CS215 Assignment 1 Solutions

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Question 1. There are two friends playing a dice-roll game. Friend A has (n + 1) fair dice and Friend B has n fair dice (a fair die has equal probability of every face). On every roll, a win is achieved if we get a prime number on the top. What is the probability that A will have more wins than B if both roll all of their dice?
[5 marks]

Solution 1. So A or B wins when they get either 2, 3 or 5, whose probability is $\frac{1}{2}$. So the probability that A wins i number of times when all of the $(n+1)$ dice are rolled is

$$p(A_i) = \binom{n+1}{i}\left(\frac{1}{2}\right)^{i}\left(\frac{1}{2}\right)^{n+1-i} \tag{1}$$

$$= \binom{n+1}{i}\left(\frac{1}{2}\right)^{n+1} \tag{2}$$

Similarly, probability that B gets i wins when all of the n are rolled is

$$p(B_i) = \binom{n}{i}\left(\frac{1}{2}\right)^{i}\left(\frac{1}{2}\right)^{n-i} \tag{3}$$

$$= \binom{n}{i}\left(\frac{1}{2}\right)^{n} \tag{4}$$

Now the probability that A has more wins than B is

$$p(i > j) = \sum_{i=0}^{n+1}\sum_{\substack{j=0\\ i>j}}^{n} p(A_i)p(B_j) \tag{5}$$

Since number of wins of A is independent from that of B. By writing each possible term $\binom{n+1}{i}$ and $\binom{n}{j}$ in a grid manner. We observe that our required terms cover exactly half of the grid. Thus equation (5) simplifies to

$$p(i > j) = \frac{\sum_{i=0}^{n+1}\binom{n+1}{i}\left(\frac{1}{2}\right)^{n+1} \cdot \sum_{j=0}^{n}\binom{n}{j}\left(\frac{1}{2}\right)^{n}}{2} = \frac{1}{2} \tag{6}$$

Thus, the probability of A winning more times than B is $\frac{1}{2}$.

Question 2. You are playing a trading game against two teams A and B (will happen in reality soon). The game is played in the form of a three-set series with A and B alternately. Also, Team B is better at trading than Team A. To encourage your trading career, the exchange (an organization responsible for managing the trades) gives you two options A-B-A (which means you play a game with Team A, then Team B and at last Team A again) or B-A-B. You will win if you win two sets in a row. Which of the two options should you choose? Justify your choice with proper calculations.
[5 marks]

Solution 2. Let p_A and p_B be the probability of winning the set against Team A and Team B respectively. Since, Team B is better than team A, we know that,

$$p_A > p_B \tag{7}$$

For the first sequence, (i.e. $A - B - A$), to result in a win, we either have to win first and second set and lose the third set, or win the second and third set and lose the first set win all three sets. Probability of this happening is, (denoted as P_1)

$$P_1 = p_A \cdot p_B \cdot (1 - p_A) + (1 - p_A) \cdot p_B \cdot p_A + p_A \cdot p_B \cdot p_A \tag{8}$$

$$P_1 = p_A p_B - p_A^2 p_B + p_A p_B - p_A^2 p_B + p_A^2 p_B \tag{9}$$

$$P_1 = 2p_A p_B - p_A^2 p_B \tag{10}$$

Similary for second sequence, (i.e, $B - A - B$), the probablity for winning will be (let it be P_2)

$$P_2 = p_B \cdot p_A \cdot (1 - p_B) + (1 - p_B) \cdot p_A \cdot p_B + p_B \cdot p_A \cdot p_B \tag{11}$$

$$P_2 = p_A p_B - p_A p_B^2 + p_A p_B - p_A p_B^2 + p_A p_B^2 \tag{12}$$

$$P_2 = 2 p_A p_B - p_A p_B^2 \tag{13}$$

Since we know that,

$$p_A > p_B$$

As, both p_A and p_B is positive, we can multiple them to both the sides without changing the inequality sign,

$$p_A \cdot p_A p_B > p_B \cdot p_A p_B \tag{14}$$

This gives us:

$$p_A^2 p_B > p_A p_B^2 \tag{15}$$

Multiplying -1 on both the sides,

$$-p_A^2 p_B < -p_A p_B^2 \tag{16}$$

Adding $2p_A p_B$ on both the sides,

$$2p_A p_B - p_A^2 p_B < 2p_A p_B - p_A p_B^2 \tag{17}$$

Which simplifies to,

$$P_1 < P_2 \tag{18}$$

So, playing in the **second** sequence ($B - A - B$) gives us better chances of winning the game.

