TODO: emptier turn skipping RIP TODO: integer stuff

1 Oblivious Filler Lower Bound

In this section we prove that, surprisingly, an oblivious filler can achieve backlog $n^{1-\varepsilon}$, although only against a certain class of "greedy-like" emptiers.

We call a cup configuration ***M-flat*** if the difference between the fill of the fullest cup and the fill of the least full is at most M; note that in an M-flat cup configuration with average fill 0 all cups have fills in $[-M, M]$. We say an emptier is ***Δ-greedy-like*** if, whenever there are two cups with fills that differ by at least Δ, the emptier never empties from the less full cup without also emptying from the more full cup. That is, if on some round t, there are cups c_1, c_2 with $\text{fill}_{I_t}(c_1) > \text{fill}_{I_t}(c_2) + \Delta$, then a Δ-greedy-like emptier doesn't empty from c_2 on round t unless it also empties from c_1 on round t. Note that a perfectly greedy emptier is 0-greedy-like. We call an emptier ***greedy-like*** if it is Δ-greedy-like for $\Delta \le O(1)$.

With an oblivious filler we are only able to prove lower bounds on backlog against greedy-like emptiers; whether or not our results can be extended to a more general class of emptiers is an interesting open question. Nonetheless, greedy-like emptiers are of great interest because all the known randomized algorithm for the cup game are greedy-like [1, 4].

As a tool in our analysis we define a new variant of the cup game: In the p-processor ***E-extra-empties*** negative-fill cup game on n cups, the filler distributes p units of water amongst the cups, and then the emptier empties from p *or more* cups. In particular the emptier is allowed to empty E extra cups over the course of the game. Note that the emptier still cannot empty from the same cup twice on a single round.

We now prove a crucial property of greedy-like emptiers:

Lemma 1. *Let $R_\Delta = 2(2+\Delta)$. Consider an M-flat cup configuration in the p-processor E-extra-empties negative-fill cup game on $n = 2p$ cups. An oblivious filler can achieve a R_Δ-flat configuration of cups against a Δ-greedy-like emptier in running time $2(M+E)$. Furthermore, throughout this process the cup configuration is M-flat on every round.*

Proof. If $M \le R_\Delta$ the algorithm does nothing, since the desired flatness is already achieved; for the rest of the proof we consider $M > R_\Delta$.

The filler's strategy is to distribute fill equally amongst all cups at every round, placing $p/n = 1/2$ fill in each cup.

Let $\ell_t = \min_{c\in S_t} \text{fill}_{S_t}(c)$, $u_t = \max_{c\in S_t} \text{fill}_{S_t}(c)$. Let L_t be the set of cups c with $\text{fill}_{S_t}(c) \le l_t + 2 + \Delta$, and let U_t be the set of cups c with $\text{fill}_{S_t}(c) \ge u_t - 2 - \Delta$.

Now we prove a key property of the sets U_t and L_t: once a cup is in U_t or L_t it is always in $U_{t'}, L_{t'}$ for all $t' > t$. This follows immediately from Claim 1.

Claim 1.

$$U_t \subseteq U_{t+1}, \quad L_t \subseteq L_{t+1}.$$

Proof. Consider a cup $c \in U_t$.

If c is not emptied from, i.e. $\text{fill}(c)$ has increased by $1/2$ from the previous round, then clearly $c \in U_{t+1}$, because backlog has increased by at most $1/2$, so $\text{fill}(c)$ must still be within $2+\Delta$ of the backlog on round $t+1$.

On the other hand, if c is emptied from, i.e. $\text{fill}(c)$ has decreased by $1/2$, we consider two cases.

Case 1: If $\text{fill}_{S_t}(c) \ge u_t - \Delta - 1$, then $\text{fill}_{S_t}(c)$ is at least 1 above the bottom of the interval defining which cups belong to U_t. The backlog increases by at most $1/2$ and the fill of c decreases by $1/2$, so $\text{fill}_{S_{t+1}}(c)$ is at least $1 - 1/2 - 1/2 = 0$ above the bottom of the interval, i.e. still in the interval.

