1 Oblivious Filler Lower Bound

In this section we prove that, surprisingly, an oblivious filler can achieve backlog $n^{1-\varepsilon}$, although only against a certain class of "greedy-like" emptiers.

The ***fill-range*** of a set of cups at a state S is $\max_c \text{fill}_S(c) - \min_c \text{fill}_S(c)$. We call a cup configuration ***R-flat*** if the fill-range of the cups less than or equal to R; note that in an R-flat cup configuration with average fill 0 all cups have fills in $[-R, R]$. We say an emptier is ***Δ-greedy-like*** if, whenever there are two cups with fills that differ by at least Δ, the emptier never empties from the less full cup without also emptying from the more full cup. That is, if on some round t, there are cups c_1, c_2 with $\text{fill}_{I_t}(c_1) > \text{fill}_{I_t}(c_2) + \Delta$, then a Δ-greedy-like emptier doesn't empty from c_2 on round t unless it also empties from c_1 on round t. Note that a perfectly greedy emptier is 0-greedy-like. We call an emptier ***greedy-like*** if it is Δ-greedy-like for $\Delta \le O(1)$.

With an oblivious filler we are only able to prove lower bounds on backlog against greedy-like emptiers; whether or not our results can be extended to a more general class of emptiers is an interesting open question. Nonetheless, greedy-like emptiers are of great interest because all the known randomized algorithms for the cup game are greedy-like [1, 4].

As a tool in our analysis we define a new variant of the cup game: In the p-processor ***E-extra-emptyings S-skip-emptyings*** negative-fill cup game on n cups, the filler distributes p units of water amongst the cups, and then the emptier empties from p *or more, or less* cups. In particular the emptier is allowed to do E extra emptyings—we say that the emptier does an extra emptying if it empties from a cup beyond the p cups it typically is allowed to empty from—and is also allowed to skip S emptyings—we say that the emptier skips an emptying if it doesn't do an emptying—over the course of the game. Note that the emptier still cannot empty from the same cup twice on a single round. Also note that the emptier is allowed to skip extra emptyings in addition to regular emptyings. Also note that a Δ-greedy-like emptier must take into account extra emptyings and skip emptyings to determine valid moves.

For a Δ-greedy-like emptier let $R_\Delta = 2(2+\Delta)$; we now prove a key property of these emptiers: an oblivious filler can attain an R_Δ-flat cup configuration against a Δ-greedy-like emptier, given cups of a known starting fill-range.

Lemma 1. *Consider an R-flat cup configuration in the p-processor E-extra-emptyings S-skip-emptyings negative-fill cup game on $n = 2p$ cups. There is an oblivious filling strategy* ***flatalg*** *that achieves an R_Δ-flat configuration of cups against a Δ-greedy-like emptier in running time $2(R + \lceil (1+1/n)(E+S) \rceil)$. Furthermore, flatalg guarantees that the cup configuration is R-flat on every round.*

Proof. If $R \le R_\Delta$ the algorithm does nothing, since the desired fill-range is already achieved; for the rest of the proof we consider $R > R_\Delta$.

The filler's strategy is to distribute fill equally amongst all cups at every round, placing $p/n = 1/2$ fill in each cup. Let $\ell_t = \min_{c \in S_t} \text{fill}_{S_t}(c)$, $u_t = \max_{c \in S_t} \text{fill}_{S_t}(c)$.

First we show that the fill-range of the cups can only increase if the fill-range is very small.