Question 3. This question has two parts:

- **3.1** Let Q_1, Q_2 be non-negative random variables. Let $P(Q_1 < q_1) \geq 1 - p_1$ and $P(Q_2 < q_2) \geq 1 - p_2$ where q_1, q_2 are non-negative. Then show that $P(Q_1 Q_2 < q_1 q_2) \geq 1 - (p_1 + p_2)$

  [3 marks]

- **3.2** Given n distinct values $\{x_i\}_{i=1}^n$ with mean μ and standard deviation σ, prove that for all i, we have $|x_i - \mu| \leq \sigma\sqrt{n-1}$. How does this inequality compare with Chebyshev's inequality as n increases? (give an informal answer)

  [3+2 marks]

Solution 3. 3.1
Define two events, E_1 and E_2:

1. $E_1 = \{Q_1 < q_1\}$
2. $E_2 = \{Q_2 < q_2\}$

So,

$$P(E_1) \geq 1 - p_1 \tag{19}$$
$$P(E_2) \geq 1 - p_2 \tag{20}$$

We need to prove that,

$$P(Q_1 Q_2 < q_1 q_2) \geq 1 - (p_1 + p_2) \tag{21}$$

Let's define another event E_3, where $E_3 = \{Q_1Q_2 < q_1q_2\}$
If we consider the complement of E_3, which is

$$\{Q_1Q_2 < q_1q_2\}^{\mathsf{c}} = \{Q_1Q_2 \geq q_1q_2\} \tag{22}$$
$$E_3^{\mathsf{c}} = \{Q_1Q_2 \geq q_1q_2\} \tag{23}$$

If $Q_1Q_2 \geq q_1q_2$, and Q_1, Q_2 are non-negative integers, it is very clear that, atleast one of the following has to be true:

$$Q_1 \geq q_1 \text{ or } Q_2 \geq q_2$$

$$\implies E_3^{\mathsf{c}} \subseteq \{Q_1 \geq q_1\} \cup \{Q_2 \geq q_2\} \tag{24}$$

$$\implies P(E_3^{\mathsf{c}}) \leq P\left(\{Q_1 \geq q_1\} \cup \{Q_2 \geq q_2\}\right) \tag{25}$$
$$\leq P(\{Q_1 \geq q_1\}) + P(\{Q_2 \geq q_2\}) \tag{26}$$

It is clear that:

$$\{Q_1 \geq q_1\} = E_1^{\mathsf{c}} \text{ and } \{Q_2 \geq q_2\} = E_2^{\mathsf{c}} \tag{27}$$

So,

$$1 - P(E_3) \leq P(E_1^{\mathsf{c}}) + P(E_2^{\mathsf{c}}) \tag{28}$$
$$\leq 1 - P(E_1) + 1 - P(E_2) \tag{29}$$
$$\leq 1 - (1 - p_1) + 1 - (1 - p_2) \tag{30}$$
$$\leq p_1 + p_2 \tag{31}$$
$$\implies P(E_3) \geq 1 - (p_1 + p_2) \tag{32}$$

Solution 3.2 We know that,

$$\frac{\sum_{i=0}^{n}(x_i - \mu)^2}{n-1} = \sigma^2 \tag{33}$$
$$\implies \sum_{i=0}^{n}(x_i - \mu)^2 = \sigma^2(n-1) \tag{34}$$

For any i,

$$(x_i - \mu)^2 \geq 0 \tag{35}$$

So, for each i,

$$(x_i - \mu)^2 \leq \sigma^2 \cdot (n-1) \tag{36}$$

Again, as both $(x_i - \mu)^2$ and $\sigma^2 \cdot (n-1)$ are greater than or equal to zero, we can take square root on both sides.

$$\sqrt[2]{(x_i - \mu)^2} \leq \sqrt[2]{\sigma^2(n-1)} |x_i - \mu| \leq \sigma\sqrt{n-1} \tag{37}$$

Chebyshev's inequality states that for any distribution, the proportion of values that lie more than k standard deviations away from the mean is bounded by:

$$P(|x - \mu| \geq k\sigma \leq \frac{1}{k^2})$$

As n increases, the deterministic bound $|x_i - \mu| \leq \sigma\sqrt{n-1}$ can potentially become less tight as $\sqrt{n-1}$ increase with n, but it remains a stronger bound for small datasets compared to Chebyshev's inequality. As Chebyshev's inequality, on the other hand, remains a probabilistic statement that applies to a portion of the distribution and becomes less restrictive for larger values of k.

Question 4. You need a new staff assistant, and you have n people to interview. You want to hire the best candidate for the position. When you interview a candidate, you can give them a score, with the highest score being the best and no ties being possible.

You interview the candidates one by one. Because of your company's hiring practices, after you interview the kth candidate, you either offer the candidate the job before the next interview or you forever lose the chance to hire that candidate. We suppose the candidates are interviewed in a random order, chosen uniformly at random from all n! possible orderings.

We consider the following strategy. First, interview m candidates but reject them all: these candidates give you an idea of how strong the field is. After the mth candidate. hire the first candidate you interview who is better than all of the previous candidates you have interviewed.

