1 Oblivious Filler Lower Bound

In this section we prove that, surprisingly, an oblivious filler can achieve backlog $n^{1-\varepsilon}$, although only against a certain class of "greedy-like" emptiers.

The ***fill-range*** of a set of cups at a state S is $\max_c \text{fill}_S(c) - \min_c \text{fill}_S(c)$. We call a cup configuration ***R-flat*** if the fill-range of the cups less than or equal to R; note that in an R-flat cup configuration with average fill 0 all cups have fills in $[-R, R]$. We say an emptier is ***Δ-greedy-like*** if, whenever there are two cups with fills that differ by at least Δ, the emptier never empties from the less full cup without also emptying from the more full cup. That is, if on some round t, there are cups c_1, c_2 with $\text{fill}_{I_t}(c_1) > \text{fill}_{I_t}(c_2) + \Delta$, then a Δ-greedy-like emptier doesn't empty from c_2 on round t unless it also empties from c_1 on round t. Note that a perfectly greedy emptier is 0-greedy-like. We call an emptier ***greedy-like*** if it is Δ-greedy-like for $\Delta \le O(1)$.

With an oblivious filler we are only able to prove lower bounds on backlog against greedy-like emptiers; whether or not our results can be extended to a more general class of emptiers is an interesting open question. Nonetheless, greedy-like emptiers are of great interest because all the known randomized algorithm for the cup game are greedy-like [1, 4].

As a tool in our analysis we define a new variant of the cup game: In the p-processor ***E-extra-emptyings S-skip-emptyings*** negative-fill cup game on n cups, the filler distributes p units of water amongst the cups, and then the emptier empties from p *or more, or less* cups. In particular the emptier is allowed to do E extra emptyings—we say that the emptier does an extra emptying if it empties from a cup beyond the p cups it typically is allowed to empty from—and is also allowed to skip S emptyings—we say that the emptier skips an emptying if it doesn't do an emptying—over the course of the game. Note that the emptier still cannot empty from the same cup twice on a single round. Also note that the emptier is allowed to skip extra emptyings in addition to regular emptyings. Also note that a Δ-greedy-like emptier must take into account extra emptyings and skip emptyings to determine valid moves.

We now prove a crucial property of greedy-like emptiers:

Lemma 1. *Let $R_\Delta = 2(2+\Delta)$. Consider an R-flat cup configuration in the p-processor E-extra-emptyings S-skip-emptyings negative-fill cup game on $n = 2p$ cups. An oblivious filler can achieve a R_Δ-flat configuration of cups against a Δ-greedy-like emptier in running time $2(R + \lceil (1+1/n)(E+S) \rceil)$. Furthermore, throughout this process the cup configuration is R-flat on every round.*

Proof. If $R \le R_\Delta$ the algorithm does nothing, since the desired fill-range is already achieved; for the rest of the proof we consider $R > R_\Delta$.

The filler's strategy is to distribute fill equally amongst all cups at every round, placing $p/n = 1/2$ fill in each cup.

Let $\ell_t = \min_{c \in S_t} \text{fill}_{S_t}(c)$, $u_t = \max_{c \in S_t} \text{fill}_{S_t}(c)$. Let L_t be the set of cups c with $\text{fill}_{S_t}(c) \le l_t + 2 + \Delta$, and let U_t be the set of cups c with $\text{fill}_{S_t}(c) \ge u_t - 2 - \Delta$.

Now we prove a key property of the sets U_t and L_t: once a cup is in U_t or L_t it is always in $U_{t'}, L_{t'}$ for all $t' > t$. This follows immediately from Claim 1.

Claim 1.

$$U_t \subseteq U_{t+1}, \quad L_t \subseteq L_{t+1}.$$

Proof. Consider a cup $c \in U_t$.

If c is not emptied from, i.e. $\text{fill}(c)$ has increased by $1/2$ from the previous round, then clearly $c \in U_{t+1}$, because backlog has increased by at most $1/2$, so $\text{fill}(c)$ must still be within $2 + \Delta$ of the backlog on round $t+1$.

On the other hand, if c is emptied from, i.e. $\text{fill}(c)$ has decreased by $1/2$, we consider two cases.

