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Inevitability on Either End of Infinity

Things happen. In life, this is a rare certainty. And mathematically, it is probably normal – or rather, is a requirement for a probability density function. A density function models the likelihood that a random variable falls within a particular range, given by the value of the integral taken over that range. Summing the entire range must equal one. And this makes sense, the likelihood that our random variable takes on *any* value *must* be certain – if we are considering *every* possible outcome, *something* happens.

For a continuous, normally distributed random variable, that density function is some parameterization of $f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$. Functionally, this integrand is fundamentally difficult to evaluate. There is not a straightforward antiderivative. But we know the improper integral taken over its entire domain must equal one, since it is defined as a probability density. By adding a dimension to this function, we can consider the entire Cartesian plane and find an indirect approach to evaluate this improper integral and show

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\,\pi}} e^{-\frac{x^2}{2}}\, dx = 1$$

First, let's examine this function and dimensional shift graphically[1].

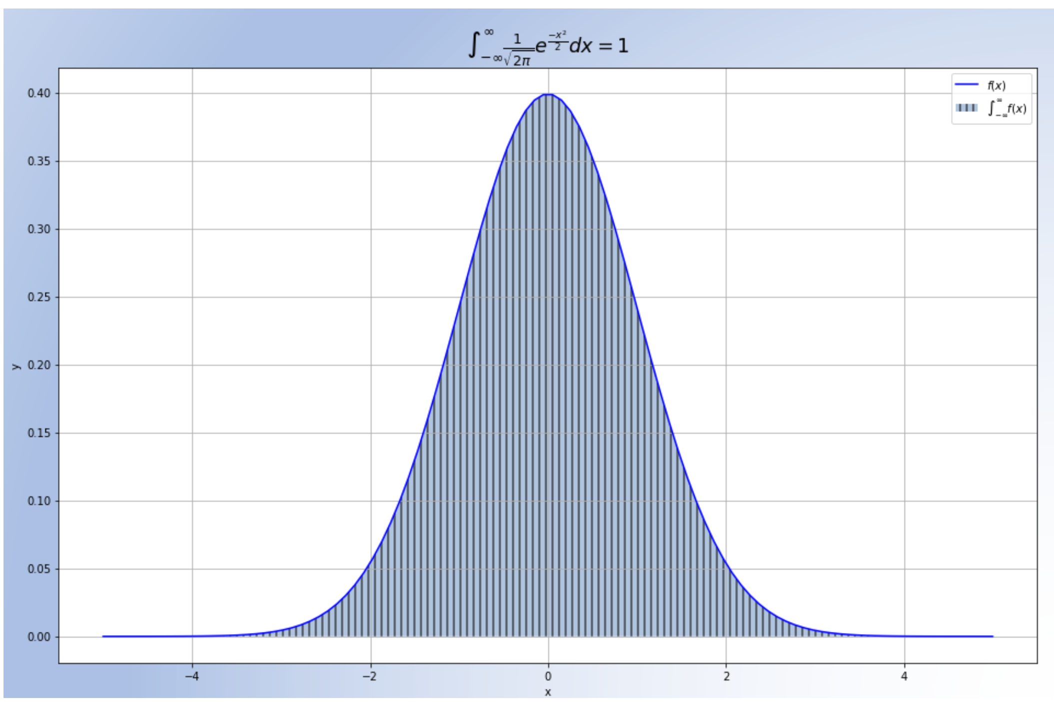

Figure 1: PDF in Two Dimensions

[1] Links to a Google Colab notebook with code for the graphics and experiments with numerical integration. While there is no obvious antiderivative we can use to integrate $f(x)$, we *can* get reasonable results from numerically integrating our function over a much smaller range – but that's for another paper, perhaps.

