7. DUALITY.

G IS A LOCALLY COMPACT ABELIAN GROUP, Γ (ALSO CALLED $\hat{G}$) ITS DUAL. ON G AND Γ HAAR MEASURES ARE CHOSEN THAT BELONG TO EACH OTHER AS INDICATED IN THE INVERSION THEOREM.

In Exercise 5.A we have discovered a natural continuous homomorphism $\alpha : G \longrightarrow \hat{\Gamma}$. In the present section we prove α to be injective, surjective and a homeomorphism.

7.1. LEMMA. <u>For every neighborhood</u> V <u>of</u> $1 \in G$ <u>there exists a compact</u> $C \subset \Gamma$ <u>and a</u> $\rho > 0$ <u>such that</u> $\{x \in G : |\gamma(x)-1| < \rho$ for all $\gamma \in C\}$ <u>is contained in</u> V.

Proof. The map $(x,y) \longmapsto xy^{-1}$ being continuous there exists a compact neighborhood W of $1 \in G$ such that $WW^{-1} \subset V$. Then $0 < m_G(W) < \infty$, m_G being the Haar measure we have selected on G. Set $f := \frac{\xi_W}{\sqrt{m_G(W)}}$. Then $f \in L^2(G)$, so $g := f * \tilde{f}$ is continuous and positive definite (Lemma 6.7 and Theorem 6.8). We have $g=0$ off WW^{-1}; this WW^{-1} is compact; and g is bounded. Therefore, $g \in L^1(G)$. By the Inversion Theorem 6.11, $1 = g(1) = \int \hat{g}(\gamma)\gamma(1)d\gamma = \int \hat{g}(\gamma)d\gamma$. Choose a compact $C \subset \Gamma$ so that $\int_{\Gamma \setminus C} |\hat{g}| < \frac{1}{3}$. We claim that $\{x \in G : |\gamma(x)-1| < \frac{1}{3}$ for all $\gamma \in C\}$ is a subset of V.

In fact, let $a \in G$ be so that $|\gamma(a)-1| < \frac{1}{3}$ for every $\gamma \in C$. Then, as $\hat{g} = \hat{f}\hat{\tilde{f}} = \hat{f}\bar{\hat{f}} \geq 0$, $|g(a)-1| = |\int \hat{g}(\gamma)\gamma(a)d\gamma - \int \hat{g}(\gamma)d\gamma| \leq$
$\leq \int_C \hat{g}(\gamma)|\gamma(a)-1|d\gamma + \int_{\Gamma\setminus C} \hat{g}(\gamma)|\gamma(a)-1|d\gamma \leq \int_\Gamma \hat{g}(\gamma)\frac{1}{3}d\gamma + \int_{\Gamma\setminus C} \hat{g}(\gamma)2d\gamma <$
$< \frac{1}{3}+\frac{2}{3}=1$, so $g(a) \neq 0$ and $a \in WW^{-1} \subset V$.

7.2. COROLLARY. (a) <u>If</u> $x,y \in G$ <u>are distinct there exists a</u> $\gamma \in \Gamma$ <u>such that</u> $\gamma(x) \neq \gamma(y)$.

(b) <u>If</u> H <u>is a closed subgroup of</u> G <u>and if</u> $x \in G$, $x \notin H$, <u>then there exists a</u> $\gamma \in \Gamma$ <u>for which</u> $\gamma(x) \neq 1$ <u>while</u> $\gamma = 1$ <u>on</u> H.

Proof. (a) By Lemma 7.1 there exist a compact $C \subset \Gamma$ and a $\rho>0$ such that $|\gamma(xy^{-1})-1| \geq \rho$ for some $\gamma \in C$. For this γ we find $\gamma(xy^{-1}) \neq 1$ and $\gamma(x) \neq \gamma(y)$.

(b) Let π be the natural surjection $G \longrightarrow G/H$. By (a) there exists a $\beta \in (G/H)^{\wedge}$ such that $\beta(\pi(x)) \neq 1$. Put $\gamma = \beta \circ \pi$.

7.3. COROLLARY. <u>Let G be compact. By definition, the linear span of the continuous characters of G is Trig(G), the set of all trigonometric polynomials on G. This Trig(G) is a dense subalgebra of C(G).</u>

Proof. If $\gamma \in \Gamma$, then $\bar{\gamma} \in \Gamma$. Hence, if $f \in \mathrm{Trig}(G)$ then $\bar{f} \in \mathrm{Trig}(G)$. Now apply Corollary 7.2 to show that Trig(G) separates the points of G, and use the Stone-Weierstrass Theorem.

