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1 span K_n with 3-valent tree

Let $d_1, \ldots, d_n \in \{1, 3\}$ denote the degrees of the vertices in our spanning tree. Because a tree on n vertices has $n-1$ edges, and because the sum of the degrees in a graph is double its number of edges, we must have

$$\sum_{i=1}^{n} d_i = 2 \cdot (n-1).$$

If n is odd, this is clearly impossible: the left hand side is odd, being a sum of an odd number of odd numbers, whereas the right side is even.

If n is even, it is possible to configure the degrees in a valid manner, but the degrees will be heavily constrained. Let k denote the number of degree 3 vertices. Then,

$$\sum_{i=1}^{n} d_i = 3k + 1(n-k) = n + 2k = 2(n-1).$$

Solving, we find

$$k = n/2 - 1.$$

Now, we recall in class that we found a formula for the number of labelled trees on vertex set $[n]$ with degrees $d_1, \ldots, d_n$, namely:

$$\binom{n-2}{d_1 - 1, \ldots, d_n - 1}.$$

Applying to the degrees that we found gives:

$$\frac{(n-2)!}{2^{n/2-1}}.$$

2 complete plus independent

This is simply a complete graph connected to an independent set. To obtain a spanning tree, we take a spanning tree for the complete graph, and then each vertex in the independent set chooses a neighbor from the complete graph. This never introduces cycles, so each vertex in the independent set is free to choose any of the n vertices in the complete graph to be its neighbor. Thus, there are

$$n^{n-2} \cdot n^n$$

spanning trees of this graph.

3 complete resistor

The Kirchoff matrix for K_n is $K = nI_n - J_n$, where I_n denotes the $n \times n$ identity matrix and J_n denotes the $n \times n$ matrix of all 1's. We compute the determinant of the reduced Kirchoff matrix, and the determinant of a cofactor of the reduced Kirchoff matrix. In general, the eigenvalues of $nI_m - J_m$ consist of $m-1$ n's, and a single $n-m$. So, the effective resistance of the circuit is:

$$\frac{n^{n-3} \cdot 2}{n^{n-2} \cdot 1} = \frac{2}{n}.$$

4 chips on a chain

Recall the Diamond Lemma, shown in class, which states that firing i and firing j commute, i.e., the result is the same regardless of what order we fire them in. Also recall the "Finiteness" Lemma, which says that all abelian sand-piles stabilize after a finite number of steps. We remark that finiteness can be easily established in this setting by analysis of the potential function $\sum_{i\in\mathbb{Z}} c_i \cdot i^2/2$, which increases by 1 each time a vertex fires, but is trivially bounded $n \cdot N^2/2$, and so the total number of firings is finite. A corollary of the Diamond Lemma combined with the Finiteness Lemma, is that the stabilization state is unique, and obtained after finitely many steps.

Definition 1. If n is odd, define $c^{(n)}_{\text{equi}} = (0, 0, 0, \dots, 1, 1, 1, 1, 1, \dots, 0, 0, 0)$, i.e. n 1's in the middle, 0's around. More formally, spots $0, \pm1, \pm2, \dots, \pm(n-1)/2$ have a single chip on them, all other locations have zero chips. If n is even, define $c^{(n)}_{\text{equi}} = (0, 0, 0, \dots, 1, 1, 0, 1, 1, \dots, 0, 0, 0)$ where there are $n/2$ 1's to directly to either side of the middle. Formally, spots $\pm1, \pm2, \dots, \pm n/2$ all have a single chip each, and no other locations have chips.

Let $c^{(n)}_{\text{init}}$ denote the configuration with n chips at 0, and no other chips.

We will omit the superscript from $c^{(n)}_{\text{init}}, c^{(n)}_{\text{equi}}$ when it is clear from context.

We claim that c_{equi} is the unique stable state obtainable from c_{init}. To prove this, we demonstrate a sequence of firings, starting from c_{init}, which results in state c_{equi}.

Proposition 1. There is a sequence of firings that, applied to state c_{init} result in state c_{equi}.

Proof. The case with an odd number of chips follows immediately from the case with one fewer chip. In particular, if we choose one chip at 0 to ignore, and then apply the firings that would cause $c^{(n-1)}_{\text{init}}$ (which is even) to become $c^{(n-1)}_{\text{equi}}$, then if we add back in the chip that we ignored we get $c^{(n)}_{\text{equi}}$. Thus, it suffices to consider the case when n is even.

We prove the claim by induction. If $n = 0$ the claim is trivially true.

