III. THE ABELIAN CASE.

5. THE DUAL GROUP.

IN THIS SECTION, G IS AN ABELIAN LOCALLY COMPACT GROUP WITH HAAR MEASURE m.

Now $L^1(G)$ is a commutative Banach algebra. We are going to investigate its structure space $\mathfrak{M}(L^1(G))$.

Let $\varphi \in \mathfrak{M}(L^1(G))$. For all $f,g \in L^1(G)$ and $x \in G$,
$\varphi(f_x)\varphi(g)=\varphi(f_x*g)=\varphi((f*g)_x)=\varphi((g*f)_x)=\varphi(g_x*f)=\varphi(g_x)\varphi(f)$.
Hence (as $\varphi\neq 0$) there exists a unique $\gamma: G \to \mathbb{C}$ such that

$$\varphi(f_x) = \overline{\gamma(x)}\varphi(f) \qquad (f \in L^1(G);\ x \in G).$$

(For historical reasons we write $\overline{\gamma(x)}$ and not $\gamma(x)$.)
By Theorem 3.13 γ is continuous. Further, for all $f \in L^1(G)$, $x,y \in G$ we find

$$\overline{\gamma(xy)}\varphi(f)=\varphi(f_{xy})=\varphi((f_y)_x)=\overline{\gamma(x)}\varphi(f_y)=\overline{\gamma(x)}\,\overline{\gamma(y)}\varphi(f).$$

Thus,

$$\gamma(xy) = \gamma(x)\gamma(y) \qquad (x,y \in G).$$

Further, $\sup_x|\gamma(x)||\varphi(f)|=\sup_x|\varphi(f_x)| \le \|\varphi\|\sup_x\|f_x\|=\|\varphi\|\,\|f\| < \infty$,
so that γ is bounded. Then for every $x \in G$ the set $\{\gamma(x)^n : n\in\mathbb{Z}\}$ must be bounded, which is only possible if

$$|\gamma(x)| = 1 \qquad (x \in G).$$

In particular,

$$\gamma(x^{-1}) = \overline{\gamma(x)} \qquad (x \in G).$$

By Lemma 4.4, for all $f,g \in L^1(G)$ it follows that

$$\varphi(f*g) = \int f(x)\varphi(g_x)dx =$$
$$= \int f(x)\overline{\gamma(x)}\varphi(g)dx = \varphi(g)\int f(x)\overline{\varphi(x)}dx.$$

As also $\varphi(f*g)=\varphi(f)\varphi(g)$ and we can choose g so that $\varphi(g)\neq 0$, we get

$$(*) \qquad \varphi(f) = \int f(x)\overline{\gamma(x)}dx \qquad (x \in G).$$

Conversely, let γ be a continuous homomorphism of G into $\{z \in \mathbb{C} : |z| = 1\}$. Now we can use (*) to define a $\varphi \in L^1(G)^*$. For $g \in L^1(G)$ and $y \in G$,

$$\varphi(g_y) = \int g_y(x)\overline{\gamma(x)}dx = \int g(y^{-1}x)\overline{\gamma(x)}dx =$$
$$= \int g(x)\overline{\gamma(yx)}dx = \overline{\gamma(y)}\int g(x)\overline{\gamma(x)}dx = \overline{\gamma(y)}\,\varphi(g).$$

Lemma 4.4 yields $\varphi(f*g) = \varphi(f)\varphi(g)$, so φ is a (non-zero, hence surjective) homomorphism $L^1(G) \longrightarrow \mathbb{C}$.

