A filling strategy that achieves backlog $\Omega(\log n)$ in the p-processor cup game on n cups against an arbitrary emptier

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The cup game is a classic problem in computer science that models work scheduling. In the cup game on n cups, a **filler** and an **emptier** take turns adding and removing water (i.e. new tasks come in and the scheduler must allocate processors to handle the incoming work) from the cups. On each round the filler will distribute some new amount of water among the cups, and the emptier will remove some amount of water from some of the cups. The filler can distribute the water however it wants (as long as it places at most 1 water in each cup), but the emptier has an added "discretization constraint": it can only remove water from some fixed number of cups. The problem is to analyze how well each player can do, that is, how much water can the filler force to be in the fullest cup, and what is the upper bound on this fill that an appropriate emptying strategy can guarantee?

Kuszmaul previously studied a variant of this problem called the "vanilla multiprocessor cup-game" in which the filler distributes p units of water among the cups (with at most 1 going to any particular cup on any particular round), and the emptier choses p cups to remove at most 1 unit of water from. Note that there is no resource augmentation in this variant of the game. Kuszmaul showed that in the p-processor cup game on n cups, a greedy emptying strategy achieves backlog bounded above by $O(\log n)$. He also provided a construction by which the filler could achieve backlog $\Omega(\log(n-p))$. For $p \leq n/2$ the upper-bound and lower-bound have no gap between them (asymptotically). However, for $p > n/2$ these could potentially be different. It was widely thought that the $O(\log n)$ upper-bound could be reduced to $O(\log(n-p))$ for $p > n/2$. We will prove the following lemma, which asserts that (surprisingly!) this is not the case!

Lemma 1. *There exists a filling strategy for the p-processor cup game on n cups that achieves backlog $\Omega(\log n)$ against any emptier.*

Proof. Kuszmaul's construction shows that there is a filling strategy that achieves backlog $\Omega(\log(n-p))$. If $p \leq n - \sqrt{n}$, $n - p \geq \sqrt{n}$, so $\log(n-p) \geq \frac{1}{2}\log(n)$, hence $\Omega(\log(n-p)) = \Omega(\log(n))$. Thus the result holds for $p \leq n - \sqrt{n}$; we proceed to consider the case where $p > n - \sqrt{n}$

Let $H_n = 1/1 + 1/2 + \cdots + 1/n$ denote the n-th harmonic number. We will show that there is a filling strategy such that if $S_t(1) < \frac{1}{2}(H_n - H_{n-p+1})$, then $\text{tot}(S_{t+n}) \geq \text{tot}(S_t) + \frac{1}{2}$. Leveraging this result, the filler can repeatedly increase the total fill by a constant amount. The filer can achieve backlog greater than $\frac{1}{2}(H_n - H_{n-p+1})$ by repeating this strategy $n \cdot (H_n - H_{n-p+1})$ times, with the added strategy that if the backlog ever becomes greater than $\frac{1}{2}(H_n - H_{n-p+1})$ in this process the filler stops, and proceeds to repeatedly put 1 unit of water in the fullest cup, thus guaranteeing backlog $\Omega(\log n)$ [1] forever after. If the backlog is less than $\frac{1}{2}(H_n - H_{n-p+1})$ at every step of this process, then the total fill must have increased by at least $\frac{1}{2}$ per each repetition of the strategy, for a total of $\frac{1}{2}n \cdot (H_n - H_{n-p+1})$ increase. This however implies that at the end (denoted t_f) $\text{av}(S_{t_f}) \geq \frac{1}{2}(H_n - H_{n-p+1})$, so $S_{t_f}(1) \geq \frac{1}{2}(H_n - H_{n-p+1})$, and again we will have achieved backlog $\Omega(\log n)$ by the end of the process.

We proceed to outline the filling strategy that is able to increase the total fill by at least $1/2$ in less than n rounds (if the backlog starts out below $\frac{1}{2}(H_n - H_{n-p+1})$.

