THE INTERNAL TEMPERATURE-DENSITY DISTRIBUTION OF MAIN SEQUENCE STARS BUILT ON THE POINT-CONVECTIVE MODEL. II. SIRIUS A[1]

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ABSTRACT

Following the procedure adopted for the sun, the equations of stellar equilibrium, are integrated for Sirius A. Expressed in solar units, the mass, luminosity, and radius of Sirius A are 2.34, 38.9, and 1.78, respectively. The point-convective model is used as before, but Morse's improved tables of the opacity replace Strömgren's. Account is taken of electron scattering which becomes as much as 17 per cent of the total opacity at the boundary of the convective core. A hydrogen content of 37 per cent by weight gives fitting of the radiative envelope to the convective core and leads to a central temperature of 30.7×10^6 degrees C., and a central density of 43 gm/cm^3. In agreement with the solar calculation the hydrogen content does not differ from the value found with the Eddington standard model, whereas the central temperature and density are considerably higher than the corresponding Eddington values. This agreement between the calculations for the sun and for Sirius A is especially striking, since Morse's values of the opacity are uniformly lower than Strömgren's—sometimes as much as 50 per cent. It is probable that both the sun and Sirius A contain appreciable amounts of helium.

The first main-sequence star investigated in detail on the basis of the point-convective model was the sun.[2] The equations of stellar equilibrium were numerically integrated, using Strömgren's tables of the opacity. The helium content was chosen as zero, and the hydrogen concentration was adjusted to secure a fit of radiative envelope onto convective core. It turned out that the predicted luminosity of the sun was about one hundred times as large as the observed luminosity. We thereupon pointed out in Paper I that the discrepancy could be removed either by (1) the use of Morse's improved tables,[3] which had just been published and which gave lower opacities than Strömgren's, or (2) the taking into account of the variation of the effective molecular weight (equation of state) throughout the star, or, finally, (3) the assumption of an appreciable admixture of helium (30–40 per cent by weight). The discrepancy could also be explained by lowering the carbon-nitrogen concentration by a factor of one hundred. However, this is an observational question (the evidence at present is against a reduction[4]) in contrast to the other three possibilities, which can be decided on theoretical grounds.

In order to throw further light on the first alternative mentioned above and, at the same time, to obtain accurate information on the internal constitution of another interesting star of the main sequence, the equations of equilibrium have been integrated for Sirius A on the basis of the point-convective model. Again, as in the sun, the molecular weight is taken as constant throughout the star, and the helium content is assumed zero. However, Morse's opacity tables are used instead of Strömgren's. Otherwise, the procedure is identical with that adopted for the sun (for details see Paper I).

[1] The results presented here are based on computations carried out by Miss E. Godefray and Mr. W. Horenstein, of the Mathematical Tables Project, with Dr. A. N. Lowan as technical director and under the immediate supervision of Dr. G. Blanch. At the time the computations were performed, in 1941, the Mathematical Tables Project was sponsored by the National Bureau of Standards, with Dr. L. J. Briggs as director, and was operated by the Work Projects Administration, New York City. These results were first reported at a meeting of the American Physical Society (cf. *Phys. Rev.*, **61**, 543(A), 1942).

[2] Cf. G. Blanch, A. N. Lowan, R. E. Marshak, and H. A. Bethe, *Ap. J.*, **94**, 37, 1941 (this paper will be referred to as "Paper I").

[3] P. Morse, *Ap. J.*, **92**, 27, 1940.

[4] Cf. H. A. Bethe, *Ap. J.*, **91**, 362, 1940; and H. N. Russell, *Scientific American*, September, 1942.

Applied to Sirius A, the equations for the radiative envelope become (cf. Paper I):

$$\frac{dp_R}{dr} = -\frac{\kappa \rho L}{4\pi r^2 c}, \tag{1}$$

$$\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho , \tag{2}$$

$$\frac{dM_r}{dr} = 4\pi r^2 \rho , \tag{3}$$

$$P = p_G + p_R = \frac{\Re}{\mu}\rho T + \tfrac{1}{3} a T^4 , \tag{4}$$

and

$$\kappa = \kappa_a + 1.5\,\kappa_s = \frac{\kappa_a'}{\tau}\frac{\rho}{T^{3.5}} + 1.5\,\kappa_s . \tag{5}$$

