SECOND OVERLAP CALCULATION

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We begin with the form of a single mode (using $z_r = \frac{\pi\omega_0^2}{\lambda}$)

$$A_{m,n}(x,y,z;k,\omega_0,z_0) = \sqrt{\frac{\left(1+i\frac{(z-z_0)}{z_r}\right)^{m+n}}{2^{m+n-1}\pi m!n!\left(1-i\frac{(z-z_0)}{z_r}\right)^{m+n+2}}}\frac{\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{(z-z_0)}{z_r}\right)}-ik(z-z_0)}}{\omega_0}$$

$$\times H_m\left(\frac{\sqrt{2}x}{\omega_0\sqrt{1+\left(\frac{z-z_0}{z_r}\right)^2}}\right)H_n\left(\frac{\sqrt{2}y}{\omega_0\sqrt{1+\left(\frac{z-z_0}{z_r}\right)^2}}\right)$$

so that if we call $P \equiv \sqrt{\omega_0\left(1+i\frac{z-z_0}{z_r}\right)}$ and $M \equiv \sqrt{\omega_0\left(1-i\frac{z-z_0}{z_r}\right)}$ we get

$$A_{m,n}(\rho,\theta,z;k,\omega_0,z_0) = \frac{P^{m+n}\mathrm{e}^{\frac{-\rho^2}{\omega_0 M^2}-ik(z-z_0)}}{\sqrt{2^{m+n-1}\pi m!n!}M^{m+n+2}}H_m\left(\frac{\sqrt{2}\rho\cos\theta}{MP}\right)H_n\left(\frac{\sqrt{2}\rho\sin\theta}{MP}\right)$$

So that we have using o's for outgoing beam and i's for incoming and $P_{i/o} = \sqrt{\omega_0\left(1+i\frac{z-z_{i/0}}{z_{ri/ro}}\right)}$

$$\langle m,n\,|\,c,d\rangle\,(z,z_o,z_i,\omega_o,\omega_i,k_o,k_i,R)$$

$$= \frac{2P_i^{c+d}M_o^{m+n}\mathrm{e}^{i(k_o(z-z_o)-k_i(z-z_i))}}{\pi\sqrt{2^{m+n+c+d}m!n!c!d!}M_i^{c+d+2}P_o^{m+n+2}}\int_0^R \mathrm{d}\rho\rho\mathrm{e}^{-\rho^2\left(\frac{1}{\omega_i M_i^2}+\frac{1}{\omega_o P_o^2}\right)}\int_0^{2\pi}\mathrm{d}\theta$$

$$\left[H_m\left(\frac{\sqrt{2}\rho\cos\theta}{M_oP_o}\right)H_n\left(\frac{\sqrt{2}\rho\sin\theta}{M_oP_o}\right)H_c\left(\frac{\sqrt{2}\rho\cos\theta}{M_iP_i}\right)H_d\left(\frac{\sqrt{2}\rho\sin\theta}{M_iP_i}\right)\right]$$

at this point we need to use the actual form of the hermite polynomials.

$$H_m(x) = m!\sum_{j=0}^{m/2}\frac{(-1)^j(2x)^{m-2j}}{j!(m-2j)!}$$

So that we have

$$\langle m,n\,|\,c,d\rangle\,(z,z_o,z_i,\omega_o,\omega_i,k_o,k_i,R)$$

$$= \frac{\sqrt{m!n!c!d!}P_i^{c+d}M_o^{m+n}\mathrm{e}^{i(k_o(z-z_o)-k_i(z-z_i))}}{\pi\sqrt{2^{m+n+c+d-2}}M_i^{c+d+2}P_o^{m+n+2}}\sum_{f=0}^{m/2}\sum_{g=0}^{n/2}\sum_{h=0}^{c/2}\sum_{l=0}^{d/2}\int_0^R \rho\mathrm{d}\rho\mathrm{e}^{-\frac{\rho^2}{\omega_o\omega_iM_i^2P_o^2}\left(\omega_oP_o^2+\omega_iM_i^2\right)}\int_0^{2\pi}\mathrm{d}\theta$$

$$\left[\frac{(-1)^{f+g+h+l}(2^{3/2}\rho)^{m+n+c+d-2(f+g+h+l)}(\sin\theta)^{n+d-2(g+l)}(\cos\theta)^{m+c-2(f+h)}}{f!g!h!l!(m-2f)!(n-2g)!(c-2h)!(d-2l)!(M_iP_i)^{c+d-2(h+l)}(M_oP_o)^{m+n-2(f+g)}}\right]$$