Let S denote the set of cups. The filler will maintain a set $U \subset S$ throughout the algorithm. The algorithm's procedure will ensure that once a cup enters U its fill never decreases for the rest of the process (the filler will maintain the fill of these cups by placing 1 unit of water in them each time). Furthermore, $|U|$ will increase by $n-p$ at each itteration of the process. U is initialized to $\emptyset$. For each of $\lfloor (n-2)/(n-p) \rfloor$ steps the filler will

- Distribute $p - |U|$ water equally among the cups

[1] Note that $\frac{1}{2}(H_n - H_{n-p+1}) \geq \Omega(\log n)$ because $n-p < \sqrt{n}$ so $H_{n-p+1} < H_{\lceil \sqrt{n} \rceil + 1} < \frac{1}{2}H_n + 100$, hence $H_n - H_{n-p+1} > H_n - \frac{1}{2}H_n - 100 = \frac{1}{2}H_n - 100 \geq \Omega(\log n)$.

in $S \setminus U$ (thus, each such cup recieves $\frac{p-|U|}{n-|U|}$ fill)

- Distribute $|U|$ water equally among the cups in U (thus, each such cup receives 1 fill)

Then the emptier must chose p cups to empty from, and hence $n-p$ cups to **neglect**. Let N be the set of neglected cups on a given step ($|N| = n-p$). The emptier appends all cups in $N \setminus U$ to U, and then appends the $(n-p) - |N \setminus U|$ fullest cups in $S \setminus U$ to U. Thus the number of elements in U at the start of step i (using 0-indexing) is $(n-p)\cdot i$.

Now, note that the fill of any cup in $S \setminus U$ at the end of round i has increased by $\frac{p-(n-p)\cdot i}{n-(n-p)\cdot i}$, and then decreased by 1 on this round. This is a net change of $-\frac{n-p}{n-(n-p)\cdot i}$.

At the end of these $\lfloor (n-2)/(n-p) \rfloor$ rounds $|U| \leq n-2$, so there are at least 2 cups in $S \setminus U$. **We claim that the cups finally in $S \setminus U$ now have 0 fill.** On the i-th round of the filler's process the fill of these cups in $S \setminus U$ has a net decrease of $\frac{n-p}{n-|U|} = \frac{n-p}{n-(n-p)\cdot i}$. Now consider the total amount that the fill of a cup that is in $S \setminus U$ at the end have decreased since the start of this process:

$$\sum_{i=0}^{\lfloor (n-2)/(n-p) \rfloor - 1} \frac{n-p}{n-(n-p)\cdot i}.$$

For $p = n-1$ this sum is easily reckognizable as a difference of harmonic numbers:

$$\sum_{i=0}^{n-3} \frac{1}{n-i} = \frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{3} = H_n - H_{n-p+1}.$$

We can achieve this as a lower bound for other values of p too by lower bounding a difference of "strided harmonic numbers" with a difference of harmonic numbers. In particular, we can lower-bound the i-th term in the sum by:

$$\frac{n-p}{n-i(n-p)} = \sum_{j=0}^{n-p-1} \frac{1}{n-i(n-p)} \geq \sum_{j=n-i(n-p)}^{n-(i-1)(n-p)-1} \frac{1}{j}.$$

Adding these up we get a lower bound on the amount that the fill decreases in these cups:

$$\sum_{i=0}^{\left\lfloor \frac{n-2}{n-p} \right\rfloor - 1} \sum_{\substack{j=n-\\(i-1)(n-p)-1}}^{n-i(n-p)} \frac{1}{j}.$$

This is now a difference of harmonic numbers. In particular, when $i=0, j = n-(i-1)(n-p)-1$ we get the smallest term in the sum $1/(n+(n-p-1))$, and when $i = \lfloor (n-2)/(n-p) \rfloor - 1, j = n - i(n-p)$ we get the largest term in the sum. This largest term is greater than or equal to $n - ((n-2)/(n-p) - 1)(n-p) = n - (n-2-(n-p)) = n-p+2$ Thus, the loss is lower bounded by $H_{n+(n-p-1)} - H_{n-p+1} \geq H_n - H_{n-p+1}$.

Because the backlog started less than $\frac{1}{2}(H_n - H_{n-p+1})$ by assumption, these cups must now have 0 fill. Hence we have 2 cups with 0 fill, so the final step of the filler's procedure is to add 1/2 fill to these 2 cups, then the total fill increases by 1/2 by necessity, as the emptier must "waste" a unit of fill on emptying one of these. That is, the amount of fill removed by the emptier is at most $p - 1/2$, which is less than the p fill that the filler added this round. Note that total fill is monotonically increasing, a simple consequence of the discretization constraint (p fill is added to the system each round, and no more than p is emptier), so the fact that total fill increases on this round implies that it increased over the whole process by at least $\frac{1}{2}$, as desired. $\square$