ASTE 586 Computer Project, Part 2

Torque-free Motion of a Rigid Body

1 Implementation

Implement a generic simulation of Euler's equations, including computation of the Euler-Rodrigues parameters (quaternions) describing the current orientation.

- Comment your code with descriptions of the variables – be accurate!
- The source code must appear in your report.

Be aware that this is a stepping stone to Part 3, where this code will be applied to non-axisymmetric rigid bodies. ★ *Avoid making the axisymmetric assumption in the software of your simulation or in post-processing your simulation.* We are only using the axisymmetric assumption so that we may compare these test cases to analytical solutions.

Do compute the Euler angles (precession, nutation, spin) using the Euler-Rodrigues parameters. Do *not* compute the Euler angles as integrals of rates that you compute based on the angular velocity. The Euler-Rodrigues parameters must be numerically integrated. The precession, nutation, and spin angles must be computed directly from simulation results.

Let the integrator report results at *all* the time nodes; do not specify them. Specify only the initial time and the final time. If you specify the time nodes, you are likely to specify them too far apart. This will create weird artifacts in your plots that may be difficult to diagnose.

1.1 Euler-Rodrigues Parameters

$$\frac{d}{dt}\begin{bmatrix}\epsilon_1(t)\\ \epsilon_2(t)\\ \epsilon_3(t)\\ \epsilon_4(t)\end{bmatrix} = \mathbf{M}\begin{bmatrix}\omega_1(t)\\ \omega_2(t)\\ \omega_3(t)\end{bmatrix} \tag{1}$$

The matrix $\mathbf{M}$ is in the notes and comes from composing the current Euler-Rodrigues parameters with a small rotation represented by $\vec{\omega}$.

1.2 Euler's Equations

$$\begin{aligned} 0 &= I_1\alpha_1 + \omega_2\omega_3\left(I_3 - I_2\right) \\ 0 &= I_2\alpha_2 + \omega_1\omega_3\left(I_1 - I_3\right) \\ 0 &= I_3\alpha_3 + \omega_1\omega_2\left(I_2 - I_1\right) \end{aligned} \tag{2}$$

Recall that $\alpha_i \equiv \frac{d}{dt}\omega_i$ and that you're looking for a set of differential equations for $\vec{\omega}(t)$

2 Test Cases

Devise your own test cases. Validate your simulation with *two* test cases of your own construction. My only stipulations are:

1. The first case must be one of *direct precession.* Simulate the axis of symmetry as $\vec{e}_2^{\,C}$ so that $I_3 = I_1$ and use

$$\begin{aligned} \dot{\psi} &= 1 \text{ rev/min} \\ 0^\circ &< \theta < 90^\circ \end{aligned}$$

   You may devise your own inputs to achieve this or you may use the example values from Section 3.1

2. The second case must be one of *retrograde precession.* Choose your moments of inertia so that the axis of symmetry is $\vec{e}_1^{\,C}$. Set the nutation $\theta = 45^\circ$. You may devise your own inputs to achieve this or you may use the example values from Section 3.2

Your simulation's duration must span at least two periods of precession and spin. Plots of precession and spin angles must each show at least two periods of oscillation. Your test cases must be validated. This validation includes *at least* the following:

1. Plot the difference between analytical and numerical solutions of $\omega_i = \vec{e}_i^{\,C}(t) \cdot {}^F\vec{\omega}^C(t)$ in rad/sec. Plot this difference for each of the three components as *numerical minus analytical.* There will be a small

error due to the numerical integration. [1] If your simulation does not pass this check, you very likely have a mistake in your coding of the analytical solution or your coding of Equation 2.

2. Plot $\left(\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2 - 1\right)$ versus time, where ϵ_i are your *simulated* Euler-Rodrigues parameters. As before, there will be a small error, but it will be fairly obvious if there's a problem. If your simulation does not pass this check, you very likely have a mistake in your coding of Equation 1.

