Introduction to Modular Forms

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1 Definitions

Throughout the note we use z^* to denote the complex conjugate of a complex number z.

Let H denote the upper half-plane of $\mathbb{C}$, i.e., the set of complex numbers with positive imaginary part. Let $SL_2(\mathbb{R})$ denote the group of 2×2 real matrices with determinant 1. We make $SL_2(\mathbb{R})$ act on $\tilde{\mathbb{C}}$ as follows: For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}), z \in \tilde{\mathbb{C}}$ we define:

$$gz = \frac{az+b}{cz+d}.$$

Proposition 1.1.

$$\mathrm{Im}(gz) = \frac{1}{|cz+d|^2}\mathrm{Im}(z).$$

Proof. Multiplying and dividing by $cz^* + d$ gives

$$gz = (az+b)(cz^*+d)\frac{1}{|cz+d|^2}.$$

Thus,

$$\mathrm{Im}(gz) = ad\mathrm{Im}(z) + bc\mathrm{Im}(z^*)\frac{1}{|zc+d|^2} = (ad-bc)\mathrm{Im}(z)\frac{1}{|zc+d|^2}.$$

Recalling that $ad - bc = 1$ we conclude the desired formula. $\square$

This implies that H is ***stable*** under the action of $SL_2(\mathbb{R})$. Observe that $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is the identity map. Define $PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\{\{-1,1\}\}$. One can now show that $PSL_2(\mathbb{R})$ acts ***faithfully*** on H; i.e., there is a unique matrix in $PSL_2(\mathbb{R})$ which is the identity action on H.

Define $SL_2(\mathbb{Z})$ as the subgroup of $SL_2(\mathbb{R})$ consisting of matrices with integer coefficients and define $PSL_2(\mathbb{Z})$ to be the image of $SL_2(\mathbb{Z})$ in $PSL_2(\mathbb{R})$. $PSL_2(\mathbb{R})$ is called the ***modular group***. We will refer to the modular group by G for the remainder of this note.

2 The Fundamental Domain of the Modular Group

We now develop some basic facts about the modular group. Let $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in G$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in G$. S is the map which does $Sz = -1/z$, i.e., first draw the radius from the origin to your point and "invert" the point about the unit circle (modify the radius as $r \mapsto 1/r$) and then reflect the angle around the imaginary axis. S satisfies $S^2 = \mathsf{id}$. T is the map which does $Tz = z+1$, i.e., translating the real part by one unit. One can show via a straightforward computation that $(ST)^3 = \mathsf{id}$.

Let $D \subseteq H$ be the set of points z with $|z| \geq 1$ and $|\mathrm{Re}(z)| \leq 1/2$. This is called the ***fundamental domain*** for the action of G on the half plane H for reasons which will be clear shortly. Let G' denote the subgroup of G generated by S, T (in fact we will later show $G' = G$).

Theorem 2.1. For every $z \in H$ there exists $g \in G'$ such that $gz \in D$.

Proof. Fix $z \in H$. Consider the lattice which is the $\mathbb{Z}$-span of $z, 1$. Because $z \in D$ this lattice is non-degenerate, i.e., its fundamental parallelpiped has non-zero area. Thus, the number of points of the lattice in any ball around the origin is finite. In particular this implies, using Proposition 1.1 that $\max_{g \in G'} \operatorname{Im}(gz)$ is well-defined.

Let $g \in G'$ maximize $\operatorname{Im}(gz)$. Choose $n \in \mathbb{Z}$ such that $|\operatorname{Re}(T^n gz)| \le 1/2$. We claim that $z' = T^n gz \in D$. Suppose not. Then we must have $|z'| < 1$. This implies that

$$\operatorname{Im}(T^n gz) = \operatorname{Im}(gz) < 1.$$

But then $\operatorname{Im}(Sz') > \operatorname{Im}(z')$. This implies that

$$\operatorname{Im}(ST^n gz) > \operatorname{Im}(T^n gz) = \operatorname{Im}(gz).$$

But this is a contradiction as g was chosen to maximize $\operatorname{Im}(gz)$ over all $g \in G'$ and $ST^n \in G'$. Hence we must actually have $z' \in D$ as desired. $\square$

Now we consider the elements of D modulo the action of G. We show that the only distinct elements of D which are congruent lie on the boundary. More specifically we have:

Proposition 2.2. Suppose that $z, z' \in D$ with $z \ne z'$ have $z' = gz$ for some $g \in G$. Then either $\operatorname{Re}(z) = \pm 1/2$ and $z' = z \pm 1$ or $|z| = 1$ and $z' = -1/z$.

