

Assignment 3

CS215: Data Structures and Algorithms

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Solutions

SOLUTION 1

Detecting Anomalous Transactions using KDE

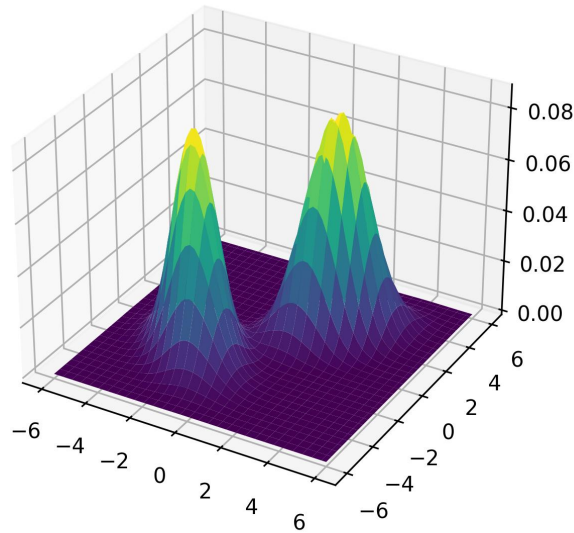


Figure 1.1: Distribution of transactions

As can be seen in the given figure, the resulting estimated distribution contains two modes.

SOLUTION 2

Higher-Order Regression

Part 1

Suppose our estimates for α and β are A and B respectively, then these values of A and B minimize

$$\sum_{i=1}^n (y_i - A - Bx_i)^2 \quad (1.1)$$

$$\Rightarrow \frac{\partial}{\partial A} \sum_{i=1}^n (y_i - A - Bx_i)^2 = 0 \quad (1.2)$$

$$\sum_{i=1}^n -2(y_i - A - Bx_i) = 0 \quad (1.3)$$

$$n\bar{y} - nA - nB\bar{x} = 0 \quad (1.4)$$

$$\bar{y} = A + B\bar{x} \quad (1.5)$$

Least square regression line is given by $y = A + Bx$. Thus by (1.5), (\bar{x}, \bar{y}) lies on the regression line.

Part 2

Suppose our estimates for β_0^* and β_1^* are A^* and B^* respectively, then A^* and B^* minimize $\sum_{i=1}^n (y_i - A^* - B^* z_i)^2$

$$\Rightarrow \frac{\partial}{\partial A^*} \sum_{i=1}^n (y_i - A^* - B^* z_i)^2 = 0 \quad \frac{\partial}{\partial B^*} \sum_{i=1}^n (y_i - A^* - B^* z_i)^2 = 0 \quad (1.6)$$

$$\sum_{i=1}^n -2(y_i - A^* - B^* z_i) = 0 \quad \sum_{i=1}^n -2z_i(y_i - A^* - B^* z_i) = 0 \quad (1.7)$$

$$n\bar{y} - nA^* - nB^*\bar{z} = 0 \quad \sum z_i y_i - A^* n\bar{z} - B^* \sum z_i^2 = 0 \quad (1.8)$$

$$\sum y_i z_i - n(\bar{y} - B^*\bar{z})\bar{z} - B^* \sum z_i^2 = 0 \quad (1.9)$$

$$B^* = \frac{\sum y_i z_i - n\bar{y}\bar{z}}{n\bar{z}^2 - \sum z_i^2} \quad A^* = \bar{y} - B^*\bar{z} \quad (1.10)$$

$$\text{Since, } z_i = x_i - \bar{x}. \bar{z} = \frac{\sum (x_i - \bar{x})}{n} = \frac{n\bar{x} - n\bar{x}}{n} = 0. \sum (x_i - \bar{x})^2 = \sum x_i^2 - n\bar{x}^2$$

$$B^* = \frac{\sum y_i (x_i - \bar{x}) - n\bar{y} \cdot 0}{n(0)^2 - (\sum x_i^2 - n\bar{x}^2)} \quad (1.11)$$

$$= \frac{\sum y_i x_i - n\bar{x}\bar{y}}{n\bar{x}^2 - \sum x_i^2} \quad (1.12)$$

This is same as B , least square estimate of β_1 i.e $B^* = B$. Also since $\bar{z} = 0$, we have $A^* = \bar{y}$ i.e $A^* = A + B\bar{x}$, where A is the least square estimate of β_0 .

