IV. THE GENERAL THEORY.

9. REPRESENTATION THEORY.

G IS A LOCALLY COMPACT GROUP WITH LEFT HAAR MEASURE m.

Let X be a locally compact space, E a Banach space. Let $\mu \in M(X)$ and let h be a bounded continuous map $X \longrightarrow E$. We want to find an element of E that can reasonably be called $\int h d\mu$.

We can define an $w \in E^{**}$ by

$$w(\varphi) := \int \varphi(h(x))d\mu(x) \qquad (\varphi \in E^*).$$

We prove now that there exists an $e \in E$ such that $w(\varphi)=\varphi(e)$ for every $\varphi \in E^*$. Let $\varepsilon>0$. There exists a compact $C \subset X$ such that $|\mu|(X \setminus C)<\varepsilon$ (regularity). Now $h(C) \subset E$ is compact. Hence there exist non-empty Borel subsets $A_1,\dots,A_m$ of C such that $A_i \cap A_j=\emptyset$ if $i\neq j$, $\cup A_i=C$ and $\|h(x)-h(y)\| \leq \varepsilon$ if x,y are contained in the same A_i. Choose $a_i \in h(A_i)$. We have

$$w(\varphi)-\varphi(\sum_i \mu(A_i)a_i)=\sum_i \left[\int_{A_i} \varphi(h(x))d\mu(x)-\varphi(a_i)\mu(A_i)\right] +$$
$$+\int_{X\setminus C}\varphi(h(x))d\mu(x) = \sum_i \int_{A_i} \left[\varphi(h(x))-\varphi(a_i)\right]d\mu(x) +$$
$$+ \int_{X\setminus C}\varphi(h(x))d\mu(x).$$

Now $|\varphi(h(x))| \leq \|\varphi\| \|h\|_\infty$ for all x; and $|\varphi(h(x))-\varphi(a_i)| \leq \|\varphi\|\varepsilon$ for $x \in A_i$. Hence,

$$|w(\varphi)-\varphi(\sum_i \mu(A_i)a_i)| \leq \sum_i \int_{A_i} \|\varphi\|\varepsilon \, d|\mu| + \int_{X\setminus C} \|\varphi\|\|h\|_\infty d|\mu| \leq$$
$$\leq \|\varphi\|\varepsilon\|\mu\|+\varepsilon\|\varphi\|\|h\|_\infty .$$

Thus, for every $n \in \mathbb{N}$ there is an $e_n \in E$ with

$$|w(\varphi)-\varphi(e_n)| \leq \frac{1}{n}\|\varphi\| \qquad (\varphi \in E^*).$$

Then

$$|\varphi(e_n)-\varphi(e_m)| \leq (\frac{1}{n}+\frac{1}{m})\|\varphi\| \qquad (n,m \in \mathbb{N};\ \varphi \in E^*).$$

By the corollary to Hahn-Banach Theorem 0.7,

$$\|e_n-e_m\| \leq \frac{1}{n}+\frac{1}{m} \qquad (n,m \in \mathbb{N}).$$

This means that the e_n form a Cauchy sequence. If e is its limit, then for all $\varphi \in E^*$, $\varphi(e) = \lim \varphi(e_n) = w(\varphi) = \int \varphi(h(x))\,d\mu(x)$. We have proved:

9.1. THEOREM. Let X be a locally compact space, E a Banach space. For every $\mu \in M(X)$ and every bounded continuous $h : X \longrightarrow E$ there exists a unique element $\int h\,d\mu$ (also denoted $\int h(x)\,d\mu(x)$) of E such that

$$\varphi\left(\int h\,d\mu\right) = \int \varphi \circ h\, d\mu \qquad (\varphi \in E^*).$$

In the notation of this theorem we have

$$\left|\varphi\left(\int h\,d\mu\right)\right| \leq \int |\varphi(h(x))|\,d|\mu|(x) \leq \int \|\varphi\|\,\|h(x)\|\,d|\mu|(x).$$

Therefore,

$$\left\|\int h\,d\mu\right\| \leq \int \|h(x)\|\,d|\mu|(x).$$

The following lemma is easy.

9.2. LEMMA. Let X be a locally compact space; let E and F be Banach spaces. If $T : E \longrightarrow F$ is linear and continuous, then

$$T\left(\int h\,d\mu\right) = \int T \circ h\, d\mu$$

for every $\mu \in M(X)$ and every bounded continuous $h : X \longrightarrow E$.

Example (compare Theorem 4.4). Let $\mu \in M(G)$, $g \in L^1(G)$. For $j \in C_\infty(G)$ define $\varphi_j \in L^1(G)^*$ by $\varphi_j(f) := \int fj\,dm$. For each j we have

$$\varphi_j(\mu * g) = (\mu * g)(j) = \iint j(xy)g(y)\,dy\,d\mu(x) = \iint j(y)g(x^{-1}y)\,dy\,d\mu(x) =$$
$$= \int \varphi_j(g_x)\,d\mu(x).$$

Now $x \longmapsto g_x$ is continuous (Theorem 3.13), so $\int g_x\,d\mu(x)$ exists and

$$\varphi_j(\mu * g) = \varphi_j\left(\int g_x\,d\mu(x)\right) \qquad (j \in C_\infty(X)).$$

The natural maps $L^1(G) \longrightarrow M(G) \longrightarrow C_\infty(G)^*$ are injective. Hence,

9.3. THEOREM. For $\mu \in M(G)$ and $g \in L^1(G)$, $\mu * g = \int g_x\,d\mu(x)$.

In a notation whose meaning is obvious, we have

$$f*g = \int f(x)g_x dx \qquad (f,g \in L^1(G)).$$

Let E be a Banach space. A <u>representation</u> of G in E is a map assigning to every $x \in G$ a surjective linear isometry $U_x : E \longrightarrow E$ in such a way that $U_x U_y = U_{xy}$ $(x,y \in G)$. Then $U_1 = I$. (If E is a Hilbert space, U_x is unitary and $U_x^* = U_x^{-1} = U_{x^{-1}}$.)