Case 2: On the other hand, if $\text{fill}_{S_t}(c) < u_t - \Delta - 1$, then every cup with fill in $[u_t - 1, u_t]$ must have been emptied from because the emptier is Δ-greedy-like. Therefore the fullest cup on round $t+1$ is the same as the fullest cup on round t, because every cup with fill in $[u_t - 1, u_t]$ has had its fill decrease by $1/2$, and no cup with fill less than $u_t - 1$ had its fill increase by more than $1/2$. Hence $u_{t+1} = u_t - 1/2$. Because both $\text{fill}(c)$ and the backlog have decreased by $1/2$, the distance between them is still at most $\Delta + 2$, hence $c \in U_{t+1}$.

The argument for why $L_t \subseteq L_{t+1}$ is symmetric. □

Now that we have shown that L_t and U_t never lose cups, we will show that they each eventually gain more than $n/2$ cups.

Claim 2. *On any round t, if $|U_t| \le n/2$ then $u_{t+1} = u_t - 1/2$. On any round t where the emptier doesn't use extra resources, if $|L_t| \le n/2$ then $\ell_{t+1} = \ell_t + 1/2$.*

Proof. If there are at least $n/2$ cups outside of U_t, i.e. cups with fills in $[\ell_t, u_t - 2 - \Delta]$, then all cups with fills in $[u_t - 2, u_t]$ must be emptied from; if one such cup was not emptied from then by the pigeon-hole principle some cup outside of U_t was emptied from, which is impossible as the emptier is Δ-greedy-like. This clearly implies that $u_{t+1} = u_t - 1/2$: no cup with fill less than $u_t - 2$ has gained enough fill to become the fullest cup, and the fullest cup from the previous rounds has lost $1/2$ units of fill.

Now consider a round where the emptier doesn't use extra resources and where $|L_t| \leq n/2$. Clearly no cup with fill in $[\ell_t, \ell_t + 2]$ can be emptied from; if one such cup were emptied from, then by the pigeonhole principle some cup outside of L_t was not emptied from, which is impossible as the emptier is Δ-greedy-like. Hence we have $\ell_{t+1} = \ell_t + 1/2$.

We remark however that we cannot guarantee that $\ell_{t+1} = \ell_t + 1/2$ on all rounds where $|L_t| \leq n/2$, because the emptier could do extra emptying; in the most extreme case the emptier could empty from every cup in which case we would have $\ell_{t+1} = \ell_t - 1/2$. □

We call a round where the emptier uses extra resources an ***emptier-extra-resource*** round. At most E of the $2(M + E)$ total rounds are emptier-extra-resource rounds. When the emptier uses extra resources it can potentially hurt the filler's efforts to achieve a flat configuration of cups, in particular by making $\ell_{t+1} < \ell_t$. However, the affect of emptier-extra-resource rounds is countered by rounds where the emptier does not use extra resources. In particular, we now define what it means for a non-emptier-extra-resource round j to cancel an emptier-extra-resource round $i < j$. For $i = 1, 2, \dots, 2(M + E)$, if round i is an emptier-extra-resource round then the first non-emptier-extra-resource round $j > i$ that has not already cancelled some emptier-extra-resource round $i' < i$ in this sequential labelling process, provided such a round exists, is said to ***cancel*** round i. Each emptier-extra-resource round is cancelled by at most one later round, some emptier-extra-resource rounds may not be cancelled at all.

Consider rounds of the form $2M + i$ for $i \in [2E + 1] - 1$. We claim there is some i such that there are $2M$ non-emptier-extra-resource rounds among rounds $[2M + i]$ that are not cancelling other rounds. Assume for contradiction that this is not so. Then every non-emptier-extra-resource round $2M + i$ is necessarily a cancelling round. Hence by round $2(M + E)$, there must have been E cancelled tasks, so on round $2(M + E)$ all emptier-extra-resource rounds are cancelled.

