

# Variable-Processor Cup Games

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## Abstract

In a *cup game* a filler and an emptier take turns adding and removing water from cups, subject to certain constraints. In one version of the cup game, the *p*-processor cup game, the filler distributes *p* units of water among the *n* cups with at most 1 unit of water to any particular cup, and the emptier chooses *p* cups to remove at most one unit of water from. Analysis of the cup game is important for applications in processor scheduling, buffer management in networks, quality of service guarantees, and deamortization.

We investigate a new variant of the classic multi-processor cup game, which we call the *variable-processor cup game*, in which the resources of the emptier and filler are variable. In particular, in the variable-processor cup game the filler is allowed to change *p* at the beginning of each round. Although the modification to allow variable resources seems small, we show that it drastically alters the outcome of the game.

We construct a filling strategy that an adaptive filler can use to get backlog  $\Omega(n)$  in running time  $2^{O(n)}$ . We also construct a filling strategy that an adaptive filler can use to get backlog  $\Omega(n^{1-\epsilon})$  for any constant  $\epsilon > 0$  in running time  $2^{O(\log^2 n)}$ . Not only is this bound shockingly large, but the steep trade off-curve between running-time and backlog is very surprising: the time required to genuinely get a 1 as the exponent instead of  $1 - \epsilon$  (e.g. 0.999) goes up from quasi-polynomial time to exponential time!

Furthermore, we demonstrate that this lower bound on backlog is tight: using a novel set of invariants we prove that a greedy emptier never lets backlog exceed  $O(n)$ .

We also investigate bounds on an oblivious filler. We show, using concentration bounds for random variables (Hoeffding's Inequality), that – surprisingly – an oblivious filler can use very similar techniques as the adaptive filler to achieve very large backlog. In particular, an oblivious filler can achieve backlog  $2^{\Omega(\sqrt{\log n})}$  in running time  $O(\text{poly}(n))$  with constant probability against any “greedy-like” emptier.

## 1 Introduction

**Definition and Motivation.** The *cup game* is a multi-round game in which the two players – the *filler* and the *emptier* – take turns adding and removing water from cups. On each round of the classic *p*-processor *cup game* on *n* cups, the filler first distributes *p* units of water among the *n* cups with at most 1 unit to any particular cup (without this restriction the filler can trivially achieve unbounded backlog by placing all of its fill in a single cup every round), and then the emptier removes at most 1 unit of water from each of *p* cups<sup>1</sup>.

The cup game naturally arises in the study of processor-scheduling. The incoming water added by the filler represents work added to the system at time steps. At each time step after the new work comes in each of *p* processors must be allocated to a task, which they will achieve 1 unit of progress on before the next time step. The assignment of processors to tasks is modeled by the emptier deciding which cups to empty from. The backlog of the system is the largest amount of work left on any given task. To model this, in the cup game, the *backlog* of the cups is defined to be the fill of the fullest cup at a given state. It is important to know bounds on how large backlog can get.

**Previous Work.** The bounds on backlog are well known for the case where *p* = 1, i.e. the *single-processor cup game*. In the single-processor cup game an adaptive filler can achieve backlog  $\Omega(\log n)$  and a greedy emptier never lets backlog exceed  $O(\log n)$ . The bounds are much better against an oblivious filler. In the randomized version of the single-processor cup game, which can be interpreted as a smoothed analysis of the deterministic version, the emptier never lets backlog exceed  $O(\log \log n)$ , and a filler can achieve backlog  $\Omega(\log \log n)$ .

Recently Kuszmaul has achieved bounds on the case where *p* > 1, i.e. the *multi-processor cup game* [2]. Kuszmaul showed that in the *p*-processor cup game on *n* cups a greedy emptier never lets backlog exceed  $O(\log n)$ . He also proved a lower bound of  $\Omega(\log(n - p))$ . Recently we showed a lower bound of  $\Omega(\log n - \log(n - p))$ . Combined these bounds imply a lower bound of  $\Omega(\log n)$ . Kuszmaul also established an upper bound of  $O(\log \log n + \log p)$  against oblivious fillers, and a lower

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<sup>1</sup>note that negative fill is not allowed, so if the emptier empties from a cup with fill below 1 the cups fill will become 0.

bound of  $\Omega(\log \log n)$  (note that tight bounds on backlog against an oblivious filler are not yet known for large  $p$ ).

**Our Variant.** We investigate a variant of the vanilla multi-processor cup game which we call the *variable-processor cup game*. In the variable-processor cup game the filler is allowed to change  $p$  (the total amount of water that the filler adds, and the emptier removes, from the cups per round) at the beginning of each round. Note that we do not allow the resources of the filler and emptier to vary separately; just like in the classic cup game we take the resources of the filler and emptier to be identical. Although this restriction may seem artificial, it is crucial; if the filler has more resources than the emptier, then the filler could trivially achieve unbounded backlog, as average fill will increase by at least some positive constant at each round. Taking the resources of the players to be identical makes the game balanced, and hence interesting.

A priori having variable resources offers neither player a clear advantage: lower values of  $p$  mean that the emptier is at more of a discretization disadvantage but also mean that the filler can “anchor” fewer cups<sup>2</sup>. We hoped that the variable-processor cup game could be simulated in the vanilla multi-processor cup game, because the extra ability given to the filler does not seem very strong.

However, we show that attempts at simulating the variable-processor cup game are futile because the variable-processor cup game is—surprisingly—vastly different from the multi-processor cup game.

**Outline and Results.** In Section 2 we establish the conventions and notations we will use to discuss the variable-processor cup game.

In Section 3 we provide an inductive proof of a lower bound on backlog in Corollary 1. The base case of the argument is a direct consequence of Proposition 1, and the inductive step follows from the “Adaptive Amplification Lemma” (Lemma 1). Corollary 1 gives a lower bound of  $\Omega(n)$  on backlog. Corollary 1 provides two algorithms: one algorithm with running time  $2^{O(n)}$  that achieves backlog  $\Omega(n)$ , and another with running time  $2^{O(\log^2 n)}$  that achieves backlog  $\Omega(n^{1-\epsilon})$  for any constant  $\epsilon > 0$ .

In Section 4 we prove a novel invariant: the average fill of the  $k$  fullest cups is at most  $2n - k$ . In particular this implies (setting  $k = 1$ ) that backlog is  $O(n)$ . Thus, our analysis is tight.

Section 5 has similar macro-structure to Section 3: We lower bound backlog in Corollary 4, using Proposition 6 as the base case of the inductive argument and the “Oblivious Amplification Lemma” (Lemma 3) to facilitate the inductive step. Corollary 4 gives the lower bound  $2^{\Omega(\sqrt{\log n})}$  on backlog, against a “greedy-like” emptier. In particular the corollary asserts that we can achieve this

<sup>2</sup>A useful part of many filling algorithms is maintaining an “anchor” set of “anchored” cups. The filler always places 1 unit of water in each anchored cup. This ensures that the fill of an anchored cup never decreases after it is placed in the anchor set.

backlog in time  $O(\text{poly}(n))$ . Note that the restriction on runtime of the filler is the main way in which this bound differs from the adaptive case.

## 2 Preliminaries

The cup game consists of a sequence of rounds. On the  $t$ -th round the state starts as  $S_t$ . The filler chooses the number of processors  $p_t$  for the round. Then the filler distributes  $p_t$  units of water among the cups (with at most 1 unit of water to any particular cup). After this, the game is in an intermediate state, which we call state  $I_t$ . Then the emptier chooses  $p_t$  cups to empty at most 1 unit of water from. Note that if the fill of a cup that the emptier empties from is less than 1 the emptier reduces the fill of this cup to 0 by emptying from it; we say that the emptier **zeros-out** a cup if it empties from a cup with fill less than 1. Note that whenever the emptier zeros-out a cup the emptier has removed less fill than the filler has added; hence the average fill will increase. This concludes the round; the state of the game is now  $S_{t+1}$ .

Denote the fill of a cup  $c$  by  $\text{fill}(c)$ . Let the **positive tilt** of a cup  $c$  be  $\text{tilt}(c) = \max(0, \text{fill}(c))$ , and let the positive tilt of a set  $X$  of cups be  $\text{tilt}(X) = \sum_{c \in X} \text{tilt}(c)$ . Let the **mass** of a set of cups  $X$  be  $m(X) = \sum_{c \in X} \text{fill}(c)$ . Denote the average fill of a set of cups  $X$  by  $\mu(X)$ . Note that  $\mu(X)|X| = m(X)$ .

Let the **rank** of a cup at a given state be its position in a list of the cups sorted by fill at the given state, breaking ties arbitrarily but consistently. For example, the fullest cup at a state has rank 1, and the least full cup has rank  $n$ .

Many of our lower bound proofs will adopt the convention of allowing for negative fill. This is because our results are used recursively, so this negative fill is really just fill below the average fill. Note that it is strictly easier for the filler to achieve high backlog when cups can zero out, because then some of the emptier’s resources are wasted. On the other hand, during the upper-bound proof we show that a greedy emptier maintains the desired invariants even if cups zero-out. This is crucial as the game is harder for the emptier when cups can zero out.

When measuring fill relative to average fill, we will reference the actual – i.e. non-relative – fills by calling them **absolute** fills.

## 3 Adaptive Filler Lower Bound

**Proposition 1.** *There exists an adaptive filling strategy for the variable-processor cup game on  $n$  cups that achieves backlog at least  $\frac{1}{4} \ln(n/2)$ , where fill is relative to the average fill of the cups, with negative fill allowed.*

*Proof.* Let  $h = \frac{1}{4} \ln(n/2)$  be the desired fill. Once a cup with fill at least  $h$  is achieved the filler stops, the process completed. Let  $A$  consist of the  $n/2$  fullest cups, and

$B$  consist of the rest of the cups. Note that  $A, B$  are implicitly functions of the round  $t$ .

If the process is not yet complete, that is  $\text{fill}(c) < h$  for all cups  $c$ , then  $\text{tilt}(A \cup B) < h \cdot n$ . Assume for sake of contradiction that there are more than  $n/2$  cups  $c$  with  $\text{fill}(c) \leq -2h$ . The mass of those cups would be at most  $-hn$  but the mass of the remaining cups cannot exceed  $hn/2$ . This implies that average fill is negative when it is in fact 0 by definition, a contradiction. Hence there are at most  $n/2$  cups  $c$  with  $\text{fill}(c) \leq -2h$ .

