# A Computational Analysis of the Riemann Zeta Function

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# Overview


1. Origin of the $\zeta$
   - Euler
   - Riemann
2. Properties
3. Ramanujan
4. MEA Method
5. Results

# Origin of the Problem

In his *Introductio in Analysin Infinitorum*, Leonhard Euler, states and solves five different series,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^8} = \frac{\pi^8}{9450}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{10}} = \frac{\pi^{10}}{93555}$$

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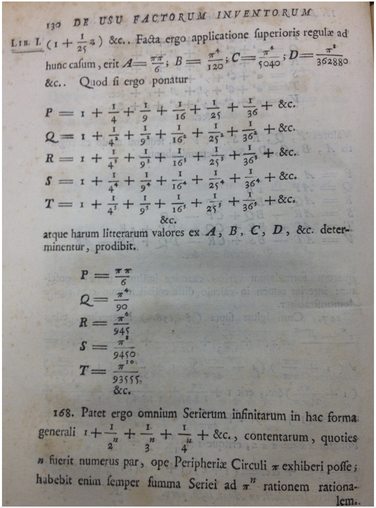

130 DE USU FACTORUM INVENTORUM

LIB. I. $(1+\frac{1}{25}z)$ &c.. Facta ergo applicatione ſuperioris regulæ ad hunc caſum, erit $A=\frac{\pi\pi}{6}$; $B=\frac{\pi^4}{120}$; $C=\frac{\pi^6}{5040}$; $D=\frac{\pi^8}{362880}$ &c.. Quod ſi ergo ponatur

$$P = 1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\&c.$$
$$Q = 1+\frac{1}{4^2}+\frac{1}{9^2}+\frac{1}{16^2}+\frac{1}{25^2}+\frac{1}{36^2}+\&c.$$
$$R = 1+\frac{1}{4^3}+\frac{1}{9^3}+\frac{1}{16^3}+\frac{1}{25^3}+\frac{1}{36^3}+\&c.$$
$$S = 1+\frac{1}{4^4}+\frac{1}{9^4}+\frac{1}{16^4}+\frac{1}{25^4}+\frac{1}{36^4}+\&c.$$
$$T = 1+\frac{1}{4^5}+\frac{1}{9^5}+\frac{1}{16^5}+\frac{1}{25^5}+\frac{1}{36^5}+\&c.$$
&c.

atque harum litterarum valores ex $A$, $B$, $C$, $D$, &c. determinentur, prodibit.

$$P = \frac{\pi\pi}{6}$$
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&c.

168. Patet ergo omnium Serierum infinitarum in hac forma generali $1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+$ &c., contentarum, quoties $n$ fuerit numerus par, ope Peripheriæ Circuli $\pi$ exhiberi poſſe; habebit enim ſemper ſumma Seriei ad $\pi^n$ rationem rationalem.

# Euler Prime Product

Thanks to Euler we also have the equality,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p_j \in \mathbb{P}} \frac{1}{1-(p_j)^{-s}}$$

where $\mathbb{P}$ is the set of all prime numbers.

# The Riemann Zeta Function

In 1859 Bernhard Riemann took on this problem and gave it its name. He begins his paper referencing Euler's work,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p_j \in \mathbb{P}} \frac{1}{1 - (p_j)^{-s}}$$

# The Riemann Zeta Function

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"The function of the complex variable s which is represented by these two expressions, wherever they converge, I denote by $\zeta(s)$."

# The Riemann Zeta Function

The Zeta Function:

$$\zeta(s) = \sum_{n \geq 1} n^{-s}$$

By Euler we have that:

$$\sum_{n \geq 1} n^{-2} = \frac{\pi^2}{6}, \sum_{n \geq 1} n^{-4} = \frac{\pi^4}{90}, \sum_{n \geq 1} n^{-6} = \frac{\pi^6}{945}$$

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And more generally that,

$$\sum_{n \geq 1} n^{-2k} = (-1)^{k+1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k}$$

wher $B_{2k}$ is the $2k^{th}$ Bernoulli number. [5] [3]

# Properties of $\zeta$

**Proposition:** For $s \geq 2$, $\zeta(s+2) < \zeta(s+1) < \zeta(s)$
*Proof:* By using the comparison test for each term, $n > 1$, and noting that

$$(n+2)^{-s} < (n+1)^{-s} < n^{-s}$$

then,

$$\zeta(s+2) < \zeta(s+1) < \zeta(s)$$

$\square$

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□

Therefore, we can at least bound all odd values of the zeta function between the surrounding even values.

# Euler Prime Product

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p_j \in \mathbb{P}} \frac{1}{1-(p_j)^{-s}}$$

Julia Code:

```
function euler(s,data)
    primes = readcsv(data)
    prod=1
    for p in primes
        prod*=(1)/(1-(1/BigFloat(p^s)))
    end
    return prod
end
```

After 1,000,000 iterations:
1.202056903159594

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1.202056903159594

Python Code:

```
from mpmath import *

def eu_iden(s, data,digits=100):
    mp.dps=digits
    primes = []
    with open(data) as inputfile:
        for line in inputfile:
            try:
                primes.append(int(line))
            except ValueError:
                pass
    prod=1
    for p in primes:
        prod*=(1-(1/mpf(p**s)))**(-1)
    inputfile.close()
    return prod
```

After 1,000,000 iterations:
1.2020569031595942

# The Ramanujan Formula

To compute $\zeta(3)$ we have a relatively fast formula thanks to Ramanujan. The Ramanujan formula,

$$\zeta(3) = \frac{8}{7}\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k \frac{1}{k^3} - \frac{4}{21}\log^3 2 + \frac{2}{21}\pi^2 \log 2$$

