realcompactification of X if dim X = 0 (not "only if", as there exist $\mathbb{N}$-compact spaces that do not have covering dimension 0 (see p.).

Trivially, $X^\delta = X^\zeta$ as soon as X is pseudocompact. Hence, every pseudocompact $\mathbb{N}$-compact space is compact.

Information on non-Archimedean uniformities is given in Refs. 56, 57, 140, 158 and 197. (In Ref. 158, they are called *equivalence uniformities* as they have nothing to do with non-Archimedean topologies.)

Measurability of cardinal numbers is studied in H. J. Keisler and A. Tarski's paper [115], where the class of all nonmeasurable cardinals is called C_1' (see, in particular, 2.34-2.48 and the list of references given on p. 272 of this paper). (We prefer the term "small" to "nonmeasurable", mainly because analysts consider measurability as a virtue, which in this context it is not.)

Chapter 3

BANACH SPACES

DEFINITIONS AND EXAMPLES

A *norm* on a vector space E over K is a map $\|\ \|$ of E into $[0,\infty)$ with the properties:

i. $\|x\| \neq 0$ if $x \in E$, $x \neq 0$

ii. $\|x + y\| \leq \max\{\|x\|,\|y\|\}$ $(x,y \in E)$

iii. $\|\alpha x\| = |\alpha|\ \|x\|$ $(\alpha \in K,\ x \in E)$

Such a norm induces an ultrametric ρ by

$$\rho(x,y) := \|x - y\| \qquad (x,y \in E)$$

and thereby a zerodimensional Hausdorff topology on E. Relative to this topology, addition and scalar multiplication are continuous maps $E \times E \to E$ and $K \times E \to E$, respectively.
A *Banach space* is a complete normed vector space. The metric completion of any normed vector space is in a natural way a Banach space.

Some of the language of the Archimedean theory of normed spaces we take over without explicit definitions (e.g., closed linear span, weak topology).

The closed linear span of a subset X of a normed vector space E is indicated by [X]. Instead of [{x}] (where $x \in E$), we also write [x] or Kx (and, in fact, the closed linear span of {x} is $\{\alpha x : \alpha \in K\}$).

If E,F are normed vector spaces, by

$$E \sim F$$

we mean that there exists a linear isometry T of E onto F; then E and F are *isomorphic* as normed spaces, or *linearly isometric*. In that case we write $T : E \sim F$ instead of $T : E \to F$.

A linear map T of a normed vector space E into a normed vector space F is easily seen to be continuous if and only if there exists a $c \in [0, \infty)$ such that

$$||Tx|| \leq c||x|| \qquad (x \in E)$$

(Here the nontriviality of the valuation is used.)
Two normed vector spaces, E and F, are called *linearly homeomorphic* if there exists a linear homeomorphism between them. Two norms, $|| \ ||_1$ and $|| \ ||_2$, on the same vector space E induce the same topology if and only if they are equivalent, i.e., if and only if there exist nonnegative real numbers c_1 and c_2 such that

$$||x||_1 \leq c_1||x||_2 \qquad ||x||_2 \leq c_2||x||_1 \qquad (x \in E)$$

We signal one difference between the Archimedean and the non-Archimedean theories: If E is a normed vector space over K, the set $\{||x|| : x \in E\}$ may not be the same as $\{|\alpha| : \alpha \in K\}$. As a consequence, a nonzero element of E may fail to have a scalar multiple of norm 1; in fact, the set $\{x \in E : ||x|| = 1\}$ may very well be empty.

Accordingly, even one-dimensional Banach spaces are not always isomorphic: Let s be any positive real numer. By $K_{(s)}$ we denote the normed vector space (which is actually a Banach space) whose underlying vector space is K itself and whose norm is $\alpha \to s|\alpha|$. Clearly, $K_{(1)} = K$. The normed spaces $K_{(s)}$ and K are isomorphic if and only if s is an element of the value group of K.

Before going into the theory of Banach spaces we present a few more examples.

3.A Example. Let X be any set. The bounded maps $X \to K$ form a linear space $l^\infty(X)$, which is a Banach space under the *sup-norm* $|| \ ||_\infty$ defined by

$$||f||_\infty := \sup\{|f(x)| : x \in X\} \qquad [f \in l^\infty(X)]$$

(When there is no danger of ambiguity, we write $|| \ ||$ instead of $|| \ ||_\infty$.)

Examples 3.B–3.G are closed linear subspaces of $l^\infty(X)$ and therefore are themselves Banach spaces. Occasionally we use the notation a_x instead of $a(x)$.

3.B Example. $c_0(X) := \{f \in l^\infty(X) :$ for every $\varepsilon > 0$, there exist only finitely many elements x of X for which $|f(x)| \geq \varepsilon\}$ is a closed linear subspace of $l^\infty(X)$. In fact, it is the closed linear span of the set of all K-valued characteristic functions of one-point subsets of X.

3.C Example. We write l^∞ and c_0 for $l^\infty(N)$ and $c_0(N)$, respectively. An element a of l^∞ is sometimes denoted as $(a_1,a_2,\ldots)$. We have $c_0 = \{a \in l^\infty : \lim a_n = 0\}$.
Further, we define $c := \{a \in l^\infty : \lim a_n$ exists$\}$. This c is also a closed linear subspace of l^∞.

3.D Example. Now we provide X with a topology. The bounded continuous functions $X \to K$ form another closed linear subspace, $BC(X)$, of $l^\infty(X)$. This $BC(X)$ is the closure in $l^\infty(X)$ of the set of all bounded locally constant functions $X \to K$ (p. 25).

3.E Example. A subset A of K is precompact if and only if for every $\varepsilon > 0$, A can be covered by finitely many balls of radius ε; equivalently, if and only if $\overline{A}$ is compact. If K is locally compact, then precompactness is the same as boundedness (Ex. 1.B).

The continuous functions $f : X \to K$ for which $f(X)$ is precompact form a closed linear subspace $PC(X)$ of $BC(X)$. This $PC(X)$ is the closed linear hull in $BC(X)$ [or in $l^\infty(X)$] of $\{\xi_U : U \in \mathcal{B}(X)\}$. Of course, if K is locally compact, then $PC(X) = BC(X)$. If X is compact, then $PC(X) = BC(X) = C(X)$. It follows from Ex. 2.K that $PC(X) \sim C(X^\zeta)$ for every X.

3.F Example. For a locally compact X, let $C_\infty(X)$ denote the set of all continuous functions $f : X \to K$ for which for every $\varepsilon > 0$, the set $\{x : |f(x)| \geq \varepsilon\}$ is compact. $C_\infty(X)$ is the closed linear subspace of $PC(X)$ that is generated by the K-valued characteristic functions of the open compact sets. If X is compact, then $C_\infty(X) = C(X)$. Observe that $C_\infty(\mathbb{N}) = c_0$.

3.G Example. Now let $\mathcal{U}$ be a non-Archimedean uniformity on X. By $BUC(X:\mathcal{U})$ we denote the closed linear subspace of $l^\infty(X)$ that consists of all bounded, $\mathcal{U}$-uniformly continuous functions $X \to K$.

If X is a zerodimensional Hausdorff space and if $\mathcal{U}$ is the strongest non-Archimedean uniformity that is compatible with the topology of X, then $BUC(X:\mathcal{U}) = BC(X)$. If $\mathcal{U}_0$ is the uniformity that is generated by the finite clopen partitions of X, then $BUC(X:\mathcal{U}_0) = PC(X)$.
Another spacial case is $BUC(X:\mathcal{U}_1)$ which consists of the bounded continuous functions $X \to K$ that have a separable range space.

For the reader who believes that there might exist measurable cardinal numbers, we include one more example.

For a topological space X, set

$$BC_s(X) := \{f \in BC(X) : f(X) \text{ is small}\}$$

If $V \subset K$ and if for every $n \in \mathbb{N}$, there exists a small set $A_n \subset K$ such that $V \subset \bigcup_{a \in A_n} B_{1/n}(a)$, then there is an obvious injection $V \to \Pi A_n$, so that V is also small (2.7). Therefore, a bounded continuous $f : X \to K$ belongs to $BC_s(X)$ if and only if for every $\varepsilon > 0$, the set of all balls of radius ε that intersect $f(X)$ is small. It follows easily that $BC_s(X)$ *is a closed linear subspace* (*and also a subring*) *of* $BC(X)$ and (use the first part of Theorem 2.10) that

$$BC_s(X) = BC(X) \textit{ if } X \textit{ is } \mathbb{N}\textit{-compact.}$$

(Of course we have the same equality if either K or X is small.)

Let $\mathcal{U}_s$ be the non-Archimedean uniformity generated by the small clopen partitions of X. The coarsest common refinement of two small clopen partitions is again small. It is now easy to prove that

$$BC_s(X) = BUC(X:\mathcal{U}_s).$$

In 3.H and 3.I we give two examples of Banach spaces that are only variations on the theme we have been playing.

3.H Example. Let X be a nonempty set and s a function $X \to (0, \infty)$. For $f : X \to K$ set

$$||f||_s := \sup\{|f(x)|s(x) : x \in X\}.$$

The functions $f : X \to K$ for which $||f||_s$ is finite form a vector space $l^\infty(X:s)$, which is a Banach space under the norm $||\ ||_s$.
By $c_0(X:s)$, we denote the closed subspace of $l^\infty(X:s)$ consisting of the functions f with the property that for every $\varepsilon > 0$, the set $\{x \in X : |f(x)|s(x) \geq \varepsilon\}$ is finite. $c_0(X:s)$ is the closed linear hull in $l^\infty(X:s)$ of the set of all characteristic functions

of the one-point subsets of X. Trivially, $l^\infty(X:1) = l^\infty(X)$; $c_0(X:1) = c_0(X)$.

3.I Example. If Y is a nonempty subset of a normed space E and $a \in E$, the *distance* between a and Y is defined as

$$\operatorname{dist}(a,Y) := \inf\{||a - y|| : y \in Y\}.$$

This terminology is convenient for the following definition of an unorthodox norm ν on l^∞:

$$\nu(x) := \max\{|| \ ||_\infty, 2 \operatorname{dist}(x, c_0)\}.$$

This ν actually is a norm on l^∞. We have $||x||_\infty \le \nu(x) \le 2||x||_\infty$ for all $x \in l^\infty$, so that $|| \ ||_\infty$ and ν determine the same topology on l^∞. By l^∞_ν, we denote the normed vector space obtained by imposing the norm ν on l^∞. This l^∞_ν is a Banach space.

BASIC OPERATIONS WITH BANACH SPACES

The following lemma is as fundamental as it is simple. It can be regarded as an extension of the implication $(\alpha) \Rightarrow (\delta)$ of Theorem 1.1.

3.1 LEMMA. *If* $a_1, \ldots, a_n$ *are elements of a normed vector space* E *and if* $||a_i|| \neq ||a_j||$ *as soon as* $i \neq j$, *then*

$$||a_1 + \ldots + a_n|| = \max\{||a_i|| : i = 1, 2, \ldots, n\} .$$

PROOF. Clearly, $||a_1 + \ldots + a_n|| \le \max\{||a_i||\}$. For the converse, assume $||a_1|| < ||a_2|| < \ldots < ||a_n||$. If $s := a_1 + \ldots + a_n$, then $a_n = s - (a_1 + \ldots + a_{n-1})$, so that $||a_n|| \le \max\{||s||, \max\{||a_i|| : i < n\}\} = \max\{||s||,||a_{n-1}||\}$. The latter term cannot be $||a_{n-1}||$ since $||a_{n-1}|| < ||a_n||$; so it must be $||s||$. Therefore, $||a_n|| \le ||s||$.

Let $a_1, a_2, \ldots,$ be a sequence of elements of a Banach space E such that $\lim_i a_i = 0$. Then the sequence $a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots,$ is Cauchy and converges.

We denote its limit by $\sum_{i=1}^\infty a_i$.

Then $||\sum_{i=1}^\infty a_i|| \le \max_i\{||a_i|| : i = 1,2,\ldots\}$. If $m \in \mathbb{N}$ is such that $||a_m|| > ||a_i||$ for all $i \neq m$, then $||\sum_{i=1}^\infty a_i|| = ||a_m||$.

(Verification of this statement is left to the reader.)

The summation is unconditional, i.e., if τ is a permutation of N, then $\sum_{i=1}^\infty a_i = \sum_{j=1}^\infty a_{\tau(j)}$ (see Ex. 3. K).

Let D be a closed linear subspace of a normed vector space E. We form the quotient vector space E/D and denote by P the quotient map $E \to E/D$. The formula

$$||Px|| := \operatorname{dist}(x,D) \qquad (x \in E)$$

is easily seen to define a norm on E/D, the so-called *quotient norm*.

3.J Exercise. If E is a Banach space, then so is E/D.

The *direct sum* $E \oplus F$ of the normed vector spaces E and F is the vector space $E \times F$ under the norm

$$||(x,y)|| := \max\{||x||,||y||\} \qquad (x \in E;\ y \in F).$$

If E and F are Banach spaces, then so is $E \oplus F$. In a similar way one can define the direct sum $E_1 \oplus \ldots \oplus E_n$ of finitely many normed spaces $E_1, \ldots, E_n$.