The filler sets the number of processors equal to 1 and plays a single processor cup game on  $n/2$  cups that each have fill at least  $-2h$  (which must exist) for  $n/2 - 1$  steps. Throughout this process the filler maintains a set of cups called the **active set**: the set of cups that the filler will place fill in. The filler initializes the active set to be  $A$ . Note that that  $\text{fill}(c) \geq -2h$  for all cups  $c \in A$ , as  $A$  consists of the  $n/2$  fullest cups. The filler removes 1 cup from the active set at each step. At each step the filler distributes water equally among the cups in its active set. Then, the emptier will choose some cup to empty from. If this cup is in the active set, the filler removes it from the active set. Otherwise, the filler chooses an arbitrary cup to remove from the active set.

After  $n/2 - 1$  steps, the active set will consist of a single cup. This cup's fill has increased by  $1/(n/2) + 1/(n/2 - 1) + \dots + 1/2 + 1/1 \geq \ln n/2 = 4h$ . This cup's fill started as at least  $-2h$ . Thus this cup has fill at least  $2h$  now, as desired.  $\square$

**Lemma 1** (The Adaptive Amplification Lemma). *Let  $f$  be an adaptive filling strategy that achieves backlog  $f(n)$  in the variable-processor cup game on  $n$  cups (relative to average fill, with negative fill allowed). Let  $n_0 \in \mathbb{N}$  be a constant such that a filler can achieve backlog 1 on  $n_0$  cups ( $n_0$  exists by Proposition 1). Let  $\delta \in (0, 1)$  be a parameter, and let  $L \in \mathbb{N}$  be a parameter such that  $n_0 \leq (1 - \delta)\delta^L$ .<sup>3</sup>*

*Then, there exists an adaptive filling strategy that achieves backlog*

$$f'(n) \geq (1 - \delta) \sum_{\ell=0}^L f((1 - \delta)\delta^\ell n)$$

*in the variable-processor cup game on  $n$  cups.*

*Proof.* The basic idea of this analysis is as follows:

- **Step 1:** Using  $f$  repeatedly, achieve average fill at least  $(1 - \delta)f(n(1 - \delta))$  in a set of  $n\delta$  cups.
- **Step 2:** Reduce the number of processors to  $n\delta$ , and recurse on the  $n\delta$  cups with high average fill.

Let  $A$ , the **anchor set**, be initialized to consist of the  $n\delta$  fullest cups, and let  $B$  the **non-anchor set** be initialized to consist of the rest of the cups (so  $|B| = (1 - \delta)n$ ). Let  $n_\ell = n\delta^{\ell-1}$ ,  $h_\ell = (1 - \delta)f(n_\ell(1 - \delta))$ ; the filler will achieve a set of at least  $n_\ell\delta$  cups with

<sup>3</sup>Note that  $n$  must be sufficiently large, and  $\delta, L$  must be chosen such that  $(1 - \delta)\delta^L n \in \mathbb{N}$  because it doesn't make sense to talk of "fractional cups".

average fill at least  $h_\ell$  on the  $\ell$ -th level of recursion. On the  $\ell$ -th level of recursion  $|A| = \delta \cdot n_\ell$ ,  $|B| = (1 - \delta) \cdot n_\ell$ .

We now elaborate on how to achieve Step 1. The filling strategy always places 1 unit of water in each anchor cup. This ensures that no cups in the anchor set ever have their fill decrease.

On the  $\ell$ -th level of recursion the filler uses the following procedure, termed a **swapping-process**, to achieve the desired average fill in  $A$ : repeatedly apply  $f$  to  $B$ , and then take the cup generated by  $f$  within  $B$  with fill at least  $f(|B|)$  and swap it with a cup in  $A$ ; repeat until  $A$  has the desired average fill. Note that

$$\mu(A) \cdot |A| + \mu(B) \cdot |B| = 0,$$

so

$$\mu(A) = -\mu(B) \cdot (1 - \delta)/\delta.$$

Thus, if at any point in the process  $B$  has average fill lower than  $-h_\ell \cdot \delta/(1 - \delta)$ , then  $A$  has average fill at least  $h_\ell$ , so the process is finished. So long as  $B$  has average fill at least  $-h_\ell \cdot \delta/(1 - \delta)$  we will apply  $f$  to  $B$ .

It is somewhat complicated to apply  $f$  to  $B$  however, because we must guarantee that in the steps that the algorithm takes while applying  $f$  the amount of mass removed from  $B$  by the emptier is the same as the amount of mass added to  $B$  by the filler. This might not be the case if the emptier does not empty from each anchor cup at each step. We say that the emptier **neglects** the anchor set on an application of  $f$  if there is some step during the application of  $f$  in which the emptier does not empty from some anchor cup.

We will apply  $f$  to  $B$  at most  $h_\ell n_\ell \delta + 1$  times, and at the end of an application of  $f$  we only swap the generated cup into  $A$  if the emptier has not neglected the anchor set during the application of  $f$ .

Note that each time the emptier neglects the anchor set the mass of the anchor set increases by 1. If the emptier neglects the anchor set  $h_\ell n_\ell \delta + 1$  times, then the average fill in the anchor set increases by more than  $h_\ell$ , so the desired average fill is achieved in the anchor set.

Otherwise, there must have been an application of  $f$  for which the emptier did not neglect the anchor set. We only swap a cup into the anchor set if this is the case. In this case we achieve fill

$-h_\ell \cdot \delta/(1 - \delta) + f(n_\ell(1 - \delta)) = (1 - \delta)f(n_\ell(1 - \delta)) = h_\ell$  in a non-anchor cup, and swap it with the smallest cup in the anchor set.

We achieve average fill  $h_\ell$  in the anchor set for  $L$  levels of recursion. Summing  $h_\ell$  for  $0 \leq \ell \leq L$  yields the desired result.  $\square$

**Corollary 1.** *There are adaptive filling strategies for the variable-processor cup game on  $n$  cups that achieve backlog  $\Omega(n)$  in time  $2^{O(n)}$  and backlog  $\Omega(n^{1-\epsilon})$  for any constant  $\epsilon > 0$  in time  $2^{O(\log^2 n)}$ .*

*Proof.*

Fix  $\epsilon \in (0, 1/2)$ , and let  $c, \delta$  be parameters, with  $c \in$

$(0, 1), 0 < \delta \ll 1/2$  – these will depend on  $\epsilon, n$ . Say that we aim to achieve backlog at least  $cn^{1-\epsilon}$ . Observe that if we apply the Amplification Lemma to a function  $f$  satisfying  $f(k) \geq ck^{1-\epsilon}$  for  $k \leq g$  then for any  $k_0$  with  $k_0(1-\delta) \leq g$  (which enforces  $k_0 \leq g/(1-\delta)$ ) we have the following:

$$\begin{aligned} f'(k_0) &\geq \\ (1-\delta) \sum_{\ell=0}^L c(((1-\delta)\delta^\ell)k_0)^{1-\epsilon} \\ &= ck_0^{1-\epsilon}(1-\delta)^{2-\epsilon} \sum_{\ell=0}^L (\delta^\ell)^{1-\epsilon}, \end{aligned}$$

where  $L$  is the greatest integer such that  $(1-\delta)\delta^L n \geq n_0$  where  $n_0$  is a constant such that a filler can achieve backlog 1 on  $n_0$  cups (this definition is identical to the definition in the statement of Lemma 1). Note that as  $\delta$  will be very small,  $\sum_{\ell=0}^L (\delta^\ell)^{1-\epsilon}$  is very well approximated by  $1 + \delta^{1-\epsilon}$ , so we will not loose much by relaxing our lower bound on  $f'(k_0)$  to only use the first 2 terms of the sum. We have the looser bound

$$f'(k_0) \geq ck_0^{1-\epsilon}(1-\delta)^{2-\epsilon}(1+\delta^{1-\epsilon}).$$

Let

$$h(\delta) = (1-\delta)^{2-\epsilon}(1+\delta^{1-\epsilon}).$$

We prove the following claim:

**Claim 1.** *There exists an appropriate choice of  $\delta$  that is small enough such that  $h(\delta) > 1$  and large enough such that  $L \geq 1$ , when  $\epsilon$  is chosen to be  $4/\lg n$ , or a positive constant. In particular, if  $\epsilon$  is chosen to be  $4/\lg n$  then we will choose  $\delta \leq O(1/n)$ , and if  $\epsilon$  is chosen to be a positive constant then we will choose  $\delta \leq O(1)$ .*

Note that if  $h(\delta) \geq 1$ , then  $f'(k_0) \geq ck_0^{1-\epsilon}$ , meaning we have constructed from  $f$  a new function  $f'$  that satisfies the inequality  $f'(k) \geq ck^{1-\epsilon}$  for  $k \leq g/(1-\delta)$ , as opposed to only for  $k \leq g$  as in the case of  $f$ .<sup>4</sup> Then by repeatedly amplifying a function, we should be able to arbitrarily extend the support, which will allow us to attain the desired backlog.

We now prove Claim 1.

*Proof.* Consider the Taylor series for  $(1-\delta)^{2-\epsilon}$  about  $\delta = 0$ :

$$(1-\delta)^{2-\epsilon} = 1 - (2-\epsilon)\delta + O(\delta^2).$$

So, to find a  $\delta$  where  $h(\delta) \geq 1$  it suffices – note that, again, we choose to neglect the  $\delta^2$  term as it does not strengthen the lower bound substantially because it is so small – to find a  $\delta$  with

$$(1 - (2-\epsilon)\delta)(1 + \delta^{1-\epsilon}) \geq 1.$$

Rearranging we have

$$\delta^{1-\epsilon} \geq (2-\epsilon)\delta + (2-\epsilon)\delta^{2-\epsilon}.$$

<sup>4</sup>Note that although  $f'(k) \geq ck^{1-\epsilon}$  holds for at least as many  $k$  as  $f(k) \geq ck^{1-\epsilon}$ , it does not necessarily hold for strictly more; in particular, if  $\lfloor g/(1-\delta) \rfloor = g$  then the inequality on  $f'$  holds for no more  $k$  than the inequality on  $f$ , as  $f$  and  $f'$  are functions on  $\mathbb{N}$ . In general we have to be somewhat careful about the fact that there are an integer number of cups throughout this proof (this issue was deferred from earlier proofs to be dealt with here).

