TEM MODES

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We have that our formula is, using $z_r = \frac{\pi\omega_0^2}{\lambda}$

$$U_j(x,z) = \frac{\left(1+i\frac{z}{z_r}\right)^{j/2}}{2^{\frac{2j-1}{4}}\pi^{1/4}(j!)^{\frac{1}{2}}\omega_0^{1/2}\left(1-i\frac{z}{z_r}\right)^{\frac{j+1}{2}}} H_j\left(\frac{\sqrt{2}x}{\omega_0\sqrt{1+\left(\frac{z}{z_r}\right)^2}}\right) \mathrm{e}^{\frac{-x^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-\frac{ikz}{2}}$$

where

$$H_0(x) = 1 \qquad H_1(x) = 2x \qquad H_2(x) = 2\left(2x^2-1\right)$$

$$H_3(x) = 4\left(2x^3-3x\right) \qquad H_4(x) = 4\left(4x^4-12x^2+3\right)$$

$$H_5(x) = 8x\left(4x^4-20x^2+15\right) \qquad H_6(x) = 8(8x^6-60x^4+90x^2-15)$$

Giving us

$$U_0(x,z) = \frac{2^{1/4}}{\pi^{1/4}\omega_0^{1/2}\left(1-i\frac{z}{z_r}\right)^{1/2}}\mathrm{e}^{\frac{-x^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-\frac{ikz}{2}}$$

$$U_1(x,z) = \frac{2^{5/4}}{\pi^{1/4}\omega_0^{3/2}\left(1-i\frac{z}{z_r}\right)^{3/2}}x\mathrm{e}^{\frac{-x^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-\frac{ikz}{2}}$$

$$U_2(x,z) = \frac{\left(1+i\frac{z}{z_r}\right)}{2^{1/4}\pi^{1/4}\omega_0^{1/2}\left(1-i\frac{z}{z_r}\right)^{3/2}}\left(\frac{2^2x^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-1\right)\mathrm{e}^{\frac{-x^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-\frac{ikz}{2}}$$

$$U_3(x,z) = \frac{2^{1/4}\left(1+i\frac{z}{z_r}\right)^{3/2}}{3^{1/2}\pi^{1/4}\omega_0^{1/2}\left(1-i\frac{z}{z_r}\right)^2}\left(\frac{2^{5/2}x^3}{\omega_0^3\left(1+\left(\frac{z}{z_r}\right)^2\right)^{3/2}}-\frac{2^{1/2}*3x}{\omega_0\left(1+\left(\frac{z}{z_r}\right)^2\right)^{1/2}}\right)\mathrm{e}^{\frac{-x^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-\frac{ikz}{2}}$$

$$= \frac{2^{3/4}\left(1+i\frac{z}{z_r}\right)}{3^{1/2}\pi^{1/4}\omega_0^{3/2}\left(1-i\frac{z}{z_r}\right)^{5/2}}\left(\frac{2^2x^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-3\right)x\mathrm{e}^{\frac{-x^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-\frac{ikz}{2}}$$

$$U_4(x,z)=\frac{\left(1+i\frac{z}{z_r}\right)^2}{2^{5/4}3^{1/2}\pi^{1/4}\omega_0^{1/2}\left(1-i\frac{z}{z_r}\right)^{5/2}}\left(\frac{2^4x^4}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{3*2^3x^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+3\right)\mathrm{e}^{\frac{-x^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-\frac{ikz}{2}}$$

$$U_5(x,z)=\frac{\left(1+i\frac{z}{z_r}\right)^2x\mathrm{e}^{\frac{-x^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-\frac{ikz}{2}}}{2^{1/4}\pi^{1/4}3^{1/2}5^{1/2}\omega_0^{3/2}\left(1-i\frac{z}{z_r}\right)^{7/2}}\left(\frac{16x^4}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{40x^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+15\right)$$

$$U_6(x,z)=\frac{\left(1+i\frac{z}{z_r}\right)^3\mathrm{e}^{\frac{-x^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-\frac{ikz}{2}}}{2^{7/4}3\pi^{1/4}5^{1/2}\omega_0^{1/2}\left(1-i\frac{z}{z_r}\right)^{7/2}}\left(\frac{64x^6}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}-\frac{240x^4}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}+\frac{180x^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-15\right)$$

1. The Combined Modes

There are 36 combined modes from these six, we write them in cylindrical coordinates. We see the general formula

$$A_{0,0}(\rho,\theta,z)=\frac{2^{1/2}}{\pi^{1/2}\omega_0\left(1-i\frac{z}{z_r}\right)}\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}$$

$$A_{0,1}(\rho,\theta,z)=\frac{2^{3/2}}{\pi^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}\rho\sin\theta\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}$$

$$A_{1,0}(\rho,\theta,z)=\frac{2^{3/2}}{\pi^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}\rho\cos\theta\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}$$

$$A_{1,1}(\rho,\theta,z)=\frac{2^{5/2}}{\pi^{1/2}\omega_0^3\left(1-i\frac{z}{z_r}\right)^3}\left(\rho^2\sin\theta\cos\theta\right)\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}$$

$$A_{0,2}\left(\rho,\theta,z\right)=\frac{\left(1+i\frac{z}{z_r}\right)}{\pi^{1/2}\omega_0\left(1-i\frac{z}{z_r}\right)^2}\left(\frac{2^2\rho^2\sin^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-1\right)\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}$$

$$A_{2,0}\left(\rho,\theta,z\right)=\frac{\left(1+i\frac{z}{z_r}\right)}{\pi^{1/2}\omega_0\left(1-i\frac{z}{z_r}\right)^2}\left(\frac{2^2\rho^2\cos^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-1\right)\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}$$

$$A_{1,2}\left(\rho,\theta,z\right)=\frac{2\left(1+i\frac{z}{z_r}\right)}{\pi^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(\frac{2^2\rho^2\sin^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-1\right)\rho\cos\theta\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}$$

$$A_{2,1}\left(\rho,\theta,z\right)=\frac{2\left(1+i\frac{z}{z_r}\right)}{\pi^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(\frac{2^2\rho^2\cos^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-1\right)\rho\sin\theta\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}$$

$$A_{2,2}\left(\rho,\theta,z\right)=\frac{\left(1+i\frac{z}{z_r}\right)^2}{2^{1/2}\pi^{1/2}\omega_0\left(1-i\frac{z}{z_r}\right)^3}\left(\frac{2^2\rho^2\cos^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-1\right)\left(\frac{2^2\rho^2\sin^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-1\right)\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}$$

$$=\frac{\left(1+i\frac{z}{z_r}\right)^2}{2^{1/2}\pi^{1/2}\omega_0\left(1-i\frac{z}{z_r}\right)^3}\left(\frac{2^4\rho^4\cos^2\theta\sin^2\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{2^2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+1\right)\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}$$

$$A_{0,3}\left(\rho,\theta,z\right)=\frac{2\left(1+i\frac{z}{z_r}\right)}{3^{1/2}*\pi^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(\frac{2^2\rho^2\sin^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-3\right)\rho\sin\theta\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}$$

$$A_{3,0}\left(\rho,\theta,z\right)=\frac{2\left(1+i\frac{z}{z_r}\right)}{3^{1/2}*\pi^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(\frac{2^2\rho^2\cos^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-3\right)\rho\cos\theta\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}$$

$$A_{1,3}\left(\rho,\theta,z\right)=\frac{2^2\left(1+i\frac{z}{z_r}\right)}{3^{1/2}*\pi^{1/2}\omega_0^3\left(1-i\frac{z}{z_r}\right)^4}\left(\frac{2^2\rho^2\sin^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-3\right)\rho^2\sin\theta\cos\theta\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}$$

$$A_{3,1}\left(\rho,\theta,z\right)=\frac{2^2\left(1+i\frac{z}{z_r}\right)}{3^{1/2}*\pi^{1/2}\omega_0^3\left(1-i\frac{z}{z_r}\right)^4}\left(\frac{2^2\rho^2\cos^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-3\right)\rho^2\sin\theta\cos\theta\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}$$

$$A_{2,3}\left(\rho,\theta,z\right)=\frac{2^{1/2}\left(1+i\frac{z}{z_r}\right)^2}{3^{1/2}\pi^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(\frac{2^2\rho^2\cos^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-1\right)\left(\frac{2^2\rho^2\sin^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-3\right)\rho\sin\theta\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}$$

$$= \frac{2^{1/2}\left(1+i\frac{z}{z_r}\right)^2}{3^{1/2}\pi^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(\frac{2^4\rho^4\cos^2\theta\sin^2\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)\right)^2}-\frac{2^2\rho^2\left(1+2\cos^2\theta\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+3\right)\rho\sin\theta \mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}$$

$$A_{3,2}\left(\rho,\theta,z\right) = \frac{2^{1/2}\left(1+i\frac{z}{z_r}\right)^2}{3^{1/2}\pi^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(\frac{2^4\rho^4\cos^2\theta\sin^2\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)\right)^2}-\frac{2^2\rho^2\left(1+2\sin^2\theta\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+3\right)\rho\cos\theta \mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}$$

$$A_{3,3}\left(\rho,\theta,z\right) = \frac{2^{3/2}\left(1+i\frac{z}{z_r}\right)^2}{3\pi^{1/2}\omega_0^3\left(1-i\frac{z}{z_r}\right)^5}\left(\frac{2^4\rho^4\cos^2\theta\sin^2\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{3*2^2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+9\right)\rho^2\cos\theta\sin\theta \mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}$$

$$A_{0,4}\left(\rho,\theta,z\right) = \frac{\left(1+i\frac{z}{z_r}\right)^2}{3^{1/2}2\pi^{1/2}\omega_0\left(1-i\frac{z}{z_r}\right)^3}\left(\frac{2^4\rho^4\sin^4\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{3*2^3\rho^2\sin^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+3\right)\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}}$$

$$A_{4,0}\left(\rho,\theta,z\right) = \frac{\left(1+i\frac{z}{z_r}\right)^2}{3^{1/2}2\pi^{1/2}\omega_0\left(1-i\frac{z}{z_r}\right)^3}\left(\frac{2^4\rho^4\cos^4\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{3*2^3\rho^2\cos^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+3\right)\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}$$

$$A_{1,4}\left(\rho,\theta,z\right) = \frac{\left(1+i\frac{z}{z_r}\right)^2}{3^{1/2}\pi^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(\frac{2^4\rho^4\sin^4\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{3*2^3\rho^2\sin^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+3\right)\rho\cos\theta \mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}$$

$$A_{4,1}\left(\rho,\theta,z\right) = \frac{\left(1+i\frac{z}{z_r}\right)^2}{3^{1/2}\pi^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(\frac{2^4\rho^4\cos^4\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{3*2^3\rho^2\cos^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+3\right)\rho\sin\theta \mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}$$

$$A_{2,4}\left(\rho,\theta,z\right) = \frac{\left(1+i\frac{z}{z_r}\right)^3}{3^{1/2}*2^{3/2}\pi^{1/2}\omega_0\left(1-i\frac{z}{z_r}\right)^4}$$

$$\times\left(\frac{2^2\rho^2\cos^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-1\right)\left(\frac{2^4\rho^4\sin^4\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{3*2^3\rho^2\sin^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+3\right)\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}$$

$$
= \frac{\left(1+i\frac{z}{z_r}\right)^3}{3^{1/2}*2^{3/2}\pi^{1/2}\omega_0\left(1-i\frac{z}{z_r}\right)^4}
$$

$$
\times\left(\frac{2^6\rho^6\cos^2\theta\sin^4\theta}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}-\frac{2^4\rho^4\sin^2\theta\left(1+5\cos^2\theta\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}+\frac{3*2^2\rho^2\left(1+\sin^2\theta\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-3\right)\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}
$$

$$
A_{4,2}\left(\rho,\theta,z\right)=\frac{\left(1+i\frac{z}{z_r}\right)^3}{3^{1/2}2^{3/2}\pi^{1/2}\omega_0\left(1-i\frac{z}{z_r}\right)^4}
$$

$$
\times\left(\frac{2^6\rho^6\sin^2\theta\cos^4\theta}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}-\frac{2^4\rho^4\cos^2\theta\left(1+5\sin^2\theta\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}+\frac{3*2^2\rho^2\left(1+\cos^2\theta\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-3\right)\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}
$$

$$
A_{3,4}\left(\rho,\theta,z\right)=\frac{\left(1+i\frac{z}{z_r}\right)^3\rho\cos\theta}{2^{1/2}3\pi^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^5}
$$

$$
\times\left(\frac{2^2\rho^2\cos^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-3\right)\left(\frac{2^4\rho^4\sin^4\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{3*2^3*\rho^2\sin^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+3\right)\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}
$$

$$
= \frac{\left(1+i\frac{z}{z_r}\right)^3\rho\cos\theta}{2^{1/2}3\pi^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^5}
$$

$$
\times\left(\frac{2^6\rho^6\cos^2\theta\sin^4\theta}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}-\frac{3*2^4\rho^4\sin^2\theta\left(1+\cos^2\theta\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}+\frac{3*2^2\rho^2\left(1+5\sin^2\theta\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-9\right)\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}
$$

$$
A_{4,3}\left(\rho,\theta,z\right)=\frac{\left(1+i\frac{z}{z_r}\right)^3\rho\sin\theta}{2^{1/2}3\pi^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^5}
$$

$$
\times\left(\frac{2^6\rho^6\sin^2\theta\cos^4\theta}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}-\frac{3*2^4\rho^4\cos^2\theta\left(1+\sin^2\theta\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}+\frac{3*2^2\rho^2\left(1+5\cos^2\theta\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-9\right)\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}
$$

$$A_{4,4}\left(\rho,\theta,z\right)=\frac{\left(1+i\frac{z}{z_r}\right)^4 \mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{2^{5/2}3\pi^{1/2}\omega_0\left(1-i\frac{z}{z_r}\right)^5}$$

$$\times\left(\frac{2^8\rho^8\cos^4\theta\sin^4\theta}{\omega_0^8\left(1+\left(\frac{z}{z_r}\right)^2\right)^4}-\frac{3*2^7\rho^6\sin^2\theta\cos^2\theta}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}+\frac{3*2^4\rho^4\left(1+10\sin^2\theta\cos^2\theta\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{3^2*2^3\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+9\right)$$

$$A_{0,5}\left(\rho,\theta,z\right)=\frac{\left(1+i\frac{z}{z_r}\right)^2\rho\sin\theta\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{3^{1/2}5^{1/2}\pi^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(\frac{16\rho^4\sin^4\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{40\rho^2\sin^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+15\right)$$

$$A_{5,0}\left(\rho,\theta,z\right)=\frac{\left(1+i\frac{z}{z_r}\right)^2\rho\cos\theta\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{3^{1/2}5^{1/2}\pi^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(\frac{16\rho^4\cos^4\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{40\rho^2\cos^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+15\right)$$

$$A_{1,5}\left(\rho,\theta,z\right)=\frac{2\left(1+i\frac{z}{z_r}\right)^2\rho^2\sin\theta\cos\theta\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{3^{1/2}5^{1/2}\pi^{1/2}\omega_0^3\left(1-i\frac{z}{z_r}\right)^5}\left(\frac{16\rho^4\sin^4\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{40\rho^2\sin^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+15\right)$$

$$A_{5,1}\left(\rho,\theta,z\right)=\frac{2\left(1+i\frac{z}{z_r}\right)^2\rho^2\sin\theta\cos\theta\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{3^{1/2}5^{1/2}\pi^{1/2}\omega_0^3\left(1-i\frac{z}{z_r}\right)^5}\left(\frac{16\rho^4\cos^4\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{40\rho^2\cos^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+15\right)$$

$$A_{2,5}\left(\rho,\theta,z\right)=\frac{\left(1+i\frac{z}{z_r}\right)^3\rho\sin\theta\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{2^{1/2}3^{1/2}5^{1/2}\pi^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^5}$$

$$\times\left(\frac{2^6\rho^6\sin^4\theta\cos^2\theta}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}-\frac{2^4\rho^4\sin^2\theta\left(9\cos^2\theta+1\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}+\frac{2^2 5\rho^2\left(\cos^2\theta+2\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-15\right)$$

$$A_{5,2}\left(\rho,\theta,z\right)=\frac{\left(1+i\frac{z}{z_r}\right)^3\rho\cos\theta\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{2^{1/2}3^{1/2}5^{1/2}\pi^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^5}$$

$$\times\left(\frac{2^6\rho^6\cos^4\theta\sin^2\theta}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}-\frac{2^4\rho^4\cos^2\theta\left(9\sin^2\theta+1\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}+\frac{2^2 5\rho^2\left(\sin^2\theta+2\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-15\right)$$

$$A_{3,5}\left(\rho,\theta,z\right)=\frac{2\left(1+i\frac{z}{z_r}\right)^3\rho^2\sin\theta\cos\theta\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{3\pi^{1/2}5^{1/2}\omega_0^3\left(1-i\frac{z}{z_r}\right)^6}\left(\frac{2^2\rho^2\cos^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-3\right)$$

$$\times\left(\frac{2^4\rho^4\sin^4\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{2^3 5\rho^2\sin^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+15\right)$$

$$=\frac{2\left(1+i\frac{z}{z_r}\right)^3\rho^2\sin\theta\cos\theta\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{3\pi^{1/2}5^{1/2}\omega_0^3\left(1-i\frac{z}{z_r}\right)^6}$$

$$\times\left(\frac{2^6\rho^6\sin^4\theta\cos^2\theta}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}-\frac{2^4\rho^4\sin^2\theta\left(3\sin^2\theta+10\cos^2\theta\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}+\frac{2^2*3*5\rho^2\left(2\sin^2\theta+\cos^2\theta\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-3^2 5\right)$$

$$=\frac{2\left(1+i\frac{z}{z_r}\right)^3\rho^2\sin\theta\cos\theta\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{3\pi^{1/2}5^{1/2}\omega_0^3\left(1-i\frac{z}{z_r}\right)^6}$$

$$\times\left(\frac{2^6\rho^6\sin^4\theta\cos^2\theta}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}-\frac{2^4\rho^4\sin^2\theta\left(3+7\cos^2\theta\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}+\frac{2^2*3*5\rho^2\left(1+\sin^2\theta\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-3^2 5\right)$$

$$A_{5,3}\left(\rho,\theta,z\right)=\frac{2\left(1+i\frac{z}{z_r}\right)^3\rho^2\sin\theta\cos\theta\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{3\pi^{1/2}5^{1/2}\omega_0^3\left(1-i\frac{z}{z_r}\right)^6}$$

$$\times\left(\frac{2^6\rho^6\cos^4\theta\sin^2\theta}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}-\frac{2^4\rho^4\cos^2\theta\left(3+7\sin^2\theta\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}+\frac{2^2*3*5\rho^2\left(1+\cos^2\theta\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-3^2 5\right)$$

$$
\begin{aligned}
A_{4,5}\left(\rho,\theta,z\right) &= \frac{\left(1+i\frac{z}{z_r}\right)^4 \rho\sin\theta \mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{2^{3/2}3\pi^{1/2}5^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^6}\left(\frac{2^4\rho^4\cos^4\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{2^3 3\rho^2\cos^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+3\right)\\
&\times\left(\frac{2^4\rho^4\sin^4\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{2^3 5\rho^2\sin^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+15\right)\\
&= \frac{\left(1+i\frac{z}{z_r}\right)^4 \rho\sin\theta \mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{2^{3/2}3\pi^{1/2}5^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^6}\\
&\times\left(\frac{2^8\rho^8\sin^4\theta\cos^4\theta}{\omega_0^8\left(1+\left(\frac{z}{z_r}\right)^2\right)^4}-\frac{2^7\rho^6\sin^2\theta\cos^2\theta\left(5\cos^2\theta+3\sin^2\theta\right)}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}\right.\\
&\left.+\frac{2^4 3\rho^4\left(5\cos^4\theta+\sin^4\theta+2^2 5\cos^2\theta\sin^2\theta\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{2^3*3*5\rho^2\left(\sin^2\theta+3\cos^2\theta\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+3^2 5\right)\\
&= \frac{\left(1+i\frac{z}{z_r}\right)^4 \rho\sin\theta \mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{2^{3/2}3\pi^{1/2}5^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^6}\left(\frac{2^8\rho^8\sin^4\theta\cos^4\theta}{\omega_0^8\left(1+\left(\frac{z}{z_r}\right)^2\right)^4}-\frac{2^7\rho^6\sin^2\theta\cos^2\theta\left(2\cos^2\theta+3\right)}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}\right.\\
&\left.+\frac{2^4 3\rho^4\left(1+4\cos^2\theta+2*7\cos^2\theta\sin^2\theta\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{2^3*3*5\rho^2\left(1+2\cos^2\theta\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+3^2 5\right)\\
A_{5,4}\left(\rho,\theta,z\right) &= \frac{\left(1+i\frac{z}{z_r}\right)^4 \rho\cos\theta \mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{2^{3/2}3\pi^{1/2}5^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^6}\left(\frac{2^8\rho^8\sin^4\theta\cos^4\theta}{\omega_0^8\left(1+\left(\frac{z}{z_r}\right)^2\right)^4}-\frac{2^7\rho^6\sin^2\theta\cos^2\theta\left(2\sin^2\theta+3\right)}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}\right.\\
&\left.+\frac{2^4 3\rho^4\left(1+4\sin^2\theta+2*7\cos^2\theta\sin^2\theta\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{2^3*3*5\rho^2\left(1+2\sin^2\theta\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+3^2 5\right)
\end{aligned}
$$

$$A_{5,5}(\rho,\theta,z)=\frac{\left(1+i\frac{z}{z_r}\right)^4\rho^2\sin\theta\cos\theta \mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{2^{1/2}3\pi^{1/2}5\omega_0^3\left(1-i\frac{z}{z_r}\right)^7}\left(\frac{2^4\rho^4\sin^4\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{2^35\rho^2\sin^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+15\right)$$

$$\times\left(\frac{2^4\rho^4\cos^4\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{2^35\rho^2\cos^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+15\right)$$

$$=\frac{\left(1+i\frac{z}{z_r}\right)^4\rho^2\sin\theta\cos\theta \mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{2^{1/2}3\pi^{1/2}5\omega_0^3\left(1-i\frac{z}{z_r}\right)^7}\left(\frac{2^8\rho^8\sin^4\theta\cos^4\theta}{\omega_0^8\left(1+\left(\frac{z}{z_r}\right)^2\right)^4}-\frac{2^75\rho^6\sin^2\theta\cos^2\theta}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}\right.$$

$$\left.+\frac{2^45\rho^4\left(3\sin^4\theta+3\cos^4\theta+2^25\cos^2\theta\sin^2\theta\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{2^335^2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+3^25^2\right)$$

$$=\frac{\left(1+i\frac{z}{z_r}\right)^4\rho^2\sin\theta\cos\theta \mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{2^{1/2}3\pi^{1/2}5\omega_0^3\left(1-i\frac{z}{z_r}\right)^7}\left(\frac{2^8\rho^8\sin^4\theta\cos^4\theta}{\omega_0^8\left(1+\left(\frac{z}{z_r}\right)^2\right)^4}-\frac{2^75\rho^6\sin^2\theta\cos^2\theta}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}\right.$$

$$\left.+\frac{2^45\rho^4\left(3+2*7\cos^2\theta\sin^2\theta\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{2^335^2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+3^25^2\right)$$

$$A_{0,6}(\rho,\theta,z)=\frac{\left(1+i\frac{z}{z_r}\right)^3\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{2^{3/2}3\pi^{1/2}5^{1/2}\omega_0\left(1-i\frac{z}{z_r}\right)^4}\left(\frac{2^6\rho^6\sin^6\theta}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}-\frac{2^4*3*5\rho^4\sin^4\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}+\frac{2^23^25\rho^2\sin^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-15\right)$$

$$A_{6,0}(\rho,\theta,z)=\frac{\left(1+i\frac{z}{z_r}\right)^3\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{2^{3/2}3\pi^{1/2}5^{1/2}\omega_0\left(1-i\frac{z}{z_r}\right)^4}\left(\frac{2^6\rho^6\cos^6\theta}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}-\frac{2^4*3*5\rho^4\cos^4\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}+\frac{2^23^25\rho^2\cos^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-15\right)$$

$$A_{1,6}(\rho,\theta,z)=\frac{\left(1+i\frac{z}{z_r}\right)^3\rho\cos\theta \mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{2^{1/2}3\pi^{1/2}5^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^5}\left(\frac{2^6\rho^6\sin^6\theta}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}-\frac{2^4*3*5\rho^4\sin^4\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}+\frac{2^2 3^2 5\rho^2\sin^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-15\right)$$

$$A_{6,1}(\rho,\theta,z)=\frac{\left(1+i\frac{z}{z_r}\right)^3\rho\sin\theta \mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{2^{1/2}3\pi^{1/2}5^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^5}\left(\frac{2^6\rho^6\cos^6\theta}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}-\frac{2^4*3*5\rho^4\cos^4\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}+\frac{2^2 3^2 5\rho^2\cos^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-15\right)$$

$$A_{2,6}(\rho,\theta,z)=\frac{\left(1+i\frac{z}{z_r}\right)^4 \mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{2^2 3\pi^{1/2}5^{1/2}\omega_0\left(1-i\frac{z}{z_r}\right)^5}\left(\frac{2^2\rho^2\cos^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-1\right)$$

$$\times\left(\frac{2^6\rho^6\sin^6\theta}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}-\frac{2^4*3*5\rho^4\sin^4\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}+\frac{2^2 3^2 5\rho^2\sin^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-15\right)$$

$$=\frac{\left(1+i\frac{z}{z_r}\right)^4 \mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{2^2 3\pi^{1/2}5^{1/2}\omega_0\left(1-i\frac{z}{z_r}\right)^5}\left(\frac{2^8\rho^8\cos^2\theta\sin^6\theta}{\omega_0^8\left(1+\left(\frac{z}{z_r}\right)^2\right)^4}-\frac{2^6\rho^6\sin^4\theta\left(1+2*7\cos^2\theta\right)}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}\right.$$

$$\left.+\frac{2^4 3*5\rho^4\sin^2\theta\left(2\cos^2\theta+1\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{2^2 3*5\rho^2\left(2\sin^2\theta+1\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+3*5\right)$$

$$A_{6,2}(\rho,\theta,z)=\frac{\left(1+i\frac{z}{z_r}\right)^4 \mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{2^2 3\pi^{1/2}5^{1/2}\omega_0\left(1-i\frac{z}{z_r}\right)^5}\left(\frac{2^8\rho^8\sin^2\theta\cos^6\theta}{\omega_0^8\left(1+\left(\frac{z}{z_r}\right)^2\right)^4}-\frac{2^6\rho^6\cos^4\theta\left(1+2*7\sin^2\theta\right)}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}\right.$$

$$\left.+\frac{2^4 3*5\rho^4\cos^2\theta\left(2\sin^2\theta+1\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{2^2 3*5\rho^2\left(2\cos^2\theta+1\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+3*5\right)$$

$$A_{3,6}(\rho,\theta,z)=\frac{\left(1+i\frac{z}{z_r}\right)^4\rho\cos\theta \mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{3^{3/2}5^{1/2}2\pi^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^6}\left(\frac{2^2\rho^2\cos^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-3\right)$$

$$\times\left(\frac{2^6\rho^6\sin^6\theta}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}-\frac{2^4*3*5\rho^4\sin^4\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}+\frac{2^23^25\rho^2\sin^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-15\right)$$

$$=\frac{\left(1+i\frac{z}{z_r}\right)^4\rho\cos\theta \mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{3^{3/2}5^{1/2}2\pi^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^6}\left(\frac{2^8\rho^8\cos^2\theta\sin^6\theta}{\omega_0^8\left(1+\left(\frac{z}{z_r}\right)^2\right)^4}-\frac{2^63\rho^6\sin^4\theta\left(\sin^2\theta+5\cos^2\theta\right)}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}\right.$$

$$\left.+\frac{2^43^25\rho^4\sin^2\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{2^23*5\rho^2\left(9\sin^2\theta+\cos^2\theta\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+3^25\right)$$

$$=\frac{\left(1+i\frac{z}{z_r}\right)^4\rho\cos\theta \mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{3^{3/2}5^{1/2}2\pi^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^6}\left(\frac{2^8\rho^8\cos^2\theta\sin^6\theta}{\omega_0^8\left(1+\left(\frac{z}{z_r}\right)^2\right)^4}-\frac{2^63\rho^6\sin^4\theta\left(1+4\cos^2\theta\right)}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}\right.$$

$$\left.+\frac{2^43^25\rho^4\sin^2\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{2^23*5\rho^2\left(8\sin^2\theta+1\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+3^25\right)$$

$$A_{6,3}\left(\rho,\theta,z\right)=\frac{\left(1+i\frac{z}{z_r}\right)^4\rho\sin\theta \mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{3^{3/2}5^{1/2}2\pi^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^6}\left(\frac{2^8\rho^8\sin^2\theta\cos^6\theta}{\omega_0^8\left(1+\left(\frac{z}{z_r}\right)^2\right)^4}-\frac{2^63\rho^6\cos^4\theta\left(1+4\sin^2\theta\right)}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}\right.$$

$$\left.+\frac{2^43^25\rho^4\cos^2\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{2^23*5\rho^2\left(8\cos^2\theta+1\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+3^25\right)$$

$$A_{4,6}\left(\rho,\theta,z\right)=\frac{\left(1+i\frac{z}{z_r}\right)^5 \mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{2^33^{3/2}\pi^{1/2}5^{1/2}\omega_0\left(1-i\frac{z}{z_r}\right)^6}\left(\frac{2^4\rho^4\cos^4\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{2^33\rho^2\cos^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+3\right)$$

$$\times\left(\frac{2^6\rho^6\sin^6\theta}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}-\frac{2^4*3*5\rho^4\sin^4\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}+\frac{2^23^25\rho^2\sin^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-15\right)$$

$$=\frac{\left(1+i\frac{z}{z_r}\right)^5 \mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{2^33^{3/2}\pi^{1/2}5^{1/2}\omega_0\left(1-i\frac{z}{z_r}\right)^6}\left(\frac{2^{10}\rho^{10}\cos^4\theta\sin^6\theta}{\omega_0^{10}\left(1+\left(\frac{z}{z_r}\right)^2\right)^5}-\frac{2^83\rho^8\cos^2\theta\sin^4\theta\left(5\cos^2\theta+2\sin^2\theta\right)}{\omega_0^8\left(1+\left(\frac{z}{z_r}\right)^2\right)^4}\right.$$

$$+\frac{2^63\rho^6\sin^2\theta\left(3*5\cos^4\theta+2*3*5\sin^2\theta\cos^2\theta+\sin^4\theta\right)}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}-\frac{2^4*3*5\rho^4\left(\cos^4\theta+3\sin^4\theta+2*3^2\sin^2\theta\cos^2\theta\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}$$

$$\left.+\frac{2^23^25\rho^2\left(2\cos^2\theta+3\sin^2\theta\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-3^25\right)$$

$$=\frac{\left(1+i\frac{z}{z_r}\right)^5 \mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{2^33^{3/2}\pi^{1/2}5^{1/2}\omega_0\left(1-i\frac{z}{z_r}\right)^6}\left(\frac{2^{10}\rho^{10}\cos^4\theta\sin^6\theta}{\omega_0^{10}\left(1+\left(\frac{z}{z_r}\right)^2\right)^5}-\frac{2^83\rho^8\cos^2\theta\sin^4\theta\left(3\cos^2\theta+2\right)}{\omega_0^8\left(1+\left(\frac{z}{z_r}\right)^2\right)^4}\right.$$

$$+\frac{2^63\rho^6\sin^2\theta\left(1+2*7\cos^2\theta+2*7\sin^2\theta\cos^2\theta\right)}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}-\frac{2^4*3*5\rho^4\left(1+2\sin^2\theta+2*7\sin^2\theta\cos^2\theta\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}$$

$$\left.+\frac{2^23^25\rho^2\left(2+\sin^2\theta\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-3^25\right)$$

$$A_{6,4}\left(\rho,\theta,z\right)=\frac{\left(1+i\frac{z}{z_r}\right)^5 \mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{2^33^{3/2}\pi^{1/2}5^{1/2}\omega_0\left(1-i\frac{z}{z_r}\right)^6}\left(\frac{2^{10}\rho^{10}\sin^4\theta\cos^6\theta}{\omega_0^{10}\left(1+\left(\frac{z}{z_r}\right)^2\right)^5}-\frac{2^83\rho^8\sin^2\theta\cos^4\theta\left(3\sin^2\theta+2\right)}{\omega_0^8\left(1+\left(\frac{z}{z_r}\right)^2\right)^4}\right.$$

$$+\frac{2^63\rho^6\cos^2\theta\left(1+2*7\sin^2\theta+2*7\sin^2\theta\cos^2\theta\right)}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}-\frac{2^4*3*5\rho^4\left(1+2\cos^2\theta+2*7\sin^2\theta\cos^2\theta\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}$$

$$+\frac{2^2 3^2 5 \rho^2 \left(2+\cos^2\theta\right)}{\omega_0^2 \left(1+\left(\frac{z}{z_r}\right)^2\right)} - 3^2 5 \Bigg)$$

$$A_{5,6}\left(\rho,\theta,z\right) = \frac{\left(1+i\frac{z}{z_r}\right)^5 \rho\cos\theta \mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{2^2 3^{3/2} 5 \pi^{1/2} \omega_0^2 \left(1-i\frac{z}{z_r}\right)^7} \left(\frac{2^4\rho^4\cos^4\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2} - \frac{2^3 5\rho^2\cos^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)} + 15\right)$$

$$\times\left(\frac{2^6\rho^6\sin^6\theta}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3} - \frac{2^4 * 3 * 5\rho^4\sin^4\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2} + \frac{2^2 3^2 5\rho^2\sin^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)} - 15\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)^5 \rho\cos\theta \mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{2^2 3^{3/2} 5 \pi^{1/2} \omega_0^2 \left(1-i\frac{z}{z_r}\right)^7} \Bigg(\frac{2^{10}\rho^{10}\sin^6\theta\cos^4\theta}{\omega_0^{10}\left(1+\left(\frac{z}{z_r}\right)^2\right)^5} - \frac{2^8 5\rho^8\cos^2\theta\sin^4\theta\left(3\cos^2\theta+2\sin^2\theta\right)}{\omega_0^8\left(1+\left(\frac{z}{z_r}\right)^2\right)^4}$$

$$\frac{2^6 3 * 5\rho^6\sin^2\theta\left(\sin^4\theta+\cos^2\theta\left(3\cos^2\theta+10\sin^2\theta\right)\right)}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3} - \frac{2^4 3 * 5\rho^4\left(\cos^4\theta+15\sin^4\theta+30\sin^2\theta\cos^2\theta\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}$$

$$+\frac{2^2 5^2 3\rho^2\left(2\cos^2\theta+3^2\sin^2\theta\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)} - 3^2 5^2\Bigg)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)^5 \rho\cos\theta \mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{2^2 3^{3/2} 5 \pi^{1/2} \omega_0^2 \left(1-i\frac{z}{z_r}\right)^7} \Bigg(\frac{2^{10}\rho^{10}\sin^6\theta\cos^4\theta}{\omega_0^{10}\left(1+\left(\frac{z}{z_r}\right)^2\right)^5} - \frac{2^8 5\rho^8\cos^2\theta\sin^4\theta\left(\cos^2\theta+2\right)}{\omega_0^8\left(1+\left(\frac{z}{z_r}\right)^2\right)^4}$$

$$\frac{2^6 3 * 5\rho^6\sin^2\theta\left(1+2\cos^2\theta\left(1+3\sin^2\theta\right)\right)}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3} - \frac{2^4 3 * 5\rho^4\left(1+2*7\sin^2\theta\left(1+\cos^2\theta\right)\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}$$

$$+\frac{2^2 5^2 3\rho^2\left(2+7\sin^2\theta\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)} - 3^2 5^2\Bigg)$$

$$A_{6,5}\left(\rho,\theta,z\right)=\frac{\left(1+i\frac{z}{z_r}\right)^5\rho\sin\theta\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}}{2^2 3^{3/2}5\pi^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^7}\left(\frac{2^{10}\rho^{10}\cos^6\theta\sin^4\theta}{\omega_0^{10}\left(1+\left(\frac{z}{z_r}\right)^2\right)^5}-\frac{2^8 5\rho^8\sin^2\theta\cos^4\theta\left(\sin^2\theta+2\right)}{\omega_0^8\left(1+\left(\frac{z}{z_r}\right)^2\right)^4}\right.$$

$$\frac{2^6 3*5\rho^6\cos^2\theta\left(1+2\sin^2\theta\left(1+3\cos^2\theta\right)\right)}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}-\frac{2^4 3*5\rho^4\left(1+2*7\cos^2\theta\left(1+\sin^2\theta\right)\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}$$

$$\left.+\frac{2^2 5^2 3\rho^2\left(2+7\cos^2\theta\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-3^2 5^2\right)$$

$$A_{6,6}\left(\rho,\theta,z\right)=\frac{\left(1+i\frac{z}{z_r}\right)^6\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}}}{2^{7/2}3^2\pi^{1/2}5\omega_0\left(1-i\frac{z}{z_r}\right)^7}\left(\frac{2^6\rho^6\sin^6\theta}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}-\frac{2^4*3*5\rho^4\sin^4\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}+\frac{2^2 3^2 5\rho^2\sin^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-15\right)$$

$$\times\left(\frac{2^6\rho^6\cos^6\theta}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}-\frac{2^4*3*5\rho^4\cos^4\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}+\frac{2^2 3^2 5\rho^2\cos^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-15\right)$$

$$=\frac{\left(1+i\frac{z}{z_r}\right)^6\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}}}{2^{7/2}3^2\pi^{1/2}5\omega_0\left(1-i\frac{z}{z_r}\right)^7}\left(\frac{2^{12}\rho^{12}\cos^6\theta\sin^6\theta}{\omega_0^{12}\left(1+\left(\frac{z}{z_r}\right)^2\right)^6}-\frac{2^{10}*3*5\rho^{10}\cos^4\theta\sin^4\theta}{\omega_0^{10}\left(1+\left(\frac{z}{z_r}\right)^2\right)^5}\right.$$

$$+\frac{2^8 3^2 5\rho^8\sin^2\theta\cos^2\theta\left(3\sin^2\theta\cos^2\theta+1\right)}{\omega_0^8\left(1+\left(\frac{z}{z_r}\right)^2\right)^4}-\frac{2^6 3*5\rho^6\left(1+2*3*7\sin^2\theta\cos^2\theta\right)}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}$$

$$\left.+\frac{2^4 3^2 5^2\rho^4\left(1+7\sin^2\theta\cos^2\theta\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{2^2 3^3 5^2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+3^2 5^2\right)$$

2. Representation of integral in Better Form

Suppose we have an integral of the form

$$F(a,b,c,d,R) = \int_0^R \mathrm{d}\rho \rho^{a+1} \mathrm{e}^{-b\rho^2} \int_0^{2\pi} \mathrm{d}\theta \sin^c\theta \cos^d\theta$$

For the angle part we first use

$$G(c,d) \equiv \int_0^{2\pi} \mathrm{d}\theta sin^c\theta \cos^d\theta = -\frac{\sin^{c-1}\theta\cos^{d+1}\theta}{d+1}\bigg|_{\theta=0}^{2\pi} + \frac{c-1}{d+1}\int_0^{2\pi} \mathrm{d}\theta \sin^{c-2}\theta\cos^{d+2}\theta$$

$$= \frac{c-1}{d+1}\int_0^{2\pi} \mathrm{d}\theta \left[\sin^{c-2}\theta\cos^d\theta - \sin^c\theta\cos^d\theta\right] = \frac{c-1}{d+1}\left[G(c-2,d) - G(c,d)\right]$$

$$\Longrightarrow G(c,d)\left(1+\frac{c-1}{d+1}\right) = G(c,d)\frac{d+c}{d+1} = \frac{c-1}{d+1}G(c-2,d)$$

$$\Longrightarrow G(c,d) = \frac{c-1}{d+c}G(c-2,d)$$

from which we can show

$$G(c,d) = \begin{cases} 0 & \text{If c is odd} \\ \dfrac{\Gamma\left(\frac{c+1}{2}\right)\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{c+d+2}{2}\right)}G(0,d) & \text{If c is even} \end{cases}$$

Then similarly for

$$H(d) = \int_0^{2\pi} \mathrm{d}\theta\cos^d\theta = \int_0^{2\pi} \mathrm{d}\theta\cos^{d-2}\theta - \int_0^{2\pi} \mathrm{d}\theta\sin^2\theta\cos^{d-2}\theta$$

$$= H(d-2) - \frac{-\sin\theta\cos^{d-1}\theta}{d-1}\bigg|_{\theta=0}^{2\pi} - \frac{1}{d-1}\int_0^{2\pi} \mathrm{d}\theta\cos^d\theta$$

$$= H(d-2) - \frac{H(d)}{d-1}$$

$$\Longrightarrow H(d)\frac{d}{d-1} = H(d-2) \Longrightarrow H(d) = \frac{d-1}{d}$$

from which we can show that

$$H(d) = \begin{cases} 0 & \text{When d is odd} \\ 2\pi\dfrac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d+2}{2}\right)\Gamma\left(\frac{1}{2}\right)} & \text{When d is even} \end{cases}$$

So that altogether we have

$$G(c,d) =$$

$$0 \qquad \text{When c or d are odd}$$

$$\frac{2\Gamma\left(\frac{c+1}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{c+d+2}{2}\right)} \qquad \text{When c and d are even}$$

So that we now have

$$F(a,b,c,d,R) = \frac{2\Gamma\left(\frac{c+1}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{c+d+2}{2}\right)} \int_0^R \mathrm{d}\rho \rho^{a+1} \mathrm{e}^{-b\rho^2}$$

making the change of variables $y = b\rho^2 \implies \rho = \sqrt{\frac{y}{b}} \implies \mathrm{d}\rho = \frac{\mathrm{d}y}{2\sqrt{by}}$

$$\implies F(a,b,c,d,R) = \frac{\Gamma\left(\frac{c+1}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}{b^{\frac{a+2}{2}}\Gamma\left(\frac{c+d+2}{2}\right)} \int_0^{bR^2} \mathrm{d}y y^{\frac{a}{2}} \mathrm{e}^{-y}$$

Now for our integrals we know that the powers of ρ were connected to the sin and cos so we since these are two even powers we are dealing with an even a. For

$$L(m,k) = \int_0^k \mathrm{d}y y^m \mathrm{e}^{-y} = -y^m \mathrm{e}^{-y}\Big|_{y=0}^{y=k} + m\int_0^k \mathrm{d}y y^{m-1}\mathrm{e}^{-y} = mL(m-1,k) - k^m \mathrm{e}^{-k}$$

Which we use to show

$$L(m,k) = \frac{\Gamma(m+1)}{\Gamma(m+1-j)} L(m-j,k) - \mathrm{e}^{-k} \sum_{l=0}^{j-1} \frac{\Gamma(m+1)}{\Gamma(m+1-l)} k^{m-l}$$

so

$$L(m,k) = \frac{\Gamma(m+1)}{\Gamma(1)} L(0,k) - \mathrm{e}^{-k} \sum_{l=0}^{m-1} \frac{\Gamma(m+1)}{\Gamma(m+1-l)} k^{m-l}$$

$$= \Gamma(m+1)\left(1 - \mathrm{e}^{-k}\right) - \mathrm{e}^{-k} \sum_{l=0}^{m-1} \frac{\Gamma(m+1)}{\Gamma(m+1-l)} k^{m-l}$$

$$= \Gamma(m+1) - \mathrm{e}^{-k} \sum_{l=0}^{m} \frac{\Gamma(m+1)}{\Gamma(m+1-l)} k^{m-l}$$

or making a change of variables

$$= \Gamma(m+1) - \mathrm{e}^{-k} \sum_{j=0}^{m} \frac{\Gamma(m+1)}{\Gamma(j+1)} k^{j}$$

$$= \Gamma(m+1)\left(1 - \sum_{j=0}^{m} \frac{\mathrm{e}^{-k}}{\Gamma(j+1)} k^{j}\right)$$

This gives

$$
\begin{aligned}
F(a,b,c,d,R) &= \frac{\Gamma\left(\frac{c+1}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}{b^{\frac{a+2}{2}}\Gamma\left(\frac{c+d+2}{2}\right)} L\left(\frac{a}{2}, bR^2\right) \\
&= \frac{\Gamma\left(\frac{c+1}{2}\right)\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{a+2}{2}\right)}{b^{\frac{a+2}{2}}\Gamma\left(\frac{c+d+2}{2}\right)} \left(1 - \sum_{j=0}^{\frac{a}{2}} \frac{\mathrm{e}^{-bR^2}}{\Gamma\left(j+1\right)} \left(bR^2\right)^j\right)
\end{aligned}
$$

when c and d are even and 0 otherwise.

