

The Variable Processor Cup Game

Alek Westover

Belmont High School

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WHAT IS THE CUP GAME?

Definition

p-processor cup-game on n cups:

multi-round game

each round:

- ▶ *filler* distributes p units of water among the cups
(with at most 1 unit to any particular cup)
- ▶ *emptier* chooses p cups and removes (at most) 1 unit of water from each

WHAT IS THE CUP GAME?

Definition

Backlog:

Amount of water in the fullest cup

- ▶ Emptier tries to *minimize* backlog
- ▶ Filler tries to *maximize* backlog

WHY IS THE CUP GAME IMPORTANT?

Models *work scheduling*:

- ▶ Cups represent tasks
- ▶ At each time step:
 - ▶ new work comes in distributed among the tasks (*filler*)
 - ▶ allocate processors to work on tasks (*emptier*)

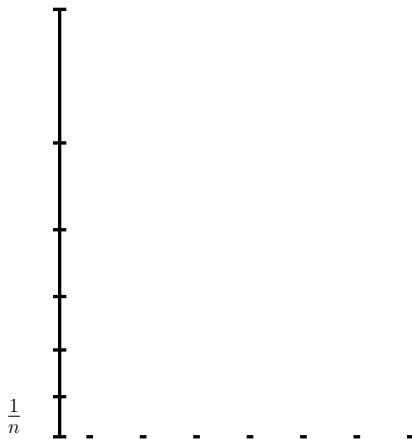
ANALYSIS OF THE CUP GAME

Prove *upper bounds* and *lower bounds* on backlog.

SINGLE-PROCESSOR LOWER BOUND

Filling strategy:

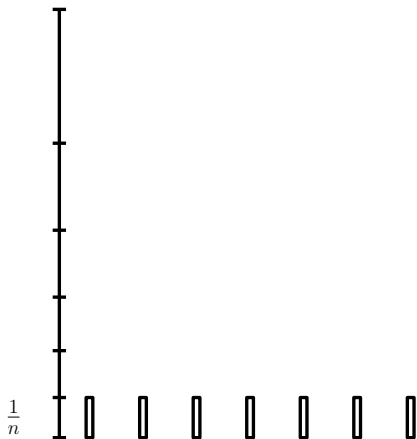
Distribute water equally amongst cups not yet emptied from.



SINGLE-PROCESSOR LOWER BOUND

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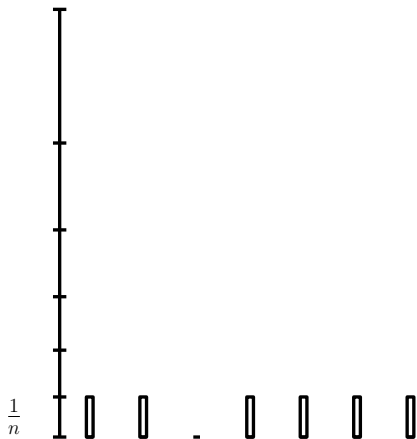
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SINGLE-PROCESSOR LOWER BOUND

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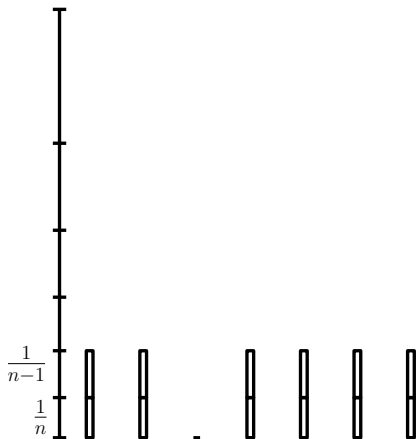
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SINGLE-PROCESSOR LOWER BOUND

