Lexicographic Fréchet Matching

Günter Rote

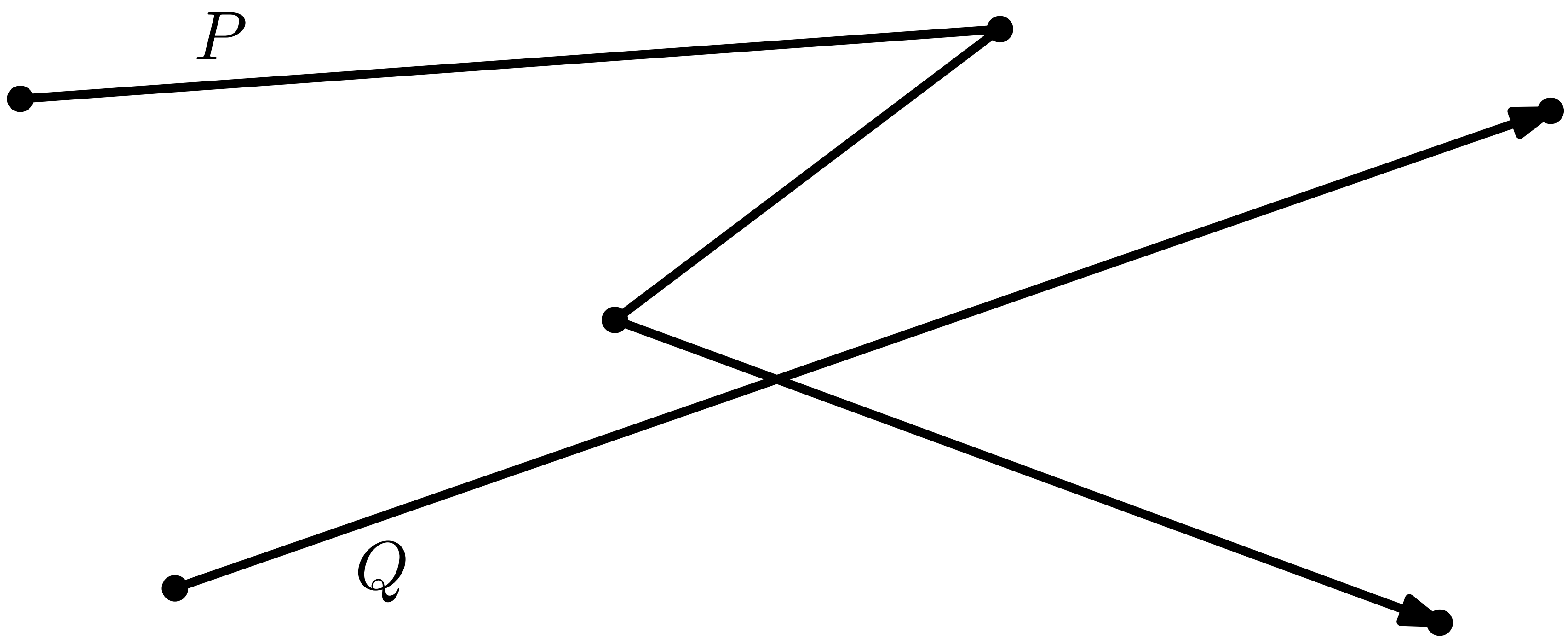
Freie Universität Berlin

Matching between two Curves

$P\colon [0, L_P] \to \mathbb{R}^2$

$Q\colon [0, L_Q] \to \mathbb{R}^2$

two curves



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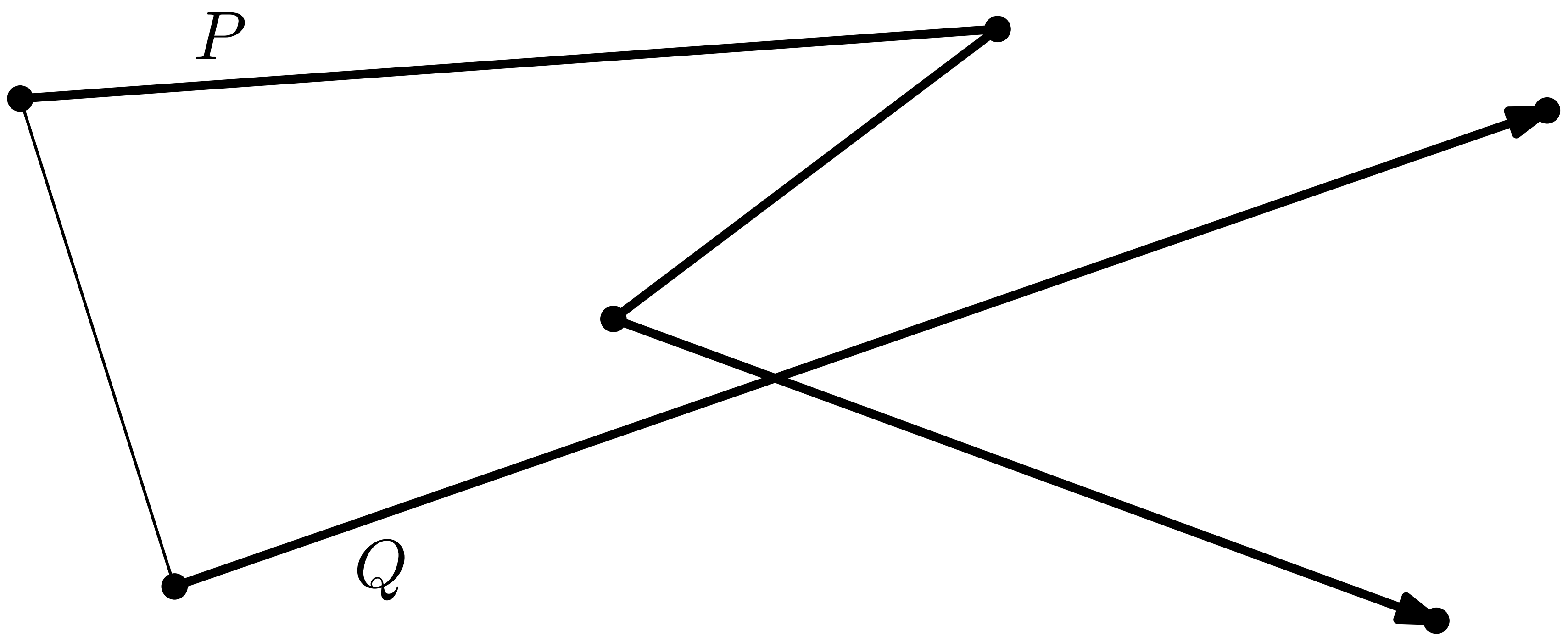

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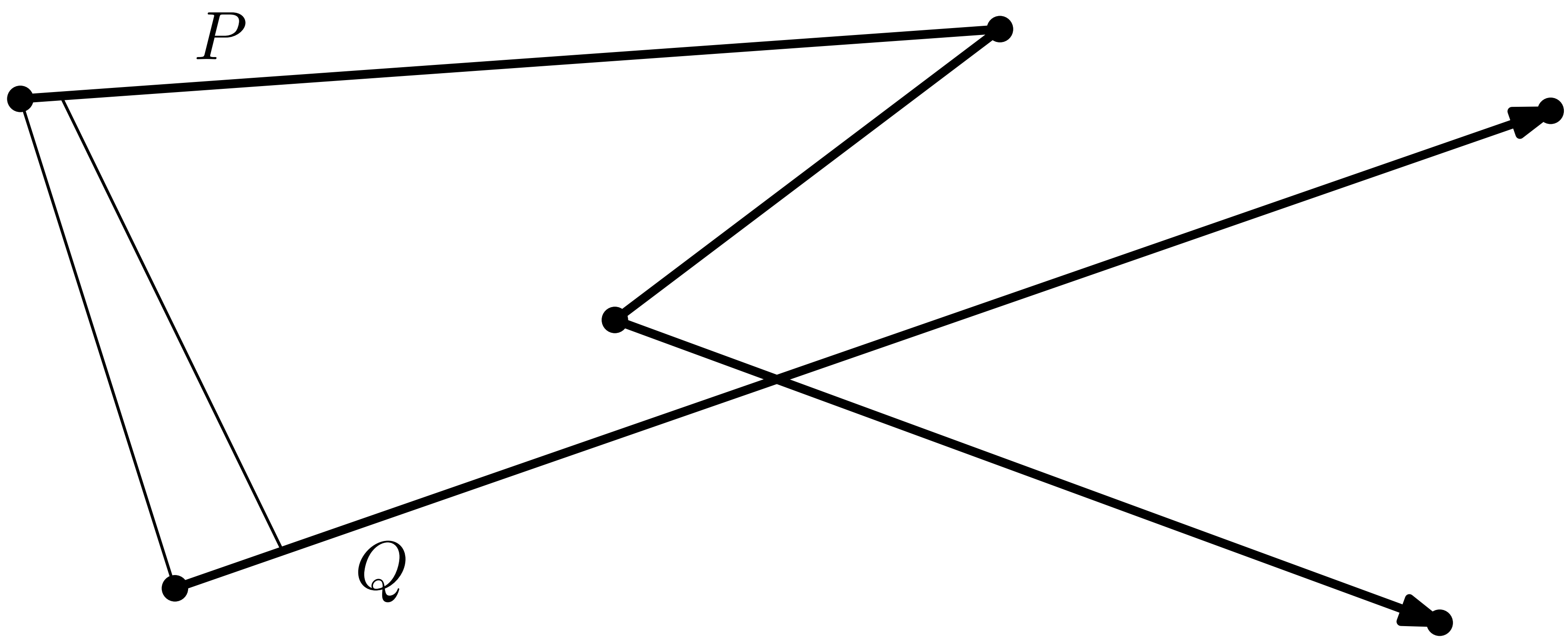

