1. THE RIESZ REPRESENTATION THEOREM.

IN THIS SECTION, X IS A LOCALLY COMPACT SPACE.

We remind the reader of a basic result from topology:

1.1. LEMMA. <u>If</u> $U \subset X$ <u>is open and</u> $x \in U$, <u>there exists an open</u> $W \subset X$ <u>such that</u> $x \in W$ <u>and</u> $\overline{W} \subset U$.

The <u>support</u> of a function f on X is

$$\operatorname{supp} f := \operatorname{clo} \{x \in X : f(x) \neq 0\}.$$

The continuous functions with compact supports form a vector space $C_{00}(X)$.

Let $C_\infty(X)$ denote the set of all continuous functions $f : X \longrightarrow \mathbb{C}$ for which $\{x \in X : |f(x)| \geq \varepsilon\}$ is compact for every $\varepsilon > 0$. This $C_\infty(X)$ is a Banach space under the supremum-norm. $C_{00}(X)$ is a dense subspace of $C_\infty(X)$. (If $f \in C_\infty^{+}(X)$ and if $\varepsilon > 0$, set $f_\varepsilon := \max(f-\varepsilon, 0)$; then $f_\varepsilon \in C_{00}(X)$ and $\|f - f_\varepsilon\| \leq \varepsilon$).

Choose any object ∞ that is not an element of X. In $X_\infty := X \cup \{\infty\}$ we introduce a topology: A set $U \subset X_\infty$ is to be <u>open</u> if

either U is an open subset of X

or $\infty \in U$ and $X_\infty \setminus U$ is a compact subset of X.

Thus, X_∞ is a compact (Hausdorff!) space, the <u>Alexandroff compactification</u> or <u>one-point compactification</u> of X. X is a dense subset of X_∞. The original topology on X coincides with the restriction of the topology of X_∞.

If $f: X_\infty \to \mathbb{C}$ is continuous and $f(\infty) = 0$, then $f|_X \in C_\infty(X)$. Conversely, every $g \in C_\infty(X)$ is the restriction of a continuous function on X whose value at ∞ is 0.

The Urysohn Lemma 0.1 has the following extension:

1.2. LEMMA. <u>If</u> $K \subset X$ <u>is compact, if</u> $C \subset X$ <u>is closed and</u> $K \cap C = \emptyset$, <u>there exists a continuous</u> $f: X \longrightarrow [0,1]$ <u>such that</u> $f|_K = 1$, $f|_C = 0$.

Proof. K and $C \cup \{\infty\}$ are disjoint closed subsets of X.

For compact $K \subset X$, open $V \subset X$ and continuous $f: X \longrightarrow [0,1]$ we write

$$K \prec f \quad \text{if } f = 1 \text{ on } K,$$
$$f \prec V \quad \text{if supp } f \subset V.$$

1.3. LEMMA. <u>If</u> $K \subset X$ <u>is compact</u>, $V \subset X$ <u>is open and</u> $K \subset V$, <u>there exists an</u> f <u>with</u> $K \prec f \prec V$.

Proof. By Lemma 1.2 there exists a continuous $g : X \to [0,1]$ such that $g = 1$ on K, $g = 0$ on $X \setminus V$. Take $f := \max(2g-1, 0)$.

1.4. LEMMA. <u>Let</u> $K \subset X$ <u>be compact</u>. <u>Let</u> $n \in \mathbb{N}$ <u>and let</u> $V_1, \ldots, V_n \subset X$ <u>be open, so that</u> $K \subset \bigcup_i V_i$. <u>Then there exist</u> $h_1, \ldots, h_n$ <u>such that</u> $h_i \prec V_i$ <u>for each</u> i <u>while</u> $K \prec \sum h_i$. (<u>The</u> h_i form a <u>partition of</u> 1 over K, subordinate to the cover $\{V_1, \ldots, V_n\}$ of K).

Proof. For each $x \in K$ choose a neighborhood W_x of x so that $\overline{W}_x$ is compact and contained in some V_i. By compactness there exist $x_1, \ldots, x_m \in K$ for which $K \subset \bigcup_j W_{x_j}$. For each i let $H_i := \bigcup \{\overline{W}_{x_j} : \overline{W}_{x_j} \subset V_i\}$. Each H_i is compact and contained in V_i while $K \subset \bigcup W_{x_j} \subset \bigcup H_i$. For each i by Lemma 1.3 we can make a g_i with $H_i \prec g_i \prec V_i$. Set $h_1 := g_1$, $h_2 := (1-g_1)g_2, \ldots,$ $h_n := (1-g_1)(1-g_2)\ldots(1-g_{n-1})g_n$. Obviously $h_i \prec V_i$ for each i. Further, $1 - \sum h_i = \prod (1-g_i)$, so $K \prec \sum h_i$.

Let μ be a linear function $C_{oo}(X) \longrightarrow \mathbb{C}$. We say that μ is <u>positive</u> if $\mu(f) \geq 0$ for every $f \geq 0$. Then $\mu(f) \in \mathbb{R}$ for every real-valued $f \in C_{oo}(X)$, and

$$\text{Re } \mu(f) = \mu(\text{Re } f), \quad \text{Im } \mu(f) = \mu(\text{Im } f)$$

for every $f \in C_{oo}(X)$.

Let $\mathcal{B}$ be the σ-algebra of all Borel subsets of X (i.e. the σ-algebra generated by the open subsets of X). If ν is a measure $\mathcal{B} \longrightarrow [0,\infty]$ such that $\nu(K) < \infty$ for every compact $K \subset X$, then $C_{oo}(X) \subset \mathcal{L}^1(\nu)$, and $f \longmapsto \int f d\nu$ is a positive linear function on $C_{oo}(X)$. We show that every positive linear function on $C_{oo}(X)$ is of this type.

Let μ be a positive linear function on $C_{oo}(X)$. For open $V \subset X$ define $\mu V \in [0,\infty]$ by

$$\mu V := \sup_{f \prec V} \mu(f).$$

Then

$$\mu \emptyset = 0,$$

$$\text{if } V_1 \subset V_2, \text{ then } \mu V_1 \le \mu V_2.$$

We extend this μ to the system $\mathcal{P}(X)$ of all subsets of X by

$$\mu E := \inf_{\substack{V \text{ open} \\ V \supset E}} \mu V.$$

(Note that, if E is open the new definition of μE coincides with the previous one).

Set

$$\mathfrak{M}_F := \{E \in \mathcal{P}(X) : \mu E < \infty ; \ \mu E = \sup_{\substack{K \subset E \\ K \text{ compact}}} \mu K\},$$

and

$$\mathfrak{M} := \{E \in \mathcal{P}(X) : E \cap K \in \mathfrak{M}_F \text{ for all compact } K\}.$$

We are now going to prove that $\mathfrak{M}$ is a σ-algebra that contains the open sets, that $\tilde{\mu} := \mu|_{\mathfrak{M}}$ is a measure, and $\mu(f) = \int f d\tilde{\mu}$ for every $f \in C_{oo}(X)$.

