II. $L^1(G)$ AND $M(G)$.

3. LOCALLY COMPACT GROUPS.

A <u>topological group</u> is a group G provided with a topology such that the maps $(x,y) \mapsto xy$ and $x \mapsto x^{-1}$ are continuous: $G \times G \to G$ and $G \to G$. This is the case if and only if $(x,y) \mapsto xy^{-1}$ is continuous. Then for every $a \in G$ the maps $x \mapsto xa$ and $x \mapsto ax$ are homeomorphisms of G, and so is the map $x \mapsto x^{-1}$.

IN THIS SECTION G IS A TOPOLOGICAL GROUP WITH UNIT ELEMENT 1. For $X, Y \in G$ set $XY := \{xy : x \in X, y \in Y\}$ and $X^{-1} := \{x^{-1} : x \in X\}$. A set $X \subset G$ is <u>symmetric</u> if $X^{-1} = X$. Note that every neighborhood X of 1 contains a symmetric neighborhood of 1 (e.g. $X \cap X^{-1}$). If $U \subset G$ is open and $a \in U$, there exists a neighborhood V of 1 such that $aV \subset U$ and $Va \subset U$.

3.1. LEMMA.

(a) <u>If $X, Y \subset G$ are compact, then XY and X^{-1} are compact.</u>

(b) <u>If $X, Y \subset G$ and if X is open, then XY and YX are open.</u>

(c) <u>Every open subgroup of G is closed. Every subgroup of G that is a neighborhood of 1 is open.</u>

(d) <u>For every neighborhood U of 1 there exists a neighborhood V of 1 such that $VV \subset U$.</u>

(e) <u>If $U \subset G$ is open and $K \subset G$ is compact, then the sets $\{x \in G : xK \subset U\}$ and $\{x \in G : Kx \subset U\}$ are open.</u>

Proof. (a) XY is the image of the compact subset $X \times Y$ of $G \times G$ under the (continuous) multiplication $G \times G \to G$. The compactness of X^{-1} follows from the continuity of $x \mapsto x^{-1}$.

(b) $XY = \bigcup\{Xy : y \in Y\}$ and $YX = \bigcup\{yX : y \in Y\}$.

(c) If H is an open subgroup of G, then $G \setminus H = \bigcup\{xH : x \in G \setminus H\}$, so $G \setminus H$ is open and H is closed. If H is a subgroup of G that contains a non-empty open set U, then $H = HU$ is open.

(d) As multiplication is a continuous map $G \times G \to G$ and $1 \cdot 1 = 1$, there exist neighborhoods V_1, V_2 of 1 such that $V_1 V_2 \subset U$. Take $V := V_1 \cap V_2$.

(e) Apply Lemma 3.2 below.

3.2. LEMMA. *If* X, Y *are topological spaces and if* $U \subset X \times Y$ *is open, then for every compact* $K \subset X$ *the set* $\{y \in Y : K \times \{y\} \subset U\}$ *is open*.

Proof. Let $K \times \{b\} \subset U$. By compactness there exist open $V_1, \ldots, V_m \subset X$ and $W_1, \ldots, W_m \subset Y$ with $K \times \{b\} \subset \bigcup_j V_j \times W_j$ and $b \in \bigcap W_j$. Then $\bigcap W_j \subset \{y : K \times \{y\} \subset U\}$.

A few examples. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$ and $\mathbb{Z}$ are abelian topological groups under addition. $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ is an abelian topological group under multiplication. Every group is a topological group if we provide it with the discrete topology. Every normed vector space is an additive topological group.