We saw before that

$$\int_0^{2\pi}\mathrm{d}\theta\sin^m\theta\cos^n\theta=\frac{2\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m+n+2}{2}\right)}$$

if both m and n were even but 0 otherwise. This tells us $n+d$ and $m+c$ must both be even, or in other words must have the same sine, so we get zero for $x-y$ indices not matching in sign. This also means we can write

$$n+d\equiv 2T\qquad m+c\equiv 2Y$$

So that

$$\langle m,n\,|\,c,d\rangle\,(z,z_o,z_i,\omega_o,\omega_i,k_o,k_i,R)=$$

$$=\frac{\sqrt{m!n!c!d!}\mathrm{e}^{i(k_o(z-z_o)-k_i(z-z_i))}}{\sqrt{2^{m+n+c+d}\pi}M_i^{2(c+d+1)}P_o^{2(m+n+1)}}\sum_{f=0}^{m/2}\sum_{g=0}^{n/2}\sum_{h=0}^{c/2}\sum_{l=0}^{d/2}\int_0^R\rho\mathrm{d}\rho\mathrm{e}^{-\frac{\rho^2}{\omega_o\omega_iM_i^2P_o^2}\left(\omega_oP_o^2+\omega_iM_i^2\right)}\int_0^{2\pi}\mathrm{d}\theta$$

$$\left[\frac{(-1)^{f+g+h+l}(M_iP_i)^{2(h+l)}(M_oP_o)^{2(f+g)}(2^{3/2}\rho)^{2(T+Y-(f+g+h+l))}(\sin\theta)^{2(T-(g+l))}(\cos\theta)^{2(Y-(f+h))}}{f!g!h!l!(m-2f)!(n-2g)!(c-2h)!(d-2l)!}\right]$$

so we have integrals of the form (with the same scripts)

$$\int_0^{2\pi}\mathrm{d}\theta(\sin\theta)^{2(T-(g+l))}(\cos\theta)^{2(Y-(f+h))}=\frac{2\Gamma\left(T-(g+l)+\frac{1}{2}\right)\Gamma\left(Y-(f+h)+\frac{1}{2}\right)}{\Gamma\left(T+Y-(f+g+h+l)+1\right)}$$

using that for integer n

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{(2n)!\sqrt{\pi}}{2^{2n}n!}$$

we can then say this integral is equal to

$$\frac{\pi(2(T-(g+l)))!(2(Y-(f+h)))!}{2^{2(T+Y-(f+g+h+l))-1}(T+Y-(f+g+h+l))!(T-(g+l))!(Y-(f+h))!}$$

$$=\frac{\pi(n+d-2(g+l))!(m+c-2(f+h))!}{2^{m+n+c+d-2(f+g+h+l)-1}\left(\frac{m+n+c+d}{2}-(f+g+h+l)\right)!\left(\frac{n+d}{2}-(g+l)\right)!\left(\frac{m+c}{2}-(f+h)\right)!}$$

so that altogether we rewrite

$$\langle m,n\,|\,c,d\rangle\,(z,z_o,z_i,\omega_o,\omega_i,k_o,k_i,R)=$$

$$\frac{4\sqrt{m!n!c!d!}\mathrm{e}^{i(k_o(z-z_o)-k_i(z-z_i))}}{M_i^{2(c+d+1)}P_o^{2(m+n+1)}}\sum_{f=0}^{m/2}\sum_{g=0}^{n/2}\sum_{h=0}^{c/2}\sum_{l=0}^{d/2}\int_0^R\mathrm{d}\rho\mathrm{e}^{-\frac{\rho^2}{\omega_o\omega_iM_i^2P_o^2}\left(\omega_oP_o^2+\omega_iM_i^2\right)}$$

$$\left[\frac{(M_iP_i)^{2(h+l)}(M_oP_o)^{2(f+g)}\rho^{m+n+c+d-2(f+g+h+l)+1}(n+d-2(g+l))!(m+c-2(f+h))!}{(-2)^{f+g+h+l}f!g!h!l!(m-2f)!(n-2g)!(c-2h)!(d-2l)!\left(\frac{m+n+c+d}{2}-(f+g+h+l)\right)!\left(\frac{n+d}{2}-(g+l)\right)!\left(\frac{m+c}{2}-(f+h)\right)!}\right]$$

