# 1 Oblivious Filler Lower Bound

In this section we prove that, surprisingly, an oblivious filler can achieve backlog $n^{1-\varepsilon}$, although only against a certain class of "greedy-like" emptiers.

We call a cup configuration $M$***-flat*** if every cup has fill in $[-M, M]$. We say an emptier is $\Delta$***-greedy-like*** if, whenever there are two cups with fills that differ by at least $\Delta$, the emptier never empties from the less full cup without also emptying from the more full cup. That is, if there are cups $c_1, c_2$ with $\text{fill}(c_1) > \text{fill}(c_2) + \Delta$, then a $\Delta$-greedy-like emptier doesn't empty from $c_2$ on this round unless it also empties from $c_1$ on this round. Note that a perfectly greedy emptier is 0-greedy-like. We call an emptier ***greedy-like*** if it is $\Delta$-greedy-like for $\Delta \leq O(1)$. In the randomized setting we are only able to prove lower bounds for backlog against greedy-like emptiers; whether or not our results can be extended to a more general class of emptiers is an interesting open question. Nonetheless, greedy-like emptiers are of great interest because all the known randomized algorithm for the cup game are $O(1)$-greedy-like [1, 4].

As a tool in our analysis we define a new variant of the cup game. In the $(p, E, T)$-***extra-emptying*** negative-fill cup game on $n$ cups the filler distributes $p$ units of water amongst the cups, and then the emptier empties from $p$ *or more* (up to $n$) cups. In particular, the game lasts for $T$ rounds, and among these rounds the emptier is allowed to empty $E$ extra cups, i.e. the sum of the number of extra cups that that the emptier empties can be as large as $E$.

We now prove a crucial property of greedy-like emptiers:

**Proposition 1.** *Consider an $M$-flat cup configuration in the $(p, E, T)$-extra-emptying negative-fill cup game on $n = 2p$ cups with average fill 0 with $T = 2(M + E)$. An oblivious filler can achieve a $2(2+\Delta)$-flat configuration of cups against a $\Delta$-greedy-like emptier in running time $T$.*

*Proof.* The filler's strategy is to distribute fill equally amongst all cups at every round, placing 1/2 fill in each cup. Let $\ell_t = \min_{c \in S_t} \text{fill}_{S_t}(c)$, $u_t = \max_{c \in S_t} \text{fill}_{S_t}(c)$. Let $L_t$ be the set of cups $c$ with $\text{fill}_{S_t}(c) \leq l_t + 2 + \Delta$, and let $U_t$ be the set of cups $c$ with $\text{fill}_{S_t}(c) \geq u_t - 2 - \Delta$.

Note the following regarding $U_t$ (symmetric properties hold for $L_t$):

**Observation 1**: If any cup with fill in $[u_t - \Delta - 2, u_t - \Delta - 1]$ is emptied from then all cups with fills in $[u_t - 1, u_t]$ must be emptied from because the emptier is $\Delta$-greedy-like.

**Observation 2**: On any round where the emptier doesn't use extra resources, if there are at least $n - p = n/2$ cups outside of $U_t$, that is cups with fills in $[\ell_t, u_t - 2 - \Delta]$, then all cups in $[u_t - 2, u_t]$ must be emptied from because the emptier is $\Delta$-greedy-like.

Now we prove a key property of the sets $U_t$ and $L_t$: once a cup is in $U_t$ or $L_t$ it is always in $U_{t'}, L_{t'}$ for all $t' > t$. This follows immediately from the following claim:

**Claim 1.**

$$U_t \subseteq U_{t+1}, \quad L_t \subseteq L_{t+1}.$$

*Proof.* Consider a cup $c \in U_t$.

If $c$ is not emptied from, i.e. $\text{fill}(c)$ has increased by 1/2, then clearly $c \in U_{t+1}$, because backlog has increased by at most 1/2, so the fill of $c$ must still be within $2 + \Delta$ of the backlog on round $t + 1$.