It is easy to see how this computation is faster than a straightforward computation of $\zeta(3)$, in that we have a $\left(\frac{1}{2}\right)^k$ term in front of $\frac{1}{k^3}$ [4]

# The Ramanujan Formula

Using the Ramanujan Formula we were able to produce the following code and results. Julia Code.

```
function ramanujan(k)
    sigma = zeros(BigFloat,k)

    for i=1:k
     sigma[i]=BigFloat(((1/2)^i)*(1/i^3))
        end
        diff1=BigFloat((4/21)*log(2)^3)
        diff2=BigFloat(2/21)*BigFloat(pi^2)*BigFloat(log(2))
        result=BigFloat(8/7)*BigFloat(sum(sigma)) - diff1 + diff2
        return result
    end
```

$$\zeta(3) = \frac{8}{7}\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k \frac{1}{k^3} - \frac{4}{21}\log^3 2 + \frac{2}{21}\pi^2 \log 2$$

# The Ramanujan Formula

Python Code

```
import numpy as np
from math import log
from mpmath import *
mp.prec+=203

def ramanujan(k):
    k=mpf(k)
    array=np.zeros_like(np.arange(k))
    for i in np.arange(1,k+1):
        array[i-1]=(mpf(0.5)**mpf(i))*(mpf(1)/mpf(i)**mpf(3))
    diff1=(mpf(4)/mpf(21))*mpf(np.log(2))**(mpf(3))
    diff2=(mpf(2)/mpf(21))*mpf(np.pi)**mpf(2)*mpf(np.log(2))
    sigma=mpf(8)/mpf(7)*np.sum(array) -diff1 + diff2
    return sigma
```

$$\zeta(3) = \frac{8}{7}\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k \frac{1}{k^3} - \frac{4}{21}\log^3 2 + \frac{2}{21}\pi^2 \log 2$$

# MEA Method

## Claim

*Given* $n$ $(n \geq 2)$ *randomly chosen positive integers* $\{k_1, \ldots, k_n\}$,

$$P\{\gcd(k_1, \ldots, k_n) = 1\} = [\zeta(n)]^{-1}$$

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## Theorem

*Let $N_n(\ell) = card\{(k_1, \ldots, k_n) \in \{1, \ldots, \ell\}^n : \gcd(k_1, \ldots, k_n) = 1\}$. Then for $n \geq 2$ we have that*

$$\lim_{\ell \to \infty} \frac{N_n(\ell)}{\ell^n} = [\zeta(n)]^{-1}$$

*[1]*

# MEA Method

Let $\lfloor x \rfloor$ denote the floor function of $x$. We will make use of the following lemma.

**Lemma**

*Let $N_n(\ell) = card\{(k_1, \ldots, k_n) \in \{1, \ldots, l\}^n : \gcd(k_1, \ldots, k_n) = 1\}$ is the number of relatively prime elements in $\{1, \ldots, l\}^n$. Then,*

$$N_n(\ell) = \ell^n - \sum_{p_i} \left( \lfloor \frac{\ell}{p_i} \rfloor \right)^n + \sum_{p_i < p_j} \left( \lfloor \frac{\ell}{p_i \cdot p_j} \rfloor \right)^n - \sum_{p_i < p_j < p_k} \left( \lfloor \frac{\ell}{p_i \cdot p_j \cdot p_k} \rfloor \right)^n + \ldots$$

*[2]*

# Results

| $n = 1,000,000$ | Julia | Python |
|---|---|---|
| Brute Force | 1.202056903159 | 1.202056903159 |
| Ramanujan Method | 1.202056903159594 | 1.2020569031595942 |

| Euler Product | Julia | Python |
|---|---|---|
| $\zeta(3)$ | 1.202056903159594 | 1.202056903159594 |
| $\zeta(5)$ | 1.036927755143369 | 1.036927755143369 |

# Results

If, for odd $s$,

$$\zeta(s) = \frac{\pi^s}{??}$$

then, $?? \approx \ldots$

| MEA Method | $\ell = 10^6$ | $\ell = 10^8$ |
|---|---|---|
| $\zeta(3)^{-1} \cdot \pi^3$ | 25.79435196830 | 25.7943501926105 |
| $\zeta(5)^{-1} \cdot \pi^5$ | 295.121515513789 | 295.121509986379 |

By our previous theorem,

$$\zeta(2) < \zeta(3) < \zeta(4) < \zeta(5) < \zeta(6)$$

So we have that,

$$\frac{\pi^2}{6} < \zeta(3) < \frac{\pi^4}{90} < \zeta(5) < \frac{\pi^6}{945}$$

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So we have that,

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By our analysis, we have,

$$\begin{aligned}
\frac{10\pi^3}{258} &< \quad \zeta(3) \quad < \frac{4\pi^3}{103} \\
\frac{20\pi^5}{5903} &< \quad \zeta(5) \quad < \frac{25\pi^5}{7378}
\end{aligned}$$

# References

[1] Stephen D. Casey. Computational strategies for computing values of the riemann zeta function. 2014.

[2] Stephen D. Casey and Brian M. Sadler. Pi, the primes, periodicities, and probability. *The American Mathematical Monthly*, 120(7):594–608, August-September 2013.

[3] Leonhard Euler. *Introductio in Analysin Infinitorum*. Tomus primus edition, 1748.

[4] Ekatherina A. Karatsuba. Fast computation of $\zeta(3)$ and some special integrals using the ramanujan formula and polylogarithms. *BIT Numerical Mathematics*, 41(4):722–730, 2001.

[5] Bernhard Riemann. Über die anzahl der primzahlen unter einer gegebenen größe. *Monatsberichte der Berliner Akademie*, November 1859.