We first make the change of variables $\gamma=\left(\frac{\rho}{R}\right)^2$

$$\implies\rho^2=R^2\gamma\qquad 2\rho\mathrm{d}\rho=R^2\mathrm{d}\gamma$$

so that we get

$$\langle m,n\,|\,c,d\rangle\,(z,z_o,z_i,\omega_o,\omega_i,k_o,k_i,R)=$$

$$\frac{2R^2\sqrt{m!n!c!d!}\mathrm{e}^{i(k_o(z-z_o)-k_i(z-z_i))}}{M_i^{2(c+d+1)}P_o^{2(m+n+1)}}\sum_{f=0}^{m/2}\sum_{g=0}^{n/2}\sum_{h=0}^{c/2}\sum_{l=0}^{d/2}\int_0^1\mathrm{d}\gamma\mathrm{e}^{-\frac{\gamma R^2}{\omega_o\omega_i M_i^2P_o^2}\left(\omega_oP_o^2+\omega_iM_i^2\right)}$$

$$\left[\frac{(M_iP_i)^{2(h+l)}(M_oP_o)^{2(f+g)}\left(\gamma R^2\right)^{\frac{m+n+c+d}{2}-(f+g+h+l)}(n+d-2(g+l))!(m+c-2(f+h))!}{(-2)^{f+g+h+l}f!g!h!l!(m-2f)!(n-2g)!(c-2h)!(d-2l)!\left(\frac{m+n+c+d}{2}-(f+g+h+l)\right)!\left(\frac{n+d}{2}-(g+l)\right)!\left(\frac{m+c}{2}-(f+h)\right)!}\right]$$

Now the integral

$$\begin{aligned}F(a,m)&\equiv\int_0^1\mathrm{d}\gamma\gamma^m\mathrm{e}^{-a\gamma}\\&=\left.\frac{-\gamma^m\mathrm{e}^{-a\gamma}}{a}\right|_0^1+\frac{m}{a}\int_0^1\mathrm{d}\gamma\gamma^{m-1}\mathrm{e}^{-a\gamma}\\&=\frac{m}{a}F(a,m-1)-\frac{\mathrm{e}^{-a}}{a}\end{aligned}$$

We inductively assume for any $j<m$ we have

$$F(a,m)=C(j)\equiv\frac{m!}{(m-j)!a^j}F(a,m-j)-\mathrm{e}^{-a}\sum_{k=0}^{j-1}\frac{m!}{(m-k)!a^{k+1}}$$

and prove using the properties of F that

$$\begin{aligned}C(j)&=\frac{m!}{(m-j)!a^j}F(a,m-j)-\mathrm{e}^{-a}\sum_{k=0}^{j-1}\frac{m!}{(m-k)!a^{k+1}}\\&=\frac{m!}{(m-j)!a^j}\left(\frac{(m-j)}{a}F(a,m-(j+1))-\frac{\mathrm{e}^{-a}}{a}\right)-\mathrm{e}^{-a}\sum_{k=0}^{j-1}\frac{m!}{(m-k)!a^{k+1}}\\&=\frac{m!}{(m-(j+1))!a^{j+1}}F(a,m-(j+1))-\mathrm{e}^{-a}\sum_{k=0}^{(j+1)-1}\frac{m!}{(m-k)!a^{k+1}}\\&=C(j+1)\end{aligned}$$

proving by induction that this works. For $j=m-1$ we get

$$F(a,m)=\frac{m!}{a^{m-1}}F(a,1)-\mathrm{e}^{-a}\sum_{k=0}^{m-2}\frac{m!}{(m-k)!a^{k+1}}$$

