Assignment 2: CS215

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Solution 1

Task A

Let $X \sim \text{Ber}(p)$, PGF of X is

$$G_{\text{Ber}}(z) = \mathbb{E}(z^X) \tag{1}$$

$$= \sum_{n=0}^{\infty} P[X = n]z^n \tag{2}$$

Since $P[X = 0] = (1 - p)$, $P[X = 1] = p$, $P[X = n] = 0$ when $n > 1$,

$$G_{\text{Ber}}(z) = P[X = 0]z^0 + P[X = 1]z^1 \tag{3}$$

$$= (1 - p) + pz \tag{4}$$

Task B

Let $X \sim \text{Bin}(n, p)$, PMF of X is

$$P[X = k] = \binom{n}{k} p^k (1 - p)^{n-k} \text{ for } k \leq n. \tag{5}$$

and $P[X = k] = 0$ for $k > n$.

$$G_{\text{Bin}}(z) = \sum_{k=0}^{\infty} P[X = k]z^k \tag{6}$$

$$= \sum_{k=0}^{n} \binom{n}{k} p^k (1 - p)^{n-k} z^k \tag{7}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (pz)^k (1 - p)^{n-k} \tag{8}$$

$$= (1 - p + pz)^n. \tag{9}$$

By equation 4, $G_{\text{Bin}}(z) = (1 - p + pz)^n = (G_{\text{Ber}}(z))^n$. Hence proved.

Task D

Let $X \sim \text{Geo}(p)$, PMF of X,

$$P[X = k] = (1 - p)^{k-1} p \tag{10}$$

for $k > 0$. $P[X = 0] = 0$. Now, PGF of X,

$$G_{\text{Geo}}(z) = \sum_{k=0}^{\infty} P[X = k]z^k \tag{11}$$

$$= \sum_{k=1}^{\infty} P[X = k]z^k \tag{12}$$

$$= \sum_{k=1}^{\infty} p(1 - p)^{k-1} z^k \tag{13}$$

$$= \sum_{k=1}^{\infty} pz(z - zp)^{k-1} \tag{14}$$

$$= pz \sum_{k=0}^{\infty} (z - zp)^k \tag{15}$$

$$= \frac{pz}{1 - z + pz} \tag{16}$$

Task E

By equation 9, $G_{\text{Bin}}(z) = (1-p+pz)^n = G_X^{(n,p)}(z)$. Let $Y \sim \text{NegBin}(n,p)$

$$P[Y=k] = \binom{k-1}{n-1} p^n (1-p)^{k-n} \text{ for } k \geq n \tag{17}$$

Otherwise, $P[Y=k]=0$. PGF of Y is

$$\begin{aligned}
G_Y^{(n,p)}(z) &= \sum_{k=0}^{\infty} P[Y=k]z^k && (18)\\
&= \sum_{k=n}^{\infty} \binom{k-1}{n-1} p^n (1-p)^{k-n} z^k && (19)\\
&= \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} p^n (1-p)^k z^{n+k} && (20)\\
&= (pz)^n \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} (z-pz)^k && (21)
\end{aligned}$$

We know $\sum_{k=0}^{\infty} \binom{k+n-1}{n-1} x^k = (1-x)^{-n}$. Thus

$$\begin{aligned}
G_Y^{(n,p)}(z) &= (pz)^n (1-z+pz)^{-n} && (22)\\
&= \left((1-p^{-1}+p^{-1}z^{-1})^n\right)^{-1} && (23)\\
&= (G_X^{(n,p^{-1})}(z^{-1}))^{-1}. && (24)
\end{aligned}$$

Hence Proved.

Task G

To prove: *Given PGF of a random variable X is $G(z)$, expectation of X i.e $\mathbb{E}[X] = G'(1)$.*
Proof:

$$\begin{aligned}
G(z) &= \mathbb{E}(z^X) = \sum_{k=0}^{\infty} P[X=k]z^k && (25)\\
G'(z) &= \sum_{k=0}^{\infty} kP[X=k]z^{k-1} && (26)\\
G'(1) &= \sum_{k=0}^{\infty} kP[X=k] && (27)\\
&= \mathbb{E}[X] && (28)
\end{aligned}$$

Hence Proved. Now, Let's derive means of Bernoulli, Binomial, Geometric and Negative Binomial distributions using this:

1. **Bernoulli Distribution:** Let $X \sim \text{Ber}(p)$,

$$\begin{aligned}
G_{\text{Ber}}(z) &= (1-p)+pz && (29)\\
G'_{\text{Ber}}(z) &= p && (30)\\
G'_{\text{Ber}}(1) &= p = \mathbb{E}[X] && (31)
\end{aligned}$$

   Thus, $\mathbb{E}[X] = p$.