First we show μ to be <u>countably subadditive</u>, i.e.

$$\mu(\bigcup E_i) \le \sum \mu E_i \quad \text{for all } E_1, E_2, \ldots \in \mathcal{P}(X).$$

The inequality is trivial if $\mu E_i = \infty$ for some i, so we may assume each μE_i to be finite. Let $\varepsilon > 0$. We prove

$$\mu(\bigcup E_i) \le \varepsilon + \sum \mu E_i.$$

For this purpose, choose open $V_1, V_2, \ldots$ with $V_i \supset E_i$, $\mu V_i \le \varepsilon 2^{-i} + \mu E_i$. Then $\mu(\bigcup E_i) \le \mu(\bigcup V_i)$ and $\sum \mu V_i \le \varepsilon + \sum \mu E_i$, so all we have to prove is

$$\mu(\bigcup V_i) \le \sum \mu V_i$$

where all V_i are open. By definition of $\mu(\bigcup V_i)$ we are done if

$$\mu(f) \le \sum \mu V_i$$

for all $f \prec \bigcup V_i$. Now for such f, by the compactness of supp f, there exists an $N \in \mathbb{N}$ with supp $f \subset V_1 \cup \ldots \cup V_n$, and (Lemma 1.4) there exist $h_1,\ldots,h_N$ such that $h_i \prec V_i$ $(i = 1,\ldots,N)$ while supp $f \prec \sum h_i$. But then $h_i f \prec V_i$ $(i = 1,\ldots,N)$ and $f = \sum h_i f$, so that

$$\mu(f) = \sum_1^N \mu(h_i f) \le \sum_1^N \mu V_i \le \sum_1^\infty \mu V_i.$$

(1) We have proved the <u>subadditivity of</u> μ.

Let $K \subset X$ be compact, $K \prec f$. Putting $V := \{x : f(x) > \frac{1}{2}\}$ we have an open set V that contains K. Now

$$\mu V = \sup_{g \prec V} \mu(g) \le \sup_{0 \le g \le 2f} \mu(g) \le \mu(2f) < \infty.$$

(2) In particular, $\mu K < \infty$ <u>for all compact</u> K.

Next, we prove

(3) $$\mu V = \sup_{\substack{K \text{ compact} \\ K \subset V}} \mu K \quad \underline{\text{for every open}}\ V \subset X.$$

Trivially, $\mu V \ge \sup \mu K$. Conversely, let $f \prec V$. Then

$$\mu(\operatorname{supp} f) = \inf_{\substack{W \text{ open} \\ W \supset \operatorname{supp} f}} \mu W = \inf_{\substack{W \text{ open} \\ f \prec W}} \mu W \ge \mu(f).$$

Hence,

$$\sup_{\substack{K \text{ compact} \\ K \subset V}} \mu K \ge \sup_{f \prec V} \mu(\operatorname{supp} f) \ge \sup_{f \prec V} \mu(f) = \mu V$$

which proves (3).

(4) From (2) and (3) we infer that $\mathfrak{M}_F$ <u>contains all compact sets and all open sets</u> V <u>for which</u> $\mu V < \infty$.

Our next step is to prove that

(5) $$\mu K_1 + \ldots + \mu K_n \le \mu(K_1 \cup \ldots \cup K_n)$$

<u>if</u> $K_1,\ldots,K_n$ <u>are compact and mutually disjoint</u>.

It suffices to consider the case $n = 2$. Let K_1, K_2 be compact, disjoint. There exist disjoint open sets V_1, V_2 for which $V_1 \supset K_1$, $V_2 \supset K_2$. For any open $W \supset K_1 \cup K_2$ the sets $W \cap V_1$ and $W \cap V_2$ are open and disjoint; it follows easily from the definition of μ that $\mu(W \cap V_1)+\mu(W \cap V_2) \leq \mu(W)$. But $\mu(W \cap V_1) \geq \mu K_1$, $\mu(W \cap V_2) \geq \mu K_2$. Therefore

$$\mu K_1 + \mu K_2 \leq \inf_{\substack{W \text{ open} \\ W \supset K_1 \cup K_2}} \mu W = \mu(K_1 \cup K_2),$$

which proves (5).

An obvious consequence is

$$\mu E_1 + \ldots + \mu E_n \leq \mu(E_1 \cup \ldots \cup E_n) \qquad (E_1, \ldots, E_n \in \mathfrak{M}_F;\ E_i \cap E_j = \emptyset \text{ if } i \neq j).$$

Therefore, if $E_1, E_2, \ldots \in \mathfrak{M}_F$ are disjoint, then

$$\sum_1^\infty \mu E_i \leq \mu(\bigcup_1^\infty E_i).$$

But the converse inequality is proved in (1). Hence,

(6) $$\sum \mu E_i = \mu(\bigcup E_i) \quad \underline{\text{if}}\ E_1, E_2, \ldots \in \mathfrak{M}_F\ \underline{\text{are}}\ \underline{\text{disjoint}}.$$

(7) We can get a little more from this, viz. <u>if</u> $E_1, E_2, \ldots \in \mathfrak{M}_F$ <u>are disjoint and</u> $\sum \mu E_i < \infty$, <u>then</u> $\bigcup E_i \in \mathfrak{M}_F$. In fact, then for every $\varepsilon > 0$ we can choose $n \in \mathbb{N}$ so that $\mu(\bigcup E_i) \leq \varepsilon + \sum_1^n \mu E_i$, and compact $K_i \subset E_i$ so that $\mu E_i \leq \varepsilon 2^{-i} + \mu K_i$. Now $K := K_1 \cup \ldots \cup K_n$ is a compact subset of $\bigcup E_i$ and $\mu(\bigcup E_i) \leq 2\varepsilon + \mu K$.

Let $E \in \mathfrak{M}_F$, $\varepsilon > 0$. There exist open $V \supset E$ and compact $K \subset E$ so that $\mu V \leq \mu E + \frac{\varepsilon}{2}$, $\mu K \geq \mu E - \frac{\varepsilon}{2}$. Now K and $V \setminus K$ are disjoint elements of $\mathfrak{M}_F$ (see (4)) so that (by (6))

(8) <u>If</u> $E \in \mathfrak{M}_F$ <u>and</u> $\varepsilon > 0$, <u>then there exist open</u> $V \supset E$ <u>and compact</u> $K \subset E$ <u>such that</u> $\mu(V \setminus K) \leq \varepsilon$.

