ASTE 586 Computer Project Part I | Andrew Gerth

diff eq's:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 200 \\ 0 & 0 & -200 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

initial cond. $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

given in form $\frac{d}{dt} \vec{x}(t) = [A] \vec{x}(t)$

use matrix eponential: $\vec{x}(t) = e^{At} \vec{x}(0)$

find $e^{At}$ by finding eig vals of A. $\det(A - \lambda I) = 0$

$$\det \begin{bmatrix} -\lambda & 2 & 0 & 0 \\ -2 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 200 \\ 0 & 0 & -200 & -\lambda \end{bmatrix} = 0 \Rightarrow \cancel{\det = -2 \begin{vmatrix} 2 & 0 & 0 \\ 0 & 0 & 200 \\ 0 & -200 & 0 \end{vmatrix}}$$

$$\cancel{= -2\left[2\left(+40000\right)\right] =}$$

$$\Rightarrow -\lambda \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 200 \\ 0 & -200 & -\lambda \end{vmatrix} + 2 \begin{vmatrix} 2 & 0 & 0 \\ 0 & -\lambda & 200 \\ 0 & -200 & -\lambda \end{vmatrix}$$

$$-\lambda\left[-\lambda\left[\lambda^2 + 40000\right]\right] + 2\left[2\left[\lambda^2 + 40000\right]\right] = \lambda^4 + 40000\lambda^2 + 4\lambda^2 + \cancel{4}\,\cancel{8}0000 = 0$$

$$\lambda^4 + 40004\lambda^2 + \cancel{4}16 0000 = 0$$

$\lambda_{1,2} = \pm(2i)$

$\lambda_{3,4} = \pm(200i)$

for $\lambda_1 = 2i$ solve $(A - 2iI)\vec{x} = \vec{0} \Rightarrow \begin{bmatrix} -2i & 2 & 0 & 0 \\ -2 & -2i & 0 & 0 \\ 0 & 0 & -2i & 200 \\ 0 & 0 & -200 & -2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$-2ix_1 + 2x_2 = 0 \Rightarrow x_2 = ix_1$ choose $x_1 = 1$, $x_3 = 0$

$-2ix_3 + 200x_4 = 0 \Rightarrow x_4 = \frac{ix_3}{100}$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ i \\ 0 \\ 0 \end{bmatrix}$$

for $\lambda_2 = -2i$ $\quad (A+2iI)\vec{v}_2 = \vec{0}$

$$\begin{bmatrix} 2i & 2 & 0 & 0 \\ -2 & 2i & 0 & 0 \\ 0 & 0 & 2i & 200 \\ 0 & 0 & -200 & 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_2 = -ix_1$

$x_4 = -\frac{ix_3}{100}$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -i \\ 0 \\ 0 \end{bmatrix}$$

for $\lambda_3 = +200i$:

$$\begin{bmatrix} -200i & 2 & 0 & 0 \\ -2 & -200i & 0 & 0 \\ 0 & 0 & -200i & 200 \\ 0 & 0 & -200 & -200i \end{bmatrix}$$

$$\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ i \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -i \end{bmatrix}$$

General Sol'n:

$$x(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + c_3 e^{\lambda_3 t}\vec{v}_3 + c_4 e^{\lambda_4 t}\vec{v}_4$$

Sub $\vec{\lambda}$ & $\vec{v}$

$$x(t) = c_1 e^{2it}\begin{bmatrix} 1 \\ i \\ 0 \\ 0 \end{bmatrix} + c_2 e^{-2it}\begin{bmatrix} 1 \\ -i \\ 0 \\ 0 \end{bmatrix} + c_3 e^{200it}\begin{bmatrix} 0 \\ 0 \\ 1 \\ i \end{bmatrix} + c_4 e^{-200it}\begin{bmatrix} 0 \\ 0 \\ 1 \\ -i \end{bmatrix}$$

$$\vec{x}(t) = \begin{bmatrix} c_1 e^{2it} \\ ic_1 e^{2it} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} c_2 e^{-2it} \\ -ic_2 e^{-2it} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ c_3 e^{200it} \\ ic_3 e^{200it} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ c_4 e^{-200it} \\ -ic_4 e^{-200it} \end{bmatrix}$$

$$\vec{x}(t) = \begin{bmatrix} c_1 e^{2it} + c_2 e^{-2it} \\ ic_1 e^{2it} - ic_2 e^{-2it} \\ c_3 e^{200it} + c_4 e^{-200it} \\ ic_3 e^{200it} - ic_4 e^{-200it} \end{bmatrix}$$

use euler's formula $\quad e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$

$x_1(t) = c_1[\cos(2t) + i\sin(2t)] + c_2[\cos(-2t) + i\sin(-2t)]$

$x_2(t) = ic_1[\cos(2t) + i\sin(2t)] - ic_2[\cos(-2t) + i\sin(-2t)]$

$x_1(t) = c_1\cos(2t) + c_1 i\sin(2t) + c_2\cos(2t) - c_2 i\sin(2t)$

$\quad = \cos(2t)[c_1 + c_2] + i\sin(2t)[c_1 - c_2]$

call $c_1 + c_2 \equiv A_1$ & $i[c_1 - c_2] \equiv A_2$

$x_2(t) = ic_1\cos(2t) + i^2 c_1\sin(2t) - ic_2\cos(2t) - i^2 c_2\sin(2t)$

$x_2(t) = \cos(2t)[ic_1 - ic_2] - \sin(2t)[c_1 + c_2]$

* same $A_1$ & $A_2$ $\longrightarrow$ continue on next page.

In terms of $A_1$ & $A_2$

$x_1(t) = A_1\cos(2t) + A_2\sin(2t)$ ... same pattern for $\lambda_3$ & $\lambda_4$

$x_2(t) = -A_1\sin(2t) + A_2\cos(2t)$

$x_3(t) = A_3\cos(200t) + A_4\sin(200t)$

$x_4(t) = -A_3(\sin(200t)) + A_4\cos(200t)$

bring back initial cond. $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$t = 0$

$1 = A_1(1) + 0$
$0 = A_2(1)$
$1 = A_3(1) + 0$
$0 = A_4(1)$

$\rightarrow \vec{A} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

now plug back into $\vec{x}(t)$

$$\boxed{\vec{x}(t) = \begin{bmatrix} \cos(2t) \\ -\sin(2t) \\ \cos(200t) \\ -\sin(200t) \end{bmatrix}}$$