In particular, for all $z \in D \setminus \left\{i, e^{2\pi i/3}, e^{2\pi i/6}\right\}$ the only $g \in G$ which fixes z is the identity matrix id. The element i is fixed by exactly S, id. The element $e^{2\pi i/3}$ is fixed by exactly $\mathsf{id}, ST, (ST)^2$. The element $e^{\pi i/3}$ is fixed by exactly $\mathsf{id}, TS, (TS)^2$.

Proof. Fix $z, z' \in D, z \ne z', g \in G$ with $z' = gz$. Because $z, gz \in D$ and $g, g^{-1} \in G$ we may assume without loss of generality that $\operatorname{Im}(gz) \ge \operatorname{Im}(z)$, i.e., that $|zc + d| \le 1$. Considering the lattice $z\mathbb{Z} + 1\mathbb{Z}$ we observe that if $|c| \ge 2$ then $|zc + d| > 1$.

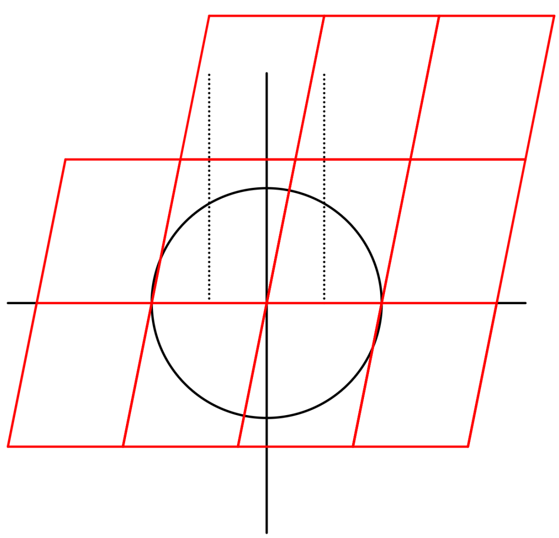

Figure 1: The lattice $z\mathbb{Z} + 1\mathbb{Z}$

Thus, it suffices to consider the cases $c = 0, 1, -1$.

Case $c = 0$: Then $d = \pm 1$. To achieve $ad - bc = 1$ we must have $a = d$. Thus, $gz = \pm z + b$.
Subcase: $b = 0$. Then we have $gz = \pm z$. Because $gz, z \in D$ we actually must have $gz = z$, which we were

assuming did not happen; thus this case does not exist. **Subcase:** $b = \pm 1$. Then (looking at the picture) we see that z must have $\mathrm{Re}(z) = \pm 1/2$. As $g \neq \mathsf{id}$ we then have that the points z, z' have real parts $1/2$ and $-1/2$ and gz is either $z + 1/2$ or $z - 1/2$ appropriately.

Summarizing, in the case $c = 0$ we found that $gz \in D$ is only possible here if $\mathrm{Re}(z) = \pm 1/2$, and in this case g does not fix z, rather g must satisfy $gz = z \pm 1$.

Case $c = 1$ To make $|z + d| \leq 1$ we must have (by inspection of the picture) that $d = 0$, unless $z = e^{2\pi i/6}$ in which case $d = 0, -1$ are both allowable, or if $z = e^{2\pi i/3}$ in which case $d = 0, 1$ are both allowable.
Subcase: $d = 0$. To make $ad - bc = 1$ we then set $b = -c$. Thus, our map becomes is

$$z \mapsto (az - c)/(cz) = a - 1/z \in D.$$

Subsubcase: First consider z with $|z| > 1$. We claim that $a - 1/z \notin D$ for any choice of $a \in \mathbb{Z}$. Indeed, $-1/z$ will be a point in the interior of the unit disk satisfying