3. Decomposition and Overlap of Modes through Circular Apertures

For this paper we are interested in integrals of the form

$$\langle i,j\,|\,k,l\rangle \equiv \int_0^R \rho \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta A^*_{i,j}(\rho,\theta,z)A_{k,l}(\rho,\theta,z)$$

which we can use the results from the last section to solve for. First we note that these results imply (due to the spherically symmetric nature of the aperture) that this is non-zero iff $i\,mod\,2 = k\,mod\,2$ and $j\,mod\,2 = l\,mod\,2$, and similarly we see via the nature of our definition

$$\langle k,l\,|\,i,j\rangle = \langle i,j\,|\,k,l\rangle^*$$

And we finally also know that because this is spherically symmetric we know that switching up both x and y contributions simultaneously should give us the same result i.e.

$$\langle i,j\,|\,k,l\rangle = \langle j,i\,|\,l,k\rangle$$

For the purposes of decomposing our beam and making sure this decompositional method can be used to measure tilt-to-length contribution to our noise we make sure we can adequately represent the currently expected model for the received beam based off the transmitted beam. We also use one more simplification from the last section,

$$G(a,b,c) \equiv F\left(a, \frac{2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}, b, c, R\right) = \int_0^R \rho^{a+1} \mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}} \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta \sin^b(\theta)\cos^c(\theta)$$

or

$$G(a,b,c) = \frac{\Gamma\left(\frac{b+1}{2}\right)\Gamma\left(\frac{c+1}{2}\right)\Gamma\left(\frac{a+2}{2}\right)\omega_0^{a+2}\left(1+\left(\frac{z}{z_r}\right)^2\right)^{\frac{a}{2}+1}}{2^{\frac{a}{2}+1}\Gamma\left(\frac{b+c+2}{2}\right)} S_{\frac{a}{2}+1}$$

where

$$S_{\frac{a}{2}+1} = 1 - \mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}} \sum_{j=0}^{\frac{a}{2}} \frac{2^j R^{2j}}{j!\omega_0^{2j}\left(1+\left(\frac{z}{z_r}\right)^2\right)^j}$$

where we recognize the sum as the first $\frac{a}{2}+1$ terms of the expansion of $\mathrm{e}^{\frac{2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}$

$$\langle 0,0\,|\,0,0\rangle = \frac{2}{\pi\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)} \int_0^R \rho \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta \mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}$$

$$= \frac{2}{\pi\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)} G(0,0,0) = S_1 = 1 - \mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}$$

$$\langle 0,0\,|\,0,1\rangle = \langle 0,0\,|\,1,0\rangle = \langle 0,0\,|\,1,1\rangle = 0$$

$$\langle 0,0\,|\,0,2\rangle = \frac{2^{1/2}}{\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}\int_0^R \rho \mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\left(\frac{4\rho^2\sin^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-1\right)$$

$$= \frac{2^{1/2}}{\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}\left(\frac{4}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}G(2,2,0)-G(0,0,0)\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)}{2^{1/2}\left(1-i\frac{z}{z_r}\right)}\left(S_2-S_1\right)$$

$$= \frac{-2^{1/2}R^2\mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}}{\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}$$

Which we know is also equal to $\langle 0,0\,|\,2,0\rangle$, and also by the rules from before

$$\langle 0,0\,|\,1,2\rangle = \langle 0,0\,|\,2,1\rangle = 0$$

$$\langle 0,0\,|\,2,2\rangle = \frac{\left(1+i\frac{z}{z_r}\right)}{\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\int_0^R \rho \mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\left(\frac{16\rho^4\cos^2\theta\sin^2\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{4\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+1\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)}{\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(\frac{16G(4,2,2)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{4G(2,0,0)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+G(0,0,0)\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)^2}{2\left(1-i\frac{z}{z_r}\right)^2}\left(S_3-2S_2+S1\right)$$

$$= \frac{R^2\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}}{\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}\right)$$

$$\langle 0,0\,|\,0,3\rangle = \langle 0,0\,|\,3,0\rangle = \langle 0,0\,|\,1,3\rangle = \langle 0,0\,|\,3,1\rangle = \langle 0,0\,|\,2,3\rangle = \langle 0,0\,|\,3,2\rangle = \langle 0,0\,|\,3,3\rangle = 0$$

$$\langle 0,0\,|\,0,4\rangle = \frac{\left(1+i\frac{z}{z_r}\right)}{2^{1/2}3^{1/2}\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\int_0^R \rho \mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta$$

$$\times\left(\frac{16\rho^4\sin^4\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{24\rho^2\sin^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+3\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)}{2^{1/2}3^{1/2}\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(\frac{16G(4,4,0)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{24G(2,2,0)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+3G(0,0,0)\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)^2}{2^{3/2}3^{1/2}\left(1-i\frac{z}{z_r}\right)^2}\left(3S_3-6S_2+3S_1\right)$$

$$= \frac{3^{1/2}\left(1+i\frac{z}{z_r}\right)^2}{2^{3/2}\left(1-i\frac{z}{z_r}\right)^2}\left(S_3-2S_2+S_1\right)$$

$$= \frac{3^{1/2}R^2\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}}{2^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}\right)$$

So that also

$$\langle 0,0\,|\,4,0\rangle = \frac{3^{1/2}R^2\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}}{2^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}\right)$$

$$\langle 0,0\,|\,1,4\rangle = \langle 0,0\,|\,4,1\rangle = 0$$

$$\langle 0,0\,|\,2,4\rangle = \frac{\left(1+i\frac{z}{z_r}\right)^2}{3^{1/2}2\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\int_0^R \rho \mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta$$

$$\times\left[\frac{64\rho^6\cos^2\theta\sin^4\theta}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}-\frac{80\rho^4\sin^2\theta\cos^2\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{16\rho^4\sin^2\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}+\frac{12\rho^2\sin^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+\frac{12\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-3\right]$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)^2}{3^{1/2}2\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left[\frac{64G(6,4,2)}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}-\frac{80G(4,2,2)+16G(4,2,0)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}+\frac{12\left(G(2,2,0)+G(2,0,0)\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-3G(0,0,0)\right]$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)^3}{3^{1/2}2^2\left(1-i\frac{z}{z_r}\right)^3}\left[3S_4-9S_3+9S_2-3S_1\right]$$

$$= \frac{3^{1/2}\left(1+i\frac{z}{z_r}\right)^3}{2^2\left(1-i\frac{z}{z_r}\right)^3}\left[S_4-3S_3+3S_2-S_1\right]$$

$$= \frac{-3^{1/2}R^2\left(1+i\frac{z}{z_r}\right)^2 \mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}}{2\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+\frac{2R^4}{3\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}\right)$$

$$\langle 0,0\,|\,4,2\rangle = \frac{-3^{1/2}R^2\left(1+i\frac{z}{z_r}\right)^2 \mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}}{2\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+\frac{2R^4}{3\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}\right)$$

$$\langle 0,0\,|\,3,4\rangle = \langle 0,0\,|\,4,3\rangle = 0$$

$$\langle 0,0\,|\,4,4\rangle = \frac{\left(1+i\frac{z}{z_r}\right)^3}{2^2 3\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^5}\int_0^R \rho \mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\int_0^{2\pi}\mathrm{d}\theta$$

$$\times\left[\frac{2^8\rho^8\cos^4\theta\sin^4\theta}{\omega_0^8\left(1+\left(\frac{z}{z_r}\right)^2\right)^4}-\frac{2^7 3\rho^6\sin^2\theta\cos^2\theta}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}+\frac{2^4 3\rho^4\left(1+10\sin^2\theta\cos^2\theta\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{2^3 3^2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+9\right]$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)^3}{12\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^5}$$

$$\times\left[\frac{2^8 G(8,4,4)}{\omega_0^8\left(1+\left(\frac{z}{z_r}\right)^2\right)^4}-\frac{2^7 3G(6,2,2)}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}+\frac{2^4 3\left(G(4,0,0)+10G(4,2,2)\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{2^3 3^2 G(2,0,0)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+9G(0,0,0)\right]$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)^4}{24\left(1-i\frac{z}{z_r}\right)^4}\left[3^2S_5 - 2^2 3^2 S_4 + 3^3 2 S_3 - 2^2 3^2 S_2 + 3^2 S_1\right]$$

$$= \frac{3\left(1+i\frac{z}{z_r}\right)^4}{8\left(1-i\frac{z}{z_r}\right)^4}\left(S_5 - 4S_4 + 6S_3 - 4S_2 + S_1\right)$$

$$= \frac{3R^2\left(1+i\frac{z}{z_r}\right)^3 \mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}}{4\omega_0^2\left(1-i\frac{z}{z_r}\right)^5}\left(1 - \frac{3R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)} + \frac{2R^4}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2} - \frac{R^6}{3\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}\right)$$

We collect all our results so far, using $\omega(z) = \omega_0\sqrt{1+\left(\frac{z}{z_r}\right)^2}$

$$\langle 0,0\,|\,0,0\rangle = 1 - \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}$$

$$\langle 0,0\,|\,0,1\rangle = \langle 0,0\,|\,1,0\rangle = \langle 0,0\,|\,1,1\rangle = 0$$

$$\langle 0,0\,|\,0,2\rangle = \langle 0,0\,|\,2,0\rangle = -\frac{\sqrt{2}R^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}$$

$$\langle 0,0\,|\,2,1\rangle = \langle 0,0\,|\,1,2\rangle = 0$$

$$\langle 0,0\,|\,2,2\rangle = \frac{\left(1+i\frac{z}{z_r}\right)R^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{R^2}{\omega(z)^2}\right)$$

$$\langle 0,0\,|\,0,3\rangle = \langle 0,0\,|\,3,0\rangle = \langle 0,0\,|\,1,3\rangle = \langle 0,0\,|\,3,1\rangle = \langle 0,0\,|\,2,3\rangle = \langle 0,0\,|\,3,2\rangle = \langle 0,0\,|\,3,3\rangle = 0$$

$$\langle 0,0\,|\,0,4\rangle = \langle 0,0\,|\,4,0\rangle = \frac{3^{1/2}R^2\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{R^2}{\omega(z)^2}\right)$$

$$\langle 0,0\,|\,1,4\rangle = \langle 0,0\,|\,4,1\rangle = 0$$

$$\langle 0,0\,|\,2,4\rangle = \langle 0,0\,|\,4,2\rangle = \frac{-3^{1/2}R^2\left(1+i\frac{z}{z_r}\right)^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{2R^2}{\omega(z)^2}+\frac{2R^4}{3\omega(z)^4}\right)$$

$$\langle 0,0\,|\,3,4\rangle = \langle 0,0\,|\,4,3\rangle = 0$$

$$\langle 0,0\,|\,4,4\rangle = \frac{3R^2\left(1+i\frac{z}{z_r}\right)^3\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{4\omega_0^2\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{3R^2}{\omega(z)^2}+\frac{2R^4}{\omega(z)^4}-\frac{R^6}{3\omega(z)^6}\right)$$

This doesn't adequately model our beam so we go to higher order terms. Now for the next set, we know

$$\langle 0,0\,|\,5,0\rangle = \langle 0,0\,|\,0,5\rangle = \langle 0,0\,|\,1,5\rangle = \langle 0,0\,|\,5,1\rangle = \langle 0,0\,|\,5,2\rangle$$
$$= \langle 0,0\,|\,3,5\rangle = \langle 0,0\,|\,5,3\rangle = \langle 0,0\,|\,4,5\rangle = \langle 0,0\,|\,5,4\rangle = \langle 0,0\,|\,5,5\rangle = 0$$

Now for the six modes,

$$\langle 0,0\,|\,0,6\rangle = \frac{\left(1+i\frac{z}{z_r}\right)^2}{2*3\pi 5^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\int_0^R \rho \mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta$$

$$\times\left(\frac{2^6\rho^6\sin^6\theta}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3} - \frac{2^4*3*5\rho^4\sin^4\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2} + \frac{2^2 3^2 5\rho^2\sin^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)} - 15\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)^2}{2*3\pi 5^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(\frac{2^6 G(6,6,0)}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3} - \frac{2^4*3*5G(4,4,0)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2} + \frac{2^2 3^2 5G(2,2,0)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)} - 15G(0,0,0)\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)^3}{2^2*3*5^{1/2}\left(1-i\frac{z}{z_r}\right)^3}\left(3*5S_4 - 3^2*5S_3 + 3^2*5S_2 - 3*5S_1\right)$$

$$= \frac{\sqrt{5}\left(1+i\frac{z}{z_r}\right)^3}{4\left(1-i\frac{z}{z_r}\right)^3}\left(S_4 - 3S_3 + 3S_2 - S_1\right)$$

$$= \frac{-\sqrt{5}R^2\left(1+i\frac{z}{z_r}\right)^2 \mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}}{2\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(1 - \frac{2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)} + \frac{2R^4}{3\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}\right)$$

so that similarly

$$\langle 0,0\,|\,6,0\rangle = \frac{-\sqrt{5}R^2\left(1+i\frac{z}{z_r}\right)^2 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(1 - \frac{2R^2}{\omega(z)^2} + \frac{2R^4}{3\omega(z)^4}\right)$$

$$\langle 0,0\,|\,1,6\rangle = \langle 0,0\,|\,6,1\rangle = 0$$

$$\langle 0,0\,|\,2,6\rangle = \frac{\left(1+i\frac{z}{z_r}\right)^3}{2^{3/2}3\pi 5^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^5}\int_0^R \rho \mathrm{e}^{\frac{-2\rho^2}{\omega(z)^2}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\left(\frac{2^8\rho^8\cos^2\theta\sin^6\theta}{\omega(z)^8}\right.$$

$$\left.-\frac{2^6\rho^6\sin^4\theta\left(1+2*7\cos^2\theta\right)}{\omega(z)^6}+\frac{2^4*3*5\rho^4\sin^2\theta\left(1+2\cos^2\theta\right)}{\omega(z)^4}-\frac{2^2*3*5\rho^2\left(1+2\sin^2\theta\right)}{\omega(z)^2}+3*5\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)^3}{2^{3/2}3*5^{1/2}\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^5}\left(\frac{2^8G(8,6,2)}{\omega(z)^8}-\frac{2^6\left(G(6,4,0)+2*7G(6,4,2)\right)}{\omega(z)^6}\right.$$

$$\left.+\frac{2^4*3*5\left(G(4,2,0)+2G(4,2,2)\right)}{\omega(z)^4}-\frac{2^2*3*5\left(G(2,0,0)+2G(2,2,0)\right)}{\omega(z)^2}+3*5G(0,0,0)\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)^4}{2^{5/2}*3*5^{1/2}\left(1-i\frac{z}{z_r}\right)^4}\left(3*5S_5-2*3\left(3+7\right)S_4+2*3*5\left(2+1\right)S_3-2*3*5\left(1+1\right)S_2+3*5S_1\right)$$

$$= \frac{5^{1/2}\left(1+i\frac{z}{z_r}\right)^4}{2^{5/2}\left(1-i\frac{z}{z_r}\right)^4}\left(S_5-2^2S_4+2*3S_3-2^2S_2+S_1\right)$$

$$= -\frac{5^{1/2}\left(1+i\frac{z}{z_r}\right)^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{5/2}\left(1-i\frac{z}{z_r}\right)^4}\left(-\frac{2R^2}{\omega(z)^2}+3\frac{2R^4}{\omega(z)^4}-\frac{4R^6}{\omega(z)^6}+\frac{2R^8}{3\omega(z)^8}\right)$$

$$= \frac{5^{1/2}R^2\left(1+i\frac{z}{z_r}\right)^3\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{3/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{3R^2}{\omega(z)^2}+\frac{2R^4}{\omega(z)^4}-\frac{R^6}{3\omega(z)^6}\right)$$

So that in the same way we have

$$\langle 0,0\,|\,6,2\rangle = \frac{5^{1/2}R^2\left(1+i\frac{z}{z_r}\right)^3\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{3/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{3R^2}{\omega(z)^2}+\frac{2R^4}{\omega(z)^4}-\frac{R^6}{3\omega(z)^6}\right)$$

Now for the next one

$$\langle 0,0\,|\,3,6\rangle = \langle 0,0\,|\,6,3\rangle = 0$$

and for the next non-zero one

$$\langle 0,0\,|\,4,6\rangle = \frac{\left(1+i\frac{z}{z_r}\right)^4}{2^{5/2}3^{3/2}\pi 5^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^6}\int_0^R \rho \mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\left(\frac{2^{10}\rho^{10}\cos^4\theta\sin^6\theta}{\omega_0^{10}\left(1+\left(\frac{z}{z_r}\right)^2\right)^5}\right.$$

$$-\frac{2^8 3\rho^8\cos^2\theta\sin^4\theta\left(2+3\cos^2\theta\right)}{\omega_0^8\left(1+\left(\frac{z}{z_r}\right)^2\right)^4}+\frac{2^6 3\rho^6\sin^2\theta\left(1+14\cos^2\theta\left(1+\sin^2\theta\right)\right)}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}$$

$$\left.-\frac{2^4*3*5\rho^4\left(1+2\sin^2\theta\left(1+7\cos^2\theta\right)\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}+\frac{2^2 3^2 5\rho^2\left(2+\sin^2\theta\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-3^2*5\right)$$

$$=\frac{\left(1+i\frac{z}{z_r}\right)^4}{2^{5/2}3^{3/2}\pi 5^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^6}\left(\frac{2^{10}G(10,6,4)}{\omega_0^{10}\left(1+\left(\frac{z}{z_r}\right)^2\right)^5}-\frac{2^8*3\left(2G(8,4,2)+3G(8,4,4)\right)}{\omega_0^8\left(1+\left(\frac{z}{z_r}\right)^2\right)^4}\right.$$

$$+\frac{2^6*3\left(G(6,2,0)+14\left(G(6,2,2)+G(6,4,2)\right)\right)}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}-\frac{2^4*3*5\left(G(4,0,0)+2\left(G(4,2,0)+7G(4,2,2)\right)\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}$$

$$\left.+\frac{2^2 3^2 5\left(2G(2,0,0)+G(2,2,0)\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-3^2*5G(0,0,0)\right)$$

$$=\frac{\left(1+i\frac{z}{z_r}\right)^5}{2^{7/2}3^{3/2}5^{1/2}\left(1-i\frac{z}{z_r}\right)^5}\left(3^2*5S_6-3^2 5^2 S_5+2*3^2 5^2 S_4-2*3^2 5^2 S_3+3^2*5^2 S_2-3^2*5S_1\right)$$

$$=\frac{3^{1/2}5^{1/2}\left(1+i\frac{z}{z_r}\right)^5}{2^{7/2}\left(1-i\frac{z}{z_r}\right)^5}\left(S_6-5S_5+10S_4-10S_3+5S_2-S_1\right)$$

$$= \frac{-3^{1/2}5^{1/2}R^2\left(1+i\frac{z}{z_r}\right)^4 \mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}}{2^{5/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^6}$$

$$\times\left(1-\frac{4R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+\frac{4R^4}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{4R^6}{3\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}+\frac{2R^8}{15\omega_0^8\left(1+\left(\frac{z}{z_r}\right)^2\right)^4}\right)$$

So that also

$$\langle 0,0\,|\,6,4\rangle = \frac{-3^{1/2}5^{1/2}R^2\left(1+i\frac{z}{z_r}\right)^4 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{5/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^6}\left(1-\frac{4R^2}{\omega(z)^2}+\frac{4R^4}{\omega(z)^4}-\frac{4R^6}{3\omega(z)^6}+\frac{2R^8}{15\omega(z)^8}\right)$$

$$\langle 0,0\,|\,5,6\rangle = \langle 0,0\,|\,6,5\rangle = 0$$

Now we have the final component to compute for all TEM modes up to order 6 polynomials.

$$\langle 0,0\,|\,6,6\rangle = \frac{\left(1+i\frac{z}{z_r}\right)^5}{2^3 3^2 5\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^7}\int_0^R \rho\mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta$$

$$\times\left(\frac{2^{12}\rho^{12}\cos^6\theta\sin^6\theta}{\omega_0^{12}\left(1+\left(\frac{z}{z_r}\right)^2\right)^6}-\frac{2^{10}*3*5\rho^{10}\cos^4\theta\sin^4\theta}{\omega_0^{10}\left(1+\left(\frac{z}{z_r}\right)^2\right)^5}\right.$$

$$+\frac{2^8 3^2 5\rho^8\sin^2\theta\cos^2\theta\left(3\sin^2\theta\cos^2\theta+1\right)}{\omega_0^8\left(1+\left(\frac{z}{z_r}\right)^2\right)^4}-\frac{2^6*3*5\rho^6\left(1+2*3*7\sin^2\theta\cos^2\theta\right)}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}$$

$$\left.+\frac{2^4 3^2 5^2\rho^4\left(1+7\sin^2\theta\cos^2\theta\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{2^2 3^3 5^2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+3^2 5^2\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)^5}{2^3 3^2 5\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^7}\left(\frac{2^{12}G(12,6,6)}{\omega_0^{12}\left(1+\left(\frac{z}{z_r}\right)^2\right)^6}-\frac{2^{10}*3*5G(10,4,4)}{\omega_0^{10}\left(1+\left(\frac{z}{z_r}\right)^2\right)^5}\right.$$

$$+\frac{2^8 3^2 5\left(3G(8,4,4)+G(8,2,2)\right)}{\omega_0^8\left(1+\left(\frac{z}{z_r}\right)^2\right)^4}-\frac{2^6*3*5\left(G(6,0,0)+2*3*7G(6,2,2)\right)}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}$$

$$\left.+\frac{2^4 3^2 5^2\left(G(4,0,0)+7G(4,2,2)\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{2^2 3^3 5^2 G(2,0,0)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+3^2 5^2 G(0,0,0)\right)$$

$$=\frac{\left(1+i\frac{z}{z_r}\right)^6}{2^4 3^2 5\left(1-i\frac{z}{z_r}\right)^6}\left(3^2 5^2 S_7-2*3^3*5^2 S_6+3^3 5^3 S_5-2^2 3^2 5^3 S_4+3^3 5^3 S_3-2*3^3 5^2 S_2+3^2 5^2 S_1\right)$$

$$=\frac{5\left(1+i\frac{z}{z_r}\right)^6}{2^4\left(1-i\frac{z}{z_r}\right)^6}\left(S_7-6S_6+15S_5-20S_4+15S_3-6S_2+S_1\right)$$

$$=\frac{5R^2\left(1+i\frac{z}{z_r}\right)^5 \mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}}{2^3\omega_0^2\left(1-i\frac{z}{z_r}\right)^7}\left(1-\frac{5R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}\right.$$

$$\left.+\frac{20R^4}{3\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{10R^6}{3\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}+\frac{2R^8}{3\omega_0^8\left(1+\left(\frac{z}{z_r}\right)^2\right)^4}-\frac{2R^{10}}{45\omega_0^{10}\left(1+\left(\frac{z}{z_r}\right)^2\right)^5}\right)$$

We gather all of our results thus far

$$\langle 0,0\,|\,0,0\rangle = 1-\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}$$

$$\langle 0,0\,|\,0,1\rangle = \langle 0,0\,|\,1,0\rangle = \langle 0,0\,|\,1,1\rangle = 0$$

$$\langle 0,0\,|\,0,2\rangle = \langle 0,0\,|\,2,0\rangle = -\frac{\sqrt{2}R^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}$$

$$\langle 0,0\,|\,2,1\rangle = \langle 0,0\,|\,1,2\rangle = 0$$

$$\langle 0,0\,|\,2,2\rangle = \frac{\left(1+i\frac{z}{z_r}\right)R^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{R^2}{\omega(z)^2}\right)$$

$$\langle 0,0\,|\,0,3\rangle = \langle 0,0\,|\,3,0\rangle = \langle 0,0\,|\,1,3\rangle = \langle 0,0\,|\,3,1\rangle = \langle 0,0\,|\,2,3\rangle = \langle 0,0\,|\,3,2\rangle = \langle 0,0\,|\,3,3\rangle = 0$$

$$\langle 0,0\,|\,0,4\rangle = \langle 0,0\,|\,4,0\rangle = \frac{3^{1/2}R^2\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{R^2}{\omega(z)^2}\right)$$

$$\langle 0,0\,|\,1,4\rangle = \langle 0,0\,|\,4,1\rangle = 0$$

$$\langle 0,0\,|\,2,4\rangle = \langle 0,0\,|\,4,2\rangle = \frac{-3^{1/2}R^2\left(1+i\frac{z}{z_r}\right)^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{2R^2}{\omega(z)^2}+\frac{2R^4}{3\omega(z)^4}\right)$$

$$\langle 0,0\,|\,3,4\rangle = \langle 0,0\,|\,4,3\rangle = 0$$

$$\langle 0,0\,|\,4,4\rangle = \frac{3R^2\left(1+i\frac{z}{z_r}\right)^3\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{4\omega_0^2\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{3R^2}{\omega(z)^2}+\frac{2R^4}{\omega(z)^4}-\frac{R^6}{3\omega(z)^6}\right)$$

$$\langle 0,0\,|\,0,5\rangle = \langle 0,0\,|\,5,0\rangle = \langle 0,0\,|\,1,5\rangle = \langle 0,0\,|\,5,1\rangle = \langle 0,0\,|\,2,5\rangle = \langle 0,0\,|\,5,2\rangle = 0$$

$$\langle 0,0\,|\,3,5\rangle = \langle 0,0\,|\,5,3\rangle = \langle 0,0\,|\,4,5\rangle = \langle 0,0\,|\,5,4\rangle = \langle 0,0\,|\,5,5\rangle = 0$$

$$\langle 0,0\,|\,0,6\rangle = \langle 0,0\,|\,6,0\rangle = \frac{-\sqrt{5}R^2\left(1+i\frac{z}{z_r}\right)^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{2R^2}{\omega(z)^2}+\frac{2R^4}{3\omega(z)^4}\right)$$

$$\langle 0,0\,|\,1,6\rangle = \langle 0,0\,|\,6,1\rangle = 0$$

$$\langle 0,0\,|\,2,6\rangle = \langle 0,0\,|\,6,2\rangle = \frac{5^{1/2}R^2\left(1+i\frac{z}{z_r}\right)^3\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{3/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{3R^2}{\omega(z)^2}+\frac{2R^4}{\omega(z)^4}-\frac{R^6}{3\omega(z)^6}\right)$$

$$\langle 0,0\,|\,3,6\rangle = \langle 0,0\,|\,6,3\rangle = 0$$

$$\langle 0,0\,|\,4,6\rangle = \langle 0,0\,|\,6,4\rangle = \frac{-3^{1/2}5^{1/2}R^2\left(1+i\frac{z}{z_r}\right)^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{5/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^6}\left(1-\frac{4R^2}{\omega(z)^2}+\frac{4R^4}{\omega(z)^4}-\frac{4R^6}{3\omega(z)^6}+\frac{2R^8}{15\omega(z)^8}\right)$$

$$\langle 0,0\,|\,5,6\rangle = \langle 0,0\,|\,6,5\rangle = 0$$

$$\langle 0,0\,|\,6,6\rangle = \frac{5R^2\left(1+i\frac{z}{z_r}\right)^5\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^3\omega_0^2\left(1-i\frac{z}{z_r}\right)^7}\left(1-\frac{5R^2}{\omega(z)^2}+\frac{20R^4}{3\omega(z)^4}-\frac{10R^6}{3\omega(z)^6}+\frac{2R^8}{3\omega(z)^8}-\frac{2R^{10}}{45\omega(z)^{10}}\right)$$

Now for the overlap of modes higher than TEM00 modes.

$$\langle 1,0\,|\,0,1\rangle = \langle 1,0\,|\,1,1\rangle = \langle 0,1\,|\,1,1\rangle = 0$$

$$\begin{aligned}
\langle 0,1\,|\,0,1\rangle = \langle 1,0\,|\,1,0\rangle &= \frac{2^3}{\pi\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}\int_0^R \rho\mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\rho^2\sin^2\theta \\
&= \frac{2^3}{\pi\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}G(2,2,0) \\
&= S_2 = 1-\mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\left(1+\frac{2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}\right)
\end{aligned}$$

$$\begin{aligned}
\langle 1,1\,|\,1,1\rangle &= \frac{2^5}{\pi\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}\int_0^R \rho\mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\int_0^{2\pi}\mathrm{d}\theta\rho^4\sin^2\theta\cos^2\theta \\
&= \frac{2^5G(4,2,2)}{\pi\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3} = S_3 \\
&= 1-\mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\left(1+\frac{2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+\frac{2R^4}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}\right)
\end{aligned}$$

$$\begin{aligned}
&\langle 0,1\,|\,0,2\rangle = \langle 1,0\,|\,0,2\rangle = \langle 0,1\,|\,2,0\rangle = \langle 1,0\,|\,2,0\rangle = \langle 0,1\,|\,1,2\rangle = \langle 1,0\,|\,2,1\rangle \\
&= \langle 1,1\,|\,1,2\rangle = \langle 1,1\,|\,2,1\rangle = \langle 1,0\,|\,2,2\rangle = \langle 0,1\,|\,2,2\rangle = \langle 1,1\,|\,2,2\rangle = 0
\end{aligned}$$

$$\begin{aligned}
\langle 0,1\,|\,2,1\rangle = \langle 1,0\,|\,1,2\rangle &= \frac{2^{5/2}}{\pi\omega_0^4\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}\int_0^R \rho\mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\rho^2\sin^2\theta\left(\frac{2^2\rho^2\cos^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-1\right) \\
&= \frac{2^{5/2}}{\pi\omega_0^4\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}\left(\frac{2^2G(4,2,2)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-G(2,2,0)\right) \\
&= \frac{\left(1+i\frac{z}{z_r}\right)}{2^{1/2}\left(1-i\frac{z}{z_r}\right)}\left(S_3-S_2\right)
\end{aligned}$$

$$= \frac{\left(1 + i\frac{z}{z_r}\right)}{\sqrt{2}\left(1 - i\frac{z}{z_r}\right)} \left(-\mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}} \frac{2R^4}{\omega_0^4\left(1 + \left(\frac{z}{z_r}\right)^2\right)^2} \right)$$

$$= \frac{-\sqrt{2}R^4\mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}}{\omega_0^4\left(1 + i\frac{z}{z_r}\right)\left(1 - i\frac{z}{z_r}\right)^3}$$

$$\langle 0,1\,|\,3,0\rangle = \langle 1,0\,|\,0,3\rangle = \langle 1,1\,|\,0,3\rangle = \langle 1,1\,|\,3,0\rangle = 0$$

$$\langle 0,1\,|\,0,3\rangle = \langle 1,0\,|\,3,0\rangle = \frac{2^{5/2}}{3^{1/2}\pi\omega_0^4\left(1 + i\frac{z}{z_r}\right)\left(1 - i\frac{z}{z_r}\right)^3} \int_0^R \rho\mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\,\mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta\rho^2\sin^2\theta$$

$$\times \left(\frac{2^2\rho^2\sin^2\theta}{\omega_0^2\left(1 + \left(\frac{z}{z_r}\right)^2\right)} - 3 \right)$$

$$= \frac{2^{5/2}}{3^{1/2}\pi\omega_0^4\left(1 + i\frac{z}{z_r}\right)\left(1 - i\frac{z}{z_r}\right)^3} \left(\frac{2^2G(4,4,0)}{\omega_0^2\left(1 + \left(\frac{z}{z_r}\right)^2\right)} - 3G(2,2,0) \right)$$

$$= \frac{\sqrt{3}\left(1 + i\frac{z}{z_r}\right)}{\sqrt{2}\left(1 - i\frac{z}{z_r}\right)} (S_3 - S_2)$$

$$= \frac{-\sqrt{6}R^4\mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}}{\omega_0^4\left(1 + i\frac{z}{z_r}\right)\left(1 - i\frac{z}{z_r}\right)^3}$$

$$\langle 0,1\,|\,1,3\rangle = \langle 1,0\,|\,1,3\rangle = \langle 0,1\,|\,3,1\rangle = \langle 1,0\,|\,3,1\rangle = 0$$

$$\langle 1,1\,|\,1,3\rangle = \langle 1,1\,|\,3,1\rangle = \frac{2^{9/2}}{3^{1/2}\pi\omega_0^6\left(1 + i\frac{z}{z_r}\right)^2\left(1 - i\frac{z}{z_r}\right)^4} \int_0^R \rho\mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\,\mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta\rho^4\sin^2\theta\cos^2\theta$$

$$\times \left(\frac{2^2\rho^2\sin^2\theta}{\omega_0^2\left(1 + \left(\frac{z}{z_r}\right)^2\right)} - 3 \right)$$

$$= \frac{2^{9/2}}{3^{1/2}\pi\omega_0^6\left(1+i\frac{z}{z_r}\right)^2\left(1-i\frac{z}{z_r}\right)^4}\left(\frac{2^2G(6,4,2)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)} - 3G(4,2,2)\right)$$

$$= \frac{3^{1/2}\left(1+i\frac{z}{z_r}\right)}{2^{1/2}\left(1-i\frac{z}{z_r}\right)}\left(S_4 - S_3\right)$$

$$= \frac{-2^{3/2}R^6\mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}}{3^{1/2}\omega_0^6\left(1+i\frac{z}{z_r}\right)^2\left(1-i\frac{z}{z_r}\right)^4}$$

$$\langle 1,1\,|\,2,3\rangle = \langle 1,1\,|\,3,2\rangle = 0$$

$$\langle 0,1\,|\,2,3\rangle = \langle 1,0\,|\,3,2\rangle = \frac{2^2}{3^{1/2}\pi\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\int_0^R \rho\mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\rho^2\sin^2\theta$$

$$\times\left(\frac{2^4\rho^4\cos^2\theta\sin^2\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2} - \frac{2^2\rho^2\left(1+2\cos^2\theta\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)} + 3\right)$$

$$= \frac{2^2}{3^{1/2}\pi\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(\frac{2^4G(6,4,2)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2} - \frac{2^2\left(G(4,2,0)+2G(4,2,2)\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)} + 3G(2,2,0)\right)$$

$$= \frac{2\left(1+i\frac{z}{z_r}\right)}{3^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(\frac{3}{4}\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)S_4 - \frac{3}{2}\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)S_3 + \frac{3}{4}\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)S_2\right)$$

$$= \frac{3^{1/2}\left(1+i\frac{z}{z_r}\right)^2}{2\left(1-i\frac{z}{z_r}\right)^2}\left(S_4 - 2S_3 + S_2\right)$$

$$= \frac{3^{1/2}R^4\mathrm{e}^{\frac{-2R}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}}{\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(1 - \frac{2R^2}{3\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}\right)$$

$$\langle 0,1\,|\,3,3\rangle = \langle 1,0\,|\,3,3\rangle = 0$$

$$\langle 1,1\,|\,3,3\rangle = \frac{2^4}{3\pi\omega_0^6\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^5}\int_0^R \rho \mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\rho^4\cos^2\theta\sin^2\theta$$

$$\times\left(\frac{2^4\rho^4\cos^2\theta\sin^2\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{3*2^2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+9\right)$$

$$=\frac{2^4}{3\pi\omega_0^6\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^5}\left(\frac{2^4G(8,4,4)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{3*2^2G(6,2,2)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+9G(4,2,2)\right)$$

$$=\frac{2^3}{3\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(\frac{3^2}{2^4}\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2S_5-\frac{3^2}{2^3}\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2S_4+\frac{3^2}{2^4}\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2S_3\right)$$

$$=\frac{3\left(1+i\frac{z}{z_r}\right)^2}{2\left(1-i\frac{z}{z_r}\right)^2}\left(S_5-2S_4+S_3\right)$$

$$=\frac{R^6\mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}}{\omega_0^6\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^5}\left(2-\frac{R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}\right)$$

$$\langle 0,1\,|\,0,4\rangle=\langle 1,0\,|\,0,4\rangle=\langle 1,1\,|\,0,4\rangle=\langle 0,1\,|\,4,0\rangle=\langle 1,0\,|\,4,0\rangle=\langle 1,1\,|\,4,0\rangle=0$$

$$\langle 0,1\,|\,1,4\rangle=\langle 1,0\,|\,4,1\rangle=\langle 1,1\,|\,1,4\rangle=\langle 1,1\,|\,4,1\rangle=0$$

$$\langle 0,1\,|\,4,1\rangle=\langle 1,0\,|\,1,4\rangle=\frac{2^{3/2}}{3^{1/2}\pi\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\int_0^R \rho \mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\rho^2\sin^2\theta$$

$$\times\left(\frac{2^4\rho^4\cos^4\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{2^3 3\rho^2\cos^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+3\right)$$

$$=\frac{2^{3/2}}{3^{1/2}\pi\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(\frac{2^4G(6,2,4)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{2^3 3G(4,2,2)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+3G(2,2,0)\right)$$

$$= \frac{2^{1/2}\left(1+i\frac{z}{z_r}\right)^2}{3^{1/2}\left(1-i\frac{z}{z_r}\right)^2}\left(\frac{3}{2^2}S_4-\frac{3}{2}S_3+\frac{3}{2^2}S_2\right)$$

$$= \frac{3^{1/2}\left(1+i\frac{z}{z_r}\right)^2}{2^{3/2}\left(1-i\frac{z}{z_r}\right)^2}\left(S_4-2S_3+S_2\right)$$

$$= \frac{3^{1/2}R^4\mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}}{2^{1/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{2R^2}{3\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}\right)$$

$$\langle 1,1\,|\,2,4\rangle = \langle 1,0\,|\,2,4\rangle = \langle 0,1\,|\,2,4\rangle = \langle 1,1\,|\,4,2\rangle = \langle 0,1\,|\,4,2\rangle$$
$$= \langle 1,0\,|\,4,2\rangle = \langle 0,1\,|\,3,4\rangle = \langle 1,0\,|\,4,3\rangle = \langle 1,1\,|\,4,3\rangle = \langle 1,1\,|\,3,4\rangle = 0$$

$$\langle 0,1\,|\,4,3\rangle = \langle 1,0\,|\,3,4\rangle = \frac{2\left(1+i\frac{z}{z_r}\right)}{3\pi\omega_0^4\left(1-i\frac{z}{z_r}\right)^5}\int_0^R \rho\mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\rho^2\sin^2\theta$$

$$\times\left(\frac{2^6\rho^6\sin^2\theta\cos^4\theta}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}-\frac{2^4 3\rho^4\cos^2\theta\left(1+\sin^2\theta\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}+\frac{2^2 3\rho^2\left(1+5\cos^2\theta\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-9\right)$$

$$= \frac{2\left(1+i\frac{z}{z_r}\right)}{3\pi\omega_0^4\left(1-i\frac{z}{z_r}\right)^5}\left(\frac{2^6G(8,4,4)}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}-\frac{2^4 3\left(G(6,2,2)+G(6,4,2)\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}+\frac{2^2 3\left(G(4,2,0)+5G(4,2,2)\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-9G(2,2,0)\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)^2}{3\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(\frac{3^2}{2^2}\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)S_5-2^4 3\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)S_4\left(\frac{3}{2^5}+\frac{3}{2^6}\right)\right.$$
$$\left.+2^2 3\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)S_3\left(\frac{1}{2^2}+\frac{5}{2^4}\right)-\frac{9}{2^2}\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)S_2\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)^3}{\left(1-i\frac{z}{z_r}\right)^3}\left(\frac{3S_5}{2^2}-\frac{3^2S_4}{2^2}+\frac{3^2S_3}{2^2}-\frac{3S_2}{2^2}\right)$$

$$= \frac{3\left(1+i\frac{z}{z_r}\right)^3}{2^2\left(1-i\frac{z}{z_r}\right)^3}\left(S_5-3S_4+3S_3-S_2\right)$$

$$= \frac{-3R^4\left(1+i\frac{z}{z_r}\right)}{2\omega_0^4\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{4R^2}{3\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+\frac{R^4}{3\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}\right)$$

$$\langle 0,1\,|\,4,4\rangle = \langle 1,0\,|\,4,4\rangle = \langle 1,1\,|\,4,4\rangle = 0$$

$$\langle 0,1\,|\,5,0\rangle = \langle 1,0\,|\,0,5\rangle = \langle 1,1\,|\,0,5\rangle = \langle 1,1\,|\,5,0\rangle = 0$$

$$\langle 0,1\,|\,0,5\rangle = \langle 1,0\,|\,5,0\rangle = \frac{2^{3/2}}{3^{1/2}5^{1/2}\pi\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\int_0^R \rho \mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\rho^2\sin^2\theta$$

$$\times\left(\frac{2^4\rho^4\sin^4\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{2^3 5\rho^2\sin^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+15\right)$$

$$= \frac{2^{3/2}}{3^{1/2}5^{1/2}\pi\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(\frac{2^4G(6,6,0)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{2^3 5G(4,4,0)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+15G(2,2,0)\right)$$

$$= \frac{2^{1/2}\left(1+i\frac{z}{z_r}\right)}{3^{1/2}5^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(\frac{3*5}{2^2}\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)S_4-\frac{3*5}{2}\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)S_3+\frac{3*5}{2^2}S_2\right)$$

$$= \frac{3^{1/2}5^{1/2}\left(1+i\frac{z}{z_r}\right)^2}{2^{3/2}\left(1-i\frac{z}{z_r}\right)^2}\left(S_4-2S_3+S_2\right)$$

$$= \frac{3^{1/2}5^{1/2}R^4\mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_R}\right)^2\right)}}}{2^{1/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{2R^2}{3\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}\right)$$

$$\langle 0,1\,|\,1,5\rangle = \langle 1,0\,|\,1,5\rangle = \langle 0,1\,|\,5,1\rangle = \langle 1,0\,|\,5,1\rangle = 0$$

$$\langle 1,1\,|\,1,5\rangle = \langle 1,1\,|\,5,1\rangle = \frac{2^{7/2}}{3^{1/2}5^{1/2}\pi\omega_0^6\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^5}\int_0^R \rho \mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\rho^4\sin^2\theta\cos^2\theta$$

$$\times\left(\frac{2^4\rho^4\sin^4\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{2^3 5\rho^2\sin^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+15\right)$$

$$= \frac{2^{7/2}}{3^{1/2}5^{1/2}\pi\omega_0^6\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^5}\left(\frac{2^4G(8,6,2)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2} - \frac{2^3 5G(6,4,2)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)} + 15G(4,2,2)\right)$$

$$= \frac{2^{5/2}}{3^{1/2}5^{1/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(\frac{3*5\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2 S_5}{2^4} - \frac{3*5\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2 S_4}{2^3} + \frac{3*5\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2 S_3}{2^4}\right)$$

$$= \frac{3^{1/2}5^{1/2}\left(1+i\frac{z}{z_r}\right)^2}{2^{3/2}\left(1-i\frac{z}{z_r}\right)^2}\left(S_5 - 2S_4 + S_3\right)$$

$$= \frac{5^{1/2}R^6 \mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}}{2^{1/2}3^{1/2}\omega_0^6\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^5}\left(2 - \frac{R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}\right)$$

$$\langle 1,1\,|\,2,5\rangle = \langle 1,1\,|\,5,2\rangle = \langle 0,1\,|\,5,2\rangle = \langle 1,0\,|\,2,5\rangle = 0$$

$$\langle 0,1\,|\,2,5\rangle = \langle 1,0\,|\,5,2\rangle = \frac{2\left(1+i\frac{z}{z_r}\right)}{3^{1/2}5^{1/2}\pi\omega_0^4\left(1-i\frac{z}{z_r}\right)^5}\int_0^R \rho \mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\rho^2\sin^2\theta$$

$$\times\left(\frac{2^6\rho^6\sin^4\theta\cos^2\theta}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3} - \frac{2^4\rho^4\sin^2\theta\left(1+9\cos^2\theta\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2} + \frac{2^2 5\rho^2\left(2+\cos^2\theta\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)} - 15\right)$$

$$= \frac{2\left(1+i\frac{z}{z_r}\right)}{3^{1/2}5^{1/2}\pi\omega_0^4\left(1-i\frac{z}{z_r}\right)^5}$$

$$\times\left(\frac{2^6G(8,6,2)}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3} - \frac{2^4\left(G(6,4,0)+9G(6,4,2)\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2} + \frac{2^2 5\left(2G(4,2,0)+G(4,2,2)\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)} - 15G(2,2,0)\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)^2}{3^{1/2}5^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}$$

$$\times\left(\frac{3*5S_5\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}{2^2}-3S_4\left(\frac{3}{2}+\frac{9}{2^2}\right)\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)\right.$$

$$\left.+5S_3\left(2+\frac{1}{2^2}\right)\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)-\frac{3*5S_2\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}{2^2}\right)$$

$$=\frac{\left(1+i\frac{z}{z_r}\right)^3}{3^{1/2}5^{1/2}\left(1-i\frac{z}{z_r}\right)^3}\left(\frac{3*5S_5}{2^2}-\frac{3^2*5S_4}{2^2}+\frac{3^2*5S_3}{2^2}-\frac{3*5S_2}{2^2}\right)$$

$$=\frac{3^{1/2}5^{1/2}\left(1+i\frac{z}{z_r}\right)^3}{2^2\left(1-i\frac{z}{z_r}\right)^3}\left(S_5-3S_4+3S_3-S_2\right)$$

$$=\frac{-3^{1/2}5^{1/2}R^4\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}}{2\omega_0^4\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{4R^2}{3\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+\frac{R^4}{3\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}\right)$$

$$\langle 0,1\,|\,3,5\rangle=\langle 0,1\,|\,5,3\rangle=\langle 1,0\,|\,3,5\rangle=\langle 1,0\,|\,5,3\rangle=0$$

$$\langle 1,1\,|\,3,5\rangle=\langle 1,1\,|\,5,3\rangle=\frac{2^3}{3*5^{1/2}\pi\omega_0^6\left(1-i\frac{z}{z_r}\right)^6}\int_0^R\rho\mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\rho^4\sin^2\theta\cos^2\theta$$

$$\times\left(\frac{2^6\rho^6\sin^4\theta\cos^2\theta}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}-\frac{2^4\rho^4\sin^2\theta\left(3+7\cos^2\theta\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}+\frac{2^2*3*5\rho^2\left(1+\sin^2\theta\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-3^25\right)$$

$$=\frac{2^3}{3*5^{1/2}\pi\omega_0^6\left(1-i\frac{z}{z_r}\right)^6}\left(\frac{2^6G(10,6,4)}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}-\frac{2^4\left(3G(8,4,2)+7G(8,4,4)\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}\right.$$

$$\left.+\frac{2^2*3*5\left(G(6,2,2)+G(6,4,2)\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-3^25G(4,2,2)\right)$$

$$= \frac{2^2\left(1+i\frac{z}{z_r}\right)}{3*5^{1/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^5}\left(\frac{3^2 5 S_6\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}{2^4} - \left(\frac{3^2}{2}+\frac{3^2 7}{2^4}\right)\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2 S_5\right.$$

$$\left.+\frac{3*5\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2 S_4}{2}\left(\frac{3}{2^2}+\frac{3}{2^3}\right) - \frac{3^2 5}{2^4}\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2 S_3\right)$$

$$= \frac{3\left(1+i\frac{z}{z_r}\right)^3}{2^2 5^{1/2}\left(1-i\frac{z}{z_r}\right)^3}\left(5S_6 - 3*5S_5 + 3*5S_4 - 5S_3\right)$$

$$= \frac{3*5^{1/2}\left(1+i\frac{z}{z_r}\right)^3}{2^2\left(1-i\frac{z}{z_r}\right)^3}\left(S_6 - 3S_5 + 3S_4 - S_3\right)$$

$$= \frac{-5^{1/2}R^6 \mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}}{\omega_0^6\left(1-i\frac{z}{z_r}\right)^6}\left(1 - \frac{R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)} + \frac{R^4}{5\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}\right)$$

$$\langle 1,1\,|\,4,5\rangle = \langle 1,1\,|\,5,4\rangle = \langle 0,1\,|\,5,4\rangle = \langle 1,0\,|\,4,5\rangle = 0$$

$$\langle 0,1\,|\,4,5\rangle = \langle 1,0\,|\,5,4\rangle = \frac{\left(1+i\frac{z}{z_r}\right)^2}{3\pi 5^{1/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^6}\int_0^R \rho \mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\rho^2\sin^2\theta$$

$$\times\left(\frac{2^8\rho^8\sin^4\theta\cos^4\theta}{\omega_0^8\left(1+\left(\frac{z}{z_r}\right)^2\right)^4} - \frac{2^7\rho^6\sin^2\theta\cos^2\theta\left(2\cos^2\theta+3\right)}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}\right.$$

$$\left.+\frac{2^4 3\rho^4\left(1+4\cos^2\theta+2*7\cos^2\theta\sin^2\theta\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2} - \frac{2^3*3*5\rho^2\left(1+2\cos^2\theta\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)} + 3^2 5\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)^2}{3\pi 5^{1/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^6}\left(\frac{2^8 G(10,6,4)}{\omega_0^8\left(1+\left(\frac{z}{z_r}\right)^2\right)^4} - \frac{2^7\left(2G(8,4,4)+3G(8,4,2)\right)}{\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}\right.$$

$$+\frac{2^4 3\left(G(6,2,0)+2^2G(6,2,2)+2*7G(6,4,2)\right)}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{2^3*3*5\left(G(4,2,0)+2G(4,2,2)\right)}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+3^2 5G(2,2,0)\Bigg)$$

$$=\frac{\left(1+i\frac{z}{z_r}\right)^3}{2*3*5^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^5}\left(\frac{3^2 5\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)S_6}{2^2}-\left(3^2+3^2 4\right)\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)S_5\right.$$

$$+3\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)S_4\left(2*3+2*3+\frac{7*3}{2}\right)-3*5\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)S_3\left(2+1\right)$$

$$\left.+\frac{3^2 5\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)S_2}{2^2}\right)$$

$$=\frac{3*5^{1/2}\left(1+i\frac{z}{z_r}\right)^4}{2\left(1-i\frac{z}{z_r}\right)^4}\left(\frac{S_6}{2^2}-S_5+\frac{3S_4}{2}-S_3+\frac{S_2}{2^2}\right)$$

$$=\frac{3*5^{1/2}\left(1+i\frac{z}{z_r}\right)^4}{2^3\left(1-i\frac{z}{z_r}\right)^4}\left(S_6-4S_5+6S_4-4S_3+S_2\right)$$

$$=\frac{3*5^{1/2}R^4\left(1+i\frac{z}{z_r}\right)^2 \mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}}{2^2\omega_0^4\left(1-i\frac{z}{z_r}\right)^6}\left(1-\frac{2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+\frac{R^4}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{2R^6}{15\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}\right)$$

From now one we will be simplifying expressions $\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)$ as $\omega(z)^2$ giving

$$G(a,b,c)=\frac{\Gamma\left(\frac{b+1}{2}\right)\Gamma\left(\frac{c+1}{2}\right)\Gamma\left(\frac{a+2}{2}\right)\omega(z)^{a+2}S_{\frac{a}{2}+1}}{2^{\frac{a+2}{2}}\Gamma\left(\frac{b+c+2}{2}\right)}$$

$$\langle 1,1\,|\,5,5\rangle=\frac{2^2\left(1+i\frac{z}{z_r}\right)}{3\pi 5\omega_0^6\left(1-i\frac{z}{z_r}\right)^7}\int_0^R \rho \mathrm{e}^{\frac{-2\rho^2}{\omega(z)^2}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\rho^4\sin^2\theta\cos^2\theta$$

$$\times\left(\frac{2^8\rho^8\sin^4\theta\cos^4\theta}{\omega(z)^8}-\frac{2^7 5\rho^6\sin^2\theta\cos^2\theta}{\omega(z)^6}+\frac{2^4 5\rho^4\left(3+2*7\cos^2\theta\sin^2\theta\right)}{\omega(z)^4}-\frac{2^3 5^2 3\rho^2}{\omega(z)^2}+3^2 5^2\right)$$

$$= \frac{2^2\left(1+i\frac{z}{z_r}\right)}{3\pi 5\omega_0^6\left(1-i\frac{z}{z_r}\right)^7}\left(\frac{2^8G(12,6,6)}{\omega(z)^8}-\frac{2^7 5G(10,4,4)}{\omega(z)^6}\right.$$

$$\left.+\frac{2^4 5\left(3G(8,2,2)+2*7G(8,4,4)\right)}{\omega(z)^4}-\frac{2^3 5^2 3G(6,2,2)}{\omega(z)^2}+3^2 5^2 G(4,2,2)\right)$$

$$= \frac{2\left(1+i\frac{z}{z_r}\right)^2}{3*5\omega_0^4\left(1-i\frac{z}{z_r}\right)^6}\left(\frac{3^2 5^2\omega(z)^4 S_7}{2^4}-\frac{3^2 5^2\omega(z)^4 S_6}{2^2}+5\omega(z)^4 S_5\left(3^2+\frac{3^2 7}{2^3}\right)-\frac{3^2 5^2\omega(z)^4 S_4}{2^2}+\frac{3^2 5^2\omega(z)^4 S_3}{2^4}\right)$$

$$= \frac{3*5\left(1+i\frac{z}{z_r}\right)^4}{2^3\left(1-i\frac{z}{z_r}\right)^4}\left(S_7-4S_6+6S_5-4S_4+S_3\right)$$

$$= \frac{3*5R^6\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^2\omega_0^6\left(1-i\frac{z}{z_r}\right)^7}\left(\frac{2}{3}-\frac{R^2}{\omega(z)^2}+\frac{2R^4}{5\omega(z)^4}-\frac{2R^6}{45\omega(z)^6}\right)$$

$$\langle 0,1\,|\,0,6\rangle = \langle 1,0\,|\,0,6\rangle = \langle 0,1\,|\,6,0\rangle = \langle 1,0\,|\,6,0\rangle = \langle 1,1\,|\,0,6\rangle = \langle 1,1\,|\,6,0\rangle = 0$$

$$\langle 0,1\,|\,1,6\rangle = \langle 1,0\,|\,6,1\rangle = \langle 1,1\,|\,1,6\rangle = \langle 1,1\,|\,6,1\rangle = 0$$

$$\langle 0,1\,|\,6,1\rangle = \langle 1,0\,|\,1,6\rangle = \frac{2\left(1+i\frac{z}{z_r}\right)}{3\pi 5^{1/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^5}\int_0^R \rho\mathrm{e}^{\frac{-2\rho^2}{\omega(z)^2}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\rho^2\sin^2\theta$$

$$\times\left(\frac{2^6\rho^6\cos^6\theta}{\omega(z)^6}-\frac{2^4*3*5\rho^4\cos^4\theta}{\omega(z)^4}+\frac{2^2 3^2 5\rho^2\cos^2\theta}{\omega(z)^2}-15\right)$$

$$= \frac{2\left(1+i\frac{z}{z_r}\right)}{3\pi 5^{1/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^5}\left(\frac{2^6G(8,2,6)}{\omega(z)^6}-\frac{2^4*3*5G(6,2,4)}{\omega(z)^4}+\frac{2^2 3^2 5G(4,2,2)}{\omega(z)^2}-15G(2,2,0)\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)^2}{3*5^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(\frac{3*5\omega(z)^2 S_5}{2^2}-\frac{3^2 5\omega(z)^2 S_4}{2^2}+\frac{3^2 5\omega(z)^2 S_3}{2^2}-\frac{3*5\omega(z)^2 S_2}{2^2}\right)$$

$$= \frac{5^{1/2}\left(1+i\frac{z}{z_r}\right)^3}{2^2\left(1-i\frac{z}{z_r}\right)^3}\left(S_5-3S_4+3S_3-S_2\right)$$

$$= -\frac{5^{1/2}R^4\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2\omega_0^4\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{4R^2}{3\omega(z)^2}+\frac{R^4}{3\omega(z)^4}\right)$$

$$\langle 0,1\,|\,2,6\rangle = \langle 1,0\,|\,2,6\rangle = \langle 1,1\,|\,2,6\rangle = \langle 0,1\,|\,6,2\rangle = \langle 1,0\,|\,6,2\rangle = \langle 1,1\,|\,6,2\rangle = 0$$

$$\langle 0,1\,|\,3,6\rangle = \langle 1,0\,|\,6,3\rangle = \langle 1,1\,|\,3,6\rangle = \langle 1,1\,|\,6,3\rangle = 0$$

$$\langle 0,1\,|\,6,3\rangle = \langle 1,0\,|\,3,6\rangle = \frac{2^{1/2}\left(1+i\frac{z}{z_r}\right)^2}{3^{3/2}\pi 5^{1/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^6}\int_0^R \rho \mathrm{e}^{\frac{-2\rho^2}{\omega(z)^2}}\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\rho^2\sin^2\theta$$

$$\times\left(\frac{2^8\rho^8\sin^2\theta\cos^6\theta}{\omega(z)^8} - \frac{2^6 3\rho^6\cos^4\theta\left(1+4\sin^2\theta\right)}{\omega(z)^6} + \frac{2^4 3^2 5\rho^4\cos^2\theta}{\omega(z)^4} - \frac{2^2*3*5\rho^2\left(8\cos^2\theta+1\right)}{\omega(z)^2} + 3^2 5\right)$$

$$= \frac{2^{1/2}\left(1+i\frac{z}{z_r}\right)^2}{\pi 3^{3/2}5^{1/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^6}\left(\frac{2^8 G(10,4,6)}{\omega(z)^8} - \frac{2^6 3\left(G(8,2,4)+4G(8,4,4)\right)}{\omega(z)^6}\right.$$

$$\left.+\frac{2^4 3^2 5G(6,2,2)}{\omega(z)^4} - \frac{2^2*3*5\left(G(4,2,0)+8G(4,2,2)\right)}{\omega(z)^2} + 3^2 5G(2,2,0)\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)^3}{2^{1/2}3^{3/2}5^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^5}\left(\frac{3^2 5\omega(z)^2 S_6}{2^2} - 2*3\omega(z)^2 S_5\left(3+\frac{3^2}{2}\right) + \frac{3^3 5\omega(z)^2 S_4}{2}\right.$$

$$\left.-3*5\omega(z)^2 S_3\left(1+2\right) + \frac{3^2 5\omega(z)^2 S_2}{2^2}\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)^4}{2^{1/2}3^{3/2}5^{1/2}\left(1-i\frac{z}{z_r}\right)^4}\left(\frac{3^2 5S_6}{2^2} - 3^2 5S_5 + \frac{3^3 5S_4}{2} - 3^2 5S_3 + \frac{3^2 5S_2}{2^2}\right)$$

$$= \frac{3^{1/2}5^{1/2}\left(1+i\frac{z}{z_r}\right)^4}{2^{5/2}\left(1-i\frac{z}{z_r}\right)^4}\left(S_6 - 4S_5 + 6S_4 - 4S_3 + S_2\right)$$

$$= \frac{3^{1/2}5^{1/2}R^4\left(1+i\frac{z}{z_r}\right)^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{3/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^6}\left(1 - \frac{2R^2}{\omega(z)^2} + \frac{R^4}{\omega(z)^4} - \frac{2R^6}{15\omega(z)^6}\right)$$

$$\langle 0,1\,|\,4,6\rangle = \langle 0,1\,|\,6,4\rangle = \langle 1,0\,|\,4,6\rangle = \langle 1,0\,|\,6,4\rangle = \langle 1,1\,|\,4,6\rangle = \langle 1,1\,|\,6,4\rangle = 0$$

$$\langle 0,1\,|\,5,6\rangle = \langle 1,0\,|\,6,5\rangle = \langle 1,1\,|\,5,6\rangle = \langle 1,1\,|\,6,5\rangle = 0$$

$$\langle 0,1\,|\,6,5\rangle = \langle 1,0\,|\,5,6\rangle = \frac{\left(1+i\frac{z}{z_r}\right)^3}{2^{1/2}3^{3/2}5\pi\omega_0^4\left(1-i\frac{z}{z_r}\right)^7}\int_0^R \rho \mathrm{e}^{\frac{-2\rho^2}{\omega(z)^2}}\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\rho^2\sin^2\theta\left(\frac{2^{10}\rho^{10}\cos^6\theta\sin^4\theta}{\omega(z)^{10}}\right.$$

$$-\frac{2^8 5\rho^8\sin^2\theta\cos^4\theta\left(\sin^2\theta+2\right)}{\omega(z)^8} + \frac{2^6*3*5\rho^6\cos^2\theta\left(1+2\sin^2\theta\left(1+3\cos^2\theta\right)\right)}{\omega(z)^6}$$

$$\left.-\frac{2^4*3*5\rho^4\left(1+2*7\cos^2\theta\left(1+\sin^2\theta\right)\right)}{\omega(z)^4} + \frac{2^2*3*5^2\rho^2\left(2+7\cos^2\theta\right)}{\omega(z)^2} - 3^2 5^2\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)^3}{2^{1/2}3^{3/2}5\pi\omega_0^4\left(1-i\frac{z}{z_r}\right)^7}\left(\frac{2^{10}G(12,6,6)}{\omega(z)^{10}} - \frac{2^8 5\left(G(10,6,4)+2G(10,4,4)\right)}{\omega(z)^8}\right.$$

$$+\frac{2^6*3*5\left(G(8,2,2)+2\left(G(8,4,2)+3G(8,4,4)\right)\right)}{\omega(z)^6} - \frac{2^4*3*5\left(G(6,2,0)+2*7\left(G(6,2,2)+G(6,4,2)\right)\right)}{\omega(z)^4}$$

$$\left.+\frac{2^2*3*5^2\left(2G(4,2,0)+7G(4,2,2)\right)}{\omega(z)^2} - 3^2 5^2 G(2,2,0)\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)^4}{2^{3/2}3^{3/2}5\omega_0^2\left(1-i\frac{z}{z_r}\right)^6}\left(\frac{3^2 5^2\omega(z)^2 S_7}{2^2} - 5\omega(z)^2 S_6\left(\frac{3^2 5}{2^2}+3^2 5\right)\right.$$

$$+3*5\omega(z)^2 S_5\left(2^2 3+2^2 3+\frac{3^3}{2}\right) - 3*5\omega(z)^2 S_4\left(2*3+3*7+\frac{3*7}{2}\right)$$

$$\left.+3*5^2\omega(z)^2 S_3\left(2+\frac{7}{2^2}\right) - \frac{3^2 5^2\omega(z)^2 S_2}{2^2}\right)$$

$$\frac{\left(1+i\frac{z}{z_r}\right)^5}{2^{3/2}3^{3/2}5\left(1-i\frac{z}{z_r}\right)^5}\left(\frac{3^2 5^2 S_7}{2^2} - \frac{3^2 5^3 S_6}{2^2} + \frac{3^2 5^3 S_5}{2^2} - \frac{3^2 5^3}{a}\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)^5}{2^{3/2}3^{3/2}5\left(1-i\frac{z}{z_r}\right)^5}\left(\frac{3^2 5^2 S_7}{2^2} - \frac{3^2 5^3 S_6}{2^2} + \frac{3^2 5^3 S_5}{2} - \frac{3^2 5^3 S_4}{2} + \frac{3^2 5^3 S_3}{2^2} - \frac{3^2 5^2 S_2}{2^2}\right)$$

$$= \frac{3^{1/2}5\left(1+i\frac{z}{z_r}\right)^5}{2^{7/2}\left(1-i\frac{z}{z_r}\right)^5}\left(S_7-5S_6+10S_5-10S_4+5S_3-S_2\right)$$

$$= -\frac{3^{1/2}5R^4\left(1+i\frac{z}{z_r}\right)^3 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{5/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^7}\left(1-\frac{2^3R^2}{3\omega(z)^2}+\frac{2R^4}{\omega(z)^4}-\frac{2^3R^6}{3*5\omega(z)^6}+\frac{2R^8}{3^2 5\omega(z)^8}\right)$$

$$\langle 0,1\,|\,6,6\rangle = \langle 1,0\,|\,6,6\rangle = \langle 1,1\,|\,6,6\rangle = 0$$

