1 Graph ramsey numbers

Let H, H' be graphs on k vertices with maximum degree d. Indentify the vertex sets of H, H' with $[k]$. Form H'' with vertex set $[k]$ and edge set $E(H) \cup E(H')$. We see that H'' is a graph on k vertices with maximum degree $2d$ which has both H and H' as subgraphs.

By the theorem we proved in class, there exists a constant C_d, depending only on d such that

$$R(H'', H'') \leq C_d \cdot k.$$

In other words, in any red-blue coloring of the complete graph on $C_d \cdot k$ vertices, we can find a red copy of H'' as a subgraph, which would contain a red copy of H as a subgraph, or we can find a blue copy of H'' as a subgraph, which contains a blue copy of H' as a subgraph. Thus $R(H, H') \leq C_d \cdot k$ as well.

2 multicolor ramsey numbers

(a)

Proposition 1. There exists a 3-coloring of K_{16} that has no monochromatic triangle.

Proof. We identify the vertices of K_{16} with the finite field $\mathbb{F}_{16}$, and let $\alpha \in \mathbb{F}_{16}$ be a generator for $\mathbb{F}_{16}$. Our colors will be residues modulo 3, i.e. $0, 1, 2$. We remark that $\mathbb{F}_{16}$ has characteristic 2, i.e. $x = -x$ in $\mathbb{F}_{16}$. For the edge $\{x, y\}$, there exists i such that $x + y = \alpha^i$; we let the color of edge $\{x, y\}$ be i mod 3.
We claim that the graph thus colored cannot have a monochromatic triangle. Assume for contradiction that x, y, z form a monochromatic triangle in color i. That is, $x+y = \alpha^{i+3j}, y+z = \alpha^{i+3k}, x+z = \alpha^{i+3\ell}$. Adding the equations yields $x + y + y + z = x + z$, so we have

$$\alpha^{i+3j} + \alpha^{i+3k} = \alpha^{i+3\ell}.$$

This is a field, so elements have inverses. Thus, defining $k' = k - j, \ell' = \ell - j$, this is equivalent to:

$$1 + \alpha^{3k'} = \alpha^{3\ell'}.$$

Raising this expression to the 5-th power yields (by the binomial formula):

$$\alpha^{3k'} + \alpha^{12k'} = 1.$$

Thus our original expression becomes

$$1 + \alpha^{3k'} = \alpha^{12k'}.$$

Multiplying by $\alpha^{3k'}$ yields

$$\alpha^{3k'} + \alpha^{6k'} = 1.$$

However, combining these two equations we arrive at

$$\alpha^{6k'} = \alpha^{12k'}.$$

But this implies that

$$k' \equiv 0 \mod 5.$$

But then our original expression becomes

$$1 + 1 = 0 = \alpha^{3\ell'},$$

which cannot be. Hence, our assumption that a monochromatic triangle exists must be false. □

Proposition 2. Any 3-coloring of K_{17} contains a monochromatic triangle.

Proof. Fix a red-blue-green-coloring of K_{17}. Choose a vertex v arbitrarily. By the pidgeon-hole principle v must have at least 6 neighbors of some color, call this color green.
If any of these 6 neighbors have a green edge between them, we have found a green triange. Otherwise, the edges between these 6 neighbors are all not green. But then we have a red-blue coloring of 6 vertices. Recalling that $R(3, 3) = 6$, we know that we can find a monochromatic triangle amongst these vertices.
Either way, we find a monochromatic triangle. □

(b) Now we generalize the constructions from part (a)

Theorem 1. There is a t-coloring of K_{2^t} which contains no monochromatic triangle.

Proof. Let $f(t)$ denote the smallest number n such that any t-coloring of K_n must contain a monochromatic triangle.
We recall the useful product-construction from class:

Lemma 1.
$$f(t_1 + t_2) \geq (f(t_1) - 1)(f(t_2) - 1) + 1.$$

Proof. We briefly review the blowup construction. Take $f(t_1) - 1$ copies of a t_2-coloring of the complete graph on $f(t_2) - 1$ vertices without any monochromatic triangles; such a graph exists by the definition of f. Then, color the edges between copies of the graph using t_1 different colors, in particular such that there is no monochromatic triangle; this is possible by the definition of $f(t_1)$. Note in particular that all edges between any two copies of the graph have the same color. Now, we clearly cannot find a monochromatic triangle within any of the little graphs, and a monochromatic triangle taking vertices from multiple different little graphs is also impossible. Hence there are no monochromatic triangles, so $f(t_1 + t_2)$ is strictly larger than the size of the graph we have constructed, which is $(f(t_1) - 1)(f(t_2) - 1)$. □

Now, we apply the lemma to prove the following claim inductively:

Claim 1. For all $t \geq 1, f(3t) \geq 2^{4t} + 1$.

