6. BOCHNER'S THEOREM.

LET G, m, Γ BE AS IN THE PRECEDING SECTION.

For $\mu \in M(G)$ define $\hat{\mu}: \Gamma \longrightarrow \mathbb{C}$ by

$$\hat{\mu}(\gamma) := \int \bar{\gamma}\, d\mu \qquad (\gamma \in \Gamma).$$

Note that $(fm)\hat{} = \hat{f}$ for $f \in L^1(G)$. The function $\hat{\mu}$ is called the <u>Fourier-Stieltjes transform</u> of μ.

6.1. THEOREM. <u>The Fourier-Stieltjes Transformation is a norm-decreasing homomorphism of</u> $M(G)$ <u>into</u> $C_u(\Gamma)$. ($C_u(\Gamma)$ is the Banach algebra of all uniformly continuous bounded functions $\Gamma \to \mathbb{C}$.)

Proof. From our Fubini Theorem 1.17 it follows that $(\mu * \nu)\hat{} = \hat{\mu}\hat{\nu}$ for all $\mu, \nu \in M(G)$. Each $\mu \mapsto \hat{\mu}(\gamma)$ is a homomorphism $M(G) \longrightarrow \mathbb{C}$, hence is a contraction (Theorem 2.4). Thus, $\hat{\mu}$ is bounded and $\|\hat{\mu}\|_\infty \le \|\mu\|$. It remains to prove $\hat{\mu} \in C_u(\Gamma)$. Here we may assume $\mu \ge 0$. Let $\varepsilon > 0$. By the regularity of the measure μ there exists a compact $K \subset G$ with $\mu(G \setminus K) < \varepsilon$. As we know, $N(K:\varepsilon)$ is a neighborhood of $1 \in \Gamma$. If now $\beta, \gamma \in \Gamma$ and $\gamma \in \beta N(K:\varepsilon)$, then $|\hat{\mu}(\gamma) - \hat{\mu}(\beta)| \le \int_K |\bar{\gamma} - \bar{\beta}|\, d\mu + \int_{G \setminus K} |\bar{\gamma} - \bar{\beta}|\, d\mu \le$

$$\le \int_K |\gamma\beta^{-1} - 1|\, d\mu + 2\mu(G \setminus K) \le \varepsilon\|\mu\| + 2\varepsilon.$$

The set of all Fourier-Stieltjes transforms of elements of $M(G)$ we call $B(\Gamma)$. It is a subalgebra of $C_u(\Gamma)$ and contains $A(\Gamma)$.

6.2. LEMMA. <u>If</u> $j \in B(\Gamma)$ <u>then</u> $\bar{j} \in B(\Gamma)$, $j_\gamma \in B(\Gamma)$ <u>for every</u> $\gamma \in \Gamma$, <u>and for each</u> $x_0 \in G$ <u>the function</u> $\gamma \longmapsto j(\gamma)\gamma(x_0)$ <u>is an element of</u> $B(\Gamma)$.

The proof of this lemma is quite analogous to the proof of Theorem 5.6.

Every element μ of $M(\Gamma)$ has a Fourier-Stieltjes transform $\hat{\mu} \in C_u(\hat{\Gamma})$. We have already found a natural continuous homomorphism α of G into $\hat{\Gamma}$. (See Exercise 5.A.) Then for every $\mu \in M(\Gamma)$, $\hat{\mu} \circ \alpha$ is an element of $C_u(G)$. We denote this element by $\check{\mu}$:

$$\check{\mu}(x) = \int \overline{\gamma(x)} d\mu(\gamma) \qquad (\mu \in M(\Gamma);\ x \in G).$$

From our Fubini Theorem 1.17 one derives

6.3. LEMMA. <u>If</u> $\mu \in M(G)$ <u>and</u> $\nu \in M(\Gamma)$, <u>then</u> $\int \hat{\mu} d\nu = \int \check{\nu} d\mu$.