The invertible complex $n \times n$-matrices form a group $GL(n, \mathbb{C})$ which can be topologized as a subset of $\mathbb{C}^{n^2}$ (the *general linear group*). The matrices with determinant 1 form a closed subgroup $SL(n, \mathbb{C})$ (the *special linear group*). Other closed subgroups are the *unitary group* $U(n)$ of all unitary matrices and the *special unitary group* $SU(n)$ which is $U(n) \cap SL(n, \mathbb{C})$. These groups are non-commutative (if $n \neq 1$) and locally compact. $U(n)$ and $SU(n)$ are compact. $GL(n, \mathbb{R})$ and $SL(n, \mathbb{R})$ are defined analogously to $GL(n, \mathbb{C})$ and $SL(n, \mathbb{C})$, respectively. The *orthogonal group* $O(n)$ of all orthogonal matrices is $U(n) \cap GL(n, \mathbb{R})$; the *special orthogonal group* $SO(n)$ is $O(n) \cap SL(n, \mathbb{R})$.

A function $f : G \to \mathbb{C}$ is *left uniformly continuous* if for every $\varepsilon > 0$ there exists a neighborhood V of 1 such that

$$\sup_{x \in G} |f(x) - f(ax)| \leq \varepsilon \quad \text{all } a \in V.$$

f is *right uniformly continuous* if for every $\varepsilon > 0$ there is a neighborhood V of 1 for which

$$\sup_{x \in G} |f(x) - f(xa)| \leq \varepsilon \quad \text{all } a \in V.$$

We call f *uniformly continuous* if it is both left and right uniformly continuous. Obviously, on an abelian group left and right uniform continuity are the same.

3.A. EXERCISE. Let G be the closed subgroup of $GL(2, \mathbb{C})$ that consists of all matrices $\begin{pmatrix}\alpha & \beta\\ 0 & 1\end{pmatrix}$ for which $|\alpha|=1$. The function

$$\begin{pmatrix}\alpha & \beta\\ 0 & 1\end{pmatrix} \longmapsto \beta$$

is right uniformly continuous but not left uniformly continuous.

3.3. THEOREM. Let G be locally compact. Every element f of $C_\infty(G)$ is uniformly continuous. (Hence if G is compact, every continuous function is uniformly continuous.)

Proof. Take $\varepsilon>0$. There exists a compact $K\subset G$ such that $|f| < \frac{\varepsilon}{2}$ off K. For every $a\in K$ the set $\{x\in G : |f(x)-f(a)|<\frac{\varepsilon}{2}\}$ is a neighborhood of a, so there exists an open set U_a containing 1 for which $U_a a \subset \{x : |f(x)-f(a)| < \frac{\varepsilon}{2}\}$. Take a symmetric neighborhood V_a of 1 such that $V_aV_a \subset U_a$. The sets $V_a a$ cover K, so $K\subset V_{a_1}a_1 \cup \dots \cup V_{a_m}a_m$ for certain $a_1,\dots,a_m\in K$. Let $V := V_{a_1}\cap\dots\cap V_{a_m}$. Then V is a symmetric neighborhood of 1. We prove $|f(x)-f(ax)|\le\varepsilon$ for all $a\in V$ and all $x\in G$.

If neither x nor ax lie in K, then $|f(x)| < \frac{\varepsilon}{2}$ and $|f(ax)| < \frac{\varepsilon}{2}$, so $|f(x)-f(ax)| < \varepsilon$. If $x\in K$, then $x\in V_{a_i}a_i$ for some i; then both x and ax are elements of $U_{a_i}a_i$, and $|f(x)-f(ax)| \le |f(x)-f(a_i)|+|f(ax)-f(a_i)| \le \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$. If $ax\in K$ then $ax\in V_{a_i}a_i$ for some i; as V is symmetric, we again have $x\in U_{a_i}a_i$, $ax\in U_{a_i}a_i$, and $|f(x)-f(ax)|\le\varepsilon$.

For a function $f : G \to \mathbb{C}$ and $a\in G$ define $f_a : G \to \mathbb{C}$ by

$$f_a(x) := f(a^{-1}x).$$

We have

$$(f_a)_b = f_{ba} \qquad (a,\ b\in G).$$

(The reader is advised to check the truth of this assertion!)