This clearly is true for sufficiently small  $\delta$ , as  $\delta^{1-\epsilon}$  will be much greater than  $\delta$  or  $\delta^{2-\epsilon}$ . However it will be beneficial to have a more explicit criterion for possible choices of  $\delta$  in terms of  $\epsilon$ . To get this, we enforce a much stronger inequality on  $\delta^{1-\epsilon}$  by vastly overestimating  $\delta^{2-\epsilon}$  as  $\delta$ . Surprisingly even with this overestimate we are still able to get the desired value of  $\epsilon$  to work, as we will demonstrate later. We have,

$$\delta \leq \frac{1}{(2(2-\epsilon))^{1/\epsilon}}. \quad (1)$$

In addition to the constraint that  $\delta$  must be small enough such that  $h(\delta) \geq 1$ , the only other constraint on  $\delta$  is that  $\delta$  must be large enough that the sum from the Amplification Lemma has at least two terms, i.e. such that  $L \geq 1$ . We need  $L \geq 1$  because otherwise the Amplification Lemma doesn't give a larger function. The condition  $L \geq 1$  enforces

$$\delta(1-\delta)n \geq n_0.$$

Recall that we choose  $\delta < 1/2$ , so  $1-\delta > 1/2$ . Thus to make  $\delta$  sufficiently big it suffices to chose  $\delta$  with

$$\delta \geq 2n_0/n. \quad (2)$$

Any choice of  $\delta$  that is sufficiently large to make  $L \geq 1$  and simultaneously small enough to make  $h(\delta) \geq 1$  is a valid choice of  $\delta$ . That is,  $\delta$  is valid if and only if it satisfies

$$\frac{2n_0}{n} \leq \delta \leq \frac{1}{(2(2-\epsilon))^{1/\epsilon}}. \quad (3)$$

To achieve the desired backlog of  $\Omega(n)$  we can use  $\epsilon = \gamma/\lg n$  for appropriate constant  $\gamma$ , as

$$n^{1-\gamma/\lg n} = n/2^\gamma = \Omega(n).$$

We show that there is a valid choice of  $\gamma$  such that the following inequality is satisfied:

$$2n_0/n \leq \frac{1}{(2(2-\gamma/\lg n))^{(1/\gamma)\lg n}}. \quad (4)$$

Note that

$$(2(2-\gamma/\lg n))^{(1/\gamma)\lg n} \leq 4^{(1/\gamma)\lg n} \leq n^{2/\gamma}.$$

Thus, clearly by choosing e.g.  $\gamma = 4$  we have the desired inequality. Inequality 4 implies that there is a valid choice of  $\delta$  when we chose  $\epsilon = \gamma/\lg n$ . When proving that we can achieve backlog  $\Omega(n)$  we use  $\epsilon = 4/\lg n$ , and  $\delta = O(1/n)$  satisfying Inequality 3 for our choice of  $\epsilon$ . When proving that we can achieve backlog  $\Omega(n^{1-\epsilon})$  for constant  $\epsilon > 0$  we choose  $\delta > 0$  to be a constant satisfying Inequality 1, and  $\delta$ , being constant, trivially satisfies Inequality 2.  $\square$

Now we proceed to show that with the appropriate values of  $\delta, \epsilon$  we can achieve a filling strategy that achieves backlog  $cn^{1-\epsilon}$  on  $n$  cups. First we present a simple existential argument to show that an algorithm achieving backlog  $\Omega(n^{1-\epsilon})$  for  $\epsilon$  constant exists. Next we modify the existential proof to achieve an algorithm achieving the same backlog, but in bounded running time, specifically running time  $2^{O(\log^2 n)}$ . Finally we present an algorithm where  $\delta$  is set extremally; doing so, along with other modifications of the approach, we can achieve backlog  $\Omega(n)$  in running time  $2^{O(n)}$ .

**Proposition 2.** *There exists a filling algorithm that achieves backlog  $\Omega(n^{1-\epsilon})$  for any constant  $\epsilon > 0$ .*

*Proof.* Let  $\epsilon > 0$  be constant. By Claim 1, there is a valid constant setting of  $\delta$ ; let  $\delta \ll 1/2$  be an appropriate constant. Let  $f^*(n)$  be the supremum over all filling strategies of the backlog achievable on  $n$  cups. Clearly  $f^*$  must be greater than or equal to the amplification of  $f^*$ . Assume for contradiction that there is some least  $n_*$  such that

$$\begin{cases} f^*(k) < ck^{1-\epsilon}, & k > n_* \\ f^*(k) > ck^{1-\epsilon}, & k \leq n_* \end{cases}$$

Note that  $n_*(1-\delta) \geq n_0/(1-\delta)$  by appropriate choice of constant  $c$ , and Proposition 1, which states that we can get backlog  $\Omega(\log n_*)$  on  $n_*$  cups<sup>5</sup>. Because  $f^*$  satisfies the Amplification Lemma we have:

$$\begin{aligned} f^*(n_*) &\geq (1-\delta) \sum_{\ell=0}^L f^*((1-\delta)^\ell n_*) \\ &\geq cn_*^{1-\epsilon} h(\delta) \\ &\geq cn_*^{1-\epsilon} \end{aligned}$$

which is a contradiction. Hence  $f^*$  achieves backlog  $cn^{1-\epsilon}$ .  $\square$

**Proposition 3.** *There is a filling strategy that achieves backlog  $\Omega(n^{1-\epsilon})$  for constant  $\epsilon > 0$  in time  $2^{O(\log^2 n)}$ .*

*Proof.* It is desirable to have an algorithm for achieving this backlog with bounded running time; we now modify the existential argument to make it constructive, which yields an algorithm for achieving backlog  $cn^{1-\epsilon}$  on  $n$  cups in finite running time. We again use constant  $\epsilon > 0$  and appropriate constant  $\delta$ .

We start with the algorithm given by Proposition 1 for achieving backlog

$$f_0(k) = \begin{cases} \lg k, & k \geq 1, \\ 0 & \text{else.} \end{cases}$$

Then we construct an algorithm that achieves better backlog using the Amplification Lemma (Lemma 1): we construct  $f_{i+1}$  as the amplification of  $f_i$ .

Define a sequence  $g_i \in \mathbb{N}^\infty$  with

$$g_i = \begin{cases} \lceil 1/\delta \rceil \gg 1, & i = 0, \\ \lceil g_{i-1}/(1-\delta) \rceil - 1 & i \geq 1 \end{cases}$$

That is,  $g_{i+1}$  is the greatest integer strictly less than  $g_i/(1-\delta)$ . Note that  $(1/\delta)/(1-\delta) > (1+\delta)/\delta = 1/\delta + 1$ . Thus  $g_1 = 1 + g_0$ , and in general,  $g_{i+1} > g_i$ , because the difference  $g_{i+1} - g_i$  can only grow as  $i$  grows.

We claim the following regarding this construction:

**Claim 2.**

$$f_i(k) \geq ck^{1-\epsilon} \text{ for all } k < g_i. \quad (5)$$

<sup>5</sup>Note: this is where it is crucial that  $\epsilon, \delta$  are constants.

*Proof.* We prove Claim 2 by induction on  $i$ . Claim 2 is true in the base case of  $f_0$  by taking  $c$  sufficiently small, in particular small enough that  $f_0(k) \geq ck^{1-\epsilon}$  holds for  $k < g_0$ .<sup>6</sup> As our inductive hypothesis we assume Claim 2 for  $f_i$ ; we aim to show that Claim 2 holds for  $f_{i+1}$ . Note the key property of  $g_i$ , that  $g_{i+1} \cdot (1-\delta) < g_i$ . Also note that (at least without loss of generality) the  $f_i$  are monotonically increasing functions: given more cups we can always achieve higher fill than with fewer cups. Thus we have, for any  $k < g_{i+1}$ ,

$$\begin{aligned} f_{i+1}(k) &\geq (1-\delta) \sum_{\ell=0}^L f_i((1-\delta)^\ell k) \\ &\geq ck^{1-\epsilon} h(\delta) \\ &\geq ck^{1-\epsilon}, \end{aligned}$$

as desired.  $\square$

Note that  $g_{i+1} \geq g_i + 1$  so by continuing this process we eventually reach some  $f_{i_*}$  such that  $f_{i_*}(n) \geq cn^{1-\epsilon}$ ; trivially  $i_* \leq n$ . In fact we can show  $i_* \leq O(\log n)$ . The recurrence is almost simply the geometric sequence  $g_i = g_{i-1}/(1-\delta)$  with  $g_0 = 1/\delta$ , for which it is trivially true that  $g_i = 1/\delta(1/(1-\delta))^i$ , hence we would have  $g_{\log_{1/(1-\delta)} n} = n \cdot 1/\delta > n$  as desired. The actual sequence is not quite this large. The sequence is lower bounded by the sequence  $g'_i$  defined as  $g'_0 = 1/\delta$ ,  $g'_i = g'_{i-1}/(1-\delta) - 1$ . Even the sequence  $g'_i$  grows exponentially though, hence only a logarithmic recursion depth is necessary. In particular,  $g'_i$  satisfies

$$g'_i = \frac{1/\delta}{(1-\delta)^i} - \frac{1/(1-\delta)^{i-1} - 1}{1/(1-\delta) - 1},$$

which simplifies to

$$g'_i = \frac{1}{(1-\delta)^i} + \frac{1-\delta}{\delta}.$$

Thus  $g'_{\log_{1/(1-\delta)} n}$ , and hence also  $g_{\log_{1/(1-\delta)} n}$  both are at least  $n$ , as desired.

However, we have not addressed the issue that the number of cups must be an integer! Although it is relatively clear that this will not affect the asymptotic results, we briefly carefully address this here. Luckily, by careful choice of  $\delta$  and a slight (constant factor) reduction of  $n$  we can get the argument to work. In particular, let  $\psi$  be the smallest natural such that  $1/2^\psi < \delta$ ; define  $\delta' = 1/2^\psi$ . Let  $\tau$  be the integer such that  $2^{\tau\psi} \leq n < 2^{(\tau+1)\psi}$ ; define  $n' = 2^{\tau\psi}$ . Note that  $O(n') = O(n)$  as  $n'$  is at worst a factor of  $2^\psi$  less than  $n$ , but  $\delta$  being constant implies that  $\psi$  is also constant. Note that

$$\tau = \left\lfloor \frac{\lg n}{\psi} \right\rfloor \geq \Omega(\log n).$$

Thus, we have that  $n'(\delta')^{i_1}(1-\delta')^{i_2} \in \mathbb{N}$  for any integers  $i_1, i_2$  satisfying  $i_1 + i_2 \leq \tau$ , which means that  $i_1, i_2 \geq \Omega(\log n)$  is possible. Now we can genuinely guarantee that throughout the levels of recursive application

<sup>6</sup>Note: this is where it is crucial that  $\epsilon, \delta$  are constants.

of the Amplification Lemma – of which there will be at most  $O(\log n)$  – the number of cups is always an integer.

