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Math 220 – Fall 2021

Applied Problem 1: Network Flow and Systems of Equations

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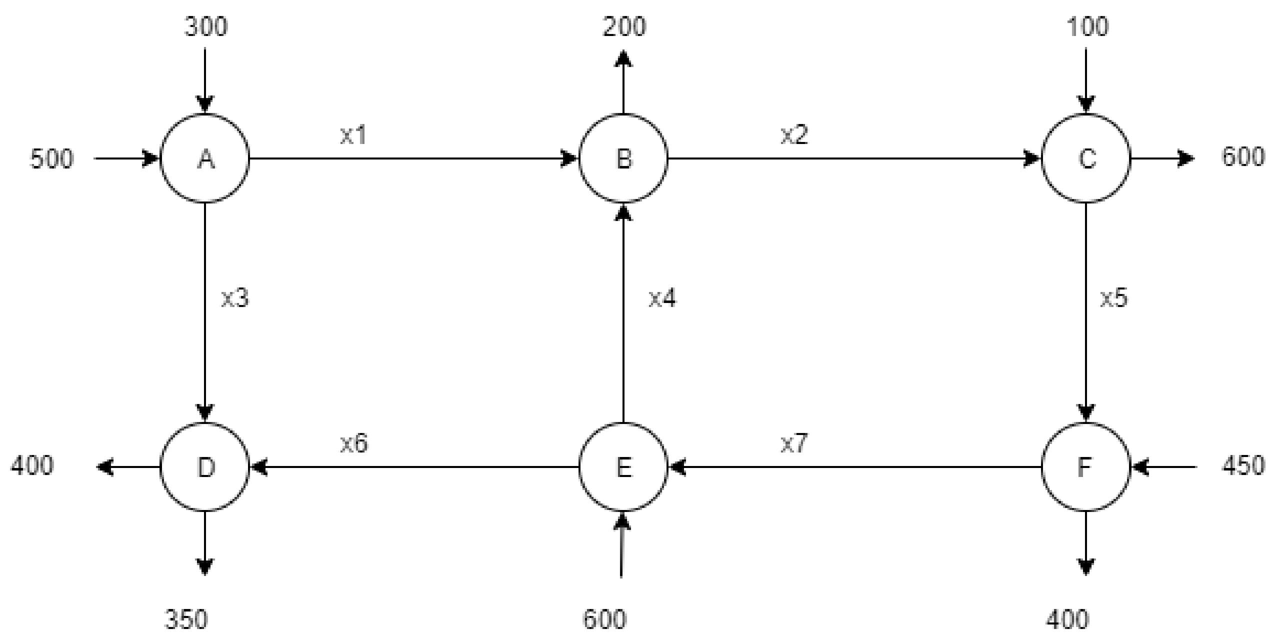
Consider a small section of a city's road system (Figure 1) Each node, labeled below with a letter, represents an intersection and each directed edge (x_n) represents one-way traffic flow along the road between the intersections it connects. Assume all traffic entering each intersection leaves that intersection (traffic does not accumulate at any node, nor stop along any of the edges). Generate a system of equations to describe the traffic flow through this network, then consider how the system responds to the closure of x_1.

Figure 1: Traffic Flow



Noting that flow into must equal the flow leaving each node, this traffic network can be described by the following system of equations:

$$x_1 + x_3 = 800$$

$$x_1 - x_2 + x_4 = 200$$

$$x_2 - x_5 = 500$$

$$x_3 + x_6 = 750$$

$$x_4 + x_6 - x_7 = 600$$

$$x_5 - x_7 = -50$$

Solution:

$$x_1 - x_6 = 50$$

$$x_2 - x_7 = 450$$

$$x_3 + x_6 = 750$$

$$x_4 + x_6 - x_7 = 600$$

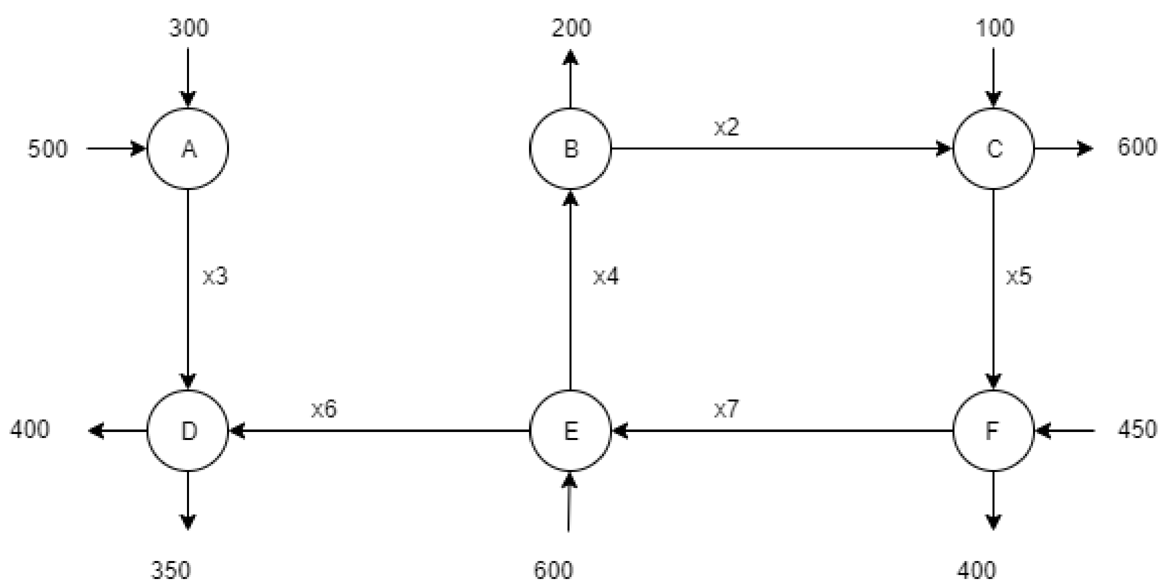
$$x_5 - x_7 = -50$$

$$x_6 \text{ and } x_7 \text{ free}; x_7 \geq 50$$

Solving our system of equations (see Matrix A and its row reduced form below), columns one through five have pivots, so x_1 through x_5 are the basic variables of the system, while x_6 and x_7 are free variables. So, our traffic network has many viable solutions, so long as $x_7 \geq 50$, to prevent negative flow along x_5.