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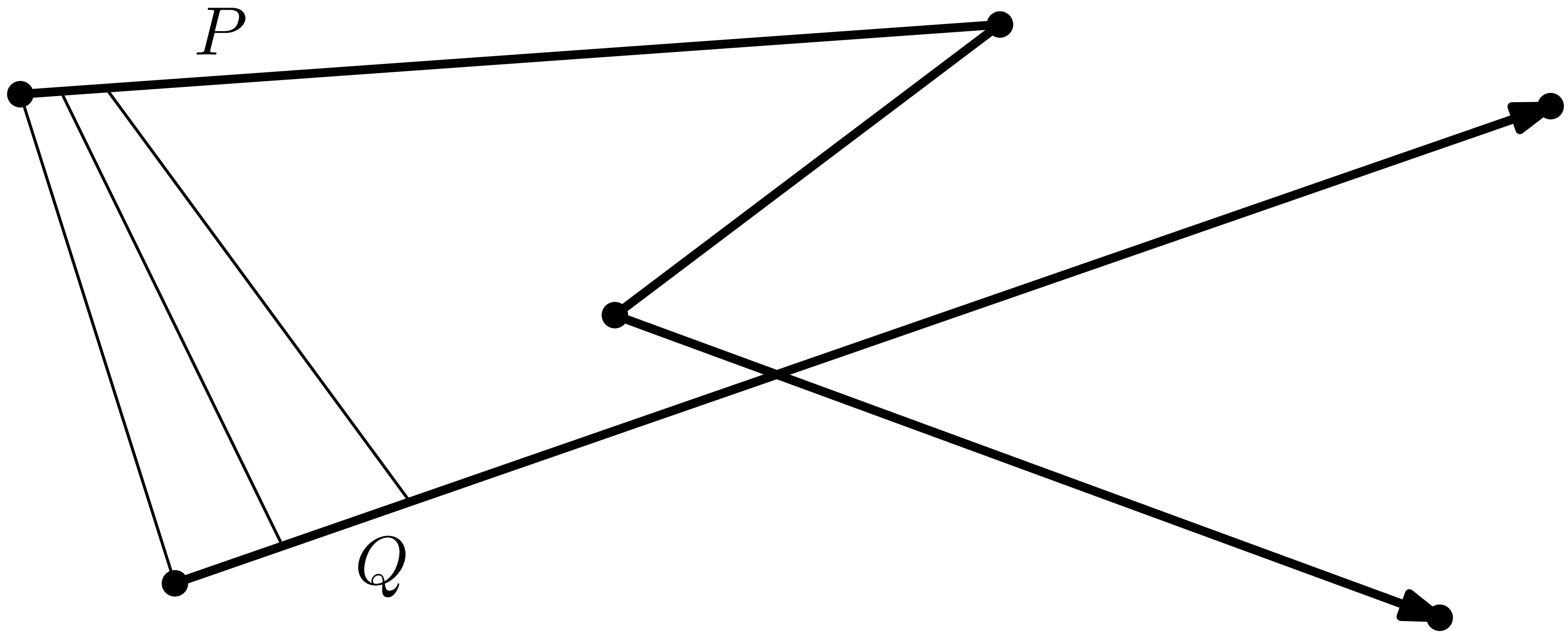

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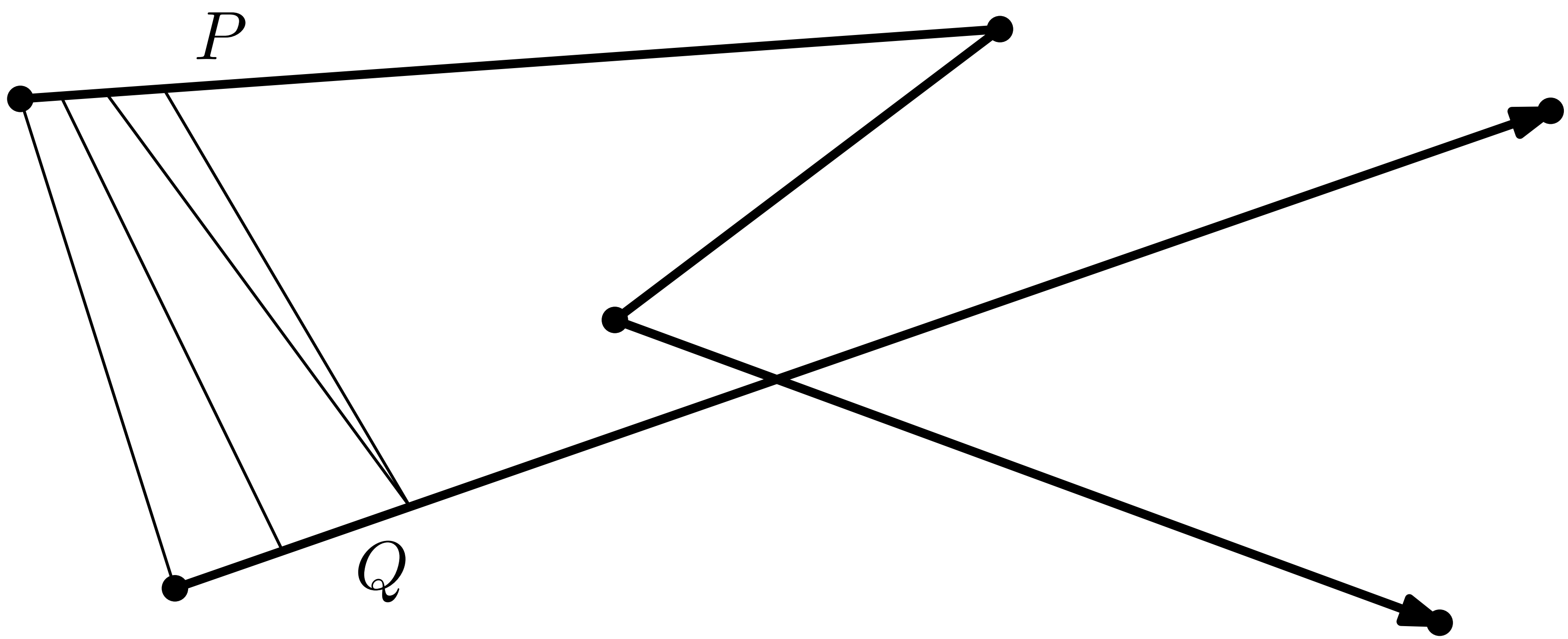

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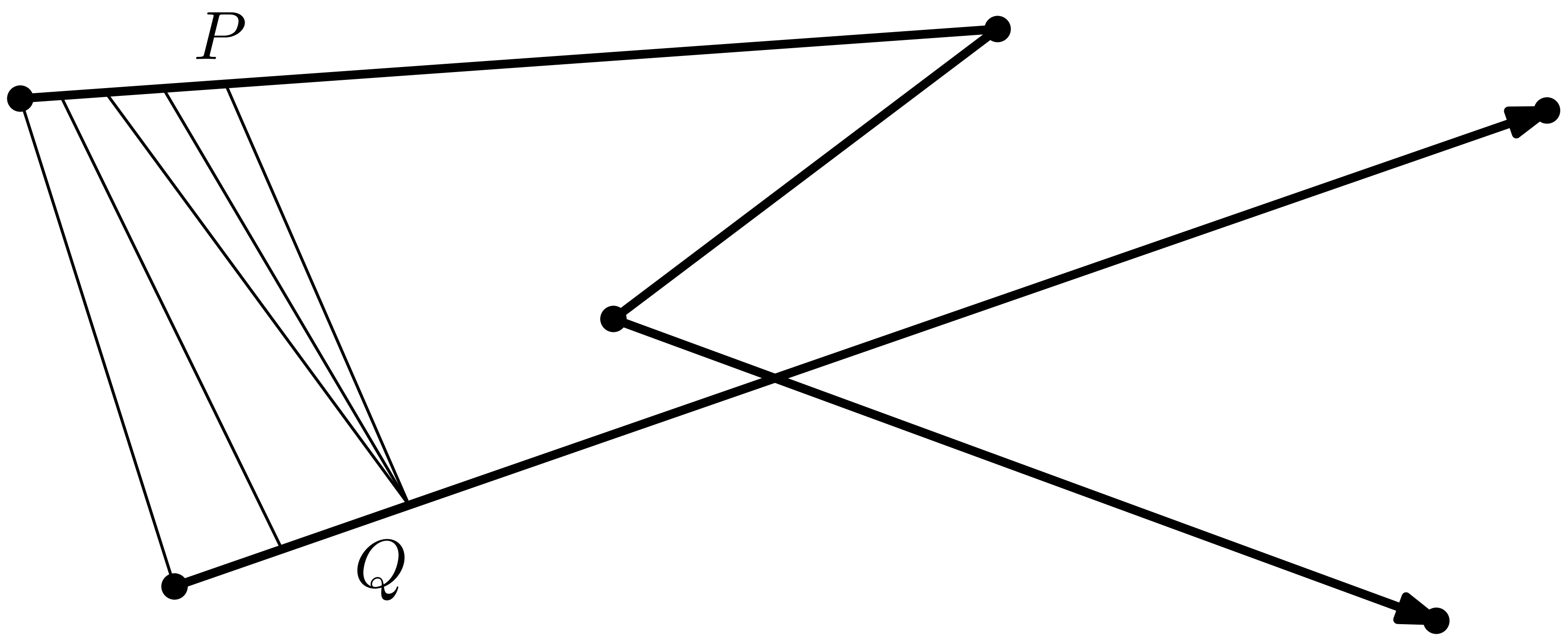

