V. VARIA.

12. LOCALLY COMPACT FIELDS.

A <u>locally compact field</u> is a field K provided with a locally compact topology such that K is a topological group under addition and $K\setminus\{0\}$ is a topological group under multiplication. IN THIS SECTION K IS A NON-DISCRETE LOCALLY COMPACT FIELD.

Let m be a Haar measure on (the additive group) K. For every $a \in K$, $X \longmapsto m(aX)$ is a left invariant measure on the Borel σ-algebra of K. By the uniqueness of the Haar measure there exists a function $D : K \longrightarrow [0,\infty)$ for which

$$m(aX) = D(a)m(X) \qquad (a \in K;\ X \subset K \text{ Borel}).$$

Evidently,

$$D(ab) = D(a)D(b) \qquad (a,b \in K)$$

and

$$D(a^{-1}) = \frac{1}{D(a)} \qquad (a \in K,\ a \neq 0).$$

$0 \in K$ has a compact neighborhood C; this C is an infinite set (K is not discrete) but has finite measure. It follows that $m(\{x\}) = 0$ for every $x \in K$. Thus,

$$D(a) = 0 \text{ if and only if } a = 0.$$

12.1. LEMMA. D <u>is continuous</u>.

Proof. Let $a \in K$, $c > D(a)$. Choose a compact $X \subset K$, $m(X) > 0$. By the regularity of m there exists an open $U \supset aX$ such that $m(U) < c\,m(X)$. According to Lemma 3.2 the set $W := \{x \in K : xX \subset U\}$ is an open set containing a. For all $x \in W$ we have $D(x)m(X) \leq m(U) < cm(X)$, i.e. $D(x) < c$.

Thus, $\{x \in K : D(x) < c\}$ is open for every $c \in \mathbb{R}$. Further, for $c > 0$ we have $\{x \in K : D(x) > c\} = \{x \in K\setminus\{0\} : D(x^{-1}) < c^{-1}\}$. Hence, D is continuous.

Apparently, $\{0\}$ is the intersection of the open sets $\{x : D(x) < n^{-1}\}$ $(n \in \mathbb{N})$. Exercise 3.L implies:

12.2. THEOREM. K <u>is metrizable</u>.

If $x_1, x_2, \ldots$ <u>is a sequence that is contained in a compact set</u> $C \subset K$ <u>and if</u> $\lim D(x_n) = 0$, <u>then</u> $\lim x_n = 0$.
In fact, assume that this conclusion is false. Then 0 has an open neighborhood U such that $C \setminus U$ contains x_n for infinitely many n. As $C \setminus U$ is compact the sequence $x_1, x_2, \ldots$ must have a subsequence $y_1, y_2, \ldots$ that lies in $C \setminus U$ and converges to some $y \in C \setminus U$. Consequently, $D(y) = \lim D(y_n) = \lim D(x_n) = 0$, so $y = 0$. But $0 \notin C \setminus U$.

12.3. THEOREM. <u>For every</u> $c \geq 0$ <u>the set</u> $\{x \in K : D(x) \leq c\}$ <u>is compact</u>.

Proof. Let $c \geq 0$; set $A := \{x : D(x) \leq c\}$. By Theorem 12.2, A is closed.

Choose an open neighborhood U of 0 with compact closure and let $W := \{x \in K : xU \subset U\}$. This W is a neighborhood of 0 because $W \supset \{x : x\overline{U} \subset U\}$ (apply Lemma 3.2); and W has compact closure because for every $a \in U \setminus \{0\}$, W is contained in the compact set $\overline{U}a^{-1}$. Furthermore,

<u>if</u> $a, b \in W$, <u>then</u> $ab \in W$.

Take $s \in W$, $0 < D(s) < 1$. (Such s exist since K is not discrete.) W is a semigroup under multiplication, so that

$$W \subset s^{-1}W \subset s^{-2}W \subset \ldots .$$

We have $s, s^2, s^3, \ldots \in \overline{W}$ and $\lim D(s^n) = \lim D(s)^n = 0$. By the remark we made before formulating this theorem it follows that $\lim s^n = 0$. Then $\lim s^n x = 0$ for all $x \in K$. A fortiori, for every x there exists an n such that $s^n x \in W$. Thus,

$$\bigcup_n s^{-n}W = K.$$

We prove that $A \subset s^{-n}W$ for some n. (Then A, being a closed subset of the compact $s^{-n}\overline{W}$, is itself compact.) If A is not contained in any $s^{-n}W$ then it must intersect infinitely many of the sets $s^{-n-1}W \setminus s^{-n}W$, i.e. there exists a subsequence $n_1, n_2, \ldots$ of $1, 2, \ldots$ and there exist $x_1, x_2, \ldots \in A$ such that $x_i \in s^{-n_i-1}W \setminus s^{-n_i}W$ for each i. Now each $x_i s^{n_i}$ lies in the compact set $s^{-1}\overline{W}$, and $\lim D(x_i)D(s)^{n_i} = 0$. Hence, $\lim x_i s^{n_i} = 0$. But each $x_i s^{n_i}$ lies in the closed set $K \setminus W$ which does not contain 0: contradiction.