Claim 1. *If $u_{t+1} - \ell_{t+1} > u_t - \ell_t$ then*

$$u_{t+1} - \ell_{t+1} \le R.$$

Proof. First we remark that the fill of any cup changes by at most $1/2$ from round to round, and in particular $|u_{t+1} - u_t| \le 1/2$, $|\ell_{t+1} - \ell_t| \le 1/2$. In order for the fill-range to increase the emptier must have emptied from some cup with fill in $[\ell_t, \ell_t + 1]$ without emptying from some cup with fill in $[u_t - 1, u_t]$; if the emptier had not emptied from every cup with fill in $[\ell_t, \ell_t + 1]$ then we would have $\ell_{t+1} = \ell_t + 1/2$ so the fill-range cannot have increased, and similarly if the emptier had emptied from every cup with fill in $[u_t - 1, u_t]$ then we would have $u_{t+1} = u_t - 1/2$ so again the fill-range could not have increased. Because the emptier is Δ-greedy-like emptying from a cup with fill at most $\ell_t + 1$ and not emptying from a cup with fill at least $u_t - 1$ implies that $u_t - 1$ and $\ell_t + 1$ differ by at most Δ. Thus,

$$u_{t+1} - \ell_{t+1} \le u_t + 1/2 - (\ell_t - 1/2) \le \Delta + 3 \le R.$$

□

Because by Claim 1 whenever the fill-range of the cups increases it increases to a value at most R, we have by induction that the fill-range of the cups never exceeds R, i.e. the cups are always R-flat.

Let L_t be the set of cups c with $\text{fill}_{S_t}(c) \le \ell_t + 2 + \Delta$, and let U_t be the set of cups c with $\text{fill}_{S_t}(c) \ge u_t - 2 - \Delta$.

Now we prove a key property of the sets U_t and L_t: if a cup is in U_t or L_t it is also in $U_{t'}, L_{t'}$ for all $t' > t$. This follows immediately from Claim 2.

Claim 2.

$$U_t \subseteq U_{t+1}, \quad L_t \subseteq L_{t+1}.$$

Proof. Consider a cup $c \in U_t$.

If c is not emptied from, i.e. fill(c) has increased by $1/2$ from the previous round, then clearly $c \in U_{t+1}$, because backlog has increased by at most $1/2$, so fill(c) must still be within $2+\Delta$ of the backlog on round $t+1$.

On the other hand, if c is emptied from, i.e. fill(c) has decreased by $1/2$, we consider two cases.

Case 1: If $\text{fill}_{S_t}(c) \geq u_t - \Delta - 1$, then $\text{fill}_{S_t}(c)$ is at least 1 above the bottom of the interval defining which cups belong to U_t. The backlog increases by at most $1/2$ and the fill of c decreases by $1/2$, so $\text{fill}_{S_{t+1}}(c)$ is at least $1 - 1/2 - 1/2 = 0$ above the bottom of the interval, i.e. still in the interval.

Case 2: On the other hand, if $\text{fill}_{S_t}(c) < u_t - \Delta - 1$, then every cup with fill in $[u_t - 1, u_t]$ must have been emptied from because the emptier is Δ-greedy-like. Therefore the fullest cup on round $t+1$ is the same as the fullest cup on round t, because every cup with fill in $[u_t - 1, u_t]$ has had its fill decrease by $1/2$, and no cup with fill less than $u_t - 1$ had its fill increase by more than $1/2$. Hence $u_{t+1} = u_t - 1/2$. Because both fill(c) and the backlog have decreased by $1/2$, the distance between them is still at most $\Delta + 2$, hence $c \in U_{t+1}$.

The argument for why $L_t \subseteq L_{t+1}$ is symmetric. □

Now we show that under certain conditions u_t decreases and ℓ_t increases.

Claim 3. *On any round t where the emptier empties from at least $n/2$ cups, if $|U_t| \leq n/2$ then $u_{t+1} = u_t - 1/2$. On any round t where the emptier empties from at most $n/2$ cups, if $|L_t| \leq n/2$ then $\ell_{t+1} = \ell_t + 1/2$.*

Proof. Consider a round t where the emptier empties from at least $n/2$ cups. If there are at least $n/2$ cups outside of U_t, i.e. cups with fills in $[\ell_t, u_t - 2 - \Delta]$, then all cups with fills in $[u_t - 2, u_t]$ must be emptied from; if one such cup was not emptied from then by the pigeon-hole principle some cup outside of U_t was emptied from, which is impossible as the emptier is Δ-greedy-like. This clearly implies that $u_{t+1} = u_t - 1/2$: no cup with fill less than $u_t - 2$ has gained enough fill to become the fullest cup, and the fullest cup from the previous round has lost $1/2$ unit of fill.