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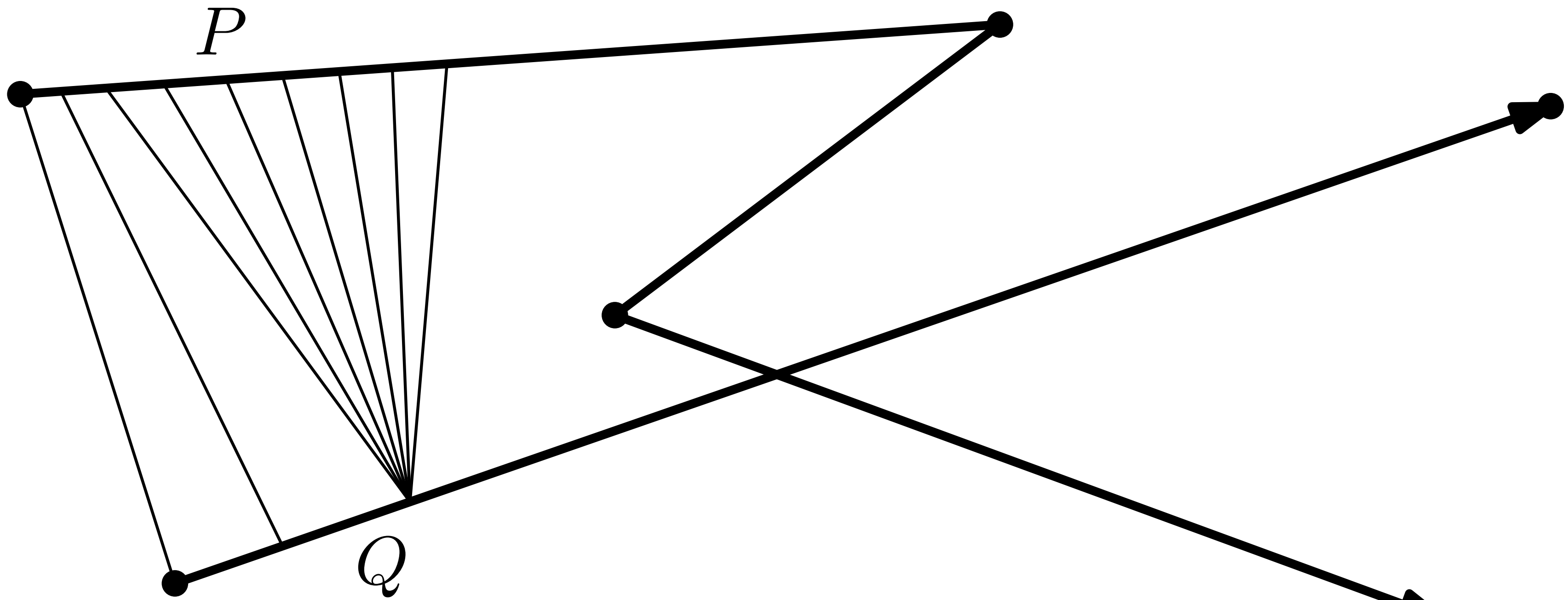

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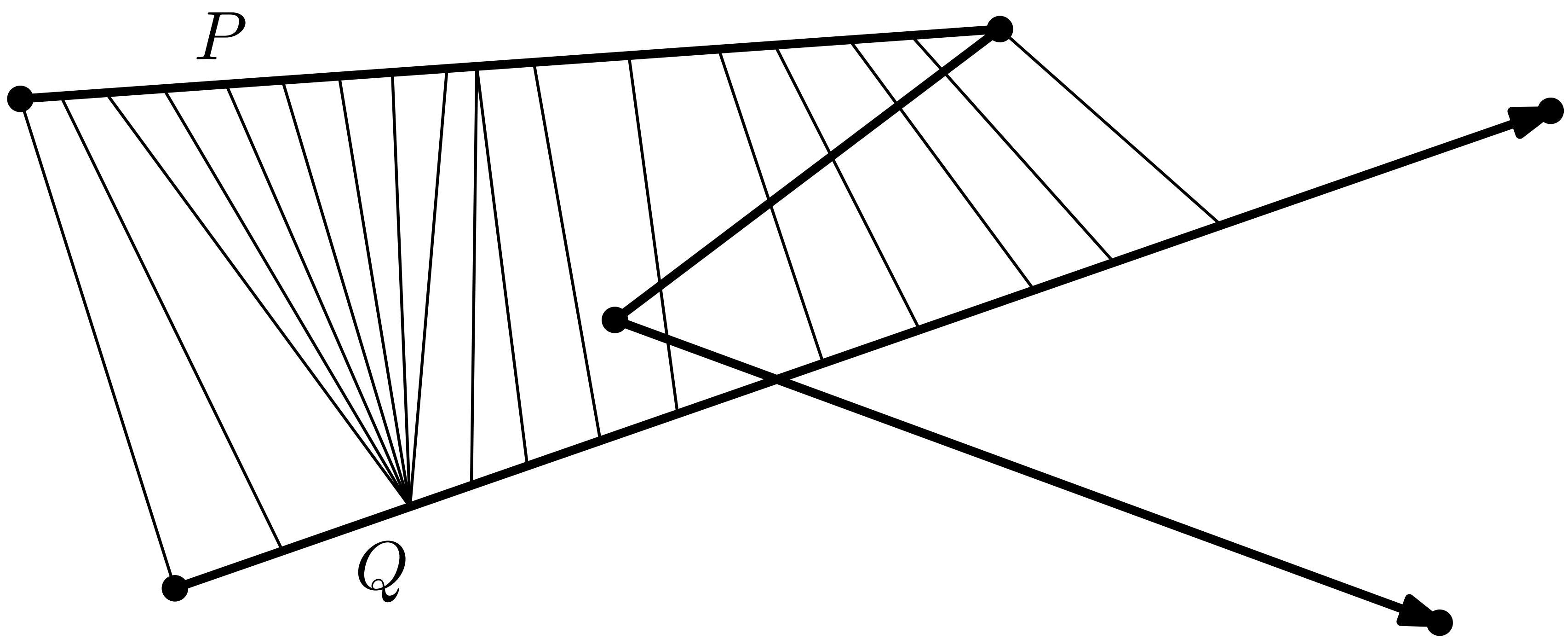

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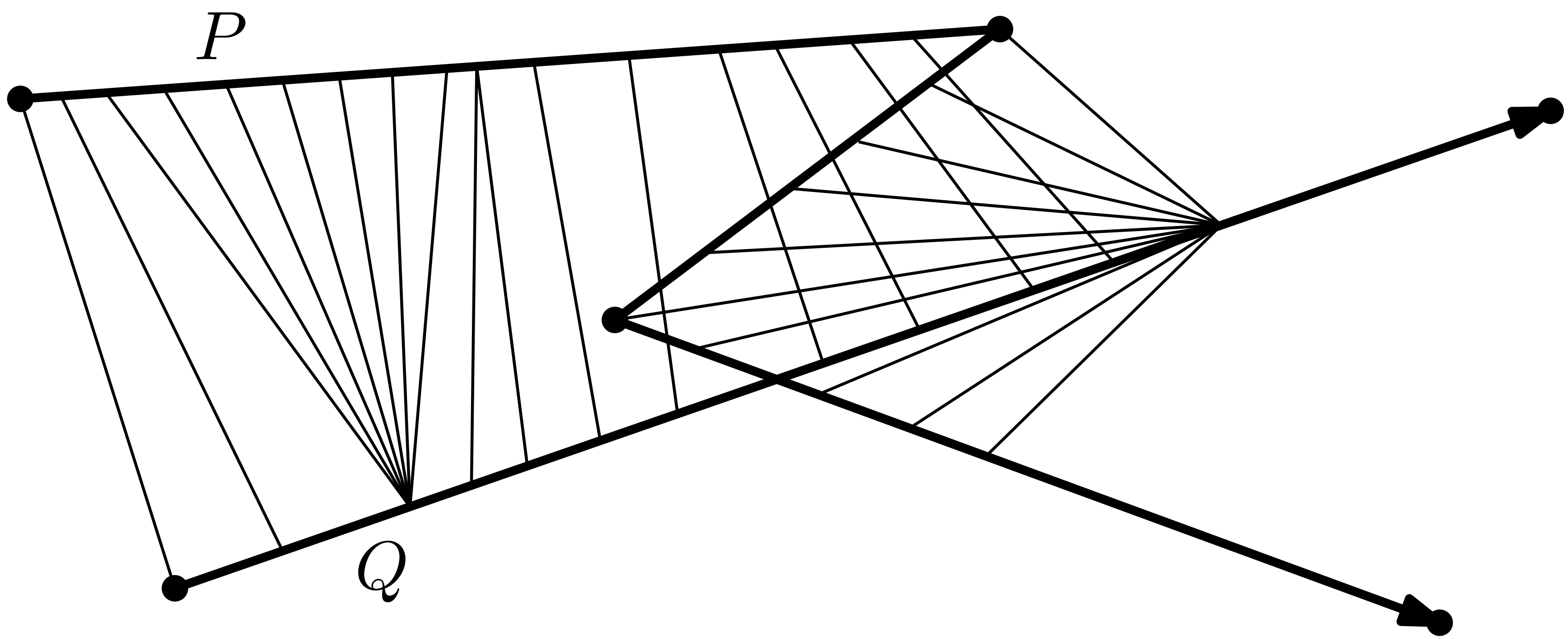

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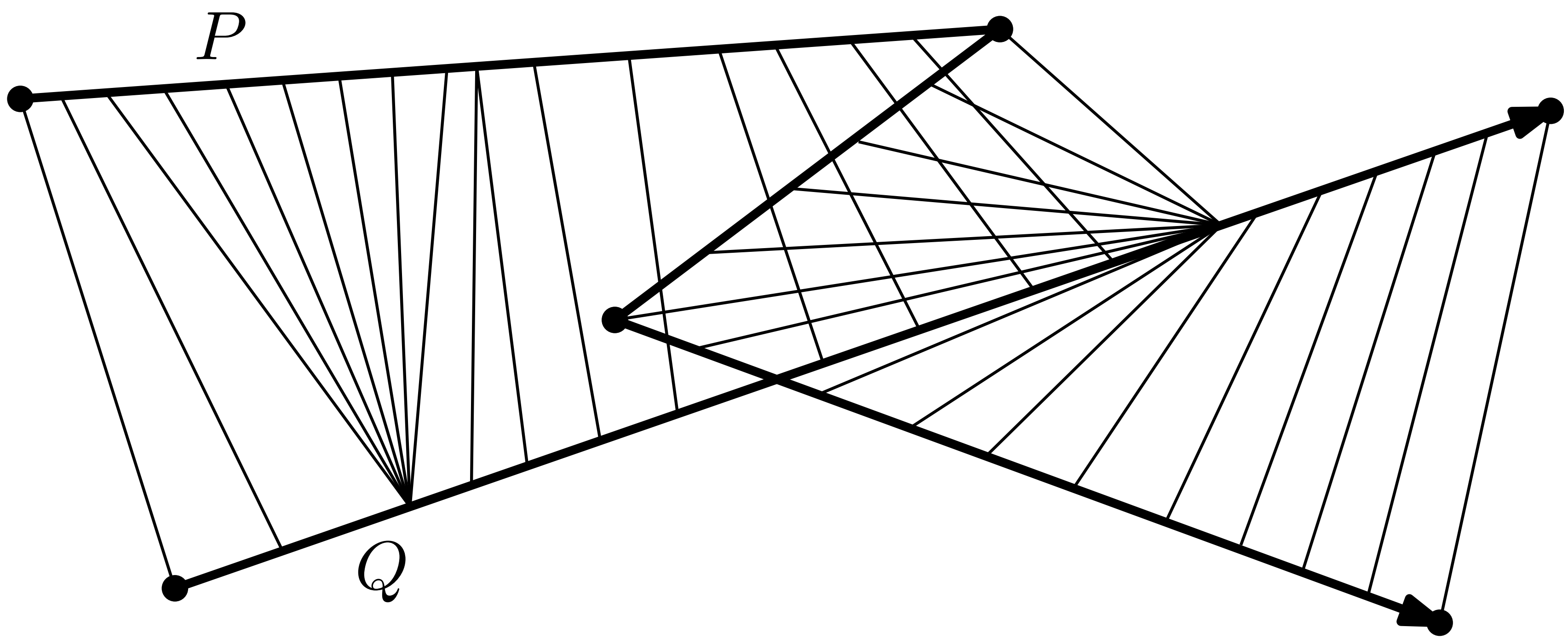