On the other hand, if $c$ is emptied from, i.e. $\text{fill}(c)$ has decreased by 1/2, we consider two cases.

**Case 1:** If $\text{fill}_{S_t}(c) \geq u_t - \Delta - 1$, then $\text{fill}_{S_t}(c)$ is at least 1 above the bottom of the interval defining which cups belong to $U_t$. The backlog increases by at most 1/2 and the fill of $c$ decreases by 1/2, so $\text{fill}_{S_{t+1}}(c)$ is at least $1 - 1/2 - 1/2 = 0$ above the bottom of the interval, i.e. still in the interval.

**Case 2:** On the other hand, if $\text{fill}_{S_t}(c) < u_t - \Delta - 1$, then every cup with fill in $[u_t - 1, u_t]$ must have been emptied from by Observation 1. The fullest cup at round $t + 1$ is the same as the fullest cup on round $t$, because the fills of all cups with fill in $[u_t - 1, u_t]$ have decreased by 1/2, and no cup with fill less than $u_t - 1$ had fill increase by more than 1/2. Hence $u_{t+1} = u_t - 1/2$. Because both the fill of $c$ and the backlog have both decreased by 1/2, the distance between them is still at most $\Delta + 2$, hence $c \in U_{t+1}$.

The argument for why $L_t \subseteq L_{t+1}$ is symmetric. $\square$

Now that we have shown that $L_t$ and $U_t$ never lose cups, we will show that they each eventually gain more than $n/2$ cups.

**Claim 2.** *On a round where the emptier doesn't use extra resources, if $|U_t| \leq n/2$ we have $u_{t+1} = u_t - 1/2$, and if $|L_t| \leq n/2$ we have $\ell_{t+1} = \ell_t + 1/2$.*

*On any round, even a round where the emptier does use extra resources, we have $u_{t+1} \leq u_t + 1/2$ and $\ell_{t+1} \geq \ell_t - 1/2$.*

*Proof.* If the emptier does not use extra resources and there are more than $n/2$ cups outside of $U_t$ then by Observation 2 the emptier must empty from every cup with fill at least $u_t - 2$. Thus $u_{t+1} = u_t - 1/2$: no cup with fill less than $u_t - 2$ could have become the fullest cup, and the previous fullest cup has lost $1/2$ units of fill.

The proof is symmetric for $L_t$.

Regardless of if the emptier uses extra resources or not no cup changes in fill by more than $1/2$, implying $u_{t+1} \leq u_t + 1/2$ and $\ell_{t+1} \geq \ell_t - 1/2$. □

We call a round where the emptier uses extra resources an ***emptier-extra-resource*** round. At most $E$ of the $2(M+E)$ total rounds are emptier-extra-resource rounds. When the emptier uses extra resources it can potentially hurt the filler's efforts to achieve a flat configuration of cups. However, the affect of emptier-extra-resource rounds is countered by rounds where the emptier does not use extra resources. In particular, we now define what it means for a non-emptier-extra-resource round $j$ to cancel an emptier-extra-resource round $i < j$. For $i = 1, 2, \ldots, 2(M+E)$, if round $i$ is an emptier-extra-resource round then the first non-emptier-extra-resource round $j > i$ that has not already cancelled some emptier-extra-resource round $i' < i$ in this sequential labelling process, provided such a round exists, is said to ***cancel*** round $i$. Each emptier-extra-resource round is cancelled by at most one later round, some emptier-extra-resource rounds may not be cancelled at all.

Consider rounds of the form $2M + i$ for $i \in [2E+1] - 1$. We claim there is some $i$ such that there are $2M$ non-emptier-extra-resource rounds among rounds $[2M+i]$ that are not cancelling other rounds. Assume for contradiction that this is not so. Then every non-emptier-extra-resource round $2M+i$ is necessarily a cancelling round. Hence by round $2(M+E)$, there must have been $E$ cancelled tasks, so on round $2(M+E)$ all emptier-extra-resource rounds are cancelled.

