

1 A Deterministic Scheduling Algorithm

In this section we present a simple deterministic scheduling algorithm that does not use preemption. We show that this algorithm is 2-competitive with OPT.

First we note that without loss of generality we may consider TAPs where the cost ratio $\pi(\tau)/\sigma(\tau)$ for any task τ is in $[1, p]$; if $\pi(\tau)/\sigma(\tau) < 1$, i.e. the parallel implementation has lower work than the serial implementation, then the scheduler clearly should never use the serial implementation of this algorithm, so we can replace the serial implementation with the parallel implementation and hence get cost ratio 1, similarly, if $\pi(\tau)/\sigma(\tau) > p$ then the scheduler should never run the parallel task and we can replace the parallel implementation of the task with the serial implementation to get cost ratio p .

We say that a time is a **verge** time for our algorithm if at this time no processors are performing tasks and there is at least one ready task.

We propose Algorithm 1, which we call **GR** (GR is short for “greedy”), as a scheduling algorithm.

Algorithm 1 GR

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while True do
  if verge time then
    schedule tasks as directed by ORACLE-1
  
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A **single-time-TAP** is a TAP where all tasks arrive at a single time. ORACLE-1 is an algorithm that yields a schedule achieving minimal awake time, i.e. the same awake time as OPT, on single-time-TAPs. In Theorem 1 we establish that ORACLE-1, which we specify in Algorithm 2, is an oracle for OPT on single-time-TAPs, hence showing that GR can actually be computed. We remark that it is not obvious that an oracle for OPT can be computed in finite time, even in this special case: there are an uncountably infinite number of possible scheduling strategies that OPT can choose from. Nevertheless, we show how to consider a finite search space that a brute force search can actually be performed on, at least in the special case of single-time-TAPs. Before analyzing ORACLE-1, which is a quite involved process, we analyze the competitive ratio of GR assuming the existence of ORACLE-1.

Consider a TAP \mathcal{T} . Let ℓ be the number of verge times for \mathcal{T} ; note that $\ell \leq n$ which in particular is finite. Let t_i be the i -th time that is a verge time, let q_i be the number of ready tasks for GR at time t_i . Let $T^{ALG}(q_1, \dots, q_\ell)$ denote the awake time of a

scheduling algorithm ALG on the truncation of the TAP \mathcal{T} that only consists of tasks arriving at times before t_ℓ .

By construction of ORACLE-1 we have that for single-time-TAPs OPT and GR achieve the same awake time, i.e.

$$T^{OPT}(q) = T^{GR}(q). \quad (1)$$

We remark GR “locally” schedules optimally, which is why we refer to GR as “greedy”.

An **ALG-gap** is an interval of time I of non-zero length where for all times in the interior of I ALG has completed every task that has arrived thus far. Additionally for an interval to be an ALG-gap the interval must contain no other intervals which are also ALG-gaps (i.e. it is a “maximal” interval satisfying our conditions). We say that a TAP is **ALG-gap-free** if it contains no ALG-gaps.

Now we prove an important property of OPT.

Claim 1. *If there is a scheduling algorithm ALG that completes all tasks by time t_* then OPT finishes all tasks by time t_* .*

Proof. Say that ALG completes all tasks by time t_* . Let $t_0 < t_*$ be the most recent time that OPT has completed all tasks that arrive before time t_0 . If OPT has not finished all tasks by time t_* then it was acting sub-optimally, as it could steal the strategy that ALG used on $[t_0, t_*]$ to achieve lower awake time. In particular, for any tasks that arrive in $[t_0, t_*]$ OPT could schedule them as ALG schedules them. We remark that OPT should not steal all of ALG. \square

As an immediate consequence of Claim 1 we have that any ALG-gap is a subset of an OPT-gap.