Let the running time  $f_{i_*}(n)$  be  $T(n)$ . Note that  $f_{i_*}(n)$  must call  $f_{i_*-1}(n(1-\delta)\delta^\ell)$  as many as  $n(1-\delta)\delta^\ell$  times, for all  $0 \leq \ell \leq L$ . However, we only need the terms of the sum where  $\ell = 0, 1$ , so we can take  $L = 1$ . Thus we have the following (loose) recurrence bounding  $T(n)$ :

$$T(n) \leq \delta n \cdot T(n(1-\delta)) + T(\delta n).$$

We can upper bound this by  $n^{\frac{\log n}{\log(1/(1-\delta))}}$ .

Continuing for  $O(\log n)$  levels of recursion should be sufficient to achieve the desired backlog. This gives running time

$$T(n) \leq ((1+\delta)n)^{O(\log n)} \leq 2^{O(\lg^2 n)}$$

as desired.  $\square$

**Proposition 4.** *There is a filling strategy that achieves backlog  $\Omega(n)$  in time  $2^{O(n)}$ .*

*Proof.* We describe a simple filling strategy that gives the desired backlog. Let  $n_0 \leq O(1)$  be a constant such that we can achieve backlog 1 on  $n_0$  cups, and note that this is possible by Proposition 1. We construct a function that achieves large backlog on  $n$  cups. To achieve large backlog on  $n$  cups we first recursively apply our function to  $(1-\delta)n$  cups repeatedly (for each of the  $\delta n$  cups that we are attempting to get high fill in), as described in the proof of the Amplification Lemma, and transfer over the cups that we get. Then we achieve backlog 1 on the  $\delta n$  cups whose average fill has been increased. The backlog we achieve satisfies the following recurrence:

$$f(n) \geq \begin{cases} (1-\delta)f((1-\delta)n) + 1, & \text{if } n\delta(1-\delta) > n_0 \\ 0, & \text{else.} \end{cases}$$

Let  $(1-\delta)^c = \delta$ , let  $\delta^2 n < n_0 < (1-\delta)^{2c-1}n$  by our choice of  $\delta = O(1/n)$ . We can get backlog

$$\sum_{i=1}^c (1-\delta)^i.$$

To see this, consider a binary tree representing our algorithm. At every branch we both proceed to recurse on a  $1-\delta$  fraction of the cups, and achieve backlog 1 on a  $\delta$  fraction of the cups.

The sum evaluates to

$$\frac{(1-\delta)^2}{\delta}$$

which, if we chose  $\delta = O(1/n)$ , becomes  $\Omega(n)$ .

The running time satisfies the recurrence

$$T(n) = \delta n T((1-\delta)n) + O(1)$$

because to achieve backlog  $f(n)$  we must achieve backlog  $f((1-\delta)n)$   $\delta n$  times, and then achieve backlog 1 on the remaining cups. Solving this recurrence yields that the running time is

$$\frac{(\delta n)^c - 1}{\delta n - 1}.$$

Recalling that  $\delta = O(1/n)$  this becomes  $2^{O(n)}$ .

We can even express this strategy more simply. Explicitly choosing  $\delta = n_0/n$ , we apply our function recursively to  $n - n_0$  cups  $n_0$  times to achieve fill  $(1 - n_0/n)f(n - n_0)$  in a set of  $n_0$  cups, and then we apply the strategy from Proposition 1 to these cups. Hence the backlog achieved satisfies the recurrence

$$f(n) = (1 - n_0/n)f(n - n_0) + 1.$$

Solving the recurrence, if we expand out  $\Omega(n)$  terms then we get a sum of  $\Omega(n)$  as desired. The recurrence for running time is

$$T(n) = n_0 T(n - n_0) + 1.$$

Solving, we again get  $2^{O(n)}$ .  $\square$

## 4 Adaptive Filler Upper Bound

Let  $\mu_S(X)$  and  $m_S(X)$  denote the average fill and mass of a set of cups  $X$  at state  $S$  (e.g.  $S = S_t$  or  $S = I_t$ ).<sup>7</sup> Let  $S_t(\{r_1, \dots, r_m\})$  and  $I_t(\{r_1, \dots, r_m\})$  denote the cups of ranks  $r_1, r_2, \dots, r_m$  at states  $S_t$  and  $I_t$  respectively. Let  $[n] = \{1, 2, \dots, n\}$ , let  $i + [n] = \{i + 1, i + 2, \dots, i + n\}$ . We will use concatenation of sets to denote unions, i.e.  $AB = A \cup B$ .

We establish the following Lemma:

**Lemma 2.** *The greedy emptier maintains the invariant*

$$\mu_{S_t}(S_t([k])) \leq 2n - k \text{ for all } t \geq 1, k \in [n].$$

*In particular, for  $k = 1$ , this implies that the greedy emptier never lets backlog exceed  $O(n)$ .*

*Proof.* We prove the invariant by induction on  $t$ . The invariant holds trivially for  $t = 1$  (the base case for the inductive proof): the cups start empty so  $\mu_{S_1}(S_1([k])) = 0 \leq 2n - k$  for all  $k \in [n]$ .

Fix a round  $t \geq 1$ , and any  $k \in [n]$ . We assume invariant for all values of  $k' \in [n]$  for state  $S_t$  (we will only explicitly use two of the invariants for each  $k$ , but the invariants that we need depend on the choice of  $p_t$  by the filler, so we actually need all of them) and show that the invariant on the  $k$  fullest cups holds on round  $t+1$ , i.e. that

$$\mu_{S_{t+1}}(S_{t+1}([k])) \leq 2n - k.$$

Note that because the emptier is greedy it always empties from the cups  $I_t([p_t])$ . Let  $A$ , with  $a = |A|$ , be  $A = I_t([\min(k, p_t)]) \cap S_{t+1}([k])$ ;  $A$  consists of cups that are among the  $k$  fullest cups in  $I_t$ , are emptied from, and are among the  $k$  fullest cups in  $S_{t+1}$ . Let  $B$ , with  $b = |B|$ , be  $B = I_t([\min(k, p_t)]) \setminus A$ ;  $B$  consists of the cups that are among the  $k$  fullest cups in state  $I_t$ , are emptied from, and aren't among the  $k$  fullest cups in  $S_{t+1}$ . Let  $C = I_t(p_t + [k - a])$ , with  $c = k - a = |C|$ .

<sup>7</sup>Note that in the lower bound proofs (e.g. Section 3) when we used the notation  $m$  (for mass) and  $\mu$  (for average fill), we omitted the subscript indicating the state at which the properties were measured. In those proofs it was sufficiently clear to leave the state implicit. However, in this section the state is crucial, and needs to be explicit in the notation.

Note that  $k - a \geq 0$  as  $a + b \leq k$ , and also  $|ABC| = k + b \leq n$ , because by definition the  $b$  cups in  $B$  must not be among the  $k$  fullest cups in state  $S_{t+1}$  so there are at least  $k + b$  cups. Note that  $a + b = \min(k, p_t)$ . We also have that  $A = I_t([a])$  and  $B = I_t(a + [b])$ , as every cup in  $A$  must have higher fill than all cups in  $B$  in order to remain above the cups in  $B$  after 1 unit of water is removed from all cups in  $AB$ . Further, note that  $S_{t+1}([k]) = AC$  because once the cups in  $B$  are emptied from the cups in  $B$  are not among the  $k$  fullest cups, so the cups in  $C$  take their places among the  $k$  fullest cups.

We now establish the following claim, which we call the *interchangeability of cups*:

**Claim 3.** *Without loss of generality*

$$S_t([r]) = I_t([r])$$

for any rank  $r \in [n]$ .

*Proof.* Say there are cups  $x, y$  with  $x \in S_t([r]) \setminus I_t([r])$ ,  $y \in I_t([r]) \setminus S_t([r])$ . Let the fills of cups  $x, y$  at state  $S_t$  be  $f_x, f_y$ ; note that  $f_x > f_y$ . Let the amount of fill that the emptier adds to these cups be  $\Delta_x, \Delta_y \leq 1$ ; note that  $f_x + \Delta_x < f_y + \Delta_y$ .

Define a new state  $S'_t$  where cup  $x$  has fill  $f_y$  and cup  $y$  has fill  $f_x$ . Let the amount of water that the filler places in these cups from the new state be  $f_x - f_y + \Delta_x$  and  $f_y - f_x + \Delta_y$  for cups  $x, y$  respectively. This is valid as both fill amounts are at most 1:  $f_x - f_y + \Delta_x < \Delta_y \leq 1$  and  $f_y - f_x + \Delta_x < \Delta_x \leq 1$ .

We can repeatedly apply this process to swap each cup in  $I_t([r]) \setminus S_t([r])$  into being one of the  $a + b$  fullest cups in the new state  $S'_t$ . At the end of this process we will have some “fake” state  $S_t^f$ . Note that  $S_t^f$  must satisfy the invariants if  $S_t$  satisfied the invariants, because our process can be thought of as just relabelling the cups; in particular  $\text{fill}(S_t^f(r)) = \text{fill}(S_t(r))$  for all ranks  $r \in [n]$ .

It is without loss of generality that we start in state  $S_t^f$  because from state  $I_t$  we could equally well have come from state  $S_t$  or state  $S_t^f$ . Thus we consider state  $I_t$  to have come from state  $S_t^f$ .  $\square$

Now we proceed with the proof of the Lemma.

**Claim 4.** *If some cup in  $A$  zeroes out then invariant holds.*

*Proof.* Say a cup in  $A$  zeroes out. Then

$$m_{S_{t+1}}(A) \leq (a - 1)(2n - (a - 1))$$

because the  $a - 1$  fullest cups must have satisfied the invariant on round  $S_t$ , and  $\text{fill}_{S_{t+1}}(I_{t+1}(a)) = 0$ . Furthermore, the fill of all cups in  $C$  must be at most 1 in  $I_t$  to be less than the fill of the cup in  $A$  that zeroed out.

