We now give a method for transforming a filling strategy for achieving large backlog into a filling strategy for achieving high fill in many cups, or high average fill in a set of cups (which of these we guarantee depends on the original filling strategy). The idea of repeating an algorithm many times is also used in the proof of the Adaptive Amplification Lemma; the construction is slightly more complicated in the randomized case however, and is much harder to analyze.

Definition 1. Let alg_0 be an oblivious filling strategy, that can get high fill in some cup against greedy-like emptiers with some probability. We construct a new filling strategy $\text{rep}_\delta(\text{alg}_0)$, which we call the δ-***repetition*** of alg_0, as follows:

Let $n_A = \lceil \delta n \rceil, n_B = \lfloor (1-\delta)n \rfloor$. Let $M \gg n$ be large, let $m = \text{poly}(M)$ be a chosen parameter. Initialize A to $\varnothing$ and B to being all of the cups. We call A the ***anchor set*** and B the ***non-anchor set***. The filler always places 1 unit of fill in each anchor cup on each round. The filling strategy consists of n_A ***donation-processes***, which are procedures that result in a cup being ***donated*** from B to A (i.e. removed from B and added to A). At the start of each donation-processes the filler chooses a value m_0 uniformly at random from $[m]$. We say that the filler ***applies*** a filling strategy alg to B if the filler uses alg on B while placing 1 unit of fill in each anchor cup. During the donation-process the filler applies alg_0 to B m_0 times, and flattens B by applying flatalg to B for $\Theta(M)$ rounds before each application of alg_0. At the end of each donation process the filler takes the cup given by the final application of alg_0 (i.e. the cup that alg_0 guarantees with some probability against a certain class of emptiers to have a certain high fill), and donates this cup to A.

A pseudocode description of this algorithm can be found in Algorithm 1.

Algorithm 1 rep

Input: $\text{alg}_0, \delta, M, m$, set of n cups
Output: Guarantees on the sets A, B (will vary based on alg_0)

```
n_A ← ⌈δn⌉, n_B ← ⌊(1 − δ)n⌋
A ← ∅, B ← all the cups
for i ∈ [n_A] do                          ▷ Donation-processes
    m_0 ← random([m])
    for j ∈ [m_0] do
        Apply flatalg to B for Θ(M) rounds
        Apply alg_0 to B
    Donate the cup given by alg_0 from B to A
```

We use the idea of repeating an algorithm in two different contexts. First in Proposition 1 we prove a result analogous to that of **??**: in particular, we show that we can achieve constant fill in a known cup by using $\text{rep}_{\Theta(1)}(\text{randalg}(\Theta(1))$ which achieves, by a Chernoff bound, $\Theta(n)$ unknown cups with constant fill, and then exploiting the emptier's greedy-like nature to achieve constant fill in a known cup. After doing this, we prove the ***Oblivious Amplification Lemma***, a result analogous to the Adaptive Amplification Lemma: in particular, we show how to take an algorithm for achieving some backlog, and then achieve higher backlog by repeating the algorithm many times. Although these results have deterministic analogues, their proofs are different and significantly more complex than the proofs for the deterministic cases.

Before proving Proposition 1 we analyze $\text{rep}_{\Theta(1)}(\text{randalg}(\Theta(1)))$ in Lemma 1.

Lemma 1. *Let $\Delta \le O(1)$, let $h \le O(1)$ with $h \ge 16 + 16\Delta$, let $k = \lceil e^{2h+1} \rceil$, let $\delta = \Theta(e^{-2h})$, let n be at least a sufficiently large constant determined by h and Δ. Let $M \gg n$ be very large. Consider an R_Δ-flat cup configuration in the variable-processor cup game on n cups with initial average fill μ_0.*

When applied to a Δ-greedy-like emptier $\text{rep}_\delta(\text{randalg}(k))$ either achieves mass at least M in the cups, or with probability at least $1 - 2^{-\Omega(n)}$ makes an (unknown) set of $\Theta(n)$ cups in A have fill at least $h + \mu_0$ while also guaranteeing that $\mu(B) \ge -h/2 + \mu_0$, where A, B are the sets defined in Definition 1. This strategy has running time poly(M).

