On variable-processor cup games (easier results)

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Oblivious Lowerbounds.

Theorem 1. *Hoeffding's Inequality Let X_i be independent bounded random variabels with $X_i \in [a, b]$. Then,*

$$P\left(\Big|\frac{1}{n}\sum_{i=1}^{n}(X_i - \mathbb{E}[X_i])\Big| \geq t\right) \leq 2\exp\left(-\frac{2nt^2}{(b-a)^2}\right)$$

Proposition 1. *There exists an oblivious filling strategy in the variabl-processor cup game on n cups that achieves backlog $\Omega(\log n)$ against a smoothed greedy emptier.*

Proof. Let A be the anchor set, randomly chosen, let B be non-anchor set, with $|A| = |B| = n/2$. Let $h = \Theta(1)$ be the fill that we will achieve at each level of our recursive procedure. Our strategy to achieve backlog $\Omega(\log n)$ is roughly as follows:

- Make each cup in the anchor set have a constant probability of having fill at least h.
- Reduce the number of processors to a constant fraction nk of n and raise the fill of nk cups to $h/2$. This step relies on the emptier being greedy.
- Recurse on the nk cups that are known to have fill $\geq h/2$.

We can perfom $\Omega(\log n)$ levels of recursion, achieving constant backlog at each step (relative to average fill); doing so yields backlog $\Omega(\log n)$.

Our strategy is somewhat complicated by the possibility of the fill being very concentrated in a few cups. We proceed as follows:

For each anchor cup i:

1. Chose an index $j \in [n^c]$
2. For n^c times (for some $c > 2$), we select a random subset $C \subset B$ of the non-anchor cups and play a single processor cup game on C.
3. On round j with $1/2$ probability we swap the winner of the single processor cup game into the anchor set, and with $1/2$ probability we swap a random cup from B into the anchor set.

Say that a cup is ***op*** if it contains fill $\geq \sqrt{\frac{n}{\log\log n}}$. If there is ever an op cup, then we win. Note that we don't even need to know which cup is op because it will take $\Omega(\text{poly}(n))$ rounds for the emptier to reduce the fill below $\text{poly}(n)$. Hence, we can assume without loss of generality that no cup is ever op.

Claim 1. *If at least $1/2$ of the non-anchor cups have fill $\geq -h/2$ then the contribution to the anchor set has fill $\geq h/2$ with constant probability.*

Proof. Playing the single-processor cup game n^c times, with only one time that we actually swap a cup into the anchor set, makes it bad for the emptier to ignore the anchor set on a more than a constant fraction of the games. In particular, if the emptier neglects the anchor set at least once for more than half of the games then the anchor set's average fill will have increased by $n^{c-1} \geq \Omega(\text{poly}(n))$. Hence we have the desired backlog.

Otherwise, we have at least a $1/2$ chance that the round j when we will perform a switch into the anchor set occurs on a round when the emptier choses not to neglect the anchor set. In this case, the round was a legitimate single processor cup game on e^h cups, and we achieved fill increase $\geq h$ by the end of the game with probability at least $1/e^h!$, the probability that we correctly guess the sequence of cups within the single processor cup game that the emptier would neglect. And we get another factor of $1/2^{e^h}$ (ish), the probability that the randomly chosen subset of the cups has all cups with fill $\geq -h/2$. In this case, which we established happens with constant probability, it is the case that the winner of the single processor cup game now has fill $\geq h/2$, as desired. ☐

Claim 2. *Let X be a uniformly randomly selected cup from B. Let Y_i be the random variable $Y_i = tilt_+(X)$ where X is a randomly selected cup from the non-anchor set at the start of the i-th round of playing single processor cups games. If less than $1/2$ of the non-anchor cups have fill $\geq -h/2$ fill, then* *with probability at least* $1 - 1/\operatorname{polylog}(n)$,

$$\frac{1}{n/2}\sum_{i=1}^{n/2} Y_i \geq h/4.$$

Proof. We assume for simplicity that the average fill of B is 0. In reality this is not the case, but by a Hoeffding bound and the fact that op cups don't exist, the fill is really tightly concentrated around 0, so this is almost WLOG.

Let the positive tilt of a cup i be $\text{tilt}_+(i) := \max(\text{fill}(i), 0)$. We have

$$\mathbb{E}[\text{tilt}_+(X)] = \frac{1}{2}\,\mathbb{E}[|\,\text{fill}(X)|] \geq h/2$$

(because negative tilt is at least $nh/4$ and positive tilt must oppose this).

Let Y_i be the random variable $Y_i = \text{tilt}_+(X)$ where X is a randomly selected cup from the non-anchor set at the start of the i-th round of playing single processor cups games. Note that the Y_i are not really independent, but it is probably ok. Note that $0 \leq Y_i \leq n/\lg\lg n$. Now we have, by Hoeffding's inequality, that

$$P\left(\Big|\frac{1}{n/2}\sum_{i=1}^{n/2}(Y_i - \mathbb{E}[Y_i])\Big| \geq h/4\right) \leq 2\exp\left(-\frac{n(h/4)^2}{(\sqrt{n/\lg\lg n})^2}\right)$$

$$P\left(\frac{1}{n/2}\sum_{i=1}^{n/2} Y_i \leq h/4\right) \leq 1/\operatorname{polylog}(n)$$

□

Now we consider two cases based on how many times we must apply Claim 1 and Claim 2. If we must apply Claim 1 at least half the time, then we achieve a constant fraction of the anchor cups with fill at least $h/2$. If on the other hand we must apply Claim 2 at least half of the time, we have that with probability $1-1/\operatorname{polylog}(n)$ the process brings $n\cdot h/8$ positive tilt to the anchor set as desired.

In either case we achieve, with probability at least $1 - 1/\operatorname{polylog} n$, positive tilt at least hn/k in the anchor set. Use the positive tilt, with one processors, we can transfer over the fill into n/k cups. (Note, we use one processor because we do not know how many cups the fill is concentrated in). The filler repeatedly distributes 1 unit of fill to each of the n/k cups in succession, and continues until $h/4$ fill has been distributed. We cannot continue beyond this point because we have used up the positive tilt. Now we recurse on this set of n/k cups.

We can perform $\Omega(\log n)$ levels of recursion, and gain $\Omega(1)$ fill at each step. Hence, overall, backlog of $\Omega(\log n)$ is achieved. □

Lemma 1. *The Oblivious Amplification Lemma Given an oblivious filling strategy for achieving backlog $f(k)$ in the variable-processor cup game on k cups that succeeds with probability at least $1/2$, there exists a strategy for achieving "amplified" fill*

$$f'(k) \geq \frac{1}{4}(f(k/2) + f(k/4) + f(k/8) + \cdots$$

that succceeds with constant probability.

Proof. Want to do: same proof as before, but there are **concerns, about using higher values of h:**

- dealing with star: problem: need to say WLOG avg fill of A,B is 0 each initially solution: no op cups wlog, so if we pick them randomly star holds by Hoeffding's. (kinda, bc stuff isnt really independent, can probably swap with replacement to fix this tho)
- dealing with start star: What if C needs to be big because we need big backlog? claim: star star isnt a problem beause the base case is the only case that needs to explicitely deal with positive and negative fill

□

Corollary 1. *There is an oblivious filling strategy for the variable-processor cup game on n cups that achieves backlog $2^{\Omega(\sqrt{\log n})}$ in running time $O(n)$*

Proof. □

Adaptive Lowerbound.

Proposition 2. *There exists an adaptive filling strategy for the variable-processor cup game on k cups that achieves backlog $\Omega(\log n)$, where fill is relative to the average fill of the cups, with negative fill allowed.*

Proof. Let $h = \frac{1}{4}\log n/2$ be the desired fill. Call a cup be ***op*** if it has fill at least h. Once there exists an op cup, the proposition is immediately satisfied. Let the ***positive tilt*** of a cup i with fill v be $\max(0, v)$, and let the positive tilt of a set S of cups be the sum of the positive tilt of each cup $i \in S$.

