2. GELFAND THEORY.

A <u>Banach algebra</u> is a Banach space A which is also a ring such that for each $a \in A$ the maps $x \longmapsto xa$ and $x \longmapsto ax$ are linear, and $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in A$. Such an A is <u>commutative</u> if $xy = yx$ for all $x, y \in A$. An element e of A is an <u>identity</u> if $e \neq 0$ and $ex = xe = x$ $(x \in A)$. (Note that then $\|e\| \geq 1$, since $\|e\|^2 \geq \|e^2\| = \|e\|$).

Let A, B be Banach algebras. A linear map $T : A \to B$ is <u>multiplicative</u> (is an <u>algebra homomorphism</u>) if $T(xy) = (Tx)(Ty)$ $(x, y \in A)$.

$\mathbf{C}$ is a Banach algebra (commutative, with an identity) under the norm $|\ |$, but also under the norm $2|\ |$. For any topological space X the bounded continuous functions $X \to \mathbf{C}$ form a commutative Banach algebra with an identity. If X is locally compact, then $C_\infty(X)$ is a commutative Banach algebra; it has an identity if and only if X is compact.

Let l^1 be the space of all sequences $a = (a_0, a_1, \ldots)$ for which $\sum |a_n|$ is finite. This l^1 is a Banach algebra under the norm $a \longmapsto \sum |a_n|$ and under the multiplication defined by

$$(a * b)_n = \sum_{j=0}^{n} a_j b_{n-j}.$$

This Banach algebra is commutative and has an identity element, viz. $(1, 0, 0, \ldots)$. The space $\{a \in l^1 : a_0 = 0\}$ is a subalgebra of l^1.

If E is a Banach space, $\dim E > 1$, then the continuous linear maps $E \longrightarrow E$ form a non-commutative Banach algebra.

In a Banach algebra A with identity e we have $\|e\| \geq 1$. The following lemma says that in a sense it suffices to consider Banach algebras in which $\|e\| = 1$.

2.1. LEMMA. <u>Let A be a Banach algebra with identity</u> e. <u>Let</u> $c = \|e\|$. <u>For</u> $x \in A$ <u>define</u>

$$|||x||| = \sup\left\{\frac{\|xy\|}{\|y\|} : y \neq 0\right\}.$$

<u>Then</u> $|||\ |||$ <u>is a norm on</u> A <u>that is equivalent to</u> $\|\ \|$:

$$|||x||| \leq \|x\| \leq c|||x||| \qquad (x \in A).$$

<u>Under this norm,</u> A <u>is a Banach algebra</u>. $|||e||| = 1$.

To any Banach algebra we can "adjoin an identity":

2.2. LEMMA. <u>Let A be a Banach algebra without identity. The following norm and multiplication turn $A \times \mathbb{C}$ into a Banach algebra A_e.</u>

$$\|(x,\lambda)\| = \|x\| + |\lambda| \qquad (x \in A;\ \lambda \in \mathbb{C}),$$
$$(x,\lambda)(y,\mu) = (xy+\mu x+\lambda y,\ \lambda\mu) \qquad (x,y \in A;\ \lambda,\mu \in \mathbb{C}).$$

<u>This A_e has an identity element of norm</u> 1 (viz. (0,1)). <u>The map</u> $x \longmapsto (x,0)$ <u>is an isometric Banach algebra homomorphism of</u> A <u>onto a subalgebra of</u> A_e. <u>If</u> A <u>is commutative, then so is</u> A_e.

For a Banach algebra A that does have an identity we define $A_e := A$.

Let A be a Banach algebra with an identity e. Let $x \in A$. An element y of A is an <u>inverse</u> of x if $xy=yx=e$. Every $x \in A$ has at most one inverse. We denote the inverse of x by x^{-1}. The invertible (or <u>regular</u>) elements of A form a subset R of A.

(1) <u>If</u> $a \in R$, <u>then</u> $\|a^{-1}\| \le \frac{1}{\|a\|}$, as $1 \le \|e\| = \|a^{-1}a\| \le \|a^{-1}\|\,\|a\|$.

(2) <u>If</u> $a, b \in R$, <u>then</u> $ab \in R$.

(3) <u>If</u> $x \in A$, $\|x\| < 1$, <u>then</u> $e-x \in R$. <u>The inverse of</u> $e-x$ <u>is</u> $e + x + x^2 + \dots$.