1. Let E be the event that we hire the best assistant, and let E_i; be the event that ith candidate is the best and we hire him. Determine $Pr(E_i)$, and show that

$$Pr(E) = \frac{m}{n}\sum_{j=m+1}^{n}\frac{1}{j-1} \tag{38}$$

[4 marks]

2. Bound $\sum_{j=m+1}^{n}\frac{1}{j-1}$ to obtain:

$$\frac{m}{n}(\ln n - \ln m) \leq Pr(E) \leq \frac{m}{n}(\ln(n-1) - \ln(m-1)) \tag{39}$$

[3 marks]

3. Show that $\frac{m}{n}(\ln(n) - \ln(m))$ is maximized when $m = \frac{n}{e}$, and explain why this means $Pr(E) \geq \frac{1}{e}$ for this choice of m. [3 marks]

Solution 4. Probability that i^{th} person is best is $\frac{1}{n}$, let him be X. All other remaining people are not better than X. Now, if $i \leq m$, $Pr(E_i) = 0$. For $i \geq m+1$, X is selected if and only if the best person from first to m^{th} person (let him be M) is better than everyone from $(m+1)^{\text{th}}$ to $(i-1)^{\text{th}}$ person. We'll try to count the number of permutations of people such that this condition is satisfied.
When all of the n people are arranged in ascending order, let j^{th} person be b_j, the last person (i.e b_n) is X. Suppose the best of first $i-1$ is b_j, then b_j should be in first m people. Remaining everyone in $i-1$ people is before b_j. The number of combinations of first $i-1$ people (except b_j) is

$$^{j-1}C_{i-1} \tag{40}$$

Total number of permutations when i^{th} person is best and the best of first m people is b_j is

$$^{j-1}C_{i-2} \cdot m \cdot (i-2)! \cdot (n-i)! \tag{41}$$

Since, j can be in $[i-1, n-1]$ So the total number of permutations is

$$P = \sum_{j=i-1}^{n-1} {}^{j-1}C_{i-2} \cdot m \cdot (i-2)! \cdot (n-i)! \tag{42}$$

$$= ^{n-1} C_{i-1} \cdot m \cdot (i-2)! \cdot (n-i)! \tag{43}$$

Total number of permutations is $(n-1)!$, probability that i^{th} person is selected given he is the best is

$$Pr(\text{selected } i | i \text{ is best}) = \frac{^{n-1}C_{i-1} \cdot m \cdot (i-2)! \cdot (n-i)!}{(n-1)!} \tag{44}$$

$$= \frac{m}{i-1} \tag{45}$$

Now, $Pr(i \text{ is best}) = \frac{1}{n}$. Therefore,

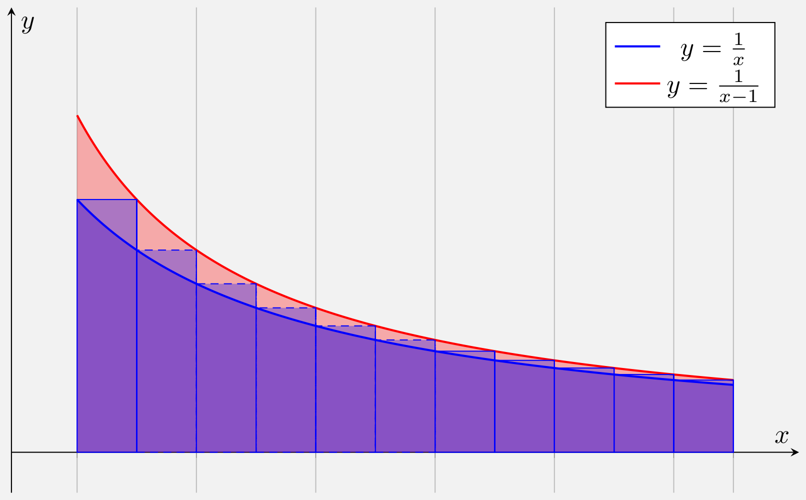
$$Pr(\text{selected } i \cap i \text{ is best}) = Pr(\text{selected } i | i \text{ is best}) \cdot Pr(i \text{ is best}) \tag{46}$$

$$= \frac{m}{i-1} \cdot \frac{1}{n} \tag{47}$$

Since each of E_i is disjoint,

$$Pr(E) = \sum_{j=0}^{n} Pr(E_j) \tag{48}$$

$$= \frac{m}{n} \sum_{j=m+1}^{n} \frac{1}{j-1} \tag{49}$$



Area under the rectangles is just $\sum_{j=m}^{n-1} \frac{1}{j}$, which is bound by the area of the two curves $\frac{1}{x}$ and $\frac{1}{x-1}$

$$\int_m^n \frac{1}{x}dx \leq \sum_{j=m}^{n-1} \frac{1}{j} \leq \int_m^n \frac{1}{x-1}dx \tag{50}$$

$$\implies \frac{m}{n}(\ln n - \ln m) \leq Pr(E) \leq \frac{m}{n}(\ln(n-1) - \ln(m-1)) \tag{51}$$

Let $n = mk$, then

$$\frac{m}{n}(\ln n - \ln m) = \frac{1}{k}(\ln mk - \ln m) \tag{52}$$

$$= \frac{\ln k}{k} = f(k) \tag{53}$$

This is maximum when $f'(k) = 0$ and $f"(k) < 0$. $f'(k) = \frac{1-\ln k}{k^2} = 0$, then $k = e$. Therefore $\frac{m}{n} = e$ i.e $n = \frac{m}{e}$. By equation 51, $Pr(E) \geq \frac{1}{e}$ for this choice of m and n.
Hence Proved.

