Assignment - 2 Solutions

Ans-2. a) $E[e^{\lambda X}] \le e^{\lambda^2\sigma^2/2}$

$$1 + \lambda E[X] + \frac{\lambda^2 E[X^2]}{2} \le 1 + \frac{\sigma^2\lambda^2}{2} + \left(\begin{array}{c}\text{higher power of } \lambda \\ (\text{say } f(\lambda))\end{array}\right)$$

So, $\frac{f(\lambda)}{\lambda^2} \to 0$ as $\lambda \to 0$

$$\Rightarrow \lambda E[X] + \frac{\lambda^2}{2} E[X^2] + \dots \le \frac{\sigma^2\lambda^2}{2} + f(\lambda) \quad \text{— (1)}$$

For $\lambda \to 0^-$, divide by $\lambda \Rightarrow$

$$E[X] \ge 0$$

For $\lambda \to 0^+$, divide by $\lambda \Rightarrow$

$$E[X] \le 0$$

$E[X] = 0$

Now as $E[X] = 0 \Rightarrow \mathrm{Var}(X) = E[X^2] - (E[X])^2 = E[X^2]$

Using (1) $\frac{\lambda^2}{2} E[X^2] + \frac{\lambda^3}{3!} E[X^3] \dots \le \frac{\sigma^2\lambda^2}{2} + f(\lambda)$

On dividing by λ^2 & $\lambda \to 0$, we get

$$E[X^2] \le \sigma^2 \Rightarrow \mathrm{Var}(X) \le \sigma^2$$

b) $E[e^{\lambda X}] \le e^{\lambda^2\sigma^2/2} \quad \forall \lambda \in \mathbb{R}$

Let $\lambda' = c\lambda \quad \forall \lambda \in \mathbb{R}$

$$\Rightarrow E[e^{\lambda' X}] \le e^{\lambda^2 c^2 \sigma^2/2}$$

$$E[e^{\lambda(cX)}] \le \exp\left(\lambda^2 \left(\frac{(|c|\sigma)^2}{2}\right)\right) \quad \forall \lambda' \in \mathbb{R}$$

$\Rightarrow cX$ is $|c|\sigma$ - sub gaussian

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Assuming X_1, X_2 independent $\Rightarrow$

$$E\left[\exp\left(\lambda(X_1+X_2)\right)\right] = E\left[\exp(\lambda X_1)\right] E\left[\exp(\lambda X_2)\right]$$

$$E\left[\exp\left(\lambda(X_1+X_2)\right)\right] \leq \exp\left(\frac{\lambda^2\sigma_1^2}{2}\right)\exp\left(\frac{\lambda^2\sigma_2^2}{2}\right)$$

$$E\left[e^{\lambda(X_1+X_2)}\right] \leq \exp\left(\frac{\lambda^2\left(\sqrt{\sigma_1^2+\sigma_2^2}\right)^2}{2}\right)$$

X_1+X_2 is $\sqrt{\sigma_1^2+\sigma_2^2}$ - subgaussian

Ans-3. a) Let $\varphi(s) = \log E\left[e^{sX}\right]$

$$\Rightarrow \quad \varphi'(s) = \frac{E\left[Xe^{sX}\right]}{E\left[e^{sX}\right]}, \quad \varphi''(s) = \frac{E\left[X^2e^{sX}\right]}{E\left[e^{sX}\right]} - \left(\frac{E\left[Xe^{sX}\right]}{E\left[e^{sX}\right]}\right)^2$$

Observe that $\varphi''(s)$ is variance under measure Q where

$$dQ = \frac{e^{sX}}{E\left[e^{sX}\right]}\,dP$$

And, $\mathrm{Var}(X) = \mathrm{Var}\left(X - \left(\frac{a+b}{2}\right)\right) \leq E\left[\left(X - \left(\frac{a+b}{2}\right)\right)^2\right]$

$$\leq (b-a)^2/4$$

Using fundamental theorem of calculus $\Rightarrow$

$$\varphi(s) = \int_0^s\int_0^u \varphi''(\rho)\,d\rho\,du \leq \frac{s^2(b-a)^2}{8}$$

Using $\varphi(0) = \log(1) = 0$, $\varphi'(0) = E[X] = 0$

$\varphi''(s) = \mathrm{Var}(X) \leq (b-a)^2/4$

$$\Rightarrow \quad E\left[\exp(sX)\right] \leq \exp\left(\frac{s^2(b-a)^2}{4}\right) \Rightarrow X \text{ is } \frac{b-a}{2} \text{ subgaussian}$$