Part 3

Let's restrict ourselves to single feature for simplicity. Suppose we have n data points

$$\{(x_1, y_1), \dots, (x_n, y_n)\} \quad (1.13)$$

and our OLS estimates for β_0, \dots, β_m be B_0, \dots, B_m . These must minimize

$$\sum_{i=1}^n (y_i - B_0 - B_1 x_i - \dots - B_m x_i^m)^2 \quad (1.14)$$

Partial differentiation w.r.t each B_i must be 0 which gives

$$\sum_{i=1}^n -2(y_i - B_0 - B_1 x_i - \dots - B_m x_i^m) = 0 \quad (1.15)$$

$$\sum_{i=1}^n -2x_i(y_i - B_0 - B_1 x_i - \dots - B_m x_i^m) = 0 \quad (1.16)$$

$$\vdots \quad (1.17)$$

$$\sum_{i=1}^n -2x_i^m(y_i - B_0 - B_1 x_i - \dots - B_m x_i^m) = 0 \quad (1.18)$$

Which give

$$\sum_{i=1}^n y_i = B_0 n + B_1 \sum_{i=1}^n x_i + \dots + B_m \sum_{i=1}^n x_i^m \quad (1.19)$$

$$\sum_{i=1}^n x_i y_i = B_0 \sum_{i=1}^n x_i + B_1 \sum_{i=1}^n x_i^2 + \dots + B_m \sum_{i=1}^n x_i^{m+1} \quad (1.20)$$

$$\vdots \quad (1.21)$$

$$\sum_{i=1}^n x_i^m y_i = \sum_{i=1}^n B_0 x_i^m + B_1 \sum_{i=1}^n x_i^{m+1} + \dots + B_m \sum_{i=1}^n x_i^{2m} \quad (1.22)$$

$$\text{Taking } X = \begin{bmatrix} 1 & x_1 & \dots & x_1^m \\ 1 & x_2 & \dots & x_2^m \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \dots & x_n^m \end{bmatrix}, Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, B = \begin{bmatrix} B_0 \\ \vdots \\ B_m \end{bmatrix} \text{ we have}$$

$$X^T Y = X^T X B \quad (1.23)$$

$$B = (X^T X)^{-1} X^T Y \quad (1.24)$$

We'll use this in our code.

SOLUTION 3

Non-parametric regression

Report Bandwidth Corresponding to Minimum Estimated Risk

After running the Nadaraya-Watson kernel regression using the Epanechnikov and Gaussian kernel and performing cross-validation for bandwidth selection, the optimal bandwidth corresponding to the minimum estimated risk is:

Optimal Bandwidth of Gaussian kernel: 0.180

Optimal Bandwidth of Gaussian kernel: 0.164

Similarities and Differences Due to Choice of Different Kernel Functions

Similarities

- **General Functionality:** Both kernels assign weights to data points based on their distance from the query point, resulting in similar predictions in regions with high data density.
- **Smoothing:** As the bandwidth increases, all kernel functions produce smoother estimates. At very large bandwidths, all kernels oversmooth the data, giving too much influence to distant points.
- **Cross-validation Behavior:** Both kernels display a similar behavior during cross-validation, and the corresponding risk curves follow the same trend with bandwidth changes.

Differences

- **Shape of the Weights:**
 - **Epanechnikov Kernel:** This kernel assigns zero weight to points farther than the bandwidth due to its quadratic form, creating a more localized effect.
 - **Gaussian Kernel:** This kernel assigns non-zero weight to every point, regardless of distance, due to its exponential decay. It results in smoother estimates, but it is more sensitive to distant points.
- **Sensitivity to Outliers:**
 - **Epanechnikov Kernel:** This kernel is more resilient to outliers because they assign zero or reduced weight to distant points, decreasing the influence of outliers on the prediction.
 - **Gaussian Kernel:** The Gaussian kernel is more prone to incorporating outliers, as it assigns non-zero weights even to far-away points, making it less resilient in the presence of outliers.
- **Plots**
 - **Epanechnikov Kernel:** This kernel produces more precise and localized estimates, with a good balance between bias and variance when using the optimal bandwidth.
 - **Gaussian Kernel:** The Gaussian kernel leads to smoother curves but gives undue influence to distant points, which can result in overfitting or oversmoothing depending on the bandwidth.