<u>Example</u>. In $C_\infty(G)$, $L^1(G)$ and $L^2(G)$ we have the <u>regular</u> representation U given by

$$U_x f = f_x \qquad (x \in G;\ f \in C_\infty(G),\ L^1(G) \text{ or } L^2(G)).$$

<u>Example</u>. Every homomorphism $\gamma : G \longrightarrow \mathbf{T}$ induces a representation U of G in $\mathbf{C}$ by

$$U_x \zeta := \gamma(x)\zeta \qquad (x \in G;\ \zeta \in \mathbf{C}).$$

Every representation of G in $\mathbf{C}$ can be obtained in this way.

In general, instead of $U_x \zeta$ we shall often write ζ_x. Notice that then

$$(\zeta_x)_y = \zeta_{yx} \qquad (x,y \in G;\ \zeta \in E).$$

A representation U in E is <u>continuous</u> if $x \longmapsto \zeta_x$ is continuous for every $\zeta \in E$. The regular representations in $C_\infty(G)$, $L^1(G)$ and $L^2(G)$ are continuous (Corollary 3.4, Theorem 3.13).

A representation U in E induces a representation U^* in E^* by

$$(U^*)_x := (U_{x^{-1}})^* \qquad (x \in G).$$

Then

$$\varphi_x(\zeta_x) = \varphi(\zeta) \qquad (\zeta \in E;\ \varphi \in E^*;\ x \in G).$$

Thus, the regular representation in $C_\infty(G)$ induces a representation in $M(G)$:

$$\mu_x(j) = \int j(xy)\,d\mu(y) \qquad (\mu \in M(G);\ j \in C_\infty(G);\ x \in G),$$

i.e.

$$\mu_x = \delta_x * \mu \qquad (\mu \in M(G);\ x \in G).$$

(See Exercise 4.C.) Then

$$(fm)_x = f_x m \qquad (f \in L^1(G);\ x \in G).$$

This representation is not continuous unless G is discrete. ($x \longmapsto \delta_x$ is not continuous!)

Viewing μ as a measure we have

$$\mu_x(A) = \mu(x^{-1}A) \qquad (A \subset G \text{ Borel}).$$

Let there be given a continuous representation of G in a Banach space E. For $\mu \in M(G)$ and $\zeta \in E$ the integral

$$T_\mu(\zeta) := \int \zeta_x \, d\mu(x)$$

exists. In this way every $\mu \in M(G)$ induces a continuous linear map $T_\mu : E \longrightarrow E$ for which $\|T_\mu\| \leq \|\mu\|$.

Let $\mu, \nu \in M(G)$. For all $\zeta \in E$ and $\varphi \in E^*$ the function $x \longmapsto \varphi(\zeta_x)$ is bounded and continuous. It follows from the Fubini Theorem that (see also Exercise 4.A)

$$\int \varphi(\zeta_z)\, d(\mu * \nu)(z) = \iint \varphi(\zeta_{xy})\, d\mu(x)\, d\nu(y) = \int \varphi\Big(\int (\zeta_y)_x \, d\mu(x)\Big)\, d\nu(y) =$$
$$= \int \varphi(T_\mu(\zeta_y))\, d\nu(y) = \varphi \circ T_\mu\Big(\int \zeta_y \, d\nu(y)\Big) = \varphi(T_\mu T_\nu \zeta).$$

Thus $T_{\mu * \nu} = T_\mu T_\nu$.

In case $E = L^1(G)$ (and the representation is the regular one), $T_\mu(f) = \mu * f$.

Let A be a Banach algebra. A <u>Banach</u> <u>module</u> over A is a Banach space E together with a map $(a, \zeta) \longmapsto a\zeta$ of $A \times E$ into E such that

(1) $(ab)\zeta = a(b\zeta) \qquad (a, b \in A;\ \zeta \in E)$;

(2) $a \longmapsto a\zeta$ and $\zeta \longmapsto a\zeta$ are linear maps $\quad (\zeta \in E;\ a \in A)$;

(3) $\|a\zeta\| \leq \|a\|\,\|\zeta\| \qquad (a \in A;\ \zeta \in E)$.

Every Banach algebra is a Banach module over itself. More generally, a closed left ideal of a Banach algebra A is a Banach module over A. In particular, $M(G)$, $M_a(G)$ and $L^1(G)$ are Banach modules over $M(G)$. Every Banach module module over $M(G)$ is also a Banach module over $M_a(G)$.

For Banach modules over $L^1(G)$ or $M(G)$ we useally write $f * \zeta$ instead of $f\zeta$ and $\mu * \zeta$ instead of $\mu\zeta$. Observe that, by the isomorphism of $L^1(G)$ and $M_a(G)$, every Banach module over $M(G)$ naturally is also a Banach module over $L^1(G)$.