In these equations a, c, G, and $\Re$ have their usual significance; their definitions and numerical values are given in Chandrasekhar's book, *An Introduction to the Study of Stellar Structure*, page 487. The quantity M_r is the mass contained within a sphere of radius r. The total pressure P consists of two parts, the gas pressure p_G and the radiation pressure p_R; in the expression for p_G, μ is the average molecular weight and equal to $1/[2X_H + 0.52(1 - X_H)]$, where 0.52 is the average number of free electrons plus nuclei per proton.[5] The opacity κ_a from photoelectric ionizations contains the guillotine factor τ and a constant κ_a', depending on X_H, which may be written as

$$\kappa_a' = 7.65 \times 10^{25} \times (1 - X_H)[X_H + 0.48(1 - X_H)] .^{6}$$

The opacity arising from electron scattering is given by

$$\kappa_S = 0.385[X_H + 0.48(1 - X_H)] .$$

If a change of variable is made as in Paper I, equations (1)–(3) become

$$\frac{d\bar{T}}{dt} = \frac{A_0 \rho^2}{\tau \bar{T}^{6.5}} + \frac{A_1 \rho}{\bar{T}^3}, \tag{6}$$

$$\frac{d\rho}{dt} = \frac{A_2 \rho \mu}{\bar{T}} - \frac{d \log \bar{T}}{dt}(\rho + A_3 \bar{T}^3), \tag{7}$$

and

$$\frac{du}{dt} = -\frac{A_4 \rho}{t^4}, \tag{8}$$

where

$$T = 10^6 \bar{T} , \qquad t = \frac{R}{r}, \qquad u = \frac{M_r}{M},$$

$$A_0 = \frac{3\kappa_a' L}{16\pi a c R} \times 10^{-45}, \qquad A_1 = \frac{4.5\,\kappa_S L}{16\pi a c R} \times 10^{-24},$$

$$A_2 = \frac{\mu G M}{\Re R} \times 10^{-6}, \qquad A_3 = \frac{4 a \mu}{3 \Re} \times 10^{18}, \qquad A_4 = \frac{4\pi R^3}{M} .$$

[5] This number (0.52), in contrast to 0.46 for the sun, was obtained in the usual way by finding the state of ionization at the point in the star where $T \approx \frac{2}{3} T_c$ (T_c is the central temperature) (cf. Strömgren, *Zs. f. Ap.*, **7**, 229, 1932; and Paper I).

[6] Only the electrons contribute to the opacity, so that 0.48 is the average number of free electrons per proton and is chosen in the same manner as 0.52 (cf. n. 5).

If the values of L, M, and R for Sirius A are inserted—namely, $M = 2.34M_{\odot}$, $R = 1.78R_{\odot}$, $L = 38.9L_{\odot}$—and if, e.g., we choose $X_H = 0.35$ (i.e., $\mu = 0.96339$), we get

$$A_0 = 1.0308 \times 10^7 , \quad A_1 = 119.55 ,$$

$$A_2 = 29.276 , \quad A_3 = 1.1769 \times 10^{-4} , \quad A_4 = 5.1243 .$$

As before, Strömgren's method[7] was used to obtain initial values of t and ρ for $T = 1.00 \times 10^6$, and the integration was continued at sufficiently small intervals in t so that the final results are probably correct to three significant figures.

TABLE 1

TEMPERATURE-DENSITY DISTRIBUTION OF SIRIUS A WITH $X_H = 0.35$

t	$T \times 10^{-6}$	ρ	$u = \frac{M_r}{M}$	τ	$n = \frac{d \log \rho}{d \log T}$
1.1435	1.000	0.00082	0.9998	1.57	5.42
1.154	1.045	0.00102	.9998	1.57	4.62
1.162	1.093	0.00122	.9998	1.56	4.07
1.170	1.140	0.00144	.9997	1.54	3.70
1.21	1.40	0.00288	.9995	1.43	3.20
1.25	1.68	0.00506	.9992	1.36	3.14
1.32	2.15	0.0112	.9982	1.35	3.22
1.40	2.69	0.0231	.9962	1.34	3.22
1.48	3.24	0.0416	.9932	1.28	3.12
1.56	3.79	0.0679	.989	1.23	3.12
1.64	4.33	0.104	.984	1.30	3.39
1.72	4.83	0.152	.977	1.41	3.50
1.80	5.32	0.215	.970	1.51	3.52
2.00	6.53	0.444	.945	1.82	3.68
2.20	7.66	0.809	.913	2.33	3.87
2.40	8.71	1.34	.874	2.93	3.91
2.70	10.2	2.48	.807	3.70	3.72
3.10	12.2	4.58	.708	4.35	3.36
3.50	14.0	7.14	.608	4.68	3.05
3.90	15.6	9.93	.515	5.16	2.94
4.30	17.0	12.7	.432	5.78	2.87
4.70	18.2	15.4	.362	6.25	2.68
5.10	19.3	17.9	.302	6.57	2.45
5.50	20.2	20.0	.253	6.79	2.22
5.90	21.1	21.9	.212	6.93	1.99
6.30	21.9	23.4	.178	7.01	1.78
6.70	22.6	24.7	.151	7.11	1.59
6.9111	22.912	25.277	0.1380	7.15	1.50

