4. CONVOLUTION.

G IS A LOCALLY COMPACT GROUP WITH HAAR MEASURE m.

Let α denote the multiplication map $G \times G \longrightarrow G$. If $k \in C_\infty(G)$, then $k \circ \alpha$ is a bounded continuous function on $G \times G$. For $\mu, \nu \in M(G)$, $\mu \times \nu$ is a finite complex measure on the Borel sets of $G \times G$, so the <u>convolution</u> of μ and ν defined by

$$(\mu * \nu)(k) := \int k \circ \alpha \, d(\mu \times \nu) \qquad (k \in C_\infty(G))$$

exists. We have

$$(\mu * \nu)(k) = \iint k(xy)\, d\mu(x)\, d\nu(y) = \iint k(xy)\, d\nu(y)\, d\mu(x)$$

so that $|(\mu * \nu)(k)| \leq \|k\|_\infty \|\mu\| \|\nu\|$. It follows that for fixed $\mu, \nu \in M(G)$, $\mu * \nu$ is an element of $M(G)$ and

$$\|\mu * \nu\| \leq \|\mu\| \|\nu\|.$$

4.A. EXERCISE. <u>For</u> $k \in C(G)$ <u>and</u> $\mu, \nu \in M(G)$,

$$\int k\, d(\mu * \nu) = \iint k(xy)\, d\mu(x)\, d\nu(y) = \iint k(xy)\, d\nu(y)\, d\mu(x).$$

<u>In particular</u>, $(\mu * \nu)(G) = \mu(G)\nu(G)$.

From Exercise 4.A and the Fubini Theorem one obtains

$$\mu * (\nu * \pi) = (\mu * \nu) * \pi \qquad (\mu, \nu, \pi \in M(G)).$$

Thus, <u>the multiplication</u> $*$ <u>turns</u> $M(G)$ <u>into a Banach algebra</u>.

If $\delta \in M(G)$ is defined by

$$\delta(k) := k(1) \qquad (k \in C_\infty(G))$$

where 1 is the identity element of G, then

$$\mu * \delta = \delta * \mu = \mu \qquad (\mu \in M(G)),$$

so that $M(G)$ <u>has an identity element</u>.

Clearly, <u>if</u> G <u>is commutative</u>, <u>then so is</u> $M(G)$. The converse is also easy to prove: if, for $x \in G$ we define $\delta_x \in M(G)$ by

$$\delta_x(k) := k(x) \qquad (k \in C_\infty(G)),$$

then

$$\delta_x * \delta_y = \delta_{xy} \qquad (x, y \in G).$$

The following formula is easy to verify.

$$(\mu * \nu)' = \nu' * \mu' \qquad (\mu, \nu \in M(G)).$$

By Lemma 1.13 we have a linear isometry $f \longmapsto fm$ of $L^1(G)$ into $M(G)$ given by

$$(fm)(k) = \int kf\,dm.$$

We indicate the set $\{fm : f \in L^1(G)\}$ by $M_a(G)$. By a general form of the Radon-Nikodym Theorem it can be proved that $M_a(G) = \{\mu \in M(G) : \mu \ll m\}$, but we shall not need this fact.

Let $\mu \in M(G)$, $g \in L^1(G)$, $\mu \geq 0$, $g \geq 0$. By Exercise 3.F, G has a σ-compact open subgroup G_o such that $g = 0$ off G_o and $\mu(G \setminus G_o) = 0$.

If $k \in C(G)$ and $k = 0$ on G_o then $\int k(xy)\,d\mu(x) = 0$ for all $y \in G_o$, so that (Exercise 4.A) $\int k\,d(\mu * gm) = \iint k(xy)\,d\mu(x)g(y)\,dy = 0$. Now take any $k \in C^+(G)$. Then

$$\int k\,d(\mu * gm) = \int k\xi_{G_o}\,d(\mu * gm) =$$

$$\text{(Exercise 4.A)} = \iint k(xy)\xi_{G_o}(xy)g(y)\,d\mu(x)\,dy =$$

$$= \iint k(y)\xi_{G_o}(y)g(x^{-1}y)\,dy\,d\mu(x) =$$

$$\text{(Fubini Theorem 1.17 (b))} = \iint k(y)\xi_{G_o}(y)g(x^{-1}y)\,d\mu(x)\,dy =$$

$$= \int k\xi_{G_o}h$$

where we define

$$h(y) := \int g(x^{-1}y)\,d\mu(x) \qquad (y \in G).$$

It is easy to see that $(x,y) \longmapsto g(x^{-1}y)$ is Borel measurable $G \times G \to [0,\infty]$. Then, according to Fubini Theorem, h is Borel measurable $G \to [0,\infty]$. Clearly h vanishes off G_o. Thus,

$$\int k\,d(\mu * gm) = \int kh \qquad (k \in C^+(G)).$$

Taking $k = 1$ we see that h is m-integrable. Apparently, $\mu * gm = hm \in M_a(G)$.