Proof. In part (a) we have already established $f(3 \cdot 1) = 17 = 2^{4 \cdot 1} + 1$. Assuming that we have already proven the claim for values up to t, we compute:

$$f(3(t+1)) \geq (f(3t) - 1)(f(3) - 1) + 1 \geq 2^{4t} \cdot 2^4 + 1 \geq 2^{4(t+1)} + 1,$$

as desired. □

Additionally, it is obvious that f is monotonically increasing: if we have more colors, then the graph size must be larger to permit always finding a monochromatic triangle.
Thus,
$$f(3t+1) \geq f(3t) \geq 2^{4t} + 1 \geq 2^{3t+1}.$$

For $3t + 2$ we apply a slightly more careful analysis, noting that $f(2) = 6$:

$$f(3t+2) \geq 2^{4t} \cdot (6-1) + 1 \geq 2^{3t+2}.$$

Summarizing our results we have
$$f(t) \geq 2^t,$$

regardless of t's residue class modulo 3.

□

Theorem 2. Any t-coloring of $K_{3 \cdot t!}$ contains a monochromatic triangle.

Proof. We proceed by induction. The base case is for $t = 2$, and this is true because $R(3, 3) = 6$. Assume we have shown the statement for $t - 1$ colors.
Consider a graph on $3 \cdot t!$ vertices. Choose a vertex v arbitrarily in the graph. v has at least $3 \cdot (t-1)!$ neighbors in some color, call the color red. Then, if any of the edges between v's red neighbors are red, we have a red triange. Elsewise, v's neighbors are colored with $t - 1$ colors, and because there are $3 \cdot (t-1)!$ such vertices, we can find a monochromatic triangle amongst those vertices.
Either way, we have a monochromatic triangle. □

3 tree: is a subgraph

First, we prove the following simple lemma:

Lemma 1. Let H be a graph on n vertices with average degree $\bar{d}$, and let $v \in V(H)$ have degree $d_* \leq \bar{d}/2$. Then $H' = H[V \setminus \{v\}]$, i.e. the graph resulting from removing v from H, has average degree at least $\bar{d}$.

Proof. The new average degree will be

$$\frac{n\bar{d} - 2d_*}{n-1} \geq \frac{n\bar{d} - \bar{d}}{n-1} \geq \bar{d},$$

where the $2d_*$ term comes from the fact that removing v not only eliminates v's contribution to the sum of degrees, but also decreases the degree of v's neighbors by 1 each. □

(a)

Theorem 1. Let T be any tree on $k \geq 2$ vertices, and G any graph with average degree strictly larger than $2k-4$. Then G contains T as a subgraph.

Proof. Note that having average degree larger than $2k-4$ implicitly also means that there are more than $2k-3$ vertices.
We repeatedly kick out vertices with degree at most half the average degree. By Lemma 1 this never decreases our graph's average degree. Let G' be the resulting graph (which is the induced subgraph on the vertices remaining after the very low degree vertices have been kicked out). Then, each vertex $v \in V(G')$ has degree larger than $k-2$ (i.e. larger than half the average degree), which in particular means that v's degree (in G'), being an integer, is at least $k-1$.
Now, it is very easy to embed T into G'. Label the vertices of T as $w_1, w_2, \ldots, w_k$, such that for each $i > 1$, w_i is connected in T to some other vertex w_j with $j < i$; this can be accomplished with, e.g. a "breadth first search" labeling. Note that for each w_i there will be a unique w_j with $j < i$ such that $w_i, w_j \in E(T)$; otherwise we would have a cycle which is impossible because T is a tree. Let $f(w_i)$ for $i > 1$ give the unique w_j with $j < i$ which is connected to w_i.
Choose $v \in G'$ arbitrarily to serve as the image of vertex w_1. For $i > 1$, we embed vertex w_i by looking at the neighborhood of $f(w_i)$, which has size at least $k-1$, and then excluding the $i-1-1$ already embedded points from this neighborhood, except we need not exclude $f(w_i)$ because it is not in its own neighborhood. Thus, there are $k-i+1$ candidates for embedding vertex w_i. And so, for any $i \leq k$ we have a non-zero number of valid candidates. We choose a candidate arbitrarily. At the end of this process we have succesfully embedded the tree T into G'. G', is a (induced) subgraph of G, so this also gives an embedding of T into G. □

(b)

Proposition 3. Any graph G with average degree $\bar{d} > k-2$ contains a star on k vertices as a subgraph.