(4) <u>If</u> $a \in R$, $x \in A$ <u>and</u> $\|x\| < \frac{1}{\|a^{-1}\|}$, <u>then</u> $a - x \in R$, since $a-x = a(e-a^{-1}x)$ and $\|a^{-1}x\| < 1$.

(5) R <u>is an open subset of</u> A. (Follows from (4)).

(6) <u>The map</u> $x \longmapsto x^{-1}$ $(x \in R)$ <u>is continuous</u>. Proof. Let $a \in R$. We prove $\|(a-x)^{-1}-a^{-1}\| \le 2\|a^{-1}\|^2\|x\|$ for all $x \in A$ with $\|x\| < \frac{1}{2\|a^{-1}\|}$. In fact, by (4) for such x, $(a-x)^{-1}$ exists and is equal to $(e-a^{-1}x)^{-1}a^{-1} = [e+a^{-1}x+(a^{-1}x)^2+\dots]a^{-1} = a^{-1} + \sum_1^\infty (a^{-1}x)^n a^{-1}$. Hence, $\|(a-x)^{-1}-a^{-1}\| = \|\sum_1^\infty (a^{-1}x)^n a^{-1}\| \le \sum_1^\infty \|a^{-1}\|^{n+1}\|x\|^n \le \|x\| \cdot \sum_1^\infty \|a^{-1}\|^{n+1}\left(\frac{1}{2\|a^{-1}\|}\right)^{n-1} = 2\|x\|\,\|a^{-1}\|^2$.

We are now ready to prove the famous

2.3. GELFAND-MAZUR THEOREM. Let A be a Banach algebra with an identity e such that every non-zero element of A is invertible. (A is a "division algebra"). Then $A = \mathbb{C}e$.

Proof. Take $a \in A$ and suppose $a \notin \mathbb{C}e$. Then $(a-\alpha e)^{-1}$ exists for all $\alpha \in \mathbb{C}$. If $\alpha, \beta \in \mathbb{C}$ and $\alpha \neq \beta$, then

$$\frac{(a-\beta e)^{-1}-(a-\alpha e)^{-1}}{\beta-\alpha} = (a-\beta e)^{-1}\frac{(a-\alpha e)-(a-\beta e)}{\beta-\alpha}(a-\alpha e)^{-1} =$$
$$= (a-\beta e)^{-1}(a-\alpha e)^{-1}.$$

By the continuity of the inversion map $x \longmapsto x^{-1}$ it follows that

$$\lim_{\beta\to\alpha} \frac{(a-\beta e)^{-1}-(a-\alpha e)^{-1}}{\beta-\alpha} = (a-\alpha e)^{-2}.$$

Thus, for every $T \in A^*$ the function $\alpha \longmapsto T((a-\alpha e)^{-1})$ is analytic $\mathbb{C} \to \mathbb{C}$. For $|\alpha| > \|a\|$, $\|(a-\alpha e)^{-1}\| \leq \|\alpha^{-1}(e-\alpha^{-1}a)^{-1}\| \leq$

$\leq |\alpha|^{-1} \sum \|\alpha^{-1}a\|^n \leq \dfrac{\|e\|}{|\alpha|-\|a\|}$. Therefore, $\lim\limits_{|\alpha|\to\infty} T((a-\alpha e)^{-1}) = 0$. By Liouville's Theorem it follows that $T((a-\alpha e)^{-1}) = 0$ for every α; in particular, $T(a^{-1}) = 0$. This, in turn, must be true for all $T \in A^*$. Then by the Hahn-Banach Theorem (0.7), $a^{-1} = 0$, which is absurd.

Now let A be a commutative Banach algebra with an identity e. Every ring ideal in A is a vector space ($\lambda a = (\lambda e)a$). If I is a closed proper ideal in A, the A/I can be made into a commutative Banach algebra with identity, in such a way that the natural map $x \longmapsto x+I$ is an algebra homomorphism, while the norm on A/I is given by

$$\|a+I\| = \inf\{\|x\| : x \in a + I\}.$$

If I is a proper ideal in A, then $I \cap R = \emptyset$ (R is the set of all invertible elements of A), i.e. $I \subset A \setminus R$. Now $A\setminus R$ is a closed set, so $\bar{I} \subset A\setminus R$ and $e \notin \bar{I}$. It is easy to see that $\bar{I}$ is an ideal. Thus, the closure of a proper ideal is a proper ideal. In particular, the closure of a maximal ideal M is a proper ideal that contains M. Hence, every maximal ideal is closed.

