Group Theory

Week 2 Exercises

Topic(s) : Cosets, Lagrange's Theorem, p^{th} roots of unity

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1 Solutions

Solution (Q1).

- Let number of elements in G be n.
- Since n is prime, $n \geq 2$, and there exists atleast one element which is not identity (e), let it be u.
- If the order of u is p, the subgroup generated by u is $\{e, u, u^2, \dots, u^{p-1}\}$.
- Clearly size of this subgroup (p) is atleast 2.
- By Lagrange's theorem size of a subgroup divides the size of the group.
- But size of G (n) is a prime, so if p divides n and $p \geq 2$ then $p = n$.
- G has an element which generates a subgroup covering entirety of G, hence G is cyclic.
- Thus, any group with prime number of elements is cyclic.

Solution (Q2).

- Yes, consider G which is infinite and H which contains just e, the identity, which in itself forms a subgroup.
- Then G has infinite number of cosets $\{\{a\} | a \in G\}$.
- Another example is, consider $\mathbb{R}^*$[1] under multiplication as G and H as $\{-1, 1\}$ then each $a \in G$, $a > 0$ forms the left coset $aH = \{-a, a\}$ which is unique.
- Thus there are infinite left cosets of G with respect to finite H.

[1] $\mathbb{R}^*$ is just $\mathbb{R} - \{0\}$