Proof. If all vertices had degree at most $k-2$, then the average degree could not be strictly larger than $k-2$. Thus at least some vertex has degree larger than $k-2$. Degree, being an integer (unlike average degree), if larger than $k-2$ is thus $k-1$. So we have a vertex v_* with degree at least $k-1$. Taking vertex v_* along with its neighbors gives a star on k vertices. □

Proposition 4. Any graph G with average degree $\bar{d} > k-2$ contains a path on k vertices as a subgraph.

Proof. We prove the claim by induction on k. For $k=2$, this theorem asserts that a graph with positive average degree has an edge; this is clear. Now, we assume we have already shown the claim to be true for $k-1$; we establish the claim for k.
We start with the following ***preprocessing*** steps:

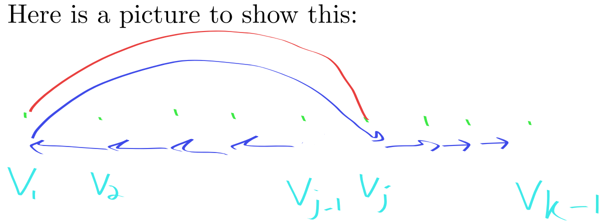
- G must have a connected component with average degree strictly larger than $k-2$ or else its overall average degree could not possibly be larger than $k-2$. We restrict to such a connected component.

- By Lemma 1 we can kick out vertices with degrees at most half the average degree to get a graph with minimum degree larger than $k/2-1$, without decreasing our average degree.

Our graph has sufficiently large average degree for us to find a path with $k-1$ vertices in it by the inductive hypothesis; fix such a chain formed from vertices $v_1 \to v_2 \to \cdots \to v_{k-1}$. We say that a vertex v_i is an ***endpoint*** if there is a path starting from v_i, traveling through all vertices $v_1, v_2, \dots, v_{k-1}$ exactly once. Apriori only v_1, v_{k-1} must be endpoints. But if, for instance, v_1 has an edge to v_j for some $j > 1$, then v_{j-1} also becomes an endpoint. In particular, we can take the path

$$v_{j-1} \to v_{j-2} \to \cdots \to v_1 \to v_j \to v_{j+1} \to v_{j+2} \cdots \to v_{k-1}.$$

Here is a picture to show this:



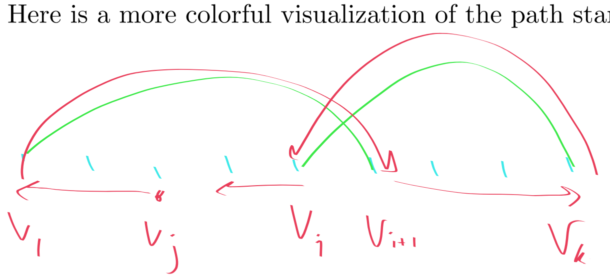
If any endpoint has a neighbor outside of the path, then we can find a path of length k by adding that neighbor to the start of the path. Now, assume for sake of contradiction that no endpoints have a neighbor outside of the path.

Lemma 2. Fix i with $1 < i < k-1$. If $\{v_1, v_{i+1}\}$ and $\{v_i, v_{k-1}\}$ are both edges in the graph, then all vertices in the path are endpoints.

Proof. Take a vertex v_j, without loss of generality let $j \le i$. Then we can do the following path:

$$v_j \to v_{j-1} \to \cdots \to v_2 \to v_1 \to v_{i+1} \to v_{i+2} \to \cdots v_k \to v_i \to v_{i-1} \to v_{j+1}.$$

Here is a more colorful visualization of the path starting from v_j:



□

Say that a vertex v_i is ***posioned*** if the edge $\{v_1, v_{i+1}\}$ exists. Because the minimum degree of a vertex in our graph is at least $\lfloor k/2 \rfloor$, v_1 poisons $\lfloor k/2 \rfloor$ elements. There are $k-1-1-\lfloor k/2 \rfloor$ options for the neighbors of v_{k-1} on the path, where the minus one is due to the fact that v_{k-1} cannot be neighbors with itself. However,

$$\lfloor k/2 \rfloor > k-2-\lfloor k/2 \rfloor,$$

so by the pigeon-hole principle v_{k-1} must have an edge to some poisoned v_i. In other words, we have some i such that both $\{v_1, v_{i+1}\}$ and $\{v_i, v_{k-1}\}$ are present as edges. So, applying **??** we find that all vertices are endpoints. By our assumption none of these endpoints have neighbors to vertices outside of the path. But this contradicts the fact that the graph is connected (by our preprocessing).