now

$$\begin{aligned}F(a,1)=\int_0^1\mathrm{d}\gamma\gamma\mathrm{e}^{-a\gamma}&=\left.\frac{-\gamma\mathrm{e}^{-a\gamma}}{a}\right|_0^1+\frac{1}{a}\int_0^1\mathrm{d}\gamma\mathrm{e}^{-a\gamma}=\left.\frac{-\gamma\mathrm{e}^{-a\gamma}}{a}\right|_0^1-\left.\frac{\mathrm{e}^{-a\gamma}}{a^2}\right|_0^1\\&=\frac{-\mathrm{e}^{-a}}{a}+\frac{1}{a^2}-\frac{\mathrm{e}^{-a}}{a^2}\end{aligned}$$

giving

$$F(a,m)=\frac{m!}{a}\left(\frac{1}{a^m}-\mathrm{e}^{-a}\sum_{k=0}^{m}\frac{1}{(m-k)!a^k}\right)$$

as m in our above expression is $\frac{m+n+c+d}{2}-(f+g+h+l)$ our integral becomes (calling $a\equiv\frac{R^2(\omega_o P_o^2+\omega_i M_i^2)}{\omega_o\omega_i M_i^2 P_o^2}$

$$\langle m,n\,|\,c,d\rangle\,(z,z_o,z_i,\omega_o,\omega_i,k_o,k_i,R)=$$

$$\frac{2R^2\sqrt{m!n!c!d!}\mathrm{e}^{i(k_o(z-z_o)-k_i(z-z_i))}}{aM_i^{2(c+d+1)}P_o^{2(m+n+1)}}\sum_{f=0}^{m/2}\sum_{g=0}^{n/2}\sum_{h=0}^{c/2}\sum_{l=0}^{d/2}$$

$$\left[\frac{(M_iP_i)^{2(h+l)}(M_oP_o)^{2(f+g)}R^{m+n+c+d-2(f+g+h+l)}(n+d-2(g+l))!(m+c-2(f+h))!}{(-2)^{f+g+h+l}f!g!h!l!(m-2f)!(n-2g)!(c-2h)!(d-2l)!\left(\frac{n+d}{2}-(g+l)\right)!\left(\frac{m+c}{2}-(f+h)\right)!}\right]$$

$$\times\left[\frac{1}{a^{\frac{m+n+c+d}{2}-(f+g+h+l)}}-\mathrm{e}^{-a}\sum_{k=0}^{\frac{m+n+c+d}{2}-(f+g+h+l)}\frac{1}{\left(\frac{m+n+c+d}{2}-(f+g+h+l+k)\right)!a^k}\right]$$

We first rewrite the form of a and then group together terms of common powers and rewrite the index k at the end as $k=\frac{m+n+c+d}{2}-(f+g+h+l)-k'$ where k' is the old index we have

$$\langle m,n\,|\,c,d\rangle\,(z,z_o,z_i,\omega_o,\omega_i,k_o,k_i,R)=$$

$$\frac{2\omega_o\omega_i\sqrt{m!n!c!d!}\mathrm{e}^{i(k_o(z-z_o)-k_i(z-z_i))}}{(\omega_oP_o^2+\omega_iM_i^2)M_i^{2(c+d)}P_o^{2(m+n)}}\sum_{f=0}^{m/2}\sum_{g=0}^{n/2}\sum_{h=0}^{c/2}\sum_{l=0}^{d/2}$$

$$\left[\frac{(M_iP_i)^{2(h+l)}(M_oP_o)^{2(f+g)}R^{m+n+c+d-2(f+g+h+l)}(n+d-2(g+l))!(m+c-2(f+h))!}{(-2)^{f+g+h+l}f!g!h!l!(m-2f)!(n-2g)!(c-2h)!(d-2l)!\left(\frac{n+d}{2}-(g+l)\right)!\left(\frac{m+c}{2}-(f+h)\right)!}\right]$$

$$\times\left[\left(\frac{\omega_o\omega_iM_i^2P_o^2}{R^2\left(\omega_oP_o^2+\omega_iM_i^2\right)}\right)^{\frac{m+n+c+d}{2}-(f+g+h+l)}\right.$$