Whenever we consider K^n $(n \in \mathbb{N})$ as a normed vector space, unless we specify otherwise *we tacitly assume that the norm is the max-norm*

$$||(\alpha_1,\ldots,\alpha_n)|| = \max\{|\alpha_i|: i = 1,\ldots,n\} \quad (\alpha_1,\ldots,\alpha_n \in K).$$

Thus, $K^n \sim K \oplus \ldots \oplus K$. In the terminology of 3.A and 3.B, $K^n = l^\infty(X) = c_0(X)$, where $X = \{1,\ldots,n\}$.

We generalize the notion of the direct sum as follows. Let I be an index set. For every $i \in I$, let E_i be a normed space. The cartesian product $\prod_i E_i$ is in a natural way a vector space. By

$$\underset{i\in I}{\times} E_i \quad \text{or} \quad \times_i E_i,$$

we denote the set of all elements a of $\prod_i E_i$ for which the set $\{\|a_i\| : i \in I\}$ is bounded. This $\times_i E_i$ can be normed by $\|a\| := \sup\{\|a_i\| : i \in I\}$. The elements a of $\prod_i E_i$ for which, for every $\varepsilon > 0$, the set $\{i \in I : \|a_i\| \geq \varepsilon\}$ is finite form a closed linear subspace of $\times_i E_i$, denoted by

$$\underset{i\in I}{\oplus} E_i \quad \text{or} \quad \oplus_i E_i .$$

We call $\times_i E_i$ and $\oplus_i E_i$ the *direct product* and the *direct sum* of the family $\{E_i\}_{i\in I}$, respectively. $\times_i E_i$ and $\oplus_i E_i$ are Banach spaces if and only if all E_i are Banach spaces. As special cases we have

$$\underset{i\in I}{\times} K = l^\infty(I) \quad \text{and} \quad \underset{i\in I}{\oplus} K = c_0(I).$$

Let E be a normed vector space, let $\{x_i\}_{i\in I}$ be a family of elements of E and let $a \in E$. We call a the *sum* of the family $\{x_i\}_{i\in I}$ (notation: $a = \sum_{i\in I} x_i = \Sigma_i x_i$) if for every $\varepsilon > 0$, there is a finite $J_0 \subset I$ such that

$$\| a - \sum_{i\in J} x_i \| \leq \varepsilon$$

for all finite subsets J and I that contain J_0. Any family of elements of E has at most one sum. If a given family has a sum, it is called *summable*.

3.K Exercise. Let $\{x_i\}_{i\in I}$ ba a family of elements of a Banach space E. Let $J := \{i \in I : x_i \neq 0\}$. To avoid notational complications that are basically trivial, assume J to be an infinite set.

i. Let the family have a sum a in E. Then for every $\varepsilon > 0$, there are only finitely many indices i with $\|x_i\| \geq \varepsilon$. Therefore, J is countable. Let $i_1, i_2, \ldots,$ be an enumeration of J. Then $a = \sum_{j=1}^{\infty} x_{i_j}$.

ii. Conversely, suppose that for every $\varepsilon > 0$, the set $\{i \in I : \|x_i\| \geq \varepsilon\}$ is finite. Again, J is countable. Let $i_1, i_2, \ldots,$ be an enumeration of J. Then $\lim_j x_{i_j} = 0$; so $\sum_{j=1}^{\infty} x_{i_j}$ exists. The family $\{x_i\}_{i\in I}$ is summable and its sum is $\sum_{j=1}^{\infty} x_{i_j}$.

Let E be a Banach space. If $X \in E$ and $0 \notin X$, then $c_0(X:\|\ \|)$ is a Banach space. With the aid of the preceding exercise one proves easily that the formula

$$Sf := \sum_{x\in X} f(x)x \qquad (f \in c_0(X:\|\ \|))$$

defines a continuous linear map $S : c_0(X:\|\ \|) \to E$. We call X a *base* for E if this S is both injective and surjective, i.e., if for every $a \in E$, there exists a unique $f : X \to K$ such that $a = \sum_{x\in X} f(x)x$. [Observe that, if $f : X \to K$ and if the family $\{f(x)x\}_{x\in X}$ is summable, then $f \in c_0(X:\|\ \|)$.] From Theorem 3.15 it follows that a subset of a finite-dimensional Banach space is a base in this sense if and only if it is a base in the sense of linear algebra.

If X is a base for E, then its elements are linearly independent and $[X] = E$. Further, if X is a base and if for every $x \in X$, x' is a nonzero scalar multiple of x, then $\{x' : x \in X\}$ is a base. Let $\pi \in K$, $0 < |\pi| < 1$. Then every $x \in E$, $x \neq 0$ has a scalar multiple x' such that $|\pi| \le \|x'\| \le 1$. Hence, *if* E *has a base, then it also has a base* Y *with the property* $|\pi| \le \|y\| \le 1$ $(y \in Y)$. (This maneuver will be executed frequently.)

A simple example : If I is any set, the characteristic functions of the one-element subsets of I form a base in $c_0(I)$. If I is infinite, they do not form a base in $l^\infty(I)$ (see Corollary 5.19). (Bases will be the subject of Chapter 5.)

The following lemma looks trivial but has far-reaching consequences.

3.2 LEMMA. *Let* x,y *be elements of a normed vector space* E. *Let* t *be a real number such that* $0 \le t \le 1$ *and* $\|x + y\| \ge t\|x\|$. *Then also* $\|x + y\| \ge t\|y\|$ *and therefore*

$$\|x + y\| \ge t \max\{\|x\|,\|y\|\}.$$

PROOF. Suppose $\|x + y\| < t\|y\|$. Then $\|x + y\| < \|y\| = \|-y\|$; so (Lemma 3.1) $\|x\| = \|y\|$ and $\|x + y\| \ge t\|x\| = t\|y\|$.

One of the most important concepts in the theory of Banach spaces over R or C is that of the inner product. There does not seem to be an immediate analogue for the non-Archimedean theory, as the inner product is based upon the order in R. However, in a real Hilbert space H, angles can be expressed in terms of distances. If x,y are nonzero elements of H, then the angle between $[x]$ and $[y]$ can be defined as $\arcsin\ (\mathrm{dist}(x,[y]) \,/\, \|x\|)$. In particular, x and y are mutually orthogonal if and only if $\|x\| = \mathrm{dist}(x,[y])$. Similar definitions turn out to have use in our theory. For us it is

not worth while to distuinguish between an angle and its sine. Accordingly, for nonzero elements x and y of a normed vector space, we set

$$\sphericalangle(x,y) := \frac{\mathrm{dist}(x,[y])}{\|x\|} .$$

It is iii. of Lemma 3.3 that makes $\sphericalangle$ interesting.

3.3 LEMMA. *Let* x,y *be nonzero elements of a normed vector space* E.

i. $0 \le \sphericalangle(x,y) \le 1$. $\sphericalangle(x,y) = 0$ *if and only if* x *and* y *are linearly dependent.*

ii. $\sphericalangle(\alpha x,\beta y) = \sphericalangle(x,y)$ *for all nonzero* $\alpha,\beta \in K$.

iii. $\sphericalangle(x,y) = \sphericalangle(y,x)$.

iv. $\sphericalangle(x,y) = \inf\{\|\alpha x+\beta y\|/\max\{\|\alpha x\|,\|\beta y\|\} : \alpha,\beta\in K, \alpha\neq 0, \beta\neq 0\}$.

v. $\|x\|\sphericalangle(x,x + y) = \|y\|\sphericalangle(y, x + y)$ ("sine rule").

PROOF. (i) and (ii) are obvious. (iii) follows from (iv). To prove (iv), take $\alpha,\beta \in K$. Then $\|\alpha x + \beta y\| \ge \mathrm{dist}(\alpha x,[y]) = |\alpha|\ \mathrm{dist}(x,[y]) = \|\alpha x\|\sphericalangle(x,y)$; so [by (i) and 3.2] $\|\alpha x + \beta y\| \ge \sphericalangle(x,y) \max\{\|\alpha x\|,\|\beta y\|\}$. This yields half of (iv). For the other half, observe that $\sphericalangle(x,y) = \inf\{\|x+\beta y\| \,/\, \|x\| : \beta \in K\} \ge$ $\inf\{\|x+\beta y\| \,/\, \max\{\|x\|,\|\beta y\|\} : \beta \in K\} =$ $\inf\{\|\alpha x+\beta y\| \,/\, \max\{\|\alpha x\|,\|\beta y\|\} : \alpha,\beta \neq 0\}$. Finally, (v) follows from the equality $\mathrm{dist}(x,[x + y]) = \mathrm{dist}(y,[x + y])$ which is easily established.

A linear map T of a normed space E into a normed space F is said to be a *similarity* if there exists a nonzero real number c such that $\|Tx\| = c\|x\|$ for all $x \in E$. Such a T is a homeomorphism of E onto a *closed* linear subspace of F.

3.4 THEOREM. *Let* E,F *be normed vector spaces and* T *a linear map* $E \to F$. *Then* T *is a similarity if and only if*

$$\sphericalangle(Tx,Ty) = \sphericalangle(x,y) \qquad (x,y \in E,\ x \neq 0,\ y \neq 0).$$

PROOF. Clearly, if c is a nonzero number such that $\|Tx\| = c\|x\|$ for all $x \in E$, then $\mathrm{dist}(Tx,[Ty]) = c\ \mathrm{dist}(x,[y])$ for all nonzero $x,y \in E$ and T preserves angles. Conversely, if T preserves angles, then (in particular) T is injective and for all linearly independent $x,y \in E$, by (v) of Lemma 3.3, we have

$$\frac{\|x\|}{\|y\|} = \frac{\sphericalangle(y,\ x+y)}{\sphericalangle(x,\ x+y)} = \frac{\sphericalangle(Ty,\ Tx+Ty)}{\sphericalangle(Tx,\ Tx+Ty)} = \frac{\|Tx\|}{\|Ty\|}$$

so that T is a similarity.

Let E be a normed vector space, $x,y \in E$. We say that x and y are *orthogonal* to each other ($x \perp y$) if either $x = 0$, $y = 0$, or $\sphericalangle(x,y) = 1$. Then $x \perp y$ if and only if

$$\mathrm{dist}(x,[y]) = \|x\|.$$

By (iv) of 3.3 this formula is equivalent to the more symmetrical

$$\|\alpha x + \beta y\| = \max\{\|\alpha x\|,\|\beta y\|\} \qquad (\alpha,\beta \in K).$$

Note that

$$x \perp y \text{ if and only if } y \perp x.$$

Obviously, $x \perp x$ if and only if $x = 0$.

In general, the set $\{y : x \perp y\}$ is not a vector space. [Example: let $E = K^2$, $x = (1,1)$, $y_1 = (1,0)$, $y_2 = (0,1)$. Then $x \perp y_1$ and $x \perp y_2$ but not $x \perp y_1 + y_2$.]

If $x \in E$ and if D is a linear subspace of E we write $x \perp D$ if $x \perp y$ for all $y \perp D$, i.e., if

$$\mathrm{dist}(x,D) = \|x\|.$$

E is called an *immediate extension* of D if 0 is the only element of E that is orthogonal to D. This is the case if and only if for every nonzero $a \in E$ there exists a $b \in D$ with $\|a - b\| < \|a\|$. Note that, *if* $D_1 \subset D_2 \subset E$, *if* E *is an immediate extension of* D_2 *and if* D_2 *is an immediate extension of* D_1, *then* E *is an immediate extension of* D_1.

For two linear subspaces, E_1 and E_2, of E we put $E_1 \perp E_2$ if $x \perp y$ for all $x \in E_1$ and $y \in E_2$. The following relations are easy to verify.

$E_1 \perp E_2$ if and only if $E_2 \perp E_1$. If $E_1 \perp E_2$ and $E_2 \supset E_3$, then $E_1 \perp E_3$. $\{0\} \perp E_1$ for every E_1. If $E_1 \perp E_2$, then $\overline{E_1} \perp \overline{E_2}$ and $E_1 \cap E_2 = \{0\}$. If $E_1 \perp E_2$, then

$$E_1 + E_2 \sim E_1 \oplus E_2 .$$

The concept of orthogonality, which is basic to large parts of our theory, is generalized in the following fashion.

Let $x_1, x_2, \ldots,$ be a finite or infinite sequence of elements of a normed space E. We say that this sequence is *orthogonal* if

$$\|\alpha_1 x_1 + \ldots + \alpha_m x_m\| = \max\{\|\alpha_i x_i\| : i = 1, \ldots, m\}$$

for all $m \in \mathbb{N}$ (or for all such m that do not exceed the length of the given sequence) and all $\alpha_1, \ldots, \alpha_m \in K$. If the sequence is infinite, it follows that

$$\left\|\sum_{i=1}^{\infty} \alpha_i x_i\right\| = \max\{\|\alpha_i x_i\| : i = 1,2,\ldots\}$$

for all $\alpha_1,\alpha_2,\ldots \in K$ for which $\lim \alpha_i x_i = 0$.

Let t be a real number, $0 < t \leq 1$. A finite or infinite sequence $x_1, x_2, \ldots,$ of elements of E is said to be *t-orthogonal* if for all m and $\alpha_1, \ldots, \alpha_m$,

$$\|\alpha_1 x_1 + \ldots + \alpha_m x_m\| \geq t \max\{\|\alpha_i x_i\| : i = 1,\ldots,m\}.$$

(If the sequence $x_1, x_2, \dots,$ is infinite, a similar inequality for infinite linear combinations follows.)