12.4. COROLLARY. *K is σ-compact.*

12.5. COROLLARY. *A sequence* $x_1, x_2, \ldots \in K$ *converges to* 0 *if and only if* $\lim D(x_n) = 0$.

Proof. The "only if" is true because of the continuity of D, the "if" because of Theorem 12.3 and the boundedness of a converging sequence of real numbers.

For the dual group $\hat{K}$ of K we use the additive notation:

$$(\gamma_1+\gamma_2)(x) = \gamma_1(x)\gamma_2(x) \qquad (\gamma_1,\gamma_2 \in \hat{K};\ x \in K)$$

and 0 denotes the constant function 1 on K. From the σ-compactness of K it follows that $\hat{K}$ *is metrizable* (Exercise 7.C).

The following definition makes $\hat{K}$ into a vector space over K:

$$(x\gamma)(y) := \gamma(xy) \qquad (\gamma \in \hat{K};\ x,y \in K).$$

The scalar multiplication $(x,\gamma) \longmapsto x\gamma$ *is continuous* $K \times \hat{K} \longrightarrow \hat{K}$. To prove this statemant, let $\lim x_n = x$ in K, $\lim \beta_n = \beta$ in $\hat{K}$, let $C \subset K$ be compact and $\varrho > 0$: we prove $x\beta - x_n\beta_n \in N(C:\varrho)$ for large n. The set $C' := \{x, x_1, x_2, \ldots\}$ is compact, so that $C'C$ is also compact. For all n,

$$\sup_{y\in C} \left|(x\beta)(y)-(x_n\beta_n)(y)\right| \le \sup_{y\in C} |\beta(xy)-\beta(x_ny)| + \sup_{y\in C} |\beta(x_ny)-\beta_n(x_ny)| \le$$

$$\le \sup_{y\in C} |\beta(xy-x_ny)-1| + \sup_{z\in C'C} |\beta(z)-\beta_n(z)|.$$

For large n, $\beta-\beta_n \in N(C'C: \frac{1}{3}\varrho)$, i.e. $\sup_{z\in C'C} |\beta(z)-\beta_n(z)| \le \frac{1}{3}\varrho$.
By Lemma 3.2 there exists a neighborhood U of 0 for which $UC \subset \{z \in K : |\beta(z)-1| < \frac{1}{3}\varrho\}$. If n is large enough, then $x-x_n \in U$, so $|\beta(xy-x_ny)-1| < \frac{1}{3}\varrho$ for all $y \in C$ and $\sup_{y\in C} |\beta(xy-x_ny)-1| \le \frac{1}{3}\varrho$. This proves that $x\beta-x_n\beta_n \in N(C:\varrho)$ for large n.

Choose $\beta \in \hat{K}$, $\beta \neq 0$. For $x \in K$ set $\Phi(x) := x\beta$. Then Φ is injective and continuous $K \longrightarrow K\beta$. We prove that Φ _is a homeomorphism_ $K \longrightarrow K\beta$ and that $K\beta$ _is closed in_ $\hat{K}$. It suffices to prove

(*) _if_ $a_1, a_2, \ldots \in K$ _and if_ $\gamma := \lim a_n\beta$ _exists, then the sequence_ $a_1, a_2, \ldots$ _converges to some_ $a \in K$, _and_ $\gamma = a\beta$.

Let $a_1, a_2, \ldots \in K$ and $\gamma = \lim a_n\beta$. If the sequence $D(a_1)$, $D(a_2), \ldots$ is bounded there is a compact set $A \subset K$ containing all the a_n (Theorem 12.3). The restriction of Φ to A is then a homeomorphism $A \longrightarrow \Phi(A) \subset K\beta$, and we are done. Otherwise the sequence $1, 2, \ldots$ has a subsequence $n_1, n_2, \ldots$ such that $\lim D(a_{n_i}) = \infty$. Then $\lim D(a_{n_i}^{-1}) = 0$, so $\lim a_{n_i}^{-1} = 0$ and $0 = 0\gamma = (\lim a_{n_i}^{-1})(\lim a_{n_i}\beta) = \lim a_{n_i}^{-1}a_{n_i}\beta =$ $= \beta$, which is a contradiction.