Case 1: If $\text{fill}_{S_t}(c) \ge u_t - \Delta - 1$, then $\text{fill}_{S_t}(c)$ is at least 1 above the bottom of the interval defining which cups belong to U_t. The backlog increases by at most $1/2$ and the fill of c decreases by $1/2$, so $\text{fill}_{S_{t+1}}(c)$ is at least $1 - 1/2 - 1/2 = 0$ above the bottom of the interval, i.e. still in the interval.

Case 2: On the other hand, if $\text{fill}_{S_t}(c) < u_t - \Delta - 1$, then every cup with fill in $[u_t - 1, u_t]$ must have been emptied from because the emptier is Δ-greedy-like. Therefore the fullest cup on round $t+1$ is the same as the fullest cup on round t, because every cup with fill in $[u_t - 1, u_t]$ has had its fill decrease by $1/2$, and no cup with fill less than $u_t - 1$ had its fill increase by more than $1/2$. Hence $u_{t+1} = u_t - 1/2$. Because both $\text{fill}(c)$ and the backlog have decreased by $1/2$, the distance between them is still at most $\Delta + 2$, hence $c \in U_{t+1}$.

The argument for why $L_t \subseteq L_{t+1}$ is symmetric. □

Now we show that under certain conditions u_t decreases and ℓ_t increases.

Claim 2. *On any round t where the emptier empties from at least $n/2$ cups, if $|U_t| \le n/2$ then $u_{t+1} = u_t - 1/2$. On any round t where the emptier empties from at most $n/2$ cups, if $|L_t| \le n/2$ then $\ell_{t+1} = \ell_t + 1/2$.*

Proof. Consider a round t where the emptier empties from at least $n/2$ cups. If there are at least $n/2$ cups outside of U_t, i.e. cups with fills in $[\ell_t, u_t - 2 - \Delta]$, then all cups with fills in $[u_t - 2, u_t]$ must be emptied from; if one such cup was not emptied from then by the pigeon-hole principle some cup outside of U_t was emptied from, which is impossible as the emptier is Δ-greedy-like. This clearly implies that $u_{t+1} = u_t - 1/2$: no cup with fill less than $u_t - 2$ has gained enough fill to become the fullest cup, and the fullest cup from the previous rounds has lost $1/2$ units of fill.

By a symmetric argument $\ell_{t+1} = \ell_t + 1/2$ if the emptier empties at most $n/2$ cups on round t and $|L_t| \leq n/2$. $\square$

Now we show that eventually there is a cup in $L_t \cap U_t$.

Claim 3. *There is a round $t_0 \leq 2(R + \lceil(1+1/n)(E+S)\rceil)$ such that $U_t \cap L_t \neq \varnothing$ for all $t \geq t_0$.*

Proof. We call a round where the emptier doesn't use $p = n/2$ resources, i.e. a round where the number of skipped emptyings and the number of extra emptyings are not equal, an ***unbalanced round***; we call a round that is not unbalanced a ***balanced*** round.

Note that there are clearly at most $E + S$ unbalanced rounds. We now associate some unbalanced rounds with balanced rounds; in particular we define what it means for a balanced round to ***cancel*** an unbalanced round. We define cancellation by a sequential process. For $i = 1, 2, \ldots, 2(R + \lceil(1+1/n)(E+S)\rceil)$ (iterating in ascending order of i), if round i is unbalanced then we say that the first balanced round $j > i$ that hasn't already been assigned (earlier in the sequential process) to cancel another unbalanced round $i' < i$, if any such round j exists, ***cancels*** round i. Note that cancellation is a one-to-one relation: each unbalanced round is cancelled by at most one balanced round and each balanced round cancels at most one unbalanced round.

Consider rounds of the form $2(R + \lceil(E+S)/n\rceil) + (E+S) + i$ for $i \in [E+S+1] - 1$. We claim that there is some such i such that among rounds $[2(R + \lceil(E+S)/n\rceil) + (E+S) + i]$ every unbalanced round has been cancelled, and such that there are $2(R + \lceil(E+S)/n\rceil)$ balanced rounds not cancelling other rounds. Assume for contradiction that such an i does not exist. Note that there are at least $2(R + \lceil(E+S)/n\rceil)$ balanced rounds in the first $2(R + \lceil(E+S)/n\rceil) + (S+E)$ rounds. Thus every balanced round $2R + (E+S) + \lceil(E+S)/n\rceil + i - 1$ for $i \in [E+S+1]$ is necessarily a cancelling round, or else there would be a round by which there are no uncancelled unbalanced rounds. Hence by round $2(R + \lceil(E+S)/n\rceil) + 2(E+S)$, there must have been $E+S$ cancelled rounds, so on round $2(R + \lceil(E+S)/n\rceil) + 2(E+S)$ all unbalanced rounds are cancelled, which leaves $2(R + \lceil(E+S)/n\rceil)$ balanced rounds that are not cancelling any rounds, as desired.

