{|\mathrm{Re}\iint d\mathbf{A}\cdot(\bar{\mathbf{E}} \times \mathbf{H})|} \tag{13.3.17}$$

This equation is an exact one if the fields involved are the true ones, satisfying boundary conditions exactly; it is an approximate one, but good to the first order in $\sqrt{\mu\omega/8\pi\sigma}$, if the fields are those for the boundary condition of infinite conductivity.

With the aid of this equation, let us now compute the attenuation of any electromagnetic wave traveling along a wave guide of uniform cross section, returning to the special case of rectangular cross section at the end. As we have seen earlier, the transverse electric waves are generated from a scalar solution $\psi_{mn}(x,y)$ of the equation $\nabla^2\psi_{mn} = -k^2_{emn}\psi_{mn}$ by the operations

$$\begin{aligned}
\mathbf{A} &= \frac{1}{ik}\mathbf{E} = \mathbf{M}_{mn} = \frac{1}{ik}\,\mathrm{curl}[\mathbf{a}_z\psi_{mn}\exp(iz\sqrt{k^2 - k^2_{emn}} - ikct)] \\
&= -(1/ik)(\mathbf{a}_z \times \mathrm{grad}\,\psi_{mn})\exp(iz\sqrt{k^2 - k^2_{emn}} - ikct) \qquad (13.3.18)\\
\mathbf{H} &= (1/ik)[\mathbf{a}_z k^2_{emn}\psi_{mn} + i\sqrt{k^2 - k^2_{emn}}\,\mathrm{grad}\,\psi_{mn}]\exp[iz\sqrt{k^2 - k^2_{emn}} - ikct]
\end{aligned}$$

where, for infinite conductivity, the normal gradient of ψ_{mn} is to be zero at the boundary. The real part of the product $\bar{\mathbf{E}} \times \mathbf{H}$ is

$$-\frac{1}{k}\sqrt{k^2 - k^2_{emn}}\,(\mathbf{a}_z \times \mathrm{grad}\,\psi_{mn}) \times \mathrm{grad}\,\psi_{mn} = \frac{1}{k}\sqrt{k^2 - k^2_{emn}}\,\mathbf{a}_z|\mathrm{grad}\,\psi_{mn}|^2$$

pointed along the axis of the tube, as expected. By use of Green's theorem and using the fact that $\partial\psi/\partial n = 0$ at the boundary, we compute the area integral

$$\mathrm{Re}[\iint d\mathbf{A}\cdot(\bar{\mathbf{E}} \times \mathbf{H})] = k^2_{emn}\sqrt{1 - (k^2_{emn}/k)^2}\,\Lambda^e_{mn}$$

where $\Lambda^e_{mn} = \iint dA|\psi_{mn}|^2$ is the normalizing constant for the scalar eigenfunction ψ_{mn}.

The square of the tangential component of $\mathbf{H}$ at the boundary is

$$|\mathbf{H}_{tan}|^2 = (k^4_{emn}/k^2)|\psi_{mn}|^2 + [1 - (k^2_{emn}/k^2)]|\partial\psi_{mn}/\partial s|^2$$

where $\partial\psi/\partial s$ is the tangential component of the gradient of ψ at the boundary. Consequently, the attenuation factor for the (m,n)th transverse electric wave is

$$\kappa_{emn} \simeq \frac{1}{2}\sqrt{\frac{\mu\omega}{4\pi\sigma}}\left\{\frac{k_{emn}^2 \oint |\psi_{mn}|^2\, ds + \left[\left(\frac{k}{k_{emn}}\right)^2 - 1\right] \oint |\partial\psi_{mn}/\partial s|^2\, ds}{k\sqrt{k^2 - k_{emn}^2}\,\Lambda_{mn}^e}\right\} \tag{13.3.19}$$

When the driving frequency is below the cutoff frequency for this wave $(k < k_{emn})$, then $\sqrt{k^2 - k_{emn}^2} = i\sqrt{k_{emn}^2 - k^2}$; κ_{emn} is imaginary and this formula does not apply. However modes below their cutoff do not transmit along the tube anyway; any effect of their absorption at the walls is not noticeable unless the absorption is large.

We can therefore say that a more correct formula for **A**, **E** and **H** is obtained by multiplying the expressions of Eq. (13.3.18) by $e^{-\kappa_{emn}z}$ to represent, to the first approximation, the attenuation. For the (m,n)th transverse electric mode, the power transmitted down the tube is then

$$S_{mn}^e = (c/4\pi)k_{emn}^2\sqrt{1 - (k_{emn}/k)^2}\,\Lambda_{mn}^e e^{-2\kappa_{emn}z} \tag{13.3.20}$$

as long as k is larger than k_{emn}.

In the case of the transverse magnetic wave, we start with another scalar solution, $\chi_{mn}(x,y)$, a solution of the two-dimensional Helmholtz equation $\nabla^2\chi_{mn} = -k_{hmn}^2\chi_{mn}$ which goes to zero at the boundary. Then

$$\begin{aligned}
\mathbf{A} = \frac{1}{ik}\mathbf{E} &= \mathbf{N}_{mn} = \frac{1}{k^2}\,\mathrm{curl}\,\mathrm{curl}[\mathbf{a}_z\chi_{mn}\exp(iz\sqrt{k^2 - k_{hmn}^2} - ikct)] \\
&= (1/k^2)[\mathbf{a}_z k_{hmn}^2\chi_{mn} + i\sqrt{k^2 - k_{hmn}^2}\,\mathrm{grad}\,\chi_{mn}] \\
&\qquad\qquad\qquad \exp[iz\sqrt{k^2 - k_{hmn}^2} - ikct] \\
\mathbf{H} &= -(\mathbf{a}_z \times \mathrm{grad}\,\chi_{mn})\exp(iz\sqrt{k^2 - k_{hmn}^2} - ikct)
\end{aligned}$$

By the same methods as were used for the other case, we find the attenuation factor and the power transmitted for these transverse magnetic waves to be

$$\begin{aligned}
\kappa_{hmn} &= \frac{1}{2}\sqrt{\frac{\mu\omega}{4\pi\sigma}}\,\frac{\oint\left|\frac{\partial\chi_{mn}}{\partial n}\right|^2 ds}{k_{hmn}^2\sqrt{1 - (k_{hmn}/k)^2}\Lambda_{mn}^h} \\
S_{mn}^h &= (c/4\pi)\sqrt{1 - (k_{hmn}/k)^2}\,\Lambda_{mn}^h e^{-2\kappa_{hmn}z}
\end{aligned} \tag{13.3.21}$$

where Λ_{mn}^h is the normalizing constant for χ_{mn} and where $\partial\chi/\partial n$ is the normal component of the gradient of χ at the boundary. Both of these formulas are only valid when $k > k_{hmn}$. We note that, for both waves, the attenuation is extremely large for frequencies just above the cutoff, then drops to a minimum and rises again, going up proportional to the square root of the frequency for very high frequencies. This increase is because of the $\sqrt{\omega}$ factor in the impedance term for a conductive surface; if σ itself changes with frequency, the attenuation factors at high frequency will depend on frequency as $\sqrt{\omega/\sigma}$.

Applying these formulas to the waves in rectangular ducts, given in Eqs. (13.3.11) and (13.3.13), where

$$\psi_{mn} = \cos\left(\frac{\pi m x}{a}\right)\cos\left(\frac{\pi n y}{b}\right);\quad \chi_{mn} = \sin\left(\frac{\pi m x}{a}\right)\sin\left(\frac{\pi n y}{b}\right)$$

we have, for the transverse electric waves

$$\kappa_{emn} = \frac{\sqrt{\mu\omega/4\pi\sigma}}{k\sqrt{k^2 - k_{mn}^2}}\left\{\frac{\epsilon_m}{a}\left[\left(\frac{\pi m}{a}\right)^2 + \left(\frac{\pi n k}{b k_{mn}}\right)^2\right] + \frac{\epsilon_n}{b}\left[\left(\frac{\pi n}{b}\right)^2 + \left(\frac{\pi m k}{a k_{mn}}\right)^2\right]\right\}$$

and for the transverse magnetic waves

$$\kappa_{hmn} = \frac{2\sqrt{\mu\omega/4\pi\sigma}}{k\sqrt{k^2 - k_{mn}^2}}\left\{\frac{1}{a}\left(\frac{\pi m k}{a k_{mn}}\right)^2 + \frac{1}{b}\left(\frac{\pi n k}{b k_{mn}}\right)^2\right\}$$

where $k_{emn}^2 = k_{mn}^2 = (\pi m/a)^2 + (\pi n/b)^2 = k_{hmn}^2$.

Reflection of Waves from End of Tube. The calculation of the reflection of electromagnetic waves from a barrier at the end of a duct is most easily carried out by means of the impedance concept. We set the barrier at the $z = 0$ plane so that the incident wave comes up from the left ($z < 0$) and is reflected back to the right. We adjust things so that the ratio of $\mathbf{a}_z \times \mathbf{E}$ to $\mathbf{H} - \mathbf{a}_z(\mathbf{a}_z \cdot \mathbf{H})$ is the same for z just smaller than zero (just outside the barrier) as it is for z just larger than zero (just inside the barrier). If this ratio is uniform across the tube at $z = 0$, then a wave of a given symmetry (given value of m and n) will be reflected with the same symmetry, but if the ratio varies from point to point across the tube, then the reflected wave will differ in symmetry from the incident wave.

Suppose first that the material to the right of $z = 0$ has dielectric constant ϵ and/or permeability μ different from unity and/or conductivity different from zero. The impedance will then, in general, depend on the kind of wave which comes down the tube. If a wave of a single mode (given m and n) is incident on the surface $z = 0$, the part of the wave which penetrates beyond $z = 0$ must have the same dependence on x and y as the incident wave (in order to fit at $z = 0$) though ts dependence on z will differ. Inside the medium ($z > 0$) the vector potential will have to satisfy the equation curl curl $\mathbf{A} = k^2\eta^2\mathbf{A}$ where $\eta^2 = \mu\epsilon + (4\pi i\mu\sigma/\omega)$ and $k = \omega/c$. The expressions for the various modes of waves traveling to the right in this medium therefore are similar to those of Eqs. (13.3.18) and (13.3.21) except that the radical $\sqrt{k^2 - k_{mn}^2}$ is everywhere to be replaced by $\sqrt{k^2\eta^2 - k_{mn}^2}$, and the expressions for $\mathbf{H}$ are to be divided by μ.

The expressions for $\mathbf{n} \times \mathbf{E}$, for $\mathbf{H}_{\tan}$ and for the ratio ζ between them, for transverse electric waves, are

$$\begin{aligned}
\mathbf{a}_z \times \mathbf{E} &= \operatorname{grad} \psi_{mn} \exp(iz\sqrt{k^2\eta^2 - k_{emn}^2} - ikct) \qquad (13.3.22)\\
\mathbf{H}_{\tan} &= (1/\mu k)\sqrt{k^2\eta^2 - k_{emn}^2} \operatorname{grad} \psi_{mn} \exp(iz\sqrt{k^2\eta^2 - k_{emn}^2} - ikct)\\
\zeta_{emn} &= [\mu/\sqrt{\eta^2 - (k_{emn}/k)^2}]
\end{aligned}$$

and the corresponding expressions for the transverse magnetic waves are

$$\begin{aligned}
\mathbf{a}_z \times \mathbf{E} &= -(1/k)\sqrt{k^2\eta^2 - k_{hmn}^2}\,\mathbf{a}_z \times \operatorname{grad} \chi_{mn} \exp(iz\sqrt{k^2\eta^2 - k_{hmn}^2} - ikct)\\
\mathbf{H}_{\tan} &= -(1/\mu)\mathbf{a}_z \times \operatorname{grad} \chi_{mn} \exp(iz\sqrt{k^2\eta^2 - k_{hmn}^2} - ikct)\\
\zeta_{hmn} &= \mu\sqrt{\eta^2 - (k_{hmn}/k)^2} \qquad (13.3.23)
\end{aligned}$$

In the open part of the duct ($z < 0$) we have a unit-amplitude incident wave, given by one of the $\mathbf{M}$ or $\mathbf{N}$ functions of Eq. (13.3.18) or (13.3.21), plus a reflected wave of the same (x,y) dependence (save that $\sqrt{k^2 - k_{mn}^2}$ is reversed in sign everywhere) of amplitude R. The expressions for $\mathbf{a}_z \times E$ and $\mathbf{H}_{\tan}$ at $z = 0$ and for the ratio there are

$$\begin{aligned}
(\mathbf{a}_z \times \mathbf{E})_0 &= (1 + R) \operatorname{grad} \psi_{mn};\quad (\mathbf{H}_{\tan})_0 = \sqrt{1 - (k_{emn}/k)^2}\,(1 - R) \operatorname{grad} \psi_{mn}\\
\zeta &= [(1 + R)/(1 - R)][1/\sqrt{1 - (k_{emn}/k)^2}];\quad \text{for transverse electric waves}\\
(\mathbf{a}_z \times \mathbf{E})_0 &= -(1 - R)\sqrt{1 - (k_{hmn}/k)^2}\,\mathbf{a}_z \times \operatorname{grad} \chi_{mn}\\
(\mathbf{H}_{\tan})_0 &= -(1 + R)\mathbf{a}_z \times \operatorname{grad} \chi_{mn}\\
\zeta &= [(1 - R)/(1 + R)]\sqrt{1 - (k_{hmn}/k)^2};\quad \text{for transverse magnetic waves}
\end{aligned}$$

The reflected amplitude R, for the different waves, may be obtained by requiring that the ζ of these last expressions equal the corresponding ζ given in Eq. (13.3.22) or (13.3.23). The results are

$$\begin{aligned}
R_{emn} &= -\frac{\sqrt{\eta^2 - (k_{emn}/k)^2} - \mu\sqrt{1 - (k_{emn}/k)^2}}{\sqrt{\eta^2 - (k_{emn}/k)^2} + \mu\sqrt{1 - (k_{emn}/k)^2}};\quad \text{transverse electric}\\
&\qquad\qquad (13.3.24)\\
R_{hmn} &= -\frac{\mu\sqrt{\eta^2 - (k_{hmn}/k)^2} - \sqrt{1 - (k_{hmn}/k)^2}}{\mu\sqrt{\eta^2 - (k_{hmn}/k)^2} - \sqrt{1 - (k_{hmn}/k)^2}};\quad \text{transverse magnetic}
\end{aligned}$$

where $\eta^2 = \mu\epsilon + (4\pi i\mu\sigma/\omega)$ and μ, ϵ, and σ have values appropriate to the region $z > 0$. When $\mu = 1$ and $\eta = 1$, all R's are zero, of course, since there is then no discontinuity at $z = 0$ to reflect waves. When η is very large, the magnitude of the R's is approximately unity; there is almost perfect reflection. When η is somewhat larger than unity and when k is just larger than its cutoff value k_{mn}, there is also strong reflection. Of course the formulas are not valid for $k < k_{mn}$, for there is no wave transmission below the cutoff frequency.

Effect of Change of Duct Size. These same impedance concepts may also be used to compute the reflection of waves from a constriction in a duct. As an example, suppose we consider that the duct for $z < 0$ is a rectangular one with cross section a by b, whereas for $z > 0$ the height is b_+, less than b, the width a remaining unchanged and the junction being symmetrical. If the incident wave from the left is the lowest mode $\mathbf{M}_{01}^-$ (we again assume $b > a$), the wave transmitted along the narrowed tube ($z > 0$) will be a mixture of several modes of the sort $\mathbf{M}_{0n}^+$, but unless the constriction is considerable, the predominant part will be the lowest mode for the narrower tube, $\mathbf{M}_{01}^+$, which has an impedance ratio $\zeta_{e01}^+ = 1/\sqrt{1 - (\pi/b_+k)^2}$ over the cross section of the narrower tube.

Consequently, to the first approximation, the impedance ratio at $z = 0$ for the waves in the larger tube ($z < 0$) is

$$\zeta \simeq \begin{cases} 0; & 0 \le y < \tfrac{1}{2}b - \tfrac{1}{2}b_+ \\ 1/\sqrt{1 - (\pi/b_+k)^2}; & \tfrac{1}{2}b - \tfrac{1}{2}b_+ < y < \tfrac{1}{2}b + \tfrac{1}{2}b_+ \\ 0; & \tfrac{1}{2}b + \tfrac{1}{2}b_+ < y \le b \end{cases}$$

where $b > b_+$ and $b > a$. This equation is of course valid only when the incident wave is the lowest mode and when $(b - b_+)$ is considerably smaller than b (that is, when the constriction is not too decided).

Even though the incident wave is just the lowest mode, the wave reflected from the "shoulders" of the constriction will not be just one mode, it will contain enough of the modes $\mathbf{M}_{0n}$ to fit the boundary condition (we need not include modes $\mathbf{M}_{mn}$ for m other than zero because the electric field everywhere is in the x direction). The wave in the wider tube ($z < 0$) is then

$$\mathbf{A} = \frac{1}{ik}\mathbf{E} = -\frac{1}{ik}\left\{\mathbf{a}_z \times \operatorname{grad}\left[\cos\left(\frac{\pi y}{b}\right)\right]\exp\left[iz\sqrt{k^2 - \left(\frac{\pi}{b}\right)^2} - ikct\right]\right.$$
$$\left. + \sum_{n=1}^{\infty} B_n \mathbf{a}_z \times \operatorname{grad}\left[\cos\left(\frac{\pi n y}{b}\right)\right]\exp\left[-iz\sqrt{k^2 - \left(\frac{\pi n}{b}\right)^2} - ikct\right]\right\}$$

where the amplitudes B_n of the reflected waves are to be computed by matching impedances.

We could calculate the coefficients B_n if we knew the dependence of $\mathbf{H}_{\text{tan}}$ on x and y at $z = 0$. To the first approximation we can assume it to be less affected by the "shoulders" than is $\mathbf{E}$ and take for an approximate form the $\mathbf{H}_{\text{tan}}$ for the incident plus lowest reflected mode,

$$\mathbf{H}_{\text{tan}} \simeq -\mathbf{a}_y(\pi/b)\sqrt{1 - (\pi/kb)^2}\,\sin(\pi y/b)e^{-ikct}(1 - B_1)$$

Consequently, the boundary condition on the B's is

$$\mathbf{a}_z \times \mathbf{E} = -\mathbf{a}_y \left\{(1 + B_1)\left(\frac{\pi}{b}\right)\sin\left(\frac{\pi y}{b}\right) + \sum_{n=2}^{\infty} B_n \left(\frac{\pi n}{b}\right)\sin\left(\frac{\pi n y}{b}\right)\right\} e^{-ikct}$$

$$= \zeta\mathbf{H}_{\tan} \simeq \begin{cases} 0; & 0 \le y < \frac{1}{2}b - \frac{1}{2}b_+ \\ \mathbf{a}_y\left(\frac{\pi}{b}\right)\sqrt{\dfrac{1 - (\pi/kb)^2}{1 - (\pi/kb_+)^2}}\sin\left(\frac{\pi y}{b}\right)(1 - B_1)e^{-ikct}; & \text{in open space} \\ 0; & \frac{1}{2}b + \frac{1}{2}b_+ < y \le b \end{cases}$$

Multiplying both sides by $\sin(\pi m y/b)$ and integrating over y provide values of the B's. We find that $B_{2n} = 0$ and

$$B_1 \simeq -\frac{\tau_1 S_+ - \tau_+}{\tau_1 S_+ + \tau_+}$$

$$B_{2n+1} \simeq \frac{(2\tau_1/\pi n)}{\tau_+ + \tau_1 S_+}\left\{\frac{1}{n}\sin\left(\frac{\pi n b_+}{b}\right) + \frac{1}{n+1}\sin\left[\frac{\pi b_+}{b}(n+1)\right]\right\}$$

$$\tau_1 = \sqrt{1 - (\pi/kb)^2};\quad \tau_+ = \sqrt{1 - (\pi/kb_+)^2};\quad S_+ = 1 + (1/\pi)\sin(\pi b_+/b)$$

If b_+ is nearly equal to b, the change of height being small, then τ_+ is nearly equal to τ_1, S_+ is nearly equal to unity and the reflected amplitude of the lowest mode is small. If k turns out to be smaller than π/b_+ (it must be larger than π/b to have waves in the larger tube), then no energy is transmitted into the smaller tube, all is reflected (τ_+ is imaginary).

We can improve on this calculation by computing the field in the narrow tube which fits onto the wave given above, and from this compute a new, somewhat more accurate magnetic field in the region of the constriction. From this again we can recompute $\zeta\mathbf{H}_{\tan}$ and obtain a better expression for $\mathbf{E}$, and so on. We then find that B_1 has a small imaginary part, indicating a certain reactive component to the over-all impedance of the junction.

If the reduction in size of the tube at $z = 0$ is in the width, not height, so that the "shoulder" comes in the sides carrying free charge rather than the sides parallel to the electric field, the distortion of the electric field is somewhat greater, and we can no longer assume that the wave in the narrower tube is nearly entirely the lowest mode. The free charge will tend to concentrate at the corners, and the electric field will not be independent of x but will increase near the "shoulders" at $x = \frac{1}{2}a - \frac{1}{2}a_+$ and $\frac{1}{2}a + \frac{1}{2}a_+$. Examination of the static field near such a "shoulder" (see Sec. 10.2) indicates that the tangential electric field at $z = 0$ should have the general shape

$$\mathbf{E}_{\tan} = ik\mathbf{A}_{\tan} \simeq ik\mathbf{a}_x A_0 \sin\left(\frac{\pi y}{b}\right)\left\{1 + 2\left[\frac{(a/a_+)^2 - 1}{1 - (2\xi/a_+)^2}\right]^{\frac{1}{3}}\right\} e^{-ikct};$$
$$-\tfrac{1}{2}a_+ < \xi < \tfrac{1}{2}a_+ \qquad (13.3.25)$$

where $\xi = x - \frac{1}{2}a$. The larger the "shoulder," the larger is a/a_+ and the more pronounced is the singularity at $\xi = \pm\frac{1}{2}a_+$. This corresponds to a wave in the narrower part of the tube ($z > 0$) of the sort

$$\mathbf{A} = \frac{\mathbf{E}}{ik} = \sum_{m=1}^{\infty} (B_m^+\mathbf{M}_{2m,1}^+ + C_m^+\mathbf{N}_{2m,1}^+) + B_0^+\mathbf{M}_{01}^+$$

where the functions $\mathbf{M}^+$, $\mathbf{N}^+$ are the suitable ones for the $z > 0$ part of the duct [with $x - \frac{1}{2}a + \frac{1}{2}a_+$ instead of x and a_+ instead of a in Eqs. (13.3.11) and (13.3.13)].

The coefficients B^+ and C^+ are computed by equating $\mathbf{E}_{\text{tan}}$ at $z = 0$ to the corresponding series. For example, multiplying both sides by $\overline{\mathbf{M}}_{2m,1}^+$ and integrating over the cross section of the narrower tube, we obtain (see page 1450)

$$B_m^+ = -i(-1)^m \frac{\pi k\epsilon_m}{b(k_{2m,1}^+)^2} A_0 \left[\frac{\pi\Gamma(\frac{2}{3})}{(\pi m/2)^{\frac{1}{6}}} J_{\frac{1}{6}}(\pi m)\right] \sqrt[3]{\left(\frac{a}{a_+}\right)^2 - 1}; \quad m > 0$$

where $(k_{2m,1}^+)^2 = (2\pi m/a_+)^2 + (\pi/b)^2$ and where we have used the expansion

$$\frac{1}{[1 - (2\xi/a_+)^2]^{\frac{1}{3}}} = \frac{a_+}{a} \sum_{m=0}^{\infty} \tfrac{1}{2}\epsilon_m \left[\frac{\pi\Gamma(\frac{2}{3})}{(\pi m a_+/2a)^{\frac{1}{6}}} J_{\frac{1}{6}}\left(\frac{\pi m a_+}{a}\right)\right] \cos\left(\frac{2\pi m\xi}{a}\right)$$

over the range $-\frac{1}{2}a_+ < \xi < \frac{1}{2}a_+$. We note that the coefficient in the brackets in the summation has the value $[\sqrt{\pi}\,\Gamma(\frac{2}{3})/\Gamma(\frac{7}{6})]$ for $m = 0$. The coefficients C for the transverse magnetic waves may be computed in the same way, multiplying by the *transverse* part of $\mathbf{N}_{2m,1}$ on both sides and integrating.

We then proceed to compute the value of $\mathbf{H}_{\text{tan}}$ at $z = 0$ in order to obtain the value of the impedance ratio over the surface of the junction ($z = 0$). At $\xi = 0$ and $z > 0$, we have

$$\begin{aligned}\mathbf{H}_{\text{tan}} = \mathbf{a}_y i A_0 \sqrt{k^2 - \left(\frac{\pi}{b}\right)^2}\,\cdot \\ \cdot\left[1 + \Omega\frac{\sqrt{\pi}\,\Gamma(\frac{2}{3})}{2\Gamma(\frac{7}{6})}\right] \sin\left(\frac{\pi y}{b}\right) \exp\left[iz\sqrt{k^2 - \left(\frac{\pi}{b}\right)^2} - ikct\right] \\ + \mathbf{a}_y \Omega A_0 \sum_{m=1}^{\infty} \frac{k^2 - (\pi/b^2)}{\sqrt{(2\pi m/a_+)^2 + (\pi/b)^2 - k^2}} \frac{\sqrt{\pi}\,\Gamma(\frac{2}{3})}{(\pi m/2)^{\frac{1}{6}}} J^{\frac{1}{6}}(\pi m)\,\cdot \\ \cdot \sin(\pi y/b) \exp[-z\sqrt{(2\pi m/a_+)^2 + (\pi/b)^2 - k^2} - ikct]\end{aligned}$$

where $\Omega = 2[(a/a_+)^2 - 1]^{\frac{1}{3}}$ and where we have assumed that $(\pi/b)^2 < k^2 < (\pi/b)^2 + (2\pi/a)^2$. If $a < b$, we can simplify the series, since it is approximately equal to

$$\mathbf{a}_y B_0 \sum_{m=1}^{\infty} \frac{1}{m} \left[\frac{\sqrt{\pi}\,\Gamma(\frac{2}{3})}{(\pi m/2)^{\frac{1}{6}}} J_{\frac{1}{6}}(\pi m) \right] \sin\left(\frac{\pi y}{b}\right) e^{-(2\pi m z/a_+) - ikct}$$

$$= \mathbf{a}_y B_0 \int_0^\pi \sin^{\frac{1}{3}}\theta\, d\theta \sum_{m=1}^{\infty} \frac{1}{m} e^{-i\pi m \cos\theta - (2\pi m z/a_+)} \sin\left(\frac{\pi y}{b}\right) e^{-ikct}$$

$$= -a_y B_0 \left[\int_0^\pi \sin^{\frac{1}{3}}\theta\, d\theta \ln(1 - e^{-i\pi\cos\theta - 2\pi z/a_+}) \right] \sin\left(\frac{\pi y}{b}\right) e^{-ikct}$$

where $B_0 = (a_+/2\pi)[k^2 - (\pi/b)^2]\Omega A_0$. We can now set $z = 0$ (analytic continuation) and show that the integral in brackets is

$$\int_0^\pi \sin^{\frac{1}{3}}\theta\, d\theta \ln[2\sin(\tfrac{1}{2}\pi\cos\theta)]$$

$$\simeq \ln(4) \int_0^{\frac{1}{2}\pi} \sin\theta\, d\theta + 2\int_0^{\frac{1}{2}\pi} \sin\theta\, d\theta \ln[\sin(\tfrac{1}{2}\pi\cos\theta)]$$

$$= \sqrt{\pi}\ln(2)\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{7}{6})} + \frac{4}{\pi}\int_0^{\frac{1}{2}\pi} \ln(\sin\varphi)\, d\varphi$$

$$= \ln(2)\left[\sqrt{\pi}\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{7}{6})} - 2\right] = 0.40696$$

so that, to this approximation, the admittance ratio $1/\zeta = (\mathbf{H}_{\tan}/\mathbf{a}_z \times \mathbf{E})$ at the center of the tube is

$$1/\zeta = \sqrt{1 - \left(\frac{\pi}{bk}\right)^2} \left\{1 + \frac{\Omega}{1+\Omega}\left[0.2936 + 0.06477i(a_+k)\sqrt{1 - \left(\frac{\pi}{bk}\right)^2}\right]\right\}$$

Our next step is to make the tangential electric field for $z < 0$ fit at $z = 0$. We have an incident wave of the $\mathbf{M}_{01}$ type and a whole sequence of reflected waves of the sort $\mathbf{M}^-_{2m,1}$ and $\mathbf{N}^-_{2m,1}$, where $\mathbf{M}^-$ and $\mathbf{N}^-$ are obtained from Eqs. (13.3.11) and (13.3.13) by changing k_z into $-k_z$. We find that the expression

$$\mathbf{A} = \mathbf{M}_{01} + \left[B\left(\frac{a_+}{a}\right) - 1 + 1.2936B\Omega\left(\frac{a_+}{a}\right)\right]\mathbf{M}^-_{01}$$
$$+ B\left(\frac{\pi^2 a_+}{b^2 a}\right)\sum_{m=1}^{\infty}\left[\frac{\Omega\sqrt{\pi}\,\Gamma(\frac{2}{3})}{(\pi m a_+/2a)^{\frac{1}{6}}} J_{\frac{1}{6}}\left(\frac{\pi m a_+}{a}\right) + \left(\frac{2a}{\pi m a_+}\right)\sin\left(\frac{\pi m a_+}{a}\right)\right]\cdot$$
$$\cdot\frac{(-1)^m}{k^2_{2m,1}}\cdot\left[\mathbf{M}^-_{2m,1} + \frac{(2mb/a)}{\sqrt{k^2 - k^2_{2m,1}}}\mathbf{N}^-_{2m,1}\right]$$

has the tangential behavior given in Eq. (13.3.25) (with $A_0 = iB\pi/bk$ at $z = 0$) for $\frac{1}{2}a - \frac{1}{2}a_+ < x < \frac{1}{2}a + \frac{1}{2}a_+$ and has zero tangential component at the "shoulders" $0 < x < \frac{1}{2}a - \frac{1}{2}a_+$ and $\frac{1}{2}a + \frac{1}{2}a_+ < x < a$. To find the value of B, we calculate $\mathbf{H}_{\tan}$ at $z = 0$ and equate admittances at the center of the tube. A quite tedious bit of algebra finally indicates that the reflected amplitude

$$R = -1 + B(a_+/a) + 1.2936B\Omega(a_+/a)$$

of the lowest mode is

$$R = -\frac{(a - a_+)(1 + 1.2936\Omega) + i(kaa_+\Omega G/\pi)\sqrt{1 - (\pi/kb)^2}}{(a + a_+)(1 + 1.2936\Omega) + i(kaa_+\Omega G/\pi)\sqrt{1 - (\pi/kb)^2}} \quad (13.3.26)$$

where $$G = \left[1.5872 + \frac{1+\Omega}{\Omega}\ln\left(\frac{a_+}{a}\right)\right]\ln(2)$$

and $$\Omega = 2[(a/a_+)^2 - 1]^{\frac{1}{2}}$$

We see (of course) that there is no reflection when $a_+ = a$. When the width a_+ of the narrow part of the tube is quite small, R approaches -1.

Reflection from Wire across Tube. As another example of the reflection of waves from an obstruction which is not uniform across the tube, we consider the case of the lowest mode $\mathbf{M}_{01}$ in a rectangular tube with $b > a$, with frequency between the lowest cutoff ω_{01} and the next lowest, incident on the metallic strip (discussed on page 1826) of width Δ with center line $y = \frac{1}{2}b$, $z = 0$. The incident wave is

$$\mathbf{A}_0 = \frac{1}{ik}\mathbf{E}_0 = i\mathbf{a}_x\left(\frac{\pi}{bk}\right)\sin\left(\frac{\pi y}{b}\right)\exp\left[iz\sqrt{k^2 - \left(\frac{\pi}{b}\right)^2} - ikct\right]$$

and the resulting electric field at the center of the strip would be

$$\mathbf{E}_0 = -\mathbf{a}_x(\pi/b)e^{-ikct}$$

if only the incident wave were present. If the strip were perfectly conducting, a current would be induced in it which would produce a back voltage just canceling this incident voltage at the surface of the strip. If the strip has resistance R per unit length, then the current induced would be such that R times I would equal the difference between incident and induced fields.

According to Eq. (13.3.16), the electric field at the center of the strip, induced by the current I, is

$$\mathbf{E}_i = -\mathbf{a}_x\left(\frac{8\pi I}{abc}\right)\left\{\frac{1}{\sqrt{1 - (\pi c/\omega b)^2}} + i\omega\left(\frac{b}{2\pi c}\right)\left[\ln\left(\frac{4b}{\pi e\Delta}\right) + 2\Sigma\right]\right\}$$

Since $IR = E_0 + E_i$, we have an equation determining I and E_i

$$I = \frac{-(ac/8)\sqrt{1 - (\pi c/\omega b)^2}}{1 + (abc/8\pi)\sqrt{1 - (\pi c/\omega b)^2}\,(R - i\omega L)}$$

where $L = (4/ac^2)[\ln(4b/\pi e\Delta) + 2\Sigma]$ is the effective inductance of the wire. This induced current produces a wave [given by the first term of

Eq. (13.3.15)] which partly cancels the incident wave in the region ($z > 0$) beyond the "wire" and which produces a local distortion of the field [given by the second term in Eq. (13.3.15)] which is not propagated along the tube. At distances far enough from the wire so that the local distortion is negligible, the wave is

$$\mathbf{A} = \begin{cases} \dfrac{(R - i\omega L)\sqrt{1 - (\pi c/\omega b)^2}}{(8\pi/abc) + (R - i\omega L)\sqrt{1 - (\pi c/\omega b)^2}}\mathbf{A}_0; & z \gg b \\ \mathbf{A}_0 - \dfrac{\mathbf{A}_0 \exp[-2iz\sqrt{k^2 - (\pi/b)^2}]}{1 + (abc/8\pi)(R - i\omega L)\sqrt{1 - (\pi c/\omega b)^2}}; & z \ll -b \end{cases} \tag{13.3.27}$$

where $\mathbf{A}_0 = (i\pi/bk)\mathbf{a}_x \sin(\pi y/b) \exp[iz\sqrt{k^2 - (\pi/b)^2} - ikct]$ is the incident wave. In the region ($z < 0$), there is in addition a reflected wave, somewhat out of phase with the incident wave (unless $L = 0$). The wire acts as though it has impedance $(R - i\omega L)$ and is coupled to the wave in the duct by a coupling constant $8\pi/abc$.

Many other specialized calculations may be made, giving fields and reflection coefficients for wave guides of various shapes and configurations. The reader is referred to books specializing in this subject for other examples.

Elastic Waves along a Bar. The waves traveling along a bar of elastic material of uniform cross section (x,y plane) have the same general form as the waves given on page 1823 and for electromagnetic waves. There is a longitudinal wave, corresponding to a wave velocity $c_c = \sqrt{(\lambda + 2\mu)/\rho}$ for an infinite medium;

$$\begin{aligned} \mathbf{L}_{mn} &= (1/ik_c)\,\mathrm{grad}[\varphi_{mn}(x,y)e^{ikz}]e^{-i\omega t} \\ &= \left[\left(\frac{1}{ik_c}\right)\mathrm{grad}\,\varphi_{mn} + \left(\frac{k}{k_c}\right)\mathbf{a}_z\varphi_{mn}\right]e^{ikz - i\omega t} \end{aligned} \tag{13.3.28}$$

where $k_c^2 = \rho\omega^2/(\lambda + 2\mu)$ and φ_{mn} is a solution of the scalar equation $\nabla^2\varphi_{mn} = -\alpha_{1mn}^2\varphi_{mn}$ (where $k^2 = k_c^2 - \alpha_{1mn}^2$) which satisfies the proper conditions at the boundary of the cross section.

We note that the velocity of waves along the bar ω/k is not, in general, equal to $c_c = \omega/k_c$ because the boundary conditions do not usually allow φ_{mn} to be a constant. Only in the case where the boundary conditions require that the displacement $\mathbf{S}$ be tangential to the boundary can one allowed φ be a constant and the wave motion be purely longitudinal. In most cases the boundary conditions result in a φ which is not constant, thus an α_{1mn} which is not zero and, consequently, a velocity along the rod which differs from c_c. In most cases, indeed, the boundary conditions cannot be solved by a longitudinal wave alone; a certain amount of transverse wave must be included. Just as with the reflection from a plane surface (discussed on page 1814) a compressional wave, reflecting back and forth from the surface of the rod as it travels along, will produce a certain amount of shear wave, and vice versa.

The two transverse waves have the form

$$\begin{aligned} \mathbf{M}_{mn} &= (1/ik_s)\,\mathrm{curl}[\mathbf{a}_z\psi_{mn}(x,y)e^{ikz}]e^{-i\omega t} \\ &= (i/k_s)[\mathbf{a}_z \times \mathrm{grad}\,\psi_{mn}]e^{ikz-i\omega t} \\ \mathbf{N}_{mn} &= \left(\frac{1}{k_s^2}\right)\mathrm{curl}\,\mathrm{curl}[\mathbf{a}_z\chi_{mn}(x,y)e^{ikz}]e^{-i\omega t} \\ &= \left[\left(\frac{ik}{k_s^2}\right)\mathrm{grad}\,\chi_{mn} + \left(\frac{\alpha_{3mn}^2}{k_s^2}\right)\mathbf{a}_z\chi_{mn}\right]e^{ikz-i\omega t} \end{aligned} \tag{13.3.29}$$

where $k_s^2 = \rho\omega^2/\mu$ and ψ, χ are the solutions of $\nabla^2\psi_{mn} = -\alpha_{2mn}^2\psi_{mn}$ $(k^2 = k_s^2 - \alpha_{2mn}^2)$ and $\nabla^2\chi_{mn} = -\alpha_{3mn}^2\chi_{mn}$ $(k^2 = k_s^2 - \alpha_{3mn}^2)$ which satisfy the correct conditions at the boundary of the cross section. The **M** waves are the torsional waves, producing motion perpendicular to the z axis. The **N** waves are the ones which combine with the **L** waves to give waves of elongation along the rod.

Suppose we have chosen two orthogonal, curvilinear coordinates, ξ_1 and ξ_2, instead of x and y, such that $\xi_1 = X$ is the boundary of the cross section and such that ξ_1, ξ_2, z form a right-handed cylindrical coordinate system (such as circular cylinder or elliptic cylinder coordinates, for example). Then in terms of these coordinates, their unit vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_z$ and their scale factors h_1, h_2, and 1, the three types of waves are

$$\begin{aligned} \mathbf{M} &= \left[-\left(\frac{i\mathbf{a}_1}{k_sh_2}\right)\left(\frac{\partial\psi}{\partial\xi_2}\right) + \left(\frac{i\mathbf{a}_2}{k_sh_1}\right)\left(\frac{\partial\psi}{\partial\xi_1}\right)\right]e^{ikz-i\omega t} \\ \mathbf{N} &= \left[\left(\frac{ik\mathbf{a}_1}{k_s^2h_1}\right)\left(\frac{\partial\chi}{\partial\xi_1}\right) + \left(\frac{ik\mathbf{a}_2}{k_s^2h_2}\right)\left(\frac{\partial\chi}{\partial\xi_2}\right) + \left(\frac{k_s^2-k^2}{k_s^2}\right)\mathbf{a}_z\chi\right]e^{ikz-i\omega t} \\ \mathbf{L} &= \left[-\left(\frac{i\mathbf{a}_1}{k_ch_1}\right)\left(\frac{\partial\varphi}{\partial\xi_1}\right) - \left(\frac{i\mathbf{a}_2}{k_ch_2}\right)\left(\frac{\partial\varphi}{\partial\xi_2}\right) + \left(\frac{k}{k_c}\right)\mathbf{a}_z\varphi\right]e^{ikz-i\omega t} \end{aligned} \tag{13.3.30}$$

If the boundary conditions are in terms of displacements, this is all we need to enable us to compute k for any given wave. Often one can adjust ψ so that the conditions are satisfied with a pure torsional wave. For example, for circular, cylindrical coordinates, r, ϕ, z, and for $\mathbf{s} = 0$ at $r = a$ (as would hold, say, for a rubber filling inside a rigid hollow cylinder), we can make $\psi = J_0(\alpha r)$, independent of ϕ $(=\xi_2)$ and then adjust α so that $dJ_0/dr = 0$ at $r = a$. On the other hand, it is not usually possible to satisfy such boundary conditions with the other two waves unless a combination of **L** and **N** is used. Since k has to be the same for both **L** and **N**, if they are to be used as parts of the same wave, this establishes a relation between α_{1mn} and α_{3mn} which, with the boundary conditions, serves to determine k.

However, in most cases of interest the boundary conditions concern the boundary traction, $\mathfrak{T} \cdot \mathbf{a}_1$ at $\xi_1 = X$, where $\mathfrak{T}$ is the stress dyadic $\lambda\mathfrak{J}$ div $\mathbf{s} + \mu(\nabla\mathbf{s} + \mathbf{s}\nabla)$. Consequently, we need the expressions for the stress dyadic for each type of wave. These are:

$$\begin{aligned}
\mathfrak{T} \text{ for } \mathbf{M} = \Big\{ &-\mathbf{a}_1\mathbf{a}_1 \frac{2i\mu}{k_s}\left[\frac{1}{h_1}\frac{\partial}{\partial\xi_1}\left(\frac{1}{h_2}\frac{\partial\psi}{\partial\xi_2}\right) - \frac{1}{h_1^2h_2}\left(\frac{\partial h_1}{\partial\xi_2}\right)\left(\frac{\partial\psi}{\partial\xi_1}\right)\right] \\
&+ \mathbf{a}_2\mathbf{a}_2 \frac{2i\mu}{k_s}\left[\frac{1}{h_2}\frac{\partial}{\partial\xi_2}\left(\frac{1}{h_1}\frac{\partial\psi}{\partial\xi_1}\right) - \frac{1}{h_1h_2^2}\left(\frac{\partial h_2}{\partial\xi_1}\right)\left(\frac{\partial\psi}{\partial\xi_2}\right)\right] \\
&+ (\mathbf{a}_1\mathbf{a}_2 + \mathbf{a}_2\mathbf{a}_1)\frac{i\mu}{k_s}\left[\frac{h_2}{h_1}\frac{\partial}{\partial\xi_1}\left(\frac{1}{h_1h_2}\frac{\partial\psi}{\partial\xi_1}\right) - \frac{h_1}{h_2}\frac{\partial}{\partial\xi_2}\left(\frac{1}{h_1h_2}\frac{\partial\psi}{\partial\xi_2}\right)\right] \\
&+ (\mathbf{a}_1\mathbf{a}_z + \mathbf{a}_z\mathbf{a}_1)\frac{\mu k}{k_sh_2}\left(\frac{\partial\psi}{\partial\xi_2}\right) - (\mathbf{a}_2\mathbf{a}_z + \mathbf{a}_z\mathbf{a}_2)\frac{\mu k}{k_sh_1}\left(\frac{\partial\psi}{\partial\xi_1}\right)\Big] e^{ikz-i\omega t} \\
\mathfrak{T} \text{ for } \mathbf{N} = \Big\{ &\mathbf{a}_1\mathbf{a}_1 \frac{2ik\mu}{k_s^2}\left[\frac{1}{h_1}\frac{\partial}{\partial\xi_1}\left(\frac{1}{h_1}\frac{\partial\chi}{\partial\xi_1}\right) + \frac{1}{h_1h_2^2}\left(\frac{\partial h_1}{\partial\xi_2}\right)\left(\frac{\partial\chi}{\partial\xi_2}\right)\right] \\
&+ \mathbf{a}_2\mathbf{a}_2 \frac{2ik\mu}{k_s^2}\left[\frac{1}{h_2}\frac{\partial}{\partial\xi_2}\left(\frac{1}{h_2}\frac{\partial\chi}{\partial\xi_2}\right) + \frac{1}{h_1^2h_2}\left(\frac{\partial h_2}{\partial\xi_1}\right)\left(\frac{\partial\chi}{\partial\xi_1}\right)\right] \\
&+ \mathbf{a}_z\mathbf{a}_z 2ik\mu(k_s^2 - k^2/k_s^2)\chi \\
&+ (\mathbf{a}_1\mathbf{a}_2 + \mathbf{a}_2\mathbf{a}_1)\frac{ik\mu}{k_s^2}\left[\frac{h_2}{h_1}\frac{\partial}{\partial\xi_1}\left(\frac{1}{h_2^2}\frac{\partial\chi}{\partial\xi_2}\right) + \frac{h_1}{h_2}\frac{\partial}{\partial\xi_2}\left(\frac{1}{h_1^2}\frac{\partial\chi}{\partial\xi_1}\right)\right] \\
&\qquad + \mu\left(\frac{k_s^2 - 2k^2}{k_s^2}\right)[\mathbf{a}_z(\text{grad }\chi) + (\text{grad }\chi)\mathbf{a}_z]\Big\} e^{ikz-i\omega t} \qquad (13.3.31) \\
\mathfrak{T} \text{ for } \mathbf{L} = \Big\{ &-\mathbf{a}_1\mathbf{a}_1 \frac{2i\mu}{k_c}\left[\frac{1}{h_1}\frac{\partial}{\partial\xi_1}\left(\frac{1}{h_1}\frac{\partial\varphi}{\partial\xi_1}\right) + \frac{1}{h_1h_2^2}\left(\frac{\partial h_1}{\partial\xi_2}\right)\left(\frac{\partial\varphi}{\partial\xi_2}\right)\right] \\
&- \mathbf{a}_2\mathbf{a}_2 \frac{2i\mu}{k_c}\left[\frac{1}{h_1}\frac{\partial}{\partial\xi_2}\left(\frac{1}{h_2}\frac{\partial\varphi}{\partial\xi_2}\right) + \frac{1}{h_1^2h_2}\left(\frac{\partial h_2}{\partial\xi_1}\right)\left(\frac{\partial\varphi}{\partial\xi_1}\right)\right] \\
&+ \mathbf{a}_z\mathbf{a}_z(2i\mu k^2/k_c)\varphi + \mathfrak{J}i\lambda k_c\varphi \\
&- (\mathbf{a}_1\mathbf{a}_2 + \mathbf{a}_2\mathbf{a}_1)\frac{i\mu}{k_c}\left[\frac{h_2}{h_1}\frac{\partial}{\partial\xi_1}\left(\frac{1}{h_2^2}\frac{\partial\varphi}{\partial\xi_2}\right) + \frac{h_1}{h_2}\frac{\partial}{\partial\xi_2}\left(\frac{1}{h_1^2}\frac{\partial\varphi}{\partial\xi_1}\right)\right] \\
&\qquad + 2\mu\left(\frac{k}{k_c}\right)[\mathbf{a}_z(\text{grad }\varphi) + (\text{grad }\varphi)\mathbf{a}_z]\Big\} e^{ikz-i\omega t}
\end{aligned}$$

Not much can be made of these equations for most cylindrical coordinates, beyond seeing the complications of fitting boundary conditions. In the case of circular cylinder coordinates, r, ϕ, z, some types of wave motion may be calculated. For example, for waves symmetric about the axis, we have

$$\varphi = J_0(r\sqrt{k_c^2 - k^2}); \quad \psi \text{ or } \chi = J_0(r\sqrt{k_s^2 - k^2})$$

For torsional waves the traction at the surface of the rod ($r = a$) is

$$(\mathfrak{T} \cdot \mathbf{a}_r)_{r=a} = \mathbf{a}_\phi\left(\frac{i\mu}{k_s}\right)(k_s^2 - k^2)J_2(a\sqrt{k_s^2 - k^2})\, e^{ikz-i\omega t} \qquad (13.3.32)$$

and for this surface to be a "free surface" (traction zero across it) $a\sqrt{k_s^2 - k^2}$ must be one of the roots $(\pi\beta_{2n})$ of the equation $J_2(\pi\beta_{2n}) = 0$. In other words the longitudinal wave number k is the following function of ω and n;

$$k = [(\rho\omega^2/\mu) - (\pi\beta_{2n}/a)^2]^{\frac{1}{2}};\quad n\text{th torsional wave} \qquad (13.3.33)$$

as long as ω is larger than its cutoff value $(\pi\beta_{2n}/a)\sqrt{\mu/\rho}$.

The lowest mode is, of course, the limiting one for $\beta \to 0$, where $\psi \to -\frac{1}{2}r^2$ and $\mathfrak{T}\cdot\mathbf{a}_r = 0$ for all values of r, though $\mathfrak{T}\cdot\mathbf{a}_z$ is not zero. In this case k is just equal to k_s; the velocity of transmission of this wave is just the velocity of shear waves in an infinite medium.