Let $A_1, A_2 \in \mathfrak{M}_F$. Take $\varepsilon > 0$. There exist compact K_1, K_2 and open V_1, V_2 such that $K_j \subset A_j \subset V_j$ and $\mu(V_j \setminus K_j) \leq \varepsilon$

$(j=1,2)$. Now $K_1 \setminus V_2$ is a compact subset of $A_1 \setminus A_2$ and
$\mu(A_1 \setminus A_2) \leq \mu(V_1 \setminus K_2) \leq \mu(V_1 \setminus K_1)+\mu(K_1 \setminus V_2)+\mu(V_2 \setminus K_2) \leq 2\varepsilon + \mu(K_1 \setminus V_2)$.
Thus, $\mu(A_1 \setminus A_2) = \sup \{\mu(K) : K \text{ compact}, K \subset A_1 \setminus A_2\}$. As $\mu(A_1 \setminus A_2) \leq \mu A_1 < \infty$, it follows that $A_1 \setminus A_2 \in \mathfrak{M}_F$.

For $A_1, A_2 \in \mathfrak{M}_F$, then $A_1 \cup A_2$ is the union of the elements A_2 and $A_1 \setminus A_2$ of $\mathfrak{M}_F$; by (7) we have $A_1 \cup A_2 \in \mathfrak{M}_F$. Also $A_1 \cap A_2 = A_1 \setminus (A_1 \setminus A_2) \in \mathfrak{M}_F$. Thus,

(9) $\mathfrak{M}_F$ <u>is a ring of sets</u>.

(10) From (7) and (2) we now infer that $\mathfrak{M}$ <u>is a σ-algebra</u>.

As $\mathfrak{M}_F$ contains all compact sets, $\mathfrak{M}$ contains all closed sets. Therefore,

(11) $\mathfrak{M}$ <u>contains all Borel sets</u>.

We can now prove the formula

(12) $$\mathfrak{M}_F = \{E \in \mathfrak{M} : \mu E < \infty\}.$$

The inclusion $\subset$ follows directly from the definition of $\mathfrak{M}$ and the fact that $\mathfrak{M}_F$ is a ring that contains all compact sets. Conversely, suppose $E \in \mathfrak{M}$, $\mu E < \infty$, $\varepsilon > 0$; we make a compact $K \subset E$ such that $\mu K \geq \mu E - \varepsilon$. There exists an open $V \supset E$ with $\mu V < \infty$, and there exists a compact $H \subset V$ with $\mu H > \mu V - \frac{\varepsilon}{2}$. Now $E \cap H \in \mathfrak{M}_F$, so $\mu K > \mu(E \cap H) - \frac{\varepsilon}{2}$ for some compact $K \subset E \cap H$. Then K is compact, $K \subset E$, and $\mu E \leq \mu(E \setminus H) + \mu(E \cap H) \leq \mu(V \setminus H) + \mu(K) + \frac{\varepsilon}{2} \leq \mu K + \varepsilon$. This proves (12).

(13) By (11), (12), (6) and (1) it follows that <u>the restriction $\tilde{\mu}$ of μ to the σ-algebra $\mathfrak{B}$ of Borel sets is σ-additive</u>.

Every element of $C_{00}(X)$ is $\mathfrak{B}$-measurable. Further, every element of $C_{00}(X)$ is bounded and the $\tilde{\mu}$-measure of its support is finite. Therefore, the elements of $C_{00}(X)$ are

$\tilde{\mu}$-integrable. We proceed to prove

(14) $\qquad \mu(f) = \int f d\tilde{\mu} \qquad (f \in C_{00}(X)).$

We may restrict ourselves to real-valued f. Furthermore, it suffices to prove $\mu(f) \leq \int f d\tilde{\mu}$. (Then, similarly, $-\mu(f) = \mu(-f) \leq \int(-f)d\tilde{\mu} = -\int f d\tilde{\mu}$, so $\mu(f) \geq \int f d\tilde{\mu}$).

Thus, take a real-valued $f \in C_{00}(X)$ and $\varepsilon > 0$. Let $K := \operatorname{supp} f$. Choose $y_0, \dots, y_n$ so that $-\|f\| - \varepsilon < y_0 < -\|f\| < y_1 < \dots < y_n = \|f\|$ $y_i - y_{i-1} \leq \varepsilon$. Set $E_i := \{x \in K : y_{i-1} < f(x) \leq y_i\}$. The E_i are mutually disjoint elements of $\mathcal{B} \cap \mathfrak{M}_F$ whose union is K. For $i = 1,\dots,n$, choose open $V_i \supset E_i$ such that $V_i \subset \{x : f(x) < y_i + \varepsilon\}$, $\mu V_i < \mu E_i + \frac{\varepsilon}{n}$. Then $K \subset \bigcup V_i$. According to Lemma 1.4 we have continuous $h_1, \dots, h_n : X \longrightarrow [0,1]$ such that $h_i \leq \xi_{V_i}$, $\sum h_i = 1$ on K. Now $\mu(f) = \sum \mu(fh_i) \leq \sum \mu((y_i+\varepsilon)h_i) = \sum (y_i+\varepsilon)\mu(h_i)$ $\leq \sum(y_i+\varepsilon)\mu V_i \leq \sum(y_i+\varepsilon)(\mu E_i + \frac{\varepsilon}{n}) \leq \sum(y_i-\varepsilon)\mu E_i + 2\varepsilon \sum \mu E_i +$ $+ \sum(\|f\| + \varepsilon)\frac{\varepsilon}{n} \leq \int \sum(y_i-\varepsilon)\xi_{E_i} d\tilde{\mu} + 2\varepsilon\mu K + (\|f\| + \varepsilon)\varepsilon \leq \int f d\tilde{\mu} + 2\varepsilon\mu K +$ $+ (\|f\| + \varepsilon)\varepsilon$. This is true for all $\varepsilon > 0$, so $\mu(f) \leq \int f d\tilde{\mu}$.

We have now proved:

1.5. RIESZ REPRESENTATION THEOREM. <u>Let</u> $\mathcal{B}$ <u>be the</u> σ<u>-algebra of all Borel subsets of</u> X. <u>For every positive linear function</u> $\mu : C_{00}(X) \longrightarrow \mathbb{C}$ <u>there exists a measure</u> $\tilde{\mu} : \mathcal{B} \longrightarrow [0,\infty]$ <u>such that</u>

(a) $\qquad \mu(f) = \int f \, d\tilde{\mu} \qquad (f \in C_{00}(X)).$

<u>We have</u>

(b) $\qquad \tilde{\mu}(K) < \infty \qquad (K \subset X \text{ compact})$

$\tilde{\mu}$ <u>can be chosen so that</u>

(c) $\qquad \tilde{\mu}(V) = \sup\{\tilde{\mu}(K) : K \subset V, K \text{ compact}\} \qquad (V \subset X \text{ open})$

<u>and</u>

(d) $\qquad \tilde{\mu}(E) = \inf\{\tilde{\mu}(V) : V \supset E, V \text{ open}\} \qquad (E \in \mathcal{B}).$

A measure $\tilde{\mu} : \mathcal{B} \longrightarrow [0,\infty]$ with properties (b), (c) and (d) is called a <u>regular Borel measure</u> or <u>positive Radon measure</u>.

