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Math 254 – Spring 2022

Writing Project 1: Economic Modeling

Mathematical models offer glimpses into the unknowable – forecasting the future or providing insights into the astonishingly complex phenomena surrounding us. And such models are only becoming more valuable. The global machine learning market is expected to grow from $21.17 billion dollars to nearly $210 billion before the close of this decade.[1] Mathematics and technology have become inextricably entangled; as computational power continues to rapidly increase, the complexity and accuracy of these models can proportionally improve, too.

However, it is important to recognize that even the most intricate and advanced models have inherent limitations. Reality is far too imperfect – too chaotic, too noisy – to be modelled completely. To quote brilliant Hungarian mathematician John von Neumann, "truth…is much too complicated to allow anything but approximations."[2] How we apply these approximations is what matters.

This report will explore economic modeling using the Cobb-Douglas production function,

$$P = bL^{\alpha}K^{1-\alpha},$$

where P is total value of annual production, L is the labor input, K is the input of capital, and b, α are determined constants relating to productivity and output elasticity, respectively.

First formulated around the 1920s, this function relates potential production output to two or more inputs. In this case, the function will be used to model the value of total annual production based on the investments in labor and capital for a given year.

Output is limited by the costs of capital and labor. Applying this constraint to the Cobb-Douglas function, the resulting model can suggest how to allocate that budget to maximize annual production. This constraint can be represented by $mL + nK = p$ where: m is the cost of one unit of labor, n is the unit cost of capital, and p is the total budget.

Solving this constrained optimization problem symbolically yields a flexible model that can determine maximum production levels based on various input parameters. The ability to forecast several economic scenarios makes this simple model much more powerful.

Performing this constrained optimization on our model (symbolic derivation follows below), production will be maximized when:

$$K = \frac{p(1-\alpha)}{n}\ ;\ L = \frac{\alpha p}{m}$$

First, let us apply some arbitrary but reasonable constants to the production function to consider the impact of raising the unit cost of labor. Assuming values of ¼ for α, 100 for b, a budget of $850,000 and a unit cost for capital of $34 - when one unit of labor costs $22, our model predicts an annual

[1] Fortune Business Insights

[2] *The Mathematician*, University of Chicago Press, 1947

production value of $1.59 million. When unit labor cost increases to $28, that value drops 5.85% to $1.50 million, a loss of nearly $93,000.

