Intro To Asymmetric Cryptography
RPISEC

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Symmetric vs Asymmetric Cryptosystems

Examples of symmetric crypto:

- classical ciphers (caesar, vigenere)
- block ciphers (DES, AES)
- stream ciphers (OTP, RC4, Salsa20)
- hash functions (MD5, SHA256)
- MACs (HMAC, Poly1305)

Examples of asymmetric crypto:

- key exchange (Diffie-Hellman)
- encryption (RSA, ElGamal)
- signatures (RSA, DSA, Schnorr)
- homomorphic encryption (Paillier, RSA, Gentry)

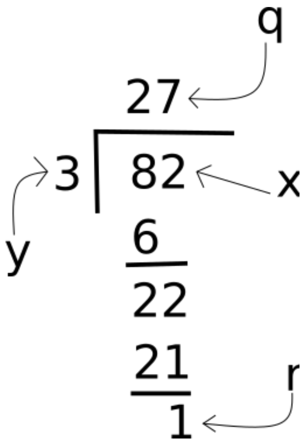
What is the significance of RSA?

- Historically the first asymmetric cryptosystem (published in 1977)
- Named for its inventors: Rivest, Shamir, Adleman
- Very flexible: can be used for both encryption and signing, is multiplicatively homomorphic
- Easy to implement
- Easy to mess up implementing, on account of its flexibility

What is RSA?

- p and q are distinct large primes
- $n = p * q$
- $\varphi(n) = (p-1) * (q-1)$
- $e * d \equiv 1 \pmod{\varphi(n)}$
- (n, e) is the "public key"
- (p, q, d) is the "private key"
- $\mathtt{enc}(x) = \mathtt{rem}(x^e, n)$
- $\mathtt{dec}(x) = \mathtt{rem}(x^d, n)$

Euclidean division



- $\forall x, y \exists q, r (y * q + r = x)$
- $q = \texttt{quot}(x, y)$
- $r = \texttt{rem}(x, y)$
- $0 \leq r < y$
- $3 * 27 + 1 = 82$

Modular congruence and primes

Divides cleanly/Is divisor of:

- $\texttt{divides}(x, y) \leftrightarrow \texttt{rem}(y, x) = 0$
- e.g. $\texttt{divides}(5, 10)$, since `10 % 5 == 0`

Modular congruence:

- $x \equiv y \pmod{n} \leftrightarrow \texttt{divides}(x - y, n)$
- $x \equiv y \pmod{n} \leftrightarrow \texttt{rem}(x, n) = \texttt{rem}(y, n)$

Primeness:

- $\texttt{prime}(p) \leftrightarrow \forall k \in [2, p-1](\neg\texttt{divides}(k, p))$
-
```
def prime(p):
    return all([p % k != 0 for k in range(2,p)])

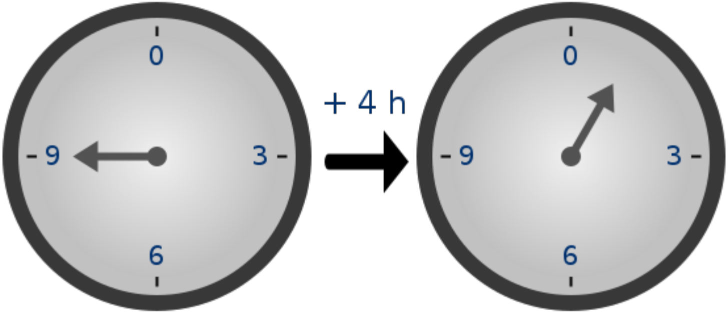
assert filter(prime, range(2, 20)) == [2, 3, 5, 7, 11, 13, 17, 19]
```

Greatest Common Divisor:

- $\texttt{gcd}(x, y) = \max\{k | \texttt{divides}(k, x) \wedge \texttt{divides}(k, y)\}$
-
```
def gcd(x, y):
    return max([1]+[k for k in range(1, x*y) if x % k == 0 and y % k == 0])
```

Modular arithmetic

- Let $\mathbb{Z}_n$ denote $\{\texttt{rem}(x, n) | x \in \mathbb{Z}\}$, or equivalently, $[0, n)$.
- We can truncate addition and multiplication to work within $\mathbb{Z}_n$ by calculating remainders after each operation
- "Clock arithmetic": in $\mathbb{Z}_{12}$, $9 + 4 = 1$, since $\texttt{rem}(9 + 4, 12) = \texttt{rem}(13, 12) = 1$