In particular, if $\check{\nu}=0$, then $\int \hat{f} d\nu = 0$ for all $f \in L^1(G)$. In other words, if $\check{\nu}=0$, then $\int j d\nu = 0$ for all $j \in A(\Gamma)$. But $A(\Gamma)$ is a dense subset of $C_\infty(\Gamma)$ (Theorem 5.5). Hence,

6.4. UNIQUENESS THOEREM. <u>If</u> $\mu \in M(\Gamma)$ <u>and if</u> $\check{\mu} = 0$, <u>then</u> $\mu = 0$.

Let H be any abelian group. A function $\varphi: H \longrightarrow \mathbb{C}$ is said to be <u>positive definite</u> if for all $p \in \mathbb{N}$, all $c_1, \ldots, c_p \in \mathbb{C}$ and all $x_1, \ldots, x_p \in H$ we have

$$\sum_{n,m=1}^{p} c_n \overline{c_m} \varphi(x_n x_m^{-1}) \geq 0.$$

An example: every group homomorphism $\varphi: H \longrightarrow \mathbb{T}$ is positive definite. In fact, for such φ and for arbitrary $c_1, \ldots, c_p \in \mathbb{C}$, $x_1, \ldots, x_p \in H$ we find $\sum_{n,m} c_n \overline{c_m} \varphi(x_n x_m^{-1}) = \sum_{n,m} c_n \varphi(x_n) \overline{c_m \varphi(x_m)} =$
$= (\sum_n c_n \varphi(x_n)) \overline{(\sum_m c_m \varphi(x_m))} = |\sum_n c_n \varphi(x_n)|^2 \geq 0$.

Another example: If $\mu \in M(G)$ is positive, then $\hat{\mu}$ is positive definite on Γ. Proof. If $c_1, \ldots, c_p \in \mathbb{C}$ and $\gamma_1, \ldots, \gamma_p \in \Gamma$, then $\sum_{n,m} c_n \overline{c_m} \hat{\mu}(\gamma_n \gamma_m^{-1}) = \sum_{n,m} \int c_n \overline{c_m} \overline{\gamma_n} \gamma_m d\mu = \int |\sum c_n \gamma_n|^2 d\mu \geq 0$.

6.5. LEMMA. <u>Let</u> H <u>be an abelian group; let</u> $\varphi: H \longrightarrow \mathbb{C}$ <u>be positive definite. Then</u> (1 <u>denoting the identity element of</u> H),

(a) $\varphi(1) = \|\varphi\|_\infty$. <u>In particular</u>, $\varphi(1) \geq 0$ <u>and</u> φ <u>is bounded</u>.

(b) $\varphi(x^{-1}) = \overline{\varphi(x)}$ <u>for all</u> $x \in H$.

(c) <u>If</u> $x, y \in H$, <u>then</u> $|\varphi(x) - \varphi(y)|^2 \leq 2\varphi(1) \mathrm{Re}(\varphi(1) - \varphi(xy^{-1}))$.

Proof. Choosing $p=1$, $c_1=1$, $x_1=1$ we find $\varphi(1)\geq 0$.

For all $x\in G$ and $c\in\mathbf{C}$,
$$(1+|c|^2)\varphi(1)+\bar{c}\varphi(x^{-1})+c\varphi(x)\geq 0.$$
(This formula is obtained by taking $p=2$, $x_1=1$, $x_2=x$, $c_1=1$, $c_2=c$.) The substitutions $c=1$ and $c=i$ show that both $\varphi(x^{-1})+\varphi(x)$ and $i(\varphi(x)-\varphi(x^{-1}))$ are real numbers. Then $\varphi(x^{-1}) = \overline{\varphi(x)}$ $(x\in H)$. Taking $c\in\mathbf{T}$ so that $c\varphi(x)=-|\varphi(x)|$ we find $2\varphi(1)-2|\varphi(x)|\geq 0$, so $|\varphi(x)|\leq\varphi(1)$, and we have proved (a) and (b).

For (c), let $x,y\in H$, $\varphi(x)\neq\varphi(y)$. Choose $p=3$, $x_1=1$, $x_2=x$, $x_3=y$, $c_1=1$, $c_2=\lambda\frac{|\varphi(x)-\varphi(y)|}{\varphi(x)-\varphi(y)}$ where $\lambda\in\mathbf{R}$, and $c_3=-c_2$. Applying (b) one obtains $(1+2\lambda^2)\varphi(1)+2\lambda|\varphi(x)-\varphi(y)|-2\lambda^2\mathrm{Re}\,\varphi(xy^{-1})\geq 0$. For given x and y this inequality is valid for all $\lambda\in\mathbf{R}$. Then the discriminant of the quadratic form $\lambda\longmapsto\lambda^2[2\varphi(1)-2\mathrm{Re}\,\varphi(xy^{-1})]$ $+\lambda\cdot 2|\varphi(x)-\varphi(y)|+\varphi(1)$ is ≤ 0, so that $|\varphi(x)-\varphi(y)|^2\leq$ $\leq 2\varphi(1)[\varphi(1) - \mathrm{Re}\,\varphi(xy^{-1})]$.

