Oblivious Lower Bound: this time its for real

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We call an emptier Δ-***greedy-like*** if, when there are two cups c_1,c_2 with fills satisfying $\text{fill}(c_1) > \text{fill}(c_2) + \Delta$ the emptier never empties from c_2 without emptying from c_1 on the same round. Intuitively, a Δ-greedy-like emptier has a $\pm\Delta$ range within it is allowed to "not be greedy". Note that a perfectly greedy emptier is 0-greedy-like. Greedy and greedy-like emptiers are of great interest.

In the randomized setting we are only able to prove lower bounds for backlog against Δ-greedy-like emptiers, for $\Delta \le O(1)$; whether or not our results can be extended to a more general class of emptiers is an interesting open question.

Let the ***anti-backlog*** of a set S of cups be $-\min_{c\in S}\text{fill}(c)$; note that anti-backlog is non-negative, and is the absolute value of the fill of the cup with lowest fill.

Let $R_\Delta = 2+\Delta$. We say that a cup configuration is Δ-***smooth*** if all cups have fill in $[-2R_\Delta - \Delta, 2R_\Delta + \Delta]$.

First we prove a key property of greedy-like emptiers.

Proposition 1. *Given a cup configuration with backlog and anti-backlog both at most T, an oblivious filler can, in running time $2T$, achieve a Δ-smooth configuration of cups against a Δ-greedy-like emptier.*

Proof. The filler's strategy is to set $p = n/2$ and distribute fill equally amongst all cups at every round, in particular placing $1/2$ units of water in each cup. Let ℓ_t be anti-backlog on round t, u_t be backlog on round t. Let L_t be the set of cups on round t with fill in $[l_t, l_t + R_\Delta]$, and let U_t be the set of cups on round t with fill in $[u_t - R_\Delta, u_t]$.

It is useful to think of U_t as the union of three disjoint intervals: an interval of length 1, followed by a "buffer" interval of length Δ, followed by another interval of length 1. Note the key property that if a cup with fill in $[u_t - R_\Delta, u_t - R_\Delta + 1]$ is emptied from, then all cups with fill in $[u_t - 1, u_t]$ must also be emptied from, because of the "buffer" interval. The symmetric property holds for L_t.

Claim 1. *Because the emptier is Δ-greedy-like we have*

$$U_t \subseteq U_{t+1}, L_t \subseteq L_{t+1}.$$

Proof. Consider a cup $c \in U_t$.

If $\text{fill}_{t+1}(c) = \text{fill}_t(c) + 1/2$, then clearly $c \in U_{t+1}$, because $u_{t+1} \le u_t + 1/2$, so $\text{fill}_{t+1}(c) \ge u_{t+1} - R_\Delta$.

On the other hand, if $\text{fill}_{t+1}(c) = \text{fill}_t(c) - 1/2$, we consider two cases.

- If $\text{fill}_t(c) \ge u_t - R_\Delta + 1$, then
$$\text{fill}_{t+1}(c) \ge u_t - R_\Delta + 1/2 \ge u_{t+1} - R_\Delta.$$
- On the other hand, if $\text{fill}_t(c) < u_t - R_\Delta + 1$, then every cup with fill in $[u_t - 1, u_t]$ must have been emptied from. The fullest cup at round $t+1$ is the same as the fullest cup on round t, because the fills of all cups with fill in $[u_t - 1, u_t]$ have decreased by $1/2$, and no cup with fill less than $u_t - 1$ had fill increase by more than $1/2$. Hence $u_{t+1} = u_t - 1/2$. Thus we again have $\text{fill}_{t+1}(c) \ge u_{t+1} - R_\Delta$.

The argument for $L_t \subseteq L_{t+1}$ is essentially identical. □

So long as there are at most $n/2$ cups c with $\text{fill}_t(c) \in [u_t - \Delta - R_\Delta, u_t]$, all cups in U_t must be emptied from; identically, for L_t, as long as there are at most $n/2$ cups c with $\text{fill}_t(c) \in [\ell_t, \ell_t + \Delta + R_\Delta]$, no cup in L_t can be emptied from.

No cup can gain $+1/2$ fill on each of $2T$ consecutive rounds, or lose $-1/2$ fill on each of $2T$ consecutive rounds: this would violate the assumption that the cups started with fills in $[-T, T]$.