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Using (1), X_t are $\left(\frac{b_t - a_t}{2}\right)$ sub-gaussian

Now, let $\sigma_t = \frac{b_t - a_t}{2}$

So, $\left(X_t - E[X_t]\right)$ are also σ_t - subgaussian $\forall\ t = 1, 2 \ldots n$

$$\Rightarrow \sum_{t=1}^{n} \left(X_t - E[X_t]\right) \text{ is } \sqrt{\sum_{t=1}^{n} \sigma_t^2} \text{ subgaussian}$$

$\Rightarrow$ If X is σ- subgaussian, then for any $\varepsilon > 0$

$$P(X > \varepsilon) \leq \exp\left(-\frac{\varepsilon^2}{2\sigma^2}\right)$$

$$\Rightarrow P\left(\sum_{t=1}^{n} \left(X_t - E[X_t]\right) \geq \varepsilon\right) \leq \exp\left(-\frac{\varepsilon^2}{2\sum_{t=1}^{n} \sigma_t^2}\right)$$

$$\Rightarrow P\left(\sum_{t=1}^{n} \left(X_t - E[X_t]\right) \geq \varepsilon\right) \leq \exp\left(\frac{-2\varepsilon^2}{\sum_{t=1}^{n} (b_t - a_t)^2}\right)$$

Ans-4.

$$R_T \le \min\left\{T\Delta,\ \Delta + \frac{4}{\Delta}\left(1 + \max\left\{0, \log\left(\frac{T\Delta^2}{4}\right)\right\}\right)\right\}$$

Lets assume $\frac{T\Delta^2}{4} < 1 \Rightarrow \Delta < \frac{2}{\sqrt{T}}$

So, $\log\left(\frac{T\Delta^2}{4}\right) < 0$

$$\Rightarrow R_T \le \min\left\{T\Delta,\ \Delta + \frac{4}{\Delta}(1+0)\right\}$$

Now, $\Delta < \frac{2}{\sqrt{T}} \Rightarrow T\Delta < 2\sqrt{T}$, but $\frac{4}{\Delta} > 2\sqrt{T}$

So, $R_T \le T\Delta$

$$\boxed{R_T \le 2\sqrt{T}}$$

Now, assume $\frac{T\Delta^2}{4} > 1 \Rightarrow \Delta > \frac{2}{\sqrt{T}}$

$\Rightarrow \log\left(\frac{T\Delta^2}{4}\right) > 0$, $\boxed{\frac{4}{\Delta} < 2\sqrt{T}}$ and $T\Delta > 2\sqrt{T}$

So Now, $R_T \le \min\left\{T\Delta,\ \Delta + \frac{4}{\Delta}\left(1 + \log\left(\frac{T\Delta^2}{4}\right)\right)\right\}$

$$R_T \le \Delta + \frac{4}{\Delta}\left(1 + \log\left(\frac{T\Delta^2}{4}\right)\right)$$

Using $\log x \le \sqrt{x} \quad \forall\ x > 0$

So, $\log\left(\frac{T\Delta^2}{4}\right) \le \sqrt{\frac{T\Delta^2}{4}} \le \frac{\Delta\sqrt{T}}{2}$

$$R_T \le \Delta + \frac{4}{\Delta} + \frac{4}{\Delta}\log\left(\frac{T\Delta^2}{4}\right) \le \Delta + 2\sqrt{T} + 2\sqrt{T}$$

$$\boxed{R_T \le \Delta + 4\sqrt{T}} \quad \forall\ T$$

Ans-5. Let $P(R_T < \varepsilon) \geq 1-\delta$

$\Rightarrow P(R_T > \varepsilon) < \delta$

Using Markov's inequality $\Rightarrow$

$$P(R_T > \varepsilon) < \frac{E[R_T]}{\varepsilon}$$

Set $\delta = \frac{E[R_T]}{\varepsilon} = \frac{1}{\varepsilon}\left(T\mu^* - \left(\sum_{i=1}^{k} P\left(\mathbb{1}_{\{mk=i\}}\right)\mu_i\right)(T-mk) - m\sum_{i=1}^{k}\mu_i\right)$

let $\Delta_i = \mu^* - \mu_i$

$$\Rightarrow \varepsilon \leq \frac{1}{\delta}\left[m\sum_{i=1}^{k}\Delta_i + (T-mk)\sum_{1}^{k}\left(\mathbb{1}_{\{mk=i\}}\right)\Delta_i\right]$$