From (c) we have

6.6. COROLLARY. If $\varphi: G\longrightarrow\mathbf{C}$ is positive definite and continuous at 1, then $\varphi\in C_u(G)$.

For the construction of an important example of a positive definite function on G we use a convolution $L^2(G)\times L^2(G)$ $\longrightarrow C_\infty(G)$. We begin with a bit of pedantry. For $f\in\mathcal{L}^2(G)$ define $\tilde{f}: G\longrightarrow\mathbf{C}$ by $\tilde{f}(x) := \overline{f(x^{-1})}$ $(x\in G)$. It follows from Theorem 3.18 that $\tilde{f}\in\mathcal{L}^2(G)$ and $\|\tilde{f}\|_2=\|f\|_2$. If $f_1,f_2\in\mathcal{L}^2(G)$ are a.e. equal, then $\tilde{f}_1=\tilde{f}_2$ a.e. (again Theorem 3.18). Thus, $f\longmapsto\tilde{f}$ defines a (conjugate linear, surjective and isometric) map of $L^2(G)$ into $L^2(G)$, and without ambiguity we can use the symbol $\tilde{f}$ not only for $f\in\mathcal{L}^2(G)$ but also for $f\in\mathbf{L}^2(G)$.

Let $(\ |\ )$ be the inner product in $L^2(G)$. For $f,g\in L^2(G)$ set
$$(f*g)(x) := (f_{x^{-1}}|g) \qquad (x\in G).$$

Then for all f,g,x

$$(f*g)(x)=\int f_{x^{-1}}(y)\overline{\tilde{g}(y)}dx =$$
$$= \int f(xy)g(y^{-1})dy = \int f(y)g(y^{-1}x)dy,$$

and for our new convolution we get the same formula we had for the convolution in $L^1(G)$. We also see that $|(f*g)(x)|\leq\|f_{x^{-1}}\|_2\|\tilde{g}\|_2=\|f\|_2\|g\|_2$, so $\|f*g\|_\infty\leq\|f\|_2\|g\|_2$. Further, by Theorem 3.13 $f*g$ is continuous.

Let $f,g\in L^2(G)$, $\varepsilon>0$. By Lemma 1.8 there exist $f_1,g_1\in C_{00}(G)$ with $\|f-f_1\|_2\leq\varepsilon$, $\|g-g_1\|_2\leq\varepsilon$. Then $\|f*g-f_1*g_1\|_\infty\leq\|f*(g-g_1)\|_\infty +$ $+\|(f-f_1)*g_1\|_\infty\leq\|f\|_2\|g-g_1\|_2+\|f-f_1\|_2\|g_1\|_2\leq\|f\|_2\varepsilon+\varepsilon(\|g\|_2+\varepsilon)$. It follows that for every $\delta>0$ we can find $f_1,g_1\in C_{00}(G)$ such that $|f*g|\leq\delta$ outside the support of f_1*g_1. This support being compact (Exercise 4.B) we obtain $f*g\in C_\infty(G)$.

6.7. LEMMA. <u>For</u> $f,g\in L^2(G)$ <u>there exists a function</u> $f*g$ <u>defined by</u>

$$(f*g)(x) := \int f(y)g(y^{-1}x)dy \qquad (x\in G).$$

<u>Then</u> $f*g\in C_\infty(G)$ <u>and</u> $\|f*g\|_\infty\leq\|f\|_2\|g\|_2$.

Returning to our positive definite functions we have

6.8. THEOREM. <u>For every</u> $f\in L^2(G)$, $f*\tilde{f}$ <u>is positive definite</u>.

Proof. For $c_1,\dots,c_p\in\mathbb{C}$ and $x_1,\dots,x_p\in G$,

$$\sum_{n,m} c_n\overline{c_m}(f*\tilde{f})(x_nx_m^{-1})=\sum_{n,m} c_n\overline{c_m}(f_{x_mx_n^{-1}}|f) =$$
$$=\sum_{n,m} c_n\overline{c_m}(f_{x_n^{-1}}|f_{x_m^{-1}})= \|\sum_n c_nf_{x_n^{-1}}\|^2\geq 0.$$

(note that $(f_{x^{-1}}|g) = (f|g_x)$ for all $f,g\in L^2(G)$, $x\in G$.)