Let t_e be some round by which there are $2M$ non-emptier-extra-resource, non-cancelling rounds. The value of u_t cannot have shrunk by more than M because the configuration started M-flat. Hence by Claim 2 there is some round $t_u \in [t_e]$ such that $|U_t| \geq n/2$. Identically, there is some round $t_\ell \in [t_e]$ such that $|L_t| \geq n/2$. Since by Claim 1 $|U_{t+1}| \geq |U_t|$ and $|L_{t+1}| \geq |L_t|$, we have that there is some round $t_0 = \max(t_u, t_\ell)$ on which both $|U_{t_0}|$ and $|L_{t_0}|$ exceed $n/2$. Then $U_{t_0} \cap L_{t_0} \neq \varnothing$. Furthermore, the sets must intersect for all $t_0 \leq t \leq [2(M + E)]$. In order for the sets to intersect it must be that the intervals $[u_t - 2 - \Delta, u_t]$ and $[\ell_t, \ell_t + 2 + \Delta]$ intersect. Hence we have that

$$\ell_t + 2 + \Delta \geq u_t - 2 - \Delta.$$

Or, rearranging,

$$u_t - \ell_t \leq 2(2 + \Delta) = R_\Delta.$$

Thus the cup configuration is R_Δ-flat.

Now we establish that throughout this flattening process the cup configuration is always M-flat. Consider a round where $u_{t+1} - \ell_{t+1} > u_t - \ell_t$. For this to happen the emptier must have used less than $n - p$ extra emptying, or else the fill of every cup would simply decrease by $1/2$ which wouldn't affect the difference $u_{t+1} - \ell_{t+1}$. In order for the difference $u_t - \ell_t$ to increase either a) some cup with fill in $[\ell_t, \ell_t + 1/2]$ was emptied from and some cup with fill in $[u_t - 1, u_t]$ was not emptied from, or b) some cup with fill in $[u_t - 1/2, u_t]$ was not emptied from and some cup with fill in $[\ell_t, \ell_t + 1]$ was emptied from. In either case, because the emptier is Δ-greedy-like, such an action implies

$$u_{t+1} - \ell_{t+1} \leq u_t + 1/2 - (\ell_t - 1/2) \leq \Delta + 5/2 \leq M.$$

Since the cup configuration starts M-flat, and after any round where the distance $u_t - \ell_t$ increases it increases to a value at most M, we have that the cups are always M-flat. □

Next we describe a simple oblivious filling strategy that will be used as a subroutine in Lemma 2; this strategy is very well-known, and similar versions of it can be found in [1, 2, 3, 4].

Proposition 1. *Consider an R-flat cup configuration in the negative-fill single-processor cup game on n cups with average fill 0. Let $d = \sum_{i=2}^n 1/i$.*

There is an oblivious filling strategy that achieves fill at least $-R + d$ in a randomly chosen cup with probability at least $1/n!$. This filling strategy guarantees that the chosen cup has fill at most $R + d$, and has running time $n - 1$. Further, when applied against a Δ-greedy-like emptier with $R = R_\Delta$, this filling strategy guarantees that the cups always remain $(R + d)$-flat.

Proof. The filler maintains an ***active set***, initialized to being all of the cups. Every round the filler distributes 1 unit of fill equally among all cups in the active set. Then the emptier removes 1 unit of fill from some cup. Finally, the filler removes a cup uniformly at random from the active set. This continues until a single cup c remains in the active set.

We now bound the probability that c has never been emptied from. On the i-th step of this process, i.e. when the size of the active set is $n-i+1$, consider the cup c' that the emptier empties. If c' is in the active-set, then with probability at least $1/(n-i+1)$ the filler removes it from the active set. If c' is not in the active set, then it is irrelevant. Hence with probability at least $1/n!$ the final cup in the active set, c, has never been emptied from. In this case, c will have gained fill $d = \sum_{i=2}^{n} 1/i$ as claimed. Because c started with fill at least $-R$, c now has fill at least $-R+d$.

Further, c has fill at most $R+d$, as c starts with fill at most R, and c gains at most $1/(n-i+1)$ fill on the i-th round of this process.