Let t_e be the first round by which there are $2(R + \lceil(E+S)/n\rceil)$ balanced non-cancelling rounds. Note that the average fill of the cups could not have decreased by more than E/n from its starting value; similarly the average fill of the cups cannot have increased by more than S/n. Because the cups start R-flat, we have that u_t cannot have decreased by more than $R + E/n$, and ℓ_t cannot have increased by more than $R + S/n$. We now claim that by Claim 2 this implies that eventually $|L_t| > n/2$ and $|U_t| > n/2$. If $|L_t| \leq n/2$ were always true, then on every balanced round ℓ_t would have increased by $1/2$, and since ℓ_t increases by at most $1/2$ on unbalanced rounds, this implies that in total ℓ_t would have increased by at least $(1/2)2(R + \lceil(E+S)/n\rceil)$, which is impossible. By a symmetric argument it is impossible that $|U_t| \leq n/2$ for all rounds.

Since by Claim 1 $|U_{t+1}| \geq |U_t|$ and $|L_{t+1}| \geq |L_t|$, we have that there is some round $t_0 \in [2(R + \lceil(1+1/n)(E+S)\rceil)]$ such that for all $t \geq t_0$ we have $|U_t| > n/2$ and $|L_t| > n/2$. But then of course we have $U_t \cap L_t \neq \varnothing$, as desired. $\square$

If there exists a cup $c \in L_t \cap U_t$, then of course $\text{fill}(c) \in [u_t - 2 - \Delta, u_t] \cap [\ell_t, \ell_t + 2 + \Delta]$. Hence we have that

$$\ell_t + 2 + \Delta \geq u_t - 2 - \Delta.$$

Rearranging,

$$u_t - \ell_t \leq 2(2 + \Delta) = R_\Delta.$$

Thus the cup configuration is R_Δ-flat.

Now we prove that $u_t - \ell_t$ can only increase if $u_t - \ell_t$ is very small.

Claim 4. *If $u_{t+1} - \ell_{t+1} > u_t - \ell_t$ then $u_t - \ell_t \leq \Delta + 3/2$.*

Proof. Consider a round t where $u_{t+1} - \ell_{t+1} > u_t - \ell_t$. The amount of resources that the emptier used on round t must be in $(0, n)$, because if it were either extreme (i.e. 0 or n) then the fills of all cups would change in the same way. In order for the difference $u_t - \ell_t$ to increase either a) some cup with fill in $[\ell_t, \ell_t + 1/2]$ was emptied from and some cup with fill in $[u_t - 1, u_t]$ was not emptied from, or b) some cup with fill in $[u_t - 1/2, u_t]$ was not emptied from and some cup with fill in $[\ell_t, \ell_t + 1]$ was emptied from.

In either case, because the emptier is Δ-greedy-like, such an action implies

$$u_t - \ell_t \leq \Delta + 3/2.$$

□

Finally we establish that throughout this flattening process the cup configuration is always R-flat. By Claim 4, on any round where $u_t - \ell_t$ increases we have

$$u_{t+1} - \ell_{t+1} \leq u_t + 1/2 - (\ell_t - 1/2) \leq \Delta + 5/2 \leq R.$$

Since the cup configuration starts R-flat, and after any round where the distance $u_t - \ell_t$ increases it increases to a value at most R, we have by induction that the cups are always R-flat.

□

Next we describe a simple oblivious filling strategy that will be used as a subroutine in Lemma 2; this strategy is very well-known, and similar versions of it can be found in [1, 2, 3, 4].

Proposition 1. *Consider an R-flat cup configuration in the single-processor E-extra-emptyings S-skip-emptyings negative-fill cup game on n cups with initial average μ_0. Let $d = \sum_{i=2}^{n} 1/i$.*

There is an oblivious filling strategy that achieves fill at least $\mu_0 - R + d$ in a randomly chosen cup c with probability at least $1/n!$ if the emptier does not use any extra emptyings. This filling strategy has running time $n-1$, and guarantees that the fill of c is at most $\mu_0 + R + d$.