6.9. BOCHNER'S THEOREM. (a) <u>If</u> $\mu\in M(\Gamma)$ <u>is positive, then</u> $\check{\mu}$ <u>is a continuous positive definite function on</u> G, <u>and</u> $\|\check{\mu}\|_\infty = \|\mu\|$.

(b) <u>Conversely, for every continuous positive definite</u>

function φ on G there exists a unique $\mu \in M(\Gamma)$ such that $\check{\mu} = \varphi$. This μ is positive, and $\|\mu\| = \|\varphi\|_\infty$.

Proof. (a) For $\mu \in M(\Gamma)$ we have already seen that $\check{\mu}$ is continuous (beginning of page 6.2); if μ is also positive, then an easy computation shows that $\check{\mu}$ is positive definite. (See the second example on page 6.2.)

(b) Let φ be continuous, positive definite on G. By the Uniqueness Theorem 6.4 there is at most one $\mu \in M(\Gamma)$ with $\check{\mu} = \varphi$; it only remains to prove existence and positivity.

For $f, g \in L^1(G)$ set

$$\begin{aligned}[f,g] := \int (f * \tilde{g})\varphi &= \iint f(x)\overline{g(y^{-1}x)}\,\varphi(y)\,dx\,dy = \\ &= \iint f(x)\overline{g(yx)}\,\varphi(y^{-1})\,dx\,dy = \\ &= \iint f(x)\overline{g(y)}\,\varphi(xy^{-1})\,dx\,dy.\end{aligned}$$

(We use the Fubini Theorem and Theorem 3.18.) Notice that

$$(*) \qquad |[f,g]| \leq \varphi(1)\|f\|\|g\| \qquad (f,g \in L^1(G)).$$

We first prove $[\ ,\ ]$ to be a semi-inner product, i.e.

(a) $f \longmapsto [f,g]$ is linear for every g;

(b) $\overline{[f,g]} = [g,f]$ for all f,g;

(c) $[f,f] \geq 0$ for every f.

Formulas (a) and (b) are clearly true. For (c), by $(*)$ it suffices to consider $f \in C_{00}(G)$. Let $f \in C_{00}(G)$; let $\delta > 0$. According to Corollary 6.6 there is a neighborhood W of 1 such that

$$|\varphi(x) - \varphi(y)| \leq \delta \quad \text{if} \quad x \in yW.$$

The continuity of the map $(x,y) \longmapsto xy^{-1}$ guarantees the existence of a neighborhood U of 1 for which $UU^{-1} \subset W$. Let $K := \operatorname{supp} f$. As K is compact there exist $a_1, \ldots a_p \in K$ and disjoint Borel subsets $E_1, \ldots, E_p$ of K such that $K = E_1 \cup \ldots \cup E_p$ and $E_n \subset a_n U$ $(n = 1, \ldots, p)$. If $x \in E_n$ and $y \in E_m$, then $xy^{-1} \in a_n a_m^{-1} UU^{-1} \subset a_n a_m^{-1} W$, so $|\varphi(xy^{-1}) - \varphi(a_n a_m^{-1})| \leq \delta$. Hence, for all n and m,

$$\left|\int_{E_n}\int_{E_m} f(x)\overline{f(y)}\,\varphi(xy^{-1})\,dx\,dy - \int_{E_n}\int_{E_m} f(x)\overline{f(y)}\,\varphi(a_n a_m^{-1})\,dx\,dy\right|$$
$$\leq \int_{E_n}\int_{E_m} |f(x)|\,|\overline{f(y)}|\,\delta\,dx\,dy,$$

i.e.

$$\left|\int_{E_n}\int_{E_m} f(x)\overline{f(y)}\varphi(xy^{-1})dxdy - \left(\int_{E_n} f\right)\left(\int_{E_m}\overline{f}\right)\varphi(a_n a_m^{-1})\right|$$

$$\leq \delta\left(\int_{E_n}|f|\right)\left(\int_{E_m}|f|\right).$$

Summation over all n,m yields (set $c_n := \int_{E_n} f$)

$$(**) \qquad \left| [f,f] - \sum_{n,m} c_n\overline{c_m}\varphi(a_n a_m^{-1})\right| \leq \delta\|f\|^2.$$

Here $\sum c_n\overline{c_m}\varphi(a_n a_m^{-1}) \geq 0$. For all δ we can find $a_1,\ldots,a_p$ and $c_1,\ldots,c_p$ that satisfy (**). This is possible only if $[f,f] \geq 0$.