Now we analyze this algorithm specifically for a Δ-greedy-like emptier. Consider a round on which the minimum fill of the cups becomes lower, i.e. where the emptier empties from some cup c with fill less than 1 above the lowest fill. On such a round, because the emptier is Δ-greedy-like, the backlog is no more than Δ greater than fill(c). Hence the cups are $(\Delta+1)$-flat on such a round. If the cups are always $(\Delta+1)$-flat then we are done. Otherwise, consider the round after the last round on which the cups are $(\Delta+1)$-flat, (or the first round of the process if the cups are never $(\Delta+1)$-flat). From this round on, the emptier cannot empty a cup with fill within 1 of the fill of the lowest cup, hence the fill of the lowest cup cannot decrease. Consider how much the backlog could increase. The backlog could increase by d from whatever value it starts at. The backlog starts as at most $\max(\Delta+1, R) = R$, and hence throughout the process the cups remain $(R+d)$-flat, as desired.

□

Now we are equipped to prove Lemma 2, which shows that we can force a constant fraction of the cups to have high fill; using Lemma 2 and exploiting the greedy-like nature of the emptier we can get a known cup with high fill (we show this in Lemma 3).

Lemma 2. *Let $\Delta \leq O(1)$, let $h \leq O(1)$ with $h \geq 16+16\Delta$, let n be at least a sufficiently large constant determined by h and Δ, and let $M \leq poly(n)$. Consider an M-flat cup configuration in the negative-fill variable-processor cup game on n cups with average fill 0. Let A, B, A' be disjoint constant-fraction-size subsets of the cups with $|A| = \Theta(n)$ sufficiently small and with $|A|+|B|+|A'| = n$. These sets will change over the course of the filler's strategy, but $|A|$ will remain fixed and $|A| \ll |A|$ will always hold.*

There is an oblivious filling strategy that makes an unknown set of $\Theta(n)$ cups in A have fill at least h with probability at least $1-2^{-\Omega(n)}$ in running time poly(n) against a Δ-greedy-like emptier. The filling strategy also guarantees that $\mu(B) \geq -h/2$.

Proof. We refer to A as the ***anchor*** set, B as the ***non-anchor*** set, and A' as the ***garbage*** set. Throughout the proof the filler uses $p = |A|+1$. The filler initializes the sets as $A' = \varnothing$, and B is all the cups besides the cups in A. The set A is chosen to satisfy

$$|A| \leq (n-2|A|)/(2e^{2h+1}+1). \tag{1}$$

We denote by *randalg* the oblivious filling strategy given by Proposition 1. We denote by *flatalg* the oblivious filling strategy given by Lemma 1. We say that the filler ***applies*** a filling strategy *alg* to a set of cups $D \subseteq B$ if the filler uses *alg* on D while placing 1 unit of fill in each anchor cup.

We now describe the filler's strategy.

The filler starts by flattening the cups, i.e. using *flatalg* on $A \cup B$ for $2M$ rounds. After this, the filling strategy always places 1 unit of water in each anchor cup. The filler performs a series of $|A|$ ***swapping-processes***, one per anchor cup, which are procedures that the filler uses to get a new cup—which will sometimes have high fill—in A. On each swapping-process the filler applies *randalg* many times to arbitrarily chosen constant-size sets $D \subset B$ with $|D| = \lceil e^{2h+1} \rceil$. The number of times that the filler applies *randalg* is chosen at the start of the swapping-process, chosen uniformly at random from $[m]$ ($m = \text{poly}(n)$ to be specified). At the end of the swapping-process, the filler does a "swap": the filler takes the cup given by *randalg* in B and puts it in A, and the filler takes the cup in A associated with the current swapping-process and moves it into A'. Before each application of *randalg* the filler flattens B by applying *flatalg* to B for poly(n) rounds (exactly how many rounds will be specified later in the proof).

We remark that this construction is similar to the construction in **??**, but has a major difference that substantially complicates the analysis: in the adaptive lower bound construction the filler halts after achieving the desired average fill in the anchor set, whereas the oblivious filler cannot halt but rather must rely on the emptier's greediness to guarantee that each application of *randalg* has constant probability of generating a cup with high fill.

We proceed to analyze our algorithm.

First note that the initial flattening of $A \cup B$ succeeds because the emptier is not allowed to do any extra emptying on $A \cup B$ so by setting $p = n/2$ the filler makes the flattening happen in the 0-extra-empties, $(n/2)$-processor cup game, which by Lemma 1 gets an R_Δ-flat cup configuration in running time $2M$.