Let $t_e$ be some round by which there are $2M$ non-emptier-extra-resource, non-cancelling rounds. The value of $u_t$ cannot have shrunk by more than $M$ because the configuration started $M$-flat. Hence by Claim 2 there is some round $t_u \in [t_e]$ such that $|U_t| \geq n/2$. Identically, there is some round $t_\ell \in [t_e]$ such that $|L_t| \geq n/2$. Since by Claim 1 $|U_{t+1}| \geq |U_t|$ and $|L_{t+1}| \geq |L_t|$, we have that there is some round $t_0 = \max(t_u, t_\ell)$ on which both $|U_{t_0}|$ and $|L_{t_0}|$ exceed $n/2$. Then $U_{t_0} \cap L_{t_0} \neq \varnothing$. Furthermore, the sets must intersect for all $t_0 \leq t \leq [2(M+E)]$. In order for the sets to intersect it must be that the intervals $[u_t - 2 - \Delta, u_t]$ and $[\ell_t, \ell_t + 2 + \Delta]$ intersect. Hence we have that

$$\ell_t + 2 + \Delta \geq u_t - 2 - \Delta.$$

Since $u_t \geq 0$ and $\ell_t \leq 0$ this implies that all cups have fill in $[-2(2+\Delta), 2(2+\Delta)]$.

□

Given a $\Delta$-greedy-like filler, let $R_\Delta = 2(2+\Delta)$. By Proposition 1, an oblivious filler can achieve a $R_\Delta$-flat configuration of cups from an $M$ flat configuration of cups against a $\Delta$-greedy-like emptier in the $(p, E, T)$-extra-emptying negative-fill cup game on $n$ cups where $n = p/2$, $T = 2(M+E)$.

Next we describe a simple oblivious filling strategy that will be used as a subroutine in Lemma 1; this strategy is very well-known, and similar versions of it can be found in [1, 2, 3, 4].

**Proposition 2.** *Consider an $R$-flat cup configuration in the negative-fill single-processor cup game on $n$ cups with average fill $0$. There is an oblivious filling strategy that achieves fill at least $-R + \sum_{i=2}^{n} 1/i$ in a randomly chosen cup with probability at least $1/n!$. Further, this filling strategy guarantees that the chosen cup has fill at most $R + \sum_{i=2}^{n} 1/i$. This filling strategy has running time $n-1$.*

*Proof.* The filler maintains an ***active set***, initialized to being all of the cups. Every round the filler distributes 1 unit of fill equally among all cups in the active set. Then the emptier removes 1 unit of fill from some cup. Finally, the filler removes a cup uniformly at random from the active set. This continues until a single cup $c$ remains in the active set. Consider the probability that $c$ has never been emptied from. On the $i$-th step of this process, i.e. when the size of the active set is $n-i+1$, consider the cup the emptier empties from. If this cup is in the active-set, with probability at least $1/(n-i+1)$ the filler removes it from the active set. If the cup is not in the active set, then it is irrelevant. Hence with probability at least $1/n!$ the final cup in the active set, $c$, has never been emptied from. In this case, $c$ will have gained fill $\sum_{i=2}^{n} 1/i$ as claimed. Because $c$ started with fill at least $-R$, $c$ now has fill at least $-R + \sum_{i=2}^{n} 1/i$.

Further, $c$ has fill at most $R + \sum_{i=2}^{n} 1/i$, as $c$ started with at most $R$ fill and $c$ gains at most $1/(n-i+1)$ fill on the $i$-th round of this process. □

Now we are equipped to prove Lemma 1, which shows that we can force a constant fraction of the cups to have high fill; using Lemma 1 and exploiting the greedy-like nature of the emptier we can get a known cup with high fill (we show this in Lemma 2).