$P\left(\mathbb{1}_{\{mk=i\}}\right) \leq e^{-m\Delta_i^2/4}$, as the rewards are 1-subgaussian

$$\Rightarrow \varepsilon \leq \frac{1}{\delta}\left(m\sum_{1}^{k}\Delta_i + (T-mk)\sum_{1}^{k}\exp\left(\frac{-m\Delta_i^2}{4}\right)\Delta_i\right)$$

Now $\delta \in (0,1)$

$$\Rightarrow \exp\left(-\frac{m\Delta_i^2}{4}\right) \leq \exp\left(-\frac{m}{\delta}\Delta_i^2/4\right)$$

$$\Rightarrow \varepsilon \leq \left(\frac{m}{\delta}\sum_{1}^{k}\Delta_i + \left(\frac{T}{\delta} - \frac{mk}{\delta}\right)\sum\exp\left(\frac{-m}{\delta}\Delta_i^2/4\right)\Delta_i\right)$$

Using similar optimization for $m/\delta \Rightarrow$

$$\frac{m}{\delta} = \max\left[1, \left\lceil\frac{4}{\Delta^2}\log\left(\frac{T\Delta^2}{4}\right)\right\rceil\right], \quad \left(\Delta = \min_{i \neq a^*}\Delta_i\right)$$

$$\Rightarrow R_T \leq \min\left\{\frac{T\Delta}{\delta}, \Delta + \frac{4}{\Delta}\left(1 + \max\left(0, \log\left(\frac{T\Delta^2}{4\delta}\right)\right)\right)\right\}$$

holds w.p. atleast $1-\delta$.

Ans-6.
$$R_T = T\mu^* - \sum_{t=1}^{T} X_t$$

Same as in Q-5, apply markov's inequality

$$P(R_T > \varepsilon) \leq \frac{E[R_T]}{\varepsilon}$$

$$E[R_T] = \left[m\sum_{1}^{K}\Delta_i + (T-mk)\sum_{1}^{K} P(\mathbb{1}_{\{mk=i\}})\Delta_i \right]$$

(same as in Q-5)

So, the bound doesn't change

$$\Rightarrow R_T \leq \min\left\{ \frac{T\Delta}{\delta},\ \Delta + \frac{4}{\Delta}\left(1+\max\left\{0, \log\left(\frac{T\Delta^2}{4}\right)\right\}\right)\right\}$$

holds w.p. atleast $1-\delta$.

So, conclusion is that random & pseudo-regret of ETC are same.

Ans-7. a)
$$N_i(t) = \sum_{n=1}^{t} \mathbb{1}_{\{A_n = i\}}$$

$$E[N_i(t)] = \sum_{n=1}^{t} P(A_n = i)$$

$$= 1 + \frac{\varepsilon T}{K} + (1-\varepsilon)\sum_{1} P\left(\hat{\mu}_{i,t-1} = \max_j \hat{\mu}_{j,(t-1)}\right)$$

Since rewards are 1-subgaussian, at $T \to \infty$

$$P\left(\hat{\mu}_{i,t-1} = \max_j \hat{\mu}_{j,(t-1)}\right) \leq \exp\left(-\frac{\varepsilon t\Delta^2}{4}\right)$$

$$\Rightarrow \lim_{T\to\infty} (1-\varepsilon)\left(P\left(\hat{\mu}_{i,t-1} > \hat{\mu}_{i^*,t-1}\right)\right) = 0$$

$$\Rightarrow N_i(t) \xrightarrow{T\to\infty} \varepsilon/K$$

$$\Rightarrow \frac{R_T}{T} = \frac{1}{T}\sum_{i=1}^{K}\Delta_i E[N_i(t)] \xrightarrow{T\uparrow\infty} \frac{\varepsilon}{K}\sum_{i=1}^{K}\Delta_i$$

b) Let $x_0 = \frac{1}{2K} \sum_{t=1}^{K} \varepsilon_t$

$$P_{\{I_t = j\}} \leq \frac{\varepsilon_t}{K} + \left(1 - \frac{\varepsilon_t}{K}\right) \underbrace{P\left(\hat{\mu}_{j\,T_j(t-1)} > \hat{\mu}_{T^*(t-1)}\right)}_{1.}$$