If M is such a maximal ideal, then A/M is a field. By the Gelfand-Mazur Theorem (2.3) for every $x \in A$ there exists a unique complex number $\varphi_M(x)$ such that $x+M = \varphi_M(x)e + M$, i.e. $x - \varphi_M(x)e \in M$. As $x \longmapsto x+M$ and $\lambda e+M \longmapsto \lambda$ are surjective algebra homomorphisms $A \longrightarrow A/M$ and $A/M \longrightarrow \mathbb{C}$, respectively, our φ_M is a surjective homomorphism $A \longrightarrow \mathbb{C}$. The kernel of φ_M is just M. The identity of A/M is $e+M$, so $\|e+M\| \geq 1$. Then for every $x \in A$, $|\varphi_M(x)| \leq |\varphi_M(x)|\,\|e+M\| = \|\varphi_M(x)e+M\| = \|x+M\| \leq \|x\|$. We obtain $\|\varphi_M\| \leq 1$.

Conversely, let $\psi: A \longrightarrow \mathbb{C}$ be any non-zero algebra homomorphism. Its kernel is a maximal ideal M. For all $x \in A$ we have $\psi(x-\psi(x)e)=\psi(x)-\psi(x)\psi(e)=\psi(x)-\psi(x)=0$, so $x-\psi(x)e \in \operatorname{Ker}\psi = M$, and $x+M=\psi(x)e+M$. But also $x+M=\varphi_M(x)e+M$. Consequently, $\psi(x)=\varphi_M(x)$. This is true for all x, so $\psi = \varphi_M$. In particular; ψ turns out to be continuous, and $\|\psi\| \leq 1$. We have proved:

2.4. THEOREM. <u>Let A be a commutative Banach algebra with identity e. Every algebra homomorphism $A \longrightarrow \mathbb{C}$ is continuous and has norm ≤ 1. The formulas</u>

$$M = \operatorname{Ker}\varphi,$$
$$x - \varphi(x)e \in M \quad (x \in A)$$

<u>yield a one-to-one correspondence between the maximal ideals M of A and the non-zero algebra homomorphisms $\varphi: A \longrightarrow \mathbb{C}$.</u>

By $\mathfrak{M}(A)$ we denote the set of all non-zero algebra homomorphisms $A \longrightarrow \mathbb{C}$. Every $x \in A$ induces a function $\hat{x}: \mathfrak{M}(A) \longrightarrow \mathbb{C}$, the <u>Gelfand transform</u> of x, by

$$\hat{x}(\varphi) := \varphi(x) \qquad (\varphi \in \mathfrak{M}(A)).$$

We provide $\mathfrak{M}(A)$ with the weakest topology that makes every $\hat{x}$ continuous: this is the <u>Gelfand topology</u> of $\mathfrak{M}(A)$.

The topology of $\mathfrak{M}(A)$ is the restriction of the w^*-topology of A^*. Further, $\mathfrak{M}(A)$ consists of all $\varphi \in A^*$ for which

$$\|\varphi\| \leq 1$$
$$\varphi(x+y)-\varphi(x)-\varphi(y) = 0 \qquad (\text{all } x,y \in A)$$
$$\varphi(xy)-\varphi(x)\varphi(y) = 0 \qquad (\text{all } x,y \in A)$$
$$\varphi(e) = 1$$

Hence, $\mathfrak{M}(A)$ is a w^*-closed subset of $\{\varphi \in A^* : \|\varphi\| \leq 1\}$. By Alaoglu's Theorem, $\mathfrak{M}(A)$ is compact.

The set $\mathfrak{M}(A)$, provided with the Gelfand topology, is called the <u>structure space</u> of A. Sometimes we shall not take the trouble to distinguish between a maximal ideal of A and the corresponding homomorphism $A \longrightarrow \mathbb{C}$. In that context, $\mathfrak{M}(A)$ consists of the maximal ideals of A, and is called the <u>maximal ideal space</u> of A.

