Assignment 2: CS215

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Solution 1

Task A

Let $X \sim \text{Ber}(p)$, PGF of X is

$$G_{\text{Ber}}(z) = \mathbb{E}(z^X) \tag{1}$$
$$= \sum_{n=0}^{\infty} P[X = n]z^n \tag{2}$$

Since $P[X = 0] = (1 - p)$, $P[X = 1] = p$, $P[X = n] = 0$ when $n > 1$,

$$G_{\text{Ber}}(z) = P[X = 0]z^0 + P[X = 1]z^1 \tag{3}$$
$$= (1 - p) + pz \tag{4}$$

Task B

Let $X \sim \text{Bin}(n, p)$, PMF of X is

$$P[X = k] = \binom{n}{k} p^k (1 - p)^{n-k} \text{ for } k \leq n. \tag{5}$$

and $P[X = k] = 0$ for $k > n$.

$$G_{\text{Bin}}(z) = \sum_{k=0}^{\infty} P[X = k]z^k \tag{6}$$
$$= \sum_{k=0}^{n} \binom{n}{k} p^k (1 - p)^{n-k} z^k \tag{7}$$
$$= \sum_{k=0}^{n} \binom{n}{k} (pz)^k (1 - p)^{n-k} \tag{8}$$
$$= (1 - p + pz)^n. \tag{9}$$

By equation 4, $G_{\text{Bin}}(z) = (1 - p + pz)^n = (G_{\text{Ber}}(z))^n$. Hence proved.

Task D

Let $X \sim \text{Geo}(p)$, PMF of X,

$$P[X = k] = (1 - p)^{k-1} p \tag{10}$$

for $k > 0$. $P[X = 0] = 0$. Now, PGF of X,

$$G_{\text{Geo}}(z) = \sum_{k=0}^{\infty} P[X = k]z^k \tag{11}$$
$$= \sum_{k=1}^{\infty} P[X = k]z^k \tag{12}$$
$$= \sum_{k=1}^{\infty} p(1 - p)^{k-1} z^k \tag{13}$$
$$= \sum_{k=1}^{\infty} pz(z - zp)^{k-1} \tag{14}$$
$$= pz \sum_{k=0}^{\infty} (z - zp)^k \tag{15}$$
$$= \frac{pz}{1 - z + pz} \tag{16}$$

Task E

By equation 9, $G_{\text{Bin}}(z) = (1 - p + pz)^n = G_X^{(n,p)}(z)$. Let $Y \sim \text{NegBin}(n, p)$

$$P[Y = k] = \binom{k-1}{n-1} p^n (1-p)^{k-n} \text{ for } k \geq n \tag{17}$$

Otherwise, $P[Y = k] = 0$. PGF of Y is

$$\begin{aligned} G_Y^{(n,p)}(z) &= \sum_{k=0}^{\infty} P[Y = k] z^k &(18)\\ &= \sum_{k=n}^{\infty} \binom{k-1}{n-1} p^n (1-p)^{k-n} z^k &(19)\\ &= \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} p^n (1-p)^k z^{n+k} &(20)\\ &= (pz)^n \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} (z - pz)^k &(21) \end{aligned}$$

We know $\sum_{k=0}^{\infty} \binom{k+n-1}{n-1} x^k = (1-x)^{-n}$. Thus

$$\begin{aligned} G_Y^{(n,p)}(z) &= (pz)^n (1 - z + pz)^{-n} &(22)\\ &= \Big((1 - p^{-1} + p^{-1} z^{-1})^n \Big)^{-1} &(23)\\ &= (G_X^{(n,p^{-1})}(z^{-1}))^{-1}. &(24) \end{aligned}$$

Hence Proved.

Task G

To prove: *Given PGF of a random variable X is $G(z)$, expectation of X i.e $\mathbb{E}[X] = G'(1)$.*
Proof:

$$\begin{aligned} G(z) &= \mathbb{E}(z^X) = \sum_{k=0}^{\infty} P[X = k] z^k &(25)\\ G'(z) &= \sum_{k=0}^{\infty} k P[X = k] z^{k-1} &(26)\\ G'(1) &= \sum_{k=0}^{\infty} k P[X = k] &(27)\\ &= \mathbb{E}[X] &(28) \end{aligned}$$

Hence Proved. Now, Let's derive means of Bernoulli, Binomial, Geometric and Negative Binomial distributions using this:

1. **Bernoulli Distribution:** Let $X \sim \text{Ber}(p)$,

$$\begin{aligned} G_{\text{Ber}}(z) &= (1-p) + pz &(29)\\ G'_{\text{Ber}}(z) &= p &(30)\\ G'_{\text{Ber}}(1) &= p = \mathbb{E}[X] &(31) \end{aligned}$$

   Thus, $\mathbb{E}[X] = p$.