Thus,

$$m_{S_{t+1}}(S_{t+1}([k])) \leq (a - 1)(2n - (a - 1)) + k - a.$$

We claim that this is less than  $a(2n - a)$ . This follows from simple manipulation of the above expression:

$$\begin{aligned} & (a - 1)(2n - (a - 1)) + k - a \\ &= a(2n - a) + a - 2n + a - 1 + k - a \\ &= a(2n - a) + (k - n) + (a - n) - 1 \\ &< a(2n - a) \end{aligned}$$

as desired.

As  $k$  increases from 1 to  $n$ ,  $k(2n - k)$  strictly increases (it is a quadratic in  $k$  that achieves its maximal value at  $k = n$ ). Thus  $a(2n - a) \leq k(2n - k)$  because  $a \leq k$ .

And we have

$$m_{S_{t+1}}(S_{t+1}([k])) \leq k(2n - k).$$

$\square$

**Claim 5.** *If no cups in  $A$  zero out and  $b = 0$  the invariant holds.*

*Proof.* First we consider the case  $b = 0$ . If  $b = 0$ , then  $S_{t+1}([k]) = S_t([k])$ .

The emptier has removed  $a$  units of fill from the cups in  $S_t([k])$ , specifically the cups in  $A$ . The filler cannot have added more than  $k$  fill to these cups, because it can add at most 1 fill to any given cup. Also, the filler cannot have added more than  $p_t$  fill to the cups because this is the total amount of fill that the filler is allowed to add. Hence the filler adds at most  $\min(p_t, k) = a + b = a + 0 = a$  fill to these cups. Thus the invariant holds:

$$m_{S_{t+1}}(S_{t+1}([k])) \leq m_{S_t}(S_t([k])) + a - a \leq k(2n - k).$$

$\square$

**Claim 6.** *If no cups in  $A$  zero out and  $b > 0$  the invariant holds.*

*Proof.* Because  $b > 0$  and  $a + b \leq k$  we have that  $a < k$ , and  $c = k - a > 0$ . Recall that  $S_{t+1}([k]) = AC$ , so the mass of the  $k$  fullest cups at  $S_{t+1}$  is the mass of  $AC$  at  $S_t$  plus any water added to cups in  $AC$  by the filler, minus any water removed from cups in  $AC$  by the emptier. The emptier removes exactly  $a$  units of water from  $AC$ . The filler adds no more than  $p_t$  units of water from  $AC$  (because the filler adds at most  $p_t$  total units of water per round) and the filler also adds no more than  $k = |AC|$  units of water from  $AC$  (because the filler adds at most 1 unit of water to each of the  $k$  cups in  $AC$ ). Thus, the filler adds no more than  $a + b = \min(p_t, k)$  units of water to  $AC$ . Combining these observations we have:

$$m_{S_{t+1}}(S_{t+1}([k])) \leq m_{S_t}(A) + m_{S_t}(C) + b.$$

The key insight necessary to bound this is to notice that larger values for  $m_{S_t}(A)$  correspond to smaller values for  $m_{S_t}(C)$  because of the invariants; the higher fill in  $A$  *pushes down* the fill that  $C$  can have. By quantifying exactly how much higher fill in  $A$  pushes down fill in  $C$  we can achieve the desired inequality. We can upper bound  $m_{S_t}(C)$  by

$$m_{S_t}(C) \leq \frac{c}{b + c} m_{S_t}(BC) = (m_{S_t}(ABC) - m_{S_t}(A)) \frac{c}{b + c}$$

because  $\mu_{S_t}(C) \leq \mu_{S_t}(B)$  without loss of generality by the interchangeability of cups. Thus we have

$$m_{S_t}(AC) \leq m_{S_t}(A) + \frac{c}{b+c} m_{S_t}(BC) \quad (6)$$

where

$$\begin{aligned} m_{S_t}(A) + \frac{c}{b+c} m_{S_t}(BC) \\ = \frac{c}{b+c} m_{S_t}(ABC) + \frac{b}{b+c} m_{S_t}(A). \end{aligned} \quad (7)$$

Note that the expression in Equation 7 is monotonically increasing in both  $\mu_{S_t}(ABC)$  and  $\mu_{S_t}(A)$ . Thus, by numerically replacing both average fills with their extremal values  $(2n - |ABC|, 2n - |A|)$  we upper bound  $m_{S_t}(A) + m_{S_t}(C)$ . At this point the inequality can be verified by straightforward algebra, however this is not elegant; instead, we combinatorially interpret the sum.

We define a new “fake” state  $F$ , which may not represent a valid configuration of cups (i.e. might not satisfy the invariants), where  $\mu_F(A) = 2n - |A|$  and  $\mu_F(ABC) = 2n - |ABC|$ , in particular with all the cups in  $A$  having identical fill, and all the cups in  $BC$  having identical fill. We can think of  $F$  as having come from a state where every cup has fill  $\mu_F(ABC) = 2n - |ABC|$ . To reach  $F$  from this state where every cup has identical fill we must increase the fill of each cup in  $A$  by some amount, and decrease the fill of each cup in  $BC$  by an amount such that the mass added to  $A$  is taken away from  $BC$ . To reach fill  $\mu_F(A) = 2n - |A|$ , the cups in  $A$  must have been increased by  $|BC|$  from their previous fill of  $2n - |ABC|$ . To equalize an increase in  $\mu_F(A)$  of  $|BC|$ , we need a corresponding decrease in  $\mu_F(BC)$  by  $|A|$ . That is,

$$\mu_F(BC) = 2n - |ABC| - |A|.$$

Thus we have the following bound:

$$\begin{aligned} m_{S_t}(A) + m_{S_t}(C) \\ \leq m_F(A) + c\mu_F(BC) \quad (*) \\ \leq a(2n - a) + c(2n - |ABC| - a) \\ \leq (a + c)(2n - a) - c(a + c + b) \\ \leq (a + c)(2n - a - c) - cb, \end{aligned}$$

where  $(*)$  follows from Equation 7.

Consider a new configuration of fills  $F$  achieved by starting with state  $S_t$ , and moving water from  $BC$  into  $A$  until  $\mu_F(A) = 2n - |A|$ .<sup>8</sup> This transformation increases (strictly increases if and only if we move a non-zero amount of water) the mass in  $AC$  because water in  $BC$  counts less towards mass in  $AC$  than water in  $A$  by Inequality 6. In particular, if mass  $\Delta \geq 0$  fill is moved from  $BC$  to  $A$ , then the mass of  $AC$  increases by  $\frac{b}{b+c}\Delta \geq 0$ .

Since  $\mu_F(A)$  is above  $\mu_F(ABC)$ , the greater than average fill of  $A$  must be counter-balanced by the lower than average fill of  $BC$ . In particular we must have  $(\mu_F(A) - \mu_F(ABC))|A| = (\mu_F(ABC) - \mu_F(BC))|BC|$ . Note that  $\mu_F(A) - \mu_F(ABC) \geq (n - |A|) - (n - |ABC|) = |BC|$ . Hence we must have

$$\mu_F(ABC) - \mu_F(BC) \geq |A|.$$

<sup>8</sup>Note that whether or not  $F$  satisfies the invariants is irrelevant.

Thus

$$\begin{aligned} \mu_F(BC) &\leq \mu_F(ABC) - |A| \leq 2n - |ABC| - |A|. \\ \text{Thus we have the following bound:} \\ m_{S_t}(A) + m_{S_t}(C) \\ &\leq m_F(A) + c\mu_F(BC) \\ &\leq a(2n - a) + c(2n - |ABC| - a) \\ &\leq (a + c)(2n - a) - c(a + c + b) \\ &\leq (a + c)(2n - a - c) - cb. \end{aligned}$$

Recall that we were considering  $b > 0$ , and since  $b > 0$  we have that  $c = k - a \geq b > 0$ , i.e.  $c \geq 1$ . Hence we have  $m_{S_t}(A) + m_{S_t}(C) \leq k(2n - k) - b$

So

$$m_{S_t}(A) + m_{S_t}(C) + b \leq k(2n - k).$$

As shown previously the left hand side of the above expression is an upper bound for  $m_{S_{t+1}}([k])$ . Hence the invariant holds.  $\square$

We have shown the inductive hypothesis for arbitrary  $k$ , so given that the invariants all hold at state  $S_t$  they also must all hold at state  $S_{t+1}$ . Thus, by induction we have the invariant for all rounds  $t \in \mathbb{N}$ .  $\square$

## 5 Oblivious Filler Lower Bound

An important theorem that we use throughout our analysis is Hoeffding’s Inequality:

**Theorem 1** (Hoeffding’s Inequality). *Let  $X_i$  for  $i = 1, 2, \dots, k$  be independent bounded random variables with  $X_i \in [a, b]$  for all  $i$ . Then,*

$$\Pr \left( \left| \frac{1}{k} \sum_{i=1}^k (X_i - \mathbb{E}[X_i]) \right| \geq t \right) \leq 2 \exp \left( - \frac{2kt^2}{(b-a)^2} \right)$$

Let  $S$  be a finite population, let  $X_i$  for  $i = 1, 2, \dots, k$  be chosen uniformly at random from  $S \setminus \{X_1, \dots, X_{i-1}\}$ , and let  $Y_i$  for  $i = 1, 2, \dots, k$  be chosen uniformly at random from  $S$ . Note that  $\{X_1, \dots, X_k\}$  represents a sample of  $S$  chosen without replacement, whereas  $\{Y_1, \dots, Y_k\}$  represents a sample with replacement. Note that as the  $Y_i$  are independent random variables Hoeffding’s Inequality provides a bound on the probability of  $\sum_{i=1}^k Y_i$  deviating from its mean by more than  $t$ .