Making the trivial observations that $\hat{x}(\varphi)\hat{y}(\varphi) = \widehat{xy}(\varphi)$ $(x,y \in A;\ \varphi \in \mathfrak{M}(A))$ and $\sup\{\,|\varphi(x)| : \varphi \in \mathfrak{M}(A)\} \leq \|x\|$ $(x \in A)$ we have:

2.5. THEOREM. <u>Let A be a commutative Banach algebra with identity. Under the Gelfand topology, the surjective algebra homomorphisms</u> $A \longrightarrow \mathbb{C}$ <u>(= the maximal ideals of A) form a compact space</u> $\mathfrak{M}(A)$. <u>The Gelfand transformation</u> $x \longmapsto \hat{x}$ <u>is an algebra homomorphism of norm</u> ≤ 1 <u>of A into</u> $C(\mathfrak{M}(A))$.

We set $\hat{A} := \{\hat{x} : x \in A\}$. If $\varphi_1, \varphi_2 \in \mathfrak{M}(A)$ are distinct, there exists an $\hat{x} \in \hat{A}$ such that $\hat{x}(\varphi_1) \neq \hat{x}(\varphi_2)$: $\hat{A}$ <u>separates the points of</u> $\mathfrak{M}(A)$. Further, $\hat{A}$ <u>contains the constant functions on</u> $\mathfrak{M}(A)$.

The kernel of the Gelfand transformation is just the intersection of all maximal ideals of A. This intersection is called the <u>radical</u> of A. If the radical is $\{0\}$, A is <u>semisimple</u>.

The <u>spectrum</u> of $x \in A$, denoted $\operatorname{Sp} x$, is $\{\alpha \in \mathbb{C} : x - \alpha e$ is not invertible$\}$ $= \{\alpha \in \mathbb{C} : x - \alpha e$ lies in a proper ideal of A$\}$ $=$ $= \{\alpha \in \mathbb{C} : x - \alpha e$ lies in a maximal ideal of A$\}$ $= \{\alpha \in \mathbb{C}$: there exists $\varphi \in \mathfrak{M}(A)$ with $\varphi(x - \alpha e) = 0\}$ $= \{\hat{x}(\varphi) : \varphi \in \mathfrak{M}(A)\}$.

2.6. COROLLARY. $\operatorname{Sp} x$ <u>is a non-empty compact subset of</u> $\mathbb{C}$. (Note that $\mathfrak{M}(A) \neq \emptyset$, as the zero ideal can be extended to a maximal ideal of A.)

Define

$$\|x\|_{\operatorname{Sp}} := \sup\{|\alpha| : \alpha \in \operatorname{Sp} x\} \qquad (x \in A).$$

Then $\|x\|_{Sp} = \|\hat{x}\|_\infty$. Thus, $\|\ \|_{Sp}$ is a seminorm; $\|\ \|_{Sp} \le \|\ \|$; $\|xy\|_{Sp} \le \|x\|_{Sp}\|y\|_{Sp}$ for all $x,y \in A$; $\|x^m\|_{Sp} = \|x\|_{Sp}{}^m$ for all $x \in A$ and $m \in \mathbb{N}$; and $\{x \in A : \|x\|_{Sp} = 0\}$ is the radical of A.

2.7. SPECTRAL RADIUS FORMULA. Let A be a commutative Banach algebra with identity. Then

$$\|x\|_{Sp} = \lim_{n\to\infty} \sqrt[n]{\|x^n\|} = \inf_n \sqrt[n]{\|x^n\|} \qquad (x \in A).$$

Proof. Let $x \in A$. For every $n \in \mathbb{N}$, $\|x\|_{Sp} = \|\hat{x}\|_\infty =$ $= \sqrt[n]{\|\hat{x}^n\|} = \sqrt[n]{\|\widehat{x^n}\|} \le \sqrt[n]{\|x^n\|}$; hence, $\|x\|_{Sp} \le \inf_n \sqrt[n]{\|x^n\|} \le \liminf_{n\to\infty} \sqrt[n]{\|x^n\|}$. We are done if we can prove $\limsup_{n\to\infty} \sqrt[n]{\|x^n\|} \le \|x\|_{Sp}$. We need a lemma from the general theory of Banach spaces.

2.8. UNIFORM BOUNDEDNESS PRINCIPLE. Let E be a Banach space, let $X \subset E$. Assume that for every $f \in E^*$, $c_f = \sup_{x\in X} |f(x)|$ is finite. Then $\sup_{x\in X} \|x\| < \infty$.