By a symmetric argument $\ell_{t+1} = \ell_t + 1/2$ if the emptier empties at most $n/2$ cups on a round t where $|L_t| \leq n/2$. □

Now we show that eventually $L_t \cap U_t \neq \varnothing$.

Claim 4. *There is a round $t_0 \leq 2(R + \lceil (1+1/n)(E+S) \rceil)$ such that $U_t \cap L_t \neq \varnothing$ for all $t \geq t_0$.*

Proof. We call a round where the emptier doesn't use $p = n/2$ resources, i.e. a round where the number of skipped emptyings and the number of extra emptyings are not equal, an ***unbalanced round***; we call a round that is not unbalanced a ***balanced*** round.

Note that there are clearly at most $E+S$ unbalanced rounds. We now associate some unbalanced rounds with balanced rounds; in particular we define what it means for a balanced round to ***cancel*** an unbalanced round. We define cancellation by a sequential process. For $i = 1, 2, \ldots, 2(R + \lceil (1+1/n)(E+S) \rceil)$ (iterating in ascending order of i), if round i is unbalanced then we say that the first balanced round $j > i$ that hasn't already been assigned (earlier in the sequential process) to cancel another unbalanced round $i' < i$, if any such round j exists, ***cancels*** round i. Note that cancellation is a one-to-one relation: each unbalanced round is cancelled by at most one balanced round and each balanced round cancels at most one unbalanced round.

Consider rounds of the form $2(R + \lceil (E+S)/n \rceil) + (E+S) + i$ for $i \in [E+S+1] - 1$. We claim that there is some such i such that among rounds $[2(R + \lceil (E+S)/n \rceil) + (E+S) + i]$ every unbalanced round has been cancelled, and such that there are $2(R + \lceil (E+S)/n \rceil)$ balanced rounds not cancelling other rounds. Assume for contradiction that such an i does not exist. Note that there are at least $2(R + \lceil (E+S)/n \rceil)$ balanced rounds in the first $2(R + \lceil (E+S)/n \rceil) + (S+E)$ rounds. Thus every balanced round $2R + (E+S) + \lceil (E+S)/n \rceil + i - 1$ for $i \in [E+S+1]$ is necessarily a cancelling round, or else there would be a round by which there are no uncancelled unbalanced rounds. Hence by round $2(R + \lceil (E+S)/n \rceil) + 2(E+S)$, there must have been $E+S$ cancelled rounds, so on round $2(R + \lceil (E+S)/n \rceil) + 2(E+S)$ all unbalanced rounds are cancelled, which leaves $2(R + \lceil (E+S)/n \rceil)$ balanced rounds that are not cancelling any rounds, as desired.

Let t_e be the first round by which there are $2(R + \lceil (E+S)/n \rceil)$ balanced non-cancelling rounds. Note that the average fill of the cups cannot have decreased by more than E/n from its starting value; similarly the average fill of the cups cannot have increased by more than S/n. Because the cups start R-flat, we have that u_t cannot have decreased by more than $R + E/n$ or else u_t would necessarily be below the average fill, and identically ℓ_t cannot have increased by more than $R + S/n$ or else it would be above the average fill. Now, by Claim 3 we have that eventually $|L_t| > n/2$: If $|L_t| \leq n/2$ were always true, then on every balanced round ℓ_t would have increased by $1/2$, and since ℓ_t increases by at most $1/2$ on unbalanced rounds, this implies that in total ℓ_t would have increased by at least $(1/2)2(R + \lceil (E+S)/n \rceil)$, which is impossible. By a symmetric argument it is impossible that $|U_t| \le n/2$ for all rounds.