7.4. COROLLARY. $\alpha: G \longrightarrow \hat{\Gamma}$ <u>is injective, and is a homeomorphism of G onto a closed subgroup of</u> $\hat{\Gamma}$.

Proof. The injectivity follows immediately from Corollary 7.2. For compact $C \subset \Gamma$ and $\rho>0$ set $N(C:\rho) := \{\varphi \in \hat{\Gamma} : |\varphi - 1| < \rho$ on $C\}$; then $\alpha^{-1}(N(C:\rho)) = \{x \in G : |\gamma(x)-1| < \rho$ for all $\gamma \in C\}$. We know (Lemma 5.8) that the sets $\varphi N(C:\rho)$ ($\varphi \in \hat{\Gamma}$; $C \subset \Gamma$ compact; $\rho>0$) form a base for the topology of $\hat{\Gamma}$. From Lemma 7.1 and the continuity of α it follows that α maps G homeomorphically onto $\alpha(G) \subset \hat{\Gamma}$. The closedness of $\alpha(G)$ now is a consequence of the following lemma.

7.5. LEMMA. <u>Every locally compact subgroup of a Hausdorff topological group is closed.</u>

Proof. Let H be a Hausdorff topological group, H_0 a subgroup that is locally compact under the relative topology. Take $a \in \overline{H_0}$. There exists an open neighborhood V of $1 \in H$ such that $V \cap H_0$ is contained in a (relatively) compact subset A of H_0. Then A is compact in H, hence closed in H, so that $\overline{V \cap H_0} \subset H_0$. aV^{-1} is a neighborhood of a, so

$aV^{-1} \cap H_0$ contains some point b. If U is any neighborhood of a, then $U \cap bV$ is a neighborhood of a, hence intersects H_0. It follows that every neighborhood of a intersects $bV \cap H_0$, so that $a \in \overline{bV \cap H_0} = \overline{b(V \cap H_0)} = b\,\overline{V \cap H_0} \subset bH_0 = H_0$.

7.6. PLANCHEREL THEOREM. <u>There is a unique linear isometry</u> $\mathcal{F}$ <u>of</u> $L^2(G)$ <u>onto</u> $L^2(\Gamma)$ <u>such that</u> $\mathcal{F}f = \hat{f}$ <u>for</u> $f \in L^1(G) \cap L^2(G)$.

Proof. Let $f \in L^1(G) \cap L^2(G)$. Then $f * \tilde{f} \in L^1(G) \cap B(G)$ (see Theorem 6.8), so that we can apply the Inversion Theorem. Apparently $(f*\tilde{f})^\wedge \in L^1(\Gamma)$ and $\int (f*\tilde{f})^\wedge(\gamma)d\gamma = (f*\tilde{f})(1)$. Now $(f*\tilde{f})^\wedge = \hat{f}\hat{\tilde{f}} = \hat{f}\bar{\hat{f}} = |\hat{f}|^2$. Therefore, $\hat{f} \in L^2(\Gamma)$ and $(\|\hat{f}\|_2)^2 =$
$= \int |\hat{f}|^2 = \int (f*\tilde{f})^\wedge = (f*\tilde{f})(1) = \int f(x)\tilde{f}(x^{-1})dx = \int f(x)\overline{f(x)}dx = \int |f|^2 = (\|f\|_2)^2$.

Thus, the restriction of the Fourier Transformation is a linear isometry of $L^1(G) \cap L^2(G)$ (under the L^2-norm) into $L^2(\Gamma)$. $L^1(G) \cap L^2(G)$ is dense in $L^2(G)$: consequently, this restriction of the Fourier Transformation has a unique continuous extension $\mathcal{F} : L^2(G) \longrightarrow L^2(\Gamma)$, and this $\mathcal{F}$ automatically is a linear isometry. The range of $\mathcal{F}$, $\mathcal{R}_{\mathcal{F}}$, is a linear subspace of $L^2(\Gamma)$. It is complete, hence closed, and we are done if we can prove $\mathcal{R}_{\mathcal{F}} = L^2(\Gamma)$. Let $\psi \in L^2(\Gamma)$, $\psi \perp \mathcal{R}_{\mathcal{F}}$: we prove $\psi = 0$. (Then $\mathcal{R}_{\mathcal{F}} = L^2(\Gamma)$ by the Projection Theorem.)