Proof. We use all definitions given in the Definition 1.

Without loss of generality we assume that the emptier does not neglect the anchor set more than M in a particular donation process; if the emptier chooses to neglect the anchor set this much then the anchor cups will have achieved mass M so Lemma 1 is already fulfilled. Similarly we assume that the emptier does not choose to skip more than M emptyings.

We say that the emptier ***neglects*** the anchor set on a round if it does not empty from each anchor cup. We say that an application of $\text{randalg}(k)$ to B is ***non-emptier-wasted*** if the emptier does not neglect the anchor set during any round of the application of $\text{randalg}(k)$. We define $d = \sum_{i=2}^{k} 1/i = \Theta(h)$. We say that an application of $\text{randalg}(k)$ to D is ***lucky*** if it achieves backlog at least $\mu(B) - R_\Delta + d$ where $\mu(B)$ is measured at the start of the application of $\text{randalg}(k)$; note that by **??** if we condition on an application of $\text{randalg}(k)$ where B started R_Δ-flat being non-emptier-wasted then the application has at least a $1/k!$ chance of being lucky.

Now we prove several important bounds on fills of cups in A and B.

Claim 1. *All applications of* flatalg *make B be R_Δ-flat and B is always $(R_\Delta + d)$-flat.*

Proof. Given that the application of flatalg immediately prior to an application of randalg(k) made B be R_Δ-flat, by **??** we have that B will stay $(R_\Delta + d)$-flat during the application of randalg(k). Given that the application of randalg(k) immediately prior to an application of flatalg resulted in B being $(R_\Delta + d)$-flat, we have that B remains $(R_\Delta + d)$-flat throughout the duration of the application of flatalg by **??**. Given that B is $(R_\Delta + d)$-flat before a donation occurs B is clearly still $(R_\Delta + d)$-flat after the donation, because the only change to B during a donation is that a cup is removed from B which cannot increase the fill-range of B. Note that B started R_Δ-flat at the beginning of the first donation-process. Note that if an application of flatalg begins with B being $(R_\Delta + d)$-flat, then by considering the flattening to happen in the $(|B|/2)$-processor M-extra-emptyings M-skip-emptyings cup game we ensure that it makes B be R_Δ-flat. Hence we have by induction that B has always been $(R_\Delta + d)$-flat and that all flattening processes have made B be R_Δ-flat. □

Now we aim to show that $\mu(B)$ is never very low, which we need in order to establish that every non-emptier-wasted lucky application of randalg(k) gets a cup with high fill. Interestingly in order to lower bound $\mu(B)$ we first must upper bound $\mu(B)$, which by greediness and flatness of B gives an upper bound on $\mu(A)$ which we use to get a lower bound on $\mu(B)$.

Claim 2. *We have always had*

$$\mu(B) \leq 2 + \mu(AB).$$

Proof. There are two ways that $\mu(B) - \mu(AB)$ can increase:
Case 1: The emptier could empty from 0 cups in B while emptying from every cup in A.
Case 2: The filler could evict a cup with fill lower than $\mu(B)$ from B at the end of a donation-process.

Note that cases are exhaustive, in particular note that if the emptier skips more than 1 emptying then $\mu(B) - \mu(AB)$ must decrease because $|B| \approx |AB|$, in particular by our choice of $\delta = \Theta(e^{-2h})$, as opposed to in Case 1 where $\mu(B) - \mu(AB)$ increases.

In Case 1, because the emptier is Δ-greedy-like,

$$\min_{a\in A} \text{fill}(a) > \max_{b\in B} \text{fill}(b) - \Delta.$$

Thus $\mu(B) \leq \mu(A) + \Delta$. As $|B| \gg |A|$, in particular by our choice of $\delta = \Theta(e^{-2h})$, this can be loosened to $\mu(B) \leq 1 + \mu(AB)$.