1.6. LEMMA. <u>Let</u> ν <u>be a regular Borel measure on</u> $\mathcal{B}$. <u>If</u> $E \in \mathcal{B}$, $\nu E < \infty$, <u>then for every</u> $\varepsilon > 0$ <u>there exist a compact</u> $K \subset E$ <u>and an open</u> $V \supset E$ <u>such that</u> $\nu(V \setminus K) < \varepsilon$.

Proof. There exist an open $V \supset E$ with $\nu V < \nu E + \frac{\varepsilon}{3}$, a compact $K_0 \subset V$ with $\nu K_0 > \nu V - \frac{\varepsilon}{3}$ and an open $U \supset V \setminus E$ with $\nu U < \nu(V \setminus E) + \frac{\varepsilon}{3}$. Set $K := K_0 \setminus U$; then K is compact, $K \subset E$ and $\nu(V \setminus K) = \nu(V \setminus K_0) + \nu(K_0 \setminus K) \leq \frac{\varepsilon}{3} + \nu(U) \leq \frac{\varepsilon}{3} + \nu(V \setminus E) + \frac{\varepsilon}{3} < \varepsilon$.

1.7. THEOREM. <u>The formula</u> $\mu \longmapsto \tilde{\mu}$ <u>defines a one-to-one correspondence between the positive linear functions on</u> $C_{00}(X)$ <u>and the positive Radon measures on</u> $\mathcal{B}$. <u>For positive functions</u> μ, μ_1, μ_2 <u>and for</u> $c \geq 0$ <u>we have</u>

$$(\mu_1 + \mu_2)^{\sim} = \tilde{\mu}_1 + \tilde{\mu}_2, \qquad (c\mu)^{\sim} = c\tilde{\mu}.$$

Proof. Let m be a positive Radon measure on $\mathcal{B}$. Every element of $C_{00}(X)$ is m-integrable, and $\mu : f \longmapsto \int f\, dm$ is a positive linear function on $C_{00}(X)$. We prove $\tilde{\mu} = m$.

Let $K \subset X$ be compact. By Lemma 1.3, $\tilde{\mu} K = \inf \{\int f dm : K \prec f\}$; so $\tilde{\mu} K \geq mK$. On the other hand, for every $\varepsilon > 0$ there exists an open $V \supset K$ with $mV \leq mK + \varepsilon$. There is an $f \in C_{00}(X)$ such that $K \prec f \prec V$. Then $\tilde{\mu} K \leq \int f dm \leq mV \leq mK + \varepsilon$. It follows that $\tilde{\mu} K = mK$ for all compact K. By regularity, for every open U we have $\tilde{\mu} U = \sup\{\tilde{\mu} K : K \text{ compact}, K \subset U\} = \sup\{mK : K \text{ compact}, K \subset U\} = mU$, and for every $E \in \mathcal{B}$, $\tilde{\mu} E = \inf\{\tilde{\mu} U : U \text{ open}, U \supset E\} = \inf\{mU : U \text{ open}, U \supset E\} = mE$.

Thus $\tilde{\mu} = m$. Consequently, $\mu \longmapsto \tilde{\mu}$ is a bijection of the set of all positive linear functions $C_{00}(X) \longrightarrow \mathbb{C}$ onto the set of all positive Radon measures on $\mathcal{B}$.

If μ_1, μ_2 are positive linear functions on $C_{00}(X)$, then $\tilde{\mu}_1 + \tilde{\mu}_2$ is a positive Radon measure, and $\int f\, d(\tilde{\mu}_1 + \tilde{\mu}_2) = (\mu_1 + \mu_2)(f)$ $(f \in C_{00}(X))$; so $\tilde{\mu}_1 + \tilde{\mu}_2 = (\mu_1 + \mu_2)^{\sim}$. Similarly, $c\tilde{\mu} = (c\mu)^{\sim}$.

1.8. LEMMA. Let m be a positive Radon measure on X. Let p = 1 or p = 2. For every $f \in L^p(m)$ and every $\varepsilon > 0$ there exists a $g \in C_{oo}(X)$ such that $\|f-g\|_{L^p(m)} < \varepsilon$.

Proof. First, let $f = \xi_E$ for some $E \in \mathcal{B}$. By Lemma 1.6 there exist a compact $K \subset E$ and an open $U \supset E$ such that $m(U\setminus K) < \varepsilon^p$ and by Lemma 1.3 there is a $g \in C_{oo}(X)$ for which $K \prec g \prec U$. Then $\|f-g\| = \sqrt[p]{\int |f-g|^p dm} \leq \sqrt[p]{\int (\xi_U - \xi_K)\, dm} < \varepsilon$.

For arbitrary $f \in L^p(m)$ there exist $E_1, \ldots, E_m \in \mathcal{B}$ and $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ such that $\|f - \sum \alpha_i \xi_{E_i}\| < \frac{\varepsilon}{2}$. We may assume $\alpha_i \neq 0$ for every i; then $\xi_{E_i} \in L^p(m)$, and we can find $g_1, \ldots, g_m \in C_{oo}(X)$ with $\|\xi_{E_i} - g_i\| < \frac{\varepsilon}{2m}|\alpha_i|^{-1}$. Setting $g := \sum \alpha_i g_i$ we have $g \in C_{oo}(X)$ and $\|f-g\| < \varepsilon$.

Identifying $f \in C_{oo}(X)$ with the class of all functions on X that are a.e. equal to f we may view $C_{oo}(X)$ as a subset of $L^p(m)$. (Note, however, that distinct elements of $C_{oo}(X)$ may determine the same element of $L^p(m)$). In this terminology, the above lemma says that $C_{oo}(X)$ is a dense subset of $L^p(m)$.

