APPENDIX.

HAAR MEASURE.

G IS A LOCALLY COMPACT GROUP. In this section we prove the existence and the essential uniqueness of the left Haar measure.

By E we denote $C_{oo}^{+}(G)\setminus\{0\}$. For $f\in E$, $\|f\|$ denotes the sup-norm of f. We first prove that there exists a map $m : E \longrightarrow (0,\infty)$ which is additive (i.e. $m(f_1+f_2)=m(f_1)+m(f_2)$ for all $f_1,f_2\in E$), positive homogeneous (i.e. $m(\alpha f)=\alpha m(f)$ if $\alpha>0$, $f\in E$) and left invariant (i.e. $m(f_a)=m(f)$ for all $f\in E$, $a\in G$).

Let $f,g\in E$. There exist $n\in\mathbb{N}$, $c_1,\dots,c_n>0$ and $a_1,\dots,a_n\in G$ such that $f\leq\sum c_i g_{a_i}$. (For every $x\in G$ there exist $c>0$ and $a\in G$ such that $f<cg_a$ on some neighborhood of x. Now use the compactness of supp f.) By (f:g) we denote the infimum of all sums $\sum_{i=1}^{n} c_i$ for which $c_i>0$ and $f\leq\sum c_i g_{a_i}$ for certain $a_1,\dots,a_n\in G$.

Thus we obtain a fumction (:) on $E\times E$ with the following properties.

$$\left.\begin{array}{l}
(f:g)\geq\frac{\|f\|}{\|g\|}>0\\
(f_a:g)=(f:g)\\
(\alpha f:g)=\alpha(f:g)\\
(f_1+f_2:g)\leq(f_1:g)+(f_2:g)\\
\text{if } f_1\leq f_2 \text{ then } (f_1:g)\leq(f_2:g)
\end{array}\right\}
\begin{array}{l}(f,g,h,f_1,f_2\in E;\\ a\in G;\ \alpha>0)\end{array}$$

$$(f:h)\leq(f:g)(g:h)$$

These assertions are all fairly evident, except maybe the first and the last one. For the first, observe that there exists an x_0 for which $f(x_0)=\|f\|$. If now $f\leq\sum c_i g_{a_i}$ then $\|f\|\leq\sum c_i g(a_i^{-1}x_0)\leq\sum c_i\|g\|$. For the last one, note that, if $f\leq\sum c_i g_{a_i}$ and $g\leq\sum d_j h_{b_j}$, then $f\leq\sum c_i d_j h_{a_i b_j}$.

Choose an $h\in E$. Put

$$m_g(f) := \frac{(f:g)}{(h:g)} \qquad (f,g \in E).$$

Every m_g is left invariant, positive homogeneous and subadditive (i.e. $m_g(f_1 + f_2) \leq m_g(f_1) + m_g(f_2)$). Further,

$$\frac{1}{(h:f)} \leq m_g(f) \leq (f:h) \qquad (f,g \in E).$$

LEMMA. <u>For all</u> $f_1, f_2 \in E$ <u>and</u> $\varepsilon > 0$ <u>there exists a neighborhood</u> V <u>of</u> 1 <u>such that</u> $(1+\varepsilon) m_g(f_1 + f_2) \geq m_g(f_1) + m_g(f_2)$ <u>for every</u> $g \in E$ <u>whose support is contained in</u> V.

For the moment let us assume that this lemma is correct. Then it is not difficult to prove the existence of a left Haar measure on G.

For every $f \in E$ the closed interval $\left[\frac{1}{(h:f)}, (f:h)\right]$ is compact. Therefore the cartesian product of these intervals is a compact subset S of $E^{\mathbb{R}}$. This S contains all the m_g. For every neighborhood V of $1 \in G$ let K_V be the closure of $\{m_g : g \in E,\ \operatorname{supp} g \subset V\}$. Then K_V is non-empty and if V_1, V_2 are neighborhoods of 1 such that $V_1 \subset V_2$, then $K_{V_1} \subset K_{V_2}$. By compactness, the intersection of all these K_V contains an element m. Obviously, m is a function $E \longrightarrow (0,\infty)$.

If V is any neighborhood of $1 \in G$ there exists a net $(g_\lambda)_{\lambda \in \Lambda}$ of elements g_λ of E having their supports in V, such that $m = \lim m_{g_\lambda}$, i.e.

$$m(f) = \lim m_{g_\lambda}(f) \qquad (f \in E).$$

It follows that m is left invariant, positive homogeneous and subadditive. For given $\varepsilon > 0$ and $f_1, f_2 \in E$ choosing V as in the lemma we find $(1+\varepsilon)m(f_1+f_2) \geq m(f_1) + m(f_2)$. It follows that m is additive. Thus m extends linearly to a left Haar integral on $C_{00}(G)$.