An $f\in C(G)$ is left uniformly continuous if and only if

$a \longmapsto f_a$ is continuous $G \longrightarrow C(G)$. If $f \in C_\infty(G)$, then $f_a \in C_\infty(G)$, and $\|f_a\| = \|f\|$. Theorem 3.3 implies

3.4. COROLLARY. <u>Let</u> G <u>be</u> <u>locally</u> <u>compact</u>. <u>For</u> <u>every</u> $f \in C_\infty(G)$, $a \longmapsto f_a$ <u>is</u> <u>a</u> <u>continuous</u> <u>map</u> $G \longrightarrow C_\infty(G)$.

Let H be a closed subgroup of G, let $\pi: G \longrightarrow G/H$ be the natural surjection. In G/H we consider the <u>quotient topology</u>: a subset W of G/H is open if and only if $\pi^{-1}(W)$ is open in G. If H is open, G/H is discrete.

3.5. LEMMA. (a) π <u>is</u> <u>an</u> <u>open</u> <u>map</u>, <u>i.e.</u> <u>if</u> $U \subset G$ <u>is</u> <u>open</u>, <u>then</u> $\pi(U)$ <u>is</u> <u>open</u> <u>in</u> G/H.

(b) G/H <u>is</u> <u>a</u> <u>Hausdorff</u> <u>space</u>. (<u>In</u> <u>particular</u>, <u>if</u> $\{1\} \in G$ <u>is</u> <u>closed</u>, <u>then</u> G <u>is</u> <u>Hausdorff</u>.)

(c) <u>The</u> <u>map</u> $(x, yH) \longmapsto xyH$ <u>is</u> <u>continuous</u> $G \times G/H \to G/H$.

(d) <u>If</u> H <u>is</u> <u>a</u> <u>normal</u> <u>subgroup</u>, <u>then</u> G/H <u>is</u> <u>a</u> <u>topological</u> <u>group</u>.

(e) <u>If</u> G <u>is</u> <u>locally</u> <u>compact</u>, <u>so</u> <u>is</u> G/H.

Proof. (a) $\pi^{-1}(\pi U) = UH = \bigcup\{Ux : x \in H\}$.

(b) Let $a, b \in G$, $\pi(a) \neq \pi(b)$. Then $b^{-1}a$ lies in the open set $G \setminus H$, so there exist open sets $V \ni a$, $W \ni b$ such that $W^{-1}V \subset G \setminus H$. Then $\pi(V)$ and $\pi(W)$ are open (see (a)!) neighborhoods of $\pi(a)$ and $\pi(b)$, respectively, and $\pi(V) \cap \pi(W) = \emptyset$.

(c) Define $\sigma: G \times G/H \longrightarrow G/H$ by $\sigma(x, yH) := xyH$ $(x,y \in G)$, i.e. $\sigma(x,\pi(y)) = \pi(xy)$ $(x,y \in G)$. If $W \subset G/H$ is open and $(a,\pi(b)) \in \sigma^{-1}(W)$ then $\pi(ab) \in W$, so $ab \in \pi^{-1}(W)$. There exist open sets $U \ni a$, $V \ni b$ with $UV \subset \pi^{-1}(W)$. Then $U \times \pi(V) \subset \sigma^{-1}(W)$.

(d) Let $W \subset G/H$ be open, $\pi(a)\pi(b)^{-1} \in W$. As $ab^{-1} \in \pi^{-1}(W)$ there exist open sets $U \ni a$, $V \ni b$ with $UV^{-1} \subset \pi^{-1}(W)$. Now $\pi(U)$ and $\pi(V^{-1})$ are open subsets of G/H, containing $\pi(a)$ and $\pi(b)^{-1}$, respectively, and $\pi(U)\pi(V)^{-1} \subset W$: we see that the map $(\pi(x), \pi(y)) \longmapsto \pi(x)\pi(y)^{-1}$ is continuous, so G/H is a topological group.

(e) There exist open $U \subset G$ and compact $K \subset G$ with $1 \in U \subset K$. For every $a \in G$, $\pi(a) \in \pi(aU) \subset \pi(aK)$, while $\pi(aU) \subset G/H$ is open and $\pi(aK) \subset G/H$ is compact.