On the other hand, the lowest elongational wave is a combination of $\mathbf{L}$ and $\mathbf{N}$. For $\mathbf{N}$ waves with $\chi = J_0(r\sqrt{k_s^2 - k^2})$ the traction across the surface $r = a$ is

$$\left\{-\mathbf{a}_r\left(\frac{ik\mu}{k_s^2}\right)(k_s^2 - k^2)[J_0(\beta_s) - J_2(\beta_s)] \right.$$
$$\left. + \mathbf{a}_z\left(\frac{\mu}{k_s^2}\right)(2k^2 - k_s^2)\sqrt{k_s^2 - k^2}\,J_1(\beta_s)\right\} e^{ikz-i\omega t}$$

where $\beta_s = a\sqrt{k_s^2 - k^2}$, whereas for $\mathbf{L}$, with $\varphi = J_0(r\sqrt{k_c^2 - k^2})$, it is

$$\left\{\mathbf{a}_r\left(\frac{i\mu}{k_c}\right)(k_c^2 - k^2)[J_0(\beta_c) - J_2(\beta_c)] + \mathbf{a}_r i\lambda k_c J_0(\beta_c)\right.$$
$$\left. - \mathbf{a}_z\left(\frac{2\mu k}{k_c}\right)\sqrt{k_c^2 - k^2}\,J_1(\beta_c)\right\} e^{ikz-i\omega t}$$

where $\beta_c = a\sqrt{k_c^2 - k^2}$. The proper combination to remove the z component of the traction at $r = a$ is

$$\left(\frac{ak_c}{2\mu k}\right)\frac{\mathbf{L}}{\beta_c J_1(\beta_c)} + \left[\frac{ak_s^2}{\mu(2k^2 - k_s^2)}\right]\frac{\mathbf{N}}{\beta_s J_1(\beta_s)}$$
$$= \mathbf{a}_z\left[\frac{a}{2\mu}\,\frac{J_0(\beta_c r/a)}{\beta_c J_1(\beta_c)} - \frac{a(k^2 - k_s^2)}{\mu(2k^2 - k_s^2)}\,\frac{J_0(\beta_s r/a)}{\beta_s J_1(\beta_s)}\right]$$
$$+ i\mathbf{a}_r\left[\frac{1}{2\mu k}\,\frac{J_1(\beta_c r/a)}{J_1(\beta_c)} - \frac{k}{\mu(2k^2 - k_s^2)}\,\frac{J_1(\beta_s r/a)}{J_1(\beta_s)}\right] \qquad (13.3.34)$$

For rods with diameter much smaller than a wave length $2\pi c_s/\omega$ or for frequencies much smaller than c_s divided by the diameter of the rod (the two phrases are equivalent), the equation corresponding to the requirement that the r component of the surface traction be zero, from which we are to compute the allowed values of k, simplifies considerably. In this case ak_c and ak_s are considerably smaller than unity. For the lowest mode ak is also small, so that β_c is small enough to neglect all but its first-order terms. For example, $J_1(\beta) \to \frac{1}{2}\beta$ and $J_0(\beta) - J_2(\beta) \to 1$, and so on.

In this case the first-order equation for the traction at the surface $r = a$, for the combination of Eq. (13.3.34), is

$$\mathbf{a}_r \frac{i}{2ak}\left[1 + \frac{\lambda k_c^2}{\mu(k_c^2 - k^2)} - \frac{k^2}{2k^2 - k_s^2}\right]$$

entirely radial, of course, as this combination was built up to suppress the z traction. For this traction to be zero, to the first order in b_c and b_s, we must have that

$$\mu(2k^2 - k_s^2)(k_c^2 - k^2) + \lambda k_c^2(2k^2 - k_s^2) - 2\mu k^2(k_c^2 - k^2) = 0$$

or

$$k^2 = \frac{(\lambda + \mu)k_s^2 k_c^2}{2\lambda k_c^2 + \mu k_s^2} = \rho\omega^2\left[\frac{\lambda + \mu}{\mu(3\lambda + 2\mu)}\right]; \quad \rho\omega^2 a^2 \ll \mu \qquad (13.3.35)$$

The modulus $[\mu(3\lambda + 2\mu)/(\lambda + \mu)]$ is called *Young's modulus.* It appears as a modulus when elastic material is stretched in one direction and not stressed at all normal to the stretch, so that the material shrinks somewhat in the normal directions. The ratio of stress to strain in this case turns out to equal Young's modulus. What we have shown by our calculation is that, for bars thin compared to a wavelength, we can neglect the transverse acceleration in comparison to the longitudinal motion and consider the wave as a pure "stretch" wave, with velocity

$$c_y = \sqrt{\mu(3\lambda + 2\mu)/\rho(\lambda + \mu)}$$

independent of frequency (as long as the rod diameter is small compared to the wavelength). For higher frequencies or larger cross sections, the approximation no longer holds, of course.

For this lowest mode, as long as the first approximation holds, the displacement vector is

$$\mathbf{s} \simeq -\frac{(3\lambda + 2\mu)^2}{a\mu k_s^2\lambda^2}\left[\mathbf{a}_z - i\mathbf{a}_r \frac{\lambda}{2(\lambda + \mu)} kr\right] e^{ikz - i\omega t}$$

or, taking the real part and multiplying by an arbitrary constant

$$\mathbf{s} \simeq A\left[\mathbf{a}_z \cos(kz - \omega t) + \mathbf{a}_r\left(\frac{\lambda/2}{\lambda + \mu}\right) kr \sin(kz - \omega t)\right]$$

This shows that at those cross sections of the rod (those values of $kz - \omega t$) where the z component of displacement is greatest, the r component is least, and vice versa. However, the place where the greatest *stretching* occurs in the z direction is at $(kz - \omega t) = (2n - \frac{1}{2})\pi$, which is the place where the greatest *shrinking* occurs in the r direction. At these regions the shrink in the r direction (minus the gradient of s_r in the r direction) is equal to $\lambda/2(\lambda + \mu)$ times the stretch in the z direction (the gradient of s_z in the z direction). The ratio $\lambda/2(\lambda + \mu)$ is called *Poisson's ratio.* We see that it has meaning only for waves along rods which are much thinner than the wavelength.

The higher modes of symmetric waves along the thin rod ($k_s a \ll 1$), β_s and β_c, cannot be small compared to unity, and consequently k is imaginary. We use the combination of Eq. (13.3.34), which removes the z component of surface traction, and then adjust the value of k/i so that the r component is zero, a rather complex equation, involving the Bessel functions J_0, J_1, and J_2 of $a\sqrt{k_s^2 - k^2}$ and of $a\sqrt{k_c^2 - k^2}$. The solutions will not be discussed here.

Nor will we discuss the *transverse vibrations* of the rod, where **s** is to a great extent pointing in the x or y direction, the restoring force comes from bending the rod and the wave velocity $c_t = \omega/k$ is much smaller than c_s.

Torsional Forced Motions of a Rod. We go further in discussing the torsional motions of a cylindrical rod, however. Suppose that a torque $Te^{-i\omega t}$ is applied through a ring of width Δ, between $z = -\frac{1}{2}\Delta$ and $z = \frac{1}{2}\Delta$, around the circumference of a rod of radius a which extends to infinity in both directions along z. The traction at $r = a$ is then

$$(\mathfrak{T} \cdot \mathbf{a}_r)_{r=a} = \mathbf{P}(z) = \begin{cases} 0; & z < -\frac{1}{2}\Delta \\ \mathbf{a}_\phi (T/a\pi a^2\Delta)e^{-i\omega t}; & -\frac{1}{2}\Delta < z < \frac{1}{2}\Delta \\ 0; & z > \frac{1}{2}\Delta \end{cases}$$

Obviously, to fit the traction given in Eq. (13.3.32) to this discontinuous form, we must use a Fourier integral over k.

The Fourier integral for the driving traction is

$$\begin{aligned} \mathbf{P}(z) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikz}\,dk \int_{-\infty}^{\infty} \mathbf{P}(\zeta)e^{-ik\zeta}\,d\zeta \\ &= \mathbf{a}_\phi \left[\frac{Te^{-i\omega t}}{2\pi^2 a^2\Delta}\right] \int_{-\infty}^{\infty} \frac{\sin(k\Delta/2)}{k}\, e^{ikz}\,dk \end{aligned} \tag{13.3.36}$$

and in order that the integral of an expression of the form of Eq. (13.3.32) reduce to this, the function ψe^{ikz} must be replaced by a Fourier integral, and the displacement field must be, according to Eq. (13.3.30),

$$\mathbf{s} = -\left[\frac{\mathbf{a}_\phi Te^{-i\omega t}}{2\pi^2\mu a^2\Delta}\right] \int_{-\infty}^{\infty} \frac{\sin(k\Delta/2)}{k\sqrt{k_s^2 - k^2}} \left[\frac{J_1(r\sqrt{k_s^2 - k^2})}{J_2(a\sqrt{k_s^2 - k^2})}\right] e^{ikz}\,dk \tag{13.3.37}$$

The singularities of the integrand are all simple poles. Two of them are at $\pm k_s$, on the real axis. The rest are along the imaginary axis, at $\pm i\sqrt{(\pi\beta_{2n}/a)^2 - k_s^2}$, where $(\pi\beta_{2n})$ is the nth root of the equation $J_2(\pi\beta_{2n}) = 0$. (This assumes that $k_s < \pi\beta_{21}/a$, otherwise the poles for the first few roots of J_2 would be on the real axis.) For k large and complex, the controlling part of the integrand is the factor $\sin(k\Delta/2)e^{ikz}$, the ratio J_1/J_0 approaches a finite constant as k goes to infinity in any direction except along the imaginary axis.

Assuming that $a < \pi\beta_{21}/k_s$, so that the only torsional mode with real wave transmission is the lowest mode, mentioned in connection with Eq. (13.3.32) (for $k = k_s$), and computing the integral for values of z much larger than $\frac{1}{2}\Delta$, the only poles which contribute an appreciable amount are the ones at $\pm k_s$. For this value of z, we must complete our contour integral by a very large semicircle around the *upper* half of the complex plane, and we thus enclose all poles in this region. To see which of the two poles on the real axis we enclose, we remember we wish an outgoing wave to the right for $z > \frac{1}{2}\Delta$, so our contour must go *above* all poles on the negative k real axis and *below* all poles on the positive k real axis and thus enclose the pole at $k = +k_s$ in this case (see discussion of page 823).

The residue at this pole, since $J_1(\epsilon) \to \frac{1}{2}\epsilon$ and $J_2(\epsilon) \to \frac{1}{8}\epsilon^2$, is such as to make

$$\mathbf{s} \to \mathbf{a}_\phi \left(\frac{iT}{\pi\mu a^3 k_s}\right) \frac{\sin(k_s\Delta/2)}{(k_s\Delta/2)} \left(\frac{r}{a}\right) e^{ik_sz-i\omega t} \tag{13.3.38}$$

where $k_s = \omega\sqrt{\rho/\mu}$ and where z is large enough ($z \gg a/\pi\beta_{21}$) so that the residues about the poles on the imaginary axis are negligible.

These other residues correspond to higher mode torsional motion which, for the frequency range we have assumed here (for the diameter rod assumed), all attenuate instead of propagate and are appreciable only near the driving ring. The energy fed into the ring, consequently, radiates away in simple torsional waves, spreading out in both directions (for the expression for z large and negative only has the sign of the z in the exponential different). The amplitude of these waves, at some distance from the driving ring, is $(1/\pi a^3\omega\sqrt{\rho\mu})$ times the amplitude T of the driving torque, provided the width Δ of the driving ring is considerably smaller than a wavelength.

To compute the driving-point impedance, we must calculate $\mathbf{s}$ at $z = \frac{1}{2}\Delta$, which means that we must compute the residues at the other poles for $z > \frac{1}{2}\Delta$ and then obtain the value at $z = \frac{1}{2}\Delta$ by analytic continuation. The nth pole along the imaginary axis is for $a\sqrt{k_s^2 - k^2} = \pi\beta_{2n}$, where $\pi\beta_{2n}$ is the nth root of $J_2(\pi\beta) = 0$. We set $k = iK_n + \epsilon$, where $K_n^2 = (\pi\beta_{2n}/a)^2 - k_s^2$; in which case $\sqrt{k_s^2 - k^2} = (\pi\beta_{2n}/a) - i\epsilon(aK_n/\pi\beta_{2n})$ for ϵ sufficiently small. Remembering the equations for the derivative of a Bessel function, we use Taylor's series to calculate the dependence of J_2 on ϵ;

$$J_2(a\sqrt{k_s^2 - k^2}) \to -ia\epsilon(aK_n/\pi\beta_{2n})J_1(\pi\beta_{2n})$$

which displays the nature of the pole at $k = iK_n$.

Collecting all the factors and computing the residues, we finally obtain the complete expression for the displacement vector

$$\mathbf{s} = \mathbf{a}_\phi \left(\frac{iTe^{-i\omega t}}{\pi a^3 \omega \sqrt{\rho\mu}}\right) \left\{ \left(\frac{r}{a}\right) \frac{\sin(k_s\Delta/2)}{(k_s\Delta/2)} e^{ik_s z} - i \sum_{n=1}^{\infty} \left(\frac{k_s}{K_n^2 \Delta}\right) \sinh(\tfrac{1}{2}K_n\Delta) e^{-K_n z} \left[\frac{J_1(\pi\beta_{2n} r/a)}{J_1(\pi\beta_{2n})}\right] \right\}; \quad z > \tfrac{1}{2}\Delta$$

The series requires manipulation before it becomes docile. We are interested in the value of $\mathbf{s}$ at $r = a$ and $z = \frac{1}{2}\Delta$, in which case the Bessel functions cancel out. For ak_s sufficiently small, K_n is nearly equal to $\pi\beta_{2n}/a$. Furthermore, we are interested primarily in the *torsional admittance* of the rod for the driving ring, the ratio between the angular displacement $\mathbf{s}/a$ of the ring and the torque T. When $ak_s \ll 1$, this admittance is

$$Y_t \simeq \left(\frac{1}{\pi a^4 \sqrt{\rho\mu}}\right) \left\{ \left[\frac{\sin(k_s\Delta/2)}{k_s\Delta/2}\right] e^{ik_s\Delta/2} - \left(\frac{i\omega a^2}{2\Delta}\right) \sqrt{\frac{\rho}{\mu}} \sum_n \frac{1}{(\pi\beta_{2n})^2} [1 - e^{-(\pi\beta_{2n}\Delta/2)}] \right\} \quad (13.3.39)$$

If Δ/a is not small but $k_s\Delta$ is small, the quantity in brackets in the first term becomes unity and the exponential part of the series will converge rapidly, only the first one being appreciable. By various methods one can show that $\sum_n [1/(\pi\beta_{mn})^2] = [1/4(m+1)]$. Consequently, for $k_s\Delta \ll 1$ but $\Delta \sim a$, the torsional admittance is

$$Y_t \simeq \frac{1}{\pi a^4 \sqrt{\rho\mu}} - i\omega \left(\frac{1}{24\pi a^2 \mu \Delta}\right) [1 - 0.455 e^{-5.14(\Delta/a)} - \cdots] \quad (13.3.40)$$

corresponding to a resistance $\pi a^4 \sqrt{\rho\mu}$ in parallel with a "stiffness" approximately equal to $24\pi a^2 \mu \Delta$.

The more interesting case, however, is the one where Δ is small compared to a and, a fortiori, $k_s\Delta$ is small. Here we use the relation

$$\sum_n \frac{1}{(\pi\beta_{2n})^2} [1 - e^{-(\pi\beta_{2n}\Delta/a)}] = \int_0^\Delta \sum_n \left(\frac{1}{\pi\beta_{2n} a}\right) e^{-(\pi\beta_{2n} x/a)}\, dx$$

But $\beta_{2n} \simeq n + \frac{3}{4}$ so that the series in the integral is, approximately,

$$\frac{1}{\pi a} e^{-\frac{3}{4}(\pi x/a)} \sum_{n=1}^{\infty} \left(\frac{1}{n + \frac{3}{4}}\right) e^{-(\pi n x/a)} \quad \text{or} \quad \left(\frac{1}{\pi a}\right) e^{\frac{1}{4}(\pi x/a)} \sum_{m=2}^{\infty} \left(\frac{1}{m - \frac{1}{4}}\right) e^{-(\pi m x/a)}$$

For Δ/a sufficiently small, an average of these series approaches

$$-\left(\frac{g}{a\pi}\right) + \sum \left(\frac{1}{\pi a n}\right) e^{-n\pi x/a} \simeq -\left(\frac{1}{a\pi}\right) - \left(\frac{1}{\pi a}\right) \ln\left(\frac{\pi x}{a}\right); \quad x \ll a$$

where the constant g is approximately unity. Performing the integration, we obtain

$$\sum_n \frac{1}{(\pi\beta_{2n})^2}\left[1 - e^{-(\pi\beta_{2n}\Delta/a)}\right] \simeq \left(\frac{\Delta}{\pi a}\right)\ln\left(\frac{a}{\pi\Delta}\right)$$

Consequently, for $\Delta \ll a \ll 1/k_s$, the admittance turns out to be approximately

$$Y \simeq \frac{1}{\pi a^4\sqrt{\rho\mu}} - i\omega\left(\frac{1}{2\pi^2 a^3\mu}\right)\ln\left(\frac{a}{\pi\Delta}\right) \tag{13.3.41}$$

corresponding to a resistance $[\pi a^4\sqrt{\rho\mu}]$ in parallel with a "stiffness" $[2\pi^2 a^3\mu/\ln(a/\pi\Delta)]$.

These two limiting approximations illustrate the nature of the coupling between the driving ring and the rod. When the second term in Y is small compared to the first term either because Δ is large or because ω is small, the coupling to the rod is good and the rotary motion of the driving ring has an amplitude only a little larger than the amplitude of motion of the torsional waves radiating away from the ring [as shown in Eq. (13.3.38)], because in this case the higher modes are very little excited.

On the other hand, at somewhat higher frequencies or for a smaller band width Δ, the higher modes are excited to an appreciable extent, the second term in Y is no longer negligible, and the coupling between the ring and the rod is less effective. The higher modes correspond to motion of the ring which does not give rise to wave motion, which thus reduces the effective coupling between ring and wave. These higher modes correspond to a twisting of the rod which is not uniform in r; if we consider the rod as made up of concentric cylindrical shells, the lowest torsional wave motion corresponds to a turning of all shells by the same angle at any given value of z, whereas the higher modes correspond to a twisting of the shells on each other, each shell turning by a different angle. The presence of the second term in Y corresponds to the fact that the ring and the outer shells of the rod near the ring turn through a larger angle than do the inner shells, for $r < a$.

This same effect of the higher modes also explains why a ring of width Δ, welded to the rod and held stationary, does not completely stop torsional waves. It will stop the parts of the wave carried by the outer shells, so to speak, but the inner core of the rod, not being completely coupled to the stationary ring, will allow part of the wave to penetrate past the ring. For example, suppose a wave

$$\mathbf{s}_i = \mathbf{a}_\phi A(r/a)e^{ik_sz - i\omega t}$$

coming from the left, is incident on a stationary ring of width Δ clamped on the rod at $z = 0$. The ring must exert an opposing torque on the surface of the rod such that the torque, without the presence of the incident wave, would produce a motion equal and opposite to that produced by the incident wave at the ring position, which torque will produce a reflected and a transmitted wave. When the coupling is good (second term in Y negligible), the amplitude of these waves is nearly equal to that of the incident wave so for $z > 0$ the two waves cancel (no transmitted wave) and for $z < 0$ the reflection is complete. If the coupling is not good, however, if some of the higher modes are present to decouple the core of the rod from its surface, the amplitude of the reflected wave is not A, but

$$\frac{A}{[1 - i\omega(a/2\pi)\sqrt{(\rho/\mu)}\ln(a/\pi\Delta)]};\quad \Delta \ll a \ll 1/k_s$$

which is nearly equal to A, and the amplitude of the transmitted wave is

$$\frac{A}{1 + (2\pi i/\omega a)\sqrt{\mu/\rho}\,[1/\ln(a/\pi\Delta)]} \tag{13.3.42}$$

which is usually quite small but is not zero. If Δ is not very much smaller than a, this last expression is more nearly equal to

$$\frac{A}{1 + (24i\Delta/\omega a^2)\sqrt{\mu/\rho}}$$

Since $\sqrt{\mu/\rho}$ is of the order of 10^5 cgs units, we see that the amount of torsional wave penetrating beyond the damped ring is quite small unless the product ωa is larger than 1,000,000 cm per sec.

Nonstationary Viscous Flow in Tube. If the pressure drop along a tube containing viscous, incompressible fluid is periodic, $-\operatorname{grad} P = \mathbf{a}_z F_0 e^{-i\omega t}$, then the flow will be periodic. As long as the velocities are below the Reynolds number, the flow will all be parallel to the axis of the tube, the z axis. The variation of flow across the tube may be expressed in terms of a set of scalar, two-dimensional eigenfunctions $\psi_{nm}(x,y)$, solutions of the equation $\nabla^2\psi_{nm} = -k_{nm}^2\psi_{nm}$ which go to zero at the boundaries of the tube cross section α

$$\mathbf{v}_\omega = \mathbf{a}_z \sum_{n,m} A_{nm}\psi_{nm}(x,y)e^{-i\omega t}$$

where the values of the coefficients A may be determined by requiring that $\mathbf{v}$ satisfy Eq. (13.3.1)

$$\nabla^2\mathbf{v} - (\rho/\eta)(\partial\mathbf{v}/\partial t) = -\mathbf{a}_z(F_0/\eta)e^{-i\omega t}$$

Substituting the series into this equation, multiplying both sides by $\bar{\psi}_{nm}$, and integrating over the cross section result in

$$\mathbf{v}_\omega = i\mathbf{a}_z e^{-i\omega t}\left(\frac{F_0}{\rho}\right)\sum_{n,m}\frac{I_{nm}\psi_{nm}(x,y)}{\omega + i(\eta k_{nm}^2/\rho)} \tag{13.3.43}$$

where $\qquad I_{nm} = [\int\int\bar{\psi}_{nm}\,dx\,dy]/[\int\int|\psi_{nm}|^2\,dx\,dy]$

The motion is thus not all in phase with the pressure drop, the (n,m)th mode having a phase lag $\tan^{-1}(\omega\rho/\eta k_{nm}^2)$.

Having computed $\mathbf{v}_\omega$, we can now calculate the flow for any transient pressure drop by means of the technique of the Laplace transform. For example, the pressure drop which is suddenly applied at $t = 0$,

$$\mathbf{a}_z F_0 u(t) = -\frac{a_z F_0}{2\pi i}\int_{-\infty+i\epsilon}^{\infty+i\epsilon} e^{-i\omega t}\,\frac{d\omega}{\omega}$$

where $u(t) = 0\ (t < 0)$, $u(t) = 1(t > 0)$, has a solution

$$\begin{aligned}
\mathbf{v}_u &= -\frac{1}{2\pi i}\int_{-\infty+i\epsilon}^{\infty+i\epsilon}\mathbf{v}_\omega\,\frac{d\omega}{\omega} = 0; && t < 0\\
&= \mathbf{a}_z\left(\frac{F_0}{\eta}\right)\sum_{n,m}\left(\frac{I_{nm}}{k_{nm}^2}\right)\psi_{nm}(x,y)[1 - e^{-(\eta k_{nm}{}^2/\rho)t}]; && t > 0 \qquad (13.3.44)
\end{aligned}$$

The velocity is zero at $t < 0$, the higher modes come into full motion first, and, finally, for $t \gg (\rho/\eta k_{01}^2)$ (where k_{01} is the lowest eigenvalue) the motion is the steady-state flow discussed in Sec. 13.2.

Electromagnetic Resonators. When both ends of a rectangular wave guide are closed off, resulting in a rectangular parallelepiped of sides l_x, l_y, and l_z, standing electromagnetic waves can exist inside the cavity. The two general types of standing waves which satisfy the boundary conditions $E_{\tan} = 0$ at the surface and which have zero divergence throughout the enclosed region, are

$$\begin{aligned}
\mathbf{M}_m(\mathbf{r}) &= \frac{1}{k_m}\operatorname{curl}\left[\mathbf{a}_z\cos\left(\frac{\pi m_x x}{l_x}\right)\cos\left(\frac{\pi m_y y}{l_y}\right)\sin\left(\frac{\pi m_z z}{l_z}\right)\right]\\
&= -\mathbf{a}_x\left(\frac{\pi m_y}{l_y k_m}\right)\cos\left(\frac{\pi m_x x}{l_x}\right)\sin\left(\frac{\pi m_y y}{l_y}\right)\sin\left(\frac{\pi m_z z}{l_z}\right)\\
&\qquad + \mathbf{a}_y\left(\frac{\pi m_x}{l_x k_m}\right)\sin\left(\frac{\pi m_x x}{l_x}\right)\cos\left(\frac{\pi m_y y}{l_y}\right)\sin\left(\frac{\pi m_z z}{l_z}\right) \qquad (13.3.45)\\
k_m^2 &= (\pi m_x/l_x)^2 + (\pi m_y/l_y)^2 + (\pi m_z/l_z)^2 = (\omega_m/c)^2; \quad m = (m_x,m_y,m_z)\\
\mathbf{N}_m(\mathbf{r}) &= \frac{1}{k_m^2}\operatorname{curl}\operatorname{curl}\left[\mathbf{a}_z\sin\left(\frac{\pi m_x x}{l_x}\right)\sin\left(\frac{\pi m_y y}{l_y}\right)\cos\left(\frac{\pi m_z z}{l_z}\right)\right]\\
&= \mathbf{a}_z\frac{1}{k_m^2}\left[\left(\frac{\pi m_x}{l_x}\right)^2 + \left(\frac{\pi m_y}{l_y}\right)^2\right]\sin\left(\frac{\pi m_x x}{l_x}\right)\sin\left(\frac{\pi m_y y}{l_y}\right)\cos\left(\frac{\pi m_z z}{l_z}\right)\\
&\qquad - \mathbf{a}_x\left(\frac{\pi m_z}{l_z k_m^2}\right)\left(\frac{\pi m_x}{l_x}\right)\cos\left(\frac{\pi m_x x}{l_x}\right)\sin\left(\frac{\pi m_y y}{l_y}\right)\sin\left(\frac{\pi m_z z}{l_z}\right)\\
&\qquad - \mathbf{a}_y\left(\frac{\pi m_z}{l_z k_m^2}\right)\left(\frac{\pi m_y}{l_y}\right)\sin\left(\frac{\pi m_x x}{l_x}\right)\cos\left(\frac{\pi m_y y}{l_y}\right)\sin\left(\frac{\pi m_z z}{l_z}\right)
\end{aligned}$$

where for $\mathbf{M}_m$, $m_z > 0$ and either m_x or m_y can be zero but not both; for $\mathbf{N}_m$, m_z may be zero but both m_x and m_y must be greater than zero.

Both types of functions satisfy the equation curl curl $\mathbf{M}_m = k_m^2\mathbf{M}_m$ or curl curl $\mathbf{N}_m = k_m^2\mathbf{N}_m$. They do not appear to be completely symmetrical, they "favor" the z axis since they are formed by operating on $\mathbf{a}_z$ times a scalar solution of the Helmholtz equation. They are, however, a complete set of transverse vector eigenfunctions for the region and are suitable for use when boundary conditions turn out to "favor" the z axis. Another, completely equivalent, set could be obtained by operating on $\mathbf{a}_x$ times a scalar wave solution, and still a third by operating on $\mathbf{a}_y$ times another function. We would find, however, that each function of the latter two sets was a simple linear combination of the sets given in Eqs. (13.3.28), so that any one of the three sets is sufficient; which of the three is used depends on the boundary conditions.

We note that the solutions for rectangular enclosures are degenerate, in that both $\mathbf{M}_m$ and $\mathbf{N}_m$ have the same natural frequency $\omega_m/2\pi$ (this will not be true for enclosures of other shape). The functions are, of course, mutually orthogonal. Their normalizing constants are

$$\int_0^{l_x} dx \int_0^{l_y} dy \int_0^{l_z} dz|\mathbf{M}_m|^2 = \int_0^{l_x} dx \int_0^{l_y} dy \int_0^{l_z} dz|\mathbf{N}_m|^2$$
$$= \frac{l_x l_y l_z}{\epsilon_{m_x}\epsilon_{m_y}\epsilon_{m_z}k_m^2}\left[\left(\frac{\pi m_x}{l_x}\right)^2 + \left(\frac{\pi m_y}{l_y}\right)^2\right] = \left[1 - \left(\frac{\pi m_z}{l_z k_m}\right)^2\right]\Lambda_m$$

As with the other eigenfunction solutions for finite enclosures, the Green's function for a driving frequency $\omega/2\pi = kc/2\pi$ may be written in two forms, the first a symmetric sum

$$\mathfrak{G}(\mathbf{r}|\mathbf{r}_0|k) = \sum_m \frac{4\pi k_m^2/\Lambda_m}{(k_m^2 - k^2)[k_m^2 - (\pi m_z/l_z)^2]}[\mathbf{M}_m(\mathbf{r}_0)\mathbf{M}_m(\mathbf{r}) + \mathbf{N}_m(\mathbf{r}_0)\mathbf{N}_m(\mathbf{r})] \tag{13.3.46}$$

which is useful for computing many forced-motion problems. The other series is obtained from Eq. (13.3.14), the complete Green's dyadic for the duct. In the present case, the function f differs from the function g, because the new boundary condition is that $E_{\tan}$ is zero at $z = 0$ and $z = l_z$. We have

$$f_{mn} = \frac{1}{K_{mn}\sin(K_{mn}l_z)}\begin{cases}\sin(K_{mn}z)\sin[K_{mn}(l_z - z_0)]; & z < z_0\\ \sin(K_{mn}z_0)\sin[K_{mn}(l - z)]; & z > z_0\end{cases}$$

and

$$g_{mn} = \frac{-1}{K_{mn}\sin(K_{mn}l_z)}\begin{cases}\cos(K_{mn}z)\cos[K_{mn}(l_z - z_0)]; & z < z_0\\ \cos(K_{mn}z_0)\cos[K_{mn}(l_z - z)]; & z > z_0\end{cases}$$

where $K_{mn}^2 = k^2 - k_{mn}^2$. (When $k_{mn} > k$, K_{mn} is positive imaginary.) The complete Green's dyadic then has the form

$$\mathfrak{G} = 4\pi \sum_{m,n} \frac{1}{k_{mn}^2 \Lambda_{mn}} \{[\mathbf{a}_z \times \text{grad } \psi_{mn}(\mathbf{r}_0)][\mathbf{a}_z \times \text{grad } \psi_{mn}(\mathbf{r})]f_{mn}$$
$$+ k_{mn}^2 \mathbf{a}_z \chi_{mn}(\mathbf{r}_0) \mathbf{a}_z \chi_{mn}(\mathbf{r}_0) g_{mn} + \text{grad } \chi_{mn}(\mathbf{r}_0) \text{ grad } \chi_{mn}(\mathbf{r}) f_{mn}\} \quad (13.3.47)$$

which is equivalent to the series of Eq. (13.3.46) plus the longitudinal part. This series also "favors" the z axis; equivalent series may be obtained by cyclic interchange of x, y, and z in the formulas.

Energy Loss at the Walls, the Q of the Cavity. In the case of resonators as with ducts of infinite length, finite conductivity of the walls produces energy loss; in the case of resonators, this is evidenced by a decay of the oscillations with time rather than with distance along the tube. The usual quantity measuring the damping of the oscillations is the "Q of the system," the number of cycles for the wave amplitude to reduce by a factor $e^{-\pi}$ (for a simple series a-c circuit $Q = \omega L/R$). In other words, the exponential factor giving the damping of the amplitude is $e^{-\frac{1}{2}(\omega/Q)t}$, and the factor giving the damping of the average energy density of the wave is $e^{-(\omega t/Q)}$. The larger the Q, the smaller the power loss and the sharper the resonance response to a driving force near the resonant frequency.

Referring to the derivation of Eq. (13.3.17), we see that the equation for Q is that ω/Q is equal to the ratio of the energy lost at the walls to the total energy of the field, the integral of $(1/8\pi)(E^2 + H^2)$ over the volume of the enclosure. In the notation of Eq. (13.3.17), therefore, we have

$$Q = \tfrac{1}{2} \sqrt{\frac{8\pi\omega\sigma}{\mu c^2}} \frac{\iiint[|E|^2 + |H|^2]\, dv}{\iint |H_{\tan}|^2\, dA} \quad (13.3.48)$$

Since Q is dimensionless, the quantity $\sqrt{\mu c^2/8\pi\sigma\omega}$ has the dimensions of a length; it is the mean distance to which the field penetrates the conductive surface and is called the *skin depth* of the conductor.

For a cylindrical cavity, closed at $z = 0$ and $z = l_z$, with cross section any of the separable cylindrical systems, the $\mathbf{M}$ standing waves are

$$\mathbf{M}_m = -(1/k_m)[\mathbf{a}_z \times \text{grad } \psi_m(x,y)] \sin(\pi m_z z/l_z) = \mathbf{E}/ik_m$$

where ψ is the solution of $\nabla_{xy}^2 \psi_m + [k_m^2 - (\pi m_z/l_z)^2]\psi_m = 0$ having zero normal gradient on the boundaries. The corresponding magnetic field is

$$\mathbf{H} = \frac{1}{k_m} \left\{ \mathbf{a}_z \left[k_m^2 - \left(\frac{\pi m_z}{l_z}\right)^2 \right] \psi_m \sin\left(\frac{\pi m_z z}{l_z}\right) + \left(\frac{\pi m_z}{l_z}\right) \text{grad } \psi_m \cos\left(\frac{\pi m_z z}{l_z}\right) \right\}$$

and
$$\iiint E^2\, dv = \iiint H^2\, dv = \tfrac{1}{2} l_z \left[k_m^2 - \left(\frac{\pi m_z}{l_z}\right)^2 \right] \Lambda_m$$

where
$$\Lambda_m = \iint |\psi_m|^2\, dx\, dy$$

so that the numerator of Eq. (13.3.48) is just twice this.

The tangential field $H_{\tan}$, on the sides of the enclosure parallel to the z axis, has a z component $(1/k_m)[k_m^2 - (\pi m_z/l_z)^2]\psi_m \sin(\pi m_z z/l_z)$, where ψ_m has its surface value, and a component perpendicular to z which is $(\pi m_z/k_m l_z)(\partial\psi_m/\partial s)\cos(\pi m_z z/l_z)$ where $\partial\psi/\partial s$ is the tangential component of grad ψ computed at the surface. The tangential $\mathbf{H}$ at the two end faces, $z = 0$ and $z = l_z$, is equal to $(\pi m_z/k_m l_z)$ grad ψ_m and $(-1)^{m_z}$ times this, respectively. Consequently, the integral of $|H_{\tan}|^2$ over the surface of the enclosure for the $\mathbf{M}$ waves is

$$\tfrac{1}{2}l_z\left\{\frac{1}{k_m^2}\left[k_m^2 - \left(\frac{\pi m_z}{l_z}\right)^2\right]^2 \oint |\psi_m|^2\,ds + \left(\frac{\pi m_z}{k_m l_z}\right)^2 \oint \left|\frac{\partial\psi_m}{\partial s}\right|^2 ds\right\} + \left(\frac{\pi m_z}{k_m l_z}\right)^2\left[k_m^2 - \left(\frac{\pi m_z}{l_z}\right)^2\right]\Lambda_m$$

where the integrations are around the perimeter of the cross section. Consequently, the value of Q for the mth transverse electric mode is

$$Q_{em} = \Lambda_m\sqrt{\frac{8\pi\omega\sigma}{\mu c^2}}\left\{\left(\frac{\pi m_z}{k_m l_z}\right)^2\Lambda_m + \left[1 - \left(\frac{\pi m_z}{k_m l_z}\right)^2\right]\oint|\psi_m|^2\,ds + \left(\frac{\pi m_z}{k_m^2 l_z}\right)^2\left[1 - \left(\frac{\pi m_z}{k_m l_z}\right)^2\right]^{-1}\oint\left|\frac{\partial\psi_m}{\partial s}\right|^2 ds\right\}^{-1} \tag{13.3.49}$$

Similar calculations may be used to obtain the Q for the mth transverse magnetic mode, for waves of the form

$$\mathbf{N}_m = \frac{1}{k_m^2}\left\{\mathbf{a}_z\left[k_m^2 - \left(\frac{\pi m_z}{l_z}\right)^2\right]\chi_m \cos\left(\frac{\pi m_z z}{l_z}\right) - \left(\frac{\pi m_z}{l_z}\right)\operatorname{grad}\chi_m \sin\left(\frac{\pi m_z z}{l_z}\right)\right\} = \frac{\mathbf{E}}{ik_m}$$

with magnetic field

$$\mathbf{H} = -[\mathbf{a}_z \times \operatorname{grad}\chi_m]\cos(\pi m_z z/l_z)$$

where χ_m is a solution of the two-dimensional Helmholtz equation which is zero at the cross-section boundary. The result is

$$Q_{hm} = \frac{\tfrac{1}{2}l_z\sqrt{8\pi\omega\sigma/\mu c^2}}{1 + (l_z/2k_m^2\Lambda_m)[1 - (\pi m_z/k_m l_z)^2]^{-1}\oint|\partial\chi_m/\partial n|^2\,ds} \tag{13.3.50}$$

A little later in our discussion we shall be computing the response of the resonator to some driving force. If we use the Green's function in the form given in Eq. (13.3.46), the various terms will have resonance denominators $(k_m^2 - k^2)$, which go to zero when the driving frequency $\omega/2\pi$ equals the resonance frequency $\omega_m/2\pi = k_m c/2\pi$ (to the approximation which assumes that the walls have infinite conductivity). Our calculations of Q indicate that, if the walls have large but finite conductivity, then these resonance denominators should be

$$\begin{aligned}&\{k_m^2[1 - (i/2Q_{em})]^2 - k^2\};\quad \text{for the term in } \mathbf{M}_m\\ &\{k_m^2[1 - (i/2Q_{hm})]^2 - k^2\};\quad \text{for the term in } \mathbf{N}_m\end{aligned} \tag{13.3.51}$$

to the first order in the small quantity $\sqrt{\mu c^2/8\pi\omega\sigma}$. The reciprocal of these factors does not go to infinity as $k \to k_m$ but goes to a maximum value of iQ_{em}/k_m^2 or iQ_{hm}/k_m^2 when $k = k_m\sqrt{1 - (1/2Q_{em})^2}$ or $k = k_m\sqrt{1 - (1/2Q_{hm})^2}$, respectively.

The other type of expression for the Green's function also has resonance factors which, near resonance, have the approximate form $A/(k_m^2 - k^2)$. To include the effect of wall conductivity to the first order in $1/Q$, these factors also should be changed to $A/\{k_m^2[1 - (i/2Q_m)]^2 - k^2\}$.

Excitation of Resonator by Driving Current. As the first example of the forced motion of an electromagnetic resonator, we consider the excitation of a rectangular enclosure of sides l_x, l_y, l_z, for which Eqs. (13.3.45) gave the standing waves, by current $a_x I e^{-i\omega t}$ in a strip of width Δ_y, parallel to the x axis and a distance Δ_z from the center of the $z = 0$ wall. (In other words, the strip lies between the line $z = \Delta_z$, $y = \frac{1}{2}l_y - \frac{1}{2}\Delta_y$ and the line $z = \Delta_z$, $y = \frac{1}{2}l_y + \frac{1}{2}\Delta_y$.) Such a current cannot couple to $\mathbf{N}$-type waves, for no $\mathbf{N}$-type wave has a component along x which is constant, whereas the $\mathbf{M}$ types for which $m_x = 0$ have this behavior. Consequently, in using the Green's function of Eq. (13.3.46), we need include only the $\mathbf{M}$ waves for which $m_x = 0$ and for which m_y is odd. (For ease in typography we will then set, for this subsection only, $m_y = 2m + 1$ and $m_z = n$.)

The equation for the excited waves in the resonator is then obtained from Eq. (13.1.52) with $\mathbf{Q} = \mathbf{a}_x\delta(z - \Delta_z)(I/c\Delta_y)e^{-i\omega t}$ for y between $\frac{1}{2}l_y - \frac{1}{2}\Delta_y$ and $\frac{1}{2}l_y + \frac{1}{2}\Delta_y$, equal zero for other values of y and with the surface integrals all zero since we assume to begin with that the walls are infinitely conductive. The equation for the vector potential is

$$\begin{aligned}\mathbf{A} &= \sum_{m,n} \frac{(16\pi k_{mn}^2/l_x l_y l_z)}{(k_{mn}^2 - k^2)[\pi(2m+1)/l_y]^2}\,\mathbf{M}_{mn}(\mathbf{r})\,\cdot\\
&\qquad\cdot\frac{I}{c\Delta_y}\int_0^{l_x} dx_0 \int_{\frac{1}{2}l_y - \frac{1}{2}\Delta_y}^{\frac{1}{2}l_y + \frac{1}{2}\Delta_y} dy_0 \int_0^{l_z} \delta(z_0 - \Delta_z)\mathbf{M}_{mn}(\mathbf{r}_0)\,dz_0 \cdot \mathbf{a}_x\\
&= \frac{16\pi I\mathbf{a}_x}{c l_y l_z}\sum_{m,n}(-1)^m \frac{\sin(\pi n\Delta_z/l_z)}{k_{mn}^2 - k^2}\left[\frac{2l_y}{\pi(2m+1)\Delta_y}\right]\cdot\\
&\qquad\cdot\sin\left[\frac{\pi(2m+1)\Delta_y}{2l_y}\right]\sin\left[\frac{\pi(2m+1)y}{l_y}\right]\sin\left[\frac{\pi n z}{l_z}\right]e^{-i\omega t}\end{aligned} \tag{13.3.52}$$

where $k_{mn}^2 = [\pi(2m+1)/l_y]^2 + [\pi n/l_z]^2$.

Near a resonance, where all terms except the resonating one may be neglected, this series is a useful form for the solution. For instance, for ω near ck_{mn} the back voltage l_xE, opposing the current in the wire, is

$$V = \frac{16\pi I l_x}{l_y l_z}\left(\frac{\pi n \Delta_z}{l_z}\right)^2 \frac{i\omega}{\omega^2 - c^2k_{mn}^2[1 - (i/2Q_{emn})]^2} \tag{13.3.53}$$

where we have included the Q term in the resonance denominator and where we have assumed that $\Delta_z \ll l_z/n$ and $\Delta_y \ll l_y/(2m + 1)$. Near this resonance frequency, the length of strip in the resonator behaves as though it were a parallel circuit, with resistance R_{mn}, inductance L_{mn}, and capacitance C_{mn} all in parallel, where

$$C_{mn} = \frac{l_y l_z}{16\pi l_x}\left(\frac{l_z}{\pi n \Delta_z}\right)^2; \quad L_{mn} = \frac{16\pi l_x}{c^2 k_{mn}^2 l_y l_z}\left(\frac{\pi n \Delta_z}{l_z}\right)^2$$

$$R_{mn} = \frac{16\pi l_x Q_{emn}}{ck_{mn} l_y l_z}\left(\frac{\pi n \Delta_z}{l_z}\right)^2$$

For calculation of the reactive voltage at nonresonant frequencies, however, the other form of Green's function is better, the one given in Eq. (13.3.47). Here again, only the terms in $\mathbf{a}_z \times \operatorname{grad} \psi_{mn}$ differ from zero, and of these only the ones for $m = 0$, n odd are nonvanishing. When $k < 2\pi/l_y$, so that (for $n > 0$) we have $\sqrt{[\pi(2n + 1)/l_y]^2 - k^2} \simeq [\pi(2n + 1)/l_y]$, we can separate the sum into the first term, containing the resonance denominator (for the low frequencies) and the rest of the terms, which represent a piling up of field near the strip and which tend to "decouple" strip and field in the resonator. The integration over the Green's function results in

$$\begin{aligned}\mathbf{A} \simeq \mathbf{a}_x \frac{8\pi I \Delta_z}{c l_y} \Bigg\{ &\sin\left(\frac{\pi y}{l_y}\right) \cos\left[z\sqrt{k^2 - \left(\frac{\pi}{l_y}\right)^2}\right] \\ &- \sin\left(\frac{\pi y}{l_y}\right) \frac{\sin[z\sqrt{k^2 - (\pi/l_y)^2}]}{\tan[l_z\sqrt{k^2 - (\pi/l_y)^2}]} \\ &+ \sum_{n=1}^{\infty} (-1)^n \frac{\sinh[\pi(2n + 1)\Delta_z/l_z]}{[\pi(2n + 1)\Delta_z/l_z]} \frac{\sin[\pi(2n + 1)\Delta_y/2l_y]}{[\pi(2n + 1)\Delta_y/2l_y]} \cdot \\ &\cdot \frac{\sinh[\pi(2n + 1)(l_z - z)/l_y]\sin[\pi(2n + 1)y/l_y]}{\sinh[\pi(2n + 1)l_z/l_y]}\Bigg\}\end{aligned} \tag{13.3.54}$$

This does not look much like Eq. (13.3.52), but it is equivalent. The resonance terms come from the tangent term. We have restricted our values of k, but within the restriction the term containing the tangent goes to infinity when $\sqrt{k^2 - (\pi/l_y)^2} = \pi/l_z$, that is, when $k^2 = k_{01}^2$ in the notation of Eq. (13.3.52). An expansion of the term for k near k_{01} shows that this term is identical with the first term of Eq. (13.3.52).

The back electromotive force on the strip, for this range of k, is

$$V \simeq \frac{8\pi I l_x \Delta_z}{l_y} \left\{ i\omega - i\omega\Delta_z \sqrt{k^2 - \left(\frac{\pi}{l_y}\right)^2} \cot\left[l_z \sqrt{k^2 - \left(\frac{\pi}{l_y}\right)^2} \right] \right.$$
$$\left. + i\omega \sum_{n=1}^{\infty} \frac{\sin[\pi(2n+1)\Delta_y/2l_y]}{[\pi(2n+1)\Delta_y/2l_y]} \frac{[1 - e^{-2\pi(2n+1)\Delta_z/l_z}]}{[2\pi(2n+1)\Delta_z/l_z]} \right\} e^{-i\omega t}$$

where we have assumed that Δ_y and Δ_z are small compared to l_y/π and l_z/π, respectively, but that l_z/l_y is large enough so that

$$\sinh[\pi(2n+1)(l_z - z)/l_y]/\sinh[\pi(2n+1)l_z/l_y] \simeq e^{-\pi(2n+1)z/l_y}$$

If $\Delta_y \geq \frac{1}{10}l_y$ and $\Delta_z \geq \frac{1}{10}l_z$, the series is fairly rapidly convergent and the first few terms suffice. But if either of the Δ's is small, a large number of terms are needed and an approximate answer may be given by changing to integration. We see that

$$\sum_{n=\alpha}^{\infty} \frac{\sin[\pi(2n+1)\Delta_y/2l_y]}{(2n+1)^2} e^{-2\pi(2n+1)\Delta_z/l_z} \simeq \int_\alpha^\infty \frac{\sin(\pi x\Delta_y/l_y)}{x^2} e^{-4\pi x\Delta_z/l_z}\, dx$$
$$= \mathrm{Im} \int_\alpha^\infty \frac{dx}{x^2} \exp\left[\left(\frac{i\pi\Delta_y}{l_y} - \frac{4\pi\Delta_z}{l_z}\right)x\right] = \mathrm{Im}\left\{\left(\frac{1}{\alpha}\right)\exp\left[\left(\frac{i\pi\Delta_y}{l_y} - \frac{4\pi\Delta_z}{l_z}\right)\alpha\right]\right.$$
$$\left. + \left(\frac{i\pi\Delta_y}{l_y} - \frac{4\pi\Delta_z}{l_z}\right)\int_\alpha^\infty \frac{dx}{x}\exp\left[\left(\frac{i\pi\Delta_y}{l_y} - \frac{4\pi\Delta_z}{l_z}\right)x\right]\right\}$$

Consequently, taking the difference between this expression and the expression with $\Delta_z = 0$ and subsequently letting α go to zero, we have

$$V \simeq i\omega(8\pi I l_x \Delta_z/l_y)\left\{1 - \Delta_z \sqrt{k^2 - (\pi/l_y)^2} \cot[l_z \sqrt{k^2 - (\pi/l_y)^2}] \right.$$
$$\left. + \left(\frac{l_z}{8\pi\Delta_z}\right)\ln\left[1 - \frac{4l_y\Delta_z}{l_z\Delta_y}\right] - \left(\frac{l_y}{\pi\Delta_y}\right)\tan^{-1}\left[\frac{4l_y\Delta_z}{l_z\Delta_y}\right]\right\} \qquad (13.3.55)$$

Thus, in the range of ω for which this is valid

$$\left[0 < k < \tfrac{3}{2}\pi\sqrt{\left(\frac{1}{l_y}\right)^2 + \left(\frac{1}{l_z}\right)^2}\right]$$

we have obtained a closed expression for V and impedance V/I which is reasonably accurate over this whole frequency range rather than just close to a resonance. Since the sum of Eq. (13.3.54) is over one index, it has been easier to sum than was the double series of Eq. (13.3.52).

Excitation by Wave Guide. Suppose the resonator is coupled to a driving generator by means of a wave guide, opening into the resonator at the end $z = 0$, with cross sections of both resonator and guide rectangular and parallel. The transverse dimensions of the wave-guide opening, a and b, are supposed to be considerably smaller than the transverse dimensions, l_x and l_y, of the resonator, and the center of the opening of the guide, in the $z = 0$ plane, is at $x = x_1$, $y = y_1$. The origin of coordinates x, y, z for the resonator is at one corner of the resonator, whereas the coordinates suitable for the wave guide are $\xi = x - x_1$, $\eta = y - y_1$, and z (taking the center of the opening as origin).

If the wave guide is driven at a frequency such that the lowest transverse electric mode ($\mathbf{E}$ along $\mathbf{a}_x$ if $b > a$) is the only true wave, then the tangential electric field across the wave-guide opening is, approximately (see discussion on page 1834):

$$\mathbf{E}_{\text{tan}} \simeq ik\mathbf{A}_{\text{tan}} = \mathbf{a}_x \frac{E_0 \cos(\pi\eta/b)}{[1 - (2\xi/a)^2]} e^{-i\omega t}; \quad 0 < |\xi| < \tfrac{1}{2}a; \quad 0 < |\eta| < \tfrac{1}{2}b$$
$$\pi/b < k < \pi/a \qquad (13.3.56)$$

To calculate the field inside the resonator, we use the Green's function of Eqs. (13.3.47) and (13.1.10).

