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HARMONIC ANALYSIS I

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1972

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LITTERATURE.

E. Hewitt-K. A. Ross, "Abstract Harmonic Analysis I"

W. Rudin, "Fourier Analysis on Groups"

H. Reiter, "Classical Harmonic Analysis and Locally Compact Groups"

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CONTENT

I. STARTING POINTS.

CONVENTIONS.

"(locally) compact" implies "Hausdorff".

If X is a set and $A \subset X$, then ξ_A denotes the characteristic function of A:

$$\xi_A(x) := 1 \quad \text{if } x \in A,$$
$$:= 0 \quad \text{if } x \notin A.$$

The complement of A is $X \setminus A$ or A^c.

Usually our scalar field is $\mathbb{C}$.

For $f: X \to \mathbb{C}$ set $\|f\|_X := \|f\|_\infty := \sup_{x \in X} |f(x)|$.

If X is a topological space, $C(X)$ denotes the vector space of all bounded continuous functions $X \to \mathbb{C}$.

If E is a space of functions on a set X, by E^r we denote the set of all real-valued elements of E, and $E^+ := \{f \in E^r : f \geqq 0\}$.

For $f: X \to \mathbb{R}$ define f^+, f^- : $X \to \mathbb{R}$ by

$$f^+ := \max(f, 0),$$
$$f^- := \max(-f, 0).$$

Then $f^+ = \frac{1}{2}(|f| + f)$, $f^- = \frac{1}{2}(|f| - f)$; $f = f^+ - f^-$, $|f| = f^+ + f^-$.

$\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.

For a normed vector space E, $\mathcal{L}(E)$ is the space of all continuous linear maps $E \longrightarrow E$ provided with the norm

$$\|T\| := \sup \frac{\|Tx\|}{\|x\|}$$
$$= \inf \{c \in [0, \infty) : \|Tx\| \leq c\|x\| \text{ for all } x \in E\}.$$

The dual of a normed vector space E is denoted E^*.

0. PRELIMINARIES.

0.1. URYSOHN'S LEMMA. If X is compact and if $C_0, C_1 \subset X$ are closed and mutually disjoint, there exists a continuous $f : X \longrightarrow [0,1]$ such that $f = 0$ on C_0, $f = 1$ on C_1.

See Kelley, "General Topology".

0.2. BAIRE CATEGORY THEOREM. Let X be a complete metric space. Let $A_1, A_2, \ldots$ be a sequence of closed subsets of X that covers X (i.e. $X = \bigcup A_n$). Then one of the A_n contains a non-empty ball.

See Kelley, "General Topology", page 200.
Rudin, "Real and Complex Analysis", Theorem 5.6.

0.3. STONE-WEIERSTRASS THEOREM. Let X be a locally compact space. Let D be a closed linear subspace of $C_\infty(X)$ so that

if $f,g \in D$, then $fg \in D$,
if $f \in D$, then $\overline{f} \in D$,
for all $a,b \in X$, $a \neq b$, there is an $f \in D$ with

$f(a) = 0$, $f(b) \neq 0$ (i.e. D separates the points of X).
Then $D = C_\infty(X)$.

See Gillman-Jerison, "Rings of Continuous Functions", 16.4.

Each of the following two theorems is known as the Hahn-Banach Theorem.

0.4. THEOREM. Let E be a vector space over $\mathbb{R}$, D a linear subspace of E, and f a linear function $D \rightarrow \mathbb{R}$. Suppose p is a function $E \rightarrow \mathbb{R}$ such that

$$(1) \quad p(\lambda x) = \lambda p(x) \qquad (x \in E;\ \lambda > 0),$$
$$(2) \quad p(x+y) \leq p(x) + p(y) \qquad (x,y \in E).$$

If $f(x) \leq p(x)$ for all $x \in D$, then f can be extended to a linear function $g : E \rightarrow \mathbb{R}$ for which $g \leq p$ everywhere.

0.5. THEOREM. (The scalar field is either $\mathbb{R}$ or $\mathbb{C}$) Let D be a linear subspace of a normed vector space E, and let $f \in D^*$. Then f can be extended to a $g \in E^*$ for which $\|g\| = \|f\|$.

If the scalar field is $\mathbb{R}$, this theorem follows directly from the preceding one. (Take $p(x) := \|f\| \|x\|$. Then f has a linear extension g for which $g(x) \leq \|f\|\|x\|$ for all x. But then also for all x, $-g(x) = g(-x) \leq \|f\|\|-x\| = \|f\|\|x\|$. Hence, $|g(x)| \leq \|f\|\|x\|$). For the complex case one applies the following lemma.

Let F be a normed vector space over $\mathbb{C}$. Its conjugate space F^* is a normed vector space over $\mathbb{C}$. But we can also view F as a normed space over $\mathbb{R}$; as such, it has a conjugate space $F^*_{\mathbb{R}}$ (consisting of all continuous $\mathbb{R}$-linear functions $F \to \mathbb{R}$).

For every $h \in F^*$ we have $\operatorname{Re} h \in F^*_{\mathbb{R}}$.

0.6. LEMMA. *The map $h \mapsto \operatorname{Re} h$ is an $\mathbb{R}$-linear surjective isometry $F^* \to F^*_{\mathbb{R}}$. h is determined by $\operatorname{Re} h$ according to the formula*

$$h(x) = \operatorname{Re} h\,(x) - i \operatorname{Re} h\,(ix) \qquad (x \in F).$$

See Rudin, "Real and Complex Analysis", Theorem 5.17.

0.7. COROLLARY. *If E is a normed vector space and $a \in E$, then*

$$\|a\| = \sup \frac{|f(a)|}{\|f\|}.$$

Proof. Clearly $\|a\| \geq \frac{|f(a)|}{\|f\|}$ for every $f \neq 0$. On the other hand, the function $\lambda a \mapsto \lambda$ ($\lambda \in \mathbb{C}$) extends to an $f \in E^*$ with $f(a) = 1$ and $\|f\| = \frac{1}{\|a\|}$.

Let E be a normed vector space (over $\mathbb{R}$ or $\mathbb{C}$). Every $a \in E$ induces a linear function a^{**} on E by

$$a^{**}(f) := f(a) \qquad (f \in E^*).$$

The w^*- *topology* (w standing for *weak*) is the weakest topology on E^* for which all these a^{**} are continuous. It is weaker than the norm-topology.

0.8. ALAOGLU THEOREM. *For every normed vector space E (over $\mathbb{R}$ or $\mathbb{C}$) the set*

$$B := \{f \in E : \|f\| \leq 1\}$$

is w^*-compact.

Proof. (for $\mathbb{R}$; the proof for $\mathbb{C}$ is almost the same). Let X be the space

$$\prod_{a \in E} [-\|a\|, \|a\|]$$

Under the product topology, X is compact. B is a subset of X. The topology of X and the w^*-topology of E induce the same topology on B. Hence, all we have to prove is that B is closed in X. But this is obvious, as

$$B = \bigcap_{\substack{\lambda,\mu\in\mathbb{R} \\ a,b\in E}} \{f \in X : f(\lambda a + \mu b) - \lambda f(a) - \mu f(b) = 0\}$$

and for all $c \in E$ the function

$$f \longmapsto f(c)$$

is continuous on X.

0.9. RADON-NIKODYM THEOREM. Let $\mathcal{A}$ be a σ-algebra of subsets of a set X. Let μ, ν be finite positive measures on $\mathcal{A}$. Suppose that $\mu(A) = 0$ for every $A \in \mathcal{A}$ for which $\nu(A) = 0$. Then there exists an $\mathcal{A}$-measurable function $h : X \to [0, \infty)$ such that $\int f\,d\mu = \int fh\,d\nu$ for every $\mathcal{A}$-measurable $f : X \to [0, \infty)$.

Let $\langle X, \mathcal{A}, \mu\rangle$ be a measure space ($\mu \geq 0$). An element Y of $\mathcal{A}$ is of σ-finite measure if Y can be written as a union of countably many sets of finite measure. If X itself is of σ-finite measure, μ and the measure space $\langle X, \mathcal{A}, \mu\rangle$ are said to be σ-finite.

0.10. THEOREM. Let $\langle X, \mathcal{A}, \mu\rangle$ be a σ-finite measure space. Every $h \in L^\infty(\mu)$ determines a $\varphi_h \in L^1(\mu)^*$ by

$$\varphi_h(f) := \int fh\,d\mu.$$

The map $h \mapsto \varphi_h$ is a surjective linear isometry $L^\infty(\mu) \to L^1(\mu)^*$.

See Rudin, "Real and Complex Analysis", Theorem 6.16.

0.11. PROJECTION THEOREM. Let D be a closed linear subspace of a Hilbert space H; let $D^\perp = \{\zeta \in H : \zeta \perp D\}$. For $\zeta \in H$ set $\operatorname{dist}(\zeta, D) := \inf\{\|\zeta - \eta\| : \eta \in D\}$. For every $\zeta \in H$ there exists a unique $P\zeta \in D$ for which $\|\zeta - P\zeta\| = \operatorname{dist}(\zeta, D)$. The map P defined this way is a linear map of H onto D. We have $P^2 = P$, $\operatorname{Ker} P = D^\perp$, $P\zeta = \zeta$ for $\zeta \in D$. It follows that $H = D + D^\perp$. (Of course $D \cap D^\perp = \{0\}$.)

P is called the projection of H onto D.

See Halmos, "Introduction to Hilbert Space", §§11-12.

A consequence of the Projection Theorem is

0.12. F. RIESZ THEOREM. Let H be a Hilbert space. For every $\varphi \in H^*$ there exists a unique $\eta \in H$ such that
$$\varphi(\zeta) = (\zeta | \eta) \qquad (\zeta \in H).$$
Then $\|\varphi\| = \|\eta\|$.

See Halmos, "Introduction to Hilbert Spaces", §17.

From the F. Riesz Theorem one infers easily:

0.13. THEOREM. Let H be a Hilbert space. By $\mathcal{L}(H)$ we denote the set of all continuous linear maps $H \longrightarrow H$. Every $T \in \mathcal{L}(H)$ induces a (unique) $T^* \in \mathcal{L}(H)$ by
$$(T\zeta | \eta) = (\zeta | T^*\eta) \qquad (\zeta, \eta \in H).$$
For S, $T \in \mathcal{L}(H)$ and $\lambda \in \mathbb{C}$ we have
$$T^{**} = T,$$
$$\|T^*\| = \|T\|,$$
$$(S+T)^* = S^* + T^*,$$
$$(\lambda T)^* = \bar{\lambda} T^*,$$
$$(ST)^* = T^* S^*.$$

Direct computation yields

0.14. POLARIZATION FORMULA. If $(\ |\ )$ is an inner product in a vector space H and if $\|\ \|$ is the norm induced by $(\ |\ )$, then
$$4(\zeta|\eta) = \|\zeta+\eta\|^2 - \|\zeta-\eta\|^2 + i\|\zeta+i\eta\|^2 - i\|\zeta-i\eta\|^2 \quad (\zeta, \eta \in H).$$

0.15. COROLLARY. Let H be a Hilbert space, T an element of $\mathcal{L}(H)$ for which $\|T\zeta\| = \|\zeta\|$ $(\zeta \in H)$. Then
$$(T\zeta | T\eta) = (\zeta | \eta) \qquad (\zeta, \eta \in H).$$
If T is surjective, then $T^{-1} = T^*$.

The $T \in \mathcal{L}(H)$ for which $T^{-1} = T^*$ (i.e. $TT^* = T^*T = I$) are called unitary.

Hilbert spaces enter our theory via

0.16. THEOREM. Let $\langle X, \mathcal{A}, \mu \rangle$ be a measure space ($\mu \geq 0$). Let $\mathcal{L}^2(\mu)$ denote the set of all measurable functions $f : X \to \mathbb{C}$ for which $\int |f|^2 d\mu < \infty$, and let $N := \{f \in \mathcal{L}^2(\mu) : f = 0$ μ-a.e.$\}$. Then $\mathcal{L}^2(\mu)$ is a vector space, N is a linear subspace of $\mathcal{L}^2(\mu)$. By $L^2(\mu)$ denote the vector space $\mathcal{L}^2(\mu)/N$. We identify $f \in \mathcal{L}^2(\mu)$ with $f \bmod N \in L^2(\mu)$. The formula

$$(f|g) := \int f\bar{g}d\mu \qquad (f,g \in L^2(\mu))$$

introduces an inner product in $L^2(\mu)$. Relative to this inner product, $L^2(\mu)$ is a Hilbert space (i.e. $L^2(\mu)$ is complete with respect to the metric determined by the inner product).

See Dieudonné, "Treatise on Analysis II", Theorem 13.11.4,
Rudin, "Real and Complex Analysis", Theorem 3.11.