SENSITIVITY ANALYSIS OF ADDITIVE MULTIATTRIBUTE VALUE MODELS

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We present two simple computational procedures for sensitivity analysis of additive multiattribute value models that yield variations in attribute weights (scaling constants). For the first, or entropy-based, procedure, the point of departure is the concept of equal weights for all attributes. For the second, or least squares procedure, the point of departure is a set of arbitrary weights for attributes. In each procedure we either (a) calculate the "closest" set of weights that equates the multiattribute value for a pair of alternatives—that alternative whose overall value is optimal for the original attribute weights and any other specific nondominated alternative—or (b) calculate the "closest" set of weights required for the specific alternative to exceed the "optimal" alternative by a specified amount.

In this paper we present two procedures—an entropy-based procedure and a "least squares" procedure—to calculate attribute weights sufficient to (1) equate, or (2) reverse by a prescribed (where feasible) amount the overall additive multiattribute value (MAV) of any pair of mutually nondominated alternatives. These procedures make a direct contribution to sensitivity analysis. The point of departure for the entropy-based procedure is equal weights for all attributes, a heavily promoted view in the psychological decision making literature (Dawes 1979, Dawes and Corrigan 1974, Wainer 1976, and Einhorn and Hogarth 1975). The point of departure for the "least squares" procedure is an arbitrary set of weights (i.e., any set of nonnegative numbers adding to one) for the specified attributes. These arbitrary weights could have resulted from any of a wide variety of assessment procedures. For example, the weights could have been assessed very carefully via tradeoffs (Dyer and Sarin 1979), or approximately via the analytic hierarchy process, AHP (Saaty, Vargas and Wendell 1983), or via the simple multiattribute rating technique, SMART (Edwards 1977), or could simply be based on equal weights.

For any multiattribute value analysis in which a single best alternative is recommended, it would be useful to know just how different the weights would have to be in order for the chosen alternative to no longer be the best. For a given pair of mutually nondominated alternatives, the least squares procedure asks and answers this simple question: what is the set of weights "closest" to the given (arbitrary) set which promotes the second (inferior in MAV) alternative so that its MAV exceeds that of the first by an amount Δ? The entropy-based procedure reframes this question, assuming the arbitrary weights are all equal; therefore, it computes the "most nearly equal" weights that promote the second alternative by an amount Δ. Weights that equate overall multiattribute values result when $\Delta = 0$.

In the following sections we present an entropy based (nearly-equal weights) procedure (Section 1), a least squares (arbitrary weights) procedure (Section 2), and examples of sensitivity analyses based on these procedures (Section 3). In Section 4, we discuss some implications for equal weights.

1. A Nearly-Equal Weights Procedure

An additive multiattribute model is of the form (1).

$$V(x_i) = \sum_{j=1}^{n} w_j v_j(x_{ij}), \qquad (1)$$

where V is overall value, $0 \leq V \leq 1$,

- x_i is a vector of attribute values $(x_{i1}, x_{i2}, \ldots, x_{in})$,
- $v_j(x_{ij})$ is a single attribute value function $0 \leq v_j(x_{ij}) \leq 1$, and
- w_j are weights reflecting the relative importance of the range of values of attribute j,

$$\sum_{j=1}^{n} w_j = 1.$$

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In practice the v_j functions may be assessed with some care, or they may simply be made linear, by using the range to standardize, as in expression (2).

$$v_j(x_{ij}) = \frac{x_{ij} - \min_i\{x_{ij}\}}{\max_i\{x_{ij}\} - \min_i\{x_{ij}\}}. \tag{2}$$

We assume that the single attribute value functions are known. The purpose of the procedure we will now describe is to compute, for two mutually nondominated multiattribute alternatives (acts), the values of the weights that make the (weighted) linear composites of the two alternatives differ by amount $\Delta > 0$ and to compute them in such a way that the weights are as "nearly equal" as possible.

Consider two multiattribute acts, x_b and x_i $i \neq b$, $x_b = (x_{b1}, x_{b2}, \ldots, x_{bn})$ and $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})$. Act x_b is the "arbitrary weights best act," that is, for arbitrary weights b_i,

$$\sum_{j=1}^{n} b_j v_j(x_{bj}) \geqslant \sum_{j=1}^{n} b_j v_j(x_{ij}) \quad \text{for all } i \neq b.$$

For this first procedure we assume $b_i = 1/n$ for all i. The problem is to find weights $w = (w_1, w_2, \ldots, w_n)$, $0 \leqslant w_j \leqslant 1$, adding to 1 so that equality (3) is satisfied, and the weights are "nearly equal."

$$\sum_{j=1}^{n} w_j v_j(x_{ij}) - \sum_{j=1}^{n} w_j v_j(x_{bj}) = \Delta > 0, \tag{3}$$

where equality (3) requires that $V(x_i)$ exceed $V(x_b)$ by the amount Δ. We operationalize the concept of nearly equal by using the maximum entropy principle, Equation 4.