The same bound can be given on the probability of  $\sum_{i=1}^k X_i$  deviating significantly from its mean, because the probability of  $\sum_{i=1}^k X_i$  deviating from its expectation by more than  $t$  is at most the probability of  $\sum_{i=1}^k Y_i$  deviating from its mean by  $t$ . Formally we can write this as

**Corollary 2.** *Let  $S$  be a finite set with  $\min(S) \geq a, \max(S) \leq b$ , and let  $X_i$  for  $i = 1, 2, \dots, k$  be chosen uniformly at random from  $S \setminus \{X_1, \dots, X_{i-1}\}$ . Then*

$$\Pr \left( \left| \frac{1}{k} \sum_{i=1}^k (X_i - \mathbb{E}[X_i]) \right| \geq t \right) \leq 2 \exp \left( - \frac{2kt^2}{(b-a)^2} \right)$$



Hoeffding proved Corollary 2 in his seminal work [1] (the result follows from his Theorem 4, combined with Hoeffding's Inequality for independent random variables). The intuition behind Corollary 2 is that samples drawn without replacement should be more tightly concentrated around the mean than samples drawn with replacement.

Another important, yet very trivial, corollary of Hoeffding's Inequality is the Chernoff Bound (i.e. Hoeffding's Inequality applied to binary random variables):

**Corollary 3.** *Let  $X_i$  for  $i = 1, 2, \dots, k$  be independent identically distributed binary random variables (i.e.  $X_i \in \{0, 1\}$ ). Then*

$$\Pr \left( \left| \frac{1}{k} \sum_{i=1}^k (X_i - \mathbb{E}[X_i]) \right| \geq t \right) \leq 2 \exp(-2kt^2)$$

We proceed with our analysis of oblivious lower bounds.

Call a cup configuration ***M-flat*** if the fill of every cup is in the interval  $[-M, M]$ .

We call an emptier  ***$\Delta$ -greedy-like*** if, when there are two cups  $c_1, c_2$  with fills satisfying  $\text{fill}(c_1) > \text{fill}(c_2) + \Delta$  the emptier never empties from  $c_2$  without emptying from  $c_1$  on the same round. Intuitively, a  $\Delta$ -greedy-like emptier has a  $\pm\Delta$  range where it is allowed to “not be greedy”. Note that a perfectly greedy emptier is 0-greedy-like. We call an emptier ***greedy-like*** if it is  $\Delta$ -greedy-like for  $\Delta \leq O(1)$ .

In the randomized setting we are only able to prove lower bounds for backlog against greedy-like emptiers; whether or not our results can be extended to a more general class of emptiers is an interesting open question.

We now prove a crucial property of greedy-like emptiers: that they are ***flattenable***, i.e.:

**Proposition 5.** *Given a cup configuration that is  $M$ -flat, an oblivious filler can, in running time  $2M$ , achieve a  $2(2 + \Delta)$ -flat configuration of cups against a  $\Delta$ -greedy-like emptier, (where fill is measured relative to average fill, with negative fill allowed).*

*Proof.* The filler sets  $p = n/2$  and distributes fill equally amongst all cups at every round, in particular placing  $1/2$  units of water in each cup. Let  $\ell_t = \min_{c \in S_t} \text{fill}_{S_t}(c)$ ,  $u_t = \max_{c \in S_t} \text{fill}_{S_t}(c)$ . Let  $L_t$  be the set of cups  $c$  with  $\text{fill}_{S_t}(c) \leq \ell_t + 2 + \Delta$ , and let  $U_t$  be the set of cups  $c$  with  $\text{fill}_{S_t}(c) \geq u_t - 2 - \Delta$ .

There are two ways to think of  $U_t$ . First we can consider  $U_t$  as capturing cups in the union of intervals of length  $1, \Delta$ , and  $1$ . Note the key property that if a cup with fill in  $[u_t - \Delta - 2, u_t - \Delta - 1]$  is emptied from, then all cups with fills in  $[u_t - 1, u_t]$  must be emptied from, because the emptier is  $\Delta$ -greedy-like. On the other hand, we can consider  $U_t$  as capturing cups with fill in the union of  $[u_t - 2, u_t]$  and  $[u_t - \Delta - 2, u_t - 2]$ . This is useful as the interval of width  $\Delta$  serves as a “buffer”. In particular, if there are more than  $n/2$  cups outside of  $U_t$  then all cups in  $[u_t - 2, u_t]$  must be emptied from because the emptier is  $\Delta$ -greedy-like.  $L_t$  is of course completely symmetric to  $U_t$ .

First we prove a key property of the sets  $U_t$  and  $L_t$ : once a cup is in  $U_t$  or  $L_t$  it is always in  $U_{t'}, L_{t'}$  for all  $t' > t$ . This follows immediately from the following claim:

**Claim 7.**

$$U_t \subseteq U_{t+1}, L_t \subseteq L_{t+1}.$$

*Proof.* Consider a cup  $c \in U_t$ .

If  $c$  is not emptied from, i.e.  $\text{fill}(c)$  has increased by  $1/2$ , then clearly  $c \in U_{t+1}$ , because backlog has increased by at most  $1/2$ , so the fill of  $c$  must still be within  $2 + \Delta$  of the backlog on round  $t + 1$ .

On the other hand, if  $c$  is emptied from, i.e.  $\text{fill}(c)$  has decreased by  $1/2$ , we consider two cases.

**Case 1:** If  $\text{fill}_{S_t}(c) \geq u_t - \Delta - 1$ , then, as  $u_{t+1} \leq u_t + 1/2$ ,  $\text{fill}_{S_{t+1}}(c) \geq u_t - \Delta - 1 - 1/2 \geq u_{t+1} - \Delta - 2$ .

**Case 2:** On the other hand, if  $\text{fill}_{S_t}(c) < u_t - \Delta - 1$ , then every cup with fill in  $[u_t - 1, u_t]$  must have been emptied from. The fullest cup at round  $t + 1$  is the same as the fullest cup on round  $t$ , because the fills of all cups with fill in  $[u_t - 1, u_t]$  have decreased by  $1/2$ , and no cup with fill less than  $u_t - 1$  had fill increase by more than  $1/2$ . Hence  $u_{t+1} = u_t - 1/2$ . Because both the fill of  $c$  and the backlog have decreased by the same amount, the distance between them is still at most  $\Delta + 2$ , hence  $c \in U_{t+1}$ . The argument for  $L_t \subseteq L_{t+1}$  is essentially identical.  $\square$

Now that we have shown that  $L_t$  and  $U_t$  never lose cups, we will show that they eventually gain a substantial number of cups.

**Claim 8.** *As long as  $|U_t| \leq n/2$  we have  $u_{t+1} = u_t - 1/2$ . Identically, as long as  $|L_t| \leq n/2$  we have  $\ell_{t+1} = \ell_t + 1/2$ .*

*Proof.* If there are more than  $n/2$  cups outside of  $U_t$  then there must be some cup with fill less than  $u_t - \Delta - 2$  that is emptied from. Because the emptier is  $\Delta$ -greedy-like this means that the emptier must empty from every cup with fill at least  $u_t - 2$ . Thus  $u_{t+1} = u_t - 1/2$ : no cup with fill less than  $u_t - 2$  could have become the fullest cup, and the previous fullest cup has lost  $1/2$  units of fill.

The proof is identical for  $L_t$ .  $\square$

By Claim 8 we see that both  $|U_t|$  and  $|L_t|$  must eventually exceed  $n/2$  at some times  $t_u, t_\ell \leq 2M$ , by the assumption that the initial configuration is  $M$ -flat. Since by Claim 7  $|U_{t+1}| \geq |U_t|$  and  $|L_{t+1}| \geq |L_t|$  we have that there is some round  $t_0 = \max(t_u, t_\ell) \leq 2M$  on which both  $|U_{t_0}|$  and  $|L_{t_0}|$  exceed  $n/2$ . Then  $U_{t_0} \cap L_{t_0} \neq \emptyset$ . Furthermore, the sets must intersect for all  $t_0 \leq t \leq 2M$ . In order for the sets to intersect it must be that the intervals  $[u_t - 2 - \Delta, u_t]$  and  $[\ell_t, \ell_t + 2 + \Delta]$  intersect. Hence we have that

$$\ell_t + 2 + \Delta \geq u_t - 2 - \Delta.$$

Since  $u_t \geq 0$  and  $\ell_t \leq 0$  this implies that all cups have fill in  $[-2(2 + \Delta), 2(2 + \Delta)]$ .  $\square$

Given a  $\Delta$ -greedy-like filler, let  $R_\Delta = 2(2 + \Delta)$ . By Proposition 5, if a filler is given a  $M$ -flat configuration of cups they can achieve a  $R_\Delta$ -flat configuration of cups.

Now we are equipped to prove the following proposition:

**Proposition 6.** *Let  $H \leq O(1)$ ,  $M \leq \text{poly}(n)$ ,  $\Delta \leq O(1)$ ,  $n \geq \Omega(1)$  at least a sufficiently large constant.*

*There exists an oblivious filling strategy for the variable-processor cup game on  $n$  cups that can achieve backlog  $H$  in a  $M$ -flat configuration of cups (relative to average fill with negative fill allowed) in running time  $\text{poly}(n)$  against a  $\Delta$ -greedy-like emptier, with probability at least  $1 - 2^{-\Omega(n)}$ .*

*Proof.* The filler starts by flattening all the cups, using the flattening procedure detailed in Proposition 5.

Let  $A$ , the **anchor** set, be a subset of the cups chosen uniformly at random from all subsets of size  $n/2$  of the cups, and let  $B$ , the **non-anchor** set, consist of the rest of the cups ( $|B| = n/2$ ). Let  $h = 16\Delta + 16$ , and let  $h' = 2$ . Note that the average fill of  $A$  and  $B$  both must start as at least  $-R_\Delta$  due to the flattening.

The filler's strategy is roughly as follows:

- **Step 1:** Make a constant fraction of the cups in  $A$  have fill at least  $h$  by playing single processor cup games on constant-size subsets of  $B$  and then swapping the cup within  $B$  that has high fill, with constant probability, into  $A$ . By a Chernoff bound this makes a constant fraction of  $A$ , say  $nc$  cups, have fill at least  $h$  with exponentially good probability. Between single-processor cup games the filler flattens  $B$ .
- **Step 2:** Reduce the number of processors to  $nc$ , and raise the fill of  $nc$  known cups to fill  $h'$ . The emptier must first empty from the cups with fill  $h$  before emptying from the cups that the filler is attempting to get fill  $h'$  in.
- **Step 3:** Recurse on the  $nc$  cups that are known to have fill at least  $h'$ .