In Eq. (13.1.10), as used here, $\mathbf{Q}$ is zero for there is no charge or current inside the enclosure. Likewise div $\mathbf{F}$ is zero everywhere, so that the first term in the surface integral is zero. It is not difficult to show that, for the $\mathfrak{G}$ of Eq. (13.3.47), $\text{div}_0\, \mathfrak{G}$ is zero on the surface $z = 0$, and also that $\mathfrak{G}$ on the surface is normal to the surface (and therefore perpendicular to $\mathbf{n} \times \text{curl}\, \mathbf{F}$ there); consequently, the second and third terms of the surface integral are zero, and

$$\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi ik} \iint dx_0\, dy_0 [\text{curl}_0\, \mathfrak{G}] \cdot [\mathbf{a}_z \times \mathbf{E}_{\text{tan}}]$$

But, for a rectangular enclosure ($\Lambda_{mn} = l_x l_y/\epsilon_m \epsilon_n$), the curl of $\mathfrak{G}$ in the x_0 coordinates at $z_0 = 0$ is

$$(\text{curl}_0\, \mathfrak{G})_{z_0=0} = 4\pi \sum_{m,n} \frac{\epsilon_m \epsilon_n}{k_{mn}^2 l_x l_y} \left\{ [-\mathbf{a}_z \times \text{grad}\, \psi_{mn}] \cdot \right.$$
$$\cdot\, \text{grad}_0(\psi_{mn}^0) \frac{\sin[K_{mn}(l_z - z)]}{\sin(K_{mn} l_z)}$$
$$+ [\mathbf{a}_z \times \text{grad}_0\, \chi_{mn}^0] \left(\frac{K_{mn}^2}{k_{mn}} \right) \mathbf{a}_z \chi_{mn} \frac{\cos[K_{mn}(l_z - z)]}{\sin(K_{mn} l_z)}$$
$$\left. + [\mathbf{a}_z \times \text{grad}_0\, \chi_{mn}^0]\, \text{grad}\, \chi_{mn} \frac{\sin[K_{mn}(l_z - z)]}{\sin(K_{mn} l_z)} \right\}$$

where superscript 0 denotes that the function depends on the x_0, y_0 coordinates. Since $\mathbf{a}_z \times \mathbf{E}_{\text{tan}} = \mathbf{a}_y E_{\text{tan}}$, since $\psi_{mn} = \cos(\pi m x/l_x) \cos(\pi n y/l_y)$, and since also $\chi_{mn} = \sin(\pi m x/l_x) \sin(\pi n y/l_y)$, we have for the field inside the resonator,

$$\mathbf{A} = \frac{1}{ik} \sum_{m,n} \frac{\epsilon_m \epsilon_n}{k_{mn}^2 l_x l_y} \left\{ \left(\frac{\pi n}{l_y}\right) [\mathbf{a}_z \times \operatorname{grad} \psi_{mn}] \frac{\sin[K_{mn}(l_z - z)]}{\sin(K_{mn} l_z)} \right.$$
$$\left. + \left(\frac{\pi m}{l_x}\right) \left[\left(\frac{k_{mn}^2}{K_{mn}}\right) \mathbf{a}_z \chi_{mn} \frac{\cos[K_{mn}(l_z - z)]}{\sin(K_{mn} l_z)} + \operatorname{grad} \chi_{mn} \frac{\sin[K_{mn}(l_z - z)]}{\sin(K_{mn} l_z)} \right] \right\} \cdot$$
$$\cdot \iint dx_0\, dy_0\, E_{\tan} \cos\left(\frac{\pi m x_0}{l_x}\right) \sin\left(\frac{\pi n y_0}{l_y}\right) e^{-i\omega t}$$

The integral in this expression becomes

$$\left(\frac{2ab}{\pi}\right) E_0 \left[\frac{\sin(\pi n y_1/l_y) \cos(\pi n b/2l_y)}{1 - (nb/l_y)^2} \right] \cos\left(\frac{\pi m x_1}{l_x}\right) B_m\left(\frac{a}{l_x}\right)$$

where (see page 1450)

$$B_0 = \tfrac{1}{2} \sqrt{\pi} \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{7}{6})}; \quad B_m\left(\frac{a}{l_x}\right) = \tfrac{1}{2} \sqrt{\pi} \frac{\Gamma(\frac{2}{3})}{(\pi m a/4l_x)^{\frac{1}{6}}} J_{\frac{1}{6}}\left(\frac{\pi m a}{2l_x}\right); \quad m > 0$$

Finally the y component of $\mathbf{H}$ at the center of the opening ($x = x_1$, $y = y_1$, $z = 0$) is

$$H_y^+ = \left(\frac{4iabE_0}{\pi l_x l_y}\right) \sum_{m,n} \frac{\epsilon_m B_m \cos(\pi n b/2l_y)}{\sqrt{1 - (\pi m/kl_x)^2 - (\pi n/kl_y)^2}} \left[\frac{1 - (\pi n/kl_y)^2}{1 - (nb/l_y)^2} \right] \cdot$$
$$\cdot \cos^2\left(\frac{\pi m x_1}{l_x}\right) \sin^2\left(\frac{\pi n y_1}{l_y}\right) \cot\left[l_z \sqrt{k^2 - \left(\frac{\pi m}{l_x}\right)^2 - \left(\frac{\pi n}{l_y}\right)^2} \right] e^{-i\omega t}$$
$$= -(4\pi E_0/c) Y_r \tag{13.3.57}$$

where Y_r is the conductance of the resonator for the guide. The (m,n)th term of this expression has a resonance infinity when $k^2 \to (\pi m/l_x)^2 + (\pi n/l_y)^2 + (\pi s/l_y)^2$ ($s = 0, 1, 2, \ldots$) (except that the terms for $m = s = 0$, $n > 0$, $k \to \pi n/l_y$ do not resonate). These, of course, are the resonance frequencies of the enclosure.

The wave in the wave guide has the $m = 0$, $n = 1$ mode incident on the opening and a sequence of reflected waves, for $n = 1$ and for all even values of m, adjusted so that its tangential behavior at $z = 0$ corresponds to Eq. (13.3.56),

$$\mathbf{A} = \mathbf{a}_r \times \operatorname{grad}\left[\cos\left(\frac{\pi\eta}{b}\right)\right] \left\{ e^{iK_{01}z} - \left[1 + \left(\frac{E_0 b}{i\pi k}\right) B_0\right] e^{-iK_{01}z} \right\} e^{-i\omega t}$$
$$- \sum_{m=1}^{\infty} \left[\frac{2E_0 B_{2m}(1)}{ikk_{2m,1}^2}\right] \left\{ \left(\frac{\pi}{b}\right) \mathbf{a}_z \times \operatorname{grad} \psi_{2m,1}^- - \left(\frac{2\pi m}{a}\right) \operatorname{grad} \chi_{2m,1}^- \right.$$
$$\left. - i \left(\frac{2\pi m k_{2m,1}^2}{aK_{2m,1}}\right) \mathbf{a}_z \chi_{2m,1}^- \right\} e^{-iK_{2m,1}z - i\omega t} \tag{13.3.58}$$

where $k_{2m,1}^2 = (2\pi m/a)^2 + (\pi/b)^2$, $\psi_{2m,1}^- = \cos(2\pi m\xi/a) \sin(\pi\eta/b)$, and $\chi_{2m,1}^- = \sin(2\pi m\xi/a) \cos(\pi\eta/b)$, and the constants B are those given earlier, with 1 instead of a/l_x. The y component of H at $\xi = \eta = z = 0$ is then

$$\begin{aligned} H_y^- &= -\sqrt{k^2 - \left(\frac{\pi}{b}\right)^2}\left[2i\left(\frac{\pi}{b}\right) + \left(\frac{E_0}{k}\right)B_0\right] \\ &\qquad - 2\left(\frac{E_0}{k}\right)\sum_{m=1}^{\infty}\frac{B_{2m}(1)}{\sqrt{k^2 - k_{2m,1}^2}}\left[k^2 - \left(\frac{\pi}{b}\right)^2\right] \\ &\simeq -\sqrt{k^2 - \left(\frac{\pi}{b}\right)^2}\left[2i\left(\frac{\pi}{b}\right) + \left(\frac{E_0}{k}\right)B_0\right] \\ &\qquad - iE_0\left(\frac{a}{2\pi k}\right)\left[k^2 - \left(\frac{\pi}{b}\right)^2\right](B_0 - 1)\ln 4 \qquad (13.3.59) \end{aligned}$$

The approximate value of E_0, the magnitude of the electric intensity at the center of the wave-guide opening, is obtained by equating H_y^- to H_y^+, since these two quantities are both supposed to represent the magnetic intensity at the center of the opening, for a unit-amplitude wave coming down the wave guide to the resonator. A glance at Eq. (13.3.58) indicates that the amplitude of the wave reflected back from the resonator is

$$-\left[1 + E_0\left(\frac{bB_0}{i\pi k}\right)\right] = R_{e01} = \frac{Y_{e01} + (1/B_0)(Y_r + Y_0)}{Y_{e01} - (1/B_0)(Y_r + Y_0)} \qquad (13.3.60)$$

where $Y_{e01} = (c/4\pi)\sqrt{1 - (\pi/kb)^2}$; $B_0 = \frac{1}{2}\sqrt{\pi}\,[\Gamma(\frac{2}{3})/\Gamma(\frac{7}{6})]$
$Y_0 = -i\omega(a/8\pi^2)[1 - (\pi/kb)^2](B_0 - 1)\ln 4$

and Y_r is given by Eq. (13.3.57). This is to be compared with Eq. (13.3.24) for the reflection of a transverse electric wave from a tube end. The quantity Y_{e01} is the conductance of the (0,1) wave in the wave guide, Y_0 is the conductance of the opening itself, and Y_r is the conductance of the resonator itself, viewed from the opening. The formula for R indicates that the wave guide is terminated by the conductance Y_r in parallel with Y_0, coupled to the guide by a coupling constant $1/B_0$.

We note that, since both Y_0 and Y_r are purely reactive (H_y^+/E_0 is purely imaginary) whereas Y_{e01} is real as long as k is larger than π/b, the magnitude R of the reflected wave is unity for driving frequency larger than the cutoff. This is not surprising, for we have not yet included any energy absorption in our expression for Y_r and so all energy incident on the resonator must be reflected out again. When $Y_r = -Y_0$, the reflection occurs with no change in phase; when $Y_r = \infty$, the reflected wave is 180° out of phase.

The electric intensity at the center of the opening, for unit amplitude incident wave (E incident $= 1$ at center of guide), is

$$bE_0/i\pi k = -2Y_{e01}/(B_0Y_{e01} - Y_0 - Y_r)$$

This can never be infinite, because Y_{e01} is real and Y_0 and Y_r are imaginary, but it can be zero, when the conductance Y_r of the resonator is infinite, as it is when the incident frequency equals one of the resonance frequencies of the enclosure. In this case the opening behaves as though it were closed by an infinitely conducting barrier.

Resonance Frequencies of a Klystron Cavity. A cavity of some interest in connection with klystron construction is the geometry shown in Fig. 13.6, rotated about the central line as axis. The lowest mode in this case is a transverse magnetic one, with free charge on two sides of the gap Δ, current oscillating back and forth along the cylinder of radius b, and the magnetic lines circles around this inner cylinder. To compute this mode and its resonance frequency, we have to calculate

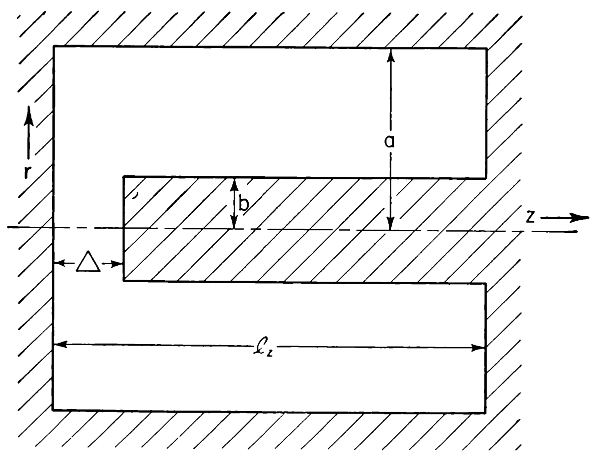

Fig. 13.6 Axial cross section of electromagnetic resonator.

the field in the region between the gap ($r < b$, $z < \Delta$) and also in the region outside the gap ($b < r < a$, $z < l_z$) and then join them at the ring-shaped surface ($r = b$, $z < \Delta$). The joining requirement will serve to fix the allowed frequencies.

If Δ is small compared to b, the field at $r = b$ will conform nearly exactly to the static shape at the "corner" at $r = b$, $z = \Delta$,

$$E_z \simeq \{E_0/[1 - (z/\Delta)^2]^{\frac{1}{3}}\}e^{-i\omega t}; \quad r = b, 0 < z < \Delta \qquad (13.3.61)$$

For the transverse magnetic field we use for the vector potential the function $\mathbf{N} = (1/k^2)$ curl curl$(\mathbf{a}_z\chi)e^{ikct}$ with χ satisfying the boundary conditions of zero normal gradient on the flat boundaries perpendicular to z and zero value on the cylindrical boundaries normal to r (and, of course, being a solution of $\nabla^2\chi + k^2\chi = 0$).

For the region inside the gap ($r \leq b$, $0 \leq z \leq \Delta$) we have to use only the Bessel functions of the first kind, J, which are finite at $r = 0$. We are interested only in the axially symmetric modes, for which χ is independent of the cylindrical angle ϕ. Consequently, χ is a combination of functions $\cos(\pi mz/\Delta)J_0(K_m r)$ where $K_m^2 = k^2 - (\pi m/\Delta)^2$. If Δ is much smaller than b or than π/k, we find that for $m > 0$, $K_m \simeq i\pi m/\Delta$ and

$$J_0(i\pi mr/\Delta) \simeq \sqrt{(\Delta/2\pi^2 mr)}\, e^{(\pi mr/\Delta)}$$

Consequently, for $0 < z < \Delta$, $0 < r < b$, we have

$$\chi \simeq A_0 J_0(kr) + \sum_{m=1}^{\infty} A_m \cos\left(\frac{\pi mz}{\Delta}\right) J_0\left(\frac{i\pi mr}{\Delta}\right)$$

and

$$\mathbf{A} \simeq \left\{ \mathbf{a}_z A_0 J_0(kr) - \mathbf{a}_z \frac{1}{k^2}\sqrt{\frac{1}{2\pi r}} \sum_{m=1}^{\infty} \left(\frac{\pi m}{\Delta}\right)^{\frac{3}{2}} A_m \cos\left(\frac{\pi mz}{\Delta}\right) e^{(\pi mr/\Delta)} \right.$$
$$\left. - i\mathbf{a}_r \frac{1}{k^2}\sqrt{\frac{1}{2\pi r}} \sum_{m=1}^{\infty} \left(\frac{\pi m}{\Delta}\right)^{\frac{3}{2}} A_m \sin\left(\frac{\pi mz}{\Delta}\right) e^{(\pi mr/\Delta)} \right\} e^{-i\omega t}$$

In order that $E_z = ikA_z$ be equal to the assumed form of Eq. (13.3.61) at $r = b$, we must have

$$A_0 = \frac{E_0 B_0}{ikJ_0(kb)}; \quad A_m = 2ik\sqrt{2\pi r}\, E_0 B_{2m}(1) \left(\frac{\Delta}{\pi m}\right)^{\frac{3}{2}} e^{-(\pi mb/\Delta)}$$

where $B_0 = \int_0^{\frac{1}{2}\pi} d\phi \sin^{\frac{1}{3}}\phi = \frac{1}{2}\sqrt{\pi}\left[\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{7}{6})}\right]$

$$B_m(u) = \int_0^{\frac{1}{2}\pi} d\phi \sin^{\frac{1}{3}}\phi \cos(\tfrac{1}{2}\pi mu \cos\phi)$$
$$= \tfrac{1}{2}\sqrt{\pi}\, \frac{\Gamma(\frac{2}{3})}{(\pi mu/4)^{\frac{1}{6}}} J_{\frac{1}{6}}(\tfrac{1}{2}\pi mu); \quad \text{as before.}$$

The magnetic field in the gap is

$$\mathbf{H} = \left\{ \mathbf{a}_\phi A_0 k J_1(kr) + i\mathbf{a}_\phi \sum_{m=1}^{\infty} \left(\frac{\pi m}{\Delta}\right) A_m J_1\left(\frac{i\pi mr}{\Delta}\right) \cos\left(\frac{\pi mz}{\Delta}\right) \right\} e^{-i\omega t}$$

and at $z = 0$, $r = b$, the ϕ component is

$$H_\phi^- \simeq -iE_0 \left\{ B_0 \frac{J_1(kb)}{J_0(kb)} + 2k \int_0^{\frac{1}{2}\pi} d\varphi \sin^{\frac{1}{3}}\varphi \sum_{m=1}^{\infty} \left(\frac{\Delta}{\pi m}\right) \cos(\pi m \cos\varphi) \right\} e^{-i\omega t}$$

$$\simeq -iE_0 \left\{ B_0 \frac{J_1(kb)}{J_0(kb)} - 2k\left(\frac{\Delta}{\pi}\right)(B_0 - 1)\ln 2 \right\} e^{-i\omega t} \quad \text{at } r = b,\ z = 0$$

which must be fitted to the value of H_0 obtained by calculating the waves from the outer, deeper enclosure.

The expression for χ in the outer region is

$$\chi = \sum_{m=0}^{\infty} C_m \cos\left(\frac{\pi m z}{l_z}\right) [N_0(k_m a)J_0(k_m r) - N_0(k_m r)J_0(k_m a)]$$

where $k_m^2 = k^2 - (\pi m/l_z)^2$. This automatically goes to zero at $r = a$; we must choose values of the C's so that χ is zero at $r = b$, except for the range of z from 0 to Δ. The component of $\mathbf{E}$ in the z direction, at $r = b$, is

$$e^{-i\omega t}\left[\frac{i}{k}\,\mathrm{curl}\;\mathrm{curl}(\mathbf{a}_z\chi)\right]_z = ik \sum_{m=0}^{\infty}\left[1 - \left(\frac{\pi m}{kl_z}\right)^2\right] C_m \cos\left(\frac{\pi m z}{l_z}\right)\cdot$$
$$\cdot [N_0(k_m a)J_0(k_m b) - N_0(k_m b)J_0(k_m a)]e^{-i\omega t}$$

which must equal the form given in Eq. (13.3.61) for $0 \le z < \Delta$ and be zero for $\Delta < z \le l_z$. This is true if

$$C_m = \left(\frac{\epsilon_m \Delta}{ikl_z}\right) \frac{E_0 B_{2m}(\Delta/l_z)/[1 - (\pi m/kl_z)^2]}{N_0(k_m a)J_0(k_m b) - N_0(k_m b)J_0(k_m a)}$$

where the quantities $B_m(\delta)$ are defined on pages 1450 and 1860. Likewise the value of H_ϕ at $r = b$, $z = 0$ is

$$H_\phi^+ \simeq -\frac{ik\,\Delta E_0}{l_z} \sum_{m=0}^{\infty} \frac{\epsilon_m B_{2m}(\Delta/l_z)}{k_m} \left[\frac{N_0(k_m a)J_1(k_m b) - N_1(k_m b)J_0(k_m a)}{N_0(k_m a)J_0(k_m b) - N_0(k_m b)J_0(k_m a)}\right]$$

which must be set equal to H_ϕ^-, given previously, by suitable choice of $k_m = \sqrt{k^2 - (\pi m/l_z)^2}$ and thus of k.

The first term in H_ϕ^- does not include the factor Δ, and if the gap spacing Δ is small compared to a, b, and l_z, then this first term is much larger than the second and for H_ϕ^+ to be equal to H_ϕ^- at least one term in the series for H_ϕ^+ must be quite large. This can occur whenever the quantity $[N_0(k_m a)J_0(k_m b) - N_0(k_m b)J_0(k_m a)]$ is quite small. Setting this quantity equal to zero results in the roots

$$k_m = \pi\gamma_n/(a - b); \quad n = 1, 2, 3, \ldots \tag{13.3.62}$$

where, over the range $a > b > \frac{1}{10}a$, the error made in letting $\gamma_n = n$ is less than 3 per cent. Consequently, we can assume that the possible resonance frequencies (for the transverse magnetic modes) occur for $k = k_{sn} + \delta_{sn}$, where

$$k_{sn}^2 = \left(\frac{\pi s}{l_z}\right)^2 + \left(\frac{\pi\gamma_n}{a - b}\right)^2 \simeq \left(\frac{\pi s}{l_z}\right)^2 + \left(\frac{\pi n}{a - b}\right)^2$$

For $\delta \ll k_{sn}$, the Bessel-Neumann fraction in the series for H^+ for $m = s$ is very much larger than all the other terms and is approximately equal to

$$\frac{\pi\gamma_n/k_{sn}(a-b)}{\delta_{sn}(b+aG_{sn})};\quad \text{where } G_{sn} = \frac{J_0(k_sb)N_1(k_sa) - J_1(k_sa)N_0(k_sb)}{J_1(k_sb)N_0(k_sa) - J_0(k_sa)N_1(k_sb)}$$

where k_s is given by Eq. (13.3.62). Finally, equating $H^+_{\phi s}$ to $H^-_{\phi s}$, we see that, approximately, the resonance frequencies are $kc/2\pi$, where

$$k \simeq \left(\frac{\pi m}{l_z}\right) + \frac{\Delta B_{2m}(\Delta/l_z)J_0(\pi mb/l_z)}{bl_zB_0\ln(a/b)J_1(\pi mb/l_z)};\quad n = 0,\ m > 0$$

Finally, for the lowest mode of all, k itself is small and we use the series expansions for J_0 and J_1 in the expression for H^-. This results in

$$k \simeq \sqrt{2\Delta/l_zb^2\ln(a/b)};\quad m = n = 0 \qquad (13.3.63)$$

where the resonance frequency is again $kc/2\pi$. When the gap is closed $(\Delta \to 0)$, this resonance vanishes and the higher frequencies become those for a ring-shaped enclosure between the cylinders $r = b$ and $r = a$ $(k = k_{mn})$. The lowest mode may be considered to correspond to the resonance of an inductance and capacitance in series; the capacitance $b^2/4\Delta$ being that across the gap Δ and the inductance $(2l_z/c^2)\ln(a/b)$ being that of the ring-shaped enclosure, so that

$$\nu_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC}} = \frac{c}{2\pi}\sqrt{\frac{2\Delta}{l_zb^2\ln(a/b)}}$$

as given by Eq. (13.3.63).

Numerous examples of standing elastic waves in various cylindrical-shaped solids could be discussed here, but the ones which can be worked out are trivial and the ones of possible practical interest are usually impossible of exact solution, because of the general intractability of boundary conditions for elastic waves.

Scattering from Cylinders. Instead, we shall turn to problems dealing with wave motion *outside* cylinders. Many of these can be dealt with by slight extensions of scalar formulas. For instance, the scattering from a circular cylinder of a plane electromagnetic wave, falling normally to the cylinder axis, may be solved in terms of two scalar cases, for the two polarizations of the wave. For the polarization having the electric intensity parallel to the cylinder axis (z axis), we use the obvious expansion in the cylindrical coordinates r, ϕ, z,

$$\mathbf{a}_ze^{ikx} = \mathbf{a}_z\sum_{m=0}^{\infty}\epsilon_mi^m\cos(m\phi)J_m(kr) \qquad (13.3.64)$$

whereas the other polarization is

$$\mathbf{a}_y e^{ikx} = \frac{1}{k}\operatorname{curl}(\mathbf{a}_z e^{ikx}) = \frac{1}{ik}\sum_{m=0}^{\infty} \epsilon_m i^m \mathbf{a}_z \times \operatorname{grad}[\cos(m\phi)J_m(kr)]$$

If the cylinder at $r = a$ is a perfect conductor, then the boundary condition on the first polarization ($\mathbf{E}$ along z) is that the amplitude must be zero at $r = a$. In this case the correct solution is

$$\mathbf{A}_\parallel = \frac{\mathbf{a}_z}{ik}\left\{e^{ikx} - i\sum_m \epsilon_m i^m e^{-i\delta_m}\sin\delta_m\, H_m(kr)\cos(m\phi)\right\} e^{-i\omega t}$$

as given in Eq. (11.2.28). The angles δ are defined on page 1564.

On the other hand, for the second polarization ($\mathbf{E}$ normal to $\mathbf{a}_z$), if the cylinder $r = a$ is a perfect conductor, the scalar function ψ involved must have zero normal gradient at $r = a$ in order that the tangential part of $\mathbf{a}_z \times \operatorname{grad}\psi$ be zero at $r = a$. Consequently,

$$\mathbf{A}_\perp = \frac{-1}{k^2}\mathbf{a}_z \times \operatorname{grad}\left\{e^{ikx} - i\sum_m \epsilon_m i^m e^{-i\delta_m'}\sin\delta_m'\, H_m(kr)\cos(m\phi)\right\} e^{-i\omega t}$$

where the angles δ' also are defined on page 1564. Consequently, the two polarizations are scattered by different amounts, indicated by the difference between the phase angles δ and δ'.

When the cylinder is conductive with conductivity σ (usually large), a certain amount of energy is lost at the surface of the cylinder, and we have an effective absorption width as well as an effective width for scattering, analogous to the case discussed on page 1489. For the parallel polarization, the tangential magnetic field at $r = a$ is

$$H_{\parallel\phi} = \frac{2i}{\pi ka}\sum_m \epsilon_m i^m \cos(m\phi)\left[\frac{e^{-i\delta_m}}{C_m(ka)}\right] e^{-i\omega t}$$

According to the derivation of Eq. (13.3.17), this magnetic field is not modified to the first order by the finite conductivity, but the electric field at the surface, instead of being zero, has a z component equal to $\sqrt{\mu\omega/4\pi\sigma}\, e^{-\frac{1}{4}\pi i}$ times $H_{\parallel\phi}$. The power absorbed per unit length of cylinder is thus

$$\frac{c}{4\pi}\sqrt{\frac{\mu\omega}{8\pi\sigma}}\int_0^{2\pi} |H_{\parallel\phi}|^2\, a\, d\phi = \frac{2c}{\pi^2 k^2 a}\sqrt{\frac{\mu\omega}{8\pi\sigma}}\sum_m\left[\frac{\epsilon_m}{C_m^2(ka)}\right].$$

The incident wave has intensity $(c/4\pi)\operatorname{Re}(\overline{\mathbf{E}} \times \mathbf{H}) = (c/4\pi)\mathbf{a}_x$, and the ratio between power lost and incident intensity is the *effective absorption* *width* of the cylinder for electromagnetic waves:

$$Q_a = \frac{8\tau_s}{\pi ka} \sum_m \left[\frac{\epsilon_m}{C_m^2(ka)}\right]$$

where $\tau_s = \sqrt{\mu c^2/8\pi\sigma\omega}$ is the *effective skin depth* of the cylinder surface for electromagnetic waves.

Referring to Eqs. (11.3.76), we see that the effect of finite conductivity of the cylinder is to produce a complex phase angle $\delta_m - i\kappa_m$, where $\delta_m = \delta_m(ka)$, defined on page 1564 and tabulated at the end of the book, and where

$$\kappa_m \simeq [2\tau_s/\pi a C_m^2(ka)]$$

The reflection coefficient R_m is then $e^{-2\kappa_m - 2i\delta_m}$ and the scattering, absorption, and total effective widths of the cylinder for electromagnetic waves are

$$\begin{aligned} Q_s &= \frac{2}{k} \sum \epsilon_m e^{-2\kappa_m}[\cosh(2\kappa_m) - \cos(2\delta_m)] \\ Q_a &= \frac{2}{k} \sum \epsilon_m e^{-2\kappa_m} \sinh(2\kappa_m) \\ Q_t &= \frac{2}{k} \sum \epsilon_m[1 - e^{-2\kappa_m} \cos(2\delta_m)] = Q_a + Q_s \end{aligned} \tag{13.3.65}$$

for the polarization where the electric vector is parallel to the cylinder axis.

A very similar sort of calculation shows that, for the polarization having the magnetic field parallel to the cylinder axis, the absorption parameter is

$$\kappa'_m \simeq 2\tau_s/\pi a[C'_m(ka)]^2$$

and the three sorts of effective widths for this polarization are given by formulas identical with the ones given in Eq. (13.3.65) except that κ'_m, δ'_m, and C'_m are used instead of κ_m, δ_m, and C_m. The scattered waves, at large distances from the cylinder are

$$\begin{aligned} \mathbf{A}_{\parallel s} &\to -\frac{\mathbf{a}_z}{ik}\sqrt{\frac{2i}{\pi kr}}\, e^{ikr - i\omega t} \sum_m \epsilon_m \cos(m\phi)(1 - e^{-2\kappa_m - 2i\delta_m}) \\ \mathbf{A}_{\perp s} &\to -\frac{\mathbf{a}_\phi}{ik}\sqrt{\frac{2i}{\pi kr}}\, e^{ikr - i\omega t} \sum_m \epsilon_m \cos(m\phi)(1 - e^{-2\kappa_m' - 2i\delta_m'}) \end{aligned} \tag{13.3.66}$$

from which the scattered intensity can be worked out.

Spherical Waves. A very important set of separable solutions is that for spherical coordinates r, ϑ, φ. The basic scalar functions, from which the vector ones are built, are the products of cosine or sine of $m\varphi$ (or $e^{\pm im\varphi}$), the spherical harmonic $P_n^m(\cos\vartheta)$, and the spherical Bessel functions $j_n(kr)$ or $n_n(kr)$. The operators used to obtain the vector solutions are, of course, the gradient, for the longitudinal solution, and the curl and curl curl for the transverse solutions. Reference to Eqs. (13.1.4) and (13.1.5) indicates that we should take the curl of $\mathbf{r}$ times the scalar solution. The resulting vectors are most easily expressed in terms of the vector harmonic functions discussed in Eqs. (13.2.18), (13.2.20) *et seq.*, and tabulated at the end of this chapter.

The vector solutions also have interesting and suggestive integral representations. As in Chap. 11, we can express our solution in terms of an integral of a plane wave $e^{i\mathbf{k}\cdot\mathbf{r}}$ over all different directions of $\mathbf{k}$. [We set $\mathbf{r} = r(\mathbf{a}_x \sin\vartheta\cos\varphi + \mathbf{a}_y \sin\vartheta\sin\varphi + \mathbf{a}_z\cos\vartheta)$ and $\mathbf{k} = k(\mathbf{a}_x \sin u\cos v + \mathbf{a}_y\sin u\sin v + \mathbf{a}_z\cos u)$ and integrate $\sin u\, du\, dv$, times $e^{i\mathbf{k}\cdot\mathbf{r}}$, times an amplitude factor, over v from 0 to 2π and over u from 0 to π.] In the present case, however, the amplitude factors are vector functions of u and v and, pleasantly enough, these amplitude factors also are the vector harmonic functions tabulated at the end of the chapter.

There must, unfortunately, be an efflorescence of sub- and superscripts surrounding the symbols for the various solutions. We must have the subscripts m and n to show the order of the underlying spherical harmonic; then we must have the subscript e or o, depending on whether we use $\cos(m\varphi)$ or $\sin(m\varphi)$ in the spherical harmonic. (We could obviate this by using the exponential $e^{im\varphi}$ instead; only in this case we need a symbol to denote complex conjugate!) Finally, we need a symbol which will indicate whether the function has or has not a pole at the origin or whether it corresponds to outgoing waves only.

The three sets of *solutions of the first kind*, finite at $r = 0$, are therefore

$$\begin{aligned}
\mathbf{L}^1_{\sigma mn}(\mathbf{r}) &= (1/k)\,\mathrm{grad}[Y_{\sigma mn}(\vartheta,\varphi)j_n(kr)] \\
&= \mathbf{P}^\sigma_{mn}(\vartheta,\varphi)\frac{1}{k}\frac{d}{dr}[j_n(kr)] + \sqrt{n(n+1)}\,\mathbf{B}^\sigma_{mn}(\vartheta,\varphi)\frac{1}{kr}[j_n(kr)] \\
&= \frac{1}{2n+1}\left\{nj_{n-1}(kr)\left[\mathbf{P}^\sigma_{mn} + \sqrt{\frac{n+1}{n}}\,\mathbf{B}^\sigma_{mn}\right]\right. \\
&\qquad\qquad \left. - (n+1)j_{n+1}(kr)\left[\mathbf{P}^\sigma_{mn} - \sqrt{\frac{n}{n+1}}\,\mathbf{B}^\sigma_{mn}\right]\right\} \\
&= \frac{i}{4\pi i^n}\int_0^{2\pi} dv \int_0^\pi \sin u\, e^{i\mathbf{k}\cdot\mathbf{r}}\mathbf{P}^\sigma_{mn}(u,v)\,du
\end{aligned} \tag{13.3.67}$$

$$\begin{aligned}
\mathbf{M}^1_{\sigma mn}(\mathbf{r}) &= \mathrm{curl}[\mathbf{r}Y_{\sigma mn}(\vartheta,\varphi)j_n(kr)] \\
&= \sqrt{n(n+1)}\,\mathbf{C}^\sigma_{mn}(\vartheta,\varphi)j_n(kr) = (1/k)\,\mathrm{curl}[\mathbf{N}^1_{\sigma mn}(\mathbf{r})] \\
&= \frac{\sqrt{n(n+1)}}{4\pi i^n}\int_0^{2\pi} dv\int_0^\pi \sin u\, e^{i\mathbf{k}\cdot\mathbf{r}}\mathbf{C}^\sigma_{mn}(u,v)\,du
\end{aligned} \tag{13.3.68}$$

$$\begin{aligned}\mathbf{N}^1_{\sigma mn}(\mathbf{r}) &= (1/k)\,\mathrm{curl}[\mathbf{M}^1_{\sigma mn}(\mathbf{r})]\\ &= n(n+1)\mathbf{P}^\sigma_{mn}(\vartheta,\varphi)\,\frac{1}{kr}\,j_n(kr)\\ &\qquad\qquad + \sqrt{n(n+1)}\,\mathbf{B}^\sigma_{mn}(\vartheta,\varphi)\,\frac{1}{kr}\,\frac{d}{dr}\,[rj_n(kr)]\\ &= \frac{n(n+1)}{2n+1}\left\{j_{n-1}(kr)\left[\mathbf{P}^\sigma_{mn} + \sqrt{\frac{n+1}{n}}\,\mathbf{B}^\sigma_{mn}\right]\right.\\ &\qquad\qquad \left. + j_{n+1}(kr)\left[\mathbf{P}^\sigma_{mn} - \sqrt{\frac{n}{n+1}}\,\mathbf{B}^\sigma_{mn}\right]\right\}\\ &= \frac{i}{4\pi i^n}\,\sqrt{n(n+1)}\int_0^{2\pi} dv \int_0^\pi \sin u\; e^{i\mathbf{k}\cdot\mathbf{r}}\mathbf{B}^\sigma_{mn}(u,v)\,du \qquad (13.3.69)\end{aligned}$$

which definitions should be compared with Eqs. (11.3.47). The order n runs from 0 to ∞, m runs from 0 to n, and σ is either e or o (even or odd). We can, of course, use the complex forms $\mathbf{L}^1_{mn} = \mathbf{L}^1_{emn} + i\mathbf{L}^1_{omn} = (1/k)\,\mathrm{grad}[X^m_n(\vartheta,\varphi)j_n(kr)]$ if we wish. There are also functions of the second kind, $\mathbf{L}^2_{\sigma mn}$, etc., obtained by using the spherical Neumann functions $n_n(kr)$ instead of $j_n(kr)$, which become infinite at the origin. The contour of integration for u in the integral representations is modified if we are to obtain n_n, in a manner analogous to that given in Eq. (11.3.48). Also there are functions of the third kind, $\mathbf{M}^3_{\sigma mn}$, etc., obtained by using the spherical Hankel functions $h_n(kr) = j_n(kr) + in_n(kr)$ instead of $j_n(kr)$. This solution represents outgoing waves.

The integral representations show clearly that **L** is a longitudinal wave, for $\mathbf{P}^\sigma_{mn}(u,v)$ points in the direction of the propagation vector **k** for every wave in the integration. The other two waves are transverse because **B** and **C** are normal to **k**. Since the angle functions **P**, **B**, and **C** are mutually orthogonal, we can use the integral representations to produce a series expansion for vector plane waves. For example, the dyadic $\mathfrak{I}e^{i\mathbf{k}\cdot\mathbf{r}}$ is a function of the angles u, v of **k** and ϑ, φ of **r**, and by expanding in terms of the u, v dependence, using the complete set **P**, **B**, **C**, we obtain

$$\begin{aligned}\mathfrak{I}e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_{\sigma,m,n} \epsilon_m i^n(2n+1)\,\frac{(n-m)!}{(n+m)!}\left\{-i\mathbf{P}^\sigma_{mn}(u,v)\mathbf{L}^1_{\sigma mn}(\mathbf{r})\right.\\ \left. + \frac{1}{\sqrt{n(n+1)}}\,[\mathbf{C}^\sigma_{mn}(u,v)\mathbf{M}^1_{\sigma mn}(r) - i\mathbf{B}^\sigma_{mn}(u,v)\mathbf{N}^1_{\sigma mn}(\mathbf{r})]\right\}\end{aligned}$$

Setting $u = 0$ (pointing **k** along z) reduces all the **P**'s, **C**'s, and **B**'s to zero except for the set $m = 0$ for **P** and $m = 1$ for **C** and **B**. Consequently, we have (see page 1799)

$$\begin{aligned}\mathfrak{I}e^{ikz} = \sum_{n=0}^\infty \frac{(2n+1)i^n}{n(n+1)}\,\{\mathbf{a}_x[\mathbf{M}^1_{o1n}(\mathbf{r}) - i\mathbf{N}^1_{e1n}(\mathbf{r})]\\ + \mathbf{a}_y[\mathbf{M}^1_{e1n}(\mathbf{r}) + i\mathbf{N}^1_{o1n}(\mathbf{r})] - in(n+1)\mathbf{a}_z\mathbf{L}^1_{eon}(\mathbf{r})\} \qquad (13.3.70)\end{aligned}$$

from which any plane wave expansion may be obtained by operating from the left by a constant vector. We see that the longitudinal wave $\mathbf{a}_z e^{ikz}$ is exclusively a series of the **L**'s, whereas both $\mathbf{a}_x e^{ikz}$ and $\mathbf{a}_y e^{ikz}$ are made up of both transverse solutions **M** and **N**.

Radiation from Dipoles and Multipoles. To explore a little further the characteristics of these rather complicated-looking vector solutions, let us compute the electromagnetic wave radiated from a sphere in which we set up (by means which need not concern us now) a simple distribution of oscillating surface currents. We usually use the full machinery of the Green's function to compute the radiation "caused" by a current distribution, but if all the current is on the surface, we can use the relation between surface current $\mathbf{J}_s$ and magnetic field at the surface $\mathbf{H}_s$ which has been used many times before,

$$\mathbf{J}_s = (c/4\pi)(\mathbf{n} \times \mathbf{H}_s)$$

where $\mathbf{n}$ is the normal to the surface, pointing *into* the boundary material. In the present case, the surface is the sphere $r = a$ and the normal $\mathbf{n}$ is $-\mathbf{a}_r$ (for waves outside the sphere) so that

$$\mathbf{J}_s = -(c/4\pi)(\mathbf{a}_r \times \mathbf{H}_{r=a})$$

By fitting the magnetic field of an outgoing wave to the surface current, we can thus calculate the radiation "caused" by the current.

Suppose we work the problem backward for a minute, taking a particular radiation field and seeing what current distribution on the sphere's surface "caused" it. We shall not have any longitudinal waves, so we need not consider the **L** waves. There are no **M** or **N** waves for $n = 0$, so the simplest **M** wave is

$$\mathbf{A} = \mathbf{M}^3_{01} = \mathbf{a}_\varphi \sin\vartheta\, h_1(kr)e^{-i\omega t} = \mathbf{E}/ik \tag{13.3.71}$$
$$\mathbf{H} = k\mathbf{N}^3_{01} = \left\{2k\mathbf{a}_r \cos\vartheta \left(\frac{1}{kr}\right) h_1(kr) - k\mathbf{a}_\vartheta \sin\vartheta \left(\frac{1}{kr}\right)\frac{d}{dr}\,[rh_1(kr)]\right\} e^{-i\omega t}$$

and the corresponding surface current on the sphere is

$$\begin{aligned}\mathbf{J}_{01} &= -\mathbf{a}_\varphi\left(\frac{\omega}{12\pi}\right)\sin\vartheta[2h_0(ka) - h_2(ka)]e^{-i\omega t} \\ &= -i\omega\mathbf{a}_\varphi\,\frac{\sin\vartheta}{4\pi}\left[\frac{1 - (ka)^2 - ika}{(ka)^3}\right] e^{ika-i\omega t}\end{aligned}$$

This sort of current oscillates parallel to the sphere's equator, as though the sphere itself were rotated about the z axis first clockwise and then counterclockwise. We do not expect free charge to pile up anywhere on the surface from this sort of current, and this expectation is confirmed when we notice that the electric field nowhere has a normal (radial) component. Such a circulatory current without free charge produces what is known as a *magnetic dipole*, and the wave $\mathbf{A} = \mathbf{M}_{01}^3$ is called *magnetic dipole radiation.* The vibrating current loop produces an oscillatory magnetic dipole, which might be said to "cause" the wave.

Turning the problem around, if the surface current is equal to $J\mathbf{a}_\varphi \cdot \sin\vartheta\, e^{-i\omega t}$, then the electric field at the surface is

$$\mathbf{E}_{r=a} = -i\omega\left[\frac{4\pi a}{c^2}\frac{1 - ika}{1 - (ka)^2 - ika}\right](-J\mathbf{a}_\varphi \sin\vartheta\, e^{-i\omega t})$$

The quantity in brackets (or rather, its limiting form for $ka \ll 1$) is the effective inductance of the sphere, per unit length parallel to the equator, for a magnetic dipole distribution of current. As the frequency is increased so that ka is no longer negligible, the impedance per unit equatorial distance ($-i\omega$ times the bracket) is no longer purely reactive. In fact, for $ka \gg 1$ the impedance is purely resistive, equal to $4\pi/c$, equal to the radiation resistance of vacuum.

Taking the first $\mathbf{N}$ wave, we have

$$\mathbf{A} = \left(\frac{\mathbf{E}}{ik}\right) = \mathbf{N}_{01}^3 = \left\{2\mathbf{a}_r\left(\frac{\cos\vartheta}{kr}\right)h_1(kr) - \mathbf{a}_\vartheta\left(\frac{\sin\vartheta}{kr}\right)\frac{d}{dr}[rh_1(kr)]\right\}e^{-i\omega t} \tag{13.3.72}$$

$$\mathbf{H} = k\mathbf{M}_{01}^3;\quad \mathbf{J}_{01} = \mathbf{a}_\vartheta\left(\frac{ck}{4\pi}\right)\sin\vartheta\left[\frac{i + ka}{(ka)^2}\right]e^{ika - i\omega t}$$

The current, in this case, oscillates back and forth from pole to pole, alternately piling up positive and negative charges at the poles. This is seen by computing the surface charge for the radial electric field $Q = (1/4\pi)(\mathbf{a}_r \cdot \mathbf{E})$. We have

$$Q = \frac{1}{2\pi a}\left[\frac{1 - ika}{(ka)^2}\right]\cos\vartheta\, e^{ika - i\omega t}$$

which is 90° out of phase with the current and is concentrated at the poles (the $\cos\vartheta$ factor) instead of at the equator (the $\sin\vartheta$ factor in $\mathbf{J}$). This simple oscillation of charge from one pole to the other is exactly equivalent to the current-charge distribution resulting if a charged sphere (neutralized by a negative point charge at its center) oscillated to and fro along the z axis, and both are just ways of describing what is called an oscillating *electric dipole* (see page 1481).

The ratio between surface current and tangential electric intensity is the admittance per unit perimeter for the electric dipole,

$$Y_{e01} = -i\omega\left[\frac{a}{4\pi}\frac{1 - ika}{1 - (ka)^2 - ika}\right]$$

The quantity in brackets, for $ka \ll 1$, is a capacitance, the effective capacitance of the two poles of the sphere for the electric dipole type of current. For large values of ka, the admittance is real, approaching the reciprocal of $c/4\pi$, the radiation resistance of free space.

In general, therefore, we have two different sorts of radiated fields, corresponding to two different sorts of "driving" current distributions on the sphere. One type, where the magnetic field is everywhere perpendicular to $\mathbf{r}$ and there is free charge on the spherical surface as well as current, is given by the formulas

$$
\begin{aligned}
\mathbf{A} &= \mathbf{N}^3_{\sigma mn}(\mathbf{r});\quad \mathbf{E}_{\tan} = -ik[\sqrt{n(n+1)}/2n+1]\mathbf{B}^\sigma_{mn}(\vartheta,\varphi)\cdot \\
&\qquad\qquad\qquad\qquad \cdot [nh_{n+1}(ka) - (n+1)h_{n-1}(ka)]e^{-i\omega t} \\
Q &= [n(n+1)/4\pi(2n+1)]Y_{\sigma mn}(\vartheta,\varphi)[h_{n+1}(ka) + h_{n-1}(ka)]e^{-i\omega t} \\
\mathbf{J} &= (ck/4\pi)\sqrt{n(n+1)}\,\mathbf{B}^\sigma_{mn}(\vartheta,\varphi)h_n(ka)e^{-i\omega t} \qquad (13.3.73) \\
Y &= -\left(\frac{\mathbf{J}}{\mathbf{E}_{\tan}}\right) = \frac{c}{4\pi}\left[\frac{-i(2n+1)h_n}{nh_{n+1} - (n+1)h_{n-1}}\right] \to \begin{cases} -i\omega(a/4\pi n); & ka \ll 1 \\ c/4\pi; & ka \gg 1 \end{cases}
\end{aligned}
$$

The current-charge combination for the case m, n corresponds to a (2^n)th *electric multipole*. Its specific impedance $1/Y$ is equal to the impedance of vacuum, $4\pi/c$, for high frequencies and is a capacitative reactance, corresponding to a specific capacitance $a/4\pi n$ esu, for very low frequencies.

The cases for $n = 2$ are called *electric quadrupoles*. The current distribution for $m = 0$, $n = 2$ is

$$\mathbf{J} = i(3c/8\pi a)\mathbf{a}_\vartheta \sin(2\vartheta)\{[3 - (ka)^2 - 3ika]/(ka)^2\}e^{ika-i\omega t}$$

corresponding to charge alternately being moved away from the equator toward both poles and then away from both poles toward the equator, free charge of one sign piling up at the poles and of opposite sign at the equator, the signs changing a half cycle later. Radiation of this type would be generated by a charged sphere alternating between a prolate and oblate spheroid, the polar distance shrinking and the equator swelling; vice versa a half cycle later.

The other sort of radiation corresponds to the electric field normal to $\mathbf{r}$, and thus there is no free charge on the surface of the sphere, only surface currents;

$$
\begin{aligned}
\mathbf{A} &= \mathbf{M}^3_{\sigma mn};\quad \mathbf{E}_{\tan} = ik\sqrt{n(n+1)}\,\mathbf{C}^\sigma_{mn}(\vartheta,\varphi)h_n(kr)e^{-i\omega t} \\
\mathbf{J} &= \left(\frac{ck}{4\pi}\right)\frac{\sqrt{n(n+1)}}{2n+1}\mathbf{C}^\sigma_{mn}(\vartheta,\varphi)[nh_{n+1}(ka) - (n+1)h_{n-1}(ka)]e^{-i\omega t} \\
Z &= -\left(\frac{\mathbf{E}_{\tan}}{\mathbf{J}}\right) = \frac{4\pi}{c}\left[\frac{-i(2n+1)h_n}{nh_{n+1} - (n+1)h_{n-1}}\right] \to \begin{cases} -i\omega(4\pi a/c^2 n); & ka \ll 1 \\ 4\pi/c; & ka \gg 1 \end{cases} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (13.3.74)
\end{aligned}
$$

The (m,n)th current distribution corresponds to a (2^n)th *magnetic multipole*. Its specific impedance Z, at high frequencies, is equal to the impedance of vacuum, at low frequencies is reactive, corresponding to a specific inductance $4\pi a/nc^2$ emu.

The magnetic quadrupole for $m = 0$ has a current distribution ($m = 0$, $n = 2$)

$$\mathbf{J} = (3ck/10\pi)\mathbf{a}_\varphi \sin(2\vartheta)[nh_{n+1}(ka) - (n+1)h_{n-1}(ka)]$$

which corresponds to current circulating parallel to the equator, going clockwise in the northern hemisphere and counterclockwise in the southern hemisphere for a half cycle and reversing in the other half cycle. The higher multipoles are, of course, still more complex patterns of current charge, for electric multipoles; of current only, for magnetic multipoles.

