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Variable Processor Cup Games

1 Lower Bound Corollary

Basic Idea. Let

$$f_0(k)=\begin{cases}\lg k, & k\geq 1,\\ 0 & \text{else.}\end{cases}$$

Note that we can achieve backlog $f_0(k)$ on k cups by Proposition **??**. Let f_{m+1} be the result of applying the Amplification Lemma to f_m with $\delta=1/2$. The function $f_{\lg n^{1/9}}(k)$ satisfies

$$\text{for } k\geq n, f_{\lg n^{1/9}}(k)\geq 2^{\lg n^{1/9}}\lg k. \tag{1}$$

In particular, using $f_{\lg n^{1/9}}(n)$ (applying the function to all of the cups) we achieve backlog $\Omega(n^{1/9}\lg n)\geq\Omega(\text{poly}(n))$ as desired. To prove Equation 1, we prove the following lower bound for f_m by induction:

$$f_m(k)\geq 2^m\lg k, \text{ for } k\geq(2^9)^m.$$

The base case follows from the definition of f_0. Assuming the property for f_m, we get the following by Lemma **??**: for $k>(2^9)^{m+1}$,

$$\begin{aligned}
&f_{m+1}(k)\\
&\geq\frac{1}{2}(f_m(k/2)+f_m(k/4)+\cdots+f_m(k/2^9)+\cdots)\\
&\geq\frac{1}{2}(f_m(k/2)+f_m(k/4)+\cdots+f_m(k/2^9))\\
&\geq\frac{1}{2}2^m(\lg(k/2)+\lg(k/4)+\cdots+\lg(k/2^9))\\
&\geq\frac{1}{2}2^m(9\lg(k)-\frac{9\cdot 10}{2})\\
&\geq 2^{m+1}\lg(k),
\end{aligned}$$

as desired. Hence the inductive claim holds, which establishes that $f_{\lg n^{1/9}}$ satisfies the desired condition, which proves that backlog can be made $\Omega(\text{poly}(n))$.

Running Time Analysis. The recursive construction requires quite a lot of steps, in fact a super-polynomial number of steps. If we consider the tree that represents computation of $f_{\log n^{1/\alpha}}(n)$ we see that each node will have at most α (some constant, e.g. $\alpha=9$, α is the number of terms that we keep in the sum) children (the children of $f_k(c)$ are $f_{k-1}(c/2), f_{k-1}(c/4), \ldots f_{k-1}(c/2^\alpha)$), and the depth of the tree is $\log n^{1/\alpha}$. Say that the running time at the node $f_{\log n^{1/\alpha}}(n)$ is $T(n)$. Then because $f_k(n)$ must call each of $f_{k-1}(n/2^i)$ $n/2^i$ times for $1\leq i\leq\alpha$, we have that $T(n)\leq\frac{\alpha n}{2}T(n/2)$. This recurrence yields $T(n)\leq\text{poly}(n)^{\log n}=O(2^{\log^2 n})$ for the running time.

Generalizing Our Approach. Generalizing our approach we can achieve a (slightly) better polynomial lower bound on backlog. In our construction the point after which we had a bound for f_m grew further out by a factor of 2^9 each time. Instead of 2^9 we now use 2^α for some $\alpha\in\mathbb{N}$, and can find a better value of α. The value of α dictates how many iterations we can perform: we can perform $\lg n^{1/\alpha}$ iterations. The parameter α also dictates the multiplicative factor that we gain upon going from f_m to f_{m+1}. For $\alpha=9$ this was 2. In general it turns out to be $\frac{\alpha-1}{4}$. Hence, we can achieve backlog $\Omega\left(\left(\frac{\alpha-1}{4}\right)^{\lg n^{1/\alpha}}\lg n\right)$. This optimizes at $\alpha=13$, to backlog $\Omega(n^{\frac{\lg 3}{13}}\log n)\approx\Omega(n^{0.122}\log n)$.

