Project: Exploring Spring-Mass Systems

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Introduction

This project serves as a culmination of sorts for Math 315, an Introduction to Differential Equations. Appropriately, it will focus on analyzing two distinct spring-mass systems, each with three masses and no damping. The first part will compare the model derived from our analytic solution to experimental observation of the system in motion. Our second system will use the same approach, but instead of modeling wheeled-masses on springs, we will consider ceilings of each floor in a three-story building.

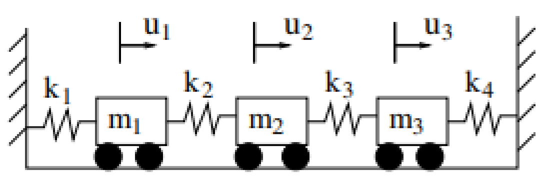
Systems of masses and springs - or more generally - systems that can be effectively modeled as masses on springs are abundant (i.e. particle interactions, population growth, electrical circuits.., etc.). Exploring these two simple cases offers insight into how these systems behave and respond to different parameters and initial conditions.

Part I

Comparing with Experiments

Consider the following spring mass system.

Spring-Mass System (Adapted from Lebl, figure 3.12.



To model this system, we will first derive the governing equations by applying Newton's second law: $F = ma$. We will assume the masses experience negligible friction for simplicity and since they are depicted on wheels. By Hooke's Law, we know the force acting on each mass equals the compression of the spring (determined by the displacement of each mass) times the spring constant. From the figure, we define positive displacement to the right.

Each mass has two springs attached on opposing sides. So, the net force acting on each cart will have at least one component from the displacement of an adjacent cart. Notice how displacement and position (left, right, center) affects the direction of the spring forces on each cart. For the central cart, its net force will be affected by the displacement of each cart, such that

$$m_2\ddot{u}_2 = -k_2(u_2 - u_1) + k_3(u_3 - u_2)$$

Distributing and solving for $\ddot{u}_2$ gives

$$\ddot{u}_2 = \frac{k_2}{m_2}u_1 - \frac{(k_2 + k_3)}{m_2}u_2 + \frac{k_3}{m_2}u_3$$

And this form, a straightforward expression of Newton's Second Law (acceleration is directly proportional to the net forces acting on an object and inversely proportional to its mass) more clearly shows the relationship between the connected masses. In this case, the displacement of mass one affects the magnitude and direction of the force from left spring, the displacement of the second cart affects the force of both springs, and the displacement of the third card affects only the right spring.

We can use this pattern to build the governing equations for the other masses - the spring constants shift in the appropriate direction and a displacement factor (the unattached mass) falls off, as we expect, since the far masses do not touch. For mass one

$$m_1\ddot{u}_1 = -k_1u_1 + k_2(u_2 - u_1)$$

From this pattern, the system of governing equations naturally emerges. We can define matrices M, and K, for all our masses and spring constants,

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \text{ and } K = \begin{bmatrix} -(k_1 + k_2) & k_2 & 0 \\ k_2 & -(k_2 + k_3) & k_3 \\ 0 & k_3 & -(k_3 + k_4) \end{bmatrix}$$

to establish a system of equations $M\ddot{\vec{u}} = K\vec{u}$ that can be similarly rearranged in terms of acceleration. For convenience, we define a matrix A that equals $M^{-1}K$ so our system becomes $\ddot{\vec{u}} = A\vec{u}$

$$\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{bmatrix} = \begin{bmatrix} \frac{-(k_1+k_2)}{m_1} & \frac{k_2}{m_1} & 0 \\ \frac{k_2}{m_2} & \frac{-(k_2+k_3)}{m_2} & \frac{k_3}{m_2} \\ 0 & \frac{k_3}{m_3} & \frac{-(k_3+k_4)}{m_3} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

And from this form, we could directly solve this second order system by the eigenvalue method, which also gives useful information about the behavior of our system, as the eigenvalues of A will equal ω^2; this solution also gives the natural frequencies for the oscillators.