We introduce notation for describing symmetric sandpiles:

$$(q_0, q_1 \times n_1, q_2 \times n_2, \dots)$$

describes a sandpile with q_0 chips at 0, followed by q_1 chips at $\pm1, \pm2, \dots, \pm n_1$, followed by q_2 chips at $\pm(n_1 + 1), \dots, \pm(n_1 + n_2)$, followed by q_3 chips at each of the next n_3 vertices (in the plus and minus directions), and so on. We need the following lemma for our induction:

Lemma 1. There is a sequence of firings that converts sandpile $(2, 1 \times m)$ to sandpile $(1 \times (m + 1))$.

Proof. We prove the claim by induction on m. We take $m = 0$ to be the base case. Here, firing 0 results in the desired state.

Now, take $m > 0$, and assume we have already established the lemma for $m - 1$. Applying the lemma for $m - 1$ we may transform $(2, 1 \times m) \to (0, 1 \times (m - 2), 2 \times 1)$. Now, we fire on the ends, and then we repeatedly fire both 2's until we obtain $(2, 0 \times 1, 1 \times (m - 1))$. Then, firing 0 again we obtain $(0, 1 \times m)$, as desired. □

Now we complete our induction. Assume we have shown the claim for n chips (where n is even). Then for $n + 2$ chips we start by firing to transform

$$(n + 2) \to (2, 1 \times n/2).$$

Then we finish by applying Lemma 1 to transform

$$(2, 1 \times n/2) \to (0, 1 \times n/2 + 1) = c^{(n+2)}_{\text{equi}}.$$

Because $N > n$, i.e., the chain is sufficiently long, all of the arguments made above are valid, because no chips are eaten by the sink ever. □

As an immediate corollary we have:

Corollary 1. $c_{\text{stab}} = c_{\text{equi}}$.

Proof. The stable state is unique, and there is some sequence of firings resulting in c_{equi}, a stable state. Thus c_{equi} is in fact the only possible stable state. $\square$

Now, we define the ***potential function***

$$\phi(c) = \sum_{i\in\mathbb{Z}} c_i i^2/2.$$

We prove a key lemma describing the behavior of the potential function:

Lemma 1. The potential function $\phi(c)$ increases by exactly one on every firing.

Proof. Consider the change to potential on firing at position $\pm i$ with $i \le N-1$: it is

$$(i+1)^2/2 + (i-1)^2/2 - 2i^2/2 = 1.$$

As claimed, the potential is one larger than the potential before firing.

If we fired at $i = N-1$, then a chip would go into sink. However, because the stable state has n chips nothing is ever fired into the sink. $\square$

Corollary 2. The number of firings until reaching the stable state is invariant, and in particular is

$$\sum_{i=1}^{\lfloor n/2 \rfloor} i^2 = \lfloor n/2 \rfloor (\lfloor n/2 \rfloor + 1/2)(\lfloor n/2 \rfloor + 1)/3.$$

Proof. The number of firings to reach the stable state is precisely $\phi(c_{\text{stab}}) - \phi(c_{\text{init}})$. $\square$

5 symmetric labeled tree

We can think of a symmetric spanning tree as a spanning tree of a certain weighted graph. In particular, we consider the complete graph on vertices $V = \{0, 1, 2, \dots, n\}$ with weights as follows: $w_{i,0} = w_{0,i} = 1$ for all $i \in [n]$, and $w_{ij} = 2$ for all $i \neq j, i, j \in [n]$. We form the reduced laplacian by crossing out the first row and column, i.e. the row and column corresponding to vertex 0. Upon doing this, we obtain the matrix $2J_n - (2n+1)I_n$ where J_n is the matrix of all 1's, and I_n is the identity matrix. The eigenvalues of $2J_n$ are $n-1$ 0's and one $2n$. Thus, the eigenvalues of $2J_n - (2n+1)I_n$ are $n-1$ $-(2n+1)$'s, and one 1. Overall, this means that the determinant of $2J_n - (2n+1)I_n$ being the product of the eigenvalues has absolulte value $\boxed{(2n+1)^{n-1}}$, which is thus the number of such trees.

6 inc complete bin tree

We say a permutation $\pi \in S_{2n+1}$ is ***alternating*** if $\pi_1 > \pi_2 < \pi_3 > \pi_4 < \cdots > \pi_{2n} < \pi_{2n+1}$. More precisely, we can recursively define an alternating sequence via the following two rules:

- Any sequence of length 1 is alternating
- If $a_1 > a_2 < a_3 > \cdots < a_{2n_1+1}$ and $b_1 > b_2 < b_3 > \cdots < b_{2n_2+1}$ are alternating sequences, and c satisfies $a_{2n_1+1} > c < b_1$, then

$$a_1 > a_2 < a_3 > \cdots < a_{2n_1+1} > c < b_1 > b_2 < b_3 > \cdots < b_{2n_2+1}$$

  is an alternating sequence.

