Oblivious Lower Bound via Flattening

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Call a cup configuration T***-flat*** if the fill of every cup is in the interval $[-T,T]$.

An emptying strategy is said to be (R,t)***-flattenable*** if, given a T-flat cup configuration, the emptier can, in running-time $t(T)$, obtain a R-flat configuration of cups. An emptying strategy is said to be ***flattenable*** if it is (R,t)-flattenable for $R \le O(1)$ and for t such that $t(T) \le O(1)$ if $T \le O(1)$.

In the randomized setting we are only able to prove lower bounds for backlog against flattenable emptiers; whether or not our results can be extended to a more general class of emptiers is an interesting open question.

However, this class of flattenable emtpiers is alredy of great interest. In particular, we will show certain emptying strategies with properties similar to a greedy emptier are flattenable; in particular, we will show that a greedy emptier is flattenable.

We call an emptier Δ***-greedy-like*** if, when there are two cups c_1,c_2 with fills satisfying $\text{fill}(c_1) > \text{fill}(c_2)+\Delta$ the emptier never empties from c_2 without emptying from c_1 on the same round. Intuitively, a Δ-greedy-like emptier has a $\pm\Delta$ range within it is allowed to "not be greedy". Note that a perfectly greedy emptier is 0-greedy-like.

We now prove that Δ-greedy-like emptiers (for $\Delta \le O(1)$) are flattenable:

Proposition 1. *Given a cup configuration that is T-flat, an oblivious filler can, in running time $2T$, achieve a $2(2+\Delta)$-flat configuration of cups against a Δ-greedy-like emptier.*

In particular, this implies that a Δ-greedy-like emptier is (R,t)-flattenable for $R = 2(2+\Delta)$ and t the function $T \mapsto 2T$; for $\Delta \le O(1)$, Δ-greedy-like emptiers are flattenable.

Proof. The filler's sets $p = n/2$ and distributes fill equally amongst all cups at every round, in particular placing $1/2$ units of water in each cup. Let $\ell_t = \min_{c\in S_t} \text{fill}_{S_t}(c)$, $u_t = \max_{c\in S_t} \text{fill}_{S_t}(c)$. Let L_t be the set of cups c with $\text{fill}_{S_t}(c) \in \le l_t+2+\Delta$, and let U_t be the set of cups c with $\text{fill}_{S_t}(c) \ge u_t - 2 - \Delta$.

There are two ways to think of U_t. First we can consider U_t as the union of intervals of length 1, Δ, and 1. Note the key property that if a cup with fill in $[u_t-\Delta-2, u_t-\Delta-1]$ is emptied from, then all cups with fills in $[u_t-1, u_t]$ must be emptied from, because the emptier is Δ-greedy-like. On the other hand, we can consider U_t as the union of $[u_t-2,u_t]$ and $[u_t-\Delta-2,u_t-2]$. This is useful as the interval of width Δ serves as a "buffer". In particular, if there are more than $n/2$ cups outside of U_t then all cups in $[u_t-2,u_t]$ must be emptied from because the emptier is Δ-greedy-like. L_t is of course completely symmetric to U_t.

First we prove a key property of the sets U_t and L_t: once a cup is in U_t or L_t it is always in $U_{t'},L_{t'}$ for all $t' > t$. This follows immediately from the following claim:

Claim 1.
$$U_t \subseteq U_{t+1}, L_t \subseteq L_{t+1}.$$

Proof. Consider a cup $c \in U_t$.

If c is not emptied from, i.e. $\text{fill}(c)$ has increased by $1/2$, then clearly $c \in U_{t+1}$, because backlog has increased by at most $1/2$, so the fill of c must still be within $2+\Delta$ of the backlog on round $t+1$.

On the other hand, if c is emptied from, i.e. $\text{fill}(c)$ has decreased by $1/2$, we consider two cases.