3. Plot $h_i(t) - h_i(0)$ where $h_i(t) = \left({}^F_{B^*}\vec{H}^B(t) \cdot \vec{e}_i^{\,F}(t)\right)$ and $i = 1, 2, 3$ versus time, in $\mathrm{kg\,m^2/sec}$. Each of these quantities must be computed from your simulation results. These plots show how the components of the angular momentum are changing, as expressed in the inertial coordinate system. As the angular momentum must be constant in the inertial coordinate system, these plots should show small values. Report the values you expected for $h_i(0)$. Did you get what you expected? Why or why not? If your simulation does not pass this check, you very likely have a mistake in your use of the quaternion (Euler-Rodrigues parameters) to rotate the angular momentum from the body coordinates to the inertial coordinates.

4. Plot the nutation angle vs time in degrees versus time. What is your expected nutation angle? Does the simulation match your expected value? The simulation's nutation angle must be computed *directly from the simulation results.* See also the supplement "Computing Precession, Nutation, and Spin".

5. Include plots of the precession and spin angles, in degrees, versus time. Again, this must be computed *directly from the simulation results.* The precession angle should grow at a constant rate. What is the expected precession rate? Does your precession rate, computed directly from the *simulation*, match what you expected it to be? See also the supplement "Computing Precession, Nutation, and Spin".

Plots should be appropriately labeled. Reported numbers and plots should include units. You should include some discussion of your results

[1] One way to observe this error is to run a few cases with various integration tolerances and compare this plot amongst the cases. The plotted error should get smaller as you reduce the integration tolerance and larger as you increase it. There is a floor, a minimum error, and the error will likely even get worse if you reduce the integration tolerance too much.

in your report. You should make sure to space your time nodes closely enough that plots are reasonably smooth. If not, your plots, at best, may appear rather jagged or, at worst, show patterns that aren't real.

★ *Take note that this validation is critical to the project.* Your benefit from this entire project relies on having the correct differential equations, proper initial conditions, and proper computation of the nutation, precession, and spin angles.

3 Help on Test Cases

I strongly recommend that you determine your own initial conditions. However, you will not be penalized for using the ones provided here.

3.1 Direct Precession

For test case 2.1, you may use the following values:

```
I1 = 5 % kg m^2
I2 = 4 % kg m^2
I3 = 5 % kg m^2
w0 = [ 0.0181844   0.128911  0 ] % rad/sec
q0 = [ 0  0  0.0871557 0.996195 ]
```

If you compute the angular momentum in the body-fixed coordinate system and transform it to the inertial system defined by `q0`, then you will obtain a vector along $\vec{e}_2^{\,F}$. However, because I've only given `q0` to six significant figures, there will be a small error.

3.2 Retrograde Precession

For test case 2.2, you may use the following values:

```
I1 = 5 % kg m^2
I2 = 4 % kg m^2
I3 = 4 % kg m^2
w0 = [ 0.0592384  0  0.0740480 ] % rad/sec
q0 = [ 0  0.382683  0  0.923880 ]
```

As in the other set of values, `q0` has been rounded to six significant figures and that will introduce a small error.

4 Table of Results for Test Cases

For clarity, fill out the table below, or a reasonable facsimile, with "YES" or "NO" in each box to indicate whether the tests have passed. Fill in each blank with the requested value.

Criteria		Test Case 1	Test Case 2
(0a)	Precession spans more than 720°	☐	☐
(0b)	Spin spans more than 720°	☐	☐
	What duration, in sec, was used?	____	____
(1)	Each $(\vec{e}_i^{\,C}(t) \cdot {}^F\vec{\omega}^C(t))$ matches analytical within 10^{-10}	☐	☐
	What was the worst error?	____	____
(2)	$\left\lvert \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2 - 1 \right\rvert < 10^{-6}$	☐	☐
	What was the worst error?	____	____
(3)	Each $\lvert h_i(t) - h_i(0) \rvert < 10^{-6}$	☐	☐
	What was the worst error?	____	____
(4)	Nutation angle is as expected	☐	☐
	What was the nutation angle value?	____	____
(5a)	Precession rate is constant	☐	☐
(5b)	Precession rate is as expected	☐	☐
	What was the precession rate value?	____	____
	What was the spin rate value?	____	____

Your angular momentum error may be limited by certain numerical issues such as machine precision or subtraction of small numbers, but you should be able to achieve 10^{-6}.