3.6. ISOMORPHISM THEOREM. *Let G_1, G_2 be topological groups, and let $\varphi: G_1 \longrightarrow G_2$ be an open surjective group homomorphism. Then $(x \bmod \operatorname{Ker}\varphi) \longmapsto \varphi(x)$ is not only a group isomorphism of $G_1/\operatorname{Ker}\varphi$ onto G_2, but also a homeomorphism.*

Proof. Let π be the natural homomorphism $G_1 \longrightarrow G_1/\operatorname{Ker}\varphi$. We know that $\varrho: \pi(x) \longmapsto \varphi(x)$ is well-defined and is a surjective isomorphism $G_1/\operatorname{Ker}\varphi \to G_2$. If $W \subset G_2$ is open, then $\varphi^{-1}(W) \subset G_1$ is open, so $\pi(\varphi^{-1}(W))$ is open in $G_1/\operatorname{Ker}\varphi$. But $\pi(\varphi^{-1}(W)) = \varrho^{-1}(W)$. Thus, ϱ is continuous. By a similar argument, ϱ^{-1} is continuous.

3.7. LEMMA. *Let H be a closed subgroup of a locally compact group G; let π be the natural map $G \longrightarrow G/H$. Then every compact subset of G/H can be written as $\pi(X)$ for some compact $X \subset G$.*

Proof. Let K be a compact neighborhood of $1 \in G$. For every $a \in G$, $\pi(aK)$ is a compact neighborhood of $\pi(a)$ (see Lemma 3.5 (a)). If $Y \subset G/H$ is compact, there exist $a_1, \ldots, a_n \in G$ such that $Y \subset \pi(a_1K) \cup \ldots \cup \pi(a_nK)$. Set $X := \pi^{-1}(Y) \cap \bigcup_j \pi(a_jK)$.

3.8. LEMMA. *Let I be any index set. For each $i \in I$ let G_i be a topological group. With coordinatewise multiplication, $\prod_i G_i$ is a group. Under the product topology, it is a topological group.*

Proof. It is easy to check that $G := \prod G_i$ is a group. Let $(x_\lambda)_{\lambda\in\Lambda}$ and $(y_\lambda)_{\lambda\in\Lambda}$ be nets in G, $x = \lim_\lambda x_\lambda$, $y = \lim_\lambda y_\lambda$. We are done if we can prove that $xy^{-1} = \lim x_\lambda(y_\lambda)^{-1}$. But this is easy: convergence in $\prod G_i$ is coordinatewise convergence and each G_i is a topological group.

We shall use two special cases:

3.9. LEMMA. (a) <u>The product of any family of compact groups is a compact group.</u>

(b) <u>A product of finitely many locally compact groups is a locally compact group.</u>

3.B. EXERCISE. Let G be a group. A metric d on G is called <u>left invariant</u> if

$$d(xa, xb) = d(a, b) \qquad (a, b, x \in G)$$

and <u>right invariant</u> if

$$d(ax, bx) = d(a, b) \qquad (a, b, x \in G).$$

It is <u>(two-sided) invariant</u> if it is both left and right invariant.

(a) Let D be a function $G \longrightarrow [0,\infty)$ with the properties

$D(x) = 0$ if and only if $x = 1$,

$D(x) = D(x^{-1})$ for every $x \in G$,

$D(xy) \leq D(x) + D(y)$ for all $x, y \in G$.

Then the formula

$$d(x,y) := D(y^{-1}x)$$

yields a left invariant metric d on G. Every left invariant metric can be obtained in this way.

(b) Let d be a left invariant metric on G. Under the topology induced by d, G is a topological group. If the topology induced by d is locally compact, then G is complete relative to d.

(c) Let G be abelian, with a (left) invariant metric d. Then the completion $\overline{G}$ of G can in a unique way be made into an (abelian) topological group, such that G is a subgroup of $\overline{G}$. (For non-abelian groups the situation is less simple.)