An element μ of $C_\infty(X)^*$ is called positive if $\mu(f) \geq 0$ for every $f \in C_\infty^+(X)$. Then the restriction of μ to $C_{oo}(X)$ is positive, so that there exists a measure $\tilde{\mu}$ on $\mathcal{B}$ with

$$\mu(f) = \int f \, d\tilde{\mu} \qquad (f \in C_{oo}(X)).$$

For $f \in C_\infty^+(X)$ and $n \in \mathbb{N}$, set $f_n := (f - \frac{1}{n})^+$; then $f_n \in C_{oo}(X)$, $f_1 \leq f_2 \leq \ldots$ and $\lim f_n = f$ uniformly. It follows that $\lim \mu(f_n) = \mu(f)$ and $\lim \int f_n \, d\tilde{\mu} = \int f \, d\tilde{\mu}$. We have proved:

1.9. THEOREM. Let $\mu \in C_\infty(X)^*$ be positive. There exists a (regular Borel) measure $\tilde{\mu}$ on the σ-algebra $\mathcal{B}$ of Borel sets of X such that

$$\mu(f) = \int f \, d\tilde{\mu} \qquad (f \in C_\infty(X)).$$

We have

$$\tilde{\mu}(X) = \|\mu\|.$$

An element μ of $C_\infty(X)^*$ is <u>real</u> if $\mu(f)\in\mathbf{R}$ for every real-valued $f\in C_\infty(X)$.

Then for arbitrary $f\in C_\infty(X)$ we have

$$\mu(f) = \mu(\mathrm{Re}\ f) + i\mu(\mathrm{Im}\ f)$$

where $\mu(\mathrm{Re}\ f)$ and $\mu(\mathrm{Im}\ f)$ are real numbers. Hence, for real $\mu\in C_\infty(X)^*$,

$$\mu(\mathrm{Re}\ f) = \mathrm{Re}\ \mu(f),\quad \mu(\mathrm{Im}\ f) = \mathrm{Im}\ \mu(f)\quad (f\in C_\infty(X)).$$

Every positive $\mu\in C_\infty(X)^*$ is real.

For arbitrary $\mu\in C_\infty(X)^*$ there exist unique linear functions $\mu_1,\mu_2 : C_\infty(X)\longrightarrow\mathbf{C}$ such that $\mu_1(f) = \mathrm{Re}\ \mu(f)$ and $\mu_2(f) = \mathrm{Im}\mu(f)$ for all <u>real-valued</u> $f\in C_\infty(X)$. Then $\mu=\mu_1+i\mu_2$ and we have

$$\begin{aligned}|\mu_1(f)| &\le |\mu_1(\mathrm{Re}f)|+|i\mu_1(\mathrm{Im}\ f)| = \\ &= |\mathrm{Re}\ \mu(\mathrm{Re}\ f)|+|\mathrm{Re}\ \mu(\mathrm{Im}\ f)| \\ &\le \|\mu\|\|\mathrm{Re}\ f\|+\|\mu\|\|\mathrm{Im}\ f\| \le 2\|\mu\|\|f\|.\end{aligned}$$

Hence, $\mu_1\in C_\infty(X)^*$. Similarly, $\mu_2\in C_\infty(X)^*$.

We see that every element of $C_\infty(X)^*$ is a linear combination of real elements of $C_\infty(X)^*$.

We shall now prove that every real element of $C_\infty(X)^*$ is a difference of two positive elements of $C_\infty(X)^*$.

Take $\mu\in C_\infty(X)^*$. For $f\in C_\infty^+(X)$ define $|\mu|(f)\in[0,\infty]$ by

$$|\mu|(f) := \sup\{|\mu(g)| : |g|\le f,\ g\in C_\infty(X)\}.$$

For $|g|\le f$ we have $|\mu(g)|\le\|\mu\|\|g\|\le\|\mu\|\|f\|$, so that

(1) $$|\mu|(f)\le\|\mu\|\|f\|<\infty\quad (f\in C_\infty^+(X)).$$

For all $g\in C_\infty(X)$ there exists a $c\in\mathbf{C}$, $|c| = 1$ so that $c\mu(g) = |\mu(g)|$. Then $cg\in C_\infty(X)$, $|cg| = |g|$ and $\mu(cg) = |\mu(g)| = |\mu(cg)|$. Hence,

(2) $$|\mu|(f) = \sup\{\mu(g) : g\in C_\infty(X);\ |g|\le f;\ \mu(g)\ge 0\}\quad (f\in C_\infty^+(X)).$$

Obviously,

(3) $$|\mu|(cf) = c|\mu|(f)\qquad (f\in C_\infty^+(X);\ c\ge 0).$$

Next, we prove that for all $f_1,f_2\in C_\infty^+(X)$

(4) $$|\mu|(f_1+f_2) = |\mu|(f_1)+|\mu|(f_2).$$

Take $\varepsilon>0$ and $g\in C_\infty(X)$ so that $|g|\le f_1+f_2$, $|\mu(g)|\ge|\mu|(f_1+f_2)-\varepsilon$. Set $g_i := g\dfrac{f_i}{f_1+f_2+\varepsilon}$ $(i=1,2)$. Then $g_i\in C_\infty(X)$ and $|g_i|\le f_i$, so $|\mu(g_i)|\le|\mu|(f_i)$. Further,

$$|g-g_1-g_2| = |g|\frac{\varepsilon}{f_1+f_2+\varepsilon} = \varepsilon\frac{|g|}{f_1+f_2+\varepsilon} \le \varepsilon$$

so that $|\mu(g)-\mu(g_1)-\mu(g_2)| \le \varepsilon\|\mu\|$. Then $|\mu|(f_1+f_2)-\varepsilon \le |\mu(g)|$ $\le |\mu(g_1)|+|\mu(g_2)|+\varepsilon\|\mu\| \le |\mu|(f_1)+|\mu|(f_2)+\varepsilon\|\mu\|$. This is true for every $\varepsilon>0$; consequently, $|\mu|(f_1+f_2) \le |\mu|(f_1)+|\mu|(f_2)$.

Conversely, for $\varepsilon>0$ choose $g_1, g_2 \in C_\infty(X)$ such that $|g_i| \le f_i$ and $\mu(g_i) \ge |\mu|(f_i)-\varepsilon$. Then $g_1+g_2 \in C_\infty(X)$, $|g_1+g_2| \le f_1+f_2$ and $|\mu|(f_1+f_2) \ge |\mu(g_1+g_2)| = \mu(g_1)+\mu(g_2)$ $\ge |\mu|(f_1)+|\mu|(f_2)-2\varepsilon$. Therefore, $|\mu|(f_1+f_2) \ge |\mu|(f_1)+|\mu|(f_2)$.

