

The FUNDAMENTAL THEOREM OF CALCULUS

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Contents

1 The derivative in \mathbb{R}

The derivative of a function $f(a, b) \rightarrow \mathbb{R}^1$ at a point x is defined as

$$f'(x) := \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

This can be interpreted geometrically as the slope of the tangent line to a curve². It can also be interpreted as the rate of change of the function. This is nice because this reflects clearly the "1-dimensional nature" of the operator.

2 The integral in \mathbb{R}

The integral is defined as

$$\int_a^b f(x)dx := \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{b-a}{n} f(x_k),$$

for any x_k satisfying $x_k - a \in \left[\frac{b-a}{n}(k-1), \frac{b-a}{n}k \right]$.

This can be interpreted geometrically as the signed area between a graph and the axis³.

¹You might ask why I chose (a, b) as the domain for f . You might also ask what a, b are. Well, a, b are extended real numbers (real numbers or $\pm\infty$). And I chose this interval because open balls form a basis for the topology on \mathbb{R} that we are using. So yeah.

²But what is the definition of $\lim_{h \rightarrow 0}$ you might ask. Also does this even exist you might ask. A function is called differentiable if the limit exists. The formal definition of the limit is

$$\lim_{h \rightarrow 0} g(h) = L \iff \forall \epsilon > 0 \exists \delta > 0 \text{ such that } |h| < \delta \implies |g(h) - L| < \epsilon.$$

³Note that

$$\lim_{n \rightarrow \infty} f(n) = L \iff \forall \epsilon > 0 \exists M \text{ such that } \forall m > M, |f(m) - L| < \epsilon.$$

3 Fundamental Theorem of Calculus

Theorem 1 (Fundamental Theorem of Calculus). *If $f : [a, b] \rightarrow \mathbb{R}$ is a continuous function with integrable derivative, then*

$$\int_a^b f'(x)dx = f(b) - f(a).$$

To prove this we must first establish the following theorem:

Theorem 2 (Mean Value Theorem). *If $f : [a, b] \rightarrow \mathbb{R}$ is a continuous, differentiable⁴ function, then*

$$\exists c \in (a, b) \text{ such that } f'(c) = \frac{f(b) - f(a)}{b - a}.$$

To prove this we must first establish the following theorem:

Theorem 3 (Rolle's Theorem). *If $f : [a, b] \rightarrow \mathbb{R}$ is a continuous, differentiable function with $f(a) = f(b)$, then*

$$\exists c \in (a, b) \text{ such that } f'(c) = 0.$$

To prove this we must first establish the following lemma and 2 theorems:

Lemma 1 (Derivative at Extrema is 0). *Let $f(x) : [a, b] \rightarrow \mathbb{R}$ be a continuous, differentiable function with a maximum at some $c \in (a, b)$. Then*

$$f'(c) = 0.$$

Lemma 2 (Intermediate Value Theorem). *If $f : [a, b] \rightarrow \mathbb{R}$ is a continuous function such that $f(a) < f(b)$, and if $k \in (f(a), f(b))$, then*

$$\exists c \in (a, b) \text{ such that } f(c) = k.$$

Theorem 4 (Extreme Value Theorem). *If $f : [a, b] \rightarrow \mathbb{R}$ is a continuous function, then*

$$f \text{ achieves extrema on } [a, b].$$

To prove this we must first establish the following theorem:

If you interpret f as a density function, the integral can be interpreted physically as the mass of the line segment from $[a, b]$. This is kind of nice because its very clearly 1-dimensional. Also note that the some restrictions must be imposed in order to ensure that this is well defined, and a convergent sum. To analyze the convergence of the sum it is useful to consider the "upper" and "lower" sums (the sums you get when you take the supremum/infimum of the values of $f(x)$ over all valid choices of x_k to be the value that you use in the sum) because these give an upper and lower bound for the integral. If the upper and lower sums agree, then the function is said to be integrable, and its integral is either of the sums.

⁴Note that this is redundant, differentiability implies continuity

Theorem 5 (Bolzano-Weierstrass Theorem). *If $\{x_i\}_{i=0}^\infty$ is a bounded infinite sequence of values in $[a, b]^5$, then there is a convergent subsequence of the sequence.*

To do this we need the following lemma.

Lemma 3 (A closed interval contains all its limit points). *Let $\{x_i\}_{i=1}^\infty$ be a sequence of values in $[a, b]$ that converges to some value in \mathbb{R} . Then*

$$\lim_{i \rightarrow \infty} x_i \in [a, b].$$

This might be kind of confusing, so see Figure ?? below for the dependency tree.

