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INTRODUCTION

Functional Analysis over valued fields different from $\mathbf{R}$ and $\mathbf{C}$ was started by A. F. Monna in a series of papers in the Proceedings of the Dutch Royal Academy of Sciences, beginning in 1943. Before then, valued fields had been investigated extensively from the algebraic point of view (W. Krull, K. Hensel, K. Rychlik, H. Hasse, A. Ostrowski, I. Kaplansky, S. MacLane) and, to a lesser extent, from the point of view of elementary analysis, theory of power series and number theory (W. Schöbe, L.Schnirelman, K. Mahler, J. de Groot, F. Loonstra, F. Kuiper).

The idea behind the whole theory is the following.

It is not unreasonable to try and generalize ordinary (functional) analysis by replacing $\mathbf{R}$ and $\mathbf{C}$ by other topological fields. This ought to give a new insight in analysis by showing what properties of the scalar field are crucial for certain "classical" theorems. For this topological field we choose a field K, provided with a (real-valued) absolute value function $|\ |$, such that K is complete relative to the metric induced by $|\ |$. Adding the condition that, as a topological field, K is neither $\mathbf{R}$ nor $\mathbf{C}$, we can prove the so-called ***strong triangle inequality***

$$|x + y| \leq \max(|x|, |y|) \qquad (x,y \in K).$$

This formula is essential to our entire theory. Among other things it implies that K is totally disconnected and cannot be made into a totally ordered field.

There is no problem (or fun) in extending the elementary theories of, say, Banach spaces and algebras to such a *non-Archimedean valued field* K. In fact, Bourbaki [27] builds up a theory of normed vector spaces over arbitrary scalar fileds. Understandably, non-Archimedean analysis has been accused of being nothing but an obvious adaptation of the classical theory to a more general situation.

The picture changes rapidly, however, as one proceeds to other subjects. Most of the functional analysis over $\mathbb{R}$ and $\mathbb{C}$ depends on ordering, connectedness, local compactness or algebraic closedness of the scalar field. Our new scalar field K in general has none of these properties. Consequently, part of the classical theory has no non-Archimedean translation, and part of the theory does translate but yields only false theorems. The latter case is not always uninteresting. In fact, quite often a very strong negation of the translation can be proved.

As a case in point, let us take the Gelfand-Mazur Theorem, "no field that properly contains $\mathbb{C}$ can be made into a normed algebra over $\mathbb{C}$". This theorem becomes a falsity if one replaces $\mathbb{C}$ by a non-Archimedean valued field K. But instead of it we have Krull's Theorem, stating that *every* field containing a non-Archimedean valued field K can be turned into a normed (even valued) algebra over K.

Another example is furnished by Liouville's Theorem, "every bounded entire function $\mathbb{C} \to \mathbb{C}$ is constant". Now let K be a non-Archimedean valued field. We call a function $K \to K$ *entire* if it is the sum of an everywhere convergent power series. It turns out that the statement "every bounded entire function $K \to K$ is constant" is false if (and only if) K is locally compact. But the story does not stop there. One can prove that for locally compact K every continuous function $K \to K$ is a uniform limit of entire functions!

There are also instances of classical theorems whose non-Archimedean analogues are not only correct but even capable

of being strengthened considerably. We have already met the strong triangle inequality.

We have the same state of things in Fourier Theory. For a locally compact space X and a non-Archimedean valued field K, the continuous functions $X \to K$ that vanish at infinity form in a natural way a Banach space $C_\infty(X)$ over K. The continuous linear maps $C_\infty(X) \to K$ may be called *measures* on X. This definition opens up a measure theory which - seen by a classical analyst - is pure surrealism. For reasonable abelian locally compact groups G it leads to a Fourier theory where the Fourier Transformation actually is an isometry of L(G) onto $C_\infty(G^\wedge)$.

These examples may serve to illustrate the nontriviality of non-Archimedean analysis. My own interest in the subject was awakened in 1965 when Marius van der Put, Wim Schikhof, Jan van Tiel and myself formed a small "p-adics and dining society". Van der Put, Schikhof and Van Tiel all wrote their theses on non-Archimedean Functional Analysis. (Function Algebras, Harmonic Analysis and Locally Convex Spaces, respectively.) The bibliography accompanying these notes has grown out of the one composed by Schikhof for his thesis.

In recent years the number of mathematicians working in non-Archimedean analysis has been increasing markedly. It is about time for someone to gather the elementary results and put them together. In 1970, Monna wrote a book [147] which deals mostly with non-Archimedean Banach space, locally convex spaces and integration theory. In the following year, Narici, Beckenstein and Bachman wrote Ref. 154 whose main subjects are normed linear spaces and normed algebras. These books were written with different intentions: with some exaggeration one may call the former an encyclopedia, the latter a text-book. Monna treats a wider spectrum of subjects and refers more often to the existing literature than Narici c.s.

The present work, which arose out of Ref. 198, is the text-book variety. By and large, there are the same subjects as in Monna's book (plus some Harmonic Analysis) but discussed more extensively and complete with all the proofs. The great diversity of the material makes this an ambitious project. Accordingly, this book will not be as easy to read as Ref. 154, which moves at a comparatively leisurely pace.

There are several parts of the non-Archimedean theory that I have avoided, sometimes because they did not seem essential to Functional Analysis, sometimes because I did not know enough about them. Thus, the reader will find little information on Valuation Theory (references are given at the end of the first chapter), on Calculus and Special Functions (Kuiper, de Groot, Loonstra: see Ref. 8), on categories of Banach spaces (Gruson), on analytic functions (Adams, Hily, Dwork, Lazard; Krasner, Robba, Escassut) and on rings of power series (school of Grauert and Remmert: for a bibliography, see page 12 of Monna's book).

I have tried to make this work self-contained. The reader will have to know something about field extensions, the algebraic closure of a field, finite abelian groups and topological and uniform spaces, but very little is necessary. No "classical" functional analysis is explicitly required, but (according to my philosophy, at least) the non-Archimedean theory can only be of any use to people who are well acquainted with its Archimedean background.

Advice to the reader.

As a first introduction to p-adic analysis I recommend Bachmann's book [8]. It deals mainly with calculus over the p-adic numbers. A very good book that moves in the direction of Functional Analysis is Mahler's work [137].

If from this book you want to an idea of what non-Archimedean Functional Analysis is all about, read pages 1-7 of Chapter 1 and then start with Chapter 3. The part you skip that way is of a very technical nature: useful but not exciting.

A. C. M. van Rooij

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Non-Archimedean Functional Analysis