Alternate grids for diffusion weighted imaging and associated reconstruction algorithms

Fernando Perez and Stéfan van der Walt
University of California Berkeley, Berkeley CA

Cory Ahrens
Colorado School of Mines, Golden CO

Introduction

We develop a systematic numerical approach for constructing nearly optimal quadratures invariant under the icosahedral group to integrate rotationally invariant subspaces of $L^2\left(\mathbb{S}^2\right)$ up to a fixed order and degree. Using these grids and a reproducing kernel, we show how to replace the standard basis of spherical harmonics by a representation formed using a single function centered at the quadrature nodes. The reproducing kernel is mostly concentrated near the corresponding grid point. In this representation, the coefficients, up to a factor, are the values on the grid of the function being represented. We may interpret this construction as an analogue of Lagrange interpolation on the sphere.

We view our approach as the first step in constructing a local and multiresolution representation of functions on the sphere that respects rotationally invariant subspaces.

Preliminaries

Here we establish notation and state some results about spherical harmonics, reproducing kernels for subspaces of spherical harmonics, the Funk-Radon transform and the Funk-Hecke theorem.

We denote the unit sphere in $\mathbb{R}^3$ as $\mathbb{S}^2$. An orthonormal basis for $L^2\left(\mathbb{S}^2\right)$ is given by the spherical harmonics,

$$Y_n^m(\theta,\phi) = \frac{1}{\sqrt{2\pi}}\overline{P}_n^m(\cos\theta)\, e^{im\phi},\ 0 \le |m| \le n, \quad n = 0, 1, \ldots, \tag{1}$$

where the polar angle $\theta \in [0,\pi]$, the azimuthal angle $\phi \in [0, 2\pi)$ and $\overline{P}_n^m$ are the normalized associated Legendre functions. We define a subspace of spherical harmonics with fixed maximum degree N as

$$\mathcal{P}_N = \operatorname{span}\left\{Y_n^m(\theta,\phi),\ |m| \le n,\ 0 \le n \le N\right\}, \tag{2}$$

which has dimension $(N+1)^2$. The reproducing kernel for $\mathcal{P}_N$,

$$K\left(\boldsymbol{\omega}\cdot\boldsymbol{\omega}'\right) = \sum_{n=0}^{N}\frac{2n+1}{4\pi}P_n\left(\boldsymbol{\omega}\cdot\boldsymbol{\omega}'\right), \tag{3}$$

satisfies

$$f(\boldsymbol{\omega}) = \int_{\mathbb{S}^2} K\left(\boldsymbol{\omega}\cdot\boldsymbol{\omega}'\right) f\left(\boldsymbol{\omega}'\right) d\Omega', \quad f \in \mathcal{P}_N. \tag{4}$$

The identity in (4) may be verified by using the addition theorem for spherical harmonics. We rely on (4) to develop a localized representation of functions in $\mathcal{P}_N$ which is an analogue of Lagrange interpolation on the sphere, known as hyper-interpolation. The Funk-Radon transform is the spherical analog of the Radon transform and is defined for functions on the sphere by

$$\mathcal{G}[f](\mathbf{u}) = \int_{\mathbb{S}^2}\delta(\mathbf{u}\cdot\mathbf{v})\, f(\mathbf{v})\, d\mathbf{v}, \tag{5}$$

where δ is the Dirac mass and effectively causes the integration to be taken over the equator defined by the intersection of $\mathbb{S}^2$ and the plane $\mathbf{u}\cdot\mathbf{v} = 0$. The last result we will need is the Funk-Hecke theorem, which states that for any continuous function $f(t)$ on $[-1,1]$ and any spherical harmonic Y_n^m

$$\int_{\mathbb{S}^2} f(\mathbf{u}\cdot\mathbf{v})\, Y_n^m(\mathbf{v})\, d\mathbf{v} = \lambda_n Y_n^m(\mathbf{u}), \tag{6}$$

where

$$\lambda_n = 2\pi\int_{-1}^{1} f(t)\, P_n(t)\, dt \tag{7}$$

and P_n is the n^{th} degree Legendre polynomial. We remark that this theorem can be extended to distributions as well.