Consider the final round on which B is skipped while A is not skipped (or consider the first round if there is no such round).

From this round onwards the only increase to $\mu(B) - \mu(AB)$ is due to B evicting cups with fill well below $\mu(B)$. We can upper bound the increase of $\mu(B) - \mu(AB)$ by the increase of $\mu(B)$ as $\mu(AB)$ is strictly increasing.

The cup that B evicts at the end of a donation-process has fill at least $\mu(B) - R_\Delta - (k-1)$, as the running time of randalg(k) is $k-1$, and because B starts R_Δ-flat by Claim 1. Evicting a cup with fill $\mu(B) - R_\Delta - (k-1)$ from B changes $\mu(B)$ by $(R_\Delta + k - 1)/(|B|-1)$ where $|B|$ is the size of B before the cup is evicted from B. Even if this happens on each of the n_A donation-processes $\mu(B)$ cannot rise higher than $n_A(R_\Delta + k - 1)/(n - n_A)$ which by design in choosing $n_B \gg n_A$, as was done in choosing $\delta = \Theta(e^{-2h})$, is at most 1.

Thus $\mu(B) \leq 2 + \mu(AB)$ is always true. □

The upper bound on $\mu(B)$ along with the guarantee that B is flat allows us to bound the highest that a cup in A could rise by greediness, which in turn upper bounds $\mu(A)$ which in turn lower bounds $\mu(B)$. In particular we have

Claim 3. *We always have*

$$\mu(B) \geq -h/2 + \mu_0.$$

Proof. By Claim 2 and Claim 1 we have that no cup in B ever has fill greater than $u_B = \mu(AB) + 2 + R_\Delta + d$. Let $u_A = u_B + \Delta + 1$. We claim that the backlog in A never exceeds u_A. Note that $\mu(AB), u_A, u_B$ are implicitly functions of the round; $\mu(AB)$ can increase from μ_0 if the emptier skips emptyings.

Consider how high the fill of a cup $c \in A$ could be. If c came from B then when it is donated to A its fill is at most $u_B < u_A$. Otherwise, c started with fill at most $R_\Delta < u_A$. Now consider how much the fill of c could increase while being in A. Because the emptier is Δ-greedy-like, if a cup $c \in A$ has fill more than Δ higher than the backlog in B then c must be emptied from, so any cup with fill at least $u_B + \Delta = u_A - 1$ must be emptied from, and hence u_A upper bounds the backlog in A.

Of course an upper bound on backlog in A also serves as an upper bound on the average fill of A as well, i.e. $\mu(A) \leq u_A$. Rearranging the expression

$$|B|\mu(B) + |A|\mu(A) = |AB|\mu(AB)$$

we have

$$\begin{aligned}
&\mu(B) \\
&= -\frac{|A|}{|B|}\mu(A) + \frac{|AB|}{|B|}\mu(AB) \\
&\geq -(\mu(AB) + 3 + R_\Delta + d + \Delta)\frac{|A|}{|B|} + \frac{|AB|}{|B|}\mu(AB) \\
&= -(3 + R_\Delta + d + \Delta)\frac{|A|}{|B|} + \mu(AB) \\
&\geq -h/2 + \mu(AB)
\end{aligned}$$

where the final inequality follows because $\mu(AB) \geq 0$, and $|B| \gg |A|$, in particular by our choice of $\delta = \Theta(e^{-2h})$. Of course $\mu(AB) \geq \mu_0$ so we have

$$\mu(B) \geq -h/2 + \mu_0.$$

□

Now we show that at least a constant fraction of the donation-processes succeed with exponentially good probability.