Hence there is some round $t_0 \le 2T$ on which we have more than $n/2$ cups in both $[u_t - \Delta - R_\Delta]$ and more than $n/2$ cups in $[\ell_t, \ell_t + \Delta + R_\Delta]$. Thus there must be a cup with fill in the intersection of these intervals, which therefore cannot be empty.

Thus, $u_{t_0} - \ell_{t_0} \le 2(R_\Delta + \Delta)$, which implies the desired property. We now prove that the desired property is maintained for rounds $t_0 < t \le 2T$.

this is sketch

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Proposition 2. *There exists an oblivious filling strategy in the variable-processor cup game on n cups that achieves backlog $\Omega(\log n)$ against a Δ-greedy-like emptier (where $\Delta \le O(1)$ is a constant known to the filler), with constant probability.*

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Proof. Let A, the ***anchor*** set, be a subset of the cups chosen uniformly at random from all subsets of size $n/2$ of the cups, and let B, the ***non-anchor*** set, consist of the rest of the cups ($|B|=n/2$). Let $h=8\Delta+8$, and let $h'=2$. Our strategy is roughly as follows:

- **Step 1:** Make a constant fraction of cups in A have fill at least h by playing single processor cup games on constant-size subsets of B. With constant probability we can attain a cup in B with constant fill by this method, that we then swap into A. By a Chernoff bound we get a constant fraction of A, say cn cups, to have fill at least h with exponentially good probability.

- **Step 2:** Reduce the number of processors to cn, and raise the fill of cn *known* cups to fill h'.

- **Step 3:** Recurse on the nc cups that are known to have fill at least h'.

By performing $\Omega(\log n)$ levels of recursion, achieving constant backlog h' at each step (relative to the average fills), the filler achieves backlog $\Omega(\log n)$.

We now provide a detailed description of the filling algorithm, to prove that the results claimed in Step 1 and Step 2 are attainable.

To attain step 2, we smooth the non-anchor set.

Now we detail how to achieve Step 1.

We perform a series of ***swapping-process***, which are procedures that we use to get a new cup in A. A swapping-process is composed of a substructure, repeated many times, which we call a ***round-block***; a round-block is a set of rounds. At the beginning of a swapping-process we choose a round-block $j\in[n^2]$ uniformly at random from all the round-blocks. The swapping-process proceeds for n^2 round-blocks; on the j-th round-block we swap a cup into the anchor set.

On each of the n^2 round-blocks, the filler selects a random subset $C\subset B$ of the non-anchor cups and plays a single processor cup game on C. In this single-processor cup game the filler employs the classic adaptive strategy for achieving backlog $\Omega(\log|B|)$ on a set of $|B|$ cups, however modified because it is an oblivious filler. In particular, the filler's strategy in the single-processor cup games is to distribute water equally among an ***active set*** of cups, and then after the emptier removes water from some cup the filler removes a random cup from the active set. There is at least constant probability that this results in the active set having a single cup at the end, with fill that has increased by at least $1/|B|+1/(|B|-1)+...+1/1\geq \ln|B|$ since the start of the round-block.

On most round-blocks – all but the j-th – the filler does nothing with the cup that it achieves in the active set at the end of the single processor cup game. However, on the j-th round-block the filler swaps the winner of the single processor cup game into the anchor set.

Claim 2. *Let $q\geq\Omega(1)$ be an appropriately small constant (q is a function of $h\leq O(1)$). In Case 1, with probability at least $1-e^{-nq^2/1024}$, we achieve fill at least h in at least $nq/16$ of the cups in A (i.e. a constant fraction of the cups in A). In particular, this implies that we achieve positive tilt $hnq/16\geq\Omega(n)$ in A.*

Proof. Consider a swapping-process where the filler does not perform a storing-operation where at least $1/2$ of the cups $c\in B$ have $\text{fill}(c)\geq -h$. Note that by assumption there are at least $n/4-3/2\gamma$ such rounds.

Say the emptier ***neglects*** the anchor set in a round-block if on at least one round of the round-block the emptier does not empty from every anchor cup. By playing the single-processor cup game for n^2 round-blocks, with only one round-block when we actually swap a cup into the anchor set, we strongly disincentive the emptier from neglecting the anchor set on more than a constant fraction of the round-blocks.