Decomposing TAPs into gap-free subsets of the TAP is very useful. Part of the reason for this is the following fact:

Claim 2. *If an algorithm ALG achieves competitive ratio r on ALG-gap-free TAPs, then ALG achieves competitive ratio r on arbitrary TAPs.*

Proof. We partition the tasks based on arrival time, splitting the tasks on the ALG-gaps. That is, we split the tasks into groups so that two tasks τ_i, τ_j are in the same group if and only if there are no gaps in between the arrival times of τ_i and τ_j . We can define an interval of time I_i for each of these ALG-gap-free subsets of the TAP, where I_i is defined so that all tasks in the i -th group start and finish at times contained in the interval I_i .

Let $T_{I_i}^{OPT}$ and $T_{I_i}^{ALG}$ denote the awake time of OPT and ALG on interval I_i . Because I_i is ALG-

gap-free we have $T^{ALG} = \sum_i T_{I_i}^{ALG}$. Further, recall that by Claim 1 any ALG-gap is also an OPT-gap, so $T^{OPT} = \sum_i T_{I_i}^{OPT}$. Hence from our assumption that ALG is r -competitive on gap-free TAPs, such as the subset of the TAP on the interval I_i , we have $T_{I_i}^{ALG} \leq r \cdot T_{I_i}^{OPT}$ for all i . Summing we get $T^{ALG} \leq r \cdot T^{OPT}$, as desired. \square

By Claim 2, in order to bound GR's competitive ratio, it suffices to consider TAPs without GR-gaps. Note however that a TAP without GR-gaps could still have OPT-gaps.

We conclude our analysis of the competitive ratio of GR in Proposition 1 with an inductive argument on the number of OPT-gaps in the TAP. First we establish the base case for the argument in Claim 3: we consider GR's competitive ratio on a TAP without OPT-gaps.

Claim 3. *GR is 2-competitive on OPT-gap-free TAPs.*

Proof. For an OPT-gap-free TAP we must have

$$T^{OPT}(q_1, \dots, q_\ell) \geq T^{GR}(q_1, \dots, q_{\ell-1}). \quad (2)$$

Because GR finishes all q_i tasks that arrive at time t_i by time t_{i+1} we can actually always decompose $T^{GR}(q_1, \dots, q_\ell)$ as

$$T^{GR}(q_1, \dots, q_\ell) = \sum_{i=1}^{\ell} T^{GR}(q_i). \quad (3)$$

By Equation (3), and Equation (1) we thus have

$$T^{GR}(q_1, \dots, q_\ell) = T^{GR}(q_1, \dots, q_{\ell-1}) + T^{OPT}(q_\ell). \quad (4)$$

Hence by Equation (2) and Equation (4) we have

$$\begin{aligned} T^{GR}(q_1, \dots, q_\ell) &\leq T^{OPT}(q_1, \dots, q_\ell) + T^{OPT}(q_\ell) \\ &\leq 2T^{OPT}(q_1, \dots, q_\ell), \end{aligned}$$

as desired. \square

Proposition 1. *GR is 2-competitive.*

Proof. The proof is by strong induction on the number of OPT-gaps. The base case of our induction is established in Claim 3, which says that if there are 0 OPT-gaps then GR is 2-competitive.

Consider a TAP that has more than 0 OPT gaps; say that its first OPT-gap starts at time t_* . Let j be the largest index such that verge time $t_j < t_*$.

Using our inductive hypothesis we have:

$$\begin{aligned} T^{OPT}(q_1, \dots, q_\ell) &\geq T^{OPT}(q_1, \dots, q_j) + T^{OPT}(q_{j+1}, \dots, q_\ell) \\ &\geq \frac{1}{2} (T^{GR}(q_1, \dots, q_j) + T^{GR}(q_{j+1}, \dots, q_\ell)) \\ &= \frac{1}{2} T^{GR}(q_1, \dots, q_\ell). \end{aligned}$$

\square

Now we analyze Algorithm 2, which we call ORACLE-1.