Since $|U_{t+1}| \ge |U_t|$ and $|L_{t+1}| \ge |L_t|$ by Claim 2, we have that there is some round $t_0 \in [2(R + \lceil (1+1/n)(E+S) \rceil)]$ such that for all $t \ge t_0$ we have $|U_t| > n/2$ and $|L_t| > n/2$. But then we have $U_t \cap L_t \ne \varnothing$, as desired. □

If there exists a cup $c \in L_t \cap U_t$, then

$$\text{fill}(c) \in [u_t - 2 - \Delta, u_t] \cap [\ell_t, \ell_t + 2 + \Delta].$$

Hence we have that

$$\ell_t + 2 + \Delta \ge u_t - 2 - \Delta.$$

Rearranging,

$$u_t - \ell_t \le 2(2 + \Delta) = R_\Delta.$$

Thus the cup configuration is R_Δ-flat by the end of this flattening process.

□

Next we describe a simple oblivious filling strategy that will be used as a subroutine in Lemma 2; this strategy is very well-known, and similar versions of it can be found in [1, 2, 3, 4].

Proposition 1. *Consider an R-flat cup configuration in the single-processor E-extra-emptyings S-skip-emptyings negative-fill cup game on n cups with initial average fill μ_0. Let $d = \sum_{i=2}^n 1/i$.*

*There is an oblivious filling strategy **randalg** with running time $n-1$ that achieves a cup with fill at most $\mu_0 + R + d$; if we condition on the emptier not performing extra emptying then randalg achieves fill at least $\mu_0 - R + d$ in a known cup c with probability at least $1/(n-1)!$.*

Furthermore, when applied against a Δ-greedy-like emptier with $R = R_\Delta$, randalg guarantees that the cup configuration is $(R+d)$-flat on every round.

Proof. First we condition on the emptier does not using extra emptying and show that in this case the filler has probability at least $1/(n-1)!$ of attaining a cup with fill at least $\mu_0 - R + d$. The filler maintains an ***active set***, initialized to being all of the cups. Every round the filler distributes 1 unit of fill equally among all cups in the active set. Next the emptier removes 1 unit of fill from some cup, or skips its emptying. Then the filler removes a random cup from the active set (chosen uniformly at random from the active set). This continues until a single cup c remains in the active set.

We now bound the probability that c has never been emptied from. Assume that on the i-th step of this process, i.e. when the size of the active set is $n-i+1$, no cups in the active set have ever been emptied from; consider the probability that after the filler removes a cup randomly from the active set there are still no cups in the active set that the emptier has emptied from. If the emptier skips its emptying on this round, or empties from a cup not in the active set then it is trivially still true that no cups in the active set have been emptied from. If the cup that the emptier empties from is in the active set then with probability $1/(n-i+1)$ it is evicted from the active set, in which case we still have that no cup in the active set has ever been emptied from. Hence with probability at least $1/(n-1)!$ the final cup in the active set, c, has never been emptied from. In this case, c will have gained fill $d = \sum_{i=2}^n 1/i$ as claimed. Because c started with fill at least $-R + \mu_0$, c now has fill at least $-R + d + \mu_0$.

Now note that regardless of if the emptier uses extra emptyings c has fill at most $\mu_0 + R + d$, as c starts with fill at most R, and c gains at most $1/(n-i+1)$ fill on the i-th round of this process.

Now we analyze this algorithm specifically for a Δ-greedy-like emptier. Consider a round t on which $\min_c \text{fill}_{S_{t+1}}(c) < \min_c \text{fill}_{S_t}(c)$, and where a cup c_0 that has $\text{fill}_{S_{t+1}}(c_0) = \max_c \text{fill}_{S_{t+1}}(c)$ was not emptied from on round t. Because the emptier is Δ-greedy-like this implies that $\text{fill}_{I_t}(c_0) - \min_c \text{fill}_{I_t}(c) \le \Delta + 1$ and then $\max_c \text{fill}_{S_{t+1}}(c) - \min_c \text{fill}_{S_{t+1}}(c) \le \Delta + 2$, i.e. the cups are $(\Delta+2)$-flat.