We can further improve over this. Note that in the proof that f_{m+1} gains a factor of 2 over f_m given above, we lower bound $9\lg k-9\cdot 10/2$ with $2\lg k$. Usually however this is very loose: for small m a significant portion of the $9\lg k$ is annihilated by the constant $1+2+\cdots+9$ (or in general $\alpha\lg k$ and $1+2+\cdots+\alpha$), but for larger values of m because k must be large we can get larger factors between steps, in theory factors arbitrarily close to α. If we could gain a factor of α at each step, then the backlog achievable would be $\Omega(\alpha^{\lg n^{1/\alpha}}\log n)=\Omega(n^{(\lg\alpha)/\alpha}\log n)$ which optimizes (over the naturals) at $\alpha=3$ to $n^{(\lg 3)/3}\approx n^{0.528}$. However, we can't actually gain a factor of α each time because of the subtracted constant. But, for any $\epsilon>0$ we can achieve a $\alpha-\epsilon$ factor increase each time (for sufficiently large m). Of course ϵ can't be made arbitrarily small because m can't be made arbitrarily large, and the "cut off" m where we start achieving the $\alpha-\epsilon$ factor increase must be a constant (not dependent on n). When the cutoff m, or equivalently ϵ, is constant then we can achieve backlog $\Omega((\alpha-\epsilon)^{\lg n^{1/\alpha}}\log n)=\Omega(n^{(\lg(\alpha-\epsilon))/\alpha}\log n)$. For instance, with this method we can get backlog $\Omega(\sqrt{n})$ for appropriate ϵ,α choice, or $\tilde{\Omega}(n^{(\lg(3-\epsilon))/3})$ for any constant $\epsilon>0$.

Existential Improvement. We now (non-constructively) demonstrate the existence of a filling strategy that achieves backlog $cn^{1-\epsilon}$ for constant $\epsilon\in(0,1)$ and $c\ll 1$.

Let $f^*(n)$ be the supremum over all filling strategies of the fill achievable on n cups. Clearly $f^*(n)$ satisfies the Amplification Lemma, i.e.

$$f^*(n)\geq(1-\delta)\sum_{\ell=0}^{M}f^*((1-\delta)\delta^{\ell}n).$$

Assume for the sake of deriving a contradiction that there is some n such that $f^*(n)<cn^{1-\epsilon}$, let n_* be the minimum such n.

Then we have

$$cn_*^{1-\epsilon}>f^*(n_*)\geq(1-\delta)\sum_{\ell=0}^{M}f^*((1-\delta)\delta^{\ell}n_*).$$

However,

$$\begin{aligned}&(1-\delta)\sum_{\ell=0}^{M}f^*((1-\delta)\delta^{\ell}n_*)\\ &\geq cn_*^{1-\epsilon}(1-\delta)\sum_{\ell=0}^{M}((1-\delta)\delta^{\ell})^{1-\epsilon}\\ &\geq cn_*^{1-\epsilon}(1-\delta)\frac{(1-\delta)^{1-\epsilon}}{1-\delta^{1-\epsilon}}.\end{aligned}$$

We will now show that there is an appropriate choice of $\delta\in(0,1)$ such that

$$\frac{(1-\delta)^{2-\epsilon}}{1-\delta^{1-\epsilon}}\geq 1,$$

which contradicts the assumption that $cn_*^{1-\epsilon}>f^*(n_*)$. Rearranging, we desire

$$(1-\delta)^{2-\epsilon}+\delta^{1-\epsilon}\geq 1.$$

For any ϵ we will show that there is an appropriate choice of $\delta\ll 1$ satisfying this inequality.

Consider the Taylor series for $(1-\delta)^{2-\epsilon}$:

$$(1-\delta)^{2-\epsilon}=1-(2-\epsilon)\delta-O(\delta^2).$$

By taking δ sufficiently small, the $O(\delta^2)$ term becomes negligible compared to the $(\alpha+1)\delta$ term. In particular, say that the $O(\delta^2)$ term is less than $c\delta^2$ for some constant c. Taking δ small enough such that $\delta^2 c<\delta$, we have that $(1-\delta)^{2-\epsilon}>1-(2-\epsilon)\delta-\delta$.

So, to find a δ where $g(\delta)\geq 1$ it suffices to find a δ with

$$\delta^{1-\epsilon}\geq(3-\epsilon)\delta.$$

The equality is achieved at $\delta=(\frac{1}{3-\epsilon})^{1/\epsilon}$.

This establishes the existence of a filling strategy that achieves backlog $\Omega(n^{1-\epsilon})$.