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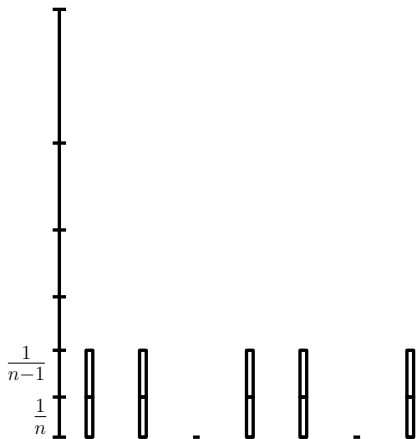
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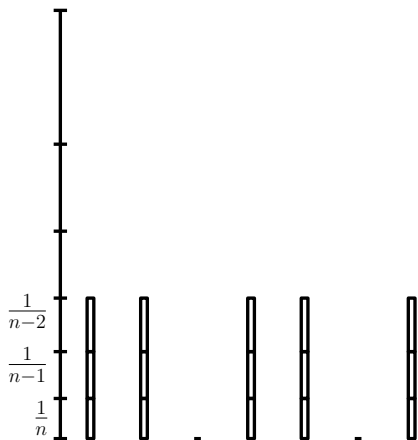
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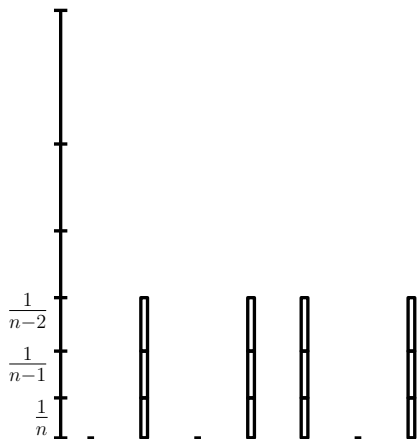
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SINGLE-PROCESSOR LOWER BOUND

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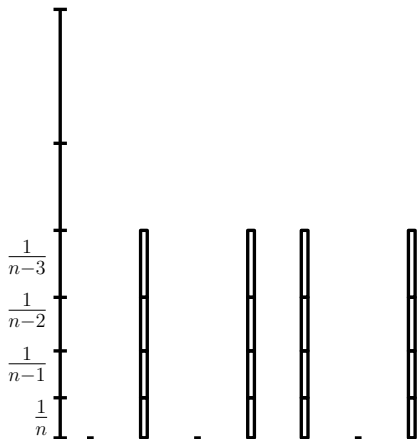
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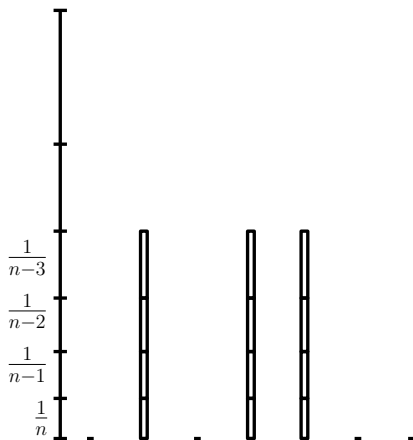
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SINGLE-PROCESSOR LOWER BOUND

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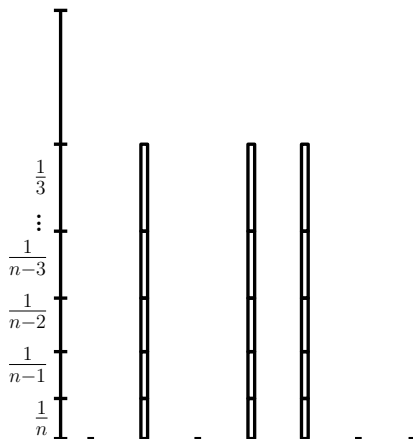
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SINGLE-PROCESSOR LOWER BOUND

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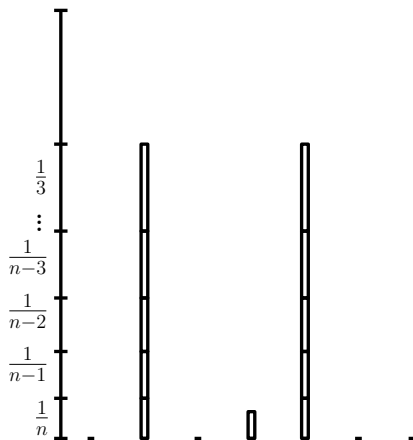
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SINGLE-PROCESSOR LOWER BOUND

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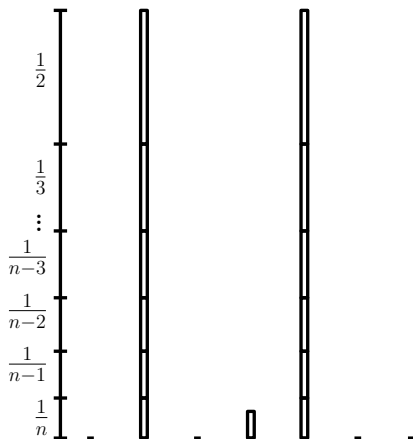
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SINGLE-PROCESSOR LOWER BOUND