We have now proved (a), (b) and (c). From these formulas one derives in the usual way Schwarz's Inequality:

$$|[f,g]|^2 \leq [f,f][g,g] \qquad (f,g \in L^1(G)).$$

In particular,

$$\left|\int(f*\tilde{g})\varphi\right|^2 \leq \varphi(1)\|g\|^2\int(f*\tilde{f})\varphi \qquad (f,g \in L^1(G)).$$

We know from Theorem 4.2 that $L^1(G)$ has an approximate identity $(e_\lambda)_{\lambda\in\Lambda}$ with $\|e_\lambda\|=1$ for each λ. For every $f \in L^1(G)$, $\lim f*e_\lambda = f$ in $L^1(G)$. φ being bounded we get

$$(***) \qquad \left|\int f\varphi\right|^2 = \lim_\lambda \left|\int(f*e_\lambda)\varphi\right|^2 \leq \sup_\lambda \varphi(1)\|\tilde{e}_\lambda\|^2\int(f*\tilde{f})\varphi =$$

$$= \varphi(1)\int(f*\tilde{f})\varphi \qquad (f \in L^1(G)).$$

Take $f \in L^1(G)$. Define $h_1, h_2, \ldots \in L^1(G)$ by

$$h_1 := f*\tilde{f},$$

$$h_{n+1} := h_n*\tilde{h}_n \qquad (n \in \mathbb{N}).$$

For all $g_1, g_2 \in L^1(G)$, $(g_1*g_2)^\sim = \tilde{g}_2*\tilde{g}_1$; consequently,

$$h_{n+1} = h_n*h_n.$$

By repeatedly applying (***) we arrive at

$$\left|\int f\varphi\right|^2 \leq \varphi(1)\int h_1\varphi \leq \varphi(1)^{1+\frac{1}{2}}\left(\int h_2\varphi\right)^{\frac{1}{2}} \leq \ldots$$

$$\leq \varphi(1)^{1+\frac{1}{2}+\frac{1}{4}+\ldots+(\frac{1}{2})^{n-1}}\cdot\left(\int h_n\varphi\right)^{(\frac{1}{2})^{n-1}}$$

$$\leq \varphi(1)^{1+\frac{1}{2}+\ldots+(\frac{1}{2})^{n-1}}\cdot\left(\|\varphi\|_\infty\|h_n\|\right)^{(\frac{1}{2})^{n-1}}$$

$$= \varphi(1)^2\|h_n\|^{(\frac{1}{2})^{n-1}}$$

By the Spectral Radius Formula 2.7,

$$\lim \|h_n\|^{(\frac{1}{2})^{n-1}} = \|\hat{h}_1\|_\infty = \|(f*\tilde{f})^\wedge\|_\infty = \|\hat{f}\hat{\tilde{f}}\|_\infty = \|\hat{f}\bar{\hat{f}}\|_\infty = \|\hat{f}\|_\infty^2.$$

We have proved

$$|\int f\varphi| \leq \varphi(1)\|\hat{f}\|_\infty \qquad (f \in L^1(G)).$$

Now we are almost done. Apparently, if $\hat{f} = 0$, the $\int f\varphi = 0$, and we can define a map $T: A(\Gamma) \longrightarrow \mathbb{C}$ by

$$T(\hat{f}) := \int f\varphi \qquad (f \in L^1(G)).$$

Clearly, T is linear, and, by what we have just proved, $|T(j)| \leq \varphi(1)\|j\|$ for all $j \in A(\Gamma)$. As we know, $A(\Gamma)$ is a dense subspace of $C_\infty(\Gamma)$ (Theorem 5.5). Therefore, T has a unique extension $\mu \in M(\Gamma)$, and $\|\mu\| \leq \varphi(1)$. For all $f \in L^1(G)$, $\int f\varphi = \int \hat{f}d\mu = \int \check{\mu}f$ (see Lemma 6.3). Hence, as both φ and $\check{\mu}$ are continuous, they must be equal. We have $\|\mu\| \leq \varphi(1) = \check{\mu}(1) = \mu(\Gamma)$ and, by Exercise 1.C, $\mu \geq 0$.