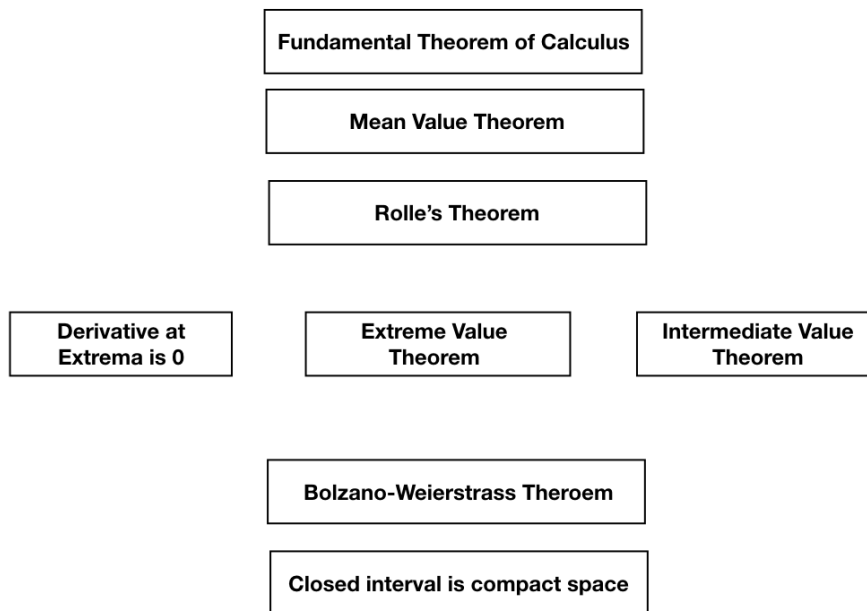


Figure 1: pretty gnarly flowchart

Enough talk. Lets fight!

Po

⁵Really any compact space, I'm just using $[a, b]$ because this is the version that we need for the proof of the version of the extreme value theorem that we need for the proof of rolles theorem that we need for the proof of the mean value theorem that we need for the proof of the FUNDAMENTAL THEOREM OF CALCULUS.

Proof of Lemma ??. Closed Interval Contains Limit Points Let x be the value that the sequence converges to. By the definition of a limit,

$$\forall \epsilon > 0 \exists M > 0 \text{ such that } \forall i > M, |x - x_i| < \epsilon.$$

Assume for contradiction that $x \notin [a, b]$. WLOG let $x > b$. But then if we set $\epsilon = (x - b)/2$,

$$\exists M > 0 \text{ such that } \forall i > M, |x - x_i| < (x - b)/2.$$

Rearranging the inequality,

$$x - \frac{x - b}{2} < x_i < x + \frac{x - b}{2}.$$

But then

$$b < \frac{x + b}{2} < x_i < \frac{3x - b}{2}.$$

This contradicts the fact that all $x_i \in [a, b]$. $\Rightarrow \Leftarrow$

□

Proof of Theorem ??. Bolzano-Weierstrass Theorem First we prove the following claim:

Every infinite sequence has a monotonic subsequence.

We consider 2 cases.

Case 1: There are infinitely many terms in the sequence that are greater than all later terms. That is, there are infinitely many n such that $x_n > x_m$ for all $m > n$. In this case we can take the sequence formed by successively taking these "dominant" terms, and this sequence will clearly be monotonically decreasing.

Case 2: There are finitely many terms in the sequence that are greater than all later terms. That is,

$$\exists N, \text{ such that } \forall n \geq N, \exists m \text{ such that } x_n < x_m.$$

In this case we can create a monotonically increasing sequence as follows: take x_N as the first term in the sequence, and then to get the $(n + 1)$ -th term in the sequence from the n -th, take m such that x_m is greater than the n -th term (this must exist by hypothesis) and use x_m as the $(n + 1)$ -th term in the sequence.

Now it is sufficient to show that a bounded monotonic sequence converges.

WLOG we consider a monotonically increasing sequence (a monotonically decreasing sequence could be negated to achieve a monotonically increasing sequence, so by negating twice this result holds for monotonically decreasing sequences too). Then consider

$$s = \sup \{x_i\}_{i=0}^{\infty}.$$

Clearly this exists because the sequence is bounded. Note that by definition of the supremum⁶,

$$\forall \epsilon > 0, \exists n \text{ such that } s - x_n < \epsilon.$$

(Because if this were not so then we could find a smaller lower bound than the supremum which is a contradiction). But in the case of a monotonically increasing sequence this actually implies:

$$s - x_n < \epsilon \implies s - x_m < \epsilon, \forall m \geq n.$$

So, we have that

$$\forall \epsilon > 0, \exists n \text{ such that } \forall m \geq n, |s - x_n| < \epsilon.$$

That is, the sequence converges. Thus, we have the result that all bounded sequences have convergent subsequences. \square

Proof of Theorem ??. Extreme Value Theorem We need only establish that all such functions achieve a maxima, and then the dual property of a minima being achieved is immediate by considering the negated function. We first establish that a continuous function on a closed interval is bounded⁷.