The emptier must have some binary function, $I(i)$ that indicates whether or not they will neglect the anchor set on round-block i if the filler has not already swapped. Note that the emptier will know when the filler perform a swap, so whether or not the emptier neglects a round-block i depends on this information. However, j is the only parameter of the swapping-process, so there is no other information that the emptier can use to decide whether or not to neglect a round-block, because on any round-block when we simply redistribute water amongst the non-anchor cups we effectively have not changed anything about the game state.

If the emptier is willing to neglect the anchor set for at least $1/2$ of the round-blocks, i.e. $\sum_{i=1}^{n^2} I(i)\geq n^2/2$, then with probability at least $1/4$, $j\in((3/4)n^2,n^2)$, in which case the emptier neglects the anchor set on at least $n^2/4$ round-blocks ($I(k)$ must be 1 for at least $n^2/4$ of the first $(3/4)n^2$ round-blocks). Each time the emptier neglects the anchor set the mass of the anchor set increases by at least 1. Thus the average fill of the anchor set will have increased by at least $(n^2/2)/(n/2)\geq\Omega(n)$ over the entire swapping-process in this case, implying that we achieve the desired backlog.

Otherwise, there is at least a $1/2$ chance that the round-block j, which is chosen uniformly at random from the round-blocks, when the filler performs a swap into the anchor set occurs on a round-block with $I(j)=0$, indicating that the emptier won't neglect the anchor set on round-block j. In this case, the round-block was a legitimate single processor cup game on C_j, the randomly chosen set of $\lceil e^{2h}\rceil$ cups on the j-th round. Then we achieve fill increase $\geq 2h$ by the end of the round-block with probability at least $1/\lceil e^{2h}\rceil!$ – the probability that we correctly guess the sequence of cups within the single processor cup game that the emptier empties from.

The probability that the random set $C_j \subset B$ contains only cups that are among the $n/4$ fullest cups in B is

$$\binom{n/2}{\lceil e^{2h}\rceil} / \binom{n}{\lceil e^{2h}\rceil} = O(1).$$

Note that because, by assumption, at least half of the cups $c \in B$ have $\text{fill}(c) \geq -h$, then the $n/4$ fullest cups in B must have fill at least $-h$. If all cups $c \in C_j$ have $\text{fill}(c) \geq -h$, then the fill of the cup in the active set at the end of the round-block is at least $-h+2h=h$, if the filler guesses the emptier's emptying sequence correctly.

Say that a swapping-process where at least half of the cups $c \in B$ have $\text{fill}(c) \geq -h$ ***succeeds*** if C_j is a subset of the $n/4$ fullest cups in B, and if the filler correctly guesses the emptier's emptying sequence. Note that if a swapping-process succeeds, then the filler is able to swap a cup with fill at least h into A. We have shown that there is a constant probability of a given swapping-process succeeding. Let X_i be the binary random variable indicating whether or not the i-th swapping process where the filler does not perform a storing-operation where at least half of the cups $c \in B$ have $\text{fill}(c) \geq -h$ succeeds. Let $q \geq \Omega(1)$ be the probability of a swapping-process succeeding, i.e. $P(X_i = 1)$. Note that the random variables X_i are clearly independent, and identically distributed.

Clearly

$$\mathbb{E}\left[\sum_{i=1}^{n/8} X_i\right] = qn/8.$$

Note that we do not use all the X_i; we know there must be at least $n/4 - 3/2\gamma$ swapping-processes that do not consist of a storing-operation, but only use $n/8$ of the X_i. We make this choice because the particular constants that we get do not matter, and because it substantially simplifies the analysis. By a Chernoff Bound (i.e. Hoeffding's Inequality applied to binary random variables),

$$P\left(\sum_{i=1}^{n/8} X_i \leq nq/16\right) \leq e^{-nq^2/1024}.$$

That is, the probability that less than $nq/16$ of the anchor cups have fill at least h is exponentially small in n, as desired.

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Lemma 1 (The Oblivious Amplification Lemma)**.** *Let f be an oblivious filling strategy that achieves backlog $f(n)$ in the variable-processor cup game on n cups with constant probability (relative to average fill, with negative fill allowed). Let $\delta \in (0,1)$ be a parameter. Then, there exists an adaptive filling strategy that, with constant probability, either achieves backlog*

$$f'(n) \geq (1-\delta)\Big(f((1-\delta)n) + f((1-\delta)\delta n)\Big)$$

or achieves backlog $\Omega(poly(n))$ in the variable processor cup game on n cups.

Proof. □