Oblivious Lower Bound via Flattening

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1 Oblivious Lower Bound

An important theorem that we use throughout our analysis is Hoeffding's Inequality:

Theorem 1 (Hoeffding's Inequality)**.** *Let X_i for $i=1,2,...,k$ be independent bounded random variables with $X_i\in[a,b]$ for all i. Then,*

$$P\left(\Big|\frac{1}{k}\sum_{i=1}^{k}(X_i-\mathbb{E}[X_i])\Big|\geq t\right)\leq 2\exp\left(-\frac{2kt^2}{(b-a)^2}\right)$$

Let S be a finite population, let X_i for $i=1,2...,k$ be chosen uniformly at random from $S\setminus\{X_1,...,X_{i-1}\}$, and let Y_i for $i=1,2,...,k$ be chosen uniformly at random from S. Note that $\{X_1,\dots,X_k\}$ represents a sample of S chosen without replacement, whereas $\{Y_1,...,Y_k\}$ represents a sample with replacement. Note that as the Y_i are independent random variables Hoeffding's Inequality provides a bound on the probability of $\sum_{i=1}^k Y_i$ deviating from its mean by more than t.

The same bound can be given on the probability of $\sum_{i=1}^k X_i$ deviating significantly from its mean, because the probability of $\sum_{i=1}^k X_i$ deviating from it's expectation by more than t is at most the probability of $\sum_{i=1}^k Y_i$ deviating from it's mean by t. Formally we can write this as

Corollary 1. *Let S be a finite set with $\min(S)\geq a,\max(S)\leq b$, and let X_i for $i=1,2...,k$ be chosen uniformly at random from $S\setminus\{X_1,...,X_{i-1}\}$. Then*

$$P\left(\Big|\frac{1}{k}\sum_{i=1}^{k}(X_i-\mathbb{E}[X_i])\Big|\geq t\right)\leq 2\exp\left(-\frac{2kt^2}{(b-a)^2}\right)$$

Hoeffding proved Corollary 1 in his seminal work [**?**] (the result follows from his Theorem 4, combined with Hoeffding's Inequality for independent random variables). The intuition behind Corollary 1 is that samples drawn without replacement should be more tightly concentrated around the mean than samples drawn with replacement.

Another important, yet very trivial, corollary of Hoeffding's Inequality is the Chernoff Bound (i.e. Hoeffding's Inequality applied to binary random variables):

Corollary 2. *Let X_i for $i=1,2,\dots,k$ be independent identically distributed binary random variables (i.e. $X_i\in\{0,1\}$). Then*

$$P\left(\Big|\frac{1}{k}\sum_{i=1}^{k}(X_i-\mathbb{E}[X_i])\Big|\geq t\right)\leq 2\exp\big(-2kt^2\big)$$

We proceed with our analysis of oblivious lower bounds.

Call a cup configuration T***-flat*** if the fill of every cup is in the interval $[-T,T]$.

We call an emptier Δ***-greedy-like*** if, when there are two cups c_1,c_2 with fills satisfying $\text{fill}(c_1)>\text{fill}(c_2)+\Delta$ the emptier never empties from c_2 without emptying from c_1 on the same round. Intuitively, a Δ-greedy-like emptier has a $\pm\Delta$ range where it is allowed to "not be greedy". Note that a perfectly greedy emptier is 0-greedy-like. We call an emptier ***greedy-like*** if it is Δ-greedy-like for $\Delta\leq O(1)$.

In the randomized setting we are only able to prove lower bounds for backlog against greedy-like emptiers; whether or not our results can be extended to a more general class of emptiers is an interesting open question.

We now prove a crucial property of greedy-like emptiers: that they are ***flattenable***, i.e.:

Proposition 1. *Given a cup configuration that is T-flat, an oblivious filler can, in running time $2T$, achieve a $2(2+\Delta)$-flat configuration of cups against a Δ-greedy-like emptier.*

Proof. The filler sets $p=n/2$ and distributes fill equally amongst all cups at every round, in particular placing $1/2$ units of water in each cup. Let $\ell_t=\min_{c\in S_t}\text{fill}_{S_t}(c)$, $u_t=\max_{c\in S_t}\text{fill}_{S_t}(c)$. Let L_t be the set of cups c with $\text{fill}_{S_t}(c)\leq l_t+2+\Delta$, and let U_t be the set of cups c with $\text{fill}_{S_t}(c)\geq u_t-2-\Delta$.