- If $\text{fill}_{S_t}(c) \ge u_t - \Delta - 1$, then, as $u_{t+1} \le u_t + 1/2$,
$$\text{fill}_{S_{t+1}}(c) \ge u_t - \Delta - 1 - 1/2 \ge u_{t+1} - \Delta - 2.$$
- On the other hand, if $\text{fill}_{S_t}(c) < u_t - \Delta - 1$, then every cup with fill in $[u_t-1, u_t]$ must have been emptied from. The fullest cup at round $t+1$ is the same as the fullest cup on round t, because the fills of all cups with fill in $[u_t-1,u_t]$ have decreased by $1/2$, and no cup with fill less than u_t-1 had fill increase by more than $1/2$. Hence $u_{t+1} = u_t - 1/2$. Because both the fill of c and the backlog have decreased by the same amount, the distance between them is still at most $\Delta+2$, hence $c \in U_{t+1}$.

The argument for $L_t \subseteq L_{t+1}$ is essentially identical. □

Now that we have shown that L_t and U_t never lose cups, we will show that they eventually gain a substantial number of cups.

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Claim 2. *As long as $|U_t| \le n/2$ we have $u_{t+1} = u_t - 1/2$. Identically, as long as $|L_t| \le n/2$ we have $\ell_{t+1} = \ell_t + 1/2$.*

Proof. If there are more than $n/2$ cups outside of U_t then there must be some cup with fill less than $u_t - \Delta - 2$ that is emptied from. Because the emptier is Δ-greedy-like this means that the emptier must empty from every cup with fill at least $u_t - 2$. Thus $u_{t+1} = u_t - 1/2$: no cup with fill less than $u_t - 2$ could have become the fullest cup, and the previous fullest cup has lost $1/2$ units of fill.

The proof is identical for L_t. □

By Claim 2 we see that both $|U_t|$ and $|L_t|$ must eventually exceed $n/2$ at some times $t_u, t_\ell \le 2T$, by the assumption that the initial configuration is T-flat. Since by Claim 1 $|U_{t+1}| \ge |U_t|$ and $|L_{t+1}| \ge |L_t|$ we have that there is some round $t_0 = \max(t_u, t_\ell) \le 2T$ on which both $|U_{t_0}|$ and $|L_{t_0}|$ exceed $n/2$. Then $U_{t_0} \cap L_{t_0} \ne \varnothing$. Furthermore, the sets must intersect for all $t_0 \le t \le 2T$. In order for the sets to intersect it must be that the intervals $[u_t - 2 - \Delta, u_t]$ and $[\ell_t, \ell_t + 2 + \Delta]$ intersect. Hence we have that

$$\ell_t + 2 + \Delta \ge u_t - 2 - \Delta.$$

Since $u_t \ge 0$ and $\ell_t \le 0$ this implies that all cups have fill in $[-2(2+\Delta), 2(2+\Delta)]$.

□

Proposition 2. *There exists an oblivious filling strategy in the variable-processor cup game on n cups that achieves backlog $\Omega(\log n)$ against a Δ-greedy-like emptier (where $\Delta \le O(1)$ is a constant known to the filler), with constant probability.*

Proof. Let A, the ***anchor*** set, be a subset of the cups chosen uniformly at random from all subsets of size $n/2$ of the cups, and let B, the ***non-anchor*** set, consist of the rest of the cups ($|B| = n/2$). Let $h = 8\Delta + 8$, and let $h' = 2$. Our strategy is roughly as follows:

- **Step 1:** Make a constant fraction of cups in A have fill at least h by playing single processor cup games on constant-size subsets of B. With constant probability we can attain a cup in B with constant fill by this method, that we then swap into A. By a Chernoff bound we get a constant fraction of A, say cn cups, to have fill at least h with exponentially good probability. Between single-processor cup games we flatten the cups.

- **Step 2:** Reduce the number of processors to cn, and raise the fill of cn *known* cups to fill h'.

- **Step 3:** Recurse on the nc cups that are known to have fill at least h'.

By performing $\Omega(\log n)$ levels of recursion, achieving constant backlog h' at each step (relative to the average fills), the filler achieves backlog $\Omega(\log n)$.