Consider some round t_1 on which the cups are not $(\Delta+2)$-flat; let t_0 be the last round on which the cups where R-flat (note that if the cups are $(\Delta+2)$-flat they are also R-flat as $\Delta + 2 < R$). Consider how the fill-range of the cups changes during the set of rounds t with $t_0 < t \le t_1$. On any such round t either $\min_c \text{fill}_{S_{t+1}}(c) \ge \min_c \text{fill}_{S_t}(c)$ in which case the fill-range increases by at most $1/(n-t+1)$ where $n-t+1$ is the size of the active set on round t, or all cups on round $t+1$ with fill equal to the backlog were emptied from, meaning that backlog decreased by at least $1 - 1/(n-t+1)$. In either case the fill-range increases by at most $1/(n-t+1)$. Thus in total the fill-range is at most $R+d$. That is, the cups are $(R+d)$-flat on round t_1, as desired.

□

Now we show that we can force a constant fraction of the cups to have high fill; using Lemma 2 and exploiting the greedy-like nature of the emptier we can get a known cup with high fill (we show this in Proposition 2).

Lemma 2. *Let $\Delta \leq O(1)$, let $h \leq O(1)$ with $h \geq 16+16\Delta$, let n be at least a sufficiently large constant determined by h and Δ, and let $R \leq poly(n)$. Consider an R-flat cup configuration in the variable-processor cup game on n cups. Let A, B be disjoint subsets of the cups with $|AB| = n$. Over the course of the algorithm B will give some cups to A, but $|A|$ will always satisfy $|A| \ll |B|$, and $|A|$ will eventually be $\Theta(n)$. Let $M \gg n$ be very large.*

There is an oblivious filling strategy that either achieves mass at least M in the cups, or makes an unknown set of $\Theta(n)$ cups in A have fill at least h with probability at least $1 - 2^{-\Omega(n)}$ in running time $poly(M)$ against a Δ-greedy-like emptier while also guaranteeing that $\mu(B) \geq -h/2$.

Proof. We refer to A as the ***anchor*** set, and B as the ***non-anchor*** set. Let $n_A = \Theta(n)$ be small enough to satisfy

$$n_A \leq (n - n_A)/(2e^{2h+1} + 1). \qquad (1)$$

The filler initializes A to $\varnothing$, and B to be all of the cups. Over the course of the algorithm B will give away n_A cups to A. Note that $|B| \geq n - n_A \gg n_A \geq |A|$.

We denote by *randalg* the oblivious filling strategy given by Proposition 1. We denote by *flatalg* the oblivious filling strategy given by Lemma 1. We say that the filler ***applies*** a filling strategy *alg* to a set of cups $D \subseteq B$ if the filler uses *alg* on D while placing 1 unit of fill in each anchor cup.

We now describe the filler's strategy.

The filler starts by flattening the cups, i.e. using *flatalg* on all of the cups for $\text{poly}(M)$ rounds (setting $p = n/2$). After this the filling strategy always places 1 unit of water in to each anchor cup on every round. The filler performs a series of n_A ***swapping-processes***, one per anchor cup, which are procedures that the filler uses to get a new cup—which will sometimes have high fill—in the anchor set. On each swapping-process the filler applies *randalg* many times to arbitrarily chosen constant-size sets $D \subset B$ with $|D| = \lceil e^{2h+1} \rceil$. The number of times that the filler applies *randalg* is chosen at the start of the swapping-process, chosen uniformly at random from $[m]$ ($m = \text{poly}(M)$ to be specified). At the end of the swapping-process, the filler does a ***swap***: the filler takes the cup given by *randalg* in B and moves it into A. After performing a swap the filler must increase p by 1 so that $p = |A| + 1$. Before each application of *randalg* the filler flattens B by applying *flatalg* to B for $\text{poly}(M)$ rounds.

We remark that this construction is similar to the construction in **??**, but has a major difference that substantially complicates the analysis: in the adaptive lower bound construction the filler halts after achieving the desired average fill in the anchor set, whereas the oblivious filler cannot halt but rather must rely on the emptier's greediness to guarantee that each application of *randalg* has constant probability of generating a cup with high fill.