By B(G) we denote $\{\check{\mu}: \mu \in M(\Gamma)\}$. By the above theorem, B(G) is the linear span of the set of all continuous positive definite functions on G. B(G) is a subset of $C_u(G)$.

For $f \in L^1(G)$, $\varphi \in C(G)$ and $x \in G$, it is easy to see that $\int f(y)\varphi(y^{-1}x)dy$ exists: we denote it by $f*\varphi(x)$. (Note that $f*\varphi$ is everywhere defined.)

6.10. LEMMA. (a) <u>If</u> $f \in L^1(G)$ <u>and</u> $\varphi \in B(G)$, <u>then</u> $f*\varphi \in B(G)$.
(b) $L^1(G) \cap B(G)$ <u>is a dense subset of</u> $L^1(G)$.

Proof. (a) By the Fubini Theorem, for $f \in L^1(G)$ and $\mu \in M(\Gamma)$ we have $f*\check{\mu} = (\hat{f}\mu)^\vee \in B(G)$.

(b) If $u \in C_{oo}(G)$, then $u*\tilde{u} \in L^1(G) \cap B(G)$. (See Theorem 6.8.) It follows from Exercise 4.B and Lemma 4.3 that $L^1(G)$ has a right approximate identity $(e_\lambda)_{\lambda \in \Lambda}$ that lies in $L^1(G) \cap B(G)$. For every $f \in L^1(G)$ we now have $f = \lim f*e_\lambda$ while (by (a)) $f*e_\lambda \in L^1(G) \cap B(G)$.

<u>For every compact</u> $K \subset \Gamma$ <u>there exists a positive definite</u> $f \in C_{oo}(G)$ <u>such that</u> $\hat{f} \geq 0$ <u>and</u> $\hat{f} \geq \xi_K$. (By compactness it suffices to consider the case $K = \{\gamma\}$ $(\gamma \in \Gamma)$. As $A(\Gamma)$ separates the points of Γ and $C_{oo}(G)$ is dense in $L^1(G)$, for $\gamma \in \Gamma$

we find $u \in C_{00}(G)$ for which $|\hat{u}(\gamma)| \geq 1$. Then $f := u * \tilde{u}$ is a positive definite element of $C_{00}(G)$, and $\hat{f}(\gamma) = \hat{u}(\gamma)\overline{\hat{u}(\gamma)} = |\hat{u}(\gamma)|^2 \geq 1$.)

6.11. INVERSION THEOREM. There exists a (unique) Haar measure m_Γ on Γ with the following property. If $f \in B(G)$ is integrable then $\hat{f} \in L^1(\Gamma)$ and

$$f(x) = \int \hat{f}(\gamma)\gamma(x)\,dm_\Gamma(\gamma) \qquad (x \in G).$$

i.e.

$$f(x) = \hat{f}^{\vee}(x^{-1}) \qquad (x \in G).$$

Proof. If $\varphi: G \longrightarrow \mathbb{C}$ is positive definite, then so is $x \longmapsto \varphi(x^{-1})$. Therefore, if $f \in B(G)$, then $x \longmapsto f(x^{-1})$ is an element of $B(G)$ and there exists a $\mu_f \in M(G)$ such that $\check{\mu}_f = f'$. By Lemma 6.3,

(a) $$\int \hat{h}\,d\mu_f = \int h(x) f(x^{-1})\,dx \qquad (h \in L^1(G);\ f \in B(G)).$$

Further, if φ is positive definite on G and if $\gamma \in \Gamma$, then $\gamma\varphi$ is positive definite. Hence, if $f \in B(G)$ and $\gamma \in \Gamma$, then $\gamma f \in B(G)$, and for all $h \in L^1(G)$,

$$\int \hat{h}\,d\mu_{\gamma f} = \int h\bar{\gamma} f'\,dm = \int (h\bar{\gamma})^\wedge\,d\mu_f = \int (\hat{h})_\gamma\,d\mu_f.$$

As $A(\Gamma)$ is dense in $C_\infty(\Gamma)$ we may infer

(b) $$\int \varphi\,d\mu_{\gamma f} = \int \varphi_\gamma\,d\mu_f \qquad (f \in B(G);\ \varphi \in C_\infty(\Gamma)).$$