Quadratures for the sphere

Quadratures are at the heart of many methods for discretely sampling functions. These methods typically result from discretizing an integral representation, e.g., in the case of the functions on the sphere Eq.(4).

While one-dimensional quadratures are well understood, developing quadratures for the sphere is an active area of research. The difficulty in constructing quadratures for the sphere is related to its topology. A variety of quadratures for the sphere already exist, ranging from exact Cartesian product quadratures to approximate quadratures found by treating nodes as charged particles and minimizing their electrostatic potential energy.

Since subspaces of spherical harmonics, $\mathcal{P}_N$, are rotationally invariant it is natural to require rotational symmetry in a quadrature for the sphere. Because of the finite nature of the quadrature, only finite rotation groups can be considered. In [1], a numerical method was presented to construct exact quadratures which are invariant under the icosahedral rotation group. These quadratures are exact for subspaces of spherical harmonics and are nearly perfectly efficient. The high efficiency of these quadratures, as compared to other more standard ones, arises because of their Gaussian nature. Moreover, these quadratures are quite uniform and have no clustering of nodes, like other spherical quadratures.

To visualize how well a point set covers the sphere we assign at each point a Gaussian density $\exp\left(-s^2/2\sigma^2\right)$, where s is the arclength on the sphere and $\sigma^2 = 2/n\log(2)$ with n the number of points. Figure 1 shows a coverage map of symmeterized version of a 64 point grid regularly used for diffusion tensor imaging at UC Berkeley. The displayed version has 128 points. Figure 2 shows a coverage map for a quadrature with 72 nodes taken from Ref. [1].

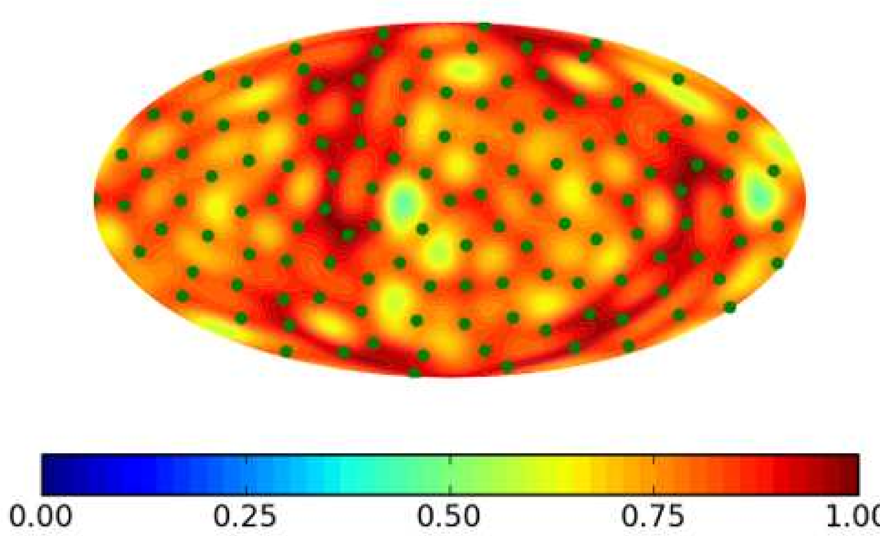

FIGURE 1: Coverage map for 64 measurement directions used at UC-Berkeley.