The computations in the middle of the last page lead to

9.4. THEOREM. *Let* E *be a Banach space with a continuous representation of* G. *The formula*

$$\mu*\zeta := \int \zeta_x d\mu(x) \qquad (\mu \in M(G);\ \zeta \in E)$$

renders E *a Banach module over* M(G). E *is a Banach module over* $L^1(G)$ *by*

$$f*\zeta = \int f(x)\zeta_x dx \qquad (f \in L^1(G);\ \zeta \in E).$$

The representation can be obtained from the module structure:

$$\zeta_x = \delta_x*\zeta \qquad (x \in G;\ \zeta \in E).$$

9.A. EXERCISE. $C_\infty(G)$ *is a Banach module over* M(G) *by*

(1) $$(\mu*h)(a) := \int h(x^{-1}a)d\mu(x) \qquad (\mu \in M(G);\ h \in C_\infty(G)).$$

If $\mu,\nu \in M(G)$ *and* $h \in C_\infty(G)$, *then*

(2) $$(\mu*\nu)(h) = \nu(\mu'*h).$$

$L^2(G)$ *is a Banach module over* M(G) *by*

(3) $$(\mu*g)(a) := \int g(x^{-1}a)d\mu(x),\ \text{almost every } a \in G \qquad (\mu \in M(G);\ g \in L^2(G)).$$

If E is a Banach module over M(G), then E^* becomes a Banach module over M(G) by

$$(\mu*\varphi)(\zeta) := \varphi(\mu'*\zeta) \qquad (\mu \in M(G);\ \varphi \in E^*;\ \zeta \in E).$$

As a special case we have formula (2) of Exercise 9.A.

In every Banach module E over M(G) we automatically have a representation U of G:

$$U_x\zeta := \delta_x*\zeta.$$

This representation may not be continuous. However,

9.5. THEOREM. *Let* E *be a Banach module over* $L^1(G)$. *Let* E_a *be the closure of the linear span of* $\{f*\zeta : f \in L^1(G);\ \zeta \in E\}$. *If* $(e_\lambda)_{\lambda\in\Lambda}$ *is a bounded left approximate identity in* $L^1(G)$, *then*

$$E_a = \{\zeta \in E : \lim e_\lambda*\zeta = \zeta\}.$$

<u>E_a is a closed submodule of E. There exists a unique representation of G in E_a for which</u>

$$(f*\zeta)_x = f_x*\zeta \qquad (f\in L^1(G);\ \zeta\in E).$$

<u>This representation is continuous. We have</u>

$$f*\zeta = \int f(x)\zeta_x dx \qquad (f\in L^1(G);\ \zeta\in E_a).$$

<u>Thus, the module structure of E_a induced by this representation is just the structure E_a obtains as a submodule of E.</u>

<u>If E is a Banach module over $M(G)$, then so is E_a, and</u>

$$\mu*\zeta = \int \zeta_x d\mu(x) \qquad (\mu\in M(G);\ \zeta\in E_a).$$

(Observe that $M(G)_a$ is just $M_a(G)$.)

Proof. Let $(e_\lambda)_{\lambda\in\Lambda}$ be a bounded left approximate identity in $L^1(G)$, $c := \sup_\lambda \|e_\lambda\|$. Obviously, $E_a \supset \{\zeta : \lim e_\lambda*\zeta = \zeta\}$. On the other hand, if $\zeta\in E_a$ and $\varepsilon>0$, there exist $\zeta_1,\ldots,\zeta_n\in E$ and $f_1,\ldots,f_n\in L^1(G)$ such that $\|\zeta-\sum f_i*\zeta_i\| \leq \frac{\varepsilon}{2(c+1)}$: there exists a $\lambda_0\in\Lambda$ such that $\|e_\lambda*f_i-f_i\|\|\zeta_i\| \leq \frac{\varepsilon}{2n}$ $(i=1,\ldots,n)$ for all $\lambda\geq\lambda_0$. For these λ we then have $\|e_\lambda*\zeta-\zeta\| \leq \|e_\lambda*(\zeta-\sum f_i*\zeta_i)\| + \sum\|(e_\lambda*f_i-f_i)*\zeta_i\| + \|\sum f_i*\zeta_i-\zeta\| \leq$
$\leq c\frac{\varepsilon}{2(c+1)} + n\frac{\varepsilon}{2n} + \frac{\varepsilon}{2(c+1)} = \varepsilon$.

Thus, $E_a = \{\zeta : \lim e_\lambda*\zeta=\zeta\}$. In particular, it follows that E_a is the closure of $L^1(G)*E := \{f*\zeta : f\in L^1(G);\ \zeta\in E\}$.

Obviously, E_a is a closed linear subspace of E. It is a module over $L^1(G)$ because it contains $L^1(G)*E$.

In the rest of this proof, $(e_\lambda)_{\lambda\in\Lambda}$ is a left approximate identity of $L^1(G)$ with $\|e_\lambda\|\leq 1$ for all λ.