Now finally onto the twos but we first collect our results for cases for $\langle 0,0|$, $\langle 1,0|$, $\langle 0,1|$, and $\langle 1,1|$ composed with all $|i,j\rangle$ for $i,j \leq 6$, (showing only non-zero elements and recalling switching kets with bras gives the complex conjugate):

$$\langle 0,0\,|\,0,0\rangle = 1 - \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}$$

$$\langle 0,0\,|\,0,2\rangle = \langle 0,0\,|\,2,0\rangle = -\frac{\sqrt{2}R^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}$$

$$\langle 0,0\,|\,2,2\rangle = \frac{\left(1+i\frac{z}{z_r}\right)R^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{R^2}{\omega(z)^2}\right)$$

$$\langle 0,0\,|\,0,4\rangle = \langle 0,0\,|\,4,0\rangle = \frac{3^{1/2}R^2\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{R^2}{\omega(z)^2}\right)$$

$$\langle 0,0\,|\,2,4\rangle = \langle 0,0\,|\,4,2\rangle = \frac{-3^{1/2}R^2\left(1+i\frac{z}{z_r}\right)^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{2R^2}{\omega(z)^2}+\frac{2R^4}{3\omega(z)^4}\right)$$

$$\langle 0,0\,|\,4,4\rangle = \frac{3R^2\left(1+i\frac{z}{z_r}\right)^3\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{4\omega_0^2\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{3R^2}{\omega(z)^2}+\frac{2R^4}{\omega(z)^4}-\frac{R^6}{3\omega(z)^6}\right)$$

$$\langle 0,0\,|\,0,6\rangle = \langle 0,0\,|\,6,0\rangle = \frac{-\sqrt{5}R^2\left(1+i\frac{z}{z_r}\right)^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{2R^2}{\omega(z)^2}+\frac{2R^4}{3\omega(z)^4}\right)$$

$$\langle 0,0\,|\,2,6\rangle = \langle 0,0\,|\,6,2\rangle = \frac{5^{1/2}R^2\left(1+i\frac{z}{z_r}\right)^3\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{3/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{3R^2}{\omega(z)^2}+\frac{2R^4}{\omega(z)^4}-\frac{R^6}{3\omega(z)^6}\right)$$

$$\langle 0,0\,|\,4,6\rangle = \langle 0,0\,|\,6,4\rangle = \frac{-3^{1/2}5^{1/2}R^2\left(1+i\frac{z}{z_r}\right)^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{5/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^6}\left(1-\frac{4R^2}{\omega(z)^2}+\frac{4R^4}{\omega(z)^4}-\frac{4R^6}{3\omega(z)^6}+\frac{2R^8}{15\omega(z)^8}\right)$$

$$\langle 0,0\,|\,6,6\rangle = \frac{5R^2\left(1+i\frac{z}{z_r}\right)^5\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^3\omega_0^2\left(1-i\frac{z}{z_r}\right)^7}\left(1-\frac{5R^2}{\omega(z)^2}+\frac{20R^4}{3\omega(z)^4}-\frac{10R^6}{3\omega(z)^6}+\frac{2R^8}{3\omega(z)^8}-\frac{2R^{10}}{45\omega(z)^{10}}\right)$$

$$\langle 0,1\,|\,0,1\rangle = \langle 1,0\,|\,1,0\rangle = 1 - \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}\left(1 + \frac{2R^2}{\omega(z)^2}\right)$$

$$\langle 1,1\,|\,1,1\rangle = 1 - \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}\left(1 + \frac{2R^2}{\omega(z)^2} + \frac{2R^4}{\omega(z)^4}\right)$$

$$\langle 0,1\,|\,2,1\rangle = \langle 1,0\,|\,1,2\rangle = -\frac{\sqrt{2}R^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{\omega_0^4\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}$$

$$\langle 0,1\,|\,0,3\rangle = \langle 1,0\,|\,3,0\rangle = -\frac{\sqrt{6}R^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{\omega_0^4\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}$$

$$\langle 1,1\,|\,1,3\rangle = \langle 1,1\,|\,3,1\rangle = -\frac{2^{3/2}R^6\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{\sqrt{3}\omega_0^6\left(1+i\frac{z}{z_r}\right)^2\left(1-i\frac{z}{z_r}\right)^4}$$

$$\langle 0,1\,|\,2,3\rangle = \langle 1,0\,|\,3,2\rangle = \frac{\sqrt{3}R^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(1 - \frac{2R^2}{3\omega(z)^2}\right)$$

$$\langle 1,1\,|\,3,3\rangle = \frac{R^6\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{\omega_0^6\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^5}\left(2 - \frac{R^2}{\omega(z)^2}\right)$$

$$\langle 0,1\,|\,4,1\rangle = \langle 1,0\,|\,1,4\rangle = \frac{\sqrt{3}R^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{\sqrt{2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(1 - \frac{2R^2}{3\omega(z)^2}\right)$$

$$\langle 0,1\,|\,4,3\rangle = \langle 1,0\,|\,3,4\rangle = \frac{-3R^4\left(1+i\frac{z}{z_r}\right)}{2\omega_0^4\left(1-i\frac{z}{z_r}\right)^5}\left(1 - \frac{4R^2}{3\omega(z)^2} + \frac{R^4}{3\omega(z)^4}\right)$$

$$\langle 0,1\,|\,0,5\rangle = \langle 1,0\,|\,5,0\rangle = \frac{\sqrt{15}R^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{\sqrt{2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(1 - \frac{2R^2}{3\omega(z)^2}\right)$$

$$\langle 1,1\,|\,1,5\rangle = \langle 1,1\,|\,5,1\rangle = \frac{\sqrt{5}R^6\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{\sqrt{6}\omega_0^6\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^5}\left(2 - \frac{R^2}{\omega(z)^2}\right)$$

$$\langle 0,1\,|\,2,5\rangle = \langle 1,0\,|\,5,2\rangle = -\frac{\sqrt{15}R^4\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2\omega_0^4\left(1-i\frac{z}{z_r}\right)^5}\left(1 - \frac{4R^2}{3\omega(z)^2} + \frac{R^4}{3\omega(z)^4}\right)$$

$$\langle 1,1\,|\,3,5\rangle = \langle 1,1\,|\,5,3\rangle = -\frac{\sqrt{5}R^6 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{\omega_0^6\left(1-i\frac{z}{z_r}\right)^6}\left(1-\frac{R^2}{\omega(z)^2}+\frac{R^4}{5\omega(z)^4}\right)$$

$$\langle 0,1\,|\,4,5\rangle = \langle 1,0\,|\,5,4\rangle = \frac{3\sqrt{5}R^4\left(1+i\frac{z}{z_r}\right)^2 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^2\omega_0^4\left(1-i\frac{z}{z_r}\right)^6}\left(1-\frac{2R^2}{\omega(z)^2}+\frac{R^4}{\omega(z)^4}-\frac{2R^6}{15\omega(z)^6}\right)$$

$$\langle 1,1\,|\,5,5\rangle = \frac{15R^6\left(1+i\frac{z}{z_r}\right) \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^2\omega_0^6\left(1-i\frac{z}{z_r}\right)^7}\left(\frac{2}{3}-\frac{R^2}{\omega(z)^2}+\frac{2R^4}{5\omega(z)^4}-\frac{2R^6}{45\omega(z)^6}\right)$$

$$\langle 0,1\,|\,6,1\rangle = \langle 1,0\,|\,1,6\rangle = -\frac{\sqrt{5}R^4\left(1+i\frac{z}{z_r}\right) \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2\omega_0^4\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{4R^2}{3\omega(z)^2}+\frac{R^4}{3\omega(z)^4}\right)$$

$$\langle 0,1\,|\,6,3\rangle = \langle 1,0\,|\,3,6\rangle = \frac{\sqrt{15}R^4\left(1+i\frac{z}{z_r}\right)^2 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{\sqrt{8}\omega_0^4\left(1-i\frac{z}{z_r}\right)^6}\left(1-\frac{2R^2}{\omega(z)^2}+\frac{R^4}{\omega(z)^4}-\frac{2R^6}{15\omega(z)^6}\right)$$

$$\langle 0,1\,|\,6,5\rangle = \langle 1,0\,|\,5,6\rangle = -\frac{5\sqrt{3}R^4\left(1+i\frac{z}{z_r}\right)^3 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{\sqrt{32}\omega_0^4\left(1-i\frac{z}{z_r}\right)^7}\left(1-\frac{8R^2}{3\omega(z)^2}+\frac{2R^4}{\omega(z)^4}-\frac{8R^6}{15\omega(z)^6}+\frac{2R^8}{45\omega(z)^8}\right)$$

Now we move to the second modes

$$\begin{aligned}
\langle 0,2\,|\,0,2\rangle = \langle 2,0\,|\,2,0\rangle &= \frac{1}{\pi\omega(z)^2}\int_0^R \rho \mathrm{e}^{\frac{-2\rho^2}{\omega(z)^2}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\left(\frac{2^4\rho^4\sin^4\theta}{\omega(z)^4}-\frac{2^3\rho^2\sin^2\theta}{\omega(z)^2}+1\right)\\
&= \frac{1}{\pi\omega(z)^2}\left(\frac{2^4G(4,4,0)}{\omega(z)^4}-\frac{2^3G(2,2,0)}{\omega(z)^2}+G(0,0,0)\right)\\
&= \frac{1}{2}\left(3S_3-2S_2+S_1\right)\\
&= 1-\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}\left(1+\frac{R^2}{\omega(z)^2}+\frac{3R^4}{\omega(z)^4}\right)
\end{aligned}$$

$$\begin{aligned}
\langle 0,2\,|\,2,0\rangle = \langle 2,0\,|\,0,2\rangle &= \frac{1}{\pi\omega(z)^2}\int_0^R \rho \mathrm{e}^{\frac{-2\rho^2}{\omega(z)^2}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\left(\frac{2^4\rho^4\sin^2\theta\cos^2\theta}{\omega(z)^4}-\frac{2^2\rho^2}{\omega(z)^2}+1\right)\\
&= \frac{1}{\pi\omega(z)^2}\left(\frac{2^4G(4,2,2)}{\omega(z)^4}-\frac{2^2G(2,0,0)}{\omega(z)^2}+G(0,0,0)\right)\\
&= \frac{1}{2}\left(S_3-2S_2+S_1\right)
\end{aligned}$$

$$= \frac{R^2 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{\omega(z)^2}\left(1 - \frac{R^2}{\omega(z)^2}\right)$$

$$\langle 0,2 \,|\, 1,2 \rangle = \langle 0,2 \,|\, 2,1 \rangle = \langle 2,0 \,|\, 1,2 \rangle = \langle 2,0 \,|\, 2,1 \rangle = \langle 1,2 \,|\, 2,1 \rangle = \langle 2,1 \,|\, 1,2 \rangle = 0$$

$$\langle 1,2 \,|\, 1,2 \rangle = \langle 2,1 \,|\, 2,1 \rangle = \frac{4}{\pi\omega(z)^4}\int_0^R \rho \mathrm{e}^{\frac{-2\rho^2}{\omega(z)^2}}\,\mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta \rho^2 \cos^2\theta \left(\frac{2^4\rho^4\sin^4\theta}{\omega(z)^4} - \frac{2^3\rho^2\sin^2\theta}{\omega(z)^2} + 1\right)$$

$$= \frac{4}{\pi\omega(z)^4}\left(\frac{2^4 G(6,4,2)}{\omega(z)^4} - \frac{2^3 G(4,2,2)}{\omega(z)^2} + G(2,0,2)\right)$$

$$= \frac{2}{\omega(z)^2}\left(\frac{3\omega(z)^2 S_4}{2^2} - \frac{\omega(z)^2 S_3}{2} + \frac{\omega(z)^2 S_2}{2^2}\right)$$

$$= \frac{1}{2}\left(3S_4 - 2S_3 + S_2\right)$$

$$= 1 - \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}\left(1 + \frac{2R^2}{\omega(z)^2} + \frac{R^4}{\omega(z)^4} + \frac{2R^6}{\omega(z)^6}\right)$$

$$\langle 0,2 \,|\, 2,2 \rangle = \langle 2,0 \,|\, 2,2 \rangle = \frac{1}{2^{1/2}\pi\omega_0^2\left(1 - i\frac{z}{z_r}\right)^2}\int_0^R \rho \mathrm{e}^{\frac{-2\rho^2}{\omega(z)^2}}\,\mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta$$

$$\times \left(\frac{2^6\rho^6\cos^2\theta\sin^4\theta}{\omega(z)^6} - \frac{2^4\rho^4\cos^2\theta\left(1+\sin^2\theta\right)}{\omega(z)^4} + \frac{2^2\rho^2\left(1+\cos^2\theta\right)}{\omega(z)^2} - 1\right)$$

$$= \frac{1}{2^{1/2}\pi\omega_0^2\left(1 - i\frac{z}{z_r}\right)^2}\left(\frac{2^6 G(6,4,2)}{\omega(z)^6} - \frac{2^4\left(G(4,0,2) + G(4,2,2)\right)}{\omega(z)^4} + \frac{2^2\left(G(2,0,0) + G(2,0,2)\right)}{\omega(z)^2} - G(0,0,0)\right)$$

$$= \frac{\left(1 + i\frac{z}{z_r}\right)}{2^{3/2}\left(1 - i\frac{z}{z_r}\right)}\left(3S_4 - S_3\left(2^2+1\right) + S_2\left(2+1\right) - S_1\right)$$

$$= \frac{\left(1 + i\frac{z}{z_r}\right)}{2^{3/2}\left(1 - i\frac{z}{z_r}\right)}\left(3S_4 - 5S_3 + 3S_2 - S_1\right)$$

$$= -\frac{R^2 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{1/2}\omega_0^2\left(1 - i\frac{z}{z_r}\right)^2}\left(1 - \frac{2R^2}{\omega(z)^2} + \frac{2R^4}{\omega(z)^4}\right)$$

$$\langle 1,2 \,|\, 2,2 \rangle = \langle 2,1 \,|\, 2,2 \rangle = 0$$

$$\langle 2,2 \,|\, 2,2 \rangle = \frac{1}{2\pi\omega(z)^2}\int_0^R \rho \mathrm{e}^{\frac{-2\rho^2}{\omega(z)^2}}\,\mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta$$

$$\times \left(\frac{2^8\rho^8\cos^4\theta\sin^4\theta}{\omega(z)^8} - \frac{2^7\rho^6\cos^2\theta\sin^2\theta}{\omega(z)^6} + \frac{2^4\rho^4\left(1 + 2\cos^2\theta\sin^2\theta\right)}{\omega(z)^4} - \frac{2^3\rho^2}{\omega(z)^2} + 1\right)$$

$$= \frac{1}{2\pi\omega(z)^2}\left(\frac{2^8 G(8,4,4)}{\omega(z)^8} - \frac{2^7 G(6,2,2)}{\omega(z)^6} + \frac{2^4\left(G(4,0,0)+2G(4,2,2)\right)}{\omega(z)^4} - \frac{2^3 G(2,0,0)}{\omega(z)^2} + G(0,0,0)\right)$$

$$= \frac{1}{2^2}\left(3^2 S_5 - 2^2 3 S_4 + S_3\left(2^3+2\right) - 2^2 S_2 + S_1\right)$$

$$= \frac{1}{2^2}\left(3^2 S_5 - 2^2 3 S_4 + 2*5 S_3 - 2^2 S_2 + S_1\right)$$

$$= 1 - \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}\left(1 + \frac{3R^2}{2\omega(z)^2} + \frac{7R^4}{2\omega(z)^4} - \frac{R^6}{\omega(z)^6} + \frac{3R^8}{2\omega(z)^8}\right)$$

$$\langle 0,2 \,|\, 0,3\rangle = \langle 1,2 \,|\, 0,3\rangle = \langle 2,2 \,|\, 0,3\rangle = \langle 2,0 \,|\, 0,3\rangle = 0$$

$$\langle 0,2 \,|\, 3,0\rangle = \langle 2,0 \,|\, 3,0\rangle = \langle 2,1 \,|\, 3,0\rangle = \langle 2,2 \,|\, 3,0\rangle = 0$$

$$\langle 1,2 \,|\, 3,0\rangle = \langle 2,1 \,|\, 0,3\rangle = \frac{4}{3^{1/2}\pi\omega(z)^4}\int_0^R \rho \mathrm{e}^{\frac{-2\rho^2}{\omega(z)^2}}\,\mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta \rho^2 \sin^2\theta$$

$$\times\left(\frac{2^4\rho^4\cos^2\theta\sin^2\theta}{\omega(z)^4} - \frac{2^2\rho^2\left(1+2\cos^2\theta\right)}{\omega(z)^2} + 3\right)$$

$$= \frac{4}{3^{1/2}\pi\omega(z)^4}\left(\frac{2^4 G(6,4,2)}{\omega(z)^4} - \frac{2^2\left(G(4,2,0)+2G(4,2,2)\right)}{\omega(z)^2} + 3G(2,2,0)\right)$$

$$= \frac{2}{3^{1/2}\omega(z)^2}\left(\frac{3\omega(z)^2 S_4}{2^2} - \omega(z)^2 S_3\left(1+\frac{1}{2}\right) + \frac{3\omega(z)^2 S_2}{2^2}\right)$$

$$= \frac{3^{1/2}}{2}\left(S_4 - 2S_3 + S_2\right)$$

$$= \frac{3^{1/2} R^4 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{\omega(z)^4}\left(1 - \frac{2R^2}{3\omega(z)^2}\right)$$

$$\langle 0,2 \,|\, 1,3\rangle = \langle 2,0 \,|\, 1,3\rangle = \langle 1,2 \,|\, 1,3\rangle = \langle 2,1 \,|\, 1,3\rangle = \langle 2,2 \,|\, 1,3\rangle = 0$$

$$\langle 0,2 \,|\, 3,1\rangle = \langle 2,0 \,|\, 3,1\rangle = \langle 1,2 \,|\, 3,1\rangle = \langle 2,1 \,|\, 3,1\rangle = \langle 2,2 \,|\, 3,1\rangle = 0$$

$$\langle 0,2 \,|\, 2,3\rangle = \langle 2,0 \,|\, 2,3\rangle = \langle 1,2 \,|\, 2,3\rangle = \langle 2,2 \,|\, 2,3\rangle = 0$$

$$\langle 0,2 \,|\, 3,2\rangle = \langle 2,0 \,|\, 3,2\rangle = \langle 2,1 \,|\, 3,2\rangle = \langle 2,2 \,|\, 3,2\rangle = 0$$

$$\langle 1,2 \,|\, 3,2\rangle = \langle 2,1 \,|\, 2,3\rangle = \frac{2^{3/2}}{3^{1/2}\pi\omega_0^4\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}\int_0^R \rho \mathrm{e}^{\frac{-2\rho^2}{\omega(z)^2}}\,\mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta \rho^2 \cos^2\theta$$

$$\times\left(\frac{2^4\rho^4\cos^2\theta\sin^2\theta}{\omega(z)^4} - \frac{2^2\rho^2\left(1+2\sin^2\theta\right)}{\omega(z)^2} + 3\right)\left(\frac{2^2\rho^2\sin^2\theta}{\omega(z)^2} - 1\right)$$

$$= \frac{2^{3/2}}{3^{1/2}\pi\omega_0^4\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}\left(\frac{2^6 G(8,4,4)}{\omega(z)^6} - \frac{2^4\left(G(6,2,4)+G(6,2,2)+2G(6,4,2)\right)}{\omega(z)^4}\right.$$

$$\left. + \frac{2^2\left(3G(4,2,2)+G(4,0,2)+2G(4,2,2)\right)}{\omega(z)^2} - 3G(2,0,2)\right)$$

$$= \frac{2^{1/2}}{3^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}\left(\frac{3^2\omega(z)^2S_5}{2^2}-\omega(z)^2S_4\left(\frac{3}{2^2}+\frac{3}{2}+\frac{3}{2}\right)+\omega(z)^2S_3\left(\frac{3}{2^2}+1+\frac{1}{2}\right)-\frac{3\omega(z)^2S_2}{2^2}\right)$$

$$= \frac{2^{1/2}\left(1+i\frac{z}{z_r}\right)}{3^{1/2}\left(1-i\frac{z}{z_r}\right)}\left(\frac{3^2S_5}{2^2}-\frac{3*5S_4}{2^2}+\frac{3^2S_3}{2^2}-\frac{3S_2}{2^2}\right)$$

$$= \frac{3^{1/2}\left(1+i\frac{z}{z_r}\right)}{2^{3/2}\left(1-i\frac{z}{z_r}\right)}\left(3S_5-5S_4+3S_3-S_2\right)$$

$$= -\frac{3^{1/2}R^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{1/2}\omega_0^4\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{4R^2}{3\omega(z)^2}+\frac{R^4}{\omega(z)^4}\right)$$

$$\langle 0,2\,|\,3,3\rangle = \langle 2,0\,|\,3,3\rangle = \langle 1,2\,|\,3,3\rangle = \langle 2,1\,|\,3,3\rangle = \langle 2,2\,|\,3,3\rangle = 0$$

$$\langle 0,2\,|\,0,4\rangle = \langle 2,0\,|\,4,0\rangle = \frac{1}{2\sqrt{3}\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}\int_0^R \rho\mathrm{e}^{\frac{-2\rho^2}{\omega(z)^2}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta$$

$$\times\left(\frac{2^2\rho^2\sin^2\theta}{\omega(z)^2}-1\right)\left(\frac{2^4\rho^4\sin^4\theta}{\omega(z)^4}-\frac{3*2^3\rho^2\sin^2\theta}{\omega(z)^2}+3\right)$$

$$= \frac{1}{2\sqrt{3}\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}\left(\frac{2^6G(6,6,0)}{\omega(z)^6}-\frac{2^47G(4,4,0)}{\omega(z)^4}+\frac{2^23^2G(2,2,0)}{\omega(z)^2}-3G(0,0,0)\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)}{4\sqrt{3}\left(1-i\frac{z}{z_r}\right)}\left(3*5S_4-3*7S_3+3^2S_2-3S_1\right)$$

$$= \frac{3^{1/2}\left(1+i\frac{z}{z_r}\right)}{4\left(1-i\frac{z}{z_r}\right)}\left(5S_4-7S_3+3S_2-S_1\right)$$

$$= -\frac{3^{1/2}R^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}\left(1-\frac{2R^2}{\omega(z)^2}+\frac{10R^4}{3\omega(z)^4}\right)$$

$$\langle 0,2\,|\,4,0\rangle = \frac{1}{2\sqrt{3}\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}\int_0^R \rho\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta$$

$$\times\left(\frac{2^2\rho^2\sin^2\theta}{\omega(z)^2}-1\right)\left(\frac{2^4\rho^4\cos^4\theta}{\omega(z)^4}-\frac{3*2^3\rho^2\cos^2\theta}{\omega(z)^2}+3\right)$$

$$= \frac{1}{2\sqrt{3}\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}\left(\frac{2^6G(6,2,4)}{\omega(z)^6}-\frac{2^4\left(G(4,0,4)+6G(4,2,2)\right)}{\omega(z)^4}+\frac{2^23\left(G(2,2,0)+2G(2,0,2)\right)}{\omega(z)^2}-3G(0,0,0)\right)$$

$$= \frac{\sqrt{3}\left(1+i\frac{z}{z_r}\right)}{4\left(1-i\frac{z}{z_r}\right)}\left(S_4-3S_3+3S_2-S_1\right)$$

$$= -\frac{\sqrt{3}R^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}\left(1-\frac{2R^2}{\omega(z)^2}+\frac{4R^4}{3\omega(z)^4}\right)$$

$$\langle 1,2\,|\,0,4\rangle = \langle 2,1\,|\,0,4\rangle = \langle 1,2\,|\,4,0\rangle = \langle 2,1\,|\,4,0\rangle = 0$$

$$\langle 2,2\,|\,0,4\rangle = \langle 2,2\,|\,4,0\rangle = \frac{1}{2^{3/2}3^{1/2}\pi\omega(z)^2}\int_0^R \rho\mathrm{e}^{\frac{-2\rho^2}{\omega(z)^2}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta$$

$$\left(\frac{2^4\rho^4\cos^2\theta\sin^2\theta}{\omega(z)^4}-\frac{2^2\rho^2}{\omega(z)^2}+1\right)\left(\frac{2^4\rho^4\sin^4\theta}{\omega(z)^4}-\frac{3*2^3\rho^2\sin^2\theta}{\omega(z)^2}+3\right)$$

$$= \frac{1}{2^{3/2}3^{1/2}\pi\omega(z)^2}\left(\frac{2^8G(8,6,2)}{\omega(z)^8}-\frac{2^6\left(G(6,4,0)+6G(6,4,2)\right)}{\omega(z)^6}\right.$$

$$\left.+\frac{2^4\left(G(4,4,0)+3\left(G(4,2,2)+2G(4,2,0)\right)\right)}{\omega(z)^4}-\frac{2^2 3\left(G(2,0,0)+2G(2,2,0)\right)}{\omega(z)^2}+3G(0,0,0)\right)$$

$$= \frac{1}{2^{5/2}3^{1/2}}\left(3*5S_5-S_4\left(18+18\right)+S_3\left(3+3\left(1+8\right)\right)-2*3S_2\left(1+1\right)+3S_1\right)$$

$$= \frac{1}{2^{5/2}3^{1/2}}\left(3*5S_5-2^2 3^2 S_4+2*3*5S_3-2^2 3S_2+3S_1\right)$$

$$= \frac{3^{1/2}}{2^{5/2}}\left(5S_5-12S_4+10S_3-4S_2+S_1\right)$$

$$= \frac{3^{1/2}R^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{\omega(z)^2}\left(1-\frac{3R^2}{\omega(z)^2}+\frac{14R^4}{3\omega(z)^4}-\frac{5R^6}{3\omega(z)^6}\right)$$

$$\langle 0,2\,|\,1,4\rangle = \langle 2,0\,|\,1,4\rangle = \langle 2,2\,|\,1,4\rangle = \langle 0,2\,|\,4,1\rangle = \langle 2,0\,|\,4,1\rangle = \langle 2,2\,|\,4,1\rangle = 0$$

$$\langle 1,2\,|\,4,1\rangle = \langle 2,1\,|\,1,4\rangle = 0$$

$$\langle 1,2\,|\,1,4\rangle = \langle 2,1\,|\,4,1\rangle = \frac{2}{3^{1/2}\pi\omega_0^4\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}\int_0^R \rho\mathrm{e}^{\frac{-2\rho^2}{\omega(z)^2}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\rho^2\cos^2\theta$$

$$\times\left(\frac{2^2\rho^2\sin^2\theta}{\omega(z)^2}-1\right)\left(\frac{2^4\rho^4\sin^4\theta}{\omega(z)^4}-\frac{3*2^3\rho^2\sin^2\theta}{\omega(z)^2}+3\right)$$

$$= \frac{2}{3^{1/2}\pi\omega_0^4\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}\left(\frac{2^6G(8,6,2)}{\omega(z)^6}-\frac{2^4\left(G(6,4,2)+6G(6,4,2)\right)}{\omega(z)^4}\right.$$

$$\left.+\frac{2^2 3\left(G(4,2,2)+2G(4,2,2)\right)}{\omega(z)^2}-3G(2,0,2)\right)$$

$$= \frac{1}{3^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}\left(\frac{3*5\omega(z)^2S_5}{2^2}-\frac{3*7\omega(z)^2S_4}{2^2}+\frac{3^2\omega(z)^2S_3}{2^2}-\frac{3\omega(z)^2S_2}{2^2}\right)$$

$$= \frac{\sqrt{3}\left(1+i\frac{z}{z_r}\right)}{2^2\left(1-i\frac{z}{z_r}\right)}\left(5S_5-7S_4+3S_3-S_2\right)$$

$$= -\frac{\sqrt{3}R^4 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2\omega_0^4\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{4R^2}{3\omega(z)^2}+\frac{5R^4}{3\omega(z)^4}\right)$$

$$\langle 1,2\,|\,2,4\rangle = \langle 2,1\,|\,2,4\rangle = \langle 1,2\,|\,4,2\rangle = \langle 2,1\,|\,4,2\rangle = 0$$

$$\langle 0,2\,|\,2,4\rangle = \langle 2,0\,|\,4,2\rangle = \frac{\left(1+i\frac{z}{z_r}\right)}{2^{3/2}3^{1/2}\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\int_0^R \rho \mathrm{e}^{\frac{-2\rho^2}{\omega(z)^2}}\,\mathrm{d}\rho\int_0^{2\pi}\,\mathrm{d}\theta$$

$$\times\left(\frac{2^2\rho^2\sin^2\theta}{\omega(z)^2}-1\right)\left(\frac{2^6\rho^6\cos^2\theta\sin^4\theta}{\omega(z)^6}-\frac{2^4\rho^4\sin^2\theta\left(1+5\cos^2\theta\right)}{\omega(z)^4}+\frac{2^2 3\rho^2\left(1+\sin^2\theta\right)}{\omega(z)^2}-3\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)}{2^{3/2}3^{1/2}\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(\frac{2^8G(8,6,2)}{\omega(z)^8}-\frac{2^6\left(G(6,4,2)+G(6,4,0)+5G(6,4,2)\right)}{\omega(z)^6}\right.$$

$$\left.+\frac{2^4\left(G(4,2,0)+5G(4,2,2)+3\left(G(4,2,0)+G(4,4,0)\right)\right)}{\omega(z)^4}-\frac{2^2 3\left(G(2,2,0)+G(2,0,0)+G(2,2,0)\right)}{\omega(z)^2}+3G(0,0,0)\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)}{2^{3/2}3^{1/2}\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(\frac{2^8G(8,6,2)}{\omega(z)^8}-\frac{2^6\left(6G(6,4,2)+G(6,4,0)\right)}{\omega(z)^6}\right.$$

$$\left.+\frac{2^4\left(4G(4,2,0)+5G(4,2,2)+3G(4,4,0)\right)}{\omega(z)^4}-\frac{2^2 3\left(2G(2,2,0)+G(2,0,0)\right)}{\omega(z)^2}+3G(0,0,0)\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)^2}{2^{5/2}3^{1/2}\left(1-i\frac{z}{z_r}\right)^2}\left(3*5S_5-2S_4\left(9+9\right)+S_3\left(2^4+5+3^2\right)-3S_2\left(2+2\right)+3S_1\right)$$

$$= \frac{3^{1/2}\left(1+i\frac{z}{z_r}\right)^2}{2^{5/2}\left(1-i\frac{z}{z_r}\right)^2}\left(5S_5-2^2 3S_4+2*5S_3-2^2S_2+S_1\right)$$

$$= \frac{\sqrt{3}R^2\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{3/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{3R^2}{\omega(z)^2}+\frac{2*7R^4}{3\omega(z)^4}-\frac{5R^6}{3\omega(z)^6}\right)$$

$$\langle 0,2\,|\,4,2\rangle = \langle 2,0\,|\,2,4\rangle = \frac{\left(1+i\frac{z}{z_r}\right)}{2^{3/2}3^{1/2}\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\int_0^R \rho \mathrm{e}^{\frac{-2\rho^2}{\omega(z)^2}}\,\mathrm{d}\rho\int_0^{2\pi}\,\mathrm{d}\theta$$

$$\times\left(\frac{2^2\rho^2\sin^2\theta}{\omega(z)^2}-1\right)\left(\frac{2^6\rho^6\sin^2\theta\cos^4\theta}{\omega(z)^6}-\frac{2^4\rho^4\cos^2\theta\left(1+5\sin^2\theta\right)}{\omega(z)^4}+\frac{2^23\rho^2\left(1+\cos^2\theta\right)}{\omega(z)^2}-3\right)$$

$$=\frac{\left(1+i\frac{z}{z_r}\right)}{2^{3/2}3^{1/2}\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(\frac{2^8G(8,4,4)}{\omega(z)^8}-\frac{2^6\left(G(6,2,4)+G(6,2,2)+5G(6,4,2)\right)}{\omega(z)^6}\right.$$

$$\left.+\frac{2^4\left(G(4,2,0)+5G(4,2,2)+3\left(G(4,2,0)+G(4,2,2)\right)\right)}{\omega(z)^4}-\frac{2^23\left(G(2,2,0)+G(2,0,0)+G(2,0,2)\right)}{\omega(z)^2}+3\right)$$

$$=\frac{\left(1+i\frac{z}{z_r}\right)}{2^{3/2}3^{1/2}\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(\frac{2^8G(8,4,4)}{\omega(z)^8}-\frac{2^6\left(6G(6,4,2)+G(6,2,2)\right)}{\omega(z)^6}\right.$$

$$\left.+\frac{2^4\left(4G(4,2,0)+8G(4,2,2)\right)}{\omega(z)^4}-\frac{2^23\left(2G(2,2,0)+G(2,0,0)\right)}{\omega(z)^2}+3G(0,0,0)\right)$$

$$=\frac{\left(1+i\frac{z}{z_r}\right)^2}{2^{5/2}3^{1/2}\left(1-i\frac{z}{z_r}\right)^2}\left(3^2S_5-2^33S_4+2^33S_3-2^23S_2+3S_1\right)$$

$$=\frac{\sqrt{3}\left(1+i\frac{z}{z_r}\right)^2}{2^{5/2}\left(1-i\frac{z}{z_r}\right)^2}\left(3S_5-2^3S_4+2^3S_3-2^2S_2+S_1\right)$$

$$=\frac{\sqrt{3}R^2\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{\sqrt{8}\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{3R^2}{\omega(z)^2}+\frac{10R^4}{3\omega(z)^4}-\frac{R^6}{\omega(z)^6}\right)$$

$$\langle 2,2\,|\,2,4\rangle=\langle 2,2\,|\,4,2\rangle=\frac{1}{2^23^{1/2}\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}\int_0^R\rho\mathrm{e}^{\frac{-2\rho^2}{\omega(z)^2}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\left(\frac{2^4\rho^4\cos^2\theta\sin^2\theta}{\omega(z)^4}-\frac{2^2\rho^2}{\omega(z)^2}+1\right)$$

$$\times\left(\frac{2^6\rho^6\cos^2\theta\sin^4\theta}{\omega(z)^6}-\frac{2^4\rho^4\sin^2\theta\left(1+5\cos^2\theta\right)}{\omega(z)^4}+\frac{3*2^2\rho^2\left(1+\sin^2\theta\right)}{\omega(z)^2}-3\right)$$

$$=\frac{1}{2^23^{1/2}\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}\left(\frac{2^{10}G(10,6,4)}{\omega(z)^{10}}-\frac{2^8\left(G(8,4,2)+G(8,4,2)+5G(8,4,4)\right)}{\omega(z)^8}\right.$$

$$+\frac{2^6\left(G(6,4,2)+G(6,2,0)+5G(6,2,2)+3(G(6,2,2)+G(6,4,2))\right)}{\omega(z)^6}$$

$$-\frac{2^4\left(G(4,2,0)+5G(4,2,2)+3G(4,2,2)+3\left(G(4,0,0)+G(4,2,0)\right)\right)}{\omega(z)^4}$$

$$\left.+\frac{2^23\left(G(2,0,0)+G(2,0,0)+G(2,2,0)\right)}{\omega(z)^2}-3G(0,0,0)\right)$$

$$= \frac{1}{2^2 3^{1/2}\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}\left(\frac{2^{10}G(10,6,4)}{\omega(z)^{10}} - \frac{2^8\left(2G(8,4,2)+5G(8,4,4)\right)}{\omega(z)^8} + \frac{2^6\left(4G(6,4,2)+8G(6,2,2)+G(6,2,0)\right)}{\omega(z)^6}\right.$$

$$\left. - \frac{2^4\left(4G(4,2,0)+8G(4,2,2)+3G(4,0,0)\right)}{\omega(z)^4} + \frac{2^2 3\left(2G(2,0,0)+G(2,2,0)\right)}{\omega(z)^2} - 3G(0,0,0)\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)}{2^3 3^{1/2}\left(1-i\frac{z}{z_r}\right)}\left(3^2 5S_6 - 3S_5\left(2^4+3*5\right) + 2^2 3S_4\left(1+2^2+2\right) - 2^3 S_3\left(2+1+3\right) + 3S_2\left(2^2+1\right) - 3S_1\right)$$

$$\frac{3^{1/2}\left(1+i\frac{z}{z_r}\right)}{2^3\left(1-i\frac{z}{z_r}\right)}\left(3*5S_6 - 31S_5 + 2^2*7S_4 - 2^4 S_3 + 5S_2 - S_1\right)$$

$$= -\frac{3^{1/2}R^2 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^2\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}\left(1 - \frac{4R^2}{\omega(z)^2} + \frac{8R^4}{\omega(z)^4} - \frac{16R^6}{3\omega(z)^6} + \frac{2R^8}{\omega(z)^8}\right)$$

$$\langle 0,2\,|\,3,4\rangle = \langle 2,0\,|\,3,4\rangle = \langle 2,2\,|\,3,4\rangle = \langle 2,1\,|\,3,4\rangle = 0$$

$$\langle 0,2\,|\,4,3\rangle = \langle 2,0\,|\,4,3\rangle = \langle 2,2\,|\,4,3\rangle = \langle 1,2\,|\,4,3\rangle = 0$$

$$\langle 1,2\,|\,3,4\rangle = \langle 2,1\,|\,4,3\rangle = \frac{2^{1/2}}{3\pi\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\int_0^R \rho\mathrm{e}^{\frac{-2\rho^2}{\omega(z)^2}}\int_0^{2\pi}\mathrm{d}\theta\rho^2\cos^2\theta\left(\frac{2^2\rho^2\sin^2\theta}{\omega(z)^2} - 1\right)$$

$$\times\left(\frac{2^6\rho^6\cos^2\theta\sin^4\theta}{\omega(z)^6} - \frac{3*2^4\rho^4\sin^2\theta\left(1+\cos^2\theta\right)}{\omega(z)^4} + \frac{2^2*3\rho^2\left(1+5\sin^2\theta\right)}{\omega(z)^2} - 9\right)$$

$$= \frac{2^{1/2}}{3\pi\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(\frac{2^8 G(10,6,4)}{\omega(z)^8} - \frac{2^6\left(G(8,4,4)+3(G(8,4,2)+G(8,4,4)\right)}{\omega(z)^6}\right.$$

$$\left. + \frac{2^4 3\left(G(6,2,2)+G(6,2,4)+G(6,2,2)+5G(6,4,2)\right)}{\omega(z)^4} - \frac{2^2 3\left(3G(4,2,2)+G(4,0,2)+5G(4,2,2)\right)}{\omega(z)^2} + 9G(2,0,2)\right)$$

$$= \frac{2^{1/2}}{3\pi\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(\frac{2^8 G(10,6,4)}{\omega(z)^8} - \frac{2^6\left(4G(8,4,4)+3G(8,4,2)\right)}{\omega(z)^6} + \frac{2^4 3\left(2G(6,2,2)+6G(6,4,2)\right)}{\omega(z)^4}\right.$$

$$\left. - \frac{2^2 3\left(8G(4,2,2)+G(4,0,2)\right)}{\omega(z)^2} + 9G(2,0,2)\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)}{2^{1/2}3\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(\frac{3^2 5\omega(z)^2 S_6}{2^2} - 3^2\omega(z)^2 S_5\left(1+2\right) + 3^2\omega(z)^2 S_4\left(1+\frac{3}{2}\right) - 3\omega(z)^2 S_3\left(2+1\right) + \frac{9\omega(z)^2 S_2}{2^2}\right)$$

$$= \frac{3\left(1+i\frac{z}{z_r}\right)^2}{2^{1/2}\left(1-i\frac{z}{z_r}\right)^2}\left(\frac{5S_6}{2^2} - 3S_5 + \frac{5}{2}S_4 - S_3 + \frac{S_2}{2^2}\right)$$

$$= \frac{3\left(1+i\frac{z}{z_r}\right)^2}{2^{5/2}\left(1-i\frac{z}{z_r}\right)^2}\left(5S_6-12S_5+10S_4-4S_3+S_2\right)$$

$$= \frac{3R^4 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{3/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{2R^2}{\omega(z)^2}+\frac{7R^4}{3\omega(z)^4}-\frac{2R^6}{3\omega(z)^5}\right)$$

$$\langle 1,2\,|\,4,4\rangle = \langle 2,1\,|\,4,4\rangle = 0$$

$$\langle 0,2\,|\,4,4\rangle = \langle 2,0\,|\,4,4\rangle = \frac{\left(1+i\frac{z}{z_r}\right)^2}{2^{5/2}3\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\int_0^R \rho \mathrm{e}^{\frac{-2\rho^2}{\omega(z)^2}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\left(\frac{2^2\rho^2\sin^2\theta}{\omega(z)^2}-1\right)$$

$$\times\left(\frac{2^8\rho^8\cos^4\theta\sin^4\theta}{\omega(z)^8}-\frac{3*2^7\rho^6\sin^2\theta\cos^2\theta}{\omega(z)^6}+\frac{3*2^4\rho^4\left(1+10\sin^2\theta\cos^2\theta\right)}{\omega(z)^4}-\frac{2^3 3^2\rho^2}{\omega(z)^2}+9\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)^2}{2^{5/2}3\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(\frac{2^{10}G(10,6,4)}{\omega(z)^{10}}-\frac{2^8\left(G(8,4,4)+2*3G(8,4,2)\right)}{\omega(z)^8}\right.$$

$$+\frac{2^6 3\left(2G(6,2,2)+G(6,2,0)+2*5G(6,4,2)\right)}{\omega(z)^6}-\frac{2^4 3\left(2*3G(4,2,0)+G(4,0,0)+10G(4,2,2)\right)}{\omega(z)^4}$$

$$\left.+\frac{2^2 3^2\left(G(2,2,0)+2G(2,0,0)\right)}{\omega(z)^2}-9G(0,0,0)\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)^3}{2^{7/2}3\left(1-i\frac{z}{z_r}\right)^3}\left(3^2 5S_6-S_5\left(3^2+2^4 3^2\right)+3S_4\left(2^2 3+2^3 3+2*3*5\right)\right.$$

$$\left.-3S_3\left(2^3 3+2^3+2*5\right)+3^2S_2\left(1+2^2\right)-3^2\right)$$

$$= \frac{3\left(1+i\frac{z}{z_r}\right)^3}{2^{7/2}\left(1-i\frac{z}{z_r}\right)^3}\left(5S_6-17S_5+22S_4-14S_3+5S_2-S_1\right)$$

$$= -\frac{3R^2\left(1+i\frac{z}{z_r}\right)^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{5/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{4R^2}{\omega(z)^2}+\frac{20R^4}{\omega(z)^4}-\frac{4R^6}{\omega(z)^6}+\frac{2R^8}{3\omega(z)^8}\right)$$

$$\langle 2,2\,|\,4,4\rangle = \frac{\left(1+i\frac{z}{z_r}\right)}{2^3 3\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\int_0^R \rho \mathrm{e}^{\frac{-2\rho^2}{\omega(z)^2}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\left(\frac{2^4\rho^4\cos^2\theta\sin^2\theta}{\omega(z)^4}-\frac{2^2\rho^2}{\omega(z)^2}+1\right)$$

$$\times\left(\frac{2^8\rho^8\cos^4\theta\sin^4\theta}{\omega(z)^8}-\frac{3*2^7\rho^6\sin^2\theta\cos^2\theta}{\omega(z)^6}+\frac{3*2^4\rho^4\left(1+10\sin^2\theta\cos^2\theta\right)}{\omega(z)^4}-\frac{2^3 3^2\rho^2}{\omega(z)^2}+9\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)}{2^3 3\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(\frac{2^{12}G(12,6,6)}{\omega(z)^{12}} - \frac{2^{10}\left(G(10,4,4)+2*3G(10,4,4)\right)}{\omega(z)^{10}}\right.$$

$$+\frac{2^8\left(G(8,4,4)+2*3G(8,2,2)+3G(8,2,2)+2*3*5G(8,4,4)\right)}{\omega(z)^8}$$

$$-\frac{2^6 3\left(2G(6,2,2)+2*3G(6,2,2)+G(6,0,0)+2*5G(6,2,2)\right)}{\omega(z)^6}$$

$$\left.+\frac{2^4 3\left(3G(4,2,2)+2*3G(4,0,0)+G(4,0,0)+2*5G(4,2,2)\right)}{\omega(z)^4} - \frac{2^2 3^2\left(G(2,0,0)+2G(2,0,0)\right)}{\omega(z)^2} + 3^2 G(0,0,0)\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)}{2^3 3\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(\frac{2^{12}G(12,6,6)}{\omega(z)^{12}} - \frac{2^{10}7G(10,4,4)}{\omega(z)^{10}} + \frac{2^8\left(31G(8,4,4)+9G(8,2,2)\right)}{\omega(z)^8}\right.$$

$$\left.-\frac{2^6 3\left(2*3^2G(6,2,2)+G(6,0,0)\right)}{\omega(z)^6} + \frac{2^4 3\left(13G(4,2,2)+7G(4,0,0)\right)}{\omega(z)^4} - \frac{2^2 3^3 G(2,0,0)}{\omega(z)^2} + 3^2G(0,0,0)\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)^2}{2^4 3\left(1-i\frac{z}{z_r}\right)^2}\left(3^2 5^2 S_7 - 2*3^2*5*7S_6 + S_5\left(3^2 31 + 2^4 3^3\right) - 3S_4\left(2^2 3^3 + 2^4 3\right)\right.$$

$$\left.+3S_3\left(13+2^3 7\right) - 2*3^3 S_2 + 3^2 S_1\right)$$

$$= \frac{3\left(1+i\frac{z}{z_r}\right)^2}{2^4\left(1-i\frac{z}{z_r}\right)^2}\left(25S_7 - 70S_6 + 79S_5 - 52S_4 + 23S_3 - 6S_2 + S_1\right)$$

$$= -\frac{3\left(1+i\frac{z}{z_r}\right)^2 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^4\left(1-i\frac{z}{z_r}\right)^2}\left(-\frac{2R^2}{\omega(z)^2} + 5\frac{2R^4}{\omega(z)^4} - 18\frac{4R^6}{3\omega(z)^6} + 34\frac{2R^8}{3\omega(z)^8} - 45\frac{4R^{10}}{15\omega(z)^{10}} + 25\frac{4R^{12}}{45\omega(z)^{12}}\right)$$

$$= \frac{3\left(1+i\frac{z}{z_r}\right)^2 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^4\left(1-i\frac{z}{z_r}\right)^2}\left(\frac{2R^2}{\omega(z)^2} - \frac{10R^4}{\omega(z)^4} + \frac{24R^6}{\omega(z)^6} - \frac{68R^8}{3\omega(z)^8} + \frac{12R^{10}}{\omega(z)^{10}} - \frac{20R^{12}}{9\omega(z)^{12}}\right)$$

$$= \frac{3R^2\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^3\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(1 - \frac{5R^2}{\omega(z)^2} + \frac{12R^4}{\omega(z)^4} - \frac{34R^6}{3\omega(z)^6} + \frac{6R^8}{\omega(z)^8} - \frac{10R^{10}}{9\omega(z)^{10}}\right)$$

$$\langle 0,2\,|\,0,5\rangle = \langle 2,0\,|\,0,5\rangle = \langle 1,2\,|\,0,5\rangle = \langle 2,2\,|\,0,5\rangle = 0$$

