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IE613 : Online Machine Learning
Assignment - 8.

Ans — 19.1 $\hat{\theta} = \underset{\theta \in \mathbb{R}^d}{\arg\min} \left(\sum_{s=1}^{t} \left(X_s - \langle A_s, \theta \rangle\right)^2 + \lambda \|\theta\|_2^2 \right)$

Differentiating, we get $\Rightarrow$

$$0 = 2 \sum_{s=1}^{t} \left(X_s - \langle A_s, \hat{\theta} \rangle\right)(-A_s) + 2\lambda\hat{\theta}$$

$$\lambda\hat{\theta} + \sum_{s=1}^{t} \left(\langle A_s, \hat{\theta} \rangle\right) A_s = \sum_{s=1}^{t} X_s A_s$$

$$\Rightarrow \lambda\hat{\theta} + \sum_{s=1}^{t} \left(\langle A_s, \hat{\theta} \rangle\right) A_s = \sum_{s=1}^{t} X_s A_s$$

$$\lambda\hat{\theta} + \sum_{s=1}^{t} A_s \left(A_s^T \hat{\theta}\right) = \sum_{s=1}^{t} X_s A_s$$

$$\hat{\theta} \left(\lambda I + \sum_{s=1}^{t} A_s A_s^T\right) = \sum_{s=1}^{t} A_s X_s$$

$$\hat{\theta} = \left(\lambda I + \sum_{s=1}^{t} A_s A_s^T\right)^{-1} \left(\sum_{s=1}^{t} X_s A_s\right)$$

Let $(V_t)_t$ be $d \times d$ matrices given by

$V_0 = \lambda I$ and $V_t = V_0 + \sum_{s=1}^{t} A_s A_s^T$

then, $\boxed{\hat{\theta} = V_t^{-1} \sum_{s=1}^{t} A_s X_s}$

Ans-19.2 Let T be the set of rounds t when $\|x_t\|_{V_{t-1}^{-1}} \geq 1$ and $G_t = V_0 + \sum_{s=1}^{t} \mathbb{I}_t(s) x_s x_s^T$

Then,
$$\left(\frac{d\lambda + |T| L^2}{d}\right)^d \geq \left(\frac{\text{trace}(G_n)}{d}\right)^d$$
$$\geq \det(G_n)$$
$$= \det(V_0) \prod_{t \in T} \left(1 + \|x_t\|^2_{G_{t-1}^{-1}}\right)$$
$$\geq \det(V_0) \prod_{t \in T} \left(1 + \|x_t\|^2_{V_{t-1}^{-1}}\right)$$
$$\geq \lambda^d 2^{|T|}$$

Rearranging and taking the logarithm shows that
$$|T| \leq \frac{d}{\log(e)} \log\left(1 + \frac{|T| L^2}{\lambda d}\right)$$

Abbreviate $a = \frac{d}{\log(2)}$ and $b = \frac{L^2}{d\lambda}$, which are both positive, Then $\Rightarrow$
$$a \log\left(1 + b\left(3a \log(1+ab)\right)\right) \leq a \log\left(1 + 3a^2 b^2\right)$$
$$\leq a \log(1+ab)^3 = 3a \log(1+ab)$$

Since $x - a\log(1+bx)$ is decreasing for $x \geq 3a\log(1+ab)$ it follows that
$$|T| \leq 3a \log(1+ab) = \frac{3d}{\log(2)} \log\left(1 + \frac{L^2}{\lambda \log(2)}\right)$$

Ans-19

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Apr-19.5 Partition C into m equal length subintervals, call these $C_1, \ldots C_m$. Associate a bandit algorithm with each of these subintervals. In round t, upon seeing $C_t \in C$, play the bandit algorithm associated with the unique subinterval that C_t belongs to.

We can use Exp3 or UCB, then regret is.

$$R_n = E\left[\sum_{t=1}^{n} \max_a r(C_t, a) - \sum_{t=1}^{n} X_t\right]$$

$$= E\left[\sum_{t=1}^{n} \max_a r(C_t, a) - \max_a \tilde{r}([C_t], a)\right] \rightarrow \mathrm{I}$$

$$+ \; E\left[\sum_{t=1}^{n} \max_a \tilde{r}([C_t], a) - X_t\right] \rightarrow \mathrm{II}$$

where for $c \in C$, $[c]$ is the index of the unique part of C, that c belongs to and for $i \in [m]$, $\tilde{r}(i, a) = E[r(C_t, a) \mid [C_t] = i]$

is the average reward when C_t falls into C_i. If $P([C_t] = i) = 0$, we define $\tilde{r}(i, a)$ arbitrarily to take the value of zero. Otherwise, we can write

$$\tilde{r}(i, a) = \int_{C_t} r(x, a) \, dP_i(x)$$

where $P_i(\cdot) = P(C_t \in \cdot \mid C_t \in C_i)$ is the conditional distribution of contexts when the context falls into C_i. First term in the regret decomposition is approximation error, second is error due to learning.

