8. STRUCTURE THEOREMS.

A topological group G_1 is said to be <u>locally isomorphic</u> to a topological group G_2 if there exists a homeomorphism φ of a neighborhood U of $1 \in G_1$ onto a neighborhood of $1 \in G_2$ such that $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a,b \in U$ for which $ab \in U$.

Then G_2 is locally isomorphic to G_1. This (not quite trivial) fact we prove as follows. Let U,φ be as above. Let W_1 be a neighborhood of $1 \in G_1$ such that $W_1W_1 \subset U$; set $W_2 := \varphi(W_1)$ and let ψ be the restriction of φ^{-1} to W_2. Now W_2 is a neighborhood of $1 \in G_2$ and ψ is a homeomorphism of W_2 onto W_1. If $x,y \in W_2$ and $xy \in W_2$ then $\psi(x),\psi(y) \in W_1 \subset U$ and $\psi(x)\psi(y) \in W_1W_1 \subset U$; then $xy=\varphi(\psi(x))\varphi(\psi(y))=\varphi(\psi(x)\psi(y))$ so that $\psi(xy)=\psi(x)\psi(y)$. It follows that G_2 is locally isomorphic to G_1.

Apparently, local isomorphism is an equivalence relation among topological groups.

<u>Examples</u>. $\mathbb{R}$ and $\mathbb{T}$ are locally isomorphic. All discrete groups are locally isomorphic.

8.A. EXERCISE. <u>Let</u> G <u>be a topological group</u>, H <u>a subgroup of</u> G. <u>Suppose that under the relative topology</u> H <u>is discrete</u>. <u>Then</u> H <u>is closed</u>. <u>If</u> H <u>is a normal subgroup of</u> G, <u>then</u> G <u>and</u> G/H <u>are locally isomorphic</u>.

We shall need the discrete subgroups of $\mathbb{R}^p$:

8.B. EXERCISE. <u>Let</u> $p \in \mathbb{N}$ <u>and let</u> H <u>be a discrete subgroup of</u> $\mathbb{R}^p$. <u>There exist linearly independent</u> $a_1,\dots,a_m \in \mathbb{R}^p$ <u>such that</u> $H = \sum \mathbb{Z}a_i$. <u>Then</u> H <u>is isomorphic to</u> $\mathbb{Z}^m$ <u>and</u> $\mathbb{R}^p/H$ <u>is</u> (<u>as a topological group</u>) <u>isomorphic to</u> $\mathbb{T}^m \times \mathbb{R}^{p-m}$.

Hint. Choose $a \in H$. One may assume that $H \cap \mathbb{R}a=\mathbb{Z}a$. If π is the surjection $\mathbb{R}^p \to \mathbb{R}^p/\mathbb{R}a$, then $\pi(H)$ is a discrete subgroup of $\mathbb{R}^p/\mathbb{R}a$.

FROM HERE ON, G IS A LOCALLY COMPACT ABELIAN GROUP. In view of the crucial role that $\mathbb{R}$ and $\mathbb{Z}$ are going to play, WE USE THE ADDITIVE NOTATIONS.

A subset X of G is said to generate G if G does not have proper closed subgroups that contain X. It is compactly generated (finitely generated, monothetic) if it is generated by a compact subset (a finite subset, a one-point subset). Examples of monothetic groups are cyclic groups and the group $\mathbb{Z}_p$ and P mentioned in Exercise 7.B.

8.1. FUNDAMENTAL THEOREM ON MONOTHETIC GROUPS. Let G be monothetic. Then either G is isomorphic (as a topological group) to $\mathbb{Z}$, or G is compact.

Proof. If G is discrete, it is either finite or isomorphic to $\mathbb{Z}$. Assume G is not discrete. Choose $a \in G$ such that $G = \overline{\mathbb{Z}a}$.

If W is a symmetric neighborhood of 0 and $N \in \mathbb{N}$, then, as G is not discrete there exists an $n \in \mathbb{Z}$, $|n| > N$ for which $na \in W$; hence there is an $n \in \mathbb{N}$, $n > N$ such that $na \in W$. It follows that $ka \in \overline{\mathbb{N}a}$ $(k \in \mathbb{Z})$, so that $\mathbb{N}a$ is dense in G.