$$\langle 0,2\,|\,5,0\rangle = \langle 2,0\,|\,5,0\rangle = \langle 2,1\,|\,5,0\rangle = \langle 2,2\,|\,5,0\rangle = 0$$

$$\langle 1,2\,|\,5,0\rangle = \langle 2,1\,|\,0,5\rangle = \frac{2}{3^{1/2}5^{1/2}\pi\omega_0^4\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}\int_0^R \rho \mathrm{e}^{\frac{-2\rho^2}{\omega(z)^2}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\rho^2\cos^2\theta$$

$$\times\left(\frac{2^2\rho^2\sin^2\theta}{\omega(z)^2}-1\right)\left(\frac{2^4\rho^4\cos^4\theta}{\omega(z)^4}-\frac{2^3 5\rho^2\cos^2\theta}{\omega(z)^2}+15\right)$$

$$\frac{2}{3^{1/2}5^{1/2}\pi\omega_0^4\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}\left(\frac{2^6 G(8,6,2)}{\omega(z)^6}-\frac{2^4\left(G(6,6,0)+2*5G(6,2,4)\right)}{\omega(z)^4}\right.$$

$$\left.+\frac{2^2 5\left(3G(4,2,2)+2G(4,4,0)\right)}{\omega(z)^2}-3*5G(2,0,2)\right)$$

$$=\frac{1}{3^{1/2}5^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}\left(\frac{3*5\omega(z)^2S_5}{2^2}-\omega(z)^2S_4\left(\frac{3*5}{2^2}+\frac{3*5}{2}\right)+5\omega(z)^2S_3\left(\frac{3}{2^2}+\frac{3}{2}\right)-\frac{3*5\omega(z)^2S_2}{2^2}\right)$$

$$=\frac{3^{1/2}5^{1/2}\left(1+i\frac{z}{z_r}\right)}{2^2\left(1-i\frac{z}{z_r}\right)}\left(S_5-3S_4+3S_3-S_2\right)$$

$$=-\frac{3^{1/2}5^{1/2}\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^2\left(1-i\frac{z}{z_r}\right)}\left(\frac{2R^4}{\omega(z)^4}-\frac{8R^6}{3\omega(z)^6}+\frac{2R^8}{3\omega(z)^8}\right)$$

$$=-\frac{3^{1/2}5^{1/2}R^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2\omega_0^4\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{4R^2}{3\omega(z)^2}+\frac{R^4}{3\omega(z)^4}\right)$$

$$\langle 0,2\,|\,1,5\rangle = \langle 2,0\,|\,1,5\rangle = \langle 1,2\,|\,1,5\rangle = \langle 2,1\,|\,1,5\rangle = \langle 2,2\,|\,1,5\rangle = 0$$

$$\langle 0,2\,|\,5,1\rangle = \langle 2,0\,|\,5,1\rangle = \langle 1,2\,|\,5,1\rangle = \langle 2,1\,|\,5,1\rangle = \langle 2,2\,|\,5,1\rangle = 0$$

$$\langle 0,2\,|\,2,5\rangle = \langle 2,0\,|\,2,5\rangle = \langle 1,2\,|\,2,5\rangle = \langle 2,2\,|\,2,5\rangle = 0$$

$$\langle 0,2\,|\,5,2\rangle = \langle 2,0\,|\,5,2\rangle = \langle 2,1\,|\,5,2\rangle = \langle 2,2\,|\,5,2\rangle = 0$$

$$\langle 1,2\,|\,5,2\rangle = \langle 2,1\,|\,2,5\rangle = \frac{2^{1/2}}{3^{1/2}5^{1/2}\pi\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\int_0^R \rho \mathrm{e}^{\frac{-2\rho^2}{\omega(z)^2}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\rho^2\cos^2\theta\left(\frac{2^2\rho^2\sin^2\theta}{\omega(z)^2}-1\right)$$

$$\times\left(\frac{2^6\rho^6\cos^4\theta\sin^2\theta}{\omega(z)^6}-\frac{2^4\rho^4\cos^2\theta\left(9\sin^2\theta+1\right)}{\omega(z)^4}+\frac{2^2 5\rho^2\left(\sin^2\theta+2\right)}{\omega(z)^2}-15\right)$$

$$=\frac{2^{1/2}}{3^{1/2}5^{1/2}\pi\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(\frac{2^8 G(10,4,6)}{\omega(z)^8}-\frac{2^6\left(G(8,6,2)+G(8,4,2)+9G(8,4,4)\right)}{\omega(z)^6}\right.$$

$$+\frac{2^4\left(G(6,0,4)+9G(6,2,4)+5\left(G(6,4,2)+2G(6,2,2)\right)\right)}{\omega(z)^4}-\frac{2^2 5\left(3G(4,2,2)+G(4,2,2)+2G(4,0,2)\right)}{\omega(z)^2}$$

$$\left.+15G(2,0,2)\right)$$

$$= \frac{2^{1/2}}{3^{1/2}5^{1/2}\pi\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(\frac{2^8G(10,4,6)}{\omega(z)^8}-\frac{2^6\left(G(8,6,2)+G(8,4,2)+9G(8,4,4)\right)}{\omega(z)^6}\right.$$

$$\left.+\frac{2^4\left(G(6,0,4)+14G(6,4,2)+10G(6,2,2)\right)}{\omega(z)^4}-\frac{2^25\left(4G(4,2,2)+2G(4,0,2)\right)}{\omega(z)^2}+15G(2,0,2)\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)}{2^{1/2}3^{1/2}5^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(\frac{3^25\omega(z)^2S_6}{2^2}-\omega(z)^2S_5\left(\frac{3*5}{2^2}+2*3+\frac{3^4}{2^2}\right)\right.$$

$$\left.+\omega(z)^2S_4\left(\frac{3^2}{2}+\frac{3*7}{2}+3*5\right)-5\omega(z)^2S_3\left(1+2\right)+\frac{3*5\omega(z)^2S_2}{2^2}\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)^2}{2^{1/2}3^{1/2}5^{1/2}\left(1-i\frac{z}{z_r}\right)^2}\left(\frac{3^2*5S_6}{2^2}-\frac{2^3*3*5S_5}{2^2}+\frac{2^3*3*5S_4}{2^2}-\frac{2^2*3*5S_3}{2^2}+\frac{3*5S_2}{2^2}\right)$$

$$= \frac{3^{1/2}5^{1/2}\left(1+i\frac{z}{z_r}\right)^2}{2^{5/2}\left(1-i\frac{z}{z_r}\right)^2}\left(3S_6-8S_5+8S_4-4S_3+S_2\right)$$

$$= -\frac{3^{1/2}5^{1/2}\left(1+i\frac{z}{z_r}\right)^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{5/2}\left(1-i\frac{z}{z_r}\right)^2}\left(-\frac{2R^4}{\omega(z)^4}+3\frac{4R^6}{3\omega(z)^6}-5\frac{2R^8}{3\omega(z)^8}+3\frac{4R^{10}}{15\omega(z)^{10}}\right)$$

$$= \frac{3^{1/2}5^{1/2}R^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{3/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{2R^2}{\omega(z)^2}+\frac{5R^4}{3\omega(z)^4}-\frac{2R^6}{5\omega(z)^6}\right)$$

$$\langle 0,2\,|\,3,5\rangle=\langle 2,0\,|\,3,5\rangle=\langle 1,2\,|\,3,5\rangle=\langle 2,1\,|\,3,5\rangle=\langle 2,2\,|\,3,5\rangle=0$$

$$\langle 0,2\,|\,5,3\rangle=\langle 2,0\,|\,5,3\rangle=\langle 1,2\,|\,5,3\rangle=\langle 2,1\,|\,5,3\rangle=\langle 2,2\,|\,5,3\rangle=0$$

$$\langle 0,2\,|\,4,5\rangle=\langle 2,0\,|\,4,5\rangle=\langle 1,2\,|\,4,5\rangle=\langle 2,2\,|\,4,5\rangle=0$$

$$\langle 0,2\,|\,5,4\rangle=\langle 2,0\,|\,5,4\rangle=\langle 2,1\,|\,5,4\rangle=\langle 2,2\,|\,5,4\rangle=0$$

$$\langle 1,2\,|\,5,4\rangle=\langle 2,1\,|\,4,5\rangle=\frac{\left(1+i\frac{z}{z_r}\right)}{2^{1/2}*3*5^{1/2}\pi\omega_0^4\left(1-i\frac{z}{z_r}\right)^5}\int_0^R\rho\mathrm{e}^{\frac{-2\rho^2}{\omega(z)^2}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\rho^2\cos^2\theta\left(\frac{2^2\rho^2\sin^2\theta}{\omega(z)^2}-1\right)$$

$$\left(\frac{2^8\rho^8\sin^4\theta\cos^4\theta}{\omega(z)^8}-\frac{2^7\rho^6\sin^2\theta\cos^2\theta\left(2\sin^2\theta+3\right)}{\omega(z)^6}+\frac{2^43\rho^4\left(1+2\sin^2\theta\left(2+7\cos^2\theta\right)\right)}{\omega(z)^4}\right.$$

$$\left.-\frac{2^3*3*5\rho^2\left(1+2\sin^2\theta\right)}{\omega(z)^2}+3^25\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)}{2^{1/2}*3*5^{1/2}\pi\omega_0^4\left(1-i\frac{z}{z_r}\right)^5}\left(\frac{2^{10}G(12,6,6)}{\omega(z)^{10}}-\frac{2^8\left(G(10,4,6)+2\left(2G(10,6,4)+3G(10,4,4)\right)\right)}{\omega(z)^8}\right.$$

$$+\frac{2^6\left(2\left(2G(8,4,4)+3G(8,2,4)\right)+3\left(G(8,2,2)+2\left(2G(8,4,2)+7G(8,4,4)\right)\right)\right)}{\omega(z)^6}$$

$$-\frac{2^4 3\left(G(6,0,2)+2\left(2G(6,2,2)+7G(6,2,4)\right)+2*5\left(G(6,2,2)+2G(6,4,2)\right)\right)}{\omega(z)^4}$$

$$\left.+\frac{2^2*3*5\left(3G(4,2,2)+2\left(G(4,0,2)+2G(4,2,2)\right)\right)}{\omega(z)^2}-3^2 5G(2,0,2)\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)}{2^{1/2}*3*5^{1/2}\pi\omega_0^4\left(1-i\frac{z}{z_r}\right)^5}\left(\frac{2^{10}G(12,6,6)}{\omega(z)^{10}}-\frac{2^8\left(5G(10,6,4)+6G(10,4,4)\right)}{\omega(z)^8}\right.$$

$$+\frac{2^6\left(2*3^2G(8,4,2)+2*23G(8,4,4)+3G(8,2,2)\right)}{\omega(z)^6}-\frac{2^4 3\left(G(6,2,0)+2*7G(6,2,2)+2*17G(6,4,2)\right)}{\omega(z)^4}$$

$$\left.+\frac{2^2*3*5\left(7G(4,2,2)+2G(4,0,2)\right)}{\omega(z)^2}-3^2 5G(2,0,2)\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)^2}{2^{3/2}*3*5^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(\frac{3^2 5^2\omega(z)^2 S_7}{2^2}-\omega(z)^2 S_6\left(\frac{3^2 5^2}{2^2}+3^3 5\right)+\omega(z)^2 S_5\left(2^2 3^3+\frac{3^2*23}{2}+2^2 3^2\right)\right.$$

$$\left.-3\omega(z)^2 S_4\left(2*3+3*7+\frac{3*17}{2}\right)+3*5\omega(z)^2 S_3\left(\frac{7}{2^2}+2\right)-\frac{3^2 5S_2}{2^2}\right)$$

$$= \frac{3\left(1+i\frac{z}{z_r}\right)^3}{2^{3/2}5^{1/2}\left(1-i\frac{z}{z_r}\right)^3}\left(\frac{5^2 S_7}{2^2}-\frac{5*17S_6}{2^2}+\frac{5*11S_5}{2}-\frac{5*7S_4}{2}+\frac{5^2 S_3}{2^2}-\frac{5S_2}{2^2}\right)$$

$$= \frac{3*5^{1/2}\left(1+i\frac{z}{z_r}\right)^3}{2^{7/2}\left(1-i\frac{z}{z_r}\right)^3}\left(5S_7-17S_6+22S_5-14S_4+5S_3-S_2\right)$$

$$= -\frac{3*5^{1/2}\left(1+i\frac{z}{z_r}\right)^3 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{7/2}\left(1-i\frac{z}{z_r}\right)^3}\left(\frac{2R^4}{\omega(z)^4}-4\frac{4R^6}{3\omega(z)^6}+10\frac{2R^8}{3\omega(z)^8}-12\frac{4R^{10}}{15\omega(z)^{10}}+\frac{10R^{12}}{45\omega(z)^{12}}\right)$$

$$= -\frac{3*5^{1/2}R^4\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{5/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{8R^2}{3\omega(z)^2}+\frac{10R^4}{3\omega(z)^4}-\frac{8R^6}{5\omega(z)^6}+\frac{2R^8}{9\omega(z)^8}\right)$$

$$\langle 0,2\,|\,5,5\rangle=\langle 2,0\,|\,5,5\rangle=\langle 1,2\,|\,5,5\rangle=\langle 2,1\,|\,5,5\rangle=\langle 2,2\,|\,5,5\rangle=0$$

$$\langle 0,2\,|\,0,6\rangle = \langle 2,0\,|\,6,0\rangle = \frac{\left(1+i\frac{z}{z_r}\right)}{2^{3/2}*3*5^{1/2}\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\int_0^R \rho \mathrm{e}^{\frac{-2\rho^2}{\omega(z)^2}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\left(\frac{2^2\rho^2\sin^2\theta}{\omega(z)^2}-1\right)$$

$$\times\left(\frac{2^6\rho^6\sin^6\theta}{\omega(z)^6}-\frac{2^4*3*5\rho^4\sin^4\theta}{\omega(z)^4}+\frac{2^2 3^2 5\rho^2\sin^2\theta}{\omega(z)^2}-15\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)}{2^{3/2}*3*5^{1/2}\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(\frac{2^8G(8,8,0)}{\omega(z)^8}-\frac{2^6\left(G(6,6,0)+3*5G(6,6,0)\right)}{\omega(z)^6}\right.$$

$$\left.+\frac{2^4*3*5\left(G(4,4,0)+3G(4,4,0)\right)}{\omega(z)^4}-\frac{2^2*3*5\left(G(2,2,0)+3G(2,2,0)\right)}{\omega(z)^2}+3*5G(0,0,0)\right)$$

$$\frac{\left(1+i\frac{z}{z_r}\right)^2}{2^{5/2}*3*5^{1/2}\left(1-i\frac{z}{z_r}\right)^2}\left(3*5*7S_5-2^4*3*5S_4+2^2*3^2*5S_3-2^2*3*5S_2+3*5S_1\right)$$

$$= \frac{5^{1/2}\left(1+i\frac{z}{z_r}\right)^2}{2^{5/2}\left(1-i\frac{z}{z_r}\right)^2}\left(7S_5-16S_4+12S_3-4S_2+S_1\right)$$

$$= \frac{5^{1/2}\left(1+i\frac{z}{z_r}\right)^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{5/2}\left(1-i\frac{z}{z_r}\right)^2}\left(\frac{2R^2}{\omega(z)^2}-3\frac{2R^4}{\omega(z)^4}+9\frac{4R^6}{3\omega(z)^6}-7\frac{2R^8}{3\omega(z)^8}\right)$$

$$= \frac{5^{1/2}R^2\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{3/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{3R^2}{\omega(z)^2}+\frac{6R^4}{\omega(z)^4}-\frac{7R^6}{3\omega(z)^6}\right)$$

$$\langle 0,2\,|\,6,0\rangle = \langle 2,0\,|\,0,6\rangle = \frac{\left(1+i\frac{z}{z_r}\right)}{2^{3/2}*3*5^{1/2}\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\int_0^R \rho \mathrm{e}^{\frac{-2\rho^2}{\omega(z)^2}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\left(\frac{2^2\rho^2\sin^2\theta}{\omega(z)^2}-1\right)$$

$$\times\left(\frac{2^6\rho^6\cos^6\theta}{\omega(z)^6}-\frac{2^4*3*5\rho^4\cos^4\theta}{\omega(z)^4}+\frac{2^2 3^2 5\rho^2\cos^2\theta}{\omega(z)^2}-15\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)}{2^{3/2}*3*5^{1/2}\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(\frac{2^8G(8,2,6)}{\omega(z)^8}-\frac{2^6\left(G(6,0,6)+3*5G(6,2,4)\right)}{\omega(z)^6}\right.$$

$$\left.+\frac{2^4*3*5\left(G(4,0,4)+3G(4,2,2)\right)}{\omega(z)^4}-\frac{2^4*3*5G(2,2,0)}{\omega(z)^2}+3*5G(0,0,0)\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)^2}{2^{5/2}*3*5^{1/2}\left(1-i\frac{z}{z_r}\right)^2}\left(3*5S_5-\left(3*5+3^2*5\right)S_4+3*5\left(3+3\right)S_3-2^2*3*5S_2+3*5S_1\right)$$

$$= \frac{5^{1/2}\left(1+i\frac{z}{z_r}\right)^2}{2^{5/2}\left(1-i\frac{z}{z_r}\right)^2}\left(S_5-4S_4+6S_3-4S_2+S_1\right)$$

$$= \frac{5^{1/2}\left(1+i\frac{z}{z_r}\right)^2 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{5/2}\left(1-i\frac{z}{z_r}\right)^2}\left(\frac{2R^2}{\omega(z)^2}-3\frac{2R^4}{\omega(z)^4}+\frac{4R^6}{\omega(z)^6}-\frac{2R^8}{3\omega(z)^8}\right)$$

$$= \frac{5^{1/2}R^2\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{3/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{3R^2}{\omega(z)^2}+\frac{2R^4}{\omega(z)^4}-\frac{R^6}{3\omega(z)^6}\right)$$

$$\langle 1,2\,|\,0,6\rangle = \langle 2,1\,|\,0,6\rangle = \langle 1,2\,|\,6,0\rangle = \langle 2,1\,|\,6,0\rangle = 0$$

$$\langle 2,2\,|\,0,6\rangle = \langle 2,2\,|\,6,0\rangle = \frac{1}{2^2*3*5^{1/2}\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}\int_0^R \rho \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta$$

$$\times\left(\frac{2^4\rho^4\cos^2\theta\sin^2\theta}{\omega(z)^4}-\frac{2^2\rho^2}{\omega(z)^2}+1\right)\left(\frac{2^6\rho^6\sin^6\theta}{\omega(z)^6}-\frac{2^4*3*5\rho^4\sin^4\theta}{\omega(z)^4}+\frac{2^2 3^2 5\rho^2\sin^2\theta}{\omega(z)^2}-15\right)$$

$$= \frac{1}{2^2*3*5^{1/2}\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}\left(\frac{2^{10}G(10,8,2)}{\omega(z)^{10}}-\frac{2^8\left(G(8,6,0)+3*5G(8,6,2)\right)}{\omega(z)^8}\right.$$

$$+\frac{2^6\left(G(6,6,0)+3*5G(6,4,0)+3^2*5G(6,4,2)\right)}{\omega(z)^6}-\frac{2^4*3*5\left(G(4,4,0)+G(4,2,2)+3G(4,2,0)\right)}{\omega(z)^4}$$

$$\left.+\frac{2^2*3*5\left(G(2,0,0)+3G(2,2,0)\right)}{\omega(z)^2}-3*5G(0,0,0)\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)}{2^3*3*5^{1/2}\left(1-i\frac{z}{z_r}\right)}\left(3*5*7S_6-\left(2^3*3*5+3^2*5^2\right)S_5+\left(3*5+2*3^3*5+3^3*5\right)S_4\right.$$

$$\left.-3*5\left(3+1+2^2 3\right)S_3+\left(2*3*5+3^2*5\right)S_2-3*5S_1\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)}{2^3*3*5^{1/2}\left(1-i\frac{z}{z_r}\right)}\left(3*5*7S_6-3*5*23S_5+2^2*3*5*7S_4-2^4*3*5S_3+3*5^2S_2-3*5S_1\right)$$

$$= \frac{5^{1/2}\left(1+i\frac{z}{z_r}\right)}{2^3\left(1-i\frac{z}{z_r}\right)}\left(7S_6-23S_5+28S_4-16S_3+5S_2-S_1\right)$$

$$= -\frac{5^{1/2}\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^3\left(1-i\frac{z}{z_r}\right)}\left(\frac{2R^2}{\omega(z)^2}-4\frac{2R^4}{\omega(z)^4}+12\frac{4R^6}{3\omega(z)^6}-16\frac{2R^8}{3\omega(z)^8}+7\frac{4R^{10}}{15\omega(z)^{10}}\right)$$

$$= -\frac{5^{1/2}R^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^2\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}\left(1-\frac{4R^2}{\omega(z)^2}+\frac{8R^4}{\omega(z)^4}-\frac{16R^6}{3\omega(z)^6}+\frac{14R^{10}}{15\omega(z)^{10}}\right)$$

$$\langle 0,2\,|\,1,6\rangle = \langle 2,0\,|\,1,6\rangle = \langle 2,1\,|\,1,6\rangle = \langle 2,2\,|\,1,6\rangle = 0$$

$$\langle 0,2\,|\,6,1\rangle = \langle 2,0\,|\,6,1\rangle = \langle 1,2\,|\,6,1\rangle = \langle 2,2\,|\,6,1\rangle = 0$$

$$\langle 1,2\,|\,1,6\rangle = \langle 2,1\,|\,6,1\rangle = \frac{2^{1/2}}{3\pi 5^{1/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\int_0^R \rho\mathrm{e}^{\frac{-2\rho^2}{\omega(z)^2}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\rho^2\cos^2\theta\left(\frac{2^2\rho^2\sin^2\theta}{\omega(z)^2}-1\right)$$

$$\times\left(\frac{2^6\rho^6\sin^6\theta}{\omega(z)^6}-\frac{2^4*3*5\rho^4\sin^4\theta}{\omega(z)^4}+\frac{2^2 3^2 5\rho^2\sin^2\theta}{\omega(z)^2}-15\right)$$

$$= \frac{2^{1/2}}{3\pi 5^{1/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(\frac{2^8G(10,8,2)}{\omega(z)^8}-\frac{2^6\left(G(8,6,2)+3*5G(8,6,2)\right)}{\omega(z)^6}+\frac{2^4*3*5\left(G(6,4,2)+3G(6,4,2)\right)}{\omega(z)^4}\right.$$

$$\left.-\frac{2^2*3*5\left(G(4,2,2)+3G(4,2,2)\right)}{\omega(z)^2}+15G(2,0,2)\right)$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)}{2^{1/2}*3*5^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(\frac{3*5*7\omega(z)^2S_6}{2^2}-2^2*3*5\omega(z)^2S_5\right.$$

$$\left.+3^2*5\omega(z)^2S_4-3*5\omega(z)^2S_3+\frac{3*5\omega(z)^2S_2}{2^2}\right)$$

$$= \frac{5^{1/2}\left(1+i\frac{z}{z_r}\right)^2}{2^{1/2}\left(1-i\frac{z}{z_r}\right)^2}\left(\frac{7S_6}{2^2}-4S_5+3S_4-S_3+\frac{S_2}{2^2}\right)$$

$$= \frac{5^{1/2}\left(1+i\frac{z}{z_r}\right)^2}{2^{5/2}\left(1-i\frac{z}{z_r}\right)^2}\left(7S_6-16S_5+12S_4-4S_3+S_2\right)$$

$$= \frac{5^{1/2}\left(1+i\frac{z}{z_r}\right)^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{5/2}\left(1-i\frac{z}{z_r}\right)^2}\left(\frac{2R^4}{\omega(z)^4}-3\frac{4R^6}{3\omega(z)^6}+9\frac{2R^8}{3\omega(z)^8}-7\frac{4R^{10}}{15\omega(z)^{10}}\right)$$

$$= \frac{5^{1/2}R^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{3/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{2R^2}{\omega(z)^2}+\frac{6R^4}{\omega(z)^4}-\frac{14R^6}{15\omega(z)^6}\right)$$

$$\langle 0,2\,|\,2,6\rangle = \langle 2,0\,|\,6,2\rangle = \frac{\left(1+i\frac{z}{z_r}\right)^2}{2^2*3*5^{1/2}\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\int_0^R \rho \mathrm{e}^{\frac{-2\rho^2}{\omega(z)^2}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\left(\frac{2^2\rho^2\sin^2\theta}{\omega(z)^2}-1\right)$$

$$\times\left(\frac{2^8\rho^8\cos^2\theta\sin^6\theta}{\omega(z)^8}-\frac{2^6\rho^6\sin^4\theta\left(1+2*7\cos^2\theta\right)}{\omega(z)^6}+\frac{2^4*3*5\rho^4\sin^2\theta\left(2\cos^2\theta+1\right)}{\omega(z)^4}\right.$$

$$\left.-\frac{2^2*3*5\rho^2\left(2\sin^2\theta+1\right)}{\omega(z)^2}+3*5\right)$$

$$\frac{\left(1+i\frac{z}{z_r}\right)^2}{2^2*3*5^{1/2}\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(\frac{2^{10}G(10,8,2)}{\omega(z)^{10}}-\frac{2^8\left(G(8,6,2)+G(8,6,0)+2*7G(8,6,2)\right)}{\omega(z)^8}\right.$$

$$+\frac{2^6\left(G(6,4,0)+2*7G(6,4,2)+3*5G(6,4,0)+2*3*5G(6,4,2)\right)}{\omega(z)^6}$$

$$-\frac{2^4*3*5\left(G(4,2,0)+2G(4,2,2)+G(4,2,0)+2G(4,4,0)\right)}{\omega(z)^4}$$

$$\left.+\frac{2^2*3*5\left(G(2,0,0)+2G(2,2,0)+G(2,2,0)\right)}{\omega(z)^2}-3*5G(0,0,0)\right)$$

$$=\frac{\left(1+i\frac{z}{z_r}\right)^3}{2^3*3*5^{1/2}\left(1-i\frac{z}{z_r}\right)^3}\left(3*5*7S_6-\left(3^2*5^2+2^3*3*5\right)S_5+\left(2^5*3^2+2^2*3*11\right)S_4\right.$$

$$\left.-3*5\left(2^3+2+2*3\right)S_3+3*5\left(2+3\right)S_2-3*5S_1\right)$$

$$=\frac{\left(1+i\frac{z}{z_r}\right)^3}{2^3*3*5^{1/2}\left(1-i\frac{z}{z_r}\right)^3}\left(3*5*7S_6-3*5*23S_5+2^2*3*5*7S_4-2^4*3*5S_3+3*5^2S_2-3*5S_1\right)$$

$$=\frac{5^{1/2}\left(1+i\frac{z}{z_r}\right)^3}{2^3\left(1-i\frac{z}{z_r}\right)^3}\left(7S_6-23S_5+28S_4-16S_3+5S_2-S_1\right)$$

$$=-\frac{5^{1/2}\left(1+i\frac{z}{z_r}\right)^3\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^3\left(1-i\frac{z}{z_r}\right)^3}\left(\frac{2R^2}{\omega(z)^2}-4\frac{2R^4}{\omega(z)^4}+12\frac{4R^6}{3\omega(z)^6}-16\frac{2R^8}{3\omega(z)^8}+7\frac{4R^{10}}{15\omega(z)^{10}}\right)$$

$$=-\frac{5^{1/2}R^2\left(1+i\frac{z}{z_r}\right)^2}{2^2\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{4R^2}{\omega(z)^2}+8\frac{R^4}{\omega(z)^4}-\frac{16R^6}{3\omega(z)^6}+\frac{14R^8}{15\omega(z)^8}\right)$$

$$\langle 0,2\,|\,6,2\rangle = \langle 2,0\,|\,2,6\rangle = \frac{\left(1+i\frac{z}{z_r}\right)^2}{2^2*3*5^{1/2}\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\int_0^R \rho \mathrm{e}^{\frac{-2\rho^2}{\omega(z)^2}}\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\left(\frac{2^2\rho^2\sin^2\theta}{\omega(z)^2}-1\right)$$
$$\times\left(\frac{2^8\rho^8\sin^2\theta\cos^6\theta}{\omega(z)^8}-\frac{2^6\rho^6\cos^4\theta\left(1+2*7\sin^2\theta\right)}{\omega(z)^6}+\frac{2^4*3*5\rho^4\cos^2\theta\left(2\sin^2\theta+1\right)}{\omega(z)^4}\right.$$
$$\left.-\frac{2^2*3*5\rho^2\left(2\cos^2\theta+1\right)}{\omega(z)^2}+3*5\right)$$
$$= \frac{\left(1+i\frac{z}{z_r}\right)^2}{2^2*3*5^{1/2}\pi\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(\frac{2^{10}G(10,6,4)}{\omega(z)^{10}}-\frac{2^8\left(G(8,6,2)+G(8,4,2)+2*7G(8,4,4)\right)}{\omega(z)^8}+\frac{2^6\left(G(6,4,0)+2*7G(\right.}{}\right.$$
at this point I began using Mathematica.
$$= -\frac{5^{1/2}R^2\left(1+i\frac{z}{z_r}\right)^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^2\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{4R^2}{\omega(z)^2}+\frac{16R^4}{3\omega(z)^4}-\frac{8R^6}{3\omega(z)^6}+\frac{2R^8}{5\omega(z)^8}\right)$$
$$\langle 1,2\,|\,2,6\rangle = \langle 2,1\,|\,2,6\rangle = \langle 1,2\,|\,6,2\rangle = \langle 2,1\,|\,6,2\rangle = 0$$
$$\langle 2,2\,|\,2,6\rangle = \langle 2,2\,|\,6,2\rangle = \frac{5^{1/2}R^2\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{5/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{5R^2}{\omega(z)^2}+\frac{12R^4}{\omega(z)^4}-\frac{34R^6}{3\omega(z)^6}+\frac{82R^8}{15\omega(z)^8}-\frac{14R^{10}}{15\omega(z)^{10}}\right)$$
$$\langle 0,2\,|\,3,6\rangle = \langle 2,0\,|\,3,6\rangle = \langle 0,2\,|\,6,3\rangle = \langle 2,0\,|\,6,3\rangle = 0$$
$$\langle 1,2\,|\,6,3\rangle = \langle 2,1\,|\,3,6\rangle = \langle 2,2\,|\,3,6\rangle = \langle 2,2\,|\,6,3\rangle = 0$$
$$\langle 1,2\,|\,3,6\rangle = \langle 2,1\,|\,6,3\rangle = -\frac{3^{1/2}5^{1/2}R^4\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^2\omega_0^4\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{8R^2}{\omega(z)^2}+\frac{12R^4}{\omega(z)^4}-\frac{32R^6}{5\omega(z)^6}+\frac{14R^8}{15\omega(z)^8}\right)$$
$$\langle 0,2\,|\,4,6\rangle = \langle 2,0\,|\,6,4\rangle = \frac{3^{1/2}5^{1/2}R^2\left(1+i\frac{z}{z_r}\right)^3\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^3\omega_0^2\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{5R^2}{\omega(z)^2}+\frac{32R^4}{3\omega(z)^4}-\frac{28R^6}{3\omega(z)^6}+\frac{46R^8}{15\omega(z)^8}-\frac{14R^{10}}{45\omega(z)^{10}}\right)$$
$$\langle 0,2\,|\,6,4\rangle = \langle 2,0\,|\,4,6\rangle = \frac{3^{1/2}5^{1/2}R^2\left(1+i\frac{z}{z_r}\right)^3\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^3\omega_0^2\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{5R^2}{\omega(z)^2}+\frac{28R^4}{3\omega(z)^4}-\frac{22R^6}{3\omega(z)^6}+\frac{34R^8}{15\omega(z)^8}-\frac{2R^{10}}{9\omega(z)^{10}}\right)$$
$$\langle 1,2\,|\,4,6\rangle = \langle 2,1\,|\,4,6\rangle = \langle 1,2\,|\,6,4\rangle = \langle 2,1\,|\,6,4\rangle = 0$$

$$\langle 2,2\,|\,4,6\rangle = \langle 2,2\,|\,6,4\rangle =$$

$$-\frac{3^{1/2}5^{1/2}R^2\left(1+i\frac{z}{z_r}\right)^2 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{7/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{6R^2}{\omega(z)^2}+\frac{50R^4}{3\omega(z)^4}-\frac{20R^6}{\omega(z)^6}+\frac{66R^8}{5\omega(z)^8}-\frac{188R^{10}}{45\omega(z)^{10}}+\frac{4R^{12}}{9\omega(z)^{12}}\right)$$

$$\langle 0,2\,|\,5,6\rangle = \langle 2,0\,|\,5,6\rangle = \langle 2,1\,|\,5,6\rangle = \langle 2,2\,|\,5,6\rangle = 0$$

$$\langle 0,2\,|\,6,5\rangle = \langle 2,0\,|\,6,5\rangle = \langle 1,2\,|\,6,5\rangle = \langle 2,2\,|\,6,5\rangle = 0$$

$$\langle 1,2\,|\,5,6\rangle = \langle 2,1\,|\,6,5\rangle$$

$$= \frac{3^{1/2}5R^4\left(1+i\frac{z}{z_r}\right)^2 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^3\omega_0^4\left(1-i\frac{z}{z_r}\right)^6}\left(1-\frac{10R^2}{3\omega(z)^2}+\frac{16R^4}{3\omega(z)^4}-\frac{56R^6}{15\omega(z)^6}+\frac{46R^8}{45\omega(z)^8}-\frac{4R^{10}}{45\omega(z)^{10}}\right)$$

$$\langle 0,2\,|\,6,6\rangle = \langle 2,0\,|\,6,6\rangle$$

$$= -\frac{5\left(1+i\frac{z}{z_r}\right)^4 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{7/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^6}\left(1-\frac{6R^2}{\omega(z)^2}+\frac{14R^4}{\omega(z)^4}-\frac{44R^6}{3\omega(z)^6}+\frac{34R^8}{5\omega(z)^8}-\frac{4R^{10}}{3\omega(z)^{10}}+\frac{4R^{12}}{45\omega(z)^{12}}\right)$$

$$\langle 1,2\,|\,6,6\rangle = \langle 2,1\,|\,6,6\rangle = 0$$

$$\langle 2,2\,|\,6,6\rangle$$

$$= \frac{5\left(1+i\frac{z}{z_r}\right)^3 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{16\omega_0^2\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{7R^2}{\omega(z)^2}+\frac{22R^4}{\omega(z)^4}-\frac{95R^6}{3\omega(z)^6}+\frac{382R^8}{15\omega(z)^8}-\frac{166R^{10}}{15\omega(z)^{10}}+\frac{20R^{12}}{9\omega(z)^{12}}-\frac{7R^{14}}{45\omega(z)^{14}}\right)$$

We now collect all our results temporarily.

$$\langle 0,0\,|\,0,0\rangle = 1-\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}$$

$$\langle 0,0\,|\,0,2\rangle = \langle 0,0\,|\,2,0\rangle = -\frac{2^{1/2}R^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}$$

$$\langle 0,0\,|\,2,2\rangle = \frac{\left(1+i\frac{z}{z_r}\right)R^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{R^2}{\omega(z)^2}\right)$$

$$\langle 0,0\,|\,0,4\rangle = \langle 0,0\,|\,4,0\rangle = \frac{3^{1/2}R^2\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{R^2}{\omega(z)^2}\right)$$

$$\langle 0,0\,|\,2,4\rangle = \langle 0,0\,|\,4,2\rangle = \frac{-3^{1/2}R^2\left(1+i\frac{z}{z_r}\right)^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{2R^2}{\omega(z)^2}+\frac{2R^4}{3\omega(z)^4}\right)$$

$$\langle 0,0\,|\,4,4\rangle = \frac{3R^2\left(1+i\frac{z}{z_r}\right)^3\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{4\omega_0^2\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{3R^2}{\omega(z)^2}+\frac{2R^4}{\omega(z)^4}-\frac{R^6}{3\omega(z)^6}\right)$$

$$\langle 0,0\,|\,0,6\rangle = \langle 0,0\,|\,6,0\rangle = \frac{-5^{1/2}R^2\left(1+i\frac{z}{z_r}\right)^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{2R^2}{\omega(z)^2}+\frac{2R^4}{3\omega(z)^4}\right)$$

$$\langle 0,0\,|\,2,6\rangle = \langle 0,0\,|\,6,2\rangle = \frac{5^{1/2}R^2\left(1+i\frac{z}{z_r}\right)^3\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{8^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{3R^2}{\omega(z)^2}+\frac{2R^4}{\omega(z)^4}-\frac{R^6}{3\omega(z)^6}\right)$$

$$\langle 0,0\,|\,4,6\rangle = \langle 0,0\,|\,6,4\rangle = -\frac{15^{1/2}R^2\left(1+i\frac{z}{z_r}\right)^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{32^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^6}\left(1-\frac{4R^2}{\omega(z)^2}+\frac{4R^4}{\omega(z)^4}-\frac{4R^6}{3\omega(z)^6}+\frac{2R^8}{15\omega(z)^8}\right)$$

$$\langle 0,0\,|\,6,6\rangle = \frac{5R^2\left(1+i\frac{z}{z_r}\right)^5\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{8\omega_0^2\left(1-i\frac{z}{z_r}\right)^7}\left(1-\frac{5R^2}{\omega(z)^2}+\frac{20R^4}{3\omega(z)^4}-\frac{10R^6}{3\omega(z)^6}+\frac{2R^8}{3\omega(z)^8}-\frac{2R^{10}}{45\omega(z)^{10}}\right)$$

$$\langle 0,1\,|\,0,1\rangle = \langle 1,0\,|\,1,0\rangle = 1-\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}\left(1+\frac{2R^2}{\omega(z)^2}\right)$$

$$\langle 1,1\,|\,1,1\rangle = 1-\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}\left(1+\frac{2R^2}{\omega(z)^2}+\frac{2R^4}{\omega(z)^4}\right)$$

$$\langle 0,1\,|\,2,1\rangle = \langle 1,0\,|\,1,2\rangle = -\frac{2^{1/2}R^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{\omega_0^4\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}$$

$$\langle 0,1\,|\,0,3\rangle = \langle 1,0\,|\,3,0\rangle = -\frac{6^{1/2}R^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{\omega_0^4\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}$$

$$\langle 1,1\,|\,1,3\rangle = \langle 1,1\,|\,3,1\rangle = -\frac{8^{1/2}R^6 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{3^{1/2}\omega_0^6\left(1+i\frac{z}{z_r}\right)^2\left(1-i\frac{z}{z_r}\right)^4}$$

$$\langle 0,1\,|\,2,3\rangle = \langle 1,0\,|\,3,2\rangle = \frac{3^{1/2}R^4 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{2R^2}{3\omega(z)^2}\right)$$

$$\langle 1,1\,|\,3,3\rangle = \frac{R^6 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{\omega_0^6\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^5}\left(2-\frac{R^2}{\omega(z)^2}\right)$$

$$\langle 0,1\,|\,4,1\rangle = \langle 1,0\,|\,1,4\rangle = \frac{3^{1/2}R^4 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{1/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{2R^2}{3\omega(z)^2}\right)$$

$$\langle 0,1\,|\,4,3\rangle = \langle 1,0\,|\,3,4\rangle = \frac{-3R^4\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2\omega_0^4\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{4R^2}{3\omega(z)^2}+\frac{R^4}{3\omega(z)^4}\right)$$

$$\langle 0,1\,|\,0,5\rangle = \langle 1,0\,|\,5,0\rangle = \frac{15^{1/2}R^4 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{1/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{2R^2}{3\omega(z)^2}\right)$$

$$\langle 1,1\,|\,1,5\rangle = \langle 1,1\,|\,5,1\rangle = \frac{5^{1/2}R^6 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{6^{1/2}\omega_0^6\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^5}\left(2-\frac{R^2}{\omega(z)^2}\right)$$

$$\langle 0,1\,|\,2,5\rangle = \langle 1,0\,|\,5,2\rangle = -\frac{15^{1/2}R^4\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2\omega_0^4\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{4R^2}{3\omega(z)^2}+\frac{R^4}{3\omega(z)^4}\right)$$

$$\langle 1,1\,|\,3,5\rangle = \langle 1,1\,|\,5,3\rangle = -\frac{5^{1/2}R^6 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{\omega_0^6\left(1-i\frac{z}{z_r}\right)^6}\left(1-\frac{R^2}{\omega(z)^2}+\frac{R^4}{5\omega(z)^4}\right)$$

$$\langle 0,1\,|\,4,5\rangle = \langle 1,0\,|\,5,4\rangle = \frac{45^{1/2}R^4\left(1+i\frac{z}{z_r}\right)^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{4\omega_0^4\left(1-i\frac{z}{z_r}\right)^6}\left(1-\frac{2R^2}{\omega(z)^2}+\frac{R^4}{\omega(z)^4}-\frac{2R^6}{15\omega(z)^6}\right)$$

$$\langle 1,1\,|\,5,5\rangle = \frac{15R^6\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{4\omega_0^6\left(1-i\frac{z}{z_r}\right)^7}\left(\frac{2}{3}-\frac{R^2}{\omega(z)^2}+\frac{2R^4}{5\omega(z)^4}-\frac{2R^6}{45\omega(z)^6}\right)$$

$$\langle 0,1\,|\,6,1\rangle = \langle 1,0\,|\,1,6\rangle = -\frac{5^{1/2}R^4\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2\omega_0^4\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{4R^2}{3\omega(z)^2}+\frac{R^4}{3\omega(z)^4}\right)$$

$$\langle 0,1\,|\,6,3\rangle = \langle 1,0\,|\,3,6\rangle = \frac{15^{1/2}R^4\left(1+i\frac{z}{z_r}\right)^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{8^{1/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^6}\left(1-\frac{2R^2}{\omega(z)^2}+\frac{R^4}{\omega(z)^4}-\frac{2R^6}{15\omega(z)^6}\right)$$

$$\langle 0,1\,|\,6,5\rangle = \langle 1,0\,|\,5,6\rangle = -\frac{75^{1/2}R^4\left(1+i\frac{z}{z_r}\right)^3 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{32^{1/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^7}\left(1-\frac{8R^2}{3\omega(z)^2}+\frac{2R^4}{\omega(z)^4}-\frac{8R^6}{15\omega(z)^6}+\frac{2R^8}{45\omega(z)^8}\right)$$

$$\langle 0,2\,|\,0,2\rangle = \langle 2,0\,|\,2,0\rangle = 1-\mathrm{e}^{\frac{2R^2}{\omega(z)^2}}\left(1+\frac{R^2}{\omega(z)^2}+\frac{3R^4}{\omega(z)^4}\right)$$

$$\langle 0,2\,|\,2,0\rangle = \langle 2,0\,|\,0,2\rangle = \frac{R^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{\omega(z)^2}\left(1-\frac{R^2}{\omega(z)^2}\right)$$

$$\langle 1,2\,|\,1,2\rangle = \langle 2,1\,|\,2,1\rangle = 1-\mathrm{e}^{\frac{2R^2}{\omega(z)^2}}\left(1+\frac{2R^2}{\omega(z)^2}+\frac{R^4}{\omega(z)^4}+\frac{2R^6}{\omega(z)^6}\right)$$

$$\langle 0,2\,|\,2,2\rangle = \langle 2,0\,|\,2,2\rangle = -\frac{R^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}\left(1-\frac{2R^2}{\omega(z)^2}+\frac{2R^4}{\omega(z)^4}\right)$$

$$\langle 2,2\,|\,2,2\rangle = 1-\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}\left(1+\frac{3R^2}{2\omega(z)^2}+\frac{7R^4}{2\omega(z)^4}-\frac{R^6}{\omega(z)^6}+\frac{3R^8}{2\omega(z)^8}\right)$$

$$\langle 1,2\,|\,3,0\rangle = \langle 2,1\,|\,0,3\rangle = \frac{3^{1/2}R^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{\omega(z)^4}\left(1-\frac{2R^2}{3\omega(z)^2}\right)$$

$$\langle 1,2\,|\,3,2\rangle = \langle 2,1\,|\,2,3\rangle = -\frac{3^{1/2}R^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{1/2}\omega_0^4\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{4R^2}{3\omega(z)^2}+\frac{R^4}{\omega(z)^4}\right)$$

$$\langle 0,2\,|\,0,4\rangle = \langle 2,0\,|\,4,0\rangle = -\frac{3^{1/2}R^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}\left(1-\frac{2R^2}{\omega(z)^2}+\frac{10R^4}{3\omega(z)^4}\right)$$

$$\langle 0,2\,|\,4,0\rangle = \langle 2,0\,|\,0,4\rangle = -\frac{3^{1/2}R^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}\left(1-\frac{2R^2}{\omega(z)^2}+\frac{2R^4}{3\omega(z)^4}\right)$$

$$\langle 2,2\,|\,0,4\rangle = \langle 2,2\,|\,4,0\rangle = \frac{3^{1/2}R^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{\omega(z)^2}\left(1-\frac{3R^2}{\omega(z)^2}+\frac{14R^4}{3\omega(z)^4}-\frac{5R^6}{3\omega(z)^6}\right)$$

$$\langle 1,2\,|\,1,4\rangle = \langle 2,1\,|\,4,1\rangle = -\frac{3^{1/2}R^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2\omega_0^4\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{4R^2}{3\omega(z)^2}+\frac{5R^4}{3\omega(z)^4}\right)$$

$$\langle 0,2\,|\,2,4\rangle = \langle 2,0\,|\,4,2\rangle = \frac{3^{1/2}R^2\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{8^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{3R^2}{\omega(z)^2}+\frac{14R^4}{3\omega(z)^4}-\frac{5R^6}{3\omega(z)^6}\right)$$

$$\langle 0,2\,|\,4,2\rangle = \langle 2,0\,|\,2,4\rangle = \frac{3^{1/2}R^2\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{8^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{3R^2}{\omega(z)^2}+\frac{10R^4}{3\omega(z)^4}-\frac{R^6}{\omega(z)^6}\right)$$

$$\langle 2,2\,|\,2,4\rangle = \langle 2,2\,|\,4,2\rangle = -\frac{3^{1/2}R^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{4\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}\left(1-\frac{4R^2}{\omega(z)^2}+\frac{8R^4}{\omega(z)^4}-\frac{16R^6}{3\omega(z)^6}+\frac{R^8}{\omega(z)^8}\right)$$

$$\langle 1,2\,|\,3,4\rangle = \langle 2,1\,|\,4,3\rangle = \frac{3R^4 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{8^{1/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{2R^2}{\omega(z)^2}+\frac{7R^4}{3\omega(z)^4}-\frac{2R^8}{3\omega(z)^8}\right)$$

$$\langle 0,2\,|\,4,4\rangle = \langle 2,0\,|\,4,4\rangle = -\frac{3R^2\left(1+i\frac{z}{z_r}\right)^2 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{32^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{4R^2}{\omega(z)^2}+\frac{20R^4}{3\omega(z)^4}-\frac{4R^6}{\omega(z)^6}+\frac{2R^8}{3\omega(z)^8}\right)$$

$$\langle 2,2\,|\,4,4\rangle = \frac{3R^2\left(1+i\frac{z}{z_r}\right) \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{8\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{5R^2}{\omega(z)^2}+\frac{12R^4}{\omega(z)^4}-\frac{34R^6}{3\omega(z)^6}+\frac{6R^8}{\omega(z)^8}-\frac{10R^{10}}{9\omega(z)^{10}}\right)$$

$$\langle 1,2\,|\,5,0\rangle = \langle 2,1\,|\,0,5\rangle = -\frac{15^{1/2}R^4 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2\omega_0^4\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{4R^2}{3\omega(z)^2}+\frac{R^4}{3\omega(z)^4}\right)$$

$$\langle 1,2\,|\,5,2\rangle = \langle 2,1\,|\,2,5\rangle = \frac{15^{1/2}R^4 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{8^{1/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{2R^2}{\omega(z)^2}+\frac{5R^4}{3\omega(z)^4}-\frac{2R^6}{5\omega(z)^6}\right)$$

$$\langle 1,2\,|\,5,4\rangle = \langle 2,1\,|\,4,5\rangle = -\frac{45^{1/2}R^4\left(1+i\frac{z}{z_r}\right) \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{32^{1/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{8R^2}{3\omega(z)^2}+\frac{10R^4}{3\omega(z)^4}-\frac{8R^6}{5\omega(z)^6}+\frac{2R^8}{9\omega(z)^8}\right)$$

$$\langle 0,2\,|\,0,6\rangle = \langle 2,0\,|\,6,0\rangle = \frac{5^{1/2}R^2\left(1+i\frac{z}{z_r}\right) \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{8^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{3R^2}{\omega(z)^2}+\frac{6R^4}{\omega(z)^4}-\frac{7R^6}{3\omega(z)^6}\right)$$

$$\langle 0,2\,|\,6,0\rangle = \langle 2,0\,|\,0,6\rangle = \frac{5^{1/2}R^2\left(1+i\frac{z}{z_r}\right) \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{8^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{3R^2}{\omega(z)^2}+\frac{2R^4}{\omega(z)^4}-\frac{R^6}{3\omega(z)^6}\right)$$

$$\langle 2,2\,|\,0,6\rangle = \langle 2,2\,|\,6,0\rangle = -\frac{5^{1/2}R^2 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{4\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}\left(1-\frac{4R^2}{\omega(z)^2}+\frac{8R^4}{\omega(z)^4}-\frac{16R^6}{3\omega(z)^6}+\frac{14R^{10}}{15\omega(z)^{10}}\right)$$

$$\langle 1,2\,|\,1,6\rangle = \langle 2,1\,|\,6,1\rangle = \frac{5^{1/2}R^4 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{8^{1/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{2R^2}{\omega(z)^2}+\frac{3R^4}{\omega(z)^4}-\frac{14R^6}{15\omega(z)^6}\right)$$

$$\langle 0,2\,|\,2,6\rangle = \langle 2,0\,|\,6,2\rangle = -\frac{5^{1/2}R^2\left(1+i\frac{z}{z_r}\right)^2}{4\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{4R^2}{\omega(z)^2}+\frac{8R^4}{\omega(z)^4}-\frac{16R^6}{3\omega(z)^6}+\frac{14R^8}{15\omega(z)^8}\right)$$

$$\langle 0,2\,|\,6,2\rangle = \langle 2,0\,|\,2,6\rangle = -\frac{5^{1/2}R^2\left(1+i\frac{z}{z_r}\right)^2 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{4\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{4R^2}{\omega(z)^2}+\frac{16R^4}{3\omega(z)^4}-\frac{8R^6}{3\omega(z)^6}+\frac{2R^8}{5\omega(z)^8}\right)$$

$$\langle 2,2\,|\,2,6\rangle = \langle 2,2\,|\,6,2\rangle = \frac{5^{1/2}R^2\left(1+i\frac{z}{z_r}\right) \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{32^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{5R^2}{\omega(z)^2}+\frac{12R^4}{\omega(z)^4}-\frac{34R^6}{3\omega(z)^6}+\frac{82R^8}{15\omega(z)^8}-\frac{14R^{10}}{15\omega(z)^{10}}\right)$$

$$\langle 1,2\,|\,3,6\rangle = \langle 2,1\,|\,6,3\rangle = -\frac{15^{1/2}R^4\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{4\omega_0^4\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{8R^2}{3\omega(z)^2}+\frac{4R^4}{\omega(z)^4}-\frac{32R^6}{15\omega(z)^6}+\frac{14R^8}{45\omega(z)^8}\right)$$

$$\langle 0,2\,|\,4,6\rangle = \langle 2,0\,|\,6,4\rangle = \frac{15^{1/2}R^2\left(1+i\frac{z}{z_r}\right)^3\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{8\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{5R^2}{\omega(z)^2}+\frac{32R^4}{3\omega(z)^4}-\frac{28R^6}{3\omega(z)^6}+\frac{46R^8}{15\omega(z)^8}-\frac{14R^{10}}{45\omega(z)^{10}}\right)$$

$$\langle 0,2\,|\,6,4\rangle = \langle 2,0\,|\,4,6\rangle = \frac{15^{1/2}R^2\left(1+i\frac{z}{z_r}\right)^3\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{8\omega_0^2\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{5R^2}{\omega(z)^2}+\frac{28R^4}{3\omega(z)^4}-\frac{22R^6}{3\omega(z)^6}+\frac{34R^8}{15\omega(z)^8}-\frac{2R^{10}}{9\omega(z)^{10}}\right)$$

$$\langle 2,2\,|\,4,6\rangle = \langle 2,2,\,|\,6,4\rangle$$
$$= -\frac{15^{1/2}R^2\left(1+i\frac{z}{z_r}\right)^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{128^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{6R^2}{\omega(z)^2}+\frac{50R^4}{3\omega(z)^4}-\frac{20R^6}{\omega(z)^6}+\frac{66R^8}{5\omega(z)^8}-\frac{188R^{10}}{45\omega(z)^{10}}+\frac{4R^{12}}{9\omega(z)^{12}}\right)$$

$$\langle 1,2\,|\,5,6\rangle = \langle 2,1\,|\,6,5\rangle = \frac{75^{1/2}R^4\left(1+i\frac{z}{z_r}\right)^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{8\omega_0^4\left(1-i\frac{z}{z_r}\right)^6}\left(1-\frac{10R^2}{3\omega(z)^2}+\frac{16R^4}{3\omega(z)^4}-\frac{56R^6}{15\omega(z)^6}+\frac{46R^8}{45\omega(z)^8}-\frac{4R^{10}}{45\omega(z)^{10}}\right)$$

$$\langle 0,2\,|\,6,6\rangle = \langle 2,0\,|\,6,6\rangle = -\frac{5\left(1+i\frac{z}{z_r}\right)^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{128^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^6}\left(1-\frac{6R^2}{\omega(z)^2}+\frac{14R^4}{\omega(z)^4}-\frac{44R^6}{3\omega(z)^6}+\frac{34R^8}{5\omega(z)^8}-\frac{4R^{10}}{3\omega(z)^{10}}+\frac{4R^{12}}{45\omega(z)^{12}}\right)$$

$$\langle 2,2\,|\,6,6\rangle = \frac{5\left(1+i\frac{z}{z_r}\right)^3\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{16\omega_0^2\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{7R^2}{\omega(z)^2}+\frac{22R^4}{\omega(z)^4}-\frac{95R^6}{3\omega(z)^6}+\frac{382R^8}{15\omega(z)^8}-\frac{166R^{10}}{15\omega(z)^{10}}+\frac{20R^{12}}{9\omega(z)^{12}}-\frac{7R^{14}}{45\omega(z)^{14}}\right)$$

$$\langle 0,3\,|\,0,3\rangle = \langle 3,0\,|\,3,0\rangle = 1-\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}\left(1+\frac{2R^2}{\omega(z)^2}-\frac{R^4}{\omega(z)^4}+\frac{10R^6}{3\omega(z)^6}\right)$$

$$\langle 2,3\,|\,0,3\rangle = \langle 3,2\,|\,3,0\rangle = -\frac{3R^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{1/2}\omega_0^4\left(1+i\frac{z}{z_r}\right)^3\left(1-i\frac{z}{z_r}\right)}\left(1-\frac{4R^2}{3\omega(z)^2}+\frac{5R^4}{9\omega(z)^4}\right)$$

$$\langle 1,3\,|\,1,3\rangle = \langle 3,1\,|\,3,1\rangle = 1-\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}\left(1+\frac{2R^2}{\omega(z)^2}+\frac{2R^4}{\omega(z)^4}-\frac{2R^6}{3\omega(z)^6}+\frac{5R^8}{3\omega(z)^8}\right)$$

$$\langle 1,3\,|\,3,1\rangle = \langle 3,1\,|\,1,3\rangle = \frac{2R^6\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{\omega(z)^6}\left(1-\frac{R^2}{2\omega(z)^2}\right)$$

$$\langle 3,3\,|\,1,3\rangle = \langle 3,3\,|\,3,1\rangle = -\frac{6^{1/2}R^6\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{\omega_0^6\left(1+i\frac{z}{z_r}\right)^4\left(1\ 67\ i\frac{z}{z_r}\right)^2}\left(1-\frac{R^2}{\omega(z)^2}+\frac{R^4}{3\omega(z)^4}\right)$$

$$\langle 0,3\,|\,2,3\rangle = \langle 3,0\,|\,3,2\rangle = -\frac{3R^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2^{1/2}\omega_0^4\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{4R^2}{3\omega(z)^2}+\frac{5R^4}{9\omega(z)^4}\right)$$

$$\langle 2,3\,|\,2,3\rangle = \langle 3,2\,|\,3,2\rangle = 1-\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}\left(1+\frac{2R^2}{\omega(z)^2}+\frac{R^4}{2\omega(z)^4}+\frac{13R^6}{3\omega(z)^6}-\frac{13R^8}{6\omega(z)^8}+\frac{R^{10}}{\omega(z)^{10}}\right)$$

$$\langle 3,3\,|\,3,3\rangle = 1-\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}\left(1+\frac{2R^2}{\omega(z)^2}+\frac{2R^4}{\omega(z)^4}-\frac{5R^6}{3\omega(z)^6}+\frac{31R^8}{6\omega(z)^8}-\frac{7R^{10}}{3\omega(z)^{10}}+\frac{5R^{12}}{9\omega(z)^{12}}\right)$$

$$\langle 2,3\,|\,4,1\rangle = \langle 3,2\,|\,1,4\rangle = \frac{3R^4 e^{\frac{-2R^2}{\omega(z)^2}}}{8^{1/2}\omega(z)^4}\left(1-\frac{2R^2}{\omega(z)^2}+\frac{7R^4}{3\omega(z)^4}-\frac{2R^6}{3\omega(z)^6}\right)$$

$$\langle 0,3\,|\,4,3\rangle = \langle 3,0\,|\,3,4\rangle = \frac{27^{1/2}R^4 e^{\frac{-2R^2}{\omega(z)^2}}}{8^{1/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{2R^2}{\omega(z)^2}+\frac{11R^4}{9\omega(z)^4}-\frac{2R^6}{9\omega(z)^6}\right)$$

$$\langle 2,3\,|\,4,3\rangle = \langle 3,2\,|\,3,4\rangle = -\frac{27^{1/2}R^4 e^{\frac{-2R^2}{\omega(z)^2}}}{4\omega_0^4\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{8R^2}{3\omega(z)^2}+\frac{32R^4}{9\omega(z)^4}-\frac{16R^6}{9\omega(z)^6}+\frac{10R^8}{27\omega(z)^8}\right)$$

$$\langle 0,3\,|\,0,5\rangle = \langle 3,0\,|\,5,0\rangle = -\frac{45^{1/2} e^{\frac{-2R^2}{\omega(z)^2}}}{2\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{4R^2}{3\omega(z)^2}+\frac{7R^4}{9\omega(z)^4}\right)$$

$$\langle 2,3\,|\,0,5\rangle = \langle 3,2\,|\,5,0\rangle = \frac{45^{1/2}R^4 e^{\frac{-2R^2}{\omega(z)^2}}}{8^{1/2}\omega(z)^4}\left(1-\frac{2R^2}{\omega(z)^2}+\frac{13R^4}{9\omega(z)^4}-\frac{14R^6}{45\omega(z)^6}\right)$$

$$\langle 1,3\,|\,1,5\rangle = \langle 3,1\,|\,5,1\rangle = -\frac{5^{1/2}R^6 e^{\frac{-2R^2}{\omega(z)^2}}}{\omega_0^6\left(1+i\frac{z}{z_r}\right)^2\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{R^2}{\omega(z)^2}+\frac{7R^4}{15\omega(z)^4}\right)$$

$$\langle 1,3\,|\,5,1\rangle = \langle 3,1\,|\,1,5\rangle = -\frac{5^{1/2}R^6 e^{\frac{-2R^2}{\omega(z)^2}}}{\omega_0^6\left(1+i\frac{z}{z_r}\right)^2\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{R^2}{\omega(z)^2}+\frac{R^4}{5\omega(z)^4}\right)$$

$$\langle 3,3\,|\,1,5\rangle = \langle 3,3\,|\,5,1\rangle = \frac{15^{1/2}R^6 e^{\frac{-2R^2}{\omega(z)^2}}}{2^{1/2}\omega(z)^6}\left(1-\frac{3R^2}{2\omega(z)^2}+\frac{13R^4}{15\omega(z)^4}-\frac{7R^6}{45\omega(z)^6}\right)$$

$$\langle 0,3\,|\,2,5\rangle = \langle 3,0\,|\,5,2\rangle = \frac{45^{1/2}R^4 e^{\frac{-2R^2}{\omega(z)^2}}}{8^{1/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{2R^2}{\omega(z)^2}+\frac{13R^4}{9\omega(z)^4}-\frac{14R^6}{45\omega(z)^6}\right)$$

$$\langle 2,3\,|\,2,5\rangle = \langle 3,2\,|\,5,2\rangle = -\frac{45^{1/2}R^4 e^{\frac{-2R^2}{\omega(z)^2}}}{4\omega_0^4\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{8R^2}{3\omega(z)^2}+\frac{28R^4}{9\omega(z)^4}-\frac{64R^6}{45\omega(z)^6}+\frac{14R^8}{45\omega(z)^8}\right)$$

$$\langle 1,3\,|\,3,5\rangle = \langle 3,1\,|\,5,3\rangle = \frac{15^{1/2}R^6 e^{\frac{-2R^2}{\omega(z)^2}}}{2^{1/2}\omega_0^6\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{3R^2}{2\omega(z)^2}+\frac{13R^4}{15\omega(z)^4}-\frac{7R^6}{45\omega(z)^6}\right)$$

$$\langle 3,1\,|\,3,5\rangle = \langle 1,3\,|\,5,3\rangle = \frac{15^{1/2}R^6 e^{\frac{-2R^2}{\omega(z)^2}}}{2^{1/2}\omega_0^6\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{3R^2}{2\omega(z)^2}+\frac{11R^4}{15\omega(z)^4}-\frac{R^6}{9\omega(z)^6}\right)$$

$$\langle 3,3\,|\,3,5\rangle = \langle 3,3\,|\,5,3\rangle = -\frac{45^{1/2}R^6 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2\omega_0^6\left(1+i\frac{z}{z_r}\right)^2\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{2R^2}{\omega(z)^2}+\frac{8R^4}{5\omega(z)^4}-\frac{8R^6}{15\omega(z)^6}+\frac{2R^8}{27\omega(z)^8}\right)$$

$$\langle 0,3\,|\,4,5\rangle = \langle 3,0\,|\,5,4\rangle = -\frac{135^{1/2}R^4\left(1+\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{32^{1/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{8R^2}{3\omega(z)^2}+\frac{22R^4}{9\omega(z)^4}-\frac{8R^6}{9\omega(z)^6}+\frac{14R^8}{135\omega(z)^8}\right)$$

$$\langle 2,3\,|\,4,5\rangle = \langle 3,2\,|\,5,4\rangle = \frac{135^{1/2}R^4 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{8\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{10R^2}{3\omega(z)^2}+\frac{46R^4}{9\omega(z)^4}-\frac{52R^6}{15\omega(z)^6}+\frac{158R^8}{135\omega(z)^8}-\frac{4R^{10}}{27\omega(z)^{10}}\right)$$

$$\langle 1,3\,|\,5,5\rangle = \langle 3,1\,|\,5,5\rangle = -\frac{75^{1/2}R^6 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{8^{1/2}\omega_0^6\left(1-i\frac{z}{z_r}\right)^6}\left(1-\frac{2R^2}{\omega(z)^2}+\frac{22R^4}{15\omega(z)^4}-\frac{4R^6}{9\omega(z)^6}+\frac{2R^8}{45\omega(z)^8}\right)$$

$$\langle 3,3\,|\,5,5\rangle = \frac{15R^6 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{4\omega_0^6\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{5R^2}{2\omega(z)^2}+\frac{38R^4}{15\omega(z)^4}-\frac{6R^6}{5\omega(z)^6}+\frac{38R^8}{135\omega(z)^8}-\frac{7R^{10}}{270\omega(z)^{10}}\right)$$

$$\langle 0,3\,|\,6,1\rangle = \langle 3,0\,|\,1,6\rangle = \frac{15^{1/2}R^4 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{8^{1/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{2R^2}{\omega(z)^2}+\frac{R^4}{\omega(z)^4}-\frac{2R^6}{15\omega(z)^6}\right)$$

$$\langle 2,3\,|\,6,1\rangle = \langle 3,2\,|\,1,6\rangle = -\frac{15^{1/2}R^4 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{4\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{8R^2}{3\omega(z)^2}+\frac{4R^4}{\omega(z)^4}-\frac{32R^6}{15\omega(z)^6}+\frac{14R^8}{45\omega(z)^8}\right)$$

$$\langle 0,3\,|\,6,3\rangle = \langle 3,0\,|\,3,6\rangle = -\frac{45^{1/2}R^4\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{4\omega_0^4\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{8R^2}{3\omega(z)^2}+\frac{20R^4}{9\omega(z)^4}-\frac{32R^6}{45\omega(z)^6}+\frac{2R^8}{27\omega(z)^8}\right)$$

$$\langle 2,3\,|\,6,3\rangle = \langle 3,2\,|\,3,6\rangle = \frac{45^{1/2}R^4 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{32^{1/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{10R^2}{3\omega(z)^2}+\frac{50R^4}{9\omega(z)^4}-\frac{4R^6}{\omega(z)^6}+\frac{58R^8}{45\omega(z)^8}-\frac{4R^{10}}{27\omega(z)^{10}}\right)$$

$$\langle 0,3\,|\,6,5\rangle = \langle 3,0\,|\,5,6\rangle = \frac{15R^4\left(1+i\frac{z}{z_r}\right)^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{8\omega_0^4\left(1-i\frac{z}{z_r}\right)^6}\left(1-\frac{10R^2}{3\omega(z)^2}+\frac{34R^4}{9\omega(z)^4}-\frac{28R^6}{15\omega(z)^6}+\frac{2R^8}{5\omega(z)^8}-\frac{4R^{10}}{135\omega(z)^{10}}\right)$$

$$\langle 2,3\,|\,6,5\rangle = \langle 3,2\,|\,5,6\rangle =$$

$$-\frac{15R^4\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{128^{1/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{4R^2}{\omega(z)^2}+\frac{67R^4}{9\omega(z)^4}-\frac{296R^6}{45\omega(z)^6}+\frac{134R^8}{45\omega(z)^6}-\frac{88R^{10}}{135\omega(z)^{10}}+\frac{7R^{12}}{135\omega(z)^{12}}\right)$$

$$\langle 0,4\,|\,0,4\rangle = 1-\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}\left(1+\frac{5R^2}{4\omega(z)^2}+\frac{17R^4}{4\omega(z)^4}-\frac{25R^6}{6\omega(z)^6}+\frac{35R^8}{12\omega(z)^8}\right)$$

$$\langle 0,4\,|\,4,0\rangle = \langle 4,0\,|\,0,4\rangle = \frac{3R^2 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{4\omega(z)^2}\left(1-\frac{3R^2}{\omega(z)^2}+\frac{2R^4}{\omega(z)^4}-\frac{R^6}{3\omega(z)^6}\right)$$

$$\langle 2,4\,|\,0,4\rangle = \langle 4,2\,|\,4,0\rangle = -\frac{3R^2 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{32^{1/2}\omega_0^2\left(1+i\frac{z}{z_r}\right)^2}\left(1-\frac{4R^2}{\omega(z)^2}+\frac{28R^4}{3\omega(z)^4}-\frac{20R^6}{3\omega(z)^6}+\frac{14R^8}{9\omega(z)^8}\right)$$

$$\langle 2,4\,|\,4,0\rangle = \langle 4,2\,|\,0,4\rangle = -\frac{3R^2 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{32^{1/2}\omega_0^2\left(1+i\frac{z}{z_r}\right)^2}\left(1-\frac{4R^2}{\omega(z)^2}+\frac{20R^4}{3\omega(z)^4}-\frac{4R^6}{\omega(z)^6}+\frac{2R^8}{3\omega(z)^8}\right)$$

$$\langle 4,4\,|\,0,4\rangle = \langle 4,4\,|\,4,0\rangle = \frac{27^{1/2}R^2\left(1-i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{128^{1/2}\omega_0^2\left(1+i\frac{z}{z_r}\right)^3}\left(1-\frac{5R^2}{\omega(z)^2}+\frac{12R^4}{\omega(z)^4}-\frac{34R^6}{3\omega(z)^6}+\frac{38R^8}{9\omega(z)^8}-\frac{14R^{10}}{27\omega(z)^{10}}\right)$$

$$\langle 1,4\,|\,1,4\rangle = \langle 4,1\,|\,4,1\rangle = 1-\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}\left(1+\frac{2R^2}{\omega(z)^2}+\frac{5R^4}{4\omega(z)^4}+\frac{17R^6}{6\omega(z)^6}-\frac{25R^8}{12\omega(z)^8}+\frac{7R^{10}}{6\omega(z)^{10}}\right)$$

$$\langle 3,4\,|\,1,4\rangle = \langle 4,3\,|\,4,1\rangle = -\frac{27^{1/2}R^4 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{32^{1/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)\left(1+i\frac{z}{z_r}\right)^3}\left(1-\frac{8R^2}{3\omega(z)^2}+\frac{14R^4}{3\omega(z)^4}-\frac{8R^6}{3\omega(z)^6}+\frac{14R^8}{27\omega(z)^8}\right)$$

$$\langle 2,4\,|\,2,4\rangle = \langle 4,2\,|\,4,2\rangle$$
$$= 1-\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}\left(1+\frac{13R^2}{8\omega(z)^2}+\frac{31R^4}{8\omega(z)^4}-\frac{11R^6}{3\omega(z)^6}+\frac{17R^8}{3\omega(z)^8}-\frac{29R^{10}}{12\omega(z)^{10}}+\frac{7R^{12}}{12\omega(z)^{12}}\right)$$

$$\langle 2,4\,|\,4,2\rangle = \langle 4,2\,|\,2,4\rangle = \frac{3R^2 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{8\omega(z)^2}\left(1-\frac{5R^2}{\omega(z)^2}+\frac{12R^4}{\omega(z)^4}-\frac{34R^6}{3\omega(z)^6}+\frac{6R^8}{\omega(z)^8}-\frac{10R^{10}}{9\omega(z)^{10}}\right)$$

$$\langle 4,4\,|\,2,4\rangle = \langle 4,4\,|\,4,2\rangle$$
$$= -\frac{27^{1/2}R^2 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{16\omega_0^2\left(1+i\frac{z}{z_r}\right)^2}\left(1-\frac{6R^2}{\omega(z)^2}+\frac{18R^4}{\omega(z)^4}-\frac{68R^6}{3\omega(z)^6}+\frac{146R^8}{9\omega(z)^8}-\frac{148R^{10}}{27\omega(z)^{10}}+\frac{20R^{12}}{27\omega(z)^{12}}\right)$$

$$\langle 3,4\,|\,3,4\rangle = \langle 4,3\,|\,4,3\rangle$$
$$= 1-\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}\left(1+\frac{2R^2}{\omega(z)^2}+\frac{7R^4}{8\omega(z)^4}+\frac{61R^6}{12\omega(z)^6}-\frac{19R^8}{3\omega(z)^8}+\frac{17R^{10}}{3\omega(z)^{10}}-\frac{67R^{12}}{36\omega(z)^{12}}+\frac{5R^{14}}{18\omega(z)^{14}}\right)$$

$$\langle 4,4\,|\,4,4\rangle = 1-\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}\left(1+\frac{55R^2}{32\omega(z)^2}+\frac{127R^4}{32\omega(z)^4}-\frac{269R^6}{48\omega(z)^6}+\frac{1099R^8}{96\omega(z)^8}\right.$$
$$\left.-\frac{463R^{10}}{48\omega(z)^{10}}+\frac{713R^{12}}{144\omega(z)^{12}}-\frac{85R^{14}}{72\omega(z)^{14}}+\frac{35R^{16}}{288\omega(z)^{16}}\right)$$

$$\langle 1,4\,|\,5,0\rangle = \langle 4,1\,|\,0,5\rangle = \frac{45^{1/2}R^4 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{4\omega(z)^4}\left(1-\frac{2R^2}{\omega(z)^2}+\frac{R^4}{\omega(z)^4}-\frac{2R^6}{15\omega(z)^6}\right)$$

$$\langle 3,4\,|\,5,0\rangle = \langle 4,3\,|\,0,5\rangle = -\frac{135^{1/2}R^4 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{32^{1/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)\left(1+i\frac{z}{z_r}\right)^3}\left(1-\frac{8R^2}{3\omega(z)^2}+\frac{22R^4}{9\omega(z)^4}-\frac{8R^6}{9\omega(z)^6}+\frac{14R^8}{135\omega(z)^8}\right)$$

$$\langle 1,4\,|\,5,2\rangle = \langle 4,1\,|\,2,5\rangle = -\frac{45^{1/2}R^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{32^{1/2}\omega_0^4\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{8R^2}{3\omega(z)^2}+\frac{10R^4}{3\omega(z)^4}-\frac{8R^6}{5\omega(z)^6}+\frac{2R^8}{9\omega(z)^8}\right)$$

$$\langle 3,4\,|\,5,2\rangle = \langle 4,3\,|\,2,5\rangle = \frac{135^{1/2}R^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{8\omega(z)^4}\left(1-\frac{10R^2}{3\omega(z)^2}+\frac{46R^4}{9\omega(z)^4}-\frac{52R^6}{15\omega(z)^6}+\frac{158R^8}{135\omega(z)^8}-\frac{4R^{10}}{27\omega(z)^{10}}\right)$$

$$\langle 1,4\,|\,5,4\rangle = \langle 4,1\,|\,4,5\rangle = \frac{135^{1/2}R^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{128^{1/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{10R^2}{3\omega(z)^2}+\frac{6R^4}{\omega(z)^4}-\frac{68R^6}{15\omega(z)^6}+\frac{38R^8}{27\omega(z)^8}-\frac{4R^{10}}{27\omega(z)^{10}}\right)$$

$$\langle 3,4\,|\,5,4\rangle = \langle 4,3\,|\,4,5\rangle$$

$$= -\frac{405^{1/2}R^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{16\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{4R^2}{\omega(z)^2}+\frac{73R^4}{9\omega(z)^2}-\frac{344R^6}{45\omega(z)^6}+\frac{506R^8}{135\omega(z)^8}-\frac{8R^{10}}{9\omega(z)^{10}}+\frac{7R^{12}}{81\omega(z)^{12}}\right)$$

$$\langle 0,4\,|\,0,6\rangle = \langle 4,0\,|\,6,0\rangle = -\frac{15^{1/2}R^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{32^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}\left(1-\frac{4R^2}{\omega(z)^2}+\frac{12R^4}{\omega(z)^4}-\frac{28R^6}{3\omega(z)^6}+\frac{14R^8}{5\omega(z)^8}\right)$$

$$\langle 4,0\,|\,0,6\rangle = \langle 0,4\,|\,6,0\rangle = -\frac{15^{1/2}R^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{32^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}\left(1-\frac{4R^2}{\omega(z)^2}+\frac{4R^4}{\omega(z)^4}-\frac{4R^6}{3\omega(z)^6}+\frac{2R^8}{15\omega(z)^8}\right)$$

$$\langle 2,4\,|\,0,6\rangle = \langle 4,2\,|\,6,0\rangle = \frac{15^{1/2}R^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{8\omega(z)^2}\left(1-\frac{5R^2}{\omega(z)^2}+\frac{44R^4}{3\omega(z)^4}-\frac{46R^6}{3\omega(z)^6}+\frac{98R^8}{15\omega(z)^8}-\frac{14R^{10}}{15\omega(z)^{10}}\right)$$

$$\langle 4,2\,|\,0,6\rangle = \langle 2,4\,|\,6,0\rangle = \frac{15^{1/2}R^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{8\omega(z)^2}\left(1-\frac{5R^2}{\omega(z)^2}+\frac{32R^4}{3\omega(z)^4}-\frac{28R^6}{3\omega(z)^6}+\frac{46R^8}{15\omega(z)^8}-\frac{14R^{10}}{45\omega(z)^{10}}\right)$$

$$\langle 4,4,\,|\,0,6\rangle = \langle 4,4\,|\,6,0\rangle = -\frac{45^{1/2}R^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{16\omega_0^2\left(1+i\frac{z}{z_r}\right)^2}\left(1-\frac{6R^2}{\omega(z)^2}+\frac{18R^4}{\omega(z)^4}-\frac{68R^6}{3\omega(z)^6}+\frac{38R^8}{3\omega(z)^8}-\frac{28R^{10}}{9\omega(z)^{10}}+\frac{4R^{12}}{15\omega(z)^{12}}\right)$$

$$\langle 1,4\,|\,1,6\rangle = \langle 4,1\,|\,6,1\rangle = -\frac{15^{1/2}R^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{32^{1/2}\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{8R^2}{3\omega(z)^2}+\frac{6R^4}{\omega(z)^4}-\frac{56R^6}{15\omega(z)^6}+\frac{14R^8}{15\omega(z)^8}\right)$$

$$\langle 3,4\,|\,1,6\rangle = \langle 4,3\,|\,6,1\rangle = \frac{45^{1/2}R^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{8\omega(z)^4}\left(1-\frac{10R^2}{3\omega(z)^2}+\frac{22R^4}{3\omega(z)^4}-\frac{92R^6}{15\omega(z)^6}+\frac{98R^8}{45\omega(z)^8}-\frac{4R^{10}}{15\omega(z)^{10}}\right)$$

$$\langle 0,4\,|\,2,6\rangle = \langle 4,0\,|\,6,2\rangle$$

$$= \frac{15^{1/2}R^2\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{8\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{5R^2}{\omega(z)^2}+\frac{44R^4}{3\omega(z)^4}-\frac{46R^6}{3\omega(z)^6}+\frac{98R^8}{15\omega(z)^8}-\frac{14R^{10}}{15\omega(z)^{10}}\right)$$

$$\langle 4,0\,|\,2,6\rangle = \langle 0,4\,|\,6,2\rangle = \frac{15^{1/2}R^2\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{8\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{5R^2}{\omega(z)^2}+\frac{28R^4}{3\omega(z)^4}-\frac{22R^6}{3\omega(z)^6}+\frac{34R^8}{15\omega(z)^8}-\frac{2R^{10}}{9\omega(z)^{10}}\right)$$

$$\langle 2,4\,|\,2,6\rangle = \langle 4,2\,|\,6,2\rangle$$

$$= -\frac{15^{1/2}R^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{128^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}\left(1-\frac{6R^2}{\omega(z)^2}+\frac{58R^4}{3\omega(z)^4}-\frac{76R^6}{3\omega(z)^6}+\frac{262R^8}{15\omega(z)^8}-\frac{28R^{10}}{5\omega(z)^{10}}+\frac{4R^{12}}{5\omega(z)^{12}}\right)$$

$$\langle 4,4\,|\,2,6\rangle = \langle 4,4\,|\,6,2\rangle = \frac{45^{1/2}R^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{512^{1/2}\omega(z)^2}\left(1-\frac{7R^2}{\omega(z)^2}+\frac{74R^4}{3\omega(z)^4}-\frac{115R^6}{3\omega(z)^6}\right.$$
$$\left.+\frac{502R^8}{15\omega(z)^8}-\frac{698R^{10}}{45\omega(z)^{10}}+\frac{164R^{12}}{45\omega(z)^{12}}-\frac{R^{14}}{3\omega(z)^{14}}\right)$$

$$\langle 1,4\,|\,3,6\rangle = \langle 4,1\,|\,6,3\rangle = \frac{45^{1/2}R^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{8\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{10R^2}{3\omega(z)^2}+\frac{22R^4}{3\omega(z)^4}-\frac{92R^6}{15\omega(z)^6}+\frac{98R^8}{45\omega(z)^8}-\frac{4R^{10}}{15\omega(z)^{10}}\right)$$

$$\langle 3,4\,|\,3,6\rangle = \langle 4,3\,|\,6,3\rangle = -\frac{135^{1/2}R^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{128^{1/2}\omega_0^4\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{4R^2}{\omega(z)^2}+\frac{83R^4}{9\omega(z)^4}-\frac{424R^6}{45\omega(z)^6}\right.$$
$$\left.+\frac{214R^8}{45\omega(z)^8}-\frac{152R^{10}}{135\omega(z)^{10}}+\frac{R^{12}}{9\omega(z)^{12}}\right)$$

$$\langle 0,4\,|\,4,6\rangle = \langle 4,0\,|\,6,4\rangle = -\frac{45^{1/2}R^2\left(1+i\frac{z}{z_r}\right)^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{16\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{6R^2}{\omega(z)^2}+\frac{18R^4}{\omega(z)^4}-\frac{68R^6}{3\omega(z)^6}\right.$$
$$\left.+\frac{38R^8}{3\omega(z)^8}-\frac{28R^{10}}{9\omega(z)^{10}}+\frac{4R^{12}}{15\omega(z)^{12}}\right)$$

$$\langle 4,0\,|\,4,6\rangle = \langle 0,4\,|\,6,4\rangle = -\frac{45^{1/2}R^2\left(1+i\frac{z}{z_r}\right)^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{16\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{6R^2}{\omega(z)^2}+\frac{46R^4}{3\omega(z)^4}\right.$$
$$\left.-\frac{52R^6}{3\omega(z)^6}+\frac{394R^8}{45\omega(z)^8}-\frac{52R^{10}}{27\omega(z)^{10}}+\frac{4R^{12}}{27\omega(z)^{12}}\right)$$

$$\langle 2,4\,|\,4,6\rangle = \langle 4,2\,|\,6,4\rangle = \frac{45^{1/2}R^2\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{512^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{7R^2}{\omega(z)^2}+\frac{74R^4}{3\omega(z)^4}-\frac{115R^6}{3\omega(z)^6}+\frac{502R^8}{15\omega(z)^8}\right.$$
$$\left.-\frac{698R^{10}}{45\omega(z)^{10}}+\frac{164R^{12}}{45\omega(z)^{12}}-\frac{R^{14}}{3\omega(z)^{14}}\right)$$

$$\langle 4,2\,|\,4,6\rangle = \langle 2,4\,|\,6,4\rangle = \frac{45^{1/2}R^2\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{512^{1/2}\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{7R^2}{\omega(z)^2}+\frac{70R^4}{3\omega(z)^4}-\frac{35R^6}{\omega(z)^6}\right.$$
$$\left.+\frac{1354R^8}{45\omega(z)^8}-\frac{626R^{10}}{45\omega(z)^{10}}+\frac{28R^{12}}{9\omega(z)^{12}}-\frac{7R^{14}}{27\omega(z)^{14}}\right)$$

$$\langle 4,4\,|\,4,6\rangle = \langle 4,4\,|\,6,4\rangle = -\frac{135^{1/2}R^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2048^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}\left(1-\frac{8R^2}{\omega(z)^2}+\frac{32R^4}{\omega(z)^4}-\frac{176R^6}{3\omega(z)^6}+\frac{572R^8}{9\omega(z)^8}\right.$$
$$\left.-\frac{5296R^{10}}{135\omega(z)^{10}}+\frac{608R^{12}}{45\omega(z)^{12}}-\frac{64R^{14}}{27\omega(z)^{14}}+\frac{14R^{16}}{81\omega(z)^{16}}\right)$$

$$\langle 1,4\,|\,5,6\rangle = \langle 4,1\,|\,6,5\rangle =$$

$$-\frac{15R^4\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{16\omega_0^4\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{4R^2}{\omega(z)^2}+\frac{9R^4}{\omega(z)^4}-\frac{136R^6}{15\omega(z)^6}+\frac{38R^8}{9\omega(z)^8}-\frac{8R^{10}}{9\omega(z)^{10}}+\frac{R^{12}}{15\omega(z)^{12}}\right)$$

$$\langle 3,4\,|\,5,6\rangle = \langle 4,3\,|\,6,5\rangle =$$

$$\frac{675^{1/2}R^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{512^{1/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{14R^2}{3\omega(z)^2}+\frac{103R^4}{9\omega(z)^4}-\frac{122R^6}{9\omega(z)^6}+\frac{1154R^8}{135\omega(z)^8}-\frac{388R^{10}}{135\omega(z)^{10}}+\frac{67R^{12}}{135\omega(z)^{12}}-\frac{14R^{14}}{405\omega(z)^{14}}\right)$$

$$\langle 0,4\,|\,6,6\rangle = \langle 4,0\,|\,6,6\rangle = \frac{75^{1/2}R^2\left(1+i\frac{z}{z_r}\right)^3\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{512^{1/2}\omega_0^2\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{7R^2}{\omega(z)^2}+\frac{22R^4}{\omega(z)^4}-\frac{95R^6}{3\omega(z)^6}+\frac{326R^8}{15\omega(z)^8}\right.$$
$$\left.-\frac{22R^{10}}{3\omega(z)^{10}}+\frac{52R^{12}}{45\omega(z)^{12}}-\frac{R^{14}}{15\omega(z)^{14}}\right)$$

$$\langle 2,4\,|\,6,6\rangle = \langle 4,2\,|\,6,6\rangle = -\frac{75^{1/2}R^2\left(1+i\frac{z}{z_r}\right)^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{32\omega_0^2\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{8R^2}{\omega(z)^2}+\frac{92R^4}{3\omega(z)^4}-\frac{164R^6}{3\omega(z)^6}+\frac{56R^8}{\omega(z)^8}\right.$$
$$\left.-\frac{1472R^{10}}{45\omega(z)^{10}}+\frac{472R^{12}}{45\omega(z)^{12}}-\frac{76R^{14}}{45\omega(z)^{14}}+\frac{14R^{16}}{135\omega(z)^{16}}\right)$$

$$\langle 4,4\,|\,6,6\rangle = \frac{15R^2\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{64\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{9R^2}{\omega(z)^2}+\frac{40R^4}{\omega(z)^4}-\frac{84R^6}{\omega(z)^6}+\frac{316R^8}{3\omega(z)^8}\right.$$
$$\left.-\frac{3532R^{10}}{45\omega(z)^{10}}+\frac{1552R^{12}}{45\omega(z)^{12}}-\frac{388R^{14}}{45\omega(z)^{14}}+\frac{154R^{16}}{135\omega(z)^{16}}-\frac{14R^{18}}{225\omega(z)^{18}}\right)$$

$$\langle 0,5\,|\,0,5\rangle = \langle 5,0\,|\,5,0\rangle = 1-\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}\left(1+\frac{2R^2}{\omega(z)^2}-\frac{7R^4}{4\omega(z)^4}+\frac{53R^6}{6\omega(z)^6}-\frac{77R^8}{12\omega(z)^8}+\frac{21R^{10}}{10\omega(z)^{10}}\right)$$

$$\langle 2,5\,|\,0,5\rangle = \langle 5,2\,|\,5,0\rangle = -\frac{15R^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{32^{1/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)\left(1+i\frac{z}{z_r}\right)^3}\left(1-\frac{8R^2}{3\omega(z)^2}+\frac{26R^4}{9\omega(z)^4}-\frac{56R^6}{45\omega(z)^6}+\frac{14R^8}{75\omega(z)^8}\right)$$

$$\langle 4,5\,|\,0,5\rangle = \langle 5,4\,|\,5,0\rangle = \frac{675^{1/2}R^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{128^{1/2}\omega_0^4\left(1+i\frac{z}{z_r}\right)^4}\left(1-\frac{10R^2}{3\omega(z)^2}+\frac{38R^4}{9\omega(z)^4}-\frac{12R^6}{5\omega(z)^6}+\frac{406R^8}{675\omega(z)^8}-\frac{4R^{10}}{75\omega(z)^{10}}\right)$$

$$\langle 1,5\,|\,1,5\rangle = \langle 5,1\,|\,5,1\rangle$$
$$= 1-\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}\left(1+\frac{2R^2}{\omega(z)^2}+\frac{2R^4}{\omega(z)^4}-\frac{7R^6}{6\omega(z)^6}+\frac{53R^8}{12\omega(z)^8}-\frac{77R^{10}}{30\omega(z)^{10}}+\frac{7R^{12}}{10\omega(z)^{12}}\right)$$

$$\langle 1,5\,|\,5,1\rangle = \langle 5,1\,|\,1,5\rangle = \frac{5R^6\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2\omega(z)^6}\left(1-\frac{3R^2}{2\omega(z)^2}+\frac{3R^4}{5\omega(z)^4}-\frac{R^6}{15\omega(z)^6}\right)$$

$$\langle 3,5\,|\,1,5\rangle = -\frac{75^{1/2}R^6\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{8^{1/2}\omega_0^6\left(1-i\frac{z}{z_r}\right)^2\left(1+i\frac{z}{z_r}\right)^4}\left(1-\frac{2R^2}{\omega(z)^2}+\frac{26R^4}{15\omega(z)^4}-\frac{28R^6}{45\omega(z)^6}+\frac{2R^8}{25\omega(z)^8}\right)$$

$$\langle 5,3\,|\,1,5\rangle = \langle 3,5\,|\,5,1\rangle = -\frac{75^{1/2}R^6\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{8^{1/2}\omega_0^6\left(1-i\frac{z}{z_r}\right)^2\left(1+i\frac{z}{z_r}\right)^4}\left(1-\frac{2R^2}{\omega(z)^2}+\frac{22R^4}{15\omega(z)^4}-\frac{4R^6}{9\omega(z)^6}+\frac{2R^8}{45\omega(z)^8}\right)$$

$$\langle 5,5\,|\,1,5\rangle = \langle 5,5\,|\,5,1\rangle$$

$$= \frac{375^{1/2}R^6\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{32^{1/2}\omega_0^6\left(1-i\frac{z}{z_r}\right)\left(1+i\frac{z}{z_r}\right)^5}\left(1-\frac{5R^2}{2\omega(z)^2}+\frac{38R^4}{15\omega(z)^4}-\frac{6R^6}{5\omega(z)^6}+\frac{58R^8}{225\omega(z)^8}-\frac{R^{10}}{50\omega(z)^{10}}\right)$$

$$\langle 2,5\,|\,2,5\rangle = \langle 5,2\,|\,5,2\rangle =$$

$$1-\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}\left(1+\frac{2R^2}{\omega(z)^2}+\frac{R^4}{8\omega(z)^4}+\frac{91R^6}{12\omega(z)^6}-\frac{17R^8}{2\omega(z)^8}+\frac{94R^{10}}{15\omega(z)^{10}}-\frac{119R^{12}}{60\omega(z)^{12}}+\frac{3R^{14}}{10\omega(z)^{14}}\right)$$

$$\langle 4,5\,|\,2,5\rangle = \langle 5,4\,|\,5,2\rangle = -\frac{675^{1/2}R^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{16\omega_0^4\left(1-i\frac{z}{z_r}\right)\left(1+i\frac{z}{z_r}\right)^3}\left(1-\frac{4R^2}{\omega(z)^2}+\frac{65R^4}{9\omega(z)^4}-\frac{56R^6}{9\omega(z)^6}\right.$$
$$\left.+\frac{662R^8}{225\omega(z)^8}-\frac{472R^{10}}{675\omega(z)^{10}}+\frac{R^{12}}{15\omega(z)^{12}}\right)$$

$$\langle 3,5\,|\,3,5\rangle = \langle 5,3\,|\,5,3\rangle = 1-\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}\left(1+\frac{2R^2}{\omega(z)^2}+\frac{2R^4}{\omega(z)^4}-\frac{29R^6}{12\omega(z)^6}+\frac{241R^8}{24\omega(z)^8}\right.$$
$$\left.-\frac{146R^{10}}{15\omega(z)^{10}}+\frac{229R^{12}}{45\omega(z)^{12}}-\frac{109R^{14}}{90\omega(z)^{14}}+\frac{R^{16}}{8\omega(z)^{16}}\right)$$

$$\langle 3,5\,|\,5,3\rangle = \langle 5,3\,|\,3,5\rangle = \frac{15R^6\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{4\omega(z)^6}\left(1-\frac{5R^2}{2\omega(z)^2}+\frac{38R^4}{15\omega(z)^4}-\frac{6R^6}{5\omega(z)^6}+\frac{38R^8}{135\omega(z)^8}-\frac{7R^{10}}{270\omega(z)^{10}}\right)$$

$$\langle 5,5\,|\,3,5\rangle = \langle 5,5\,|\,5,3\rangle = -\frac{1125^{1/2}R^6\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{8\omega_0^6\left(1-i\frac{z}{z_r}\right)^2\left(1+i\frac{z}{z_r}\right)^4}\left(1-\frac{3R^2}{\omega(z)^2}+\frac{19R^4}{5\omega(z)^4}-\frac{12R^6}{5\omega(z)^6}\right.$$
$$\left.+\frac{554R^8}{675\omega(z)^8}-\frac{97R^{10}}{675\omega(z)^{10}}+\frac{7R^{12}}{675\omega(z)^{12}}\right)$$

$$\langle 4,5\,|\,4,5\rangle = \langle 5,4\,|\,5,4\rangle = 1-\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}\left(1+\frac{2R^2}{\omega(z)^2}+\frac{19R^4}{32\omega(z)^4}+\frac{379R^6}{48\omega(z)^6}-\frac{1361R^8}{96\omega(z)^8}\right.$$
$$\left.+\frac{4039R^{10}}{240\omega(z)^{10}}-\frac{1483R^{12}}{144\omega(z)^{12}}+\frac{1303R^{14}}{360\omega(z)^{14}}-\frac{185R^{16}}{288\omega(z)^{16}}+\frac{7R^{18}}{144\omega(z)^{18}}\right)$$

$$\langle 5,5\,|\,5,5\rangle = 1-\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}\left(1+\frac{2R^2}{\omega(z)^2}+\frac{2R^4}{\omega(z)^4}-\frac{161R^6}{48\omega(z)^6}+\frac{1639R^8}{96\omega(z)^8}-\frac{5861R^{10}}{240\omega(z)^{10}}\right.$$
$$\left.+\frac{13939R^{12}}{720\omega(z)^{12}}-\frac{1031R^{14}}{120\omega(z)^{14}}+\frac{3187R^{16}}{1440\omega(z)^{16}}-\frac{217R^{18}}{720\omega(z)^{18}}+\frac{7R^{20}}{400\omega(z)^{20}}\right)$$

$$\langle 0,5\,|\,6,1\rangle = \langle 5,0\,|\,1,6\rangle = -\frac{75^{1/2}R^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{32^{1/2}\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{8R^2}{3\omega(z)^2}+\frac{2R^4}{\omega(z)^4}-\frac{8R^6}{15\omega(z)^6}+\frac{2R^8}{45\omega(z)^8}\right)$$

$$\langle 2,5\,|\,6,1\rangle = \langle 5,2\,|\,1,6\rangle = \frac{75^{1/2}R^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{8\omega(z)^4}\left(1-\frac{10R^2}{3\omega(z)^2}+\frac{16R^4}{3\omega(z)^4}-\frac{56R^6}{15\omega(z)^6}+\frac{46R^8}{45\omega(z)^8}-\frac{4R^{10}}{45\omega(z)^{10}}\right)$$

$$\langle 4,5\,|\,6,1\rangle = \langle 5,4\,|\,1,6\rangle$$
$$= -\frac{15R^4 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{16\omega_0^4\left(1-i\frac{z}{z_r}\right)\left(1+i\frac{z}{z_r}\right)^3}\left(1-\frac{4R^2}{\omega(z)^2}+\frac{9R^4}{\omega(z)^4}-\frac{136R^6}{15\omega(z)^6}+\frac{38R^8}{9\omega(z)^8}-\frac{8R^{10}}{9\omega(z)^{10}}+\frac{R^{12}}{15\omega(z)^{12}}\right)$$

$$\langle 0,5\,|\,6,3\rangle = \langle 5,0\,|\,3,6\rangle = \frac{15R^4 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{8\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{10R^2}{3\omega(z)^2}+\frac{34R^4}{9\omega(z)^4}-\frac{28R^6}{15\omega(z)^6}+\frac{2R^8}{5\omega(z)^8}-\frac{4R^{10}}{135\omega(z)^{10}}\right)$$

$$\langle 2,5\,|\,6,3\rangle = \langle 5,2\,|\,3,6\rangle$$
$$= -\frac{15R^4 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{128^{1/2}\omega_0^4\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{4R^2}{\omega(z)^2}+\frac{67R^4}{9\omega(z)^4}-\frac{296R^6}{45\omega(z)^6}+\frac{134R^8}{45\omega(z)^8}-\frac{88R^{10}}{135\omega(z)^{10}}+\frac{7R^{12}}{135\omega(z)^{12}}\right)$$

$$\langle 4,5\,|\,6,3\rangle = \langle 5,4\,|\,3,6\rangle = \frac{675^{1/2}R^4 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{512^{1/2}\omega(z)^4}\left(1-\frac{14R^2}{3\omega(z)^2}+\frac{103R^4}{9\omega(z)^4}-\frac{122R^6}{9\omega(z)^6}+\frac{1154R^8}{135\omega(z)^8}\right.$$
$$\left.-\frac{388R^{10}}{135\omega(z)^{10}}+\frac{67R^{12}}{135\omega(z)^{12}}-\frac{14R^{14}}{405\omega(z)^14}\right)$$

$$\langle 0,5\,|\,6,5\rangle = \langle 5,0\,|\,5,6\rangle = -\frac{1125^{1/2}R^4\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{16\omega_0^4\left(1-i\frac{z}{z_r}\right)^5}\left(1-\frac{4R^2}{\omega(z)^2}+\frac{53R^4}{9\omega(z)^4}-\frac{184R^6}{45\omega(z)^6}\right.$$
$$\left.+\frac{946R^8}{675\omega(z)^8}-\frac{152R^{10}}{675\omega(z)^{10}}+\frac{R^{12}}{75\omega(z)^{12}}\right)$$

$$\langle 2,5\,|\,6,5\rangle = \langle 5,2\,|\,5,6\rangle = \frac{1125^{1/2}R^4 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{512^{1/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)^4}\left(1-\frac{14R^2}{3\omega(z)^2}+\frac{89R^4}{9\omega(z)^4}-\frac{94R^6}{9\omega(z)^6}\right.$$
$$\left.+\frac{1382R^8}{225\omega(z)^8}-\frac{1348R^{10}}{675\omega(z)^{10}}+\frac{221R^{12}}{675\omega(z)^{12}}-\frac{14R^{14}}{675\omega(z)^{14}}\right)$$

$$\langle 4,5\,|\,6,5\rangle = \langle 5,4\,|\,5,6\rangle = -\frac{3375^{1/2}R^4 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2048^{1/2}\omega_0^4\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}\left(1-\frac{16R^2}{3\omega(z)^2}+\frac{128R^4}{9\omega(z)^4}-\frac{96R^6}{5\omega(z)^6}\right.$$
$$\left.+\frac{9916R^8}{675\omega(z)^8}-\frac{4384R^{10}}{675\omega(z)^{10}}+\frac{1112R^{12}}{675\omega(z)^{12}}-\frac{448R^{14}}{2025\omega(z)^{14}}+\frac{14R^{16}}{1125\omega(z)^{16}}\right)$$

$$\langle 0,6\,|\,0,6\rangle = \langle 6,0\,|\,6,0\rangle = 1-\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}\left(1+\frac{11R^2}{8\omega(z)^2}+\frac{41R^4}{8\omega(z)^4}-\frac{31R^6}{3\omega(z)^6}+\frac{14R^8}{\omega(z)^8}-\frac{133R^{10}}{20\omega(z)^{10}}+\frac{77R^{12}}{60\omega(z)^{12}}\right)$$

$$\langle 0,6\,|\,6,0\rangle = \langle 6,0\,|\,0,6\rangle = \frac{5R^2 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{8\omega(z)^2}\left(1-\frac{5R^2}{\omega(z)^2}+\frac{20R^4}{3\omega(z)^4}-\frac{10R^6}{3\omega(z)^6}+\frac{2R^8}{3\omega(z)^8}-\frac{2R^{10}}{45\omega(z)^{10}}\right)$$

$$\langle 2,6\,|\,0,6\rangle = \langle 6,2\,|\,6,0\rangle = -\frac{5R^2 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{128^{1/2}\omega_0^2\left(1+i\frac{z}{z_r}\right)^2}\left(1-\frac{6R^2}{\omega(z)^2}+\frac{22R^4}{\omega(z)^4}-\frac{92R^6}{3\omega(z)^6}+\frac{98R^8}{5\omega(z)^8}-\frac{28R^{10}}{5\omega(z)^{10}}+\frac{44R^{12}}{75\omega(z)^{12}}\right)$$

$$\langle 6,2\,|\,0,6\rangle = \langle 2,6\,|\,6,0\rangle$$

$$= -\frac{5R^2 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{128^{1/2}\omega_0^2\left(1+i\frac{z}{z_r}\right)^2}\left(1-\frac{6R^2}{\omega(z)^2}+\frac{14R^4}{\omega(z)^4}-\frac{44R^6}{3\omega(z)^6}+\frac{34R^8}{5\omega(z)^8}-\frac{4R^{10}}{3\omega(z)^{10}}+\frac{4R^{12}}{45\omega(z)^{12}}\right)$$

$$\langle 4,6\,|\,0,6\rangle = \langle 6,4\,|\,6,0\rangle = \frac{75^{1/2}R^2\left(1-i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{512^{1/2}\omega_0^2\left(1+i\frac{z}{z_r}\right)^3}\left(1-\frac{7R^2}{\omega(z)^2}+\frac{26R^4}{\omega(z)^4}-\frac{125R^6}{3\omega(z)^6}\right.$$
$$\left.+\frac{478R^8}{15\omega(z)^8}-\frac{182R^{10}}{15\omega(z)^{10}}+\frac{164R^{12}}{75\omega(z)^{12}}-\frac{11R^{14}}{75\omega(z)^{14}}\right)$$

$$\langle 6,4\,|\,0,6\rangle = \langle 4,6\,|\,6,0\rangle = \frac{75^{1/2}R^2\left(1-i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{512^{1/2}\omega_0^2\left(1+i\frac{z}{z_r}\right)^3}\left(1-\frac{7R^2}{\omega(z)^2}+\frac{22R^4}{\omega(z)^4}-\frac{95R^6}{3\omega(z)^6}\right.$$
$$\left.+\frac{326R^8}{15\omega(z)^8}-\frac{22R^{10}}{3\omega(z)^{10}}+\frac{52R^{12}}{45\omega(z)^{12}}-\frac{R^{14}}{15\omega(z)^{14}}\right)$$

$$\langle 6,6\,|\,0,6\rangle = \langle 6,6\,|\,6,0\rangle = -\frac{125^{1/2}R^2\left(1-i\frac{z}{z_r}\right)^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{32\omega_0^2\left(1+i\frac{z}{z_r}\right)^4}\left(1-\frac{8R^2}{\omega(z)^2}+\frac{92R^4}{3\omega(z)^4}-\frac{164R^6}{3\omega(z)^6}\right.$$
$$\left.+\frac{728R^8}{15\omega(z)^8}-\frac{1024R^{10}}{45\omega(z)^{10}}+\frac{424R^{12}}{75\omega(z)^{12}}-\frac{52R^{14}}{75\omega(z)^{14}}+\frac{22R^{16}}{675\omega(z)^{16}}\right)$$

$$\langle 1,6\,|\,1,6\rangle = \langle 6,1\,|\,6,1\rangle = 1-\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}\left(1+\frac{2R^2}{\omega(z)^2}+\frac{11R^4}{8\omega(z)^4}+\frac{41R^6}{12\omega(z)^6}-\frac{31R^8}{6\omega(z)^8}\right.$$
$$\left.+\frac{28R^{10}}{5\omega(z)^{10}}-\frac{133R^{12}}{60\omega(z)^{12}}+\frac{11R^{14}}{30\omega(z)^{14}}\right)$$