We note that using proof by contradiction is merely an aesthetic choice. It is potentially clearer to just say: if either of v_1, v_{k-1} has a neighbor off the path we find a path of k vertices, otherwise, all vertices are endpoints, and one must be connected to something off the path because the graph is connected, so we can find a path of k vertices. Either way, we have finished the proof.

□

4 All the colors

Definition 1. Consider an edge-coloring of G, the complete graph on $S \subset \mathbb{N}$ vertices. We say:
- G is ***monochromatic*** if all edges in G have the same color.
- G is ***locally-rainbow*** if any two edges which are incident to a common vertex have different colors.
- G is ***globally-rainbow*** if all edges have distinct colors.
- G is ***up-ordered*** if for all $i \in S$, there is a color $c(i)$ such that for all $j > i$, edge ij has color $c(i)$.
- G is ***down-ordered*** if for all $i \in S$, there is a color $c(i)$ such that for all $j < i$, edge ij has color $c(i)$.

Theorem 3. Let $k \geq 4$, $s = 8k^3$, and N be a power of 2 larger than s^{s^s}. Consider any edge coloring of the complete graph K on vertices $[N]$. Then, K must have an induced subgraph of size k which is either monochromatic, globally-rainbow, up-ordered, or down-ordered.

Proof. To start, we prove a key lemma, allowing us to convert from locally-rainbow to globally-rainbow. We remark that this lemma is the same as a problem on an earlier pset.

Lemma 3. Let G be a locally-rainbow clique on $4t^3$ vertices. Then, G has a globally-rainbow clique of size t as a subgraph.

Proof. We find the globally-rainbow subgraph greedily. To start, we choose v_1 as an arbitrary vertex of G. Now we itteratively find an appropriate v_{i+1} given the set $\{v_1, v_2, \dots, v_i\}$. In particular, we apply the following claim $t-1$ times:

Claim 2. Let $i \in [t]$, and let $V_i = \{v_1, v_2, \dots, v_i\} \subset V(G)$ be i distinct vertices such that $G[V_i]$ is a globally-rainbow i-clique. Then, there exists $v_{i+1} \in V(G) \setminus V_i$ such that $G[V_i \sqcup \{v_{i+1}\}]$ is a globally-rainbow $(i+1)$-clique.

Proof. Let $C(V_i)$ denote the set of colors represented in the edges of $G[V_i]$. Note that $|C(V_i)| \leq \binom{i}{2}$. For each $v_j \in V_i$, let $B(v_j)$ denote the set of neighbors w of v_j via an edge of color $c \in C(V_i)$, i.e. w such that $\{w, v_j\}$ has color c. Let

$$B(V_i) = \bigcup_{v_j \in V_i} B(V_i),$$

be the set of all "bad" vertices, i.e. vertices which if added to V_i would not result in a globally-rainbow clique. Fortunately, $|B(V_i)| \leq i \cdot \binom{i}{2} \leq i^3/2$. Thus,

$$|V(G) \setminus (B(V_i) \cup V_i)| \geq 4t^3 - t^3/2 - t \geq 1,$$

i.e. $V(G) \setminus (B(V_i) \cup V_i)$ is non-empty; we choose v_{i+1} arbitrarily from this non-empty set. Clearly, because we have removed all the bad vertices, v_{i+1}'s edges to each $v_j \in V_i$ are of colors which do not appear in the edges of $G[V_i]$, and because G is locally-rainbow, the edges adjacent to v_{i+1} also do not contain any duplicate colors. Thus $G[V_i \sqcup \{v_{i+1}\}]$ is globally-rainbow, as desired. $\square$

Using the claim t times gives t vertices forming a globally-rainbow clique, proving the lemma. $\square$

Lemma 2. Let $S \subset \mathbb{N}$ with $|S| = n$, and consider a coloring of the complete graph on vertices S. Fix $i \in S$. Then, at least one of the following holds:
1. There are $\sqrt{n}$ edges[1] of distinct colors incident to vertex i.
2. There is a color c and $\sqrt{n}/2$ vertices $j < i$ such that edge $\{j, i\}$ has color c.
3. There is a color c and $\sqrt{n}/2$ vertices $j > i$ such that edge $\{j, i\}$ has color c.

Proof. Let $i \leq n/2$; a symmetric argument will work for $i \geq n/2$. There are at least $n/2$ vertices $j > i$. If no color repeats at least $\sqrt{n}/2$ times amongst edges incident to i, then there are at least $(n/2)/(\sqrt{n}/2) = \sqrt{n}$ colors represented in the edges incident to i. Thus, we either satisfy condition 1 or condition 3. $\square$

[1]Technically, we should have round this number, however N was chosen to be an extremely large power of 2; thus, I believe that for the numbers we encounter when applying the lemma they will be integers. Either way, this is a unimportant detail.