Question 5. Imagine an infinitely long line of traders waiting outside a brokerage firm to place their trades. Each trader is assigned an ID number from 1 to 200 (both inclusive, obviously these IDs are not unique). The firm's director announces a special offer: the first trader in the queue whose ID number matches the ID of any trader who has already placed a trade will receive a free trade (i.e., a trade without any margins). You have the option to choose your position in this queue. However, you don't know the ID numbers of the traders ahead of you or behind you. Your goal is to maximize your chances of being the first trader whose ID matches someone who has already placed a trade. Given this situation, what position in the queue should you choose to maximize your chances of receiving the free trade? [6 marks]

Solution 5. Note: To run the code for this, execute `python3 ./code/q5.py`.
We have infinite number of trader where each trader is assigned an ID number from 1 to 200 waiting in a queue. We need to find the position which maximises the chance of getting a free trade, where

free trade is awarded to a person whose ID number matches with someone ahead in the queue, that is, person who has traded. Since there are 200 unique IDs, the last position one can win is 201 (by pigeon hole principle). 1st person can never win as it has no one ahead of him. Therefore for i^{th} person $i \in \{2, 3, 4, \cdots, 201\}$ to win, the probability of winning a free trade is given by

$$P(i) = \frac{200}{200} \times \frac{199}{200} \times \cdots \times \frac{202-i}{200} \times \frac{i-1}{200} \tag{54}$$

$$= \prod_{j=2}^{i} \left(\frac{202-j}{200} \right) \times \frac{i-1}{200} \tag{55}$$

The probability is derived by the logic that if i^{th} person is going to win, then all of the people ahead him need to loose and ith person has i-1 choices to secure victory. For maximising this probability I have used python and I have attached a file for the code. The position which maximises the probability of winning a free trade is 15 with a probability of 0.0439.

Question 6. Suppose that you have computed the mean, median and standard deviation of a set of n numbers stored in array A where n is very large. Now, you decide to add another number to A. Write a python function to update the previously computed mean, another python function to update the previously computed median, and yet another python function to update the previously computed standard deviation. Note that you are not allowed to simply recompute the mean, median or standard deviation by looping through all the data. You may need to derive formulae for this. Include the formulae and their derivation in your report. Note that your python functions should be of the following form:

```
function newMean = UpdateMean(OldMean, NewDataValue, n, A),
function newMedian = UpdateMedian(OldMedian, NewDataValue, n, A),
function newStd = UpdateStd(OldMean, OldStd, NewMean, NewDataValue, n, A).
```

Also explain, how would you update the histogram of A, if you received a new value to be added to A? (Only explain, no need to write code.) Please specify clearly if you are making any assumptions. [10 marks]

Solution 6. Note: To run the code for this, execute `python3 ./code/q6.py`.
For computing new mean from previously computed mean and the new value, we just need to multiply the older mean with the size of the set and add the new value and divide the whole sum by n+1. The mean is given by

$$\texttt{NewMean} = \frac{\texttt{OldMean} * n + \texttt{newDataVal}}{n+1} \tag{56}$$

For computing the new median, we need to append the new value to the array and then sort it. If the older n was even, we need to return the key at $\left(\frac{n+2}{2}\right)$. If older n was odd, we need to return the average of key at $\left(\frac{n+1}{2}, \frac{n+2}{2}\right)$. This approach is used if the array is not sorted. If we make an assumption that the array is sorted, then we need to compare the new element with just the current median, one element just larger than the median and one element just smaller than the median and we can apply the same approach as before to compute the median.
For computing the new standard deviation, first we get new variance using the contribution of old variance and mean and then add the effect of new data and mean to it. The formula is given by

$$\texttt{newVar} = \frac{((n-1) * \texttt{oldVar} + (\texttt{NewDataVal} - \texttt{OldMean}) * (\texttt{NewDataVal} - \texttt{NewMean}))}{n} \tag{57}$$

If we want to update the histogram of A,we can directly find the bin range in which the value lies in and increment the value of that particular bin by 1. The assumptions we are making include that we already have a histogram that tracks the count of each bin and the bin ranges/width are fixed.

Question 7. Read about the following plots:

1. Violin Plot

2. Pareto Chart
3. Coxcomb Chart
4. Waterfall Plot

Describe the uses of these plots. Take some sample data and generate one example plot for each of them.