Assume the contrary. Then there must be a sequence of values in $[a, b]$, say $\{x_n\}_{n=1}^{\infty}$ such that $f(x_n) > n, \forall n$. Because we have a bounded sequence, we the Bolzano-Weierstrass Theorem tells us that there must be some convergent subsequence, say $\{x_{n_k}\}_{k=1}^{\infty}$ that converges to some value. By Lemma ??, we know that the point that the sequence converges to must lie in $[a, b]$. Call this point α . Now, note that because f is a continuous function on $[a, b]$, in particular f is continuous at α , that is:

$$\forall \epsilon > 0, \exists \delta \text{ such that } |x - \alpha| < \delta \implies |f(x) - f(\alpha)| < \epsilon.$$

Take some $\epsilon > 0$. For all $\delta > 0, \exists N$ such that $\forall n_k > N, |x_{n_k} - \alpha| < \delta$ because the sequence converges to α . But, $f(x_{n_k}) > n_k$, so $\exists n_k > N$ such that $f(x_{n_k}) - f(\alpha) \geq \epsilon$. This contradicts the continuity of f at α . Thus we have established that a continuous function on a closed interval must be bounded⁸.

Let $M = \sup \{f(x) : x \in [a, b]\}$. We now aim to show that M , which exists because the function is bounded, is actually achieved. We know that by the definition of the supremum,

$$\forall \epsilon > 0, \exists x \in [a, b] \text{ such that } M - f(x) < \epsilon.$$

⁶It occurs to me that I probably have not defined the supremum. Here is the definition in words: the supremum of a set is the least upper bound for the set.

⁷You might be wondering "did you ever define continuity?". I don't know. So here is a definition: A function f is continuous at a point x if and only if $\forall \epsilon > 0, \exists \delta$ such that $\forall y$ such that $|x - y| < \delta, |f(x) - f(y)| < \epsilon$.

⁸Note that if the interval were open but finite, the point of convergence generated by Bolzano-Weierstrass may have not been in the domain of the function, which would wreck the proof, and if the interval were unbounded then Bolzano-Weierstrass wouldn't have applied, so we really do need these conditions (there are lots of easy counterexamples to show this as well, think about what happens if you try to find the minima of $1/x$ on \mathbb{R}^+ or try to find the maxima of $1/x$ on $(0, 1]$).

Form a sequence of x_n satisfying $M - f(x_n) < \frac{1}{n}$. Take a convergent subsequence x_{n_k} of the sequence. Let the value that the subsequence converges to be α . Note that $\alpha \in [a, b]$. Then, because f is continuous at α , if we take some $\epsilon > 0$,

$$\exists \delta > 0 \text{ such that } |x - \alpha| < \delta \implies |f(x) - f(\alpha)| < \epsilon.$$

By the convergence of the subsequence, $\exists N$ such that $\forall n_k > N, |x_{n_k} - \alpha| < \delta$. Thus $|f(x_{n_k}) - f(\alpha)| < \epsilon, \forall n_k > N$. Recall that $f(x_{n_k}) > M - 1/n_k$. Thus, $f(\alpha) > M - \epsilon - 1/n_k$. But the only value of $f(\alpha)$ that satisfies this for all values of ϵ and n_k is M , so $f(\alpha) = M$. □

Proof of ??. (Derivative at Extrema is 0) WLOG say that the extrema is a maxima⁹.

We proceed by contradiction, i.e. by assuming that $f'(c) \neq 0$. There are 2 cases.

Case: $f'(c) > 0$ By an interpretation of the definition of the derivative's defining limit,¹⁰

$$\forall \epsilon > 0, \exists N \text{ such that } \forall n > N, \left| \frac{f(c + \frac{1}{n}) - f(c)}{\frac{1}{n}} - f'(c) \right| < \epsilon.$$

We can set $\epsilon = \frac{1}{2}f'(c) > 0$. This implies:

$$0 < \frac{1}{2}f'(c) \frac{f(c + \frac{1}{n}) - f(c)}{\frac{1}{n}} < \frac{3}{2}f'(c).$$

But then $f(c + \frac{1}{n}) > f(c)$, which is not possible because $f(c)$ is a maxima.