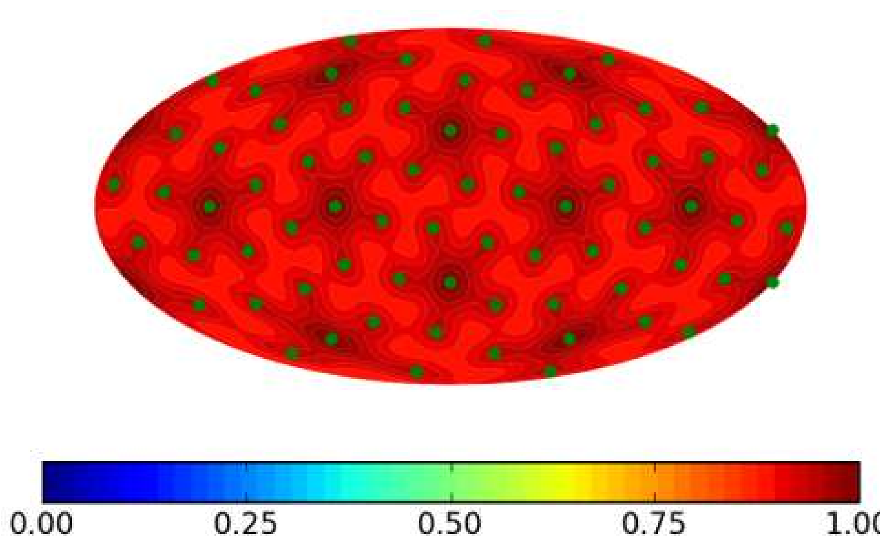

FIGURE 2: Coverage map for quadrature with 72 nodes from Ref. [1].

The two figures clearly show that the quadrature nodes are more uniformly distributed than the nodes used in diffusion tensor imaging at UC-Berkeley.

The quadratures from Ref. [1] will now be used to develop a localized representation of functions on the sphere, ideally suited for reconstruction of HARDI signals.

Local representations

Using the new quadratures, we construct an alternative representation for functions on invariant subspaces of $L^2\left(\mathbb{S}^2\right)$ by discretizing Eq. (4)

$$f(\boldsymbol{\omega}) = \sum_{j=1}^{M} K\left(\boldsymbol{\omega}\cdot\boldsymbol{\omega}_j\right) w_j f\left(\boldsymbol{\omega}_j\right). \tag{8}$$

If $f \in \mathcal{P}_N$, then (8) provides an exact reconstruction of the function f from its values $f(\boldsymbol{\omega}_1), f(\boldsymbol{\omega}_2), \ldots, f(\boldsymbol{\omega}_M)$. The functions $\left\{K\left(\boldsymbol{\omega}\cdot\boldsymbol{\omega}_j\right) w_j\right\}_{j=1}^{M}$ play a role similar to that of Lagrange interpolating polynomials and, therefore, we may think of (8) as an analogue of Lagrange interpolation on the sphere. Figure 3 shows a plot of one of the functions $K\left(\boldsymbol{\omega}\cdot\boldsymbol{\omega}_j\right)$ with the point $\boldsymbol{\omega}_j$ taken as the north pole, $\mathbf{e}_z$.

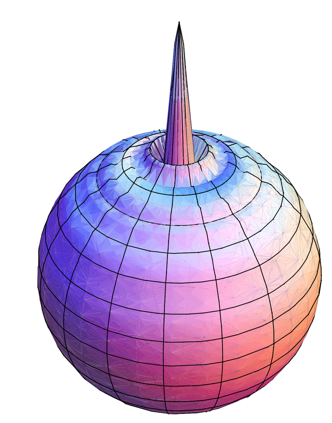

FIGURE 3: The function $K\left(\boldsymbol{\omega}\cdot\mathbf{e}_z\right)$.

HARDI Reconstruction

The diffusion weighted MR signal has the form

$$s(\mathbf{q}) = s_0\int_{\mathbb{R}^3} p(\mathbf{r})\, e^{-2\pi i\mathbf{q}\cdot\mathbf{r}} d^3 r, \tag{9}$$

where $p(\mathbf{r})$ is the ensemble averaged probability distribution function. To reconstruct directional information, the radial projection of $s(\mathbf{q})$, the so-called orientation distribution function (ODF), can be used and is defined by

$$\psi(\mathbf{u}) = \int_0^\infty p(\mathbf{u}r)\, dr, \quad \mathbf{u} \in \mathbb{S}^2 \tag{10}$$

Tuch [3] showed that

$$\psi(\mathbf{u}) \approx \mathcal{G}[s](\mathbf{u}). \tag{11}$$

Similar to [2]

Conclusions

We introduced a numerical method for constructing quadratures invariant under the icosahedral group which integrate rotationally invariant subspaces of $L^2\left(\mathbb{S}^2\right)$. Using these quadratures, an exact representation of functions on rotationally invariant subspaces of $L^2\left(\mathbb{S}^2\right)$, similar to Lagrange interpolation, was developed. The results of this paper are the first step to develop practical computational methods for applications that deal with the sphere. See [1] for more details.

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