The continuous homomorphisms $G \longrightarrow \{z \in \mathbb{C} : |z| = 1\}$ are called the <u>characters</u> of G. Under pointwise multiplication they form an abelian group denoted Γ or $\hat{G}$. We have

$$\gamma^{-1} = \bar{\gamma} \qquad (\gamma \in \Gamma).$$

If $\gamma_1, \gamma_2 \in \Gamma$ are distinct, one easily finds an $f \in L^1(G)$ with $\int f\bar{\gamma}_1 \neq \int f\bar{\gamma}_2$. Thus, we have proved:

5.1. THEOREM. <u>Formula</u> (*) <u>establishes a bijection</u> $\Gamma \longrightarrow \mathfrak{M}(L^1(G))$.

We endow Γ with the locally compact topology such that this bijection is a homeomorphism. Note that we have not yet proved that Γ is a topological group (although for each $\beta \in \Gamma$ the map $\gamma \longmapsto \beta\gamma$ is easily seen to be a homeomorphism).

We shall view the Gelfand Transformation as a map $L^1(G) \longrightarrow C_\infty(\Gamma)$. For $f \in L^1(G)$, $\hat{f} : \Gamma \to \mathbb{C}$ is then given by

$$\hat{f}(\gamma) = \int f(x)\overline{\gamma(x)}dx \qquad (\gamma \in \Gamma).$$

If G is discrete, then $M_a(G) = M(G)$ (see Theorem 3.11), so $L^1(G)$ has an identity, and $\mathfrak{M}(L^1(G))$ and Γ are compact.

Conversely, suppose G is compact. If $\gamma \in \Gamma$ and $x \in G$, then $\gamma_x = \overline{\gamma(x)}\,\gamma$, so $\int \gamma\, dm = \int \gamma_x dm = \overline{\gamma(x)}\int \gamma\, dm$. Therefore:

5.2. LEMMA. <u>Let</u> G <u>be compact</u>. <u>Then</u>

$$\int \gamma\, dm = 0 \quad \underline{\text{if}}\ \gamma \in \Gamma,\ \gamma \neq 1$$
$$= m(G) \quad \underline{\text{if}}\ \gamma = 1.$$

Therefore, if G is compact, then $\hat{\gamma} = m(G)\,\xi_{\{\gamma\}}$ for every

$\gamma \in \Gamma$. But $\hat{\gamma}$ has to be a continuous function on $\mathfrak{M}(L^1(G))$. It follows that Γ is discrete. We have proved the following theorem.

5.3. THEOREM. *If* G *is discrete*, Γ *is compact*. *If* G *is compact*, Γ *is discrete*.

Let $A(\Gamma) := \{\hat{f} : f \in L^1(G)\}$. $A(\Gamma)$ is a subalgebra of $C_\infty(\Gamma)$, separating the points of Γ.

For $f \in L^1(G)$ define $\tilde{f} : G \longrightarrow \mathbb{C}$ by

$$\tilde{f}(x) = \overline{f(x^{-1})} \qquad (x \in G).$$ (See page 3.12.)

By Theorem 3.18, $\tilde{f} \in L^1(G)$ and $\int \tilde{f}\, dm = \overline{\int f dm}$. We have $\tilde{f} * \tilde{g} = (f*g)^\sim$ for all $f, g \in L^1(G)$.

5.4. LEMMA. *For every* $f \in L^1(G)$,

$$\hat{\tilde{f}} = \overline{\hat{f}} \quad \underline{\text{and}} \quad \tilde{\hat{f}} = \tilde{\bar{f}}.$$

Hence, if $j \in A(\Gamma)$, then $\bar{j} \in A(\Gamma)$. By the Stone-Weierstrass Theorem it follows:

5.5. THEOREM. $A(\Gamma)$ *is a dense subalgebra of* $C_\infty(\Gamma)$.

For $h : \Gamma \longrightarrow \mathbb{C}$, $\gamma \in \Gamma$, $h_\gamma : \Gamma \longrightarrow \mathbb{C}$ is

$$h_\gamma(\beta) := h(\gamma^{-1}\beta).$$

5.6. THEOREM. (a) *If* $j \in A(\Gamma)$, *then* $j_\gamma \in A(\Gamma)$ *for every* $\gamma \in \Gamma$. (In fact, if $f \in L^1(G)$ and $\gamma \in \Gamma$, then $\hat{f}_\gamma = (f\gamma)\hat{} \in A(\Gamma)$.)