If $f \in L^1(G) \cap L^2(G)$ and $x \in G$, then $\widehat{f_x} \in \mathcal{R}_{\mathcal{F}}$, so that $0 = \int \widehat{f_x}(\gamma)\overline{\psi(\gamma)}d\gamma = \int \hat{f}(\gamma)\overline{\psi(\gamma)}\,\overline{\gamma(x)}d\gamma = (\hat{f}\bar{\psi})^\vee(x)$. Hence, if $f \in L^1(G) \cap L^2(G)$, then $(\hat{f}\bar{\psi})^\vee = 0$. By the Uniqueness Theorem 6.4 $\hat{f}\bar{\psi} = 0$ a.e. For every compact $C \subset \Gamma$ there exists an $f \in C_{00}(G)$ such that $\hat{f} > 0$ everywhere on C. (See last paragraph of page 6.7.) Hence, for every compact $C \subset \Gamma$ we have $\psi = 0$ a.e. on C. By Exercise 3.F (e), $\psi = 0$ a.e. on Γ.

We usually write $\hat{f}$ instead of $\mathcal{F}f$, also if $f \in L^2(G)$ and $f \notin L^1(G)$. An immediate consequence of the Plancherel Theorem is

7.7. PARSEVAL FORMULA. *If* $f, g \in L^2(G)$, *then*

$$\int f(x)\overline{g(x)}dx = \int \hat{f}(\gamma)\overline{\hat{g}(\gamma)}d\gamma$$

and

$$\int f(x)g(x^{-1})dx = \int \hat{f}(\gamma)\hat{g}(\gamma)d\gamma.$$

Proof. Use the Polarization Formula 0.14.

7.8. LEMMA. *If* $g \in L^2(G)$ *and* $\gamma \in \Gamma$ *then* $\gamma g \in L^2(G)$ *and* $\widehat{\gamma g} = (\hat{g})_\gamma$.

Proof. The announced formula is certainly valid for $g \in L^1(G) \cap L^2(G)$ (see Theorem 5.6 (a)), hence, by continuity, for all $g \in L^2(G)$.

7.9. LEMMA. *If* $f, g \in L^2(G)$, *then* $fg \in L^1(G)$ *and* $\widehat{fg} = \hat{f} * \hat{g}$.

Proof. For $\beta \in \Gamma$, $(\widehat{fg})(\beta) = \int f(x)\overline{(\beta\bar{g})(x)}dx = \int \hat{f}(\gamma)\overline{(\widehat{\beta\bar{g}})(\gamma)}d\gamma =$
$= \int \hat{f}(\gamma)\overline{\hat{\bar{g}}}(\beta^{-1}\gamma)d\gamma = \int \hat{f}(\gamma)\hat{g}(\gamma^{-1}\beta)d\gamma = (\hat{f} * \hat{g})(\beta)$. (See Lemma 5.4.)

7.10. COROLLARY. $A(\Gamma) = \{ f_1 * f_2 : f_1, f_2 \in L^2(\Gamma) \}$.

Proof. The inclusion $\supset$ follows from the Plancherel Theorem 7.6 and Lemma 7.9. For the converse inclusion, let $h \in A(\Gamma)$. Then $h = \hat{g}$ for some $g \in L^1(G)$. There exist $g_1, g_2 \in L^2(G)$ such that $g_1 g_2 = g$. Then $\hat{g}_1, \hat{g}_2 \in L^2(\Gamma)$ and $\hat{g}_1 * \hat{g}_2 = h$ (Lemma 7.9).

7.11. LEMMA. *If* $K \subset \Gamma$ *is compact and* $V \subset \Gamma$ *is open, such that* $K \subset V$, *there exists an* $h \in A(\Gamma)$, $0 \leq h \leq 1$, *for which* $h = 1$ *on* K, $h = 0$ *outside* V.

Proof. Assume $K \neq \emptyset$. One easily constructs an open set W containing K such that $\overline{W}$ is compact and contained in V. By Lemmas 3.2 and 5.7, $\{ \gamma \in \Gamma : K\gamma^{-1} \subset W \}$ and $\{ \gamma \in \Gamma : \overline{W}\gamma \subset V \}$ are open neighborhoods of $1 \in \Gamma$, so there exists a compact neighborhood A of $1 \in \Gamma$ with $KA^{-1} \subset W$ and $WA \subset V$. Both W and A have finite positive measure, so that ξ_W and ξ_A are elements of $L^2(\Gamma)$ and $\xi_W * \xi_A \in A(\Gamma)$. Now $(\xi_W * \xi_A)(\gamma) = m_\Gamma(W \cap \gamma A^{-1})$

($\gamma \in \Gamma$), where m_Γ is the Haar measure on Γ. Then $h := m_\Gamma(A)^{-1}\xi_W * \xi_A$ satisfies the conditions.

Now we are ready for the big theorem of this section.

7.12. PONTRYAGIN DUALITY THEOREM. $\alpha : G \longrightarrow \hat{\hat{G}}$ <u>is an isomorphism of topological groups</u>. (See also Exercise 7.G.)

Proof. After Corollary 7.4 we only have to prove that α is surjective. Suppose $\hat{\Gamma}\setminus\alpha(G) \neq \emptyset$. As $\hat{\Gamma}\setminus\alpha(G)$ is an open subset of $\hat{\Gamma}$, by Lemma 7.11 there exists an $f \in L^1(\Gamma)$ such that $f \neq 0$ but $\hat{f} = 0$ on $\alpha(G)$, i.e. an $f \in L^1(\Gamma)$ with $f \neq 0$, $\check{f} = 0$. But now we have a contradiction with the Uniqueness Theorem 6.4.

The Duality Theorem has a number of interesting consequences.

7.13. COROLLARY. $L^1(G)$ <u>has an identity element if and only if</u> G <u>is discrete</u>.

Proof. If G is discrete, then $L^1(G)$ is isomorphic to $M(G)$, which has an identity element. Conversely, if $L^1(G)$ has an identity, then $\mathfrak{M}(L^1(G))$ is compact. Then Γ is compact, so G is discrete.

7.14. COROLLARY. <u>If</u> $\mu \in M(G)$ <u>and</u> $\hat{\mu} = 0$, <u>then</u> $\mu = 0$. <u>Therefore</u>, $M(G)$ <u>and</u> $L^1(G)$ <u>are semi-simple</u>.

Proof. Use Duality Theorem and Uniqueness Theorem 6.4.

7.15. COROLLARY. <u>Let</u> $\mu \in M(G)$. <u>If</u> $\hat{\mu} \in L^1(\Gamma)$, <u>then</u> $\mu \in M_a(G)$ <u>and</u> $\mu = fm$ <u>for some</u> $f \in L^1(G) \cap B(G)$.

Proof. We have $\hat{\mu} \in B(\Gamma)$, so (Inversion Theorem and Duality Theorem) $\hat{\hat{\mu}} \in L^1(\hat{\Gamma})$ and $\check{\hat{\hat{\mu}}} = \mu$. Set $f(x) := \hat{\hat{\mu}}(\alpha(x)^{-1})$ ($x \in G$). Then $f \in L^1(G) \cap B(G)$ and $\hat{f} = \hat{\mu}$. By Corollary 7.14, $fm = \mu$.

For a closed subgroup H of G, $H^{\perp} := \{\gamma \in \Gamma : \gamma = 1 \text{ on } H\}$ is called the <u>annihilator</u> of H. It is a closed subgroup of Γ. Similarly, the <u>annihilator</u> of a closed subgroup Λ of Γ is the closed subgroup $\Lambda_{\perp} := \{x \in G : \gamma(x) = 1 \text{ for every } \gamma \in \Lambda\}$. Trivially, $(H^{\perp})_{\perp} \supset H$. But Corollary 7.2 (b) now implies

$$(H^{\perp})_{\perp} = H.$$

By duality,

$$(\Lambda_{\perp})^{\perp} = \Lambda$$

for closed subgroups Λ of Γ. Thus, we have a one-to-one correspondence $H \longrightarrow H^{\perp}$ between the closed subgroups of G and those of Γ.

Let H be a closed subgroup of G and let π be the natural surjection $G \longrightarrow G/H$. This π induces a continuous homomorphism $\hat{\pi} : (G/H)^{\wedge} \longrightarrow \hat{G}$ by

$$\hat{\pi}(\gamma) := \gamma \circ \pi \qquad (\gamma \in (G/H)^{\wedge})$$

and it is easy to see that $\hat{\pi}$ maps $(G/H)^{\wedge}$ onto $H^{\perp}$. Further, if $Y \subset G/H$ is compact there exists a compact $X \subset G$ such that $Y = \pi(X)$ (Lemma 3.7). Then for every $\varrho > 0$, $\pi(N(X:\varrho)) = N(Y:\varrho)$. It follows from Lemma 5.8 that $\hat{\pi}$ is a homeomorphism $(G/H)^{\wedge} \longrightarrow H^{\perp}$.