We have now proved (4). It follows easily from (3) and (4) that $|\mu|$ can be extended to a linear function $C_\infty(X) \to \mathbb{C}$. We use $|\mu|$ also to denote this extension. For all $f \in C_\infty(X)$ we have $|\mu|(\mathrm{Re}\, f) = \mathrm{Re}\,|\mu|(f)$. Now for $f \in C_\infty(X)$ choose $c \in \mathbb{C}$, $|c| = 1$, such that $c|\mu|(f) \ge 0$. Then $\big||\mu|(f)\big| = c|\mu|(f) = |\mu|(cf)$. Thus, $|\mu|(cf)$ is a real number, and $\big||\mu|(f)\big| = \mathrm{Re}\,|\mu|(cf) =$ $= |\mu|(\mathrm{Re}\, cf)$. Furthermore, $\mathrm{Re}\, cf \le |f|$, so $|\mu|(\mathrm{Re}\, cf) \le$ $\le |\mu|(|f|) \le \|\mu\|\|f\|$ (see (1)). We have found $\big||\mu|(f)\big| \le \|\mu\|\|f\|$ for all f, i.e. $\||\mu|\| \le \|\mu\|$.

Trivially, $|\mu(f)| \le |\mu|(|f|)$ for every f, so that $\|\mu\| \le \||\mu|\|$. We have proved

1.10. THEOREM. <u>Let</u> $\mu \in C_\infty(X)^*$. <u>Define</u> $|\mu| \in C_\infty(X)^*$ <u>as above</u>. <u>Then</u> $|\mu|$ <u>is positive</u>, $\||\mu|\| = \|\mu\|$; <u>and</u>

$$|\mu(f)| \le |\mu|(|f|) \qquad (f \in C_\infty(X)).$$

In particular, let μ be real. For $f \in C_\infty^+(X)$, $\mu(f) \le |\mu(f)|$ $\le |\mu|(f)$, so $|\mu|-\mu$ is positive. As $|\mu|$ is also positive we obtain

1.11. THEOREM. <u>Every real element of</u> $C_\infty(X)^*$ <u>is a difference of two positive ones. Every</u> $\mu \in C_\infty(X)^*$ <u>can be written in the form</u> $\mu_1-\mu_2+i\mu_3-i\mu_4$ <u>where</u> $\mu_1, \mu_2, \mu_3, \mu_4 \in C_\infty(X)^*$ <u>are positive</u>.

Then for $h \in C(X)$ we define $\int h\, d\mu = \int h d\mu_1 - \int h d\mu_2 + i\int h d\mu_3 - i\int h d\mu_4$; it is not hard to prove that $|\int h d\mu| \le \int |h| d|\mu|$.

Let $M(X)$ denote the vector space generated by the finite positive Radon measures on $\mathcal{B}$. If $\nu \in M(X)$, there exists a finite positive Radon measure π such that $|\nu(E)| \leq \pi(E)$ for every E. Therefore, every $\nu \in M(X)$ that is positive is a finite positive Radon measure. This observation justifies the following definition. The elements of $M(X)$ are called the finite Radon measures on X. The Riesz Representation Theorem, Theorem 1.11 and Lemma 1.7 imply

1.12. THEOREM. The correspondence $\mu \mapsto \tilde{\mu}$ mentioned in Lemma 1.7 induces a linear bijection $\mu \mapsto \tilde{\mu}$ of $C_\infty(X)^*$ onto $M(X)$.

From now on we shall usually identify μ with $\tilde{\mu}$.

If $\mu \in M(X)$, then $\bar{\mu} \in M(X)$, where

$$\bar{\mu}(E) := \overline{\mu(E)} \qquad (E \in \mathcal{B}).$$

We have

$$\|\bar{\mu}\| = \|\mu\|.$$

For $f \in C_\infty(X)$ one obtains

$$\int f \, d\bar{\mu} = \overline{\int \bar{f} \, d\mu}.$$

1.13. LEMMA. Let μ be a positive Radon measure on $\mathcal{B}$. For every $f \in L^1(\mu)$,

$$f\mu : h \longmapsto \int fh \, d\mu$$

is an element of $C_\infty(X)^*$. The map $f \longmapsto f\mu$ is a linear isometry of $L^1(\mu)$ into $C_\infty(X)^*$. Further, $\bar{f}\mu = (f\mu)^-$.

Proof. Let $f \in L^1(\mu)$. For all $h \in C_\infty(X)$, $|(f\mu)(h)| \leq \int |f| \, \|h\|_\infty \, d\mu = \|h\|_\infty \|f\|_1$. Hence, $f\mu \in C_\infty(X)^*$ and $\|f\mu\| \leq \|f\|_1$.

For the converse inequality, first assume $f \in C_{oo}(X)$. Then for every $n \in \mathbb{N}$, $\|f\mu\| \geq \left|(f\mu)\left(\frac{\bar{f}}{|f| + \frac{1}{n}}\right)\right| = \int \frac{f\bar{f}}{|f| + \frac{1}{n}} \, d\mu = \int |f| \frac{|f|}{|f| + \frac{1}{n}} \, d\mu$. The latter integral tends to $\int |f| \, d\mu$ as $n \to \infty$ (Lebesgue!). Hence, $\|f\mu\| \geq \|f\|_1$.

For arbitrary $f \in L^1(\mu)$, by Lemma 1.8 there exist $f_1, f_2, \ldots \in C_{oo}(X)$ such that $\lim f_n = f$ in $L^1(\mu)$. Then $\lim \|f_n\|_1 = \|f\|_1$. On the other hand, for every n, $\|f_n\mu - f\mu\| = \|(f_n - f)\mu\| \leq \|f_n - f\|_1$.

It follows that $\lim f_n\mu = f\mu$ in $C_\infty(X)^*$ and $\lim \|f_n\mu\| = \|f\mu\|$. Consequently, $\|f\mu\| = \lim \|f_n\mu\| \geq \lim \|f_n\|_1 = \|f\|_1$.

1.A. EXERCISE. Let μ be a positive Radon measure on $\mathcal{B}$; let $f \in L^1(\mu)$. Then for all $E \in \mathcal{B}$, $(f\mu)(E) = \int f\xi_E \, d\mu$. Further, $|f\mu| = |f|\mu$.

1.B. EXERCISE. Let μ be a finite positive Radon measure on X. Let φ be a continuous map of X into a locally compact space Y. The formula

$$\nu(A) := \mu(\varphi^{-1}A)$$

defines a (finite positive) measure ν on the Borel subsets of Y. This ν is again a Radon measure.

Hint. A finite positive measure π on a locally compact space Z is regular if and only if $\pi(A) = \sup\{\pi(K) : K \subset A,\ K \text{ compact}\}$.

(If μ is not finite, ν may fail to be regular. For an example, let $Y = \mathbb{R}$, let X be $\mathbb{R}$ under the discrete topology, let μ be the counting measure and let φ be the identity map. Then ν is again the counting measure, which is not regular on $\mathbb{R}$.)

1.C. EXERCISE. If $\mu \in M(X)$ and $\mu(X) = \|\mu\|$, then $\mu \geq 0$.