Further, when applied against a Δ-greedy-like emptier with $R = R_\Delta$, this filling strategy guarantees that the cups always remain $(R+d)$-flat.

Proof. First we assume that the emptier does not use extra emptying, and show that in this case the filler has probability at least $1/n!$ of attaining a cup with fill at least $\mu_) - R + d$. The filler maintains an ***active set***, initialized to being all of the cups. Every round the filler distributes 1 unit of fill equally among all cups in the active set. Next the emptier removes 1 unit of fill from some cup. Then the filler removes a cup uniformly at random from the active set. This continues until a single cup c remains in the active set.

We now bound the probability that c has never been emptied from. Assume that on the i-th step of this process, i.e. when the size of the active set is $n-i+1$, no cups in the active set have ever been emptied from; consider the probability that after the filler removes a cup randomly from the active set there are still no cups in the active set that the emptier has emptied from. If the emptier skips its emptying on this round, or empties from a cup not in the active set then it is trivially still true that no cups in the active set have been emptied from. If the cup that the emptier empties from is in the active set then with probability $1/(n-i+1)$ it is evicted from the active set, in which case we would still have that no cup in the active set had ever been emptied from. Hence with probability at least $1/(n-1)! \geq 1/n!$ the final cup in the active set, c, has never been emptied from. In this case, c will have gained fill $d = \sum_{i=2}^{n} 1/i$ as claimed. Because c started with fill at least $-R + \mu_0$, c now has fill at least $-R + d + \mu_0$.

Further, c has fill at most $\mu_0 + R + d$, as c starts with fill at most R, and c gains at most $1/(n-i+1)$ fill on the i-th round of this process.

Now we analyze this algorithm specifically for a Δ-greedy-like emptier. Consider a round t on which $\min_c \mathrm{fill}_{S_t}(c) > \min_c \mathrm{fill}_{S_{t+1}}(c)$, where a cup c_0 that has $\mathrm{fill}_{S_{t+1}}(c_0) = \max_c \mathrm{fill}_{S_{t+1}}(c)$ was not emptied from on round t. This implies that $\mathrm{fill}_{I_t}(c_0) - \min_c \mathrm{fill}_{I_t}(c) \leq \Delta + 1$ and then $\max_c \mathrm{fill}_{S_{t+1}}(c) - \min_c \mathrm{fill}_{S_{t+1}}(c) \leq \Delta + 2$, i.e. the cups are $(\Delta+2)$-flat.

Consider some round t_1 on which the cups are not $(\Delta+2)$-flat; let t_0 be the last round on which the cups where R-flat (note that if the cups are $(\Delta+2)$-flat they are also R-flat as $\Delta + 2 < R$). Consider how the fill-range of the cups changes during the set of rounds t with $t_0 < t \leq t_1$. On any such round t either $\min_c \mathrm{fill}_{S_t}(c) > \min_c \mathrm{fill}_{S_{t+1}}(c)$ in which case the fill-range increases by at most $1/(n-t+1)$ where $n-t+1$ is the size of the active set on round t, or all cups on round $t+1$ with fill equal to the backlog were emptied from, meaning that backlog decreased by at least $1 - 1/(n-t+1)$. In either case the fill-range increases by at most $1/(n-t+1)$. Thus in total the fill-range is at most $R+d$. That is, the cups are $(R+d)$-flat on round t_1, as desired.

□

Now we are equipped to prove Lemma 2, which shows that we can force a constant fraction of the cups to have high fill; using Lemma 2 and exploiting the greedy-like nature of the emptier we can get a known cup with high fill (we show this in Proposition 2).

Lemma 2. *Let $\Delta \leq O(1)$, let $h \leq O(1)$ with $h \geq 16 + 16\Delta$, let n be at least a sufficiently large constant determined by h and Δ, and let $R \leq poly(n)$. Consider an R-flat cup configuration in the variable-processor cup game on n cups. Let A, B, A' be disjoint constant-fraction-size subsets of the cups with*

$|A| = \Theta(n)$ *sufficiently small and with* $|A| + |B| + |A'| = n$. *These sets will change over the course of the filler's strategy, but* $|A|$ *will remain fixed and* $|A| \ll |A|$ *will always hold.*

Let $M \gg n$ *be very large.*

There is an oblivious filling strategy that either achieves mass at least M *in the cups, or makes an unknown set of* $\Theta(n)$ *cups in* A *have fill at least* h *with probability at least* $1 - 2^{-\Omega(n)}$ *in running time* poly(M) *against a* Δ*-greedy-like emptier while also guaranteeing that* $\mu(B) \geq -h/2$.

