Final Review

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1 Moment Generation Functions

Definition. For a real-valued random variable X, its *moment-generating function* is $M_X : \mathbb{R}[0, \infty]$,

$$M_X(t) = \mathbb{E}[e^{tX}], t \in \mathbb{R}$$

Proposition.
(a) $M_X(0) = 1$.
(b) $M_{X+Y} = M_X M_Y$ for X, Y independent,
(c) M_X is fully determined by the moments of X so long as $\mathbb{E}X^n \leq (cn)^n$ for some $c > 0$.

Definition. For random variables $X_1, \ldots, X_n$, they have moment generating function $M_{X_{1,\ldots,n}}(t_1, \ldots, t_n) = \mathbb{E}[e^{t_1 X_1 + \cdots + t_n X_n}]$.

Definition. $X_1, X_2, \ldots, X_n$ are *multivariate normal random variable* if for any $X_i \in X_1, \ldots, X_n$ there exist independent normal random variables $Y_1, Y_2, \ldots, Y_m$ and $a_{ij} \in \mathbb{R}$ such that

$$\sum_{0 \leq j \leq m} a_{ij} Y_j = X_i$$

2 Law of Large Numbers

Definition. X_n *converges in probability* to X if for any $\varepsilon > 0$,

$$\lim_{n \to \infty} \mathbb{P}(|X_n - X| \geq \varepsilon) = 0$$

denoted by $X_n \xrightarrow{p} X$.

Definition. X_n *converges almost surely* to X if

$$\mathbb{P}(\lim_{n\to\infty} X_n = X) = 1.$$

denoted by $X_n \xrightarrow{a.s.} X$.

Theorem. (Weak and Strong Law of Large Numbers) Suppose $\{X_n\}$ are i.i.d random variables with $X_n \xrightarrow{d} X, \mathbb{E}[X] < \infty$, then

$$\lim_{n\to\infty} \frac{X_1 + X_2 + \cdots + X_n}{n} \xrightarrow{p \ \& \ a.s.} \mathbb{E}[X].$$

3 Central Limit Theorem

Definition. $X_n \in \mathbb{R}$ *converges in distribution* to $X \in \mathbb{R}$ if

$$\lim_{n\to\infty} \mathbb{P}(X_n \leq a) = \mathbb{P}(X \leq a)$$

for all $a \in \mathbb{R}$ at which the CDF of X is continuous. This is denoted as

$$X_n \xrightarrow{d} X.$$

Theorem. (Central Limit Theorem) Suppose $X_n \in \mathbb{R}$ are i.i.d. with $\mathbb{E}[X^2] < \infty$, $Var(X) > 0$. Then

$$\frac{X_1 + \cdots + X_n - n\mathbb{E}[X]}{\sqrt{n \cdot Var(X)}} \Rightarrow \mathcal{N}(0, 1).$$