Clearly, 1-orthogonality is the same as orthogonality. Any permutation of a t-orthogonal sequence is t-orthogonal.

The following observation, which is a direct consequence of 3.1, will recur several times.

3.L Example. *If $x_1, \dots, x_n$ are elements of a normed space E such that the nonzero entries in the sequence $||x_1||, \dots, ||x_n||$ lie in distinct cosets of the value group of K, then $x_1, \dots, x_n$ is an orthogonal sequence.*

It should be noted that *orthogonality of a sequence is not the same thing as pairwise orthogonality of its elements.* (As on p. 56, consider the elements (1,1), (1,0), and (0,1) of K^2.)

It is easy to extend the definition of (t-)orthogonality to uncountable families or unindexed sets (we come to that in Chapter 5).

A closed linear subspace E_1 of a normed vector space E is *complemented* if E has a closed linear subspace E_2 such that $E_1 \cap E_2 = \{0\}$ and $E_1 + E_2 = E$. Such an E_2 is called a complement of E_1. We say that E_2 is an *orthocomplement* of E_1 if it is a complement and $E_1 \perp E_2$. (Then $E \sim E_1 \oplus E_2$.)

LINEAR OPERATORS

Let E,F be normed vector spaces. $L(E,F)$ will be the vector space of all continuous linear maps $E \to F$.
We put $E' := L(E,K)$ and $L(E) := L(E,E)$. E' is called the *conjugate space* or *dual Banach space* of E.

For any linear $T : E \to F$, let

$$||T|| := \inf\{c \in [0, \infty) : ||Tx|| \le c\, ||x|| \text{ for all } x \in E\}.$$

Just as in the Archimedean theory, T is continuous if and only if $||T||$ is finite, and $||\ ||$ is a norm on $L(E,F)$. If $E \neq \{0\}$, then for all linear $T : E \to F$

$$||T|| = \sup \{ \frac{||Tx||}{||x||} : x \in E,\ x \neq 0 \}.$$

<u>Note</u>: These formulas are less innocent than they look. The formula

$$||T||_o := \sup\{||Tx|| : x \in E,\ ||x|| \le 1\}$$

defines a decent norm $||\ ||_o$ on $L(E,F)$, that is equivalent to $||\ ||$ but need not be identical with it. As an example, let $F = \mathbb{Q}_3$ with the valuation as the norm, and let also $E = \mathbb{Q}_3$, but with $||x|| = 2|x|$ for $x \in E$. For the identity map $I : E \to F$, we find $||I|| = 1/2$, but $||I||_o = 1/3$. (The same example shows that $||T||$ may not be $\sup\{||Tx|| : ||x|| = 1\}$.) One reason for preferring $||\ ||$ to $||\ ||_o$ is Theorem 3.15(v). Another is the convenient formula

$$||Tx|| \le ||T||\ ||x|| \qquad [x \in E,\ T \in L(E,F)].$$

Some elementary properties of continuous linear operators are shown in the following exercises.

3.M Exercise.

i. If F is a Banach space, then so is $L(E,F)$.
ii. Let E,F be normed vector spaces, D a closed linear subspace of E. Let $P:E \to E/D$ be the canonical surjection. Then $T \to TP$ is a linear isometry of $L(E/D,F)$ onto $\{S \in L(E,F) : S(D) = \{0\}\}$.

3.N Exercise. Let E,F be normed vector spaces.

i. Every $T \in L(E,F)$ determines $T' \in L(F',E')$ by

$$T'(f) := fT \qquad (f \in F')$$

$\|T'\| \le \|T\|$.

ii. The formula

$$\hat{a}(f) := f(a) \qquad (a \in E,\ f \in E')$$

yields a linear map $a \to \hat{a}$ of E into E''; we have $\|\hat{a}\| \le \|a\|$ for all $a \in E$. If $T \in L(E,F)$, then $T''\hat{a} = (Ta)^\wedge$. We call the map $J_E: a \to \hat{a}$ the *natural map* $E \to E''$.

From the preceding exercise we see that for every normed vector space E the natural map $J_E: E \to E''$ is continuous and, in fact, that $\|J_E\| \le 1$. Contrary to what one might expect, J_E need not be isometric. For certain K, we shall construct infinite-dimensional Banach spaces E for which $E' = \{0\}$, hence $E'' = \{0\}$. Then obviously J_E cannot be an isometry (see Corollary 4.3).

These phenomena will occupy us in Chapter 4, but this is as good a place as any to introduce some of the relevant terminology.

A normed space E is said to be *reflexive* if J_E is a surjective isometry. E is said to be *pseudoreflexive* if J_E is isometric: E is *topologically pseudoreflexive* if J_E is a homeomorphism of E onto a subspace of E''. Clearly, the latter is the case if and only if there exists a positive real number c such that $\|\hat{a}\| \ge c\|a\|$ for all $a \in E$.

3.P Exercise. Let E be topologically pseudoreflexive. The formula $\nu(x) := \|J_E x\|$ defines a norm ν on E that is equivalent to $\|\ \|$. Giving E this norm ν we obtain a a normed vector space E_ν for which $E' \sim (E_\nu)'$. The space E_ν is pseudoreflexive.

3.Q Exercise.

i. Let $\{E_i\}_{i\in I}$ be a family of normed spaces. For $f \in \oplus_i E_i$ and $g \in \times_i E_i'$, $i \mapsto g_i(f_i)$ is an element of $c_0(I)$, so that

$$\langle f,g\rangle := \sum_{i\in I} g_i(f_i)$$

exists. Moreover, $|\langle f,g\rangle| \le \|f\|\,\|g\|$. The formula

$$(Tg)(f) := \langle f,g\rangle \qquad (f \in \oplus_i E_i,\ g \in \times_i E_i')$$

yields a linear isometry

$$T : \times_i E_i' \sim (\oplus_i E_i)'$$

ii. If X is a set and s a function $X \to (0,\infty)$, then

$$[c_0(X : s)]' \sim l^\infty(X : \tfrac{1}{s})$$

$c_0(X : s)$ is pseudoreflexive. If X is finite, $c_0(X : s)$ is reflexive.

Theorems 3.5, 3.11 and 3.12 are analogues of Archimedean theorems that are usually proved by means of category arguments. The same techniques work here.

3.5 CLOSED GRAPH THEOREM. *Let* T *be a linear map of a Banach space* E *into a Banach space* F *such that its graph* $\{(x,Tx) : x \in E\}$ *is a closed subset of* $E \times F$. *Then* T *is continuous.*

PROOF. Take $\pi \in K$, $0 < |\pi| < 1$. Let $B := \mathrm{clo}\{x \in E : \|Tx\| \le 1\}$. The sets $\pi^{-n}B$ $(n \in \mathbb{N})$ are closed and their union is E. By the Baire category theorem there exist an $m \in \mathbb{N}$, an $\varepsilon > 0$ and an $a \in E$ such that $B_\varepsilon(a) \subset \pi^{-m}B$. Then $B_\varepsilon(0) \subset a + \pi^{-m}B \subset \pi^{-m}B + \pi^{-m}B = \pi^{-m}B$. With $\delta := |\pi|^m\varepsilon$ we have $B_\delta(0) \subset B$. Thus, for any $x \subset E$ with $\|x\| \le \delta$ there exists $x' \in E$, $\|Tx'\| \le 1$, $\|x - x'\| \le |\pi|\delta$, hence $\|\pi^{-1}x - \pi^{-1}x'\| \le \delta$. For such an x we can inductively construct a sequence $x_0, x_1, \ldots,$ such that for all n,

$$||Tx_n|| \leq 1,$$
$$||\pi^{-n-1}x - \pi^{-n-1}x_0 - \pi^{-n}x_1 - \ldots - \pi^{-1}x_n|| \leq \delta.$$

Then $||x - (x_0 + \pi x_1 + \ldots + \pi^n x_n)|| \leq |\pi|^{n+1}\delta$ for all n; so $x = \Sigma\, \pi^n x_n$. Furthermore, $\Sigma\, \pi^n Tx_n$ converges. By our assumptions, this means that $Tx = \Sigma\, \pi^n Tx_n$, so $||Tx|| \leq 1$. We see that $||Tx|| \leq 1$ for every $x \in B_\delta(0)$. Then T is continuous.

We need the following consequences.

3.6 COROLLARY. *If* T *is a continuous linear bijection of a Banach space onto a Banach space, then* T *is a homeomorphism.*

PROOF. The graph of T^{-1} is closed.

3.7 COROLLARY. *If* X *is a base of a Banach space* E, *there exists a* $t > 0$ *such that*

$$||\sum_{x\in X} a_x x|| \geq t \max\{||a_x x|| : x \in X\}$$

for all $a \in c_0(X : ||\ ||)$.

3.8 COROLLARY. *A Banach space* E *has a base if and only if is linearly homeomorphic to some* $c_0(X)$.

PROOF. The "if" is obvious. Conversely, let E have a base. Take $\pi \in K$, $0 < |\pi| < 1$. As we know (see page 54), E has a base X such that $|\pi| \leq ||x|| \leq 1$ for all $x \in X$. By the preceding corollary, E is linearly homeomorphic to $c_0(X : ||\ ||)$. The identity map is a linear homeomorphism of the latter space onto $c_0(X)$.

Let E be a Banach space. A *projection* in E is an element P $L(E)$ for which $P^2 = P$. If P is such a projection, then P(E)

is the kernel of I - P so that P(E) is a closed linear subspace of E. As the kernels of P and I - P have trivial intersection and the kernel of P is just (I - P)(E), we see that P(E) *and* $P^{-1}(0)$ *are complements to each other*. The projection P is called an *orthoprojection* if $P(E) \perp P^{-1}(0)$.

3.R Exercise. Let P be a projection in a Banach space E.

i. I - P is a projection.
ii. If $P \neq 0$, then $||P|| \geq 1$.
iii. If $P \neq 0, I$, then $||P|| = ||I - P||$.
iv. P is an orthoprojection if and only if $||P|| \leq 1$.
v. If P is an orthoprojection, then so is I - P.
vi. Let Q be a projection and PQ = QP. Then PQ is a projection. If both P and Q are orthoprojections, then PQ is an orthoprojection too.

The following theorem is now a direct consequence of Corollary 3.6.

3.9 THEOREM.

i. *If* E_1, E_2 *are closed linear subspaces of a Banach space* E *that are complements to each other, then the natural map* $E_1 \oplus E_2 \to E$ *is a homeomorphism, i.e., there exists a positive number* t *such that*

$$||x + y|| \geq t \max\{\, ||x||\,,\ ||y||\, \} \qquad (x \in E_1, y \in E_2)$$

ii. *For a closed linear subspace* D *of a Banach space* E *the following conditions are equivalent.*

α. D *is complemented.*
β. *There exists a projection of* E *onto* D.
γ. *If* F *is any Banach space, every* $S \in L(D,F)$ *has an extension* $\overline{S} \in L(E,F)$.

iii. *If* D *is a closed linear subspace of a Banach space* E, *then every complement of* D *is linearly homeomorphic to* E/D.

PROOF. (i) follows from Corollary 3.6, (ii) from (i). (For the implication $(\alpha) \Rightarrow (\beta)$, extend the identity map of D.) To prove (iii), apply Corollary 3.6 to the restriction of the quotient map $E \to E/D$ to a complement of D.

Similarly, we have the following result.

3.10 THEOREM. *Let* D *be a closed linear subspace of a Banach space* E.

i. *The conditions* (α)-(γ) *are equivalent.*
   - α. D *has an orthocomplement.*
   - β. *There exists an orthoprojection of* E *onto* D.
   - γ. *If* F *is any Banach space, then every* $S \in L(D,F)$ *has an extension* $\bar{S} \in L(E,F)$ *for which* $||\bar{S}|| = ||S||$.

ii. *All orthocomplements of* D *are linearly isometric to* E/D.

Returning to the Closed Graph Theorem we have one more consequence.

3.11 OPEN MAPPING THEOREM. *If* E *and* F *are Banach spaces and if* $T \in L(E,F)$ *is surjective, then the image under* T *of any open subset of* E *is open in* F.

PROOF. Let $D := T^{-1}(0)$ and let P be the quotient map $E \to E/D$. We have an induced map $\bar{T} : E/D \to F$ such that $T = \bar{T}P$. By Corollary 3.6, $\bar{T}$ is a homeomorphism. Now if $U \subset E$ is open, it is easy to see that P(U) is open in E/D, so that $\bar{T}P$ (U) [which is T(U)] is open in F.

The proof of the following theorem is also a simple adaption of its Archimedean counterpart.

3.12 UNIFORM BOUNDEDNESS THEOREM (BANACH-STEINHAUS). *Let* E *be a Banach space,*F *a normed vector space. If* $\mathcal{S}$ *is a subset of* $L(E,F)$ *such that for every* $x \in E$ *the set* $\{Sx : S \in \mathcal{S}\}$ *is bounded in* F, *then* $\mathcal{S}$ *is a bounded set in* $L(E,F)$.