But for practice (and insight into how coupled oscillators behave), we can reduce this to a system of first-order differential equations. We can introduce another variable, $\vec{v} = \dot{\vec{u}}$ to generate a 6x6 matrix, so that $\dot{\vec{v}} = A\vec{v}$. For the generic first-order system,

$$\begin{bmatrix} v_1 \\ v_2 \\ 3_3 \\ \dot{v_1} \\ \dot{v_2} \\ \dot{v_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-(k_1+k_2)}{m_1} & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & \frac{k_2}{m_2} & \frac{-(k_2+k_3)}{m_2} & \frac{k_3}{m_2} \\ 0 & 0 & 0 & 0 & \frac{k_3}{m_3} & \frac{-(k_3+k_4)}{m_3} \end{bmatrix}$$

We will consider the simplest case, where all masses and spring constants are identical, giving

$$\begin{bmatrix} v_1 \\ v_2 \\ 3_3 \\ \dot{v_1} \\ \dot{v_2} \\ \dot{v_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} \dot{u_1} \\ \dot{u_2} \\ \dot{u_3} \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Using technology, we can find the eigenvalues and eigenvectors of this matrix,

$$\vec{\lambda} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -3.41 \\ -2.00 \\ -0.586 \end{bmatrix} \text{ and } \vec{v} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -0.5 & 0.707 & -0.5 & 0 & 0 & 0 \\ 0.707 & 1.24 & -0.707 & 0 & 0 & 0 \\ -0.5 & -0.707 & -0.5 & 0 & 0 & 0 \end{bmatrix}$$

For our purposes, we only need the real solutions for $\vec{u}$. which we get from the lower half of the eigenvalue and eigenvector matrices. Borrowing notation from Dr. Strang's MIT OpenCourseWare (Lecture on Second Order Systems, our solution will have the form

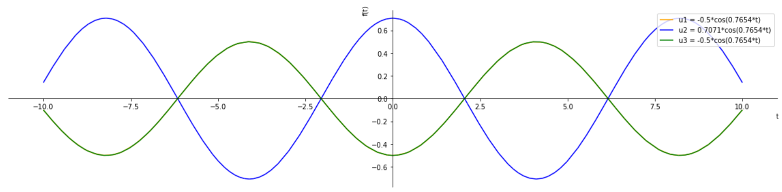
$$u_n = A_n(\cos \omega_n t)v_n + B_n(\sin w_n t)v_n$$

where n pertains to one mode shape of our system (since we have three masses, there are three possible shapes (one for each natural frequency), A and B coefficients correspond to initial displacement and velocity, respectively, and ω and $\vec{v}$ come from our eigenvalues ($\omega^2 = \lambda$) and eigenvectors, calculated above numerically using technology[1].

[1] And, technology allows us to play around with these systems even more. A bit tangential to this report, but I explore two different 'routines' to find solutions for these problems in this Google Colab notebook - one, using the 'SciPy solve_ivp' Python module to numerically solve the second-order system directly, another essentially following the steps outlined above to obtain the periodic solutions described by the eigenvalues and eigenvectors (with some symbolic math from the SymPy Python module).

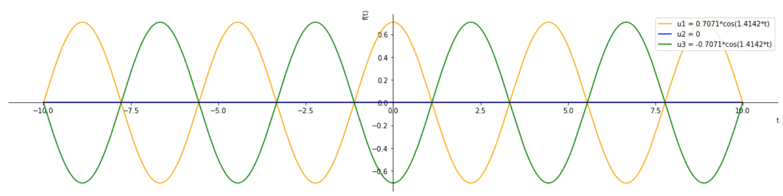
So our general solution for u will be a linear combination of these three mode shapes. For our simplified case (all masses and springs being equivalent) those modes look like:

Figure 1: Lowest frequency mode, $\omega = 0.765$



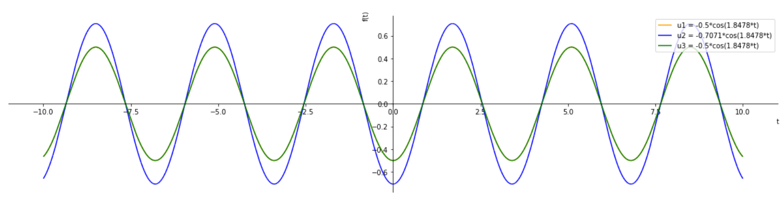
In this first mode, the central mass oscillates with the largest amplitude and out of phase with the other two masses.

Figure 2: Mode 2, $\omega = 1.41$



The central mass stays at equilibrium in this central mode, the other two masses oscillate out of phase with each other.