Proof. We refer to A as the ***anchor*** set, B as the ***non-anchor*** set, and A' as the ***garbage*** set. The filler initializes A' to $\varnothing$, and B to be all the cups besides the cups in A. The set A is chosen to satisfy

$$|A| \leq (n - 2|A|)/(2e^{2h+1} + 1). \qquad (1)$$

We denote by *randalg* the oblivious filling strategy given by Proposition 1. We denote by *flatalg* the oblivious filling strategy given by Lemma 1. We say that the filler ***applies*** a filling strategy *alg* to a set of cups $D \subseteq B$ if the filler uses *alg* on D while placing 1 unit of fill in each anchor cup.

We now describe the filler's strategy.

The filler starts by flattening the cups, i.e. using *flatalg* on $A \cup B$ for poly(M) rounds (by setting $p = n/2$). After this the filling strategy always places 1 unit of water in to each anchor cup on every round. The filler performs a series of $|A|$ ***swapping-processes***, one per anchor cup, which are procedures that the filler uses to get a new cup—which will sometimes have high fill—in A. On each swapping-process the filler applies *randalg* many times to arbitrarily chosen constant-size sets $D \subset B$ with $|D| = \lceil e^{2h+1} \rceil$. The number of times that the filler applies *randalg* is chosen at the start of the swapping-process, chosen uniformly at random from $[m]$ ($m =$ poly(M) to be specified). At the end of the swapping-process, the filler does a "swap": the filler takes the cup given by *randalg* in B and puts it in A, and the filler takes the cup in A associated with the current swapping-process and moves it into A'. Before each application of *randalg* the filler flattens B by applying *flatalg* to B for poly(M) rounds.

We remark that this construction is similar to the construction in **??**, but has a major difference that substantially complicates the analysis: in the adaptive lower bound construction the filler halts after achieving the desired average fill in the anchor set, whereas the oblivious filler cannot halt but rather must rely on the emptier's greediness to guarantee that each application of *randalg* has constant probability of generating a cup with high fill.

We proceed to analyze our algorithm.

If the emptier skips more than M emptyings then the filler has achieved mass M. Otherwise, the filler cannot skip more than M emptyings. We assume that the emptier skips fewer than M rounds for the rest of the analysis. Also note that if the emptier neglects the anchor set for more than M times without ever decreasing the fill of the anchor set then the anchor set has mass at least M. We also assume that the emptier never does this for the remainder of the analysis.

First note that the initial flattening of $A \cup B$ makes $A \cup B$ be R_Δ-flat by Lemma 1. In particular, note that the flattening happens in the $(n/2)$-processor 0-extra-emptyings M-skip-emptyings variable-processor cup game on n cups.

We say that a property of the cups has ***always*** held if the property has held since the start of the first swapping-process; i.e. from now on we only consider rounds after the initial flattening of $A \cup B$.

We say that the emptier ***neglects*** the anchor set on a round if it does not empty from each anchor cup. We say that an application of *randalg* to $D \subset B$ is ***successful*** if the emptier does not neglect the anchor set during any round of the application of *randalg*. We define $d = \sum_{i=2}^{|D|} 1/i$ (recall that $|D| = \lceil e^{2h+1} \rceil$). We say that an application of *randalg* to D is ***lucky*** if it achieves backlog at least $\mu(B) - R_\Delta + d$; note that by Proposition 1 any successful application of *randalg* where B started R_Δ-flat has at least a $1/|D|!$ chance of being lucky.

Now we prove several important bounds on fills of cups in A and B.