Since the functions **B** and **C** form a complete set of orthogonal vector functions for the surface of a sphere, it is possible to set up solutions for the radiation from any arbitrarily specified distribution of current on the sphere or for any arbitrarily specified distribution of electric intensity. The series expansion thus automatically separates the resulting radiation field into terms corresponding to the set of multipoles which is equivalent to the actual current distribution. (See also the discussion on page 1278 *et seq.*, concerning the equivalent expansion for a static distribution of charge.) As a matter of fact, as we shall see later, an easy way to compute the radiation from a vibrating collection of charge current, at some distance from the collection, is to express the vibrations in terms of equivalent electric and magnetic dipoles, pair of quadrupoles, and so on, in a manner analogous to the expansion of the static collection of charge given in Eqs. (10.3.42) and (10.3.43). The radiated field then comes out naturally in terms of the functions **M** and **N**.

Standing Waves in a Spherical Enclosure. We can also use the functions **M** and **N** to compute the standing electromagnetic waves *inside* a spherical enclosure. The standing waves of vector potential **A** and the allowed frequencies for a perfectly conducting sphere of radius a are

$$\begin{aligned}
\mathbf{M}^\sigma_{mln}(\mathbf{r}) &= \sqrt{l(l+1)}\;\mathbf{C}^\sigma_{ml}(\vartheta,\varphi)j_l(\pi\beta_{ln}r/a) \\
\nu_{hln} &= \tfrac{1}{2}c(\beta_{ln}/a) \\
\mathbf{N}^\sigma_{mln}(\mathbf{r}) &= l(l+1)(a/\pi\gamma_{ln}r)j_l(\pi\gamma_{ln}r/a)\mathbf{P}^\sigma_{ml}(\vartheta,\varphi) \\
&\qquad + \sqrt{l(l+1)}\left(\frac{a}{\pi\gamma_{ln}r}\right)\frac{d}{dr}\left[rj_l\left(\frac{\pi\gamma_{ln}r}{a}\right)\right]\mathbf{B}^\sigma_{ml}(\vartheta,\varphi) \\
\nu_{eln} &= \tfrac{1}{2}c(\gamma_{ln}/a)
\end{aligned} \tag{13.3.75}$$

where β_{ln} is the nth root of the equation $j_l(\pi\beta) = 0$ and γ_{ln} is the nth root of the equation $d[\pi\gamma j_l(\pi\gamma)]/d\gamma = 0$. In both cases we have adjusted k (and thus ν) so that the tangential components of **A** and **E** are zero at $r = a$.

The normalization constants for these eigenfunctions are not so complicated as they might seem. For the magnetic case, the integral of the square of $\mathbf{M}$ is

$$\Lambda^h_{mln} = \frac{2\pi a^3}{\epsilon_m}\left[\frac{l(l+1)}{2l+1}\right]\frac{(l+m)!}{(l-m)!}[j_l^2(\pi\beta_{ln}) - j_{l-1}(\pi\beta_{ln})j_{l+1}(\pi\beta_{ln})]$$
$$= \frac{2\pi a^3}{\epsilon_m}\left[\frac{l(l+1)}{2l+1}\right]\frac{(l+m)!}{(l-m)!}j_{l+1}^2(\pi\beta_{ln}) \tag{13.3.76}$$

Since $j_l(\pi\beta_{ln}) = 0$, then $j_{l+1}(\pi\beta_{ln}) = -j_{l-1}(\pi\beta_{ln})$. The integration of the square of the $\mathbf{N}$ function is somewhat more difficult, but use of a form of the equation for the spherical Bessel function

$$l(l+1)j_l(kr) - \frac{d}{dr}\left[r^2\frac{d}{dr}j_l(kr)\right] = k^2r^2j_l(kr)$$

eventually produces a simple answer.

$$\Lambda^e_{mln} = \left(\frac{a}{\pi\gamma_{ln}}\right)^2\frac{4\pi}{\epsilon_m}\left[\frac{l(l+1)}{2l+1}\right]\frac{(l+m)!}{(l-m)!}\int_0^a\left\{l(l+1)j_l^2 + \left[\frac{d}{dr}rj_l\right]^2\right\}dr$$

Integration of the second term in the braces by parts yields

$$\left[rj_l\frac{d}{dr}(rj_l)\right]_0^a - \int_0^a rj_l\frac{d^2}{dr^2}(rj_l)\,dr = -\int_0^a j_l\frac{d}{dr}\left[r^2\frac{d}{dr}j_l\right]dr$$

since for $ka = \pi\gamma_{ln}$, we have $[d(rj_l)/dr] = 0$. Consequently,

$$\Lambda^e_{mln} = \frac{2\pi a^3}{\epsilon_m}\left[\frac{l(l+1)}{2l+1}\right]\frac{(l+m)!}{(l-m)!}[j_l^2(\pi\gamma_{ln}) - j_{l-1}(\pi\gamma_{ln})j_{l+1}(\pi\gamma_{ln})]$$
$$= \frac{2\pi a^3}{\epsilon_m}\left[\frac{l(l+1)}{2l+1}\right]\frac{(l+m)!}{(l-m)!}\left[1 - \frac{l(l+1)}{(\pi\gamma_{ln})^2}\right]j_l^2(\pi\gamma_{ln}) \tag{13.3.77}$$

We can also show that the integral of E^2 over the interior equals the integral of H^2 equals $k^2\Lambda_{mln}$, where $k = \pi\beta_{ln}/a$ for the $\mathbf{M}$ waves, $= \pi\gamma_{ln}/a$ for the $\mathbf{N}$ waves.

If the conductivity of the interior surface of the sphere is large but not infinite, we can compute the energy loss in terms of the dimensionless factor Q defined in Eq. (13.3.48). Using the equations for Λ^e and Λ^h and the corresponding expressions for $\mathbf{H}_{\tan}$ for the $\mathbf{N}$ and $\mathbf{M}$ waves, we obtain for the Q factor for the various standing waves,

$$Q^e_{ln} = \frac{a}{2\tau_s}\left[1 - \frac{l(l+1)}{(\pi\gamma_{ln})^2}\right];\quad \text{for } \mathbf{A} = \mathbf{N}_{mln}$$

$$Q^h_{ln} = (a/2\tau_s);\quad \text{for } \mathbf{A} = \mathbf{M}_{mln};\quad \tau_s = \sqrt{\mu c^2/8\pi\omega\sigma}$$

a remarkably simple result.

The Green's function for the interior of the sphere, for nearly perfect conductivity, for driving frequency $kc/2\pi$, is then

$$\mathfrak{G}(\mathbf{r}|\mathbf{r}_0|k) = 4\pi \sum_{\sigma,m,l,n} \left\{ \frac{\mathbf{N}^\sigma_{mln}(\mathbf{r}_0)\mathbf{N}^\sigma_{mln}(\mathbf{r})}{(\Lambda^e_{mln})^2 \left[\left(\frac{\pi\gamma_{ln}}{a}\right)^2 \left(1 - \frac{i}{2Q^e_{ln}}\right)^2 - k^2\right]} + \frac{\mathbf{M}^\sigma_{mln}(\mathbf{r}_0)\mathbf{M}^\sigma_{mln}(\mathbf{r})}{(\Lambda^h_{mln})^2 \left[\left(\frac{\pi\beta_{ln}}{a}\right)^2 \left(1 - \frac{i}{2Q^h_{ln}}\right)^2 - k^2\right]} \right\}$$

from which many problems concerned with the excitation of waves by various driving forces may be calculated.

Vibrations of an Elastic Sphere. The vibrations of an elastic sphere may also be expressed in terms of the functions **L**, **M**, and **N** given in Eqs. (13.3.67) to (13.3.69). The k for the longitudinal function **L** is $k_c = \omega \sqrt{\rho/(\lambda + 2\mu)}$, corresponding to pure compressional waves, and the k for both transverse functions is $k_s = \omega \sqrt{\rho/\mu}$, corresponding to shear waves. In order to calculate the frequencies and shapes of the free vibrations, we have first to compute the traction $\mathbf{T}_r = \lambda \mathbf{a}_r \operatorname{div} \mathbf{s} + \mu \mathbf{a}_r \cdot (\Delta\mathbf{s} + \mathbf{s}\Delta)$ at the surface $r = a$. This is

$$\begin{aligned}
&\text{For } \mathbf{L}^1_{ml}, && \mathbf{T}_r = \frac{1}{k_c} \mathbf{P}^\sigma_{ml} \left[-\lambda k_c^2 j_l(k_c a) + 2\mu \frac{d^2}{da^2} j_l(k_c a) \right] \\
& && \quad + \frac{1}{k_c} \sqrt{l(l+1)}\, \mathbf{B}^\sigma_{ml} \left[\frac{2\mu}{a} \frac{d}{da} j_l(k_c a) - \frac{\mu}{a^2} j_l(k_c a) \right] \\
&\text{For } \mathbf{M}^1_{\sigma ml}, && \mathbf{T}_r = \mu \sqrt{l(l+1)}\, \mathbf{C}^\sigma_{ml} \left[\frac{d}{da} j_l(k_s a) - \frac{1}{a} j_l(k_s a) \right] \qquad (13.3.78) \\
&\text{For } \mathbf{N}^1_{\sigma ml}, && \mathbf{T}_r = 2\mu l(l+1) \frac{1}{k_s} \mathbf{P}^\sigma_{ml} \frac{d}{da} \left[\frac{j_l(k_s a)}{a} \right] \\
& && \quad + \frac{\mu}{k_s} \sqrt{l(l+1)}\, \mathbf{B}^\sigma_{ml} \left[\frac{d^2}{da^2} j_l(ka) + \frac{l^2 + l - 2}{a^2} j_l(ka) \right]
\end{aligned}$$

If the outer surface of the sphere ($r = a$) is free, we have to choose the values of the k's such that these **T**'s are zero. The simplest types of waves are the torsional set $\mathbf{s} = \mathbf{M}^1_{\sigma mn}(\mathbf{r})$ with $k = k_s = \omega \sqrt{\rho/\mu}$. The allowed values of k_s and thence the allowed values of frequency $= (1/2\pi)k_s \sqrt{\mu/\rho}$ are those for which $j'_l(k_s a) = (1/k_s a) j_l(k_s a)$ where the prime indicates differentiation by the argument. The modes are degenerate, being independent of m, so to compute the frequencies of free vibration we need consider only the cases for $m = 0$. Using the formulas for dj_l/dz and j_l/z given at the end of Chap. 11, we see that the equation for k_s becomes $(l - 1)j_{l-1}(k_s a) = (l + 2)j_{l+1}(k_s a)$.

There is a "lowest mode," for $k_s = 0$, corresponding to

$$\mathbf{s} = \mathbf{a}_\varphi r \sin \vartheta$$

representing free rotation about the z axis. (The $\mathbf{C}^e_{11}$ mode for $k_s = 0$ represents rotation about the x axis; the $\mathbf{C}^o_{11}$, rotation about the y axis.) These modes must, of course, be included when we consider the free motion of the elastic sphere, though they do not represent elastic vibration.

The vibrational modes with the simplest angle dependence are the ones for $l = 1$ ($\mathbf{C}_{ml} = 0$ for $l = 0$), and for these modes the equation for k_s is $j_2(k_s a) = 0$ or

$$k_s = \pi\beta_{2n}/a; \quad \text{resonance frequency} = (\beta_{2n}/2a)\sqrt{\mu/\rho}; \quad l = 1$$

where β_{2n} are the roots given on page 1576. The displacement of the point (r, ϑ, φ) in the sphere for the lowest of these modes is

$$\mathbf{s} = \mathbf{a}_\varphi \sin\vartheta\, j_1(\pi\beta_{21}r/a)\, e^{-i(\pi\beta_{21}/a)c_s t}; \quad c_s = \sqrt{\mu/\rho}$$

the interior of the sphere rotating about the z axis in one direction and the outer part rotating about the z axis in the opposite direction.

The modes for $l = 2$ have another sequence of allowed frequencies, the lowest allowed value of k_s being somewhat smaller than $\pi\beta_{11}/a$ as an examination of the equation for k_s will show. The corresponding displacement is

$$\mathbf{s} = \tfrac{3}{2}\mathbf{a}_\varphi \sin(2\vartheta) j_2(\pi\delta_{31}r/a)e^{-i(\pi\delta_{31}/a)c_s t}$$

where δ_{31} is the lowest root of $j_1(\pi\delta) = 3j_3(\pi\delta)$, a little smaller than β_n. For this mode the northern hemisphere rotates in one direction about the z axis and the southern hemisphere rotates in the other direction, and vice versa for the other half cycle of the motion.

The only purely compressional modes are completely symmetric ones, $\mathbf{L}$ modes for $m = l = 0$. For $r = a$ to be a free surface, we must have

$$\lambda j_0(k_c a) = 2\mu j_0''(k_c a) = \mu j_2(k_c a) - \mu j_0(k_c a)$$

or

$$k_c = \pi\epsilon_{0n}/a; \quad \text{where } j_0(\pi\epsilon_{0n}) = [4\mu/(3\lambda + 2\mu)]j_2(\pi\epsilon_{0n})$$

The corresponding displacements are

$$\mathbf{s} = -\mathbf{a}_r j_1(\pi\epsilon_{0n}r/a)e^{-i(\pi\epsilon_{0n}/a)c_c t}; \quad c_c = \sqrt{(\lambda + 2\mu)/\rho}$$

the modes corresponding to completely radial vibrations.

No other modes can be completely compressional; for $l > 0$ we must use a combination of $\mathbf{L}$ and $\mathbf{N}$ functions to satisfy the boundary condition that $\mathbf{T}_r = 0$. For these combinations we set

$$\mathbf{s} = \mathbf{L}^1_{ml}(\mathbf{r})_{k=k_c} + A\mathbf{N}^1_{ml}(\mathbf{r})_{k=k_s}$$

and adjust A and ω so that $\mathbf{T}_r = 0$. This gives rise to a pair of simultaneous equations in $j_l(k_c a)$ and $j_l(k_s a)$ and their derivatives, together with λ, μ, and A, which can be solved in any specific case but for which a general solution cannot be written in finite form. For the modes for $m = 0$, $l = 1$, the vibration is such as to change the sphere from a prolate to an oblate spheroid during a cycle of the vibration.

The Green's Function for Free Space. Returning now to the electromagnetic waves, a comparison between Eqs. (13.3.79), (13.3.67), (13.3.69), and (11.3.53) shows that we can express the transverse Green's dyadic for a spherical enclosure in terms of the scalar Green's function for the same enclosure, operated on by vector operators for both $\mathbf{r}$ and $\mathbf{r}_0$. For the $\mathbf{M}$ part of Eq. (13.3.79) we start with the scalar Green's function which goes to zero at $r = a$ which may be written as a sum of terms for different order l of the spherical harmonics.

$$G^0(\mathbf{r}|\mathbf{r}_0|k) = \sum_{l=0}^{\infty} g_l^0(\mathbf{r}|\mathbf{r}_0|k); \quad g_l^0 = \sum_{m,n} \frac{4\pi\psi_{mln}(\mathbf{r}_0)\psi_{mln}(\mathbf{r})}{\Lambda_{mln}^2(k_{ln}^2 - k^2)}$$

where $k_{ln} = \pi\beta_{ln}/a$, $\psi_{mln} = Y_{ml}j_l$, and where Λ_{mln} is $[1/l(l+1)]$ times the normalizing factor used in Eq. (13.3.79). Consequently, the $\mathbf{M}$ part of Eq. (13.3.79) (omitting the small terms in Q, that is, for a perfectly conducting sphere) may be written symbolically

$$\sum_{l=0}^{\infty} \frac{1}{l(l+1)} \operatorname{curl}_0 \operatorname{curl}[\mathbf{r}_0\mathbf{r}g_l^0]$$

Similarly, the $\mathbf{N}$ part of the dyadic function may be written

$$\frac{1}{k^2}\sum_{l=0}^{\infty} \frac{1}{l(l+1)} (\operatorname{curl}_0 \operatorname{curl}_0)(\operatorname{curl} \operatorname{curl})[\mathbf{r}_0\mathbf{r}g_l^1]$$

where g_l^1 is the lth term in the series for the scalar Green's function, having zero value of the normal gradient of r times g at $r = a$. The operator expression for the longitudinal part uses the operators grad_0 grad and does not need the factor $1/l(l+1)$.

When we consider the Green's dyadic for infinite space, the distinction between g^0 and g^1 vanishes, but the operator notation is still valid. According to Eq. (11.3.44)

$$\frac{e^{ikR}}{R} = ik\sum_{l=0}^{\infty}(2l+1)\sum_{m,\sigma}\epsilon_m\frac{(l-m)!}{(l+m)!}\psi_{\sigma ml}^1(\mathbf{r}_0)\psi_{\sigma ml}^3(\mathbf{r}); \quad r > r_0$$

where $\quad \psi_{\sigma ml}^1 = Y_{ml}^\sigma(\vartheta,\varphi)j_l(kr) \quad$ and $\quad \psi_{\sigma ml}^3 = Y_{ml}^\sigma(\vartheta,\varphi)h_l(kr)$

Using the operators as outlined above, we can immediately obtain the Green's dyadic for free space:

$$\mathfrak{G}(\mathbf{r}|\mathbf{r}_0|k) = \mathfrak{J}\frac{e^{ikR}}{k} = ik\sum_{l=0}^{\infty}\frac{2l+1}{l(l+1)}\sum_{m,\sigma}\epsilon_m\frac{(l-m)!}{(l+m)!}\{\mathbf{M}^1_{\sigma ml}(\mathbf{r}_0)\mathbf{M}^3_{\sigma ml}(\mathbf{r}) + \mathbf{N}^1_{\sigma ml}(\mathbf{r}_0)\mathbf{N}^3_{\sigma ml}(\mathbf{r}) + l(l+1)\mathbf{L}^1_{\sigma ml}(\mathbf{r}_0)\mathbf{L}^3_{\sigma ml}(\mathbf{r})\} \quad (13.3.79)$$

where m goes from 0 to l and σ is e or o, where the vector functions are those defined in Eqs. (13.3.67) to (13.3.69), the functions with superscript 3 corresponding to outgoing waves, and where we assume that $r > r_0$. When $r < r_0$, we interchange r and r_0 in the above series. We could have obtained this series by going through the usual technique for calculating Green's functions, but since the scalar form was already available, it has been desirable to show how operator symbolism enables one to go from the scalar to the dyadic forms. We have included the longitudinal part, although it will usually disappear in the electromagnetic case.

Radiation from a Vibrating Current Distribution. The electromagnetic radiation into free space from a vibrating collection of charge current, confined within a sphere of radius a, is a problem which encompasses many important cases, from atomic radiation to the production of radio waves. We assume, to begin with, that the vibration is simple harmonic, of frequency $\nu = \omega/2\pi = kc/2\pi$. We can, later, use the Laplace transform method to compute the radiation from transient motions of the charge current. Consequently, we take the charge and current distribution to be $\rho(r,\vartheta,\varphi)e^{-i\omega t}$ and $\mathbf{J}(r,\vartheta,\varphi)e^{-i\omega t}$, where both ρ and $\mathbf{J}$ are zero for $r > a$. These two quantities are not completely independent, for the longitudinal part of $\mathbf{J}$ is related to ρ by the equation of continuity

$$\operatorname{div}\mathbf{J}_l = -(\partial\rho/\partial t) = i\omega\rho$$

The present problem is a good one to use to review again the results of gauge transformations and the corresponding relations between longitudinal and transverse Green's functions. In the gauge which uses both vector and scalar potentials, the equation for $\mathbf{A}$ is $\nabla^2\mathbf{A} - (1/c^2)(\partial^2\mathbf{A}/\partial t^2) = -(4\pi\mathbf{J}/c)$ and the complete Green's dyadic is used; for the infinite domain the two potentials being

$$\mathbf{A} = \frac{e^{-i\omega t}}{c}\iiint\frac{e^{ikR}}{R}\mathbf{J}(\mathbf{r}_0)\,dv_0; \quad \varphi = \iiint\frac{e^{ikR}}{R}\rho(\mathbf{r}_0)\,dv_0\cdot e^{-i\omega t}$$

The vector potential has a longitudinal part (which would be expressed in terms of a series of functions $\mathbf{L}^3_{ml}$) which cancels out wh-n we compute the fields. To obtain $\mathbf{H}$ we take the curl of $\mathbf{A}$ which, of course, cancels the longitudinal part of $\mathbf{A}$; to obtain $\mathbf{E}$ we use the combination $-$ grad φ $+ ik\mathbf{A}$, and the contribution from φ just cancels the longitudinal part of $ik\mathbf{A}$ in the region $r > a$. To demonstrate this last, we take the divergence of $-\text{grad}\,\varphi + ik\mathbf{A}$, which should equal zero if the longitudinal parts cancel out.

$$\begin{aligned} ik \operatorname{div} \mathbf{A} &= i\omega e^{-i\omega t} \iiint \left[\text{grad} \frac{e^{ikR}}{R}\right] \cdot \mathbf{J}(\mathbf{r}_0)\, dv_0 \\ &= -i\omega e^{-i\omega t} \iiint \left[\text{grad}_0 \frac{e^{ikR}}{R}\right] \cdot \mathbf{J}(\mathbf{r}_0)\, dv_0 \end{aligned}$$

But

$$\left[\text{grad}_0 \frac{e^{ikR}}{R}\right] \cdot \mathbf{J}(\mathbf{r}_0) = \text{div}_0\left[\frac{e^{ikR}}{R} \mathbf{J}(\mathbf{r}_0)\right] - \frac{e^{ikR}}{R} \text{div}_0\, \mathbf{J}(\mathbf{r}_0)$$

and the integral of the first divergence term over the infinite volume equals a surface integral over the infinite sphere, which is zero, leaving the integral of the second term;

$$\begin{aligned} ik \operatorname{div} \mathbf{A} &= i\omega \iiint \frac{e^{ikR}}{R} \text{div}_0\, \mathbf{J}(\mathbf{r}_0)\, dv_0 \cdot e^{-i\omega t} \\ &= -\omega^2 \iiint \frac{e^{ikR}}{R} \rho(\mathbf{r}_0)\, dv_0 \cdot e^{-i\omega t} \\ &= -\omega^2\varphi = \text{div grad}\,\varphi + 4\pi\rho = \nabla^2\varphi; \quad \text{when } r > a \end{aligned}$$

using the equation of continuity for ρ and the equation for φ. This, for $r > a$, shows that grad φ equals ik times the longitudinal part, $\mathbf{A}_l$, of the vector potential. Consequently, $\mathbf{E}$ involves only the transverse part of $\mathbf{A}$, except when $r < a$.

If, on the other hand, we take the gauge having $\varphi = 0$, the vector potential satisfies the equation

$$\text{curl curl}\,\mathbf{A} + (1/c^2)(\partial^2\mathbf{A}/\partial t^2) = 4\pi\mathbf{J}/c$$

which differs from the other case by the absence of the grad div $\mathbf{A}$ part of $\nabla^2\mathbf{A}$. As we showed in the discussion preceding Eq. (13.1.37), the solution of this equation is given in terms of the transverse Green's dyadic,

$$\mathbf{A} = \frac{1}{c} \iiint \mathfrak{G}_t(\mathbf{r}|\mathbf{r}_0|k) \cdot \mathbf{J}(\mathbf{r}_0)\, dv_0 e^{-i\omega t} - \frac{4\pi}{k^2} \mathbf{J}_l(\mathbf{r}) \qquad (13.3.80)$$

where $\mathbf{J}_l$ is the longitudinal part of the current (the part which "produces" the free charge). This, for $r > a$, is just the transverse part of the vector potential for the previous gauge. In the present case, $\mathbf{E}$ is $ik\mathbf{A}$ which, again for $r > a$, is ik times the transverse part of the former $\mathbf{A}$. In the former case, we got rid of the longitudinal part, in computing $\mathbf{E}$ in the charge-free space $r > a$, by canceling out the longitudinal part of $ik\mathbf{A}$ with the term $-$ grad φ; in the present case, we get rid of the longitudinal part in computing $\mathbf{E}$ by using the transverse Green's function and never having a longitudinal part of $\mathbf{A}$ in the first place. The latter technique seems more efficient as long as we can get an expression for $\mathfrak{G}_t$.

In the case of the expansion of the Green's function in spherical coordinates given in Eq. (13.3.79), it is easy to obtain the transverse Green's dyadic; one simply leaves out the terms in $\mathbf{L}$. Consequently, the complete expression for the vector potential in the gauge where $\varphi = 0$ is, for $r > a$,

$$\mathbf{A}(\mathbf{r}) = \frac{ik}{c} \sum_{l=1}^{\infty} \frac{2l+1}{l(l+1)} \sum_{m,\sigma} \epsilon_m \frac{(l-m)!}{(l+m)!} \cdot \{m^\sigma_{ml}\mathbf{M}^3_{\sigma ml}(\mathbf{r}) + n^\sigma_{ml}\mathbf{N}^3_{\sigma ml}(\mathbf{r})\} e^{-i\omega t} \tag{13.3.81}$$

where the constants

$$m^\sigma_{ml} = \iiint \mathbf{J}(\mathbf{r}_0) \cdot \mathbf{M}^1_{\sigma ml}(\mathbf{r}_0)\, dv_0; \quad n^\sigma_{ml} = \iiint \mathbf{J}(\mathbf{r}_0) \cdot \mathbf{N}^1_{\sigma ml}(\mathbf{r}_0)\, dv_0$$

might be called the (m,l)th components of the current distribution.

When $ka \ll l$ (when the wavelength λ is much larger than $1/l$ times the radius outside which $\mathbf{J} = 0$), we are justified in taking the first term in the expansion of the spherical Bessel function entering into $\mathbf{M}^1$ and $\mathbf{N}^1$,

$$j_l(kr) \to \frac{2^l l!}{(2l+1)!} (kr)^l$$

so that, for instance,

$$n^\sigma_{ml} \to \frac{2^l(l+1)!k^{l-1}}{(2l+1)!} \iiint \{lr^{l-1}Y^\sigma_{lm}\mathbf{a}_r \cdot \mathbf{J} + r^l\mathbf{J} \cdot \operatorname{grad} Y^\sigma_{lm}\}\, dv$$

But $r^l\mathbf{J} \cdot \operatorname{grad} Y = \operatorname{div}(r^lY\mathbf{J}) - lr^{l-1}Y\mathbf{a}_r \cdot \mathbf{J} - r^lY \operatorname{div} \mathbf{J}$. The integral of the first term is a surface integral which is zero. The second term just cancels the first term in the integrand for n^σ_{ml}, and the third term, because of the continuity equation relating $\mathbf{J}$ and ρ, is $i\omega r^l Y\rho$, so that

$$n^\sigma_{ml} = -ik^lc \left[\frac{2^l(l+1)!}{(2l+1)!}\right] p^\sigma_{ml}$$

where

$$p^\sigma_{ml} = \frac{i(2l+1)!}{ck^l2^l(l+1)!} \iiint \mathbf{J} \cdot \mathbf{N}^1_{\sigma ml}\, dv \to \iiint \rho(\mathbf{r}) Y^\sigma_{ml}(\vartheta,\varphi) r^l\, dv; \quad ka \ll l \tag{13.3.82}$$

The constants p for a given value of l are called the *electric multipole parameters* of the lth order of the charge distribution. In the limit of wavelength much larger than a, they are combinations of the multipole strength for the static field, treated in Eqs. (10.3.42) *et seq.* For shorter wavelengths, they differ from these limiting values but they may as well be still called multipole parameters.

From Eqs. (10.3.36) and Prob. 10.29 we see that, for $ka \ll l$,

$$\begin{gathered} p_{01} = \iiint z\rho\, dv = D_z^e;\quad p_{11}^e = D_x^e;\quad p_{11}^o = D_y^e \\ p_{02} = Q_{zz}^e - \tfrac{1}{2}Q_{xx}^e - \tfrac{1}{2}Q_{yy}^e;\quad p_{12}^e = 3Q_{xz}^e;\quad p_{12}^o = 3Q_{yz}^e \\ p_{22}^e = 3Q_{xx}^e - 3Q_{yy}^e;\quad p_{22}^o = 6Q_{xy}^e \end{gathered} \tag{13.3.83}$$

where the vector $\mathbf{D}^e$ is the vector *electric dipole strength* $\iiint \mathbf{r}\rho\, dv$ of the charge distribution and $\mathfrak{Q}^e$ is the dyadic *electric quadrupole strength* $\iiint (\mathbf{rr})\rho\, dv$ of the distribution. The fact that only five quadrupole parameters p_{m2} are needed to describe the field, whereas there are six independent components of the quadrupole dyadic, is explained in the discussion on page 1279.

To find the expressions for the higher multipole parameters in terms of the components of higher multipole strength, we use Eq. (10.3.34),

$$\begin{aligned} r^l X_{ml} &= r^l[Y_{ml}^e + iY_{ml}^0] = e^{im\varphi}P_l^m(\cos\vartheta)r^l \\ &= \frac{(l+m)!}{2\pi i^m l!}\int_0^{2\pi} [z + \tfrac{1}{2}i(x - iy)e^{iu} + \tfrac{1}{2}i(x+iy)e^{-iu}]^l e^{imu}\, du \\ &= \frac{(l+m)!}{2^m}\sum_s \frac{(-1)^s}{4^s s!(m+s)!(l-m-2s)!} z^{l-m-2s}(x^2+y^2)^s(x+iy)^m \end{aligned}$$

so that

$$\begin{aligned} P_{ml} &= p_{ml}^e + ip_{ml}^0 \\ &= \frac{(l+m)!}{2^m(l-m)!m!} \\ &\quad \iiint (x+iy)^m \left\{ z^{l-m} - \frac{(l-m)(l-m-1)}{4(m+1)} z^{l-m-2}(x^2+y^2) \right. \\ &\quad + \frac{(l-m)(l-m-1)(l-m-2)(l-m-3)}{8(2!)(m+1)(m+2)} z^{l-m-4}(x^2+y^2)^2 \\ &\quad \left. - \cdots \right\} dv \end{aligned}$$

For the constants m_{ml}^σ, when $ka \ll l + 1$, we have

$$\begin{aligned} m_{ml}^\sigma \to \frac{-2^l l! k^l}{(2l+1)!} &\iiint r^l \mathbf{J}\cdot[\mathbf{r}\times \operatorname{grad}(Y_{ml})]\, dv \\ &= \frac{2^l l! k^l}{(2l+1)!}\iiint r^l(\mathbf{r}\times\mathbf{J})\cdot\operatorname{grad}(Y_{ml})\, dv \end{aligned}$$

where $(\mathbf{r} \times \mathbf{J})$ is the current circulation vector, which produces magnetic multipoles. These magnetic multipoles may be analyzed in exactly the same way that we analyzed the electric multipole; we can set

$$m_{ml}^\sigma = [2^l l l! k^l/(2l+1)!]h_{ml}^\sigma$$

where the constants

$$h^\sigma_{ml} = \frac{(2l+1)!}{2^l l! l! k^l} \iiint j_l(kr)(\mathbf{r} \times \mathbf{J}) \cdot \operatorname{grad}(Y_{ml})\, dv$$
$$\rightarrow \frac{1}{l} \iiint \operatorname{grad}(r^l Y_{ml}) \cdot (\mathbf{r} \times \mathbf{J})\, dv; \quad ka \ll l \qquad (13.3.84)$$

may be called the *magnetic multipole parameters* for the charge-current distribution. They are combinations of the *magnetic multipole strengths,* defined by analogy with the electric multipole strengths. For instance, for $ka \ll l$,

$$\begin{aligned} h_{01} &= \textstyle\iiint \operatorname{grad} z \cdot (\mathbf{r} \times \mathbf{J})\, dv = D^h_z \\ h^e_{11} &= D^h_x; \quad h^0_{11} = D^h_y; \quad \mathbf{D}^h = \textstyle\iiint (\mathbf{r} \times \mathbf{J})\, dv \\ h_{02} &= Q^h_{zz} - \tfrac{1}{2}Q^h_{xx} - \tfrac{1}{2}Q^h_{yy}; \quad h^e_{12} = 3Q^h_{xz}; \quad h^0_{12} = 3Q^h_{yz} \\ h^e_{22} &= 3Q^h_{xx} - 3Q^h_{yy}; \quad h^0_{22} = 6Q^h_{xy} \end{aligned} \qquad (13.3.85)$$

where $\mathbf{D}^h$ is the vector *magnetic dipole strength* $\iiint(\mathbf{r} \times \mathbf{J})\, dv$ and the Q's are the components of the symmetric dyadic $\mathfrak{G}^h$, the *magnetic quadrupole strength* $\frac{1}{2}\iiint[\mathbf{r}(\mathbf{r} \times \mathbf{J}) + (\mathbf{r} \times \mathbf{J})\mathbf{r}]\, dv$. The relations between the h's and the higher magnetic multipole strengths may be worked out from the relationship between X_{ml} and the powers of x, y, and z as given for the electric multipole case.

Collecting all these terms, we have for the electric and magnetic fields radiated by the vibrating charge-current distribution, for $r > a$,

$$\mathbf{E} = ik^2 e^{-i\omega t} \sum_{l=1}^{\infty} \frac{2^l(l-1)!}{(2l)!} k^l \sum_{m,\sigma} \epsilon_m \frac{(l-m)!}{(l+m)!} \left\{ p^\sigma_{ml} \mathbf{N}^3_{\sigma ml}(\mathbf{r}) + \frac{i}{c}\frac{l}{l+1} h^\sigma_{ml} \mathbf{M}^3_{\sigma ml}(\mathbf{r}) \right\} \qquad (13.3.86)$$

$$\xrightarrow[kr \to \infty]{} ik \frac{e^{i(kr-\omega t)}}{r} \sum_{l=1}^{\infty} \frac{2^l(l-1)!}{(2l)!} (-ik)^l \sum_{m,\sigma} \epsilon_m \frac{(l-m)!}{(l+m)!} \cdot \sqrt{l(l+1)} \left\{ p^\sigma_{ml} \mathbf{B}^\sigma_{ml}(\vartheta,\varphi) + \frac{l}{l+1} \frac{h^\sigma_{ml}}{c} \mathbf{C}^\sigma_{ml}(\vartheta,\varphi) \right\}$$

$$\mathbf{H} = ik^2 e^{-i\omega t} \sum_{l=1}^{\infty} \frac{2^l(l-1)!}{(2l)!} k^l \sum_{m,\sigma} \epsilon_m \frac{(l-m)!}{(l+m)!} \cdot \left\{ \frac{1}{c}\frac{l}{l+1} h^\sigma_{ml} \mathbf{N}^3_{\sigma ml}(\mathbf{r}) - i p^\sigma_{ml} \mathbf{M}^3_{\sigma ml}(\mathbf{r}) \right\}$$

$$\xrightarrow[kr \to \infty]{} ik \frac{e^{i(kr-\omega t)}}{r} \sum_{l=1}^{\infty} \frac{2^l(l-1)!}{(2l)!} (-ik)^l \sum_{m,\sigma} \epsilon_m \frac{(l-m)!}{(l+m)!} \cdot \sqrt{l(l+1)} \cdot \left\{ \frac{l}{l+1} \frac{h^\sigma_{ml}}{c} \mathbf{B}^\sigma_{ml}(\vartheta,\varphi) - p^\sigma_{ml} \mathbf{C}^\sigma_{ml}(\vartheta,\varphi) \right\}$$

The real part of $(c/4\pi)\bar{\mathbf{E}} \times \mathbf{H}$ is the intensity of radiation and the integral of this over the surface of a sphere of radius considerably larger than a,

$$W = k^2c \sum_{l=1}^{\infty} \frac{2^{2l}(l-1)!(l+1)!}{(2l)!(2l+1)!} k^{2l} \sum_{m,\sigma} \epsilon_m \frac{(l-m)!}{(l+m)!} \cdot \left[(p^\sigma_{ml})^2 + \frac{l^2}{(l+1)^2}\left(\frac{h^\sigma_{ml}}{c}\right)^2 \right]$$

$$= \frac{k^2}{c} \sum_{l=1}^{\infty} \frac{2l+1}{l(l+1)} \sum_{m,\sigma} \epsilon_m \frac{(l-m)!}{(l+m)!} [(n^\sigma_{ml})^2 + (m^\sigma_{ml})^2] \tag{13.3.87}$$

is the total power emitted by the charge-current collection. Since the quantities p and h are, for $ka \ll l$, independent of k, being equal to the static parameters, we see that the radiation from the lth equivalent multipole is proportional to the $2(l + 1)$ power of the frequency for low frequencies. Also, since $|p^\sigma_{ml}|$ can be shown to be smaller than a^lQ, where Q is some constant, of the dimensions of charge, independent of m and l, and since $|h^\sigma_{ml}|$ is smaller than $a^{l+1}I$ where the constant I has the dimensions of current, then the series converges quite rapidly as long as $ka = \omega a/c$ is considerably smaller than unity. The electric dipole term, for p^σ_{m1}, is usually the largest term, the next terms, an order $(ka)^2$ smaller, are the electric quadrupole and the magnetic dipole, and so on. It is important to note, however, that this general order of magnitudes only holds when the longitudinal part of the current is of the same magnitude as the transverse part. It may not hold, for example, in atomic nuclei, where the transverse part of the current often predominates.

Radiation from a Half-wave Antenna. As an example of the radiation from an object about the same size as the wavelength, we compute the components of the radiation and the total power radiated from a straight wire of length $2a$, used as a half-wave antenna in free space. Setting the origin at the center of the wire and the z axis along the wire, we have, to the first approximation, $\mathbf{J} = \mathbf{a}_z I_0 \delta(x)\delta(y) \cos(kz)\, e^{-ikct}$ $(|z| < a)$ where, for resonance, we must have $k = \pi/2a$, or the frequency $\nu = c/4a$. We say that this is to the first approximation, because we have no assurance that it would be the actual distribution of current in a resonating wire antenna; to do the problem exactly, we should study the free vibrations of the system of current in the wire plus radiation field. The distribution quoted is the correct one for free vibrations of current in the wire if the current were not coupled to the radiation field. We can hope that the effect of the actual coupling to the field is not to modify appreciably the current distribution.

Investigation of the resulting integrals m^σ_{ml} and n^σ_{ml} indicates that all the m's are zero (consequently, there are no equivalent magnetic multipole fields) and that the only n's which are not zero are those for $m = 0$ and $l = 2n + 1$, an odd number. We have

$$n_{0,\,2n+1} = -(2n+1)(2n+2)\frac{2aI_0}{\pi}\int_0^{\frac{1}{2}\pi} n_0(z)j_{2n+1}(z)\,dz = aI_0 j_{2n+1}(\tfrac{1}{2}\pi)$$
$$n_{01} = (4aI_0/\pi^2);\quad n_{03} = (48aI_0/\pi^5)(10-\pi^2);\quad \ldots$$

so that the series for the field is

$$\mathbf{A} = \left(\frac{3iI_0}{\pi c}\right)\left\{\mathbf{N}^3_{01}(\mathbf{r}) + \left(\frac{14}{3\pi^3}\right)(10-\pi^2)\mathbf{N}^3_{03}(\mathbf{r}) + \cdots\right\}e^{-i(\pi c/2a)t}$$

The effective dipole amplitude is $12a^2I_0/\pi^3c$, the quadrupole amplitude is zero, and the higher multipoles are small enough to neglect in calculating the energy radiated, although the effect of the octopole term is just noticeable in the distribution in angle of the radiated wave. The power radiated by the dipole term is $6I_0^2/\pi^2c$, which is approximately equal to the power radiated by the half-wave antenna.

To the approximation which includes the octopole term but no higher multipole, the fields at large distances from the antenna ($r \gg a$) are

$$\mathbf{E} \to -i\left(\frac{3I_0}{\pi cr}\right)e^{i(\pi/2a)(r-ct)}\mathbf{a}_\vartheta \sin\vartheta\left[1 - \left(\frac{7}{\pi^3}\right)(10-\pi^2)(5\cos^2\vartheta - 1)\right]$$
$$\mathbf{H} \to -i\left(\frac{3I_0}{\pi cr}\right)e^{i(\pi/2a)(r-ct)}\mathbf{a}_\varphi \sin\vartheta\left[1 - \left(\frac{7}{\pi^3}\right)(10-\pi^2)(5\cos^2\vartheta - 1)\right]$$

being perpendicular to each other and to the radius vector. The fields are greatest in the plane at right angles to the axis ($\vartheta = \frac{1}{2}\pi$) and are zero along the axis ($\vartheta = 0$) of the antenna.

Radiation from a Current Loop. The case where a uniform current circulates in a loop of radius a about the z axis is also of interest. The current in this case is

$$\mathbf{J} = \mathbf{a}_\varphi I_0\delta(r-a)(1/r)\delta(\vartheta - \tfrac{1}{2}\pi)e^{-ikct}$$

so that the parameters n_{ml} are all zero and the magnetic parameters m_{ml} are zero except for the cases $m = 0$, l an odd integer. Using Eq. (13.3.81), we obtain

$$m_{0,\,2n+1} = 2\pi aI_0 j_{2n+1}(ka)\frac{(-1)^n(2n+2)!}{2^{2n+1}n!(n+1)!}$$

where we have used the expression for $P_n^1(0)$, given at the end of Chap. 10. When ka is small, the circumference of the loop being small compared to the wavelength, the magnitudes of the first few effective magnetic multipoles are

$$h_{01} \simeq \pi a^2I_0;\quad h_{03} \simeq -\tfrac{1}{4}\pi a^4I_0;\quad h_{05} \simeq \tfrac{1}{8}\pi a^6I_0;\quad \cdots$$

The vector potential is then (for $r > a$)

$$\mathbf{A} = \frac{2\pi i k I_0}{c} \sum_{n=0}^{\infty} (-1)^n \frac{(4n+3)(2n)!}{2^{2n+1}n!(n+1)!} \cdot \\ \cdot \mathbf{a}_\varphi P^1_{2n+1}(\cos\vartheta) j_{2n+1}(ka) h_{2n+1}(kr) e^{-i\omega t} \quad (13.3.88)$$

and the fields at large distances from the loop are

$$\mathbf{E} \to \left(\frac{2\pi I_0}{cr}\right) \mathbf{a}_\varphi e^{ik(r-ct)} \sum_{n=0}^{\infty} \frac{(4n+3)(2n)!}{2^{2n+1}n!(n+1)!} j_{2n+1}(ka) P^1_{2n+1}(\cos\vartheta)$$

$$\mathbf{H} \to -\left(\frac{2\pi I_0}{cr}\right) \mathbf{a}_\vartheta e^{ik(r-ct)} \sum_{n=0}^{\infty} \frac{(4n+3)(2n)!}{2^{2n+1}n!(n+1)!} j_{2n+1}(ka) P^1_{2n+1}(\cos\vartheta)$$

and the total energy radiated by the loop is

$$W = \left(\frac{2\pi^2 I_0^2}{c}\right) \sum_{n=0}^{\infty} \frac{(4n+3)(2n)!(2n+2)!}{4^{2n+1}[n!(n+1)!]^2} [j_{2n+1}(ka)]^2 \quad (13.3.89)$$

For $ka < 1$, we can use the series expansion for the Bessel function j_{2n+1} and, retaining the first two terms, we have $\mathbf{E} = \mathbf{a}_\varphi F$, $\mathbf{H} = -\mathbf{a}_\vartheta F$,

$$F \simeq \frac{\pi k I_0 a}{cr} e^{ik(r-ct)} \sin\vartheta[1 + \tfrac{1}{40}(ka)^2(5\cos^2\vartheta - 1) + \cdots]; \\ ka < 1; \quad r \gg a$$

the fields having much the same angle dependence as for the dipole antenna, except that the electric and magnetic fields have been interchanged, $\mathbf{E}$ doing the looping around the z axis and $\mathbf{H}$ pointing in the axial plane. At these long wavelengths, the total energy radiated comes almost entirely from the magnetic dipole term, $\pi^2 k^2 I_0^2 a^2/3c$. This quantity is smaller than the corresponding expression for the dipole antenna, for the same magnitude of I_0, by the factor $(ka)^2$. A loop antenna, thus, is a less effective radiator than is a dipole antenna of the same general size and for the same driving current. (The "catch" to this is that for low frequencies it requires a higher driving voltage to produce a given current in a dipole than it does in a loop.)

Scattering from a Sphere. Finally, we can compute the scattering of a plane electromagnetic wave from a perfectly conducting sphere of radius a. If the electric vector is to be in the x direction in the plane wave, we obtain the expression

$$\mathbf{A}_i = \mathbf{a}_x e^{ikz} = \sum_{n=0}^{\infty} \frac{(2n+1)i^n}{n(n+1)} [\mathbf{M}^1_{o1n}(\mathbf{r}) - i\mathbf{N}^1_{e1n}(\mathbf{r})] \quad (13.3.90)$$

by multiplying Eq. (13.3.70) by $\mathbf{a}_x$. We must now add to this enough of a series of outgoing waves $\mathbf{M}^3_{o1n}$ and $\mathbf{N}^3_{e1n}$ to make $\mathbf{A}_{\tan}$ go to zero at $r = a$. The scattered wave and its asymptotic form, for $kr \gg 1$, are

$$\mathbf{A}_s = \sum_{n=1}^{\infty} \frac{(2n+1)i^{n-1}}{n(n+1)} [e^{-i\delta_n} \sin \delta_n \mathbf{M}^3_{o1n}(\mathbf{r}) - ie^{-i\epsilon_n} \sin \epsilon_n \mathbf{N}^3_{e1n}(\mathbf{r})] \tag{13.3.91}$$

$$\to -\frac{e^{ik(r-ct)}}{kr} \sum_{n=1}^{\infty} \frac{(2n+1)}{\sqrt{n(n+1)}} [e^{-i\delta_n} \sin \delta_n \, \mathbf{C}^o_{1n}(\vartheta,\varphi) + e^{-i\epsilon_n} \sin \epsilon_n \, \mathbf{B}^e_{1n}(\vartheta,\varphi)]$$

where the angles δ_n and ϵ_n are defined by the equations

$$j_n(ka) = D_n \sin \delta_n; \quad h_n(ka) = -iD_n e^{i\delta_n}$$
$$\left[\frac{1}{z}\frac{d}{dz} zj_n(z)\right]_{z=ka} = -E_n \sin \epsilon_n; \quad \left[\frac{1}{z}\frac{d}{dz} zh_n(z)\right]_{z=ka} = iE_n e^{i\epsilon_n}$$

The δ's are tabulated at the end of the book and discussed at the end of Chap. 10.

The last form of the series shows how $\mathbf{A}$ (and, therefore, $\mathbf{E}$) depends on the angle of scattering. The field is entirely transverse, when $kr \gg 1$ and $r \gg a$, and because of the difference between δ_n and ϵ_n, the amplitude scattered in the plane of the incident electric vector ($\varphi = 0, \pi$) differs from that scattered in the perpendicular plane ($\varphi = \pm\frac{1}{2}\pi$). In the plane $\varphi = 0, \pi$, the electric vector lies in that plane ($\mathbf{A}_s$ is proportional to $\mathbf{a}_\vartheta$); in the plane $\varphi = \pm\frac{1}{2}\pi$, $\mathbf{E}$ is again parallel to the x, z plane ($\mathbf{A}_s$ proportional to $\mathbf{a}_\vartheta$). For intermediate angles φ, the scattered electric vector is not everywhere parallel to the x, y plane, but it is always perpendicular to the radius vector r.

The total cross section for scattering of electromagnetic waves from a perfectly conducting sphere of radius a, the ratio of power scattered by the sphere to incident intensity, is equal to the square of $|\mathbf{A}_s|$ (why?) integrated over a sphere of large radius r,

$$Q_s = 2\pi a^2 \sum_{n=1}^{\infty} \frac{2n+1}{k^2a^2} [\sin^2 \delta_n + \sin^2 \epsilon_n] \tag{13.3.92}$$

The difference between this expression and the one of Eq. (11.3.72), for the scattering of a scalar wave from a sphere, lies in the effect of the polarization of the vector wave. The vector wave arrives at the spherical surface in two parts: a transverse electric part ($\mathbf{M}$ waves), which are scattered with the phase angles δ_n, and the transverse magnetic part ($\mathbf{N}$ waves), which are scattered with the phase angles ϵ_n. The total cross section is equal to the average of the two separate cross sections.

The angular distribution of the scattered intensity, for a unit intensity, incident wave is, of course, just the magnitude $|\mathbf{A}_s|^2$ for large values of r.

For long wavelengths, when $ka \ll 1$, the angles δ_n and ϵ_n go rapidly to zero for increasing n, and only the $n = 1$ term needs to be retained. The scattered field for an incident wave $\mathbf{E} = \mathbf{a}_x E_0 e^{ik(z-ct)} (\mathbf{E} = ik\mathbf{A})$ and the scattering cross section, for $ka \ll 1$ ($\lambda \gg 2\pi a$), are

$$\begin{aligned} \mathbf{A}_s &\simeq E_0 k^2 a^3 [\mathbf{N}^3_{e11} - \tfrac{1}{2} i \mathbf{M}^3_{o11}] e^{-ikct} \\ \mathbf{E}_s &\simeq E_0 (a/r) e^{ik(r-ct)} (\tfrac{1}{2} k^2 a^2) [\mathbf{a}_\vartheta (2\cos\vartheta - 1)\cos\varphi \\ &\qquad + \mathbf{a}_\varphi (\cos\vartheta - 2)\sin\varphi]; \quad r \gg a \\ Q_s &\simeq \tfrac{10}{3}\pi a^2 (ka)^4 = (160\pi^5 a^6 / 3\lambda^4) \end{aligned} \tag{13.3.93}$$

for a perfectly conducting sphere of radius a. This limiting case is called *Rayleigh scattering*, after the first investigator. The amount scattered varies with the fourth power of the frequency of the incident wave.