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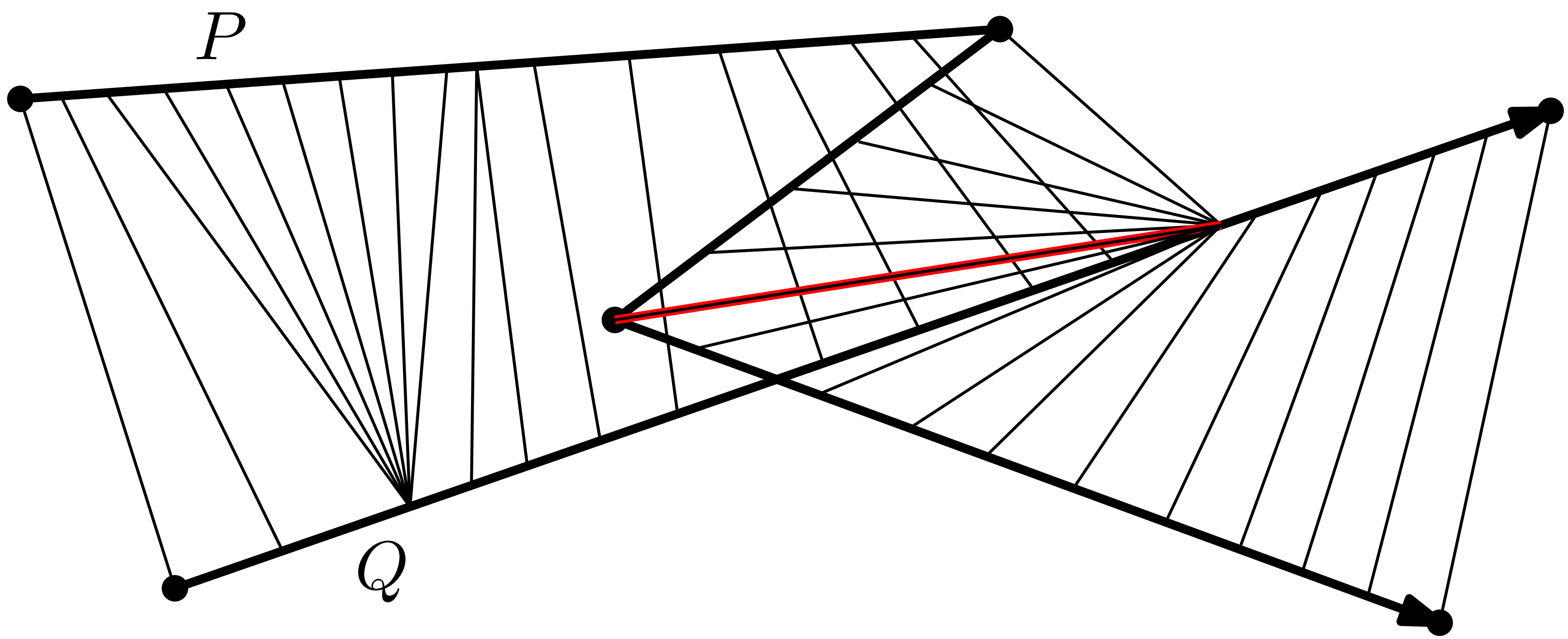

Fréchet distance [Alt Godau 1995]:

$$\max\{\, \|P(\alpha(t)) - Q(\beta(t))\| : 0 \le t \le M \,\} \to \text{MIN!}$$

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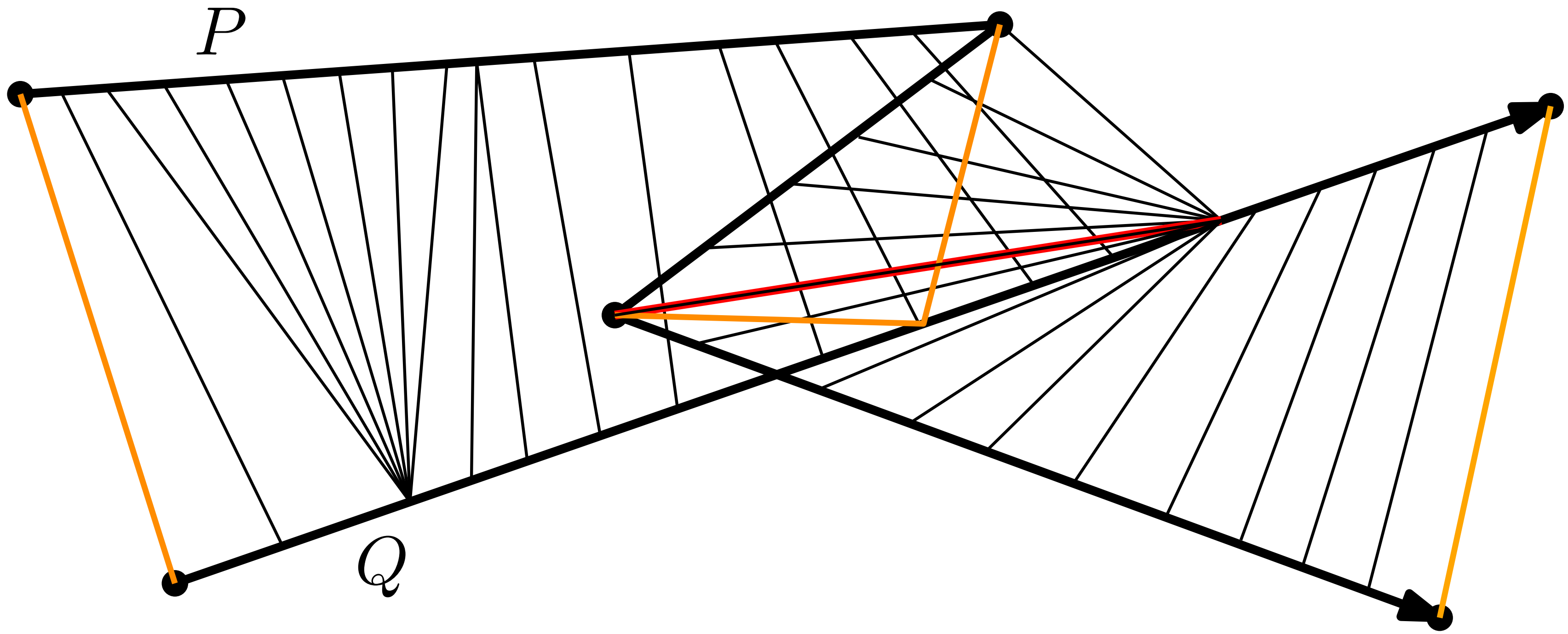

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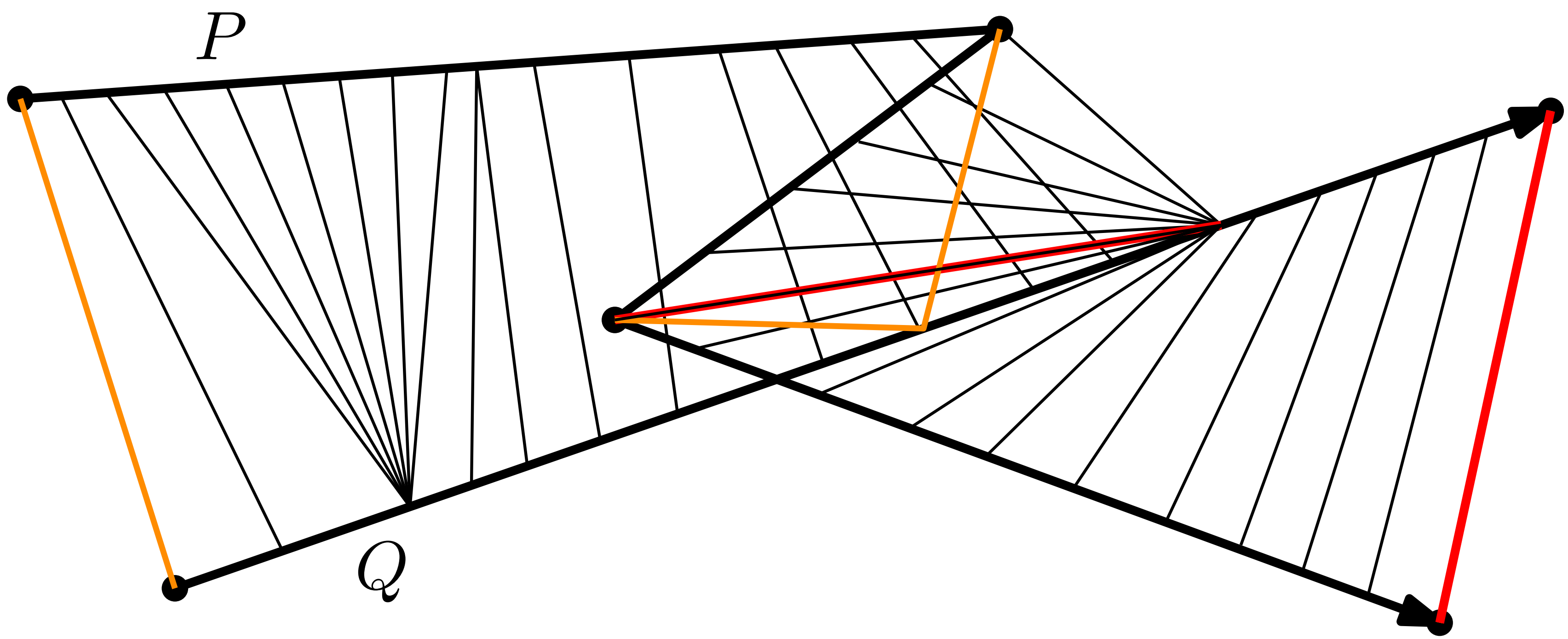