Proof. For $m \in \mathbb{N}$ set $F_m := \{f \in E^* : c_f \le m\}$. Each F_m is closed, as $F_m = \bigcap_{x\in X}\{f \in E^* : |f(x)| \le m\}$. Together the F_m cover E which is a complete metric space. By the Baire Category Theorem there is an $N \in \mathbb{N}$ such that F_N contains a ball $\{f \in E^* : \|f-f_0\| \le r\}$ (where $r>0$). Now if $f \in E^*$ and $\|f\| \le r$, then $f=(f+f_0)-f_0$ and $f+f_0 \in F_N$, $f_0 \in F_N$. Hence, for such f we have $c_f \le 2N$. It follows that for arbitrary $f \in E$, $c_f \le \frac{\|f\|}{r}2N$. By the Hahn-Banach Theorem, $\|x\| \le \frac{2N}{r}$ for all $x \in X$.

We now continue our proof of 2.7; it remains to be shown that $\limsup_{n\to\infty} \sqrt[n]{\|x^n\|} \le \|x\|_{Sp}$.

Set $D := \{\alpha \in \mathbb{C} : |\alpha|\|x\|_{Sp} < 1\}$. If $\alpha \in D, \alpha \ne 0$, then

$\alpha^{-1} \notin \mathrm{Sp}\,x$, so $x-\alpha^{-1}e$ is invertible. Hence, $e-\alpha x$ is invertible for all $\alpha \in D$ (including, of course, $\alpha = 0$). For $T \in A^*$ the function $\alpha \mapsto T((e-\alpha x)^{-1})$ is analytic on the disc D (see the proof of the Gelfand-Mazur Theorem), hence is the sum of a power series:

$$\text{(i)} \qquad T((e-\alpha x)^{-1}) = \sum_0^\infty \lambda_n \alpha^n \qquad (\alpha \in D).$$

We know that

$$(e-\alpha x)^{-1} = e+\alpha x+\alpha^2 x^2+\ldots \qquad (\|\alpha x\|<1),$$

so

$$\text{(ii)} \qquad T((e-\alpha x)^{-1}) = \sum_0^\infty T(x^n)\alpha^n \qquad \left(|\alpha| < \frac{1}{\|x\|}\right).$$

By (i) and (ii), $\lambda_n = T(x^n)$ for every n. Thus,

$$T((e-\alpha x)^{-1}) = \sum_0^\infty T(x^n)\alpha^n = \sum_0^\infty T(\alpha^n x^n) \qquad (\alpha \in D).$$

In particular, for every $\alpha \in D$ we find

$$\sup\{|T(\alpha^n x^n)| : n=0,1,2,\ldots\} < \infty \qquad (T \in A^*).$$

The Uniform Boundedness Principle now implies the existence of a number $C(\alpha)$ (for each $\alpha \in D$) such that $\sup_n \|\alpha^n x^n\| \le C(\alpha)$.

It follows that

$$\limsup_{n\to\infty} \sqrt[n]{\|x^n\|} \le \limsup_{n\to\infty} \sqrt[n]{|\alpha^{-n}|\,C(\alpha)} \le |\alpha|^{-1} \qquad (\alpha \in D, \alpha \neq 0).$$

From the definition of D it is apparent that $\limsup_{n\to\infty} \sqrt[n]{\|x^n\|} \le \|x\|_{Sp}$.

Now let us consider a commutative Banach algebra A without identity. Let A_e be as in Lemma 2.2. We identify $x \in A$ with $(x,0) \in A_e$, so that A becomes a maximal ideal in A_e. Every algebra homomorphism $A \to \mathbb{C}$ extends to a homomorphism $A_e \to \mathbb{C}$, hence is continuous and has norm ≤ 1.

We no longer have the simple connection between the maximal ideals of A and the surjective homomorphisms $A \to \mathbb{C}$. As an example, let A be any Banach space. The multiplication $xy := 0$ $(x,y \in A)$ turns A into a commutative Banach algebra. There are no surjective homomorphisms $A \to \mathbb{C}$, but the kernel of any linear map $A \to \mathbb{C}$ is a maximal ideal of A!