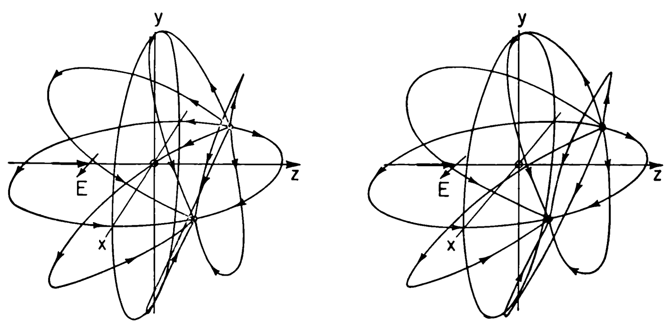

Fig. 13.7 Electric lines of force from an electromagnetic wave scattered from a conductive sphere, at great distances from the sphere, for $\lambda \gg a$ (Rayleigh scattering).

Several comments of interest may be made concerning these limiting formulas for long wavelengths. In the first place, comparison with Eqs. (13.3.81) to (13.3.84) shows that the scattered wave may be considered to be produced by the induced electric and magnetic dipoles,

$$\mathbf{D}^e = a_x E_0 a^3; \quad \mathbf{D}^h = -a_x E_0 a^3 c$$

The lines of electric force on a sphere of radius much larger than a wavelength, at one instant of the cycle, are plotted in Fig. 13.7. We see that the combination of electric and magnetic dipole terms produces a sort of lopsided dipole field with poles at $\cos\vartheta = \frac{1}{2}$, $\varphi = 0, \pi$. In these directions, there is no scattered intensity to this approximation. In other directions, the polarization of the scattered electric vector is along the lines of force shown in the figure. To this approximation, more is scattered backward ($\vartheta = \pi$) then forward ($\vartheta = 0$). Of course these results hold only for a perfectly conductive sphere. A dielectric sphere would give quite different results, although in the great majority of cases the scattering cross section at low frequencies varies as the fourth power of the frequency.

Energy Absorption by the Sphere. To determine the loss of energy because of the surface currents in the sphere, we need to compute the surface currents from the tangential component of the magnetic field, as was done on page 1863. While we are at it, we may as well calculate the electric charge induced on the surface. From Eq. (13.3.91) we compute the electric field normal to the surface of the sphere at $r = a$, for an incident wave $\mathbf{E} = \mathbf{a}_x E_0 e^{ik(z-ct)}$ (here we include both incident and scattered field),

$$(E_r)_{r=a} = -\frac{iE_0}{ka}\sum_{n=1}^{\infty}(2n+1)i^n D_n e^{-i\epsilon_n}\sin(\delta_n - \epsilon_n)\cos\varphi\, P_n^1(\cos\vartheta)e^{-i\omega t}$$
$$\rightarrow 3E_0\cos\varphi\sin\vartheta e^{-ikct};\quad ka \ll 1 \qquad (13.3.94)$$

where the quantities D_n are defined just below Eq. (13.3.91). The charge density $E_r/4\pi$ on the surface corresponds, in the long wave limit, to an electric dipole $\mathbf{D}^e = \mathbf{a}_x E_0 a^3$, as mentioned before. The tangential magnetic field at $r = a$ is

$$(\mathbf{H}_{\tan})_{r=a} = -E_0\sum_{n=1}^{\infty}\frac{(2n+1)i^n}{\sqrt{n(n+1)}}[D_n e^{-i\epsilon_n}\sin(\delta_n - \epsilon_n)\mathbf{C}_{1n}^e(\vartheta,\varphi)$$
$$+ iE_n e^{-i\delta_n}\sin(\delta_n - \epsilon_n)\mathbf{B}_{1n}^0(\vartheta,\varphi)]e^{-i\omega t}$$
$$\rightarrow \tfrac{3}{2}kaE_0[\mathbf{a}_\vartheta\sin\varphi(1 + i\cos\vartheta) + \mathbf{a}_\varphi\cos\varphi(\cos\vartheta + i)]e^{-ikct} \qquad (13.3.95)$$

The surface current $c\mathbf{H}_{\tan}/4\pi$, in the long wave limit, corresponds to both electric and magnetic dipoles, as mentioned earlier. The power lost at the surface is, according to the discussion of page 1829, just $(c/4\pi)\sqrt{\mu\omega/8\pi\sigma}\iint|\mathbf{H}_{\tan}|^2\,dA$, to the first approximation in the small quantity $\sqrt{\mu\omega/8\pi\sigma}$. Since the incident intensity is $(c/4\pi)E_0^2$, the ratio of power absorbed to incident intensity, the absorption cross section, is just

$$Q_a = 2\pi a^2\sqrt{\frac{\mu\omega}{8\pi\sigma}}\sum_{n=1}^{\infty}(2n+1)(D_n^2 + E_n^2)\sin^2(\delta_n - \epsilon_n) \qquad (13.3.96)$$
$$\rightarrow 2\pi a^2\sqrt{\mu\omega/8\pi\sigma};\quad ka \ll 1$$

To put this into the form of Eq. (11.3.65), we set up the reflection coefficients for the nth electric and magnetic waves,

$$R_n^e = e^{-2(\tau_n + i\epsilon_n)};\quad R_n^h = e^{-2(\kappa_n + \delta_n)}$$

where

$$\kappa_n = \tfrac{1}{2}k^2a^2\sqrt{\mu\omega/8\pi\sigma}\,D_n^2\sin^2(\delta_n - \epsilon_n)$$
$$\tau_n = \tfrac{1}{2}k^2a^2\sqrt{\mu\omega/8\pi\sigma}\,E_n^2\sin^2(\delta_n - \epsilon_n)$$

The scattering, absorption, and total cross sections are then

$$Q_a = \frac{\pi}{2k^2}\sum_{n=1}^{\infty}(2n+1)[|1 - R_n^e|^2 + |1 - R_n^h|^2]$$

$$Q_s = \frac{\pi}{2k^2}\sum_{n=1}^{\infty}(2n+1)[2 - |R_n^e|^2 - |R_n^h|^2]$$

$$Q_e = \frac{\pi}{2k^2}\sum_{n=1}^{\infty}(2n+1)[4 - R_n^e - \bar{R}_n^e - R_n^h - \bar{R}_n^h]$$

analogous to Eq. (11.3.65) except that in the present case we have two types of waves, the transverse electric and transverse magnetic, the cross sections being the average of the individual cross sections, for each type wave.

Distortion of Field by Small Object. Any object, dielectric or conductor, will have electric and magnetic multipoles induced in it by an incident electric field and, if the size of the object is small enough compared to the wavelength of the incident field, the induced effect can be expressed in terms of an electric and a magnetic dipole only, to a satisfactory degree of approximation. These dipoles may usually be calculated in terms of the incident electric and magnetic fields.

For example, to the object, an electric field of wavelength considerably longer than its dimensions appears as an alternating field $\mathbf{E}(\mathbf{r}_0)e^{-i\omega t}$, where $\mathbf{r}_0$ is the radius vector of the center of gravity of the body. This field produces an electric dipole which may be written, to a good approximation, as the electric dipole produced by a static electric field $\mathbf{E}(\mathbf{r}_0)$, times the factor $e^{-i\omega t}$. We can always write the dipole induced by a static field as

$$\mathbf{D}^e = \mathfrak{D}_e \cdot \mathbf{E}(\mathbf{r}_0)$$

where $\mathfrak{D}_e$ is a dyadic determined by the shape and electrical properties of the object. For instance, if the object is a conductive sphere of radius a, $\mathfrak{D}_e = a^3\mathfrak{J}$; if it is a thin wire (length l small compared to λ, diameter 2ρ much smaller than length) pointed in the direction $\mathbf{a}_d$, then

$$\mathfrak{D}_e = [l^3/16\ln(l/\rho)]\mathbf{a}_d\mathbf{a}_d$$

Similarly, the magnetic field about the object is $\mathbf{H}(\mathbf{r}_0)e^{-i\omega t}$, which induces a magnetic dipole to oppose the rate of change of $\mathbf{H}$

$$\mathbf{D}^h = -\mathfrak{D}_h \cdot \mathbf{H}(\mathbf{r}_0)$$

where $\mathfrak{D}_h$ is also a symmetric dyadic depending on the characteristics of the body. The corresponding dyadic for a conductive sphere of radius a

is $ca^3\mathfrak{J}$, for example; for other cases, the dyadic may be more directional. (For a ring or disk with axis along $\mathbf{a}_d$, the form is $cl^3\mathbf{a}_d\mathbf{a}_d$, where l is a length characteristic of the disk or ring.)

By rearranging the terms of Eq. (13.3.93), we can see that the field scattered from a small object at the origin, having induced dipoles $\mathbf{D}^e e^{-i\omega t}$ and $\mathbf{D}^h e^{-i\omega t}$, is

$$\mathbf{A}_s = ik\,\frac{e^{ik(r-ct)}}{r}\left[\mathbf{a}_r \times (\mathbf{a}_r \times \mathbf{D}^e) - \frac{1}{2c}\,\mathbf{a}_r \times \mathbf{D}^h\right]; \quad kr \ll 1$$

Consequently, if the incident electric and magnetic fields at the origin are $\mathbf{E}_0 e^{-i\omega t}$ and $\mathbf{H}_0 e^{-i\omega t}$, the scattered wave several wavelengths away from the origin is

$$\mathbf{A}_s = ik\,\frac{e^{ik(r-ct)}}{r}\left\{\mathbf{a}_r \times [\mathbf{a}_r \times (\mathfrak{D}_e \cdot \mathbf{E}_0)] + \frac{1}{2c}\,[\mathbf{a}_r \times (\mathfrak{D}_h \cdot \mathbf{H}_0)]\right\} \tag{13.3.97}$$

to the first approximation in k times the longest dimension of the object.

Another method of expressing the induced field is to find a simple expression for the equivalent current induced in the body by the incident field. An examination of the definitions of the parameters m and n, together with the definitions of the equivalent dipoles [Eqs. (13.3.82) and (13.3.84)], shows that the scattered field produced by the dipoles $\mathbf{D}^e$ and $\mathbf{D}^h$ is the same as that produced by a current

$$\mathbf{J} = -i\omega\mathbf{D}^e\delta(\mathbf{r}_0 - \mathbf{r}) - \mathbf{D}^h \times \operatorname{grad}_0[\delta(\mathbf{r}_0 - \mathbf{r})]$$

where we have placed the object at $\mathbf{r}_0$. In other words, outside the body at $\mathbf{r}_0$ the scattered field, according to Eq. (13.3.80), is

$$\mathbf{A}_s(r) = -\frac{1}{c}\left\{ikc\mathbf{D}^e \cdot \mathfrak{G}_t(\mathbf{r}|\mathbf{r}_0|k) + \mathbf{D}^h \cdot \operatorname{curl}_0[\mathfrak{G}_t(\mathbf{r}|\mathbf{r}_0|k)]\right\} e^{-i\omega t} \tag{13.3.98}$$

where the dipoles are induced by the incident field. This equation will hold for any sort of incident field, producing any sort of dipoles.

Let us apply this equation to the problem of a small body at a point $\mathbf{r}_0$ $(r_0,\vartheta_0,\varphi_0)$ inside a hollow, conducting sphere of inner radius a. There is no loss in generality if we place the small body along the z axis ($\vartheta_0 = 0$); in fact it reduces to zero a great many terms in the Green's function of Eq. (13.3.79). We, of course, leave out the $\mathbf{L}$ terms, since we use the transverse Green's function. The only functions $\mathbf{M}$, $\mathbf{N}$ which are not zero along the z axis, $\vartheta \to 0$, are

$$\begin{aligned}
&\mathbf{N}^1_{e0l} \to l(l+1)(1/kr_0)j_l(kr_0)\mathbf{a}_z\\
&\mathbf{N}^1_{e1l} \to \tfrac{1}{2}l(l+1)(1/kr_0)(d/dr_0)[r_0 j_l(kr_0)]\mathbf{a}_x\\
&\mathbf{N}^1_{o1l} \to \tfrac{1}{2}l(l+1)(1/kr_0)(d/dr_0)[r_0 j_l(kr_0)]\mathbf{a}_y\\
&\mathbf{M}^1_{e1l} \to -\tfrac{1}{2}l(l+1)j_l(kr_0)\mathbf{a}_y\\
&\mathbf{M}_{o1l} \to \tfrac{1}{2}l(l+1)j_l(kr_0)\mathbf{a}_x
\end{aligned}$$

Consequently, the field scattered by the induced dipoles at ($r = r_0$, $\vartheta = 0$) is

$$\mathbf{A}_s(\mathbf{r}) = -\frac{ik}{c}\sum_{l=1}^{\infty}\frac{(2l+1)}{l(l+1)}\left\{ikc\left[\frac{j_l(kr_0)}{kr_0}\,l(l+1)D_z^e\mathbf{N}_{e0l}^3(\mathbf{r})\right.\right.$$
$$\left. + \frac{1}{kr_0}\frac{d}{dr_0}(r_0j_l)D_x^e\mathbf{N}_{e1l}^3(\mathbf{r}) + \cdots - j_l(kr_0)D_y^e\mathbf{M}_{e1l}^3(\mathbf{r})\right]$$
$$\left.+ k\left[\frac{j_l(kr_0)}{kr_0}\,l(l+1)D_z^h\mathbf{M}_{o1l}^3(\mathbf{r}) + \cdots - j_l(kr_0)D_y^h\mathbf{N}_{e1l}^3(\mathbf{r})\right]\right\}e^{-i\omega t}$$

But this cannot be the field inside the sphere, for two reasons: (1) this field does not have its tangential part zero at $r = a$ and (2) we have not specified what incident field "causes" the induced dipoles. Both these questions are answered by one statement.

In the first place, we must add to the field $\mathbf{A}_s$ a field, everywhere finite inside $r = a$, reflected from the inner surface of the sphere, which is adjusted to make the tangential part of the total field zero at $r = a$. For instance, to the first term of $\mathbf{A}_s$, we must subtract a term

$$-\frac{ik}{c}(2l+1)ikc\frac{j_l(kr_0)}{kr_0}D_z^e[-ie^{i\epsilon_l}\csc\epsilon_l]\mathbf{N}_{e0l}^1(\mathbf{r})$$

where, as before,

$$iE_le^{i\epsilon_l} = \left[\frac{1}{z}\frac{d}{dz}zh_l(z)\right]_{z=ka}$$

A similar reflected field is subtracted from the $\mathbf{M}$ terms, except that we use the phase angle δ_l defined, as before, by

$$-iD_le^{i\delta_l} = h_l(ka)$$

Next we see that this reflected wave may be considered as being the incident wave, inducing the dipoles, in the absence of any other driving field. For resonance inside the sphere, the field must drive the induced dipole, which must, in turn, drive the field. The reflected electric wave at $r = r_0$, $\vartheta = 0$ is

$$\mathbf{E}_r(\mathbf{r}_0) = \frac{ik^2}{c}\sum_{l=1}^{\infty}(2l+1)\left\{ikc\left[l(l+1)\left(\frac{j_l}{kr_0}\right)^2\frac{e^{i\epsilon_l}}{\sin\delta_l}D_z^e\mathbf{a}_z + \cdots\right.\right.$$
$$\left. + \tfrac{1}{2}(j_l)^2\frac{e^{i\delta_l}}{\sin\delta_l}D_y^e\mathbf{a}_y\right] + \tfrac{1}{2}kj_l\frac{1}{kr_0}\frac{d}{dr_0}(r_0j_l)\left[-\frac{e^{i\delta_l}}{\sin\delta_l}D_x^h\mathbf{a}_y + \cdots\right.$$
$$\left.\left. - \frac{e^{i\epsilon_l}}{\sin\epsilon_l}D_y^h\mathbf{a}_x\right]\right\}$$

and the reflected magnetic wave is similar, except that δ's and ϵ's have been interchanged, as well as $\mathbf{D}^e$'s and $\mathbf{D}^h$'s. From these fields we can compute the induced dipoles and, by equating both sides, we can eventually compute the resonance frequencies of a spherical enclosure plus a small object at $\mathbf{r}_0$ inside the enclosure.

To carry out the calculation we need to have the components of the dyadic dipole functions $\mathfrak{D}_e$ and $\mathfrak{D}_h$, mentioned earlier. To show how the calculation goes, we assume the body to be a conducting sphere of radius b, much smaller than a. In this case $\mathfrak{D}_e = b^3\mathfrak{J}$ and $\mathfrak{D}_h = cb^3\mathfrak{J}$. We can then equate components of $\mathbf{D}^e$ and $\mathbf{D}^h$. The equation for D_z^e is, for example,

$$D_z^e = \left\{-k^3b^3 \sum_{l=1}^{\infty} (2l+1)l(l+1) \left[\frac{j_l(kr_0)}{kr_0}\right]^2 \frac{e^{i\epsilon_l}}{\sin \epsilon_l}\right\} D_z^e$$

This equation will serve to compute those modes where $\mathbf{D}^h$ is zero and $\mathbf{D}^e$ is along z.

Since k^3b^3 is very small, one of the terms in the series must be quite large, which can be true only when ϵ_l is nearly equal to $n\pi$ ($n = 1, 2, \ldots$) in other words when $(1/ka)(d/da)[aj_l(ka)]$ is nearly zero, but not quite. As indicated on page 1576, when $\epsilon_l = n\pi$, $ka = \pi\gamma_{ln}$, so that we can set $ka = \pi\gamma_{ln} + e_{ln}$, where e_{ln} is small. Expanding the small quantities, we find, to the first approximation, that if

$$ka = \pi\gamma_{ln} + e_{ln}; \quad e^{i\epsilon_n} \simeq (-1)^n$$

then

$$\sin \epsilon_l \simeq -\frac{e_{ln}}{E_l(\pi\gamma_{ln})}\left[\frac{l(l+1)}{(\pi\gamma_{ln})^2} - 1\right] j_l(\pi\gamma_{ln})$$

Consequently, the allowed frequencies of vibration which excite only the z component of the electric dipole in the sphere are $\omega/2\pi$, where

$$\omega = (\pi c\gamma_{ln}/a) + (ce_{ln}/a)$$

where

$$e_{ln} \simeq (-1)^n \frac{(\pi\gamma_{ln})^3b^3}{ar_0^2}\left[\frac{(2l+1)l(l+1)E_l(\pi\gamma_{ln})}{l(l+1) - (\pi\gamma_{ln})^2}\right]\frac{j_l^2(\pi\gamma_{ln}r_0/a)}{j_l(\pi\gamma_{ln})}$$

and the standing wave has the general form $\mathbf{N}_{e0l}^1(\mathbf{r})e^{-i\omega t}$ except near the dipole.

The frequencies at which the magnetic dipole is excited in the z direction are

$$\omega = (\pi c\beta_{ln}/a) + (cg_{ln}/a)$$

$$g_{ln} \simeq (-1)^n(2l+1)l(l+1)\frac{\pi\beta_{ln}b^3}{ar_0^2} D_l(\pi\beta_{ln})\left[\frac{j_l^2(\pi\beta_{ln}r_0/a)}{j_{l+1}(\pi\beta_{ln})}\right]$$

where $j_l(\pi\beta_{ln}) = 0$. The standing wave is approximately equal to $\mathbf{M}_{e0l}^1(\mathbf{r})e^{-i\omega t}$, except near the dipole.

The other modes of vibration involve both electric and magnetic dipoles together. The equations relating them are

$$D_x^e + iD_y^e = -\tfrac{1}{2}k^3b^3 \sum (2l+1) \left\{ \left[q_l^2 \frac{e^{i\epsilon_l}}{\sin \epsilon_l} + j_l^2 \frac{e^{i\delta_l}}{\sin \delta_l} \right] (D_x^e + iD_y^e) + \frac{1}{c} j_l q_l \left[\frac{e^{i\epsilon_l}}{\sin \epsilon_l} - \frac{e^{i\delta_l}}{\sin \delta_l} \right] (D_x^h + iD_y^h) \right\}$$

$$D_x^h + iD_y^h = -\tfrac{1}{2}k^3b^3 \sum (2l+1) \left\{ \left[q_l^2 \frac{e^{i\delta_l}}{\sin \delta_l} + j_l^2 \frac{e^{i\epsilon_l}}{\sin \epsilon_l} \right] (D_x^h + iD_y^h) - c j_l q_l \left[\frac{e^{i\delta_l}}{\sin \delta_l} - \frac{e^{i\epsilon_l}}{\sin \epsilon_l} \right] (D_x^e + iD_y^e) \right\}$$

where $j_l = j_l(kr_0)$ and $q_l(kr_0) = (1/kr_0)(d/dr_0)[r_0 j_l(kr_0)]$. These are solved by letting either δ_l or ϵ_l be nearly equal to πn and by taking a combination of electric and magnetic dipoles. For example, when

$$D_x^h + iD_y^h = -c \left[\tfrac{1}{2}(\pi\beta_{ln})^3 \left(\frac{b}{a}\right)^3 (2l+1)(-1)^n j_l^2 \left(\frac{\pi\beta_{ln} r_0}{a}\right) \cdot \frac{D_l(\pi\beta_{ln})}{j_{l+1}(\pi\beta_{ln})} \right]^{\frac{1}{2}} (D_x^e - iD_y^e)$$

then δ_l can be exactly $n\pi$, the allowed frequencies are exactly

$$\omega = \pi c \beta_{ln}/a$$

and the standing wave is approximately equal to $\mathbf{M}_{e1l}^1(\mathbf{r}) + i\mathbf{M}_{o1l}^1(\mathbf{r})$, except near the object. On the other hand, another set of allowed frequencies is

$$\omega = (\pi e \beta_{ln}/a) + (c d_{ln}/a)$$

where

$$d_{ln} \simeq \tfrac{1}{2}(-1)^n(2l+1) \left(\pi\beta_{ln} \frac{b}{a}\right)^3 \left[\frac{D_l(\pi\beta_{ln})}{j_{l+1}(\pi\beta_{ln})} \right] \left[j_l^2(z) + \left(\frac{1}{z} \frac{d}{dz} z j_l \right)^2 \right]_{z=\pi\gamma_{ln}r_0/a}$$

corresponding to

$$D_x^h + iD_y^h \simeq c \left[\tfrac{1}{2}(-1)^n(2l+1) \left(\pi\beta_l \frac{b}{a}\right)^3 \frac{D_l(\pi\beta_{ln})}{j_{l+1}(\pi\beta_{ln})} \cdot \left(\frac{1}{z} \frac{d}{dz} z j_l \right)^2_{z=\pi\gamma_{ln}r/a} \right]^{\frac{1}{2}} (D_x^e + iD_y^e)$$

and to a wave shape that is also approximately equal to $\mathbf{M}_{e1l}^1 + i\mathbf{M}_{o1l}^1$. Finally there is a similar pair of sets of eigenfunctions and frequencies near $ka = \pi\gamma_{ln}(\epsilon_l \simeq \pi n)$, involving combinations of $D_x^e + iD_y^e$ and $D_x^h + iD_y^h$ and corresponding to standing waves approximately equal to $\mathbf{N}_{e1l}^1 + i\mathbf{N}_{o1l}^1$. [We can, of course, have standing waves of a single polarization, $\mathbf{N}_{e1l}^1$ or $\mathbf{M}_{o1l}^1$, etc., instead of the circularly polarized sets $(\mathbf{N}_e + i\mathbf{N}_o)$, $(\mathbf{M}_e + i\mathbf{M}_o)$.]

The presence of the small sphere has thus removed some of the degeneracies in the set of eigenfunctions for the hollow sphere. Without the extra object each value of l had two different sets of frequencies $\omega = \pi c\beta_{ln}/a$ and $\omega = \pi c\gamma_{ln}/a$, corresponding to $2(2l + 1)$ different sets of eigenfunctions, $\mathbf{M}^1_{\sigma ml}$ and $\mathbf{N}^1_{\sigma ml}$. With the small sphere each value of l has five different sets of eigenfrequencies.

Recapitulation. We have now reached an end to our discussion of vector solutions of the Laplace and the wave equation. No doubt many more instructive examples could be worked out and many more techniques, discussed earlier in the book with respect to scalar fields, could be modified to apply to vector fields. But it seems likely that it is not really necessary to discuss in detail these other possibilities, that the examples given already give enough of a hint so that the reader can make the other extensions from scalar to vector if and when it is required. Consequently, we close this section, as all books must be closed, by stopping.

Problems for Chapter 13

13.1 Show that the function $\mathbf{M}_{mn} = \operatorname{curl}[\mathbf{a}_z S_{mn}(h,\eta) he_{mn}(h,\xi)]$ is tangential to the spheroid ξ = constant only for the cases $m = 0$ and that the function $\mathbf{N}_{mn} = \operatorname{curl} \mathbf{M}_{mn}$ has a tangential curl only for $m = 0$ [see Eqs. (11.3.83) and (11.3.91) for definitions of the functions S and he]. Use the function $\mathbf{N}_{01}$ to discuss the electric dipole radiation from a prolate spheroid, plotting the surface current and angle dependence of the asymptotic radiation as function of $\vartheta = \cos^{-1}\eta$ for $h = 0, \frac{1}{4}, \frac{1}{2}$, using the series expansions of pages 1504 and 1510.

13.2 The Proca equation, analogous to the Klein-Gordon and Dirac equations, for a particle of unit spin, relates a vector potential $\mathbf{A}$ with a scalar potential φ by the equations

$$\begin{aligned} -\operatorname{curl}\operatorname{curl}\mathbf{A} - \kappa^2\mathbf{A} + (k - V)^2\mathbf{A} &= (k - V)\operatorname{grad}\varphi \\ \nabla^2\varphi - \kappa^2\varphi &= \operatorname{div}[(k - V)\mathbf{A}] \end{aligned}$$

where V is the potential energy of the particle and k its total energy in appropriate units; κ is proportional to its rest mass. Discuss the separability of this set of equations for $V = 0$, for $V = V(z)$, for $V = V(r)$. Show that with the **M**-type solutions [see Eq. (13.1.4)]φ may be set zero; obtain the equation for the scalar ψ. Discuss also the **L** and **N** types.

13.3 Write out the vector eigenfunction sets, corresponding to Eqs. (13.1.15), (13.1.17), (13.1.19), and (13.1.20), for the interior of a circular cylinder (surfaces $r = a$, $z = 0$, $z = b$) and for the interior of a sphere (surface $r = a$).

13.4 A vector source function

$$\mathbf{Q} = \mathbf{a}_z Q_0[1 - (1/a^2)(x^2 + y^2 + z^2)]e^{-ikct}; \quad r < a$$

which is zero for $r > a$, radiates out into free space. Using Eqs. (13.1.31), (13.1.32), and (13.1.39) calculate the solutions of

$$\begin{aligned} \nabla^2\mathbf{A} + k^2\mathbf{A} &= -4\pi\mathbf{Q}/c^2 \\ -\text{curl curl }\mathbf{A} + k^2\mathbf{A} &= -4\pi\mathbf{Q}/c^2 \\ c_e^2 \text{ grad div }\mathbf{A} - c_t^2 \text{ curl curl }\mathbf{A} + \omega^2\mathbf{A} &= -4\pi\mathbf{Q} \end{aligned}$$

for $r \gg a$ and for $ak \ll 1$. What are the differences between the solutions for r just larger than a? For r less than a? (Assume $ka \ll 1$ in each case.)

13.5 Incompressible fluid, of viscosity η, is originally at rest inside a circular pipe of inner radius a. Beginning at $t = 0$, a uniform pressure drop F is applied along the pipe. Use the Laplace transform to show that the fluid velocity at time t, at point r, φ, z (cylindrical coordinates) is

$$v = k\left(\frac{2Fa^2}{\eta}\right)\sum_{n=0}^{\infty}\frac{J_0(\pi\beta_{0n}r/a)}{(\pi\beta_{0n})^3J_0(\pi\beta_{0n})}\left\{1 - \exp\left[-\left(\frac{\eta}{\rho}\right)\left(\frac{\pi\beta_{0n}}{a}\right)^2 t\right]\right\}$$

Compare with the steady-state solution of Eq. (13.2.10).

13.6 A disk, of elastic material of thickness h and radius a, is held fastened at its hub to an axle of radius b, held rigid. Electric current I is sent radially out from the axle to the outer perimeter, where it is taken off by brushes which do not impose any mechanical traction on the perimeter. Torque on the disk is provided by a uniform magnetic field B, parallel to the axis of the disk. Show that the displacement of a point of the disk a distance r from its axis is

$$\mathbf{s} = \left(\frac{2IB}{\mu c}\right)\left[a\left(\frac{r}{b} - 1\right) - r\ln\left(\frac{r}{b}\right)\right]\mathbf{a}_\varphi$$

What is the stress in the disk as function of r? What is the torque on the hub axle?

13.7 Show that $-$ curl curl$[\mathbf{a}_\varphi\psi(\xi_1, \xi_2)] = \mathbf{a}_\varphi[\nabla^2\psi - (1/r^2)\psi]$ where ξ_1, ξ_2, φ are curvilinear rotational coordinates defined by $x = r(\xi_1,\xi_2)\,\cdot$ $\cdot\cos\varphi$, $y = r(\xi_1,\xi_2)\sin\varphi$, $z = z(\xi_1,\xi_2)$. Consequently show that a solution of the vector equation $-$ curl curl $\mathbf{F} + k^2\mathbf{F} = 0$ is $\mathbf{F} = \mathbf{a}_\varphi\psi_n(\xi_1,\xi_2)$, where $\cos\varphi\ \psi_n(\xi_1,\xi_2)$ is a solution of the scalar Helmholtz equation. Use this result to obtain the "torsional part" of the Green's function for prolate spheroidal coordinates, ξ, η, φ [see Eq. (10.3.53)], *i.e.*, show that a solution of

$$\text{curl curl } \mathbf{G}_\varphi(\xi,\eta|\xi_0,\eta_0) = [32\pi\delta(\xi - \xi_0)\delta(\eta - \eta_0)/a^3(\xi^2 - \eta^2)]\mathbf{a}_\varphi$$

is

$$\mathbf{G}_\varphi = \mathbf{a}_\varphi\left(\frac{4\pi}{ia}\right)\sum_{n=1}^{\infty}\frac{2n+1}{n^2(n+1)^2}P_n^1(\eta)P_n^1(\eta_0)\begin{cases}P_n^1(\xi)Q_n^1(\xi_0); & \xi_0 > \xi\\ P_n^1(\xi_0)Q_n^1(\xi); & \xi_0 < \xi\end{cases}$$

Obtain the corresponding series for oblate spheroidal coordinates.

13.8 Use the results of Prob. 13.7 to solve the following problems:

a. An oblate spheroid, uniformly charged on its surface, $\xi = \xi_0$, is spun about its axis with angular velocity ω. Find the generated magnetic field.

b. Suppose the oblate spheroid is conducting and the charge arranges itself to make the surface $\xi = \xi_0$ an equipotential surface. What is then the magnetic field when the spheroid is rotated?

c. A prolate spheroid of outer surface $\xi = \xi_0$ has its two ends, from $\eta = 1$ to $\eta = 1 - \eta_0$ and from $\eta = -1 + \eta_0$ to $\eta = -1$, soldered to two rigid, coaxial rods. Compute the torque required to rotate one rod an angle α with respect to the other. Compute also the displacement of various parts of the spheroid.

13.9 A plane shear wave is incident on a free, plane surface (traction zero at surface). Compute the angles of reflection and relative amplitudes of the reflected compressional and shear waves as function of angle of incidence for the two "polarizations," one for $\mathbf{s}$ in the plane of incidence, the other for $\mathbf{s}$ perpendicular to the plane of incidence. What happens when the angle of incidence gets large enough so that the angle of reflection for the reflected compressional wave is complex?

13.10 Show that the series (13.3.14) will represent the Green's dyadic for a cylindrical duct, satisfying the boundary conditions that $(n \times \mathfrak{G}) = 0$ on the duct walls and also on the surface $z = 0$, if the functions f_n and g_n in the series are

$$f_n = \frac{1}{\gamma}\begin{cases}\sin(\gamma z)e^{i\gamma z_0}; & z_0 > z;\\ \sin(\gamma z_0)e^{i\gamma z}; & z_0 < z\end{cases} \quad \begin{aligned}\gamma &= \sqrt{k^2 - k_{mn}^2}\\ &= i\sqrt{k_{mn}^2 - k^2}\end{aligned}$$

$$g_n = \frac{i}{\gamma}\begin{cases}\cos(\gamma z)e^{i\gamma z_0}; & z_0 > z\\ \cos(\gamma z_0)e^{i\gamma z}; & z_0 < z\end{cases}$$

The dyadic is useful for the part of the duct, for $z > 0$, and so may be called $\mathfrak{G}^+$. What are the functions f_n and g_n for $\mathfrak{G}^-$, useful for $z < 0$? Use these dyadics to solve the following problem: A lowest mode wave is started from $-\infty$ along the duct, strikes a diaphragm with a hole in it at $z = 0$, part is reflected and part transmitted through the hole. Show that the equation for the vector potential is

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \dfrac{1}{4\pi} \oint_O \operatorname{curl}_0[\mathfrak{G}^+(\mathbf{r}|\mathbf{r}_0^s)] \cdot [\mathbf{a}_z \times \mathbf{A}(\mathbf{r}_0^s)]\, dS_0; & z > 0 \\ \mathbf{B}_{01} \sin[z\sqrt{k^2 - k_{01}^2}] - \dfrac{1}{4\pi} \oint_O \operatorname{curl}_0 \mathfrak{G}^- \cdot [\mathbf{a}_z \times \mathbf{A}]\, dS_0; & z < 0 \end{cases}$$

where the integral is over the area of opening in the diaphragm at $z = 0$. What is the asymptotic amplitude of the transmitted beam if k is larger than k_{01} but smaller than k_{10}? Obtain the integral equation

$$\sqrt{1 - \left(\frac{k_{01}}{k}\right)^2} \operatorname{grad} \psi_{01}(x,y) = \frac{1}{2\pi} \oint_O \mathfrak{F}(xy|x_0y_0) \cdot [\mathbf{a}_z \times \mathbf{A}(x_0y_0)]\, dx_0\, dy_0; \qquad z,\, z_0 = 0$$

where $\mathfrak{F} = \operatorname{curl} \operatorname{curl}_0 \mathfrak{G}^0$, for the tangential component of $\mathbf{A}$ in the opening. Obtain the variational principle from which the best value of $\mathbf{A}$ and of the transmitted amplitude can be obtained.

13.11 Use the variational principle obtained in Prob. 13.10 to compute the transmission through a slit of width Δ, parallel to the x axis in the center of a diaphragm across $z = 0$, in a rectangular duct of sides a and b $(b > a)$. Use $A_0\mathbf{a}_x \sqrt{1 - [2(y - \frac{1}{2}b)/\Delta]^2}$ for the trial function for $\mathbf{A}$ in the opening.

13.12 Compute the radiation from a strip carrying current $Ie^{-i\omega t}$, the strip being across the diameter of a duct of circular cross section. Obtain a formula, analogous to Eq. (11.3.16), for the driving voltage across the strip.

13.13 Compute and plot the attenuation factor κ (in appropriate units) of Eq. (11.3.17) as function of ω or k (in appropriate units) for the lowest three modes for a rectangular duct ($b = 1.5a$). For the lowest three modes for a circular duct.

13.14 Discuss the modes of microwave transmission along a duct of elliptic cross section. Write out the expressions for the **M** and **N** modes and the attenuation of the lowest three modes.

13.15 A useful duct for transmitting microwaves is the concentric line, consisting of the space between two concentric circular cylinders. Show that the lowest mode of this duct has zero cutoff frequency (see page 1820). What is the attenuation constant for this wave?

13.16 Discuss the modes of transmission in an "elliptic concentric line," a strip surrounded by an elliptic cylinder. Write out expressions for the lowest three modes of transmission, their cutoff frequencies, and their attenuation constants.

13.17 Work out graphically or numerically the value of k for the lowest elongational wave in a circular rod [see Eq. (13.3.34)] when $k_sa = 2$ and $k_ca = 1$. Plot the amplitudes of the r and z components of $\mathbf{s}$ as a function of r/a.

13.18 Calculate the torsional and elongational wave motion in an elastic medium filling the interior of a rigid, circular tube of inner radius a.

13.19 A sphere of radius a, originally at rest in an infinite viscous fluid, is suddenly started into rotation about the z axis at $t = 0$, with an angular velocity ω. Show that the steady-state motion of the fluid (for $t \to \infty$) is

$$\mathbf{v} = \mathbf{a}_\varphi \omega a(a/r)^2 \sin \vartheta; \quad r > a$$

Compute the transient motion of the fluid by the Laplace transform method.

13.20 Set down the expressions for the **M** and **N** standing waves for an electromagnetic resonator in the form of a hollow, circular "can" of inner radius a and distance between flat ends b. What are the expressions for the Q of the resonator for these various waves if the can walls are metal of conductivity σ? Write down the Green's dyadic for this resonator in forms similar to Eqs. (13.3.46) and (13.3.47).

13.21 A rectangular electromagnetic resonator is driven by a current loop consisting of a wire of square-cornered U shape, projecting out from the center of the $z = 0$ face carrying current $Ie^{-i\omega t}$. (The three parts of the wire inside the resonator are the line $x = \frac{1}{2}l_x - \frac{1}{2}\Delta$, $y = \frac{1}{2}l_y$, from $z = 0$ to $z = \Delta$, the line $y = \frac{1}{2}l_y$, $z = \Delta$ from $x = \frac{1}{2}l_x - \frac{1}{2}\Delta$ to $\frac{1}{2}l_x + \frac{1}{2}\Delta$, and the line $x = \frac{1}{2}l_x + \frac{1}{2}\Delta$, $y = \frac{1}{2}l_y$ from $z = \Delta$ to $z = 0$.) Write out the series for the field excited by the loop. What modes are missing? What are the resonance frequencies?

13.22 An electromagnetic resonator has eigenvector standing waves $\mathbf{F}_n(\mathbf{r})$ ($= \mathbf{M}_n$ or $\mathbf{N}_n$) which are solutions of

$$\text{curl curl } \mathbf{F}_n = k_n^2 \mathbf{F}_n$$

A small, conducting object, with surface S_0 is placed inside the resonator. Show that the integral equation for the perturbed wave which is closest in form to $\mathbf{F}_m$ is

$$\mathbf{A}(\mathbf{r}) = \mathbf{F}_m(\mathbf{r}) - \frac{1}{4\pi} \oint \mathfrak{G}_m(\mathbf{r}|\mathbf{r}_0^s) \cdot \mathbf{K}(\mathbf{r}_0^s)\, dS_0$$

for $\mathbf{r}$ on and outside S_0, inside the resonator. The dyadic $\mathfrak{G}_m$ is the series of Eq. (13.3.46), with the single term $\bar{\mathbf{F}}_m(r)\mathbf{F}_m(r_0)$ omitted, and the vector $\mathbf{K}$ is the surface current ($\mathbf{n} \times \text{curl } \mathbf{A}$) induced on S_0 by the perturbed standing wave $\mathbf{A}$. Show that a variational principle, from which the best form of $\mathbf{K}$ can be obtained (and thus $\mathbf{A}$ can be obtained by use of the integral equation), is

$$J = \frac{[\oint \mathbf{U}(\mathbf{r}^s) \cdot \mathbf{F}_m(\mathbf{r}^s)\, dS]^2}{\oint \oint \mathbf{U}(\mathbf{r}^s) \cdot \mathfrak{G}_m(\mathbf{r}^s|\mathbf{r}_0^s) \cdot \mathbf{U}(\mathbf{r}_0^s)\, dS\, dS_0}; \quad \delta J = 0$$

where the "best" form for $\mathbf{U}$ is proportional to the correct surface current $\mathbf{K}$ and the stationary value of J is equal to $\oint \mathbf{K} \cdot \mathbf{F}_m\, dS$. Show that this "best" value of J is approximately equal to the difference between the correct eigenvalue k^2 and k_n^2, $(k^2 - k_n^2)$, divided by the integral of $\mathbf{F}_n \cdot \mathbf{F}_n$ over the interior of the resonator.

13.23 In a rectangular resonator $(l_z > l_y > l_x)$ the lowest mode of electromagnetic oscillation is

$$\mathbf{M}_{011}(x,y,z) = -\mathbf{a}_x(\pi/l_y k_{011}) \sin(\pi y/l_y) \sin(\pi z/l_z)$$

A small conducting sphere of radius $a(a \ll l_x)$ is placed inside the resonator at the point (x_1,y_1,z_1). Use the variational principle of Prob. 13.22 to compute the perturbed frequency of oscillation of the resonator for its lowest mode. Use the trial function $\mathbf{U} = \mathbf{a}_\vartheta \sin \vartheta$, where r, ϑ, φ are spherical coordinates centered on the sphere, with polar axis parallel to the x axis. Compute also the approximate change in $\mathbf{A}$ caused by the sphere, by using the integral equation for $\mathbf{A}$.

13.24 A hydrogen atom, with wave functions ψ_{mln}, given in Eq. (12.3.40), makes a transition between state $m = 0$, $l = 1$, $n = 2$, and state $m = l = 0$, $n = 1$, giving up its excess energy into electromagnetic radiation. During the transition the effective oscillating charge and current are given by the formulas

$$\rho = \bar{\psi}_{001}\psi_{012} \exp[-i(E_2 - E_1)t/\hbar]$$
$$\mathbf{J} = (\hbar/M) \operatorname{Im}(\psi_{001} \operatorname{grad} \psi_{012}) \exp[-i(E_2 - E_1)t/\hbar]$$

Utilize Eqs. (13.3.83) and (13.3.87) to compute the effective dipole moment for the transition and the total rate of radiation of energy. [We can assume that the mean radius of the atom is much smaller than the wavelength, so we can use Eqs. (13.3.83.)] Compute the quadrupole moment for the transition from $m = 0$, $l = 2$, $n = 3$ to $m = 0$, $l = 0$, $n = 1$ and the corresponding rate of radiation. Is this smaller or larger than the rate for the $012 \to 001$ transition? What transition gives rise to a magnetic dipole?

13.25 Find the combination of $\mathbf{L}_{01}^1$ amd $\mathbf{N}_{01}^1$ [see Eq. (13.3.78)] for which the traction at the surface of the sphere $(r = a)$ is zero (*i.e.*, find the axially symmetric compressional waves in a free sphere). What are the allowed frequencies of the lowest three such modes? What are the modifications of shape of the sphere and its internal displacements for these same three modes?

13.26 A rigid sphere of mass M and radius a, embedded in an infinite elastic medium, is moved back and forth with displacement $\mathbf{a}_z De^{-i\omega t}$. Show that as long as the wavelength of the elastic waves produced is long compared to $2\pi a$, the displacement of the elastic medium at point r, ϑ, φ $(r > a)$ is

$$\mathbf{S} \simeq \frac{-3iDe^{-i\omega t}}{2(k_s a)^2 + (k_c a)^2} [(k_c a)^3 \mathbf{L}_{01}^3 + (k_s a)^3 \mathbf{N}_{01}^3]$$
$$\to \frac{-3De^{-i\omega t}a/r}{2(k_s a)^2 + (k_c a)^2} [\mathbf{a}_r (k_c a)^2 e^{ik_c r} \cos \vartheta - \mathbf{a}_\vartheta (k_s a)^2 e^{ik_s r} \sin \vartheta]; \quad r \to \infty$$

where $\mathbf{L}$ [given in Eq. (13.3.67)] uses $k = k_c = \omega\sqrt{\rho/(\lambda + 2\mu)}$ and $\mathbf{N}$ [given in Eq. (13.3.69)] uses $k = k_s = \omega\sqrt{\rho/\mu}$. Show also that the force required to displace the sphere by an amount $\mathbf{a}_z z$, with a velocity $\mathbf{a}_z \dot{z}$ and an acceleration $\mathbf{a}_z \ddot{z}$ is

$$\mathbf{F} = \mathbf{a}_z\{M\ddot{z} + 4\pi a z[3\mu(\lambda + 2\mu)/(2\lambda + 5\mu)]\}$$

as long as all the frequencies entering into the Fourier analysis of z are small compared with $(1/2\pi a)\sqrt{\mu/\rho}$.

13.27 A plane shear wave $\mathbf{s} = s_0\mathbf{a}_x \exp(ik_s z - i\omega t)$ is incident on the sphere of radius a discussed in Prob. 13.26. Show that the force on the sphere, caused by the incident wave, is $\frac{4}{3}\pi\mu k_s^2 a^3 s_0 \mathbf{a}_x$. Using the results of Prob. 13.26, calculate the amplitude of motion of the sphere and the asymptotic form for the scattered wave, for $k_s a \ll 1$. Show that the driving force caused by a plane compressional wave $\mathbf{s} = s_0\mathbf{a}_z \exp(ik_c z - i\omega t)$ is $\frac{4}{3}\pi(\lambda + 2\mu)k_c^2 a^3 s_0 \mathbf{a}_z$. Calculate the corresponding amplitude of motion and asymptotic form for the scattered wave. Does an incident shear wave produce a scattered compressional wave and vice versa? Why?

13.28 Set up the series giving the electromagnetic wave scattered by a dielectric sphere of index of refraction $n > 1$, the incident wave being a plane wave in the z direction with $\mathbf{E}$ pointing in the x direction. Compute the scattered fields and the scattering cross section for the radius of the sphere small compared to λ, the wavelength.

13.29 In Prob. 13.28 the results, for different polarizations of the incident plane wave, can be combined into a dyadic form

$$\mathfrak{A}(\mathbf{r}) = \mathfrak{J}_i \exp(i\mathbf{k}_i \cdot \mathbf{r}) + \mathfrak{A}_s(\mathbf{r}) \to \mathfrak{J}_i \exp(i\mathbf{k}_i \cdot \mathbf{r}) + \frac{e^{ikr}}{r}\mathfrak{F}(\mathbf{k}_s|\mathbf{k}_i); \quad r \to \infty$$

where $\mathfrak{J}_i = \mathfrak{J} - \mathbf{a}_i\mathbf{a}_i$, $\mathbf{a}_i$ being the unit vector in the direction of $\mathbf{k}_i(\mathbf{k}_i = k\mathbf{a}_i)$. The plane plus scattered wave for an incident wave $\mathbf{a}_e A_0 \cdot \exp(i\mathbf{k} \cdot \mathbf{r})$ is then obtained by multiplying both sides of the equation by $\mathbf{a}_e A_0$ (where $\mathbf{a}_e$ is a unit vector perpendicular to $\mathbf{a}_i$)$\mathfrak{A} \cdot \mathbf{a}_e A_0$ being then $\mathbf{A}$ and $\mathfrak{A}_s \cdot \mathbf{a}_e A_0$ the scattered wave. The dyadic $\mathfrak{F}$ in the asymptotic form is the generalization of the angle-distribution factor f of Eq. (11.4.58), the direction and amplitude of the scattered wave at large r being given by $\mathfrak{F} \cdot \mathbf{a}_e A_0$. On the other hand, $\mathbf{a}_s$ is the unit vector in the scattered direction, $\mathbf{k}_s = \mathbf{a}_s k$. Show that for a dielectric scattering object, of index of refraction n (different from unity only inside a region enclosed in a sphere of radius a), the integral equation for $\mathfrak{A}$ is

$$\mathfrak{A}(\mathbf{r}) = \mathfrak{J}_i \exp(i\mathbf{k}_i \cdot \mathbf{r}) + \frac{k^2}{4\pi}\int \mathfrak{G}_t(\mathbf{r}|\mathbf{r}_0) \cdot \mathfrak{A}(\mathbf{r}_0)U(\mathbf{r}_0)\, dv_0$$

where $U = n^2 - 1$, different from zero only inside the scatterer and where $\mathfrak{G}_t$ is the transverse Green's dyadic, suitable for electromagnetic fields. Show that $\mathfrak{F} = \left(\frac{k^2}{4\pi}\right) \int U(\mathbf{r}_0)\mathfrak{J}_s \cdot \mathfrak{A}(\mathbf{r}_0) \exp(-i\mathbf{k}_s \cdot \mathbf{r}_0)\, dv_0$ where $\mathfrak{J}_s = \mathfrak{J} - \mathbf{a}_s\mathbf{a}_s$. Show that a variational principle for $\mathfrak{F}$ may be written

$$\frac{4\pi}{k^2}[\mathfrak{F}] = \frac{\int U(\mathbf{r}_0)\mathfrak{J}_s \cdot \mathfrak{B}(\mathbf{r}_0)e^{-i\mathbf{k}_s\cdot\mathbf{r}_0}\, dv_0 \int |\tilde{\mathfrak{B}}(\mathbf{r}) \cdot \mathfrak{J}_i| U(\mathbf{r})e^{i\mathbf{k}_i\cdot\mathbf{r}}\, dv}{\int |\tilde{\mathfrak{B}} \cdot \mathfrak{B}| U\, dv - \left(\frac{k^2}{4\pi}\right)\iint U(r)|\tilde{\mathfrak{B}}(\mathbf{r}) \cdot \mathfrak{G}_t \cdot \mathfrak{B}(\mathbf{r}_0)| U(\mathbf{r}_0)\, dv\, dv_0};$$
$$\delta|\mathfrak{F}| = 0$$

where $|\mathfrak{B}|$ is the spur of dyadic $\mathfrak{B}$, etc. (see page 60), where $\mathfrak{B}$ is the trial function for $\mathfrak{A}$, and where $\tilde{\mathfrak{B}}$ is the trial function for the adjoint of $\mathfrak{A}$ (see page 1546).

