

Assignment 2: CS215

Satyam Sinoliya, 23B0958
Vaibhav Singh, 23B1068
Shaik Awez Mehtab, 23B1080

Solution 1

Task A

When $X \sim \text{Ber}(p)$, PGF of X is

$$G_{\text{Ber}}(z) = \mathbb{E}(z^X) \quad (1)$$

$$= \sum_{n=0}^{\infty} P[X = n]z^n \quad (2)$$

Since $P[X = 0] = (1 - p)$, $P[X = 1] = p$, $P[X = n] = 0$ when $n > 1$,

$$G_{\text{Ber}}(z) = P[X = 0]z^0 + P[X = 1]z^1 \quad (3)$$

$$= (1 - p) + pz \quad (4)$$

Task B

When $X \sim \text{Bin}(n, p)$, PMF of X is

$$P[X = k] = \binom{n}{k} p^k (1 - p)^{n-k} \text{ for } k \leq n. \quad (5)$$

and $P[X = k] = 0$ for $k > n$.

$$G_{\text{Bin}}(z) = \sum_{k=0}^{\infty} P[X = k]z^k \quad (6)$$

$$= \sum_{k=0}^n \binom{n}{k} p^k (1 - p)^{n-k} z^k \quad (7)$$

$$= \sum_{k=0}^n \binom{n}{k} (pz)^k (1 - p)^{n-k} \quad (8)$$

$$= (1 - p + pz)^n. \quad (9)$$

By equation 4, $G_{\text{Bin}}(z) = (1 - p + pz)^n = (G_{\text{Ber}}(z))^n$. Hence proved.

Task D

When $X \sim \text{Geo}(p)$, PMF of X ,

$$P[X = k] = (1 - p)^{k-1} p \quad (10)$$

for $k > 0$. $P[X = 0] = 0$. Now, PGF of X ,

$$G_{\text{Geo}}(z) = \sum_{k=0}^{\infty} P[X = k]z^k \quad (11)$$

$$= \sum_{k=1}^{\infty} P[X = k]z^k \quad (12)$$

$$= \sum_{k=1}^{\infty} p(1 - p)^{k-1} z^k \quad (13)$$

$$= \sum_{k=1}^{\infty} pz(z - zp)^{k-1} \quad (14)$$

$$= pz \sum_{k=0}^{\infty} (z - zp)^k \quad (15)$$

$$= \frac{pz}{1 - z + pz} \quad (16)$$

Task E

By equation 9, $G_{\text{Bin}}(z) = (1 - p + pz)^n = G_X^{(n,p)}(z)$. For $Y \sim \text{NegBin}(n, p)$

$$P[Y = k] = \binom{k-1}{n-1} p^n (1-p)^{k-n} \text{ for } k \geq n \quad (17)$$

Otherwise, $P[Y = k] = 0$. PGF of Y is

$$G_Y^{(n,p)}(z) = \sum_{k=0}^{\infty} P[Y = k] z^k \quad (18)$$

$$= \sum_{k=n}^{\infty} \binom{k-1}{n-1} p^n (1-p)^{k-n} z^k \quad (19)$$

$$= \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} p^n (1-p)^k z^{n+k} \quad (20)$$

$$= (pz)^n \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} (z-pz)^k \quad (21)$$

We know $\sum_{k=0}^{\infty} \binom{k+n-1}{n-1} x^k = (1-x)^{-n}$. Thus

$$G_Y^{(n,p)}(z) = (pz)^n (1 - z + pz)^{-n} \quad (22)$$

$$= \left((1 - p^{-1} + p^{-1}z^{-1})^n \right)^{-1} \quad (23)$$

$$= (G_X^{(n,p^{-1})}(z^{-1}))^{-1}. \quad (24)$$

Hence Proved.

Solution 2

Task A

To prove:

Let X be a continuous real-valued random variable with CDF $F_X : \mathbb{R} \rightarrow [0, 1]$. Assume that F_X is invertible. Then the random variable $Y := F_X(X) \in [0, 1]$ is uniformly distributed in $[0, 1]$

Proof:

F_X by definition can also be written as

$$F_X(x) = P(X \leq x)$$

Define a new random variable Y ,

$$Y = F_X(X)$$

Y is the result of applying CDF F_X to the random variable X . To prove the theorem, assume $y \in [0, 1]$. So, the probability that $Y \leq y$ is:

$$P(Y \leq y) = P(F_X(X) \leq y)$$

It is assumed that $F_X(x)$ is invertible, so,

$$P(Y \leq y) = P(X \leq F_X^{-1}(y))$$

which is basically, probability that X is less than or equal to $F_X^{-1}(y)$. This can be written in the CDF form, which is $F_X(F_X^{-1}(y))$. So,

$$P(Y \leq y) = P(X \leq F_X^{-1}(y)) = F_X(F_X^{-1}(y)) = y$$

So,

$$P(Y \leq y) = y$$

where $y \in [0, 1]$, which is the CDF of uniform distribution in $[0, 1]$. So, Y is a uniform distribution in $[0, 1]$ regardless of X .

Solution 3

Solution 4

Solution 5