2. **Binomial Distribution:** Let $X \sim \text{Bin}(n, p)$,

$$G_{\text{Bin}}(z) = (1 - p + pz)^n \tag{32}$$
$$G'_{\text{Bin}}(z) = np(1 - p + pz)^{n-1} \tag{33}$$
$$G'_{\text{Bin}}(1) = np = \mathbb{E}[X] \tag{34}$$

Thus, $\mathbb{E}[X] = np$.

3. **Geometric Distribution:** Let $X \sim \text{Geo}(p)$,

$$G_{\text{Geo}}(z) = \frac{pz}{1 - z + pz} \tag{35}$$
$$G'_{\text{Geo}}(z) = \frac{p(1 - z + pz) - pz(p - 1)}{(1 - z + pz)^2} \tag{36}$$
$$= \frac{p}{(1 - z + pz)^2} \tag{37}$$
$$G'_{\text{Geo}}(1) = \frac{p}{p^2} = \frac{1}{p} = \mathbb{E}[X] \tag{38}$$

Thus, $\mathbb{E}[X] = \frac{1}{p}$.

4. **Negative Binomial Distribution:** Let $X \sim \text{NegBin}(n, p)$,

$$G_{\text{NegBin}}(z) = \left(\frac{pz}{1 - z + pz}\right)^n \tag{39}$$
$$G'_{\text{NegBin}}(z) = n\left(\frac{pz}{1 - z + pz}\right)^{n-1}\left(\frac{p}{(1 - z + pz)^2}\right) \tag{40}$$
$$G'_{\text{NegBin}}(1) = \frac{n}{p} = \mathbb{E}[X] \tag{41}$$

Thus, $\mathbb{E}[X] = \frac{n}{p}$.

Solution 2

Task A

To prove:
Let X be a continuous real-valued random variable with CDF : $\mathbb{R} \to [0, 1]$. Assume that F_X is invertible. Then the random variable $Y := F_X(X) \in [0, 1]$ is uniformly distributed in $[0, 1]$

Proof:
F_X by definition can also be written as

$$F_X(x) = P(X \leq x)$$

Define a new random variable Y,

$$Y = F_X(X)$$

Y is the result of applying CDF F_X to the random variable X. To prove the theorem, assume $y \in [0, 1]$. So, the probablity that $Y \leq y$ is:

$$P(Y \leq y) = P(F_X(X) \leq y)$$

It is assumed that $F_X(x)$ is invertible, so,

$$P(Y \leq y) = P(X \leq F_X^{-1}(y))$$

which is basically, probablity that X is less that or equal to $F_X^{-1}(y)$. This can be written in the CDF form, which is $F_X(F_X^{-1}(y))$. So,

$$P(Y \leq y) = P(X \leq F_X^{-1}(y)) = F_X(F_X^{-1}(y)) = y$$

So,

$$P(Y \leq y) = y$$

where $y \in [0,1]$, which is the CDF of uniform distributon in $[0,1]$. So, Y is a uniform distributon in $[0,1]$ regardless of X.

Task B

According to the theorem proved above, CDF of any random variable X mapped with itself gives a uniform random variable Y in $[0,1]$. So, let $Y \sim \text{Uniform}(0,1)$. Then for any random variable X,

$$F_X(X) = Y \tag{42}$$
$$X = F_X^{-1}(Y) \tag{43}$$

Algorithm $\mathcal{A}$:

1. Input: A sample y from the uniform distributon on $[0,1]$.
2. Transformation:
   - Apply the inverse CDF to y to compute a sample u.
   - Define $\mathcal{A}(u) = u = F_X^{-1}(y)$
3. Output: The random variable $U = F_X^{-1}(Y)$

This gives us the correct required random variables as, CDF of U is $F_U(u)$,

$$P(U \leq u) = P(F_X(Y) \leq u) \tag{44}$$
$$F_U(u) = P(F_X(F_X^{-1}(X) \leq u)) \tag{45}$$
$$F_U(u) = P(X \leq u) \tag{46}$$
$$F_U(u) = F_X(u) \tag{47}$$
$$\tag{48}$$

U and X have the same CDF, which was initially required.