13.30 Calculate the scattering dyadic $\mathfrak{F}(\mathbf{k}_s|\mathbf{k}_i)$ by the variational technique of Prob. 13.29 for a sphere of index of refraction n and radius a. Use the functions $\mathfrak{B} = \mathfrak{J}_i \exp(i\mathbf{k}_i \cdot \mathbf{r})$; $\tilde{\mathfrak{B}} = \mathfrak{J}_s \exp(i\mathbf{k}_s \cdot \mathbf{r})$ as trial functions. (These are both transverse, so $\mathfrak{G}$ may be used in the integral in the denominator instead of $\mathfrak{G}_t$, to simplify the calculations.) Work out the scattered fields, $\mathbf{E}_s$ and $\mathbf{H}_s$ and cross section Q for $\lambda \gg 2\pi a$, and compare them with the corresponding expressions for scattering from a conducting sphere [see Eq. (13.3.93)].

Table of Spherical Vector Harmonics

Solutions of the vector Laplace and Helmholtz equations in the spherical coordinates r, ϑ, φ involve amplitude functions of the radius, multiplied by the following three sets of vector functions of the angles ϑ and φ: **P**, **B**, and **C**. These functions are given in terms of the complex scalar spherical harmonics

$$\begin{aligned} X_n^m(\vartheta,\varphi) &= e^{im\varphi} P_n^m(\cos\vartheta); \quad m = 0, 1, 2, \ldots, n \\ &= [\cos(m\varphi) + i\sin(m\varphi)] \sin^m\vartheta\, T_{n-m}^m(\cos\varphi) \end{aligned}$$

The real part of the vector function will be denoted by the superscript e (for even) and the imaginary part will be denoted by the superscript o (for odd); that is, $\mathbf{P}_{mn} = \mathbf{P}_{mn}^e + i\mathbf{P}_{mn}^o$, etc. The functions are as follows:

$$\begin{aligned} \mathbf{P}_{mn}(\vartheta,\varphi) &= \mathbf{a}_r X_n^m(\vartheta,\varphi) \\ &= \frac{1}{2n+1}[r^{1-n}\operatorname{grad}(r^n X_n^m) - r^{n+2}\operatorname{grad}(r^{-n-1}X_n^m)] \\ &= \frac{1}{2n+1}\{\tfrac{1}{2}\mathbf{i}[(1-\delta_{0m})(n+m)(n+m-1)X_{n-1}^{m-1} - (1+\delta_{0m})X_{n-1}^{m+1} \\ &\quad - (1-\delta_{0m})(n-m+1)(n-m+2)X_{n+1}^{m-1} + (1+\delta_{0m})X_{n+1}^{m+1}] \\ &\quad + \tfrac{1}{2}\mathbf{j}[(1-\delta_{0m})(n+m)(n+m-1)iX_{n-1}^{m-1} + (1+\delta_{0m})iX_{n-1}^{m+1} \\ &\quad - (1-\delta_{0m})(n-m+1)(n-m+2)iX_{n+1}^{m-1} - (1+\delta_{0m})iX_{n+1}^{m+1}] \\ &\quad + \mathbf{k}[(n+m)X_{n-1}^m + (n-m+1)X_{n+1}^m]\} \end{aligned}$$

where $\delta_{mn} = 0(m \neq n) = 1(m = n)$.

$$\mathbf{B}_{mn}(\vartheta,\varphi) = \frac{r}{\sqrt{n(n+1)}} \operatorname{grad}[X_n^m(\vartheta,\varphi)] = \mathbf{a}_r \times \mathbf{C}_{mn}$$

$$= \frac{1}{2n+1}\left[\sqrt{\frac{n+1}{n}}\, r^{1-n} \operatorname{grad}(r^n X_n^m) + \sqrt{\frac{n}{n+1}}\, r^{n+2} \operatorname{grad}(r^{-n-1}X_n^m)\right]$$

$$= \frac{\sqrt{n(n+1)}}{(2n+1)\sin\vartheta}\left\{\mathbf{a}_\vartheta\left[\left(\frac{n-m+1}{n+1}\right)X_{n+1}^m - \left(\frac{n+m}{n}\right)X_{n-1}^m\right] + \mathbf{a}_\varphi \frac{m(2n+1)}{n(n+1)}\, iX_n^m\right\}$$

$$= \frac{\tfrac{1}{2}\mathbf{i}}{2n+1}\left\{\sqrt{\frac{n+1}{n}}\,[(1-\delta_{0m})(n+m)(n+m-1)X_{n-1}^{m-1} - (1+\delta_{0m})X_{n-1}^{m+1}] + \sqrt{\frac{n}{n+1}}\,[(1-\delta_{0m})(n-m+1)(n-m+2)X_{n+1}^{m-1} - (1+\delta_{0m})X_{n+1}^{m+1}]\right\}$$

$$+ \frac{\tfrac{1}{2}\mathbf{j}}{2n+1}\left\{\sqrt{\frac{n+1}{n}}\,[(1-\delta_{0m})(n+m)(n+m-1)iX_{n-1}^{m-1} + (1+\delta_{0m})iX_{n-1}^{m+1}] + \sqrt{\frac{n}{n+1}}\,[(1-\delta_{0m})(n-m+1)(n-m+2)iX_{n+1}^{m-1} + (1+\delta_{0m})iX_{n+1}^{m+1}]\right\}$$

$$+ \frac{\mathbf{k}}{2n+1}\left\{\sqrt{\frac{n+1}{n}}\,(n+m)X_{n-1}^m - \sqrt{\frac{n}{n+1}}\,(n-m+1)X_{n+1}^m\right\}$$

$$\mathbf{C}_{mn}(\vartheta,\varphi) = \frac{1}{\sqrt{n(n+1)}} \operatorname{curl}[\mathbf{r}X_n^m(\vartheta,\varphi)] = -\mathbf{a}_r \times \mathbf{B}_{mn}$$

$$= \frac{\sqrt{n(n+1)}}{(2n+1)\sin\vartheta}\left\{\mathbf{a}_\vartheta \frac{m(2n+1)}{n(n+1)}\, iX_n^m - \mathbf{a}_\varphi\left[\left(\frac{n-m+1}{n+1}\right)X_{n+1}^m - \left(\frac{n+m}{n}\right)X_{n-1}^m\right]\right\}$$

$$= \frac{1}{\sqrt{n(n+1)}}\{\tfrac{1}{2}\mathbf{i}[(1-\delta_{0m})(n+m)(n-m+1)iX_n^{m-1} + (1+\delta_{0m})iX_n^{m+1}] - \tfrac{1}{2}\mathbf{j}[(1-\delta_{0m})(n+m)(n-m+1)X_n^{m-1} - (1+\delta_{0m})X_n^{m+1}] - \mathbf{k}miX_n^m\}$$

$$\mathbf{P}_{mn}^s \cdot \mathbf{B}_{mn}^s = \mathbf{P}_{mn}^s \cdot \mathbf{C}_{mn}^s = \mathbf{B}_{mn}^s \cdot \mathbf{C}_{mn}^s = 0; \quad s = e, o$$

We note that $\mathbf{P}_{on} = \mathbf{P}_{on}^e$, etc.; only the real parts of the expressions in $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, are to be considered in the special case $m = 0$.

Special Cases:

$$\mathbf{P}_{00} = \mathbf{a}_r; \quad \mathbf{P}_{01} = \mathbf{a}_r \cos\vartheta; \quad \mathbf{P}_{11} = \mathbf{a}_r e^{i\varphi} \sin\vartheta$$
$$\mathbf{B}_{00} = 0; \quad \mathbf{B}_{01} = -\tfrac{1}{2}\sqrt{2}\,\mathbf{a}_\vartheta \sin\vartheta; \quad \mathbf{B}_{11} = \tfrac{1}{2}\sqrt{2}\, e^{i\varphi}(\mathbf{a}_\vartheta \cos\vartheta + i\mathbf{a}_\varphi)$$
$$\mathbf{C}_{00} = 0; \quad \mathbf{C}_{01} = \tfrac{1}{2}\sqrt{2}\,\mathbf{a}_\varphi \sin\vartheta; \quad \mathbf{C}_{11} = \tfrac{1}{2}\sqrt{2}\, e^{i\varphi}(i\mathbf{a}_\vartheta - \mathbf{a}_\varphi \cos\vartheta)$$

Zonal Vector Harmonics $(m = 0, n > 0)$:

$$\begin{aligned}
\mathbf{P}_{0n}(\vartheta,\varphi) &= \mathbf{a}_r P_n(\cos\vartheta) \\
&= \frac{1}{2n+1}\{[\mathbf{i}\cos\varphi + \mathbf{j}\sin\varphi][P^1_{n+1}(\cos\vartheta) - P^1_{n-1}(\cos\vartheta)] \\
&\qquad + \mathbf{k}[(n+1)P_{n+1}(\cos\vartheta) + nP_{n-1}(\cos\vartheta)]\} \\
&\to \mathbf{k}; \quad \vartheta \to 0
\end{aligned}$$

$$\begin{aligned}
\mathbf{B}_{0n}(\vartheta,\varphi) &= -\frac{1}{\sqrt{n(n+1)}}\,\mathbf{a}_\vartheta P^1_n(\cos\vartheta) \\
&= -\frac{1}{2n+1}\left\{\frac{\mathbf{i}\cos\varphi + \mathbf{j}\sin\varphi}{\sqrt{n(n+1)}}[nP^1_{n+1}(\cos\vartheta) + (n+1)P^1_{n-1}(\cos\vartheta)]\right. \\
&\qquad \left. + \mathbf{k}\sqrt{n(n+1)}\,[P_{n+1}(\cos\vartheta) - P_{n-1}(\cos\vartheta)]\right\}
\end{aligned}$$

$$\begin{aligned}
\mathbf{C}_{0n}(\vartheta,\varphi) &= \frac{1}{\sqrt{n(n+1)}}\,\mathbf{a}_\varphi P^1_n(\cos\vartheta) \\
&= \frac{1}{\sqrt{n(n+1)}}(-\mathbf{i}\sin\varphi + \mathbf{j}\cos\varphi)P^1_n(\cos\vartheta)
\end{aligned}$$

Integral and Differential Interrelations:

$$\iint \mathbf{P}^s_{mn}\cdot\mathbf{B}^\sigma_{\mu\nu}\,d\Omega = \iint \mathbf{P}^s_{mn}\cdot\mathbf{C}^\sigma_{\mu\nu}\,d\Omega = \iint \mathbf{B}^s_{mn}\cdot\mathbf{C}^\sigma_{\mu\nu}\,d\Omega = 0$$

$$\begin{aligned}
\iint \mathbf{P}^s_{mn}\cdot\mathbf{P}^\sigma_{\mu\nu}\,d\Omega &= \iint \mathbf{B}^s_{mn}\cdot\mathbf{B}^\sigma_{\mu\nu}\,d\Omega \\
&= \iint \mathbf{C}^s_{mn}\cdot\mathbf{C}^\sigma_{\mu\nu}\,d\Omega = \frac{(4\pi/\epsilon_m)}{2n+1}\frac{(n+m)!}{(n-m)!}\,\delta_{s\sigma}\delta_{m\mu}\delta_{n\nu}
\end{aligned}$$

where $d\Omega = \sin\vartheta\,d\vartheta\,d\varphi$ and the integration is over the ranges $0 \le \vartheta \le \pi$, $0 \le \varphi \le 2\pi$ $(s, \sigma = e, o;\ \nu, n = 1, 2, 3, \ldots;\ \mu, m = 0, 1, \ldots, n)$.

$$\begin{aligned}
&\operatorname{div}[r^{n+1}\mathbf{P}_{mn}(\vartheta,\varphi)] = (n+3)r^n X^m_n(\vartheta,\varphi) \\
&\operatorname{div}(r^{n-1}\mathbf{P}_{mn}) = (n+1)r^{n-2}X^m_n; \quad \operatorname{div}(r^{-n}\mathbf{P}_{mn}) = -(n-2)r^{-n-1}X^m_n \\
&\operatorname{div}(r^{-n-2}\mathbf{P}_{mn}) = -nr^{-n-3}X^m_n \\
&\operatorname{div}(r^{n+1}\mathbf{B}_{mn}) = -\sqrt{n(n+1)}\,r^n X^m_n \\
&\operatorname{div}(r^{n-1}\mathbf{B}_{mn}) = -\sqrt{n(n+1)}\,r^{n-2}X^m_n \\
&\operatorname{div}(r^{-n}\mathbf{B}_{mn}) = -\sqrt{n(n+1)}\,r^{-n-1}X^m_n \\
&\operatorname{div}(r^{-n-2}\mathbf{B}_{mn}) = -\sqrt{n(n+1)}\,r^{-n-3}X^m_n \\
&\operatorname{div}(r^{n+1}\mathbf{C}_{mn}) = \operatorname{div}(r^{n-1}\mathbf{C}_{mn}) = \operatorname{div}(r^{-n}\mathbf{C}_{mn}) = \operatorname{div}(r^{-n-2}\mathbf{C}_{mn}) = 0 \\
&\operatorname{curl}(r^{n+1}\mathbf{P}_{mn}) = \sqrt{n(n+1)}\,r^n\mathbf{C}_{mn}; \quad \operatorname{curl}(r^{n-1}\mathbf{P}_{mn}) = \sqrt{n(n+1)}\,r^{n-2}\mathbf{C}_{mn} \\
&\operatorname{curl}(r^{-n}\mathbf{P}_{mn}) = \sqrt{n(n+1)}\,r^{-n-1}\mathbf{C}_{mn}
\end{aligned}$$

$$\operatorname{curl}(r^{-n-2}\mathbf{P}_{mn}) = \sqrt{n(n+1)}\, r^{-n-3}\mathbf{C}_{mn}$$
$$\operatorname{curl}(r^{n+1}\mathbf{B}_{mn}) = -(n+2)r^{n}\mathbf{C}_{mn};\quad \operatorname{curl}(r^{n-1}\mathbf{B}_{mn}) = -nr^{n-2}\mathbf{C}_{mn}$$
$$\operatorname{curl}(r^{-n}\mathbf{B}_{mn}) = (n-1)r^{-n-1}\mathbf{C}_{mn};\quad \operatorname{curl}(r^{-n-2}\mathbf{B}_{mn}) = (n+1)r^{-n-3}\mathbf{C}_{mn}$$
$$\operatorname{curl}(r^{n}\mathbf{C}_{mn}) = (n+1)r^{n-1}\mathbf{B}_{mn} + \sqrt{n(n+1)}\, r^{n-1}\mathbf{P}_{mn}$$
$$\operatorname{curl}(r^{-n-1}\mathbf{C}_{mn}) = -nr^{-n-2}\mathbf{B}_{mn} + \sqrt{n(n+1)}\, r^{-n-2}\mathbf{P}_{mn}$$

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Appendix

Having finished our exposition of the techniques of calculating potentials, waves, and diffusion fields, scalar and vector, we have now only certain tabulations to append. With all the quantities which have been discussed, it is not surprising that we have had to make each symbol do multiple duty. For those quantities mentioned in only a single section we have used whatever symbol happens to be handiest, but for the quantities dealt with throughout the book we have attempted to be consistent in our notation. Duplication has been unavoidable, but we have tried to keep it from being confusing; to further reduce confusion we list, in the Glossary, each of the symbols used in several sections, with its definition and a reference to the equation or page where it is first defined, if this is appropriate. In some cases the symbolism used is at variance with other usage (or there are several different usages). These differences are pointed out in the Glossary, and a reference is given where the other notation is used or both notations are compared.

The purpose of all the calculation discussed in this work is to arrive at numbers, to check with or to predict measured values. It is thus appropriate to include numerical tables, so that the formulas derived can be converted into numbers. Naturally a selection had to be made; to have tables of all the functions discussed would have added an extra volume or so. The functions chosen are those most often encountered and are almost exclusively solutions of the sorts of differential equations discussed in Chap. 5; they are nearly all factors for a separated solution of one of the partial differential equations of physics. Trigonometric and hyperbolic functions have a singularity at infinity; they enter as axial factors in all rotational coordinates and as z factors in all cylindrical coordinates. The Legendre functions have singularities at ± 1 and ∞; they enter as angle factors for spherical coordinates and, for the Laplace equation, are factors for spheroidal, bispherical, and toroidal coordinates. Bessel functions, having singularities at zero and infinity, are used in spherical and circular cylindrical coordinates. Mathieu functions are used for solutions in elliptic cylinder coordinates. We have not included a table of Weber functions or of spheroidal wave functions, nor have we tabulated certain solutions of the Schroedinger equation; we did not believe these functions important enough to our main subject to merit the labor of tabulating these functions and others, which have a large number of parameters and would require extensive tables to be of actual use.

Glossary of Symbols Used

In general, italic or Greek type is used for scalar quantities (real or complex), boldface roman is used for vectors in ordinary space or in abstract space, German gothic type is used for linear vector operators (dyadics or abstract vector operators), special script is used for differential operators, and Hebrew characters for dyadic operators (tetradics).

The symbols $|\quad|$ in general mean "the magnitude of"; $|\mathbf{A}| = A$ is equal to the square root of the sum of the squares of the components of the vector $\mathbf{A}$; $|f|$ is the square root of the sum of the squares of the real and the imaginary parts of the complex number $f = u + iv$; $|\mathfrak{A}|$, the spur or expansion factor of operator $\mathfrak{A}$, is the sum of the diagonal elements of the matrix of $\mathfrak{A}$ (see, however, page 1021). The line over a symbol indicates the complex conjugate; $\bar{f} = u - iv$; the vector conjugate to $\mathbf{A}$ is $\bar{\mathbf{A}}$, having components $\bar{A}_n$ which are complex conjugates of the components A_n of $\mathbf{A}$. The operator $\mathfrak{A}$ has a conjugate $\bar{\mathfrak{A}}$, with matrix components which are the complex conjugates of those of $\mathfrak{A}$, has an adjoint $\tilde{\mathfrak{A}}$, with matrix reversed with respect to the main diagonal (that is, $\tilde{A}_{mn} = A_{nm}$), and has a Hermitian conjugate $\mathfrak{A}^* = \bar{\tilde{\mathfrak{A}}}$, which is the adjoint of its conjugate (that is, $A^*_{mn} = \bar{A}_{nm}$) (see pages 773 and 880). The adjoint $\tilde{\mathcal{A}}$ of a differential operator $\mathcal{A}$ is defined in Eqs. (5.2.11) and (7.5.4).

The symbol $\oint$ represents an integral over a closed path or a closed surface: $\oint f\,dz$ is an integral along a closed contour in the complex plane; $\oint \mathbf{F} \cdot d\mathbf{s}$ is a net circulation integral (see page 18); $\oint \mathbf{F} \cdot d\mathbf{A}$ is a net outflow integral (see page 16).

The list of symbols given here includes those used in more than a single section. The usual meaning or meanings are given, together with the equation or page number where the symbol is defined. Books having further formulas using these symbols, or using alternate symbolism, or comparing symbols, are listed here, together with the abbreviations used to refer to them in the Glossary:

JE—Jahnke, E., and F. Emde: "Tables of Functions," Dover, New York, 1945.

MO—Magnus, W., and F. Oberhettinger: "Special Functions of Mathematical Physics," Chelsea, New York, 1949.

BS—NBS, Computation Laboratory: "Tables Relating to Mathieu Functions," Columbia University Press, New York, 1951.
WW—Whittaker, E. T., and G. N. Watson: "Modern Analysis," Cambridge, New York, 1927.

Glossary

a Diffusion constant [(2.4.3)].
$\mathbf{a}$ Unit vector in ordinary space ($\mathbf{a}_r$ is the unit vector along r, etc.).
$\mathbf{A}$ Vector potential [page 202, (13.2.1)].
$d\mathbf{A}$ Vector element of surface area, outward-pointing if surface is a closed one (page 16)

$B(p,q) = [\Gamma(p)\Gamma(q)/\Gamma(p+q)]$ Beta function [(4.5.54)].
$\mathbf{B} = \text{curl } \mathbf{A}$ Magnetic induction [(2.5.3)].
$\mathbf{B}_{mn}(\vartheta,\varphi)$ Vector spherical harmonic of the second kind (page 1899).

c Velocity of light [(1.7.1)] or of general wave motion [(2.1.9)].
$\text{cn}(z,k)$ Elliptic function [(4.5.75)].
$\cos z$ Cosine function (page 1320).
$\cosh z$ Hyperbolic cosine function (page 1321).
C_m,C'_m Amplitude of Hankel function and its derivative (page 1564).
$\mathbf{C}_{mn}(\vartheta,\varphi)$ Vector spherical harmonic of the third kind (page 1899).

$\text{dn}(z,k)$ Elliptic function [(4.5.75)].
$D_m(z)$ Weber function [(11.2.63) and page 1565].
D_m,D'_m Amplitude of spherical Hankel function and its derivative (page 1575).
$\mathbf{D}$ Electric displacement [(2.5.1)].
$\mathfrak{D}$ Displacement dyadic [(1.6.20)].

$e = 2.71828 \ldots$ Base of natural logarithms or electric charge of elementary particle.
$= 4.80 \times 10^{-10}$ esu for electron.
E Energy.
$E_m^{(n)}(z)$ Semicylindrical function (page 1757).
$Ei(z)$ Exponential integral (page 434).
$E_n(x)$ Generalized exponential integral [(12.2.14)].
$\mathbf{e}$ Unit vector in abstract vector space [(1.6.31)].
$\mathbf{E}$ Electric intensity [(2.5.1)].

f Distribution function [page 175 and (12.2.1)].
$f_n(\xi_n)$ Separation factor for Laplacian operator [(5.1.32)].

$f(\vartheta)$ Angle-distribution factor (9.3.2), (11.4.58), and (12.3.56)].
$F(a|c|z)$ Confluent hypergeometric function [(5.2.59) and (5.3.47)] $= z^{-\frac{1}{2}c}e^{+\frac{1}{2}z}M_{\frac{1}{2}c-a,\ \frac{1}{2}c-\frac{1}{2}}(z)$ where $M_{k,m}(z)$ is defined in *WW*, Chap. 16.
$F(a,b|c|z)$ Hypergeometric series [(5.2.44) and (5.3.14)].
$\mathbf{F}$ Force.
$\mathfrak{F}$ Electromagnetic four-dyadic [(2.5.20)].

$g_k(\mathbf{r}|\mathbf{r}_0)$ Green's function for infinite domain [(7.2.18)].
$g(\mathbf{r}|\mathbf{r}_0|t)$ Impulse function, Laplace transform of Green's function [(7.3.8) and (11.1.15)].
$G_k(\mathbf{r}|\mathbf{r}_0)$ General Green's function [(2.1.7), (7.2.5), and (8.2.27)].
$G(a|c|z)$ Confluent hypergeometric function of second kind [(5.3.59)].
$\mathbf{G}_n$ Third eigenvector field (neither transverse nor longitudinal) for vector Laplace equation (13.2.13) (see $\mathbf{L}_n$ and $\mathbf{M}_n$).
$\mathfrak{G}(\mathbf{r}|\mathbf{r}_0)$ Dyadic Green's function for vector solutions [(13.1.8)].
$\mathcal{G}$ Factorized differential operator (page 730).

h_n Scale factor for curvilinear coordinates [(1.3.4)].
$\hbar = 1.054 \times 10^{-27}$ erg · sec. Planck's constant divided by 2π.
$h_n(z) = \sqrt{\pi/2z}\, H^{(1)}_{n+\frac{1}{2}}(z)$ Spherical Hankel function [(11.3.42)].
H Hamiltonian energy density [(3.2.7)].
$H_m(z)$ Hankel function (5.3.69), $H^{(1)}_m$ is the function $J_m + iN_m$ and $H^{(2)}_m = J_m - iN_m$. If no superscript is used, the function $H^{(1)}_m = J_m + iN_m$ is meant.
$H_m(x)$ Hermite polynomial (page 786).
$\mathbf{H}$ Magnetic intensity [(2.5.8) and (13.2.1)].
$\mathfrak{H}$ Hamiltonian operator [(2.6.27)].
$\mathcal{H}$ Hamiltonian differential operator (12.3.1).

$i = \sqrt{-1} = -j$ See Eq. (4.1.1).
i, j, k Unit quaternions (page 74).
$\mathrm{Im}\, f = v$, if $f = u + iv$ Imaginary part of f.
$I_n(x) = i^{-n}J_n(ix)$ Hyperbolic Bessel function (page 1323).
$\mathbf{i}, \mathbf{j}, \mathbf{k}$ Unit vectors along x, y, z, respectively [(1.2.1)].
$\mathfrak{I}$ Idemfactor. Unit dyadic (page 57).

$j = -i$ See page 128.
$j_n(x) = \sqrt{\pi/2x}\, J_{n+\frac{1}{2}}(x)$ Spherical Bessel function [(5.3.67), (11.3.42)].
$je_m(h,z)$ Spheroidal radial function [(5.3.96), (11.3.91)].
$J_n(x)$ Bessel function of the first kind [(5.3.63)].
$J^{(m)}_n(z)$ Semicylindrical function (page 1756).
$Je_m(h,z)$, $Jo_m(h,z)$ Radial Mathieu functions (5.3.90). For relations with other notations see BS, page xxxviii.

J Current density or mass flow vector [(2.4.1) and (13.2.1)].

$k = 2\pi/\lambda$ Wave number.
$K_n(z) = \frac{1}{2}\pi i e^{\frac{1}{2}\pi i n} H_n^{(1)}(iz)$ Hyperbolic Bessel function (page 1323).
$K(x|t)$ General kernel of integral representation (5.3.1) or of integral equation (8.1.16).
k Wave vector, magnitude equal to $2\pi/\lambda$, direction normal to wave fronts [(9.4.69)].

l $\begin{cases} \text{Length.} \\ \text{Quantum number for orbital angular momentum [(12.3.33)].} \end{cases}$
$\ln z$ Natural logarithm of z.
$L = T - V$ Lagrange density [(3.1.1)].
$L_n^a(x)$ Laguerre polynomial [page 784 and (12.3.35)].
$\mathbf{L}_n = \text{grad } \varphi_n$ Longitudinal eigenvector field [(13.1.14)].
$\mathfrak{L}$ Lagrange differential operator or Lagrange integral (3.1.1).

m $\begin{cases} \text{Mass of particle.} \\ \text{Quantum number for } z \text{ component of angular momentum (page 1661).} \end{cases}$
M $\begin{cases} \text{Mass of particle [(12.3.1)].} \\ \text{Mach number [(2.3.23)].} \end{cases}$
M_m^e, M_m^0 Normalizing factor for Mathieu function (page 1409).
M Angular momentum density (page 321).
$\mathbf{M}_n = \text{curl}(\mathbf{a}_1 w \psi_n)$ First transverse eigenvector field [(13.1.14)].
$\mathfrak{M}$ Angular momentum operator [(1.6.42)].
$\mathfrak{M}_t(K) = \mathfrak{L}_z(K)$ Related differential operator [(5.3.7)].

n $\begin{cases} \text{Any integer.} \\ \text{Radial quantum number (12.3.36) and (12.3.39)].} \end{cases}$
$n_m(x) = \sqrt{\pi/2x}\, N_{m+\frac{1}{2}}(x)$ Spherical Neumann function [(5.3.67),(11.3.42)].
$N_m(z)$ Neumann function (5.3.75). Equals the function $Y_m(z)$ used in WW, Chap. 17.
$Ne_m(h,z)$, $No_m(h,z)$ Radial Mathieu functions of second kind [(5.3.91)]. For relation to other notations see BS, page xxxviii.
$\mathbf{N}_n = (1/k)\, \text{curl curl}(\mathbf{a}_1 w \chi)$ Second transverse eigenvector field [(13.1.14)].
n Unit vector normal to surface (page 11).

p Pressure [(2.3.10)].
p_n Momentum, conjugate to coordinate q_n [(2.6.3)].
$P(u,v)$ Bilinear concomitant [(5.2.12)].
$P_n^m(z)$ Legendre function of first kind [(5.2.47) and (5.3.36)]. The function used here is $(-1)^m$ times the one defined in MO, page 60.

$P\begin{Bmatrix} a & b & c & \\ \lambda & \mu & \nu & w \\ \lambda' & \mu' & \nu' & \end{Bmatrix}$ Riemann symbol [(5.2.37)].

$\mathbf{p}$ Momentum vector.

$\mathbf{P}$ Field momentum vector [(3.4.5)].

$\mathbf{P}_{mn}(\vartheta,\varphi)$ Vector spherical harmonic of the first kind (page 1898).

$\mathfrak{p}$, $\mathfrak{P}$ Momentum operator [(1.6.42) and (1.6.32)].

$\mathcal{P}$ Differentiation operator [(6.3.67)]. / Principal value of an integral [(4.2.9)].

q Source density [(1.4.8)].

q_n Coordinate, conjugate to p_n [(2.6.3)].

Q Magnitude of charge [(2.5.25)]. / Total flow of fluid in cubic centimeters per second (page 1186). / Scattering cross section [(2.4.12), (11.3.72), and (11.4.64)].

$Q_n^m(z)$ Legendre function of the second kind (page 1327). Corresponds to $(-1)^m$ times the function defined in WW, page 325, to the $\mathfrak{Q}_n^m$ of MO page 60.

$\mathfrak{q}$ Position operator (page 235).

$\mathfrak{Q}$ Integration operator [(6.3.67)].

r Radius, distance from origin.

R Reflection coefficient [(9.3.10)]. / Resistance. / Distance from point (x,y,z) to point (x_0,y_0,z_0).

$\operatorname{Re} f = u$ if $f = u + iv$ Real part of f.

$\mathbf{r} = r\mathbf{a}_r = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$ Radius vector.

$\mathbf{R} = \mathbf{r} - \mathbf{r}_0$ Vector distance between (x,y,z) and (x_0,y_0,z_0) [(7.2.13)].

$\mathfrak{R}$ Rotation operator [(1.6.21)].

$\sin z$ Sine of z (page 1320).

$\sinh z$ Hyperbolic sine of z (page 1321).

$\operatorname{sn}(z,k)$ Elliptic function [(4.5.77)].

ds Element of length along a curve.

S Cross-sectional area. / Stäckel determinant [(5.1.25)].

dS Element of surface area.

$Se_m(h,z)$, $So_m(h,z)$ Periodic Mathieu functions [(5.2.73)]. Proportional to functions ce_m and se_m of WW, but normalized differently; for proportionality constant see BS, page xxxviii.

$S_{ml}(h,z)$ Spheroidal function [(5.3.95)].

$\mathbf{s}$ Displacement of elastic solid [(1.6.20) and (13.2.3)].

$d\mathbf{s}$ Element of integral along line in space.

$\mathbf{S}$ Energy flow (Poynting) vector [(2.2.20) and (3.3.15)].
$\mathfrak{S}$ Strain dyadic [(1.6.21)].
$\mathfrak{s}$ General Mathieu function [(5.2.72)].

t Time.
$\mathrm{tn}(z,k)$ Elliptic function (page 487).
T $\begin{cases}\text{Transmission coefficient [(9.3.11)].}\\ \text{Tension [(2.1.1)].}\\ \text{Kinetic energy [(3.2.1)].}\end{cases}$
$T(\mathbf{k}_s|\mathbf{k}_i) = -(4\pi)f(\vartheta)$ Angle-distribution factor for scattering of waves [(9.3.44) and (12.3.56)].
$T_n^m(z)$ Gegenbauer function [(5.2.52) and (5.3.35)].
$= (2^m/\sqrt{\pi})\Gamma(m + \frac{1}{2})C_n^{m+\frac{1}{2}}(z)$ where C_n^ν is the Gegenbauer polynomial discussed in WW, page 329, and in MO, page 76.
$\mathfrak{T}$ $\begin{cases}\text{Stress dyadic [page 70 and (13.2.4)].}\\ \text{Kinetic energy operator (page 243).}\end{cases}$

u, v, w Components of velocity.
$u(x) = 0$ and $x < 0$, $= 1$ when $x > 0$. Unit step function [(2.1.6)].
$U(a|c|z)$ Whittaker function; confluent hypergeometric functions of the third kind [(5.3.52)]. Related to the functions $W_{k,m}(z)$ of WW, page 339, by the relations,

$$W_{k,m}(z) = e^{-\frac{1}{2}z}z^{m+\frac{1}{2}}e^{i\pi(k-m-\frac{1}{2})}U_2(m - k + \tfrac{1}{2}|2m + 1|z)$$
$$W_{-k,m}(-z) = e^{-\frac{1}{2}z}z^{m+\frac{1}{2}}U_1(m - k + \tfrac{1}{2}|2m + 1|z)$$

$\mathfrak{U}$ $\begin{cases}\text{Stress part of stress-energy dyadic [(3.4.6)].}\\ \text{Rate of change of strain (page 158).}\end{cases}$

v Speed.
dv Volume element.
V $\begin{cases}\text{Voltage (page 219).}\\ \text{Potential energy [(3.2.2)].}\end{cases}$
$V_n^m(z)$ Gegenbauer function of the second kind [(5.3.41)].
$\mathbf{v}$ Velocity vector.

$w = \xi_1 + i\xi_2$ Generalized two-dimensional coordinates [(5.1.5)].
W Energy [(2.1.11)]. Energy density.
$\mathbf{w} = \frac{1}{2}$ curl $\mathbf{v}$ Vorticity vector [(2.3.3) and (13.2.9)].
$\mathfrak{W}$ Stress-energy dyadic [(3.3.7)].

x, y, z Rectangular coordinates.
X Reactance [(4.2.20)].
$\mathfrak{x}$ Quantum mechanical position operator [(1.6.32)].

$Y = 1/Z$ Admittance [(3.3.16)].
$\mathfrak{Y} = (\mathfrak{Z})^{-1}$ Admittance dyadic [(3.4.24)].

$z = x + iy$ Complex number.
z Acoustic impedance [(11.1.35)].
$Z = R + iX$ Impedance.
$\mathfrak{Z}$ Impedance dyadic [(3.4.24)].

α Angle in Lorentz transformation [(1.7.2)].
α, β, γ Direction cosines.
$\boldsymbol{\alpha}$ Dirac operator [(2.6.53)].

γ $\begin{cases} \text{Second viscosity coefficient [(2.3.10)].} \\ \text{Ratio of specific heats [(2.3.22)].} \end{cases}$
$\Gamma(x)$ Gamma function [(4.5.29)].

δ Variation symbol [(3.1.2)].
δ_m Phase angle of Hankel function (pages 1564 and 1575).
$\delta_{mn} = 0$ when $m \neq n$, $= 1$ when $m = n$ Kronecker delta function [(1.3.1)].
$\delta(x)$ Dirac delta function [(2.1.4), (2.6.23), and (6.3.58)].
∂ Partial derivative symbol.
Δ Determinant.
$\Delta(y_1,y_2) = y_1y_2' - y_2y_1'$ Wronskian of the solutions y_1 and y_2 [(5.2.2)].

ϵ Dielectric constant [(2.5.1)].
$\epsilon_m = 1$ when $m = 0$, $= 2$ when $m > 0$ Neumann factor [(10.1.13)].

η Coefficient of viscosity [(2.3.10) and (13.2.5)].

ϑ Spherical angle.
$\vartheta_n(u,q)$ $(n = 1, 2, 3, 4)$ Theta function [(4.5.68)].

$\lambda = c/\nu$ Wavelength.
λ Modulus of elasticity [(1.6.28) and (13.2.3)].
λ_n Eigenvalue for integral equation [(8.2.24)].
Λ Normalization constant.

μ $\begin{cases} \text{Permeability [(2.5.5) and (13.2.1)].} \\ \text{Shear modulus [(1.6.28) and (13.2.3)].} \end{cases}$

$\nu = \omega/2\pi = c/\lambda$ Frequency [(2.1.10)].

ξ Curvilinear coordinate [(1.3.4)].

$\pi = 3.14159\ .\ .\ .$ Ratio circumference to diameter of circle.

Π Product symbol. $\prod_{n=0}^{N} f_n = f_0 \cdot f_1 \cdot f_2 \cdot \cdot \cdot f_{N-1} \cdot f_N$.

ρ Density [(2.4.6) and (2.5.1)].

$\boldsymbol{\rho}$ Dirac operator [(2.6.53)].

σ $\begin{cases} \text{Conductivity [(2.5.19)].} \\ \text{Real part of variable } k \text{ in Fourier transform (page 460).} \end{cases}$

$\boldsymbol{\sigma}$ Spin operator [(1.7.17)].

$\sum$ Summation symbol. $\sum_{n=0}^{N} f_n = f_0 + f_1 + f_2 + \cdot \cdot \cdot + f_{N-1} + f_N$.

τ $\begin{cases} \text{Proper time [(1.7.1)].} \\ \text{Imaginary part of } k \text{ in Fourier transform [(4.8.18) and (8.5.13)].} \end{cases}$

Υ Angle factor for two-particle wave function [(12.3.81)].

ϕ Cylindrical angle.

φ $\begin{cases} \text{Spherical angle.} \\ \text{Phase angle of complex number.} \end{cases}$

φ_n Eigenfunction for unperturbed state [(9.1.2)].

Φ_{mn} Element of Stäckel determinant [(5.1.25)].

ψ Scalar field, wave function.

$\psi_m(z)$ Polygamma function [(4.5.46)].

ψ_n Eigenfunction [(6.3.16)].

$\Psi = \psi e^{-iEt/h}$ Time-dependent wave function [(12.3.1)].

$\omega = 2\pi\nu$ Angular frequency [(2.1.10)].

$\boldsymbol{\omega}$ Rotation vector [(1.4.4)].

∇ Gradient operator, pronounced "del" [(1.4.1)].

∇^2 Laplacian operator [(1.1.4)].

$\square$ Four-dimensional gradient, pronounced "quad."

$\square^2$ d'Alembertian operator [(1.7.6)].

$\times$ Vector multiplication of two vectors [(1.2.3)].

$\cdot$ Scalar multiplication of two vectors [(1.2.2)]. Also ordinary multiplication sign.

$\left\{ {i \atop j\ k} \right\}$ Christoffel symbols [(1.5.4)].

$[\lambda]$ Function of one or more variational parameters, the stationary value of which is the eigenvalue λ [(9.4.6) and (11.4.6)].

$[n,m]$ Matrix components $\int \varphi_n M \varphi_m \, dv$ where φ_n and φ_m are nth and mth approximations to a correct solution ψ [(9.4.96)].

$|A|$ Magnitude of A (see page 350).

$|A_{nm}|$ Determinant with elements A_{nm}.

$$\binom{n}{m} = \begin{cases} [\Gamma(n+1)/m!\Gamma(n-m+1)]; & n > 0 \\ [(-1)^m\Gamma(m-n)/m!\Gamma(-n)]; & n < 0 \end{cases}$$

Coefficients in the binomial expansion $(1 + x)^n = \sum_m \binom{n}{m} x^m$.

ג Gimel, ד Daleth $\Big\}$ general tetradics [(1.6.29)].

י Yod, unity tetradic (page 72).

ע Ayin, diagonalizing tetradic (page 72).

Table I. Trigonometric and Hyperbolic Functions

(See page 1320)

x	sin x	cos x	tan x	sinh x	cosh x	tanh x	e^x	e^{-x}
0.0	0.0000	1.0000	0.0000	0.0000	1.0000	0.0000	1.0000	1.0000
0.2	0.1987	0.9801	0.2027	0.2013	1.0201	0.1974	1.2214	0.8187
0.4	0.3894	0.9211	0.4228	0.4108	1.0811	0.3799	1.4918	0.6703
0.6	0.5646	0.8253	0.6841	0.6367	1.1855	0.5370	1.8221	0.5488
0.8	0.7174	0.6967	1.0296	0.8881	1.3374	0.6640	2.2255	0.4493
1.0	0.8415	0.5403	1.5574	1.1752	1.5431	0.7616	2.7183	0.3679
1.2	0.9320	0.3624	2.5722	1.5095	1.8106	0.8337	3.3201	0.3012
1.4	0.9854	+0.1700	+5.7979	1.9043	2.1509	0.8854	4.0552	0.2466
1.6	0.9996	−0.0292	−34.233	2.3756	2.5775	0.9217	4.9530	0.2019
1.8	0.9738	−0.2272	−4.2863	2.9422	3.1075	0.9468	6.0496	0.1653
2.0	0.9093	−0.4161	−2.1850	3.6269	3.7622	0.9640	7.3891	0.1353
2.2	0.8085	−0.5885	−1.3738	4.4571	4.5679	0.9757	9.0250	0.1108
2.4	0.6755	−0.7374	−0.9160	5.4662	5.5569	0.9837	11.023	0.0907
2.6	0.5155	−0.8569	−0.6016	6.6947	6.7690	0.9890	13.464	0.0742
2.8	0.3350	−0.9422	−0.3555	8.1919	8.2527	0.9926	16.445	0.0608
3.0	+0.1411	−0.9900	−0.1425	10.018	10.068	0.9951	20.086	0.0498
3.2	−0.0584	−0.9983	+0.0585	12.246	12.287	0.9967	24.533	0.0407
3.4	−0.2555	−0.9668	0.2643	14.965	14.999	0.9978	29.964	0.0331
3.6	−0.4425	−0.8968	0.4935	18.285	18.313	0.9985	36.598	0.0273
3.8	−0.6119	−0.7910	0.7736	22.339	22.362	0.9990	44.701	0.0223
4.0	−0.7568	−0.6536	1.1578	27.290	27.308	0.9993	54.598	0.0183
4.2	−0.8716	−0.4903	1.7778	33.335	33.351	0.9996	66.686	0.0150
4.4	−0.9516	−0.3073	3.0963	40.719	40.732	0.9997	81.451	0.0123
4.6	−0.9937	−0.1122	+8.8602	49.737	49.747	0.9998	99.484	0.0100
4.8	−0.9962	+0.0875	−11.385	60.751	60.759	0.9999	121.51	0.0082
5.0	−0.9589	0.2837	−3.3805	74.203	74.210	0.9999	148.41	0.0067
5.2	−0.8835	0.4685	−1.8856	90.633	90.639	0.9999	181.27	0.0055
5.4	−0.7728	0.6347	−1.2175	110.70	110.71	1.0000	221.41	0.0045
5.6	−0.6313	0.7756	−0.8139	135.21	135.22	1.0000	270.43	0.0037
5.8	−0.4646	0.8855	−0.5247	165.15	165.15	1.0000	330.30	0.0030
6.0	−0.2794	0.9602	−0.2910	201.71	201.71	1.0000	403.43	0.0025
6.2	−0.0831	0.9965	−0.0834	246.37	246.37	1.0000	492.75	0.0020
6.4	+0.1165	0.9932	+0.1173	300.92	300.92	1.0000	601.85	0.0016
6.6	0.3115	0.9502	0.3279	367.55	367.55	1.0000	735.10	0.0013
6.8	0.4941	0.8694	0.5683	448.92	448.92	1.0000	897.85	0.0011
7.0	0.6570	0.7539	0.8714	548.32	548.32	1.0000	1096.6	0.0009
7.2	0.7937	0.6084	1.3046	669.72	669.72	1.0000	1339.4	0.0007
7.4	0.8987	0.4385	2.0493	817.99	817.99	1.0000	1636.0	0.0006
7.6	0.9679	0.2513	3.8523	999.10	999.10	1.0000	1998.2	0.0005
7.8	0.9985	+0.0540	+18.507	1220.3	1220.3	1.0000	2440.6	0.0004
8.0	0.9894	−0.1455	−6.7997	1490.5	1490.5	1.0000	2981.0	0.0003

Table II. Trigonometric and Hyperbolic Functions

(See page 1320)

x	$\sin \pi x$	$\cos \pi x$	$\tan \pi x$	$\sinh \pi x$	$\cosh \pi x$	$\tanh \pi x$	$e^{\pi x}$	$e^{-\pi x}$
0.00	0.0000	1.0000	0.0000	0.0000	1.0000	0.0000	1.0000	1.0000
0.05	0.1564	0.9877	0.1584	0.1577	1.0124	0.1558	1.1701	0.8546
0.10	0.3090	0.9511	0.3249	0.3194	1.0498	0.3042	1.3691	0.7304
0.15	0.4540	0.8910	0.5095	0.4889	1.1131	0.4392	1.6019	0.6242
0.20	0.5878	0.8090	0.7265	0.6705	1.2040	0.5569	1.8745	0.5335
0.25	0.7071	0.7071	1.0000	0.8687	1.3246	0.6558	2.1933	0.4559
0.30	0.8090	0.5878	1.3764	1.0883	1.4780	0.7363	2.5663	0.3897
0.35	0.8910	0.4540	1.9626	1.3349	1.6679	0.8003	3.0028	0.3330
0.40	0.9511	0.3090	3.0777	1.6145	1.8991	0.8502	3.5136	0.2846
0.45	0.9877	+0.1564	6.3137	1.9340	2.1772	0.8883	4.1111	0.2432
0.50	1.0000	0.0000	∞	2.3013	2.5092	0.9171	4.8105	0.2079
0.55	0.9877	−0.1564	−6.3137	2.7255	2.9032	0.9388	5.6287	0.1777
0.60	0.9511	−0.3090	−3.0777	3.2171	3.3689	0.9549	6.5861	0.1518
0.65	0.8910	−0.4540	−1.9626	3.7883	3.9180	0.9669	7.7062	0.1298
0.70	0.8090	−0.5878	−1.3764	4.4531	4.5640	0.9757	9.0170	0.1109
0.75	0.7071	−0.7071	−1.0000	5.2280	5.3228	0.9822	10.551	0.09478
0.80	0.5878	−0.8090	−0.7265	6.1321	6.2131	0.9870	12.345	0.08100
0.85	0.4540	−0.8910	−0.5095	7.1879	7.2572	0.9905	14.445	0.06922
0.90	0.3090	−0.9511	−0.3249	8.4214	8.4806	0.9930	16.902	0.05916
0.95	+0.1564	−0.9877	−0.1584	9.9632	9.8137	0.9949	19.777	0.05056
1.00	0.0000	−1.0000	0.0000	11.549	11.592	0.9962	23.141	0.04321
1.05	−0.1564	−0.9877	0.1584	13.520	13.557	0.9973	27.077	0.03693
1.10	−0.3090	−0.9511	0.3249	15.825	15.857	0.9980	31.682	0.03156
1.15	−0.4540	−0.8910	0.5095	18.522	18.549	0.9985	37.070	0.02697
1.20	−0.5878	−0.8090	0.7265	21.677	21.700	0.9989	43.376	0.02305
1.25	−0.7071	−0.7071	1.0000	25.367	25.387	0.9992	50.753	0.01970
1.30	−0.8090	−0.5878	1.3764	29.685	29.702	0.9994	59.387	0.01683
1.35	−0.8910	−0.4540	1.9626	34.737	34.751	0.9996	69.488	0.01438
1.40	−0.9511	−0.3090	3.0777	40.647	40.660	0.9997	81.307	0.01230
1.45	−0.9877	−0.1564	6.3137	47.563	47.573	0.9998	95.137	0.01051
1.50	−1.0000	0.0000	∞	55.654	55.663	0.9998	111.32	0.00898
1.55	−0.9877	+0.1564	−6.3137	65.122	65.130	0.9999	130.25	0.00767
1.60	−0.9511	0.3090	−3.0777	76.200	76.206	0.9999	152.41	0.00656
1.65	−0.8910	0.4540	−1.9626	89.161	89.167	0.9999	178.33	0.00561
1.70	−0.8090	0.5878	−1.3764	104.32	104.33	1.0000	208.66	0.00479
1.75	−0.7071	0.7071	−1.0000	122.07	122.08	1.0000	244.15	0.00409
1.80	−0.5878	0.8090	−0.7265	142.84	142.84	1.0000	285.68	0.00350
1.85	−0.4540	0.8910	−0.5095	167.13	167.13	1.0000	334.27	0.00299
1.90	−0.3090	0.9511	−0.3249	195.56	195.56	1.0000	391.12	0.00256
1.95	−0.1564	0.9877	−0.1584	228.82	228.82	1.0000	457.65	0.00219
2.00	0.0000	1.0000	0.0000	267.75	267.75	1.0000	535.49	0.00187