We say a labeled plane complete binary tree is ***increasing*** if the labels increase in any path down from the root.

Proposition 1. Fix $n \in \mathbb{N}$ Alternating permutations in S_{2n+1} are in bijection with labeled plane complete binary trees on $2n+1$ vertices.

Proof. Let $T(V, E)$ be an increasing labeled plane complete binary tree on $2n+1$ vertices, with (distinct) labels denoted $\ell : V \to \mathbb{N}$ and root v_0. We turn T into a sequence π as follows:

- If v_0 is a leaf, return its label $\ell(v_0)$
- Else, recursively turn the left sub-tree into a sequence π_1, and the right sub-tree in to a sequence π_2. Then, form the sequence for T by concatenating $\pi_1, \ell(v_0), \pi_2$.

This is commonly known as an "in-order traversal" of T. We claim that the generated sequence π is an alternating sequence. We establish this via an inductive argument. For a base case, we consider $n = 0$. Here, T consists only of the root. Thus, π is a single element sequence, and is therefore an alternating sequence. For $n > 0$, the root will have a left and right sub-tree, by virtue of being a complete binary tree. Furthermore, the left and right sub-trees of the root are themselves complete binary trees, and in particular have an odd number of vertices. Thus, we may write the sequences π_1, π_2 generated by recursively applying our algorithm to the sub-trees as $a_1 > a_2 < a_3 > \cdots < a_{2n_1+1}$ and $b_1 > b_2 < b_3 > \cdots < b_{2n_2+1}$, where $2n_1+1, 2n_2+1$ denote the size of the left and right sub-tree. The root has minimal label in the whole tree, call it $c = \ell(v_0)$. Thus, upon concatenating the sequences we obtain:

$$a_1 > a_2 < a_3 > \cdots < a_{2n_1+1} > c < b_1 > b_2 < b_3 > \cdots < b_{2n_2+1},$$

which is also an alternating sequence, as desired. Applied to the specific case where the labels form a permutation of $[2n+1]$, we get an alternating permutation.

Now we show the inverse transformation, i.e., how to transform an alternating permutation into a tree. It is much the same:

- If the sequence π consists of a single element π_1, transform π into a tree with a single vertex, labeled by π_1.
- Otherwise, split the sequence π into π_1 and π_2 where π_1 is the numbers in the sequence to the left of the minimal element, and π_2 is the numbers in the sequence to the right of the minimal element, and c is the label of the minimal element. Then, we recursively transform π_1 into a labeled tree T_1, and π_2 into a labeled tree T_2. Finally, we form T by making the root have label c, and giving it T_1 as a left sub-tree and T_2 as a right sub-tree.

It is evident by induction that this produces an increasing labeled plane complete binary tree. Indeed, if labels increase on any path down from the root of a sub-tree, then labels also increase on any path down from the root of the entire tree.

$\square$

7 xtreme point

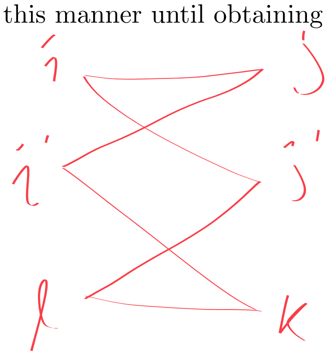
We associate with each matrices a weighted bipartite subgraph of $K_{n,n+1}$, where the weights incident to any vertex in the left sum to $n+1$, weights of edges incident to any vertex on the right part sum to n.

We start by reviewing the key lemma discussed in class:

Lemma 2. A matrix A is extreme if and only if its graph G is a spanning forest of $K_{n,n+1}$ with appropriate weights. Equivalenlty, A is not extreme iff its graph G contains a cycle.

Proof. First, assume that the graph G representing matrix A contains a cycle $\ell_1 \to r_1 \to \ell_2 \to r_2 \cdots \to r_m \to \ell_m = \ell_1$, where the ℓ_i denote vertices on the left, and the r_i denote vertices on the right. Let $\varepsilon > 0$ be smaller than any value in our matrix. Define the matrix B by placing $+\varepsilon$ in entry ℓ_i, r_i and $-\varepsilon$ in entry r_i, ℓ_{i+1}. Then, we can express our matrix A as the mid-point of $A+B, A-B$, so A is not extremal.