[8 marks]

Solution 7. Note: To get graphs for each of these execute the following:

1. `python3 ./code/q7-violin.py`
2. `python3 ./code/q7-pareto.py`
3. `python3 ./code/q7-coxcomb.py`
4. `python3 ./code/q7-waterfall.py`

Here are the descriptions and usages of the given plots along with their examples:

1. **Violin plot:** It's a hybrid of box plot and kernel density plot. A box plot represents data in a linear fashion. It's made of a straight line from lowest value to highest value along with a box from first to third quartiles, marking all the quartiles of the dataset. Here's an example in figure 1:

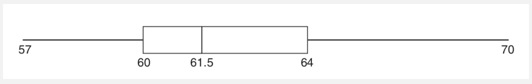


Figure 1: Box plot

A kernel density plot represents the density/frequency of data points. It's similar to a histogram, but smooth. In a violin graph, this is kept vertical, with two mirror images of it reflected along y-axis. This is useful in the sense that we can look into both centrel tendencies of the data (like mean, median etc.) but also how the data is distributed. Both at once. We can visualise the following data which I've taking from (*) containing the average number of hours a person studies given the number of courses taken

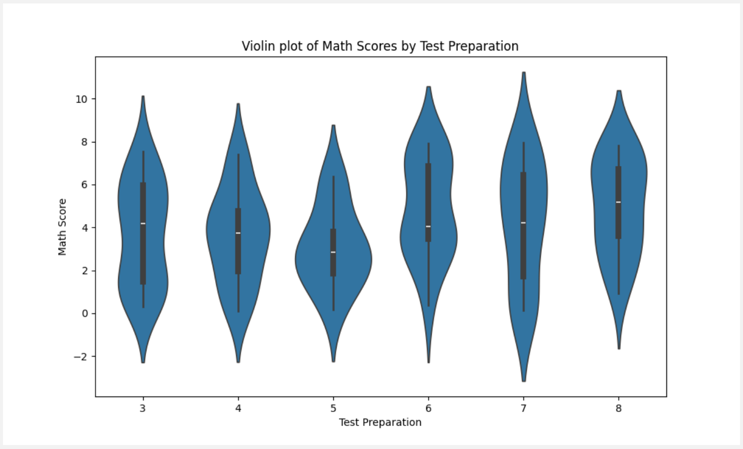


Figure 2: Violin Plot

As mentioned before, a violin plot can show both statistical summary along with distribution, which a normal plot can't. Here in figure 2, the gray line represents the box plot component of it. And the plot you get by rotating it by 90° is the distribution plot.

2. **Pareto Chart:** A pareto chart consists of both a bars and a line graph in the same plot. The bar graph represents the individual data like frequency, cost, height, weight of individual values. These values are arranged in decreasing order of the height of corresponding graphs. The line graph represents the cumulative data like frequency, cost, etc. till that point. Since the values decreases, this line graph is concave.

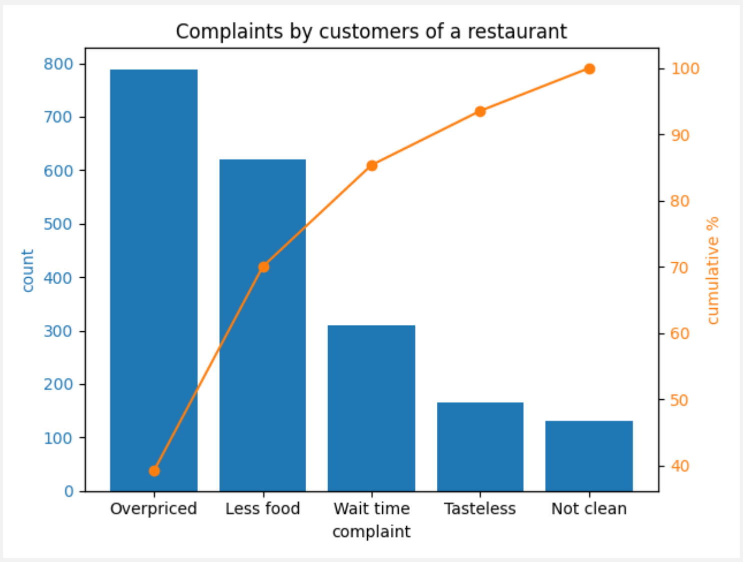


Figure 3: Pareto Plot

An example of a pareto plot is shown where I've collected data of complaints from customers about a given restaurant. It's useful in this case because it is able to show the proportion of the problem particular category is causing as well as the magnitude of the problem.

3. **Coxcomb Chart:** A coxcomb chart is similar to a pie chart. It's also known as a polar chart. But unlike a pie chart, here the area of each sector represents the proportion of the problem caused by a particular category of data unlike a pie chart, where the angle represents the proportion.