Claim 5. *All applications of flatalg succeed and* B *is always* $(R_\Delta + d)$*-flat.*

Proof. Given that the application of *flatalg* immediately prior to an application of *randalg* made B be R_Δ-flat, by Proposition 1 we have that B will stay $(R_\Delta + d)$-flat during the application of *randalg*. Given that the application of *randalg* immediately prior to an application of *flatalg* resulted in B being $(R_\Delta + d)$-flat, we have that B remains $(R_\Delta + d)$-flat throughout the duration of the application of *flatalg* by Lemma 1. Given that B is $(R_\Delta + d)$-flat before a swap occurs B is clearly still $(R_\Delta + d)$-flat after the swap, because the only change to B during a swap is that a cup is removed from B which cannot increase the backlog in B or decrease the fill of the least full cup in B. Note that B started R_Δ-flat before the first application of *flatalg* because $A \cup B$ was flattened. Note that if an application of *flatalg* begins with B being $(R_\Delta + d)$-flat, then by considering the flattening to happen in the $(|B|/2)$-processor M-extra-emptyings

M-skip-emptyings cup game we ensure that it makes B be R_Δ-flat, because the emptier cannot skip more than M emptyings and also cannot do more than M extra emptyings or the mass would be at least M. Hence we have by induction that B has always been $(R_\Delta + d)$-flat and that all flattening processes have made B be R_Δ-flat. □

Claim 6. *We have always had*

$$\mu(B) \le 2 + \mu(A \cup B).$$

Proof. Consider how high $\mu(B) - \mu(A \cup B)$ could rise. There are two ways that it can rise: a) B could be skipped while A is not skipped, or b) a cup with fill lower than $\mu(B)$ could be evicted from B at the end of a swapping-process.

If B is skipped while A is not skipped then this implies that

$$\min_{a \in A} \mathrm{fill}(a) > \max_{b \in B} \mathrm{fill}(b) - \Delta.$$

Thus $\mu(B) \le \mu(A) + \Delta$. As $|B| \gg |A|$ this can be loosened to $\mu(B) \le 1 + \mu(A \cup B)$.

Consider the final round on which B is skipped while A is not skipped (or consider the first round if there is no such round).

From this round on, the only increase to $\mu(B) - \mu(A \cup B)$ is due to B evicting cups with fill well below $\mu(B)$. We can upper bound the increase of $\mu(B) - \mu(A \cup B)$ by the increase of $\mu(B)$ as $\mu(A \cup B)$ is strictly increasing.

The cup that B evicts at the end of a swapping-process has fill at least $\mu(B) - R_\Delta - (|D| - 1)$, as the running time of *randalg* is $|D| - 1$, and because B starts R_Δ-flat by Claim 5. Evicting a cup with fill $\mu(B) - R_\Delta - (|D| - 1)$ from B changes $\mu(B)$ by $(R_\Delta + |D| - 1)/(|B| - 1)$ where $|B|$ is the size of B before the cup is evicted from B. Even if this happens on each of the $|A|$ swapping processes $\mu(B)$ cannot rise higher than $|A|(R_\Delta + |D| - 1)/(n - 2|A|)$ which by design in choosing $|B| \gg |A|$, as was done in (1), is at most 1.

Thus it always is the case that

$$\mu(B) \le 2 + \mu(A \cup B).$$

□

The upper bound on $\mu(B)$ along with the guarantee that B is flat allows us to bound the highest that a cup in A could rise by greediness, which in turn upper bounds $\mu(A)$ which in turn lower bounds $\mu(B)$. In particular we have

Claim 7. *We always have*

$$\mu(B) \ge -h/2.$$

Proof. By Claim 6 and Claim 5 we have that no cup in B ever has fill greater than $u_B = \mu(A \cup B) + 2 + R_\Delta + d$.

Let $u_A = u_B + \Delta + 1$. We claim that the backlog in A never exceeds u_A.

Consider how high the fill of a cup $c \in A$ could be. If c came from B then when it is swapped into A its fill is at most $u_B < u_A$. Otherwise, c started with fill at most $R_\Delta < u_A$. Now consider how much the fill of c could increase while being in A. Because the emptier is Δ-greedy-like, if a cup $c \in A$ has fill more than Δ higher than the backlog in B then c must be emptied from, so any cup with fill at least $u_B + \Delta = u_A - 1$ must be emptied from, and hence u_A upper bounds the backlog in A.