Let $x\in G$. If $f\in L^1(G)$ and $\zeta\in E$, then $f_x*\zeta=\lim(e_\lambda*f)_x*\zeta=$
$=\lim(\delta_x*e_\lambda*f)*\zeta=\lim((e_\lambda)_x*f)*\zeta=\lim(e_\lambda)_x*(f*\zeta)$. Thus, $\lim(e_\lambda)_x*\eta$ exists for all $\eta\in L^1(G)*E$ and consequently, for all η in the linear hull $[\![L^1(G)*E]\!]$ of $L^1(G)*E$. As $\|(e_\lambda)_x\|=$
$=\|e_\lambda\|\leq 1$, $\eta\longmapsto\lim(e_\lambda)_x*\eta$ is a continuous linear map

$[\![L^1(G)*E]\!] \longrightarrow E_a$ whose norm is ≤ 1. By continuity this map extends to a linear $U_x : E_a \longrightarrow E_a$ whose norm is ≤ 1. As we have already seen,

$$U_x(f*\zeta) = f_x*\zeta \qquad (f\in L^1(G);\ \zeta\in E).$$

For every $x\in G$ we have such a U_x. We have $U_1=I$, and, for all $x,y\in G$, $U_xU_y=U_{xy}$ on $L^1(G)*E$, hence $U_xU_y=U_{xy}$ on all of E_a. In particular, $U_xU_{x^{-1}}=U_{x^{-1}}U_x=I$. As $\|U_x\|\leq 1$ and $\|U_{x^{-1}}\|\leq 1$ it follows that U_x is a (surjective) isometry. Thus, U is a representation of G in E_a.

For all $f\in L^1(G)$ and $\zeta\in E$, $x\longmapsto f_x*\zeta$ is continuous. It follows easily that the representation U is continuous. Furthermore, if $f\in L^1(G)$, then for all $g\in L^1(G)$ and $\eta\in E$ we obtain

$$\int f(x)U_x(g*\eta)dx=\int f(x)g_x*\eta\, dx=(\int f(x)g_x dx)*\eta =$$
$$=(f*g)*\eta=f*(g*\eta).$$

Hence, $\int f(x)U_x\zeta dx= f*\zeta$ for $\zeta\in L^1(G)*E$. By continuity the formula is valid for all $\zeta\in E_a$.

The proof of the rest of the theorem presents no difficulty.

If $\zeta\in E_a$ then $\zeta=\lim e_\lambda*\zeta$, so $\zeta\in(E_a)_a$ by the definition of E_a. Therefore, $(E_a)_a=E_a$.

E_a is called the <u>(absolutely)</u> <u>continuous</u> or <u>essential</u> part of E. We call E <u>continuous</u> if $E=E_a$. For any Banach module E over $L^1(G)$ or $M(G)$, E_a is continuous.

9.6. THEOREM. <u>Let E be a Banach space with a continuous representation U of G. Make E into a module over $L^1(G)$ and $M(G)$ as was done in Theorem 9.4. Then E is continuous, and the representation of G that is induced by the module structure (see Theorem 9.5) is just U. (Thus, $C_\infty(G)$, $L^1(G)$ and $L^2(G)$ are continuous modules.)</u>

<u>An $S\in\mathcal{L}(E)$ commutes with every U_a $(a\in G)$ if and only if S is an $L^1(G)$-module homomorphism, and if and only if S is an</u>

M(G)-*module homomorphism*.

For a closed linear subspace D *of* E *the following conditions are equivalent*.

(a) $U_a(D) \subset D$ *for all* $a \in G$.

(b) D *is an* $L^1(G)$-*submodule of* E.

(c) D *is an* M(G)-*submodule of* E.

We leave the proof to the reader.

The situation becomes particularly simple if we restrict ourselves to Hilbert spaces. A *Hilbert module* over a Banach algebra A is a Banach module that is a Hilbert space.

9.7. THEOREM. *Let* H *be a Hilbert module over* $L^1(G)$. *Then*

$$(f*\zeta|\eta) = (\zeta|\tilde{f}*\eta) \qquad (f \in L^1(G);\ \zeta,\eta \in H).$$

The spaces H_a *and* $\{\zeta : L^1(G)*\zeta=\{0\}\}$ *are each other's orthocomplements*.

Let $(e_\lambda)_{\lambda\in\Lambda}$ *be a bounded left approximate identity in* $L^1(G)$. *For every* $\zeta \in H$,

$$P\zeta := \lim e_\lambda * \zeta$$

exists in H. P *is the projection of* H *onto* H_a. *We have*

$$P(f*\zeta)=f*P\zeta=f*\zeta \qquad (f \in L^1(G);\ \zeta \in H).$$

Proof. Instead of $L^1(G)$ we write L. Let $H_0 := \{\zeta : L*\zeta = \{0\}\}$. Like H_a, H_0 is a closed linear subspace of H.

Let $\zeta \in H_a$. By Theorem 4.2 we may assume that 1 is a bound for some left approximate identity $(u_\lambda)_{\lambda\in\Lambda}$ of L. If ζ_0 is the projection of ζ in H_0, then $\zeta_0 \perp \zeta-\zeta_0$, so $\|\zeta\|^2=\|\zeta_0\|^2+\|\zeta-\zeta_0\|^2$. For each λ, $\|\zeta-\zeta_0\| \geq \|u_\lambda*(\zeta-\zeta_0)\|=\|u_\lambda*\zeta\|$. But by Theorem 9.5, $\lim u_\lambda*\zeta=\zeta$, so $\|\zeta-\zeta_0\| \geq \|\zeta\|$. It follows that $\zeta_0=0$:

$$(1^*) \qquad H_0 \perp H_a.$$

For all $f \in L$ and $\zeta \in H$, $\eta \longmapsto (\widetilde{f}*\eta|\zeta)$ is a continuous linear function on H, so there exists a unique $f \square \zeta \in H$ with