We say that a property of the cups has ***always*** held if the property has held since the start of the first swapping-process; i.e. from now on we only consider rounds after the initial flattening of $A \cup B$.

We say that the emptier ***neglects*** the anchor set on a round if it does not empty from each anchor cup. We say that an application of *randalg* to $D \subset B$ is ***successful*** if the emptier does not neglect the anchor set during any round of the application of *randalg*. We define $d = \sum_{i=2}^{|D|} 1/i$ (recall that $|D| = \lceil e^{2h+1} \rceil$). We say that an application of *randalg* to D is ***lucky*** if it achieves backlog at least $\mu(B) - R_\Delta + d$; note that by Proposition 1 any successful application of *randalg* where B started R_Δ-flat has at least a $1/|D|!$ chance of being lucky.

Now we prove several important bounds on fills of cups in A and B.

Claim 3. *If all applications of flatalg so far have made B be R_Δ-flat, then B has always been $(R_\Delta + d)$-flat and $\mu(B)$ has always been at most* 1*.*

Proof. Given that the application of *flatalg* immediately prior to an application of *randalg* made B be R_Δ-flat, by Proposition 1 we have that B will stay $(R_\Delta + d)$-flat during the application of *randalg*. Given that the application of *randalg* immediately prior to an application of *flatalg* resulted in B being $(R_\Delta + d)$-flat, we have that B remains $(R_\Delta + d)$-flat throughout the duration of the application of *flatalg* by Lemma 1. Given that B is $(R_\Delta + d)$-flat before a swap occurs B is clearly still $(R_\Delta + d)$-flat after the swap, because the only change to B during a swap is that a cup is removed from B which cannot increase the backlog in B or decrease the fill of the least full cup in B. Note that B started R_Δ-flat before the first application of *flatalg* because $A \cup B$ was flattened. Hence we have by induction that B has always been $(R_\Delta + d)$-flat.

Now consider how high $\mu(B)$ could rise. The only time when $\mu(B)$ rises is at the end of a swapping-process. The cup that B evicts at the end of a swapping-process has fill at least $\mu(B) - R_\Delta - (|D| - 1)$, as the running time of *randalg* is $|D| - 1$, and B started R_Δ-flat by assumption. The highest that $\mu(B)$ can rise is clearly if a cup with fill as far below $\mu(B)$ as possible is evicted at every swapping-process. Evicting a cup with fill $\mu(B) - R_\Delta - (|D| - 1)$ from B changes $\mu(B)$ by $(R_\Delta + |D| - 1)/(|B| - 1)$ where $|B|$ is the size of B before the cup is evicted from B. Even if this happens on each of the $|A|$ swapping processes $\mu(B)$ cannot rise higher than $|A|(R_\Delta + |D| - 1)/(n - 2|A|)$ which by design in choosing $|B| \gg |A|$, as was done in (1), is at most 1. $\square$

Let $u_A = 1 + (R_\Delta + d) + \Delta + 1$, $\ell_A = -|B| - u_A \cdot (|A| - 1)$.

Claim 4. *If B has always been $(R_\Delta + d)$-flat and $\mu(B)$ has never exceeded* 1*, then the fills of cups in A have always been in $[\ell_A, u_A]$ and we have always had $\mu(B) > -h/2$.*

Proof. Because the emptier is Δ-greedy-like, if a cup $c \in A$ has fill more than Δ higher than the backlog in B then c must be emptied from. Our assumptions on B imply that no cup in B ever has fill more than $1 + (R_\Delta + d)$, so any cup with fill at least $1 + (R_\Delta + d) + \Delta = u_A - 1$ must be emptied from, and hence u_A upper bounds the backlog in A.

Of course an upper bound on backlog in A also serves as an upper bound on the average fill of A as well, i.e. $\mu(A) \leq u_A$. Then, because $A \cup B$ has average fill 0, we have that

$$\mu(B) = -\frac{|A|}{|B|}\mu(A) \geq -u_A \frac{|A|}{|B|}.$$

Note that $|B| \gg |A|$ so this quantity, so in particular by (1) this lower bound on $\mu(B)$ can be loosened to $\mu(B) \geq -h/2$.