This makes sense for actual road networks, drivers have choices at each intersection, different drivers will take different routes depending on their destination. The average vehicles per hour could maneuver through the network in a wide variety of configurations – so long as at least fifty vehicles per hour move from node F to E along x_7. Looking at our network, traveling along x_7 allows a driver to reach every other intersection, except for node A, giving cars passing along this road the option to loop back around the network (E, then return to B, C, F with the option to exit at any) or continue to D (and exit the system).

Figure 2 Traffic Flow Without x_1



Without x_1 (see Figure 2), the traffic network stops functioning, $x_6 = -50$, this model requires positive traffic along the one-way roads. However, even if the model collapses mathematically, the real world needs to find a way to continue operating - smoothly, if possible. The *way* this model fails is informative, with x_1 closed, all the traffic into A (800) can only exit through Node D, which only has capacity* for 750 cars to exit, so traffic will need to find another way out of the system – since x_7 is free, traffic could be rerouted along this route (maybe with some flaggers to manage two-way traffic along the one-way road).

*Realistically, the constraints on the system are more to aid in the modeling than actual limitations of network. For traffic, the average vehicles per hour are *not* requirements or restrictions. The fifty extra vehicles needing to exit Node D's southern intersection amount to less than one extra vehicle each minute, which in practice should not drastically alter the traffic flow of the system. Practical interpretation of the model is as valuable as the model's accuracy.

Data:

Table 1 Matrices Used

Matrix A

Node	x_1	x_2	x_3	x_4	x_5	x_6	x_7	c
A	1	0	1	0	0	0	0	800
B	1	-1	0	1	0	0	0	200
C	0	1	0	0	-1	0	0	500
D	0	0	1	0	0	1	0	750
E	0	0	0	1	0	1	-1	600
F	0	0	0	0	1	0	-1	-50

no x_1

Node	x_2	x_3	x_4	x_5	x_6	x_7	c
A	0	1	0	0	0	0	800
B	-1	0	1	0	0	0	200
C	1	0	0	-1	0	0	500
D	0	1	0	0	1	0	750
E	0	0	1	0	1	-1	600
F	0	0	0	1	0	-1	-50

solution

rref	x_1	x_2	x_3	x_4	x_5	x_6	x_7	c
A	1	0	0	0	0	-1	0	50
B	0	1	0	0	0	0	-1	450
C	0	0	1	0	0	1	0	750
D	0	0	0	1	0	1	-1	600
E	0	0	0	0	1	0	-1	-50
F	0	0	0	0	0	0	0	0

solution

rref	x_2	x_3	x_4	x_5	x_6	x_7	c
A	1	0	0	0	0	-1	450
B	0	1	0	0	0	0	800
C	0	0	1	0	0	-1	650
D	0	0	0	1	0	-1	-50
E	0	0	0	0	1	0	-50
F	0	0	0	0	0	0	0

Commentary:

As mentioned above, the constraints are more for ease of analysis than they are true limitations of the system. The flow rates are based on average vehicles per hour – just because the average flow in or out of the network happens to be 1950, it's not a hard requirement. Perhaps certain branches could be narrower city road (or under construction) thus limiting their capacity for cars, but these numbers used in the above matrices are more for simplifying the analysis than determining the actual flow capacity of this section of city streets.

Compared to say, a similar setup for an electrical circuit, or plumbing flow, where exceeding any of those capacities along one of the edges *would* cause some failure of the network – in the form of a burst pipe or wires catching fire. Even if our streets become overloaded from the capacities predicted by this model, practically that means traffic slows down. Its useful to know how to mitigate these to prevent further disruption throughout the larger network of roads (a small slowdown in one neighborhood is normal, traffic clogging the entire city becomes more disruptive).

We could extend this analysis to consider larger and larger portions of the entire city network of streets, seeing how local slowdowns effect other parts of the system. This would require much larger and more complex matrices but would also make for a more practically useful model. Or, instead of using average vehicles per hour for node and edge, we could make more complex hourly models, which would more accurately illustrate how, and of particular importance to all things traffic, *when* cars are moving through this system. The average daytime hourly rates are informative but driving through a city between 6:00 – 9:00 AM is a much different landscape (generally) than 11:00 – 2:00 PM, weekend day times, will be rather different than weekday traffic.

Solving these much more complex matrices by hand with a calculator quickly becomes unwieldy. But, applying these same principles (and a few more robust properties of matrices beyond their usefulness in solving systems of equations) this rather complex problem can be made more manageable programmatically.

The larger challenges, particularly where traffic is involved, is the human element. Not all drivers entering a section of city roads will leave, at least, not right away – perhaps they live in that neighborhood - eventually, though, any car will ultimately leave (unless one of the destinations along the route is a junkyard – and eventually that car, too, will be distributed elsewhere), but the 'when' gets increasingly more complicated to keep track of, and timing is everything for the flow of traffic networks. Incorporating real time traffic data to continuously refine our traffic network matrices would generate a very robust, and presumably rather valuable, model – particularly if it were designed to be flexible enough to handle *any* given traffic network.