Addition and Multiplication mod 7

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

*	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

Addition and Multiplication mod 8

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	0
2	2	3	4	5	6	7	0	1
3	3	4	5	6	7	0	1	2
4	4	5	6	7	0	1	2	3
5	5	6	7	0	1	2	3	4
6	6	7	0	1	2	3	4	5
7	7	0	1	2	3	4	5	6

$*$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

Multiplicative inverses

- In $\mathbb{Q}$ and $\mathbb{R}$, inverses exist everywhere except zero: $x * \frac{1}{x} = 1$
- In $\mathbb{Z}$, inverses only exist at 1
- In $\mathbb{Z}_p$, inverses exist everywhere except zero, and can be found via fermat's little theorem
  (e.g. $3 * 4 \equiv 12 \equiv 1 \pmod{11}$)
- In $\mathbb{Z}_n$, multiples of factors of n act as additional zeros for the purposes of not having an inverse; when they exist, they can be found via an extension of the algorithm for euclidean division

Fermat's little theorem

- $x^{p-1} \equiv 1 \pmod{p}$
- $x * x^{p-2} \equiv 1 \pmod{p}$
- $x^{p-2} = \texttt{invert}(x, p)$

Multiplication mod 13

$x * x^{p-2} \equiv 1 \pmod{p}$

$*$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11	12
2	0	2	4	6	8	10	12	1	3	5	7	9	11
3	0	3	6	9	12	2	5	8	11	1	4	7	10
4	0	4	8	12	3	7	11	2	6	10	1	5	9
5	0	5	10	2	7	12	4	9	1	6	11	3	8
6	0	6	12	5	11	4	10	3	9	2	8	1	7
7	0	7	1	8	2	9	3	10	4	11	5	12	6
8	0	8	3	11	6	1	9	4	12	7	2	10	5
9	0	9	5	1	10	6	2	11	7	3	12	8	4
10	0	10	7	4	1	11	8	5	2	12	9	6	3
11	0	11	9	7	5	3	1	12	10	8	6	4	2
12	0	12	11	10	9	8	7	6	5	4	3	2	1

Exponentiation mod 13

$x * x^{p-2} \equiv 1 \pmod{p}$

x^y	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	4	8	3	6	12	11	9	5	10	7	1
3	1	3	9	1	3	9	1	3	9	1	3	9	1
4	1	4	3	12	9	10	1	4	3	12	9	10	1
5	1	5	12	8	1	5	12	8	1	5	12	8	1
6	1	6	10	8	9	2	12	7	3	5	4	11	1
7	1	7	10	5	9	11	12	6	3	8	4	2	1
8	1	8	12	5	1	8	12	5	1	8	12	5	1
9	1	9	3	1	9	3	1	9	3	1	9	3	1
10	1	10	9	12	3	4	1	10	9	12	3	4	1
11	1	11	4	5	3	7	12	2	9	8	10	6	1
12	1	12	1	12	1	12	1	12	1	12	1	12	1

Euler's totient theorem and phi

- $\gcd(x, n) = 1 \to x^{\varphi(n)} \equiv 1 \pmod{n}$
- $\varphi(n) = |\{k | 1 \leq k < n \wedge \gcd(k, n) = 1\}|$
- ```
  def phi(n):
      return len([k for k in range(1, n) if gcd(k, n) == 1])
  ```
- This definition is inefficient to calculate (linear in the value of n, so exponential in the bitlength of n)
  ```

Fundamental Theorem of Arithmetic

- Any integer can be decomposed (uniquely) into a product of prime powers
- $\forall n \exists \vec{v} (n = \Pi_{i=1}^{\texttt{len}(v)} p_i^{v_i})$
- e.g. for $n = 84 = 2 * 42 = 2^2 * 21 = 2^2 * 3 * 7$, $\vec{v} = [2, 1, 0, 1]$
- Computing $\vec{v}$ from n is expensive (algorithms like GNFS and ECM find $\vec{v}$ slower than polytime in the bitlength of n)
- Computing n from $\vec{v}$ is fast (p_i can be found via sieving, and multiplication is cheap)
- When coding, a sparse representation of the prime vector is more convenient:

```
histogram = lambda s: (lambda d: [[d.__setitem__(c, d.get(c, 0)+1) for c in s], d][-1])(dict())
product = lambda xs: reduce(__import__('operator').mul, xs, 1)
assert product([2,2,3,7]) == 84
assert histogram([2,2,3,7]) == {2: 2, 3: 1, 7: 1}
```

Calculating Euler's phi via FTA

If we know the factors of n, we can compute $\varphi(n)$ efficiently:

1. $\varphi(p^k) = (p-1) * p^{k-1}$
2. $\gcd(n, m) = 1 \rightarrow \varphi(n * m) = \varphi(n) * \varphi(m)$

- $\varphi(n) = \varphi(\prod_{i=1}^{\mathtt{len}(v)} p_i^{v_i}) =^{2.} \prod_{i=1}^{\mathtt{len}(v)} \varphi(p_i^{v_i}) =^{1.} \prod_{i=1}^{\mathtt{len}(v)} (p_i - 1) * p_i^{v_i - 1}$
- e.g. $\varphi(82) = \varphi(2^2 * 3 * 7) = \varphi(2^2) * \varphi(3) * \varphi(7) = (2-1) * 2^1 * (3-1) * (7-1) = 1 * 2 * 2 * 6 = 24$
-
```
histogram = lambda s: (lambda d: [[d.__setitem__(c, d.get(c, 0)+1) for c in s], d][-1])(dict())
product = lambda xs: reduce(__import__('operator').mul, xs, 1)
def phi(factors_of_n):
    return product([(p-1)*(p**(k-1)) for (p, k) in histogram(factors_of_n).items()])

assert product([2,2,3,7]) == 84 and phi([2,2,3,7]) == 24
```

Inverses modulo a composite

- ```
  from gmpy import gcd, invert
  def eee(x, y):
      x, y = min(x, y), max(x, y)
      r, s, t = x, 1, 0
      R, S, T = y, 0, 1
      while r > 0 and R > 0:
          q = r/R
          new = r-q*R, s-q*S, t-q*T
          r, s, t = R, S, T
          R, S, T = new
      assert gcd(x, y) == r # gcd from euclidean algorithm
      assert r == x*s + y*t # s and t are the bezout coefficients
      inv = s + y*(s < 0) # modular inverse from bezout coefficient
      if r == 1:
          assert (x * inv) % y == 1
      return (r, s, t, inv)

  assert [eee(i, 7)[3] for i in range(1, 7)] == [1, 4, 5, 2, 3, 6]
  assert [invert(i, 7) for i in range(1, 7)] == [1, 4, 5, 2, 3, 6]
  ```
  ```

Correctness of RSA[1]

- $e * d \equiv 1 \pmod{\varphi(n)}$
- $\mathtt{enc}(x) \equiv x^e \pmod{n}$
- $\mathtt{dec}(x) \equiv x^d \pmod{n}$
- $\mathtt{dec}(\mathtt{enc}(x)) \equiv (x^e)^d \equiv x^{e*d} \equiv^{\text{Euler}} x^1 \equiv x \pmod{n}$

[1] Assumes $\mathtt{gcd}(x, n) = 1$, full proof is a little bit more involved

Exploiting RSA: Cube root of small message

- ```
  from codecs import encode
  from gmpy import invert, next_prime
  import os
  d = 0
  while d == 0:
      p = next_prime(int(encode(os.urandom(1024/8), 'hex'), 16))
      q = next_prime(int(encode(os.urandom(1024/8), 'hex'), 16))
      n = p * q
      phi = (p - 1) * (q - 1)
      e = 3
      d = invert(e, phi)

  message = int(encode('hello', 'hex'), 16)
  ciphertext = pow(message, e, n)
  assert pow(ciphertext, d, n) == message
  assert round(ciphertext**(1.0/3)) == message
  ```
  ```

Resources

- https://en.wikipedia.org/wiki/RSA_(cryptosystem)
- https://en.wikipedia.org/wiki/Extended_Euclidean_Algorithm
- https://en.wikipedia.org/wiki/Modular_arithmetic
- https://crypto.stanford.edu/~dabo/papers/RSA-survey.pdf
- https://cryptopals.com/, Sets 5 and 6