Claim 4. *There exists choice of $m = \Theta(M)$ such that with probability at least $1 - 2^{-\Omega(n)}$, the filler achieves fill at least $h + \mu_0$ in $\Theta(n)$ of the cups in A.*

Proof. If the emptier was not allowed to neglect the anchor set ever then the claim would clearly be true as each application of randalg(k) would unconditionally succeed with constant probability, so a Chernoff bound would give that $\Theta(n)$ of the donation-processes donate a cup with fill at least $\mu(B) - R_\Delta + d \geq h + \mu_0$, where the inequality follows from Claim 3 which asserts that $\mu(B) \geq -h/2 + \mu_0$, and from the facts $d \geq 2h$ and $h \geq 16(1 + \Delta)$. However, the emptier is allowed to neglect the anchor set, and worse, the emptier can choose to neglect the anchor set conditional on the filler's progress during randalg(k). However, by applying randalg(k) a random number of times, chosen from $[m]$ (where $m = \Theta(M)$ which is quite large), we guarantee that with exponentially good probability the filler succeeds many times, in particular $\Theta(M)$ times. But since the emptier cannot neglect the anchor set more than M times, by appropriately large choice of m we can make it so that the filler succeeds at least $2M$ times with exponentially good probability. Then the emptier would have at best a $1/2$ chance of preventing the donation-process from giving away a cup with fill $h + \mu_0$ whenever one such cup is achieved. We now formalize this reasoning.

We can lower bound the probability of getting $\Theta(n)$ cups with fills all at least $h + \mu_0$ by considering an augmented emptier that is allowed to interfere with M applications of randalg(k) per donation-process that only interferes with applications of randalg(k) that would otherwise donate a cup with fill at least $h + \mu_0$ into A. The optimal strategy for such an emptier, given our filler's strategy of randomly choosing how many times to apply randalg(k) before donating a cup, is to simply interfere with the first M applications of randalg(k) that would have achieved a cup with fill $h + \mu_0$ without interference. Let $m = 4Mk! = \Theta(M)$. Recall that conditional on the emptier not interfering, each of these applications of randalg(k) has at least a $1/k!$ chance of getting a cup with fill h. Hence, by a Chernoff bound with exponentially good probability at least $2M$ of m applications of randalg(k) have the potential to donate a cup with fill $h + \mu_0$ to A, if the emptier does not interfere. The filler chooses an application uniformly at random from $[m]$ on which to donate a cup. With probability at least $1/k!$ this is on an application where the filler could get a cup with fill $h + \mu_0$ in A if the emptier does not interfere, and with probability at least $1/2$ the emptier does not interfere on this application of randalg(k), because the emptier can interfere on at most M of the applications of randalg(k).

Against this augmented emptier whether or not donation-processes achieve a cup with fill $h + \mu_0$ in A are independent events. As each happens with at least constant probability, by a Chernoff bound there is exponentially high probability that at least a constant fraction of them succeed.

Note that we used the Chernoff bound $\Theta(n)$; by a union bound there is exponentially good probability that all of the desired events occur.

□

We now analyze the running time of the filling strategy. There are n_A donation-processes. Each donation-process consists of $\Theta(M)$ applications of randalg(k), which each take constant time, and $\Theta(M)$ applications of flatalg, which each take poly(M) time. Thus overall the algorithm takes poly(M) time, as desired.

□

Now, using Lemma 1 we show in Proposition 1 that an oblivious filler can achieve constant backlog.

Proposition 1. *Let $H \leq O(1)$, let $\Delta \leq O(1)$, let n be at least a sufficiently large constant determined by H and Δ. Let $M \gg n$ be very large. Consider an R_Δ-flat cup configuration in the variable-processor cup game on n cups with average fill μ_0. There is an oblivious filling strategy that either achieves mass M among the cups, or achieves fill at least $\mu_0 + H$ in a chosen cup in running time poly(M) against a Δ-greedy-like emptier with probability at least $1 - 2^{-\Omega(n)}$.*

Proof. The filler starts by using $\text{rep}_\delta(\text{randalg}(k))$ with parameter settings as in Lemma 1 where $h = H \cdot 16(1+\Delta)$, i.e. $k = \lceil e^{2h+1} \rceil$, $\delta = \Theta(e^{-2h})$. Let the number of cups which must now exist by Lemma 1 with fill at least $h + \mu_0$ be of size $nc = \Theta(n)$.

