# Complexity of Art Gallery Variants

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Computational Geometry

# The Art Gallery Problem

**Input**: Polygon $P$, number of guards $g$.
**Output**: Is there a placement of $g$ guards that can see all of $P$?

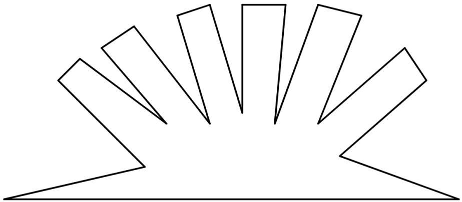

Figure: An Art Gallery

Polygon representation: $P$ is represented as a list of $n$ vertices which are pairs of $B$ bit binary numbers in $\left\{ i \cdot 2^{-B} \mid i \in [2^B] \right\}$.

# Can't Always Place Guards at Vertices

**Theorem (Chvatal)**

$\lfloor n/3 \rfloor$ *vertex guards always suffice.*

We can efficiently compute a set of $\lfloor n/3 \rfloor$ guards that suffice by triangulating $P$ and 3-coloring the resulting graph.

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But, if we want to *minimize* the number of guards, it's useful to place guards off of vertices.

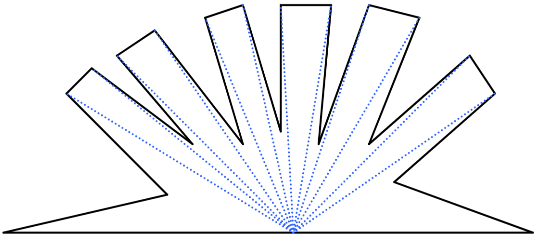

Figure: One guard suffices, but need many vertex guards.

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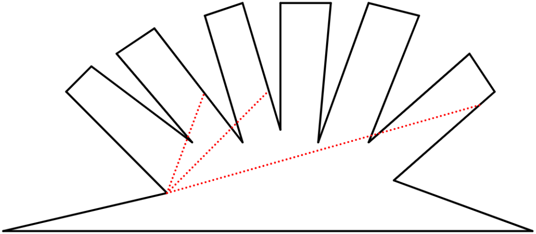

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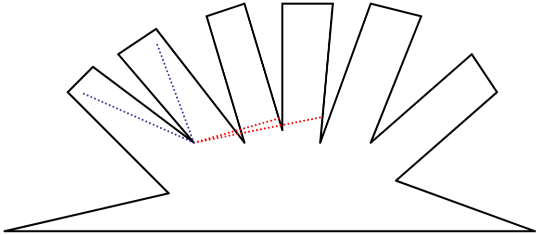

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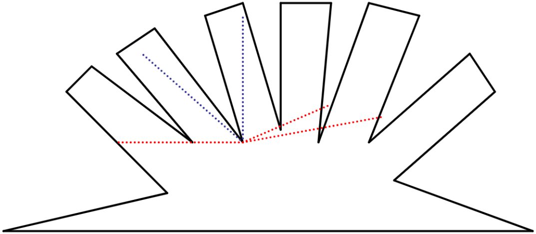

Figure: One guard suffices, but need many vertex guards.

# Complexity of Art Gallery

**Question**: Is Art Gallery in NP?
**Answer**: Probably not.

### Definition ($\exists\mathbb{R}$ formula)

$\exists\mathbb{R}$ is the set of all true formulas

$$\exists X_1, X_2, \ldots, X_n \in \mathbb{R} \mid \Phi(X_1, \ldots, X_n)$$

where $\Phi$ is a well-formed sentence involving variables $X_1, \ldots, X_n$ and symbols $\vee, \wedge, \neg, 0, 1, +, -, \cdot, (, ), =, <, \leq$.

**Example**:

$$\exists(x, y) \mid x > 0 \wedge y > 0 \wedge x + y < 1.$$

# Complexity of Art Gallery

**Question**: Is Art Gallery in NP?
**Answer**: Probably not.

**Theorem (Abrahamsen, Adamaszek, Miltzow STOC'18)**

*Art Gallery is $\exists\mathbb{R}$-Complete.*

**Conjecture**

NP $\subsetneq \exists\mathbb{R}$.

Why is $\exists\mathbb{R}$ potentially larger than NP?
Problems in NP have short certificates, but the variables we quantify over can be real, seems hard to write down $X_1, \ldots, X_n$ in a way that makes checking $\phi(X_1, \ldots, X_n)$ easy.

# Art Gallery Variants

- **Line Guard**: is it possible to guard an art gallery with $g$ line segments?
  Requirement: every point in the art gallery can be seen by some point on some line.
- **Shape Guard**: is it possible to guard an art gallery with $g$ (fixed) shapes (e.g., radius $\varepsilon$ circles)?
  Requirement: every point in the art gallery can be seen by some point in some shape.
- **3D Line Guard**: Line guard, but in 3D.
- **Promise Point Shape Guard**: Distinguish between two cases:
  1. $P$ can be guarded by $g$ point guards.
  2. $P$ can't be guarded by $g$ $\varepsilon$-radius circle guards.

# Results

**Theorem**

*Line Guard and Shape Guard are in $\exists\mathbb{R}$.*

**Theorem**

*Promise Point Shape Guard is in NP.*

**Conjecture (In Progress)**

3D Line Guard is $\exists\mathbb{R}$-hard.

# LineGuard $\in \exists\mathbb{R}$ Proof Sketch

**Try 1**:

$\exists k$ line segments such that $\forall (x, y) \in P$,
$\exists (x', y')$ a point on one of our line segments such that
$(x', y')$ can see $(x, y)$.

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Kind of...

# LineGuard $\in \exists\mathbb{R}$ Proof Sketch

**Definition**

$$X = \{\text{corners of polygon}\} \cup \{\text{guard segment endpoints}\}$$

$$\mathcal{L} = \{\text{Lines joining points in } X\}$$

$$\mathcal{R} = \{\text{regions in the arrangement defined by } \mathcal{L}\}$$

$$C = \{\text{Centroids of regions in } R\}$$

Claim: If the guard segments can see all points in $C$ then they can see the entire polygon $P$.

Note: $C$ depends on the guard locations. This is to be expected, because we do not believe that LineGuard $\in$ NP.

# LineGuard $\in \exists\mathbb{R}$ Proof Sketch

**Lemma**

*If guard segment $\ell$ can see any point inside region $R \in \mathcal{R}$ then $\ell$ can see all of $R$.*

**Corollary**

*If the guard segments can all points in $C$ then they can see all points in $P$.*

**Corollary**

LineGuard $\in \exists\mathbb{R}$.

# LineGuard $\in \exists\mathbb{R}$ Proof Sketch

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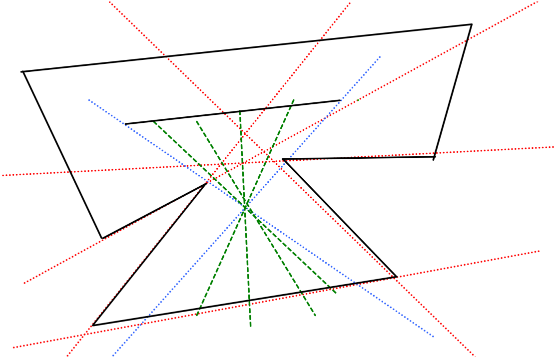

Figure: Proof of Key Lemma: any segments in the middle of the cone would further subdivide it.