Thus the completed formulation of the problem is

$$\text{maximize} \quad -\sum_{j=1}^{n} w_j \log w_j \tag{4}$$

$$\begin{aligned} \text{subject to} \quad & \sum_{j=1}^{n} w_j a_j = \Delta \\ & \sum_{j=1}^{n} w_j = 1 \\ & w_j \geqslant 0 \quad j = 1, 2, \ldots, n, \end{aligned} \tag{5}$$

where $a_j = v_j(x_{ij}) - v_j(x_{bj})$.

The solution (Appendix 1) is

$$w_j = \exp(\lambda^* a_j) \Big/ \sum_{j=1}^{n} \exp(\lambda^* a_j), \tag{6}$$

where λ^* is determined interatively from Equation 7:

$$\sum_{j=1}^{n} (a_j - \Delta)\exp(\lambda a_j) = 0. \tag{7}$$

The motivation for maximizing entropy is pragmatic—the maximum value of (4) *without* constraint (3) is $w_i = 1/n$ for all i. That is, equal weights is the maximum entropy solution. The mathematical approach for this procedure is similar to that of Jaynes (1968). That article, while dealing with objective procedures for establishing prior probability distributions, stimulated us to work on "nearly equal" attribute weights.

2. A Least Squares Procedure

The purpose of this procedure is to compute, for two mutually nondominated multiattribute alternatives (acts), the weights that make the MAV of act x_i exceed the MAV of act x_b by an amount Δ, and to compute these weights so that they are as "close" as possible to the given (arbitrary, previously determined) weights. The concept of "close" is operationalized using the minimum squared deviation principle, Equation 8. Except for the fact that it substitutes the least squares-based objective function, Equation 8, for the entropy-based objective function, Equation 4, the problem formulation is identical:

$$\text{minimize} \quad \sum_{j=1}^{n} (w_i - b_i)^2. \tag{8}$$

To solve this problem, form the Lagrangian and ignore nonnegativity conditions. The solution, *provided all weights have a computed value greater than or equal to zero,* is

$$w_j = \frac{(\Delta - \sum_{j=1}^{n} a_j b_j)(na_j - \sum_{j=1}^{n} a_j)}{n \sum_{j=1}^{n} a_j^2 - (\sum_{j=1}^{n} a_j)^2} + b_j. \tag{9}$$

Why have two procedures? The least squares procedure is more general (it accommodates an arbitrary set of weights, including equal weights), and its solution values, Equation 9, are quite easily computed. The problem is that the values computed by Equation 9 ignore the nonnegativity constraints, $w_j \geqslant 0$. When the computed values are, in fact, positive, the solution is fine. However, when some of the computed w_j values are negative, a somewhat complicated combinatorial problem results.

The entropy-based procedure *always* produces nonnegative weights. However, it has two limitations: (1) the procedure seeks "nearly equal" weights, rather than "closest to arbitrary weights," and (2) the weight computation requires that Equation 7 be solved iteratively.

A complete formulation of the least squares procedure, including the nonnegativity conditions, is a quadratic programming problem—one of the simplest possible nonlinear programming problems. It is rather easily solved using the software package GINO (for General INteractive Optimizer; Liebman et al. 1986). Thus, the least squares procedure is preferred because it accommodates an arbitrary set of weights, is often easily and directly solved using Equation 9 and, when necessary because of negative values from Equation 9, can be solved using standard nonlinear programming packages.

While not the focus of this paper, the quadratic programming problem just described can easily be extended to provide a basis for eliciting weights. Given a preliminary set of weights, possibly equal weights, the extension is to incorporate, possibly sequentially, additional constraints, linear in the weights, that represent partial information provided by the decision maker. Additional constraints may include any of the following types of linear expressions:

(a) rank orders of weights: $w_1 \geqslant w_2 \geqslant \ldots \geqslant w_n$;
(b) ranks of combined weights: $w_2 + w_3 + w_4 \geqslant w_1 \geqslant w_2 + w_3$, and so forth;
(c) bounded intervals: $\alpha_i \leqslant w_i \leqslant \beta_i$, where α_i, β_i are specified constants;
(d) fixed weights $w_i = \beta_i$ for selected attributes;
(e) MAV equality $V(x_i) = V(x_j)$;
(f) MAV inequality $V(x_i) \geqslant V(x_j)$;
(g) MAV absolute difference $V(x_i) \geqslant V(x_j) + \Delta_{ij}$, where $\Delta_{ij} \geqslant 0$ is a specified difference; and
(h) MAV relative difference $[V(x_i) - V(x_j)]/V(x_j) \geqslant \Delta_{ij}$.