Now, we repeatedly pass to a subgraph via any of the conditions of Lemma 2. In particular, we will form a sequence $v_1, v_2, v_3, v_4, \ldots,$, along with a sequence of refinements of our set $[N] = S_0 \supset S_1 \supset S_2 \supset S_3 \cdots$, along with a sequence of ***passes***. There are 3 types of "passes", which correspond to the 3 conditions which may be satisfied in Lemma 2. Our sequence will terminate once any of the following conditions holds:

- We perform $4k^3$ passes via condition (1).
- We perform k passes via condition (2) with a single color.
- We perform k passes via condition (2) in k distinct colors.
- We perform k passes via condition (3) with a single color.
- We perform k passes via condition (3) in k distinct colors.

When we perform a "pass" we will guarantee that $|S_i| \geq \sqrt{|S_{i-1}|}/2$. From our description of our termination conditions, we see that we will perform less than $4k^3 + 2k^2$ passes total overall; by the pidgeon-hole principle if we perform more than this many passes we certainly have satisfied one of the termination conditions. We observe that, by our choice of $N = s^{s^s}$ where $s = 8k^3 \geq 4k^3 + 2k^2$, performing $4k^3 + 2k^2$ passes starting from a set of size N will not produce the empty set, so all of our passes will be valid. Furthermore, after $i \leq 4k^3 + 2k^2$ passes, we see that the size of the remaining set is at least $2^{2^{k^3}} > 8k^6$ since $k \geq 4$.

Now we describe how the sequences are generated. After describing them, we will argue that the procedure is well defined. On step $i \geq 1$, while S_{i-1} is non-empty, take an arbitrary v_i from S_{i-1}.

- If v_i satisfies condition (1) of Lemma 2 with respect to the induced graph on vertices S_{i-1}, i.e. v_i has many neighbors with distinct colors among vertices in S_{i-1}, then we define A to be the set of neighbors of v_i via these edges of all distinct colors. Let $v_{j_1}, v_{j_2}, \ldots$ with $j_\ell < i$ be the indices of all previous times when we have passed from S_{i-1} to S_i via condition (1). Let C denote the set of colors that occur on edges between any pair of distinct vertices $v_{j_\ell}, v_{j'_\ell}$. Then, let A' be A, except we remove from A any vertex $a \in A$ which has an edge to v_i of color $c \in C$. We stop if we have performed $4k^3$ passes via condition (1), so before this point we have not yet performed $4k^3$ passes via condition (1), hence $|C| < \binom{4k^3}{2} < 4k^6$. Thus, we need to remove at most $4k^6$ elements from A. Recall that we have guaranteed earlier that at any point in this process the set which we are working with has size at least $8k^6$. Hence, at least half of the elements in A need not be discarded, i.e. $|A'| \geq |A|/2$. Finally let $S_i \subset A'$ be A' except throw out elements until we have $|S_i| = \sqrt{|S_{i-1}|}/2$.

  Note that in doing so, we will preserve the invariant that $v_{j_1}, v_{j_2}, \ldots,$ form a locally-rainbow clique. If we perform $4k^3$ of this type of pass, then we will have a $4k^3$-sized locally-rainbow clique, and thus by Lemma 2 we can find a k-clique which is globally-rainbow.
- If v_i satisfies condition (2) or (3) of Lemma 2, we simply set S_i to be the set guaranteed to exist in S_{i-1} of neighbors of v_i all of the same color; by Lemma 2 we have $|S_i| \geq \sqrt{|S_{i-1}|}/2$.

We have already observed that $4k^3$ passes via condition (1) suffices to find a k-rainbow-clique. Now we argue that the other termination conditions result in either a k-monochromatic-clique or a k-up-ordered/k-down-ordered set:

- If we pass via condition (2) in the same color, call it "red", k times on vertices $v_{j_1}, v_{j_2}, \ldots, v_{j_k}$ then all edges from v_{j_ℓ} to $v_{j_{\ell'}}$ are red, and we have found a red monochromatic k-clique.
- By identical reasoning, passing k times via condition (3) in a single color yields a monochromatic k-clique.
- If we pass via condition (2) in k distinct colors then we have an up-ordered clique, defined by the vertices that we pass on.
- Similarly, if we pass via condition (3) in k distinct colors then we have an down-ordered clique.

Our process is well defined and continues until terminating at which point it yields a clique of one of the desired forms. Hence, N is demonstrated to be large enough that it must contain a clique of one of the desired forms.

□