$$\langle 3,6\,|\,1,6\rangle = \langle 6,3\,|\,6,1\rangle$$

$$= -\frac{75^{1/2}R^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{128^{1/2}\omega_0^4\left(1-i\frac{z}{z_r}\right)\left(1+i\frac{z}{z_r}\right)^3}\left(1-\frac{4R^2}{\omega(z)^2}+\frac{11R^4}{\omega(z)^4}-\frac{184R^6}{15\omega(z)^6}+\frac{98R^8}{15\omega(z)^8}-\frac{8R^{10}}{5\omega(z)^{10}}+\frac{11R^{12}}{75\omega(z)^{12}}\right)$$

$$\langle 5,6\,|\,1,6\rangle = \langle 6,5\,|\,6,1\rangle = \frac{375^{1/2}R^4\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{512^{1/2}\omega_0^4\left(1+i\frac{z}{z_r}\right)^4}\left(1-\frac{14R^2}{3\omega(z)^2}+\frac{13R^4}{\omega(z)^4}-\frac{50R^6}{3\omega(z)^6}\right.$$
$$\left.+\frac{478R^8}{45\omega(z)^8}-\frac{52R^{10}}{15\omega(z)^{10}}+\frac{41R^{12}}{75\omega(z)^{12}}-\frac{22R^{14}}{675\omega(z)^{14}}\right)$$

$$\langle 2,6\,|\,2,6\rangle = \langle 6,2\,|\,6,2\rangle = 1-\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}\left(1+\frac{27R^2}{16\omega(z)^2}+\frac{67R^4}{16\omega(z)^4}-\frac{173R^6}{24\omega(z)^6}+\frac{707R^8}{48\omega(z)^8}\right.$$
$$\left.-\frac{481R^{10}}{40\omega(z)^{10}}+\frac{679R^{12}}{120\omega(z)^{12}}-\frac{79R^{14}}{60\omega(z)^{14}}+\frac{11R^{16}}{80\omega(z)^{16}}\right)$$

$$\langle 2,6\,|\,6,2\rangle = \langle 6,2\,|\,2,6\rangle = \frac{5R^2\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{16\omega(z)^2}\left(1-\frac{7R^2}{\omega(z)^2}+\frac{22R^4}{\omega(z)^4}-\frac{95R^6}{3\omega(z)^6}+\frac{382R^8}{15\omega(z)^8}\right.$$
$$\left.-\frac{166R^{10}}{15\omega(z)^{10}}+\frac{20R^{12}}{9\omega(z)^{12}}-\frac{7R^{14}}{45\omega(z)^{14}}\right)$$

$$\langle 4,6\,|\,2,6\rangle = \langle 6,4\,|\,6,2\rangle = -\frac{75^{1/2}R^2 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{32\omega_0^2\left(1+i\frac{z}{z_r}\right)^2}\left(1-\frac{8R^2}{\omega(z)^2}+\frac{100R^4}{3\omega(z)^4}-\frac{188R^6}{3\omega(z)^6}\right.$$
$$\left.+\frac{992R^8}{15\omega(z)^8}-\frac{352R^{10}}{9\omega(z)^{10}}+\frac{328R^{12}}{25\omega(z)^{12}}-\frac{172R^{14}}{75\omega(z)^{14}}+\frac{22R^{16}}{135\omega(z)^{16}}\right)$$

$$\langle 6,4\,|\,2,6\rangle = \langle 4,6\,|\,6,2\rangle = -\frac{75^{1/2}R^2 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{32\omega_0^2\left(1+i\frac{z}{z_r}\right)^2}\left(1-\frac{8R^2}{\omega(z)^2}+\frac{92R^4}{3\omega(z)^4}-\frac{164R^6}{3\omega(z)^6}+\frac{56R^8}{\omega(z)^8}\right.$$
$$\left.-\frac{1472R^{10}}{45\omega(z)^{10}}+\frac{472R^{12}}{45\omega(z)^{12}}-\frac{76R^{14}}{45\omega(z)^{14}}+\frac{14R^{16}}{135\omega(z)^{16}}\right)$$

$$\langle 6,6\,|\,2,6\rangle = \langle 6,6\,|\,6,2\rangle = \frac{125^{1/2}\left(1-i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{2048^{1/2}\omega_0^2\left(1+i\frac{z}{z_r}\right)^3}\left(1-\frac{9R^2}{\omega(z)^2}+\frac{40R^4}{\omega(z)^4}-\frac{84R^6}{\omega(z)^6}+\frac{508R^8}{5\omega(z)^8}\right.$$
$$\left.-\frac{1084R^{10}}{15\omega(z)^{10}}+\frac{6832R^{12}}{225\omega(z)^{12}}-\frac{548R^{14}}{75\omega(z)^{14}}+\frac{206R^{16}}{225\omega(z)^{16}}-\frac{154R^{18}}{3375\omega(z)^{18}}\right)$$

$$\langle 3,6\,|\,3,6\rangle = \langle 6,3\,|\,6,3\rangle = 1-\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}\left(1+\frac{2R^2}{\omega(z)^2}+\frac{17R^4}{16\omega(z)^4}+\frac{137R^6}{24\omega(z)^6}-\frac{563R^8}{48\omega(z)^8}\right.$$
$$\left.+\frac{1957R^{10}}{120\omega(z)^{10}}-\frac{3833R^{12}}{360\omega(z)^{12}}+\frac{227R^{14}}{60\omega(z)^{14}}-\frac{161R^{16}}{240\omega(z)^{16}}+\frac{11R^{18}}{216\omega(z)^{18}}\right)$$

$$\langle 5,6\,|\,3,6\rangle = \langle 6,5\,|\,6,3\rangle = -\frac{1125^{1/2}R^4 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{32\omega_0^4\left(1-i\frac{z}{z_r}\right)\left(1+i\frac{z}{z_r}\right)^3}\left(1-\frac{16R^2}{3\omega(z)^2}+\frac{142R^4}{9\omega(z)^4}-\frac{344R^6}{15\omega(z)^6}\right.$$
$$\left.+\frac{272R^8}{15\omega(z)^8}-\frac{1088R^{10}}{135\omega(z)^{10}}+\frac{458R^{12}}{225\omega(z)^{12}}-\frac{184R^{14}}{675\omega(z)^{14}}+\frac{154R^{16}}{10125\omega(z)^{16}}\right)$$

$$\langle 4,6\,|\,4,6\rangle = \langle 6,4\,|\,6,4\rangle = 1-\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}\left(1+\frac{113R^2}{64\omega(z)^2}+\frac{263R^4}{64\omega(z)^4}-\frac{401R^6}{48\omega(z)^6}+\frac{2059R^8}{96\omega(z)^8}-\frac{391R^{10}}{15\omega(z)^{10}}\right.$$
$$\left.+\frac{7117R^{12}}{360\omega(z)^{12}}-\frac{629R^{14}}{72\omega(z)^{14}}+\frac{1081R^{16}}{480\omega(z)^{16}}-\frac{265R^{18}}{864\omega(z)^{18}}+\frac{77R^{20}}{4320\omega(z)^{20}}\right)$$

$$\langle 4,6\,|\,6,4\rangle = \langle 6,4\,|\,4,6\rangle = \frac{15R^2 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{64\omega(z)^2}\left(1-\frac{9R^2}{\omega(z)^2}+\frac{40R^4}{\omega(z)^4}-\frac{84R^6}{\omega(z)^6}+\frac{316R^8}{3\omega(z)^8}\right.$$
$$\left.-\frac{3532R^{10}}{45\omega(z)^{10}}+\frac{1552R^{12}}{45\omega(z)^{12}}-\frac{388R^{14}}{45\omega(z)^{14}}+\frac{154R^{16}}{135\omega(z)^{16}}-\frac{14R^{18}}{225\omega(z)^{18}}\right)$$

$$\langle 6,6\,|\,4,6\rangle = \langle 6,6\,|\,6,4\rangle = -\frac{375^{1/2}R^2 \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}}{8192^{1/2}\omega_0^2\left(1+i\frac{z}{z_r}\right)^2}\left(1-\frac{10R^2}{\omega(z)^2}+\frac{50R^4}{\omega(z)^4}-\frac{120R^6}{\omega(z)^6}+\frac{2596R^8}{15\omega(z)^8}\right.$$
$$\left.-\frac{152R^{10}}{\omega(z)^{10}}+\frac{18472R^{12}}{225\omega(z)^{12}}-\frac{2024R^{14}}{75\omega(z)^{14}}+\frac{394R^{16}}{75\omega(z)^{16}}-\frac{1876R^{18}}{3375\omega(z)^{18}}+\frac{28R^{20}}{1125\omega(z)^{20}}\right)$$

$$\langle 5,6\,|\,5,6\rangle = \langle 6,5\,|\,6,5\rangle = 1-\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}\left(1+\frac{2R^2}{\omega(z)^2}+\frac{53R^4}{64\omega(z)^4}+\frac{803R^6}{96\omega(z)^6}-\frac{687R^8}{32\omega(z)^8}+\frac{8639R^{10}}{240\omega(z)^{10}}\right.$$
$$\left.-\frac{2957R^{12}}{90\omega(z)^{12}}+\frac{3251R^{14}}{180\omega(z)^{14}}-\frac{957R^{16}}{160\omega(z)^{16}}+\frac{2549R^{18}}{2160\omega(z)^{18}}-\frac{2737R^{20}}{21600\omega(z)^{20}}+\frac{7R^{22}}{1200\omega(z)^{22}}\right)$$
$$\langle 6,6\,|\,6,6\rangle = 1-\mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}\left(1+\frac{231R^2}{128\omega(z)^2}+\frac{531R^4}{128\omega(z)^4}-\frac{673R^6}{64\omega(z)^6}+\frac{12481R^8}{384\omega(z)^8}-\frac{24847R^{10}}{480\omega(z)^{10}}+\frac{76253R^{12}}{1440\omega(z)^{12}}\right.$$
$$\left.-\frac{8137R^{14}}{240\omega(z)^{14}}+\frac{13243R^{16}}{960\omega(z)^{16}}-\frac{30469R^{18}}{8640\omega(z)^{18}}+\frac{2639R^{20}}{4800\omega(z)^{20}}-\frac{343R^{22}}{7200\omega(z)^{22}}+\frac{77R^{24}}{43200\omega(z)^{24}}\right)$$

4. Calculation of Power Received by Satellite

We expect that before taking into account the efficiency of all our systems the total transmitted power between spacecrafts should be approximately

$$P_{recieved} = \frac{.5D^4}{\lambda^2 L^2} P_{emitted}$$

with D the diameter of emitting/receiving telescopes, λ the wavelength, L the distance between the spacecrafts. This was calculated assuming an initial waist size at $z = 0$ (at the emitting telescope) of $\omega_0 = .446D$. For our current LISA specifications this means we use $D = .3\,m$, $\lambda = 1.064 * 10^{-6}\,m$, $L = 2.5 * 10^9\,m$, and the power emitted $P_{emitted} = 2\,W$ giving for the received power (again not taking into account efficiencies)

$$P_{received} = 1.1448 * 10^{-9}\,W$$

or the power received we hope to approach with our approximation as being about 1, 1448 Picowatts. We see how many terms we need to use in our expansion to get the power to around this value. Our initial beam is broken into a laser with amplitude approximated to the 6th order TEM modes as (using the nomenclature developed in the last section)

$$A(x,y,z) \approx \sum_{i=0}^{6}\sum_{j=0}^{6} \sqrt{2} A_{i,j}(x,y,z) \left\langle i,j \,|\, 0,0 \right\rangle_{z=0}$$

so that for our approximating beam we have the non zero coefficients (using $\omega_0 = .446 * .3$)

$$\begin{aligned}
\left\langle 0,0 \,|\, 0,0 \right\rangle_{z=0} &= .919 \\
\left\langle 0,2 \,|\, 0,0 \right\rangle_{z=0} = \left\langle 2,0 \,|\, 0,0 \right\rangle_{z=0} &= -.1439 \\
\left\langle 2,2 \,|\, 0,0 \right\rangle_{z=0} &= -.0261 \\
\left\langle 0,4 \,|\, 0,0 \right\rangle_{z=0} = \left\langle 4,0 \,|\, 0,0 \right\rangle_{z=0} &= -.032 \\
\left\langle 2,4 \,|\, 0,0 \right\rangle_{z=0} = \left\langle 4,2 \,|\, 0,0 \right\rangle_{z=0} &= .0406 \\
\left\langle 4,4 \,|\, 0,0 \right\rangle_{z=0} &= -.0208 \\
\left\langle 0,6 \,|\, 0,0 \right\rangle_{z=0} = \left\langle 6,0 \,|\, 0,0 \right\rangle_{z=0} &= .0524 \\
\left\langle 2,6 \,|\, 0,0 \right\rangle_{z=0} = \left\langle 6,2 \,|\, 0,0 \right\rangle_{z=0} &= -.0220 \\
\left\langle 4,6 \,|\, 0,0 \right\rangle_{z=0} = \left\langle 6,4 \,|\, 0,0 \right\rangle_{z=0} &= .0016 \\
\left\langle 6,6 \,|\, 0,0 \right\rangle_{z=0} &= .0097
\end{aligned}$$

Giving for our amplitude after being transmitted as (we have have left out explicit dependence of functions on x, y, z, simply assume A is a function of x, y, z because $A_{i,j}$ is the function $A_{i,j}(x,y,z)$)

$$\begin{aligned}
A \approx \sqrt{2}\,(&.919A_{0,0} - .1439\,(A_{0,2} + A_{2,0}) - .0261A_{2,2} - .032\,(A_{0,4} + A_{4,0}) + .0406\,(A_{2,4} + A_{4,2}) \\
&-.0208A_{4,4} + .0524\,(A_{0,6} + A_{6,0}) - .0220\,(A_{2,6} + A_{6,2}) + .0016\,(A_{4,6} + A_{6,4}) + .0097A_{6,6})
\end{aligned}$$

So that for our intensity up to only second order TEM modes we have

$$I = AA^* = 2\left(.8446A_{0,0}A^*_{0,0} - .1323\left(A_{0,0}\left(A^*_{0,2} + A^*_{2,0}\right) + A^*_{0,0}\left(A_{0,2} + A_{2,0}\right)\right) - .024\left(A_{0,0}A^*_{2,2} + A^*_{0,0}A_{2,2}\right)\right.$$

$$+.0207\left(A_{0,2}A_{0,2}^{*}+A_{2,0}A_{2,0}^{*}+A_{0,2}A_{2,0}^{*}+A_{2,0}A_{0,2}^{*}\right)+.0006831A_{2,2}A_{2,2}^{*}$$
$$+.0038\left(A_{2,2}\left(A_{0,2}^{*}+A_{2,0}^{*}\right)+A_{2,2}^{*}\left(A_{2,0}+A_{0,2}\right)\right)\Big)$$

which when integrated over the receiving telescope gives a total power

$$P=2\,(.8446\,\langle 0,0\,|\,0,0\rangle_{z=L}-.1323\,(\langle 0,2\,|\,0,0\rangle_{z=L}+\langle 2,0\,|\,0,0\rangle_{z=L}+\langle 0,0\,|\,0,2\rangle_{z=L}+\langle 0,0\,|\,2,0\rangle_{z=L})$$
$$-.024\,(\langle 2,2\,|\,0,0\rangle_{z=L}+\langle 0,0\,|\,2,2\rangle_{z=L})$$
$$+.0207\,(\langle 0,2\,|\,0,2\rangle_{z=L}+\langle 2,0\,|\,2,0\rangle_{z=L}+\langle 0,2\,|\,2,0\rangle_{z=L}+\langle 2,0\,|\,0,2\rangle_{z=L})$$
$$+.0006831\,\langle 2,2\,|\,2,2\rangle_{z=L}+.0038\,(\langle 0,2\,|\,2,2\rangle_{z=L}+\langle 2,0\,|\,2,2\rangle_{z=L}+\langle 2,2\,|\,2,0\rangle_{z=L}+\langle 2,2\,|\,0,2\rangle_{z=L}))$$

or utilizing the symmetries found in the brackets

$$P=2\,(.8446\,\langle 0,0\,|\,0,0\rangle_{z=L}-.2645\,(\langle 0,2\,|\,0,0\rangle_{z=L}+\langle 0,0\,|\,0,2\rangle_{z=L})$$
$$-.024\,(\langle 2,2\,|\,0,0\rangle_{z=L}+\langle 0,0\,|\,2,2\rangle_{z=L})+.0414\,(\langle 0,2\,|\,0,2\rangle_{z=L}+\langle 0,2\,|\,2,0\rangle_{z=L})$$
$$+.0006831\,\langle 2,2\,|\,2,2\rangle_{z=L}+.0075\,(\langle 0,2\,|\,2,2\rangle_{z=L}+\langle 2,2\,|\,0,2\rangle_{z=L}))$$

so that we must now calculate

$$\langle 0,0\,|\,0,0\rangle_{z=L}=1.1237*10^{-9}$$
$$\langle 0,2\,|\,0,0\rangle_{z=L}+\langle 0,0\,|\,0,2\rangle_{z=L}=1.5892*10^{-9}$$
$$\langle 0,0\,|\,2,2\rangle_{z=L}+\langle 2,2\,|\,0,0\rangle_{z=L}=1.1237*10^{-9}$$
$$\langle 0,2\,|\,0,2\rangle_{z=L}+\langle 0,2\,|\,2,0\rangle_{z=L}=1.1237*10^{-9}$$
$$\langle 2,2\,|\,2,2\rangle_{z=L}=2.8093*10^{-10}$$
$$\langle 0,2\,|\,2,2\rangle_{z=L}+\langle 2,2\,|\,0,2\rangle_{z=L}=7.946*10^{-10}$$

so that after plugging in for the total power of this up to second order TEM mode expansion the total power on the receiving end we have

$$P_{received}\approx 1.1089*10^{-9}\,W$$

or in other words $1,108.9$ Picowatts, less than 40 picowatts away from the predicted amount. We can expect higher order terms to further hone us in but we show so with using simulation.The approximation to each order is given by

$$P_{0th-ord}=1.8982*10^{-9}\,W$$
$$P_{2nd-ord}=1.1089*10^{-9}\,W$$
$$P_{4th-ord}=1.0717*10^{-9}\,W$$
$$P_{6th-ord}=1.2172*10^{-9}\,W$$
$$P_{8th-ord}=1.1284*10^{-9}\,W$$
$$P_{10th-ord}=1.1440*10^{-9}\,W$$

5. Solving with full generality

By writing, where m is an integer and $l \in \{0,1\}$ we have

$$H_{2m+l}(x) = (2m+l)! \sum_{j=0}^{m} \frac{(-1)^j (2x)^{2(m-j)+l}}{j!(2(m-j)+l)!}$$

and by writing (with $\Delta z = z - z_0$

$$A_{m,j}(x,z;\omega_0,z_0,k) = \frac{\omega_0^{m+j+1}\left(1+i\frac{\Delta z}{z_r}\right)^{m+j+1}}{\sqrt{2^{m+j-1}\pi m! j!}\omega^{m+j+2}} H_m\left(\frac{\sqrt{2}x}{\omega}\right) H_j\left(\frac{\sqrt{2}y}{\omega}\right) \mathrm{e}^{-\frac{\rho^2}{\omega_0^2\left(1-i\frac{\Delta z}{z_r}\right)}-ik\Delta z}$$

we then get that, using $\Delta z_i = z - z_{0,i}$, that m_1, m_2, $j_1\, j_2$ are integers, l_1, $l_2 \in \{0,1\}$, and that $\omega_i = \omega_{0,i}\sqrt{1+\left(\frac{\Delta z_i}{z_{ri}}\right)^2}$, we get the only non-zero terms are of the form

$$\begin{aligned}
&\langle 2m_1+l_1, 2j_1+l_2 \,|\, 2m_2+l_1, 2j_2+l_2\rangle (\omega_{0,1},\omega_{0,2},z_1,z_2,\Delta z_1,\Delta z_2,k_1,k_2) \\
&= \int_0^R \mathrm{d}\rho\rho \int_0^{2\pi} \mathrm{d}\theta A^*_{2m_1+l_1,2j_1+l_2}(x,y,z;\omega_{0,1},z_1,k_1) A_{2m_2+l_1,2j_2+l_2}(x,y,z;\omega_{0,2},z_2,k_2) \\
&= \frac{\omega_{0,1}^{2(m_1+j_1)+l_1+l_2+1}\omega_{0,2}^{2(m_2+j_2)+l_1+l_2+1}\left(1-i\frac{\Delta z_1}{z_{r1}}\right)^{2(m_1+j_1)+l_1+l_2+1}\left(1+i\frac{\Delta z_2}{z_{r2}}\right)^{2(m_2+j_2)+l_1+l_2+1}}{\pi 2^{m_1+m_2+j_1+j_2+l_1+l_2-1}\omega_1^{2(m_1+j_1+1)+l_1+l_2}\omega_2^{2(m_2+j_2+1)+l_1+l_2}} \\
&\quad\times\sqrt{(2m_1+l_1)!(2j_1+l_2)!(2m_2+l_1)!(2j_2+l_2)!}\sum_{c_1=0}^{m_1}\sum_{c_2=0}^{m_2}\sum_{d_1=0}^{j_1}\sum_{d_2=0}^{j_2} \\
&\left[\frac{(-1)^{c_1+c_2+d_1+d_2}2^{3(m_1+m_2+j_1+j_2-(c_1+c_2+d_1+d_2)+l_1+l_2)}}{c_1!c_2!d_1!d_2!\,(2(m_1-c_1)+l_1)!\,(2(m_2-c_2)+l_1)!\,(2(j_1-d_1)+l_2)!\,(2(j_2-d_2)+l_2)!}\right. \\
&\times\int_0^R \rho\mathrm{d}\rho \mathrm{e}^{i(k_1(z-z_{0,1})-k_2(z-z_{0,2}))-\frac{\rho^2}{\omega_1^2\omega_2^2}\left(\left(\omega_1^2+\omega_2^2\right)+i\left(\frac{\omega_1^2\Delta z_2}{z_{r2}}-\frac{\omega_2^2\Delta z_1}{z_{r1}}\right)\right)}\left(\frac{\rho^{2(m_1+m_2+j_1+j_2-(c_1+c_2+d_1+d_2)+l_1+l_2)}}{\omega_1^{2(m_1+j_1-(c_1+d_1))+l_1+l_2}\omega_2^{2(m_2+j_2-(c_2+d_2))+l_1+l_2}}\right) \\
&\quad\left.\times\int_0^{2\pi}\mathrm{d}\theta\sin^{2(j_1+j_2-(d_1+d_2)+l_2)}\theta\cos^{2(m_1+m_2-(c_1+c_2)+l_1)}\theta\right]
\end{aligned}$$

Now earlier we showed

$$\int_0^{2\pi}\mathrm{d}\theta\sin^c\theta\cos^d\theta = \frac{2\Gamma\left(\frac{c+1}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{c+d+2}{2}\right)}$$

for even c, d and 0 for odd so that we get

$$\int_0^{2\pi}\mathrm{d}\theta\sin^{2c}\theta\cos^{2d}\theta = \frac{2\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(d+\frac{1}{2}\right)}{\Gamma\left(c+d+1\right)}$$

for integers c, d and after some manipulation we can show for any integer n

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!\sqrt{\pi}}{2^{2n}n!}$$

so that we have

$$\int_0^{2\pi} \mathrm{d}\theta \sin^{2c}\theta\cos^{2d}\theta = \pi\frac{(2c)!(2d)!}{2^{2(c+d)-1}c!d!(c+d)!}$$

and then substituting in $j_1+j_2-(d_1+d_2)+l_2$ for c and $m_1+m_2-(c_1+c_2)+l_1$ for d and writing $\omega_{0,*/+}=\omega_{0,1}(*/+)\omega_{0,2}$, $\omega_*=\omega_1*\omega_2$, $\omega_+=\omega_1^2+\omega_2^2$, and $\Delta kz=k_1\Delta z_1-k_2\Delta z_2=k_2z_{0,2}-k_1z_{0,1}$ as well as making the changes for the ordering of the sums leads to

$$\langle 2m_1+l_1,2j_1+l_2\,|\,2m_2+l_1,2j_2+l_2\rangle\,(\omega_{0,1},\omega_{0,2},z_1,z_2,\Delta z_1,\Delta z_2,k_1,k_2)$$

$$=\frac{\omega_{0,1}^{2(m_1+j_1)}\omega_{0,2}^{2(m_2+j_2)}\omega_{0,*}^{l_1+l_2+1}\left(1-i\frac{\Delta z_1}{z_{r1}}\right)^{2(m_1+j_1)+l_1+l_2+1}\left(1+i\frac{\Delta z_2}{z_{r2}}\right)^{2(m_2+j_2)+l_1+l_2+1}}{2^{m_1+m_2+j_1+j_2+l_1+l_2-2}\omega_1^{2(m_1+j_1)}\omega_2^{2(m_2+j_2)}\omega_*^2}$$

$$\times\sqrt{(2m_1+l_1)!(2j_1+l_2)!(2m_2+l_1)!(2j_2+l_2)!}\sum_{c_1=0}^{m_1}\sum_{c_2=0}^{m_2}\sum_{d_1=0}^{j_1}\sum_{d_2=0}^{j_2}\frac{(-1)^{m_1+m_2+j_1+j_2-(c_1+c_2+d_1+d_2)}}{(m_1-c_1)!(m_2-c_2)!(j_1-d_1)!(j_2-d_2)!}$$

$$\left[\frac{2^{c_1+c_2+d_1+d_2+l_1+l_2}(2(d_1+d_2+l_2))!(2(c_1+c_2+l_1))!\mathrm{e}^{i\Delta kz}}{(2c_1+l_1)!(2c_2+l_1)!(2d_1+l_2)!(2d_2+l_2)!(d_1+d_2+l_2)!(c_1+c_2+l_1)!(c_1+c_2+d_1+d_2+l_1+l_2)!}\right.$$

$$\left.\times\int_0^R\rho\mathrm{d}\rho\mathrm{e}^{-\frac{\rho^2}{\omega_*^2}\left(\omega_++i\left(\frac{\omega_1^2\Delta z_2}{z_{r2}}-\frac{\omega_2^2\Delta z_1}{z_{r1}}\right)\right)}\frac{\rho^{2(c_1+c_2+d_1+d_2+l_1+l_2)}}{\omega_1^{2(c_1+d_1)}\omega_2^{2(c_2+d_2)}\omega_*^{2(l_1+l_2)}}\right]$$

We first simplify some terminology to make this more compact, we call $m_i+j_i\equiv S_i$, $l_1+l_2=L$, $P_{b,\pm}=1\pm i\frac{\Delta z_b}{z_{rb}}$, $T\equiv S_1+S_2$, and $w_++i\left(\frac{\omega_1^2\Delta z_2}{z_{r2}}-\frac{\omega_2^2\Delta z_1}{z_{r1}}\right)\equiv W$ as well as using double factorials so that our equation becomes

$$=\frac{\omega_{0,1}^{2S_1}\omega_{0,2}^{2S_2}\omega_{0,*}^{L+1}P_{1,-}^{2S_1+L+1}P_{2,+}^{2S_2+L+1}}{2^{T+L-2}\omega_1^{2S_1}\omega_2^{2S_2}\omega_*^2}\sqrt{(2m_1+l_1)!(2j_1+l_2)!(2m_2+l_1)!(2j_2+l_2)!}\mathrm{e}^{i\Delta kz}\sum_{c_1=0}^{m_1}\sum_{c_2=0}^{m_2}\sum_{d_1=0}^{j_1}\sum_{d_2=0}^{j_2}$$

$$\left[\frac{2^{2(c_1+c_2+d_1+d_2+l_1+l_2)}\omega_1^{2(c_2+d_2)}\omega_2^{2(c_1+d_1)}(-1)^{T-(c_1+c_2+d_1+d_2)}(2(d_1+d_2+l_2)-1)!!(2(c_1+c_2+l_1)-1)!!}{(m_1-c_1)!(m_2-c_2)!(j_1-d_1)!(j_2-d_2)!(2c_1+l_1)!(2c_2+l_1)!(2d_1+l_2)!(2d_2+l_2)!(c_1+c_2+d_1+d_2+L)!}\right.$$

$$\left.\int_0^R\rho\mathrm{d}\rho\mathrm{e}^{-\frac{W\rho^2}{\omega_*^2}}\left(\frac{\rho^2}{\omega_*^2}\right)^{c_1+c_2+d_1+d_2+L}\right]$$

looking at the integral

$$\int_0^R\rho\mathrm{d}\rho\mathrm{e}^{-\frac{W\rho^2}{\omega_*^2}}\left(\frac{\rho^2}{\omega_*^2}\right)^{c_1+c_2+d_1+d_2+L}=\frac{\omega_*^2}{2W^{c_1+c_2+d_1+d_2+L+1}}\int_0^R\frac{2W\rho\mathrm{d}\rho}{\omega_*^2}\left(\frac{W\rho^2}{\omega_*^2}\right)^{c_1+c_2+d_1+d_2+L}\mathrm{e}^{-\frac{W\rho^2}{\omega_*^2}}$$

Now using the substitution $q=\frac{W\rho^2}{\omega_*^2}\implies \mathrm{d}q=\frac{2W\rho\mathrm{d}\rho}{\omega_*^2}$

$$\implies=\frac{\omega_*^2}{2W^{c_1+c_2+d_1+d_2+L+1}}\int_0^{\frac{WR^2}{\omega_*^2}}\mathrm{d}q\mathrm{e}^{-q}q^{c_1+c_2+d_1+d_2+L}$$

which we can show via induction is equal to

$$= \frac{(c_1+c_2+d_1+d_2+L)!\omega_*^2}{2W^{c_1+c_2+d_1+d_2+L+1}}\left(1-\mathrm{e}^{\frac{-WR^2}{\omega_*^2}}\sum_{h=0}^{c_1+c_2+d_1+d_2+L}\frac{W^hR^{2h}}{\omega_*^{2h}h!}\right)$$

so that altogether we have

$$= \frac{\omega_{0,1}^{2S_1}\omega_{0,2}^{2S_2}\omega_{0,*}^{L+1}P_{1,-}^{2S_1+L+1}P_{2,+}^{2S_2+L+1}}{2^{T+L-1}\omega_1^{2S_1}\omega_2^{2S_2}W}\sqrt{(2m_1+l_1)!(2j_1+l_2)!(2m_2+l_1)!(2j_2+l_2)!}\mathrm{e}^{i\Delta kz}\sum_{c_1=0}^{m_1}\sum_{c_2=0}^{m_2}\sum_{d_1=0}^{j_1}\sum_{d_2=0}^{j_2}$$

$$\left[\frac{2^{2(c_1+c_2+d_1+d_2+l_1+l_2)}\omega_1^{2(c_2+d_2)}\omega_2^{2(c_1+d_1)}(-1)^{T-(c_1+c_2+d_1+d_2)}(2(d_1+d_2+l_2)-1)!!(2(c_1+c_2+l_1)-1)!!}{(m_1-c_1)!(m_2-c_2)!(j_1-d_1)!(j_2-d_2)!(2c_1+l_1)!(2c_2+l_1)!(2d_1+l_2)!(2d_2+l_2)!}\right.$$

$$\left.\frac{1}{W^{c_1+c_2+d_1+d_2+L}}\left(1-\mathrm{e}^{-\frac{WR^2}{\omega_*^2}}\sum_{h=0}^{c_1+c_2+d_1+d_2+L}\frac{W^hR^{2h}}{h!\omega_*^{2h}}\right)\right]$$

Noting that

$$\sigma \equiv \frac{W}{\omega_*^2} = \frac{\omega_{0,1}^2P_{1,+}+\omega_{0,2}^2P_{2,-}}{\omega_{0,1}^2\omega_{0,2}^2P_{1,+}P_{2,-}} = \frac{\omega_{0,1}^2P_{1,+}+\omega_{0,2}^2P_{2,-}}{\omega_{0,*}^2P_{1,+}P_{2,-}} = \frac{1}{\omega_{0,1}^2P_{1,+}}+\frac{1}{\omega_{0,2}^2P_{2,-}} = \frac{P_{1,-}}{\omega_1^2}+\frac{P_{2,+}}{\omega_2^2}$$

allowing us to simplify

$$\langle 2m_1+l_1, 2j_1+l_2 \,|\, 2m_2+l_1, 2j_2+l_2\rangle(\omega_{0,1},\omega_{0,2},z_1,z_2,\Delta z_1,\Delta z_2,k_1,k_2)$$

$$= \frac{\omega_{0,1}^{2S_1}\omega_{0,2}^{2S_2}\omega_{0,*}^{L+1}P_{1,-}^{2S_1+L+1}P_{2,+}^{2S_2+L+1}}{2^{T+L-1}\omega_1^{2S_1}\omega_2^{2S_2}}\sqrt{(2m_1+l_1)!(2j_1+l_2)!(2m_2+l_1)!(2j_2+l_2)!}\mathrm{e}^{i\Delta kz}\sum_{c_1=0}^{m_1}\sum_{c_2=0}^{m_2}\sum_{d_1=0}^{j_1}\sum_{d_2=0}^{j_2}$$

$$\left[\frac{2^{2(c_1+c_2+d_1+d_2+l_1+l_2)}\omega_1^{2(c_2+d_2)}\omega_2^{2(c_1+d_1)}(-1)^{T-(c_1+c_2+d_1+d_2)}(2(d_1+d_2+l_2)-1)!!(2(c_1+c_2+l_1)-1)!!}{(m_1-c_1)!(m_2-c_2)!(j_1-d_1)!(j_2-d_2)!(2c_1+l_1)!(2c_2+l_1)!(2d_1+l_2)!(2d_2+l_2)!}\right.$$

$$\left.\frac{1}{(\sigma\omega_*^2)^{c_1+c_2+d_1+d_2+L+1}}\left(1-\mathrm{e}^{-\sigma R^2}\sum_{h=0}^{c_1+c_2+d_1+d_2+L}\frac{(\sigma R^2)^h}{h!}\right)\right]$$

we can simplify this to get

$$= \frac{\omega_{0,*}^{L+1}P_{1,-}^{S_1+L+1}P_{2,+}^{S_2+L+1}}{2^{T+L-1}P_{1,+}^{S_1}P_{2,-}^{S_2}}\sqrt{(2m_1+l_1)!(2j_1+l_2)!(2m_2+l_1)!(2j_2+l_2)!}\mathrm{e}^{ik\Delta z_0}\sum_{c_1=0}^{m_1}\sum_{c_2=0}^{m_2}\sum_{d_1=0}^{j_1}\sum_{d_2=0}^{j_2}$$

$$\left[\frac{2^{2(c_1+c_2+d_1+d_2+l_1+l_2)}\omega_1^{2(c_2+d_2)}\omega_2^{2(c_1+d_1)}(-1)^{T-(c_1+c_2+d_1+d_2)}(2(d_1+d_2+l_2)-1)!!(2(c_1+c_2+l_1)-1)!!}{(m_1-c_1)!(m_2-c_2)!(j_1-d_1)!(j_2-d_2)!(2c_1+l_1)!(2c_2+l_1)!(2d_1+l_2)!(2d_2+l_2)!}\right.$$

$$\left.\frac{1}{(\sigma\omega_*^2)^{c_1+c_2+d_1+d_2+L+1}}\left(1-\mathrm{e}^{-\sigma R^2}\sum_{h=0}^{c_1+c_2+d_1+d_2+L}\frac{(\sigma R^2)^h}{h!}\right)\right]$$

We first recognize that it makes no sense to match up modes of different wavelengths, so we've rewritten $\Delta kz = k\Delta z_0$ where k is the common wavelength and $\Delta z_0 = z_{0,2} - z_{0,1}$ is the difference in waste positions of the beams. We now quickly check for some simple cases against our calculations when $\omega_{0,1} = \omega_{0,2}$, $\omega_1 = \omega_2$, $z_1 = z_2$ $\Delta z_1 = \Delta z_2$ so that we have $P_{1,\pm} = P_{2,\pm}$, $\sigma = \frac{2}{\omega(z)^2}$, $\omega_* = \omega(z)^2$,

$$\langle 0,0 \,|\, 0,0 \rangle$$

that $S_1 = S_2 = L = m_1 = m_2 = j_1 = j_2 = 0$ giving us

$$\langle 0,0 \,|\, 0,0 \rangle = \frac{2\omega_0^2 \left(1 + \frac{\Delta z}{z_r}\right)}{\frac{2\omega(z)^4}{\omega(z)^2}} \left(1 - \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}\right)$$

$$= 1 - \mathrm{e}^{\frac{-2R^2}{\omega(z)^2}}$$

as expected. We note that this shows a symmetry we had not noticed before, but that

6. TTL Calculation

We start off by putting the coordinate system for the first satellite in terms of the coordinate system for the second satellite, so that we can write out the integration as necessary. We define the z-axis of each coordinate system to be the direction in which the telescope points, and then due to the rotational symmetry of the telescope we can change the orientation of the telescopes x-y axis to be in any direction. In the absolute (external) system we have the z-axis as the axis of the initial optical axis, with each satellite perturbed from this transversely by

$$\begin{pmatrix} \delta x_i \\ \delta y_i \\ \delta z_i \end{pmatrix}$$

And where these have distance L along the z axis. This gives that the coordinate system relative to the first telescope whose telescope is pointed θ_1 away from the z-axis and ϕ_1 away from the x-axis as

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \hat{\mathbf{R}}(\gamma_1 \hat{k}) \hat{\mathbf{R}}(\theta_1 \hat{\jmath}) \hat{\mathbf{R}}(\phi_1 \hat{k}) \left(\vec{r} - \vec{r}_1\right)$$

where these are change of basis matrices $\hat{\mathbf{R}}(\vec{\omega})$ for a basis rotated about the vector $\vec{\omega}$

$$\hat{\mathbf{R}}(a\hat{k}) = \begin{pmatrix} \cos a & \sin a & 0 \\ -\sin a & \cos a & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \hat{\mathbf{R}}(a\hat{\jmath}) = \begin{pmatrix} \cos a & 0 & -\sin a \\ 0 & 1 & 0 \\ \sin a & 0 & \cos a \end{pmatrix}$$

and x_1, y_1 z_1 are components of vector $\vec{r} = (x,\, y,\, z)$ (these are components in absolute frame), γ_1 is a rotation orienting the first SC determined by the position of its other Test Mass, and $\vec{r}_1$ is the location of the first spacecraft in the global frame

$$\vec{r}_1 = \vec{r}_{1o} + \begin{pmatrix} \delta x_1 \\ \delta y_1 \\ \delta z_1 \end{pmatrix}$$

where $\vec{r}_0$ is the location the SC is expected to be at in it's orbit with no motion along the line between TM.

Then we similarly have

$$\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \hat{\mathbf{R}}(\gamma_2 \hat{k}) \hat{\mathbf{R}}(\theta_2 \hat{\jmath}) \hat{\mathbf{R}}(\phi_2 \hat{k}) \left(\vec{r} - \vec{r}_2\right)$$

where these variables hold the same values of corresponding quantities in the second SC and

$$\vec{r}_2 = \vec{r}_{1o} + \begin{pmatrix} \delta x_2 \\ \delta y_2 \\ L + \delta z_2 \end{pmatrix}$$

where L is the distance between the spacecrafts intended positions, 2.5 million kilometers, along the z-axis which is the intended optical axis. This then gives us the function of the first SC's components in terms of the coordinates of the second SC, but there is still one change to make.

We want to expand into functions at the second SC by evaluating their overlap on the plane at the second SC perpendicular to its telescope's incoming orientation axis. This gives us for the evaluation of the first coordinate system in terms of the second set of coordinates. Inverting the equation for the coordinates relative to the second spacecraft we have

$$\begin{aligned}\vec{r} &= \vec{r}_2 + \hat{\mathbf{R}}^{-1}(\phi_2\hat{k})\hat{\mathbf{R}}^{-1}(\theta_2\hat{j})\hat{\mathbf{R}}^{-1}(\gamma_2\hat{k})\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}\\ &= \vec{r}_{1o} + \begin{pmatrix}\delta x_2\\ \delta y_2\\ L+\delta z_2\end{pmatrix} + \hat{\mathbf{R}}(-\phi_2\hat{k})\hat{\mathbf{R}}(-\theta_2\hat{j})\hat{\mathbf{R}}(-\gamma_2\hat{k})\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}\end{aligned}$$

which gives for the coordinates from the first telescope in terms of the second telescope system as

$$\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix} = \hat{\mathbf{R}}(\gamma_1\hat{k})\hat{\mathbf{R}}(\theta_1\hat{j})\hat{\mathbf{R}}(\phi_1\hat{k})\left[\begin{pmatrix}\Delta x\\ \Delta y\\ L+\Delta z\end{pmatrix} + \hat{\mathbf{R}}(-\phi_2\hat{k})\hat{\mathbf{R}}(-\theta_2\hat{j})\hat{\mathbf{R}}(-\gamma_2\hat{k})\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}\right]$$

where we have used $\Delta x = \delta x_2 - \delta x_1$, $\Delta y = \delta y_2 - \delta y_1$, $\Delta z = \delta z_2 - \delta z_1$. This gives us that in the plane $z_2 = 0$, or at the second telescope, we have

$$\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix} = \hat{\mathbf{R}}(\gamma_1\hat{k})\hat{\mathbf{R}}(\theta_1\hat{j})\hat{\mathbf{R}}(\phi_1\hat{k})\begin{pmatrix}\Delta x\\ \Delta y\\ L+\Delta z\end{pmatrix} + \hat{\mathbf{R}}(\gamma_1\hat{k})\hat{\mathbf{R}}(\theta_1\hat{j})\hat{\mathbf{R}}((\phi_1-\phi_2)\hat{k})\hat{\mathbf{R}}(-\theta_2\hat{j})\hat{\mathbf{R}}(-\gamma_2\hat{k})\begin{pmatrix}x_2\\y_2\\0\end{pmatrix}$$

We also now expand to first order in θ_1 and θ_2, the angles from the z-axis which by assumption are small since we expect spacecraft orientation misalignment to be on the order of 10s of nanoradians, giving

$$\hat{\mathbf{R}}(\theta_i\hat{j}) = \begin{pmatrix}1 & 0 & -\theta_i\\ 0 & 1 & 0\\ \theta_i & 0 & 1\end{pmatrix} = \mathbb{I} + \theta_i\begin{pmatrix}0 & 0 & -1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}$$

so that

$$\begin{aligned}\hat{\mathbf{R}}(\gamma_1\hat{k})\hat{\mathbf{R}}(\theta_1\hat{j})\hat{\mathbf{R}}(\phi_1\hat{k}) &\longrightarrow \hat{\mathbf{R}}(\gamma_1\hat{k})\left(\mathbb{I} + \theta_1\begin{pmatrix}0 & 0 & -1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}\right)\hat{\mathbf{R}}(\phi_1\hat{k})\\
&= \hat{\mathbf{R}}\left((\gamma_1+\phi_1)\hat{k}\right) + \theta_1\begin{pmatrix}\cos\gamma_1 & \sin\gamma_1 & 0\\ -\sin\gamma_1 & \cos\gamma_1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0 & -1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}\begin{pmatrix}\cos\phi_1 & \sin\phi_1 & 0\\ -\sin\phi_1 & \cos\phi_1 & 0\\ 0 & 0 & 1\end{pmatrix}\\
&= \begin{pmatrix}\cos(\gamma_1+\phi_1) & \sin(\gamma_1+\phi_1) & 0\\ -\sin(\gamma_1+\phi_1) & \cos(\gamma_1+\phi_1) & 0\\ 0 & 0 & 1\end{pmatrix} + \theta_1\begin{pmatrix}\cos\gamma_1 & \sin\gamma_1 & 0\\ -\sin\gamma_1 & \cos\gamma_1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0 & -1\\ 0 & 0 & 0\\ \cos\phi_1 & \sin\phi_1 & 0\end{pmatrix}\\
&= \begin{pmatrix}\cos(\gamma_1+\phi_1) & \sin(\gamma_1+\phi_1) & 0\\ -\sin(\gamma_1+\phi_1) & \cos(\gamma_1+\phi_1) & 0\\ 0 & 0 & 1\end{pmatrix} + \theta_1\begin{pmatrix}0 & 0 & -\cos\gamma_1\\ 0 & 0 & \sin\gamma_1\\ \cos\phi_1 & \sin\phi_1 & 0\end{pmatrix}\end{aligned}$$

$$= \begin{pmatrix} \cos(\gamma_1+\phi_1) & \sin(\gamma_1+\phi_1) & -\theta_1\cos\gamma_1 \\ -\sin(\gamma_1+\phi_1) & \cos(\gamma_1+\phi_1) & \theta_1\sin\gamma_1 \\ \theta_1\cos\phi_1 & \theta_1\sin\phi_1 & 1 \end{pmatrix}$$

Similarly we have using that θ_2 is really a small angle about π radians since the second telescope opening faces the first so we rewrite the old θ_2 as $\pi - \theta_2$

$$\hat{\mathbf{R}}(\gamma_1\hat{k})\hat{\mathbf{R}}(\theta_1\hat{j})\hat{\mathbf{R}}((\phi_1-\phi_2)\hat{k})\hat{\mathbf{R}}(-\theta_2\hat{j})\hat{\mathbf{R}}(-\gamma_2\hat{k})$$

$$= \hat{\mathbf{R}}(\gamma_1\hat{k})\left(\mathbb{I}+\theta_1\begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix}\right)\begin{pmatrix}\cos(\phi_1-\phi_2) & \sin(\phi_1-\phi_2) & 0\\ -\sin(\phi_1-\phi_2) & \cos(\phi_1-\phi_2) & 0\\ 0&0&1\end{pmatrix}\left(\mathbb{I}+\theta_2\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix}\right)\hat{\mathbf{R}}(-\gamma_2\hat{k})$$

$$= \hat{\mathbf{R}}\left(\gamma_1\hat{k}\right)\left(\begin{pmatrix}\cos(\phi_1-\phi_2) & \sin(\phi_1-\phi_2) & 0\\ -\sin(\phi_1-\phi_2) & \cos(\phi_1-\phi_2) & 0\\ 0&0&1\end{pmatrix}+\theta_1\begin{pmatrix}0&0&-1\\0&0&0\\ \cos(\phi_1-\phi_2) & \sin(\phi_1-\phi_2) & 0\end{pmatrix}\right)$$

$$\times\left(\mathbb{I}+\theta_2\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix}\right)\hat{\mathbf{R}}(-\gamma_2\hat{k})$$

and since for small θ_1, θ_2 we have $\theta_1\theta_2 \longrightarrow 0$ for first order approximations we have

$$= \hat{\mathbf{R}}\left(\gamma_1\hat{k}\right)\left(\begin{pmatrix}\cos(\phi_1-\phi_2) & \sin(\phi_1-\phi_2) & 0\\ -\sin(\phi_1-\phi_2) & \cos(\phi_1-\phi_2) & 0\\ 0&0&1\end{pmatrix}+\theta_1\begin{pmatrix}0&0&-1\\0&0&0\\ \cos(\phi_1-\phi_2) & \sin(\phi_1-\phi_2) & 0\end{pmatrix}\right.$$

$$\left.+\theta_2\begin{pmatrix}0&0&\cos(\phi_1-\phi_2)\\0&0&-\sin(\phi_1-\phi_2)\\-1&0&0\end{pmatrix}\right)\hat{\mathbf{R}}\left(-\gamma_2\hat{k}\right)$$

$$= \hat{\mathbf{R}}\left(\gamma_1\hat{k}\right)\left(\hat{\mathbf{R}}\left((\phi_1-\phi_2)\hat{k}\right)-\begin{pmatrix}0&0&\theta_1-\theta_2\cos(\phi_1-\phi_2)\\0&0&\theta_2\sin(\phi_1-\phi_2)\\ \theta_2-\theta_1\cos(\phi_1-\phi_2) & -\theta_1\sin(\phi_1-\phi_2) & 0\end{pmatrix}\right)\hat{\mathbf{R}}\left(-\gamma_2\hat{k}\right)$$

$$= \hat{\mathbf{R}}\left[\left((\gamma_1+\phi_1)-(\gamma_2+\phi_2)\right)\hat{k}\right]$$

$$-\begin{pmatrix}\cos\gamma_1&\sin\gamma_1&0\\-\sin\gamma_1&\cos\gamma_1&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&0&\theta_1-\theta_2\cos(\phi_1-\phi_2)\\0&0&\theta_2\sin(\phi_1-\phi_2)\\ \theta_2-\theta_1\cos(\phi_1-\phi_2) & -\theta_1\sin(\phi_1-\phi_2) & 0\end{pmatrix}\begin{pmatrix}\cos\gamma_2&-\sin\gamma_2&0\\ \sin\gamma_2&\cos\gamma_2&0\\0&0&1\end{pmatrix}$$

$$= \hat{\mathbf{R}}\left[\left((\gamma_1+\phi_1)-(\gamma_2+\phi_2)\right)\hat{k}\right]-\begin{pmatrix}\cos\gamma_1&\sin\gamma_1&0\\-\sin\gamma_1&\cos\gamma_1&0\\0&0&1\end{pmatrix}$$

$$\times\begin{pmatrix}0&0&\theta_1-\theta_2\cos(\phi_1-\phi_2)\\0&0&\theta_2\sin(\phi_1-\phi_2)\\ \theta_2\cos\gamma_2-\theta_1\cos(\phi_1-(\phi_2+\gamma_2)) & -(\theta_2\sin\gamma_2+\theta_1\sin(\phi_1-(\phi_2+\gamma_2))) & 0\end{pmatrix}$$

$$= \hat{\mathbf{R}}\left[\left((\gamma_1+\phi_1)-(\gamma_2+\phi_2)\right)\hat{k}\right]$$

$$
-\begin{pmatrix} 0 & 0 & \theta_1\cos\gamma_1-\theta_2\cos\left((\phi_1+\gamma_1)-\right. \\ 0 & 0 & -\left(\theta_1\sin\gamma_1-\theta_2\sin\left((\phi_1+\gamma_1)+\right.\right. \\ \theta_2\cos\gamma_2-\theta_1\cos\left(\phi_1-(\phi_2+\gamma_2)\right) & -\left(\theta_2\sin\gamma_2+\theta_1\sin\left(\phi_1-(\phi_2+\gamma_2)\right)\right) & 0 \end{pmatrix}
$$

$$
=\begin{pmatrix} \cos\left((\phi_1+\gamma_1)-(\phi_2+\gamma_2)\right) & \sin\left((\phi_1+\gamma_1)-(\phi_2+\gamma_2)\right) & \theta_2\cos\left((\phi_1+\gamma_1)-\phi_2\right)-\theta_1\cos\gamma_1 \\ -\sin\left((\phi_1+\gamma_1)-(\phi_2+\gamma_2)\right) & \cos\left((\phi_1+\gamma_1)-(\phi_2+\gamma_2)\right) & \theta_1\sin\gamma_1-\theta_2\sin\left((\phi_1+\gamma_1)-\phi_2\right) \\ \theta_1\cos\left(\phi_1-(\phi_2+\gamma_2)\right)-\theta_2\cos\gamma_2 & \theta_2\sin\gamma_2+\theta_1\sin\left(\phi_1-(\phi_2+\gamma_2)\right) & 1 \end{pmatrix}
$$

we create several quantities to make the writing of this easier,

$$\lambda_i \equiv \phi_i + \gamma_i$$

$$\Delta\lambda \equiv \lambda_1 - \lambda_2$$

$$\delta_1 = \lambda_1 - \phi_2$$

$$\delta_2 = \lambda_2 - \phi_1$$

we can rewrite this all as

$$
\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} \cos\lambda_1 & \sin\lambda_1 & -\theta_1\cos\gamma_1 \\ -\sin\lambda_1 & \cos\lambda_1 & \theta_1\sin\gamma_1 \\ \theta_1\cos\phi_1 & \theta_1\sin\phi_1 & 1 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ L+\Delta z \end{pmatrix}
$$

$$
+\begin{pmatrix} \cos\Delta\lambda & \sin\Delta\lambda & \theta_2\cos\delta_1-\theta_1\cos\gamma_1 \\ -\sin\Delta\lambda & \cos\Delta\lambda & \theta_1\sin\gamma_1-\theta_2\sin\delta_1 \\ \theta_1\cos\delta_2-\theta_2\cos\gamma_2 & \theta_2\sin\gamma_2-\theta_1\sin\delta_2 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ 0 \end{pmatrix}
$$

$$x_1 = \Delta x\cos\lambda_1 + \Delta y\sin\lambda_1 - (L+\Delta z)\theta_1\cos\gamma_1 + x_2\cos\Delta\lambda + y_2\sin\Delta\lambda$$

$$y_1 = -\Delta x\sin\lambda_1 + \Delta y\cos\lambda_1 + (L+\Delta z)\,\theta_1\sin\gamma_1 - x_2\sin\Delta\lambda + y_2\cos\Delta\lambda$$

$$z_1 = \Delta x\theta_1\cos\phi_1 + \Delta y\theta_1\sin\phi_1 + L + \Delta z + x_2\left(\theta_1\cos\delta_2 - \theta_2\cos\gamma_2\right) + y_2\left(\theta_2\sin\gamma_2 - \theta_1\sin\delta_2\right)$$