How an interactive solution sequence would proceed would be determined by the decision maker's (DM) specific reactions. First a problem is solved. That problem may include, initially, a number of constraints of the type (a)–(h). The results would consist of (1) attributes and their weights, (2) alternatives and their MAVs, and (3) perhaps a list of tight constraints. DM might react to further constrain a "high" weight, e.g., $w_i \leqslant \beta_i$, or a "low" weight, e.g., $w_i \geqslant \alpha_i$; or to explore the implications of promoting an alternative, e.g., $V(x_i) \geqslant V(x_j) + \Delta_{ij}$. Alternatively, previously imposed constraints could be relaxed. These changes would be solved directly using a nonlinear programming software package such as GINO.

3. Examples of Sensitivity Analysis

In this section, we use expressions (6) and (9) alone as a basis for sensitivity analysis of attribute weights. We will further assume the arbitrary weights b_j are all equal ($b_j = 1/9$). A multiattribute data set consisting of 15 alternatives (U.S. cities), each described by 9 attributes—climate, housing, health care, crime, transportation, education, recreation, arts and economics—provides the $v_j(x_{ij})$ values in Table I. The numerical value for each attribute is based on a composite index from fairly "objective" components, which were rescaled to the 0 (worst level) to 1 (best level) intervals. No claim is made that these attributes or scale values actually operationalize any individual's objectives per se. The data source is Boyer and Savageau (1981). The particular group is 1 of 18 groups of 15 cities clustered alphabetically. This particular group was the only one analyzed using the procedures of this paper; thus it was not selected for any particular reason. Under the assumption of equal weights, alternative B has the highest MAV.

Table I
$V(x)$ For 15 Alternatives of 9 Attributes Having Equal Weights

Alt	Attr 1	Attr 2	Attr 3	Attr 4	Attr 5	Attr 6	Attr 7	Attr 8	Attr 9	V
A	1.000	0.674	0.162	0.808	0.373	0.578	0.121	0.020	0.432	0.463
B	1.000	0.198	0.573	0.087	1.000	0.500	0.738	0.519	0.571	0.576
C	0.751	0.234	0.196	0.200	0.016	0.321	0.685	0.000	0.208	0.290
D	0.437	0.807	0.383	0.394	0.102	0.155	0.000	0.092	0.370	0.304
E	0.000	0.491	0.108	0.503	0.282	0.212	0.278	0.390	1.000	0.363
F	0.479	0.354	0.030	0.106	0.000	0.999	0.473	0.069	0.130	0.293
G	0.625	0.000	1.000	0.000	0.674	0.734	1.000	1.000	0.109	0.571
H	0.303	0.817	0.061	0.146	0.087	0.120	0.291	0.115	0.755	0.299
I	0.421	0.594	0.136	0.617	0.321	0.031	0.452	0.013	0.230	0.313
J	0.502	0.495	0.053	0.560	0.034	0.304	0.545	0.090	0.182	0.307
K	0.360	0.997	0.068	0.489	0.014	0.000	0.081	0.028	0.622	0.295
L	0.096	0.120	0.000	0.667	0.413	0.538	0.589	0.031	0.499	0.328
M	0.418	1.000	0.204	0.631	0.001	0.236	0.235	0.009	0.381	0.346
N	0.490	0.182	0.173	1.000	0.153	1.000	0.502	0.208	0.000	0.412
O	0.640	0.705	0.282	0.391	0.304	0.150	0.307	0.262	0.509	0.394

Table II
Attribute Weights Equating Equal Weights Alternative and Alternative in Column 1 with Corresponding Best Alternative