Figure 1 suggests that our normally distributed variable *likely* falls between -2 and 2, and even more probably between -4 and 4, as the graph quickly approaches zero beyond those bounds. Using its symmetry about the y-axis, we could get a reasonable approximation of the integral by doubling the area of the trapezoid defined by the x-axis and the curve $f(x)$ $\{0 \leq \mathrm{x} \leq 2\}$ (estimating values graphically):

$$\frac{0.05 + 0.4}{2} 2 = 0.45 * 2 = 0.9$$

By adding more trapezoids, we could improve our approximation and see the value of the integral quickly approaches 1. With an infinite number of trapezoids, this should converge to one – a classical calculus approach - but leaves us right where we began, trying to evaluate a clunky improper integral.[2] Since our goal is to confirm that the value of this integral *over the entire range* is equal to one, let's instead consider a similar function, $g(x, y) = e^{-x^2 - y^2}$ over all of R^2.

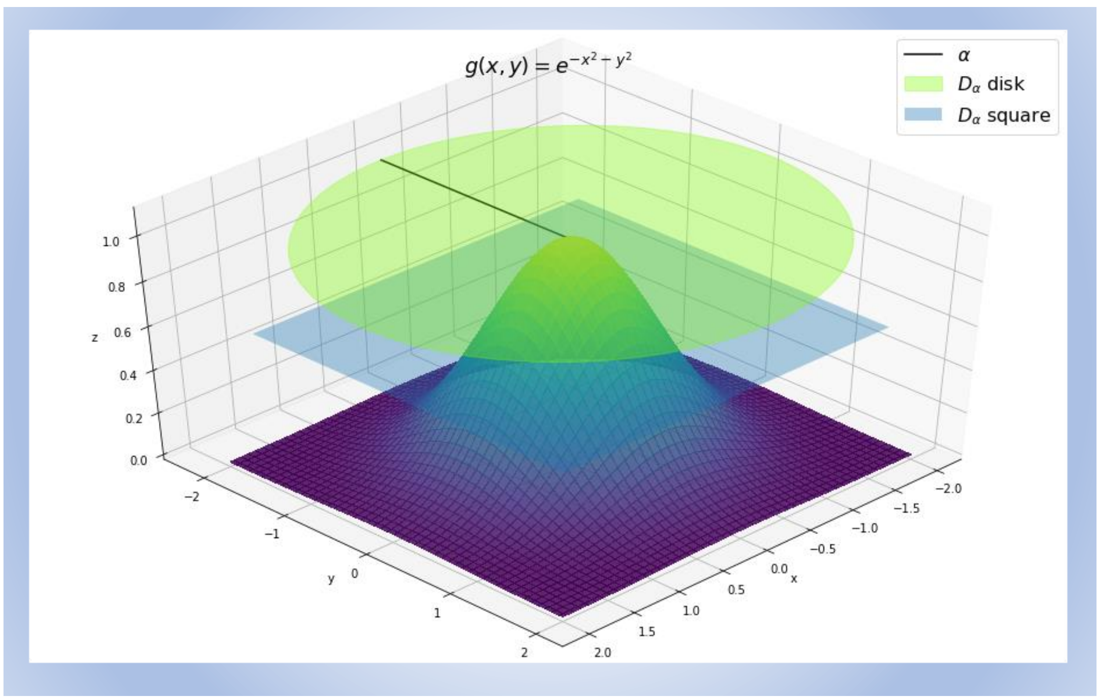

Figure 2: PDF in Three Dimensions

As illustrated by the overlays in Figure 2 on the surface plot of $g(x, y)$, we can use different geometries to cover the entire Cartesian plane. By transforming $g(x, y)$ into polar coordinates, we can simplify the integral. And from the graph, we can see a relationship between the area of the disk with radius α to area of the square with vertices at $(\pm\alpha, \pm\alpha)$ as α approaches ∞. We can leverage this symmetry to compute our desired integral.

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[2] Although, with modern computing technology, these calculations can be made much more efficiently, making a numerical approximation to a high degree of accuracy much more manageable. Further still, since this path has already been tread by many a bright mathematician, a special function exists to handle this complicated integral, the Gauss error function, but that, too, must wait for another paper.