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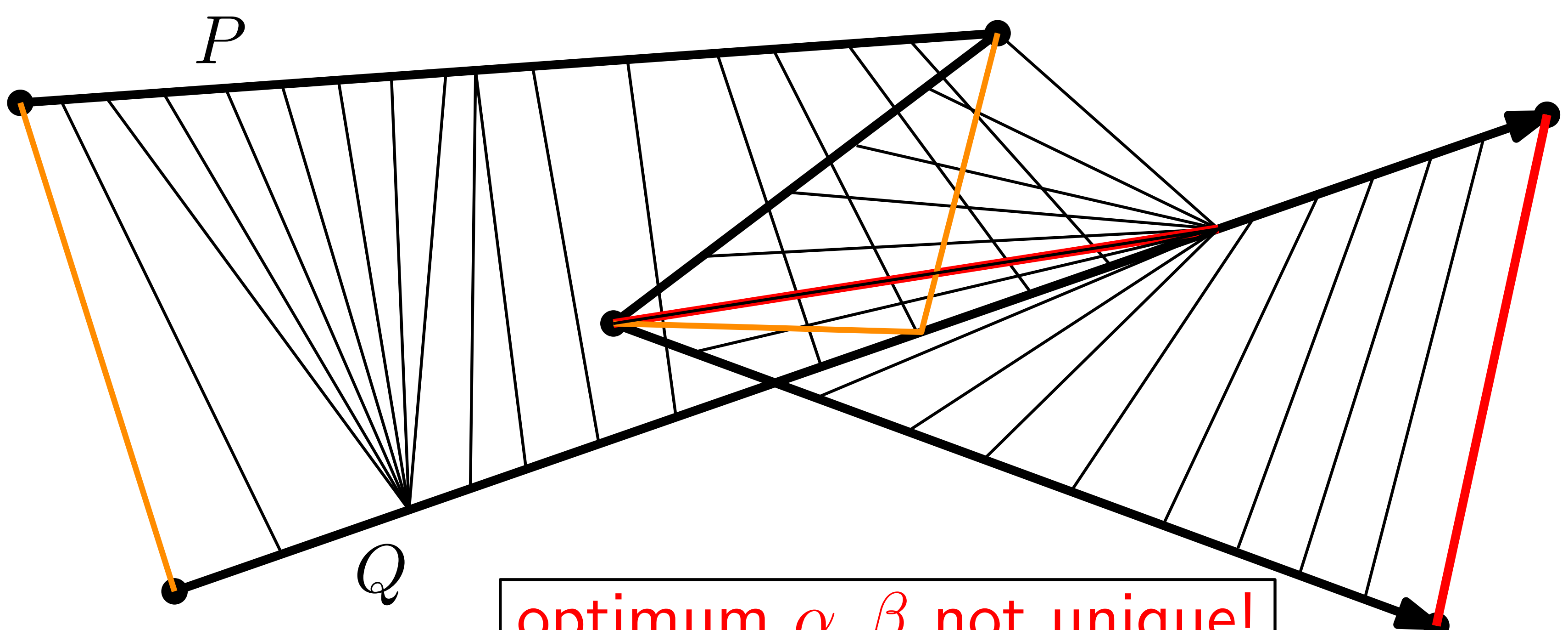

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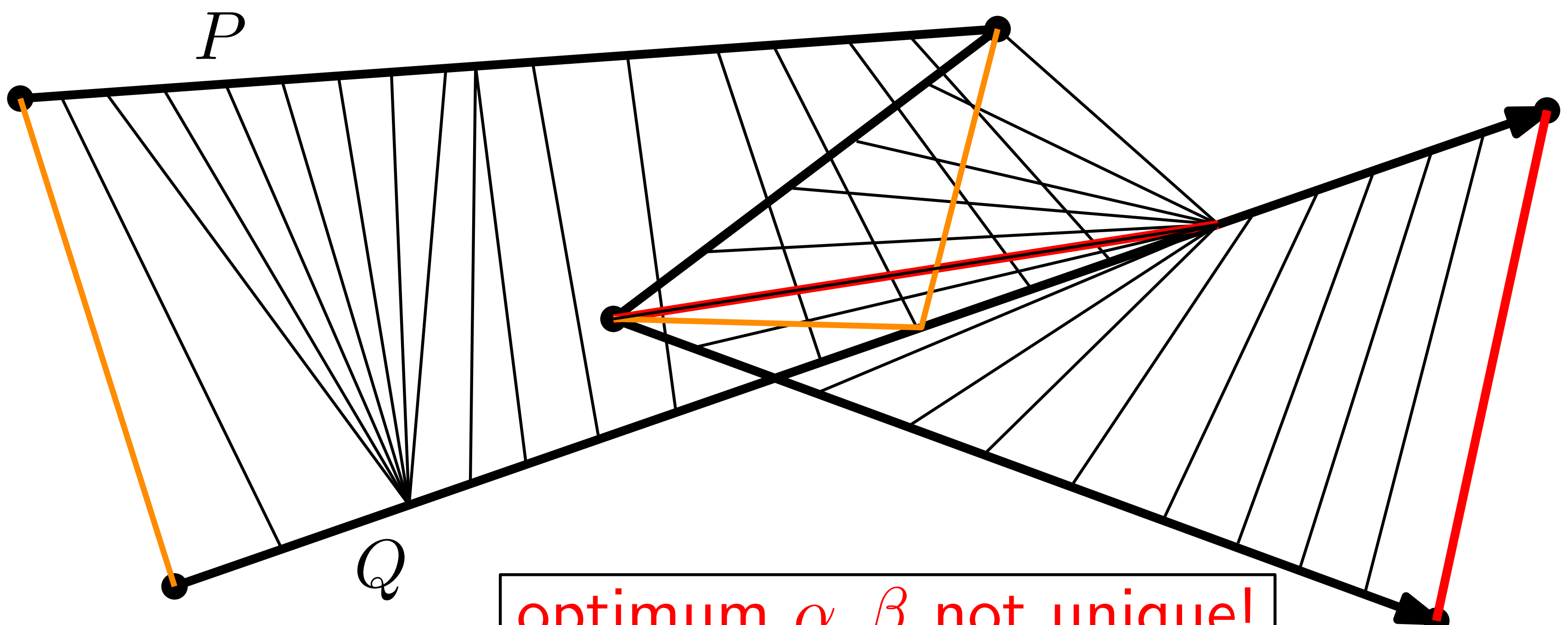

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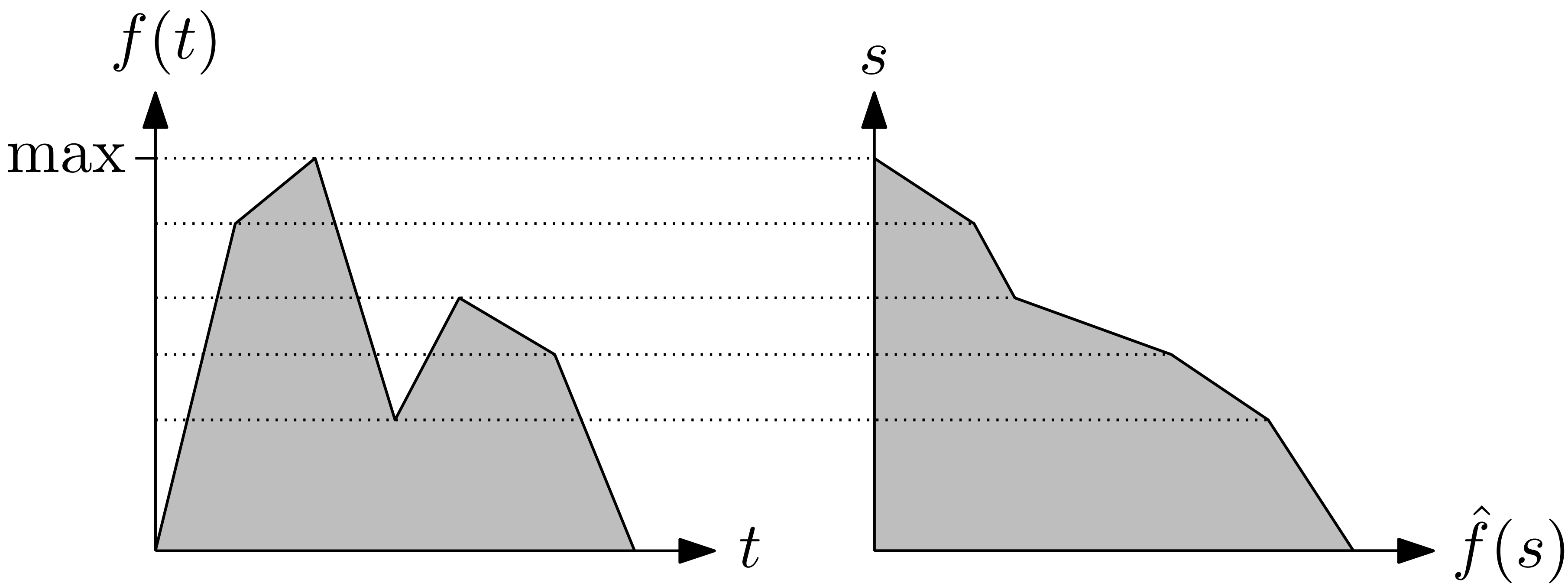
Fréchet distance [Alt Godau 1995]:

$$\max\{\, \underbrace{\|P(\alpha(t)) - Q(\beta(t))\|}_{f(t) \,=\, \text{distance function}} : 0 \le t \le M \,\} \to \text{MIN!}$$