To achieve Step 1 the filler performs a series of **swapping-processes**, which are procedures that the filler uses to get a new cup –hopefully with high fill– in  $A$ . A swapping-process is composed of a substructure, repeated many times, which we call a **round-block**; a round-block is a set of rounds. A swapping-process will consist of  $n \cdot c_\Delta$  round-blocks ( $c_\Delta = \Theta(1)$  a function of  $\Delta$  to be specified); at the beginning of each swapping-process the filler chooses a round-block  $j$  uniformly at random from  $[n \cdot c_\Delta]$ .

For each round-block  $i \in [n \cdot c_\Delta]$ , the filler selects a random subset  $D_i \subset B$  of the non-anchor cups and plays a single processor cup game on  $D_i$ . In this single-processor cup game the filler essentially employs the classic adaptive strategy for achieving backlog  $\Omega(\log |B|)$  on a set of  $|B|$  cups, with slight modifications for the fact that it is oblivious. In particular, the filler will only achieve this fill with constant probability. While doing this, the filler always places 1 unit of fill in each cup in the anchor set. Note that the filler sets  $p = n/2 + 1$ .

At the end of each round-block the filler applies the flattening procedure to flatten the non-anchor set. Note that this will not affect the running-time beyond a multiplicative factor (of e.g. 3).

On most round-blocks – all but the  $j$ -th – the filler does nothing with the cup that it achieves with constant probability in its single processor cup game. However, on the  $j$ -th round-block the filler swaps the “winner” of the single processor cup game into the anchor set (with constant probability there is a winner).

**Claim 9.** *With probability at least  $1 - 2^{-\Omega(n)}$ , the filler achieves fill at least  $h$  in at least  $nc = \Theta(n)$  of the cups in  $A$ .*

*Proof.* Consider a particular swapping-process. Let  $j$ , the round-block on which the filler will perform the swap, be chosen uniformly randomly from  $[n \cdot c_\Delta]$  ( $c_\Delta$  to be determined).

Say the emptier **neglects** the anchor set during a round-block if on at least one round of the round-block the emptier does not empty from every cup in the anchor set. By playing the single-processor cup game for many round-blocks with only one round-block when the filler actually swaps a cup into the anchor set, the filler prevents the emptier from neglecting the anchor set too often.

The fill of any cup in the anchor set can clearly never exceed  $R_\Delta + \Delta$  because  $B$  is  $R_\Delta$ -flat at the start of each round-block (a cup with fill this high would necessarily be emptied from). Let  $\mu_\Delta = 2R_\Delta + \Delta$ ; the emptier can neglect the anchor set no more than  $(n/2)\mu_\Delta$  times. **might need to mess with the size of  $A$  and  $B$  a bit to make this more true...** Furthermore, the average fill of  $B$  is thus always at least  $-\mu_\Delta$ . **actually this isn't quite true, the average fill sinks a (very) little bit (a very small constant).** As  $B$  is  $R_\Delta$ -flat this also means that the fills of cups in  $B$  at the start of each round-block are at least  $-\mu_\Delta - R_\Delta$ .

On each round-block the filler chooses a random subset  $D_i \subset B$  of  $\lceil e^{2h} \rceil$  cups. If the emptier does not neglect the anchor set on round-block  $i$  then the filler plays a legitimate single-processor cup game on  $n$  cups. The filler maintains an **active-set** of cups, which is a subset of  $D_i$  initialized to  $D_i$ . On each round of the round-block the filler distributes 1 unit of fill equally among all cups in the active set. Then the emptier removes fill from some cup in  $B$ . The filler chooses a random cup to remove from the active set. The probability that the cup the emptier emptied from is not in the active set after a random cup is removed from the active set by the filler is at least constant. By the end of the round-block the active-set will consist of a single cup. With constant probability, in particular probability at least

$$q_0 = 1/\lceil e^{2h} \rceil!,$$

this cup has gained fill at least  $\ln \lceil e^{2h} \rceil \geq 2h$ . Recalling that the cups fill started as at least  $-\mu_\Delta - R_\Delta$ , we have that this cup now has fill at least  $2h - \mu_\Delta - R_\Delta$ ; by design in choosing  $h$  this quantity is at least  $h$ .

Now we shall choose  $c_\Delta$ , choosing it large enough such that with constant probability there is some round-block

on which the emptier doesn't neglect the anchor set on which the filler succeeds.

We choose

$$c_\Delta = 2 \frac{1}{q_0} \mu_\Delta.$$

By having  $n \cdot c_\Delta$  round-blocks, we make it so that there should be at least  $n\mu_\Delta$  round-blocks on which the filler correctly guesses the emptier's emptying sequence. Formally this is due to a Chernoff bound: the expectation of the number of rounds when the filler correctly guesses the emptier's emptying sequence is at least  $2n\mu_\Delta$ , and the probability that it deviates from its expectation by more than  $n\mu_\Delta$  is hence exponentially small in  $n$ . As shown before, the emptier cannot neglect the anchor set more than  $(n/2)\mu_\Delta$  times. The filler correctly guesses the emptiers emptying sequence on the  $j$ -th round-block. Conditioned on this event, the  $j$  is chosen uniformly randomly from all the round-blocks on which the filler correctly guesses the emptiers emptying sequence. Since the emptier can neglect the anchor set on at most half of these round-blocks there is at least a  $1/2$  chance that  $j$  is chosen on a round-block where the filler does not neglect the anchor set. Thus, overall, there is at least a constant probability of achieving fill  $h$  in a cup in  $A$ .

Say that a swapping-process *succeeds* if the filler is able to swap a cup with fill at least  $h$  into  $A$ . We have shown that there is a constant probability of a given swapping-process succeeding. Let  $X_i$  be the binary random variable indicating whether or not the  $i$ -th swapping process succeeds. Let  $q \geq \Omega(1)$  be the probability of a swapping-process succeeding, i.e.  $\Pr(X_i = 1)$ . Note that the random variables  $X_i$  are clearly independent, and identically distributed.

Clearly

$$\mathbb{E} \left[ \sum_{i=1}^{n/4} X_i \right] = qn/4.$$

By a Chernoff Bound (i.e. Hoeffding's Inequality applied to binary random variables),

$$\Pr \left( \sum_{i=1}^{n/4} X_i \leq nq/8 \right) \leq e^{-nq^2/128}.$$

That is, the probability that less than  $nq/8$  of the anchor cups have fill at least  $h$  is exponentially small in  $n$ , as desired.  $\square$

Hence Step 1 is possible.

Step 2 is easily achieved by setting  $p = nc$  and uniformly distributing the fillers fill among a chosen set  $S$  of  $nc$  cups. The greedy nature of the emptier will force it to focus on the cups which must exist in  $A$  with large positive fill until the cups in  $S$  have sufficiently high fill. In particular, the fills of the cups in  $S$  must start as at least  $-\mu_\Delta - \Delta \geq -h/2$  by design in choosing  $h$ . After removing from the very full cups for  $\lceil h/2 + h' \rceil$  rounds the fills of these new cups are clearly at least  $h'$ . Note that throughout this

process the emptier cannot have emptied from the cups in  $S$  until they attained fill  $h'$  because there would be  $p = nc$  cups at least  $\lceil h/2 + h' \rceil \geq h' + \Delta$  by design in choice of  $h$ .

Step 3, which is to recurse, is of course possible. By performing  $H$  levels of recursion, increasing the fill by  $h' = 2$  and reducing the problem size by a factor of  $c$  at each level of recursion, the filler achieves backlog at least  $2H$ . Say the probability of Step 1 succeeding is at least  $1 - e^{-nk}$ . Then (by the union bound) the probability that any of  $H$  levels of recursion fail is bounded above by

$$e^{-nk} + e^{-nck} + e^{-nc^2k} + \dots + e^{-nc^Hk} \leq 2^{-\Omega(n)}.$$

Hence the probability that every level of recursion succeeds is at least  $1 - 2^{-\Omega(n)}$ .  $\square$

**Lemma 3** (The Oblivious Amplification Lemma). *Let  $\Delta \leq O(1)$ ,  $M, M' \geq R_\Delta$ ,  $q \geq \Omega(1)$ ,  $f$  be an oblivious filling strategy that achieves backlog  $f(n)$  (relative to average fill with negative fill allowed) in the variable-processor cup game on  $n$  cups with probability at least  $1 - 2^{-qn}$  in running time  $T(n) \leq \text{poly}(n)$  when given a  $M$ -flat configuration, against a  $\Delta$ -greedy-like emptier.*

*Let  $0 < \delta \ll 1/2$ ,  $L \in \mathbb{N}$ , and  $\eta \in \mathbb{N}$  ( $\eta$  chosen sufficiently large as a function of  $\delta, \Delta$ ), be appropriate constant parameters. Let  $0 \ll \phi < 1$  be a constant which will be defined in terms of  $\eta$  satisfying  $\lim_{\eta \rightarrow \infty} \phi = 1$ .*

*There exists an oblivious filling strategy that achieves backlog*

$$f'(n) \geq \phi \cdot (1 - \delta) \sum_{\ell=0}^L f((1 - \delta)\delta^\ell n)$$

*in the variable processor cup game on  $n$  cups given a  $M'$ -flat configuration of cups in running time*

$$T'(n) \leq O(M') + (\delta L)n^{\eta+1}T((1 - \delta)n)$$

*against a  $\Delta$ -greedy-like emptier with probability at least  $1 - 2^{-\Omega(n)}$ .*

*Proof.* The proof is quite similar to the proof of Lemma 1, but more complicated because the filler's strategy must be randomized.

The filler starts by flattening all the cups, using the flattening procedure detailed in Proposition 5.

Let  $A$ , the **anchor** set, be a subset of the cups chosen uniformly at random from all subsets of size  $n\delta$  of the cups, and let  $B$ , the **non-anchor** set, consist of the rest of the cups ( $|B| = n(1 - \delta)$ ). Let  $n_\ell = n\delta^{\ell-1}$ ,  $h_\ell = (1 - \delta)f(n_\ell(1 - \delta))$ ; the filler will achieve average fill  $h_\ell$  on a set of  $n_\ell\delta$  cups on the  $\ell$ -th level of its recursive process. Note that the average fill of  $A$  and  $B$  both must start as at least  $-R_\Delta$  due to the flattening.