PROOF. Let $\mathcal{S} \subset L(E,F)$ be so that for every $x \in E$ the set $\{Sx : S \in \mathcal{S}\}$ is bounded. For every $n \in \mathbb{N}$ the set $E_n := \{x \in E : \text{for all } S \in \mathcal{S},\ ||Sx|| \leq n\}$ is closed in E. As $\bigcup_n E_n = E$, one of the E_n has nonempty interior. Moreover, each E_n is a group. Hence, there exist $n \in \mathbb{N}$ and $\varepsilon > 0$ such that $B_\varepsilon(0) \subset E_n$. Let $\pi \in K$, $0 < |\pi| < 1$. For every $x \in E$, there is an $m \in \mathbb{Z}$ for which $\varepsilon|\pi| \leq |\pi|^m||x|| \leq \varepsilon$: then $\pi^m x \in B_\varepsilon(0)$, so that for all $S \in \mathcal{S}$ we have $||Sx|| = |\pi|^{-m}||S(\pi^m x)|| \leq |\pi|^{-m}n \leq (n / \varepsilon|\pi|)\ ||x||$. Thus, $\mathcal{S} \subset \{T \in L(E,F) : ||T|| \leq n / \varepsilon|\pi|\}$.

We have the familiar corollary.

3.13 COROLLARY. *If* $T_1, T_2, \ldots,$ *is a sequence in* $L(E,F)$ *such that* $Tx := \lim T_n x$ *exists for every* $x \in E$, *then* $T \in L(E,F)$.

PROOF. By the uniform boundedness theorem, there exists a number c such that $||T_n|| \leq c$ for all $n \in \mathbb{N}$; then $||Tx|| \leq c||x||$ for all $x \in E$.

BANACH SPACES OF COUNTABLE TYPE

In the Archimedean theory of Banach spaces a special role is played by the separable ones. A closer investigation brings to light, however, that the useful property of these spaces quite often is not the existence of a countable subset that is dense but of a countable subset whose linear span is dense. These properties are equivalent, thanks to the separability of the scalar field.

In our case the situation is different if K is not separable. Then the only separable Banach space is $\{0\}$ but many Banach spaces have countable generating subsets (e.g., every finite-dimensional Banach space and c_0).

We say that a Banach space E is *of countable type* if it is the closed linear span of a countable set. If K is separable, a Banach space is of countable type and only if it is separable.

Fundamental in our theory of Banach spaces of countable type is the following lemma.

3.14 LEMMA. *Let* F *be a closed linear subspace of a normed vector space* E *and let* $a \in E$, $a \notin F$.

i. $[a] + F$ *is closed. If* F *is complete, so is* $[a] + F$.

ii. *For every* $t \in R$ *with* $0 < t < 1$, *there exists an* $e \in E$ *such that* $[a] + F = [e] + F$ *and*

$$\mathrm{dist}(e,F) \geq t\|e\|.$$

For every such e, *we have*

$$\|\alpha e + x\| \geq \max\{\|\alpha e\| \ \|x\|\} \qquad (\alpha \in K,\ x \in F).$$

With π *as on* *p. 20*, *we can choose* e *such that*

$$|\pi| \leq \|e\| \leq 1.$$

iii. *If* F *is spherically complete, then the conclusions of* (ii) *are also valid for* $t = 1$. (*Then* $e \perp F$.)

PROOF. Put $r := \mathrm{dist}(a,F)$. There exists a $z \in F$ with $\|a - z\| \leq t^{-1}r$. Set $e_0 := a - z$. Then clearly $[a] + F = [e] + F$ and $t\|e_0\| \leq r = \mathrm{dist}(e_0 + z, F) = \mathrm{dist}(e_0,F)$.

If e is any nonzero element of E such that $t\|e\| \leq \mathrm{dist}(e,F)$, then for every $\alpha \in K$ and $x \in F$ we have $\|\alpha e + x\| \geq \mathrm{dist}(\alpha e,F) \geq t\|\alpha e\|$ and therefore (see Lemma 3.2) $\|\alpha e + x\| \geq t \max\{\|\alpha e\|,\|x\|\}$. It follows that the map $(\alpha,x) \to \alpha e + x$ is a linear homeomorphism of $K \oplus F$ onto $[a] + F$, so $[a] + F$ is metrically complete and closed in E. The last part of (ii) is obvious since our e has a scalar multiple whose norm is between $|\pi|$ and 1.

Now assume F to be spherically complete. We only have to construct a $z \in F$ such that $\|a - z\| = r$ (the rest follows easily). For every $s > r$, the set $B_s(a) \cap F$ is a ball in F of radius s. Hence, $\bigcap_{s>r} B_s(a) \cap F$ contains an element z. Now $\|a - z\| \leq \inf\{s : s > r\}$ and $\|a - z\| \geq \mathrm{dist}(a,F) = r$. Hence, $\|a - z\| = r$.

As an application, let us consider a normed vector space E that is two-dimensional over K. Choose $e_1 \in E$, $e_1 \neq 0$. As K is metrically complete, so is $[e_1]$ and, by the lemma, so is E. Further, the lemma says that for every $t \in (0, 1)$ we can choose an $e_2 \in E$ such that

$$\|\alpha_1 e_1 + \alpha_2 e_2\| \geq t \max\{\|\alpha_1 e_1\|,\|\alpha_2 e_2\|\} \qquad (\alpha_1,\alpha_2 \in K).$$

Apparently, e_1 and e_2 form a base for E (both in the sense of the linear algebra and in the sense of the Banach space theory). The map $(\alpha_1,\alpha_2) \mapsto \alpha_1 e_1 + \alpha_2 e_2$ is a linear homeomorphism of K^2 onto E.

We can do better if K is spherically complete. In that case, by the last part of the lemma we can choose e_2 so that $e_2 \perp e_1$. Setting $s_1 = \|e_1\|$, $s_2 = \|e_2\|$ we see that the map $(\alpha_1,\alpha_2) \to \alpha_1 e_1 + \alpha_2 e_2$ is now actually an isometry

$$(*) \qquad K_{(s_1)} \oplus K_{(s_2)} \sim E$$

It is reasonable to ask if the same formula can be proved if K fails to be spherically complete. The answer is negative: whenver K is not spherically complete there is a two-dimensional Banach space E for which no isomorphism of the type (*) exists. It is worth the trouble to construct such an E explicitly.

Assume that K is not spherically complete. Then we have a sequence $B_{\varepsilon_1}(\lambda_1) \supset B_{\varepsilon_2}(\lambda_2) \supset \ldots$, of balls in K whose intersection is empty. Take $\lambda \in K$. There is an i with $\lambda \notin B_{\varepsilon_i}(\lambda_i)$. Then for all $j > i$, $|\lambda - \lambda_i| > \varepsilon_i \geq |\lambda_i - \lambda_j|$, so that $|\lambda - \lambda_j| = |\lambda - \lambda_i|$, by 1.1($\delta$). Thus, for every $\lambda \in K$ the sequence $j \to |\lambda - \lambda_j|$ is eventually constant. In particular, $\lim_j |\lambda - \lambda_j|$ exists and is an element of the value group of K. (Note that the limit is not 0.)

It follows that for all $\alpha_1, \alpha_2 \in K$,

$$\nu(\alpha_1,\alpha_2) := \lim_j |\alpha_1 - \alpha_2\lambda_j|$$

exists and is either 0 (viz., if and only if $\alpha_1 = \alpha_2 = 0$) or an element of the value group of K. It is now almost trivial that ν is a norm on K^2. By 3.14(i), under this norm K^2 is a Banach space; we denote this Banach space by K^2_ν. We proceed to prove that K^2_ν *does not contain any two nonzero elements that are orthogonal to each other* [it follows that no decomposition (*) exists for $E = K^2_\nu$].

Let $e := (1,0) \in K^2$. We fir show that for every nonzero $x \in K^2_\nu$ there is an $\alpha \in K$ such that $\nu(x - \alpha e) < \nu(x)$. This conclusion is trivial for $x \in [e]$. Now let $x \in K^2_\nu$, $x \notin [e]$: then $x = (\alpha_1,\alpha_2)$ for certain $\alpha_1,\alpha_2 \in K$ for which $\alpha_2 \neq 0$. Without loss of generality we assume $\alpha_2 = 1$. There is an $i \in \mathbb{N}$ such that $\alpha_1 \notin B_{\varepsilon_i}(\lambda_i)$. From the reasoning that led to the definition of ν we infer that $\lim_j |\alpha_1 - \lambda_j| = |\alpha_1 - \lambda_i| > \varepsilon_i$, so $\nu(x) > \varepsilon_i \geq$

$\limsup_j |\lambda_i - \lambda_j| = \nu((\lambda_i,1)) = \nu(x - (\alpha_1 - \lambda_i)e)$. Thus, we can take $\alpha := \alpha_1 - \lambda_i$.

Let now x,y be nonzero elements of K^2_ν: then they are not orthogonal to each other. In fact, there exist $\alpha,\beta \in K$ such that $\nu(x - \alpha e) < \nu(x)$ and $\nu(y - \beta e) < \nu(y)$. Obviously, $\alpha,\beta \neq 0$. Then $\nu(\beta x - \alpha y) \leq \max\{\nu(\beta x - \alpha\beta e), \nu(\alpha y - \alpha\beta e)\}$ $< \max\{\nu(\beta x), \nu(\alpha y)\}$.

The construction above is essentially the only possible alternative to (*) :

3.S Exercise. Let E be a two-dimensional normed vector space. Suppose there do not exist numbers s_1 and s_2 such that (*) holds. Then there exist linearly independent elements e_1 and e_2 of E, a positive real number s and a sequence $B_{\varepsilon_1}(\lambda_1) \supset B_{\varepsilon_2}(\lambda_2) \supset \ldots$ of balls in K whose intersection is empty, such that for all $\alpha_1,\alpha_2 \in K$,

(**) $\quad ||\alpha_1 e_1 + \alpha_2 e_2|| = s \lim_j |\alpha_1 + \alpha_2\lambda_j|$.

[<u>Hint</u>: Take $e_2 \in E$, $e_2 \neq 0$. If E contains a nonzero element that is orthogonal to e_2, we have (*). Otherwise, take any $e_1 \in E\backslash[e_2]$). There exist $\lambda_1,\lambda_2,\ldots \in K$ such that $\mathrm{dist}(e_1,[e_2]) = \lim ||e_1 - \lambda_j e_2||$. If $\lambda \in K$, then for large j we have $||e_1 - \lambda e_2|| > ||e_1 - \lambda_j e_2||$, whence $||e_1 - \lambda e_2|| = |\lambda - \lambda_j|\ ||e_2||$. Prove now that (**) follows for a suitable s.]

By induction, the reasoning we gave on p. 67 carries over to all finite-dimensional spaces. We obtain the following theorem which looks deceptively familiar.

3.15 THEOREM. *Let* E *be an* n-*dimensional normed vector space* ($n \in \mathbb{N}$).

i. E *is linearly homeomorphic to* K^n.

ii. E *is a Banach space. All linear functions* $E \to K$ *are continuous. All linear subspaces of* E *are closed and complemented.*

iii. *For every* $t \in (0, 1)$ *there exists a* t*-orthogonal sequence* $e_1, \ldots, e_n$ *in* E *that forms a base for* E.

iv. *For every* $t \in (0, 1)$ *there exist positive real numbers* $s_1, \ldots, s_n$ *and a linear bijection* $T : K_{(s_1)} \oplus \ldots \oplus K_{(s_n)} \to E$ *such that*

$$t||x|| \le ||Tx|| \le ||x|| \qquad (x \in K_{(s_1)} \oplus \ldots \oplus K_{(s_n)}).$$

v. E *is reflexive.*

PROOF. We leave (i) and (ii) to the reader and instead of proving (iii), (iv), and (v) we merely mention that their proofs are very similar to the ones of (i), (iii), and (iv) of the following theorem, respectively.

3.16 THEOREM. *Let* E *be an infinite-dimensional Banach space of countable type.*

i. *For every sequence* $t_1, t_1, \ldots,$ *of elements of the real interval* $(0, 1)$, E *has a base* $\{e_i : i \in \mathbb{N}\}$ *such that*

$$||\sum_{i=1}^{m} \alpha_i e_i|| \ge \max\{t_i||\alpha_i e_i|| : i = 1,\ldots,m\}$$

for all $m \in \mathbb{N}$ *and* $\alpha_1, \ldots \alpha_m \in K$.

ii. *For every* $t \in (0, 1)$, E *contains a* t*-orthogonal sequence that forms a base for* E. E *is linearly homeomorphic to* c_0.

iii. *For every* $t \in (0, 1)$ *there exist a function* $s : \mathbb{N} \to (0, 1)$ *and a linear bijection* $S : c_0(\mathbb{N} : s) \to E$ *such that*

$$t||x|| \le ||Sx|| \le ||x|| \qquad [x \in c_0(\mathbb{N} : s)].$$

iv. E *is pseudoreflexive.*

v. *Let* D *be any closed linear subspace of* E. *Then both* D *and* E/D *are of countable type.* D *is complemented. For every* $\varepsilon > 0$ *there exists a projection of* E *onto* D *of norm less than or equal to* $1 + \varepsilon$.

vi. *Let* D *be a linear subspace of* E, *let* $f \in D'$ *and* $\varepsilon > 0$. *Then* f *can be extended to an* $\bar{f} \in E'$ *with* $||\bar{f}|| \le (1 + \varepsilon)||f||$.

(There is some redundancy in the statement of this theorem, but the formulation given above is convenient for quoting.)