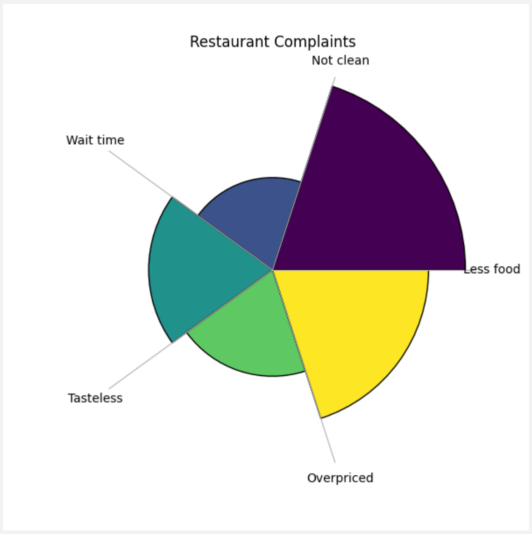


Figure 4: Coxcomb Plot

Here in figure 4 I've shown the same data shown before in figure 3 in a coxcomb chart. Like a pie chart, a coxcomb chart isn't useful when it comes to to large number of categories of data. It's useful when we have a small number of data which is categorical.

4. **Waterfall plot:** A waterfall plot shows how two-dimensional data changes over time, this two-dimensional data is typically a spectra. This results in a 3D graph. But this 3D graph is drawn in a 2D manner, by staggering the curves both accross the screen and vertically. Due to this, the curves in the front "hide" the ones behind.

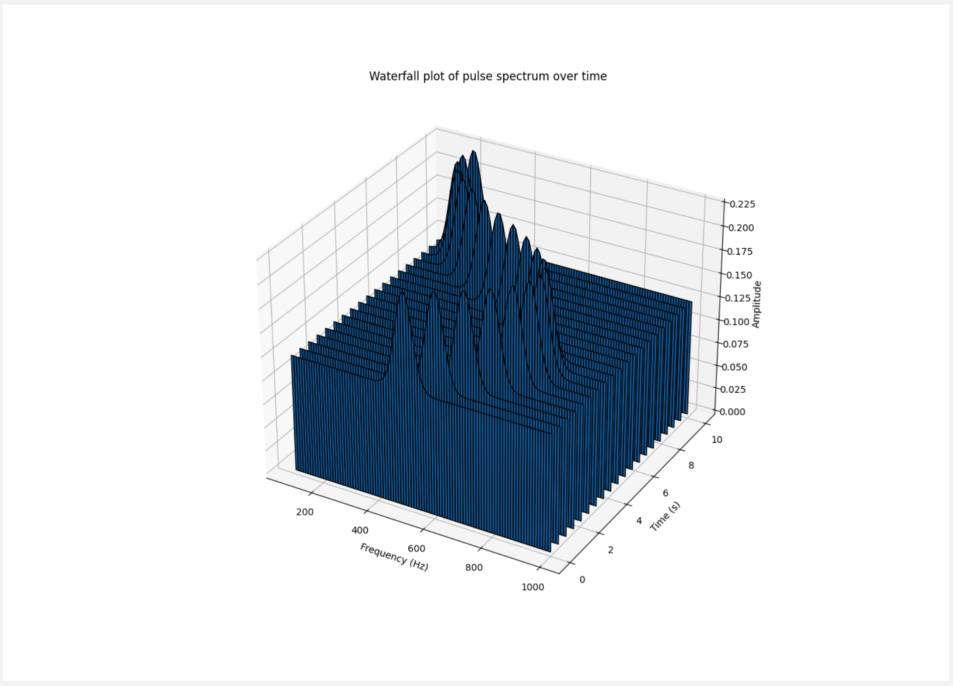


Figure 5: Waterfall Plot

This is very useful in studying data which has a spectrum, for example, showing spectrum of a given signal at different intervals of time, spectra at different engine speeds when testing engines etc.

Question 8. Download the image of Monalisa from here. Read the image using matplotlib (example). Write a piece of python code to shift the image along the X direction by tx pixels where tx is an integer ranging from -10 to $+10$ (so, in total you need to do this for 20 values). While doing so, assign a value of 0 to unoccupied pixels. For each shift, compute the correlation coefficient between the original image and its shifted version. Make a plot of correlation coefficients across the shift values. Also, generate a normalized histogram for the original image. You might need to refer to section 3.3 from this book. You are not allowed to use any inbuilt function for generating the histogram. If you are using any other libraries, then please mention about them in the pdf.

[8 marks]

Solution 8. Note: The code for this is written in `./code/q8.ipynb`, you can run it through google colab/vscode.
The image of Mona Lisa was read using the `matplotlib` library. The image was then shifted horizontally by `tx` pixels for each value of `tx` in the range of -10 to +10. The shifting operation was implemented manually, ensuring that unoccupied pixels were assigned a value of 0.
A custom function `shiftimg` was created to handle the shifting process:

- If `tx > 0`, pixels were shifted rightwards by `tx` units.
- If `tx < 0`, pixels were shifted leftwards by `tx` units.
- If `tx = 0`, the function returned the original image.