If $f, g \in L^1(G) \cap B(G)$, then (by the proof of Lemma 6.10 (a)), $(\hat{f}\mu_g)^\vee = f' * \check{\mu}_g = f' * g'$ and $(\hat{g}\mu_f)^\vee = g' * f'$. But G is commutative, so that $f' * g' = g' * f'$. Applying Theorem 6.4 we find

(c) $$\hat{f}\mu_g = \hat{g}\mu_f \qquad (f, g \in L^1(G) \cap B(G)).$$

If $k, h \in C_{00}(\Gamma)$ and $k \neq 0$ on $\operatorname{supp} h$, by (h/k) we denote the unique $\varphi \in C_{00}(\Gamma)$ for which $k\varphi = h$ and $\varphi = 0$ on $\{\gamma \in \Gamma : k(\gamma) = 0\}$. For $h \in C_{00}(\Gamma)$ and $f \in C_{00}(G)$ we write $h \prec f$ if f is positive definite and $\hat{f} \geq 1$ on $\operatorname{supp} h$: by the remark after Lemma 6.10, for every h such an f exists.

We want to define a function T on $C_{oo}(\Gamma)$ by

(d) $$T(h) := \int (h/\hat{f})\,d\mu_f \qquad (h \prec f)$$

and then prove T to be a Haar integral on Γ.

First we have to prove that the right hand member of (d) does not depend on the choice of f. Suppose $h \in C_{oo}(\Gamma)$ and let $h \prec f$, $h \prec g$. Then by (c),

$$\int (h/\hat{f})\,d\mu_f = \int (h/\hat{f}\hat{g})\hat{g}\,d\mu_f = \int (h/\hat{f}\hat{g})\hat{f}\,d\mu_g = \int (h/\hat{g})\,d\mu_g.$$

Thus, T is well-defined.

If $h_1, h_2 \in C_{oo}(\Gamma)$, there exists a positive definite $f \in C_{oo}(G)$ such that $\hat{f} \geq 1$ on $(\operatorname{supp} h_1) \cup (\operatorname{supp} h_2)$. Then $h_1 \prec f$, $h_2 \prec f$, $h_1 + h_2 \prec f$ and $((h_1+h_2)/\hat{f}) = (h_1/\hat{f}) + (h_2/\hat{f})$. It follows that $T(h_1+h_2) = T(h_1) + T(h_2)$, so that T is <u>linear</u>.

If $h \in C_{oo}^{+}(G)$ and $h \prec f$, then $(h/\hat{f}) \geq 0$. As $\mu_f \geq 0$ we have $T(h) \geq 0$: T is <u>positive</u>.

Finally we prove T to be <u>invariant</u>. Let $h \in C_{oo}(\Gamma)$, $\gamma \in \Gamma$. Choose a positive definite $f \in C_{oo}(G)$ such that $\hat{f} \geq 1$ on supp h. Then (as we have seen) γf is positive definite and $(\gamma f)^\wedge = (\hat{f})_\gamma \geq 1$ on supp h_γ. Hence,

$$T(h_\gamma) = \int (h_\gamma/(\hat{f})_\gamma)\,d\mu_{\gamma f} = \int (h/\hat{f})_\gamma\,d\mu_{\gamma f} = \text{(by (b))}$$
$$= \int (h/\hat{f})\,d\mu_f = T(h).$$

We see that T is a Haar interal. Let m_Γ denote the corresponding Haar measure. We have

(e) $$\int \varphi\,d\mu_f = \int \varphi \hat{f}\,dm_\Gamma \qquad (\varphi \in C_{oo}(\Gamma);\ f \in L^1(G) \cap B(G)).$$

To prove this formula, choose g so that $\varphi \prec g$; applying (c) one obtains

$$\int \varphi\,d\mu_f = \int (\varphi/\hat{g})\hat{g}\,d\mu_f = \int (\varphi/\hat{g})\hat{f}\,d\mu_g = \int (\varphi\hat{f}/\hat{g})\,d\mu_g = \int \varphi\hat{f}\,dm_\Gamma .$$

From (e) we infer that $\mu_f = \hat{f} m_\Gamma$. Therefore, $\hat{f}$ is m_Γ-integrable and

$$f(x) = \int \gamma(x)\,d\mu_f(\gamma) = \int \hat{f}(x)\gamma(x)\,dm_\Gamma(\gamma) \qquad (x \in G).$$

In the following examples we let m_G denote the Haar measure on G and m_Γ the corresponding Haar measure on Γ.