$$|\mathrm{Re}(-1/z)| = |\mathrm{Re}(z)|/|z|^2 < 1/2.$$

Thus, $-1/z \notin D$, and for any non-zero a the point $a - 1/z$ is translated such that $\mathrm{Re}(a - 1/z) \notin [-1/2, 1/2]$ making $a - 1/z \notin D$.
Subsubcase: Now consider z with $|z| = 1$. Then $-1/z$ is simply the reflection of z around the imaginary axis, and in particular $-1/z \in D$. If we take $a - 1/z$ for any z with a non-zero, $|z| = 1$ and $|\mathrm{Re}(z)| < 1/2$ then it will still land outside of D. For any z with $|z| = 1$ we have $z = -1/z$ if and only if $z = i$; in particular, S stabilizes i. If $|z| = 1, \mathrm{Re}(z) = 1/2$ then we are dealing with $z = e^{2\pi i/6}$. In this case $a - 1/z \in D$ if and only if $a \in \{0, 1\}$. The transformation $z \mapsto 1 - 1/z$ fixes $e^{2\pi i/6}$. If $|z| = 1, \mathrm{Re}(z) = -1/2$ then we are dealing with $z = e^{2\pi i/3}$. In this case $a - 1/z \in D$ if and only if $a \in \{0, -1\}$. The transformation $z \mapsto -1 - 1/z$ fixes $e^{2\pi i/3}$.

Summarizing the subcase of $d = 0$ we have found that $z \mapsto a - 1/z$ acting on any z with $|z| > 1$ lands outside of D, whereas for any z with $|z| = 1, |\mathrm{Re}(z)| < 1/2$, $z \mapsto a - 1/z$ lands in D if and only if $a = 0$ i.e., $gz = -1/z$. In particular, none of these z are stablized by any $g \neq \mathsf{id}$ except for $z = i$, which is stablized only by S. Finally, we have found that in this case for $z = e^{2\pi i/3}, e^{2\pi i/6}$ we have that $(ST)^2$ stabilizes $e^{2\pi i/3}$ and that $(TS)^2$ stabilizes $e^{2\pi i/6}$, but no other transformations of the form considered in this case stablize these points.

Subcase $d \neq 0$
Subsubcase $z = e^{2\pi i/3}, d = 1$**.** We have the map

$$z \mapsto \frac{az + b}{z + 1}.$$

We must have $a - b = 1$. Simplifying we have:

$$gz = a + e^{2\pi i/3}.$$

Thus we must have $a = 0, b = -1$ or $a = 1, b = 0$.
Subsubcase $z = e^{2\pi i/6}, d = -1$**.** This computation is similar to the subsubcase $z = e^{2\pi i/3}, d = 1$ and is omitted.

Sumarizing our analysis of $e^{2\pi i/3}, e^{2\pi i/6}$ we have observed that if $g(e^{2\pi i/3}) \neq e^{2\pi i/3}$ then it must be the case that $g(e^{2\pi i/3}) = e^{2\pi i/6}$. We have also observed that, in terms of transformations of the form considered in this case the only transformation that fixes $e^{2\pi i/3}$ is ST. We have made symmetric observations about $e^{2\pi i/6}$.

Case $c = -1$ Recall that g is only defined modulo $\{-1, 1\}$. Thus, we can just trade this case for the case $c = 1$.

We have exhaustively considered all cases, so we conclude the theorem.

$\square$

An immediate corollary is:

Corollary 2.3. The canonical map $D \to H/G$ is surjective and its restriction to the interior of D is injective.

Finally we show:

Theorem 2.4. $G = G'$.

Proof. Fix $g \in G$. Consider $2i$, an arbitrary point in the interior of D. By Theorem 2.1 there exists $g' \in G'$ such that $g'(g(2i))$ lies in D. Because $g'(g(2i)), 2i$ are equivalent modulo G and both lie in the interior of D Proposition 2.2 implies that $g'(g(2i)) = 2i$. So we have $g'g = \mathsf{id}$ and thus $g \in G'$. Thus, $G' = G$. $\square$