All the shifted images are displayed in the Jupyter notebook (q8.ipynb). Some of them are displayed below.

Figure 6: Image with `tx=10`

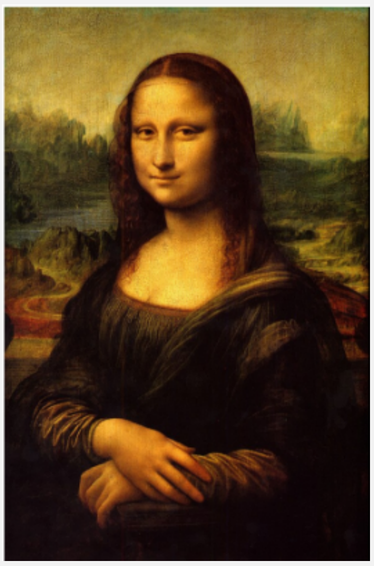


Figure 7: Image with `tx=-5`

For each shifted image, the correlation coefficient between the original and shifted image was calculated. This coefficient quantifies the linear relationship between the two images, with values ranging from -1 to 1, where 1 indicates a perfect positive correlation, -1 indicates a perfect negative correlation, and 0 indicates no correlation. This done using a function `corrcoef` in `numpy` library. This function require 2 1-D arrays to determine the correlation coeffiecient. The flattenning of arrays was done using `.flatten()`. Then correlation for each shift is plotted on a graph shown below.

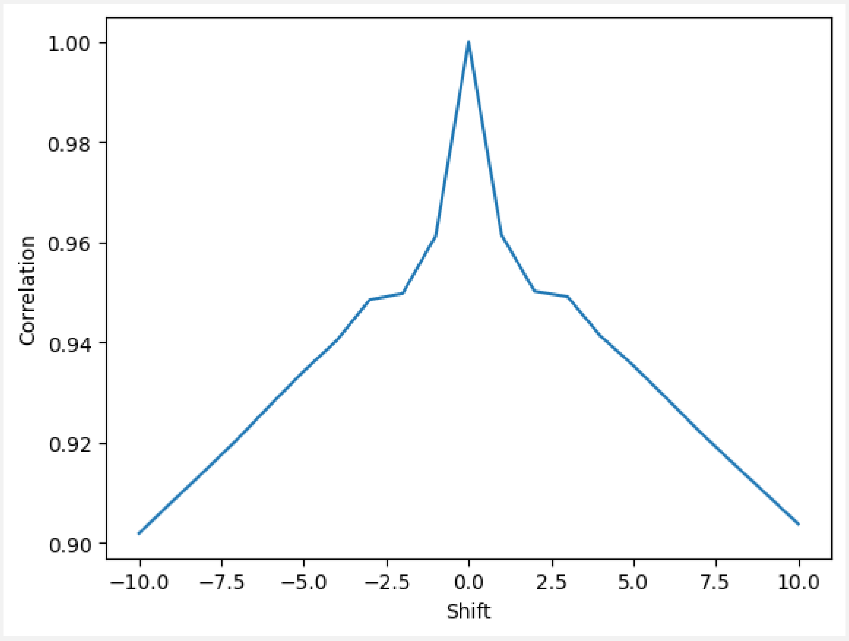


Figure 8: Correlation Coefficient vs Shift Values

As observed in Figure 8, the correlation decreases as the shift increases in either direction. This is expected as the more the image is shifted, the less it resembles the original, resulting in lower correlation values.
Following this, normalized histogram is plotted for each channel (RGB). This is done without use of any library, by creating a function `draw_hist` which takes the image of a single channel, transverses through each pixel taking the note of its value and finally each value is divided by the total number of

pixels(width*height) to normalize the histogram.