It remains to prove the lemma. Let $f_1, f_2 \in E$ and $\varepsilon > 0$. Set $f = f_1 + f_2$. Let $f_0 \in E$ be such that $f_0 = 1$ on $\operatorname{supp} f$ and let

$\delta := \frac{\varepsilon}{3}\frac{(f:g)}{(f_o:g)}$. Define $h_1, h_2 : G \longrightarrow [0,\infty)$ by

$$h_i := \frac{f_i}{f + \delta f_o} \qquad (i = 1,2).$$

h_i is continuous and $\neq 0$ at every point of G where f_i is not 0 while $0 \leq h_i \leq \delta^{-1} f_i$. Thus, $h_i \in E$ and $\operatorname{supp} h_i = \operatorname{supp} f_i$. Choose a neighborhood V of $1 \in G$ such that

$$|h_i(x) - h_i(y)| < \frac{\varepsilon}{6} \qquad (i = 1,2;\ x,y \in G \text{ so that } x^{-1}y \in V).$$

Now let $g \in E$ have its support in V. If $f \leq \sum c_j g_{a_j}$ then

$$f_i = fh_i + \delta f_o h_i \leq fh_i + \delta f_o \leq \sum c_j h_i g_{a_j} + \delta f_o \leq \sum c_j [h_i(a_j) + \tfrac{\varepsilon}{6}] g_{a_j} + \delta f_o,$$

so that $(f_i:g) \leq \sum c_j [h_i(a_j) + \frac{\varepsilon}{6}] + \delta(f_o:g) \quad (i = 1,2)$. Consequently, $(f_1:g) + (f_2:g) \leq \sum c_j(1 + \frac{\varepsilon}{3}) + 2\delta(f_o:g)$. From the definition of $(f:g)$ we conclude that $(f_1:g) + (f_2:g) \leq (f:g)(1 + \frac{\varepsilon}{3}) + 2\delta(f_o:g) =$ $= (f:g)(1 + \frac{\varepsilon}{3}) + \frac{2\varepsilon}{3}(f:g) = (f:g)(1 + \varepsilon)$. The lemma follows.

We now have an existence proof. The uniqueness is easy to prove for the abelian case. More generally (without a commutativity condition on G), let n be a non-zero left invariant and m a non-zero two-sided invariant Radon measure on G: we prove that they are multiples of each other. For all $f,h \in C_{oo}(G)$ a few applications of the Fubini Theorem lead to

$$\begin{aligned} m(f)n(h) &= \int h(y)[\int f(x)dm(x)]dn(y) = \\ &= \int h(y)[\int f(xy)dm(x)]dn(y) = \\ &= \int[\int h(y)f(xy)dn(y)]dm(x) = \\ &= \int[\int h(x^{-1}y)f(y)dn(y)]dm(x) = \\ &= \int f(y)[\int h(x^{-1}y)dm(x)]dn(y) = \\ &= \int f(y)[\int h(x^{-1})dm(x)]dn(y) = n(f)m(h^o) \end{aligned}$$

(where $h^o(x) = h(x^{-1})$). Choosing h so that $m(h) \neq 0$ we see that $m = c\,n$ where $c = \frac{m(h^o)}{m(h)}$.

The general case requires more caution. Let m,n be left Haar measures (or integrals) on G. By Lemma 3.12, if $f \in C_{oo}^{+}(G)$

and $f \neq 0$ then $m(f) > 0$ and $n(f) > 0$. Out of the set $\mathcal{H} := \{h \in C_{00}{}^{+}(G) : h(1) \neq 0,\ h^{o} = h\}$ we make a net by defining

$$h_1 \leq h_2 \quad \text{if} \quad \operatorname{supp} h_2 \subset \operatorname{supp} h_1 .$$

We prove that

$$(*) \qquad \frac{n(f)}{m(f)} = \lim_{h \in \mathcal{H}} \frac{n(h)}{m(h)} \qquad (f \in C_{00}{}^{+}(G),\ f \neq 0).$$

Then clearly n will be a multiple of m.

Take $f \in C_{00}{}^{+}(G)$, $f \neq 0$ and take $\varepsilon > 0$. For every $h \in C_{00}(G)$ we have

$$\begin{aligned} m(h)n(f) &= \iint h(y)f(x)\,dn(x)\,dm(y) = \\ &= \iint h(y)f(yx)\,dn(x)\,dm(y) \end{aligned}$$

while

$$\begin{aligned} n(h)m(f) &= \iint h(x)f(y)\,dn(x)\,dm(y) = \\ &= \iint h(y^{-1}x)f(y)\,dn(x)\,dm(y) = \\ &= \iint h^{o}(x^{-1}y)f(y)\,dm(y)\,dn(x) = \\ &= \iint h^{o}(y)f(xy)\,dm(y)\,dn(x). \end{aligned}$$

Thus, for $h \in \mathcal{H}$ we obtain

$$m(h)n(f) - n(h)m(f) = \iint h(y)\left[f(yx) - f(xy)\right] dn(x)\,dm(y)$$

and

$$\begin{aligned} \left|\frac{n(f)}{m(f)} - \frac{n(h)}{m(h)}\right| &\leq \frac{1}{m(f)m(h)} \iint h(y)\,|f(yx) - f(xy)|\,dn(x)\,dm(y) \\ &\leq \frac{1}{m(f)} \sup_{y \in \operatorname{supp} h} \int |f(yx) - f(xy)|\,dn(x) \\ &= \frac{1}{m(f)} \sup_{y \in \operatorname{supp} h} \int |L_y f - R_y f|\,dn. \end{aligned}$$

We know $y \longrightarrow L_y f$ and $y \longrightarrow R_y f$ to be continuous as maps $G \longrightarrow L^1(n)$. (See Theorem 3.13 for "L"; "R" can be proved similarly.) It follows that there exists a neighborhood U of $1 \in G$ such that $\int |L_y f - R_y f|\,dn \leq \int |L_y f - f|\,dn + \int |R_y f - f|\,dn \leq \varepsilon\, m(f)$ for all $y \in U$. Then

$$\left|\frac{n(f)}{m(f)} - \frac{n(h)}{m(h)}\right| \leq \varepsilon \quad \text{for all } h \in \mathcal{H} \text{ for which } \operatorname{supp} h \subset U.$$

We have proved (*).