TODO: simpler description of frosting

Algorithm 2 ORACLE-1: oracle for OPT on single-time-TAPs

Input: tasks τ_1, \dots, τ_n all with $t(\tau_i) = 0$

Output: a way to schedule the tasks to processors ρ_1, \dots, ρ_p that achieves minimal awake time

minAwakeTime $\leftarrow \infty$

bestSchedule \leftarrow schedule everything in serial on ρ_1

for $I \in \{0, 1\}^n$ **do**

$x \leftarrow \sum_{i=1}^n I_i$

for $J \in \{1, \dots, n\}^x$ **do**

$j \leftarrow 0$

for $i \in \{1, 2, \dots, n\}$ **do**

if $I_i = 1$ **then**

$j \leftarrow j + 1$

schedule task τ_i in serial on ρ_{J_j}

for $i \in \{1, 2, \dots, n\}$ **do**

if $I_i = 0$ **then**

$f \leftarrow \pi(\tau_i)$

while $f > 0$ **do**

$b_1 \leftarrow \min_{\rho_i} (\text{work}(\rho_i))$

$B_1 \leftarrow \{\rho_i \mid \text{work}(\rho_i) = b_1\}$

$b_2 \leftarrow \inf_{\rho_i \notin B_1} (\text{work}(\rho_i))$

$\Delta_f \leftarrow \min(f / |B_1|, b_2 - b_1)$

give fill Δ_f to all $\rho_i \in B_1$

$f \leftarrow f - \Delta_f$

if awakeTime(I, J) \leq minAwakeTime **then**

minAwakeTime \leftarrow awakeTime(I, J)

bestSchedule \leftarrow schedule(I, J)

The algorithm is a straightforward brute-force approach to computing what to do. The algorithm tries every possible way of assigning some tasks to be run in parallel and others to be run in serial, and for each of these the algorithm tries every possible way to arrange the serial tasks, and for each of these the algorithm then **frosts** the configuration with parallel tasks, meaning, the algorithm distributes the parallel tasks so as to minimally increase awake time.

We remark that the running time of ORACLE-1 for n tasks is at least $\sum_{i=1}^n n^{(i)} \geq 2^{2^n}$, which is pretty big. Nevertheless the existence of our algorithm is very interesting, and the running time can probably be reduced.

We now prove that ORACLE-1 actually does compute a schedule with awake time the same as that of OPT.

Theorem 1 (Cake Frosting Theorem). *ORACLE-1 is an oracle for OPT on TAPs where all tasks arrive at a single time.*

Proof. Consider a TAP where all tasks arrive at a single time. We claim that the awake time achieved by OPT is not better than the awake time achieved by ORACLE-1, or equivalently GR (which in this case is the same as ORACLE-1).

TODO: make this not specific to the single-task case Say OPT schedules x of the q tasks in serial, and schedules the remaining $q - x$ in parallel. Say that an optimal assignment of the tasks, under the constraint that x of the tasks are scheduled in serial and $q - x$ are scheduled in parallel, achieves awake time M .

We claim that there must exist an optimal assignment of tasks such that each processor is assigned either $\lfloor x/p \rfloor$ or $\lceil x/p \rceil$ serial tasks. To prove this, we start from some optimal assignment, and modify it in such a way as to make the configuration “closer” to our desired configuration. Let $n_\sigma(\rho_i)$ be the number of serial tasks scheduled on processor ρ_i .