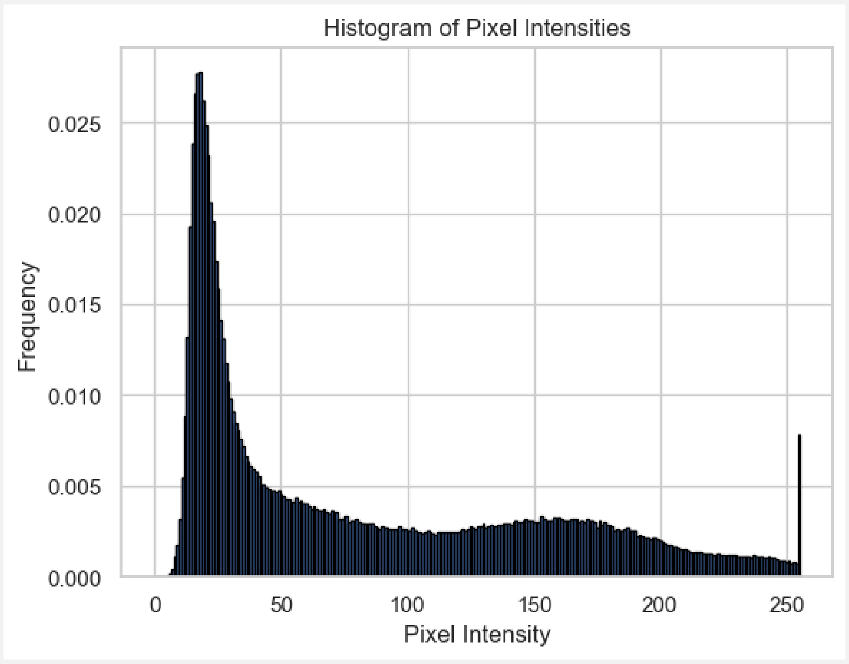


Figure 9: Normalized Histogram of red channel of the Original Image

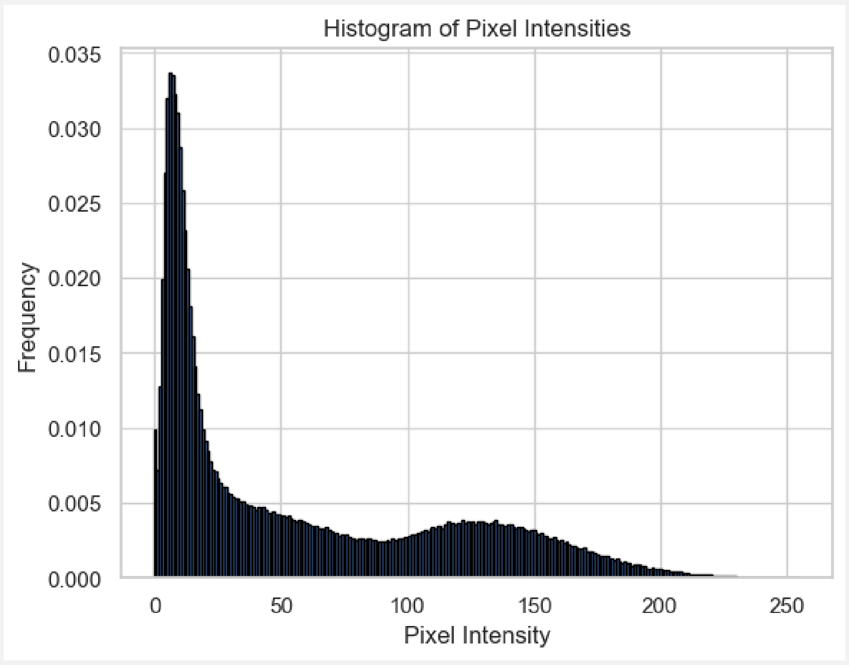


Figure 10: Normalized Histogram of green channel of the Original Image

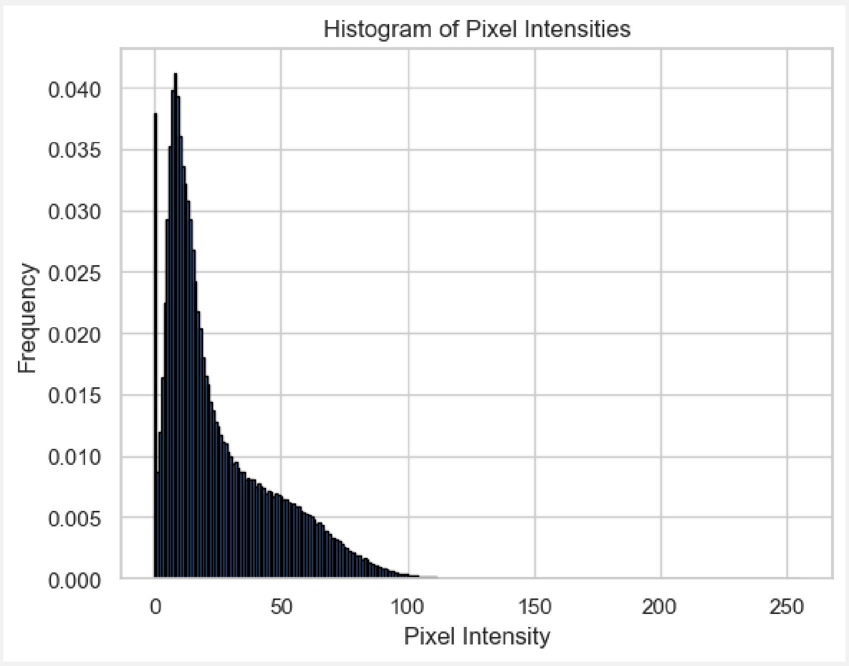


Figure 11: Normalized Histogram of blue channel of the Original Image