Group Theory

Week 1 Exercises

Topics : Groups, Subgroups, Homomorphisms

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1 Exercise 1

Solution (Q2.4.1). $f(x) = x$ and $g(x) = x$, even though f and g are surjective $f \cdot g$ is not.

Solution (Q2.4.2). Let S be the set of functions from $[0, 1]$ to $[2, 3]$. For a function to be a binary operation on set T, it's domain is defined to be $T \times T$ and it should return a value from T, but when we take two functions from S, their composition might not be in S, it might not even be defined, for example take $f, g : [0, 1] \to [2, 3]$ and $f(x) = x + 2$ then $g(f(x))$ is not even defined, hence composition is not a binary operation.

Solution (Q2.4.3). No, because a binary operation on set S is a function with domain $S \times S$, but here it's not defined $\forall (a, b) \in S$, for eg. (b, a).

Solution (Q3.3.1).

1. Identity is 0, for $x \in \mathbb{Z}$, $-x$ is it's inverse.
2. Identity is the identity matrix I, for an invertible matrix A, inverse is A^{-1}.
3. Identity is $f(x) = x$, inverse is $f^{-1}(x)$.
4. Identity is $\phi(x, y) = (x, y)$, inverse is $\phi^{-1}(x, y)$, for a symmetry $\phi(x, y)$.

Solution (Q3.3.2). Since for a group G, identity $e \in G$ is such that $\forall a \in G$, $a \cdot e = e \cdot a = a$, suppose e and f both are identities, then

$$\begin{aligned} e \cdot f &= f && (1) \\ &= e && (2) \\ \implies e &= f. && (3) \end{aligned}$$

Hence proved that identity is unique.

Solution (Q3.3.3). In a group G, a' is said to be the inverse of a iff $a \cdot a' = a' \cdot a = e$ where e is identity. Suppose we have two identities a_1 and a_2 of a, then

$$\begin{aligned} a_1 \cdot a \cdot a_2 &= (a_1 \cdot a)a_2 && (4) \\ &= e \cdot a_2 && (5) \\ &= a_2 && (6) \\ a_1 \cdot a \cdot a_2 &= a_1 \cdot (a \cdot a_2) && (7) \\ &= a_1 \cdot e && (8) \\ &= a_1 \implies a_1 = a_2. && (9) \end{aligned}$$

Hence proved that inverse of any $a \in G$ is unique.

Solution (Q3.3.4). Since there exists unique inverse a' of a,

$$\begin{aligned} a \cdot b &= a \cdot c && (10) \\ \implies a' \cdot a \cdot b &= a' \cdot a \cdot c && (11) \\ \implies e \cdot b &= e \cdot c && (12) \\ \implies b &= c. && (13) \end{aligned}$$

Hence proved.

Note: In the following solutions, to prove H (a subset of G) can form a subgroup of G, which is a group itself I've just proved H is closed and it contains identity, i.e for $a, b \in H$, $a \cdot b \in H$. This is sufficient since all $a, b \in H$ satisfy the other three conditions required for H to be a group since they are already in G which is a group. First, $a \cdot (b \cdot c) = (a \cdot b) \cdot c, \forall a, b, c \in H$ which is true since a, b, c are also in group G. The other two are true since H does contain the identity.

Solution (Q4.4.1). Since H is a subgroup of G, by definition H has same operation as G, hence $\forall a \in H$, $a \in G$. Since H is a group, it also has an identity, let it be $e' \in H$. Now $\forall a \in H$, $a \cdot e' = e' \cdot a = e'$, but $\forall a \in G$, $a \cdot e = e \cdot a = e$, since they're the same operation, cancellation gives $e = e'$. Hence $e \in H$.

Solution (Q4.4.2). Assuming addition under integers, the set of odd integers isn't a subgroup, it isn't even a group since two odd numbers upon adding don't give an odd number. Whereas the set of even numbers is a subgroup, since it's a group, under the same operation as of integers. The set $\{kn \mid k \in \mathbb{Z}\}$ are all subgroups for each $n \in \mathbb{Z}$ of integers.