(2) $$(\widetilde{f}*\eta|\zeta) = (\eta|f \square \zeta)$$

for all η. Under $\square$, H is a Hilbert module over L. In analogy with (1^*) we have

($1^\square$) $$\text{if } L\square\zeta = \{0\}, \text{ then } \zeta \perp L\square H.$$

Now take $\zeta \in H$, $\zeta \perp H_a$. For all $f \in L$ and $\eta \in H$ the left hand member of (2) vanishes. Hence $L\square\zeta = \{0\}$, so $\zeta \perp L\square H$. But then for all $f \in L$ and $\eta \in H$ we have $(f*\zeta|\eta) = (\zeta|\widetilde{f}\square\eta) = 0$, so $L*\zeta = \{0\}$. Thus,

(3) $$\text{if } \zeta \perp H_a, \text{ then } \zeta \in H_0.$$

By (1^*) and (3), H_a and H_0 are each other's orthocomplements.

Every $\zeta \in H$ can in a unique way be written as $\zeta_a + \zeta_0$ with $\zeta_a \in H_a$, $\zeta_0 \in H_0$. Since $\zeta_a = \lim e_\lambda * \zeta_a$ and $e_\lambda * \zeta_0 = 0$ for every λ, we have $\zeta_a = \lim e_\lambda * \zeta$. Thus, for all $\zeta \in H$, $P\zeta := \lim e_\lambda * \zeta$ exists and P is the projection onto H_a.

Let $f \in L$, $\zeta, \eta \in H$. It remains to prove

(4) $$(f*\zeta|\eta) = (\zeta|\widetilde{f}*\eta).$$

If $\zeta \in H_0$, then $\zeta \perp L*\eta$; so both members of (4) vanish: we may assume $\zeta \in H_a$. Similarly we may assume $\eta \in H_a$. If U is the representation of G in H_a induced by the module structure (Theorem 9.5), then each U_x is a linear isometry of H_a onto itself and is therefore unitary. It follows that $(U_x\zeta|\eta) = (\zeta|U_{x^{-1}}\eta)$, or $(\zeta_x|\eta) = (\zeta|\eta_{x^{-1}})$. Thus, $(f*\zeta|\eta) = \int f(x)(\zeta_x|\eta)dx = \int f(x)(\zeta|\eta_{x^{-1}})dx = \int f(x)\overline{(\eta_{x^{-1}}|\zeta)}dx = \int \widetilde{f}(x)(\eta_x|\zeta)dx = (\widetilde{f}*\eta|\zeta) = (\zeta|\widetilde{f}*\eta)$.

A simple consequence of the above theorem:

9.8. COROLLARY. If D is a closed submodule of a Hilbert module H over $L^1(G)$, then so is $D^\perp$. The projection P of H onto D is a module homomorphism: $P(f*\zeta) = f*P\zeta$ ($f \in L^1(G)$; $\zeta \in H$).

Proof. If $f \in L^1(G)$ and $\zeta \in D^\perp$, then $f*\zeta \in D^\perp$ since $(f*\zeta | \eta) = (\zeta | \tilde{f}*\eta) = 0$ for all $\eta \in D$; so $D^\perp$ is a module. For For $f \in L^1(G)$ and $\zeta \in H$ we have $f*P\zeta \in f*D \subset D$, $f*(I-P)\zeta \in f*D^\perp \subset D^\perp$, and $f*\zeta = f*P\zeta + f*(I-P)\zeta$: so $P(f*\zeta) = f*P\zeta$.

For a Banach space E denote by $\mathcal{L}(E)$ the Banach algebra of all continuous linear maps $E \longrightarrow E$. A <u>representation</u> of a Banach algebra A in E is an algebra homomorphism $T : a \longrightarrow T_a$ of A into $\mathcal{L}(E)$ for which $\|T_a\| \leq \|a\|$ $(a \in A)$. Such a representation turns E into a Banach module over A by

$$a\zeta := T_a\zeta \qquad (a \in A;\ \zeta \in E).$$

Conversely, every module structure on E corresponds to such a representation. There is a one-to-one correspondence between the representations T of A in $\mathbb{C}$ and the algebra homomorphisms $\varphi : A \longrightarrow \mathbb{C}$ given by the formula

$$T_a\zeta = \varphi(a)\zeta \qquad (a \in A;\ \zeta \in \mathbb{C}).$$

A representation T of $L^1(G)$ or $M(G)$ in H is a <u>*representation</u> if $T_{\tilde{f}} = (T_f)^*$ $(f \in L^1(G))$ or $T_{\tilde{\mu}} = (T_\mu)^*$ $(\mu \in M(G))$, respectively. The first part of Theorem 9.7 says that <u>every representation of $L^1(G)$ in a Hilbert space is a *representation</u>.

We conclude this section by showing that the same is not true for representations of $M(G)$, even if G is extremely decent.

9.9. LEMMA. <u>There exists a</u> $\mu \in M(\mathbb{R})$ <u>such that</u> $\mu \geq 0$, $\|\mu\| = 1$, $\tilde{\mu} = \mu$, μ <u>is continuous</u> (<u>i.e.</u> $\mu(\{a\}) = 0$ <u>for every</u> $a \in \mathbb{R}$) <u>and such that</u> $\delta_0, \mu, \mu*\mu, \mu*\mu*\mu, \ldots$ <u>are mutually singular</u>.

Before proving this lemma we mention some its consequences. By $\mathfrak{G}$ we denote the Gelfand Transformation $M(\mathbb{R}) \longrightarrow C(\mathfrak{M}(M(\mathbb{R})))$. We view $\hat{\mathbb{R}}$ as a subset of $\mathfrak{M}(M(\mathbb{R}))$, (see Exercise 6.B,) so that $\hat{\mu}$ is the restriction of $\mathfrak{G}\mu$ to $\hat{\mathbb{R}}$. By the Uniqueness Theorem 6.4, if $\mathfrak{G}\mu = \mathfrak{G}\nu$ on $\hat{\mathbb{R}}$ then $\mathfrak{G}\mu = \mathfrak{G}\nu$ everywhere.

9.10. THEOREM. <u>There exists a</u> $\mu \in M(\mathbb{R})$ <u>for which</u> $\mathfrak{G}\tilde{\mu} \neq \overline{\mathfrak{G}\mu}$

Proof. Let μ be as in Lemma 9.9; set $\pi := \delta_0 - \mu*\mu$

Define $\mu^{(0)}, \mu^{(1)},\ldots$ by

$$\mu^{(0)} := \delta_0, \quad \mu^{(n)} := \mu * \mu^{(n-1)} \qquad (n \in \mathbb{N})$$

and define $\pi^{(0)}, \pi^{(1)},\ldots$ in a similar way. For every $n \in \mathbb{N}$, $\pi^{(n)} = \sum_{p=0}^{n} \binom{n}{p}(-1)^p \mu^{(2p)}$. As $\mu^{(0)}, \mu^{(2)}, \mu^{(4)},\ldots$ are mutually singular, $\|\pi^{(n)}\| = \sum_p \binom{n}{p}\|\mu^{(2p)}\| = \sum\binom{n}{p} = 2^n$. By the Spectral Radius Formula 2.7, $\|\zeta\pi\|_\infty = \lim \sqrt[n]{\|\pi^{(n)}\|} = 2$. The spectrum of π is compact, so there must exist a $\varphi \in \mathfrak{M}(M(\mathbb{R}))$ with $|\varphi(\pi)|=2$. Now $\varphi(\pi)=\varphi(\delta_0 - \mu*\mu) = 1-\varphi(\mu)^2$ and $|\varphi(\mu)| \leq \|\mu\|=1$. Consequently, $\varphi(\mu)^2=-1$ and $\varphi(\mu)=\pm i$. As $\mu=\tilde{\mu}$, certainly $\varphi(\tilde{\mu}) \neq \overline{\varphi(\mu)}$.

9.11. COROLLARY. $\hat{\mathbb{R}}$ <u>is not dense in</u> $\mathfrak{M}(M(\mathbb{R}))$.

Proof. Let μ, φ be as above. The restriction of $\zeta\mu$ to $\hat{\mathbb{R}}$ is $\hat{\mu}$, which is real-valued since $\mu=\tilde{\mu}$. But $(\zeta\mu)(\varphi) \notin \mathbb{R}$. Therefore φ does not lie in the closure of $\hat{\mathbb{R}}$ in $\mathfrak{M}(M(\mathbb{R}))$.

9.12. COROLLARY. <u>There exist representations of</u> $M(\mathbb{R})$ <u>in</u> (one-dimensional) <u>Hilbert spaces that are not</u> <u>*representations</u>.

Proof. Let μ and φ be as in Theorem 9.10 and its proof. The formula

$$T_\nu \varphi := \varphi(\nu)\zeta \qquad (\nu \in M(\mathbb{R});\ \zeta \in \mathbb{C})$$

defines a representation T of $M(\mathbb{R})$ in $\mathbb{C}$, and $T_{\tilde{\mu}} = T_\mu \neq T_\mu^*$.

9.13. COROLLARY (WIENER-PITT). <u>There exists a function</u> j <u>on</u> $\mathbb{R}$ <u>with</u> $j \geq 1$ <u>such that</u> j <u>is, but</u> $\frac{1}{j}$ <u>is not a Fourier-Stieltjes transform</u>.

NOTE. This result should be viewed in contrast with Exercise 2.H.

Proof. Let again μ, φ be as in Theorem 9.10 and its proof and set $\nu := \delta + \mu*\mu$, $j := \hat{\nu}$. Then $j = 1+(\mu*\tilde{\mu})\hat{} = 1+\hat{\mu}\bar{\hat{\mu}} \geq 1$. If $\pi \in M(\mathbb{R})$ and $\hat{\pi} = \frac{1}{j}$, then $(\pi*\nu)\hat{} = \hat{\delta}_0$, so $\pi*\nu = \delta_0$ (Uniqueness Theorem) and $(\zeta\pi)(\zeta\nu)=1$. But $(\zeta\nu)(\varphi)=0$.

9.B. EXERCISE. <u>The Fourier-Stieltjes Transformation $M(\mathbf{R}) \longrightarrow C_u(\mathbf{R})$ is not surjective.</u> (See also Exercise 7.F.)

The proof of Lemma 9.9 is somewhat involved. We start off with a lemma.