**Lemma 1.** *Let $\Delta \leq O(1)$, let $h \leq O(1)$ with $h \geq 16+16\Delta$, let $n$ be at least a sufficiently large constant determined by $h$ and $\Delta$, and let $M \leq poly(n)$.*

*Consider an $M$-flat cup configuration in the negative-fill variable-processor cup game on $n$ cups with average fill 0. Let $A$ be a chosen subset of $n/32$ cups, and let $B$ consist of the rest of the cups ($|B| = n \cdot 31/32$).*

*There is an oblivious filling strategy that makes an unknown set of $\Theta(n)$ cups in $A$ have fill at least $h$ with probability at least $1 - 2^{-\Omega(n)}$ in running time poly($n$) against a $\Delta$-greedy-like emptier. The filling strategy also guarantees that $\mu(B) \geq -h/2$.*

*Proof.* We refer to $A$ as the ***anchor*** set, and $B$ as the ***non-anchor*** set. Throughout the proof the filler uses $p = |A| + 1$.

We denote by *randalg-1* the oblivious filling strategy given by Proposition 2. We denote by *flatalg* the oblivious filling strategy given by Proposition 1. We say that the filler ***applies*** a filling strategy *alg* to a set of cups $D \subset B$ if the filler uses *alg* on $D$ while placing 1 unit of fill in each anchor cup.

We now describe the filler's strategy.

The filler starts by flattening the cups, i.e. using *flatalg* on $A \cup B$ for $2M$ rounds. After this, the filling strategy always places 1 unit of water in each anchor cup. The filler performs a series of $|A|$ ***swapping-processes***, one per anchor cup, which are procedures that the filler uses to get a new cup—which will sometimes have high fill—in $A$. On each swapping-process the filler applies *randalg-1* many times to arbitrarily chosen constant-size sets $D \subset B$ with $|D| = \lceil e^{2h+1} \rceil$. The number of times that the filler applies *randalg-1* is chosen at the start of the swapping-process, chosen uniformly at random from $[m]$ ($m = \Theta(n)$ to be specified). At the end of the swapping-process the filler swaps the cup given by *randalg-1* with the cup in the anchor set associated with this swapping-process. Before each application of *randalg-1* the filler flattens $B$ by applying *flatalg* to $B$ for $r$ rounds ($r = \Theta(n)$ to be specified).

We remark that this construction is similar to the construction in **??**, but has a major difference: in the adaptive lower bound construction the filler halts after achieving the desired average fill in the anchor set, whereas the oblivious filler cannot halt but rather must rely on flattening to guarantee that each application of *randalg-1* has constant probability of generating a cup with high fill.

We proceed to analyze the algorithm.

First note that the initial flattening of $A \cup B$ succeeds by Proposition 1. In particular, the game is the $(n/2, 0, 2M)$-extra-emptying cup game, where $E = 0$ so the emptier is actually not allowed to ever do extra emptying and this is equivalent to the $(n/2)$-processor cup game that lasts for $2M$ rounds, and hence the filler can achieve this by setting $p = n/2$.

We say that the emptier ***neglects*** the anchor set on a round if it does not empty from each anchor cup. We say that an application of *randalg-1* to $D \subset B$ is ***successful*** if the emptier does not neglect the anchor set during any round of the application. We define $d = \sum_{i=2}^{|D|} 1/i$ (recall that $|D| = \lceil e^{2h+1} \rceil$). Note that by Proposition 2 on any successful application of *randalg-1* where $B$ started $R_\Delta$-flat there is a $1/|D|!$ chance of getting a cup with fill at least $\mu(B) - R_\Delta + d$.

Now we prove an important claim:

**Claim 3.** *Fix any swapping-process $i \in [|A|]$. There exists $r = \Theta(n)$ such that on the $i$-th swapping-process every flattening makes the cups $R_\Delta$-flat, and such that the following bounds hold for any cups $a \in A, b \in B$ where fill is measured after any application of randalg-1, and where $\mu_0(B)$ refers to the average fill of $B$ prior to the application of randalg-1:*

$$-R_\Delta - |D| \leq \text{fill}(a) - \mu_0(B) \leq R_\Delta + 1 + d + \Delta \tag{1}$$

$$-R_\Delta - |D| \leq \text{fill}(b) - \mu_0(B) \leq R_\Delta + d. \tag{2}$$

*Proof.* We prove Claim 3 by induction on $i$. Note that the statements we aim to prove are interconnected: having the bounds (1), (2) on the fills of cups in $A, B$ makes it possible for the flattening to succeed, which in turn leads to the bounds (1) (2) being maintained.