2. **Binomial Distribution:** Let $X \sim \text{Bin}(n, p)$,

$$G_{\text{Bin}}(z) = (1 - p + pz)^n \tag{32}$$
$$G'_{\text{Bin}}(z) = np(1 - p + pz)^{n-1} \tag{33}$$
$$G'_{\text{Bin}}(1) = np = \mathbb{E}[X] \tag{34}$$

Thus, $\mathbb{E}[X] = np$.

3. **Geometric Distribution:** Let $X \sim \text{Geo}(p)$,

$$G_{\text{Geo}}(z) = \frac{pz}{1 - z + pz} \tag{35}$$
$$G'_{\text{Geo}}(z) = \frac{p(1 - z + pz) - pz(p-1)}{(1 - z + pz)^2} \tag{36}$$
$$= \frac{p}{(1 - z + pz)^2} \tag{37}$$
$$G'_{\text{Geo}}(1) = \frac{p}{p^2} = \frac{1}{p} = \mathbb{E}[X] \tag{38}$$

Thus, $\mathbb{E}[X] = \frac{1}{p}$.

4. **Negative Binomial Distribution:** Let $X \sim \text{NegBin}(n, p)$,

$$G_{\text{NegBin}}(z) = \left(\frac{pz}{1 - z + pz}\right)^n \tag{39}$$
$$G'_{\text{NegBin}}(z) = n\left(\frac{pz}{1 - z + pz}\right)^{n-1}\left(\frac{p}{(1 - z + pz)^2}\right) \tag{40}$$
$$G'_{\text{NegBin}}(1) = \frac{n}{p} = \mathbb{E}[X] \tag{41}$$

Thus, $\mathbb{E}[X] = \frac{n}{p}$.

Solution 2

Task A

To prove:
Let X be a continuous real-valued random variable with CDF : $\mathbb{R} \to [0, 1]$. Assume that F_X is invertible. Then the random variable $Y := F_X(X) \in [0, 1]$ is uniformly distributed in $[0, 1]$
Proof:
F_X by definition can also be written as

$$F_X(x) = P(X \le x)$$

Define a new random variable Y,

$$Y = F_X(X)$$

Y is the result of applying CDF F_X to the random variable X. To prove the theorem, assume $y \in [0, 1]$. So, the probablity that $Y \le y$ is:

$$P(Y \le y) = P(F_X(X) \le y)$$

It is assumed that $F_X(x)$ is invertible, so,

$$P(Y \le y) = P(X \le F_X^{-1}(y))$$

which is basically, probablity that X is less that or equal to $F_X^{-1}(y)$. This can be written in the CDF form, which is $F_X(F_X^{-1}(y))$. So,

$$P(Y \leq y) = P(X \leq F_X^{-1}(y)) = F_X(F_X^{-1}(y)) = y$$

So,

$$P(Y \leq y) = y$$

where $y \in [0, 1]$, which is the CDF of uniform distributon in $[0, 1]$. So, Y is a uniform distributon in $[0, 1]$ regardless of X.

Task B

According to the theorem proved above, CDF of any random variable X mapped with itself gives a uniform random variable Y in $[0, 1]$. So, let $Y \sim \text{Uniform}(0, 1)$. Then for any random variable X,

$$F_X(X) = Y \tag{42}$$

$$X = F_X^{-1}(Y) \tag{43}$$

Algorithm $\mathcal{A}$:

1. Input: A sample y from the uniform distributon on $[0, 1]$.
2. Transformation:
   - Apply the inverse CDF to y to compute a sample u.
   - Define $\mathcal{A}(u) = u = F_X^{-1}(y)$
3. Output: The random variable $U = F_X^{-1}(Y)$

This gives us the correct required random variables as, CDF of U is $F_U(u)$,

$$P(U \leq u) = P(F_X(Y) \leq u) \tag{44}$$

$$F_U(u) = P(F_X(F_X^{-1}(X) \leq u)) \tag{45}$$

$$F_U(u) = P(X \leq u) \tag{46}$$

$$F_U(u) = F_X(u) \tag{47}$$

(48)

U and X have the same CDF, which was initially required.

Solution 3

Solution 4

Solution 5