Two independent integrations were performed for Sirius A, namely, for $X_H = 0.35$ and $X_H = 0.40$. The results are given in Tables 1 and 2, respectively, at intervals at least four times as large as those used in the actual computations. The integrations are carried along until $n \equiv d \log \rho / d \log T$ becomes equal to 1.5. At this point, radiative equilibrium gives way to convective equilibrium, and one must try to fit the radiative envelope onto an incomplete Emden polytrope of index $n = 1.5$.[8] For the case $X_H =$

[7] Cf. *Zs. f. Ap.*, **2**, 350, 1931, and Paper I.

[8] The value of n is altered if the radiation pressure becomes large; for Sirius A the radiation pressure was taken into account in the integration, but its effect on n is inappreciable.

0.35, $\Delta\xi \equiv \xi(U) - \xi(V) = 0.068$,[9] where U and V are the two associated Emden functions of polytropic index 1.5 (defined in Paper I). For the case $X_H = 0.40$, $\Delta\xi \equiv \xi(U) - \xi(V) = -0.113$.[9] It follows that $X_H = 0.369$ would lead to $\Delta\xi = 0$[10] and, would therefore, fit the radiative envelope onto the convective core. To obtain the corresponding central temperature and density, we must interpolate between the two cases which have been integrated. From footnote 9 we see that $\xi_{av} = 1.225$ for the first case; this leads to $T_c = 29.5_5 \times 10^6$ and $\rho_c = 36.9$. For the second case, $\xi_{av} = 1.001$, which

TABLE 2

TEMPERATURE-DENSITY DISTRIBUTION OF SIRIUS A WITH $X_H = 0.40$

t	$T \times 10^{-6}$	ρ	$u = \frac{M_r}{M}$	τ	$n = \frac{d \log \rho}{d \log T}$
1.1624	1.000	0.00081	0.9997	1.69	5.88
1.166	1.015	0.00088	.9997	1.59	4.98
1.174	1.05	0.00104	.9997	1.57	4.37
1.182	1.09	0.00122	.9997	1.55	3.95
1.190	1.14	0.00142	.9997	1.53	3.67
1.198	1.19	0.00164	.9996	1.51	3.48
1.206	1.23	0.00188	.9996	1.49	3.36
1.23	1.38	0.00272	.9995	1.43	3.20
1.27	1.64	0.00465	.9992	1.36	3.13
1.34	2.08	0.0100	.9983	1.35	3.20
1.42	2.59	0.0201	.9967	1.35	3.24
1.50	3.10	0.0357	.994	1.30	3.17
1.58	3.61	0.0577	.991	1.24	3.10
1.75	4.67	0.133	.981	1.37	3.49
1.95	5.82	0.289	.963	1.61	3.55
2.15	6.92	0.544	.940	1.98	3.77
2.35	7.96	0.928	.911	2.50	3.93
2.55	8.94	1.47	.878	3.05	3.91
2.75	9.89	2.17	.841	3.55	3.80
3.10	11.5	3.80	.769	4.17	3.51
3.50	13.3	6.20	.684	4.60	3.23
3.90	15.0	9.01	.601	5.02	3.11
4.30	16.5	12.1	.525	5.62	3.09
4.70	17.8	15.2	.457	6.20	2.99
5.10	19.0	18.4	.397	6.62	2.82
5.50	20.1	21.3	.345	6.90	2.63
5.90	21.0	24.1	.301	7.12	2.46
6.30	21.9	26.6	.264	7.32	2.31
6.70	22.8	28.9	.232	7.47	2.16
7.10	23.5	31.0	.204	7.57	2.02
7.50	24.2	32.8	.181	7.63	1.88
7.90	24.9	34.4	.162	7.67	1.76
8.30	25.5	35.9	.145	7.95	1.72
8.70	26.0	37.2	.130	8.23	1.67
9.40	26.9	39.2	.110	8.76	1.59
10.00	27.5	40.7	0.096	9.16	1.50