The filler's strategy is essentially as follows:

- **Step 1:** Using  $f$  repeatedly, achieve fill  $(1 - \delta)f(n(1 - \delta))$  in cups in the non-anchor set and then swap these cups into the anchor set.
- **Step 2:** Decrease the number of processors to  $p = \delta n$  and recurse on the anchor set.

First we show how to achieve Step 1. The filler's strategy will be to always place 1 fill in each cup in the anchor-set while applying  $f$  to  $B$ . As always, the filler cannot directly apply  $f$  to  $B$ ; the filler must ensure that the emptier is using the appropriate amount of resources on  $B$ .

For each cup in  $A$  the filler performs a procedure called a **swapping-process**, which consists of a sub-structure repeated many times that we call a **round-block**. Each round-block consists of an **attempt** to apply  $f$ . We say that the emptier **neglects** the anchor set on a round-block if there is at least 1 round on which the emptier does not empty from each cup in the anchor set. The mass of the anchor set increases by at least 1 on each round-block that the anchor set is neglected. This cannot happen more than  $n\delta(2R_\Delta + \Delta) = n\delta\mu_\Delta$  times. Thus, by making each swapping-process consist of  $n^\eta$  round-blocks (note: we do  $n^\eta$  round-blocks on all levels of recursion, everything else changes to  $n_\ell$  but not this) and then choosing a single round-block among these (uniformly at random) to swap a cup in to  $A$ , we guarantee that with probability at least  $\delta\mu_\Delta/n^\eta$  this round-block occurs on a round-block when the emptier does not neglect the anchor set. On this round-block  $f$  is legitimately applied, and succeeds with probability at least  $1 - 2^{-qn}$ . At the end of each round-block the filler flattens  $B$ , so that  $f$  is receiving as input a flattened set of cups as needed. Over the course of this process the average fill of  $B$  will decrease a little. In the most extreme case  $f$  may have succeeded up to  $\delta n$  times, in which case the mass transferred from  $B$  to  $A$  would be  $\delta n f((1 - \delta)n)$ . In order for there to be an increase in the difference of the average fills of  $A$  and  $B$  by this amount  $B$  would have had to contribute  $|A|/n = \delta$  of the difference, with  $A$  contributing  $|B|/n = (1 - \delta)$  of the difference. Hence the average fill of  $A$  would have actually only increased by  $(1 - \delta)f((1 - \delta)n)$ . **the whole preceding part is a little bit sketchy.**

For Step 2 the filler simply recurses. The run-time bound is clear: The initial smoothing takes time  $O(M')$ , and after that, at each level of recursion for each cup in the anchor set the filler applies  $f$  to the non-anchor set  $n^\eta$  times. Hence the running time due to this is

$$\sum_{\ell=0}^L (\delta^\ell n \delta) n^\eta T((1 - \delta)\delta^\ell n).$$

This is quite complicated, a simpler bound suffices; bounding each term in the sum with the first term, which is clearly the largest, we have

$$T'(n) \leq O(M') + (L\delta)n^{\eta+1}T((1 - \delta)n).$$

It is almost clear that the desired backlog is achieved; if every swapping process succeeded then we would achieve fill  $(1 - \delta)f((1 - \delta)\delta^\ell n)$  in each cup in the anchor set at each level of recursion hence achieving backlog

$$(1 - \delta) \sum_{\ell=0}^L f((1 - \delta)\delta^\ell n)$$

overall. However each swapping process has some (very small) probability of failing; we computed probability of failure this to be at most  $\delta\mu_\Delta/n^\eta$ . Consider the

probability that more than a constant fraction  $w = \Theta(1)$  of the  $s = \sum_{\ell=0}^L n\delta^{\ell+1}$  swapping-processes fail. Let  $X_i$  be the random variable indicating whether the  $i$ -th swapping-process succeeds (note: this is swapping-processes on all levels of recursion), and let  $X = \sum_{i=1}^s X_i$ . Clearly  $\mathbb{E}[X] = s(1 - \delta\mu_\Delta/n^\eta)$ . Success of the swapping-processes are not independent events: a swapping-process is in-fact more likely to succeed given that previous swapping-processes have failed. Hence we can upper bound the probability of more than a  $w$ -fraction of the swapping-processes failing by a Chernoff Bound:

$$\Pr\left(\frac{1}{s}X \geq \frac{1}{s}\mathbb{E}[X] - w/2\right) \geq 1 - 2e^{-sw^2/2} \geq 1 - 2^{\Omega(n)}$$

By appropriately large choice for  $\eta \leq O(1)$ ,

$$\delta\mu_\Delta/n^\eta \leq w/2$$

no matter how small  $w \geq \Omega(1)$  is chosen. In particular this implies that  $\Pr[X \geq s(1 - w)] \geq 1 - 2^{\Omega(n)}$ .

Now we will define  $\phi$  such that success of  $s(1 - w)$  of the swapping-processes guarantees backlog

$$\phi \cdot (1 - \delta) \sum_{\ell=0}^L f(n(1 - \delta)\delta^\ell).$$

In the worst case the failed swapping-processes bring very negative cups into the anchor-set, potentially as negative as  $-\delta f((1 - \delta)\delta^\ell n)$  on the  $\ell$ -th level of recursion. However, clearly this is equivalent to removing at most 2 cups worth of mass from the anchor set. Overall we thus remove at most  $2w$  cups worth of mass from the anchor set. Hence choosing  $\phi = 1 - 2w$  works.

In order to achieve this backlog however, not only does the filler need to be able to swap over  $s(1 - w)$  cups on rounds where the emptier neglects the anchor set, but no applications of  $f$  can fail; failure happens with probability  $2^{-n(1 - \delta)\delta^\ell q}$  for an application of  $f$  to  $n(1 - \delta)\delta^\ell$  cups. Taking a union bound over the  $\text{poly}(n)$  applications of  $f$  clearly still gets probability failure at most  $2^{-\Omega(n)}$ .

Thus overall the filler succeeds with at least probability  $1 - 2^{-\Omega(n)}$ .  $\square$

**Corollary 4.** *There is an oblivious filling strategy for the variable-processor cup game on  $n$  cups that achieves backlog at least  $2^{\Omega(\sqrt{\log n})}$  in running time  $O(n)$  with constant probability.*

*Proof.* Given the Oblivious Amplification Lemma we could try to apply the same strategy as outlined in the proof of Corollary 1 to achieve backlog  $\Omega(n^{1-\epsilon})$  for constant  $\epsilon > 0$  in time  $2^{O(\log^2 n)}$ . Because of our definition of an overpowered cup as a cup with fill at least  $\tilde{\Omega}(\sqrt{n})$ , we can't get quite as close to linear as an adaptive filler could. However, the more pressing problem is that of running time: randomized algorithms are traditionally supposed to have polynomial-running time. By artificially reducing  $n$ , i.e. ignoring some portion of the cups, we can get an algorithm that achieves high backlog, but in polynomial time. In particular, we want

to choose a subset of  $n'$  of the cups to focus on, where  $2^{O(\log^2 n')} = O(n)$ . An appropriate choice is  $n' = 2^{\sqrt{\log n}}$ .

With  $n'$  chosen, we apply the exact same strategy as given in the paragraph in the proof of Corollary on the  $2^{O(\log^2 n)}$ -time construction for achieving backlog  $\Omega(n^{1-\epsilon})$  for constant  $\epsilon > 0$ , but using repeated application of the Oblivious Amplification Lemma rather than the Adaptive Amplification Lemma, which yields the disclaimer that the backlog is only achieved with constant probability. Thus, we achieve backlog  $\Omega(n')$  in running time  $2^{O(\log^2 n')}$ . By design, expressing this in terms of  $n$  the filler achieves backlog  $2^{\Omega(\sqrt{\log n})}$  in running time  $O(n)$ .

For completeness we – briefly (as they are nearly identical) – present the ideas from Proposition 3 in the randomized context.

Fix constant  $\epsilon > 0$  and choose appropriate constant  $\delta$ , as mandated by Claim 1. Choose constant  $c$ , according to an inequality to be specified later. We aim to achieve backlog  $c(n')^{1-\epsilon}$ . As before, we define a sequence of functions. First,

$$f_0(k) = \begin{cases} \lg k, & k \geq 1, \\ 0 & \text{else.} \end{cases}$$

Then, we define  $f_i$  as the amplification of  $f_{i-1}$  for  $i \geq 1$ , where amplification is as defined in the Oblivious Amplification Lemma. However, this time the filling strategy  $f_i$  achieves backlog  $f_i(k)$  on  $k$  cups only with constant probability. As before we define a sequence  $g_i$  as

$$g_i = \begin{cases} \lceil 1/\delta \rceil \gg 1, & i = 0, \\ \lceil g_{i-1}/(1-\delta) \rceil - 1 & i \geq 1. \end{cases}$$

Claim 2, which states that

$$f_i(k) \geq ck^{1-\epsilon} \text{ for all } k < g_i,$$

holds with no modifications required.

Again, because  $g_i$  is increasing, we achieve the desired backlog  $c(n')^{1-\epsilon}$  in finite time. In particular, applying identical arguments to those in Corollary 1, we find that the running time is  $2^{O(\log^2 n')}$ .

As stated earlier, by design of  $n'$ , this means we get backlog  $2^{\Omega(\sqrt{\log n})}$  in time  $O(n)$ .

□

**Claim 10.** *Without loss of generality the cup state is always  $(h\sqrt{n}/\lg \lg n)$ -flat.*

*Proof.* In order to flatten a set of cups we must have a bound on the magnitude of the fills of the cups. We claim that without loss of generality no cup has fill larger in magnitude than  $h\sqrt{n}/\log \log n$ . If a cup has more than  $h\sqrt{n}/\log \log n$  fill we call the cup **overpowered**. If there ever is an overpowered cup then we are automatically done: the emptier has achieved  $\text{poly}(n)$  backlog and will maintain it for at least  $\text{poly}(n)$  rounds. If a cup ever has fill less than  $-h\sqrt{n}/\log \log n$  then the absolute average fill must be large enough such that the absolute fill of this cup is at least 0. Thus there is an overpowered cup. From now on we assume that there are no overpowered cups. □

## 6 Conclusion

Many important open questions remain open.

## References

- [1] Wassily Hoeffding. Probability inequalities for sums of bounded random variables. *Journal of the American Statistical Association*, page 28, 1962.
- [2] William Kuszmaul. Achieving optimal backlog in the vanilla multi-processor cup game. *SIAM*, 2020.