Case: $f'(c) < 0$ Now we will use an alternate definition of the derivative:¹¹

$$\forall \epsilon > 0, \exists N \text{ such that } \forall n > N, \left| \frac{f(c - \frac{1}{n}) - f(c)}{-\frac{1}{n}} - f'(c) \right| < \epsilon.$$

We set $\epsilon = -\frac{1}{2}f'(c) > 0$. This implies:

$$\frac{3}{2}f'(c) \frac{f(c - \frac{1}{n}) - f(c)}{-\frac{1}{n}} < \frac{1}{2}f'(c) < 0.$$

⁹Note: I love using the duality of min/max or inf/sup under negation (i.e. reversing set orderings)!

¹⁰Note that by assumption the limit (i.e. derivative exists). Also note that we can have $h \rightarrow 0$ using any sequence that converges to 0. I chose the sequence $(1/n)_{n=1}^{\infty}$ here, but could have just as well have chosen $(1/n^2)_{n=1}^{\infty}$. Because the limit exists these all give the same value for the derivative.

¹¹Note that the sequences I chose in this and the previous paragraph effectively implement approaching from below and approaching from above according to a nice regular path.

But then $f(c - \frac{1}{n}) > f(c)$, which is not possible because $f(c)$ is a maxima.

We have shown that assuming either $f'(c) < 0$ or assuming $f'(c) > 0$ leads to a contradiction. Thus $0 \leq f'(c) \leq 0$, i.e. $f'(c) = 0$. \square

Proof of Theorem ??. Intermediate Value Theorem Consider $\{x \in [a, b] : f(x) < k\}$. Let $s = \sup\{x \in [a, b] : f(x) < k\}$. I claim that $f(s) = k$. First of all note that $f(s) \geq k$. This is easily shown by contradiction: if $f(s) < k$ then by the continuity of f at s , if we consider the open ball of radius $\frac{k-f(s)}{2}$ around $f(s)$, then the preimage of this ball consists of points x that all satisfy $f(x) < k$, but some of these points have $x > s$ (note that it is possible that some points in this ball are not in $[a, b]$, but $s \neq b$, so there is at least one point $x \in [a, b]$ in this ball with $x > s$) which contradicts the fact that s is the supremum. Now, by the definition of the supremum,

$$\forall x \in (a, s), \exists y \in [a, b] \text{ such that } f(y) < k, y > x.$$

Then, consider the sequence x_n defined as follows: to get x_n let $x = \max(a, s - \frac{1}{n})$ ¹², and then take $x_n \in [a, b]$ such that $x_n > x, f(x_n) < k$ which is guaranteed to exist. This sequence x_n satisfies $f(x_n) < k$ for all k , and also, as shown before, a continuous function on a closed interval must be bounded, so $f(x_n)$ is a sequence with values in $[-M, k)$ for some M , so $f(x_n)$, which must converge because x_n is continuous, converges to a value in $[-M, k]$ (the closure of the set that its values are in). Because the sequence x_n converges to s , we have that $f(s) \leq k$. Now,

$$k \leq f(s) \leq k \implies f(s) = k.$$

\square

Proof of Theorem ??. Rolle's Theorem Firstly, note that if f is a constant function, then the result is trivially true. Now, consider a non-constant function f . By the Extreme Value Theorem it achieves a maxima somewhere, and by the fact that extrema implies derivative is 0, we know that there is some point where the derivative is 0. \square

Proof of Theorem ??. Mean Value Theorem Consider the function

$$L(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a).$$

Note that $L(a) = f(a), L(b) = f(b)$. Now consider the function $g(x) = f(x) - L(x)$. Note that $g(a) = g(b) = 0$. All the hypothesis for Rolle's theorem apply. Thus $\exists c \in [a, b]$ such that $g'(c) = 0$. But $g'(x)$ is simply, by linearity of the derivative,

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

¹²Note the slightly annoying fact that $s - \frac{1}{n}$ might not be in $[a, b]$, but this is easily fixable as $\max(a, s - \frac{1}{n})$ is definitely in the interval. Also, after a while the max condition is not necessary, because $s - \frac{1}{n} > a$ for some n

Thus

$$g'(c) = 0 = f'(c) - \frac{f(b) - f(a)}{b - a} \implies f'(c) = \frac{f(b) - f(a)}{b - a}.$$