Because $\mu(B) \leq 1$ we have $\mu(A) \geq -|B|/|A|$. The mass of a subset of $|A| - 1$ of the cups is at most $(|A| - 1)u_A$, so we can lower bound the fill of any particular cup $c \in A$ by

$$\text{fill}(c) \geq -|B| - u_A \cdot (|A| - 1).$$

$\square$

Let $r = |A|(\ell_A + u_A)$.

Claim 5. *If at the start of an application of flatalg B is $(R_\Delta + d)$-flat, $\mu(B) \leq 1$, and all cups in A have fills in $[\ell_A, u_A]$ then by applying flatalg for $2((R_\Delta + d) + r)$ rounds, the filler guarantees that B will be R_Δ-flat at the end of the application of flatalg.*

Proof. If all cups in A start the application with fill at least ℓ_A and the emptier uses $r + 1$ extra empties during the application of *flatalg*, then by the pigeonhole principle some cup in A will have fill higher than u_A by the end of the application, as no cup in A loses water during the application of *flatalg*.

Lemma 1 says that B will remain $(R_\Delta + d)$-flat throughout the application of *flatalg*, and $\mu(B)$ obviously cannot rise during the application of *flatalg*. Hence throughout the entire of the process it will still be the case that $\mu(B) \leq 1$ and that B is $(R_\Delta + d)$-flat. Then Claim 4 says that the backlog in A cannot have grown larger than u_A. Hence it is impossible for the emptier to have used $r+1$ extra empties.

Thus, we can consider the application of *flatalg* as happening in the $(|B|/2)$-processor r-extra-empties cup game. By Lemma 1 we thus have that the cup configuration at the end of the application of *flatalg* will be R_Δ-flat by applying *flatalg* for $2((R_\Delta + d) + r)$ rounds. □

Now we combine Claim 3 Claim 4 and Claim 5 to get the following:

Claim 6. *All applications of flatalg make B be R_Δ-flat.*

Proof. This follows by induction on the flattening processes. Assume that all previous flattening processes have made B be R_Δ-flat. Then by Claim 3 we have that $\mu(B) \leq 1$ has always held, and that B has always been $(R_\Delta + d)$-flat. Thus by Claim 4 the fills of cups in A have always been in $[\ell_A, u_A]$. Thus by Claim 5 the next flattening successfully makes B be R_Δ-flat.

Note that the first application of *flatalg* makes B be R_Δ-flat because $A \cup B$ is flattened. Hence by induction all applications of *flatalg* make B be R_Δ-flat. □

TODO: this proof feels sketchy Now we show that this guarantees that with constant probability the final application of *randalg* a swapping-process is both lucky and successful.

Claim 7. *There exists choice of $m = poly(n)$ such that with at least constant probability the final application of randalg on any fixed swapping-process is both lucky and successful.*

Proof. Fix some swapping-process. By Claim 6 we have that the fill of each cup in A starts the swapping-process with fill at least ℓ_A, and never exceeds u_A throughout the course of the swapping-process. This places an upper bound of r on the number of applications of *randalg* on which A can be neglected.

The filler chooses $m = 4|D|!r$. By a Chernoff bound, there is exponentially high probability that of $4|D|!r$ applications of *randalg* at least $2r$ are lucky. The emptier can choose at most r of these to neglect, so there is at least a $1/2$ chance that the randomly chosen final application of *randalg* is successful, conditioning on it lucky. The final application is lucky with probability $1/|D|!$. Hence overall this choice of m makes the final round lucky and successful with constant probability $1/(2|D|!)$. □

Claim 8. *With probability at least $1 - 2^{-\Omega(n)}$, the filler achieves fill at least h in at least $\Theta(n)$ of the cups in A.*

Proof. By Proposition 1 using *randalg* on $|D| = \lceil e^{2h+1} \rceil$ gives the filler a cup that has fill at least $\mu(B) + d - R_\Delta$ with probability $1/|D|!$. By **??** we have that $\mu(B) \geq -h/2$, so with probability $1/|D|!$ this generated cup has fill at least h if it wasn't neglected.