We proceed to analyze our algorithm.

Note that if the emptier skips more than M emptyings, or neglects the anchor set more than M times without decreasing the fills of any anchor cups in between these times, then the filler has achieved mass M in the cups, in which case the filler has fulfilled the statement of Lemma 2. Throughout the remainder of the analysis we consider the case of an emptier that chooses to not skip more than M times, and chooses to not neglect the anchor set more than M times without decreasing the fill of anchor cups in between these times.

First note that the initial flattening makes the cups R_Δ-flat by Lemma 1. In particular, note that the flattening happens in the $(n/2)$-processor 0-extra-emptyings M-skip-emptyings variable-processor cup game on n cups.

We say that a property of the cups has ***always*** held if the property has held since the start of the first swapping-process; i.e. from now on we only consider rounds after the initial flattening.

We say that the emptier ***neglects*** the anchor set on a round if it does not empty from each anchor cup. We say that an application of *randalg* to $D \subset B$ is ***non-emptier-wasted*** if the emptier does not neglect the anchor set during any round of the application of *randalg*. We define $d = \sum_{i=2}^{|D|} 1/i$ (recall that $|D| = \lceil e^{2h+1} \rceil$). We say that an application of *randalg* to D is ***lucky*** if it achieves backlog at least $\mu(B) - R_\Delta + d$; note that by Proposition 1 if we condition on an application of *randalg* where B started R_Δ-flat being non-emptier-wasted then the application has at least a $1/|D|!$ chance of being lucky.

Now we prove several important bounds on fills of cups in A and B.

Claim 5. *All applications of flatalg make B be R_Δ-flat and B is always $(R_\Delta + d)$-flat.*

Proof. Given that the application of *flatalg* immediately prior to an application of *randalg* made B be R_Δ-flat, by Proposition 1 we have that B will stay $(R_\Delta + d)$-flat during the application of *randalg*. Given that the application of *randalg* immediately prior to an application of *flatalg* resulted in B being $(R_\Delta + d)$-flat, we have that B remains $(R_\Delta + d)$-flat throughout the duration of the application of *flatalg* by Lemma 1. Given that B is $(R_\Delta + d)$-flat before a swap occurs

B is clearly still $(R_\Delta + d)$-flat after the swap, because the only change to B during a swap is that a cup is removed from B which cannot increase the fill-range of B. Note that B started R_Δ-flat before the first application of *flatalg* because all the cups was flattened. Note that if an application of *flatalg* begins with B being $(R_\Delta + d)$-flat, then by considering the flattening to happen in the $(|B|/2)$-processor M-extra-emptyings M-skip-emptyings cup game we ensure that it makes B be R_Δ-flat, because the emptier cannot skip more than M emptyings and also cannot do more than M extra emptyings or the mass would be at least M. Hence we have by induction that B has always been $(R_\Delta + d)$-flat and that all flattening processes have made B be R_Δ-flat. □

Now we aim to show that $\mu(B)$ is never too low, which we need in order to establish that every non-emptier-wasted lucky application of *randalg* gets a cup with high fill. Interestingly in order to lower bound $\mu(B)$ we first must upper bound $\mu(B)$, which by greediness and flatness of B gives an upper bound on $\mu(A)$ which we use to get a lower bound on $\mu(B)$.

Claim 6. *We have always had*

$$\mu(B) \leq 2 + \mu(AB).$$

Proof. There are two ways that $\mu(B) - \mu(AB)$ can increase:

Case 1: The emptier could empty from 0 cups in B while emptying from every cup in A.

Case 2: The filler could evict a cup with fill lower than $\mu(B)$ from B at the end of a swapping-process.

Note that cases are exhaustive, in particular note that if the emptier skips more than 1 emptying then $\mu(B) - \mu(AB)$ must decrease because $|A| \approx |AB|$, in particular (1), as opposed to in Case 1 where $\mu(B) - \mu(AB)$ increases.