In order to flatten a set of cups we must have a bound on the magnitude of the fills of the cups. We claim that without loss of generality no cup has fill larger in magnitude than $h\sqrt{n/\log\log n}$. If a cup has more than $h\sqrt{n/\log\log n}$ fill we call the cup ***overpowered***. If there ever is an overpowered cup then we are automatically done: the emptier has achieved $\mathrm{poly}(n)$ backlog and will maintain it for at least $\mathrm{poly}(n)$ rounds. If a cup ever has fill less than $-h\sqrt{n/\log\log n}$ then the absolute average fill must be large enough such that the absolute fill of this cup is at least 0. Thus there is an overpowered cup. From now on we assume that there are no overpowered cups.

At the start of our procedure we flatten the cups. By assumption they start $h\sqrt{n/\log\log n}$-flat, and by Proposition 1 we can make them $2(2+\Delta)$-flat.

We now describe how to achieve Step 1.

We perform a series of ***swapping-process***, which are procedures that we use to get a new cup in A. A swapping-process is composed of a substructure, repeated many times, which we call a ***round-block***; a round-block is a set of rounds. At the beginning of a swapping-process we choose a round-block $j \in [n^2]$ uniformly at random from all the round-blocks. The swapping-process proceeds for n^2 round-blocks; on the j-th round-block we swap a cup into the anchor set.

On each of the n^2 round-blocks, the filler selects a random subset $C \subset B$ of the non-anchor cups and plays a single processor cup game on C. In this single-processor cup game the filler employs the classic adaptive strategy for achieving backlog $\Omega(\log|B|)$ on a set of $|B|$ cups, however modified because it is an oblivious filler. In particular, the filler's strategy in the single-processor cup games is to distribute water equally among an ***active set*** of cups, and then after the emptier removes water from some cup the filler removes a random cup from the active set. There is at least constant probability that this results in the active set having a single cup at the end, with fill that has increased by at least $1/|B| + 1/(|B|-1) + ... + 1/1 \ge \ln|B|$ since the start of the round-block.

Between each round block we do flattening.

On most round-blocks – all but the j-th – the filler does nothing with the cup that it achieves in the active set at the end of the single processor cup game. However, on the j-th round-block the filler swaps the winner of the single processor cup game into the anchor set.

Claim 3. *Let $q \ge \Omega(1)$ be an appropriately small constant (q is a function of $h \le O(1)$). With probability at least $1 - e^{-nq^2/1024}$, we achieve fill at least h in at least $nq/16$ of the cups in A (i.e. a constant fraction of the cups in A).*

Proof. The cups start $2(2+\Delta)$-flat.

Say the emptier ***neglects*** the anchor set in a round-block if on at least one round of the round-block the emptier does not empty from every anchor cup. By playing the single-processor cup game for n^2 round-blocks, with only one round-block when we actually swap a cup into the anchor set, we strongly disincentive the emptier from neglecting the anchor set on more than a constant fraction of the round-blocks.

The emptier must have some binary function, $I(i)$ that indicates whether or not they will neglect the anchor set on round-block i if the filler has not already swapped. Note that the emptier will know when the filler perform a swap, so whether or not the emptier neglects a round-block i depends on this information. However, j is the only parameter of the swapping-process, so there is no other information that the emptier can use to decide whether or not to neglect a round-block, because on any round-block when we simply redistribute water amongst the non-anchor cups we effectively have not changed anything about the game state.

If the emptier is willing to neglect the anchor set for at least 1/2 of the round-blocks, i.e. $\sum_{i=1}^{n^2} I(i) \geq n^2/2$, then with probability at least 1/4, $j \in ((3/4)n^2, n^2)$, in which case the emptier neglects the anchor set on at least $n^2/4$ round-blocks ($I(k)$ must be 1 for at least $n^2/4$ of the first $(3/4)n^2$ round-blocks). Each time the emptier neglects the anchor set the mass of the anchor set increases by at least 1. Thus the average fill of the anchor set will have increased by at least $(n^2/2)/(n/2) \geq \Omega(n)$ over the entire swapping-process in this case, implying that we achieve the desired backlog.