Of course an upper bound on backlog in A also serves as an upper bound on the average fill of A as well, i.e. $\mu(A) \le u_A$. Then we have

$$\begin{aligned}
&\mu(B) \\
&\quad = -\frac{|A|}{|B|}\mu(A) + \frac{|AB|}{|B|}\mu(AB) \\
&\quad \ge -(\mu(AB) + 2 + R_\Delta + d + \Delta + 1)\frac{|A|}{|B|} + \frac{|A| + |B|}{|B|}\mu(AB) \\
&\quad = -(2 + R_\Delta + d + \Delta + 1)\frac{|A|}{|B|} + \mu(AB) \\
&\quad \ge -h/2
\end{aligned}$$

where the final inequality follows because $\mu(AB) \ge 0$, and $|B| \gg |A|$, in particular by (1).

□

Now we show that this guarantees that with constant probability the final application of *randalg* on a swapping-process is both lucky and successful.

Claim 8. *There exists choice of $m = poly(M)$ such that with at least constant probability the final application of randalg on any fixed swapping-process is both lucky and successful.*

Proof. Fix some swapping-process. The filler chooses $m = 4M|D|!$. By a Chernoff bound, there is exponentially high probability that of $4M|D|!$ applications of *randalg* at least $2M$ would be lucky if the emptier didn't neglect A during the application. The emptier can choose at most M of these to neglect, so there is at least a $1/2$ chance that the randomly chosen final application of *randalg* is successful, conditioning on it lucky. The final application is lucky with probability $1/|D|!$. Hence overall this choice of m makes the final round lucky and successful with constant probability $1/(2|D|!)$. □

Claim 9. *With probability at least $1 - 2^{-\Omega(n)}$, the filler achieves fill at least h in at least $\Theta(n)$ of the cups in A.*

Proof. By Claim 8 each swapping-process has at least constant probability of swapping a cup with fill at least $\mu(B) + d - R_\Delta$ into A. The events that the swapping-processes swap such a cup into A are independent, so by a Chernoff bound there is exponentially high probability that at least a constant fraction of them succeed. By **??** $\mu(B) \geq -h/2$. Recalling that $d \geq 2h$ and $h \geq 16(1 + \Delta)$, we have that such a swapped cup has fill at least h, as desired. □

We now briefly analyze the running time of the filling strategy. There are $|A|$ swapping-processes. Each swapping-process consists of poly(M) applications of *randalg*, which take constant time, and the poly(M) flattening procedure, which take poly(M) time. Clearly overall the algorithm takes poly(M) time, as desired.

□

Finally, using Lemma 2 we can show in Proposition 2 that an oblivious filler can achieve constant backlog. We remark that Proposition 2 plays a similar role in the proof of the lower bound on backlog as **??** does in the adaptive case, but is vastly more complicated to prove (in particular, **??** is trivial, whereas we have already proved several lemmas and propositions as preparation for the proof of Proposition 2).

Proposition 2. *Let $H \leq O(1)$, let $\Delta \leq O(1)$, let n be at least a sufficiently large constant determined by H and Δ, and let $R \leq$ poly(n). Consider an R-flat cup configuration in the negative-fill variable-processor cup game on n cups with average fill 0. Given this configuration, an oblivious filler can achieve fill H in a chosen cup in running time poly(n) against a Δ-greedy-like emptier with probability at least $1 - 2^{-\Omega(n)}$.*

Proof. The filler starts by performing the procedure detailed in Lemma 2, using $h = H \cdot 16(1 + \Delta)$. Let the number of cups which must now exist with fill h be of size $nc = \Theta(n)$.

The filler reduces the number of processors to $p = nc$. Now the filler exploits the filler's greedy-like nature to to get fill H in a set $S \subset B$ of nc chosen cups.

The filler places 1 unit of fill into each cup in S. Because the emptier is greedy-like it must focus on the nc cups in A with fill at least h until the cups in S have sufficiently high fill. In particular, $(5/8)h$ rounds suffice. Over $(5/8)h$ rounds the nc high cups in A cannot have their fill decrease below $(3/8)h \geq h/8 + \Delta$. Hence, any cups with fills less than $h/8$ must not be emptied from during these rounds. The fills of the cups in S must start as at least $-h/2$ as $\mu(B) \geq -h/2$. After $(5/8)h$ rounds the fills of the cups in S are at least $h/8$, because throughout this process the emptier cannot have emptied from them until they got fill at least $h/8$, and if they are never emptied from then they achieve fill $h/8$.

Thus the filling strategy achieves backlog $h/8 \geq H$ in some known cup (in fact in all cups in S, but a single cup suffices), as desired.

□

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