If $\varphi: A \to \mathbb{C}$ is a non-zero (algebra) homomorphism, then it is surjective, so that its kernel is a maximal ideal in A. If $v \in \varphi^{-1}(1)$, then $vx-x \in \operatorname{Ker}\varphi$ for every $x \in A$. This observation leads to the following definition. An ideal I of A is <u>regular</u> if there exists a $v \in A$ such that $vx-x \in I$ for all $x \in A$. Here by "<u>ideal</u>" we mean an <u>algebra</u> <u>ideal</u>, i.e. a ring ideal which is also a vector space.

The kernel of a non-zero homomorphism $A \to \mathbb{C}$ is a regular maximal ideal. On the other hand, if $M \subset A$ is a <u>closed</u> regular maximal ideal, then A/M is isomorphic to $\mathbb{C}$ (Gelfand-Mazur), so M is kernel of a continuous surjective homomorphism $A \to \mathbb{C}$.

Now it turns out that every regular maximal ideal is closed. Let M be a regular maximal ideal. There exists a $v \in A$ such that $vx-x \in M$ for all $x \in A$. For an $y \in M$ with $\|v-y\|<1$, set $z := \sum_1^\infty (v-y)^n$; then $z=v-y+(v-y)z$, so $v=-(vz-z)+y+yz \in M$, and $x=(x-vx)+vx \in M$ for every $x \in A$. We have a contradiction. It follows that there does not exist any $y \in M$ with $\|v-y\|<1$. Consequently, the closure $\overline{M}$ of M doesnot contain v. Then $\overline{M}$ is a proper ideal of A. By the maximality of M, we have $M = \overline{M}$, and M is closed.

2.9. THEOREM. <u>Let A be a commutative Banach algebra. Every non-zero (algebra) homomorphism $A \to \mathbb{C}$ is continuous and has norm ≤ 1, every regular maximal ideal of A is closed. There is a one-to-one correspondence between the regular maximal ideals M of A and the non-zero homomorphisms $\varphi: A \to \mathbb{C}$ given by the formula $M = \operatorname{Ker}\varphi$.</u>

The <u>structure space</u> $\mathfrak{M}(A)$ of A is again the set of all non-zero homomorphisms $A \to \mathbb{C}$, under the weakest topology that makes all the functions $\hat{x} : \varphi \mapsto \varphi(x)$ continuous.

2.A. EXERCISE. <u>Define $\varphi_0 \in \mathfrak{M}(A_e)$ by $\varphi_0(a,\lambda) := \lambda$; then $\operatorname{Ker}\varphi_0 = A$. The formula $\varphi \mapsto \varphi|_A$ defines a homeomorphism of $\mathfrak{M}(A_e)\setminus\{\varphi_0\}$ onto $\mathfrak{M}(A)$. Consequently, $\mathfrak{M}(A)$ is locally compact.</u>

For $x \in A$, the function $\hat{x}$ is the <u>Gelfand transform</u> of x.

2.B. EXERCISE. <u>The Gelfand Transformation</u> $x \longmapsto \hat{x}$ <u>is</u>

<u>a norm-decreasing algebra homomorphism of A into $C_\infty(\mathfrak{M}(A))$. The set $\hat{A}$ of all Gelfand transforms separates the points of</u> $\mathfrak{M}(A)$.

The <u>spectrum</u> $\mathrm{Sp}\,x$ of $x \in A$ is $\{0\} \cup \{\varphi(x) : \varphi \in \mathfrak{M}(A)\}$. Thus, if we again view A as a subalgebra of A_e, the spectrum of x relative to A is, by definition, the spectrum of x relative to A_e. Therefore, the spectral radius formula still holds:

$$\|x\|_{Sp} := \|\hat{x}\|_\infty = \lim_{n\to\infty} \sqrt[n]{\|x^n\|} \qquad (x \in A).$$

2.C. EXERCISE. $x, y \in A$ <u>are called each other's adverses (or quasi-inverses) if</u> $x+y=xy$. <u>This is true if and only if</u> $e-x$ <u>and</u> $e-y$ <u>are each other's inverses in</u> A_e. <u>For every</u> $x \in A$ <u>we have</u> $\mathrm{Sp}\,x = \{\alpha \in \mathbb{C} : \alpha = 0$ or $\alpha^{-1}x$ has no adverse$\}$.