The filler sets $p = 1$, i.e. uses a single processor. Now the filler exploits the emptier's greedy-like nature to to get fill H in a chosen cup c_0. Specifically, for $(5/8)h$ rounds the filler places 1 unit of fill into c_0. Because the emptier is Δ-greedy-like it must empty from the nc cups in A with fill at least $h + \mu_0$ until c_0 has large fill. Over $(5/8)h$ rounds the cups in A cannot have their fill decrease below $(3/8)h \geq h/8 + \Delta + \mu_0$. Hence, any cups with fills less than $h/8 + \mu_0$ must not be emptied from during these rounds. The fill of c_0 started as at least $-h/2 + \mu_0$ as $\mu(B) \geq -h/2 + \mu_0$. After $(5/8)h$ rounds c_0 has fill at least $h/8 + \mu_0$, because the emptier cannot have emptied c_0 until it attained fill $h/8 + \mu_0$, and if c_0 is never emptied from then it achieves fill $h/8 + \mu_0$. Thus the filling strategy achieves backlog $h/8 + \mu_0 \geq H + \mu_0$ in c_0, a known cup, as desired. □

Next we prove the ***Oblivious Amplification Lemma***.

Lemma 2 (Oblivious Amplification Lemma)**.** *Let $\delta \in (0, 1/2)$ be a parameter. Let $\Delta \leq O(1)$. Let M be very large. Consider a cup configuration in the variable-processor cup game on n cups with average fill μ_0 that is R_Δ-flat. Let* $\text{alg}(f)$ *be an oblivious filling strategy that either achieves mass M or achieves backlog $\mu_0 + f(n)$ on such cups with probability at least $1 - 2^{-\Omega(n)}$ in running time $T(n)$ against a Δ-greedy-like emptier.*

Consider a cup configuration in the variable-processor cup game on n cups with average fill μ_0 that is R_Δ-flat. There exists an oblivious filling strategy $\text{alg}(f')$ *that either achieves mass M or achieves backlog $f'(n)$ satisfying*

$$f'(n) \geq (1-\delta) f(\lfloor (1-\delta)n \rfloor) + f(\lceil \delta n \rceil) + \mu_0$$

and $f'(n) \geq f(n)$, with probability at least $1 - 2^{-\Omega(n)}$ in running time

$$T'(n) \leq M \cdot n \cdot T(\lfloor (1-\delta)n \rfloor) + T(\lceil \delta n \rceil)$$

against a Δ-greedy-like emptier.

Proof. We use the notation from Lemma 1, and from Definition 1. To summarize, we define $n_A = \lceil \delta n \rceil$, $n_B = \lfloor (1-\delta)n \rfloor$, we refer to A as the anchor set and B as the non-anchor set, we say that the filler applies $\text{alg}(f)$ to B if it uses $\text{alg}(f)$ on B while placing 1 fill into each cup in A, and we say that A is neglected during an application of $\text{alg}(f)$ to B if there is some round during the application where the emptier does not empty from all anchor cups.

The filler defaults to using $\text{alg}(f)$ on all the cups if

$$f(n) \geq (1-\delta) f(n_B) + f(n_A).$$

In this case our strategy trivially has the desired guarantees. In the rest of the proof we consider the case where we cannot simply fall back on $\text{alg}(f)$ to achieve the desired backlog.

The filler's strategy is roughly as follows:

Step 1: Use $\text{rep}_\delta(\text{alg}(f))$ on all the cups; this will get A to have high average fill. **Step 2:** Flatten A using flatalg, and then use $\text{alg}(f)$ on A.

Now we analyze Step 1, and show that by appropriately choosing parameters it can be made to succeed.

Note that, exactly as in the proof of Lemma 1, the emptier cannot neglect the anchor set more than M times per donation-process, and the emptier cannot skip more than M emptyings, without causing the mass of the cups to be at least M; we assume the emptier chooses not to do this.