9.14. LEMMA. <u>A finite sequence $x_1,\dots,x_n$ of real numbers is linearly independent over $\mathbf{Q}$ if and only if the set $\{(\gamma(x_1),\dots,\gamma(x_n)) : \gamma\in\hat{\mathbf{R}}\}$ is dense in $\mathbf{T}^n$.</u>

Proof. Set $\Phi(\gamma) := (\gamma(x_1),\dots,\gamma(x_n))$ $(\gamma\in\hat{\mathbf{R}})$ and $\Phi(\hat{\mathbf{R}}) := \{\Phi(\gamma) : \gamma\in\hat{\mathbf{R}}\}$. $\Phi(\hat{\mathbf{R}})$ and its closure are subgroups of $\mathbf{T}^n$. The following propositions are equivalent. (For (1) $\Rightarrow$ (2) and (4) $\Rightarrow$ (5), apply Corollary 7.2; (2) and (3) are equivalent by Exercise 5.B.)

(1) $\Phi(\hat{\mathbf{R}})$ is not dense in $\mathbf{T}^n$.

(2) There exists a $\beta\in\widehat{\mathbf{T}^n}$, $\beta\neq 1$, $\Phi(\hat{\mathbf{R}})\subset\operatorname{Ker}\beta$.

(3) There exist $p_1,\dots,p_n\in\mathbf{Z}$ such that $\prod_j \gamma(x_j)^{p_j} = 1$ for all $\gamma\in\hat{\mathbf{R}}$ while not every p_j is 0.

(4) There exist $p_1,\dots,p_n\in\mathbf{Z}$ such that $\gamma(\sum p_jx_j)=1$ for all $\gamma\in\hat{\mathbf{R}}$ while not every p_j is 0.

(5) There exist $p_1,\dots,p_n\in\mathbf{Z}$ such that $\sum p_jx_j=0$ while not every p_j is 0.

(6) $x_1,\dots,x_n$ are not linearly independent over $\mathbf{Q}$.

Now we come to the proof of Lemma 9.9.

We first show that for every n $\{0,1,2,\dots\}$ we can choose closed intervals $S_1^n,S_2^n,\dots,S_{2^n}^n$ in $\mathbf{R}$ and a finite $\Gamma_n\subset\hat{\mathbf{R}}$ with the following properties:

(a) <u>$S_1^n,\dots,S_{2^n}^n$ are pairwise disjoint.</u>

(b) <u>Each S_j^n has length >0 but $\leq 3^{-n}$.</u>

(c) <u>If $n\neq 0$, then $S_{2j-1}^n\subset S_j^{n-1}$ and $S_{2j}^n\subset S_j^{n-1}$ for $j=1,\dots,2^{n-1}$.</u>

(d) <u>If $n\neq 0$ and $\alpha_1,\dots,\alpha_{2^n}\in\mathbf{T}$ there exists a $\gamma\in\Gamma_n$ such that $|\gamma-\alpha_j|<\frac{1}{n}$ on S_j^n $(j=1,\dots,2^n)$.</u>

For n=0 we take $S_1^0 := [0,1]$, $\Gamma_0 = \emptyset$. Now let $N \in \mathbb{N}$ and suppose that for every $m<N$ we have S_j^m and Γ_m such that (a)-(d) are true for $n=0,1,\dots,N-1$. In each S_j^{N-1} choose x_{2j-1} and x_{2j} in such a way that $x_1,\dots,x_{2^N}$ are linearly independent over Q. As $\{(\gamma(x_1),\dots,\gamma(x_{2^N})) : \gamma \in \hat{\mathbb{R}}\}$ is a dense subset of of the compact group $\mathbb{T}^{2^N}$ there exists a finite set $\Gamma_N \subset \hat{\mathbb{R}}$ such that for every $(\alpha_1,\dots,\alpha_{2^N}) \in \mathbb{T}^{2^N}$ there is a $\gamma \in \hat{\mathbb{R}}$ with

(*) $$\max_j |\alpha_j - \gamma(x_j)| < \frac{1}{2N}.$$

Next, for $j=1,\dots,2^N$ we can choose a closed interval S_j^N containing x_j in such a way that (a)-(c) hold for n=N while

$$S_j^N \subset \{x \in \mathbb{R} : |\gamma(x)-\gamma(x_j)| < \frac{1}{2N} \text{ for each } \gamma \in \Gamma_N\}.$$

If now $(\alpha_1,\dots,\alpha_{2^N}) \in \mathbb{T}^{2^N}$ and if $\gamma \in \Gamma_N$ satisfies (*), then $|\gamma - \alpha_j| < \frac{1}{N}$ on S_j^N $(j=1,\dots,2^N)$. Thus, $S_1^N,\dots,S_{2^N}^N$ and Γ_N fulfil (d) for n=N.

We see that we can make our closed intervals S_j^n such that (a)-(d) are true for every n. Set $P_n := S_1^n \cup \dots \cup S_{2^n}^n$, then P_n is compact and $P_1 \supset P_2 \supset \dots$. Hence, $P := \bigcap P_n$ is compact.

We next prove P to be independent over $\mathbb{Q}$. Let $a_1,\dots,a_m$ be distinct points of P. Let $\alpha_1,\dots,\alpha_m \in \mathbb{T}$, $\varepsilon>0$; we make a $\gamma \in \mathbb{R}$ such that $|\gamma(a_j)-\alpha_j| < \varepsilon$ for each j. (The independence of $a_1,\dots,a_m$ then follows by Lemma 9.14.) Take $n \in \mathbb{N}$ so that $n > \frac{1}{\varepsilon}$ and $|a_i - a_{i'}| > 3^{-n}$ for all i,i' with $i \neq i'$. For $i=1,\dots,m$ let $j(i) \in \{1,\dots,2^n\}$ be so that $a_i \in S_{j(i)}^n$. As the length of each S_j^n is $\leq 3^{-n}$ we have $j(i) \neq j(i')$ as soon as $i \neq i'$. By property (d) there is a $\gamma \in \mathbb{R}$ such that $|\gamma - \alpha_i| < \frac{1}{n}$ on $S_{j(i)}^n$ $(i=1,\dots,m)$. Then surely $|\gamma(a_i)-\alpha_i| < \varepsilon$ for each i.