Simplifying using that we are taking to first order in θ_i and Δx, Δy, this puts several of the quantities at zero and we also simplify the process of expanding to a certain order. Define $c = max\{\theta_1,\, \theta_2,\, \Delta x,\, \Delta y\}$. Then we can rewrite each of these as

$$c_1\sin a_1 \equiv \theta_1 \qquad c_1\sin a_2 \equiv \theta_2$$

$$c_2\sin a_x \equiv \Delta x \qquad c_2\sin a_y \equiv \Delta y$$

where a_j for any j is an angle within $[-\pi/2, \pi/2]$. Then we need only worry about taking to first order in a single variable, c so that we get

$$
\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = c\begin{pmatrix} \sin a_x\cos\lambda_1 + \sin a_y\sin\lambda_1 - \sin a_1\left(L+\Delta z\right)\cos\gamma_1 \\ \sin a_y\cos\lambda_1 - \sin a_x\sin\lambda_1 + \sin a_1\left(L+\Delta z\right)\sin\gamma_1 \\ x_2\left(\sin a_1\cos\delta_2 - \sin a_2\cos\gamma_2\right) + y_2\left(\sin a_2\sin\gamma_2 - \sin a_1\sin\delta_2\right) \end{pmatrix} + \begin{pmatrix} x_2\cos\Delta\lambda + y_2\sin\Delta\lambda \\ y_2\cos\Delta\lambda - x_2\sin\Delta\lambda \\ L+\Delta z \end{pmatrix}
$$

Now we can specify the values of the angles ϕ_i and γ_i somewhat by using that if the telescopes we speak about are along their respective z-axis then the second telescope on each SC should be $\pi/3$ away from the z-axis, towards the respective x-axis. We assume all three spacecrafts equilibrium positions are in the x-z axis originally. This means for the angle of deviation of the second telescope to be small we introduce a second subscript for θ_i to denote the first telescope so $\theta_i \longrightarrow \theta_{i,1}$ and we must have $\gamma_1 = -\phi_1 + \theta_{1,2}$ where $\theta_{1,2}$ is the small angle deviation of the second telescope from the direction intended, while for the second telescope since the z-axis is rotated near π radians it flips the sign of the x-axis so that we get $\gamma_2 = \pi - \phi_2 + \theta_{2,2}$ so that we get

$$\lambda_1 = \theta_{1,2} = c \sin a_{1,2}$$
$$\lambda_2 = \pi + \theta_{2,2} = \pi + c \sin a_{2,2}$$
$$\delta_1 = c \sin a_{1,2} - \phi_2$$
$$\delta_2 = \pi + c \sin a_{2,2} - \phi_1$$
$$\Delta\lambda = c\left(\sin a_{1,2} - \sin a_{2,2}\right) - \pi$$

giving us finally

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$
$$\approx c \begin{pmatrix} \sin a_x - \sin a_{1,1} \cos \phi_1 \left(L + \Delta z\right) \\ - \sin a_{1,1} \sin \phi_1 \left(L + \Delta z\right) \\ -x_2 \left(\sin a_{1,1} \cos \phi_1 - \sin a_{2,1} \cos \phi_2\right) - y_2 \left(\sin a_{2,1} \sin \phi_2 - \sin a_{1,1} \sin \phi_1\right) \end{pmatrix} + \begin{pmatrix} -x_2 - c\left(\sin a_{1,2} - \sin a_{2,2}\right) y_2 \\ -y_2 + c\left(\sin a_{1,2} - \sin a_{2,2}\right) x_2 \\ L + \Delta z \end{pmatrix}$$

so that further simplifying all terms and exploiting that all the non-zero overlap function for a circular aperture are symmetric in x-y axis we have

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \approx \begin{pmatrix} x_2 \\ y_2 \\ L + \Delta z \end{pmatrix} + c \begin{pmatrix} 2y_2 \sin a + \sin b \left(L + \Delta z\right) \\ 2x_2 \sin d + \sin f \left(L + \Delta z\right) \\ 2\left(x_2 \sin g + y_2 \sin h\right) \end{pmatrix}$$

Where we've used the new angles a, b, d, f, g, h simply to show the quantities are less than 1. We disregard all small prior angles use and now use the variables found in the coordinate form. Putting this in a form we can use this gives us

$$\begin{pmatrix} \rho_1 \cos \theta_1 \\ \rho_1 \sin \theta_1 \\ z_1 \end{pmatrix} \approx \begin{pmatrix} \rho_2 \cos \theta_2 \\ \rho_2 \sin \theta_2 \\ L + \Delta z \end{pmatrix} + c \begin{pmatrix} 2\rho_2 \sin \theta_2 \sin a + \sin b \left(L + \Delta z\right) \\ 2\rho_2 \cos \theta_2 \sin d + \sin f \left(L + \Delta z\right) \\ 2\rho_2 \left(\cos \theta_2 \sin g + \sin \theta_2 \sin h\right) \end{pmatrix}$$

so that to first order in c we have

$$\rho_1^2 = \rho_2^2 + 2c\rho_2 \left(2\rho_2 \sin \theta_2 \cos \theta_2 \left(\sin a + \sin d\right) + \left(L + \Delta_z\right)\left(\cos \theta_2 \sin b + \sin \theta_2 \sin f\right)\right)$$

so that to first order in c we have

$$\rho_1^{2n} = \rho_2^{2n} + 2nc\rho_2^{2(n-1)} \left(2\rho_2^2 \sin \theta_2 \cos \theta_2 \left(\sin a + \sin d\right) + \rho_2 \left(L + \Delta z\right)\left(\cos \theta_2 \sin b + \sin \theta_2 \sin f\right)\right)$$

We also have that to first order in c

$$\left(\rho_1 \cos \theta_1\right)^n \approx \left(\rho_2 \cos \theta_2\right)^n + nc\left(\rho_2 \cos_2 \theta\right)^{n-1}\left(2\rho_2 \sin \theta_2 \sin a + \sin b \left(L + \Delta z\right)\right)$$
$$\left(\rho_1 \sin \theta_1\right)^n \approx \left(\rho_2 \sin \theta_2\right)^n + nc\left(\rho_2 \sin \theta_2\right)^{n-1}\left(2\rho_2 \cos \theta_2 \sin d + \sin f \left(L + \Delta z\right)\right)$$
$$\left(\frac{z_1}{z_r}\right)^2 \approx \left(\frac{L + \Delta z}{z_r}\right)^2 + \frac{4\rho_2 c \left(L + \Delta z\right)}{z_r^2}\left(\cos \theta_2 \sin g + \sin \theta_2 \sin h\right)$$
$$\implies \frac{1}{\omega(z_1)^2} = \frac{1}{\omega_0^2 \left(1 + \left(\frac{z}{z_r}\right)^2\right)} = \frac{1}{\omega_0^2 \left(1 + \left(\frac{L+\Delta z}{z_r}\right)^2 + \frac{4c(L+\Delta z)}{z_r^2}\left(x_2 \sin g + y_2 \sin h\right)\right)}$$

$$\approx \frac{1}{\omega\left(L+\Delta z\right)^2} - \frac{\frac{4c\rho_2(L+\Delta z)}{z_r^2}\left(\cos\theta_2\sin g+\sin\theta_2\sin h\right)}{\omega_0^2\left(1+\left(\frac{L+\Delta z}{z_r}\right)^2\right)^2}$$

$$= \frac{1}{\omega(L+\Delta z)^2} - \frac{4c\rho_2\omega_0^2\left(L+\Delta z\right)\left(\cos\theta_2\sin g+\sin\theta_2\sin h\right)}{z_r^2\omega(L+\Delta z)^4}$$

so that we have

$$\frac{1}{\omega(z_1)^{2n}} \approx \frac{1}{\omega(L+\Delta z)^{2n}} - \frac{4nc\rho_2\omega_0^2\left(L+\Delta z\right)\left(\cos\theta_2\sin g+\sin\theta_2\sin h\right)}{z_r^2\omega\left(L+\Delta z\right)^{2(n+1)}}$$

The final term we will need to expand in terms of c for our analysis is

$$\frac{1}{\omega_0^2\left(1-i\frac{z_1}{z_r}\right)} \approx \frac{1}{\omega_0^2\left(1-i\frac{L+\Delta z}{z_r}\right)} + \frac{2ic\rho_2\left(\cos\theta_2\sin g+\sin\theta_2\sin h\right)}{z_r\omega_0^2\left(1-i\frac{L+\Delta z}{z_r}\right)^2}$$

so that for the exponential term

$$-\frac{\rho_1^2}{\omega_0^2\left(1-i\frac{z_1}{z_r}\right)} \approx -\left(\rho_2^2+2c\left(2\rho_2^2\sin\theta_2\cos\theta_2\left(\sin a+\sin d\right)+\rho_2\left(L+\Delta z\right)\left(\cos\theta_2\sin b+\sin\theta_2\sin f\right)\right)\right)$$

$$\times\left(\frac{1}{\omega_0^2\left(1-i\frac{L+\Delta z}{z_r}\right)} + \frac{2ic\rho_2\left(\cos\theta_2\sin g+\sin\theta_2\sin h\right)}{z_r\omega_0^2\left(1-i\frac{L+\Delta z}{z_r}\right)^2}\right)$$

$$= -\frac{\rho_2^2}{\omega_0^2\left(1-i\frac{L+\Delta z}{z_r}\right)}$$

$$-\frac{2c\rho_2}{\omega_0^2\left(1-i\frac{L+\Delta z}{z_r}\right)}\left(\left(2\rho_2\sin\theta_2\cos\theta_2\left(\sin a+\sin d\right)+\left(L+\Delta z\right)\left(\cos\theta_2\sin b+\sin\theta_2\sin f\right)\right)\right.$$

$$\left.+i\frac{\rho_2^2\left(\cos\theta_2\sin g+\sin\theta_2\sin h\right)}{\left(z_r-i\left(L+\Delta z\right)\right)}\right)$$

so that for

$$-\frac{\rho_2^2}{\omega_0^2\left(1-i\frac{z_1}{z_r}\right)} - ikz_1 = -\frac{\rho_2^2}{\omega_0^2\left(1-i\frac{L+\Delta z}{z_r}\right)} - ik\left(L+\Delta z\right)$$

$$-2c\rho_2\left(\frac{2\rho_2\sin\theta_2\cos\theta_2\left(\sin a+\sin d\right)+\left(L+\Delta z\right)\left(\cos\theta_2\sin b+\sin\theta_2\sin f\right)}{\omega_0^2\left(1-i\frac{L+\Delta z}{z_r}\right)}\right.$$

$$\left.+i\left(k\left(\cos\theta_2\sin g+\sin\theta_2\sin h\right)+\frac{\rho_2^2\left(\cos\theta_2\sin g+\sin\theta_2\sin h\right)}{z_r\omega_0^2\left(1-i\frac{L+\Delta z}{z_r}\right)^2}\right)\right)$$

so that the exponential factor in each of the HG modes is

$$\mathrm{e}^{-\frac{\rho_1^2}{\omega_0^2\left(1-i\frac{z_1}{z_r}\right)}-ikz} \approx \mathrm{e}^{-\frac{\rho_2^2}{\omega_0^2\left(1-i\frac{L+\Delta z}{z_r}\right)}-ik(L+\Delta z)}$$

$$\times\left(1-2c\rho_2\left(\frac{2\rho_2\sin\theta_2\cos\theta_2\left(\sin a+\sin d\right)+\left(L+\Delta z\right)\left(\cos\theta_2\sin b+\sin\theta_2\sin f\right)}{\omega_0^2\left(1-i\frac{L+\Delta z}{z_r}\right)}\right.\right.$$

$$\left.\left.+i\left(k\left(\cos\theta_2\sin g+\sin\theta_2\sin h\right)+\frac{\rho_2^2\left(\cos\theta_2\sin g+\sin\theta_2\sin h\right)}{z_r\omega_0^2\left(1-i\frac{L+\Delta z}{z_r}\right)^2}\right)\right)\right)$$

We now generalize after replacing $\sin a+\sin d=2\sin\alpha$

$$G(m,n,p,q)\equiv\frac{\rho_1^{2m+n+p}\sin^n\theta_1\cos^p\theta_1}{\omega(z_1)^{2q}}\mathrm{e}^{-\frac{\rho_1^2}{\omega_0^2\left(1-i\frac{z_1}{z_r}\right)}-ikz_1}$$

$$\approx\left(\rho_2^{2m}+2cm\rho_2^{2(m-1)}\left(4\rho_2^2\sin\theta_2\cos\theta_2\sin\alpha+\rho_2\left(L+\Delta z\right)\left(\cos\theta_2\sin b+\sin\theta_2\sin f\right)\right)\right)$$

$$\times\left(\rho_2\sin\theta_2\right)^{n-1}\left(\rho_2\sin\theta_2+nc\left(2\rho_2\cos\theta_2\sin d+\sin f\left(L+\Delta z\right)\right)\right)$$

$$\times\left(\rho_2\cos\theta_2\right)^{p-1}\left(\rho_2\cos\theta_2+pc\left(2\rho_2\sin\theta_2\sin a+\sin b\left(L+\Delta z\right)\right)\right)$$

$$\times\frac{1}{\omega\left(L+\Delta z\right)^{2q}}\left(1-\frac{4qc\rho_2\omega_0^2\left(L+\Delta z\right)\left(\cos\theta_2\sin g+\sin\theta_2\sin h\right)}{z_r^2\omega\left(L+\Delta z\right)^2}\right)$$

$$\times\mathrm{e}^{-\frac{\rho_2^2}{\omega_0^2\left(1-i\frac{L+\Delta z}{z_r}\right)}-ik(L+\Delta z)}\left(1-2c\rho_2\left(\frac{4\rho_2\sin\theta_2\cos\theta_2\sin\alpha+\left(L+\Delta z\right)\left(\cos\theta_2\sin b+\sin\theta_2\sin f\right)}{\omega_0^2\left(1-i\frac{L+\Delta z}{z_r}\right)}\right.\right.$$

$$\left.\left.+i\left(\cos\theta_2\sin g+\sin\theta_2\sin h\right)\left(k+\frac{\rho_2^2}{z_r\omega_0^2\left(1-i\frac{L+\Delta z}{z_r}\right)^2}\right)\right)\right)$$

$$\approx\mathrm{e}^{-\frac{\rho_2^2}{\omega_0^2\left(1-i\frac{L+\Delta z}{z_r}\right)}-ik(L+\Delta z)}\frac{\rho_2^{2m+n+p}\sin^n\theta_2\cos^p\theta_2}{\omega\left(L+\Delta z\right)^{2q}}$$

$$\times\left(1+c\left(2m\left(4\sin\theta_2\cos\theta_2\sin\alpha+\frac{L+\Delta z}{\rho_2}\left(\cos\theta_2\sin b+\sin\theta_2\sin f\right)\right)\right.\right.$$

$$+\frac{n}{\sin\theta_2}\left(2\cos\theta_2\sin d+\frac{\sin f}{\rho_2}\left(L+\Delta z\right)\right)+\frac{p}{\cos\theta_2}\left(2\sin\theta_2\sin a+\frac{\sin b}{\rho_2}\left(L+\Delta z\right)\right)$$

$$\left.\left.-\frac{4q\rho_2\omega_0^2\left(L+\Delta z\right)\left(\cos\theta_2\sin g+\sin\theta_2\sin h\right)}{z_r^2\omega\left(L+\Delta z\right)^2}\right)\right)$$

So that when it comes to rewriting HG TEM modes expanded out in terms of another and these variables to first order in c. We write down the TEM mode first

$$A_{n,m}(x,y,z;\omega_0,k) = \frac{\left(1+i\frac{z}{z_r}\right)^{\frac{m+n}{2}}}{\omega_0\sqrt{\pi n!m!}2^{\frac{m+n-1}{2}}\left(1-i\frac{z}{z_r}\right)^{\frac{m+n+2}{2}}} H_n\left(\frac{\sqrt{2}x}{\omega_0\sqrt{\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\right)$$

$$H_m\left(\frac{\sqrt{2}y}{\omega_0\sqrt{\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\right)\mathrm{e}^{-\frac{(x^2+y^2)}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}$$

reducing to the parts that depend upon orientation angles and locations as well as coefficients we can rewrite this as

$$A_{n,m} = \frac{C_{n,m}D(z)^{m+n}}{\left(1-i\frac{z}{z_r}\right)}H_n\left(\frac{x}{F(z)}\right)H_m\left(\frac{y}{F(z)}\right)\mathrm{e}^{-\frac{(x^2+y^2)}{\omega_0^2\left(1-i\frac{z}{z_r}\right)}-ikz}$$

where $D(z) = \sqrt{\frac{\left(1+i\frac{z}{z_r}\right)}{\left(1-i\frac{z}{z_r}\right)}}$, $C_{n,m} = \frac{1}{\omega_0\sqrt{\pi n!m!2^{m+n-1}}}$, and $F(z) = \frac{\omega_0\sqrt{1+\left(\frac{z}{z_r}\right)^2}}{\sqrt{2}}$.

We use the recursion relationships for Hermitian polynomials

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

and

$$H_{n+1}(x) = 2xH_n(x) - H_n'(x)$$

using these we write down

$$H_n'(x) = 2nH_{n-1}(x)$$

$$xH_n(x) = \frac{H_{n+1}(x)}{2} + nH_{n-1}(x)$$

$$x^2H_n(x) = x\left[\frac{H_{n+1}(x)}{2} + nH_{n-1}(x)\right] = \frac{H_{n+2}(x)}{4} + \frac{n+1}{2}H_n(x) + \frac{n}{2}H_n(x) + n(n-1)H_{n-2}(x)$$

$$x^2H_n(x) = \frac{H_{n+2}(x)}{4} + \left(n+\frac{1}{2}\right)H_n(x) + n(n-1)H_{n-2}(x)$$

We take the derivative of these elements to find the expanded solution,

$$A_{n,m}(x_1,y_1,z_1) \approx A_{n,m}(x_2,y_2,L+\Delta z)$$

$$+c\partial_c\left[\frac{C_{n,m}D(z_1(c))^{m+n}}{\left(1-i\frac{z_1(c)}{z_r}\right)}H_n\left(\frac{x_1(c)}{F(z_1(c))}\right)H_m\left(\frac{y_1(c)}{F(z_1(c))}\right)\mathrm{e}^{-\frac{(x_1(c)^2+y_1(c)^2)}{\omega_0^2\left(1-i\frac{z_1(c)}{z_r}\right)}-ikz_1(c)}\right]$$

$$= A_{n,m}(x_2, y_2, L+\Delta z) + c\left(\frac{(m+n)A_{n,m}(x_2, y_2, L+\Delta z)}{D(L+\Delta z)}\right.$$

$$+\frac{2iC_{n,m}D(L+\Delta z)^{n+m}\left(x_2\sin g + y_2\sin h\right)H_n\left(\frac{x_2}{F(L+\Delta z)}\right)H_m\left(\frac{y_2}{F(L+\Delta z)}\right)}{z_r\left(1-i\frac{L+\Delta z}{z_r}\right)^2}\mathrm{e}^{-\frac{x_2^2+y_2^2}{\omega_0^2\left(1-i\frac{L+\Delta z}{z_r}\right)}-ik(L+\Delta z)}$$

$$+\frac{C_{n,m}D(L+\Delta z)^{m+n}}{1-i\frac{L+\Delta z}{z_r}}\mathrm{e}^{-\frac{x_2^2+y_2^2}{\omega_0^2\left(1-i\frac{L+\Delta z}{z_r}\right)}-ik(L+\Delta z)}$$

$$\times\left(H_n'\left(\frac{x_2}{F\left(L+\Delta z\right)}\right)H_m\left(\frac{y_2}{F\left(L+\Delta z\right)}\right)\left(\frac{2y_2\sin a+\sin b\left(L+\Delta z\right)}{F\left(L+\Delta z\right)}+x_2\partial_c\left(\frac{1}{F(z_1(c))}\right)\right)\right.$$

$$\left.+H_n\left(\frac{x_2}{F\left(L+\Delta z\right)}\right)H_m'\left(\frac{y_2}{F\left(L+\Delta z\right)}\right)\left(\frac{2x_2\sin d+\sin f\left(L+\Delta z\right)}{F\left(L+\Delta z\right)}+y_2\partial_c\left(\frac{1}{F(z_1(c))}\right)\right)\right)$$

$$-\frac{C_{n,m}D(L+\Delta z)^{m+n}}{1-i\frac{L+\Delta z}{z_r}}H_n\left(\frac{x_2}{F(L+\Delta z)}\right)H_m\left(\frac{y_2}{F(L+\Delta z)}\right)\mathrm{e}^{-\frac{x_2^2+y_2^2}{\omega_0^2\left(1-i\frac{L+\Delta z}{z_r}\right)}-ik(L+\Delta z)}$$

$$\times\left(2ik\left(x_2\sin g+y_2\sin h\right)-\frac{2\left(x_2\left(2y_2\sin a+\sin b\left(L+\Delta z\right)\right)+y_2\left(2x_2\sin d+\sin f\left(L+\Delta z\right)\right)\right)}{\omega_0^2\left(1-i\frac{L+\Delta z}{z_r}\right)}\right.$$

$$\left.\left.+\frac{x_2^2+y_2^2}{\omega_0^2}\partial_c\left(\frac{1}{1-i\frac{z_1(c)}{z_r}}\right)\right)\right)$$

where all the derivative terms are taken and then have $c \longrightarrow 0$. We first evaluate these derivative terms,

$$\partial_c\frac{1}{1-i\frac{z_1(c)}{z_r}} = \frac{2i\left(x_2\sin g+y_2\sin h\right)}{z_r\left(1-i\frac{L+\Delta z}{z_r}\right)^2}$$

$$\partial_c\frac{1}{F(z_1(c))} = \partial_c\frac{\sqrt{2}}{\omega_0\sqrt{1+\left(\frac{z_1(c)}{z_r}\right)^2}} = \frac{-2\sqrt{2}\left(L+\Delta z\right)\left(x_2\sin g+y_2\sin h\right)}{\omega_0\left(1+\left(\frac{L+\Delta z}{z_r}\right)^2\right)^{3/2}}$$

$$\partial_c D(z_1(c))|_{c=0} = \partial_{z_1}D(z_1(c))\partial_c z_1(c)$$

$$= 2\rho_2\left(\cos\theta_2\sin g+\sin\theta_2\sin h\right)\left(\frac{1}{2}\sqrt{\frac{1-i\frac{L+\Delta z}{z_r}}{1+i\frac{L+\Delta z}{z_r}}}\left(\frac{i}{z_r\left(1-i\frac{L+\Delta z}{z_r}\right)}+\frac{i\left(1+i\frac{L+\Delta z}{z_r}\right)}{z_r\left(1-i\frac{L+\Delta z}{z_r}\right)^2}\right)\right)$$

$$= \frac{i\rho_2}{z_r\left(1-i\frac{L+\Delta z}{z_r}\right)}\left(\cos\theta_2\sin g+\sin\theta_2\sin h\right)\left(\frac{2}{\sqrt{\left(1+\left(\frac{L+\Delta z}{z_r}\right)^2\right)}}\right)+$$

7. New attempt at TTL contribution

We assume the expected position for the two satellites are the origin for the first and $L+\Delta z$ in the z-direction for the second, with the first oriented with first telescope in the z-direction and second telescope on the first SC pointing into the x-z plane rotated $\pi/3$ towards the x-axis while the first telescope for the second SC is pointing towards the first SC or in the negative z direction while the second telescope in the second SC is rotated into the x-z plane. We allow a small deviation of the angular orientation of the SC using euler angles and find that for the dot product of each SC reference frame axis to be nearly one with the corresponding global axis (SC x-axis nearly along actual x-axis, etc) and allowing for some slight positional offset from the equilibrium lengths $(\Delta x_i, \Delta y_i, \Delta z_i)$ where the x-y changes are very small (the z is due to gravitational waves as well as non-inertial motion so can be larger), then we find the euler angles are restricted so that the coordinate of a point in the first SC reference frame can be written in terms of the coordinates of the second SC reference frame as

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \hat{\mathbf{R}}_z(\delta_2-\phi_1)\hat{\mathbf{R}}_y(\delta 1)\hat{\mathbf{R}}_z(\phi_1)\left[\begin{pmatrix} \Delta x \\ \Delta y \\ L+\Delta z \end{pmatrix} + \hat{\mathbf{R}}_z(-\phi_2)\hat{\mathbf{R}}_y(\delta 3)\hat{\mathbf{R}}_z(\phi_2-\delta_4)\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\right]$$

where

$$\Delta x = \Delta x_2 - \Delta x_1 \quad \Delta y = \Delta y_2 - \Delta y_1 \quad \Delta z = \Delta z_2 - \Delta z_1$$

and

$$\hat{\mathbf{R}}_z(a) = \begin{pmatrix} \cos a & \sin a & 0 \\ -\sin a & \cos a & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R_y(a) = \begin{pmatrix} \cos a & \end{pmatrix}$$

L is the length between SC in equilibrium (2.5 million kilometers), ϕ_1 is the first euler angle of the first SC, ϕ_1 and δ_1 the ϕ, θ spherical component giving the direction of the first telescope $(\delta_1, \delta_2 \ll 1)$ and ϕ_2 and $\pi - \delta_3$ the ϕ, θ spherical components of direction of second telescope.

8. Actual Decomposition of Top Hat Beam

To find the contribution $P_{n,m}$ of mode $A_{n,m}$ to the top hat beam, we calculate

$$P_{n,m} = \int_0^R \int_0^{2\pi} \mathrm{d}\theta \mathrm{d}\rho \rho A^*_{n,m}(\rho, \theta, z_s)\, I_0$$

where I_0 is the intensity or square root of the power (magnitude of electric field) over the top hat and z_s is the value of z at the surface of the tophat This gives

$$P_{0,0} = \frac{2^{1/2} I_0 \mathrm{e}^{ikz_s}}{\pi^{1/2}\omega_0\left(1+i\frac{z_s}{z_r}\right)} F\left(0, \frac{1}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}, 0, 0, R\right) = \sqrt{2\pi}\omega_0 I_0 \mathrm{e}^{ikz_s}\left(1-\mathrm{e}^{-\frac{R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\right)$$

$$P_{1,0} = P_{0,1} = P_{1,1} = P_{2,1} = P_{1,2} = 0$$

$$P_{0,2} = \frac{I_0 \mathrm{e}^{ikz_s}\left(1-i\frac{z_s}{z_r}\right)}{(2\pi)^{1/2}\,\omega_0\left(1+i\frac{z_s}{z_r}\right)^2}\left(\frac{2^2}{\omega_0^2\left(1+\left(\frac{z_s}{z_r}\right)^2\right)}F\left(2,\frac{1}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)},2,0,R\right)\right.$$

$$\left.-F\left(0,\frac{1}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)},0,0,R\right)\right)$$

$$= \frac{I_0 \mathrm{e}^{ikz_s}\left(1-i\frac{z_s}{z_r}\right)}{(2\pi)^{1/2}\,\omega_0\left(1+i\frac{z_s}{z_r}\right)^2}\left(\frac{2^2}{\omega_0^2\left(1+\left(\frac{z_s}{z_r}\right)^2\right)}\frac{\sqrt{\pi}}{2}\sqrt{\pi}\omega_0^4\left(1+i\frac{z_s}{z_r}\right)^2\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\left(1+\frac{R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}\right)\right)\right.$$

$$\left.-\pi\omega_0^2\left(1+i\frac{z_s}{z_r}\right)\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\right)\right)$$

$$= \frac{\sqrt{\pi}\omega_0 I_0 \mathrm{e}^{ikz_s}\left(1-i\frac{z_s}{z_r}\right)}{\sqrt{2}\left(1+i\frac{z_s}{z_r}\right)}\left(\frac{2}{\left(1-i\frac{z_s}{z_r}\right)}\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\left(1+\frac{R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}\right)\right)-\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\right)\right)$$

$$= \frac{\sqrt{\pi}\omega_0 I_0 \mathrm{e}^{ikz_s}}{\sqrt{2}\left(1+i\frac{z_s}{z_r}\right)}\left(2\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\left(1+\frac{R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}\right)\right)-\left(1-i\frac{z_s}{z_r}\right)\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\right)\right)$$

so altogether

$$P_{0,0} = \sqrt{2\pi}\omega_0 I_0 \mathrm{e}^{ikz_s}\left(1-\mathrm{e}^{-\frac{R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\right)$$

$$P_{0,2} = P_{2,0} = \frac{\sqrt{\pi}\omega_0 I_0 \mathrm{e}^{ikz_s}}{\sqrt{2}\left(1+i\frac{z_s}{z_r}\right)}\left(2\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\left(1+\frac{R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}\right)\right)-\left(1-i\frac{z_s}{z_r}\right)\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\right)\right)$$

Now for the next one we have

$$P_{2,2} = \frac{I_0 \mathrm{e}^{ikz_s}\left(1-i\frac{z_s}{z_r}\right)^2}{2^{3/2}\pi^{1/2}\omega_0\left(1+i\frac{z_s}{z_r}\right)^3}\left(\frac{2^4}{\omega_0^4\left(1+\left(\frac{z_s}{z_r}\right)^2\right)^2}F\left(4,\frac{1}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)},2,2,R\right)\right.$$

$$\left.-\frac{2^2}{\omega_0^2\left(1+\left(\frac{z_s}{z_r}\right)^2\right)}F\left(2,\frac{1}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)},0,0,R\right)+F\left(0,\frac{1}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)},0,0,R\right)\right)$$

$$
\begin{aligned}
&= \frac{I_0 \mathrm{e}^{ikz_s}\left(1 - i\frac{z_s}{z_r}\right)^2}{2^{3/2}\pi^{1/2}\omega_0\left(1 + i\frac{z_s}{z_r}\right)^3} \\
&\times \left(\frac{2^4}{\omega_0^4\left(1+\left(\frac{z_s}{z_r}\right)^2\right)^2}\left(\frac{\pi\omega_0^6\left(1+i\frac{z_s}{z_r}\right)^3}{4}\right)\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\left(1+\frac{R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}+\frac{R^4}{2\omega_0^4\left(1+i\frac{z_s}{z_r}\right)^2}\right)\right)\right. \\
&\quad - \frac{2^2}{\omega_0^2\left(1+\left(\frac{z_s}{z_r}\right)^2\right)}\left(\pi\omega_0^4\left(1+i\frac{z_s}{z_r}\right)^2\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\left(1+\frac{R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}\right)\right)\right) \\
&\quad \left. + \left(\pi\omega_0^2\left(1+i\frac{z_s}{z_r}\right)\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\right)\right)\right)
\end{aligned}
$$

$$= \frac{\sqrt{\pi}\omega_0 I_0 \mathrm{e}^{ikz}\left(1-i\frac{z_s}{z_r}\right)^2}{2^{3/2}\left(1+i\frac{z_s}{z_r}\right)^2}$$

$$\times\left(\frac{4}{\left(1-i\frac{z_s}{z_r}\right)^2}\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\left(1+\frac{R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}+\frac{R^4}{2\omega_0^4\left(1+i\frac{z_s}{z_r}\right)^2}\right)\right)\right.$$

$$\left.-\frac{4}{\left(1-i\frac{z_s}{z_r}\right)}\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\left(1+\frac{R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}\right)\right)+\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\right)\right)$$

Then we have

$$P_{3,0}=P_{0,3}=P_{3,1}=P_{1,3}=P_{3,2}=P_{2,3}=P_{3,3}=P_{3,4}=P_{4,3}=0$$

The next difficult one is

$$P_{0,4}=\frac{I_0\left(1-i\frac{z_s}{z_r}\right)^2\mathrm{e}^{ikz_s}}{3*2^{5/2}\pi^{1/2}\omega_0\left(1+i\frac{z_s}{z_r}\right)^3}\left(\frac{2^4}{\omega_0^4\left(1+\left(\frac{z_s}{z_r}\right)^2\right)^2}F\left(4,\frac{1}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)},4,0,R\right)\right.$$

$$\left.-\frac{3*2^3}{\omega_0^2\left(1+\left(\frac{z_s}{z_r}\right)^2\right)}F\left(2,\frac{1}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)},2,0,R\right)+3F\left(0,\frac{1}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)},0,0,R\right)\right)$$

$$=\frac{I_0\left(1-i\frac{z_s}{z_r}\right)^2\mathrm{e}^{ikz_s}}{3*2^{5/2}\pi^{1/2}\omega_0\left(1+i\frac{z_s}{z_r}\right)^3}$$

$$\times\left(\frac{2^4}{\omega_0^4\left(1+\left(\frac{z_s}{z_r}\right)^2\right)^2}\left(\frac{3\pi\omega_0^6\left(1+i\frac{z_s}{z_r}\right)^3}{4}\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\left(1+\frac{R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}+\frac{R^4}{2\omega_0^4\left(1+i\frac{z_s}{z_r}\right)^2}\right)\right)\right)\right.$$

$$-\frac{3*2^3}{\omega_0^2\left(1+\left(\frac{z_s}{z_r}\right)^2\right)}\left(\frac{\pi\omega_0^4\left(1+i\frac{z_s}{z_r}\right)^2}{2}\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\left(1+\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}\right)\right)\right)$$

$$\left.+3\pi\omega_0^2\left(1+i\frac{z_s}{z_r}\right)\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\right)\right)$$

or

$$P_{0,4} = P_{4,0} = \frac{\sqrt{\pi}\omega_0 I_0 \left(1 - i\frac{z_s}{z_r}\right)^2 \mathrm{e}^{ikz_s}}{2^{5/2}\left(1 + i\frac{z_s}{z_r}\right)^2} \left(\frac{2^2}{\left(1 - i\frac{z_s}{z_r}\right)^2} \left(1 - \mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}} \left(1 + \frac{R^2}{\omega_0^2\left(1 + i\frac{z_s}{z_r}\right)} + \frac{R^4}{2\omega_0^4\left(1 + i\frac{z_s}{z_r}\right)^2}\right)\right)\right.$$

$$\left. - \frac{2^2}{\left(1 - i\frac{z_s}{z_r}\right)} \left(1 - \mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}} \left(1 + \frac{R^2}{\omega_0^2\left(1 + i\frac{z_s}{z_r}\right)}\right)\right) + \left(1 - \mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\right)\right)$$

Now for the next one

$$P_{2,4} = \frac{I_0 \mathrm{e}^{ikz_s}\left(1 - i\frac{z_s}{z_r}\right)^3}{3 * 2^{7/2}\pi^{1/2}\omega_0\left(1 + i\frac{z_s}{z_r}\right)^4}$$

$$\times \left(\frac{2^6}{\omega_0^6\left(1 + \left(\frac{z_s}{z_r}\right)^2\right)^3} F\left(6, \frac{1}{\omega_0^2\left(1 + i\frac{z_s}{z_r}\right)}, 4, 2, R\right)\right.$$

$$- \frac{2^4}{\omega_0^4\left(1 + \left(\frac{z_s}{z_r}\right)^2\right)^2} \left(F\left(4, \frac{1}{\omega_0^2\left(1 + i\frac{z_s}{z_r}\right)}, 2, 0, R\right) + 5 * F\left(4, \frac{1}{\omega_0^2\left(1 + i\frac{z_s}{z_r}\right)}, 2, 2, R\right)\right)$$

$$+ \frac{3 * 2^2}{\omega_0^2\left(1 + \left(\frac{z_s}{z_r}\right)^2\right)} \left(F\left(2, \frac{1}{\omega_0^2\left(1 + i\frac{z_s}{z_r}\right)}, 0, 0, R\right) + F\left(2, \frac{1}{\omega_0^2\left(1 + i\frac{z_s}{z_r}\right)}, 2, 0, R\right)\right) - 3F\left(0, \frac{1}{\omega_0^2\left(1 + i\frac{z_s}{z_r}\right)}, \right.$$

$$= \frac{I_0 \mathrm{e}^{ikz_s}\left(1 - i\frac{z_s}{z_r}\right)^3}{3 * 2^{7/2}\pi^{1/2}\omega_0\left(1 + i\frac{z_s}{z_r}\right)^4}$$

$$\times \left(\frac{2^6}{\omega_0^6\left(1 + \left(\frac{z_s}{z_r}\right)^2\right)^3} \frac{3\pi\omega_0^8\left(1 + i\frac{z_s}{z_r}\right)^4}{2^3} \left(1 - \mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}} \left(1 + \frac{R^2}{\omega_0^2\left(1 + i\frac{z_s}{z_r}\right)} + \frac{R^4}{2\omega_0^4\left(1 + i\frac{z_s}{z_r}\right)^2} + \frac{R^6}{\omega_0^6\left(1 + i\frac{z_s}{z_r}\right)} \right.\right.\right.$$

$$- \frac{2^4}{\omega_0^4\left(1 + \left(\frac{z_s}{z_r}\right)^2\right)^2} \left(1 - \mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}} \left(1 + \frac{R^2}{\omega_0^2\left(1 + i\frac{z_s}{z_r}\right)} + \frac{R^4}{2\omega_0^4\left(1 + i\frac{z_s}{z_r}\right)^2}\right)\right) \pi\omega_0^6\left(1 + i\frac{z_s}{z_r}\right)^3\left(1 + \frac{5}{4}\right)$$

$$+\frac{3*2^2}{\omega_0^2\left(1+\left(\frac{z_s}{z_r}\right)^2\right)}\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\left(1+\frac{R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}\right)\right)\pi\omega_0^4\left(1+i\frac{z_s}{z_r}\right)^2\left(\frac{1}{2}+1\right)$$

$$-3\pi\omega_0^2\left(1+i\frac{z_s}{z_r}\right)\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\right)\Bigg)$$

$$=\frac{\sqrt{\pi}\omega_0 I_0\mathrm{e}^{ikz_s}\left(1-i\frac{z_s}{z_r}\right)^3}{2^{7/2}\left(1+i\frac{z_s}{z_r}\right)^3}$$

$$\times\left(\frac{2^3}{\left(1-i\frac{z_s}{z_r}\right)^3}\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\left(1+\frac{R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}+\frac{R^4}{2\omega_0^4\left(1+i\frac{z_s}{z_r}\right)^2}+\frac{R^6}{2*3\omega_0^6\left(1+i\frac{z_s}{z_r}\right)^3}\right)\right)\right.$$

$$-\frac{3*2^2}{\left(1-i\frac{z_s}{z_r}\right)^2}\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\left(1+\frac{R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}+\frac{R^4}{2\omega_0^4\left(1+i\frac{z_s}{z_r}\right)^2}\right)\right)$$

$$\left.+\frac{2*3}{\left(1-i\frac{z_s}{z_r}\right)}\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\left(1+\frac{R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}\right)\right)-\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\right)\right)$$

So that we finally have

$$P_{2,4}=P_{4,2}=\frac{\sqrt{\pi}\omega_0 I_0\mathrm{e}^{ikz_s}\left(1-i\frac{z_s}{z_r}\right)^3}{2^{7/2}\left(1+i\frac{z_s}{z_r}\right)^3}$$

$$\times\left(\frac{2^3}{\left(1-i\frac{z_s}{z_r}\right)^3}\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\left(1+\frac{R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}+\frac{R^4}{2\omega_0^4\left(1+i\frac{z_s}{z_r}\right)^2}+\frac{R^6}{2*3\omega_0^6\left(1+i\frac{z_s}{z_r}\right)^3}\right)\right)\right.$$

$$-\frac{3*2^2}{\left(1-i\frac{z_s}{z_r}\right)^2}\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\left(1+\frac{R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}+\frac{R^4}{2\omega_0^4\left(1+i\frac{z_s}{z_r}\right)^2}\right)\right)$$

$$\left.+\frac{2*3}{\left(1+i\frac{z_s}{z_r}\right)}\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\left(1+\frac{R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}\right)\right)-\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\right)\right)$$

For our final coefficient we have

$$P_{4,4}=\frac{I_0\mathrm{e}^{ikz_s}\left(1-i\frac{z_s}{z_r}\right)^4}{3*2^{11/2}\pi^{1/2}\omega_0\left(1+i\frac{z_s}{z_r}\right)^5}$$

$$\times\left(\frac{2^8}{\omega_0^8\left(1+\left(\frac{z_s}{z_r}\right)^2\right)^4}F\left(8,\frac{1}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)},4,4,R\right)-\frac{3*2^7}{\omega_0^6\left(1+\left(\frac{z_s}{z_r}\right)^2\right)^3}F\left(6,\frac{1}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)},2,2,R\right)\right.$$

$$+\frac{3*2^4}{\omega_0^4\left(1+\left(\frac{z_s}{z_r}\right)^2\right)^2}\left(F\left(4,\frac{1}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)},0,0,R\right)+10F\left(4,\frac{1}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)},2,2,R\right)\right)$$

$$\left.-\frac{3^2 2^3}{\omega_0^2\left(1+\left(\frac{z_s}{z_r}\right)^2\right)}F\left(2,\frac{1}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)},0,0,R\right)+9F\left(0,\frac{1}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)},0,0,R\right)\right)$$

Filling these out we get

$$P_{4,4}=\frac{I_0\mathrm{e}^{ikz_s}\left(1-i\frac{z_s}{z_r}\right)^4}{3*2^{11/2}\pi^{1/2}\omega_0\left(1+i\frac{z_s}{z_r}\right)^5}$$

$$\times\left(\frac{\pi 3^2 2^4\omega_0^2\left(1+i\frac{z_s}{z_r}\right)^5}{\left(1+\left(\frac{z_s}{z_r}\right)^2\right)^4}\right.$$

$$\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\left(1+\frac{R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}+\frac{R^4}{2\omega_0^4\left(1+i\frac{z_s}{z_r}\right)^2}+\frac{R^6}{2*3\omega_0^6\left(1+i\frac{z_s}{z_r}\right)^3}+\frac{R^8}{2^3*3\omega_0^8\left(1+i\frac{z_s}{z_r}\right)^4}\right)\right)$$

$$-\frac{\pi 3^2*2^5\omega_0^2\left(1+i\frac{z_s}{z_r}\right)^4}{\left(1+\left(\frac{z_s}{z_r}\right)^2\right)^3}\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\left(1+\frac{R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}+\frac{R^4}{2\omega_0^4\left(1+i\frac{z_s}{z_r}\right)^2}+\frac{R^6}{2*3\omega_0^6\left(1+i\frac{z_s}{z_r}\right)^3}\right)\right)$$

$$+\frac{\pi 3*2^5\omega_0^2\left(1+i\frac{z_s}{z_r}\right)^3}{\left(1+\left(\frac{z_s}{z_r}\right)^2\right)^2}\left(1+\frac{5}{4}\right)\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\left(1+\frac{R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}+\frac{R^4}{2\omega_0^4\left(1+i\frac{z_s}{z_r}\right)^2}\right)\right)$$

$$\left.-\frac{\pi 3^2 2^3\omega_0^2\left(1+i\frac{z_s}{z_r}\right)^2}{\left(1+\left(\frac{z_s}{z_r}\right)^2\right)}\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\left(1+\frac{R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}\right)\right)+3^2\pi\omega_0^2\left(1+i\frac{z_s}{z_r}\right)\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\right)\right)$$

Simplifying we get

$$P_{4,4} = \frac{3\sqrt{\pi} I_0 \omega_0 \mathrm{e}^{ikz_s}}{2^{11/2}\left(1+i\frac{z_s}{z_r}\right)^4}\times$$

$$\left(2^4\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\left(1+\frac{R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}+\frac{R^4}{2\omega_0^4\left(1+i\frac{z_s}{z_r}\right)^2}+\frac{R^6}{2*3\omega_0^6\left(1+i\frac{z_s}{z_r}\right)^3}+\frac{R^8}{2^3*3\omega_0^8\left(1+i\frac{z_s}{z_r}\right)^4}\right)\right)\right)$$

$$-2^5\left(1-i\frac{z_s}{z_r}\right)\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\left(1+\frac{R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}+\frac{R^4}{2\omega_0^4\left(1+i\frac{z_s}{z_r}\right)^2}+\frac{R^6}{2*3\omega_0^6\left(1+i\frac{z_s}{z_r}\right)^3}\right)\right)\right)$$

$$+3*2^3\left(1-i\frac{z_s}{z_r}\right)^2\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\left(1+\frac{R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}+\frac{R^4}{2\omega_0^4\left(1+i\frac{z_s}{z_r}\right)^2}\right)\right)\right)$$

$$-2^3\left(1-i\frac{z}{z_r}\right)^3\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\left(1+\frac{R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}\right)\right)\right)+\left(1-i\frac{z_s}{z_r}\right)^4\left(1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\right)\right)\right)$$

We now simplify these expressions using the notation $\left(\mathrm{e}^{\frac{R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\right)_n$ to be the first n elements of the tailor expansion for $\mathrm{e}^{\frac{R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}$ and then defining

$$E_n \equiv 1-\mathrm{e}^{\frac{-R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\left(\mathrm{e}^{\frac{R^2}{\omega_0^2\left(1+i\frac{z_s}{z_r}\right)}}\right)_n$$

so that these become

$$P_{0,0} = \sqrt{2\pi}\omega_0 I_0 \mathrm{e}^{ikz_s} E_1$$

$$P_{0,1} = P_{1,0} = P_{1,1} = P_{1,2} = P_{2,1} = 0$$

$$P_{0,2} = P_{2,0} = \frac{\sqrt{\pi}\omega_0 I_0 \mathrm{e}^{ikz_s}}{\sqrt{2}\left(1+i\frac{z_s}{z_r}\right)}\left(2E_2-\left(1-i\frac{z_s}{z_r}\right)E_1\right)$$

$$P_{2,2} = \frac{\sqrt{\pi}\omega_0 I_0 \mathrm{e}^{ikz_s}}{2^{3/2}\left(1+i\frac{z_s}{z_r}\right)^2}\left(2^2E_3-2^2\left(1-i\frac{z_s}{z_r}\right)E_2+\left(1-i\frac{z_s}{z_r}\right)^2E_1\right)$$

$$P_{1,3} = P_{3,1} = P_{2,3} = P_{3,2} = P_{3,3} = P_{3,4} = P_{4,3} = 0$$

$$P_{0,4} = P_{4,0} = \frac{\sqrt{\pi}\omega_0 I_0 \mathrm{e}^{ikz_s}}{2^{5/2}\left(1+i\frac{z_s}{z_r}\right)^2}\left(2^2E_3-2^2\left(1-i\frac{z_s}{z_r}\right)E_2+\left(1-i\frac{z_s}{z_r}\right)^2E_1\right)$$

$$P_{2,4} = P_{4,2} = \frac{\sqrt{\pi}\omega_0 I_0 \mathrm{e}^{ikz_s}}{2^{7/2}\left(1+i\frac{z_s}{z_r}\right)^3}\left(2^3 E_4 - 3*2^2\left(1-i\frac{z_s}{z_r}\right)E_3 + 3*2\left(1-i\frac{z_s}{z_r}\right)^2 E_2 - \left(1-i\frac{z_s}{z_r}\right)^3 E_1\right)$$

$$P_{4,4} = \frac{3\sqrt{\pi}I_0\omega_0 \mathrm{e}^{ikz_s}}{2^{11/2}\left(1+i\frac{z_s}{z_r}\right)^4}\left(2^4 E_5 - 2^5\left(1-i\frac{z_s}{z_r}\right)E_4 + 3*2^3\left(1-i\frac{z_s}{z_r}\right)^2 E_3 - 2^3\left(1-i\frac{z_s}{z_r}\right)^3 E_2 + \left(1-i\frac{z_s}{z_r}\right)^4 .\right.$$

9. Calculating Overlap after Propagation.

We first calculate the deviation in placement necessary to bring the beam's center along to a plane. After some calculation, if we are talking about 1-dimensional angular perturbation and displacement of satellites, we have (where x_0, y_0, and z_0 is the position on the plane (of either the aperture or where the top hat forms) that the coordinates of the equation to use are

$$z(x_0, y_0, z_0) = \frac{z_0}{\cos\Delta\theta} + (y_0 - \Delta y - z_0\tan\Delta\theta)\sin\Delta\theta$$
$$x(x_0, y_0, z_0) = x_0$$
$$y(x_0, y_0, z_0) = (y_0 - \Delta y - z_0\tan\Delta\theta)\cos\Delta\theta$$

10. Other Type of Overlap

So this was one description of the top hat beam, but it appears we were actually going to achieve the top hat by truncating or clipping a Gaussian Beam. We start off with the lowest order TEM modes, using a clipping of radius R. We get integrals of the form

$$P_{j,k} = \int_0^R \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta U_j^*(\rho\cos\theta, z_0)U_k^*(\rho\sin\theta, z_0)\sqrt{\frac{2}{\pi}}\frac{I_0 \mathrm{e}^{\frac{-\left(x(\rho,\theta,z_0)^2 + y(\rho,\theta,z_0)^2\right)}{\omega_0^2\left(1-i\frac{z(\rho,\theta,z_0)}{z_r}\right)} - ikz(\rho,\theta,z_0)}}{\omega_0\left(1-i\frac{z(\rho,\theta,z_0)}{z_r}\right)}$$

Rewriting in terms of ρ and θ we have

$$z(\rho, \theta, z_0) = \frac{z_0}{\cos\Delta\theta} + (\rho\sin\theta - \Delta y - z_0\tan\Delta\theta)\sin\Delta\theta$$
$$x(\rho, \theta, z_0) = \rho\cos\theta$$
$$y(\rho, \theta, z_0) = (\rho\sin\theta - \Delta y - z_0\tan\Delta\theta)\cos\Delta\theta$$

Note that z_0 is millions of kilometers. We first look at the dependence on $\Delta\theta$ and Δy, expanding to low orders since Δy is many orders of magnitude lower than z_0 and Δy must be small for it to make it through the aperture for reasonable R. We expand to first order (for now) in each of these.

$$f(\Delta\theta, \Delta y) = \frac{\mathrm{e}^{\frac{-\left(\rho^2\cos^2\theta + y^2(\Delta\theta,\Delta y)\right)}{\omega_0^2\left(1-i\frac{z(\Delta\theta,\Delta y)}{z_r}\right)} - ikz(\Delta\theta,\Delta y)}}{1 - i\frac{z(\Delta\theta,\Delta y)}{z_r}}$$

Where we understand that y and z still depend on z_0, ρ, and θ but we don't include these explicitly since we aren't expanding in them. We start out with the multivariate expansion

$$f(\Delta\theta, \Delta y) \approx f(0,0) + \Delta\theta \left(\partial_{\Delta\theta} f\right)(0,0) + \Delta y \left(\partial_{\Delta y} f\right)(0,0)$$