A posteriori we see that $\mu * gm \in M_a(G)$ for all $\mu \in M(G)$ and $g \in L^1(G)$; if $h \in L^1(G)$ and $\mu * gm = hm$ then $h(y) = \int g(x^{-1}y)\,d\mu(x)$ for almost every y. We have proved part of the following theorem, the rest of which we leave to the reader.

4.1. THEOREM. $M_a(G)$ <u>is a (closed) two-sided ideal in</u> $M(G)$. <u>For</u> $\mu, \nu \in M(G)$ <u>and</u> $f, g \in L^1(G)$ <u>we can define elements</u> $\mu * g$, $f * \nu$ <u>and</u> $f * g$ <u>of</u> $L^1(G)$ <u>by</u>

$$(\mu * g)m := \mu * gm,$$
$$(f * \nu)m := fm * \nu,$$
$$(f * g)m := fm * gm.$$

<u>Then</u>

$$(\mu * g)(y) = \int g(x^{-1}y)\,d\mu(x) \qquad \text{(almost all } y \in G)$$

$$(f * \nu)(y) = \int f(yx^{-1})\,\Delta(x^{-1})\,d\nu(x) \quad \text{(almost all } y \in G\text{)},$$

$$(f*g)(y) = \int f(x)g(x^{-1}y)\,dx = \int f(yx)g(x^{-1})\,dx \quad \text{(almost all } y \in G\text{)}.$$

For $f, g \in L^1(G)$, $a \in G$, we have

$$(f*g)' = g' * f', \quad f_a = \delta_a * f, \quad (f*g)_a = f_a * g.$$

The mapping $(f,g) \longmapsto f*g$ defines a multiplication (the convolution) in $L^1(G)$ that turns $L^1(G)$ into a Banach algebra.

If G is commutative, so is $L^1(G)$. The converse statement is also true, but harder to prove. (See Exercise 4.G. or 4.B.)

In general, $L^1(G)$ will not have an identity. But it does have an approximate identity. A net $(e_\lambda)_{\lambda \in \Lambda}$ in a Banach algebra A is a left (right) approximate identity if $\lim e_\lambda x = x$ ($\lim xe_\lambda = x$) for all $x \in A$. A net that is both a left and a right approximate identity is a two-sided approximate identity. An approximate identity $(e_\lambda)_{\lambda \in \Lambda}$ is bounded if there exists a number s such that $\|e_\lambda\| \leq s$ for every λ; such an s is called a bound for the approximate identity.

4.2. THEOREM. $L^1(G)$ has a two-sided approximate identity of bound 1. The elements of this approximate identity can be chosen from $C_{oo}{}^+(G)$.

This theorem is an easy consequence of Lemma 4.3.

4.3. LEMMA. Let $f \in L^1(G)$, $\varepsilon > 0$. There exists a neighborhood U of 1 with the following property. If $u \in L^1(G)$, $u \geq 0$, $\|u\| = 1$ and $u = 0$ off U, then $\|u*f - f\| \leq \varepsilon$.

Proof. By Theorem 3.13 there is a neighborhood U of 1 such that $\|f_x - f\| \leq \varepsilon$ for all $x \in U$. Now let $u \in L^1(G)$, $u \geq 0$, $\|u\| = 1$, $u = 0$ on $G \setminus U$. Then $\int u(x)\,dx = 1$, and

$$\|u*f-f\| = \int\left|\int u(x)f(x^{-1}y)dx - f(y)\right| dy =$$
$$= \int\left|\int u(x)\,[f(x^{-1}y)-f(y)]\,dx\right|dy.$$

By Exercise 3.F, G has a σ-compact open subgroup G_o outside of which $u=f=0$. Now $(x,y) \longmapsto u(x)|f(x^{-1}y)-f(y)|$ is a measurable function on $G \times G$ that vanishes off $G_o \times G_o$. By the Fubini Theorem 1.14 we obtain

$$\begin{aligned}\|u*f-f\| &\le \iint u(x)\,|f(x^{-1}y)-f(y)|\,dxdy = \\ &= \iint u(x)\,|f_x(y)-f(y)|\,dydx = \\ &= \int u(x)\,\|f_x-f\|\,dx \le \left(\int u(x)dx\right) \sup_{x\in U}\|f_x-f\| \le \varepsilon.\end{aligned}$$

Proof of THEOREM 4.2. Let Λ be the set of all compact neighborhoods of 1. For each $V\in\Lambda$ choose $e_V \in C_{oo}^{+}(G)$ so that $e_V=0$ outside V and $\int e_V(x)dx=1$. For $V_1,V_2 \in\Lambda$ define $V_1 \prec V_2$ if $V_1 \supset V_2$. Thus, $\{e_V : V\in\Lambda\}$ is a net. From Lemma 4.3 it follows that $\lim_V e_V*f = f$; and also that $\lim_V e_V'*f' = f'$, so that $\lim_V f*e_V = f$.