1.14. LEMMA. Let μ be a positive Radon measure on X. Let $\mathcal{F} \subset C_{00}{}^+(X)$, so that for all $f, g \in \mathcal{F}$ there exists an $h \in \mathcal{F}$ for which $h \geq f$, $h \geq g$. Let $F(x) := \sup\{f(x) : f \in \mathcal{F}\}$. Then F is a Borel measurable function $X \to [0,\infty]$, and

$$\int F \, d\mu = \sup\{\int f \, d\mu : f \in \mathcal{F}\}.$$

Proof. For every $a \in \mathbb{R}$, $\{x : F(x) > a\} = \bigcup_{f \in \mathcal{F}} \{x : f(x) > a\}$, so that $\{x : F(x) > a\}$ is open. It follows that F is Borel measurable. Choose $c \in \mathbb{R}$, $c < \int F \, d\mu$: we make an $f \in \mathcal{F}$ for which $\mu(f) > c$. Let $\lambda_1, \lambda_2, \ldots$ be an enumeration of $\mathbb{Q}$. For every i let U_i be the open set $\{x : F(x) > \lambda_i\}$. Then $F = \sup \lambda_i \xi_{U_i}$ (pointwise). By the Monotone Convergence Theorem there exists an $n \in \mathbb{N}$ such that

$$\int(\sup_{i\leq n}\lambda_i\xi_{U_i})d\mu > c.$$

For each i choose a compact $K_i \subset U_i$ so that $\int G\,d\mu > c$ where $G := \sup_{i\leq n}\lambda_i\xi_{K_i}$. (Here we use the regularity of μ).

Choose $\delta > 0$ so small that

$$\int G\,d\mu > c + \delta\mu(\bigcup_{i\leq n} K_i).$$

For every $a \in \bigcup K_i$ there exists an $f_a \in \mathcal{F}$ for which $f_a(a) > G(a) - \delta$. As G is upper semicontinuous, the inequality $f_a > G - \delta$ holds in a neighborhood of a. By compactness, we can choose $f_1,\dots,f_N \in \mathcal{F}$ such that $\max_j f_j > G - \delta$ on $\bigcup K_i$. There exists an $f \in \mathcal{F}$ which is $\geq f_j$ for each j; then

$$f \in \mathcal{F}, \quad f > G - \delta \text{ on } \bigcup K_i.$$

Consequently, $\mu(f) \geq \mu(G) - \delta\mu(\bigcup K_i) > c$.

1.15. THEOREM. Let X,Y be locally compact, let μ,ν be positive Radon measures on X,Y, respectively.

(a) If $f \in C_{oo}(X \times Y)$ then $x \longmapsto \int f(x,y)d\nu(y)$ and $y \longmapsto \int f(x,y)d\mu(x)$ are elements of $C_{oo}(X)$ and of $C_{oo}(Y)$, respectively. Moreover,

$$\iint f(x,y)d\nu(y)d\mu(x) = \iint f(x,y)d\mu(x)d\nu(y).$$

(b) There is a unique positive Radon measure $\mu\times\nu$ on $X \times Y$ such that

$$\int f d(\mu\times\nu) = \iint f(x,y)d\nu(y)d\mu(x) =$$
$$= \iint f(x,y)d\mu(x)d\nu(y) \qquad (f \in C_{oo}(X\times Y)).$$

Proof. (a) There exist compact $A \subset X$ and $B \subset Y$ such that $f = 0$ outside $A \times B$. The assertions of (a) do not change if we replace μ by $\xi_A\mu$ and ν by $\xi_B\nu$. Therefore we may assume μ and ν to be finite.

For $g \in C_{oo}(X)$ and $h \in C_{oo}(Y)$ define $g\otimes h \in C_{oo}(X\times Y)$ by

$$(g\otimes h)(x,y) := g(x)h(y) \qquad (x \in X;\ y \in Y).$$

Let E be the linear span in $C_{oo}(X\times Y)$ of $\{g\otimes h : g \in C_{oo}(X);\ h \in C_{oo}(Y)\}$. (a) is certainly valid if f is of the form $g\otimes h$, and consequently, for all $f \in E$. Now take any $f \in C_{oo}(X\times Y)$. By the Stone-Weierstrass Theorem there exist $f_1, f_2, \dots \in E$

such that $\lim \|f - f_i\|_\infty = 0$. Now for all $x \in X$ and i,

$$|\int f(x,y)d\nu(y) - \int f_i(x,y)d\nu(y)| \leq \int |f(x,y) - f_i(x,y)| d\nu(y) \leq$$
$$\leq \|f-f_i\|_\infty \, \nu(X).$$

Thus, $x \mapsto \int f(x,y)d\nu(y)$ is continuous, being a uniform limit of continuous functions. It is an element of $C_{oo}(X)$ since it vanishes off A. Similarly, $y \mapsto \int f(x,y)d\mu(x)$ is an element of $C_{oo}(Y)$. Furthermore,

$$\iint f(x,y)d\nu(y)d\mu(x) = \lim \iint f_i(x,y)d\nu(y)d\mu(x) =$$
$$= \lim \iint f_i(x,y)d\mu(x)d\nu(y) = \iint f(x,y)d\mu(x)d\nu(y).$$

(b) is a simple consequence of (a) and of Theorem 1.7.

Let $X, Y, \mu, \nu, \mu \times \nu$ be as above.

For a function f on $X \times Y$ and for $x \in X$ define $f_x : Y \longrightarrow \mathbb{C}$ by

$$f_x(y) := f(x,y) \qquad (y \in Y).$$

If f is sufficiently decent we can define $f_\nu : X \longrightarrow \mathbb{C}$ by

$$f_\nu(x) = \int f_x \, d\nu \qquad (x \in X).$$

Let $U \subset X \times Y$ be open. Set $\mathcal{F} := \{f \in C_{oo}(X \times Y) : f \prec U\}$, $\mathcal{F}_x := \{f_x : f \in \mathcal{F}\}$, $\mathcal{F}_\nu = \{f_\nu : f \in \mathcal{F}\}$. Then $\mathcal{F}_x \subset C_{oo}(Y)$, $\mathcal{F}_\nu \subset C_{oo}(X)$. We have

(*) $$(\mu \times \nu)(U) = \sup\{ (\mu \times \nu)(f) : f \in \mathcal{F} \}$$

by the definition of $\mu \times \nu$. We also have

$$\sup\{f_x : f \in \mathcal{F}\} = (\xi_U)_x \qquad (x \in X),$$

so that, by Lemma 1.14,

$$\sup\{\int f_x d\nu : f \in \mathcal{F}\} = \int (\xi_U)_x d\nu ,$$

i.e.

$$\sup\{g : g \in \mathcal{F}_\nu\} = (\xi_U)_\nu .$$

Applying Lemma 1.14 again we find

$$\sup\{\int g \, d\mu : g \in \mathcal{F}_\nu\} = \int (\xi_U)_\nu \, d\mu .$$

In other words,

$$\sup\{\int f(x,y)d\nu(y)d\mu(x) : f \in \mathcal{F}\} = \iint \xi_U(x,y)d\nu(y)d\mu(x),$$

which means (use (*))

$$(\mu \times \nu)(U) = \iint \xi_U(x,y)d\nu(y)d\mu(x).$$

Now let us assume that μ and ν are finite. Let $Z := X \times Y$. The subsets A of Z for which $(\xi_A)_x$ is Borel measurable for all $x \in X$ form a σ-algebra containing the open sets. Hence, <u>if</u> $A \subset Z$ <u>is a Borel set, then</u> $(\xi_A)_x$ <u>is Borel measurable</u> $(x \in X)$, so that $(\xi_A)_\nu$ exists.