Example. G is compact. Choose m_G so that $m_G(G)=1$. Every $\gamma_0 \in \Gamma$ is positive definite and integrable, so that by the Inversion Theorem, for all x we must have $\gamma_0(x) = = \int \hat{\gamma}_0(\gamma)\gamma(x)dm_\Gamma(\gamma) = \int \xi_{\{\gamma_0\}}(\gamma)\gamma(x)dm_\Gamma(\gamma) = \gamma_0(x)m_\Gamma(\{\gamma_0\})$. Consequently, m_Γ is the counting measure. (Remember that Γ is discrete!)

Example. G is discrete. For m_G we choose the counting measure. We prove that then $m_\Gamma(\Gamma)=1$. In fact, $\xi_{\{1\}}$ is a continuous positive definite function on G and an element of $L^1(G)$. Its Fourier transform is the constant function 1 on Γ. Hence, $1=\xi_{\{1\}}(1)=\int \widehat{\xi_{\{1\}}}(\gamma)\gamma(1)dm_\Gamma(\gamma)=m_\Gamma(\Gamma)$.

Example. $G = \mathbf{R}$. We have already found a homeomorphic isomorphism $\mathbf{R} \longrightarrow \hat{\mathbf{R}}$. We shall identify $\mathbf{R}$ with $\hat{\mathbf{R}}$ without, however, mixing up $m_{\mathbf{R}}$ and $m_{\hat{\mathbf{R}}}$. Both of these measures are multiples of the Lebesgue measure m: there exist $a,b>0$ such that $m_{\mathbf{R}}=am$, $m_{\hat{\mathbf{R}}}=bm$. If a function f is Lebesgue-integrable, then $\int_{\mathbf{R}} f = a\int_{-\infty}^{\infty} f(x)dx$, $\int_{\hat{\mathbf{R}}} f = b\int_{-\infty}^{\infty} f(x)dx$. If g is a positive integrable function on $\hat{\mathbf{R}}$ then $\check{g} : \mathbf{R} \longrightarrow \mathbf{C}$ is positive definite. Assume $\check{g} \in L^1(\mathbf{R})$. Then $\check{g}\hat{\ } \in L^1(\hat{\mathbf{R}})$ and $\check{g}(x)=\check{g}\hat{\ }\check{\ }(-x)$ $(x \in \mathbf{R})$. By the Uniqueness Theorem 6.4 it follows that $g(y)=\check{g}\hat{\ }(-y)$ $(y \in \hat{\mathbf{R}})$. In particular, $g(0)=\int_{\mathbf{R}} \check{g}$.

We apply the above to the function $g : y \longmapsto e^{-|y|}$. By a simple computation, $\check{g}(x) = \frac{2b}{1+x^2}$ $(x \in \mathbf{R})$, so $\check{g} \in L^1(\mathbf{R})$. Then $1 = g(0) = \int_{\mathbf{R}} \check{g} = 2ab\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2ab\pi$. We conclude that for am and bm to be Haar measures $m_{\mathbf{R}}$ and $m_{\hat{\mathbf{R}}}$, respectively, it is necessary (and sufficient) that $ab = \frac{1}{2\pi}$.

6.A. EXERCISE. Use these computations to prove that

$$\int_{-\infty}^{\infty} \frac{e^{ixy}}{1+x^2}\, dx = \pi e^{-|y|} \qquad (y \in \mathbf{R}).$$

6.B. EXERCISE. For $\mu \in M(G)$ let $\zeta\mu$ be its Gelfand transform $\mathfrak{M}(M(G)) \longrightarrow \mathbb{C}$.

<u>Every</u> $\gamma \in \Gamma$ <u>induces</u> <u>a</u> $j(\gamma) \in \mathfrak{M}(M(G))$ <u>by</u>

$$j(\gamma)(\mu) = \hat{\mu}(\gamma) \qquad (\mu \in M(G)).$$

j <u>is</u> <u>a</u> <u>homeomorphism</u> <u>of</u> Γ <u>onto</u> <u>a</u> <u>subset</u> <u>of</u> $\mathfrak{M}(M(G))$, <u>and</u>

$$\hat{\mu} = \zeta\mu \circ j.$$

Hint. Consider convergence of nets in Γ and in $j(\Gamma)$.