Choose a compact neighborhood V of 0. Every $y-V$ $(y \in G)$ intersects $\mathbb{N}a$. Therefore

$$G = \bigcup_{n \in \mathbb{N}} (V+na).$$

Similarly, $G = \bigcup_{n \in \mathbb{N}} (V-na)$. By compactness there exists an $N \in \mathbb{N}$ such that

$$V \subset \bigcup_{n=1}^{N} (V-na).$$

We prove that $G = \bigcup_{n=1}^{N} (V+na)$: then G must be compact. Take $x \in G$. There exist $k \in \mathbb{N}$ with $x \in V+ka$. Then $x-ka \in V$, so $x-ka \in V-na$ for some $n \in \{1,\dots,N\}$, i.e. $x \in V+(k-n)a$ for some $n \in \{1,\dots,N\}$. It follows that, for every x, $\min\{k \in \mathbb{N} : x \in V+ka\} \leq N$. Thus, $G \subset \bigcup_{n=1}^{N} (V+na)$.

8.C. EXERCISE. Every compact subsemigroup of a topological group is itself a group. (No commutativity required.)

8.D. EXERCISE. Let $\mathbf{T}_d$ denote the group $\mathbf{T}$ provided with the discrete topology. $\bar{\mathbf{Z}}$ is the Bohr compactification of Z. (See Exercise 7.D.) The following conditions on G are equivalent.

(a) G is monothetic and not isomorphic to $\mathbf{Z}$.

(b) There exists a continuous homomorphism of $\bar{\mathbf{Z}}$ onto G.

(c) Γ is discrete and isomorphic to a subgroup of $\mathbf{T}_d$.

The main theorem of this section is the following.

8.2. STRUCTURE THEOREM FOR COMPACTLY GENERATED LOCALLY COMPACT ABELIAN GROUPS. <u>If G is compactly generated, it is isomorphic (as a topological group) to $\mathbf{Z}^k \times \mathbf{R}^m \times C$ for certain $k,m \in \{0,1,2,\dots\}$ and some compact abelian group C.</u>

Before proving the theorem we mention a consequence.

8.3. COROLLARY. G <u>contains an open subgroup that is isomorphic (as a topological group) to $\mathbf{R}^m \times C$ for some $m \in \{0,1,2,\dots\}$ and some compact abelian group C.</u>

Proof. Let X be a compact symmetric neighborhood of 0, and let H be the closure of $X \cup (X+X) \cup (X+X+X) \cup \dots$. Then H is a subgroup of G, generated by X. (See also Lemma 3.17.) As $H=\bigcup_{x\in H}(x+X)$, H is an open subset of G. By the structure theorem, H is isomorphic to some $\mathbf{Z}^k \times \mathbf{R}^m \times C$. Then H contains an open subgroup isomorphic to $\mathbf{R}^m \times C$.

Now we turn to the proof of the Structure Theorem.

Let G be generated by a compact symmetric neighborhood V of 0. There exist $x_1,\dots,x_m \in G$ such that $V+V \subset \bigcup_i (x_i+V)$. Let H be the smallest subgroup of G containing $x_1,\dots,x_m$ (not necessarily closed). Then G=H+V.

Call a finite sequence $(z_1,\dots,z_n)$ of elements of G <u>independent</u> if $(i_1,\dots,i_n) \longmapsto i_1z_1+\dots+i_nz_n$ is an isomomorphism of $\mathbf{Z}^n$ onto a discrete subgroup of G. Out of the x_i $(1 \le i \le m)$

one can build a maximal independent sequence, $(x_1,\ldots,x_p)$, say. Set $Z_p := \sum_{i=1}^{p} \mathbb{Z}x_i$. Then Z_p <u>is isomorphic and homeomorphic to $\mathbb{Z}^p$, and is closed in</u> G (Exercise 8.A). We prove G/Z_p to be compact. Let π be the natural homomorphism $G \longrightarrow G/Z_p$. If each $\overline{\mathbb{Z}\pi(x_i)}$ $(i=1,\ldots,m)$ is compact in G/Z_p then $G/Z_p =$ $= \pi(V) + \sum \overline{Z\pi(x_i)}$ is compact and we are done. Suppose that among $x_1,\ldots,x_m$ there is an x for which $\overline{Z\pi(x)}$ is not compact. Then $n \longmapsto n\pi(x)$ is an isomorphism of $\mathbb{Z}$ onto a discrete subgroup of G/Z_p. The map