For $i = 1$, because $A \cup B$ starts $R_\Delta$-flat, the bounds on the fill of cups in $A$ and $B$ (1) (2) trivially hold.

Now we assume (1) and (2), and show that the flattening can be made to succeed by large enough choice of $r$. Consider how much $\mu(B)$ can change by over the course of an application of *randalg-1* to an $R_\Delta$-flat cup configuration. If the emptier neglected every anchor cup on every round of the application of *randalg-1*, then the average fill of $B$ would have decreased by $|A||D|/|B| = |D|/31$. The mass of $B$ cannot have increased during the application of *randalg-1*. Hence the bounds (1) (2) can be rewritten in terms of $\mu(B)$.

Now note that during a flattening process the backlog in $B$ cannot increase until the cups are $R_\Delta$-flat around their current average. Now we note that the number of times that the emptier can neglect the anchor set during a flattening process is limited by virtue of the emptier being $\Delta$-greedy-like. In particular, the emptier can't neglect any anchor cup more than $2R_\Delta + |D| + 1 + d + 2\Delta$ times, because a cup with this much fill has fill more than $\Delta$ larger than the backlog in $B$. We set $r = |A|(2R_\Delta + |D| + 1 + d + 2\Delta)$. Thus the application of *flatalg* can be seen as an instance of the $(n/2, r, 2(M+r))$ where $M$ is a bound on the flatness of the cups guaranteed by the bounds (1) (2), and thus by Proposition 1 the flattening process successfully makes $\mu(B)$ $R_\Delta$-flat around its average.

Now we show that an application of *randalg-1* to a $R_\Delta$-flat set of cups guarantees the bounds (1) (2). This is clearly true by Proposition 2 and the emptier's $\Delta$-greedy-like nature.

$\square$

**Claim 4.** *There exists $m = \Theta(n)$ such that with probability at least $1/2$ the emptier does not neglect the anchor set on the final application of randalg-1 on the $i$-th swapping-process.*

*Proof.* This follows immediately from the arguments made in the previous proof. $\square$

**Claim 5.** *Throughout the entire process*

$$\mu(B) \geq -h/2.$$

*Proof.* At the start $\mu(B)$ is at least $-R_\Delta|A|/|B|$ due to flattening of all of the cups.

There are two ways that $\mu(B)$ can decrease: $\mu(B)$ decreases when the emptier neglects the anchor-set, and when a good swap occurs.

Because the emptier is greedy-like, the emptier must empty from a cup $c \in A$ if the fill of $c$ is more than $\Delta$ greater than the most full cup in $B$. Within each swapping-process no cup in $B$ ever is raised to have fill above $\mu(B) + R_\Delta + d$ by Proposition 2 (and because $B$ is flattened before the filler applies *randalg-1* to $B$). Thus, no cup in $A$ has fill greater than $\mu(B) + d + R_\Delta + \Delta$ must be emptied from. If $\mu(A) \leq \mu(B) + d + R_\Delta + \Delta$ then

$$\mu(B) = -\frac{|A|}{|B|}\mu(A) \geq -\frac{|A|}{|B|}\left(\mu(B) + d + R_\Delta + \Delta\right).$$

Rearranging this we have

$$\mu(B) \geq -\frac{1}{30}(d + R_\Delta + \Delta) \geq -h/4.$$

$\square$

**Claim 6.** *With probability at least $1 - 2^{-\Omega(n)}$, the filler achieves fill at least $h$ in at least $\Theta(n)$ of the cups in $A$.*

*Proof.* By Proposition 2 using *randalg-1* on $|D| = \lceil e^{2h+1} \rceil$ gives the filler a cup that has fill at least $\mu(B) + d - R_\Delta$ with probability $1/|D|!$. By Claim 5 we have that $\mu(B) \geq -h/2$, so with probability $1/|D|!$ this generated cup has fill at least $h$ if it wasn't neglected.