PROOF. Let $\pi \in K$, $0 < |\pi| < 1$.

i. We may assume $t_1 < t_2 < \ldots$. Choose a sequence $\{0\} = E_0 \subset E_1 \subset E_2 \subset \ldots$ of linear subspaces of E such that each E_n is n-dimensional and $\overline{\cup E_n} = E$. By lemma 3.14, for each $n \in \mathbb{N}$ we can choose an $e_n \in E_n$ for which

$$|\pi| \le ||e_n|| \le 1$$

$$||\alpha e_n + x|| \ge \frac{t_n}{t_{n+1}} \max\{||\alpha e_n||, ||x||\} \quad (\alpha \in K,\ x \in E_{n-1}).$$

It follows inductively that for all $\alpha_1, \ldots, \alpha_n \in K$,

$$(*) \qquad ||\alpha_n e_n + \ldots + \alpha_1 e_1|| \ge \max_{i \le n}\{\frac{t_i}{t_{n+1}}||\alpha_i e_i|| : i \le n\}$$

As $t_{n+1} < 1$, we have the inequality announced in (i). It remains to prove that $\{e_i : i \in \mathbb{N}\}$ is a base. Let $s : \mathbb{N} \to (0, \infty)$ be the function $i \mapsto ||e_i||$ $(i \in \mathbb{N})$. We have a map $S : c_0(\mathbb{N} : s) \to E$, defined by

$$S(\alpha_1,\alpha_2,\ldots) := \sum_{i=1}^{\infty} \alpha_i e_i \qquad [(\alpha_1,\alpha_2,\ldots) \in c_0(\mathbb{N} : s)].$$

Trivially, S is linear and $||S|| \le 1$. It follows from $(*)$ that

$$(**) \qquad ||\alpha_1 e_1 + \ldots + \alpha_n e_n|| \ge t_1 \max\{||\alpha_i e_i||\}$$

for all $n \in \mathbb{N}$ and $\alpha_1,\dots,\alpha_n \in K$, so that $\|Sx\| \geq t_1\|x\|$ $(x \in c_0(\mathbb{N} : s))$. Hence, S is a linear homeomorphism of $c_0(\mathbb{N} : s)$ onto its range space. Then this range space is closed in E. But it contains every e_i : so S must be surjective and $\{e_i : i \in \mathbb{N}\}$ is a base for E.

ii. Apply the above to $t_i = \sqrt[i]{t}$ $(i \in \mathbb{N})$. By (**), the sequence $e_1, e_2, \dots,$ is t-orthogonal. As $|\pi| \leq s(i) \leq 1$ for every i, $c_0(\mathbb{N} : s)$ and c_0 are identical as sets and the identity map between them is a homeomorphism.

iii. Again, apply (i) with $t_i = {}^{i}\!\sqrt{t}$.

iv. Let $a \in E$. Take $t \in (0, 1)$: we make a nonzero $f \in E'$ such that $|f(a)| \geq t\|f\|\,\|a\|$. [Then (iv) follows.]

Let s and S be as in (iii). There exists a $b = (\beta_1,\beta_2,\dots) \in c_0(\mathbb{N} : s)$ such that $Sb = a$ and there is an $i \in \mathbb{N}$ such that $\|b\| = s(i)|\beta_i|$. Let $f : E \to K$ be the function that to each $x \in E$ assigns the i-th coordinate of $S^{-1}x$. Then $f \in E'$, $\|f\| \leq s(i)^{-1}\|S^{-1}\| \leq s(i)^{-1}t^{-1}$ while on the other hand $|f(a)| = |\beta_i| = s(i)^{-1}\|b\| \geq s(i)^{-1}\|Sb\| = s(i)^{-1}\|a\|$.

v. It is clear that $F := E/D$ is of countable type. Let Q be the quotient map $E \to F$. Let $\varepsilon > 0$, $t := (1+\varepsilon)^{-\frac{1}{2}}$. F has a base consisting of the elements of a (finite or infinite) t-orthogonal sequence $e_1, e_2, \dots$. For each i choose $a_i \in E$ such that $Qa_i = e_i$ and $\|e_i\| \geq t\|a_i\|$. If $\lambda_1,\lambda_2,\dots \in K$, then

$$\|\Sigma\, \lambda_i a_i\| \leq \max\{|\lambda_i| t^{-1}\|e_i\|\} \leq t^{-2}\|\,\Sigma\, \lambda_i e_i\|.$$

Thus, there exists a $T \in L(F,E)$ with $Te_i = a_i$ for all i (that are less than or equal to dim F) and necessarily $\|T\| \leq t^{-2} = 1 + \varepsilon$. Note that $QT = id_F$. Set $P := I - TQ$. Then $P \in L(E)$. As $\|T\| \leq 1 + \varepsilon$ and $\|Q\| \leq 1$ we see that $\|P\| \leq 1 + \varepsilon$. From the identities $QP = Q - QTQ = Q - id_F \circ Q = Q - Q = 0$ we infer that $P(E)$ is contained in the kernel of Q, which is D. But on D we have $Q = 0$ and therefore $P = 1$. Consequently, P is a projection of E onto D of norm $\leq 1 + \varepsilon$.

In particular, P is a continuous linear surjection $E \to D$, so D is of countable type.

vi. We may assume D to be closed. Let P be as in (v) and set $\tilde{f} := f \circ P$.

Even in c_0 itself, a closed linear subspace may fail to have an *ortho*complement. As an example, let K have a dense valuation. Take $\lambda_1,\lambda_2,\dots \in K$ such that $1 \leq |\lambda_1| < |\lambda_2| < \dots < 2$, and let D be the set of all elements $(\alpha_1,\alpha_2,\dots)$ of c_0 for which $\Sigma\, \alpha_i\lambda_i = 0$. Being the kernel of an element of $(c_0)'$, D is a closed linear subspace of c_0 and, trivially, $D \neq c_0$. We prove that no nonzero element of c_0 can be $\perp D$.

Let $a = (\alpha_1,\alpha_2,\dots) \in c_0$, $a \neq 0$. To prove that the formula $a \perp D$ is false, it suffices to construct a $b \in D$ with $\|a - b\| < \|a\|$. Let $m := \max\{\, i : |\alpha_i| = \|a\|\}$ and define $b = (\beta_1,\beta_2,\dots) \in c_0$ by

$$\beta_i := \alpha_i \qquad (i \leq m),$$

$$\beta_{m+1} := -(\alpha_1\lambda_1 + \dots + \alpha_m\lambda_m)\lambda_{m+1}^{-1},$$

$$\beta_i := 0 \qquad (i \geq m + 2).$$

Then $b \in D$; as $|\beta_{m+1}| < \|a\|$ we have $\|a - b\| < \|a\|$.

[Later on, we shall see that, if the valuation is discrete, every closed subspace of c_0 does have an orthocomplement (Corollaries 2.4 and 4.7) and that, for all K, every finite-dimensional subspace of c_0 has an orthocomplement (Lemma 4.35).]

3.17 COROLLARY. *A Banch space E is locally compact if and only if K is locally compact and E is finite-dimensional.* (In Ex. 3.U we study the structure of a locally compact K.)

PROOF. The "if" follows from Theorem 3.15. Now let E be locally compact. E contains a closed subset that is homeomorphic to K, thus K must be locally compact. If E is infinite-dimensional, then it contains a closed set that is homeomorphic to c_0, but c_0 is not locally compact.

3.18 COROLLARY. *In a Banach space that has a base, every closed linear subspace of countable type is complemented.*

PROOF. Let E be a Banach space with a base X. Let D be a closed linear subspace of E that is of countable type. D contains a countable set Y with [Y] = D. Every $y \in Y$ can be written as

$$y = \sum_{x \in X} \lambda_{xy} x$$

for suitable $\lambda_{xy} \in K$. For every $y \in Y$ there are only countably many x for which $\lambda_{xy} \neq 0$. Hence, the set $Z := \{x \in X : \text{there is a } y \in Y \text{ such that } \lambda_{xy} \neq 0\}$ is countable. Now clearly $[X \backslash Z]$ is a complement of [Z] in E, while by 3.16(v), D has a complement in [Z]. From the equivalence of (α) and (β) in Theorem 3.9 it follows easily that D is complemented in E.

3.T Exercise. The following conditions on a compact zero-dimensional Hausdorff space X are equivalent.

α. C(X) is of countable type.

β. X is ultrametrizable.

γ. The ring $\mathcal{B}(X)$ of all clopen subsets of X is countable.

EXTENSIONS OF VALUATIONS

This section is devoted to the following theorem.

3.19 THEOREM (Krull). *Let L_1 be a subfield of a field L_2. Then every non-Archimedean valuation on L_1 can be extended to a (non-Archimedean) valuation on L_2.*

There exist several purely algebraic proofs of this theorem [5a],[26],[243]. We present a proof that uses the language of normed vector spaces, which makes it more interesting from the point of view of a functional analyst.

It should be noted that this theorem does not have an Archimedean pendant. To the contrary, the famous Gelfand-Mazur Theorem (which is fundamental to the Archimedean theory of Banach algebras) asserts the impossibility of extending the natural valuation of $\mathbb{C}$ to any larger field.

For our proof we need some knowledge of the Banach space $K_\rho\{X\}$ introduced in 1.K. Recall in particular that the norm on $K_\rho\{X\}$ has the property that $\|fg\| = \|f\|\,\|g\|$ for all f and g.

3.20 THEOREM (Lazard). *Let $\rho > 0$. Let $f = \sum \alpha_i X^i$ be a nonzero element of $K_\rho\{X\}$ and let $m := \max\{i : \|f\| = |\alpha_i|\rho^i\}$. Then for every $g \in K_\rho\{X\}$ there exist unique $q, r \in K_\rho\{X\}$ such that r is a polynomial of degree $<m$ and $g = qf + r$. For these q and r we have $\|g\| = \max\{\|q\|\,\|f\|, \|r\|\}$.*

PROOF. Let P_m be the set of all polynomials of degree less then m, normed as a subspace of $K_\rho\{X\}$. Let $(f) := \{qf : q \in K_\rho\{X\}\}$. Both P_m and (f) are closed linear subspaces of $K_\rho\{X\}$. [P_m is finitedimensional, hence complete: and (f), being the range of the similarity (!) $q \mapsto qf$, is also complete.] What we have to prove is that P_m and (f) are orthocomplements to each other. We prove first that $(f) \perp P_m$ and then that $(f) + P_m = K_\rho\{X\}$.

A. Let $q = \Sigma\, \beta_i X^i \in K_\rho\{X\}$, $n := \max\{i : ||q|| = |\beta_i|\rho^i\}$. If $\gamma_0, \gamma_1, \gamma_2, \ldots$, are the coefficients of qf, then by what we proved in 1.K, $||qf|| = |\gamma_{m+n}|\rho^{m+n}$. It follows that $||qf + r|| \geq ||qf||$ for every $r \in K_\rho\{X\}$ for which the coefficient of X^{m+n} vanishes. In particular this is true for all $r \in P_m$. Then $qf \perp P_m$. Thus, $(f) \perp P_m$.

B. It follows that $(f) + P_m$ is a closed linear subspace D of $K_\rho\{X\}$. Suppose $D \neq K_\rho\{X\}$. By lemma 3.14, there exists an $h \in K_\rho\{X\}$ such that $\mathrm{dist}(h,D) > \sqrt{t}\,||h||$, where $t := \max\{|\alpha_i|\rho^i||f||^{-1} : i > m\}$. As the polynomials form a dense set in $K_\rho\{X\}$, we may assume that h actually is a polynomial. Let $f_0 := \alpha_0 + \alpha_1 X + \ldots + \alpha_m X^m$. Then $f_0 \in K[X]$ and $||f - f_0|| = t||f||$. There exist polynomials q_0, r_0, such that $r_0 \in P_m$ and $h = q_0 f_0 + r_0$. But then $q_0 f + r_0 \in D$. Thus, $\sqrt{t}\,||h|| \leq ||h - (q_0 f + r_0)||$, whence (by Lemma 3.2) $\sqrt{t}\,||q_0 f + r_0|| \leq ||h - (q_0 f + r_0)|| = ||q_0(f - f_0)|| = ||q_0||\;||f - f_0|| = t||q_0||\;||f|| = t||q_0 f|| = t\,\mathrm{dist}(q_0 f, P_m) \leq t||q_0 f + r_0||$, which is a contradiction.