$$\left.-\left(\frac{\omega_o\omega_iM_i^2P_o^2}{R^2\left(\omega_oP_o^2+\omega_iM_i^2\right)}\right)^{\frac{m+n+c+d}{2}-(f+g+h+l)}\mathrm{e}^{-\frac{R^2\left(\omega_oP_o^2+\omega_iM_i^2\right)}{\omega_o\omega_iM_i^2P_o^2}}\sum_{k=0}^{\frac{m+n+c+d}{2}-(f+g+h+l)}\frac{\left(R^2\left(\omega_oP_o^2+\omega_iM_i^2\right)\right)^k}{k!\left(\omega_o\omega_iP_o^2M_i^2\right)^k}\right]$$

$$=\frac{2\sqrt{m!n!c!d!}\mathrm{e}^{i(k_o(z-z_o)-k_i(z-z_i))}}{\left(\frac{P_o^2}{\omega_i}+\frac{M_i^2}{\omega_o}\right)^{\frac{m+n+c+d+2}{2}}M_i^{c+d-(m+n)}P_o^{m+n-(c+d)}}\sum_{f=0}^{m/2}\sum_{g=0}^{n/2}\sum_{h=0}^{c/2}\sum_{l=0}^{d/2}$$

$$\left[\frac{\left(\frac{-M_o^2}{2\omega_o}\left(1+\frac{\omega_oP_o^2}{\omega_iM_i^2}\right)\right)^{(f+g)}\left(\frac{-P_i^2}{2\omega_i}\left(1+\frac{\omega_iM_i^2}{\omega_oP_o^2}\right)\right)^{(h+l)}(n+d-2(g+l))!(m+c-2(f+h))!}{f!g!h!l!(m-2f)!(n-2g)!(c-2h)!(d-2l)!\left(\frac{n+d}{2}-(g+l)\right)!\left(\frac{m+c}{2}-(f+h)\right)!}\right]$$

$$\times\left[1-\mathrm{e}^{-\frac{R^2\left(\omega_o P_o^2+\omega_i M_i^2\right)}{\omega_o\omega_i M_i^2 P_o^2}}\sum_{k=0}^{\frac{m+n+c+d}{2}-(f+g+h+l)}\frac{1}{k!}\left(\frac{R^2\left(\omega_o P_o^2+\omega_i M_i^2\right)}{\omega_i\omega_o M_i^2 P_o^2}\right)^k\right]$$

To improve computation we use binomial coefficients if possible

$$=\frac{2\sqrt{m!n!c!d!}\mathrm{e}^{i(k_o(z-z_o)-k_i(z-z_i))}}{\left(\frac{P_o^2}{\omega_i}+\frac{M_i^2}{\omega_o}\right)^{\frac{m+n+c+d+2}{2}}}\left(\frac{P_0}{M_i}\right)^{c+d-(m+n)}\sum_{f=0}^{m/2}\sum_{g=0}^{n/2}\sum_{h=0}^{c/2}\sum_{l=0}^{d/2}$$

$$\left[\frac{\left(\frac{-M_o^2}{2\omega_o}\left(1+\frac{\omega_o P_o^2}{\omega_i M_i^2}\right)\right)^{f+g}\left(\frac{-P_i^2}{2\omega_i}\left(1+\frac{\omega_i M_i^2}{\omega_o P_o^2}\right)\right)^{h+l}}{f!g!h!l!\left(\frac{n+d}{2}-(g+l)\right)!\left(\frac{m+c}{2}-(f+h)\right)!}\binom{n+d-2(g+l)}{n-2g}\binom{m+c-2(f+h)}{m-2f}\right]$$

$$\times\left[1-\mathrm{e}^{-\frac{R^2\left(\omega_o P_0^2+\omega_i M_i^2\right)}{\omega_o\omega_i M_i^2 P_o^2}}\sum_{k=0}^{\frac{m+n+c+d}{2}-(f+g+h+l)}\frac{1}{k!}\left(\frac{R^2\left(\omega_o P_o^2+\omega_i M_i^2\right)}{\omega_o\omega_i M_i^2 P_o^2}\right)^k\right]$$