There are two ways to think of U_t. First we can consider U_t as capturing cups in the union of intervals of length 1, Δ, and 1. Note the key property that if a cup with fill in $[u_t-\Delta-2,u_t-\Delta-1]$ is emptied from, then all cups with fills in $[u_t-1,u_t]$ must be emptied from, because the emptier is Δ-greedy-like. On the other hand, we can consider U_t as capturing cups with

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fill in the union of $[u_t-2,u_t]$ and $[u_t-\Delta-2,u_t-2]$. This is useful as the interval of width Δ serves as a "buffer". In particular, if there are more than $n/2$ cups outside of U_t then all cups in $[u_t-2,u_t]$ must be emptied from because the emptier is Δ-greedy-like. L_t is of course completely symmetric to U_t.

First we prove a key property of the sets U_t and L_t: once a cup is in U_t or L_t it is always in $U_{t'},L_{t'}$ for all $t'>t$. This follows immediately from the following claim:

Claim 1.

$$U_t\subseteq U_{t+1}, L_t\subseteq L_{t+1}.$$

Proof. Consider a cup $c\in U_t$.

If c is not emptied from, i.e. fill(c) has increased by $1/2$, then clearly $c\in U_{t+1}$, because backlog has increased by at most $1/2$, so the fill of c must still be within $2+\Delta$ of the backlog on round $t+1$.

On the other hand, if c is emptied from, i.e. fill(c) has decreased by $1/2$, we consider two cases.

- If $\text{fill}_{S_t}(c)\geq u_t-\Delta-1$, then, as $u_{t+1}\leq u_t+1/2$,

$$\text{fill}_{S_{t+1}}(c)\geq u_t-\Delta-1-1/2\geq u_{t+1}-\Delta-2.$$

- On the other hand, if $\text{fill}_{S_t}(c) < u_t-\Delta-1$, then every cup with fill in $[u_t-1,u_t]$ must have been emptied from. The fullest cup at round $t+1$ is the same as the fullest cup on round t, because the fills of all cups with fill in $[u_t-1,u_t]$ have decreased by $1/2$, and no cup with fill less than u_t-1 had fill increase by more than $1/2$. Hence $u_{t+1}=u_t-1/2$. Because both the fill of c and the backlog have decreased by the same amount, the distance between them is still at most $\Delta+2$, hence $c\in U_{t+1}$.

The argument for $L_t\subseteq L_{t+1}$ is essentially identical. □

Now that we have shown that L_t and U_t never lose cups, we will show that they eventually gain a substantial number of cups.

Claim 2. *As long as $|U_t|\leq n/2$ we have $u_{t+1}=u_t-1/2$. Identically, as long as $|L_t|\leq n/2$ we have $\ell_{t+1}=\ell_t+1/2$.*

Proof. If there are more than $n/2$ cups outside of U_t then there must be some cup with fill less than $u_t-\Delta-2$ that is emptied from. Because the emptier is Δ-greedy-like this means that the emptier must empty from every cup with fill at least u_t-2. Thus $u_{t+1}=u_t-1/2$: no cup with fill less than u_t-2 could have become the fullest cup, and the previous fullest cup has lost $1/2$ units of fill.

The proof is identical for L_t. □

By Claim 2 we see that both $|U_t|$ and $|L_t|$ must eventually exceed $n/2$ at some times $t_u,t_\ell\leq 2T$, by the assumption that the initial configuration is T-flat. Since by Claim 1 $|U_{t+1}|\geq|U_t|$ and $|L_{t+1}|\geq|L_t|$ we have that there is some round $t_0=\max(t_u,t_\ell)\leq 2T$ on which both $|U_{t_0}|$ and $|L_{t_0}|$ exceed $n/2$. Then $U_{t_0}\cap L_{t_0}\neq\varnothing$. Furthermore, the sets must intersect for all $t_0\leq t\leq 2T$. In order for the sets to intersect it must be that the intervals $[u_t-2-\Delta,u_t]$ and $[\ell_t,\ell_t+2+\Delta]$ intersect. Hence we have that

$$\ell_t+2+\Delta\geq u_t-2-\Delta.$$

Since $u_t\geq 0$ and $\ell_t\leq 0$ this implies that all cups have fill in $[-2(2+\Delta),2(2+\Delta)]$.

□

Given a Δ-greedy-like filler, let $R_\Delta=2(2+\Delta)$. By Proposition 1, if a filler is given a T-flat configuration of cups they can achieve a R_Δ-flat configuration of cups.