Let f, α_i, m, P_m be as above. By the theorem, there exist $q \in K_\rho\{X\}$ and $r \in P_m$ such that $X^m = qf + r$ and $||r|| \leq ||X^m|| = \rho^m$, so $||X^m - r|| \leq \rho^m$. Applying Lazard's theorem again, this time with $X^m - r$ instead of f, we obtain $q' \in K_\rho\{X\}$ and $r' \in P_m$ such that $f = q'(X^m - r) + r'$. Then $f = q'qf + r'$. By the uniqueness part of the theorem, we must have $q'q = 1$ (and $r' = 0$). In particular, $f = q'(X^m - r)$. Letting the coefficients of r be $-\beta_0, -\beta_1, \ldots, -\beta_{m-1}$ and taking into account the inequality $||r|| \leq \rho^m$, we see that we have proved

3.21 COROLLARY. *Let $\rho > 0$. Let f and m be as in Lazard's theorem* (3.20). *Then there exist an invertible element g of $K_\rho\{X\}$ and elements $\beta_0, \ldots, \beta_{m-1}$ of K such that*

$$\begin{cases} f = g(X^m + \beta_{m-1}X^{m-1} + \ldots + \beta_1 X + \beta_0) \\ |\beta_i|\rho^i \leq \rho^m \qquad (i = 0, 1, \ldots, m-1). \end{cases}$$

3.22 COROLLARY. *If $\alpha_0 + \alpha_1 X + \ldots + \alpha_n X^n$ is an irreducible polynomial over K (with $\alpha_n \neq 0$), then for all $\rho > 0$,*

$$|\alpha_i|\rho^i \leq \max\{\,|\alpha_0|\,,\,|\alpha_n|\rho^n\,\} \qquad (i = 0,\ldots,n)$$

and

$$|\alpha_i|^n \leq |\alpha_0|^{n-i}|\alpha_n|^i \qquad (i = 0,\ldots,n).$$

In particular,

$$|\alpha_i| \leq \max\{\,|\alpha_0|\,,\,|\alpha_n|\,\} \qquad (i = 0,\ldots,n).$$

(It should be kept in mind that our K is complete. For noncomplete non-Archimedean valued fields the statements above are in general false.)

PROOF. Let $f := \alpha_0 + \alpha_1 X + \ldots + \alpha_n X^n$ be irreducible, let $\rho > 0$. We regard f as an element of $K_\rho\{X\}$. Let m, g, β_i be as above. Let $b := \beta_0 + \beta_1 X + \ldots + \beta_{m-1}X^{m-1} + X^m$. There exist polynomials q,r such that the degree of r is $< m$ and $f = qb + r$. As also $f = gb$, from the uniqueness part of 3.20 (applied to b instead of f) we have $q = g$ (and $r = 0$). In particular, g is a polynomial. But the irreducibility of f now implies that either g or b is constant. In the first case, $m = \deg f = n$, the constant value of g is α_m, and for every i we have $|\alpha_i|\rho^i = |\alpha_m||\beta_i|\rho^i \leq |\alpha_m|\rho^n$. In the second case, $m = 0$, so that for all i (by the definition of m) $|\alpha_i|\rho^0 = |\alpha_0|$.

We have now proved the first of the three formulas announced in 3.22. The second and third are special cases: take $\rho = \sqrt[n]{|\alpha_0|/|\alpha_n|}$ and $\rho := 1$, respectively.

PROOF OF THEOREM 3.19 (Krull). Without loss of generality we make the following assumptions.

A. The valuation on L_1 is not trivial.

B. There exists an $a_0 \in L_2$ such that $L_2 = L_1(a_0)$. (Apply Zorn's lemma.)

C. This a_0 is algebraic over L_1. [Otherwise L_2 is isomorphic to the field $L_1(X)$ of all rational functions over L_1 and the valuation can be extended according to 2.F.]

D. L_2 has no proper subfield that properly contains L_1.

E. L_1 is complete relative to its valuation. [Otherwise, let $\overline{L}_1$ be the completion of L_1 (see 2.C). Then L_2 is isomorphic to a subfield of a finite extension of $\overline{L}_1$.]

We assume in particular that L_1 is complete and nontrivially valued. Thus, without getting into conflict with our conventions, we may write K instead of L_1. We further simplify the notations by dropping the subscript of L_2.

Every $a \in L$ induces a K-linear map $M_a : x \to ax$ of L into L. Let n be the dimension of L as a vector space over K. Define a function $\nu : L \to [0, \infty)$ by

$$\nu(a) := \sqrt[n]{|\det M_a|} \qquad (a \in L).$$

Clearly, $\nu(a) = |a|$ if a happens to be an element of K. We proceed to prove that ν is a valuation.

Of course, $\nu(a) \geq 0$ for all $a \in L$ and $\nu(ab) = \nu(a)\nu(b)$ for all $a,b \in L$. Furthermore, $\nu(a) = 0$ only if $a = 0$. It remains to prove the strong triangle inequality. Let $a,b \in L$, $\nu(a) \leq \nu(b)$; we want to prove $\nu(a + b) \leq \nu(b)$. Set $c := ab^{-1}$. Since ν is multiplicative, $\nu(c) \leq 1$ and we are done if we can show that $\nu(1 + c) \leq 1$. In other words, we know that $|\det M_c| \leq 1$ and we want to prove $|\det(I + M_c)| \leq 1$. We may assume $c \notin K$. By our assumption (D), the minimum polynomial of c over K has degree n. Let this minimum polynomial be $\alpha_0 + \alpha_1 X + \ldots + \alpha_{n-1}X^{n-1} + X^n$. By considering the matrices of M_c and $I + M_c$ relative to the base $1, c, c^2, \ldots, c^{n-1}$ of L, one easily proves that

$$\det M_c = (-1)^n \alpha_0,$$

$$\det(I + M_c) = (-1)^n(\alpha_0 - \alpha_1 + \alpha_2 - \ldots \pm \alpha_{n-1}).$$

The desired formula now follows from Corollary 3.22.

In general, the extension of the valuation is not unique. For transcendental field extensions this is easy to see: for every $\rho > 0$ the embedding $K[X] \to K_\rho\{X\}$ yields a valuation on K(X) that extends the valuation of K. For algebraic extensions the situation is complicated. We refer the reader to Refs. 187 and 243. For algebraic extensions of a complete ground field, however, we have a simple uniqueness theorem that also produces an explicit formula for the extended valuation.

3.23 THEOREM. *Let* L *be a field that is an algebraic extension of* K. *Then there exists a unique valuation* ν *on* L *whose restriction to* K *is* $|\ |$. *If* $a \in L$ *and if* $\alpha_0 + \alpha_1 X + \ldots + \alpha_{n-1}X^{n-1} + X^n$ *is the minimum polynomial of* a *over* K, *then*

$$\nu(a) = \sqrt[n]{|\alpha_0|}.$$

In particular, the value group of this valuation is contained in $\{ \sqrt[n]{s} : n \in \mathbb{N};$ s is an element of the value group of K $\}$. *If* L *is a finite extension of* K *and if the valuation of* K *is discrete, then so is the valuation of* L.

PROOF. Let ν be a valuation on L that extends the given valuation of K. Take $a \in L$, $a \neq 0$. Let $\alpha_0 = \alpha_1 X + \ldots + \alpha_{n-1}X^{n-1} + X^n$ be the minimum polynomial of a. Set $\alpha_n := 1$. Putting $s := \nu(a)$ and $t := \sqrt[n]{|\alpha_0|}$, by the second formula of Corollary 3.22 we have $|\alpha_i| \leq t^{n-i}$ for $i = 0, \ldots, n$. Now

$$s^n = \nu\left(\sum_{i=0}^{n-1} \alpha_i a^i\right)$$

$$\leq \max\{|\alpha_i| s^i : i < n\} \leq t^n \max\{(\tfrac{s}{t})^i : i < n\}$$

whence $(s/t)^n \leq \max\{(s/t)^i : i < n\}$ and $s/t \leq 1$. On the other hand,

$$t^n = |\alpha_0| = \nu\left(\sum_{j=1}^{n} \alpha_j a^j\right)$$

$$\leq \max\{|\alpha_{n-i}| s^{n-i} : i < n\}$$

$$\leq s^n \max\{(\tfrac{t}{s})^i : i < n\}$$

from which it follows that $t/s \leq 1$. Thus, $s = t$, which means that $\nu(a) = \sqrt[n]{|\alpha_0|}$.

This proves the theorem except for the last part. Now suppose that L is an m-dimensional vector space over K. If $a \in L$ and if n is as above, then by adjoining a to K one obtains a subfield of L that is n-dimensional over K. Hence, n is divisor of m and $|a|^m$ lies in the value group of K. The last part of the theorem follows.

Apparently, the algebraic closure of K carries a natural valuation. By $\widetilde{K}$ we denote the metric completion of this algebraic closure. (By Corollary 3.25, $\widetilde{K}$ is itself algebraically closed.) It follows from the uniqueness part of Theorem 3.23 that every continuous K-linear isomorphism of K into itself is a surjective isometry.

$\widetilde{\mathbb{Q}}_p$ is usually called Ω_p or C_p.

We can now describe the locally compact nontrivially non-Archimedean valued fields (the so-called *local fields*).

3.U Exercise. Let K be locally compact. From 1.B we know that the residue class field k of K has characteristic $p \neq 0$. The characteristic of K itself is either 0 or p.

i. Let K have characteristic 0. Let K_0 be the closure of the prime field of K. Choose $\tau > 0$ so that $|p| = p^{-\tau}$. Then there exists an isomorphism T of $\mathbb{Q}_p$ onto K_0 such that $|Tx| = (|x|_p)^\tau$ for all $x \in \mathbb{Q}_p$, while K is a finite algebraic extension of K_0 (with its natural valuation).

ii. Let K have characteristic p. There exists an isomorphism θ of k into K [1.F(vii)]. Let $\pi \in K$ be so that $|\pi| < 1$ and so that the value group of K consists of the powers of $|\pi|$ (1.B). Let $k((X))$ denote the field of all formal Laurent series over k with a valuation defined by

$$\left\| \sum_{n=N}^{\infty} \lambda_n X^n \right\| := |\pi|^N$$

(where $\lambda_N \neq 0$) (see 1.I). The map $\Sigma \lambda_n X^n \to \Sigma \lambda_n \pi^n$ is an isometric isomorphism of $k((X))$ onto K.

Theorems 3.19 and 3.23 have many interesting consequences. Two of these are Corollaries 3.24 and 3.25; we shall find other ones later on.

3.24 COROLLARY. *Let f be a nonzero polynomial over K.*

i. *If $\lambda_1, \lambda_2, \ldots,$ is a sequence of elements of K such that $\lim f(\lambda_i) = 0$, then $\lambda_1, \lambda_2, \ldots,$ has a subsequence that converges to a root of K.*

ii. *$f(K)$ is a closed subset of K.*

iii. *If X is a compact subset of K, then $f^{-1}(X)$ is compact.*

PROOF.

i. Over $\widetilde{K}$, f is a product of linear factors, i.e., there exist $\alpha \in K$, $\alpha \neq 0$, and $\xi_1, \ldots, \xi_n \in \widetilde{K}$ such that

$f = \alpha\Pi(X - \xi_j)$ in $\widetilde{K}[X]$. Then $\lim_i \alpha\Pi(\lambda_i - \xi_j) = 0$, so that the sequence $\lambda_1, \lambda_2, \ldots$ has a subsequence which (in $\widetilde{K}$) converges to one of the ξ_j. Then this ξ_j lies in K.

ii. Let $a \in \text{clo}\, f(K)$. Now apply (i) to $f - a$.

iii. Let $\alpha_1, \alpha_2, \ldots,$ be a sequence in $f^{-1}(X)$: we prove that it has a convergent subsequence. As all $f(\alpha_i)$ are elements of the compact set X we may assume that $\lim f(\alpha_i)$ exists. Apply (i) to the polynomial $f - \lim f(\alpha_i)$.

3.25 COROLLARY. *$\widetilde{K}$ is algebraically closed. If K contains a dense algebraically closed subfield, then K is algebraically closed.*

PROOF. As $\widetilde{K}$ is a complete nontrivially non-Archimedean valued field with a dense algebraically closed subfield, the first part of the corollary follows from the second.

Suppose K has a dense algebraically closed subfield L. Let $f = X^n + \alpha_1 X^{n-1} + \ldots + \alpha_{n-1}X + \alpha_n$ be a polynomial over K. We prove that it has a root in K. Set

$$c := \max_j \sqrt[j]{|\alpha_j|}\ .$$

For every $i \in \mathbb{N}$, choose $\alpha_{1i}, \ldots, \alpha_{ni} \in L$ such that $|\alpha_j - \alpha_{ji}| \leq 1/i$ and $|\alpha_{ji}| = |\alpha_j|$ $(j = 1,\ldots,n)$. Then $X^n + \alpha_{1i}X^{n-1} + \ldots + \alpha_{ni}$ has a root λ_i in L, hence in K. For each i, $|\lambda_i|^n = |\Sigma_{j=1}^n \alpha_{ji}\lambda_i^{n-j}| \leq \max_j |\alpha_{ji}||\lambda_i|^{n-j} = \max_j |\alpha_j||\lambda_i|^{n-j}$, and therefore $|\lambda_i| \leq c$. Thus,

$$f(\lambda_i) = |\Sigma(\alpha_j - \alpha_{ji})\lambda_i^{n-j}| \leq \max_j |\alpha_j - \alpha_{ji}||\lambda_i|^{n-j} \leq$$
$$\leq (1/i) \max_j c^{n-j}.$$ Now apply Corollary 3.24(i).

We close the section with some miscellaneous facts about $\widetilde{K}$.

3.26 THEOREM. *If K is separable (as a metric space) then $\widetilde{K}$ is a Banach space of countable type over K.*

PROOF. K contains a countable dense subfield L. As L[X] is countable and every polynomial has only finitely many roots in $\widetilde{K}$, the algebraic closure L_{alg} of L is a countable subfield of $\widetilde{K}$. By Corollary 3.25, the closure M of L_{alg} is algebraically closed. As $M \supset K$, it follows that $M = \widetilde{K}$. Hence, in the topological sense, $\widetilde{K}$ is separable. Then K is of countable type.

<u>Note</u>: Assume, in addition to the above, that the characteristic of K = 0. We view $\mathbb{Q}$ as a subfield of K. Let c be the cardinal number of the continuum. Let S be a maximal subset of $\widetilde{K}$ whose elements are algebraically independent over $\mathbb{Q}$. Then $\widetilde{K}$ is the algebraic closure of $\mathbb{Q}(S)$.
Therefore, $\#\mathbb{Q}(S) = \#\widetilde{K}$. It is not difficult to see that $\#\widetilde{K} = c$ ($\#K \leq c$ because $\widetilde{K}$ is a separable metric space; $\#\widetilde{K} \geq c$ because, with π as on p. 20, the map

$$A \mapsto \sum_{n\in A} \pi^n$$

is an injection of $\{A : A \subset \mathbb{N}\}$ into $\widetilde{K}$). Thus, $\#\mathbb{Q}(S) = c$, whence $\#S = c$. Now by a similar argument, the field C of complex numbers in the algebraic closure of $\mathbb{Q}(T)$ where T is a subset of $\mathbb{C}$ of cardinality c whose elements are algebraically independent over $\mathbb{Q}$. Then $\#S = \#T$, so that the fields Q(S) and $\mathbb{Q}(T)$ are isomorphic. Then so are the fields $\widetilde{K}$ and $\mathbb{C}$. We have proved the following: *If K is separable as a metric space and has characteristic 0, then $\widetilde{K}$ and $\mathbb{C}$ are isomorphic fields.*
In particular, for every prime number p, Ω_p is algebraically isomorphic to $\mathbb{C}$. We see that there are many valuations on $\mathbb{C}$ that turn $\mathbb{C}$ into a complete valued field.

3.27 THEOREM. *If* R *is a closed linear subspace of* $\widetilde{K}$ *that is also a subring of* $\widetilde{K}$, *then* R *is a field.*

PROOF. Take $a \in R$, $a \neq 0$: we prove $a^{-1} \in R$. We may assume $|a| < 1$. Take $\varepsilon > 0$, $\varepsilon < 1$. Choose a $b \in \widetilde{K}$ such that b is algebraic over K and $|a - b| < \varepsilon|a|$. Then $|a - b| \leq |a|$, so $|b| = |a| < 1$. Let $\beta_0 + \beta_1 X + \ldots + \beta_{n-1}X^{n-1} + X^n$ be the minimum polynomial of b over K. For simplicity of notations we set $\beta_n := 1$. If $b_1, \ldots, b_n$ are the conjugates of b in $\widetilde{K}$, then $\Sigma_i \beta_i X^i = \Pi(X - b_j)$, while, by Theorem 3.23, for every j we have $|b_j| = |b| = |a|$. Writing the β_i as symmetric function of $b_1, \ldots, b_n$ one sees that $|\beta_i| \leq |a|^{n-i}$ for each i and that $|\beta_0| = |a|^n$. (This follows also from Theorem 3.23 and Corollary 3.22). Now

$$\begin{aligned} \left| \sum_i \beta_i a^i \right| &= \left| \sum_i \beta_i (a^i - b^i) \right| \\ &\leq \max_i |\beta_i||a - b|\,|a^{i-1} + a^{i-2}b + \ldots + b^{i-1}| \\ &\leq \max_i |a|^{n-i}|a - b||a|^{i-1} \\ &= |a|^{n-1}|a - b| \leq \varepsilon|\beta_0|. \end{aligned}$$

Hence,

$$(*) \quad \left|1 + \sum_{i=1}^n \beta_0^{-1} \beta_i a^i\right| \leq \varepsilon$$

Now $\Sigma_{i=1}^n \beta_0^{-1}\beta_i a^i \in R$ and ε was arbitrary between 0 and 1. It follows that $1 \in R$. But $(*)$ also yields

$$\left|a^{-1} + \sum_{i=1}^n \beta_0^{-1}\beta_i a^{i-1}\right| \leq \frac{\varepsilon}{|a|}$$

and now we know thet $\Sigma_{i=1}^n \beta_0^{-1}\beta_i a^{i-1} \in R$. Hence, $a^{-1} \in R$.

If L is a valued field that contains K (both valuations coinciding on K), then we have a natural embedding of k into the residue class field l of L.

The canonical map $\{\alpha \in K : |\alpha| \leq 1\} \to k$ is the restriction of the map $\{\alpha \in L : |\alpha| \leq 1\} \to l$.

3.28 THEOREM. *Let* L *be an algebraic extension of* K. *Then the residue class field* l *of* L *is an algebraic extension of* k. *The residue class field of* $\widetilde{K}$ *is the algebraic closure of* k. (See also Theorem 4.50.)

PROOF. As we already have Theorem 1.5, we only have to prove l to be algebraic over k. For $\lambda \in L$, $|\lambda| \leq 1$ let $\overline{\lambda}$ be the corresponding element of l. Take $\alpha \in L$, $|\alpha| = 1$: we show $\overline{\alpha}$ to be algebraic over k. Let $P(X) = X^m + \lambda_1 X^{m-1} + \ldots + \lambda_m$ be the minimum polynomial of α over K. In L there exist $\alpha_1, \ldots, \alpha_m$ such that $P(X) = \Pi(X - \alpha_i)$. By Theorem 3.23, $|\alpha_i| = |\alpha| = 1$. It follows that $|\lambda_j| \leq 1$ for each j. Then $\overline{\alpha}$ is a root of $X^m + \overline{\lambda_1}X^{m-1} + \ldots + \overline{\lambda_m} \in k[X]$.

Notes

Except for the section on extension of a valuation, most of the material presented in this chapter can be found in Monna's papers [139, 141, 142, 149]. Other sources are Refs. 27, 29, 30, 65, 87, 102, 154, 168, and 169 and especially Gruson and Van der Put's paper [89] (which also covers much of Chapters 4 and 5 of this text). A general reference is Monna's book [147] which also contains an extensive bibliography. For categories of Banach spaces, see Gruson's papers [87, 88].

To a large extent one can build up the Archimedean and the non-Archimedean theories of normed vector spaces simultaneously. This is actually done by Bourbaki [27] who proves, among other results the "category theorems" (3.5, 3.11, and 3.12) and also the theorem (3.15) on finite-dimensional normed spaces for all complete nontrivially valued scalar fields at the same time. (It is interesting to note that the nontrivialness of the valuation is crucial for the category theorems and not only for their traditionally given proofs. Let us for just one minute go against our conventions and consider a field K with the trivial valuation $|\ |$. For n = 1,2 let E_n be the vector space $K^{\mathbb{N}}$ under the norm $a \mapsto \max \{n^{-i}|a_i| : i \in \mathbb{N}\}$. Then E_1 and E_2 are Banach spaces. The identity map $E_1 \to E_2$ is a continuous linear bijection, but is not a homeomorphism. See Ref. 59a.)

The idea of a normed vector space over K can be varied in many ways. To start with the less conventional ones, K. Mathiak [138] considers (orthogonality in) normed spaces over a valued scalar field that may not be commutative, while K. Iseki [103] and T. Konda [119] study a K-vector space E with a set of functions $\nu : E \to [0, \infty)$ for each of which there exists a positive number r such that

$$\nu(\alpha x) = |\alpha|^r \nu(x) \qquad (\alpha \in K,\ x \in E).$$

Bornological K-vector spaces are investigated in Refs. 3 and 167. (A bornological K-vector space is essentially a K-vector space E with a covering $\mathcal{U}$ of E by sets that are "not too large". For instance, in a normed space E for $\mathcal{U}$ one can take the collection of all bounded sets or the collection of all precompact sets or the collection of all small sets in the sense of p. 31).

More "classical" generalizations are topological vector spaces and locally convex spaces.

A topological vector space over K is, of course, a K-vector space E endowed with a topology such that addition and scalar multiplication are continuous maps $E \times E \to E$ and $K \times E \to E$, respectively. See Monna [145], Bourbaki [27], and Ellis [52]. In Refs. 176, 178, T. T. Raghunathan works at the

topological vector space of all entire functions $K \to K$.

Before defining "locally convex space", we introduce convexity.

A subset X of a K-vector space E is said to be *convex* if $\alpha x + \beta y + \gamma z \in X$ as soon as $x,y,z \in X$ and $\alpha,\beta,\gamma \in B_1(0) \subset K$, $\alpha + \beta + \gamma = 1$. (In Chapter 4 we shall deal with "absolutely convex" sets. A subset in E is absolutely convex if and only if it is convex and contains 0.)

The intersection of any system of convex sets is convex: in a normed space the closure of a convex set is convex. A study of convex sets in a Banach space is made in Ref. 199a.

A *locally convex space* is defined as a topological vector space whose topology has a base consisting of convex sets. These base sets are automatically clopen, so that a locally convex space is zerodimensional. (A topological vector space may not be!)

One can also start from the notion of a seminorm. A *seminorm* on a K-vector space E is a function $p : E \to [0, \infty)$ satisfying the requirements

$$p(x + y) \leq \max \{p(x),p(y)\} \qquad (x,y \in E),$$

$$p(\alpha x) = |\alpha| p(x) \qquad (\alpha \in K,\ x \in E).$$

Then $p^{-1}(0)$ is a vector space and p induces a norm on $E/p^{-1}(0)$.

If $\mathcal{P}$ is a collection of seminorms on a vector spece E, then the weakest topology that makes all elements of $\mathcal{P}$ continuous renders E a locally convex space. Conversely, every locally convex space can be obtained in this way from a set of seminorms. (The connection between convex sets and seminorms, however, is just a little more complicated than in the Archimedean theory.)

The original definition of local convexity is due to A. F. Monna [143, 144]. Further study was made by J. van Tiel [232, 234], N. de Grande de Kimpe [77 - 80], K. Madlener [135].

Under a very reasonable condition on K (viz., spherical completeness: see Chapter 4) the theory turns out to be very similar to the Archimedean one. A major difference is pointed out by M. van der Put and J. van Tiel in Ref. 175: if K is spherically complete, then every locally convex space is nuclear. Examples are the weak topology (p. 158) and the strict topology (p. 274).

In the definition of "norm" the strong triangle inequality [(ii) on p. 45] is a natural but no means logically unavoidable requirement. Let us say that a function $\|\ \|$ defined on a vector space E and with values in $[0, \infty)$ is a *A-norm* if it has properties (i) and (ii) of a norm while in addition to that one has

$$\|x + y\| \leq \|x\| + \|y\| \qquad (x,y \in E).$$

Trivially, every norm is an A-norm. The following statements are easily verified.

Let $\|\ \|$ be an A-norm on a vector space E. The formula $\rho(x,y) := \|x - y\|$ defines a metric on E, rendering E a topological vector space. The continuous linear functions $E \to K$ form a vector space E^∇. On this E^∇ we can introduce a norm (not merely an A-norm!) by

$$\|f\| := \sup_{x\neq 0} \frac{|f(x)|}{\|x\|} \qquad (f \in E^\nabla).$$

Among all seminorms on E that are $\leq \|\ \|$, there is a largest one, ν, say. Set $N := \{x \in E : \nu(x) = 0\}$. Then N is a linear subspace of E, ν induces in a natural way a norm on E/N. We have $(E/N)' \sim E^\nabla$.

A non-trivial example. Let E be the vector space of all K-valued sequences $(\alpha_1,\alpha_2,\ldots)$ for which $\Sigma|\alpha_i| < \infty$. For $a = (\alpha_1,\alpha_2,\ldots) \in E$, set $\|a\| := \Sigma\ |\alpha_i|$. Then $\|\ \|$ is an A-norm but not a norm. We have $E \subset c_0$. In the above terminology, ν is the restriction of the sup-norm.

Slightly more complicated is the following. Let L be the set of all Borel measurable functions $f : [0, 1] \to K$ for which the Lebesgue integral $_0\int^1 |f(x)|\ dx$ is finite. L is a vector space. $N := \{f \in L : f = 0 \text{ a.e.}\}$ is a linear subspace. On $E := L/N$ we can introduce an A-norm by

$$\|f\| \bmod N\| := {}_0\int^1 |f(x)|\ dx \qquad (f \in L).$$

The topology induced by this A-norm is actually connected! (Every $f \in E$ can be joined to 0 by the arc $s \mapsto f\xi_{[0,s]}$.) It follows that in this case $\nu = 0$.

Let Γ be the value group of K. For a normed space E set $\|E^*\| := \{\|x\| : x \in E, x \neq 0\}$. This $\|E^*\|$ is always a union of cosets of Γ in the multiplicative group $(0, \infty)$: it is equal to Γ if and only if every nonzero element of E has a scalar multiple of norm 1. We have seen (p. 46) that, in contrast to the Archimedean case, $\|E^*\|$ may be different from Γ. Although the property $\|E^*\| = \Gamma$ does not seem to be particularly nice, it is intriguing to note that the following is unknown: PROBLEM [Serre, [217]; *see also* Ref.89,3.22(iii) . *If* E *is a normed space, can one always introduce a norm* ν *on* E *that is equivalent to the given norm and such that* $\{\nu(x) : x \in E, x \neq 0\} = \Gamma$?

There are two partial solutions.

3.29 THEOREM. *Let* E *be a normed space. In each of the following cases* (i) *and* (ii) *there exists a norm* ν *on* E, *equivalent to the given norm and with* $\{\nu(x) : x \in E, x \neq 0\} = \Gamma$.

i. *The valuation of* K *is discrete.*

ii. $\|E^*\|$ *is a union of countably many cosets of* Γ. (This is easily seen to be the case if E is of countable type.)