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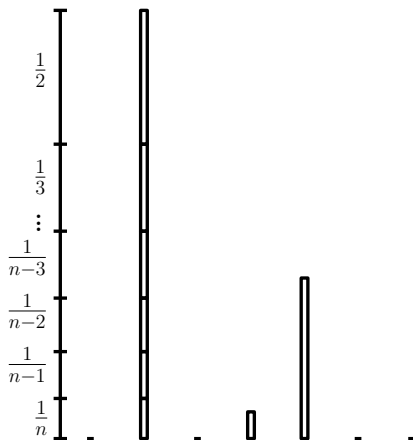
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SINGLE-PROCESSOR LOWER BOUND

Filling strategy:

Distribute water equally amongst cups not yet emptied from.



SINGLE-PROCESSOR LOWER BOUND

Filling strategy:

Distribute water equally amongst cups not yet emptied from.

Achieves backlog:

$$\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{2} = \Omega(\log n).$$

SINGLE-PROCESSOR UPPER BOUND

A *greedy emptier* – an emptier that always empties from the fullest cup – never lets backlog exceed $O(\log n)$.

Definitions

- ▶ S_t : state at start of round t
- ▶ I_t : state after the filler adds water on round t , but before the emptier removes water
- ▶ $\mu_k(S)$: average fill of k fullest cups at state S .

SINGLE-PROCESSOR UPPER BOUND PROOF

Proof: Inductively prove a set of invariants:

$$\mu_k(S_t) \leq \frac{1}{k+1} + \dots + \frac{1}{n}.$$

Let a be the cup that the emptier empties from on round t

If a is one of the k fullest cups in S_{t+1} :

$$\mu_k(S_{t+1}) \leq \mu_k(S_t).$$

Otherwise:

$$\mu_k(S_{t+1}) \leq \mu_{k+1}(I_t) \leq \mu_{k+1}(S_t) + \frac{1}{k+1}.$$

OUR VARIANT OF THE CUP GAME

Definition

Variable-Processor Cup Game:

Resources are variable

i.e. filler can change p

Modification may seem small, but it drastically alters the game

Adaptive Filler Lower Bound

NEGATIVE FILL

In lower bound proofs we allow *negative fill*

- ▶ Measure fill relative to average fill
- ▶ Important for recursion
- ▶ Strictly easier for the filler if cups can zero out

AMPLIFICATION LEMMA

Lemma

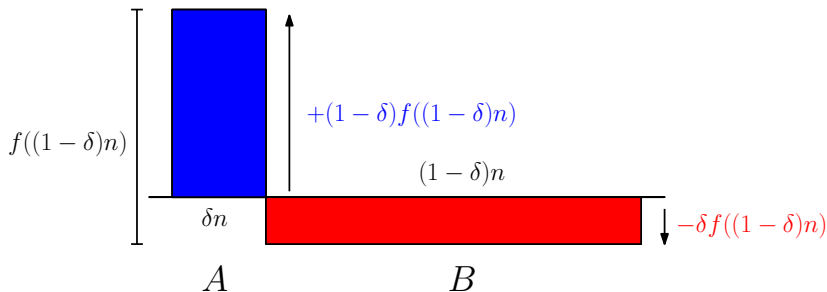
Given a strategy for achieving backlog $f(n)$ on n cups, we can construct a new strategy that achieves backlog

$$f'(n) \geq (1 - \delta) \sum_{\ell=0}^L f((1 - \delta)\delta^\ell n)$$

for appropriate parameters $L \in \mathbb{N}, 0 < \delta \ll 1/2$.

AMPLIFICATION LEMMA PROOF SKETCH

- ▶ A starts as the δn fullest cups, B as the $(1 - \delta)n$ other cups.
- ▶ Repeatedly apply f to B and swap generated cup into A .
- ▶ Decrease p , recurse on A .



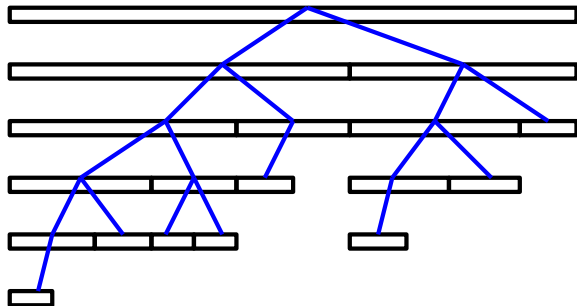
ADAPTIVE FILLER LOWER BOUND

Let $\epsilon > 0$ be any constant.