Now we are equipped to prove the following proposition:

Proposition 2. *There exists an oblivious filling strategy for the variable-processor cup game on n cups – where $n\geq\Omega(1)$ is sufficiently large – that, given a T-flat configuration of cups with $T\leq poly(n)$, can achieve backlog $\Omega(\log n)$ in running time $poly(n)$ against a Δ-greedy-like emptier where $\Delta\leq O(1)$ is a constant known to the filler, with constant probability.*

Proof. The filler starts by flattening all the cups, using the flattening procedure detailed in Proposition 1.

Let A, the ***anchor*** set, be a subset of the cups chosen uniformly at random from all subsets of size $n/2$ of the cups, and let B, the ***non-anchor*** set, consist of the rest of the cups ($|B|=n/2$). Let $h=8\Delta+8$, and let $h'=2$. Note that the average fill of A and B both must start as at least $-R_\Delta$ due to the flattening.

The filler's strategy is roughly as follows:

- **Step 1:** Make a constant fraction of the cups in A have fill at least h by playing single processor cup games on constant-size subsets of B and then swapping the cup within B that has high fill, with constant probability, into A. By a Chernoff bound this makes a constant fraction of A, say nc cups, have fill at least h with exponentially good probability. Between single-processor cup games the filler flattens B.

- **Step 2:** Reduce the number of processors to nc, and raise the fill of nc *known* cups to fill h'. The emptier must first empty from the cups with fill h before emptying from the cups that the filler is attempting to get fill h' in.

- **Step 3:** Recurse on the nc cups that are known to have fill at least h'.

By performing $\Omega(\log n)$ levels of recursion, increasing the fill by a constant amount at each level of recursion, the filler achieves backlog $\Omega(\log n)$. Say the probability of Step 1 succeeding is at least $1-e^{-nk}$. Then the probability that any of $(1/2)\log_{1/c} n$ levels of recursion fail is bounded above by (by the union bound)

$$e^{-nk}+e^{-nck}+e^{-nc^2k}+\cdots+e^{-nc^{\log_{1/c}\sqrt{n}}k}$$

which is bounded above by

$$O\left(\frac{\log n}{e^{k\sqrt{n}}}\right)$$

which, for sufficiently large n can clearly be made constant. Hence the probability that every level of recursion succeeds is at least constant. Hence, once we show that Step 1 succeeds with the desired probability and that Step 2 is possible, we have that the entire process successfully achieves backlog $\Omega(\log n)$ with constant probability.

We now describe how to achieve Step 1.

The filler performs a series of ***swapping-process***, which are procedures that the filler uses to get a new cup in A. A swapping-process is composed of a substructure, repeated many times, which we call a ***round-block***; a round-block is a set of rounds. A swapping-process will consists of $n\cdot c_\Delta$ round-blocks ($c_\Delta \le O(1)$ a function of Δ to be specified); at the beginning of each swapping-process the filler chooses a round-blocks j uniformly at random from $[n\cdot c_\Delta]$.

For each round-block $i\in[n\cdot c_\Delta]$, the filler selects a random subset $D_i\subset B$ of the non-anchor cups and plays a single processor cup game on D_i. In this single-processor cup game the filler essentially employs the classic adaptive strategy for achieving backlog $\Omega(\log|B|)$ on a set of $|B|$ cups, with slight modifications for the fact that it is oblivious. In particular, the filler will only achieve this fill with constant probability. While doing this, the filler always places 1 unit of fill in each cup in the anchor set. Note that the filler sets $p=n/2+1$.

At the end of each round-block the filler applies the flattening procedure to flatten the non-anchor set. Note that this will not affect the running-time beyond a multiplicative factor (of, say, 3).

On most round-blocks – all but the j-th – the filler does nothing with the cup that it achieves with constant probability in its single processor cup game. However, on the j-th round-block the filler swaps the "winner" of the single processor cup game into the anchor set (with constant probability there is a winner).

Claim 3. *With probability at least $1-e^{-O(n)}$, the filler achieves fill at least h in at least $nc=O(n)$ of the cups in A.*

Proof. Consider a particular swapping-process. Let j, the round-block on which the filler will perform the swap, be chosen uniformly randomly from $[n\cdot c_\Delta]$ (c_Δ to be determined).

Say the emptier ***neglects*** the anchor set during a round-block if on at least one round of the round-block the emptier does not empty from every cup in the anchor set. By playing the single-processor cup game for many round-blocks with only one round-block when the filler actually swaps a cup into the anchor set, the filler prevents the emptier from neglecting the anchor set too often.

The fill of any cup in the anchor set can clearly never exceed $R_\Delta+\Delta$ because B is R_Δ-flat at the start of each round-block (a cup with fill this high would necessarily be emptied from). Let $\mu_\Delta=2R_\Delta+\Delta$; the emptier can neglect the anchor set no more than $(n/2)\mu_\Delta$ times. Furthermore, the average fill of B is thus always at least $-\mu_\Delta$. As B is R_Δ-flat this also means that the fills of cups in B at the start of each round-block are at least $-\mu_\Delta-R_\Delta$.

On each round-block the filler chooses a random subset $D_i\subset B$ of $\lceil e^{2h}\rceil$ cups. If the emptier does not neglect the anchor set on round-block i then the filler plays a legitimate single-processor cup game on n cups. The filler maintains an ***active-set*** of cups, which is a subset of D_i initialized to D_i. On each round of the round-block the filler distributes 1 unit of fill equally among all cups in the active set. Then the emptier removes fill from some cup in B. The filler chooses a random cup to remove from the active set. The probability that the cup the emptier emptied from is not in the active set after a random cup is removed from the active set by the filler is at least constant. By the end of the round-block the active-set will consist of a single cup. With constant probability, in particular probability at least

$$q_0=1/\lceil e^{2h}\rceil!,$$

this cup has gained fill at least $\ln\lceil e^{2h}\rceil\ge 2h$. Recalling that the cups fill started as at least $-\mu_\Delta-R_\Delta$, we have that this cup now has fill at least $2h-\mu_\Delta-R_\Delta$; by design in choosing h this quantity is at least h.

Now we shall choose c_Δ, choosing it large enough such that with constant probability there is some round-block on which the emptier doesn't neglect the anchor set on which the filler succeeds.

We choose

$$c_\Delta=2\frac{1}{q_0}\mu_\Delta.$$

By having $n\cdot c_\Delta$ round-blocks, we make it so that there should be at least $n\mu_\Delta$ round-blocks on which the filler correctly guesses the emptier's emptying sequence. Formally this is due to a Chernoff bound: the expectation of the number of rounds when the filler correctly guesses the emptier's emptying sequence is at least $2n\mu_\Delta$, and the probability that it deviates from its expectation by more than $n\mu_\Delta$ is hence exponentially small in n. As shown before, the emptier cannot neglect the anchor set more than $(n/2)\mu_\Delta$ times. The filler correctly guesses the emptiers emptying sequence on the j-th round-block. Conditioned on this event, the j is chosen uniformly randomly from all the round-blocks on which the filler correctly guesses the emptiers emptying sequence. Since the emptier can neglect the anchor set on at most half of these round-blocks there is at least a $1/2$ chance that j is chosen on a round-block where the filler does not neglect the anchor set. Thus, overall, there is at least a constant probability of

Say that a swapping-process ***succeeds*** if the filler is able to swap a cup with fill at least h into A. We have shown that there is a constant probability of a given swapping-process succeeding. Let X_i be the binary random variable indicating whether or not the i-th swapping process succeeds. Let $q \geq \Omega(1)$ be the probability of a swapping-process succeeding, i.e. $P(X_i = 1)$. Note that the random variables X_i are clearly independent, and identically distributed.

Clearly

$$\mathbb{E}\left[\sum_{i=1}^{n/4} X_i\right] = qn/4.$$

By a Chernoff Bound (i.e. Hoeffding's Inequality applied to binary random variables),

$$P\left(\sum_{i=1}^{n/4} X_i \leq nq/8\right) \leq e^{-nq^2/128}.$$

That is, the probability that less than $nq/8$ of the anchor cups have fill at least h is exponentially small in n, as desired.

□

Hence Step 1 is possible.

Step 2 is easily achieved by setting $p = nc$ and uniformly distributing the fillers fill among a chosen set of nc cups. The greedy nature of the emptier will force it to focus on the cups which must exist in A with large positive fill until the new cups have sufficiently high fill. In particular, the fills of the cups in nc must be at least $-\mu_\Delta - \Delta \geq -h$. After removing from the very full cups for $h+2$ rounds the fills of these new cups are clearly at least 2.