(b) *If* $j \in A(\Gamma)$ *and* $a \in G$, *then* $\gamma \longmapsto \gamma(a)j(\gamma)$ *is an element of* $A(\Gamma)$. (In fact, if $f \in L^1(G)$ and $a \in G$, then $\gamma(a)f(\gamma) = \widehat{f_{a^{-1}}}(\gamma)$ for every $\gamma \in \Gamma$.)

5.7. LEMMA. $(x, \gamma) \longmapsto \gamma(x)$ *is a continuous map* $G \times \Gamma \longrightarrow \mathbb{C}$.

Proof. Let $(x_0, \gamma_0) \in G \times \Gamma$; choose $f \in L^1(G)$ so that $\hat{f}(\gamma_0) \neq 0$. For all $(x, \gamma) \in G \times \Gamma$,

$$|\hat{f}_x(\gamma)-\hat{f}_{x_0}(\gamma_0)| \le |(f_x-f_{x_0})\hat{}\,(\gamma)|+|\hat{f}_{x_0}(\gamma)-\hat{f}_{x_0}(\gamma_0)| \le$$
$$\le \|f_x-f_{x_0}\|+|\hat{f}_{x_0}(\gamma)-\hat{f}_{x_0}(\gamma_0)|.$$

From the continuity of $x \longmapsto f_x$ (see Theorem 3.13) and of $\hat{f}_{x_0}$ we infer that $(x,\gamma)\longmapsto \hat{f}_x(\gamma)$ is continuous at (x_0,γ_0). But by Theorem 5.6 (b), $\hat{f}_x(\gamma)=\overline{\gamma(x)}\hat{f}(\gamma)$, while $(x,\gamma)\longmapsto \hat{f}(\gamma)$ is continuous and does not take the value 0 at (x_0,γ_0). The lemma follows.

For compact $K\subset G$ and $\rho>0$ we set

$$N(K:\rho) := \{\gamma\in\Gamma : |\gamma-1|<\rho \text{ on } K\}.$$

From the above lemma and from Lemma 3.2 it follows that $N(K:\rho)$ is an open set in Γ.

5.8. LEMMA. A set $W\subset\Gamma$ is open if and only if for every $\gamma\in W$ there exist a compact $K\subset G$ and a $\rho>0$ such that $\gamma N(K:\rho)\subset W$. (Thus, the $N(K:\rho)$ form a base for the topology of Γ.)

Proof. The "if" is obvious; we are done if every neighborhood of $1\in\Gamma$ contains some $N(K:\rho)$.

Let $W\subset\Gamma$ be a neighborhood of 1. By the definition of the topology of Γ there exist $n\in\mathbb{N}$, $f_1,\dots,f_n\in L^1(G)$ and $\delta>0$ such that $W\supset\{\gamma : |\hat{f}_i(\gamma)-\hat{f}_i(1)| < 3\delta \text{ for each } i\}$. There exist $g_1,\dots,g_n\in C_{00}(G)$ with $\|f_i-g_i\|<\delta$ (Lemma 1.8). Then $|\hat{f}_i(\gamma)-\hat{g}_i(\gamma)|\le\delta$ for all i and γ, so that $W\supset\{\gamma : |\hat{g}_i(\gamma)-\hat{g}_i(1)| < \delta$ for every $i\}$. Set $K:=\bigcup_i \operatorname{supp} g_i$ and choose $\rho>0$ so that $\|g_i\|_\infty\cdot m(K)\rho<\delta$ $(i=1,\dots,n)$. We prove $N(K:\rho)\subset W$. Let $\gamma\in N(K:\rho)$ and $i\in\{1,\dots,n\}$; we have to show that $|\hat{g}_i(\gamma)-\hat{g}_i(1)|<\delta$. Now in fact $|\hat{g}_i(\gamma)-\hat{g}_i(1)| \le \int|g_i||\gamma-1|\,dm =$
$= \int_K|g_i||\gamma-1|\,dm \le \|g_i\|_\infty\int_K|\gamma-1|\,dm \le \|g_i\|\rho m(K) < \delta$.

5.9. THEOREM. Γ is a (locally compact, abelian) topological group.

Proof. Define $\pi : \Gamma \times \Gamma \longrightarrow \Gamma$ by $\pi(\beta,\gamma) := \beta\gamma^{-1}$. We have to prove π to be continuous. Let $W \subset \Gamma$ be open, $(\beta,\gamma) \in \pi^{-1}(W)$. Then $\beta\gamma^{-1} \in W$, so there exist K, ϱ with $\beta\gamma^{-1}N(K:\varrho) \subset W$. Now $\beta N(K:\frac{1}{2}\varrho) \times \gamma N(K:\frac{1}{2}\varrho)$ is a neighborhood of (β,γ) in $\Gamma\times\Gamma$ that is easily seen to be contained in $\pi^{-1}(W)$.

The topological group Γ (or $\hat{G}$) is called the <u>dual group</u> of G. The function $\hat{f} : \Gamma \longrightarrow \mathbb{C}$ is the <u>Fourier transform</u> of $f \in L^1(G)$.

For an example, let $G = \mathbb{R}$. Every $a \in \mathbb{R}$ determines a character $\gamma^{(a)}$ by

$$\gamma^{(a)}(x) := e^{iax} \qquad (x \in \mathbb{R}).$$

We shall prove that every character is of this form. Let $\gamma \in \hat{\mathbb{R}}$. Then γ is continuous and $\gamma(0)=1$, so there is an $r>0$ with $\int_0^r \gamma(t)dt \neq 0$. For every $x \in \mathbb{R}$,

$$\int_x^{x+r}\gamma(t)dt = \int_0^r \gamma(x+t)dt = \int_0^r \gamma(x)\gamma(t)dt = \gamma(x)\int_0^r \gamma(t)dt.$$

The left hand member is a differentiable function of x, so γ must be differentiable, and for all $x \in \mathbb{R}$ we have

$$\gamma'(x) = \frac{d}{dt}\gamma(x+t)\Big]_{t=0} = \frac{d}{dt}\gamma(x)\gamma(t)\Big]_{t=0} = \gamma'(0)\gamma(x).$$

It follows that

$$\gamma(x) = \gamma(0)e^{\gamma'(0)x} = e^{\gamma'(0)x} \qquad (x \in \mathbb{R}).$$

As $|\gamma(x)|=1$ for all x, the exponent must be purely imaginary, i.e. $\gamma'(0) = ia$ for some $a \in \mathbb{R}$. But then $\gamma = \gamma^{(a)}$.

Thus, $a \longmapsto \gamma^{(a)}$ is a surjection $\mathbb{R} \longrightarrow \hat{\mathbb{R}}$. It clearly is injective and a group homomorphism, and with the aid of Lemma 5.8 one proves it to be a homeomorphism.

The case $G=\mathbb{T}$ is now easy. By considering the homomorphism $x \longmapsto e^{ix}$ of $\mathbb{R}$ onto $\mathbb{T}$ one derives from the above that all characters of $\mathbb{T}$ are of the form $z \longmapsto z^n$ for some $n \in \mathbb{Z}$. Conversely, these functions $z \longmapsto z^n$ are characters, and one obtains a bijective group homomorphism $\mathbb{Z} \longrightarrow \hat{\mathbb{T}}$ which, by Theorem 5.3, turns out to be a homeomorphism.

It is even easier to prove $\hat{\mathbb{Z}}$ to be isomorphic (as a group) to $\mathbb{T}$. Again one uses Lemma 5.8 to prove that the isomorphism preserves the topology. We content ourselves by merely mentioning the results, leaving the proofs to the reader.