For $n \in \mathbb{N}$ and $j=1,\dots,2^N$ let f_j^n be the characteristic function of $S_j^n \cap P$, restricted to P. Then $f_j^n \in C(P)$ and

$f_j^n = f_{2j-1}^{n+1} + f_{2j}^{n+1}$. There exists a linear function J defined on the linear hull of $\{f_j^n : n \in \mathbb{N};\ j=1,\dots,2^n\}$ for which

$$J\Big(\sum_{j=1}^{2^n} \lambda_j f_j^n\Big) = \sum \lambda_j 2^{-n} \qquad (n \in \mathbb{N};\ \lambda_1,\dots,\lambda_{2^n} \in \mathbb{C}).$$

J can be extended to a continuous linear function on $C(P)$ and induces a Borel measure μ_1 on P with

$$\mu_1(S_j^n \cap P) = 2^{-n} \qquad (n \in \mathbb{N};\ j=1,\dots,2^n).$$

We have $\mu_1 \geq 0$, $\mu_1(P)=1$, and $\mu_1(\{a\})=0$ for every a. This μ_1 determines a $\mu_2 \in M(\mathbb{R})$ by

$$\mu_2(X) := \mu_1(X \cap P) \qquad (X \subset \mathbb{R} \text{ Borel}),$$

and finally we make $\mu \in M(\mathbb{R})$:

$$\mu := \tfrac{1}{2}(\mu_2 + \tilde{\mu}_2).$$

The crucial properties of μ are the following:

$$\mu \in M(\mathbb{R}),\ \ \mu \geq 0,\ \mu(\mathbb{R}) = 1,\ \ \mu = \tilde{\mu}\ ,\ \mu(\{x\}) = 0 \quad (x \in \mathbb{R})$$

and

$$(\mathbb{R} \setminus Q) = 0$$

where Q is the compact set $P \cup -P$.

Let $n,m \in \{0,1,\dots\}$, $n<m$. We want to prove that $\mu^{(n)}$ and $\mu^{(m)}$ are mutually singular. (For $\mu^{(n)},\mu^{(m)}$, see the proof of Theorem 9.10.) Define the compact sets $Q_0,Q_1,Q_2\dots$ by

$$Q_0 := \{0\},\ \ Q_1 := Q,\ \ \ Q_{n+1} := Q_n + Q \quad (n \in \mathbb{N}).$$

According to Exercise 4.B (b), $\mu^{(n)}(\mathbb{R} \setminus Q_n) = 0$. We are done if we can prove $\mu^{(m)}(Q_n) = 0$.

Set $A := \{(x_1,\dots,x_m) \in Q^m : x_1+\dots+x_m \in Q_n\}$. As $\mu = \xi_Q \mu$, from Exercise 4.B (a) we obtain

$$
\begin{aligned}
(1)\qquad \mu^{(m)}(Q_n) &= \int\dots\int \xi_{Q_n}(x_1+\dots+x_m)\,d\mu(x_1)\dots d\mu(x_m) = \\
&= \int\dots\int \xi_Q(x_1)\dots\xi_Q(x_m)\,\xi_{Q_n}(x_1+\dots+x_m)\,d\mu(x_1)\dots d\mu(x_m) = \\
&= \int\dots\int \xi_A(x_1,\dots,x_m)\,d\mu(x_1)\dots d\mu(x_m).
\end{aligned}
$$

Let $(x_1,\dots,x_m) \in A$. Set $I := \{i : x_i \in P\}$, $I' := \{j : -x_j \in P\}$; then $I \cap I' = \emptyset$ and $I \cup I' = \{1,\dots,m\}$. As $x_1+\dots+x_m \in Q_n$ there exist $y_1,\dots,y_s,z_1,\dots,z_t \in P$ such that $s+t=n$ and

$x_1+\ldots+x_m = y_1+\ldots+y_s-z_1-\ldots-z_t$, i.e.

$$\sum_{i\in I} x_i + \sum_{k=1}^{t} z_k = \sum_{j\in I'} (-x_j) + \sum_{l=1}^{s} y_l.$$

The linear independence of P and the fact that $s+t<m$ imply that there must exist an $i\in I$ and a $j\in I'$ such that $x_i=-x_j$.

Thus,

$$A \subset \bigcup_{i<j} \{(x_1,\ldots,x_m) \ : \ x_i+x_j=0\},$$

so that

(2) $$\xi_A(x_1,\ldots,x_m) \le \sum_{\substack{i,j \text{ so} \\ \text{that } i<j}} \xi_{\{-x_j\}}(x_i)$$

But if $i<j$ then

$$\int\ldots\int \xi_{\{-x_j\}}(x_i)\,d\mu(x_1)\ldots d\mu(x_n) =$$
$$= \int\ldots\int \mu(\{-x_j\})\,d\mu(x_{i+1})\ldots d\mu(x_n) = 0$$

By (1), (2) and (3), $\mu^{(m)}(Q_n)=0$.