To formalize a notion of “closeness” we define a potential function ϕ of the assignment S of tasks:

$$\phi(S) = \sum_i \min(|n_\sigma(\rho_i) - \lfloor x/p \rfloor|, |n_\sigma(\rho_i) - \lceil x/p \rceil|).$$

Note that $\phi(S)$ is non-negative. We desire a configuration with $\phi(S) = 0$. Consider a configuration of tasks achieving awake time M with $\phi(S) > 0$. We first apply the following procedure to the configuration: while the difference between the maximum work assigned to a processor and the minimum work assigned to a processor is more than 1 swap 1 unit of work from a processor with the maximum amount of work to a processor with the minimum amount of work. This swap cannot increase the range of works assigned to processors, and the process must eventually terminate. Now, while $\phi(S) > 0$ there must exist ρ_i, ρ_j with $n_\sigma(\rho_i) < \lfloor x/p \rfloor$ and $n_\sigma(\rho_j) > \lfloor x/p \rfloor$ or there exists ρ_i, ρ_j with $n_\sigma(\rho_i) < \lceil x/p \rceil$ and $n_\sigma(\rho_j) > \lceil x/p \rceil$. By construction the range of the works is at most 1; hence ρ_i has at least 1 unit of parallel work to have work within 1 of ρ_j despite having at least 2 fewer serial tasks than ρ_j . Then ρ_i gives this 1

unit of parallel work to ρ_j , and in exchange ρ_j gives ρ_i a serial task. This swapping operation decreases $\phi(S)$ by exactly 1, and does not change the amount of work assigned to each processor, which importantly means that the swap does not increase the range of the works. Hence, we can repeat this swapping process to eventually achieve a configuration with awake time M where each processor has $\lfloor x/p \rfloor$ or $\lceil x/p \rceil$ serial tasks.

If $x \bmod p = 0$ then the awake time of OPT is clearly at least

$$x/p + (q - x)k/p. \quad (5)$$

Note that if x increases by p then (5) changes by $1 - k < 0$. That is, (5) is monotonically decreasing in x , and is thus minimized by setting $x = p \lfloor q/p \rfloor$ and hence getting $q - x = q \bmod p$. This gives awake time $\lfloor q/p \rfloor + (q \bmod p)k/p \geq T(q)$.

Now we consider the case where $x \bmod p \neq 0$. Here $(p - (x \bmod p))/k$ tasks can be added in serial without increasing the work at all. Thus the awake time is at least

$$\lfloor x/p \rfloor + \frac{k}{p} \left(\max \left(0, q - x - \frac{p - (x \bmod p)}{k} \right) \right). \quad (6)$$

Consider when the max in (6) yields 0. For this to happen we must have

$$q - x \leq \frac{p - x \bmod p}{k}.$$

As $p - x \bmod p \leq p$ we get the following bound on $q - x$:

$$q - x \leq p/k,$$

which must be met in order for the max to yield 0. Clearly as $q \geq x \geq q - p/k$, $\lfloor x/p \rfloor \geq \lfloor q/p \rfloor - 1$; we claim that this inequality is actually strict. Imagine that $\lfloor x/p \rfloor = \lfloor q/p \rfloor - 1$. Because the max expression yields 0, meaning that the parallel tasks add nothing to the awake time, if all processors only have $\lfloor q/p \rfloor - 1$ or $\lfloor q/p \rfloor < q/p$ serial tasks, then the total work is at most $\lfloor q/p \rfloor < q/p$ which is impossible: the total work in the system must be at least q and it can be distributed in the best case perfectly equally which makes q/p as a lower bound on the time to complete q tasks. Hence $\lfloor x/p \rfloor = \lfloor q/p \rfloor$. But then the awake time is at least $\lfloor x/p \rfloor = \lfloor q/p \rfloor + 1 \geq T(q)$.

Now we consider the case when the max in (6) yields some positive number. Note that if x increases by p (but x is still sufficiently small so that the max yields a positive number) then (6) changes by $1 - k < 0$. Further, if x increases by 1 (but x is still sufficiently small so that the max yields a positive number) without $\lfloor x/p \rfloor$ changing, then (6) changes

by $(k/p)(1/k - 1) < 0$. That is, (6) is monotonically decreasing in x . Hence we still have that the awake time is at least $T(q)$.

We have considered all cases, and shown that no matter what choice of x OPT makes, and no matter how OPT schedules given that choice of x , OPT must incur awake time at least $T(q)$, as desired. \square

References