2.D. EXERCISE. Let X be a compact space. C(X) is a commutative Banach algebra. Every $a \in X$ determines a maximal ideal M_a. Every maximal ideal of C(X) is of the form M_a. (If I is an ideal that is not contained in any M_a, there exist $f_1, \ldots, f_n \in I$ such that, for every $a \in X$, some $f_i(a)$ is $\neq 0$; then $g := \sum f_i \overline{f}_i \in I$, $1 \in I$, and $I = C(X)$).

For $a \in X$ let φ_a be the homomorphism $f \longmapsto f(a)$. Then $a \longmapsto \varphi_a$ is a homeomorphism of X onto $\mathfrak{M}(C(X))$.

2.E. EXERCISE. <u>Let</u> X <u>be a set</u>, B(X) <u>the Banach algebra of all bounded complex functions on</u> X, <u>and</u> A <u>a closed subalgebra of</u> B(X) <u>so that</u>

1) $1 \in A$,

2) <u>if</u> $f \in A$ <u>then</u> $\overline{f} \in A$.

<u>Let</u> $M = \mathfrak{M}(A)$. <u>We denote by</u> f^* <u>the Gelfand transform of</u> $f \in A$.

<u>The Gelfand Transformation is a linear isometry of</u> A <u>onto</u> C(M). <u>Furthermore,</u>

$$\overline{f^*} = (\overline{f})^* \qquad (f \in A). \tag{*}$$

<u>If</u> $f \in A$ <u>is real-valued, then so is</u> f^*; <u>if</u> $f \geq 0$, <u>then</u> $f^* \geq 0$.

(Hint. First prove that the Gelfand Transformation is isometric. Now let $f \in A$ be real-valued and let $\varphi \in M$. For all $c \in \mathbb{R}$, $\|f+ci\| \leq \sqrt{\|f\|^2+c^2}$, so $|\varphi(f)+ci| \leq \sqrt{\|f\|^2+c^2}$. Show that, consequently, $\varphi(f) \in \mathbb{R}$. Then (*) follows easily. Use the Stone-Weierstrass Theorem to prove surjectivity of the Gelfand Transformation.)

2.F. EXERCISE. Let X be any topological space. The bounded continuous functions $X \to \mathbb{C}$ form a commutative Banach algebra $C(X)$ with identity. The structure space of this Banach algebra is the <u>Stone-Čech compactification</u> of X, denoted by X^β (or βX).

The Gelfand Transformation is a surjective isometric isomorphism of $C(X)$ onto $C(X^\beta)$. (Use the Stone-Weierstrass Theorem.)

For $x \in X$ let $\beta(x) \in X^\beta$ be the homomorphism $f \mapsto f(x)$. Then β is a continuous map of X onto a dense subset of X^β. If X is compact, β is a homeomorphism of X onto X .

If X,Y are topological spaces and $\sigma: X \to Y$ is continuous, then $f \mapsto f \circ \sigma$ is an algebra homomorphism of $C(Y)$ into $C(X)$, inducing a continuous map $\sigma^\beta: X^\beta \to Y^\beta$ so that the diagram

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[d, "\beta"'] & Y \arrow[d, "\beta"] \\ X^\beta \arrow[r, "\sigma^\beta"] & Y^\beta \end{tikzcd}$$

is commutative.

If σ is a continuous map of a topological space X into a compact space Y, there exists a unique continuous $\bar{\sigma}: X^\beta \to Y$ such that $\bar{\sigma} \circ \beta = \sigma$:

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[d, "\beta"'] & Y \\ X^\beta \arrow[ur, "\bar{\sigma}"'] & \end{tikzcd}$$

2.G. EXERCISE. Make l^1 into a Banach algebra as was done on the first page of this chapter. Set $a := (0,1,0,0,\ldots)$. Show that $\varphi \mapsto \varphi(a)$ is a homeomorphism of $\mathfrak{M}(l^1)$ onto $D := \{\alpha \in \mathbb{C} : |\alpha| \leq 1\}$, and that for all $x \in l^1$ one has $\hat{x}(\varphi) = \sum_0^\infty x_n \varphi(a)^n$.

What is the maximal ideal space of the subalgebra $\{x \in l^1 : x_0 = 0\}$ of l^1?

2.H. EXERCISE. Let A be a commutative Banach algebra with an identity. If $x \in A$ and if the Gelfand transform $\hat{x}$ of x does not take the value 0, then x is invertible.