Modifying the Existential Argument to achieve backlog $n^{1-\epsilon}$ in finite time. We can modify the existential argument to get a guarantee on how long it will take to achieve the desired backlog. Fix an $\epsilon>0$, and choose a $\delta \in (0,1)$ satisfying $(1-\delta)^{2-\epsilon}/1-\delta^{1-\epsilon} \ge 1$. Fix $c \ll 1$. Say we aim to achieve backlog at least $cn^{1-\epsilon}$. Note that the choice of δ is motivated by the fact that

$$(1-\delta)\sum_{\ell=0}^{M}((1-\delta)\delta^i)^{1-\epsilon} \approx \frac{(1-\delta)^{2-\epsilon}}{1-\delta^{1-\epsilon}},$$

and, as in the existential argument it will be useful to assert that this quantity is at least 1. ok I'm kind of worried about things not being integers being a problem. We start with

$$f_0(k) = \begin{cases} \lg k, & k \ge 1, \\ 0 & \text{else.} \end{cases}$$

Then we construct f_n as the amplification of f_{n-1}. We claim the following regarding this construction:

$$f_\ell(k) \ge cn^{1-\epsilon} \text{ for all } k > n/(1-\delta)^\ell.$$

This is clearly true in the base case with f_0. If $f_\ell(k) \ge cn^{1-\epsilon}$ for all k then we are already done. Otherwise, let k_*+1 be the smallest k such that $f_\ell(k) < cn^{1-\epsilon}$. Note that by assumption we have $k_* > n/(1-\delta)^\ell$. Now consider the amplification $f_{\ell+1}$ of f_ℓ.

$$\begin{aligned} &f_{\ell+1}(k_*/(1-\delta)) \\ &\qquad \ge (1-\delta)\sum f_\ell((1-\delta)\delta^i n) \\ &\qquad \ge cn^{1-\epsilon}\frac{(1-\delta)^{2-\epsilon}}{1-\delta^{1-\epsilon}} \\ &\qquad \ge cn^{1-\epsilon}. \end{aligned}$$

This is as desired. Thus, by taking $f_{(\log n)/\log(1/(1-\delta))}$ we achieve backlog $cn^{1-\epsilon}$.

Achieving backlog $\boldsymbol{\Omega(n^{\lg 3/2})}$. Recall the recursive procedure that we use in the proof of the Amplification Lemma: to achieve the desired fill we must call $f(n/2^\ell)$ for $\ell = 0,1,2,\ldots$. As f_{m+1} recursively calls f_m, there is even more recursion.

Let $\#(m,i)$ denote the number of times $f_m(n/2^i)$ occurs in the recursive construction. Let there be $M = \lg(n/2)$ levels of recursion. The first level in the tree has $\#(M,i) = 1$ for all i. Note that we have

$$\#(m-1,i) = \sum_{j>i}\#(m,j)$$

for any level m, because any expression $f_m(n/2^j)$ will call $f_{m-1}(n/2^i)$ for $j > i$.

This is very reminiscent of the hockey stick identity:

$$\binom{n}{i} = \sum_{i-1 \le j \le n-1}\binom{j}{i-1}.$$

In fact we claim that if you look at it right (i.e. sideways) the $\#(m,i)$'s form Pascal's triangle! Specifically the bijection is

$$\#(m,i) = \binom{i}{M-m}.$$

This is true because of the Hockey Stick Identity and the base case like $\#(M,i) = 1$ for all i. We induct on the diagonals of Pascal's triangle. The inductive hypothesis is that $\#(m,i) = \binom{i}{M-m}$ for all i for some m. Then by the Hockey Stick Identity we get

$$\begin{aligned} \#(m-1,i) &= \sum_{j>i}\#(m,j) \\ &= \sum_{j>i}\binom{j}{M-m} = \binom{i}{M-(m-1)} \end{aligned}$$

as desired.

We can also prove this with a simple combinatorial argument: there is a bijection between terms of the form $f_{M-m}(n/2^{m+i-m})$ and integer partitions of $i-m$ into m integers, as you must divide up the array subdivisions among the different levels of recursion. This demonstrates that

$$\#(m,i) = \binom{i-m+m}{M-m} = \binom{i}{M-m}.$$

We know that $f_m(n/2^M) \geq 1$ by design in Lemma **??**, so to determine the total backlog we add up the occurrences of $f_m(n/2^M)$ on each level, weighted by the 1/2 decay factor. Then the backlog we get is

$$\sum_{i=0}^{M} \binom{M}{i} \frac{1}{2^i} = (3/2)^M = n^{\lg(3/2)}.$$

This is optimal for $\delta = 1/2$.

Constructively achieving backlog $\Omega(n^{1-\epsilon})$ The existential proof that backlog $\Omega(n^{1-\epsilon})$ suggests that we will need to take $\delta \ll 1$ to achieve this backlog. The analysis from the case $\delta = 1/2$ doesn't immediately apply here; that analysis was significantly simplified by the fact that $\delta = 1-\delta$ for $\delta = 1/2$. However, we use some similar ideas.