Figure 3: Highest frequency mode, $\omega = 1.85$

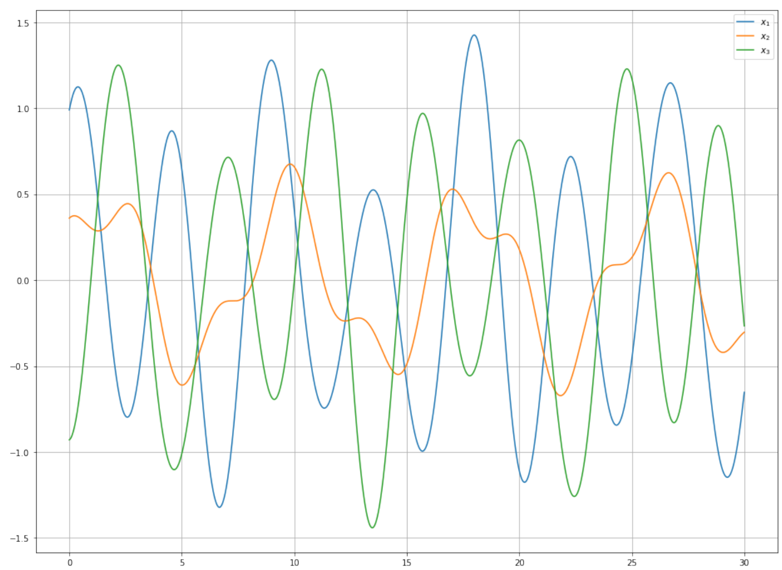


For the highest frequency mode, all three mass oscillate in phase, the central mass with slightly greater amplitude.

Each of the above modes match what we observe experimentally for a Three Degrees of Freedom Mass-Spring System. And this makes sense from a physical standpoint, too. With three masses attached to four springs, there are three different ways for the system to oscillate - all masses moving in the same direction, the two on the ends moving together (which stabilizes the central mass, keeping it at equilibrium), or the masses trying to move in opposing directions.

With randomized initial conditions (initial displacements and velocities between -1 and 1), we verify that the particular solution for any given set of initial conditions is some linear combination of these three modes.

Figure 4: Randomized Initial Conditions



The outside masses have similar amplitudes but opposing phases and the central mass moves almost in between the phases of the other two.

Part II
Earthquake!

This model easily extends beyond our simplified carts on springs example. Let's consider, instead, a 3-story building. The physics of this system are nearly identical to our cart model, but with one fewer spring. This seems a little counter-intuitive, that the motion of our carts could describe the motion of different floors of a physical, vertical structure.

But, in both cases, the movement of each mass (here, our mass will be the ceiling of each floor of the building) depends strictly on its own displacement and the displacement of the floor(s) attached to it. Only instead of springs, the exterior structure of the building applies the restorative force on each mass toward its equilibrium position. And, since the top of the building is not attached to anything, we have one fewer spring, but the equations that describe our system will be essentially the same.

Without the fourth spring, the only equation that changes is for $\ddot{u_3}$, which now only experiences a force from the third spring. The direction and magnitude of that force is dependent on the displacement of the second floor and the displacement of the third floor - if both floors move in the same direction the net force will balance, if they move opposite directions, the net force will increase.

$$\begin{bmatrix} \ddot{u_1} \\ \ddot{u_2} \\ \ddot{u_3} \end{bmatrix} = \begin{bmatrix} \frac{-(k_1+k_2)}{m_1} & \frac{k_2}{m_1} & 0 \\ \frac{k_2}{m_2} & \frac{-(k_2+k_3)}{m_2} & \frac{k_3}{m_2} \\ 0 & \frac{k_3}{m_3} & \frac{-k_3}{m_3} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

We'll start with the simplest case of equivalent masses and springs. We set up a 6x6 matrix for our system of first-order equations:

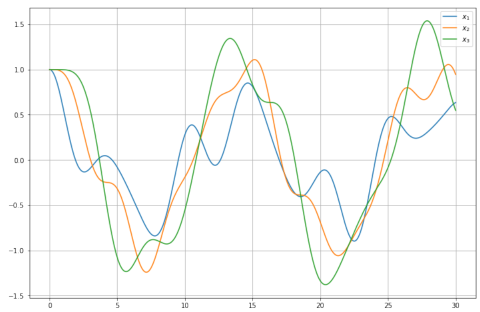
$$\begin{bmatrix} v_1 \\ v_2 \\ 3_3 \\ \dot{v_1} \\ \dot{v_2} \\ \dot{v_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \dot{u_1} \\ \dot{u_2} \\ \dot{u_3} \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

And solving that system as an eigenvalue problem, we determine the three natural frequencies of for our building, as our eigenvalues, $\vec{\lambda}$ will equal $\vec{\omega^2}$

$$\vec{\omega} = \begin{bmatrix} 0.445 \\ 1.25 \\ 1.80 \end{bmatrix}$$

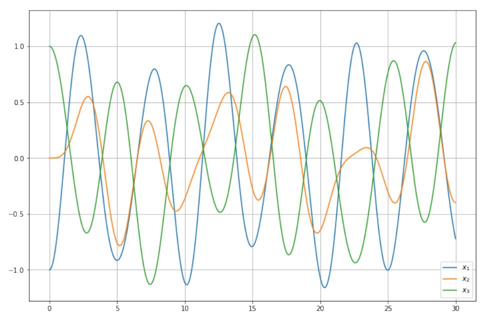
Which gives the following modal shapes:

Figure 5: Lowest frequency mode, $\omega = 0.445$



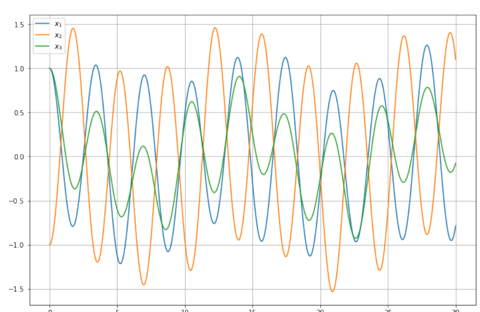
All three masses moving in the same direction

Figure 6: Mode 2, $\omega = 1.25$



Outside masses alternate about the central mass

Figure 7: Highest frequency mode, $\omega = 1.80$



Outsides masses move in phase with each other, alternate with central mass

As before, we can compare our solution to real world experiments. In this video, a shaking table shows vibrations over a range of frequencies for a toy 3-story building. If we wanted to improve out model, we could even attempt to define a forcing function that matches changing frequencies of the table. Still, we can compare our simplified model to what we observe in the video. Both systems seem to agree - at lower frequencies, the three masses tend to all oscillate together, as frequency increases, the central mass starts to stabilize while the exterior two alternate with each other, then at higher frequencies, the outside masses tend to synchronize, switching places with the central mass.

For our buildings, the natural frequencies have important, real-world implications. In an earthquake, we want the lowest natural frequency of our building to be as high as possible. If the earthquake matches the natural frequency our system, the resonance response would have catastrophic consequences for our structure. So, using technology to model this system also provides a convenient way to adjust various parameters - the mass and/or stiffness matrices (to tune the natural frequency), or the initial conditions (to explore various system responses).

For example, by increasing the stiffness of the first floor spring to $k_1 = 10$, our new natural frequencies become

$$\vec{\omega} = \begin{bmatrix} 0.596 \\ 1.59 \\ 3.33 \end{bmatrix}$$

Figure 8: Lowest Frequency Mode $k_1 = 10$

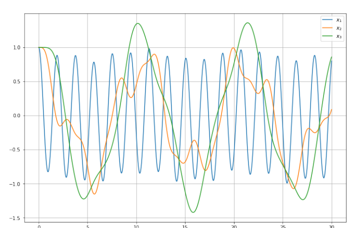
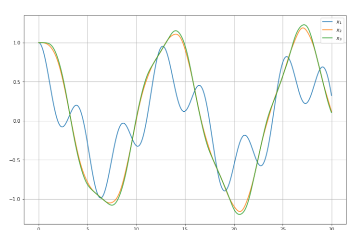


Figure 9: Lowest Frequency Mode $k_3 = 10$



Increasing the stiffness of the first floor 'spring' raises the frequency of the lowest mode. Conversely, increasing k_3 has little effect on the lower modes(but greatly increases the frequency of the highest mode

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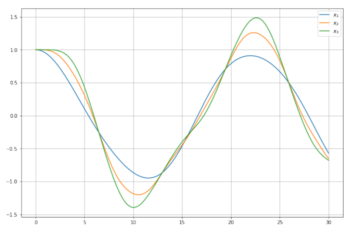
Figure 10: Increasing Mass, $m_1 = 10$



We can also try tuning other parameters. However, increasing the bottom mass lowers the frequency of the lowest mode, $\omega = 0.099$, suggesting that only increasing the bottom mass is not an effective approach to prevent an earthquake from triggering a resonance response of our building.

Conclusion

This project was a fitting finale to this differential equations course. Mass-spring systems can effectively model so many real world applications - from atomic interactions, population dynamics, and structural engineering and just about everything in between. Both of our models were straightforward and relatively simply - neither had any damping to consider or any forcing function driving the oscillation. But having set up some technological tools as a foundation, adding those terms is almost as simple as tuning our parameters for our building model. Changing a few lines of code, we can build increasingly elaborate models and explore their behavior.

Most importantly, this project built a foundation for how to functionally apply differential equations to describe real-world behavior. Using Newton's second law to build our system of equations is the most valuable takeaway from this project. Translating those equations into a tun-able system that explains experimental results feels like a watershed moment for me. Even though we explored very simple models, this project felt like a profound step away from abstraction and drilling practice problems. And a step, however small and tentative, toward a deeper understanding of the world around us.