Comparison of Distance Functions

Goal: a finer criterion than $\max\{\, f(t) : 0 \le t \le M \,\}$



profile function $\hat{f}\colon \mathbb{R} \to \mathbb{R}_{\ge 0}$:

$\hat{f}(s) =$ the amount of time that $f(t)$ is at least s

$= \mu(\{\, t \mid f(t) \ge s \,\})$ (μ = Lebesgue measure)

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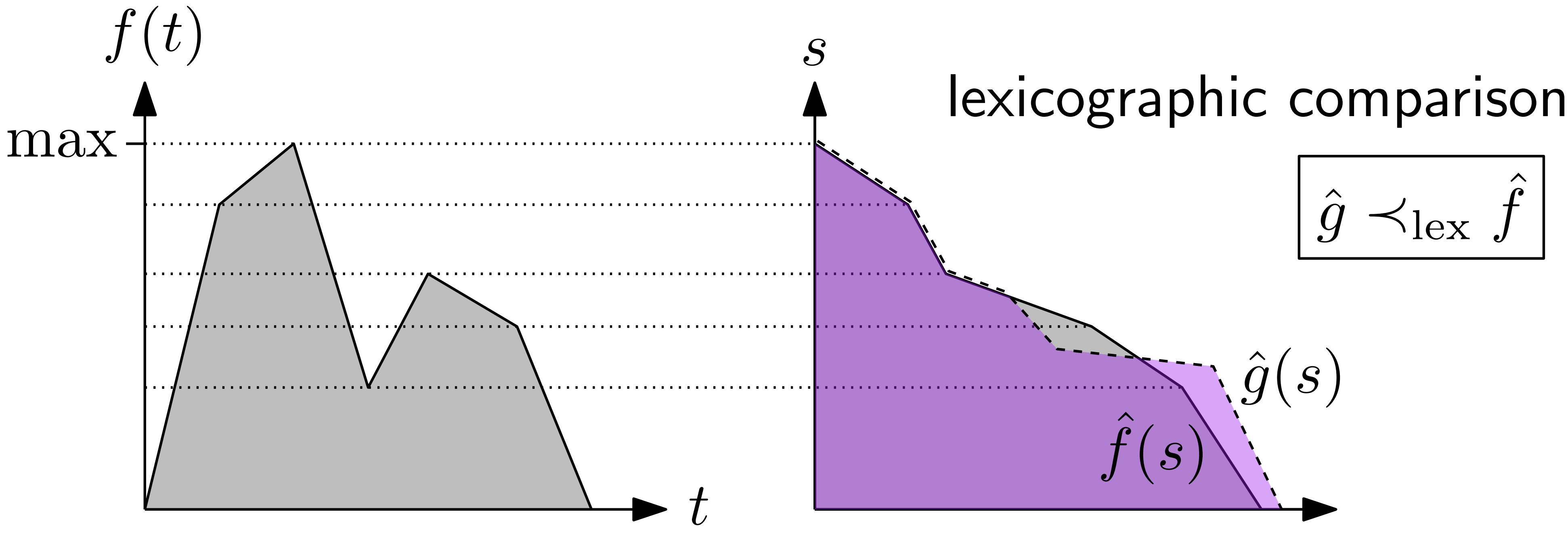

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Main Assumption:

The speed at which the curves P and Q are traversed by the parametrizations α and β is bounded by 1.

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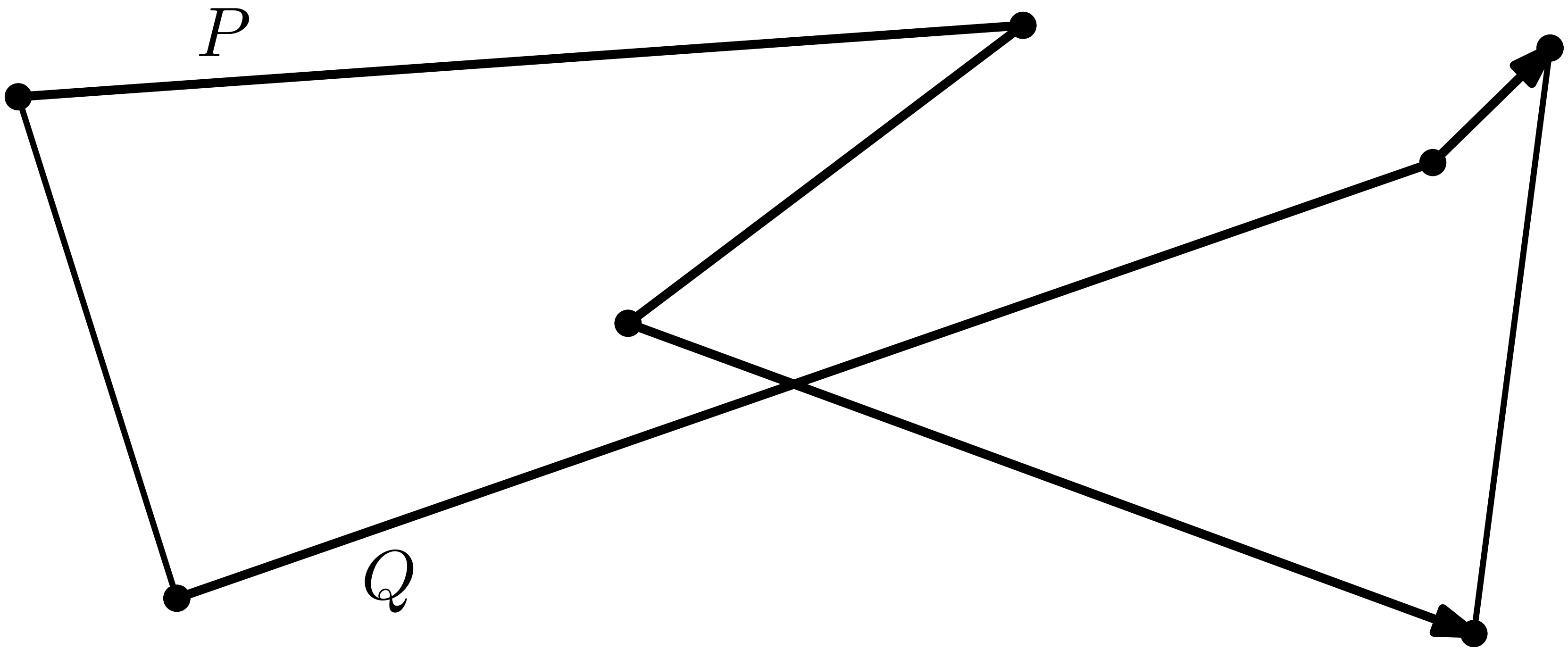
Assume arc-length parametrization for P and Q.

PROBLEM STATEMENT:
Minimize the profile $\hat{f}$ of the distance function

$$f(t) = \|P(\alpha(t)) - Q(\beta(t))\|$$

with respect to $\prec_{\text{lex}}$
under the constraints $\alpha'(t), \beta'(t) \leq 1$.

Example



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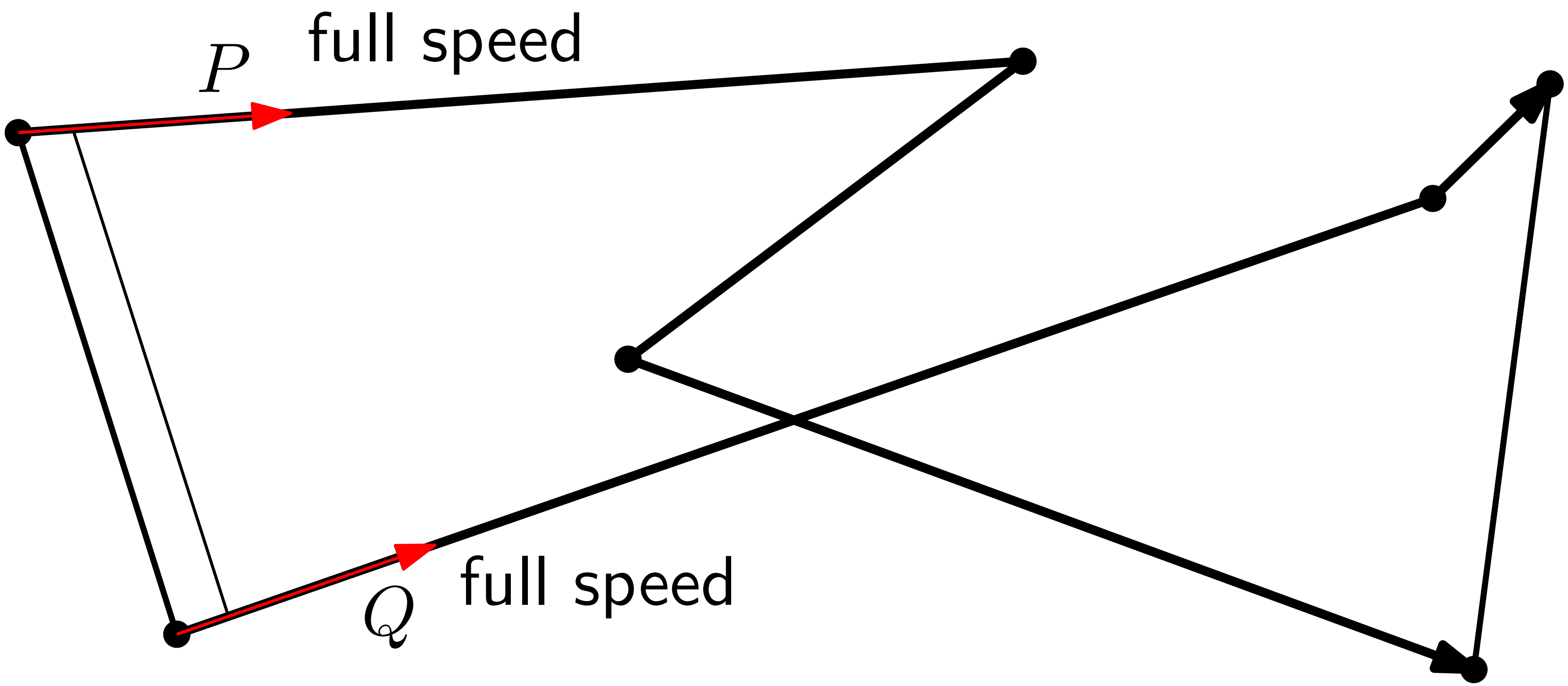