Roughly speaking, by choosing m (recall that we choose how many times to apply $\text{alg}(f)$ in a donation-process uniformly at random from $[m]$) sufficiently large, because each application of $\text{alg}(f)$ where the anchor set is not neglected succeeds with exponentially good probability, we can take a union bound and guarantee that all applications of $\text{alg}(f)$ succeed. Each successful application of $\text{alg}(f)$ removes from B a cup with fill

TODO: that's basically like $\mu(B) + f(n_B)$ but just not quite because skips lol. if you do this often enough either $\mu(B)$ gets small enough that we're good (note that it's monotonically decreasing) or $\mu(B)$ stays pretty high so all the cups we're transferring over are pretty good.

If an application of $\text{alg}(f)$ is successful, then with probability at least $1 - 2^{-\Omega(n)}$ it generates a cup with fill $f(|B|) + \mu(B)$ in B, because equal resources were put into B on each round while $\text{alg}(f)$ was used, and the cup state started as R_Δ-flat and hence also started as M-flat (as $M \geq R_\Delta$).

Now we aim to show that $\mu(A)$ is large; we do so by showing that $\mu(B)$ is small (i.e. very negative). Because the probability of an application of $\text{alg}(f)$ being successful is only $1 - 1/\text{poly}(n)$, which is in particular not as good as the $1 - 2^{-\Omega(n)}$ that we will guarantee, we will not be able to actually assume that every such application of $\text{alg}(f)$ is successful. However, (as we will show later) we can guarantee that at least a constant fraction ϕ of the swapping-processes are successful with exponentially good probability.

The filler swaps $|A|$ cups into B. Consider how $\mu(B \cup A \setminus A_0)$ changes when a new cup is swapped into A and placed in A_0. Let the initial value of $\mu(B \cup A \setminus A_0)$ be μ_0. Say that initially $|A_0| = i$ (i.e. i swapping-processes have occured so far). If the swapping-process is successful then the swapped cup has fill at least $\mu_0 - R_\Delta + f(|B|)$. Hence the new average fill of $B \cup A \setminus A_0$ after the swap is

$$\frac{\mu_0 \cdot (n-i) - (\mu_0 - R_\Delta + f(|B|))}{n-i-1} = \mu_0 - \frac{f(|B|) - R_\Delta}{n-i-1}.$$

This recurrence relation allows us to find the value of $\mu(B \cup A \setminus A_0) = \mu(B)$ after $|A|$ swapping processes (i.e. once $A \setminus A_0 = \varnothing$):

$$\mu(B) \leq -\sum_{i=1}^{|A|\phi} \frac{f(|B|) - R_\Delta}{n-i}.$$

Now we bound $H_{n-1} - H_{n-|A|\phi-1}$ where H_i is the i-th harmonic number. Using the fact that

$$H_n = \ln n + \gamma + 1/(2n) - 1/(12n^2) + 1/(120n^4) - \ldots$$

we have,

$$\begin{aligned}
&H_{n-1} - H_{n-|A|\phi-1} \\
&\geq \ln \frac{n-1}{n-|A|\phi-1} - \frac{1}{2(n-|A|\phi-1)} \\
&\geq \ln \frac{n}{n-|A|\phi} - \frac{1}{n} \\
&= \ln \frac{n}{n-\lceil \delta n \rceil \phi} - \frac{1}{n} \\
&\geq \ln \frac{1}{1-\delta\phi} - \frac{1}{n} \\
&\geq \delta\phi - \frac{1}{n}.
\end{aligned}$$

Hence we have,

$$\mu(A) \geq \frac{(1-\delta)}{\delta} \left(\delta\phi - \frac{1}{n} \right) (f(|B|) - R_\Delta). \qquad (1)$$

We have shown that in Step 1 the filler achieves average fill $\mu_0 + (1-\delta) f(n_B)$ in A. Now the filler flattens A and uses $\operatorname{alg}(f)$ on A.A This gets a cup with fill

$$\mu_0 + (1-\delta) f(n_B) + f(n_A)$$

in A, as desired.

□

References