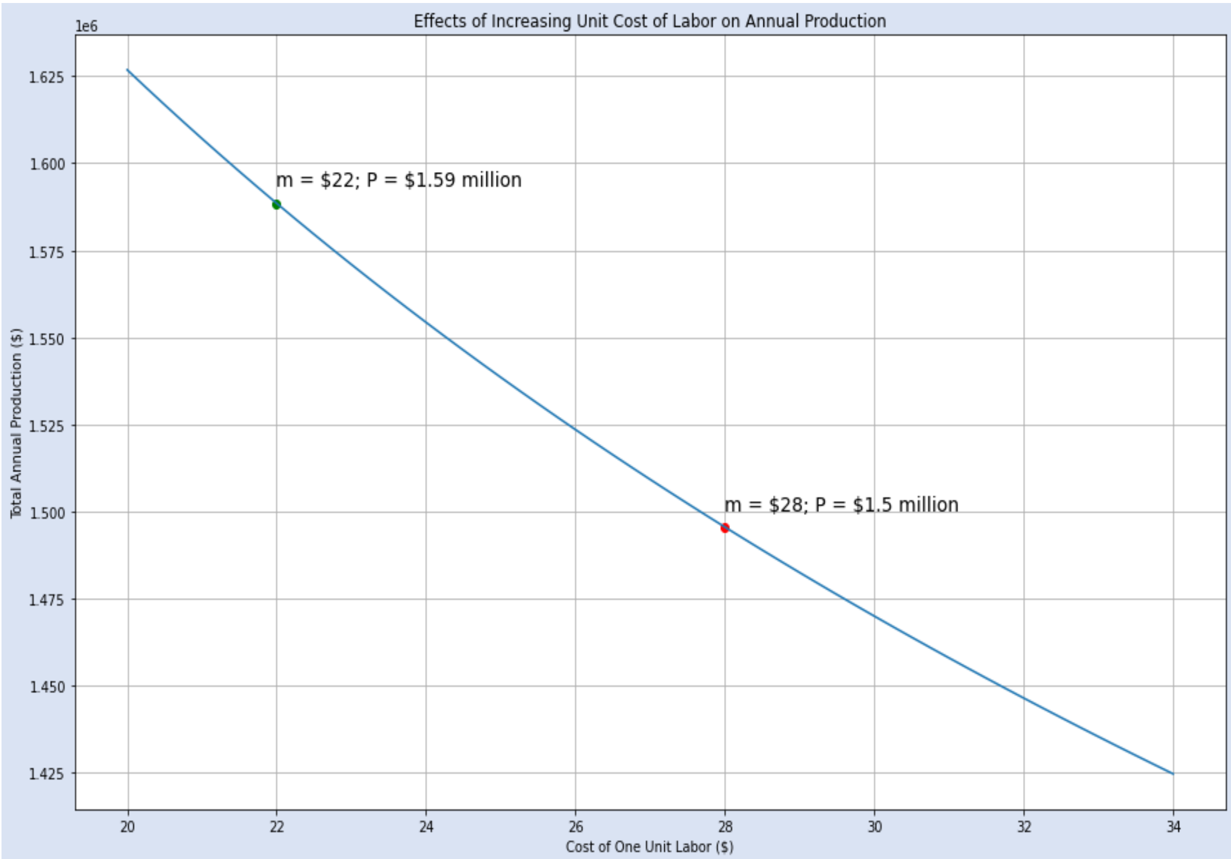

Figure 1: Overall Production as Labor Cost Increases with other Parameters Fixed

As seen on the above graph, when all other variables are kept constant, increasing the cost of labor negatively affects production. This result is not particularly illuminating. When the cost of one of the production inputs increases, less of that input can be purchased on the same budget, so annual output drops. But, as mentioned, the real power of this model comes from its flexibility.

In the earlier scenario, the other parameters do not respond to the increase in labor costs. The hypothetical company/country accepts the loss in production. This feels like an unreasonable assumption. Even if the cost of the inputs cannot be directly controlled, or are the result of some change in policy, the other parts of the system would still respond to this increase.

Consider, perhaps, the additional unit labor cost is due to an increase in the minimum wage. On its own, this additional cost lowers production, but increasing worker compensation could also boost worker productivity. If true, the total productivity factor of the production function (b) might increase from 100 to 110. In this situation – total production *increases* to $1.65 million.

Another possibility, given the increasing cost per unit labor, the company/country could try to lower capital costs (renegotiate rates for equipment, tax breaks for certain industries affected heavily by the wage increase, etc.). Lowering capital cost from $34 per unit to $30 also boosts production (to $1.64

million). In both scenarios, overall production can go up, even with higher unit costs for labor. The real usefulness of this model is the ability to explore several possible outcomes.[3]