In Case 1, because the emptier is Δ-greedy-like,

$$\min_{a \in A} \text{fill}(a) > \max_{b \in B} \text{fill}(b) - \Delta.$$

Thus $\mu(B) \leq \mu(A) + \Delta$. As $|B| \gg |A|$, in particular by (1), this can be loosened to $\mu(B) \leq 1 + \mu(AB)$.

Consider the final round on which B is skipped while A is not skipped (or consider the first round if there is no such round).

From this round onwards the only increase to $\mu(B) - \mu(AB)$ is due to B evicting cups with fill well below $\mu(B)$. We can upper bound the increase of $\mu(B) - \mu(AB)$ by the increase of $\mu(B)$ as $\mu(AB)$ is strictly increasing.

The cup that B evicts at the end of a swapping-process has fill at least $\mu(B) - R_\Delta - (|D| - 1)$, as the running time of *randalg* is $|D| - 1$, and because B starts R_Δ-flat by Claim 5. Evicting a cup with fill $\mu(B) - R_\Delta - (|D| - 1)$ from B changes $\mu(B)$ by $(R_\Delta + |D| - 1)/(|B| - 1)$ where $|B|$ is the size of B before the cup is evicted from B. Even if this happens on each of the n_A swapping processes $\mu(B)$ cannot rise higher than $n_A(R_\Delta + |D| - 1)/(n - n_A)$ which by design in choosing $|B| \gg |A|$, as was done in (1), is at most 1.

Thus it always is the case that $\mu(B) \leq 2 + \mu(AB)$. □

The upper bound on $\mu(B)$ along with the guarantee that B is flat allows us to bound the highest that a cup in A could rise by greediness, which in turn upper bounds $\mu(A)$ which in turn lower bounds $\mu(B)$. In particular we have

Claim 7. *We always have*

$$\mu(B) \geq -h/2.$$

Proof. By Claim 6 and Claim 5 we have that no cup in B ever has fill greater than $u_B = \mu(AB) + 2 + R_\Delta + d$.

Let $u_A = u_B + \Delta + 1$. We claim that the backlog in A never exceeds u_A.

Consider how high the fill of a cup $c \in A$ could be. If c came from B then when it is swapped into A its fill is at most $u_B < u_A$. Otherwise, c started with fill at most $R_\Delta < u_A$. Now consider how much the fill of c could increase while being in A. Because the emptier is Δ-greedy-like, if a cup $c \in A$ has fill more than Δ higher than the backlog in B then c must be emptied from, so any cup with fill at least $u_B + \Delta = u_A - 1$ must be emptied from, and hence u_A upper bounds the backlog in A.

Of course an upper bound on backlog in A also serves as an upper bound on the average fill of A as well, i.e. $\mu(A) \leq u_A$. Rearranging the expression

$$|B|\mu(B) + |A|\mu(A) = |AB|\mu(AB)$$

we have

$$\begin{aligned}
\mu(B) & \\
&= -\frac{|A|}{|B|}\mu(A) + \frac{|AB|}{|B|}\mu(AB) \\
&\geq -(\mu(AB) + 3 + R_\Delta + d + \Delta)\frac{|A|}{|B|} + \frac{|AB|}{|B|}\mu(AB) \\
&= -(3 + R_\Delta + d + \Delta)\frac{|A|}{|B|} + \mu(AB) \\
&\geq -h/2
\end{aligned}$$

where the final inequality follows because $\mu(AB) \geq 0$, and $|B| \gg |A|$, in particular by (1). □

Now we show that this guarantees that with constant probability the final application of *randalg* on a swapping-process is both lucky and successful.

Claim 8. *There exists choice of $m = poly(M)$ such that with at least constant probability the final application of randalg on any fixed swapping-process is both lucky and successful.*

Proof. Fix some swapping-process. The emptier can neglect the anchor set a limited number of times in this swapping-process, in particular at most M times. Hence, the emptier must have some policy for deciding whether or not to neglect the i-th application of *randalg*. The emptier's policy can of course depend on whether or not the application of *randalg* looks like it is going to be lucky. The emptier must come up with some function $I : [m] \to \{0, 1\}$ where $I(i) = 1$ means that the emptier will thwart the i-th application of *randalg* if it looks like it is going to be lucky, and if the emptier has not already neglected the anchor set M times.