Alt[a]	Attr 1	Attr 2	Attr 3	Attr 4	Attr 5	Attr 6	Attr 7	Attr 8	Attr 9	Best Alt.
G	.1095	.1102	.1130	.1107	.1097	.1122	.1123	.1133	.1091	*G, B*
A	.1145	.1469	.0924	.1668	.0825	.1192	.0830	.0882	.1065	*A, B*
N	.0883	.1153	.0937	.1904	.0736	.1523	.1024	.0983	.0855	*G*
M	.0795	.2160	.0927	.1793	.0589	.1000	.0842	.0838	.1056	*A*
E	.0467	.1632	.0784	.1839	.0614	.0930	.0788	.1085	.1861	*A*
O	.0789	.2402	.0863	.1853	.0513	.0799	.0721	.0901	.1157	*A*
K	.0735	.2444	.0823	.1755	.0551	.0826	.0725	.0832	.1309	*A*
D	.0671	.2704	.1096	.1890	.0451	.0870	.0546	.0790	.1032	*A*
L	.0390	.1186	.0609	.2876	.0598	.1385	.1078	.0683	.1195	*N*
H	.0564	.2816	.0708	.1421	.0433	.0830	.0766	.0807	.1654	*A*
I	.0624	.2236	.0752	.2664	.0548	.0721	.0916	.0686	.0852	*A*
J	.0644	.2133	.0622	.2777	.0318	.1014	.1019	.0715	.0758	*A*
F	.0617	.1763	.0596	.1427	.0293	.2999	.0917	.0689	.0698	*N*
C	.0479	.2594	.0224	.4097	.0006	.0726	.1533	.0097	.0244	*N*

[a] Alternatives (rows) in order of decreasing entropy of attribute weights.

With each procedure, for each of the 14 alternatives paired with alternative *B*, we ask what the weights must be to equate the MAV of each alternative with that of alternative *B*. To equate values, we set $\Delta = 0$. Table II presents the 14 sets of weights, computed using the entropy procedure, together with the best overall alternative for each set of weights. Table III provides the same information, though different weights, computed using the least squares procedures. Table III originally contained negative weights, based on Equation 9, for three alternatives (*J, F, C*); the actual values in Table III are those computed using GINO.

A sensitivity analysis based on Table III might proceed as follows: The decision maker may directly reject some sets of weights as too extreme. For example, assume the decision maker considers only the optimal alternatives for each set of weights in Table III for which $w_1 \geqslant 0.08$. This implies that only alternatives *A, B, G* would be considered further. Thus, for discussion purposes, attention is now focused only on alternatives *A, B* and *G*.

We next use expression (9) to compute the least squared weights required for alternative *A* to exceed alternative *B* in MAV by a preassigned value loss level of $\Delta = 0.05$. For this set of weights the MAV of each

Table III
Least Squares Attribute Weights Equating MAV of Row Alternative and MAV of Best Alternative (*B*) Assuming Equal Weights, with Corresponding Best Alternative

Alt[a]	Attr 1	Attr 2	Attr 3	Attr 4	Attr 5	Attr 6	Attr 7	Attr 8	Attr 9	Best Alt.
G	.109	.110	.113	.111	.110	.112	.112	.113	.109	*B, G*
A	.118	.147	.093	.162	.080	.123	.080	.088	.109	*A, B*
N	.088	.121	.095	.183	.065	.155	.106	.101	.084	*G*
M	.079	.204	.098	.181	.042	.108	.086	.086	.115	*A*
E	.030	.163	.085	.176	.059	.103	.086	.120	.177	*A*
O	.082	.222	.094	.189	.028	.084	.071	.099	.130	*A*
K	.073	.225	.087	.183	.036	.088	.071	.089	.146	*A*
D	.068	.241	.123	.196	.019	.100	.042	.088	.121	*A*
L	.006	.138	.059	.244	.057	.157	.127	.073	.139	*N*
H	.047	.247	.075	.162	.014	.095	.085	.092	.181	*A*
I	.055	.229	.080	.253	.037	.074	.107	.068	.097	*A*
J[b]	.066	.214	.062	.247	.000	.122	.123	.079	.086	*A*
F[b]	.062	.192	.058	.166	.000	.257	.111	.076	.078	*N*
C[b]	.064	.276	.000	.333	.000	.116	.210	.000	.000	*N*

[a] Alternatives (rows) in same order as in Table II.
[b] Equation 9 produces some negative weights; weights calculated by GINO.

Table IV
Attribute Weights Associated with Choices (Optimality) of Alternatives A, B or G

Optimal Act	A^a	A or B	B or G	G^b
Attr 1	0.121	0.118	0.109	0.092
Attr 2	0.163	0.147	0.110	0.101
Attr 3	0.085	0.093	0.113	0.133
Attr 4	0.184	0.162	0.111	0.107
Attr 5	0.066	0.080	0.110	0.094
Attr 6	0.128	0.123	0.112	0.124
Attr 7	0.067	0.080	0.112	0.125
Attr 8	0.077	0.088	0.113	0.136
Attr 9	0.109	0.109	0.109	0.087
	1.000	1.000	0.999	0.999

[a] $V(A) - V(B) = 0.05$, $V(A) - V(G) = 0.094$.
[b] $V(G) - V(B) = 0.05$, $V(G) - V(A) = 0.187$.

alternative is also exceeded by the MAV of alternative A. We employ the same procedure, using expression (9), to compute weights so that alternative G exceeds alternative B by 0.05.