$$\varphi : (i_0,i_1,\ldots,i_p) \longmapsto i_0x+i_1x_1+\ldots+i_px_p$$

is a homomorphism of $\mathbb{Z}^{p+1}$ onto the subgroup $\mathbb{Z}x+Z_p$ of G. If for certain $i_0,\ldots,i_p$, $\varphi(i_0,i_1,\ldots,i_p)=0$, then $i_0\pi(x)=$ $= -\pi(i_1x_1+\ldots+i_px_p)=0$, so $i_0=0$. But then $i_1x_1+\ldots+i_px_p=0$, so $i_1=\ldots=i_p=0$. Hence φ is injective. Further, there exist open neighborhoods W of $0 \in G$ and W' of $0 \in G/Z_p$ such that $W \cap Z_p=\{0\}$ and $W' \cap \mathbb{Z}\pi(x)=\{0\}$. Then $W \cap \pi^{-1}(W')$ is a neighborhood of $0 \in G$ and $(W \cap \pi^{-1}(W')) \cap (\mathbb{Z}x+Z_p)=W \cap \pi^{-1}(W') \cap \pi^{-1}(\mathbb{Z}\pi(x))=$ $=W \cap \pi^{-1}(W' \cap \mathbb{Z}\pi(x))=W \cap \pi^{-1}(0)=W \cap Z_p=\{0\}$. Therefore, $\mathbb{Z}x+Z_p$ is discrete, i.e. $\varphi(\mathbb{Z}^{p+1})$ is discrete, and we have a contradiction with the maximality of the sequence $(x_1,\ldots,x_p)$.

We see that G/Z_p <u>is compact</u>.

Before we can draw useful conclusions from this fact we need another fact:

8.4. THEOREM. <u>Every connected locally compact abelian group that is locally isomorphic to some $\mathbb{R}^p$ $(p \in \mathbb{N})$ is (as a topological group) isomorphic to $\mathbb{R}^m \times \mathbb{T}^{p-m}$ for certain</u> $m \in \{0,1,\ldots,p\}$.

Proof. Let G be a connected locally compact abelian group. Let U be an open neighborhood of $0 \in \mathbb{R}^p$, V a neighborhood of $0 \in G$, and φ a homeomorphism of U onto V such that

$\varphi(x+y)=\varphi(x)+\varphi(y)$ for all $x,y\in U$ for which $x+y \in U$. Without restriction we may assume $\frac{1}{2}x \in U$ for every $x \in U$; then $\varphi(x)=2\varphi(\frac{1}{2}x)$ $(x \in U)$. For every $x \in \mathbb{R}^p$ we have $(\frac{1}{2})^k x \in U$ for large $k \in \mathbb{N}$. It follows that there exists a unique $\Phi : \mathbb{R}^p \longrightarrow G$ for which

$$\begin{cases} \Phi = \varphi & \text{on } U, \\ \Phi(2x) = 2\,\Phi(x) & (x \in \mathbb{R}^p). \end{cases}$$

This Φ is easily seen to be a homomorphism. Every $x \in \mathbb{R}^p$ has an open neighborhood (e.g. $x+U$) which by Φ is mapped homeomorphically onto a neighborhood of $\Phi(x)$. Then Φ is continuous and open. Therefore $\Phi(\mathbb{R}^p)$ is an open subgroup of G. Applying the Isomorphism Theorem 3.6 we find $\Phi(\mathbb{R}^p)$ to be isomorphic to $\mathbb{R}^p/\mathrm{Ker}\,\Phi$. But $\mathrm{Ker}\,\Phi$ is a discrete subgroup of $\mathbb{R}^p$, since $U \cap \mathrm{Ker}\,\Phi = \{0\}$. Hence, $\Phi(\mathbb{R}^p)$ is isomorphic to some $\mathbb{R}^m \times \mathbb{T}^{p-m}$ (Exercise 8.B).

$\Phi(\mathbb{R}^p)$ was an open subgroup of G. Hence it is also closed (Lemma 3.1 (c)). By the connectedness of G, $\Phi(\mathbb{R}^p)=G$.

We return to the Structure Theorem.