Let G be a locally compact group. A positive linear function $m : C_{00}(G) \to \mathbb{C}$ is <u>left invariant</u> if

$$m(f_a) = m(f) \qquad (f \in C_{00}(G);\ a \in G).$$

By the Riesz Representation Theorem such an m determines a regular measure m on the σ-algebra $\mathcal{B}$ of all Borel subsets of G that is <u>left invariant</u>, i.e. for which

$$m(aX) = m(X) \qquad (X \in \mathcal{B};\ a \in G).$$

A non-zero left invariant regular measure on $\mathcal{B}$ is a (<u>left</u>)

Haar measure on G. The induced integral is a (left) Haar integral.

3.10. THEOREM. Let G be a locally compact group.
(a) There exists a Haar measure on G.
(b) If m_1, m_2 are Haar measures on G, then $m_1 = cm_2$ for some $c \in (0,\infty)$.

Although this theorem is basic to harmonic analysis, we are not going to prove it in this course. We refer the reader to Hewitt-Ross, "Abstract Harmonic Analysis I", 15.5.

The basic example is, of course, Lebesgue measure on $\mathbb{R}$. The following observation is almost trivial.

3.11. THEOREM. On a discrete group the counting measure is a Haar measure. On a non-discrete group every one-point set is negligible relative to every Haar measure.

On a compact group G there exists a unique Haar measure m for which $m(G) = 1$: we call this m the normalized Haar measure of G.

IN THE REST OF THIS SECTION G IS A LOCALLY COMPACT GROUP WITH A LEFT HAAR MEASURE m (OR m_G). Instead of $\int f(x)dm(x)$ we usually write $\int f(x)dx$, or $\int f$.

3.12. LEMMA. $m(U) > 0$ for every non-empty open $U \subset G$.

Proof. Suppose U is open, non-empty and $m(U)=0$. Every compact $K \subset G$ can be covered by finitely many left translates $x_1U,\dots,x_mU$ of U. Then $m(K) \leq \sum m(x_iU) = 0$, so $m(K)=0$ for all compact K. By the regularity of m, $m(G)= \sup\{m(K) : K \subset G \text{ compact}\} = 0$.

Instead of $L^p(m)$ $(p=1,2)$ we write $L^p(G)$. (This notation is not quite honest, as Haar measure on G is not unique. However, if m_1 and m_2 are Haar measures, then $m_1 = cm_2$ for some $c>0$. Then $L^p(m_1)$ and $L^p(m_2)$ are the same sets, and $\| \ \|_{L^p(m_1)} = \sqrt[p]{c}\, \| \ \|_{L^p(m_2)}$.)

3.13. THEOREM. <u>Let</u> $p = 1, 2$, $f \in L^p(G)$. <u>Then</u> $a \longmapsto f_a$ <u>is a continuous map</u> $G \longrightarrow L^p(G)$.

Proof. Clearly if $f \in L^p(G)$, then $f_a \in L^p(G)$ for every a G and $\|f_a\|_p = \|f\|_p$.

Take $\varepsilon>0$. By Lemma 1.8 there exists a $k \in C_{oo}(G)$ for which $\|f-k\|_p < \frac{\varepsilon}{3}$. Then $\|f_x-k_x\|_p < \frac{\varepsilon}{3}$ for every $x \in G$. Let $K :=$ supp k. By the regularity of the Haar measure m, there exists an open $U \supset K$ with $m(U) < \infty$. By Lemma 3.1 (e) and Corollary 3.4 there exists a neighborhood V of 1 such that $VK \subset U$ and such that $\|k-k_x\|_p < \frac{\frac{\varepsilon}{3}}{\sqrt[p]{m(U)}}$. For all $x \in V$ we now have $k=0$ and $k_x=0$ outside U, so $\|k-k_x\|_p \leq \sqrt[p]{(\|k-k_x\|_\infty)^p\, m(U)} < \frac{\varepsilon}{3}$. Then for such x, $\|f-f_x\|_p < \|f-k\|_p + \|k-k_x\|_p + \|k_x-f_x\|_p < \varepsilon$.