(Case (i) (see Ref.217) is easy to prove: Define

$\nu(x) := \inf \{x \in \Gamma : s \geq ||x||\} \qquad (x \in E)$

(This is essentially Lemma 4.13.) For case (ii), see Ref.99.

The definition of angle and orthogonality make perfectly good sense in the Archimedean theory (R. C. James, [105]) but they seem to be useful only for Hilbert spaces. This is probably due to their lack of symmetry. In fact, the following theorem was proved by G. Birkhoff in Ref.21. *Let* E *be a Banach space over* $\mathbb{R}$ *such that* $\dim_{\mathbb{R}} E \neq 2$ *and such that* $\perp$ *is a symmetric relation in* $E\backslash\{0\}$. *If, in addition for all nonzero* $a,b \in E$ *there exists at most one* $\lambda \in \mathbb{R}$ *such that* $||a - \lambda b|| = \mathrm{dist}(a,[b])$, *then* E *is a Hilbert space.* He also shows that the condition $\dim_{\mathbb{R}} E \neq 2$ is necessary.

The importance of the symmetry of $\perp$ for us will become abundantly clear in Chapter 5.

Theorem 3.4 is in the same vein as the following result obtained by N. Shilkret [221].

3.30 THEOREM (Shilkret). *Let* E,F *be normed spaces: assume that there exist* $a,b \in E\backslash\{0\}$ *with* $a \perp b$. *For a linear* $T : E \to F$ *the following conditions* (α) *and* (β) *are equivalent.*

α. *If* $x,y \in E$ *and* $x \perp y$, *then* $Tx \perp Ty$.

β. *If* $x,y \in E$ *and* $||x|| \leq ||y||$, *then* $||Tx|| \leq ||Ty||$.

If the valuation of K *is dense,* (α) *and* (β) *are equivalent to* T *being a similarity. In general, from* (α) *it follows that either* $T = 0$ *or* T *is a homeomorphism.*

Let us say that a Banach space E has the *complementation property* if every closed linear subspace of E is complemented. J. Lindenstrauss and L. Tzafriri have proved [127] that every real Banach space with the complementation property is linearly homeomorphic to a Hilbert space.

PROBLEM. *What Banach spaces* (over K) *have the complementation property ?*

If the valuation of K is discrete, every Banach space does: see Corollary 4.14. For densely valued K every Banach space of countable type has the property: see Theorem 3.15(ii) and Theorem 3.16(v), but l^∞ does not (5.19). For a dense valuation a Banach space with the complementation property cannot be arbitrarily large.

3.31 THEOREM [199]. *Let the valuation of* K *be dense. Let* E *be a Banach space with the complementation property. Then there exists a subset* X *of* E *such that* $[X] = E$ *and for which* $\#X \leq \#K$. *If* I *is a set with* $\#I = \#K$, *then* E *is (as a Banach space) isomorphic to a quotient space of* $c_0(I)$.

A related question is, whether Corollary 3.18 may be inverted: PROBLEM. *Let the valuation of* K *be dense: let* E *be a Banach space in which every closed linear subspace of countable type is complemented. Does* E *necessarily have a base?* (From the assumptions one can prove the existence of a real number s such that for every closed subspace D of E that is of countable type there exists a projection of E onto D whose norm is at most s.)

Conspicuously absent in this chapter are the Hahn-Banach Theorem and the subject of reflexivity. These will concern us in Chapter 4, But we already have enough material to see that not all is well. Let us assume that the following weak form of the Hahn-Banach theorem is valid:

(*) For every 2-dimensional normed space E, every linear subspace D of E and every $f \in D'$, there exists a $g \in E'$ for which $f = g|_D$ and $||g|| \leq ||f||$.

Assume that K is not spherically complete. Then in particular we let E be the Banach space, K_ν^2 previously constructed (p. 68). Set $e_1 := (1,0) \in E$, $D := [e_1]$ and define $f \in D'$ by

$$f(\lambda e_1) := \lambda \qquad (\lambda \in K).$$

Then $||f|| = 1$. By (*), f can be extended to a $g \in E'$ with $||g|| = 1$. Choose a nonzero e_2 with $g(e_2) = 0$. Then

$$\begin{aligned} \mathrm{dist}(e_1,[e_2]) &= \inf \{||e_1 - e_2||: \lambda \in K\} \\ &\geq \inf \{|g(e_1 - e_2)|: \lambda \in K\} = |g(e_1)| = 1 \\ &= ||e_1|| \end{aligned}$$

so that $e_1 \perp e_2$. But this contradicts the knowledge we have about K_ν^2.

It follows that (*) implies that K has to be spherically complete. In particular, the full Hahn-Banach theorem will hold only if K is spherically complete. In Chapter 4 we shall see that spherical completeness of K is also sufficient for the Hahn-Banach theorem. (See also Theorem 4.54.)

Like their Archimedean counterparts, the finite-dimensional normed spaces over K have been sadly neglected. As topological vector spaces they may not be very interesting (3.15) but geometrically they are. We mention a few results that illustrate both their good and their bad behavior (see Ref.199, part I and 236a for details).

3.32 THEOREM. *Let* $n \in \mathbb{N}$: *let* E *be an n-dimensional Banach space. Define* $V : E^n \to [0, \infty)$ *by*

$$V(x_1,\ldots,x_n) := \prod_{k=1}^{n} \mathrm{dist}(x_k, \sum_{i>k} [x_i]) \qquad (x_1,\ldots,x_n \in E).$$

Then V *is a symmetric function.* $V(x_1,\ldots,x_n) = 0$ *if and only if the* x_i *are linearly dependent. If* $T : E \to E$ *is linear and if* $x_1,\ldots,x_n \in E$, *then*

$$V(Tx_1,\ldots,Tx_n) = |\det T| V(x_1,\ldots,x_n).$$

For every linear $T : E \to E$ *one has* $|\det T| \leq ||T||^n$, *while* $|\det T| = ||T||^n$ *if and only if* T *is a similarity.*

3.33 THEOREM. *Let* E *be a two-dimensional Banach space. Then there exists a bijective similarity of* E *onto* E'. *More precisely, if* e_1, e_2 *is a base for* E, *if* f_1, f_2 *is the dual base in* E' *and if* T *is the linear bijection* $E \to E'$ *defined by*

$$T(\lambda_1 e_1 + \lambda_2 e_2) := \lambda_2 f_1 - \lambda_1 f_2 \qquad (\lambda_1, \lambda_2 \in K),$$

then

$$||Tx|| = V(e_1,e_2)^{-1} ||x|| \qquad (x \in E).$$

3.34 . THEOREM. *Let* p *be a prime number and* $K = \mathbb{C}_p$. *Then there exists a three-dimensional Banach space* E *with the following properties.*

i. *Every two-dimensional subspace is isomorphic to* K^2.
ii. E *is not isomorphic to* K^3. E' *contains no nonzero mutually orthogonal elements.* (Hence, there is no similarity of E onto E'.)

By Theorem 3.16, all infinite-dimensional Banach spaces of countable type are linearly homeomorphic. This situation may remind the reader of M. Kadec's theorem, according to which all infinite-dimensional Banach spaces of countable type over $\mathbb{R}$ are homeomorphic (see Refs.107 and 118). The resemblance is, however, superficial: in the Archimedean case there need exist no *linear* homeomorphism.

The topological classification of the Banach spaces over K is extremely simple.

3.35 THEOREM. *Let* E *be a Banach space.*

i. *If* E *is locally compact and* $E \neq \{0\}$, *then* E *is homeomorphic to* $\mathbb{N} \times D$, *where* D *is the Cantor set.*

ii. *Let* E *not be locally compact. Let* S *be a discrete topological space whose cardinality is just the weight of* E. *Then* E *is homeomorphic to* $S^{\mathbb{N}}$.

Here the weight of E is the smallest among the cardinal numbers of the bases for the topology of E. It is equal to the smallest of the cardinal numbers of the dense subsets of E. (See W. W. Comfort's survey paper [371].)

Attempts to introduce inner product spaces have met with little success.

Let E be a K-vector space. A quadratic form $f : E \to K$ is called *definite* if $f(x) \neq 0$ for all $x \neq 0$. Then $|f|$ is a norm on E. It was shown by T. Springer (Ref.224,Part I) that, if $K = \mathbb{Q}_p$, no definite quadratic form exists on a vector space of dimension greater than or equal to 4. Thus, along these lines a reasonable theory of inner product spaces does not seem feasible. [See, however, Ref. 16.b.]

A different approach was taken by G. Kalisch [108] who axiomatically defines an inner product that behaves like the bilinear form $(x,y) \to \Sigma x_i y_i$ that can be defined on c_0.

(Of course, the orthogonality relation is an ersatz for an inner product.)

A few words about the extension of the valuation.

We have already pointed out that purely algebraic proofs of Krull's theorem (3.19) can be found in Bourbaki [26] and Van der Waerden (ref.243,Part I). [See also Endler [58], Ribenboim [187], Zariski-Samuel (Ref.256,Part II).]

The proof presented above is due to Van der Put (private

communication). Lazard's theorem (3.20) appears in Ref.124. (It is one of the many analogues of the Weierstrass preparation theorem; see Ref.1.)

In Ex. 3.U we classify the locally compact non-Archimedean valued fields. In Theorem 21-22 of Ref.165, Pontryagin gives a description of all locally compact topological fields. (See Kaplansky's paper [112], and also Refs.26, 120, 246, and 253.) It turns out that for any locally compact field the topology can be obtained from a valuation. The connected, locally compact fields are $\mathbb{R}$ and $\mathbb{C}$; the other ones are the discrete fields and the fields mentioned in 3.U.

For a local field K it is natural to choose a valuation such that the value group is $\{c^n : n \in \mathbb{Z}\}$, c being the number of elements of the residue class field (see 1.B). For this valuation one has $m(A) = |x|m(A)$, where $A \subset K$ is any Borel set, x is any element of K and m is the $\mathbb{R}$-valued Haar measure of the locally compact additive group K.

The observation that $\mathbb{C}$ is complete under many non-Archimedean valuations was made in Ref.171. The following theorem is proved in Ref.215 (p.4 and Theorem IV). Let us call two non-Archimedean valuations, μ and ν, on a field L *equivalent* if there exists an $r \in (0, \infty)$ such that

$$\mu(\alpha) = \nu(\alpha)^r \qquad (\alpha \in L).$$

3.36 THEOREM (Schmidt).

i. *The conditions* (α) *and* (β) *are equivalent.*

α. *There exists a non-Archimedean valuation* ν *on* K *that is not equivalent to* $|\ |$ *and relative to which* K *is complete.*

β. K *is algebraically closed.*

ii. *If the given valuation* $|\ |$ *of* K *is discrete, then every discrete valuation of* K *is equivalent to* $|\ |$.

We conclude by mentioning one famous theorem from valuation theory, which is known as Hensel's lemma. Although the theorem itself is algebraic, we introduce it because it may well become a useful tool for analysts also.

3.3 THEOREM (Hensel-Rychlik). *Let* $B := \{\lambda \in K : |\lambda| \leq 1\}$. *Let* k *be the residue class field of* K. *Let* $\lambda \to \overline{\lambda}$ *denote the natural quotient map* $B \to k$. *For a polynomial* $P = \Sigma\, \lambda_i X^i$ *with coefficients in* B *define* $\overline{P} \in k[X]$ *by* $\overline{P} := \Sigma\, \overline{\lambda}_i X^i$.

Now let $P \in K[X]$ *be irreducible and assume that all coefficients of* P *lie in* B. *Then either* $\overline{P}$ *is constant or* $\overline{P}$ *and* P *have the same degree. In the second case there exists a* $\mu \in k$ *such that* $\mu\overline{P}$ *is a power of an irreducible polynomial over* k.

The first part of this theorem follows directly from Corollary 3.22. A simple proof of the second part is given by D. S. Rim in Ref.191.

Chapter 4

SPHERICALLY COMPLETE BANACH SPACES

In the Notes section to Chapter 3, we had a first glimpse of the connection between the non-Archimedean Hahn-Banach Theorem and spherical completeness of K. In this chapter we study this connection. The main theorem is the equivalence of conditions (α) and (γ) of Theorem 4.15. This equivalence is the simplest version of the famous Ingleton theorem.

BASIC PROPERTIES AND EXAMPLES

4.A The following statements are easy to verify. If E,F are spherically complete Banach spaces, then $E \oplus F$ is also spherically complete. More generally, if $\{E_i\}_{i\in I}$ is a family of Banach spaces, then $\times_i E_i$ is spherically complete if and only if each E_i is spherically complete. In particular, l^∞ and $l^\infty(X{:}s)$ are spherically complete if and only if K is spherically complete.

4.B c_0 is spherically complete if and only if the valuation of K is discrete. [If the valuation is discrete, then the spherical completeness of c_0 follows from Corollary 2.4. If the valuation is dense, take a sequence $\alpha_1,\alpha_2,\ldots \in K$ such that $2 \geq |\alpha_1| > |\alpha_2| > \ldots > 1$; for $n \in \mathbb{N}$, set

$$a_n := (\alpha_1,\alpha_2,\ldots,\alpha_n,0,0,\ldots)$$