In G we have found a discrete subgroup Z_p, isomorphic to $\mathbb{Z}^p$, such that G/Z_p is compact. Let Γ be the dual group of G and $D := \{\gamma \in \Gamma : \gamma = 0 \text{ on } \mathbb{Z}_p\}$. By Theorem 7.16, Γ/D is isomorphic to $\hat{Z}_p$, hence to $\mathbb{T}^p$. Then Γ/D is locally isomorphic to $\mathbb{R}^p$. But D is discrete, being isomorphic to $(G/Z_p)\hat{}$. Hence, Γ is locally isomorphic to $\mathbb{R}^p$. Our proof of Theorem 8.4 yields an isomorphism (of topological groups) of $\mathbb{R}^m \times \mathbb{T}^{p-m}$ onto some open subgroup Γ_0 of Γ.

In Lemma 8.5 we shall see that there exists a homomorphism $\sigma : \Gamma/\Gamma_0 \longrightarrow \Gamma$ such that $\sigma(\gamma+\Gamma_0) \in \gamma + \Gamma_0$ $(\gamma \in \Gamma)$. It is not hard to prove that the formula

$$(\gamma, \beta + \Gamma_0) \longmapsto \gamma + \sigma(\beta + \Gamma_0) \qquad (\gamma \in \Gamma_0; \beta + \Gamma_0 \in \Gamma/\Gamma_0)$$

defines an isomorphism (of topological groups) of $\Gamma_0 \times (\Gamma/\Gamma_0)$ onto Γ. Then $\hat{\Gamma}$ is isomorphic to $\hat{\Gamma}_0 \times (\Gamma/\Gamma_0)\hat{}$. As Γ_0

was isomorphic to $\mathbb{R}^m \times \mathbb{T}^{p-m}$, $\hat{\Gamma}_o$ is isomorphic to $\mathbb{R}^m \times \mathbb{Z}^{p-m}$. Further, $C := (\Gamma/\Gamma_o)^\wedge$ is compact (Γ_o is open in Γ). Now G is isomorphic to $\mathbb{R}^m \times \mathbb{Z}^{p-m} \times C$.

It remains to prove the existence of σ.

An abelian group A is called <u>divisible</u> if for every $a \in A$ and every $k \in \mathbb{N}$ there exists an $x \in A$ for which $kx=a$. Examples of divisible groups are $\mathbb{R}$, $\mathbb{T}$, $\mathbb{Q}$ and $\mathbb{R}^m \times \mathbb{T}^{p-m}$ $(0 \leq m \leq p)$.

8.5. LEMMA. <u>If A is a divisible subgroup of an abelian group B, then the groups B and $A \times (B/A)$ are isomorphic. If $\pi: B \longrightarrow B/A$ is the natural surjection there exists a homomorphism $\sigma: B/A \longrightarrow B$ such that</u> $\pi \circ \sigma = id_{B/A}$. (For a set X we denote by id_X the identity map $X \longrightarrow X$.)

Proof. We only prove that such a σ exists: the rest of the lemma follows easily. By Zorn's Lemma it suffices to prove: if J is a subgroup of B/A, if $\rho: J \longrightarrow B$ is a homomorphism for which $\pi \circ \rho = id_J$ and if $u \in B/A$, $u \notin J$, then ρ can be extended to a homomorphism ρ' of $J+\mathbb{Z}u$ into B such that $\pi \circ \rho' = id_{J+\mathbb{Z}u}$.

For such J, ρ, u, choose $x \in \pi^{-1}(u)$. If $mu \notin J$ for all $m \in \mathbb{N}$, we can simply define

$$\rho'(v+mu) := \rho(v)+mx \qquad (v \in J;\ m \in \mathbb{Z}).$$

Otherwise there exists a smallest $k \in \mathbb{N}$ with $ku \in J$. Now $kx-\rho(ku) \in A$, so $kx-\rho(ku)=ky$ for some $y \in A$. Put

$$\rho'(v+mu) := \rho(v)+m(x-y) \qquad (v \in J;\ m \in \mathbb{Z}).$$

It is easy to prove that ρ' is well-defined and satisfies the requirements.

8.E. EXERCISE. Let G be a locally compact abelian group. The union of the compact subgroups of G is a closed subgroup of G.