4.4. LEMMA. <u>For all</u> $\varphi\in L^1(G)^*$ <u>we have</u>

$$\varphi(f*g) = \int f(x)\varphi(g_x)dx \qquad (f,g \in L^1(G)).$$

Proof. Let $\varphi\in L^1(G)^*$ and $f,g \in L^1(G)$. Take a σ-compact open subgroup G_o of G outside which $f=g=0$. Then $\xi_{G_o} m$ is a σ-finite measure on the Borel sets of G. From Theorem 0.10 it follows that there exists a bounded Borel function j such that

$$\varphi(h) = \int hjdm \qquad (h \in L^1(G);\ h=0 \text{ off } G_o).$$

Therefore,

$$\varphi(f*g) = \int (f*g)jdm$$

and

$$\varphi(g_x) = \int g_x jdm \qquad (x \in G_o).$$

Applying the Fubini Theorem 1.14 to the function $(x,y) \longmapsto f(x)g(x^{-1}y)j(y)$ (which vanishes outside $G_o \times G_o$) we obtain

$$\varphi(f*g) = \iint f(x)g(x^{-1}y)j(y)dxdy =$$
$$= \int f(x)\int g(x^{-1}y)j(y)dydx = \int f(x)\varphi(g_x)dx.$$

4.B. EXERCISE. <u>If</u> $f,g \in C_{oo}(G)$, <u>then</u> $f*g \in C_{oo}(G)$, <u>and</u> $\operatorname{supp} f*g \subset (\operatorname{supp} f)(\operatorname{supp} g)$.

4.C. EXERCISE. <u>If</u> $\mu,\nu \in M(G)$ <u>then</u>

$$\int h\; d(\mu*\nu) = \iint h(xy)d\mu(x)d\nu(y)$$

<u>for every bounded Borel measurable function</u> h <u>on</u> G.

Hint. Use Exercise 1.B.

4.D. EXERCISE. <u>For</u> $\mu \in M(G)$ <u>and</u> $x \in G$ <u>set</u> $\mu_x := \delta_x * \mu$.

(a) $\mu_x(A) = \mu(x^{-1}A)$ $(\mu \in M(G);\; x \in G;\; A \subset G$ <u>Borel</u>).

(b) $(fm)_x = f_x m$ $(f \in L^1(G);\; x \in G)$.

(c) <u>For every</u> x, $\mu \mapsto \mu_x$ <u>is a surjective linear isometry</u> $M(G) \longrightarrow M(G)$. <u>For all</u> x,y <u>and</u> μ, $\mu_{xy} = (\mu_y)_x$.

(d) <u>For</u> $\mu \in M(G)$ <u>the following are equivalent</u>.

1) $\mu \in M_a(G)$.

2) <u>For all</u> $\varepsilon > 0$ <u>there exists a neighborhood</u> U <u>of</u> 1 <u>such that</u> $\|\mu_x - \mu\| \le \varepsilon$ $(x \in U)$.

3) $x \mapsto \mu_x$ <u>is a continuous map</u> $G \longrightarrow M(G)$.

4.E. EXERCISE. <u>If</u> H <u>is a closed subgroup of</u> G <u>that is of positive</u> σ-<u>finite</u> (<u>Haar</u>) <u>measure</u>, <u>then</u> H <u>is open in</u> G.

(Hint. Let U be a compact neighborhood of $1 \in G$. As H is contained in a σ-compact set (Exercise 3.F (d)), $0 < m(U \cap H) < \infty$. Put $B := U \cap H$ and $f := \xi_B * \xi_B$. Then

$$f(x) = \int \xi_B (\xi_{B^{-1}})_x dm \qquad (x \in G).$$

f is continuous, $f(1)\neq 0$, and f vanishes off H.)

4.F. EXERCISE. Every $\mu \in M(G)$ induces a linear $T_\mu : L^1(G) \longrightarrow L^1(G)$ by

$$T_\mu(f) := \mu * f.$$

We have $T_{\mu * \nu} = T_\mu T_\nu$ and $\|T_\mu\| = \|\mu\|$.

Hint for the last part: Let $\varepsilon > 0$, choose $k \in C_\infty(G)$ so that $\|k\|=1$ and $|\mu(k)| \geq \|\mu\| - \varepsilon$. The function $j : x \longmapsto \mu(k_x)$ is an element of $C_\infty(G)$ and $(\mu * f)(k) = \int fj\,dm$ for all $f \in L^1(G)$. Choose f so that $\|f\|=1$ and $|\int fj\,dm - j(1)| < \varepsilon$.

4.G. EXERCISE. If $L^1(G)$ is abelian, then G is abelian.

Hint. Let $a, b \in G$. Show that $\delta_{ab} * f * g = \delta_{ba} * f * g$ for all $f, g \in L^1(G)$, and apply Exercise 4.F with $\mu = \delta_{ab} - \delta_{ba}$.