Table III. Hyperbolic Tangent of Complex Quantity

$$\tanh\,[\pi(\alpha - i\beta)] = \theta - i\chi = |\zeta|e^{-i\varphi}$$

α	$\tanh \pi\alpha$	θ	χ	$\lvert\zeta\rvert$	φ	θ	χ	$\lvert\zeta\rvert$	φ
		$\beta = 0.00$				$\beta = 0.05$			
0.0000	0.00	0.0000	0.0000	0.0000	0–90°	0.0000	0.1584	0.1584	90.00°
0.0159	0.05	0.0500	0.0000	0.0500	0.00	0.0512	0.1580	0.1660	72.03
0.0319	0.10	0.1000	0.0000	0.1000	0.00	0.1025	0.1567	0.1872	56.82
0.0481	0.15	0.1500	0.0000	0.1500	0.00	0.1537	0.1547	0.2180	45.20
0.0645	0.20	0.2000	0.0000	0.2000	0.00	0.2048	0.1519	0.2549	36.56
0.0813	0.25	0.2500	0.0000	0.2500	0.00	0.2558	0.1482	0.2956	30.09
0.0985	0.30	0.3000	0.0000	0.3000	0.00	0.3068	0.1438	0.3388	25.12
0.1163	0.35	0.3500	0.0000	0.3500	0.00	0.3577	0.1386	0.3836	21.18
0.1349	0.40	0.4000	0.0000	0.4000	0.00	0.4084	0.1325	0.4293	17.98
0.1543	0.45	0.4500	0.0000	0.4500	0.00	0.4589	0.1256	0.4758	15.32
0.1748	0.50	0.5000	0.0000	0.5000	0.00	0.5093	0.1181	0.5228	13.05
0.1968	0.55	0.5500	0.0000	0.5500	0.00	0.5596	0.1096	0.5702	11.08
0.2207	0.60	0.6000	0.0000	0.6000	0.00	0.6095	0.1005	0.6177	9.36
0.2468	0.65	0.6500	0.0000	0.6500	0.00	0.6593	0.0905	0.6654	7.82
0.2761	0.70	0.7000	0.0000	0.7000	0.00	0.7088	0.0798	0.7133	6.43
0.3097	0.75	0.7500	0.0000	0.7500	0.00	0.7581	0.0683	0.7612	5.15
0.3497	0.80	0.8000	0.0000	0.8000	0.00	0.8070	0.0561	0.8090	3.97
0.3999	0.85	0.8500	0.0000	0.8500	0.00	0.8558	0.0432	0.8569	2.88
0.4686	0.90	0.9000	0.0000	0.9000	0.00	0.9041	0.0295	0.9047	1.87
0.5831	0.95	0.9500	0.0000	0.9500	0.00	0.9523	0.0151	0.9524	0.91
		$\beta = 0.10$				$\beta = 0.15$			
0.0000	0.00	0.0000	0.3249	0.3249	90.00°	0.0000	0.5095	0.5095	90.00°
0.0159	0.05	0.0553	0.3240	0.3286	80.32	0.0629	0.5079	0.5118	82.93
0.0319	0.10	0.1104	0.3213	0.3398	71.03	0.1256	0.5031	0.5186	75.98
0.0481	0.15	0.1655	0.3169	0.3575	62.43	0.1878	0.4951	0.5296	69.22
0.0645	0.20	0.2202	0.3106	0.3808	54.67	0.2493	0.4841	0.5445	62.75
0.0813	0.25	0.2746	0.3027	0.4087	47.78	0.3099	0.4700	0.5629	56.61
0.0985	0.30	0.3286	0.2929	0.4402	41.72	0.3692	0.4531	0.5845	50.82
0.1163	0.35	0.3820	0.2815	0.4745	36.39	0.4273	0.4333	0.6085	45.40
0.1349	0.40	0.4349	0.2684	0.5110	31.68	0.4838	0.4110	0.6347	40.35
0.1533	0.45	0.4871	0.2537	0.5492	27.51	0.5385	0.3860	0.6626	35.63
0.1748	0.50	0.5386	0.2374	0.5886	23.79	0.5914	0.3589	0.6917	31.25
0.1968	0.55	0.5893	0.2196	0.6289	20.44	0.6423	0.3295	0.7219	27.16
0.2207	0.60	0.6390	0.2003	0.6697	17.41	0.6911	0.2982	0.7527	23.34
0.2468	0.65	0.6880	0.1796	0.7110	14.63	0.7378	0.2652	0.7840	19.76
0.2761	0.70	0.7358	0.1576	0.7525	12.08	0.7822	0.2305	0.8155	16.43
0.3097	0.75	0.7827	0.1342	0.7941	9.73	0.8243	0.1945	0.8469	13.28
0.3497	0.80	0.8285	0.1096	0.8357	7.54	0.8642	0.1573	0.8783	10.32
0.3999	0.85	0.8731	0.0838	0.8771	5.48	0.9015	0.1191	0.9094	7.52
0.4686	0.90	0.9166	0.0569	0.9184	3.56	0.9367	0.0800	0.9401	4.88
0.5831	0.95	0.9589	0.0289	0.9594	1.73	0.9695	0.0403	0.9704	2.38

Table III. Hyperbolic Tangent of Complex Quantity.—(*Continued*)

α	$\tanh \pi\alpha$	θ	χ	$\lvert\zeta\rvert$	φ	θ	χ	$\lvert\zeta\rvert$	φ
		$\beta = 0.20$				$\beta = 0.25$			
0.0000	0.00	0.0000	0.7265	0.7265	90.00°	0.0000	1.0000	1.0000	90.00°
0.0159	0.05	0.0763	0.7238	0.7278	83.98	0.0998	0.9950	1.0000	84.28
0.0319	0.10	0.1520	0.7155	0.7315	78.01	0.1980	0.9802	1.0000	78.58
0.0481	0.15	0.2265	0.7019	0.7375	72.12	0.2934	0.9560	1.0000	72.93
0.0645	0.20	0.2993	0.6831	0.7458	66.34	0.3846	0.9230	1.0000	67.38
0.0813	0.25	0.3698	0.6593	0.7560	60.72	0.4706	0.8824	1.0000	61.93
0.0985	0.30	0.4376	0.6312	0.7680	55.27	0.5504	0.8348	1.0000	56.60
0.1163	0.35	0.5023	0.5989	0.7816	50.01	0.6236	0.7818	1.0000	51.42
0.1349	0.40	0.5635	0.5627	0.7964	44.96	0.6896	0.7241	1.0000	46.40
0.1543	0.45	0.6212	0.5235	0.8123	40.12	0.7484	0.6632	1.0000	41.55
0.1748	0.50	0.6749	0.4814	0.8290	35.50	0.8000	0.6000	1.0000	36.87
0.1968	0.55	0.7247	0.4370	0.8462	31.09	0.8446	0.5355	1.0000	32.38
0.2207	0.60	0.7703	0.3907	0.8637	26.89	0.8824	0.4706	1.0000	28.07
0.2468	0.65	0.8120	0.3430	0.8815	22.91	0.9139	0.4060	1.0000	23.95
0.2761	0.70	0.8497	0.2943	0.8992	19.11	0.9395	0.3423	1.0000	20.02
0.3097	0.75	0.8835	0.2451	0.9169	15.51	0.9600	0.2800	1.0000	16.27
0.3497	0.80	0.9136	0.1955	0.9343	12.08	0.9757	0.2195	1.0000	12.68
0.3999	0.85	0.9401	0.1459	0.9514	8.82	0.9869	0.1611	1.0000	9.27
0.4686	0.90	0.9632	0.0967	0.9681	5.73	0.9945	0.1050	1.0000	6.03
0.5831	0.95	0.9831	0.0480	0.9843	2.79	0.9986	0.0512	1.0000	2.93
		$\beta = 0.30$				$\beta = 0.35$			
0.0000	0.00	0.0000	1.3764	1.3764	90.00°	0.0000	1.9626	1.9626	90.00°
0.0159	0.05	0.1440	1.3664	1.3740	83.98	0.2403	1.9391	1.9539	82.93
0.0319	0.10	0.2841	1.3373	1.3671	78.01	0.4672	1.8708	1.9283	75.98
0.0481	0.15	0.4164	1.2904	1.3559	72.12	0.6697	1.7653	1.8882	69.22
0.0645	0.20	0.5382	1.2282	1.3410	66.34	0.8408	1.6326	1.8365	62.75
0.0813	0.25	0.6470	1.1537	1.3228	60.72	0.9776	1.4829	1.7762	56.61
0.0985	0.30	0.7419	1.0701	1.3021	55.27	1.0809	1.3261	1.7109	50.82
0.1163	0.35	0.8223	0.9803	1.2794	50.01	1.1538	1.1701	1.6432	45.40
0.1349	0.40	0.8885	0.8873	1.2556	44.96	1.2007	1.0200	1.5755	40.35
0.1543	0.45	0.9413	0.7933	1.2311	40.12	0.2267	0.8793	1.5092	35.63
0.1748	0.50	0.9820	0.7005	1.2063	35.50	1.2359	0.7500	1.4457	31.25
0.1968	0.55	1.0120	0.6102	1.1819	31.09	1.2324	0.6323	1.3852	27.16
0.2207	0.60	1.0326	0.5237	1.1578	26.89	1.2198	0.5263	1.3284	23.34
0.2468	0.65	1.0449	0.4415	1.1344	22.91	1.2002	0.4313	1.2755	19.76
0.2761	0.70	1.0507	0.3640	1.1121	19.11	1.1762	0.3467	1.2262	16.43
0.3097	0.75	1.0509	0.2916	1.0906	15.51	1.1493	0.2713	1.1808	13.28
0.3497	0.80	1.0465	0.2240	1.0703	12.08	1.1202	0.2039	1.1386	10.32
0.3999	0.85	1.0387	0.1612	1.0511	8.82	1.0902	0.1440	1.0996	7.52
0.4686	0.90	1.0278	0.1032	1.0330	5.73	1.0599	0.0905	1.0637	4.88
0.5831	0.95	1.0148	0.0494	1.0160	2.79	1.0296	0.0428	1.0305	2.38

Table III. Hyperbolic Tangent of Complex Quantity.—(*Continued*)

α	tanh $\pi\alpha$	θ	χ	$\|\zeta\|$	φ	θ	χ	$\|\zeta\|$	φ
		β = 0.40				β = 0.45			
0.0000	0.00	0.0000	3.0777	3.0777	90.00°	0.0000	6.3138	6.3138	90.00°
0.0159	0.05	0.5115	2.9990	3.0423	80.32	1.8580	5.7272	6.0211	72.03
0.0319	0.10	0.9565	2.7833	2.9431	71.03	2.9217	4.4691	5.3394	56.82
0.0481	0.15	1.2948	2.4799	2.7976	62.43	3.2313	3.2535	4.5855	45.20
0.0645	0.20	1.5189	2.1427	2.6265	54.67	3.1500	2.3362	3.9217	36.56
0.0813	0.25	1.6444	1.8124	2.4473	47.78	2.9260	1.6953	3.3816	30.09
0.0985	0.30	1.6960	1.5119	2.2720	41.72	2.6722	1.2524	2.9511	25.12
0.1163	0.35	1.6966	1.2501	2.1074	36.39	2.4310	0.9417	2.6070	21.18
0.1349	0.40	1.6652	1.0277	1.9569	31.68	2.2154	0.7188	2.3291	17.98
0.1543	0.45	1.6149	0.8411	1.8208	27.51	2.0269	0.5550	2.1015	15.32
0.1748	0.50	1.5547	0.6853	1.6989	23.79	1.8633	0.4319	1.9126	13.05
0.1968	0.55	1.4901	0.5554	1.5901	20.44	1.7210	0.3371	1.7538	11.08
0.2207	0.60	1.4247	0.4466	1.4932	17.41	1.5972	0.2632	1.6187	9.36
0.2468	0.65	1.3609	0.3553	1.4065	14.63	1.4887	0.2044	1.5026	7.82
0.2761	0.70	1.2994	0.2781	1.3289	12.08	1.3931	0.1568	1.4019	6.43
0.3097	0.75	1.2412	0.2129	1.2593	9.73	1.3083	0.1180	1.3137	5.15
0.3497	0.80	1.1862	0.1569	1.1966	7.54	1.2331	0.0857	1.2361	3.97
0.3999	0.85	1.1349	0.1090	1.1401	5.48	1.1655	0.0587	1.1670	2.88
0.4686	0.90	1.0867	0.0674	1.0888	3.56	1.1047	0.0360	1.1053	1.87
0.5831	0.95	1.0418	0.0314	1.0423	1.73	1.0499	0.0167	1.0500	0.91
		β = 0.475				β = 0.50			
0.0000	0.00	0.0000	12.706	12.706	90.00°	∞	0.0000	∞	0–90°
0.0159	0.05	5.787	9.030	10.725	57.35	20.000	0.0000	20.000	0.00
0.0319	0.10	6.213	4.811	7.859	37.75	10.000	0.0000	10.000	0.00
0.0481	0.15	5.260	2.682	5.905	27.01	6.6667	0.0000	6.6667	0.00
0.0645	0.20	4.356	1.636	4.653	20.58	5.0000	0.0000	5.0000	0.00
0.0813	0.25	3.662	1.075	3.817	16.35	4.0000	0.0000	4.0000	0.00
0.0985	0.30	3.138	0.7445	3.225	13.35	3.3333	0.0000	3.3333	0.00
0.1163	0.35	2.736	0.5367	2.789	11.10	2.8571	0.0000	2.8571	0.00
0.1349	0.40	2.422	0.3979	2.454	9.33	2.5000	0.0000	2.5000	0.00
0.1543	0.45	2.170	0.3008	2.190	7.90	2.2222	0.0000	2.2222	0.00
0.1748	0.50	1.9638	0.2304	1.9773	6.70	2.0000	0.0000	2.0000	0.00
0.1968	0.55	1.7927	0.1778	1.8014	5.67	1.8182	0.0000	1.8182	0.00
0.2207	0.60	1.6486	0.1375	1.6543	4.76	1.6667	0.0000	1.6667	0.00
0.2468	0.65	1.5257	0.1060	1.5293	3.97	1.5385	0.0000	1.5385	0.00
0.2761	0.70	1.4194	0.0809	1.4217	3.27	1.4286	0.0000	1.4286	0.00
0.3097	0.75	1.3269	0.0605	1.3284	2.61	1.3333	0.0000	1.3333	0.00
0.3497	0.80	1.2458	0.0438	1.2465	2.02	1.2500	0.0000	1.2500	0.00
0.3999	0.85	1.1737	0.0300	1.1741	1.46	1.1765	0.0000	1.1765	0.00
0.4686	0.90	1.1095	0.0183	1.1097	0.94	1.1111	0.0000	1.1111	0.00
0.5831	0.95	1.0518	0.0085	1.0519	0.46	1.0526	0.0000	1.0526	0.00

Table IV. Inverse Hyperbolic Tangent of Complex Quantity

$$\pi(\alpha - i\beta) = \tanh^{-1}(\theta - i\chi)$$

(See page 130)

θ	α ($\chi = 0$)	β ($\chi = 0$)	α ($\chi = 0.2$)	β ($\chi = 0.2$)	α ($\chi = 0.4$)	β ($\chi = 0.4$)	α ($\chi = 0.6$)	β ($\chi = 0.6$)	α ($\chi = 0.8$)	β ($\chi = 0.8$)
0.0	0.0000	0.0000	0.0000	0.0628	0.0000	0.1211	0.0000	0.1720	0.0000	0.2148
0.2	0.0645	0.0000	0.0619	0.0653	0.0552	0.1250	0.0468	0.1762	0.0386	0.2186
0.4	0.1349	0.0000	0.1281	0.0738	0.1118	0.1379	0.0931	0.1894	0.0760	0.2302
0.6	0.2206	0.0000	0.2041	0.0936	0.1703	0.1640	0.1373	0.2135	0.1103	0.2500
0.8	0.3497	0.0000	0.2955	0.1426	0.2255	0.2110	0.1749	0.2500	0.1386	0.2776
1.0	∞	0–0.5	0.3672	0.2659	0.2593	0.2814	0.1985	0.2964	0.1576	0.3106
1.2	0.3816	0.5000	0.3271	0.3894	0.2562	0.3524	0.2041	0.3436	0.1661	0.3445
1.4	0.2852	0.5000	0.2681	0.4394	0.2322	0.4013	0.1962	0.3826	0.1655	0.3750
1.6	0.2334	0.5000	0.2255	0.4610	0.2060	0.4307	0.1823	0.4111	0.1593	0.3999
1.8	0.1994	0.5000	0.1950	0.4723	0.1832	0.4488	0.1674	0.4312	0.1505	0.4198
2.0	0.1748	0.5000	0.1721	0.4792	0.1644	0.4605	0.1536	0.4454	0.1409	0.4341
2.2	0.1561	0.5000	0.1542	0.4837	0.1490	0.4686	0.1411	0.4557	0.1317	0.4454
2.4	0.1412	0.5000	0.1399	0.4868	0.1361	0.4743	0.1302	0.4634	0.1230	0.4541
2.6	0.1291	0.5000	0.1281	0.4890	0.1252	0.4786	0.1208	0.4692	0.1151	0.4610
2.8	0.1188	0.5000	0.1181	0.4908	0.1159	0.4819	0.1124	0.4737	0.1079	0.4665
3.0	0.1103	0.5000	0.1097	0.4921	0.1080	0.4845	0.1052	0.4773	0.1016	0.4709
3.2	0.1029	0.5000	0.1025	0.4931	0.1010	0.4865	0.0988	0.4802	0.0959	0.4744
3.4	0.0965	0.5000	0.0961	0.4940	0.0950	0.4881	0.0931	0.4826	0.0907	0.4774
3.6	0.0908	0.5000	0.0905	0.4947	0.0895	0.4895	0.0880	0.4845	0.0860	0.4799
3.8	0.0858	0.5000	0.0855	0.4953	0.0847	0.4906	0.0834	0.4862	0.0817	0.4820
4.0	0.0813	0.5000	0.0812	0.4958	0.0804	0.4916	0.0793	0.4876	0.0778	0.4838

θ	α ($\chi = 1.0$)	β ($\chi = 1.0$)	α ($\chi = 1.2$)	β ($\chi = 1.2$)	α ($\chi = 1.4$)	β ($\chi = 1.4$)	α ($\chi = 1.6$)	β ($\chi = 1.6$)	α ($\chi = 2.0$)	β ($\chi = 2.0$)
0.0	0.0000	0.2500	0.0000	0.2789	0.0000	0.3026	0.0000	0.3222	0.0000	0.3524
0.2	0.0316	0.2532	0.0259	0.2814	0.0213	0.3046	0.0178	0.3238	0.0127	0.3534
0.4	0.0619	0.2627	0.0506	0.2890	0.0417	0.3106	0.0348	0.3285	0.0249	0.3564
0.6	0.0892	0.2783	0.0729	0.3012	0.0602	0.3201	0.0503	0.3360	0.0362	0.3612
0.8	0.1118	0.2993	0.0916	0.3173	0.0760	0.3326	0.0639	0.3459	0.0464	0.3675
1.0	0.1281	0.3238	0.1058	0.3360	0.0885	0.3472	0.0749	0.3574	0.0552	0.3750
1.2	0.1373	0.3493	0.1150	0.3558	0.0974	0.3628	0.0832	0.3699	0.0623	0.3833
1.4	0.1403	0.3734	0.1197	0.3750	0.1028	0.3783	0.0890	0.3826	0.0679	0.3920
1.6	0.1386	0.3944	0.1207	0.3926	0.1054	0.3930	0.0924	0.3949	0.0719	0.4008
1.8	0.1341	0.4120	0.1190	0.4080	0.1056	0.4064	0.0938	0.4064	0.0745	0.4093
2.0	0.1281	0.4262	0.1157	0.4211	0.1042	0.4182	0.0937	0.4169	0.0760	0.4174
2.2	0.1216	0.4376	0.1114	0.4321	0.1017	0.4284	0.0926	0.4262	0.0766	0.4249
2.4	0.1150	0.4468	0.1067	0.4412	0.0985	0.4372	0.0906	0.4344	0.0764	0.4318
2.6	0.1087	0.4542	0.1019	0.4488	0.0950	0.4446	0.0883	0.4416	0.0756	0.4381
2.8	0.1027	0.4602	0.0973	0.4551	0.0913	0.4510	0.0855	0.4478	0.0743	0.4437
3.0	0.0974	0.4652	0.0927	0.4604	0.0878	0.4564	0.0828	0.4531	0.0729	0.4488
3.2	0.0924	0.4693	0.0884	0.4648	0.0843	0.4610	0.0799	0.4579	0.0712	0.4533
3.4	0.0877	0.4727	0.0844	0.4686	0.0808	0.4650	0.0771	0.4619	0.0695	0.4573
3.6	0.0835	0.4756	0.0807	0.4718	0.0776	0.4650	0.0743	0.4655	0.0676	0.4609
3.8	0.0796	0.4781	0.0773	0.4746	0.0745	0.4714	0.0717	0.4686	0.0657	0.4641
4.0	0.0760	0.4802	0.0739	0.4769	0.0716	0.4740	0.0691	0.4713	0.0639	0.4670

Table V. Logarithmic and Inverse Hyperbolic Functions

x	ln x	$\sinh^{-1} x$	$\cosh^{-1} x$	x	ln x	$\sinh^{-1} x$	$\cosh^{-1} x$
0.0	$-\infty$	0.0000	......	4.0	1.3863	2.0947	2.0634
0.1	−2.3026	0.0998	......	4.2	1.4351	2.1421	2.1137
0.2	−1.6094	0.1987	......	4.4	1.4816	2.1874	2.1616
0.3	−1.2040	0.2957	......	4.6	1.5261	2.2308	2.2072
0.4	−0.9163	0.3900	......	4.8	1.5686	2.2724	2.2507
0.5	−0.6931	0.4812	......	5.0	1.6094	2.3124	2.2924
0.6	−0.5108	0.5688	......	5.2	1.6487	2.3509	2.3324
0.7	−0.3567	0.6527	......	5.4	1.6864	2.3880	2.3709
0.8	−0.2231	0.7327	......	5.6	1.7228	2.4238	2.4078
0.9	−0.1054	0.8089	......	5.8	1.7579	2.4584	2.4435
1.0	0.0000	0.8814	0.0000	6.0	1.7918	2.4918	2.4779
1.1	0.0953	0.9503	0.4436	6.2	1.8245	2.5241	2.5111
1.2	0.1823	1.0160	0.6224	6.4	1.8563	2.5555	2.5433
1.3	0.2624	1.0785	0.7564	6.6	1.8871	2.5859	2.5744
1.4	0.3365	1.1380	0.8670	6.8	1.9169	2.6154	2.6046
1.5	0.4055	1.1948	0.9624	7.0	1.9459	2.6441	2.6339
1.6	0.4700	1.2490	1.0470	7.2	1.9741	2.6720	2.6624
1.7	0.5306	1.3008	1.1232	7.4	2.0015	2.6992	2.6900
1.8	0.5878	1.3504	1.1929	7.6	2.0281	2.7256	2.7169
1.9	0.6419	1.3980	1.2572	7.8	2.0541	2.7514	2.7431
2.0	0.6931	1.4436	1.3170	8.0	2.0794	2.7765	2.7687
2.1	0.7419	1.4875	1.3729	8.2	2.1041	2.8010	2.7935
2.2	0.7885	1.5297	1.4254	8.4	2.1282	2.8249	2.8178
2.3	0.8329	1.5703	1.4750	8.6	2.1518	2.8483	2.8415
2.4	0.8755	1.6094	1.5221	8.8	2.1748	2.8711	2.8647
2.5	0.9163	1.6472	1.5668	9.0	2.1972	2.8934	2.8873
2.6	0.9555	1.6837	1.6094	9.2	2.2192	2.9153	2.9094
2.7	0.9933	1.7191	1.6502	9.4	2.2407	2.9367	2.9310
2.8	1.0296	1.7532	1.6892	9.6	2.2618	2.9576	2.9522
2.9	1.0647	1.7863	1.7267	9.8	2.2824	2.9781	2.9729
3.0	1.0986	1.8184	1.7627	10.0	2.3026	2.9982	2.9932
3.1	1.1314	1.8496	1.7975	10.2	2.3224	3.0179	3.0131
3.2	1.1632	1.8799	1.8309	10.4	2.3418	3.0373	3.0326
3.3	1.1939	1.9093	1.8633	10.6	2.3609	3.0562	3.0518
3.4	1.2238	1.9379	1.8946	10.8	2.3795	3.0748	3.0705
3.5	1.2528	1.9657	1.9248	11.0	2.3979	3.0931	3.0890
3.6	1.2809	1.9928	1.9542	11.2	2.4159	3.1110	3.1071
3.7	1.3083	2.0193	1.9827	11.4	2.4336	3.1287	3.1248
3.8	1.3350	2.0450	2.0104	11.6	2.4510	3.1460	3.1423
3.9	1.3610	2.0702	2.0373	11.8	2.4681	3.1630	3.1594
4.0	1.3863	2.0947	2.0634	12.0	2.4849	3.1798	3.1763

Table VI. Spherical Harmonic Functions

$P_0^0(x) = 1.0000; \quad P_1^0(x) = x$

(See pages 782 and 1325)

x	$P_1^1(x)$	$P_2^0(x)$	$P_2^1(x)$	$P_2^2(x)$	$P_3^0(x)$	$P_3^1(x)$	$P_3^2(x)$	$P_3^3(x)$
1.00	0.0000	1.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000
0.95	0.3122	0.8538	0.8899	0.2925	0.7184	1.6452	1.3894	0.4567
0.90	0.4359	0.7150	1.1769	0.5700	0.4725	1.9942	2.5650	1.2423
0.85	0.5268	0.5838	1.3433	0.8325	0.2603	2.0643	3.5381	2.1927
0.80	0.6000	0.4600	1.4400	1.0800	0.0800	1.9800	4.3200	3.2400
0.75	0.6614	0.3438	1.4882	1.3125	−0.0703	1.7983	4.9219	4.3407
0.70	0.7141	0.2350	1.4997	1.5300	−0.1925	1.5533	5.3550	5.4632
0.65	0.7599	0.1338	1.4819	1.7325	−0.2884	1.2681	5.6306	6.5829
0.60	0.8000	+0.0400	1.4400	1.9200	−0.3600	0.9600	5.7600	7.6800
0.55	0.8352	−0.0462	1.3780	2.0925	−0.4091	0.6420	5.7544	8.7379
0.50	0.8660	−0.1250	1.2990	2.2500	−0.4375	0.3248	5.6250	9.7428
0.45	0.8930	−0.1962	1.2056	2.3925	−0.4472	+0.0167	5.3831	10.683
0.40	0.9165	−0.2600	1.0998	2.5200	−0.4400	−0.2750	5.0400	11.548
0.35	0.9367	−0.3162	0.9836	2.6325	−0.4178	−0.5445	4.6069	12.330
0.30	0.9539	−0.3650	0.8585	2.7300	−0.3825	−0.7870	4.0950	13.021
0.25	0.9682	−0.4062	0.7262	2.8125	−0.3359	−0.9985	3.5156	13.616
0.20	0.9798	−0.4400	0.5879	2.8800	−0.2800	−1.1758	2.8800	14.109
0.15	0.9887	−0.4662	0.4449	2.9325	−0.2166	−1.3162	2.1994	14.497
0.10	0.9950	−0.4850	0.2985	2.9700	−0.1475	−1.4179	1.4850	14.776
0.05	0.9987	−0.4962	0.1498	2.9925	−0.0747	−1.4794	0.7481	14.944
0.00	1.0000	−0.5000	0.0000	3.0000	0.0000	−1.5000	0.0000	15.000

Table VII. Legendre Functions for Large Arguments

(See pages 1286 and 1325)

x	$P_0^0(x)$	$P_1^0(x)$	$iP_1^1(x)$	$P_2^0(x)$	$iP_2^1(x)$	$-P_2^2(x)$
1.0	1.0000	1.0000	0.0000	1.0000	0.0000	0.0000
1.2	1.0000	1.2000	0.6633	1.6600	2.3880	1.3200
1.4	1.0000	1.4000	0.9798	2.4400	4.1151	2.8800
1.6	1.0000	1.6000	1.2490	3.3400	5.9952	4.6800
1.8	1.0000	1.8000	1.4967	4.3600	8.0820	6.7200
2.0	1.0000	2.0000	1.7321	5.5000	10.392	9.0000
2.2	1.0000	2.2000	1.9596	6.7600	12.933	11.520
2.4	1.0000	2.4000	2.1817	8.1400	15.708	14.280
2.6	1.0000	2.6000	2.4000	9.6400	18.720	17.280
2.8	1.0000	2.8000	2.6153	11.260	21.969	20.520
3.0	1.0000	3.0000	2.8284	13.000	25.456	24.000
3.5	1.0000	3.5000	3.3541	17.875	35.218	33.750
4.0	1.0000	4.0000	3.8730	23.500	46.476	45.000
4.5	1.0000	4.5000	4.3875	29.875	59.231	57.750
5.0	1.0000	5.0000	4.8990	37.000	73.485	72.000
5.5	1.0000	5.5000	5.4083	44.875	89.237	87.750
6.0	1.0000	6.0000	5.9161	53.500	106.49	105.00
6.5	1.0000	6.5000	6.4226	62.875	125.24	123.75
7.0	1.0000	7.0000	6.9282	73.000	145.49	144.00
7.5	1.0000	7.5000	7.4330	83.875	167.24	165.75
8.0	1.0000	8.0000	7.9372	95.500	190.49	189.00

x	$Q_0^0(x)$	$Q_1^0(x)$	$Q_1^1(x)$	$Q_2^0(x)$	$Q_2^1(x)$	$Q_2^2(x)$
1.1	1.5223	0.6745	1.7028	0.3518	1.2549	8.1352
1.2	1.1990	0.4387	1.0138	0.19025	0.6345	3.4372
1.4	0.8959	0.2542	0.5511	0.08595	0.2733	1.2968
1.6	0.7332	0.17307	0.3653	0.04878	0.15215	0.6825
1.8	0.6264	0.12749	0.2652	0.03102	0.09574	0.4164
2.0	0.5493	0.09861	0.2033	0.02118	0.06495	0.2771
2.2	0.4904	0.07891	0.16167	0.01520	0.04640	0.19541
2.4	0.4437	0.06476	0.13210	0.01132	0.03446	0.14375
2.6	0.4055	0.05421	0.11022	0.00868	0.02636	0.10922
2.8	0.3736	0.04610	0.09350	0.00682	0.02066	0.08513
3.0	0.3466	0.03972	0.08040	0.00546	0.01651	0.06777
3.5	0.2939	0.02863	0.05775	0.00334	0.01009	0.04112
4.0	0.2554	0.02165	0.04359	0.00220	0.00663	0.02691
4.5	0.2260	0.01697	0.03411	0.00153	0.00460	0.01860
5.0	0.2027	0.01366	0.02744	0.00110	0.00332	0.01341
5.5	0.1839	0.01124	0.02256	0.00082	0.00248	0.00999
6.0	0.1682	0.00942	0.01889	0.00063	0.00190	0.00765
6.5	0.1551	0.00800	0.01605	0.00050	0.00149	0.00599
7.0	0.1438	0.00689	0.01380	0.00040	0.00119	0.00478
7.5	0.1341	0.00599	0.01200	0.00032	0.00096	0.00387
8.0	0.1257	0.00526	0.01053	0.00026	0.00079	0.00318

Table VIII. Legendre Functions for Imaginary Arguments

(See pages 1292 and 1328)

x	$P_0^0(ix)$	$-iP_1^0(ix)$	$P_1^1(ix)$	$-P_2^0(ix)$	$-iP_2^1(ix)$	$P_2^2(ix)$
0.0	1.0000	0.0000	1.0000	0.5000	0.0000	3.0000
0.2	1.0000	0.2000	1.0198	0.5600	0.6119	3.1200
0.4	1.0000	0.4000	1.0770	0.7400	1.2924	3.4800
0.6	1.0000	0.6000	1.1662	1.0400	2.0991	4.0800
0.8	1.0000	0.8000	1.2806	1.4600	3.0735	4.9200
1.0	1.0000	1.0000	1.4142	2.0000	4.2426	6.0000
1.2	1.0000	1.2000	1.5621	2.6600	5.6234	7.3200
1.4	1.0000	1.4000	1.7205	3.4400	7.2260	8.8800
1.6	1.0000	1.6000	1.8868	4.3400	9.0566	10.680
1.8	1.0000	1.8000	2.0591	5.3600	11.119	12.720
2.0	1.0000	2.0000	2.2361	6.5000	13.416	15.000
2.5	1.0000	2.5000	2.6926	9.8750	20.194	21.750
3.0	1.0000	3.0000	3.1623	14.000	28.461	30.000
3.5	1.0000	3.5000	3.6401	18.875	38.221	39.750
4.0	1.0000	4.0000	4.1231	24.500	49.477	51.000
4.5	1.0000	4.5000	4.6098	30.875	62.232	63.750
5.0	1.0000	5.0000	5.0990	38.000	76.485	78.000
5.5	1.0000	5.5000	5.5902	45.875	92.238	93.750
6.0	1.0000	6.0000	6.0828	54.500	109.49	111.00
6.5	1.0000	6.5000	6.5765	63.875	128.24	129.75
7.0	1.0000	7.0000	7.0711	74.000	148.49	150.00
7.5	1.0000	7.5000	7.5664	84.875	170.24	171.75

x	$iQ_0^0(ix)$	$-Q_1^0(ix)$	$-Q_1^1(ix)$	$-iQ_2^0(ix)$	$-iQ_2^1(ix)$	$-iQ_2^2(ix)$
0.0	1.5708	1.0000	1.5708	0.7854	2.0000	4.7124
0.2	1.3734	0.7253	1.2045	0.4691	1.2385	3.3004
0.4	1.1903	0.5239	0.9106	0.2808	0.7642	2.2526
0.6	1.0304	0.3818	0.6871	0.17159	0.4782	1.5216
0.8	0.8961	0.2832	0.5228	0.10824	0.3070	1.0330
1.0	0.7854	0.2146	0.4036	0.07079	0.20337	0.7124
1.2	0.6947	0.1663	0.3170	0.04800	0.13919	0.5019
1.4	0.6202	0.13165	0.2534	0.03366	0.09826	0.3619
1.6	0.5586	0.10624	0.2060	0.02432	0.07137	0.2670
1.8	0.5071	0.08722	0.1700	0.01805	0.05316	0.2012
2.0	0.4636	0.07270	0.14232	0.01371	0.04050	0.15471
2.5	0.3805	0.04873	0.09607	0.00750	0.02227	0.08636
3.0	0.3218	0.03475	0.06878	0.00451	0.01342	0.05251
3.5	0.2783	0.02595	0.05150	0.00291	0.00867	0.03411
4.0	0.2450	0.02009	0.03993	0.00198	0.00591	0.02332
4.5	0.2187	0.01599	0.03183	0.00140	0.00420	0.01662
5.0	0.19740	0.01302	0.02594	0.00103	0.00309	0.01224
5.5	0.17985	0.01081	0.02154	0.00078	0.00233	0.00927
6.0	0.16515	0.00911	0.01816	0.00060	0.00180	0.00718
6.5	0.15265	0.00778	0.01552	0.00047	0.00143	0.00567
7.0	0.14190	0.00672	0.01341	0.00038	0.00114	0.00456
7.5	0.13255	0.00586	0.01171	0.00031	0.00093	0.00372

Table IX. Legendre Functions of Half-integral Degree

(See pages 1302 and 1329)

x	$P^0_{-\frac{1}{2}}(x)$	$-iP^1_{-\frac{1}{2}}(x)$	$-P^2_{-\frac{1}{2}}(x)$	$P^0_{\frac{1}{2}}(x)$	$iP^1_{\frac{1}{2}}(x)$	$P^0_{\frac{3}{2}}(x)$
1.0	1.0000	0.0000	0.0000	1.0000	0.0000	1.0000
1.2	0.9763	0.07447	0.02536	1.0728	0.2344	1.3910
1.4	0.9549	0.09968	0.04613	1.1416	0.3283	1.8126
1.6	0.9355	0.11603	0.06340	1.2070	0.3986	2.2630
1.8	0.9177	0.12778	0.07794	1.2694	0.4567	2.7406
2.0	0.9013	0.1367	0.0903	1.3291	0.5072	3.2439
2.2	0.8861	0.1436	0.1009	1.3866	0.5523	3.7719
2.4	0.8719	0.1491	0.1101	1.4419	0.5933	4.3236
2.6	0.8587	0.1536	0.1181	1.4954	0.6311	4.8979
2.8	0.8463	0.1572	0.1251	1.5472	0.6664	5.4941
3.0	0.8346	0.1602	0.1313	1.5974	0.6996	6.1113
3.5	0.8082	0.1657	0.1438	1.7169	0.7753	7.7427
4.0	0.7850	0.1692	0.1533	1.8290	0.8432	9.4930
4.5	0.7643	0.1715	0.1606	1.9349	0.9052	11.355
5.0	0.7457	0.1728	0.1663	2.0356	0.9627	13.322
5.5	0.7289	0.1736	0.1708	2.1316	1.0165	15.389
6.0	0.7136	0.1739	0.1744	2.2237	1.0673	17.552
6.5	0.6995	0.1739	0.1772	2.3122	1.1156	19.806
7.0	0.6864	0.1737	0.1795	2.3975	1.1616	22.148
7.5	0.6743	0.1734	0.1813	2.4799	1.2058	24.575
8.0	0.6631	0.1729	0.1828	2.5598	1.2482	27.083

x	$Q^0_{-\frac{1}{2}}(x)$	$Q^1_{-\frac{1}{2}}(x)$	$Q^2_{-\frac{1}{2}}(x)$	$Q^0_{\frac{1}{2}}(x)$	$Q^1_{\frac{1}{2}}(x)$	$Q^0_{\frac{3}{2}}(x)$
1.1	2.8612	2.3661	10.644	0.9788	1.9471	0.4818
1.2	2.5010	1.7349	5.6518	0.6996	1.2524	0.2856
1.4	2.1366	1.2918	3.1575	0.4598	0.7618	0.14609
1.6	1.9229	1.0943	2.3230	0.3430	0.5501	0.09080
1.8	1.7723	0.9748	1.9018	0.2720	0.4285	0.06214
2.0	1.6566	0.8918	1.6454	0.2240	0.3489	0.04516
2.2	1.5634	0.8293	1.4712	0.18932	0.29263	0.03422
2.4	1.4856	0.7798	1.3441	0.16312	0.25076	0.02676
2.6	1.4193	0.7391	1.2465	0.14266	0.21842	0.02143
2.8	1.3617	0.7048	1.1687	0.12628	0.19274	0.01751
3.0	1.3110	0.6753	1.1048	0.11289	0.17189	0.01454
3.5	1.2064	0.6163	0.9846	0.08824	0.13380	0.00966
4.0	1.1242	0.5713	0.8990	0.07154	0.10819	0.00682
4.5	1.0572	0.5353	0.8339	0.05957	0.08993	0.00503
5.0	1.0011	0.5057	0.7820	0.05063	0.07634	0.00384
5.5	0.9532	0.4806	0.7393	0.04374	0.06588	0.00301
6.0	0.9117	0.4591	0.7033	0.03829	0.05764	0.00241
6.5	0.87524	0.44025	0.67231	0.03389	0.05099	0.00197
7.0	0.84288	0.42362	0.64530	0.03028	0.04553	0.00163
7.5	0.81389	0.40877	0.62144	0.02727	0.04099	0.00137
8.0	0.78772	0.39542	0.60015	0.02473	0.03716	0.00116

Table X. Bessel Functions for Cylindrical Coordinates

(See pages 1321 and 1563)

x	$J_0(x)$	$N_0(x)$	$J_1(x)$	$N_1(x)$	$J_2(x)$	$N_2(x)$
0.0	1.0000	$-\infty$	0.0000	$-\infty$	0.0000	$-\infty$
0.1	0.9975	−1.5342	0.0499	−6.4590	0.0012	−127.64
0.2	0.9900	−1.0811	0.0995	−3.3238	0.0050	−32.157
0.4	0.9604	−0.6060	0.1960	−1.7809	0.0197	−8.2983
0.6	0.9120	−0.3085	0.2867	−1.2604	0.0437	−3.8928
0.8	0.8463	−0.0868	0.3688	−0.9781	0.0758	−2.3586
1.0	0.7652	+0.0883	0.4401	−0.7812	0.1149	−1.6507
1.2	0.6711	0.2281	0.4983	−0.6211	0.1593	−1.2633
1.4	0.5669	0.3379	0.5419	−0.4791	0.2074	−1.0224
1.6	0.4554	0.4204	0.5699	−0.3476	0.2570	−0.8549
1.8	0.3400	0.4774	0.5815	−0.2237	0.3061	−0.7259
2.0	0.2239	0.5104	0.5767	−0.1070	0.3528	−0.6174
2.2	0.1104	0.5208	0.5560	+0.0015	0.3951	−0.5194
2.4	+0.0025	0.5104	0.5202	0.1005	0.4310	−0.4267
2.6	−0.0968	0.4813	0.4708	0.1884	0.4590	−0.3364
2.8	−0.1850	0.4359	0.4097	0.2635	0.4777	−0.2477
3.0	−0.2601	0.3768	0.3391	0.3247	0.4861	−0.1604
3.2	−0.3202	0.3071	0.2613	0.3707	0.4835	−0.0754
3.4	−0.3643	0.2296	0.1792	0.4010	0.4697	+0.0063
3.6	−0.3918	0.1477	0.0955	0.4154	0.4448	0.0831
3.8	−0.4026	+0.0645	+0.0128	0.4141	0.4093	0.1535
4.0	−0.3971	−0.0169	−0.0660	0.3979	0.3641	0.2159
4.2	−0.3766	−0.0938	−0.1386	0.3680	0.3105	0.2690
4.4	−0.3423	−0.1633	−0.2028	0.3260	0.2501	0.3115
4.6	−0.2961	−0.2235	−0.2566	0.2737	0.1846	0.3425
4.8	−0.2404	−0.2723	−0.2985	0.2136	0.1161	0.3613
5.0	−0.1776	−0.3085	−0.3276	0.1479	+0.0466	0.3677
5.2	−0.1103	−0.3312	−0.3432	0.0792	−0.0217	0.3617
5.4	−0.0412	−0.3402	−0.3453	+0.0101	−0.0867	0.3439
5.6	+0.0270	−0.3354	−0.3343	−0.0568	−0.1464	0.3152
5.8	0.0917	−0.3177	−0.3110	−0.1192	−0.1989	0.2766
6.0	0.1507	−0.2882	−0.2767	−0.1750	−0.2429	0.2299
6.2	0.2017	−0.2483	−0.2329	−0.2223	−0.2769	0.1766
6.4	0.2433	−0.2000	−0.1816	−0.2596	−0.3001	0.1188
6.6	0.2740	−0.1452	−0.1250	−0.2858	−0.3119	+0.0586
6.8	0.2931	−0.0864	−0.0652	−0.3002	−0.3123	−0.0019
7.0	0.3001	−0.0259	−0.0047	−0.3027	−0.3014	−0.0605
7.2	0.2951	+0.0339	+0.0543	−0.2934	−0.2800	−0.1154
7.4	0.2786	0.0907	0.1096	−0.2731	−0.2490	−0.1645
7.6	0.2516	0.1424	0.1592	−0.2428	−0.2097	−0.2063
7.8	0.2154	0.1872	0.2014	−0.2039	−0.1638	−0.2395
8.0	0.1716	0.2235	0.2346	−0.1581	−0.1130	−0.2630

Table XI. Hyperbolic Bessel Functions

(See page 1323)

$I_m(z) = i^{-m}J_m(iz)$

z	$I_0(z)$	$I_1(z)$	$I_2(z)$
0.0	1.0000	0.0000	0.0000
0.1	1.0025	0.0501	0.0012
0.2	1.0100	0.1005	0.0050
0.4	1.0404	0.2040	0.0203
0.6	1.0921	0.3137	0.0464
0.8	1.1665	0.4329	0.0843
1.0	1.2661	0.5652	0.1358
1.2	1.3937	0.7147	0.2026
1.4	1.5534	0.8861	0.2876
1.6	1.7500	1.0848	0.3940
1.8	1.9895	1.3172	0.5260
2.0	2.2796	1.5906	0.6890
2.2	2.6292	1.9141	0.8891
2.4	3.0492	2.2981	1.1342
2.6	3.5532	2.7554	1.4338
2.8	4.1574	3.3011	1.7994
3.0	4.8808	3.9534	2.2452
3.2	5.7472	4.7343	2.7884
3.4	6.7848	5.6701	3.4495
3.6	8.0278	6.7926	4.2540
3.8	9.5169	8.1405	5.2325
4.0	11.302	9.7594	6.4222
4.2	13.443	11.705	7.8683
4.4	16.010	14.046	9.6259
4.6	19.093	16.863	11.761
4.8	22.794	20.253	14.355
5.0	27.240	24.335	17.505
5.2	32.584	29.254	21.332
5.4	39.010	35.181	25.978
5.6	46.738	42.327	31.621
5.8	56.039	50.945	38.470
6.0	67.235	61.341	46.788
6.2	80.717	73.888	56.884
6.4	96.963	89.025	69.141
6.6	116.54	107.31	84.021
6.8	140.14	129.38	102.08
7.0	168.59	156.04	124.01
7.2	202.92	188.25	150.63
7.4	244.34	227.17	182.94
7.6	294.33	274.22	222.17
7.8	354.68	331.10	269.79
8.0	427.57	399.87	327.60

Table XII. Bessel Functions for Spherical Coordinates

(See pages 622 and 1573)

$$j_n(x) = \sqrt{\pi/2x}\, J_{n+\frac{1}{2}}(x);\ n_n(x) = \sqrt{\pi/2x}\, N_{n+\frac{1}{2}}(x)$$

x	$j_0(x)$	$n_0(x)$	$j_1(x)$	$n_1(x)$	$j_2(x)$	$n_2(x)$
0.0	1.0000	$-\infty$	0.0000	$-\infty$	0.0000	$-\infty$
0.1	0.9983	−9.9500	0.0333	−100.50	0.0007	−3005.0
0.2	0.9933	−4.9003	0.0664	−25.495	0.0027	−377.52
0.4	0.9735	−2.3027	0.1312	−6.7302	0.0105	−48.174
0.6	0.9411	−1.3756	0.1929	−3.2337	0.0234	−14.793
0.8	0.8967	−0.8709	0.2500	−1.9853	0.0408	−6.5740
1.0	0.8415	−0.5403	0.3012	−1.3818	0.0620	−3.6050
1.2	0.7767	−0.3020	0.3453	−1.0283	0.0865	−2.2689
1.4	0.7039	−0.1214	0.3814	−0.7906	0.1133	−1.5728
1.6	0.6247	+0.0183	0.4087	−0.6133	0.1416	−1.1682
1.8	0.5410	0.1262	0.4268	−0.4709	0.1703	−0.9111
2.0	0.4546	0.2081	0.4354	−0.3506	0.1985	−0.7340
2.2	0.3675	0.2675	0.4346	−0.2459	0.2251	−0.6028
2.4	0.2814	0.3072	0.4245	−0.1534	0.2492	−0.4990
2.6	0.1983	0.3296	0.4058	−0.0715	0.2700	−0.4121
2.8	0.1196	0.3365	0.3792	+0.0005	0.2867	−0.3359
3.0	+0.0470	0.3300	0.3457	0.0630	0.2986	−0.2670
3.2	−0.0182	0.3120	0.3063	0.1157	0.3054	−0.2035
3.4	−0.0752	0.2844	0.2623	0.1588	0.3066	−0.1442
3.6	−0.1229	0.2491	0.2150	0.1921	0.3021	−0.0890
3.8	−0.1610	0.2082	0.1658	0.2158	0.2919	−0.0378
4.0	−0.1892	0.1634	0.1161	0.2300	0.2763	+0.0091
4.2	−0.2075	0.1167	0.0673	0.2353	0.2556	0.0514
4.4	−0.2163	0.0699	+0.0207	0.2321	0.2304	0.0884
4.6	−0.2160	+0.0244	−0.0226	0.2213	0.2013	0.1200
4.8	−0.2075	−0.0182	−0.0615	0.2037	0.1691	0.1456
5.0	−0.1918	−0.0567	−0.0951	0.1804	0.1347	0.1650
5.2	−0.1699	−0.0901	−0.1228	0.1526	0.0991	0.1781
5.4	−0.1431	−0.1175	−0.1440	0.1213	0.0631	0.1850
5.6	−0.1127	−0.1385	−0.1586	0.0880	+0.0278	0.1856
5.8	−0.0801	−0.1527	−0.1665	0.0538	−0.0060	0.1805
6.0	−0.0466	−0.1600	−0.1678	+0.0199	−0.0373	0.1700
6.2	−0.0134	−0.1607	−0.1629	−0.0124	−0.0654	0.1547
6.4	+0.0182	−0.1552	−0.1523	−0.0425	−0.0896	0.1353
6.6	0.0472	−0.1440	−0.1368	−0.0690	−0.1094	0.1126
6.8	0.0727	−0.1278	−0.1172	−0.0915	−0.1243	0.0875
7.0	0.0939	−0.1077	−0.0943	−0.1092	−0.1343	0.0609
7.2	0.1102	−0.0845	−0.0692	−0.1220	−0.1391	0.0337
7.4	0.1215	−0.0593	−0.0429	−0.1294	−0.1388	+0.0068
7.6	0.1274	−0.0331	−0.0163	−0.1317	−0.1338	−0.0189
7.8	0.1280	−0.0069	+0.0095	−0.1289	−0.1244	−0.0427
8.0	0.1237	+0.0182	0.0336	−0.1214	−0.1111	−0.0637

Table XIII. Legendre Functions for Spherical Coordinates
(See pages 782 and 1325)

ϑ	$P_{-1} = P_0$	$P_1(\cos\vartheta)$	$P_2(\cos\vartheta)$	$P_3(\cos\vartheta)$	$P_4(\cos\vartheta)$
0°	1.0000	1.0000	1.0000	1.0000	1.0000
5	1.0000	0.9962	0.9886	0.9773	0.9623
10	1.0000	0.9848	0.9548	0.9106	0.8532
15	1.0000	0.9659	0.8995	0.8042	0.6847
20	1.0000	0.9397	0.8245	0.6649	0.4750
25	1.0000	0.9063	0.7321	0.5016	0.2465
30	1.0000	0.8660	0.6250	0.3248	0.0234
35	1.0000	0.8192	0.5065	0.1454	−0.1714
40	1.0000	0.7660	0.3802	−0.0252	−0.3190
45	1.0000	0.7071	0.2500	−0.1768	−0.4063
50	1.0000	0.6428	0.1198	−0.3002	−0.4275
55	1.0000	0.5736	−0.0065	−0.3886	−0.3852
60	1.0000	0.5000	−0.1250	−0.4375	−0.2891
65	1.0000	0.4226	−0.2321	−0.4452	−0.1552
70	1.0000	0.3420	−0.3245	−0.4130	−0.0038
75	1.0000	0.2588	−0.3995	−0.3449	+0.1434
80	1.0000	0.1736	−0.4548	−0.2474	0.2659
85	1.0000	0.0872	−0.4886	−0.1291	0.3468
90	1.0000	0.0000	−0.5000	0.0000	0.3750