Suppose $K\beta \neq \hat{K}$. Then $\hat{K}$ has a non-trivial character that is 1 on K (Theorem 7.2 (b)). By the Duality Theorem there is an $a \in K$, $a \neq 0$ such that $(x\beta)(a) = 1$ $(x \in K)$. Then $\beta(xa) = 1$ for all x, so $\beta = 0$, and we again have a contradiction.

We have thus proved:

12.6. THEOREM. _Choose_ $\beta \in \hat{K}$, $\beta \neq 0$. _The map_ $x \longmapsto x\beta$ _is a homeomorphism and a group isomorphism of_ K _onto_ $\hat{K}$.

Suppose K contains a non-zero compact subgroup C (under addition). Take $c \in C$, $c \neq 0$. Then every $a \in K$ lies in a compact subgroup of K (viz. $ac^{-1}C$).

It follows now from the corollary (8.3) to the Structure Theorem for locally compact abelian groups that K contains an open subgroup that is either isomorphic to $\mathbb{R}^m$ (some $m \in \mathbb{N}$) or compact.

Case I. _Let_ Φ _be a homeomorphism-and-isomorphism of an_ m-_dimensional vector space_ E _onto an open subgroup of_ K. Choose $a \in K$, $a \neq 0$ and $b \in \Phi(E)$, $b \neq 0$. Then $a^{-1}b\,\Phi(E)$ is an open subgroup of K, so $\Phi(E) \cap a^{-1}b\,\Phi(E)$ is an open subgroup of $\Phi(E)$. But E has no proper open subgroups, so $\Phi(E) \subset a^{-1}b\,\Phi(E)$, and $a=ab^{-1}b \in ab^{-1}a^{-1}b\,\Phi(E)=\Phi(E)$: Φ maps E onto K.

Let $e:=\Phi^{-1}(1)$. Define $\varphi: \mathbb{R} \longrightarrow K$ by $\varphi(\lambda):=\Phi(\lambda e)$. Then φ is an isomorphism of $\mathbb{R}$ onto a subfield R of K. It is not hard to prove that

$$\Phi(\sum_{i=1}^{n}\lambda_i a_i) = \sum_{i=1}^{n}\varphi(\lambda_i)\,\Phi(a_i) \qquad (n \in \mathbb{N};\ \lambda_1,\dots,\lambda_n \in \mathbb{R};\ a_1,\dots,a_n \in E).$$

(Prove the formula for rational $\lambda_1,\dots,\lambda_n$ first.) Then K is a finite dimensional vector space over R, hence is an algebraic extension of R. But R is isomorphic to $\mathbb{R}$, so $\dim_R K \leq 2$. Thus, for E we may take either $\mathbb{R}$ or $\mathbb{C}$. It is easy to see that in either case Φ is multiplicative ($\Phi(xy)=\Phi(x)\,\Phi(y)$), so that K, as a topological field, is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$.

Case II. Let U be a compact open subgroup of K. Set $W:=\{x \in K : xU \subset U\}$. As in the proof of Theorem 12.3 W is a neighborhood of 0, $\overline{W}$ is compact, and $ab \in W$ for all $a,b \in W$. But this time U is closed and a subgroup of K: consequently W is a compact open subring of K.

We prove

(*) if $a \in K$ and $D(a) < 1$, then $D(1-a) = 1$.

Let $a \in K$, $D(a) < 1$. Then $\lim_{n\to\infty} a^n = 0$ (Corollary 12.5), so $a^N \in W$ for some $N \in \mathbb{N}$.

$$W' := W+aW+a^2W+\dots+a^{N-1}W$$

is a compact open subgroup and $aW' \subset W'$. Therefore, $(1-a)W' \subset W'$, so $D(1-a) \leq 1$. But also $(1+a+a^2+\dots+a^n)W' \subset W'$, so $D(1+a+\dots+a^n) \leq 1$ $(n \in \mathbb{N})$. Now $\lim_{n\to\infty}(1+a+\dots+a^n) = (1-a)^{-1}$. Hence, $D((1-a)^{-1}) \leq 1$, i.e. $D(1-a) \geq 1$.

This proves (*). But (*) leads to

(**) if $a,b \in K$ and $D(a) < D(a+b)$, then $D(a+b) = D(b)$.

In fact, if $D(a) < D(a+b)$, then $D(\frac{a}{a+b}) < 1$, so $1 = D(1-\frac{a}{a+b}) = D(\frac{b}{a+b})$, whence $D(a+b) = D(b)$.