Now

$$f(0,0) = \frac{\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z_0}{z_r}\right)} - ikz_0}}{1 - i\frac{z_0}{z_r}}$$

$$\partial_{\Delta\theta} f = (\partial_y f)(\partial_{\Delta\theta} y) + (\partial_z f)(\partial_{\Delta\theta} z)$$

$$= \left[\frac{-2y\mathrm{e}^{\frac{-(\rho^2\cos^2\theta + y^2)}{\omega_0^2\left(1-i\frac{z}{z_r}\right)} - ikz}}{\omega_0^2\left(1 - i\frac{z}{z_r}\right)^2}\right]\left[\sin\Delta\theta\left(\Delta y + z_0\tan\Delta\theta - \rho\sin\theta\right) - \frac{z_0}{\cos\Delta\theta}\right]$$

$$+ \left[\frac{-ik\mathrm{e}^{\frac{-(\rho^2\cos^2\theta + y^2)}{\omega_0^2\left(1-i\frac{z}{z_r}\right)} - ikz}}{1 - i\frac{z}{z_r}} + \frac{i\mathrm{e}^{\frac{-(\rho^2\cos^2\theta + y^2)}{\omega_0^2\left(1-i\frac{z}{z_r}\right)} - ikz}}{z_r\left(1 - i\frac{z}{z_r}\right)^2} - \frac{i\left(\rho^2\cos^2\theta + y^2\right)\mathrm{e}^{\frac{-(\rho^2\cos^2\theta + y^2)}{\omega_0^2\left(1-i\frac{z}{z_r}\right)} - ikz}}{z_r\omega_0^2\left(1 - i\frac{z}{z_r}\right)^3}\right]$$

$$\times \left[\frac{-z_0\sin\Delta\theta}{\cos^2\Delta\theta} + \cos\Delta\theta\left(\rho\sin\theta - \Delta y - z_0\tan\Delta\theta\right) - \frac{z_0}{\cos\Delta\theta}\right]$$

$$= \left[\frac{-2y\mathrm{e}^{\frac{-(\rho^2\cos^2\theta + y^2)}{\omega_0^2\left(1-i\frac{z}{z_r}\right)} - ikz}}{\omega_0^2\left(1 - i\frac{z}{z_r}\right)^2}\right]\left[\sin\Delta\theta\left(\Delta y - \rho\sin\theta\right) - z_0\cos\Delta\theta\right]$$

$$+ \left[\frac{i\mathrm{e}^{\frac{-(\rho^2\cos^2\theta + y^2)}{\omega_0^2\left(1-i\frac{z}{z_r}\right)} - ikz}}{1 - i\frac{z}{z_r}}\left(\frac{1}{z_r\left(1 - i\frac{z}{z_r}\right)} - k - \frac{\left(\rho^2\cos^2\theta + y^2\right)}{z_r\omega_0^2\left(1 - i\frac{z}{z_r}\right)^2}\right)\right]$$

$$\times \left[\cos\Delta\theta\left(\rho\sin\theta - \Delta y\right) - z_0\left(\sin\Delta\theta + \frac{1}{\cos\Delta\theta} + \frac{\tan\Delta\theta}{\cos\Delta\theta}\right)\right]$$

while

$$\partial_{\Delta y} f = (\partial_y f)(\partial_{\Delta y} y) + (\partial_z f)(\partial_{\Delta y} z)$$

$$= \left[\frac{-2y\mathrm{e}^{\frac{-(\rho^2\cos^2\theta + y^2)}{\omega_0^2\left(1-i\frac{z}{z_r}\right)} - ikz}}{\omega_0^2\left(1 - i\frac{z}{z_r}\right)^2}\right]\left[-\cos\Delta\theta\right]$$

$$+ \left[\frac{i\mathrm{e}^{\frac{-(\rho^2\cos^2\theta + y^2)}{\omega_0^2\left(1-i\frac{z}{z_r}\right)} - ikz}}{1 - i\frac{z}{z_r}}\left(\frac{1}{z_r\left(1 - i\frac{z}{z_r}\right)} - k - \frac{\left(\rho^2\cos^2\theta + y^2\right)}{z_r\omega_0^2\left(1 - i\frac{z}{z_r}\right)^2}\right)\right]\left[-\sin\Delta\theta\right]$$

So that

$$(\partial_{\Delta\theta} f)(0,0) = \left[\frac{2z_0\rho\sin\theta \mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z_0}{z_r}\right)}-ikz_0}}{\omega_0^2\left(1-i\frac{z_0}{z_r}\right)^2}\right] + \left[\frac{i\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z_0}{z_r}\right)}-ikz_0}}{1-i\frac{z_0}{z_r}}\left(\frac{1}{z_r\left(1-i\frac{z_0}{z_r}\right)} - k - \frac{\rho^2}{z_r\omega_0^2\left(1-i\frac{z_0}{z_r}\right)^2}\right)(\rho\sin$$

$$= \frac{\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z_0}{z_r}\right)}-ikz_0}}{1-i\frac{z_0}{z_r}}\left[\frac{2\rho z_0\sin\theta}{\omega_0^2\left(1-i\frac{z_0}{z_r}\right)} + i\left(z_0-\rho\sin\theta\right)\left(k - \frac{1}{z_r\left(1-i\frac{z_0}{z_r}\right)} + \frac{\rho^2}{z_r\omega_0^2\left(1-i\frac{z_0}{z_r}\right)^2}\right)\right]$$

$$(\partial_{\Delta y} f)(0,0) = \frac{2\rho\sin\theta \mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z_0}{z_r}\right)}-ikz_0}}{\omega_0^2\left(1-i\frac{z_0}{z_r}\right)^2}$$

This gives for our total function to first order:

$$f \approx \frac{\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z_0}{z_r}\right)}-ikz_0}}{1-i\frac{z_0}{z_r}}\left[1 + \frac{2\rho\sin\theta\Delta y}{\omega_0^2\left(1-i\frac{z_0}{z_r}\right)} + \Delta\theta\left[\frac{2\rho z_0\sin\theta}{\omega_0^2\left(1-i\frac{z_0}{z_r}\right)} + i\left(z_0-\rho\sin\theta\right)\left(k - \frac{1}{z_r\left(1-i\frac{z_0}{z_r}\right)} + \frac{\rho^2}{z_r\omega_0^2\left(1-\right.}\right.\right.$$

So that the term we multiply by the complex conjugates to the TEM modes is

$$\frac{\sqrt{2}I_0\mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z_0}{z_r}\right)}-ikz_0}}{\sqrt{\pi}\omega_0\left(1-i\frac{z_0}{z_r}\right)}\left[1 + \frac{2\rho\sin\theta\Delta y}{\omega_0^2\left(1-i\frac{z_0}{z_r}\right)} + \Delta\theta\left[\frac{2\rho z_0\sin\theta}{\omega_0^2\left(1-i\frac{z_0}{z_r}\right)} + i\left(z_0-\rho\sin\theta\right)\left(k - \frac{1}{z_r\left(1-i\frac{z_0}{z_r}\right)} + \frac{\rho^2}{z_r\omega_0^2\left(1-\right.}\right.\right.$$

We now plug this into our integral, using basis functions having same parameters, so that we get

$$P_{0,0} = \frac{2I_0}{\pi\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)}\int_0^R \rho\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta \mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)}}$$

$$\times\left[1 + \frac{2\rho\sin\theta\Delta y}{\omega_0^2\left(1-i\frac{z_0}{z_r}\right)} + \Delta\theta\left[\frac{2\rho z_0\sin\theta}{\omega_0^2\left(1-i\frac{z_0}{z_r}\right)} + i\left(z_0-\rho\sin\theta\right)\left(k - \frac{1}{z_r\left(1-i\frac{z_0}{z_r}\right)} + \frac{\rho^2}{z_r\omega_0^2\left(1-i\frac{z_0}{z_r}\right)^2}\right)\right]\right]$$

using the F notation from before, we can ignore all sin terms

$$= \frac{2I_0}{\pi\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)}\left[F\left(0, \frac{2}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)}, 0, 0, R\right)\right.$$

$$+iz_0\Delta\theta\left[\left(k-\frac{1}{z_r\left(1-i\frac{z_0}{z_r}\right)}\right)F\left(0,\frac{2}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)},0,0,R\right)+\frac{1}{z_r\omega_0^2\left(1-i\frac{z_0}{z_r}\right)}F\left(2,\frac{2}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)},0,0,R\right)\right]\Bigg]$$

$$= \frac{2I_0}{\pi\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)}\left[\left(1+iz_0\Delta\theta\left(k-\frac{1}{z_r\left(1-i\frac{z_0}{z_r}\right)}\right)\right)F\left(0,\frac{2}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)},0,0,R\right)\right.$$

$$\left.+\frac{iz_0\Delta\theta}{z_r\omega_0^2\left(1-i\frac{z_0}{z_r}\right)}F\left(2,\frac{2}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)},0,0,R\right)\right]$$

$$= I_0\left[\left(1+iz_0\Delta\theta\left(k-\frac{1}{z_r\left(1-i\frac{z_0}{z_r}\right)}\right)\right)\left(1-\mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)}}\right)\right.$$

$$\left.+\frac{iz_0\Delta\theta\left(1+i\frac{z_0}{z_r}\right)}{2z_r}\left(1-\mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)}}\left(1+\frac{2R^2}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)}\right)\right)\right]$$

We do not go through much actual simplification but for notational simplification we should use the notation

$$S_i = 1-\mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)}}E_i$$

where we use E_i to denote the first i terms of the Maclaurin series for e^x (so $1+x+\frac{x^2}{2}+\ldots$) at $x=\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)}$, so that

$$P_{0,0} = I_0\left[\left(1+iz_0\Delta\theta\left(k-\frac{1}{z_r-iz_0}\right)\right)S_1+\frac{iz_0\Delta\theta\left(1+i\frac{z_0}{z_r}\right)}{2z_r}S_2\right]$$

the but we note with the nature of z_0 and R that certain terms here may be neglected.
For the next term we have

$$P_{1,0} = \frac{4I_0}{\pi\omega_0^3\left(1-i\frac{z_0}{z_r}\right)\left(1+i\frac{z_0}{z_r}\right)^2}\int_0^R \mathrm{d}\rho\rho\int_0^{2\pi}\mathrm{d}\theta\mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)}}\rho\cos\theta\left(1+\frac{2\rho\sin\theta\Delta y}{\omega_0^2\left(1-i\frac{z_0}{z_r}\right)}+\Delta\theta\left[\ldots\right.\right.$$

$$= 0$$

because there was either a single power of cos, a single power of sin, or a combination. However, for the other term

$$P_{0,1} = \frac{4I_0}{\pi\omega_0^3\left(1-i\frac{z_0}{z_r}\right)\left(1+i\frac{z_0}{z_r}\right)^2}\int_0^R \mathrm{d}\rho\rho\int_0^{2\pi}\mathrm{d}\theta\mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)}}\rho\sin\theta\left(1+\frac{2\rho\sin\theta\Delta y}{\omega_0^2\left(1-i\frac{z_0}{z_r}\right)}\right.$$

$$+\Delta\theta\left[\frac{2\rho z_0\sin\theta}{\omega_0^2\left(1-i\frac{z_0}{z_r}\right)}+i\left(z_0-\rho\sin\theta\right)\left(k-\frac{1}{z_r-iz_0}+\frac{\rho^2}{z_r\omega_0^2\left(1-i\frac{z_0}{z_r}\right)^2}\right)\right]\Bigg)$$

$$=\frac{4I_0}{\pi\omega_0^3\left(1+\left(\frac{z_0}{z_r}\right)^2\right)\left(1+i\frac{z_0}{z_r}\right)}\left[F(1,\frac{2}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)},1,0,R)+\frac{2\Delta y}{\omega_0^2\left(1-i\frac{z_0}{z_r}\right)}F\left(2,\frac{2}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)},2,0,R\right)\right.$$

$$+\Delta\theta\left[\frac{2z_0}{\omega_0^2\left(1-i\frac{z_0}{z_r}\right)}F\left(2,\frac{2}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)},2,0,R\right)+iz_0\left(k-\frac{1}{z_r-iz_0}\right)F\left(1,\frac{2}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)},1,0,R\right)\right.$$

$$+\frac{iz_0}{z_r\omega_0^2\left(1-i\frac{z_0}{z_r}\right)^2}F\left(3,\frac{2}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)},1,0,R\right)-i\left(k-\frac{1}{z_r-iz_0}\right)F\left(2,\frac{2}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)},2,0,R\right)$$

$$\left.\left.-\frac{i}{z_r\omega_0^2\left(1-i\frac{z_0}{z_r}\right)^2}F\left(4,\frac{2}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)},2,0,R\right)\right]\right]$$

$$=\frac{4I_0}{\pi\omega_0^3\left(1+\left(\frac{z_0}{z_r}\right)^2\right)\left(1+i\frac{z_0}{z_r}\right)}\left[\frac{2\Delta y}{\omega_0^2\left(1-i\frac{z_0}{z_r}\right)}F\left(2,\frac{2}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)},2,0,R\right)\right.$$

$$+\Delta\theta\left[\frac{2z_0}{\omega_0^2\left(1-i\frac{z_0}{z_r}\right)}F\left(2,\frac{2}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)},2,0,R\right)-i\left(k-\frac{1}{z_r-iz_0}\right)F\left(2,\frac{2}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)},2,0,R\right)\right.$$

$$\left.\left.-\frac{i}{z_r\omega_0^2\left(1-i\frac{z_0}{z_r}\right)^2}F\left(4,\frac{2}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)},2,0,R\right)\right]\right]$$

$$=\frac{4I_0}{\pi\omega_0^3\left(1+\left(\frac{z_0}{z_r}\right)^2\right)\left(1+i\frac{z_0}{z_r}\right)}\left[\left(\frac{2\left(\Delta y+z_0\Delta\theta\right)}{\omega_0^2\left(1-i\frac{z_0}{z_r}\right)}-i\Delta\theta\left(k-\frac{1}{z_r-iz_0}\right)\right)F\left(2,\frac{2}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)},2,0,R\right)\right.$$

$$-\frac{i\Delta\theta}{z_r\omega_0^2\left(1-i\frac{z_0}{z_r}\right)^2}F\left(4,\frac{2}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)},2,0,R\right)\Bigg]$$

$$=\frac{I_0\omega_0\left(1-i\frac{z_0}{z_r}\right)}{2}\left[\left(\frac{2\left(\Delta y+z_0\Delta\theta\right)}{\omega_0^2\left(1-i\frac{z_0}{z_r}\right)}-i\Delta\theta\left(k-\frac{1}{z_r-iz_0}\right)\right)S_2\right.$$

$$\left.-\frac{i\Delta\theta\left(1+i\frac{z_0}{z_r}\right)}{z_r-iz_0}S_3\right]$$

Now we know $P_{1,1}$ is zero as well because of the odd $\cos\theta$ term. Now, for the next set

$$P_{0,2}=\frac{I_0}{\pi\omega_0^2\left(1+i\frac{z_0}{z_r}\right)^2}\int_0^R \mathrm{d}\rho\rho\int_0^{2\pi}\mathrm{d}\theta \mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)}}\left(\frac{4\rho^2\sin^2\theta}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)}-1\right)$$

$$\times\left[1+\frac{2\rho\sin\theta\Delta y}{\omega_0^2\left(1-i\frac{z_0}{z_r}\right)}+\Delta\theta\left[\frac{2\rho z_0\sin\theta}{\omega_0^2\left(1-i\frac{z_0}{z_r}\right)}+i\left(z_0-\rho\sin\theta\right)\left(k-\frac{1}{z_r-iz_0}+\frac{\rho^2}{z_r\omega_0^2\left(1-i\frac{z_0}{z_r}\right)^2}\right)\right]\right]$$

so that since all odd sin terms are zero we have

$$=\frac{I_0}{\pi\omega_0^2\left(1+i\frac{z_0}{z_r}\right)^2}\int_0^R \mathrm{d}\rho\rho\int_0^{2\pi}\mathrm{d}\theta \mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)}}\left(\frac{4\rho^2\sin^2\theta}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)}-1\right)$$

$$\times\left[1+iz_0\Delta\theta\left(k-\frac{1}{z_r-iz_0}+\frac{\rho^2}{z_r\omega_0^2\left(1-i\frac{z_0}{z_r}\right)^2}\right)\right]$$

$$=\frac{I_0}{\pi\omega_0^2\left(1+i\frac{z_0}{z_r}\right)^2}\left[\left(iz_0\Delta\theta\left(\frac{1}{z_r-iz_0}-k\right)-1\right)G(0,0,0)-\frac{iz_0\Delta\theta}{z_r\omega_0^2\left(1-i\frac{z_0}{z_r}\right)^2}G(2,0,0)\right.$$

$$\left.+\frac{4}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)}\left[\left(1+iz_0\Delta\theta\left(k-\frac{1}{z_r-iz_0}\right)\right)G(2,2,0)+\frac{iz_0\Delta\theta}{z_r\omega_0^2\left(1-i\frac{z_0}{z_r}\right)^2}G(4,2,0)\right]\right]$$

where to save space we have used the notation

$$G(a,b,c)=F\left(a,\frac{2}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)},b,c,R\right)$$

so that

$$G(a,b,c)=\frac{\Gamma\left(\frac{b+1}{2}\right)\Gamma\left(\frac{c+1}{2}\right)\Gamma\left(\frac{a+2}{2}\right)\omega_0^{a+2}\left(1+\left(\frac{z_0}{z_r}\right)^2\right)^{\frac{a+2}{2}}}{2^{\frac{a+2}{2}}\Gamma\left(\frac{b+c+2}{2}\right)}S_{\frac{a}{2}+1}$$

where we defined S_j before so that now we expand

$$P_{0,2}=\frac{I_0\omega_0\left(1-i\frac{z_0}{z_r}\right)}{2\pi\omega_0\left(1+i\frac{z_0}{z_r}\right)}\left[\left(iz_0\Delta\theta\left(\frac{1}{z_r-iz_0}-k\right)-1\right)\pi S_1-\frac{iz_0\Delta\theta}{z_r\omega_0^2\left(1-i\frac{z_0}{z_r}\right)^2}\left(\frac{\pi\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)}{2}\right)S_2\right.$$

$$+\frac{4}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)}\left[\left(1+iz_0\Delta\theta\left(k-\frac{1}{z_r-iz_0}\right)\right)\left(\frac{\pi\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)}{4}\right)S_2\right.$$

$$\left.\left.+\frac{iz_0\Delta\theta}{z_r\omega_0^2\left(1-i\frac{z_0}{z_r}\right)^2}\left(\frac{\pi\omega_0^4\left(1+\left(\frac{z_0}{z_r}\right)^2\right)^2}{4}\right)S_3\right]\right]$$

$$=\frac{I_0\left(1-i\frac{z_0}{z_r}\right)}{2\left(1+i\frac{z_0}{z_r}\right)}\left[\left(iz_0\Delta\theta\left(\frac{1}{z_r-iz_0}-k\right)-1\right)S_1-\frac{iz_0\Delta\theta\left(1+i\frac{z_0}{z_r}\right)}{2\left(z_r-iz_0\right)}S_2\right.$$

$$\left.+\left(1+iz_0\Delta\theta\left(k-\frac{1}{z_r-iz_0}\right)\right)S_2+\frac{iz_0\Delta\theta\left(1+i\frac{z_0}{z_r}\right)}{z_r-iz_0}S_3\right]$$

So for now we have

$$=\frac{I_0}{2}\left[\left(S_2-S_1\right)\frac{z_r-iz_0}{z_r+iz_0}+iz_0\Delta\theta\left(\frac{S_2-S_1}{z_r+iz_0}\left(k\left(z_r-iz_0\right)-1\right)+\frac{2S_3-S_2}{2z_r}\right)\right]$$

$$
\begin{aligned}
P_{0,0} &= I_0 \left[S1 + iz_0 \Delta\theta \left[\left(k - \frac{1}{z_r - iz_0} \right) S_1 + \frac{1 + i\frac{z_0}{z_r}}{2z_r} S_2 \right] \right] \\
P_{1,0} &= 0 \\
P_{0,1} &= \frac{I_0}{\omega_0} \left[\Delta y S_2 + \Delta\theta \left[\left(z_0 - \frac{i\omega_0^2}{2z_r} \left(k \left(z_r - iz_0 \right) - 1 \right) \right) S_2 - \frac{i\omega_0^2 \left(1 + i\frac{z_0}{z_r} \right)}{2z_r} S_3 \right] \right] \\
P_{0,2} &= \frac{I_0}{2} \left[\left(S_2 - S_1 \right) \frac{z_r - iz_0}{z_r + iz_0} + iz_0 \Delta\theta \left(\frac{S_2 - S_1}{z_r + iz_0} \left(k \left(z_r - iz_0 \right) - 1 \right) + \frac{2S_3 - S_2}{2z_r} \right) \right]
\end{aligned}
$$

. The next term is $P_{2,0}$, and as $A_{2,0}$ has the same form as $A_{0,2}$ except with cos instead of sin, and as the only terms that were non-zero were those not coupling to the sin term in our expansion, and as each of our expressions for the final terms are symmetric under the exchange of sin and cos terms, which has to do with the fact we are integrating from 0 to 2π, this yields,

$$
P_{2,0} = P_{0,2} = \frac{I_0}{2} \left[\left(S_2 - S_1 \right) \frac{z_r - iz_0}{z_r + iz_0} + iz_0 \Delta\theta \left(\frac{S_2 - S_1}{z_r + iz_0} \left(k \left(z_r - iz_0 \right) - 1 \right) + \frac{2S_3 - S_2}{2z_r} \right) \right]
$$

Now for the next two terms $P_{1,2}$ and $P_{2,1}$ these will be different since $P_{2,1}$ couples with the sin terms while $P_{1,2}$ doesn't. Looking at these,

$$
P_{1,2} = \frac{2I_0}{\pi\omega_0^3 \left(1 + i\frac{z_0}{z_r} \right)^3} \int_0^R \mathrm{d}\rho\rho \int_0^{2\pi} \mathrm{d}\theta \mathrm{e}^{\frac{-2\rho^2}{\omega_0^2 \left(1 + \left(\frac{z_0}{z_r} \right)^2 \right)}} \rho \cos\theta \left(\frac{4\rho^2 \sin^2\theta}{\omega_0^2 \left(1 + \left(\frac{z_0}{z_r} \right)^2 \right)} - 1 \right)
$$

$$
\times \left[1 + \frac{2\rho \sin\theta \Delta y}{\omega_0^2 \left(1 - i\frac{z_0}{z_r} \right)} + \Delta\theta \left[\frac{2\rho z_0 \sin\theta}{\omega_0^2 \left(1 - i\frac{z_0}{z_r} \right)} + i \left(z_0 - \rho \sin\theta \right) \left(k - \frac{1}{z_r - iz_0} + \frac{\rho^2}{z_r \omega_0^2 \left(1 - i\frac{z_0}{z_r} \right)^2} \right) \right] \right]
$$

but because of the odd cos this evaluates to zero so

$$
P_{1,2} = 0
$$

, as for the next one, this will not be zero

$$
P_{2,1} = \frac{2I_0}{\pi\omega_0^3 \left(1 + i\frac{z_0}{z_r} \right)^3} \int_0^R \mathrm{d}\rho\rho \int_0^{2\pi} \mathrm{d}\theta \mathrm{e}^{\frac{-2\rho^2}{\omega_0^2 \left(1 + \left(\frac{z_0}{z_r} \right)^2 \right)}} \rho \sin\theta \left(\frac{4\rho^2 \cos^2\theta}{\omega_0^2 \left(1 + \left(\frac{z_0}{z_r} \right)^2 \right)} - 1 \right)
$$

$$
\times \left[1 + \frac{2\rho \sin\theta \Delta y}{\omega_0^2 \left(1 - i\frac{z_0}{z_r} \right)} + \Delta\theta \left[\frac{2\rho z_0 \sin\theta}{\omega_0^2 \left(1 - i\frac{z_0}{z_r} \right)} + i \left(z_0 - \rho \sin\theta \right) \left(k - \frac{1}{z_r - iz_0} + \frac{\rho^2}{z_r \omega_0^2 \left(1 - i\frac{z_0}{z_r} \right)^2} \right) \right] \right]
$$

which picks out only the sin terms yielding

$$= \frac{2I_0}{\pi\omega_0^3\left(1+i\frac{z_0}{z_r}\right)^3}\int_0^R \mathrm{d}\rho\rho\int_0^{2\pi}\mathrm{d}\theta \mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)}}\left(\frac{4\rho^2\cos^2\theta}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)}-1\right)$$

$$\times\left[\frac{2\rho^2\sin^2\theta\Delta y}{\omega_0^2\left(1-i\frac{z_0}{z_r}\right)}+\Delta\theta\left[\frac{2\rho^2 z_0\sin^2\theta}{\omega_0^2\left(1-i\frac{z_0}{z_r}\right)}-i\rho^2\sin^2\theta\left(k-\frac{1}{z_r-iz_0}+\frac{\rho^2}{z_r\omega_0^2\left(1-i\frac{z_0}{z_r}\right)^2}\right)\right]\right]$$

$$= \frac{2I_0}{\pi\omega_0^3\left(1+i\frac{z_0}{z_r}\right)^3}\int_0^R \mathrm{d}\rho\rho\int_0^{2\pi}\mathrm{d}\theta \mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)}}\rho^2\sin^2\theta\left(\frac{4\rho^2\cos^2\theta}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)}-1\right)$$

$$\times\left[\frac{2\Delta y}{\omega_0^2\left(1-i\frac{z_0}{z_r}\right)}+\Delta\theta\left[\frac{2z_0}{\omega_0^2\left(1-i\frac{z_0}{z_r}\right)}-i\left(k-\frac{1}{z_r-iz_0}+\frac{\rho^2}{z_r\omega_0^2\left(1-i\frac{z_0}{z_r}\right)^2}\right)\right]\right]$$

$$= \frac{2I_0}{\pi\omega_0^5\left(1+i\frac{z_0}{z_r}\right)^3\left(1-i\frac{z_0}{z_r}\right)}\left[\left(\frac{4G(4,2,2)}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)}-G(2,2,0)\right)\left(2\Delta y+\Delta\theta\left[2z_0-i\omega_0^2\left(k\left(1-i\frac{z_0}{z_r}\right)-\frac{1}{z_r}\right)\right]\right)\right.$$

$$\left.+\frac{i\Delta\theta}{z_r-iz_0}\left(G(4,2,0)-\frac{4G(6,2,2)}{\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)}\right)\right]$$

$$= \frac{2I_0}{\pi\omega_0^5\left(1+i\frac{z_0}{z_r}\right)^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)}$$

$$\times\left[\left(\frac{\pi\omega_0^6\left(1+\left(\frac{z_0}{z_r}\right)^2\right)^3 S_3}{8\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)}-\frac{\pi\omega_0^4\left(1+\left(\frac{z_0}{z_r}\right)^2\right)^2 S_2}{8}\right)\left(2\Delta y+\Delta\theta\left[2z_0-i\omega_0^2\left(k\left(1-i\frac{z_0}{z_r}\right)-\frac{1}{z_r}\right)\right]\right)\right.$$

$$\left.+\frac{i\Delta\theta}{z_r-iz_0}\left(\frac{\pi\omega_0^6\left(1+\left(\frac{z_0}{z_r}\right)^2\right)^3 S_3}{8}-\frac{3\pi\omega_0^8\left(1+\left(\frac{z_0}{z_r}\right)^2\right)^4 S_4}{16\omega_0^2\left(1+\left(\frac{z_0}{z_r}\right)^2\right)}\right)\right]$$

$$= \frac{I_0\left(1-i\frac{z_0}{z_r}\right)}{4\omega_0}\left[\frac{S_3-S_2}{1+i\frac{z_0}{z_r}}\left(2\Delta y+\Delta\theta\left[2z_0-i\omega_0^2\left(k\left(1-i\frac{z_0}{z_r}\right)-\frac{1}{z_r}\right)\right]\right)+\frac{i\omega_0^2\Delta\theta}{z_r}\left(S_3-\frac{3}{2}S_4\right)\right]$$

We finish with the last of the order two elements So that now we have

$$P_{0,0} = I_0\left[S1 + iz_0\Delta\theta\left[\left(k - \frac{1}{z_r - iz_0}\right)S_1 + \frac{1 + i\frac{z_0}{z_r}}{2z_r}S_2\right]\right]$$

$$P_{1,0} = 0$$

$$P_{0,1} = \frac{I_0}{\omega_0}\left[\Delta y S_2 + \Delta\theta\left[\left(z_0 - \frac{i\omega_0^2}{2z_r}\left(k\left(z_r - iz_0\right) - 1\right)\right)S_2 - \frac{i\omega_0^2\left(1 + i\frac{z_0}{z_r}\right)}{2z_r}S_3\right]\right]$$

$$P_{1,1} = 0$$

$$P_{0,2} = \frac{I_0}{2}\left[\left(S_2 - S_1\right)\frac{z_r - iz_0}{z_r + iz_0} + iz_0\Delta\theta\left(\frac{S_2 - S_1}{z_r + iz_0}\left(k\left(z_r - iz_0\right) - 1\right) + \frac{2S_3 - S_2}{2z_r}\right)\right]$$

$$P_{2,0} = \frac{I_0}{2}\left[\left(S_2 - S_1\right)\frac{z_r - iz_0}{z_r + iz_0} + iz_0\Delta\theta\left(\frac{S_2 - S_1}{z_r + iz_0}\left(k\left(z_r - iz_0\right) - 1\right) + \frac{2S_3 - S_2}{2z_r}\right)\right]$$

$$P_{1,2} = 0$$

$$P_{2,1} = \frac{I_0\left(1 - i\frac{z_0}{z_r}\right)}{4\omega_0}\left[\frac{S_3 - S_2}{1 + i\frac{z_0}{z_r}}\left(2\Delta y + \Delta\theta\left[2z_0 - i\omega_0^2\left(k\left(1 - i\frac{z_0}{z_r}\right) - \frac{1}{z_r}\right)\right]\right) + \frac{i\omega_0^2\Delta\theta}{z_r}\left(S_3 - \frac{3}{2}S_4\right)\right]$$

using that $z_0 = 2.5 \cdot 10^9\, m$, $\omega_0 = 1 \cdot 10^{-3}\, m$, $R = 3 \cdot 10^{-1}\, m$, and $\lambda = 1.064 \cdot 10^{-6}$, which gives $z_r = 2.9526$, we then have divergence angle $\frac{\omega_0}{z_r} = \frac{\lambda}{\pi\omega_0} \approx 3.387 \cdot 10^{-4}$ radians, which we use to compare with the magnitudes that lead to relevant phase shifts

$$P_{0,0} = I_0\left[2.51 \cdot 10^{-13} - \left(1.13 \cdot 10^{-8} - 3.71 \cdot 10^{3} i\right)\Delta\theta\right]$$

$$P_{1,0} = 0$$

$$P_{0,1} = I_0\left[3.15 \cdot 10^{-23}\Delta y + \left(3.78 \cdot 10^{-34} - 9.31 \cdot 10^{-23} i\right)\Delta\theta\right]$$

$$P_{0,2} = I_0\left[\left(1.26 \cdot 10^{-13} + 2.97 \cdot 10^{22} i\right) - \Delta\theta\left(8.76 \cdot 10^{-6} + 3.71 \cdot 10^{3} i\right)\right]$$

$$P_{2,0} = I_0\left[\left(1.26 \cdot 10^{-13} + 2.97 \cdot 10^{22} i\right) - \Delta\theta\left(8.76 \cdot 10^{-6} + 3.71 \cdot 10^{3} i\right)\right]$$

$$P_{1,2} = 0$$

$$P_{2,1} = I_0\left[\left(1.58 \cdot 10^{-23} + 3.72 \cdot 10^{-32} i\right)\Delta y + \left(1.1 \cdot 10^{-31} - 4.65 \cdot 10^{-23} i\right)\Delta\theta\right]$$

Actually it turns out ω_0 is $17.8 * 3/4$ We see the the relative orders of magnitude of the perturbation, leaving a much larger fraction of the power in the first mode rather than the zeroth order.

11. Checking initial Parameters

We first make sure our parameters (30 cm telescope entrance/exit for beam with initial width $w_0 = 17.8$ cm) give the amount of power we need. For a gaussian beam with initial power over the total beam of two watts we get that the total power over the receiving telescope entrance aperture is

$$P_t = \int_0^{.3} \mathrm{d}\rho\rho \int_0^{2\pi} \mathrm{d}\theta U^*(\rho, \theta, z) U(\rho, \theta, z)$$

where $U(\rho, \theta)$ is the intensity of the beam,

$$U(\rho, \theta, z) = \frac{2}{\pi^{1/2}\omega_0 \left(1 - i\frac{z}{z_r}\right)} \mathrm{e}^{\frac{-\rho^2}{\omega_0^2\left(1-i\frac{z}{z_r}\right)} - ikz}$$

Giving the total power over an aperture of size R begin

$$P_T(R) = \frac{4}{\pi\omega_0^2\left(1 + \left(\frac{z}{z_r}\right)^2\right)} \int_0^R \mathrm{d}\rho\rho \int_0^{2\pi} \mathrm{d}\theta \mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}$$

$$= \frac{8}{\omega_0^2\left(1 + \left(\frac{z}{z_r}\right)^2\right)} \int_0^R \mathrm{d}\rho\rho \mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}$$

$$= -2\, \mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\Bigg|_0^R$$

$$= 2\left[1 - \mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\right]$$

So that for the receiving telescope with $R = 30$ cm, initial beam width $\omega_0 = 17.8$ cm, over a distance $z = 2.5 \cdot 10^9$ m we get that the exponent is $-7.9552 \cdot 10^9$ giving a value

$$P_t(30\,cm) \approx 1.6 \cdot 10^{-8} W$$

which is far greater than the $7 \cdot 10^{-10}$ Watts that were expected. I also tried the best fit ω mentioned, $\omega_0 = 17.8 \cdot (3/4)$ cm, which yields a total received power of $8.9 \cdot 10^{-9}$ Watts, still an order of magnitude too great. If perhaps it was 17.8 mm instead of centimeters, this yields too low of a total power received gives one that at least agrees with the order of magnitude (a bit greater than 1/7 the expected 700 picoWatts). However, there is another explanation, which is simply that when it goes through the exit of the primary aperture it is broken into eigenmodes other than the 00 mode. Instead what we do is break it up into the initial modes, and then find the total power. We have that the contributions are the contributions found before with $z_0 = \Delta\theta = \Delta y = 0$

$$
\begin{aligned}
I_{00} &= \frac{2\sqrt{2}}{\pi\omega_0^2}\int_0^R \mathrm{d}\rho\rho \int_0^{2\pi} \mathrm{d}\theta \mathrm{e}^{\frac{-2\rho^2}{\omega_0^2}} \\
&= \frac{4\sqrt{2}}{\omega_0^2}\int_0^R \mathrm{d}\rho\rho \mathrm{e}^{\frac{-2\rho^2}{\omega_0^2}} \\
&= -\sqrt{2}\left[\mathrm{e}^{\frac{-2\rho^2}{\omega_0^2}}\right|_0^R = \sqrt{2}\left[1-\mathrm{e}^{\frac{-2R^2}{\omega_0^2}}\right]
\end{aligned}
$$

so that with $R = 30\,cm$ and $\omega_0 = 17.8 * 3/4$ (actually slightly off if using $.446 * D$ where D is the telescope diameter we get a lower value. We need to calculate overlaps between the modes since they are no longer orthonormal.

11.1. **Overlaps.** We denote

$$
\langle i,j \,|\, k,l\rangle = \int_0^R \rho \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta A_{i,j}^*(\rho,\theta,z) A_{k,l}(\rho,\theta,z)
$$

Where we recognize as usual

$$
\langle k,l \,|\, i,j\rangle = \langle i,j \,|\, k,l\rangle^*
$$

We see that we have already calculated the overlaps $\langle i,j \,|\, 0,0\rangle$ for $i,j \leq 2$ above by setting in our results for the tilt-to-length coupling $\Delta\theta$ and Δy to zero and I_0 to 1. This gives

$$
\begin{aligned}
\langle 0,0 \,|\, 0,0\rangle &= S_1 = 1-\mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}} \\
\langle 0,0 \,|\, 0,1\rangle = \langle 0,0 \,|\, 1,0\rangle = \langle 1,0 \,|\, 0,0\rangle = \langle 0,1 \,|\, 0,0\rangle &= 0 \\
\langle 1,1 \,|\, 0,0\rangle = \langle 0,0 \,|\, 1,1\rangle &= 0
\end{aligned}
$$

$$
\langle 0,2 \,|\, 0,0\rangle = \langle 2,0 \,|\, 0,0\rangle = \frac{(S_2-S_1)(z_r-iz)}{\sqrt{2}(z_r+iz)} = \frac{-\sqrt{2}\,(z_r-iz)\,R^2\mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}}{\omega_0^2\,(z_r+iz)\left(1+\left(\frac{z}{z_r}\right)^2\right)} = \frac{-\sqrt{2}R^2\mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}}{\omega_0^2\left(1+i\frac{z}{z_r}\right)^2}
$$

$$
\begin{aligned}
\langle 0,0 \,|\, 0,2\rangle = \langle 0,0 \,|\, 2,0\rangle &= \frac{-\sqrt{2}R^2\mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}}{\omega_0^2\left(1-i\frac{z}{z_r}\right)^2} \\
\langle 0,0 \,|\, 1,2\rangle = \langle 0,0 \,|\, 2,1\rangle = \langle 1,2 \,|\, 0,0\rangle = \langle 2,1 \,|\, 0,0\rangle &= 0
\end{aligned}
$$

We now calculate

$$
\langle 2,2 \,|\, 0,0\rangle = \frac{\left(1-i\frac{z}{z_r}\right)}{\pi\omega_0^2\left(1+i\frac{z}{z_r}\right)^3}\int_0^R \rho\mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta \mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}} \left(\frac{16\rho^4\cos^2\theta\sin^2\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2} - \frac{4\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)} + 1\right)
$$

$$= \frac{\left(1 - i\frac{z}{z_r}\right)}{\pi\omega_0^2\left(1 + i\frac{z}{z_r}\right)^3}\left(\frac{16}{\omega_0^4\left(1 + \left(\frac{z}{z_r}\right)^2\right)^2}G(4,2,2) - \frac{4}{\omega_0^2\left(1 + \left(\frac{z}{z_r}\right)^2\right)}G(2,0,0) + G(0,0,0)\right)$$

$$= \frac{\left(1 - i\frac{z}{z_r}\right)}{\pi\omega_0^2\left(1 + i\frac{z}{z_r}\right)^3}\left(\frac{16}{\omega_0^4\left(1 + \left(\frac{z}{z_r}\right)^2\right)^2}\frac{\pi\omega_0^6\left(1 + \left(\frac{z}{z_r}\right)^2\right)^3}{2^5}S_3 - \frac{4}{\omega_0^2\left(1 + \left(\frac{z}{z_r}\right)^2\right)}\frac{\pi\omega_0^4\left(1 + \left(\frac{z}{z_r}\right)^2\right)^2}{2^2}S_2 \right.$$
$$\left. + \frac{\pi\omega_0^2\left(1 + \left(\frac{z}{z_r}\right)^2\right)}{2}\right)$$

$$= \frac{\left(1 - i\frac{z}{z_r}\right)^2}{2\left(1 + i\frac{z}{z_r}\right)^2}\left(S_3 + S_1 - 2S_2\right)$$

$$= \frac{\left(1 - i\frac{z}{z_r}\right)^2}{2\left(1 + i\frac{z}{z_r}\right)^2}\mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\left(2 + \frac{2R^2}{\omega_0^2\left(1 + \left(\frac{z}{z_r}\right)^2\right)} + \frac{2R^4}{\omega_0^4\left(1 + \left(\frac{z}{z_r}\right)^2\right)^2} - 2 - \frac{4R^2}{\omega_0^2\left(1 + \left(\frac{z}{z_r}\right)^2\right)}\right)$$

$$= \frac{R^2\left(1 - i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)\right)}}}{\omega_0^2\left(1 + i\frac{z}{z_r}\right)^3}\left(\frac{R^2}{\omega_0^2\left(1 + \left(\frac{z}{z_r}\right)^2\right)} - 1\right)$$

Now we know odd terms have zero overlap with even terms in the same index so that

$$\langle 1,0 \,|\, 0,1\rangle = \langle 1,0 \,|\, 0,2\rangle = \langle 1,0 \,|\, 1,1\rangle = \langle 1,0 \,|\, 2,0\rangle$$
$$= \langle 1,0 \,|\, 2,1\rangle = \langle 1,0 \,|\, 2,2\rangle = \langle 0,1 \,|\, 0,2\rangle = \langle 0,1 \,|\, 1,1\rangle$$
$$= \langle 0,1 \,|\, 0,2\rangle = \langle 0,1 \,|\, 0,2\rangle = \langle 0,1 \,|\, 1,1\rangle = \langle 0,1 \,|\, 1,2\rangle$$
$$= \langle 0,1 \,|\, 2,0\rangle = \langle 0,1 \,|\, 2,2\rangle = \langle 1,1 \,|\, 0,2\rangle = \langle 1,1 \,|\, 2,0\rangle$$
$$= \langle 1,1 \,|\, 1,2\rangle = \langle 1,1 \,|\, 2,1\rangle = \langle 1,1 \,|\, 2,2\rangle = 0$$

In fact the only non-zero ones are (along with their conjugates)

$$\langle 1,0 \,|\, 1,0\rangle = \langle 0,1 \,|\, 0,1\rangle \quad \langle 1,2 \,|\, 1,2\rangle = \langle 2,1 \,|\, 2,1\rangle$$
$$\langle 0,2 \,|\, 0,2\rangle = \langle 2,0 \,|\, 2,0\rangle \quad \langle 0,2 \,|\, 2,0\rangle = \langle 2,0 \,|\, 0,2\rangle$$
$$\langle 1,0 \,|\, 1,2\rangle = \langle 0,1 \,|\, 2,1\rangle \quad \langle 1,1 \,|\, 1,1\rangle \quad \langle 2,2 \,|\, 2,2\rangle$$

where we used that because the integration is over the entire angular range so that cos and sin are interchangeable to show the equalities above. Calculating these

$$\langle 1,0\,|\,1,0\rangle = \frac{2^3}{\pi\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}\int_0^R \rho \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta \rho^2 \cos^2\theta \mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}$$

$$= \frac{2^3}{\pi\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2} G(2,0,2)$$

$$= \frac{2^3}{\pi\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}\frac{\pi\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}{2^3} S_2 = S_2$$

$$= 1-\mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\left(1+\frac{2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}\right)$$

Now we calculate the next term

$$\langle 1,1\,|\,1,1\rangle = \frac{2^5}{\pi\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}\int_0^R \rho \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta \mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}} \rho^4 \sin^2\theta\cos^2\theta$$

$$= \frac{2^5}{\pi\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3} G(4,2,2)$$

$$= \frac{2^5}{\pi\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}\frac{\pi\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}{2^5} S_3 = S_3$$

$$= 1-\mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\left(1+\frac{2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+\frac{2R^4}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}\right)$$

And now for the next one

$$\langle 1,0\,|\,1,2\rangle = \frac{4\sqrt{2}}{\pi\omega_0^4\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}\int_0^R \rho\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\rho^2\cos^2\theta\left(\frac{4\rho^2\sin^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-1\right)$$

$$= \frac{4\sqrt{2}}{\pi\omega_0^4\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}\left[\frac{4}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}G(4,2,2)-G(2,0,2)\right]$$

$$= \frac{4\sqrt{2}}{\pi\omega_0^4\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}\left[\frac{4}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}\frac{\pi\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}{2^5}S_3-\frac{\pi\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}{2^3}S_2\right]$$

$$= \frac{\left(1+i\frac{z}{z_r}\right)}{\sqrt{2}\left(1-i\frac{z}{z_r}\right)}\left[S_3-S_2\right]$$

$$= -\sqrt{2}\frac{\left(1+i\frac{z}{z_r}\right)}{\left(1-i\frac{z}{z_r}\right)}\frac{R^4\mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}$$

$$= \frac{-\sqrt{2}R^4\mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}}{\omega_0^4\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}$$

Grouping together the results so far so as not to lose track (not showing conjugates)

$$\langle 0,0\,|\,0,0\rangle = 1-\mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}$$

$$\langle 0,0\,|\,0,1\rangle = \langle 0,0\,|\,1,0\rangle = \langle 0,0\,|\,1,1\rangle = \langle 0,0\,|\,1,2\rangle = \langle 0,0\,|\,2,1\rangle = 0$$

$$\langle 0,0\,|\,0,2\rangle = \langle 0,0\,|\,2,0\rangle = \frac{-\sqrt{2}R^2\mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}}{\omega_0^2\left(1-i\frac{z}{z_r}\right)^2}$$

$$\langle 0,0\,|\,2,2\rangle = \frac{R^2\left(1+i\frac{z}{z_r}\right)\mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}}{\omega_0^2\left(1-i\frac{z}{z_r}\right)^3}\left(\frac{R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}-1\right)$$

$$\langle 1,0\,|\,1,0\rangle = \langle 0,1\,|\,0,1\rangle = 1-\mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\left(1+\frac{2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}\right)$$

$$\langle 1,0\,|\,0,1\rangle = \langle 1,0\,|\,1,1\rangle = \langle 1,0\,|\,0,2\rangle = \langle 1,0\,|\,2,0\rangle = \langle 1,0\,|\,2,1\rangle = \langle 1,0\,|\,2,2\rangle = 0$$

$$\langle 1,0\,|\,1,2\rangle = \frac{-R^2\mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}}{\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}$$

$$\langle 0,1\,|\,1,1\rangle = \langle 0,1\,|\,1,2\rangle = \langle 0,1\,|\,2,0\rangle = \langle 0,1\,|\,0,2\rangle = \langle 0,1\,|\,2,2\rangle = 0$$

$$\langle 0,1\,|\,2,1\rangle = \frac{-R^2\mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}}{\left(1+i\frac{z}{z_r}\right)\left(1-i\frac{z}{z_r}\right)^3}$$

$$\langle 1,1\,|\,1,1\rangle = 1-\mathrm{e}^{\frac{-2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\left(1+\frac{2R^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+\frac{2R^4}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}\right)$$

$$\langle 1,1\,|\,1,2\rangle = \langle 1,1\,|\,2,1\rangle = \langle 1,1\,|\,2,2\rangle = 0$$

We have several more terms to calculate,

$$\langle 0,2\,|\,0,2\rangle = \frac{1}{2\pi\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}\int_0^R \rho\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\mathrm{e}^{\frac{-2\rho^2}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}}\left(\frac{16\rho^4\sin^4\theta}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}-\frac{8\rho^2\sin^2\theta}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}+1\right)$$

$$= \frac{1}{2\pi\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}\left[\frac{16}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}G(4,4,0) - \frac{8}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}G(2,2,0) + G(0,0,0)\right]$$

$$= \frac{1}{2\pi\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}\left[\frac{16}{\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}\frac{3\pi\omega_0^6\left(1+\left(\frac{z}{z_r}\right)^2\right)^3}{2^5}S_3\right.$$

$$\left. - \frac{8}{\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}\frac{\pi\omega_0^4\left(1+\left(\frac{z}{z_r}\right)^2\right)^2}{2^3}S_2 + \frac{\pi\omega_0^2\left(1+\left(\frac{z}{z_r}\right)^2\right)}{2}S_1\right]$$

$$= \frac{1}{4}$$

12. Deriving ABCD Matrices

We have 8 ABCD Matrices to consider for initial ray vector $(\Delta y, \Delta\theta)$,

$$M_1 = \begin{pmatrix} 1 & l_1 \\ 0 & 1 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 1 & 0 \\ \frac{-2}{|r_1|} & 1 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 1 & l_2 \\ 0 & 1 \end{pmatrix}$$

$$M_4 = \begin{pmatrix} 1 & 0 \\ \frac{-2}{r_2} & 1 \end{pmatrix}$$

$$M_5 = \begin{pmatrix} 1 & l_3 \\ 0 & 1 \end{pmatrix}$$

$$M_6 = \begin{pmatrix} 1 & 0 \\ \frac{-2}{|r_3|} & 1 \end{pmatrix}$$

$$M_7 = \begin{pmatrix} 1 & l_4 \\ 0 & 1 \end{pmatrix}$$

$$M_8 = \begin{pmatrix} 1 & 0 \\ \frac{-2}{r_4} & 1 \end{pmatrix}$$

Where l_1 is the length from SC to SC, l_2 from the first mirror to the second along the gut ray, l_3 from the second to the third along the gut ray, and l_4 from the third to the fourth along the gut ray. r_i is the radius of curvature of the i-th mirror. We have also taken into account the direction the mirrors are facing as well as the type of mirror (concave vs convex) when writing out the signs. We calculate out

$$l_1 = 2.5 \cdot 10^9 \, m$$
$$l_2 = 3.70274 \cdot 10^{-1} \, m$$
$$l_3 = 4.586502 \cdot 10^{-1} \, m$$
$$l_4 = 9.65655 \cdot 10^{-2} \, m$$

There may be one final matrix after the last mirror one specifying distance to the image plane, but we are not including this in case it isn't the proper distance to the optical bench. For future reference this distance is $2.485057 \cdot 10^{-1}$ meters. As for the mirrors we have

$$r_1 = -7.504961362098647 \cdot 10^{-1}$$
$$r_2 = -2.065070165594815 \cdot 10^{-2}$$
$$r_3 = -7.321036159457827 \cdot 10^{-1}$$
$$r_4 = 5.518125318885137 \cdot 10^{-1}$$

This gives us the matrices

$$M_1 = \begin{pmatrix} 1 & 2.5 \cdot 10^9 \\ 0 & 1 \end{pmatrix}$$
$$M_2 = \begin{pmatrix} 1 & 0 \\ -2.6649037929766646 & 1 \end{pmatrix}$$
$$M_3 = \begin{pmatrix} 1 & 3.70274 \cdot 10^{-1} \\ 0 & 1 \end{pmatrix}$$
$$M_4 = \begin{pmatrix} 1 & 0 \\ 96.8490094584232963 & 1 \end{pmatrix}$$
$$M_5 = \begin{pmatrix} 1 & 4.586502 \cdot 10^{-1} \\ 0 & 1 \end{pmatrix}$$
$$M_6 = \begin{pmatrix} 1 & 0 \\ 2.7318537382393067 & 1 \end{pmatrix}$$
$$M_7 = \begin{pmatrix} 1 & 9.65655 \cdot 10^{-2} \\ 0 & 1 \end{pmatrix}$$
$$M_8 = \begin{pmatrix} 1 & 0 \\ 3.6244193171098065 & 1 \end{pmatrix}$$

(we have left as many digits as we know in the calculations but will round for results). We combine the last 7 of these for the purpose of making sure that this somewhat preserves collimated beams, as we know the real telescope should so that if it does not these are most likely incorrect. The last seven comprise the parts of the telescope so they should keep the beams collimated. We get

$$M8 \times M7 \times M6 \times M5 \times M4 \times M3 \times M2 = \begin{pmatrix} -.9172 & 25.3935 \\ -6.3997 & 176.0939 \end{pmatrix}$$

So that we see for beams parallel to each other and at least close in displacement from the optical axis the angular contribution is two orders of magnitude greater to the final outgoing angle, or comparing

$$\begin{pmatrix} -.9172 & 25.3935 \\ -6.3997 & 176.0939 \end{pmatrix} \begin{pmatrix} y \\ \theta \end{pmatrix} = \begin{pmatrix} 25.3935\theta - .9172y \\ 176.0939\theta - 6.3997y \end{pmatrix}$$

to

$$\begin{pmatrix} -.9172 & 25.3935 \\ -6.3997 & 176.0939 \end{pmatrix} \begin{pmatrix} y + \delta \\ \theta \end{pmatrix} = \begin{pmatrix} 25.3935\theta - .9172(y + \delta) \\ 176.0939\theta - 6.3997(y + \delta) \end{pmatrix}$$

we see the angles now differ by a percent error

13. Q-FACTOR

Using that after the ABCD matrix $q' = \frac{Aq+B}{Cq+D}$, using that $q_0 = z_0 + iz_r = 2.5 \cdot 10^9 + 2.9526i$, we then plug in using the matrices derived before to find that the imaginary part (or radius) is almost gone and we are left with a negative q factor

$$q_{final} = -.24$$