If $a \in G$ is arbitrary, then $\|f_a-f_{xa}\|_p = \|f-f_x\|_p < \varepsilon$ for all $x \in V$. Thus, $a \longmapsto f_a$ is continuous.

For every $a \in G$, the measure $X \longmapsto m(Xa)$ is a left Haar measure, so there exists a number $\Delta(a)>0$ such that

$$m(Xa) = \Delta(a)m(X) \qquad (X \subset G \text{ Borel}).$$

Obviously, Δ is a homomorphism of G into the multiplicative group $(0,\infty)$, and it doesnot depend on the choice of m. Δ is called the <u>modular function</u> of G.

The following is obvious.

3.14. THEOREM. $\int f(xa)dx = \Delta(a)^{-1} \int f(x)dx \qquad (f \in L^1(G);\ a \in G)$.

3.C. EXERCISE. Δ <u>is continuous</u>. (Let $X \subset G$ be a compact neighborhood of 1; then $0<m(X)<\infty$. Use the regularity of m and Lemma 3.1 (e) to prove that for every $\varepsilon>0$ there exists a neighborhood of 1 on which $\Delta < 1+\varepsilon$.)

G is called <u>unimodular</u> if $\Delta = 1$.

3.15. THEOREM. <u>All abelian groups, all discrete groups and all compact groups are unimodular. For any G, the modular function is identically 1 on the center of G (i.e. $\{a \in G : xa = ax$ for all $x \in G\}$) and on all compact subgroups of G.</u>

Proof. The abelian case is trivial, the discrete case follows from Theorem 3.11. Further, if H is a compact subgroup of G and if Δ is the modular function of G, then (use Exercise 3.C) $\Delta(H)$ is a compact subgroup of $(0,\infty)$, so $\Delta(H) = \{1\}$ and $\Delta = 1$ on H.

(Another proof for compact G: For all $a \in G$, $m(G) = m(Ga) = \Delta(a)m(G)$, so $\Delta(a) = 1$.)

<u>Example. The closed subgroup</u>

$$H := \{ \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} : x > 0;\ y \in \mathbb{R} \}$$

<u>of</u> $SL(2,\mathbb{R})$ <u>is not unimodular. In fact, for all</u> x,y,

$$\Delta(\begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix}) = x^{-2}.$$

Proof. The map $(x,y) \longmapsto \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix}$ is a homeomorphism of $(0,\infty)\times\mathbb{R}$ onto H. For simplicity we write (x,y) instead of $\begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix}$.

For $f \in C_{00}(H)$ set

$$I(f) = \int_{-\infty}^{\infty}\int_{0}^{\infty} x^{-2} f(x,y)\,dx\,dy .$$

If $(a,b) \in H$ and if we put

$$g(x,y) = f((a,b)(x,y)) = f(ax, ay+bx^{-1}),$$

$$h(x,y) = f((x,y)(a,b)) = f(ax, bx+a^{-1}y),$$

then, by direct computation, $I(g) = I(f)$, $I(h) = a^2 I(f)$. Thus, I is a left Haar integral, and $\Delta(a,b) = a^{-2}$.

In §11 we shall see that $SL(2,\mathbb{R})$ is unimodular. Thus, closed subgroups of unimodular groups may fail to be unimodular. (See also Exercise 3.I and Corollary 13.6.)

3.D. EXERCISE. On the subgroup $G = \{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : x>0,\ y \in \mathbf{R} \}$ of $GL(2, \mathbf{R})$ the modular function is $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \mapsto x$.

3.E. EXERCISE. The group mentioned in Exercise 3.A is not abelian, discrete or compact, but it is unimodular. (For every $\beta \in \mathbf{C}$, $\{ \begin{pmatrix} \alpha & \beta(1-\alpha) \\ 0 & 1 \end{pmatrix} : \alpha \in \mathbf{T} \}$ is a compact subgroup, and on compact subgroups Δ must be 1.)