By **??** there is a choice of c_Δ large enough that the probability of this cup having been neglected is at most $1/2$. In particular, we choose $c_\Delta = 4r|D|!$. By applying *randalg* $|A| \cdot c_\Delta$ times we have by a Chernoff bound that with exponentially good probability in $|A| \cdot c_\Delta = \Theta(n)$ there are at least $2|A|r$ applications where the filler would succeed if the emptier doesn't neglect the anchor set. As shown, the emptier cannot neglect the anchor set more than $|A|r$ times. Hence, there is at least a $(1/2)/|D|!$ chance that on the j-th application of *randalg* the emptier doesn't neglect the anchor set and the filler correctly guesses the emptier's emptying sequence. Thus, overall, there is at least a constant probability of achieving fill h in a cup in A.

Say that a swapping-process is ***victorious*** if the filler is able to swap a cup with fill at least h into A. The events that swapping-processes are victorious are independent events; each happens with constant probability. Hence by a Chernoff bound with exponentially good probability in n at least a constant fraction of them succeed, as desired.

□

We now briefly analyze the running time of the filling strategy. There are $|A|$ swapping-processes. Each swapping-process consists of $|A| \cdot c_\Delta$ applications of *randalg*, and the flattening procedure before each application. Clearly this all takes poly(n) time, as desired.

□

Finally, using Lemma 2 we can show in Lemma 3 that an oblivious filler can achieve constant backlog. We remark that Lemma 3 plays a similar role in the proof of the lower bound on backlog as **??** does in the adaptive case, but is vastly more complicated to prove (in particular, **??** is trivial, whereas we have already proved several lemmas and propositions as preperation for the proof of Lemma 3).

Lemma 3. *Let $H \leq O(1)$, let $\Delta \leq O(1)$, let n be at least a sufficiently large constant determined by H and Δ, and let $M \leq poly(n)$. Consider an M-flat cup configuration in the negative-fill variable-processor* *cup game on n cups with average fill* 0*. Given this configuration, an oblivious filler can achieve fill H in a chosen cup in running time poly(n) against a Δ-greedy-like emptier with probability at least* $1-2^{-\Omega(n)}$.

Proof. The filler starts by performing the procedure detailed in 2, using $h = H \cdot 16(1+\Delta)$. Let the number of cups which must now exist with fill h be of size $nc = \Theta(n)$.

The filler reduces the number of processors to $p = nc$. Now the filler exploits the filler's greedy-like nature to to get fill H in a set $S \subset B$ of nc chosen cups.

The filler places 1 unit of fill into each cup in S. Because the emptier is greedy-like it must focus on the nc cups in A with fill at least h until the cups in S have sufficiently high fill. In particular, $(5/8)h$ rounds suffice. Over $(5/8)h$ rounds the nc high cups in A cannot have their fill decrease below $(3/8)h \geq h/8 + \Delta$. Hence, any cups with fills less than $h/8$ must not be emptied from during these rounds. The fills of the cups in S must start as at least $-h/2$. After $(5/8)h$ rounds the fills of the cups in S are at least $h/8$, because throughout this process the emptier cannot have emptied from them until they got fill at least $h/8$, and if they are never emptied from then they achieve fill $h/8$.

Thus the filling strategy achieves backlog $h/8 \geq H$ in some known cup (in fact in all cups in S, but a single cup suffices), as desired.

□

References

[1] Michael A Bender, Martín Farach-Colton, and William Kuszmaul. Achieving optimal backlog in multi-processor cup games. In *Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing*, pages 1148–1157, 2019.

[2] Michael A Bender, Sándor P Fekete, Alexander Kröller, Vincenzo Liberatore, Joseph SB Mitchell, Valentin Polishchuk, and Jukka Suomela. The minimum backlog problem. *Theoretical Computer Science*, 605:51–61, 2015.

[3] Paul Dietz and Rajeev Raman. Persistence, amortization and randomization. 1991.

[4] William Kuszmaul. Achieving optimal backlog in the vanilla multi-processor cup game. *SIAM*, 2020.