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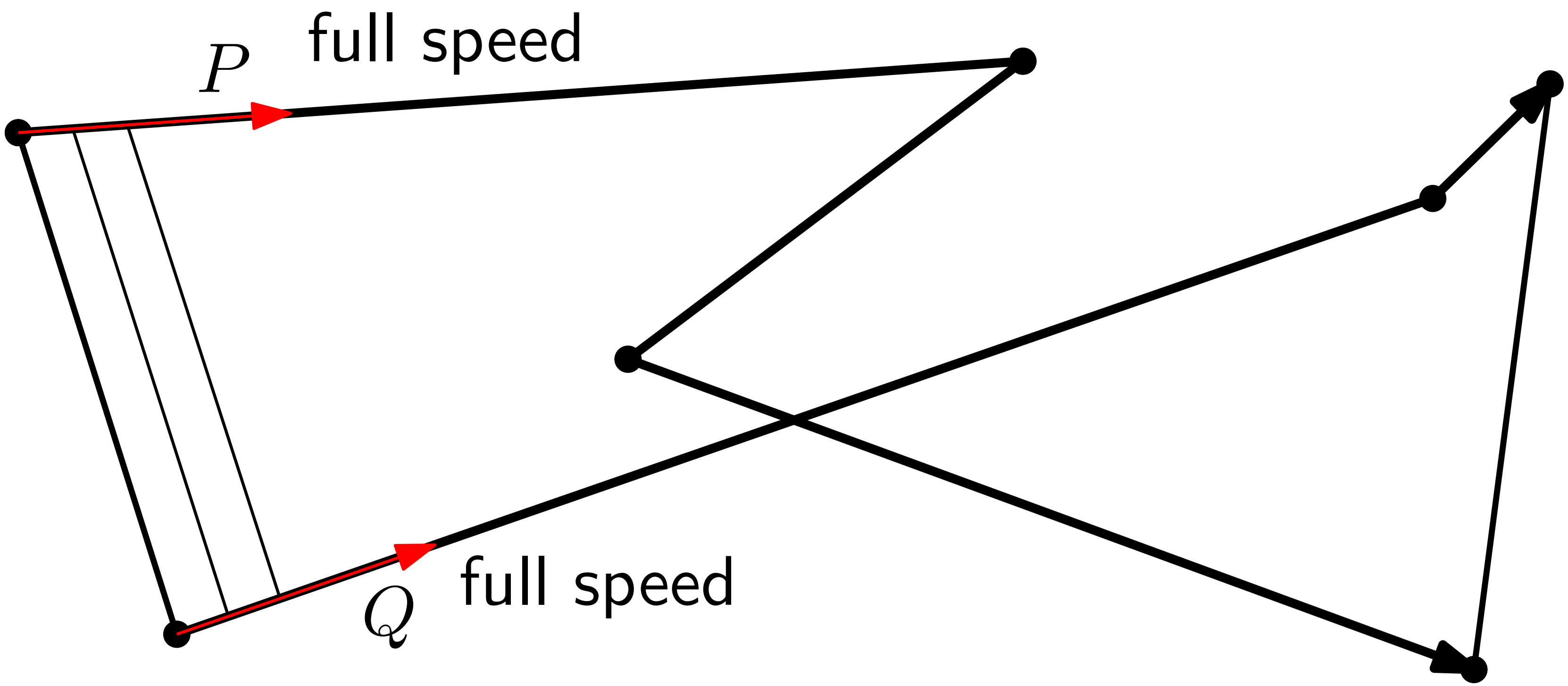

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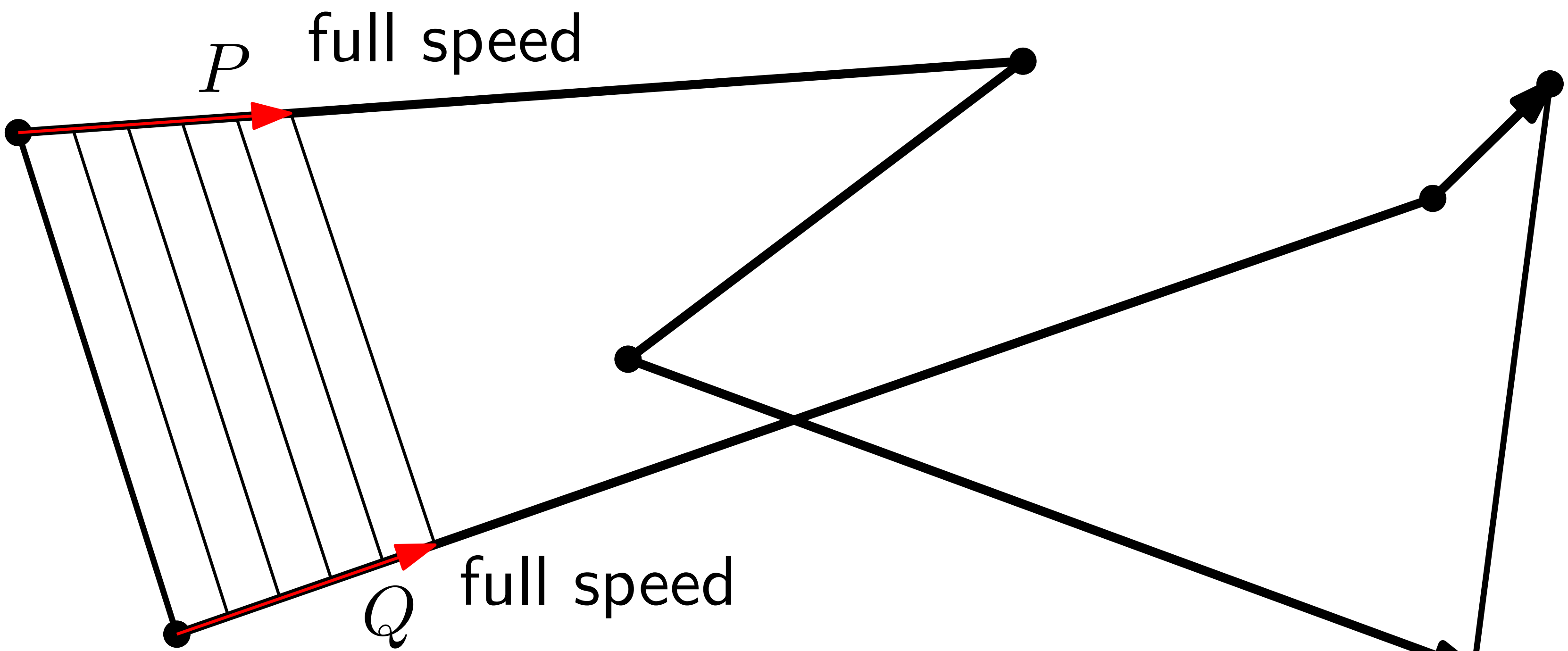

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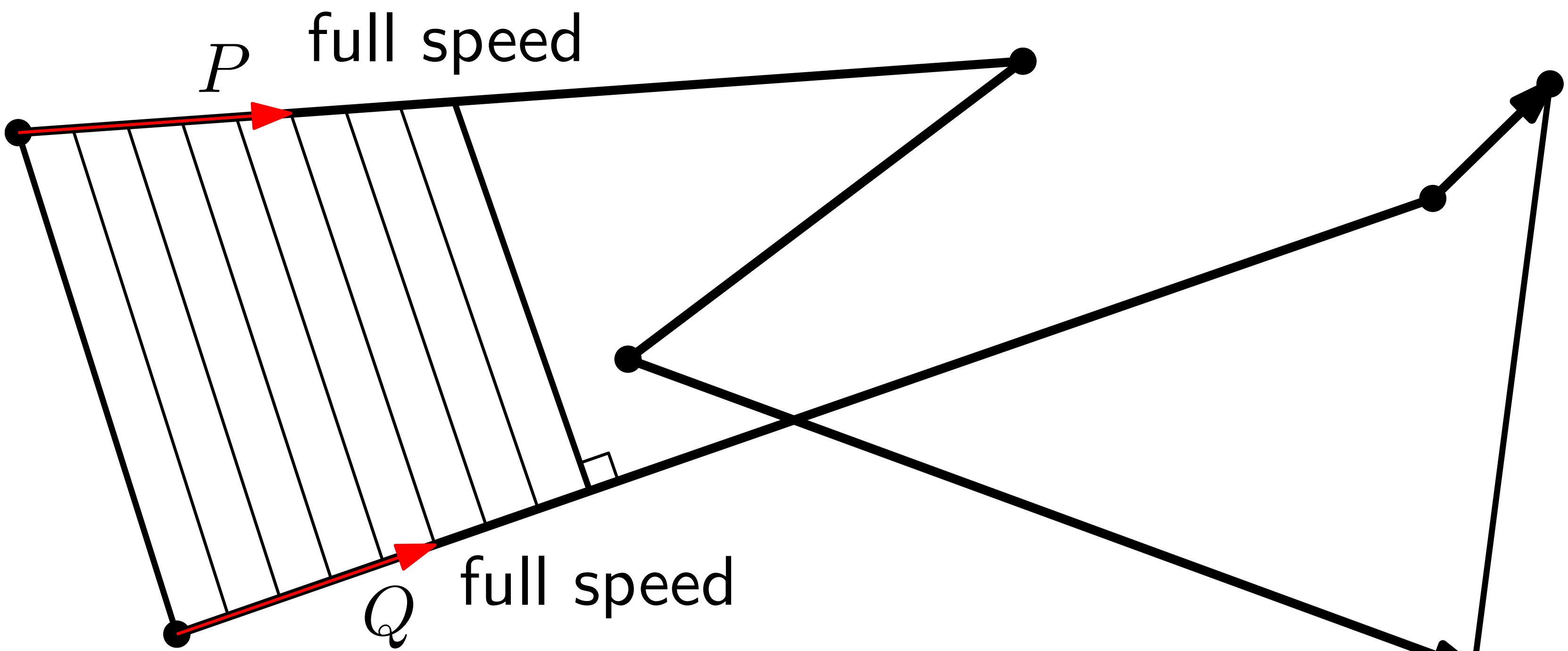