□

Proof of Theorem ??. THE FUNDAMENTAL THEOREM OF CALCULUS Consider the expression of $f(b) - f(a)$ as a telescoping sum¹³:

$$f(b) - f(a) = \sum_{i=1}^n f\left(a + \frac{i \cdot (b - a)}{n}\right) - f\left(a + \frac{(i - 1) \cdot (b - a)}{n}\right).$$

Taking the limit as $n \rightarrow \infty$ of both sides cannot change the equality because neither side ultimately depends on n .¹⁴

$$\lim_{n \rightarrow \infty} f(b) - f(a) = \lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(a + \frac{i \cdot (b - a)}{n}\right) - f\left(a + \frac{(i - 1) \cdot (b - a)}{n}\right).$$

Now we apply the Mean Value Theorem to each interval $\left(a + \frac{i \cdot (b - a)}{n}, a + \frac{(i - 1) \cdot (b - a)}{n}\right)$, which yields

$$f(b) - f(a) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{b - a}{n} f'(x_k)$$

for some x_k in the appropriate interval. This is the definition of the integral of f' , i.e.

$$f(b) - f(a) = \int_a^b f'(x) dx.$$

Q.E.D.

□

4 Vector fields

A vector field¹⁵ is a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. This can be interpreted geometrically as assigning to every point in the plane a vector.

¹³One might well ask, "what is the motivation behind doing this?" The motivation is the awesomeness of the mean value theorem.

¹⁴Another way (possibly more rigorous) of seeing why taking limits is valid is to note that for any n , the sum is $f(b) - f(a)$, so really $\forall \epsilon > 0, \exists N = 1$, such that $\forall n > N$,

$$\left| \sum_{i=1}^n f\left(a + \frac{i \cdot (b - a)}{n}\right) - f\left(a + \frac{(i - 1) \cdot (b - a)}{n}\right) - (f(b) - f(a)) \right| = 0 < \epsilon.$$

¹⁵for our purposes here today

5 Parametric Functions

A parametric function¹⁶ is a function $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$. This can be interpreted geometrically as a correspondence between time and position, i.e. at any time t , $\gamma(t)$ specifies the position of the object.

6 Integrals over regions in \mathbb{R}^2

The integral of some function $f(x, y)$ over a region $D \subset \mathbb{R}^2$ is defined to be

$$\int_D f(x, y) dx dy := \sum_{\text{rect}_i \in \text{all rects}} f(x_i, y_i) \cdot \text{area of rect}_i.$$

This can be interpreted geometrically as the signed volume between a surface in \mathbb{R}^3 and the xy -plane. It can be interpreted physically as the mass of something when f is a density. Note that this can be separated into 2 single integrals in 2 ways, that turn out to be equivalent by Fubini's Theorem.

7 Line integrals in \mathbb{R}^2

Work is a physicsy thing. Line integrals look kinda like

$$\int_C \vec{F}(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t) dt$$

They represent work. Note that there is no "new definition" here, this is just a single variable integral (albiet a potentially complicated one). What's that derrivative thing? Well note that $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ such that

$$t \mapsto \begin{bmatrix} \gamma_x(t) \\ \gamma_y(t) \end{bmatrix}.$$

Then,

$$\gamma'(t) := \begin{bmatrix} \gamma'_x(t) \\ \gamma'_y(t) \end{bmatrix}.$$

8 Green's theorem

Theorem 6 (Green's theorem). *Let $S \subset \mathbb{R}^2$, and let $f, g : S \rightarrow \mathbb{R}$*

$$\int_{\partial S} f dx + g dy = \int_S \frac{dg}{dx} - \frac{df}{dy} dx dy.$$

¹⁶for our purposes here today

Proof. First we will show that it is true for a square. Then we will piece together squares, like in the motivation behind the definition of the integral, and we will discover that stuff on the boundary cancels, which yields the formula.

For the square: explicitly compute the line integral, parameterize each side separately. You'll need the mean value theorem to do this. You'll get $h^2(\frac{dg}{dx} - \frac{df}{dy})$ where those derivatives are evaluated somewhere on the square. But the square is small, so how wrong could it possibly be?

When you have adjacent squares the integrals add, and the shared side cancels.

□

Corollary 1 (Area from line integral). *Let $S \subset \mathbb{R}^2$, then*

$$[S] = \int_S dx dy = \frac{1}{2} \int_{\partial S} -y dx + x dy.$$

Proof. This is trivial from Theorem ??: simply verify that the exterior derivative of $x dy - y dx$ is $dx \wedge dy$. □

This is kinda awesome (imo) because it means that knowing the function on the boundary (and that the function is well-behaved, i.e. continuous, differentiable, that stuff) tells you about the area.