Ultimately the decision depends on the decision maker's choice of weights. Table IV presents the weights/choice issue directly: the leftmost weights imply that A exceeds B by at least 0.05 in MAV; the next set is for MAV(A) = MAV(B); the third set is where MAV(B) = MAV(G); and the rightmost weights imply G exceeds B by at least 0.05. For example, for A to be selected, attributes 4 and 2 must be acknowledged as relatively most important, and attributes 3, 5, 7 and 8 as relatively least important. Thus, in this example, the least squares procedure has directed the attention of the decision maker. Either a choice of weights implies choice of alternative, or a choice of alternative implies relationships among weights.

More comprehensive sensitivity analyses would go beyond the calculations of Equations 6 and/or 9. Relationships implied by the Table IV weights, such as the requirement that w_2 and w_4 be the largest, could be explored. For example, the decision maker could restrict $w_2 \leq 0.12$ and then use GINO to find the "closest" set of weights for MAV(A) = MAV(B) + 0.05 and MAV(A) $\geq$ MAV(G). Such procedures again involve a more comprehensive quadratic programming problem.

4. Implications for Equal Weights

One by-product of this analysis is that, in selecting a single best alternative, based on an additive MAV model, the weights do matter. For the example problem of Table I, the weights of Tables II and III clearly indicate that for a small perturbation of weights, the optimal alternative changes. The loss in MAV for such a small change is itself small. Table IV shows explicitly the relationship between size of weight perturbation and value loss.

The procedures from which the weights of Tables II, III and IV were calculated will always produce a solution (provided that, in the least squares procedure, the nonnegativity conditions are enforced), requiring only that one act not dominate the other acts. The specific values for the weights will, of course, depend on the specific value functions of Table I.

Appendix

The "nearly equal" weights optimization model may be expressed as follows.

$$\text{Minimize} \quad \sum_{j=1}^{n} w_j \log w_j \tag{1a}$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_j w_j = \Delta \tag{2a}$$

$$\sum_{j=1}^{n} w_j = 1 \tag{3a}$$

$$w_j \geq 0 \quad j = 1, \ldots, n, \tag{4a}$$

where the a_j are known constants, $a_j = v_j(x_{ij}) - v_j(x_{ej})$, and Δ is a nonnegative parameter. Since the objective function is strictly convex and the feasible region is bounded and convex, the optimization model must have a unique optimal solution when the feasible region is nonempty. Moreover, the Kuhn-Tucker conditions are necessary and sufficient for optimality. It is helpful to consider 3 cases in determining the optimal solution to (1a) through (4a).

Case 1. $\Delta > \max_{1 \leq j \leq n} a_j$.

In this case the optimization model is infeasible. It is not possible for act i to exceed act e in value by Δ for any set of weights.

Case 2. $\Delta = \max_{1 \leq j \leq n} a_j$.

Since Δ is a nonnegative parameter, this case can only occur when $\max_{1 \leq j \leq n} a_j \geq 0$. The optimal solution is trivial, namely,

$$w_k = \frac{1}{O(K)} \quad \text{for } k \in K,$$

$$w_k = 0 \quad \text{for } k \notin K, \tag{5a}$$

where $K = \{k \mid a_k = \max_{1 \leq j \leq n} a_j\}$ and $O(K)$ is the number of elements of K.

Case 3. $\Delta < \max_{1 \leq j \leq n} a_j$.

This case can occur only when $\max_{1 \leq j \leq n} a_j > 0$, that is, when act e does not dominate act i. The optimal solution is

$$w_k = \frac{\exp(\lambda a_k)}{\sum_{j=1}^{n} \exp(\lambda a_j)}, \tag{6a}$$

where λ is the unique solution of the equation

$$\sum_{j=1}^{n} (a_j - \Delta)\exp(\lambda a_j) = 0. \tag{7a}$$

Note that in this case $w_k > 0$ for $k = 1, \ldots, n$, so that every attribute receives some weight.

Case 2 follows easily from the fact that a feasible solution must satisfy $w_k = 0$ for $k \notin K$ and that the optimal values of w_k for $k \in K$ solve the following problem.

$$\begin{aligned} \text{Minimize} \quad & \sum_{k \in K} w_k \log w_k \\ \text{subject to} \quad & \sum_{k \in K} w_k = 1 \\ & w_k \geq 0 \quad k \in K. \end{aligned}$$

We can now prove that (6a) and (7a) determine the optimal solution for Case 3 by showing that the Kuhn-Tucker conditions are satisfied. For details, please write to the authors.

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