Solution (Q4.4.3). Any subgroup H of G containing all the elements of set S must have atleast all the elements of S, since H is closed under it's operation. Thus the smallest possible subgroup of G containing all the elements of S can be S.

Solution (Q4.4.4). $H \cap K$ is a subgroup of G because let $a, b \in H \cap K$, then $a \cdot b$ is also in H, similarly $a \cdot b$ is also in K. Thus $a \cdot b \in H \cap K$. Thus $H \cap K$ forms a group and is a subgroup of G. $H \cup K$ is not a subgroup of G because if $a \in H - K \subseteq H$ and $b \in K - H \subseteq K$ then we can't even guarantee $a \cdot b$ is in $H \cup K$ thus it's not even a group in first place. With similar arguments we can prove $T = \cap_{i \in I} H_i$ is a subgroup of G by taking $a, b \in T$. Since a,b belong to each of H_i, $a \cdot b$ also belongs to each of H_i thus is in T. There for T is a group with same operator as G and thus a subgroup of G.

Solution (Q5.3.1).

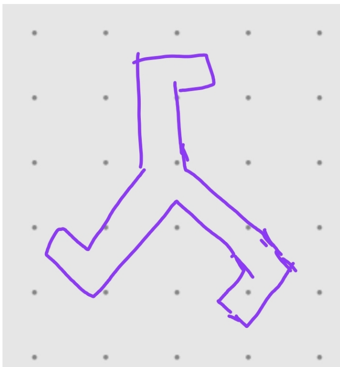

Figure 1: The group of symmetries associated with this are identity, rotation about center by $120°$, by $240°$

Solution (Q6.4.1). Let U and V be groups with operations $\cdot_U$ and $\cdot_V$ respectively and $a \in U$ and a' be it's inverse. Let the function $\phi : U \to V$ be a homomorphism. Then

$$\phi(a \cdot_U a') = \phi(a) \cdot_V \phi(a') \tag{14}$$
$$\phi(e_U) = \phi(a) \cdot_V \phi(a') \tag{15}$$
$$e_V = \phi(a) \cdot_V \phi(a') \tag{16}$$
$$\tag{17}$$

since we know that $\phi(e_U) = e_V$ where e is identity. Similarly we can show that

$$e_V = \phi(a') \cdot_V \phi(a) \tag{18}$$
$$\implies \phi(a) \cdot_V \phi(a') = e_V = \phi(a') \cdot_V \phi(a) \tag{19}$$
$$\implies \phi(a') = (\phi(a))' \tag{20}$$

Thus, homomorphism takes inverse of an element (a) to inverse of it's image.

Solution (Q6.4.2). Let $a, b \in K$. Then $\phi(a) = \phi(b) = e_V$. Also

$$\begin{aligned} \phi(a \cdot_U b) &= \phi(a) \cdot_V \phi(b) && (21)\\ &= e_V && (22)\\ \implies \ a \cdot b \in K & && (23) \end{aligned}$$

Thus, K is a group under same operation as of U thus is a subgroup of U.

Solution (Q6.4.3). Let $a, b \in H$, i.e $\exists a_0, b_0 \in U$ such that $\phi(a_0) = a$ and $\phi(b_0) = b$. Then

$$\begin{aligned} \phi(a_0 \cdot_U b_0) &= \phi(a_0) \cdot_V \phi(b_0) && (24)\\ &= a \cdot_V b && (25)\\ &\implies \exists c_0 \in U (a_0 \cdot_U b_0) \text{ such that } \phi(c_0) = a \cdot_V b && (26)\\ \implies \ a \cdot_V b \in V & && (27) \end{aligned}$$

Thus, K forms a subgroup of V, known as image of ϕ.