By repeated amplification using $\delta = \Theta(1)$ we get:

Theorem

There is an adaptive filling strategy that achieves backlog $\Omega(n^{1-\epsilon})$ in running-time $2^{O(\log^2 n)}$.

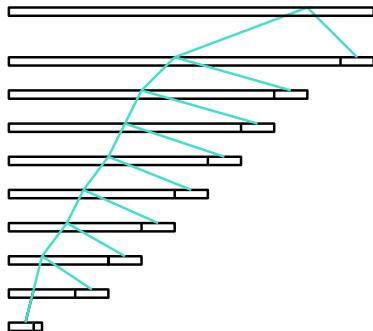


ADAPTIVE FILLER LOWER BOUND

By repeated amplification, using $\delta = \Theta(1/n)$, we get:

Theorem

There is an adaptive filling strategy that achieves backlog $\Omega(n)$ in running-time $2^{O(n)}$.



Upper Bound

UPPER BOUND

We prove a novel set of invariants:

Theorem

A greedy emptier maintains the invariant:

$$\mu_k(S_t) \leq 2n - k.$$

In particular this implies that backlog is

$$O(n).$$

Note: this matches our lower bound!

Oblivious Filler Lower Bound

OBLIVIOUS FILLER LOWER BOUND

Definition

Oblivious Filler: Can't observe the emptier's moves

Classically emptier does much better in the randomized setting.

But not in the variable-processor cup game!

OBLIVIOUS FILLER LOWER BOUND

Definition

Δ -greedy-like emptier:

Let x, y be cups. If $\text{fill}(x) > \text{fill}(y) + \Delta$ then a Δ -greedy-like emptier empties from y *only if* it also empties from x .

Oblivious filler can achieve backlog $\Omega(n^{1-\epsilon})$ for $\epsilon > 0$ constant in running time $2^{\text{polylog}(n)}$ against Δ -greedy-like emptiers for $\Delta \leq O(1)$ with probability at least $1 - 2^{-\text{polylog}(n)}$.

Same lower bound as an adaptive filler in quasi-polynomial length games!

FLATTENING

Definition

A cup configuration is R -flat if all cups have fills in $[-R, R]$.

Proposition

An oblivious filler can get a $2(2 + \Delta)$ -flat configuration from an R -flat configuration against a Δ -greedy-like emptier in running time $O(R)$.

OBLIVIOUS FILLER: CONSTANT FILL

Getting constant fill in a *known* cup is hard now. Strategy:

- ▶ Play many single-processor cup games on $\Theta(1)$ cups blindly. Each succeeds with constant probability.
- ▶ By a Chernoff Bound with probability $1 - 2^{-\Omega(n)}$ at least a constant fraction nc of these succeed
- ▶ Set $p = nc$.
- ▶ Fill nc known cups; because emptier is greedy-like it must focus on the nc cups with high fill before these cups.
- ▶ Recurse on the nc known cups with high fill

OBLIVIOUS AMPLIFICATION LEMMA

Almost identical to the Adaptive Amplification Lemma!

Lemma

Given a strategy f for achieving backlog $f(n)$ on n cups, we can construct a new strategy that achieves backlog

$$f'(n) \geq \phi \cdot (1 - \delta) \sum_{\ell=0}^L f((1 - \delta)\delta^\ell n)$$

for appropriate parameters $L \in \mathbb{N}$, $0 < \delta \ll 1/2$ and constant $\phi \in (0, 1)$ of our choice.

(Note: Lemma is actually more complicated than this.)

OBLIVIOUS FILLER LOWER BOUND

Theorem

There is an oblivious filling strategy that achieves backlog

$$\Omega(n^{1-\epsilon})$$

for constant $\epsilon > 0$ with probability at least $1 - 2^{-\text{polylog}(n)}$ in running time $2^{O(\log^2 n)}$.

Achieve this probability by a union bound on $2^{\text{polylog}(n)}$ events.

OPEN QUESTIONS

- ▶ Can we extend the oblivious lower bound construction to work with arbitrary emptiers?
- ▶ Are there shorter more simple constructions?

ACKNOWLEDGEMENTS

- ▶ My mentor William Kuszmaul
- ▶ MIT PRIMES
- ▶ My Parents

Question Slides

UPPER BOUND PROOF SKETCH

Want to show: $\mu_k(S_t) \leq 2n - k$. for all $k \in [n], t \in \mathbb{N}$

Define A, B, C . Consider extremal fill configuration:

