

# Variable Processor Cup Games

Alek Westover

Belmont High School

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# WHAT IS THE CUP GAME?

The  $p$ -processor cup game on  $n$  cups is a multi-round game in which two players take turns emptying and removing water from the cups. On each round,

- ▶ The *filler* distributes  $p$  units of water among the cups (with at most 1 unit to any particular cup).
- ▶ Then the *emptier* chooses  $p$  cups to remove (at most) one unit of water from.

The *backlog* of the system is the amount of water in the fullest cup; The emptier aims to minimize backlog whereas the filler aims to maximize backlog.

**Note:** The emptier's resources must be allocated discretely whereas the filler can continuously distribute resources.

## WHY IS IT IMPORTANT?

The cup game models *work scheduling*. The  $n$  cups represent tasks that must be performed. At each time step  $p$  new units of work come in, distributed arbitrarily among the  $n$  tasks (with the constraint that no task gets more than 1 unit of work) and  $p$  processors must be allocated to a subset the tasks, on which they will achieve 1 unit of progress.

The cup game is also an interesting mathematical object.

## PREVIOUS WORK ON THE PROBLEM

- ▶ The Single-Processor cup game ( $p = 1$ ) has been tightly analysed with *oblivious* and *adaptive* fillers (i.e. fillers that can't and can observe the emptier's actions).
- ▶ The Multi-Processor cup game ( $p > 1$ ) is substantially more difficult.
  - ▶ Kuszmaul established upper bound of  $O(\log n)$ .<sup>1</sup>
  - ▶ We established a matching lower bound of  $\Omega(\log n)$ .
- ▶ Variants where valid moves depend on a graph have been studied.

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<sup>1</sup>William Kuszmaul. Achieving optimal backlog in the vanilla multi-processor cup game. SIAM, 2020.

## SINGLE-PROCESSOR LOWER BOUND

*Filling strategy:* distribute water equally amongst cups not yet emptied by the emptier.

Achieves backlog:

$$\frac{1}{n} + \frac{1}{n-1} + \dots = \Omega(\log n).$$

## SINGLE-PROCESSOR UPPER BOUND

A *greedy* emptier (always empty from the fullest cup) never lets backlog exceed  $O(\log n)$ .

Inductively prove a set of invariants:  $\mu_k(S_t) \leq \frac{1}{k+1} + \dots \frac{1}{n}$ .

Let  $a$  be the cup that the emptier empties from in state  $S_t$ .

**Case 1:  $a$  is the fullest cup in  $S_{t+1}$**

Then  $\mu_k(S_{t+1}) \leq \mu_k(S_t)$ , as 1 unit of water was removed from the  $k$  fullest cups, while at most 1 unit of water was added to them.

**Case 2:  $a$  is not the fullest cup in  $S_{t+1}$**

$$\mu_k(S_{t+1}) \leq \mu_{k+1}(I_t) \leq \mu_{k+1}(S_{t+1}) + \frac{1}{k+1}.$$

## OUR VARIANT OF THE CUP GAME

We investigate a variant of the classic multi-processor cup game, the *variable-processor cup game*, in which the resources are variable: the filler is allowed to change  $p$ .

Although the modification to allow variable resources seems small, we will show that it drastically alters the outcome of the game.

# AMPLIFICATION LEMMA

## Lemma

*Given a filling strategy for achieving backlog  $f(n)$  on  $n$  cups, we can construct a new filling strategy that achieves backlog*

$$f'(n) \geq (1 - \delta) \sum_{\ell=0}^L f((1 - \delta)\delta^\ell n)$$

*for any parameter  $\delta$  with  $0 < \delta \ll 1/2$  of our choice, where  $L$  is the largest integer such that we can achieve backlog 1 on  $(1 - \delta)\delta^L n$  cups.*

Remark: WLOG the number of cups in the recursive calls is an integer.



## PROOF SKETCH OF AMPLIFICATION LEMMA

- ▶ Let  $A$  be the  $\delta n$  fullest cups and  $B$  be the  $(1 - \delta)n$  other cups.
- ▶ By repeatedly applying  $f$  to each cup in  $B$ , and transferring over the cup generated in  $B$  with backlog  $f((1 - \delta)n)$  to  $A$ , while maintaining the fill of cups in  $A$ , we make  $\mu(A) - \mu(B) \geq f((1 - \delta)n)$ . This is accomplished while maintaining  $\mu(A \cup B)$ . The mass of  $A$  is guaranteed to be above  $a\mu(A \cup B)$  by the same amount that the mass of  $B$  is guaranteed to be depressed from  $b\mu(A \cup B)$ , thus fraction of the difference that  $\mu(A)$  gets is  $|B|/|A \cup B| = (1 - \delta)$ . So  $\mu(A)$  is at least  $(1 - \delta)f((1 - \delta)n)$  above  $\mu(A \cup B)$ .
- ▶ We then recursively apply this procedure to  $A$ . Summing over  $\ell = 0, 1, \dots, L$  we have the desired result.

Note: we are ignoring a lot of details here, e.g. ensuring that we actually are playing a cup game on  $B$  when applying  $f$  to it.

## LOWER BOUND AGAINST ADAPTIVE FILLER IN THE VARIABLE PROCESSOR CUP GAME

Using this we derive a very surprising result: backlog can be increased as high as  $\Omega(n)$  by choosing  $\delta = 1/n$  and repeatedly applying the Amplification Lemma.

# UPPER BOUND AGAINST ADAPTIVE FILLER IN THE VARIABLE PROCESSOR CUP GAME

We prove a novel set of invariants that the greedy emptier maintains:

$$\mu_k(S_t) \leq n - k.$$

In particular this implies that backlog is  $O(n)$ .

Note that this matches our lowerbound: we have a tight analysis of the game!

## STRATEGY EVEN WORKS WITH OBLIVIOUS FILLER!

Using Hoeffding's Inequality, we can surprisingly prove the same lower-bound for an Oblivious filler as for an Adaptive filler, although only against greedy-like emptiers. <sup>2</sup>

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<sup>2</sup>Wassily Hoeffding. Probability inequalities for sums of bounded random variables. Journal of the American Statistical Association, page 28, 1962.

## OPEN QUESTIONS

- ▶ Can we extend the Oblivious lower-bound construction to work against a broader class of emptiers?
- ▶ Can we extend the Oblivious lower-bound construction to work against arbitrary emptiers?

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