Otherwise, there is at least a 1/2 chance that the round-block j, which is chosen uniformly at random from the round-blocks, when the filler performs a swap into the anchor set occurs on a round-block with $I(j)=0$, indicating that the emptier won't neglect the anchor set on round-block j. In this case, the round-block was a legitimate single processor cup game on C_j, the randomly chosen set of $\lceil e^{2h} \rceil$ cups on the j-th round. Then we achieve fill increase $\geq 2h$ by the end of the round-block with probability at least $1/\lceil e^{2h} \rceil!$ – the probability that we correctly guess the sequence of cups within the single processor cup game that the emptier empties from.

The probability that the random set $C_j \subset B$ contains only cups that are among the $n/4$ fullest cups in B is

$$\binom{n/2}{\lceil e^{2h} \rceil} \Big/ \binom{n}{\lceil e^{2h} \rceil} = O(1).$$

Note that because, by assumption, at least half of the cups $c \in B$ have $\text{fill}(c) \geq -h$, then the $n/4$ fullest cups in B must have fill at least $-h$. If all cups $c \in C_j$ have $\text{fill}(c) \geq -h$, then the fill of the cup in the active set at the end of the round-block is at least $-h+2h=h$, if the filler guesses the emptier's emptying sequence correctly.

Say that a swapping-process where at least half of the cups $c \in B$ have $\text{fill}(c) \geq -h$ ***succeeds*** if C_j is a subset of the $n/4$ fullest cups in B, and if the filler correctly guesses the emptier's emptying sequence. Note that if a swapping-process succeeds, then the filler is able to swap a cup with fill at least h into A. We have shown that there is a constant probability of a given swapping-process succeeding. Let X_i be the binary random variable indicating whether or not the i-th swapping process where the filler does not perform a storing-operation where at least half of the cups $c \in B$ have $\text{fill}(c) \geq -h$ succeeds. Let $q \geq \Omega(1)$ be the probability of a swapping-process succeeding, i.e. $P(X_i = 1)$. Note that the random variables X_i are clearly independent, and identically distributed.

Clearly

$$\mathbb{E}\left[\sum_{i=1}^{n/8} X_i\right] = qn/8.$$

Note that we do not use all the X_i; we know there must be at least $n/4 - 3/2\gamma$ swapping-processes that do not consist of a storing-operation, but only use $n/8$ of the X_i. We make this choice because the particular constants that we get do not matter, and because it substantially simplifies the analysis. By a Chernoff Bound (i.e. Hoeffding's Inequality applied to binary random variables),

$$P\left(\sum_{i=1}^{n/8} X_i \leq nq/16\right) \leq e^{-nq^2/1024}.$$

That is, the probability that less than $nq/16$ of the anchor cups have fill at least h is exponentially small in n, as desired.

□

Step 2 is easily achieved by setting $p = nc$ and uniformly distributing the fillers fill among a chosen set of nc cups. The greedy nature of the emptier will force it to focus on the cups which must exist in A with large positive fill.

We succeed on the ℓ-th level of recursion with probability at least $1 - e^{-nz^\ell}$ for some constant z. The probability that we succeed on all of $\log n$ levels of recursion is therefore pretty good.

□

Lemma 1 (The Oblivious Amplification Lemma)**.** *Let f be an oblivious filling strategy that achieves backlog $f(n)$ in the variable-processor cup game on n cups with constant probability (relative to average fill, with negative fill allowed). Let $\delta \in (0,1)$ be a parameter. Then, there exists an adaptive filling strategy that, with constant probability, either achieves backlog*

$$f'(n) \geq (1-\delta)\Big(f((1-\delta)n) + f((1-\delta)\delta n)\Big)$$

or achieves backlog $\Omega(poly(n))$ in the variable processor cup game on n cups.

Proof. ☐