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$$\max_a r(c,a) - \max_a \tilde{r}([c],a) \le \max_a \left| r(c,a) - \tilde{r}([c],a) \right|$$

$$= \max_a \int_{C_i} \left| r(x,a) - r(c,a) \right| dP_i(x) \le L/m$$

Ans :

So, $|r(x,a) - r(c,a)| \le \frac{L}{m}$, thus, approx error is bounded by $\frac{Ln}{m}$.

For bounding error due to learning :

$$E[X_t | C_t] = i = \tilde{r}(i,a)$$

So, bandit C_t satisfies the conditions of a stochastic bandit environment.

For $i \in [m]$, let $T_i = \{ t \in [n] \mid [C_t] = i \}$

$N_i = |T_i|$, and $R_{ni} = \sum_{t \in T_i} \max_a \tilde{r}([C_t], a) - X_t$

Then,

$$E[R_{ni}] \le C \sqrt{K \log(K) N_i}$$

So, error due to learning is bounded by

$$R_{n\,(\text{II.})} \le \sum_{i \in [m]} E[R_{ni}] \le C (K \log(K))^{1/2} \cdot \sum_i N_i^{1/2}$$

$$\le C \sqrt{K \log(K) m n}$$

By Cauchy-Schwartz

So,

$$R_n \le \frac{Ln}{m} + C\sqrt{K \log(K) m n}, \text{ optimise } m \Rightarrow$$

$$m = \left(\frac{L}{C}\right)^{2/3} \left(\frac{n}{K \log(K)}\right)^{1/3}; \quad \boxed{R_n \le C' n^{2/3} (LK \log(K))^{1/3}}$$

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Ans-2a.

$$E\left[\|\hat{\theta}_n - \theta^*\|^2_{V_n}\right] = E\left[(\hat{\theta}_n - \theta^*)^T V_n (\hat{\theta} - \theta^*)\right]$$

$$V_n = 0.I + \sum_{s=1}^{n} A_s A_s^T$$

$$A_s A_s^T = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ & 0 & \cdots & & \\ & \cdots & 0 & 1 & \cdots & 0 \\ 0 & \cdots & & & 0 \end{bmatrix}$$, ~~a~~ its a matrix with 1 in the $(i,i)^{th}$ position

Since, each basis vector is played m times

$$V_n = mI$$

$$\hat{\theta}_n = V_n^{-1}\left(\sum_{s=1}^{n} A_s X_s\right) \qquad [\text{as } \lambda = 0]$$

$$= \frac{1}{m}\left(\sum A_s\left(\langle A_s, \theta^*\rangle + \eta_s\right)\right)$$

$$= \frac{1}{m}\left(\sum_{s=1}^{n} A_s (A_s^T \theta^*) + \sum_{s=1}^{n} A_s \eta_s\right)$$

$$= \frac{1}{m}\left(mI\theta^* + \sum_{s=1}^{n} A_s \eta_s\right)$$

$$= \theta^* + \frac{1}{m}\sum_{s=1}^{n} A_s \eta_s$$

$$\hat{\theta}_n - \theta^* = \frac{1}{m}\left(\sum_{s=1}^{n} A_s \eta_s\right)$$

$$E\left[(\hat{\theta}_n - \theta^*)^T V_n (\hat{\theta}_n - \theta^*)\right] = E\left[\frac{1}{m}\left(\sum A_s^T \eta_s\right) m \left(\frac{1}{m}\sum A_s \eta_s\right)\right]$$

$$= E\left[\frac{1}{m}\left[d\sum_{i=1}^{dm} \eta_i^2 + \underbrace{\sum_{j=1}^{d}\sum_{i=1}^{m} \eta_i \eta_j}_{\Downarrow\, 0}\right]\right]$$

$$= \frac{1}{m} \times md$$

$$= d$$

Ans 20.7 $E[X_t \| F_{t-1}] = E[X_t]$

(Assuming X_t is independent of $X_1, X_2 \dots X_{t-1}$)

[$\because X_t$ is adapted to the filteration]

$\Rightarrow P\left(\sum(X_t - E[X_t \| F_{t-1}]) \geq \varepsilon\right)$

$= P\left(\sum(X_t - E[X_t]) \geq \varepsilon\right)$

Let $Y_t = X_t - E[X_t]$

Y_t is a 0-mean r.v, for which $\Rightarrow$

$|Y_t| \in [a_t - E[Y_t], b_t - E[Y_t]]$ almost surely (a.s.)

$M_{Y_t}(\lambda) \leq \exp\left(\frac{\lambda^2 (b_t - a_t)^2}{8}\right)$

Y_t is $\left(\frac{b_t - a_t}{2}\right)$ — sub gaussian

$\sum Y_t$ is $\sqrt{\sum_{t=1}^{n}\left(\frac{b_t - a_t}{2}\right)^2}$ — sub gaussian

Using chernoff bound, $\Rightarrow$

$$P\left(\sum_{t=1}^{n} Y_t \geq \varepsilon\right) \leq \exp\left(\frac{-\varepsilon^2 \; n}{2\left(\sum_{t=1}^{n}\frac{(b_t - a_t)^2}{4}\right)}\right)$$

$$\leq \exp\left(\frac{-2n\varepsilon^2}{\sum_{t=1}^{n}(b_t - a_t)^2}\right)$$