The filler chooses $m = 4M|D|!$. By a Chernoff bound, there is exponentially high probability that of $4M|D|!$ applications of *randalg* at least $2M$ look like they are going to be lucky. The emptier can choose at most M of these on which to neglect the anchor set and thwart the application of *randalg*. Thus, conditioning on the final round looking like it is going to be lucky, there is at least a $1/2$ chance that it is successful. The final application looks like it is going to be lucky with probability $1/|D|!$. Hence our choice of m makes the final round lucky and successful with constant probability at least $1/(2|D|!)$. □

Claim 9. *With probability at least $1 - 2^{-\Omega(n)}$, the filler achieves fill at least h in at least $\Theta(n)$ of the cups in A.*

Proof. By Claim 8 each swapping-process has at least constant probability of swapping a cup with fill at least $\mu(B) + d - R_\Delta$ into A. The events that the swapping-processes swap such a cup into A are independent, so by a Chernoff bound there is exponentially high probability that at least a constant fraction of them succeed. By Claim 7 $\mu(B) \geq -h/2$. Recalling that $d \geq 2h$ and $h \geq 16(1 + \Delta)$, we have that such a swapped cup has fill at least h, as desired. □

We now analyze the running time of the filling strategy. There are $|A|$ swapping-processes. Each swapping-process consists of $\text{poly}(M)$ applications of *randalg*, which each take constant time, and $\text{poly}(M)$ applications of *flatalg*, which each take $\text{poly}(M)$ time. Thus overall the algorithm takes $\text{poly}(M)$ time, as desired.

□

Finally, using Lemma 2 we can show in Proposition 2 that an oblivious filler can achieve constant backlog. We remark that Proposition 2 plays a similar role in the proof of the lower bound on backlog as **??** does in the adaptive case, but is vastly more complicated to prove (in particular, **??** is trivial, whereas we have already proved several lemmas and propositions as preparation for the proof of Proposition 2).

Proposition 2. *Let $H \leq O(1)$, let $\Delta \leq O(1)$, let n be at least a sufficiently large constant determined by H and Δ, and let $R \leq poly(n)$. Let $M \gg n$ be very large. Consider an R-flat cup configuration in the negative-fill variable-processor cup game on n cups with average fill 0. Given this configuration, an oblivious filler can either achieve mass M among the cups, or achieve fill H in a chosen cup in running time $poly(M)$ against a Δ-greedy-like emptier with probability at least $1 - 2^{-\Omega(n)}$.*

Proof. The filler starts by performing the procedure detailed in Lemma 2, using $h = H \cdot 16(1 + \Delta)$. Let the number of cups which must now exist with fill h be of size $nc = \Theta(n)$.

The filler reduces the number of processors to $p = nc$. Now the filler exploits the filler's greedy-like nature to to get fill H in a set $S \subset B$ of nc chosen cups.

The filler places 1 unit of fill into each cup in S. Because the emptier is greedy-like it must focus on the nc cups in A with fill at least h until the cups in S have sufficiently high fill. In particular, $(5/8)h$ rounds suffice. Over $(5/8)h$ rounds the nc high cups in A cannot have their fill decrease below $(3/8)h \geq h/8 + \Delta$. Hence, any cups with fills less than $h/8$ must not be emptied from during these rounds. The fills of the cups in S must start as at least $-h/2$ as $\mu(B) \geq -h/2$. After $(5/8)h$ rounds the fills of the cups in S are at least $h/8$, because throughout this process the emptier cannot have emptied from them until they got fill at least $h/8$, and if they are never emptied from then they achieve fill $h/8$.

Thus the filling strategy achieves backlog $h/8 \geq H$ in some known cup (in fact in all cups in S, but a single cup suffices), as desired.

□

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