Assignment 2: CS215

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Solution 1

Task A

When $X \sim \text{Ber}(p)$, PGF of X is

$$G_{\text{Ber}}(z) = \mathbb{E}(z^X) \tag{1}$$
$$= \sum_{n=0}^{\infty} P[X = n]z^n \tag{2}$$

Since $P[X = 0] = (1 - p)$, $P[X = 1] = p$, $P[X = n] = 0$ when $n > 1$,

$$G_{\text{Ber}}(z) = P[X = 0]z^0 + P[X = 1]z^1 \tag{3}$$
$$= (1 - p) + pz \tag{4}$$

Task B

When $X \sim \text{Bin}(n, p)$, PMF of X is

$$P[X = k] = \binom{n}{k} p^k (1 - p)^{n-k} \text{ for } k \leq n. \tag{5}$$

and $P[X = k] = 0$ for $k > n$.

$$G_{\text{Bin}}(z) = \sum_{k=0}^{\infty} P[X = k]z^k \tag{6}$$
$$= \sum_{k=0}^{n} \binom{n}{k} p^k (1 - p)^{n-k} z^k \tag{7}$$
$$= \sum_{k=0}^{n} \binom{n}{k} (pz)^k (1 - p)^{n-k} \tag{8}$$
$$= (1 - p + pz)^n. \tag{9}$$

By equation 4, $G_{\text{Bin}}(z) = (1 - p + pz)^n = (G_{\text{Ber}}(z))^n$. Hence proved.

Task D

When $X \sim \text{Geo}(p)$, PMF of X,

$$P[X = k] = (1 - p)^{k-1} p \tag{10}$$

for $k > 0$. $P[X = 0] = 0$. Now, PGF of X,

$$G_{\text{Geo}}(z) = \sum_{k=0}^{\infty} P[X = k]z^k \tag{11}$$
$$= \sum_{k=1}^{\infty} P[X = k]z^k \tag{12}$$
$$= \sum_{k=1}^{\infty} p(1 - p)^{k-1} z^k \tag{13}$$
$$= \sum_{k=1}^{\infty} pz(z - zp)^{k-1} \tag{14}$$
$$= pz \sum_{k=0}^{\infty} (z - zp)^k \tag{15}$$
$$= \frac{pz}{1 - z + pz} \tag{16}$$

Task E

By equation 9, $G_{\text{Bin}}(z) = (1 - p + pz)^n = G_X^{(n,p)}(z)$. For $Y \sim \text{NegBin}(n, p)$

$$P[Y = k] = \binom{k-1}{n-1} p^n (1-p)^{k-n} \text{ for } k \geq n \tag{17}$$

Otherwise, $P[Y = k] = 0$. PGF of Y is

$$G_Y^{(n,p)}(z) = \sum_{k=0}^{\infty} P[Y = k] z^k \tag{18}$$

$$= \sum_{k=n}^{\infty} \binom{k-1}{n-1} p^n (1-p)^{k-n} z^k \tag{19}$$

$$= \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} p^n (1-p)^k z^{n+k} \tag{20}$$

$$= (pz)^n \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} (z - pz)^k \tag{21}$$

We know $\sum_{k=0}^{\infty} \binom{k+n-1}{n-1} x^k = (1-x)^{-n}$. Thus

$$G_Y^{(n,p)}(z) = (pz)^n (1 - z + pz)^{-n} \tag{22}$$

$$= \left((1 - p^{-1} + p^{-1} z^{-1})^n \right)^{-1} \tag{23}$$

$$= (G_X^{(n,p^{-1})}(z^{-1}))^{-1}. \tag{24}$$

Hence Proved.

Solution 2

Solution 3

Solution 4

Solution 5