Computation of Improper Integral Probability Density Function

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx = 1$$

First, consider $g(x,y) = e^{-x^2-y^2}$

We want to integrate $g(x,y)$ over all of R^2

$$\iint_{R^2} g(x,y)\ dA$$

Using the geometry shown in Figure 2, we simplify our integrand by switching to polar coordinates

$$g(r,\theta) = e^{-(x^2+y^2)} = e^{-r^2}$$

$$\lim_{\alpha\to\infty} \int_0^{2\pi} \int_0^{\alpha} g(r,\theta)\, dr d\theta = \int_0^{2\pi} \int_0^{\alpha} r\, e^{-r^2}\, dr d\theta$$

Then perform u-substitution, letting $u = -r^2,\ du = -2dr$, cancelling the extra r term from the polar transformation

$$\int_0^{2\pi} -\frac{1}{2} \int_0^{-\alpha} e^u du\, d\theta$$

Starting with respect to θ, we can compute iterated integral,

$$\frac{1}{2} \int_{-\alpha}^{0} 2\,\pi\, e^u \mathrm{du} = \pi \int_{-\alpha}^{0} e^u du$$

undoing the u-substitution and evaluating the integral gives

$$\pi\left[e^{-r^2}\right]_{-\alpha}^{0}$$

then taking the limit as $\alpha \to \infty$ yields a solution.

$$\lim_{\alpha\to\infty} \pi\left(e^{-0^2} - e^{-\alpha^2}\right) = \pi(1-0) = \pi$$

In the spirit of taking limits, we can also integrate $g(x,y)$ over the entire Cartesian plane using squares,

$$\lim_{\alpha\to\infty} \int_0^{2\pi} \int_0^{\alpha} g(r,\theta) dr d\theta = \lim_{\alpha\to\infty} \int_{-\alpha}^{\alpha} \int_{-\alpha}^{\alpha} g(x,y)\ dx\, dy$$

then from symmetry and Fubini's theorem, we can rearrange our integrand as

$$\int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha} e^{-(x^2+y^2)}\ dx\, dy = \int_{-\alpha}^{\alpha} e^{-x^2}\ dx \int_{-\alpha}^{\alpha} e^{-y^2}\ dy = \left(\int_{-\alpha}^{\alpha} e^{-x^2}\ dx\right)^2$$

from above, we know this integral evaluates to π, so we can extend that

$$\lim_{\alpha\to\infty}\left(\int_{-\alpha}^{\alpha} e^{-x^2}\ dx\right)^2 = \pi^2$$

and thus

$$\lim_{\alpha\to\infty}\int_{-\alpha}^{\alpha} e^{-x^2}\ dx = \sqrt{\pi}$$

To bring it all together, we can perform a change of variables: let $v = \frac{x}{\sqrt{2}}, dv = \frac{dx}{\sqrt{2}}$, then

$$\lim_{\alpha\to\infty}\int_{-\alpha}^{\alpha}\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}dx = \lim_{\alpha\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-\alpha}^{\alpha}\sqrt{2}\,e^{-v^2}dv$$

this improper integral of $e^{-v^2}dv$ evaluates to $\sqrt{\pi}$, as shown above, finishing our computation.

$$\frac{1}{\sqrt{2\pi}}\left(\sqrt{2\pi}\right) = 1$$

Conclusion:

It makes sense, in the probabilistic context, that considering an infinite range of possible outcomes, *something* is bound to happen. The geometry of that infinite landscape, though, is a bit more perplexing. Our intuition balks when confronted with the infinite. A circle with an infinitely large radius seems cumbersome at best, a square with vertices at opposing ends of infinity, even more unwieldy. Calculus, however, is quite comfortable there; it is built on infinitesimals, it roams the frontiers of infinity. By recognizing the symmetry in our unbounded shapes - and using some clever coordinate transformations - calculus gives us the leverage to wrangle the incomprehensible, to wrestle with the infinite and find some certainty on the othe r side.