If no cups are op, then positive tilt $< h \cdot n$. Assume for sake of contradiction that there are $\geq n/2$ cups with fill $\leq -2h$. Then the mass of those cups would be $\leq -hn$, but there isn't enough positive tilt to oppose this. Hence there are $< n/2$ cups with fill $\leq -2h$. We set the number of processors equal to 1 and play a single processor cup game on $n/2$ cups that have fill at least $-2h$, which must exist as stated. In the single processor cup game we distribute water equally among all the cups in the active set at each step. Then the emptier will chose some cup to empty. If this cup is in our active set we remove it from the active set. At the end of this process the active set is non-empty, and any cup in the active set has gained fill at least $H_{n/2} \geq \log n/2 = 4h$. Thus such a cup has fill at least $2h$ now, so the proposition is statisfied. ◻

Lemma 2. *The Adaptive Amplification Lemma Given an adaptive filling strategy for achieving backlog $f(k)$ in the variable-processor cup game on k cups, there exists a strategy for achieving "amplified" fill*

$$f'(k) \geq \frac{1}{4}(f(k/2) + f(k/4) + f(k/8) + \cdots.$$

Proof. If, at any point in the process that will be described, backlog is greater than $f'(k)$, then the filler stops and the Lemma is satisfied as the desired backlog having been achieved. Thus we assume without loss of generality for the rest of the proof that no cup ever exceeds fill $f'(k)$ during the course of our algorithm. That is, we assume that we don't achieve the desired backlog until the end of our process.

The main idea of this analysis is as follows:

1. Using f repeatedly, achieve average fill at least $\frac{1}{2}f(n/2)$ in $n/2$ cups.

2. Halve the number of processors

3. Recurse on the $n/2$ cups with high average fill.

Let $h_l = f(k/2^l)$; the filler will achieve a set of at least $n_l/2 = n/2^l$ cups with average fill at least $h_l/2$ on the l-th level of recursion.

On the l-th level of recursion we will repeat the following at most $n_l/2$ times:

- If the non-anchor set has average fill at least $-h_l/2$ then we apply f to the non-anchor set. This gets us fill $-h_l/2 + f(n_l/2) = h_l/2$ in some non-anchor cup. We replace the lowest cup in the anchor set with this cup. A slight complication with this method is that we are anchoring the anchor set, and assuming that the emptier allways empties from each anchor cup; this may not be the case. However, the issue can be resolved by applying f up to $h_l n_l/4 + 1$ times. If on all of these applications of f the emptier doesn't always empty from each cup in the anchor set, then the average fill in the anchor set increases by more than $h_l/2$, so the desired fill was achieved. Otherwise, there must a time when we apply f and the emptier does empty from each anchor cup each time. In this case we actually do achieve fill $-h_l/2 + f(n_l/2)$ in a non-anchor cup, and swap it into the anchor set, as descibred before.

- If the non-anchor has average fill lower than $-h_l/2$, then anchor set has average fill at least $h_l/2$, so we terminate the process.

◻

Corollary 2. *There is an adaptive filling strategy for the variable-processor cup game on n cups that achieves backlog $\Omega(poly(n))$ in running time $2^{O(\log^2 n)}$*

Proof. Let

$$f_0(k) = \begin{cases} \log_2 k, & k \geq 1, \\ 0 & \text{else.} \end{cases}$$

Let f_{m+1} be the result of applying The Amplification Lemma to f_m. By repeated amplification $\log_2 n^{1/9}$ times we achieve a function $f_{\log_2 n^{1/9}}(k)$ with the property that for $k \geq n$, $f_{\log_2 n^{1/9}}(k) \geq 2^{\log_2 n^{1/9}} \log_2 k$. In particular, this gives a filling strategy that when applied to n cups gives backlog $\Omega(n^{1/9}\log_2 n) \geq \Omega(\text{poly}(n))$ as desired. To prove this, we prove the following lowerbound for f_m by induction:

$$f_m(k) \geq 2^m \log_2 k, \text{ for } k \geq (2^9)^m.$$

The base case follows from the definition of f_0. Assuming the property for f_m, we get the following:

for $k > (2^9)^{m+1}$,

$$\begin{aligned}
f_{m+1}(k) &= \frac{1}{2}(f_m(k/2) + f_m(k/4) + \cdots + f_m(k/2^9) + \cdots) \\
&\geq \frac{1}{2}(f_m(k/2) + f_m(k/4) + \cdots + f_m(k/2^9)) \\
&\geq \frac{1}{2}2^m(\log_2(k/2) + \log_2(k/4) + \cdots + \log_2(k/2^9)) \\
&\geq \frac{1}{2}2^m(9\log_2(k) - \frac{9 \cdot 10}{2}) \\
&\geq 2^{m+1}\log_2(k),
\end{aligned}$$

as desired. Hence the inductive claim holds, which establishes that $f_{\log_2 n^{1/9}}$ satisfies the desried condition, which proves that backlog can be made $\Omega(\text{poly}(n))$.

$\square$