A trivial remark: If G, H are locally compact groups with modular functions Δ_G, Δ_H, and if τ is an isomorphism of G onto H that is also a homeomorphism, then $\Delta_G = \Delta_H \circ \tau$. Thus:

3.16. THEOREM. <u>If τ is an automorphism of G that is also a homeomorphism, then</u> $\Delta(\tau(x)) = \Delta(x) \quad (x \in G)$.

For application of the Fubini Theorem we need σ-finite measures, but the Haar measure is in general not σ-finite. The following lemma 3.17 leads to Exercise 3.F which makes the Fubini Theorem easier to handle.

3.17. LEMMA. G <u>contains a (closed) open</u> σ-<u>compact subgroup</u> G_o.

Proof. $1 \in G$ has a symmetric compact neighborhood V. Let $G_o := V \cup VV \cup VVV \cup \dots$. Then G_o is a subgroup of G. As V, VV, VVV,... are compact, G_o is σ-compact. By Lemma 3.1 (c) G_o is open and closed.

3.F. EXERCISE. <u>Let</u> G <u>be a locally compact group with left Haar measure</u> m.

(a) <u>If</u> $X_1, X_2, \dots$ <u>are</u> σ-<u>compact subsets of</u> G, <u>there exists an open</u> σ-<u>compact subgroup</u> G_o <u>of</u> G <u>that contains every</u> X_n.

(b) For every $\mu \in M(G)$ there is a σ-compact $X \subset G$ such that $\mu(Y) = 0$ for every Borel set $Y \subset G \setminus X$.

(c) If $f \in C_{\infty}(G)$, there is a σ-compact $X \subset G$ outside which $f = 0$.

(d) If X is a Borel set of finite Haar measure, then X is contained in a σ-compact set. (Hint. We may assume X to be open. Let G_o be an open σ-compact subgroup of G. If S is a coset of G_o, then either $X \cap S = \emptyset$ or $m(X \cap S) > 0$.)

(e) If $f : G \to \mathbb{C}$ is m-integrable, there exists a σ-compact subset of G outside which $f = 0$.

Let $f, g \in L^1(G)^*$, $f, g \geq 0$. Choose a σ-compact open subgroup G_o of G such that $f = g = 0$ outside G_o. The functions

$$(x,y) \longmapsto f(x)g(x^{-1}y)$$

and

$$(z,y) \longmapsto f(yz)g(z^{-1})$$

are Borel measurable on $G \times G$ and vanish outside $G_o \times G_o$. By our Fubini Theorem 1.17 1.14,

$$\begin{aligned}\int f(x)dx \int g(y)dy &= \int f(x)\left[\int g(y)dy\right]dx = \int f(x)\left[\int g(x^{-1}y)dy\right]dx = \\ &= \iint f(x)g(x^{-1}y)dxdy = \iint f(yz)g(z^{-1})dzdy = \\ &= \int g(z^{-1}) \int f(yz)dydz = \int g(z^{-1})\,\Delta(z^{-1}) \int f(y)dydz = \\ &= \int f(y)dy \cdot \int g(z^{-1})\,\Delta(z^{-1})dz.\end{aligned}$$

Therefore,

3.18. THEOREM. If $g \in L^1(G)$ then $x \longmapsto \Delta(x^{-1})g(x^{-1})$ is also integrable and $\int \Delta(x^{-1})g(x^{-1})dx = \int g(x)dx$. (Hence, if $X \subset G$ is negligible then so is X^{-1}.)

In particular, if $f \in C_{oo}(G)$, then $\int f(x^{-1})dx = \int \Delta(x^{-1})f(x)dx = \int \Delta(x)^{-1}f(x)dx$. As $f \longmapsto \int f(x^{-1})dx$ is right translation invariant, we obtain

3.19. THEOREM. $\Delta^{-1}m$ is a right Haar measure.

For $\mu \in M(G)$ define $\mu' \in M(G)$ by

$$\mu'(X) := \mu(X^{-1}) \qquad (X \subset G \text{ Borel}).$$

We also introduce $\tilde{\mu} := \overline{\mu'} = \bar{\mu}'$, $\tilde{\mu}(X) = \overline{\mu(X^{-1})}$ $\qquad (X \subset G$ Borel).