ϑ	$P_5(\cos\vartheta)$	$P_6(\cos\vartheta)$	$P_7(\cos\vartheta)$	$P_8(\cos\vartheta)$	$P_9(\cos\vartheta)$
0°	1.0000	1.0000	1.0000	1.0000	1.0000
5	0.9437	0.9216	0.8962	0.8675	0.8358
10	0.7840	0.7045	0.6164	0.5218	0.4228
15	0.5471	0.3983	0.2455	0.0962	−0.0428
20	0.2715	0.0719	−0.1072	−0.2518	−0.3517
25	0.0009	−0.2040	−0.3441	−0.4062	−0.3896
30	−0.2233	−0.3740	−0.4102	−0.3388	−0.1896
35	−0.3691	−0.4114	−0.3096	−0.1154	+0.0965
40	−0.4197	−0.3236	−0.1006	+0.1386	0.2900
45	−0.3757	−0.1484	+0.1271	0.2983	0.2855
50	−0.2545	+0.0564	0.2854	0.2947	0.1041
55	−0.0868	0.2297	0.3191	0.1422	−0.1296
60	+0.0898	0.3232	0.2231	−0.0736	−0.2679
65	0.2381	0.3138	0.0422	−0.2411	−0.2300
70	0.3281	0.2089	−0.1485	−0.2780	−0.0476
75	0.3427	0.0431	−0.2731	−0.1702	+0.1595
80	0.2810	−0.1321	−0.2835	+0.0233	0.2596
85	0.1577	−0.2638	−0.1778	0.2017	0.1913
90	0.0000	−0.3125	0.0000	0.2734	0.0000

Table XIV. Amplitudes and Phases for Cylindrical Bessel Functions

(See page 1563)

For $m > 0$, $C'_m \to 0.1592m!(2/x)^{m+1} \leftarrow \frac{1}{2}C_{m+1}$ and

$$\delta_m \to \left[\frac{180m}{(m!)^2}\right]\left(\frac{x}{2}\right)^{2m} \text{ degrees} \leftarrow -\delta'_m \quad \text{as } x \text{ approaches zero}$$

$C_0 \to \sqrt{1 + (2/\pi)^2(\ln x)^2}$; $C'_0 \to 0.6366/x$; $\delta_0 \to -90/\ln x$; $\delta'_0 \to 45x^2$ degrees

As x approaches ∞, $C_m \to C'_m \to \sqrt{2/\pi x}$; $\delta_m \to x - \frac{1}{4}\pi(2m - 1)$ radians

$\delta'_m \to x - \frac{1}{4}\pi(2m + 1)$ radians

x	$C_0(x)$	$\delta_0(x)$	$C'_0(x)$	$\delta'_0(x)$	$C_1(x)$	$\delta_1(x)$	$C'_1(x)$	$\delta'_1(x)$
0.0	∞	00.00°	∞	00.00°	∞	00.00°	∞	00.00°
0.1	1.8300	33.03	6.4591	0.44	6.4591	0.44	63.057	−0.45
0.2	1.4659	42.48	3.3253	1.71	3.3253	1.71	15.546	1.82
0.3	1.2679	50.45	2.2979	3.70	2.2979	3.70	6.8535	4.04
0.4	1.1356	57.75	1.7916	6.28	1.7916	6.28	3.8748	6.97
0.5	1.0384	64.65	1.4913	9.35	1.4913	9.35	2.5393	−10.30
0.6	0.9628	71.31	1.2926	12.82	1.2926	12.82	1.8440	13.62
0.7	0.9016	77.79	1.1513	16.60	1.1513	16.60	1.4451	16.53
0.8	0.8507	84.14	1.0454	20.66	1.0454	20.66	1.1994	18.73
0.9	0.8075	90.40	0.9629	24.94	0.9629	24.94	1.0388	20.07
1.0	0.7703	96.58	0.8966	29.39	0.8966	29.39	0.9283	−20.50
1.2	0.7088	108.77	0.7963	38.74	0.7963	38.74	0.7884	18.94
1.4	0.6599	120.80	0.7234	48.52	0.7234	48.52	0.7035	14.80
1.6	0.6198	132.71	0.6675	58.62	0.6675	58.62	0.6453	8.84
1.8	0.5861	144.54	0.6230	68.96	0.6230	68.96	0.6019	−1.61
2.0	0.5573	156.31	0.5866	79.49	0.5866	79.49	0.5676	+6.52
2.2	0.5323	168.04	0.5560	90.15	0.5560	90.15	0.5392	15.31
2.4	0.5104	179.72	0.5298	100.93	0.5298	100.93	0.5152	24.57
2.6	0.4910	191.37	0.5071	111.81	0.5071	111.81	0.4944	34.20
2.8	0.4736	203.00	0.4872	122.75	0.4872	122.75	0.4760	44.11
3.0	0.4579	214.61	0.4694	133.76	0.4694	133.76	0.4597	54.24
3.2	0.4436	226.20	0.4536	144.82	0.4536	144.82	0.4450	64.55
3.4	0.4306	237.78	0.4392	155.92	0.4392	155.92	0.4317	75.01
3.6	0.4187	249.34	0.4262	167.06	0.4262	167.06	0.4195	85.58
3.8	0.4077	260.90	0.4143	178.23	0.4143	178.23	0.4084	96.25
4.0	0.3975	272.44	0.4034	189.42	0.4034	189.42	0.3980	107.01
4.2	0.3881	283.98	0.3933	200.64	0.3933	200.64	0.3885	117.83
4.4	0.2792	295.51	0.3839	211.88	0.3839	211.88	0.3796	128.72
4.6	0.3710	307.04	0.3752	223.14	0.3752	223.14	0.3713	139.65
4.8	0.3633	318.56	0.3670	234.42	0.3670	234.42	0.3635	150.64
5.0	0.3560	330.07	0.3594	245.71	0.3594	245.71	0.3562	161.66

Table XIV. Amplitudes and Phases for Cylindrical Bessel Functions. (*Continued*)

x	$C_2(x)$	$\delta_2(x)$	$C_2'(x)$	$\delta_2'(x)$	$C_3(x)$	$\delta_3(x)$	$C_3'(x)$	$\delta_3'(x)$
0.1	127.65	00.00°	2 546.4	00.00°	5 099.3	00.00°	152 852	00.00°
0.2	32.157	0.01	318.25	−0.01	639.82	0.00	9 565.1	0.00
0.4	8.2984	0.14	39.711	0.14	81.203	0.00	600.72	0.00
0.6	3.8930	0.64	11.716	0.69	24.692	0.01	119.57	−0.01
0.8	2.3598	1.84	4.9215	2.09	10.815	0.05	38.196	0.06
1.0	1.6547	3.98	2.5289	−4.77	5.8216	0.19	15.814	−0.20
1.2	1.2733	7.19	1.5025	8.91	3.5901	0.52	7.7118	0.57
1.4	1.0432	11.46	1.0117	14.06	2.4425	1.18	4.2116	1.35
1.6	0.8927	16.73	0.7627	19.03	1.7911	2.32	2.5037	2.77
1.8	0.7879	22.87	0.6309	22.49	1.3931	4.07	1.5963	5.08
2.0	0.7111	29.75	0.5573	−23.69	1.1351	6.52	1.0860	−8.44
2.2	0.6526	37.26	0.5130	22.56	0.9597	9.74	0.7898	12.71
2.4	0.6065	45.29	0.4836	19.45	0.8354	13.72	0.6158	17.32
2.6	0.5691	53.76	0.4624	14.75	0.7441	18.43	0.5136	21.41
2.8	0.5381	62.59	0.4457	8.84	0.6749	23.83	0.4535	24.15
3.0	0.5119	71.74	0.4319	−1.99	0.6209	29.85	0.4175	−25.09
3.2	0.4894	81.14	0.4198	+5.59	0.5778	36.42	0.3952	24.19
3.4	0.4698	90.77	0.4090	13.73	0.5426	43.49	0.3804	21.64
3.6	0.4525	100.58	0.3992	22.33	0.5132	50.98	0.3698	17.71
3.8	0.4371	110.55	0.3901	31.29	0.4884	58.86	0.3617	12.66
4.0	0.4233	120.67	0.3816	40.55	0.4671	67.06	0.3549	−6.72
4.2	0.4108	130.90	0.3737	50.06	0.4486	75.56	0.3489	−0.04
4.4	0.3995	141.24	0.3662	59.77	0.4322	84.32	0.3434	+7.22
4.6	0.3891	151.68	0.3592	69.66	0.4178	93.30	0.3383	14.97
4.8	0.3795	162.19	0.3525	79.70	0.4048	102.49	0.3334	23.13
5.0	0.3706	172.78	0.3462	89.87	0.3931	111.85	0.3287	31.62

x	$C_8(x)$	$\delta_8(x)$	$C_8'(x)$	$\delta_8'(x)$	$C_9(x)$	$\delta_9(x)$	$C_9'(x)$	$\delta_9'(x)$
1.6	10 485.	00.00°	51 209.	00.00°	103 635	00.00°	572 462	00.00°
1.8	4 188.9	0.00	18 068.	0.00	36 685.	0.00	179 235	0.00
2.0	1 853.9	0.00	7 144.1	0.00	14 560.	0.00	63 665.	0.00
2.2	891.96	0.00	3 099.0	0.00	6 342.5	0.00	25 055.	0.00
2.4	460.04	0.00	1 451.6	0.00	2 985.1	0.00	10 734.	0.00
2.6	251.68	0.00	725.56	0.00	1 500.0	0.00	4 940.5	0.00
2.8	144.86	0.00	383.37	0.00	797.25	0.00	2 417.7	0.00
3.0	87.150	0.00	212.56	0.00	444.96	0.00	1 247.7	0.00
3.2	54.522	0.00	122.93	0.00	259.24	0.00	674.59	0.00
3.4	35.320	0.00	73.802	0.00	156.91	0.00	380.03	0.00
3.6	23.612	0.00	45.804	0.00	98.275	0.00	222.08	0.00
3.8	16.243	0.01	29.287	−0.01	63.483	0.00	134.11	0.00
4.0	11.471	0.02	19.236	−0.02	42.178	0.00	83.430	0.00
4.2	8.3005	0.04	12.946	0.04	28.756	0.00	53.319	0.00
4.4	6.1442	0.07	8.9067	0.08	20.078	0.01	34.924	−0.01
4.6	4.6463	0.13	6.2524	0.14	14.333	0.01	23.396	0.01
4.8	3.5855	0.22	4.4705	0.25	10.446	0.02	16.001	0.02
5.0	2.8209	0.37	3.2506	−0.42	7.7639	0.04	11.154	−0.04

Table XIV. Amplitudes and Phases for Cylindrical Bessel Functions. (*Continued*)

x	$C_4(x)$	$\delta_4(x)$	$C'_4(x)$	$\delta'_4(x)$	$C_5(x)$	$\delta_5(x)$	$C'_5(x)$	$\delta'_5(x)$
0.4	1 209.7	00.00°	1 2016.	00.00°	24 114.	00.00°	300 210	00.00°
0.6	243.02	0.00	1 595.5	0.00	3 215.6	0.00	26 554.	0.00
0.8	78.751	0.00	382.94	0.00	776.70	0.00	4 775.6	0.00
1.0	33.278	0.00	127.29	0.00	260.41	0.00	1 268.8	0.00
1.2	16.686	0.02	52.031	−0.02	107.65	0.00	431.86	0.00
1.4	9.4432	0.05	24.539	0.06	51.519	0.00	174.55	0.00
1.6	5.8564	0.15	12.851	0.16	27.492	0.01	80.057	−0.01
1.8	3.9060	0.34	7.2904	0.37	15.970	0.02	40.455	0.02
2.0	2.7662	0.70	4.4045	−0.79	9.9360	0.04	22.074	−0.04
2.2	2.0609	1.32	2.8012	1.55	6.5462	0.10	12.817	0.10
2.4	1.6037	2.30	1.8612	2.80	4.5296	0.21	7.8343	0.22
2.6	1.2954	3.72	1.2872	4.73	3.2717	0.41	4.9989	0.45
2.8	1.0805	5.67	0.9265	7.46	2.4550	0.75	3.3086	0.86
3.0	0.9261	8.20	0.6965	−11.01	1.9064	1.29	2.2607	−1.53
3.2	0.8122	11.34	0.5496	15.13	1.5270	2.11	1.5896	2.59
3.4	0.7260	15.11	0.4566	19.29	1.2576	3.27	1.1486	4.18
3.6	0.6593	19.47	0.3987	22.81	1.0619	4.84	0.8534	6.41
3.8	0.6065	24.42	0.3631	25.11	0.9167	6.88	0.6539	9.35
4.0	0.5640	29.90	0.3412	−25.90	0.8067	9.42	0.5190	−12.92
4.2	0.5291	35.87	0.3275	25.14	0.7219	12.49	0.4287	16.83
4.4	0.5000	42.29	0.3187	22.95	0.6553	16.09	0.3693	20.62
4.6	0.4753	49.12	0.3126	19.54	0.6022	20.21	0.3312	23.74
4.8	0.4542	56.32	0.3081	15.10	0.5590	24.83	0.3071	25.76
5.0	0.4359	63.84	0.3044	−9.81	0.5235	29.92	0.2921	−26.44

x	$C_6(x)$	$\delta_6(x)$	$C'_6(x)$	$\delta'_6(x)$	$C_7(x)$	$\delta_7(x)$	$C'_7(x)$	$\delta'_7(x)$
1.0	2 570.8	00.00°	15 164.	00.00°	30 589.	00.00°	211 552	00.00°
1.2	880.41	0.00	4 294.4	0.00	8 696.4	0.00	49 849.	0.00
1.4	358.55	0.00	1 485.1	0.00	3 021.8	0.00	14 750.	0.00
1.6	165.97	0.00	594.89	0.00	1 217.3	0.00	5 159.6	0.00
1.8	84.816	0.00	266.75	0.00	549.47	0.00	2 052.0	0.00
2.0	46.914	0.00	130.81	0.00	271.55	0.00	903.50	0.00
2.2	27.695	0.00	68.986	0.00	144.52	0.00	432.13	0.00
2.4	17.271	0.01	38.648	−0.01	81.825	0.00	221.39	0.00
2.6	11.290	0.03	22.783	0.03	48.837	0.00	120.20	0.00
2.8	7.6918	0.06	14.028	0.06	30.510	0.00	68.583	0.00
3.0	5.4365	0.12	8.9670	−0.13	19.840	0.01	40.857	−0.01
3.2	3.9723	0.23	5.9221	0.25	13.370	0.02	25.275	0.02
3.4	2.9921	0.42	4.0245	0.47	9.3044	0.03	16.164	0.04
3.6	2.3179	0.72	2.8051	0.83	6.6677	0.07	10.647	0.07
3.8	1.8431	1.19	2.0001	1.41	4.9090	0.13	7.2002	0.15
4.0	1.5015	1.87	1.4564	−2.30	3.7063	0.23	4.9853	−0.26
4.2	1.2510	2.83	1.0822	3.60	2.8650	0.40	3.5256	0.46
4.4	1.0640	4.11	0.8211	5.42	2.2647	0.67	2.5417	0.77
4.6	0.9220	5.77	0.6374	7.85	1.8284	1.06	1.8649	1.26
4.8	0.8127	7.85	0.5081	10.88	1.5059	1.63	1.3911	2.00
5.0	0.7272	10.38	0.4177	−14.40	1.2640	2.42	1.0543	−3.06

Table XV. Amplitudes and Phases for Spherical Bessel Functions

(See pages 1477 and 1575)

As x approaches zero $D_m \to \dfrac{1 \cdot 1 \cdot 3 \cdots (2m-1)}{x^{m+1}}$; $D'_m \to \dfrac{m+1}{x} D_m$

$$\delta_m \to \frac{57.30x^{2m+1}}{1 \cdot 3 \cdot 5 \cdots (2m+1) \cdot 1 \cdot 1 \cdot 3 \cdots (2m-1)} \text{ degrees}; \quad \delta'_m \to -\frac{m}{m+1}\delta_m$$

$D_0 = 1/x$; $\delta_0 = 57.296x$ degrees; $\delta'_0 \to 19\,098x^3$

As x approaches ∞, $D_m \to 1/x \to D'_m$; $\delta_m \to x - \frac{1}{2}m\pi$ radians

$\delta'_m \to x - \frac{1}{2}\pi(m+1)$ radians

x	$D_0(x)$	$\delta_0(x)$	$D'_0(x)$	$\delta'_0(x)$	$D_1(x)$	$\delta_1(x)$	$D'_1(x)$	$\delta'_1(x)$
0.1	10.000	05.73°	100.50	00.02°	100.50	00.02°	2 000.0	−00.01°
0.2	5.0000	11.46	25.495	0.15	25.495	0.15	250.05	0.08
0.3	3.3333	17.19	11.600	0.49	11.600	0.49	74.149	0.25
0.4	2.5000	22.92	6.7315	1.12	6.7315	1.12	31.350	0.58
0.5	2.0000	28.65	4.4721	2.08	4.4721	2.08	16.125	−1.10
0.6	1.6667	34.38	3.2394	3.41	3.2394	3.41	9.4081	1.82
0.7	1.4285	40.11	2.4911	5.10	2.4911	5.11	6.0034	2.73
0.8	1.2500	45.84	2.0010	7.18	2.0010	7.18	4.1014	3.80
0.9	1.1111	51.57	1.6609	9.58	1.6609	9.58	2.9599	4.96
1.0	1.0000	57.30	1.4142	12.30	1.4142	12.30	2.2361	−6.14
1.2	0.8333	68.75	1.0848	18.56	1.0848	18.56	1.4262	8.11
1.4	0.7143	80.21	0.8778	25.75	0.8778	25.75	1.0205	8.97
1.6	0.6250	91.67	0.7370	33.68	0.7370	33.68	0.7931	8.25
1.8	0.5556	103.13	0.6355	42.19	0.6355	42.19	0.6529	5.87
2.0	0.5000	114.59	0.5590	51.16	0.5590	51.16	0.5590	−1.97
2.2	0.4545	126.05	0.4993	60.49	0.4993	60.49	0.4918	+3.21
2.4	0.4167	137.51	0.4514	70.13	0.4514	70.13	0.4411	9.44
2.6	0.3846	148.97	0.4121	80.01	0.4121	80.01	0.4011	16.50
2.8	0.3571	160.43	0.3792	90.08	0.3792	90.08	0.3686	24.23
3.0	0.3333	171.89	0.3514	100.32	0.3514	100.32	0.3415	32.49
3.2	0.3125	183.35	0.3274	110.70	0.3274	110.70	0.3184	41.18
3.4	0.2941	194.81	0.3066	121.20	0.3066	121.20	0.2985	50.23
3.6	0.2778	206.25	0.2883	131.79	0.2883	131.79	0.2811	59.57
3.8	0.2632	217.72	0.2721	142.47	0.2721	142.47	0.2657	69.15
4.0	0.2500	229.18	0.2577	153.22	0.2577	153.22	0.2519	78.92
4.2	0.2081	240.64	0.2448	164.03	0.2448	164.03	0.2396	88.88
4.4	0.2273	252.10	0.2331	174.91	0.2331	174.91	0.2285	98.97
4.6	0.2174	263.56	0.2225	185.83	0.2225	185.83	0.2184	109.20
4.8	0.2083	275.02	0.2128	196.78	0.2128	196.78	0.2091	119.55
5.0	0.2000	286.48	0.2040	207.79	0.2040	207.79	0.2006	129.98

Table XV. Amplitudes and Phases for Spherical Bessel Functions (*Continued*)

x	$D_2(x)$	$\delta_2(x)$	$D'_2(x)$	$\delta'_2(x)$	$D_3(x)$	$\delta_3(x)$	$D'_3(x)$	$\delta'_3(x)$
0.1	3 005.0	00.00°	90 050.	00.00°	150 150	00.00°	6 003 000	00.00°
0.2	377.53	0.00	5637.4	0.00	9 412.6	0.00	187 875	0.00
0.4	48.174	0.01	354.57	−0.01	595.44	0.00	5 906.2	0.00
0.6	14.793	0.09	70.730	0.06	120.04	0.00	785.47	0.00
0.8	6.5741	0.36	22.667	0.25	39.102	0.01	188.94	−0.01
1.0	3.6056	0.99	9.4340	−0.70	16.643	0.03	62.968	−0.02
1.2	2.2705	2.18	4.6457	1.59	8.4253	0.10	25.816	0.08
1.4	1.5768	4.12	2.5833	3.07	4.8265	0.28	12.217	0.22
1.6	1.1768	6.91	1.5836	5.19	3.0376	0.64	6.4256	0.51
1.8	0.9268	10.59	1.0572	7.77	2.0603	1.29	3.6669	1.05
2.0	0.7603	15.13	0.7629	−10.40	1.4856	2.34	2.2361	−1.97
2.2	0.6434	20.47	0.5901	12.49	1.1268	3.92	1.4436	3.38
2.4	0.5578	26.54	0.4837	13.51	0.8913	6.10	0.9823	5.34
2.6	0.4927	33.23	0.4148	13.14	0.7298	8.94	0.7036	7.80
2.8	0.4416	40.48	0.3676	11.31	0.6149	12.46	0.5308	10.54
3.0	0.4006	48.20	0.3333	−8.11	0.5303	16.66	0.4214	−13.16
3.2	0.3669	56.32	0.3071	−3.73	0.4661	21.52	0.3508	15.17
3.4	0.3388	64.80	0.2862	+1.65	0.4161	26.95	0.3042	16.17
3.6	0.3149	73.59	0.2688	7.86	0.3762	32.94	0.2723	15.94
3.8	0.2943	82.62	0.2540	14.75	0.3437	39.43	0.2496	14.41
4.0	0.2764	91.89	0.2411	22.20	0.3168	46.36	0.2326	−11.67
4.2	0.2607	101.36	0.2296	30.12	0.2941	53.68	0.2193	7.84
4.4	0.2468	111.00	0.2194	38.44	0.2747	61.36	0.2084	−3.08
4.6	0.2343	120.79	0.2101	47.08	0.2579	69.36	0.1992	+2.47
4.8	0.2231	130.72	0.2016	56.00	0.2433	77.63	0.1912	8.70
5.0	0.2130	140.77	0.1939	65.16	0.2303	86.16	0.1840	15.48

x	$D_8(x)$	$\delta_8(x)$	$D'_8(x)$	$\delta'_8(x)$	$D_9(x)$	$\delta_9(x)$	$D'_9(x)$	$\delta'_9(x)$
2.0	4 530.1	00.00°	19 768.	00.00°	37 889.	00.00°	184 915	00.00°
2.2	1 977.1	0.00	7 790.5	0.00	14 980.	0.00	66 113.	0.00
2.4	932.47	0.00	3 342.8	0.00	6 451.1	0.00	25 947.	0.00
2.6	469.63	0.00	1 541.1	0.00	2 986.2	0.00	11 016.	0.00
2.8	250.25	0.00	755.58	0.00	1 470.6	0.00	5 001.8	0.00
3.0	140.06	0.00	390.70	0.00	764.20	0.00	2 407.3	0.00
3.2	81.850	0.00	211.68	0.00	416.31	0.00	1 219.1	0.00
3.4	49.707	0.00	119.52	0.00	236.48	0.00	645.82	0.00
3.6	31.246	0.00	70.012	0.00	139.45	0.00	356.11	0.00
3.8	20.265	0.00	42.388	0.00	85.051	0.00	203.55	0.00
4.0	13.523	0.01	26.440	−0.00	53.485	0.00	120.19	0.00
4.2	9.2642	0.01	16.944	−0.01	34.591	0.00	73.094	0.00
4.4	6.5027	0.02	11.131	0.02	22.954	0.00	45.665	0.00
4.6	4.6692	0.04	7.4788	0.04	15.599	0.00	29.242	0.00
4.8	3.4251	0.07	5.1305	0.07	10.839	0.01	19.156	0.00
5.0	2.5638	0.13	3.5874	−0.12	7.6895	0.01	12.815	−0.01

Table XV. Amplitudes and Phases for Spherical Bessel Functions (*Continued*)

x	$D_4(x)$	$\delta_4(x)$	$D'_4(x)$	$\delta'_4(x)$	$D_5(x)$	$\delta_5(x)$	$D'_5(x)$	$\delta'_5(x)$
0.6	1 385.7	00.00°	11 427.	00.00°	20 665.	00.00°	205 264	00.00°
0.8	335.57	0.00	2 058.2	0.00	3 736.1	0.00	27 685.	0.00
1.0	112.90	0.00	547.85	0.00	999.44	0.00	5 883.7	0.00
1.2	46.879	0.00	186.90	0.00	343.17	0.00	1 668.9	0.00
1.4	22.559	0.01	75.742	−0.01	140.20	0.00	578.29	0.00
1.6	12.121	0.03	34.839	0.02	65.140	0.00	232.16	0.00
1.8	7.0994	0.08	17.661	0.06	33.437	0.00	104.36	0.00
2.0	4.4613	0.18	9.6689	−0.15	18.591	0.01	51.313	−0.01
2.2	2.9741	0.38	5.6351	0.32	11.042	0.02	27.141	0.02
2.4	2.0859	0.74	3.4594	0.64	6.9354	0.05	15.253	0.04
2.6	1.5294	1.32	2.2198	1.18	4.5716	0.11	9.0209	0.10
2.8	1.1660	2.22	1.4811	2.02	3.1446	0.22	5.5733	0.20
3.0	0.9201	3.50	1.0242	−3.27	2.2471	0.42	3.5758	−0.37
3.2	0.7483	5.25	0.7334	5.00	1.6622	0.74	2.3714	0.68
3.4	0.6248	7.51	0.5443	7.23	1.2690	1.25	1.6198	1.16
3.6	0.5336	10.33	0.4195	9.83	0.9972	1.99	1.1367	1.91
3.8	0.4646	13.72	0.3364	12.58	0.8045	3.05	0.8183	2.99
4.0	0.4112	17.68	0.2807	−15.10	0.6648	4.47	0.6043	−4.48
4.2	0.3690	22.19	0.2432	17.00	0.5613	6.30	0.4583	6.43
4.4	0.3350	27.22	0.2174	17.95	0.4831	8.60	0.3577	8.79
4.6	0.3071	32.73	0.1994	17.79	0.4228	11.38	0.2881	11.45
4.8	0.2838	38.69	0.1863	16.47	0.3755	14.66	0.2399	14.15
5.0	0.2642	45.06	0.1764	−14.04	0.3378	18.43	0.2065	−16.56

x	$D_6(x)$	$\delta_6(x)$	$D'_6(x)$	$\delta'_6(x)$	$D_7(x)$	$\delta_7(x)$	$D'_7(x)$	$\delta'_7(x)$
1.0	10 881.	00.00°	75 167.	00.00°	140 453	00.00°	1 112 740	00.00°
1.2	3 098.8	0.00	17 733.	0.00	33 227.	0.00	218 416	0.00
1.4	1 079.0	0.00	5 254.7	0.00	9 879.0	0.00	55 372.	0.00
1.6	435.72	0.00	1 841.1	0.00	3 475.1	0.00	16 940.	0.00
1.8	197.24	0.00	733.60	0.00	1 391.1	0.00	5 985.2	0.00
2.0	97.792	0.00	323.68	0.00	617.05	0.00	2 370.4	0.00
2.2	52.238	0.00	155.17	0.00	297.64	0.00	1 030.1	0.00
2.4	29.702	0.00	79.694	0.00	153.95	0.00	483.46	0.00
2.6	17.812	0.01	43.385	−0.01	84.491	0.00	242.16	0.00
2.8	11.189	0.01	24.827	0.01	48.802	0.00	128.25	0.00
3.0	7.3207	0.03	14.835	−0.03	29.476	0.00	71.282	0.00
3.2	4.9682	0.06	9.2059	0.06	18.521	0.00	41.335	0.00
3.4	3.4852	0.13	5.9066	0.11	12.057	0.01	24.884	−0.01
3.6	2.5202	0.23	3.9037	0.21	8.1040	0.02	15.489	0.02
3.8	1.8743	0.41	2.6493	0.38	5.6086	0.04	9.9334	0.03
4.0	1.4311	0.70	1.8415	−0.66	3.9878	0.07	6.5446	−0.06
4.2	1.1198	1.13	1.3083	1.09	2.9075	0.13	4.4184	0.12
4.4	0.8967	1.75	0.9486	1.73	2.1704	0.23	3.0499	0.22
4.6	0.7336	2.62	0.7015	2.65	1.6567	0.39	2.1483	0.37
4.8	0.6124	3.78	0.5290	3.92	1.2916	0.64	1.5417	0.61
5.0	0.5206	5.29	0.4072	−5.58	1.0276	1.00	1.1256	−0.98

Table XVI. Periodic Mathieu Functions

(See pages 1408 and 1568)

	$x = 0°$	10°	20°	30°	40°	50°	60°	70°	80°	90°
					$Se_0(h, \cos x)$					
$h^2 = 0$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1	1.0000	1.0080	1.0313	1.0674	1.1126	1.1617	1.2089	1.2480	1.2739	1.2829
2	1.0000	1.0170	1.0666	1.1448	1.2445	1.3550	1.4633	1.5549	1.6162	1.6379
3	1.0000	1.0268	1.1057	1.2319	1.3956	1.5812	1.7667	1.9262	2.0344	2.0728
4	1.0000	1.0373	1.1481	1.3279	1.5657	1.8408	2.1212	2.3663	2.5345	2.5946
5	1.0000	1.0485	1.1935	1.4323	1.7542	2.1340	2.5286	2.8789	3.1221	3.2094
6	1.0000	1.0601	1.2415	1.5445	1.9604	2.4610	2.9906	3.4679	3.8029	3.9238
7	1.0000	1.0721	1.2917	1.6638	2.1840	2.8222	3.5092	4.1376	4.5831	4.7447
8	1.0000	1.0845	1.3439	1.7899	2.4248	3.2183	4.0870	4.8927	5.4696	5.6799
9	1.0000	1.0972	1.3979	1.9226	2.6828	3.6501	4.7268	5.7387	6.4701	6.7379
					$Se_1(h, \cos x)$					
$h^2 = 0$	1.0000	0.9848	0.9397	0.8660	0.7660	0.6428	0.5000	0.3420	0.1736	0.0000
1	1.0000	0.9886	0.9539	0.8943	0.8076	0.6927	0.5499	0.3825	0.1963	0.0000
2	1.0000	0.9927	0.9693	0.9250	0.8535	0.7486	0.6066	0.4289	0.2225	0.0000
3	1.0000	0.9971	0.9858	0.9585	0.9042	0.8112	0.6711	0.4822	0.2527	0.0000
4	1.0000	1.0018	1.0037	0.9951	0.9603	0.8815	0.7443	0.5434	0.2877	0.0000
5	1.0000	1.0069	1.0230	1.0351	1.0224	0.9604	0.8275	0.6138	0.3282	0.0000
6	1.0000	1.0123	1.0438	1.0786	1.0910	1.0489	0.9220	0.6944	0.3748	0.0000
7	1.0000	1.0180	1.0662	1.1261	1.1668	1.1479	1.0292	0.7867	0.4286	0.0000
8	1.0000	1.0242	1.0902	1.1777	1.2503	1.2584	1.1502	0.8921	0.4904	0.0000
9	1.0000	1.0307	1.1160	1.2335	1.3419	1.3815	1.2866	1.0119	0.5610	0.0000
					$Se_2(h, \cos x)$					
$h^2 = 0$	1.0000	0.9397	0.7660	0.5000	0.1736	−0.1736	−0.5000	−0.7660	−0.9397	−1.0000
1	1.0000	0.9467	0.7917	0.5496	0.2451	−0.0882	−0.4103	−0.6794	−0.8582	−0.9208
2	1.0000	0.9530	0.8147	0.5944	0.3098	−0.0110	−0.3298	−0.6026	−0.7869	−0.8522
3	1.0000	0.9586	0.8355	0.6348	0.3681	+0.0581	−0.2589	−0.5366	−0.7275	−0.7956
4	1.0000	0.9638	0.8544	0.6716	0.4209	0.1200	−0.1968	−0.4810	−0.6797	−0.7512
5	1.0000	0.9686	0.8720	0.7057	0.4696	0.1764	−0.1420	−0.4345	−0.6424	−0.7178
6	1.0000	0.9731	0.8885	0.7379	0.5154	0.2285	−0.0929	−0.3955	−0.6142	−0.6942
7	1.0000	0.9774	0.9044	0.7689	0.5593	0.2779	−0.0481	−0.3627	−0.5937	−0.6790
8	1.0000	0.9816	0.9200	0.7992	0.6023	0.3256	−0.0062	−0.3346	−0.5798	−0.6710
9	1.0000	0.9857	0.9355	0.8294	0.6450	0.3727	+0.0338	−0.3103	−0.5715	−0.6695
					$Se_3(h, \cos x)$					
$h^2 = 0$	1.0000	0.8660	0.5000	0.0000	−0.5000	−0.8660	−1.0000	−0.8660	−0.5000	0.0000
1	1.0000	0.8732	0.5242	+0.0407	−0.4530	−0.8261	−0.9753	−0.8562	−0.4981	0.0000
2	1.0000	0.8802	0.5481	0.0815	−0.4052	−0.7842	−0.9484	−0.8443	−0.4952	0.0000
3	1.0000	0.8871	0.5717	0.1221	−0.3566	−0.7410	−0.9195	−0.8307	−0.4912	0.0000
4	1.0000	0.8938	0.5949	0.1625	−0.3078	−0.6965	−0.8891	−0.8155	−0.4863	0.0000
5	1.0000	0.9004	0.6176	0.2023	−0.2590	−0.6513	−0.8572	−0.7991	−0.4807	0.0000
6	1.0000	0.9067	0.6398	0.2415	−0.2104	−0.6057	−0.8245	−0.7820	−0.4747	0.0000
7	1.0000	0.9128	0.6612	0.2798	−0.1625	−0.5602	−0.7916	−0.7643	−0.4683	0.0000
8	1.0000	0.9187	0.6820	0.3171	−0.1154	−0.5151	−0.7587	−0.7467	−0.4620	0.0000
9	1.0000	0.9244	0.7020	0.3533	−0.0694	−0.4709	−0.7263	−0.7295	−0.4558	0.0000
					$Se_4(h, \cos x)$					
$h^2 = 0$	1.0000	0.7660	0.1736	−0.5000	−0.9397	−0.9397	−0.5000	0.1736	0.7660	1.0000
1	1.0000	0.7730	0.1944	−0.4727	−0.9214	−0.9410	−0.5181	0.1504	0.7465	0.9835
2	1.0000	0.7798	0.2153	−0.4448	−0.9019	−0.9407	−0.5349	0.1279	0.7274	0.9671
3	1.0000	0.7867	0.2361	−0.4165	−0.8813	−0.9390	−0.5505	0.1061	0.7085	0.9510
4	1.0000	0.7935	0.2570	−0.3876	−0.8595	−0.9358	−0.5648	0.0851	0.6900	0.9350
5	1.0000	0.8002	0.2779	−0.3584	−0.8367	−0.9311	−0.5778	0.0648	0.6716	0.9190
6	1.0000	0.8069	0.2988	−0.3287	−0.8127	−0.9249	−0.5895	0.0451	0.6535	0.9031
7	1.0000	0.8136	0.3197	−0.2986	−0.7877	−0.9173	−0.6000	0.0262	0.6356	0.8872
8	1.0000	0.8201	0.3405	−0.2681	−0.7617	−0.9083	−0.6093	+0.0080	0.6179	0.8714
9	1.0000	0.8267	0.3613	−0.2374	−0.7348	−0.8979	−0.6173	−0.0096	0.6004	0.8555

Table XVI. Periodic Mathieu Functions.—(*Continued*)

$So_1(h, \cos x)$

	$x = 0°$	10°	20°	30°	40°	50°	60°	70°	80°	90°
$h^2 = 0$	0.0000	0.1736	0.3420	0.5000	0.6428	0.7660	0.8660	0.9397	0.9848	1.0000
1	0.0000	0.1743	0.3471	0.5159	0.6769	0.8242	0.9507	1.0484	1.1104	1.1317
2	0.0000	0.1750	0.3523	0.5325	0.7129	0.8864	1.0424	1.1675	1.2490	1.2773
3	0.0000	0.1757	0.3577	0.5498	0.7507	0.9528	1.1415	1.2976	1.4013	1.4378
4	0.0000	0.1764	0.3632	0.5676	0.7905	1.0235	1.2484	1.4393	1.5684	1.6141
5	0.0000	0.1771	0.3688	0.5861	0.8321	1.0985	1.3634	1.5933	1.7511	1.8075
6	0.0000	0.1778	0.3745	0.6051	0.8757	1.1782	1.4869	1.7602	1.9505	2.0188
7	0.0000	0.1785	0.3804	0.6248	0.9213	1.2625	1.6192	1.9409	2.1675	2.2494
8	0.0000	0.1793	0.3864	0.6451	0.9688	1.3517	1.7609	2.1360	2.4033	2,5005
9	0.0000	0.1800	0.3925	0.6660	1.0184	1.4460	1.9122	2.3464	2.6590	2.7733

$So_2(h, \cos x)$

	$x = 0°$	10°	20°	30°	40°	50°	60°	70°	80°	90°
$h^2 = 0$	0.0000	0.1710	0.3214	0.4330	0.4924	0.4924	0.4330	0.3214	0.1710	0.0000
1	0.0000	0.1714	0.3246	0.4422	0.5098	0.5172	0.4610	0.3460	0.1854	0.0000
2	0.0000	0.1719	0.3278	0.4517	0.5279	0.5434	0.4910	0.3725	0.2010	0.0000
3	0.0000	0.1723	0.3312	0.4616	0.5470	0.5712	0.5230	0.4010	0.2179	0.0000
4	0.0000	0.1728	0.3346	0.4718	0.5669	0.6006	0.5572	0.4318	0.2362	0.0000
5	0.0000	0.1733	0.3382	0.4824	0.5877	0.6316	0.5937	0.4648	0.2560	0.0000
6	0.0000	0.1738	0.3418	0.4934	0.6095	0.6644	0.6326	0.5004	0.2774	0.0000
7	0.0000	0.1743	0.3455	0.5047	0.6322	0.6990	0.6741	0.5386	0.3005	0.0000
8	0.0000	0.1748	0.3494	0.5164	0.6560	0.7356	0.7183	0.5796	0.3254	0.0000
9	0.0000	0.1753	0.3533	0.5285	0.6807	0.7741	0.7654	0.6236	0.3523	0.0000

$So_3(h, \cos x)$

	$x = 0°$	10°	20°	30°	40°	50°	60°	70°	80°	90°
$h^2 = 0$	0.0000	0.1667	0.2887	0.3333	0.2887	0.1667	0.0000	−0.1667	−0.2887	−0.3333
1	0.0000	0.1671	0.2916	0.3411	0.3017	0.1822	+0.0135	−0.1587	−0.2866	−0.3337
2	0.0000	0.1675	0.2945	0.3489	0.3147	0.1977	0.0270	−0.1511	−0.2852	−0.3349
3	0.0000	0.1679	0.2974	0.3567	0.3277	0.2134	0.0404	−0.1438	−0.2844	−0.3369
4	0.0000	0.1683	0.3003	0.3644	0.3409	0.2292	0.0540	−0.1368	−0.2843	−0.3398
5	0.0000	0.1687	0.3032	0.3722	0.3542	0.2452	0.0676	−0.1300	−0.2848	−0.3434
6	0.0000	0.1691	0.3060	0.3801	0.3676	0.2615	0.0814	−0.1233	−0.2859	−0.3479
7	0.0000	0.1695	0.3089	0.3881	0.3813	0.2781	0.0955	−0.1168	−0.2876	−0.3532
8	0.0000	0.1699	0.3118	0.3961	0.3953	0.2951	0.1098	−0.1104	−0.2900	−0.3593
9	0.0000	0.1704	0.3148	0.4042	0.4095	0.3126	0.1246	−0.1040	−0.2929	−0.3662

$So_4(h, \cos x)$

	$x = 0°$	10°	20°	30°	40°	50°	60°	70°	80°	90°
$h^2 = 0$	0.0000	0.1607	0.2462	0.2165	0.0855	−0.0855	−0.2165	−0.2462	−0.1607	0.0000
1	0.0000	0.1611	0.2489	0.2228	0.0941	−0.0783	−0.2137	−0.2476	−0.1630	0.0000
2	0.0000	0.1615	0.2516	0.2292	0.1028	−0.0709	−0.2108	−0.2490	−0.1653	0.0000
3	0.0000	0.1619	0.2542	0.2355	0.1116	−0.0634	−0.2079	−0.2504	−0.1677	0.0000
4	0.0000	0.1623	0.2569	0.2419	0.1205	−0.0558	−0.2048	−0.2517	−0.1701	0.0000
5	0.0000	0.1627	0.2596	0.2483	0.1294	−0.0480	−0.2016	−0.2532	−0.1726	0.0000
6	0.0000	0.1631	0.2622	0.2547	0.1385	−0.0401	−0.1983	−0.2546	−0.1751	0.0000
7	0.0000	0.1635	0.2649	0.2612	0.1476	−0.0320	−0.1950	−0.2561	−0.1778	0.0000
8	0.0000	0.1639	0.2675	0.2676	0.1569	−0.0238	−0.1915	−0.2576	−0.1805	0.0000
9	0.0000	0.1643	0.2701	0.2741	0.1662	−0.0155	−0.1881	−0.2593	−0.1834	0.0000

Table XVII. Normalizing Constants for Periodic Mathieu Functions and Limiting Values of Radial Mathieu Functions

(See pages 563 and 1569)

$$M_m^e(h) = \int_0^{2\pi} [Se_m(h, \cos x)]^2\,dx; \quad Je_m(h,1) + iNe_m(h,1) = -C_m^e i e^{i\delta_m^e}$$

h^2	$M_0^e(h)$	$Je_0\ (h,\ 1)$	$C_0^e(h)$	$\delta_0^e(h)$	$M_1^e(h)$	$Je_1\ (h,\ 1)$	$C_1^e(h)$	$\delta_1^e(h)$
0	6.2832	1.2533	∞	00.00°	3.1416	0.0000	∞	00.00°
1	8.2179	1.1130	1.2544	62.53	3.3555	0.5705	2.0043	16.54
2	11.086	1.0017	1.0497	72.60	3.6100	0.7345	1.5528	28.23
3	15.262	0.9140	0.9331	78.39	3.9146	0.8192	1.3429	37.59
4	21.214	0.8448	0.8530	82.04	4.2815	0.8619	1.2106	45.40
5	29.521	0.7896	0.7933	84.43	4.7254	0.8791	1.1152	52.03
6	40.896	0.7450	0.7468	86.03	5.2646	0.8798	1.0407	57.71
7	56.211	0.7084	0.7093	87.13	5.9216	0.8698	0.9798	62.59
8	76.524	0.6779	0.6784	87.89	6.7233	0.8532	0.9283	66.79
9	103.12	0.6522	0.6524	88.44	7.7022	0.8325	0.8838	70.38

h^2	$M_2^e(h)$	$Je_2\ (h,\ 1)$	$C_2^e(h)$	$\delta_2^e(h)$	$M_3^e(h)$	$Je_3\ (h,\ 1)$	$C_3^e(h)$	$\delta_3^e(h)$
0	3.1416	0.0000	∞	00.00°	3.1416	0.0000	∞	00.00°
1	2.9205	0.08116	6.7431	0.69	3.0431	0.00663	51.926	0.01
2	2.7755	0.1660	3.6093	2.64	2.9451	0.01906	18.717	0.06
3	2.6945	0.2510	2.5874	5.57	2.8495	0.03557	10.418	0.20
4	2.6639	0.3333	2.0862	9.19	2.7582	0.05560	6.9408	0.46
5	2.6720	0.4102	1.7892	13.26	2.6729	0.07882	5.1101	0.88
6	2.7101	0.4802	1.5921	17.55	2.5953	0.1049	4.0117	1.50
7	2.7725	0.5423	1.4508	21.95	2.5266	0.1337	3.2943	2.33
8	2.8559	0.5961	1.3435	26.34	2.4679	0.1648	2.7970	3.38
9	2.9585	0.6418	1.2585	30.66	2.4197	0.1978	2.4364	4.66

h^2	$M_4^e(h)$	$Je_4\ (h,\ 1)$	$C_4^e(h)$	$\delta_4^e(h)$
0	3.1416	0.0000	∞	00.00°
1	3.0893	0.00041	618.07	0.00
2	3.0369	0.00166	155.94	0.00
3	2.9845	0.00377	69.992	0.00
4	2.9319	0.00675	39.786	0.01
8	2.8793	0.01064	25.751	0.02
6	2.8266	0.01544	18.099	0.05
7	2.7740	0.02120	13.470	0.09
8	2.7216	0.02793	10.457	0.15
9	2.6696	0.03564	8.3864	0.24

As $h \to 0$

$$C_0^e \to -\sqrt{\tfrac{1}{2}\pi}\left[1 + \left(\frac{2}{\pi}\right)^2 \ln^2(0.4453h)\right]^{\frac{1}{2}}$$

$$\delta_0^e \to \frac{-90}{\ln(0.4453h)} \text{ degrees}$$

$$Je'_m(h,\ 1) = [(d/dx)Je_m(h,\ \cos x)]_{x=0} = 0; \quad Ne'_m(h,\ 1) = [1/Je_m(h,\ 1)]$$

As $h \to 0$, $M_m^e \to (2\pi/\epsilon_m)$; $Je_m(h,\ 1) \to \sqrt{\tfrac{1}{2}\pi}\ (\epsilon_m h^m/4^m m!)$

for $m > 0$, $C_m^e \to -Ne_m(h,\ 1) \to \dfrac{4^m(m-1)!}{\sqrt{2\pi}\ h^m}$; $\delta_m \to \dfrac{360h^{2m}}{4^{2m}m!(m-1)!}$ degrees

Table XVII. Normalizing Constants for Periodic Mathieu Functions and Limiting Values of Radial Mathieu Functions.—(*Continued*)

$$M_m^0(h) = \int_0^{2\pi} [So_m(h, \cos x)]^2\,dx; \quad Jo'_m(h, 1) + iNo'_m(h, 1) = iC_m^0 e^{i\delta_m^0}$$

h^2	$M_1^0(h)$	$Jo'_1(h, 1)$	$C_1^0(h)$	$\delta_1^0(h)$	$M_2^0(h)$	$Jo'_2(h, 1)$	$C_2^0(h)$	$\delta_2^0(h)$
0	3.1416	0.0000	∞	00.00°	0.7854	0.0000	∞	00.00°
1	3.7914	0.6081	1.1995	−30.46	0.8547	0.1503	12.096	−0.70
2	4.5780	0.8361	0.9903	−57.60	0.9322	0.2887	5.6559	−2.93
3	5.5267	0.9977	1.0493	−71.95	1.0190	0.4162	3.5103	−6.81
4	6.6672	1.1243	1.1448	−79.15	1.1164	0.5338	2.4623	−12.52
5	8.0336	1.2288	1.2378	−83.06	1.2255	0.6424	1.8727	−20.06
6	9.6652	1.3177	1.3221	−85.35	1.3480	0.7428	1.5281	−29.08
7	11.608	1.3952	1.3974	−86.78	1.4854	0.8357	1.3336	−38.80
8	13.913	1.4639	1.4651	−87.71	1.6395	0.9218	1.2360	−48.23
9	16.641	1.5255	1.5262	−88.33	1.8125	1.0018	1.2003	−56.58

h^2	$M_3^0(h)$	$Jo'_3(h, 1)$	$C_3^0(h)$	$\delta_3^0(h)$	$M_4^0(h)$	$Jo'_4(h, 1)$	$C_4^0(h)$	$\delta_4^0(h)$
0	0.3491	0.0000	∞	00.00°	0.1963	0.0000	∞	00.00°
1	0.3607	0.01926	150.76	−0.01	0.1997	0.00162	2430.8	−0.00
2	0.3739	0.05349	52.391	−0.06	0.2033	0.00641	602.64	−0.00
3	0.3887	0.09635	27.983	−0.20	0.2072	0.01429	265.59	−0.00
4	0.4050	0.1452	17.800	−0.47	0.2113	0.02514	148.12	−0.01
5	0.4231	0.1985	12.448	−0.91	0.2158	0.03885	93.974	−0.02
6	0.4429	0.2548	9.2346	−1.58	0.2205	0.05530	64.675	−0.05
7	0.4645	0.3134	7.1324	−2.52	0.2256	0.07436	47.075	−0.09
8	0.4882	0.3734	5.6718	−3.77	0.2310	0.09588	35.693	−0.15
9	0.5139	0.4342	4.6125	−5.40	0.2368	0.1197	27.916	−0.25

$$Jo_m(h,1) = 0; \quad Jo'_m(h,1) = [(d/dx)Jo_m(h, \cos x)]_{x=0} = -[1/No_m(h,1)]$$

$$\text{As } h \to 0, \quad M_m^0 \to (\pi/m^2); \quad Jo'_m(h,1) \to \sqrt{2\pi}\,[h^m/4^m(m-1)!]$$

$$C_m^0 \to No'_m(h,1) \to \frac{4^m m!}{\sqrt{2\pi}\,h^m}; \quad \delta_m^0 \to \frac{-360h^{2m}}{4^{2m}m!(m-1)!} \text{ degrees}$$

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