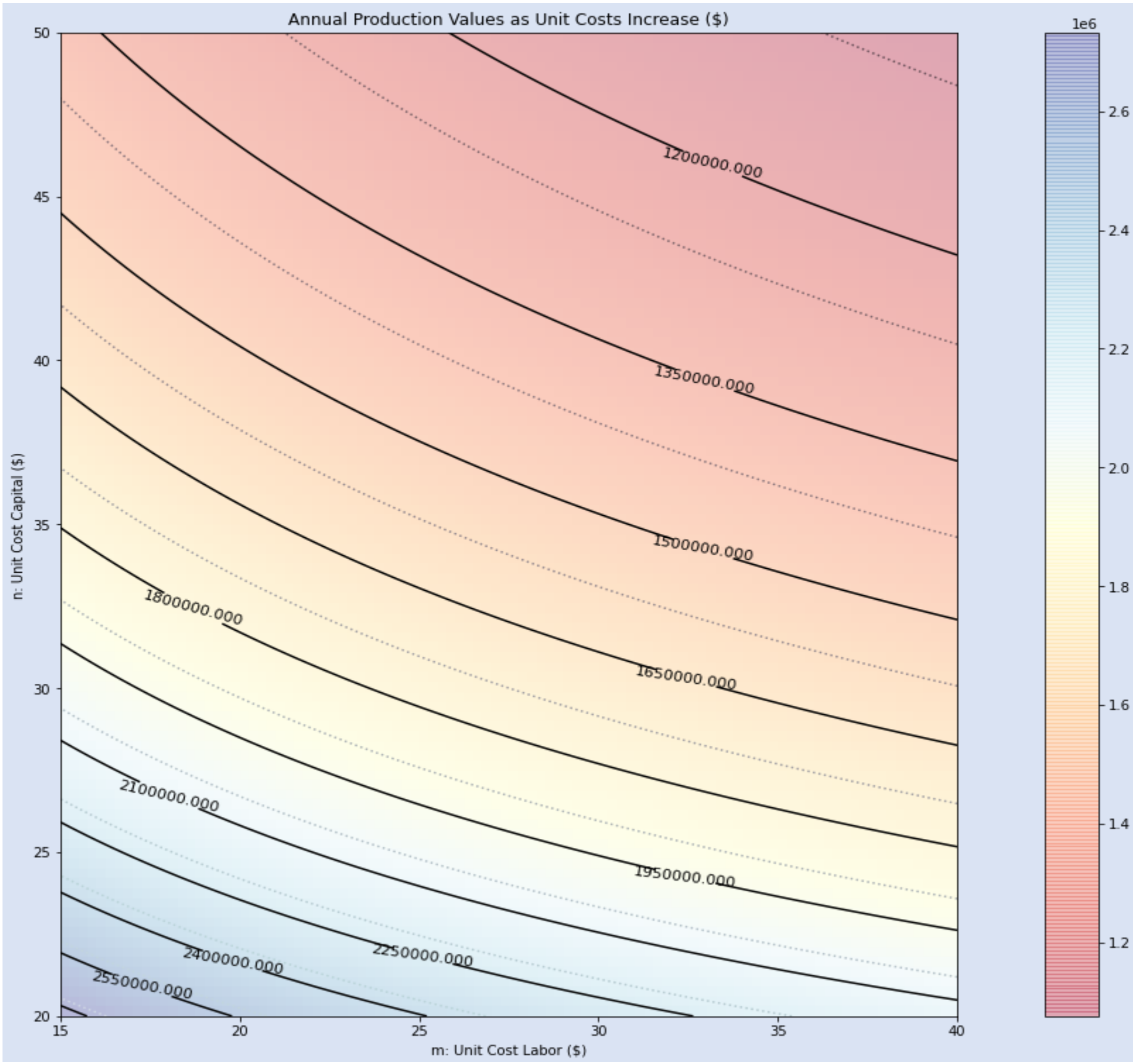

Figure 2: This level map considers a modest boost in productivity (b = 105) as a response to increased labor cost. The original levels (from the given parameters) are shown as dotted lines.

Almost any system, of natural or human design, will exceed our ability to completely model it mathematically. A successful model will allow flexibility to consider several possible outcomes, giving fuller insight into how the underlying system *might* respond to potential changes. As our technology and understanding improves, we can compensate for discrepancies between model and reality (Monte Carlo simulations or other corrective techniques made more feasible with modern computers). Yet, with great computational power comes great responsibility, even the best-tuned models are merely that – approximations of a much-too-complicated truth.

[3] To explore some of this flexibility in more depth, see supplemental Google Colab notebook, linked here

Symbolic Derivation of K and L (Lagrange Multipliers):

$$\boldsymbol{\nabla P = \lambda \nabla g}$$

Find partial derivatives and λ:

$$P_L = \lambda\, g_L \rightarrow \alpha\, b\, L^{\alpha-1} K^{1-\alpha} = \lambda\,(m) \rightarrow \lambda = \frac{\alpha\, b\, L^{\alpha-1} K^{1-\alpha}}{m}$$

$$P_K = \lambda\, g_K \rightarrow (1-\alpha)\; b\, L^{\alpha} K^{-\alpha} = \lambda\,(n) \rightarrow \lambda = \frac{(1-\alpha)\, b\, L^{\alpha} K^{-\alpha}}{n}$$

Equate λ and cross multiply:

$$n\, \alpha\, b\, L^{\alpha-1} K^{1-\alpha} = m\;(1-\alpha)\, b\, L^{\alpha} K^{-\alpha}$$

Simply (divide by $b\, L^{\alpha} K^{-\alpha}$):

$$\frac{n\,\alpha\, K}{L} = m(1-\alpha)$$

Solve for n:

$$n = \frac{m\,(1-\alpha)\, L}{\alpha\, K}$$

Substitute into constraint equation:

$$mL + \frac{mL\,(1-\alpha)}{\alpha K} K = p$$

Distribute, cancel, and solve for L:

$$L = \frac{\alpha\, p}{m}$$

Repeat above steps, solving $\frac{n\,\alpha\,K}{L} = m\,(1-\alpha)$ for m:

$$m = \frac{n\,\alpha\, K}{(1-\alpha)L}$$

Substitute into constraint equation:

$$\frac{n\,\alpha\, K}{(1-\alpha)L} L + nK = p$$

Distribute, cancel, and solve for K:

$$(p - nK)\,(1-\alpha) = n\,\alpha\, K$$

$$K = \frac{p\,(1-\alpha)}{n}$$

Maximum production values are obtained when:

$$K = \frac{p(1-\alpha)}{n}\;;\; L = \frac{\alpha p}{m}$$