For $f \in L^1(G)$ define $f' : G \longrightarrow \mathbb{C}$ by

$$f'(x) := \Delta(x^{-1}) f(x^{-1}) \qquad (x \in G).$$

Instead of $\overline{f'}$ (which is the same as $\bar{f}'$) we also write $\tilde{f}$.

3.G. EXERCISE. $\underline{\mu \longmapsto \mu'}$ $\underline{\text{and}}$ $f \longmapsto f'$ $\underline{\text{are}}$ $\underline{\text{linear}}$ $\underline{\text{isometries}}$ $M(G) \longrightarrow M(G)$ $\underline{\text{and}}$ $L^1(G) \longrightarrow L^1(G)$. $\underline{\text{If}}$ $f \in L^1(G)$ $\underline{\text{then}}$ $(fm)' = f'm$ $\underline{\text{and}}$ $(fm)^{\sim} = \tilde{f}m$. (Observe that $\mu \longmapsto \tilde{\mu}$ and $f \longmapsto \tilde{f}$ are not linear.)

3.H. EXERCISE. <u>If</u> n <u>is a right Haar measure on</u> G, <u>then</u>

$$\int g(ax)\,dn(x) = \Delta(a) \int g(x)\,dn(x) \qquad (a \in G;\ g \in C_{00}(G))$$

<u>where</u> Δ <u>is the modular function of</u> G.

3.I. EXERCISE. Let H be an open subgroup of G. Then $\Delta|_H$ is the modular function of H.

3.J. EXERCISE. <u>A countable locally compact group is discrete</u>.

3.K. EXERCISE. If G is connected, it is σ-compact.

For arbitrary locally compact group G let G_c be the union of all connected subsets of G that contain 1. Then G_c is a closed connected subgroup of G.

3.L. EXERCISE. Obviously, if G is metrizable, then $\{1\}$ is an intersection of countably many open sets.

Conversely, <u>if $\{1\}$ is an intersection of countably many open sets, then the topology of G is given by a left invariant metric</u>. (See Exercise 3.B.)

(Hint. There exist open sets $U_1, U_2, \ldots$ so that $\overline{U}_1$ is compact, $\overline{U}_{n+1} \subset U_n$ for every n, and $\bigcap U_n = \{1\}$. Then every neighborhood of 1 contains a U_n. There exists an $f \in C_{00}^{+}(G)$ such that

$$\begin{cases} \text{if } x \in G,\ x \notin U_n \text{ then } f(x) \le 1-2^{-n}, \\ f(1) = 1. \end{cases}$$

Define

$$D(x) := \|f_x - f\|_\infty .$$

D is continuous and has the three properties mentioned in Exercise 3.F (a), hence determines a left invariant metric d on G. For every n, $\{x : D(x) < 2^{-n}\} \subset U_n$. The topology induced by d is just the given topology of G.)

3.M. EXERCISE. By the preceding exercise, the topology of a metrizable locally compact group G is given by a left invariant metric d. Then, of course, G also carries a right invariant metric d' that yields the same topology (e.g. $d'(x,y) := d(x^{-1},y^{-1})$.) However, a locally compact group G whose topology is given by a two-sided invariant metric is unimodular. (Hint. If $\varepsilon > 0$ and $A := \{x : d(x,1) \le \varepsilon\}$, then $aAa^{-1} = A$ for all $a \in G$.)

It follows that the group mentioned in the example on page 3.9 does not carry a two-sided invariant metric. A fortiori this is true for the groups $SL(2,\mathbb{R})$, $GL(2,\mathbb{C})$, etc., which are metrizable and unimodular. (See Exercise 11.C.)

The topology of a compact metrizable group is given by a two-sided invariant metric.