1. holds if $\hat{\mu}_{j\,T_j(t-1)} \geq \mu_j + \frac{\Delta_j}{2}$ OR

$$\hat{\mu}^*_{T_{a^*}(t-1)} \leq \mu^* - \frac{\Delta_j}{2}$$

Analysis for both these eq^{n}s $\Rightarrow$

$$P\left(\hat{\mu}_{i, T_j(t)} \geq \mu_j + \frac{\Delta_j}{2}\right) = \sum_1^T P\left(\sum T_j(T) = t \,\middle|\, \mu_{j,t} > \mu_j + \frac{\Delta_j}{2}\right)$$

$$* \, P\left(\hat{\mu}_{j,t} \geq \mu_j + \frac{\Delta_j}{2}\right)$$

$$\leq \sum_1^T P\left(T_j(T) = t \,\middle|\, \hat{\mu}_{j,t} > \mu_j + \frac{\Delta_j}{2}\right) \cdot \exp\left(-\frac{\Delta_j^2 t}{2}\right)$$

$$\leq P\left(\sum_1^T T_j(T) = t \,\middle|\, \hat{\mu}_{j,t} \geq \mu_j + \frac{\Delta_j}{2}\right) + \frac{2}{\Delta_j^2} \exp\left(-\Delta_j^2 \frac{x_0}{2}\right)$$

$T_j^P(T) \rightarrow$ no. of arms pulled giving j

$$\leq \sum_1^T P\left(T_j^P(T) \leq \pi(\hat{\mu} \ldots)\right) + \frac{2}{\Delta_j^2} \exp\left(-\Delta_j^2 \frac{x_0}{2}\right)$$

$$\leq 1 + x_0$$

$$x_0 = \frac{1}{2K} \sum \varepsilon_t$$

$$x_0 \leq \frac{c}{\Delta^2} \ln\left(\frac{\Delta^2}{cK}\right) \quad \text{if} \quad T \geq \frac{cK}{\Delta^2}$$

Put this into the formula of regret, we get $\Rightarrow$

$$R_T \le C \sum_{i=1}^{K} \left(\Delta_i + \frac{\Delta_i}{\Delta_{min}^2} \log \max\left\{ e, \frac{T \Delta_{min}^2}{K} \right\} \right)$$

Ans-8. Modified UCB :

$$UCB_i(t-1, \delta) = \begin{cases} \infty & \text{if } T_i(t-1) = 0 \\ \hat{\mu}_i(t-1) + \sqrt{\dfrac{2\log(1/\delta)}{T_i(t-1)}} & \text{else} \end{cases}$$

$$t \in \{ 1, 2, 4 \dots 2^l \}$$
$$P_0 \quad P_1 \quad P_2 \dots P_l \leftarrow l \text{ phases}$$

So, till l^{th} phase, total rounds $= 1 + 2 + \dots 2^l = 2^{l+1} - 1$

$\hat{\mu} = \frac{\text{reward}}{\text{count}}$ & count[i] is incremented by 2^l for i^{th} chosen arm in l^{th} phase

We know, for simple UCB algo,

$$R_n \le 3\Delta + \frac{16\log(n)}{\Delta}$$

Now, for any $I_l = [2^l, 2^{l+1} - 1]$

no. of rounds $= 2^{l+1} - 2^l = 2^l$

so, $n = 2^l$

$$R_k \le \sum_{i=1}^{k} 3\Delta_i + \sum_{i=1} \frac{16\log(2^l)}{\Delta_i}$$

Now, summing the cumulative regret for (l phases), where $2^{l+1} - 1 < T$

So, sum the regret from $k=0$ to $\log(T)$

$$\therefore R_T \leq \sum_{k=0}^{\log(T)} \left(\sum_{i=1}^{K} 3\Delta_i + \ell \sum_{i=1}^{K} \frac{16}{\Delta_i} \right)$$

$$\left(\because \log_2(2^\ell) = \ell \right)$$

$$R_T \leq \log(T) \sum_{i=1}^{K} (3\Delta_i) + \frac{\log(T)\left(\log(T)+1\right)}{2} \sum_{i=1}^{K} \frac{16}{\Delta_i}$$

So, $$\boxed{R_T \leq 3\log(T) \sum_{i=1}^{K} \Delta_i + \left[\log(T)\right]^2 \sum_{i=1}^{K} \frac{16}{\Delta_i}}$$

b) Now, if ℓ^{th} phase has a length of $[\alpha^\ell]$ with $\alpha > 1$, where $[\cdot]$ is assumed as ~~g.i.f.~~ greatest integer function, then the order of the regret would still remain same just with the inclusion of term $\log(\alpha)$

$$\Rightarrow R_T < 3\log(T) \sum_{i=1}^{K} \Delta_i + \left[\log(T)\right]^2 \sum_{i=1}^{K} \frac{16 \cdot \log(\alpha)}{\Delta_i}$$

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