Now, assume that A is not extreme. Then there is a matrix $B \neq 0$ with 0 row and column sums such that $A+B, A-B$ are both matrices with appropriate row sums $n+1$ and column sums n. In particular, this means we can find an entry $b_{ij} > 0$ in B. To make B have zero-sum rows and columns, we then find $b_{i'j}, b_{ij'} < 0$ as well. Then, we can apply the same logic to find $b_{i'k}, b_{j'\ell} > 0$, and we can continue onwards in this manner until obtaining a cycle.



□

Proposition 2. A matrix is extreme if and only if its graph is a spanning forest of $K_{n,n+1}$ with appropriate weights.

Proof. For any vertex v on the left of the graph, the sum of incident weights to v is $n+1$. This is larger than the allowed sum of incident weights to each vertex w on the right. Thus, there can be no leaves on the left. In other words, there are at least $2n$ edges in our graph. Now, assume for contradiction that our graph has no cycles and is disconnected. Then, by the pidgeon hole principle there must be some connected component with at least as many vertices from the left part as from the right part. Call the number of vertices in the left part of this connected component m, and the number of vertices in the right part of this connected component m' where $m \geq m'$ by assumption. An acylic graph on $m+m'$ vertices has at most $m+m'-1$ edges. However, this connected component contains at least 2 edges per each vertex in the left part, i.e., at least $2m > m+m'-1$ total edges. This contradicts our assumption that the graph is acyclic, completing the proof. □

Corollary 3. There are $(n+1)^{n-1}n^n$ extreme matrices.

Proof. The number of spanning trees of $K_{n,n+1}$ is $(n+1)^{n-1}n^n$. In fact, there is a unique valid way to assign weights to the edges of a spanning tree of $K_{n,n+1}$ to result in a matrix with the appropriate row and column sums. To see this, first observe that the weights of the edges incident to leaves are determined. After recording these weights in the matrix, we delete the leaves. This creates new leaves, whose weights are then likewise determined. Continuing in this manner, all the weights are determined.

□

8 a catalan determinant

We are interested in computing the following determinant:

$$\det \begin{pmatrix} C_3 & C_4 & C_5 & \cdots \\ C_4 & C_5 & C_6 & \cdots \\ C_5 & C_6 & C_7 & \cdots \\ \vdots & \cdots & \cdots & \\ C_{n+2} & \cdots & \cdots & \end{pmatrix}.$$

We use Lindstrom's Lemma about lattice paths. This tells us that to compute this determinant, we can instead compute the number of n-tuples of non-crossing paths connecting n sinks to n sources in the following grid:

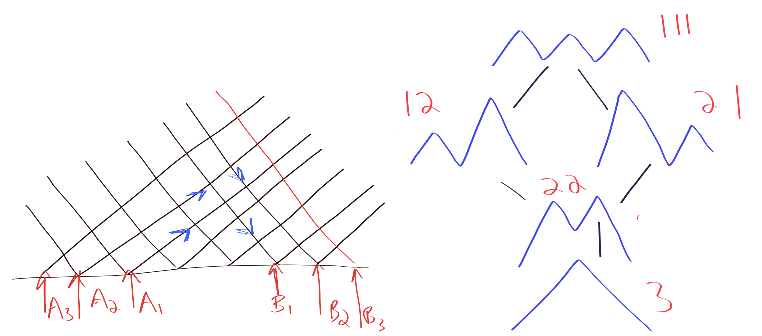


Figure 1: (left: grid), right: Dyck path Forms Poset

Consider the following poset, depicting the inclusions of the 5 Dyck paths of length 2×3:

An n-tuple of non-crossing paths can be described by a string describing which of the 2×3 Dyck paths each of the paths has in its middle. For example, the string $111, 111, 12, 22, 3, 3, 3$ would mean that the connection $A_1 \to B_1$ has form 111, $A_2 \to B_2$ has form 111, $A_3 \to B_3$ has form 12, etc. (the forms are defined in the picture, they indicate the heights of the peaks in the Dyck paths).

Note in particular the n-tuple of middle Dyck path forms is weakly increasing with respect to the poset shown in Figure 1. The number of such n-tuples is $2\binom{n+3}{3} - \binom{n+2}{2}$: there are 2 branches of the poset to go down, but we are overcounting the n-tuples that don't involve either of $12, 21$, which necessitates subtracting $\binom{n+2}{2}$. The binomial coefficient expressions are because we can think of this as deciding when to advance in the poset, so we can think of it as a problem of forming a word with 3 advance symbols and n identical symbols representing the paths.

Thus, by Lindstrom's lemma

$$2\binom{n+3}{3} - \binom{n+2}{2}$$

is the determinant of our matrix.

Acknowledgements. Thanks for such a great class! The topics were really cool, and your proofs were great. I especially appreciated your style of "proof by example", i.e., giving enough proof that it was obvious why statements were true, but not laboriously going through boring details.