Solution 3

Solution 4

Solution 5

Task A

Given, PDF of GMM variable X is $f_X = \sum_{i=1}^{K} p_i P[X_i = x]$. Let it's CDF be F_X. Then $F_X(x)$ is given by

$$F_X(x) = P[X \le x] \tag{49}$$
$$= \int_{-\infty}^{x} f_X(t)dt \tag{50}$$
$$= \int_{-\infty}^{x} \sum_{i=1}^{K} p_i P[X_i = t]dt \tag{51}$$
$$= \sum_{i=1}^{K} p_i \int_{-\infty}^{x} P[X_i = t]dt \tag{52}$$
$$= \sum_{i=1}^{K} p_i P[X_i \le x] \tag{53}$$
$$= \sum_{i=1}^{K} p_i F_{X_i}(x) \tag{54}$$

Where $F_{X_i}(x) = P[X_i \le x]$ is CDF of X_i.
Now, let CDF of output of the given algorithm be $F_{\mathcal{A}}(x) = P[\mathcal{A} \le x]$. Since the events that we choose $\mathcal{A}$ to be from the distribution i (say E_i) are disjoint for $i = 1, \ldots, k$.

$$F_{\mathcal{A}}(x) = P[\mathcal{A} \le x] \tag{55}$$
$$= \sum_{i=1}^{K} P[E_i] \cdot P[\mathcal{A} \le x | E_i] \tag{56}$$
$$= \sum_{i=1}^{K} p_i F_{X_i}(x) \tag{57}$$
$$= F_X(x) \tag{58}$$

We know that PDF of a random variable X with CDF $F_X(x)$ is $\frac{\partial F_X}{\partial x}$. Thus,

$$f_{\mathcal{A}}(x) = \frac{\partial F_{\mathcal{A}}}{\partial x} \tag{59}$$
$$= \frac{\partial F_X}{\partial x} \tag{60}$$
$$= f_X \tag{61}$$

Since x was arbitrary, for every $u \in \mathbb{R}$, $f_{\mathcal{A}}(u) = f_X(u)$. i.e the algorithm indeed samples from the GMM variable's distribution.

Task B

Since $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$, $\mathbb{E}[X_i] = \mu_i$ and $\mathrm{Var}[X_i] = \sigma_i^2$.

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} t \cdot P[X = t]dt \tag{62}$$
$$= \int_{-\infty}^{\infty} t \cdot \sum_{i=1}^{K} p_i P[X_i = t]dt \tag{63}$$
$$= \sum_{i=1}^{K} p_i \int_{-\infty}^{\infty} P[X_i = t]dt \tag{64}$$
$$= \sum_{i=1}^{K} p_i \mathbb{E}[X_i] \tag{65}$$
$$= \sum_{i=1}^{K} p_i \mu_i \tag{66}$$

Let $\mu = \mathbb{E}[X]$.

$$\text{Var}[X] = \int_{-\infty}^{\infty} (t-\mu)^2 P[X=t]dt \tag{67}$$
$$= \int_{-\infty}^{\infty} (t-\mu)^2 \sum_{i=1}^{K} p_i P[X_i = t]dt \tag{68}$$
$$= \sum_{i=1}^{K} p_i \int_{-\infty}^{\infty} (t-\mu)^2 P[X_i = t]dt \tag{69}$$
$$= \sum_{i=1}^{K} p_i \text{Var}[X_i] \tag{70}$$
$$= \sum_{i=1}^{K} p_i \sigma_i^2 \tag{71}$$

Let $\sigma^2 = \text{Var}[X]$.

$$\text{MGF}_X(t) = \int_{-\infty}^{\infty} e^{tX} P[X=x]dx \tag{72}$$
$$= \int_{-\infty}^{\infty} e^{tX} \sum_{i=1}^{K} p_i P[X_i = x]dx \tag{73}$$
$$= \sum_{i=1}^{K} p_i \int_{-\infty}^{\infty} e^{tX} P[X_i = x]dx \tag{74}$$
$$= \sum_{i=1}^{K} p_i \text{MGF}_{X_i}(t) \tag{75}$$
$$= \sum_{i=1}^{K} p_i e^{t\mu_i + \frac{1}{2}t^2\sigma_i^2} \tag{76}$$

Task C

Given $Z = \sum_{i=1}^{K} p_i X_i$, where $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$

$$\mathbb{E}[Z] = \mathbb{E}\left[\sum_{i=1}^{K} p_i X_i\right] \tag{77}$$
$$= \sum_{i=1}^{K} p_i \mathbb{E}[X_i] \tag{78}$$
$$= \sum_{i=1}^{K} p_i \mu_i \tag{79}$$