By Claim 3 there is a choice of $c_\Delta$ large enough that the probability of this cup having been neglected is at most $1/2$. In particular, we choose $c_\Delta = 4r|D|!$. By applying *randalg-1* $|A| \cdot c_\Delta$ times we have by a Chernoff bound that with exponentially good probability in $|A| \cdot c_\Delta = \Theta(n)$ there are at least $2|A|r$ applications where the filler would succeed if the emptier doesn't neglect the anchor set. As shown, the emptier cannot neglect the anchor set more than $|A|r$ times. Hence, there is at least a $(1/2)/|D|!$ chance that on the $j$-th application of *randalg-1* the emptier doesn't neglect the anchor set and the filler correctly guesses the emptier's emptying sequence. Thus, overall, there is at least a constant probability of achieving fill $h$ in a cup in $A$.

Say that a swapping-process is ***victorious*** if the filler is able to swap a cup with fill at least $h$ into $A$. The events that swapping-processes are victorious are independent events; each happens with constant probability. Hence by a Chernoff bound with exponentially good probability in $n$ at least a constant fraction of them succeed, as desired.

$\square$

We now briefly analyze the running time of the filling strategy. There are $|A|$ swapping-processes. Each swapping-process consists of $|A| \cdot c_\Delta$ applications of *randalg-1*, and the flattening procedure before each application. Clearly this all takes poly$(n)$ time, as desired.

$\square$

Finally, using Lemma 1 we can show in Lemma 2 that an oblivious filler can achieve constant backlog. We remark that Lemma 2 plays a similar role in the proof of the lower bound on backlog as **??** does in the adaptive case, but is vastly more complicated to prove (in particular, **??** is trivial).

**Lemma 2.** *Let $H \leq O(1)$, let $\Delta \leq O(1)$, let $n$ be at least a sufficiently large constant determined by $H$ and $\Delta$, and let $M \leq poly(n)$. Consider an $M$-flat cup configuration in the negative-fill variable-processor cup game on $n$ cups with average fill $0$. Given this configuration, an oblivious filler can achieve fill $H$ in a chosen cup in running time poly$(n)$ against a $\Delta$-greedy-like emptier with probability at least $1-2^{-\Omega(n)}$.*

*Proof.* The filler starts by performing the procedure detailed in 1, using $h = H \cdot 16(1+\Delta)$. Let the number of cups which must now exist with fill $h$ be of size $nc = \Theta(n)$.

The filler reduces the number of processors to $p = nc$. Now the filler exploits the filler's greedy-like nature to to get fill $H$ in a set $S \subset B$ of $nc$ chosen cups.

The filler places 1 unit of fill into each cup in $S$. Because the emptier is greedy-like it must focus on the $nc$ cups in $A$ with fill at least $h$ until the cups in $S$

have sufficiently high fill. In particular, $(5/8)h$ rounds suffice. Over $(5/8)h$ rounds the $nc$ high cups in $A$ cannot have their fill decrease below $(3/8)h \geq h/8 + \Delta$. Hence, any cups with fills less than $h/8$ must not be emptied from during these rounds. The fills of the cups in $S$ must start as at least $-h/2$. After $(5/8)h$ rounds the fills of the cups in $S$ are at least $h/8$, because throughout this process the emptier cannot have emptied from them until they got fill at least $h/8$, and if they are never emptied from then they achieve fill $h/8$.

Thus the filling strategy achieves backlog $h/8 \geq H$ in some known cup (in fact in all cups in $S$, but a single cup suffices), as desired.

□

# References


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