[9] For $X_H = 0.35$,

$$\xi(U) = \xi(1.9211) = 1.259\,,$$
$$\xi(V) = \xi(1.2186) = 1.191\,;$$

for $X_H = 0.40$,

$$\xi(U) = \xi(0.68513) = 0.94457\,,$$
$$\xi(V) = \xi(0.95291) = 1.05757\,.$$

[10] Linear interpolation was used; this is adequate (cf. Paper I).

yields $T_c = 32.5_5 \times 10^6$ and $\rho_c = 52.5$. Using linear interpolation,[11] we get, corresponding to $X_H = 0.369$, $T_c = 30.7 \times 10^6$, and $\rho_c = 42.8$. These values are to be compared with those derived for the same μ (i.e., $\mu = 0.94$) on the Eddington standard model, namely, $T_c = 24.4 \times 10^6$ and $\rho_c = 31.5$, or with the point-convective values (assuming constant guillotine factor), namely, $T_c = 25.2 \times 10^6$ and $\rho_c = 21$.

If the temperature-density distributions obtained for the cases $X_H = 0.35$ and $X_H = 0.40$ (cf. Tables 1 and 2) are inserted into the expression for the energy production due to the carbon cycle,[12] one finds for the luminosities:

$$L = \int_0^R 4\pi r^2 \epsilon \rho \, d r = \begin{cases} 1100 L_\odot & \text{for} \quad X_H = 0.35 \\ 8200 L_\odot & \text{for} \quad X_H = 0.40 \, . \end{cases} \tag{9}$$

These values are to be compared with an observed value of $38.9L_\odot$ and are, therefore, 28 and 210 times too large, respectively. For $X_H = 0.369$ it is necessary to interpolate between the above two luminosities. Because of the sensitive dependence of the luminosity on the temperature, linear interpolation is too crude. Therefore, we resort to the following device: roughly, $L \backsim \rho_c^2 T_c^n$. We fix n so that the ratio of $L(X_H = 0.40)$ to $L(X_H = 0.35)$ is correctly given when the appropriate values of ρ_c and T_c are inserted. This leads to $n = 13.5$. We then substitute the values of ρ_c and T_c corresponding to $X_H = 0.369$ and obtain $L(X_H = 0.369) = 2400L_\odot$. This is about sixty times the observed luminosity.

In agreement with the solar calculations, the theoretical luminosity turns out to be much larger than the observed value. The discrepancy is especially striking for Sirius A, since Morse's tables of the opacity were used—in contrast to Strömgren's tables for the sun—and Morse's opacities are uniformly lower than Strömgren's, sometimes as much as 50 per cent. The source of the discrepancy is, therefore, not to be found in the opacity tables used. Nor is it likely that the use of constant molecular weight in the integrations is responsible, although this point should ultimately be checked. Furthermore, considerations of the sort given by Sen and Burman[13] do not really resolve the difficulty. These authors show that they can reconcile the observed luminosity and mass of the sun with a hydrogen content of 36 per cent (zero helium content) and a central temperature of 20×10^6. However, the radius turns out to be 15 per cent larger than the sun's. This discrepancy appears small but actually contains the source of the trouble. The central temperature varies roughly as the inverse power of the radius, and the luminosity as the seventeenth power of the central temperature; also a 15 per cent decrease in radius increases the density by 50 per cent. The combination of these two factors leads to almost as serious a discrepancy as that found previously for the sun.

The most plausible conclusion to be drawn from the evidence obtained thus far is that helium is present in appreciable amounts in both the sun and Sirius A. It is not expected that the helium content would be the same in all stars of the main sequence. As a matter of fact, it would be interesting to discover whether there is a systematic variation of the helium content among main-sequence stars from the point of view of the theory of stellar interiors.

[11] In the case of the sun (cf. Paper I) linear interpolation between the $X_H = 0.30$ and $X_H = 0.40$ solutions in accordance with the above procedure leads to $T_c = 26.2 \times 10^6$ and $\rho_c = 123$, as compared with the correct values, $T_c = 25.7 \times 10^6$ and $\rho_c = 110$ (for $X_H = 0.35$). This is quite good and will be even better for Sirius A where the range covered is smaller.

[12] The carbon cycle gives $\epsilon = 3 \times 10^{21} X_H \zeta^2 e^{-\zeta}$, where $\zeta = 152/T^{1/3}$ and T is expressed in millions of degrees (cf. Paper I).

[13] *Ap. J.*, **100**, 347, 1944.