This $\hat{\pi}$ yields a homeomorphism $G/H \longrightarrow (H^{\perp})^{\wedge}$. Similarly, for a closed subgroup Λ of Γ we have a homeomorphism $\Gamma/\Lambda \longrightarrow (\Lambda_{\perp})^{\wedge}$. Taking $\Lambda = H^{\perp}$ one obtains a homeomorphism $\sigma : \Gamma/H^{\perp} \longrightarrow \hat{H}$. We also have the restriction map $\varrho : \Gamma \longrightarrow \hat{H}$, which is a continuous homomorphism whose kernel is $H^{\perp}$; this ϱ induces $\varrho' : \Gamma/H^{\perp} \longrightarrow \hat{H}$, and one easily proves $\varrho' = \sigma$.

We have proved

7.16. THEOREM. <u>Let H be a closed subgroup of G; let π be the natural homomorphism $G \longrightarrow G/H$. Then $\gamma \longmapsto \gamma \circ \pi$ is a homeomorphism and an isomorphism of $(G/H)^{\wedge}$ onto $H^{\perp}$. Further, the formula</u>

$$\sigma(\gamma H^{\perp}) = \gamma|_H \qquad (\gamma \in \hat{G})$$

<u>defines a homeomorphism and an isomorphism σ of $\hat{G}/H^{\perp}$ onto $\hat{H}$.</u>

7.17. COROLLARY. Let H be a closed subgroup of G. Then every continuous character of H can be extended to a continuous character of G.

The following is an example of the various theorems that connect properties of G with properties of Γ. (See also Exercises 7.C and 8.D.)

7.18. THEOREM. Let G be compact (and Γ discrete). G is connected if and only if Γ is torsion-free. (A group H is called torsion-free if there do not exist $n \in \mathbb{N}$ and $x \in H$, $x \neq 1$ with $x^n=1$.)

Proof. If G is not connected, it has a non-trivial closed open subset A. It follows from Lemma 3.1 (e) that $H := \{x : xA \subset A \text{ and } x^{-1}A \subset A\}$ is an open neighborhood of 1. But H is a group and $H \neq G$. Thus H is a proper open subgroup of G. Then G/H is finite, as G is compact. It follows that $(G/H)^\wedge$ is finite. Then Γ contains a finite subgroup, viz. $H^\perp$, and cannot be torsion-free.

Conversely, if Γ is not torsion-free, there exist $n \in \mathbb{N}$ and $\gamma \in \Gamma$, $\gamma \neq 1$ with $\gamma^n=1$. Then $\{1\} \neq \gamma(G) \subset \{z \in \mathbb{T} : z^n=1\}$, so $\gamma(G)$ is not connected. As γ is continuous, G itself is not connected.

7.A. EXERCISE. Let p be a prime number. For $n \in \{0,1,\ldots\}$ let $G_n := \{t \in \mathbb{T} : t^{p^n}=1\}$. Then $G_0 \subset G_1 \subset G_2 \ \ldots$. Set $G := \bigcup G_n$. This G is a group; each G_n is a finite subgroup of G. We view G under the discrete topology. In $\hat{G}$ we have $\hat{G}=G_0^\perp \supset G_1^\perp \supset \ldots$, $\bigcap G_n^\perp=\{1\}$. Further, $\hat{G}$ is a compact group.

(a) Every compact set $K \subset G$ is contained in some G_n; then $N(K:\rho) \supset N(G_n:\rho)=G_n^\perp$ for $0<\rho<1$. A set $W \subset \hat{G}$ is a neighborhood of 1 if and only if W contains a $G_n^\perp$.

(b) For $\gamma \in \hat{G}$ define

$$D(\gamma) := \inf \{p^{-n} : n \in \{0,1,2,\ldots\};\ \gamma \in G_n^\perp\}.$$

Then $D(\gamma^{-1})=D(\gamma)$, $D(\gamma_1\gamma_2) \leq \max(D(\gamma_1),D(\gamma_2)) \leq D(\gamma_1)+D(\gamma_2)$ and $G_n^\perp=\{\gamma : D(\gamma) \leq p^{-n}\}$. The formula

$$d(\gamma_1,\gamma_2) := D(\gamma_1\gamma_2^{-1})$$

defines a metric d on $\hat{G}$ which induces the topology of $\hat{G}$.