5.10. THEOREM. (a) *For* $a \in \mathbb{R}$ *define* $\gamma^{(a)} : \mathbb{R} \longrightarrow \mathbb{T}$ *by*

$$\gamma^{(a)}(x) := e^{iax} \qquad (x \in \mathbb{R}).$$

Then $a \longmapsto \gamma^{(a)}$ *is an isomorphism of topological groups* $\mathbb{R} \longrightarrow \hat{\mathbb{R}}$. *If for the Haar measure on* $\mathbb{R}$ *we take the Lebesgue measure and if we identify* $a \in \mathbb{R}$ *with* $\gamma^{(a)} \in \hat{\mathbb{R}}$, *then we can express the Fourier transform of* $f \in L^1(\mathbb{R})$ *by*

$$\hat{f}(a) = \int_{-\infty}^{\infty} f(x)e^{-iax}dx \qquad (a \in \mathbb{R}).$$

(b) *For* $n \in \mathbb{Z}$ *define* $\gamma^{(n)} : \mathbb{T} \longrightarrow \mathbb{T}$ *by*

$$\gamma^{(n)}(z) := z^n \qquad (z \in \mathbb{T}).$$

Then $n \longmapsto \gamma^{(n)}$ *is an isomorphism of topological groups* $\mathbb{Z} \longrightarrow \hat{\mathbb{T}}$. *We identify* $n \in \mathbb{Z}$ *with* $\gamma^{(n)} \in \hat{\mathbb{T}}$ *and choose the Haar measure* m *on* $\mathbb{T}$ *so that* $m(T)=1$. *Then for* $f \in L^1(\mathbb{T})$ *we have*

$$f(n) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{ix})e^{-inx}dx \qquad (n \in \mathbb{Z}).$$

(c) *For* $a \in \mathbb{T}$ *define* $\gamma^{(a)} : \mathbb{Z} \longrightarrow \mathbb{T}$ *by*

$$\gamma^{(a)}(n) := a^n \qquad (n \in \mathbb{Z}).$$

Then $a \longmapsto \gamma^{(a)}$ *is an isomorphism of topological groups* $\mathbb{T} \longrightarrow \hat{\mathbb{Z}}$. *We identify* $a \in \mathbb{T}$ *with* $\gamma^{(a)} \in \hat{\mathbb{Z}}$. *For the Haar measure on* $\mathbb{Z}$ *we choose the counting measure. Then for all* $f \in L^1(\mathbb{Z})$,

$$f(a) = \sum_{-\infty}^{\infty} f(n)a^{-n} \qquad (a \in \mathbb{T}).$$

5.A. EXERCISE. *Every* $x \in G$ *determines a continuous character* $\alpha(x)$ *of* Γ:

$$[\alpha(x)](\gamma) := \gamma(x) \qquad (x \in G;\ \gamma \in \Gamma).$$

This α *is a continuous homomorphism of* G *into* $\hat{\Gamma}$. (Use Lemma 5.8 to describe the topology of $\hat{\Gamma}$.)

5.B. EXERCISE. *For any two locally compact abelian groups* G *and* H *the formula*

$$[\Phi(\alpha,\beta)](x,y) = \alpha(x)\beta(y)$$

defines an isomorphism Φ of $\hat{G}\times\hat{H}$ onto $(G\times H)\hat{}$ which is also a homeomorphism.

5.C. EXERCISE. For a family $(H_\alpha)_{\alpha\in A}$ of abelian groups we define its direct sum $\bigoplus_{\alpha\in A} H_\alpha$ to be the group $\{x\in \prod_{\alpha\in A} H_\alpha :$ $x_\alpha = 1$ for all but finitely many $\alpha\in A\}$. If each H_α is a discrete group, we view $\bigoplus_{\alpha\in A} H_\alpha$ as a topological group under the discrete topology.

If $(G_\alpha)_{\alpha\in A}$ is a family of compact abelian groups, then $(\prod_{\alpha\in A} G_\alpha)\hat{}$ is isomorphic to $\bigoplus_{\alpha\in A} \hat{G}_\alpha$.