□

Lemma 1 (The Oblivious Amplification Lemma). *Let f be an oblivious filling strategy that achieves backlog $f(n)$ in the variable-processor cup game on n cups with constant probability (relative to average fill, with negative fill allowed). Let $\delta \in (0,1)$ be a parameter. Then, there exists an adaptive filling strategy that, with constant probability, either achieves backlog*

$$f'(n) \geq (1-\delta)\Big(f((1-\delta)n) + f((1-\delta)\delta n)\Big)$$

or achieves backlog $\Omega(poly(n))$ in the variable processor cup game on n cups.

Proof. □

Corollary 3. *There is an oblivious filling strategy for the variable-processor cup game on n cups that achieves backlog at least $2^{\Omega(\sqrt{\log n})}$ in running time $O(n)$ with constant probability.*

Proof. Given the Oblivious Amplification Lemma we could try to apply the same strategy as outlined in the proof of Corollary **??** to achieve backlog $\Omega(n^{1-\epsilon})$ for constant $\epsilon > 0$ in time $2^{O(\log^2 n)}$. Because of our definition of an overpowered cup as a cup with fill at least $\tilde{\Omega}(\sqrt{n})$, we can't get quite as close to linear as an adaptive filer could. However, the more pressing problem is that of running time: randomized algorithms are traditionally supposed to have polynomial-running time. By artificially reducing n, i.e. ignoring some portion of the cups, we can get an algorithm that achieves high backlog, but in polynomial time. In particular, we want to choose a subset of n' of the cups to focus on, where $2^{O(\log^2 n')} = O(n)$. An appropriate choice is $n' = 2^{\sqrt{\log n}}$.

With n' chosen, we apply the exact same strategy as given in the paragraph in the proof of Corollary on the $2^{O(\log^2 n)}$-time construction for achieving backlog $\Omega(n^{1-\epsilon})$ for constant $\epsilon > 0$, but using repeated application of the Oblivious Amplification Lemma rather than the Adaptive Amplification Lemma, which yields the disclaimer that the backlog is only achieved with constant probability. Thus, we achieve backlog $\Omega(n')$ in running time $2^{O(\log^2 n')}$. By design, expressing this in terms of n the filler achieves backlog $2^{\Omega(\sqrt{\log n})}$ in running time $O(n)$.

For completeness we – briefly (as they are nearly identical) – present the ideas from Proposition **??** in the randomized context.

Fix constant $\epsilon > 0$ and choose appropriate constant δ, as mandated by Claim **??**. Choose constant c, according to an inequality to be specified later. We aim to achieve backlog $c(n')^{1-\epsilon}$. As before, we define a sequence of functions. First,

$$f_0(k) = \begin{cases} \lg k, & k \geq 1, \\ 0 & \text{else.} \end{cases}$$

Then, we define f_i as the amplification of f_{i-1} for $i \geq 1$, where amplification is as defined in the Oblivious Amplification Lemma. However, this time the filling strategy f_i achieves backlog $f_i(k)$ on k cups only with constant probability. As before we define a sequence g_i as

$$g_i = \begin{cases} \lceil 1/\delta \rceil \gg 1, & i = 0, \\ \lceil g_{i-1}/(1-\delta) \rceil - 1 & i \geq 1. \end{cases}$$

Claim **??**, which states that

$$f_i(k) \geq ck^{1-\epsilon} \text{ for all } k < g_i,$$

holds with no modifications required.

Again, because g_i is increasing, we achieve the desired backlog $c(n')^{1-\epsilon}$ in finite time. In particular, applying identical arguments to those in Corollary **??**, we find that the running time is $2^{O(\log^2 n')}$.

As stated earlier, by design of n', this means we get backlog $2^{\Omega(\sqrt{\log n})}$ in time $O(n)$.

$\square$

Claim 4. *Without loss of generality the cup state is always* $(h\sqrt{n/\text{lglg}n})$*-flat.*

Proof. In order to flatten a set of cups we must have a bound on the magnitude of the fills of the cups. We claim that without loss of generality no cup has fill larger in magnitude than $h\sqrt{n/\text{loglog}n}$. If a cup has more than $h\sqrt{n/\text{loglog}n}$ fill we call the cup ***overpowered***. If there ever is an overpowered cup then we are automatically done: the emptier has achieved poly(n) backlog and will maintain it for at least poly(n) rounds. If a cup ever has fill less than $-h\sqrt{n/\text{loglog}n}$ then the absolute average fill must be large enough such that the absolute fill of this cup is at least 0. Thus there is an overpowered cup. From now on we assume that there are no overpowered cups. $\square$