We have just proved that $(\mu \times \nu)(A) = \int (\xi_A)_\nu \, d\mu$ if A is open. It follows that $\int f \, d(\mu \times \nu) = \int f_\nu \, d\mu$ for every f in the linear span E of $\{\xi_A : A \subset Z \text{ open}\}$. Let $\mathcal{W}$ be the collection of all subsets B of Z for which $\xi_B \in E$. Then

$$(\mu \times \nu)(B) = \int (\xi_B)_\nu \, d\mu \qquad (B \in \mathcal{W}).$$

$\mathcal{W}$ <u>contains all open sets. If</u> $A \in \mathcal{W}$ <u>then</u> $Z \setminus A \in \mathcal{W}$; <u>if</u> $A, B \in \mathcal{W}$ <u>then</u> $A \cap B \in \mathcal{W}$. (In fact, the product of any two elements of E is again an element of E.)

We are going to prove that

$$(**) \qquad (\xi_A)_\nu \text{ is Borel measurable, } (\mu \times \nu)(A) = \int (\xi_A)_\nu \, d\mu$$

for <u>all</u> Borel subsets A of Z.

A collection $\mathcal{A}$ of subsets of Z is <u>closed</u> if the following condition is satisfied. If $A_1, A_2, \ldots \in \mathcal{A}$ and if $A \subset Z$ is such that $\lim \xi_{A_n} = \xi_A$ pointwise, then $A \in \mathcal{A}$.

The intersection $\mathcal{A}_0$ of all closed collections that contain $\mathcal{W}$ is itself closed. By the Lebesgue Convergence Theorem one proves that the Borel sets for which (**) is true form a closed collection. As this collection contains $\mathcal{W}$ it also contains $\mathcal{A}_0$. Thus,

every $A \in \mathcal{A}_0$ has property (**).

Thus, we are done if $\mathcal{A}_0$ contains all Borel sets. This, in turn, will be be true if $\mathcal{A}_0$ is a σ-algebra. First, if $A_1, A_2, \ldots \in \mathcal{A}_0$ and if $A_1 \subset A_2 \subset \ldots$ then $\xi_{\cup A_n} = \lim \xi_{A_n}$, so $\bigcup A_n \in \mathcal{A}_0$. Second, $\{A \subset Z : Z \setminus A \in \mathcal{A}_0\}$ is a closed collection that contains $\mathcal{W}$ and therefore contains $\mathcal{A}_0$: consequently, if $A \in \mathcal{A}_0$ then $Z \setminus A \in \mathcal{A}_0$. It remains to prove that $A \cap B \in \mathcal{A}_0$ whenever $A, B \in \mathcal{A}_0$. $\{A \subset Z : A \cap B \in \mathcal{A}_0 \text{ for all } B \in \mathcal{W}\}$ is closed and contains $\mathcal{W}$. It follows that $A \cap B \in \mathcal{A}_0$ if $A \in \mathcal{A}_0$ and $B \in \mathcal{W}$. But then $\{B \subset Z : A \cap B \in \mathcal{A}_0 \text{ for all } A \in \mathcal{A}_0\}$ is closed and contains $\mathcal{W}$. Hence, $A \cap B \in \mathcal{A}_0$ for all $A, B \in \mathcal{A}_0$.

It follows now that (**) is true for all Borel sets $A \subset Z$. By interchanging the roles of μ and ν one obtains the following.

1.16. LEMMA. Let μ, ν be finite positive Radon measures on X,Y, respectively. For every Borel set $A \subset X \times Y$ the functions $x \longmapsto \int \xi_A(x,y)d\nu(y)$ and $y \longmapsto \int \xi_A(x,y)d\mu(x)$ are Borel measurable and

$$(\mu \times \nu)(A) = \iint \xi_A(x,y)d\nu(y)d\mu(x) = \iint \xi_A(x,y)d\mu(x)d\nu(y).$$

The finiteness condition can be weakened but may not be omitted entirely. The most useful generalization of the above lemma is the following.

1.17. FUBINI THEOREM. Let μ, ν be positive Radon measures on X and Y, respectively. Let $F : X \times Y \longrightarrow [0,\infty]$ be Borel measurable. Assume that one of the following three assertions is valid.

(a) μ and ν are finite.

(b) μ is finite. There is a σ-compact $Y_0 \subset Y$ such that $F = 0$ off $X \times Y_0$.

(c) There exist σ-compact $X_0 \subset X$, $Y_0 \subset Y$ such that $F = 0$ off $X_0 \times Y_0$.

Then $x \longmapsto \int F(x,y)d\nu(y)$ and $y \longmapsto \int F(x,y)d\mu(x)$ are Borel measurable maps $X \longrightarrow [0,\infty]$ and $Y \longrightarrow [0,\infty]$, respectively and

$$\int F d(\mu \times \nu) = \iint F(x,y)d\nu(y)d\mu(x) = \iint F(x,y)d\mu(x)d\nu(y).$$

Proof. (a) The assertions of the theorem are certainly true for an F that is a finite linear combination of characteristic functions of Borel sets. Then they are also true for an F that is a uniform limit of such linear combinations, i.e. for all bounded Borel functions F.

For $n \in \mathbb{N}$ set $F_n = \min(F,n)$; every F_n is a bounded Borel function. Further, $F_1 \leq F_2 \leq \dots$ and $\lim F_n = F$. The conclusions of the theorem now follow from the Monotone Convergence Theorem.

(b) There exist compact sets $Y_1, Y_2, \dots$ such that

$Y_1 \subset Y_2 \subset \ldots$ and $\cup Y_n = Y_o$. Set $\nu_n = \xi_{Y_n} \nu$. Apply (a) to μ and ν_n and again use Monotone Convergence Theorem.

We leave (c) to the reader.