(c) Let $\beta \in \hat{G}$ be the identity map $G \longrightarrow \mathbb{T}$. Then $n \longmapsto \beta^n$ is a one-to-one homomorphism of $\mathbb{Z}$ onto a dense subgroup of $\hat{G}$.

(d) For $k,l \in \mathbb{Z}$ set

$$\bar{d}(k,l) := \inf\{p^{-n} : n \in \{0,1,2,\dots\};\ k-l \text{ divisible by } p^n\}.$$

Then $\bar{d}$ is a metric on $\mathbb{Z}$. By $\mathbb{Z}_p$ we denote the completion of $\mathbb{Z}$ relative to this metric. By Exercise 3.B, $\mathbb{Z}_p$ is a topological group, and, in fact, $\mathbb{Z}_p$ is compact. $\hat{\mathbb{Z}}_p$ is isomorphic to the group $\{t \in \mathbb{T} : t^{p^n}=1 \text{ for some } n \in \mathbb{N}\}$.

The elements of $\mathbb{Z}_p$ are the p-adic integers.

7.B. EXERCISE. For $k,l \in \mathbb{Z}$ set

$$d(k,l) := \inf\{\tfrac{1}{n!} : n \in \{0,1,2,\dots\};\ k-l \text{ divisible by } n!\}.$$

Then d is a metric on $\mathbb{Z}$. Let P denote the completion of $\mathbb{Z}$ relative to this metric. P is a compact group. Its dual group is isomorphic to $\mathbb{Q}/\mathbb{Z}$ (or to $\{t \in \mathbb{T} : t^m=1 \text{ for some } m \in \mathbb{N}\}$).

The elements of P are the Prüfer numbers.

(Hint. Compare with Exercise 7.A.)

7.C. EXERCISE. Γ is metrizable if and only if G is σ-compact. (See Exercise 3.L.)

7.D. EXERCISE. Let Γ_d denote the group Γ under the discrete topology, $\bar{G}$ the dual group of Γ_d. $\bar{G}$ is a compact group. It is called the Bohr compactification of G. Every $x \in G$ determines a $b(x) \in \bar{G}$ by

$$b(x)(\gamma) := \gamma(x) \qquad (\gamma \in \Gamma).$$

(a) b is an injective continuous homomorphism of G onto a dense subgroup of $\bar{G}$. If G is compact, b is a homeomorphism $G \longrightarrow \bar{G}$ (in fact, $b=\alpha$).

(b) Every continuous homomorphism τ of G into a compact group H induces a continuous homomorphism $\bar{\tau} : \bar{G} \longrightarrow H$ such that $\tau = \bar{\tau} \circ b$:

7.E. EXERCISE. If $f \in L^1(G)$ and $\hat{f} \in L^2(\Gamma)$, then $f \in L^2(G)$.

Hint. Let $\mathcal{F}$ denote the Fourier-Plancherel Transformation $L^2(G) \longrightarrow L^2(\Gamma)$. Use Lemma 6.3 and the Plancherel Theorem to prove that $\mathcal{F}^{-1}h = \check{h}$ for all $h \in L^1(\Gamma) \cap L^2(\Gamma)$, so that $\{\check{h} : h \in L^1(\Gamma) \cap L^2(\Gamma)\}$ is a dense set in $L^2(G)$. Use Lemma 6.3 again to show that $|\int f\check{h}| \leq \|\hat{f}\|_2 \|h\|_2$ for $h \in L^1(\Gamma) \cap L^2(\Gamma)$.

7.F. EXERCISE. The Fourier Transformation $L^1(\mathbf{R}) \longrightarrow C_\infty(\mathbf{R})$ is not surjective. (Compare Exercise 9.B.) In fact, define

$$h(x) := \int_{|x|}^{\infty} \frac{\sin t}{t}\, dt \qquad (x \in \mathbf{R}),$$

where in the right hand member we have an improper Riemann integral. Then $h \in C_\infty(\mathbf{R})$ but h is not a Fourier transform.

Hint. Let $g(x) := x^{-1}$ for $|x| \geq 1$, $g(x) := 0$ for $|x| < 1$. Then $g \in L^2(\mathbf{R})$, $g \notin L^1(\mathbf{R})$ and up to a constant h is the Fourier-Plancherel transform of g. Apply the preceding exercise.

7.G. EXERCISE. In Theorem 7.12 if we choose the Haar measure on $\hat{\hat{G}}$ that belongs to the Haar measure of $\hat{G}$ as indicated in the Inversion Theorem 6.11, then α is measure-preserving.