By (**), for all $a,b \in K$,

$$D(a+b) \leq \max\,(D(a),D(b)).$$

In particular, D determines a metric d by

$$d(a,b) := D(a-b).$$

By Corollary 12.5, the topology of K is just the one induced by d.

A <u>valuation</u> on a field F is a map $x \longmapsto \|x\|$ of F into $\mathbb{R}$ such that

(i) $\|x\| \geq 0$ for all $x \in F$; $\|x\| = 0$ if and only if $x = 0$.

(ii) $\|x+y\| \leq \|x\| + \|y\|$ $(x,y \in F)$.

(iii) $\|xy\| = \|x\|\|y\|$ $(x,y \in F)$.

Such a valuation is <u>non-Archimedean</u> if (ii) can be strengthened to

(ii*) $\|x+y\| \leq \max(\|x\|,\|y\|)$ $(x,y \in F)$.

Every valuation $\| \ \|$ on F induces a metric d by

$$d(x,y) := \|x-y\|,$$

which turns F into a topological field.

In the above we have proved the following.

12.7. THEOREM. (PONTRYAGIN-KOWALSKY). *If K is connected, it is isomorphic (as a topological field) to either $\mathbb{R}$ or $\mathbb{C}$. Otherwise there is a non-Archimedean valuation on K which induces the given topology of K.*

12.A. EXERCISE. Let K be a (non-discrete) locally compact field with a non-Archimedean valuation. Then, as a locally compact abelian group, K is not compactly generated. (In particular, K is not compact.)

12.B. EXERCISE <u>The p-adic numbers</u>.

Let p be a prime number. Every non-zero rational number x can be written in the form $p^n \frac{a}{b}$ where $n,a,b \in \mathbb{Z}$ while neither a nor b is divisible by p; we then set $D(x) := p^{-n}$. (This definition is unambiguous: if $p^n \frac{a}{b} = p^{n'} \frac{a'}{b'}$ and a,b,a',b' are not divisible by p, then $n = n'$.) Further, we $D(0) := 0$.

This D is determined by the following properties.

$$D(p) = p^{-1}, \quad D(0) = 0;$$

$$D(m) = 1 \text{ if } m \in \mathbb{Z} \text{ is not divisible by } p;$$

$$D(xy) = D(x)D(y) \quad (x,y \in \mathbb{Q}).$$

D is a non-Archimedean valuation on $\mathbb{Q}$.

By $\mathbb{Q}_p$ we denote the completion of $\mathbb{Q}$ relative to the metric induced by D. Then D has a unique continuous extension to $\mathbb{Q}_p$; we denote this extension by $|\ |_p$. The metric on $\mathbb{Q}_p$ turns out to be

$$(x,y) \longmapsto |x-y|_p \qquad (x,y \in \mathbb{Q}_p).$$

Addition and multiplication have unique continuous extensions $\mathbb{Q}_p \times \mathbb{Q}_p \longrightarrow \mathbb{Q}_p$. Thus, $\mathbb{Q}_p$ <u>becomes a non-Archimedean valued field, complete relative to the metric induced by its valuation</u> $|\ |_p$.

The elements of $\mathbb{Q}_p$ are called p-<u>adic numbers</u>.

The closure of $\mathbb{Z}$ in $\mathbb{Q}_p$ is isometric and isomorphic to the group $\mathbb{Z}_p$ of p-adic integers. (See Exercise 7.A.)

We view $\mathbb{Z}_p$ as a (compact) subgroup of $\mathbb{Q}_p$. Then

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}.$$

(Hint to proving the inclusion $\supset$. Let $x \in \mathbb{Q}_p$, $|x|_p \leq 1$ and let $\varepsilon > 0$. Choose $n \in \mathbb{N}$ so that $p^{-n} < \varepsilon$. There exist $k, l \in \mathbb{Z}$, $l \neq 0$ so that $\left|x - \frac{k}{l}\right|_p < p^{-n}$; then $\left|\frac{k}{l}\right|_p \leq 1$. We may assume that l is not divisible by p. Then p^n and l are relatively prime and there exists an $m \in \mathbb{Z}$ such that $k-ml$ is divisible by p^n. Now $|x-m|_p < \varepsilon$.)

$\mathbb{Q}_p$ <u>is locally compact</u>.

<u>If</u> m <u>is a Haar measure on</u> $\mathbb{Q}_p$, <u>then</u>

$$m(aX) = |a|_p m(X) \qquad (a \in \mathbb{Q}_p;\ X \subset \mathbb{Q}_p \text{ Borel}).$$