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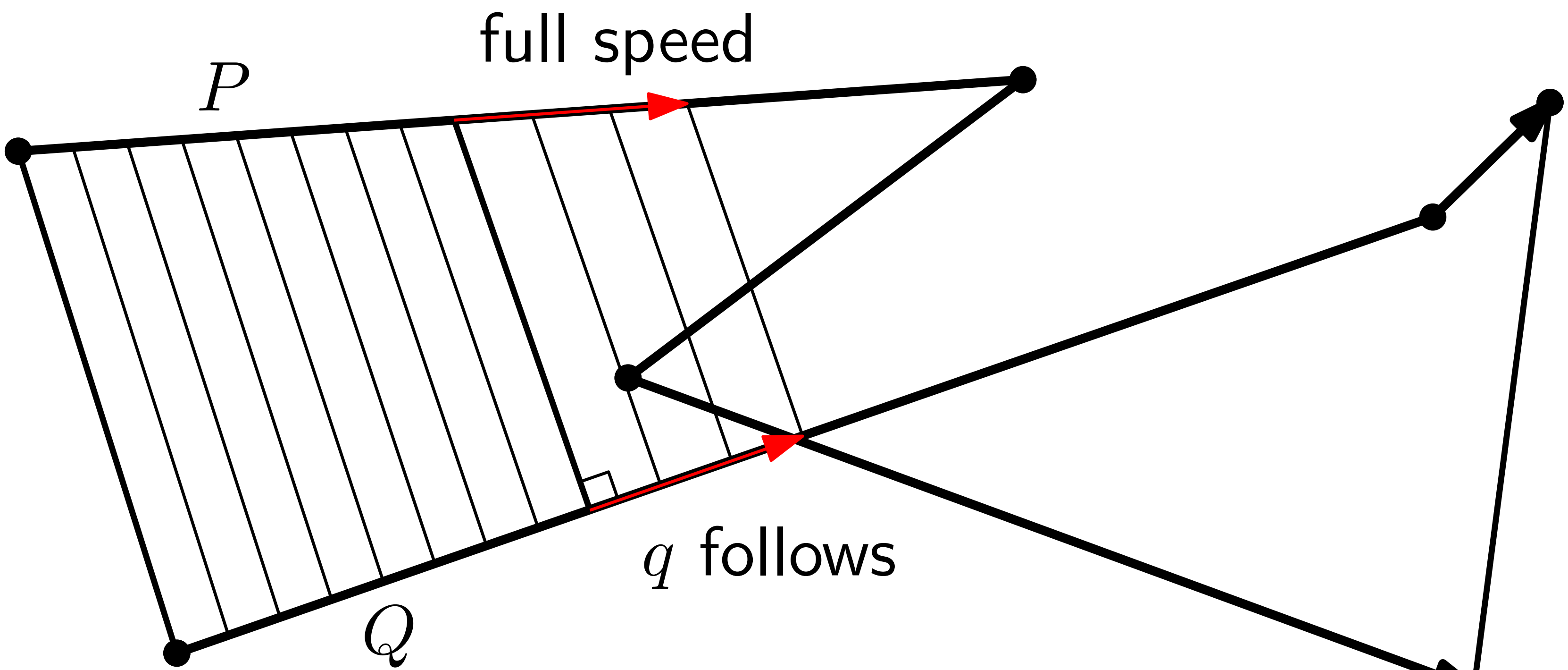

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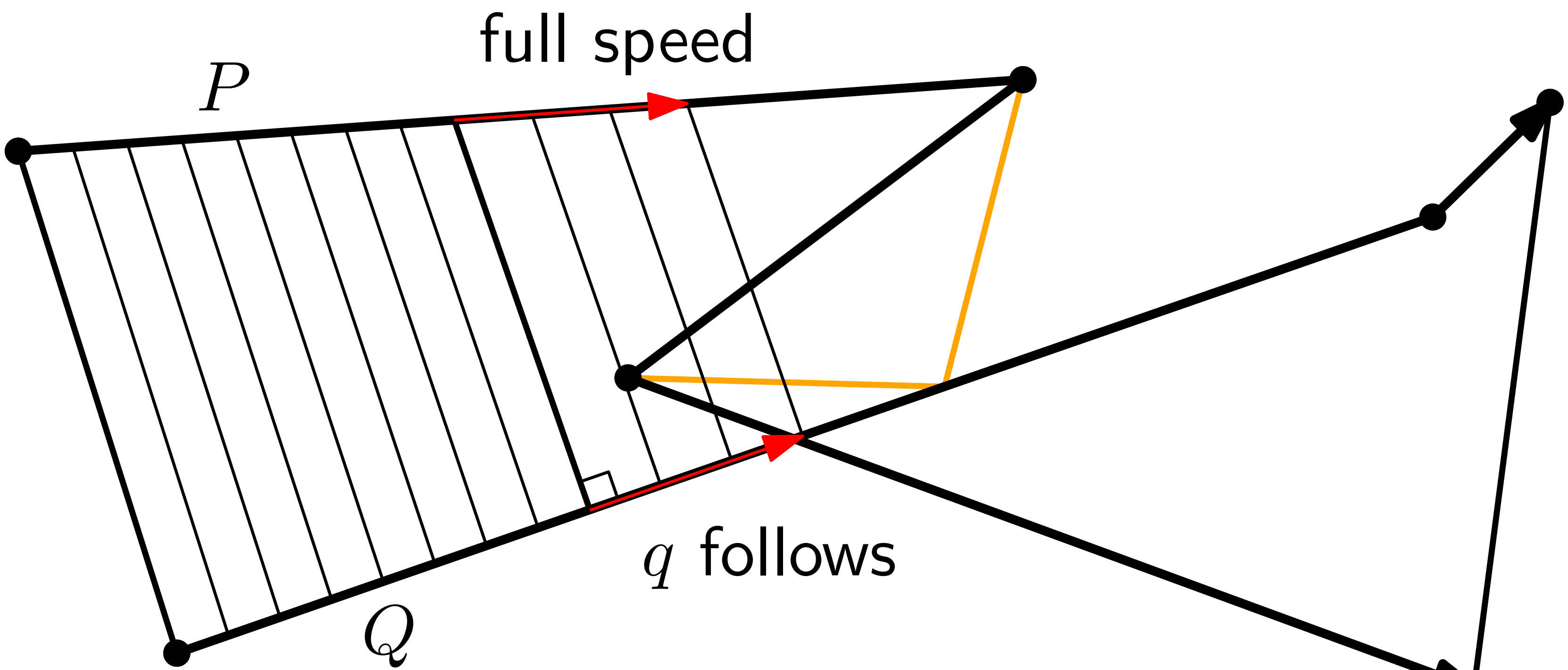

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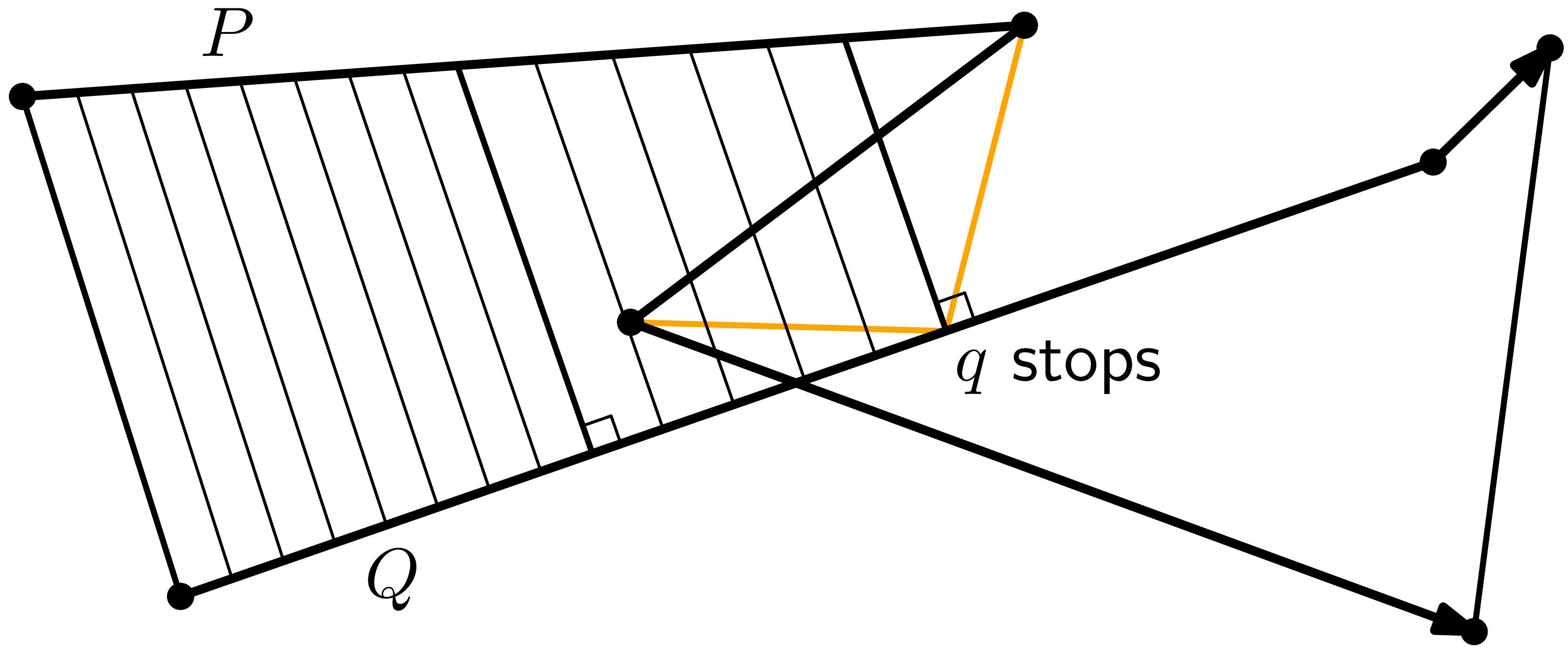

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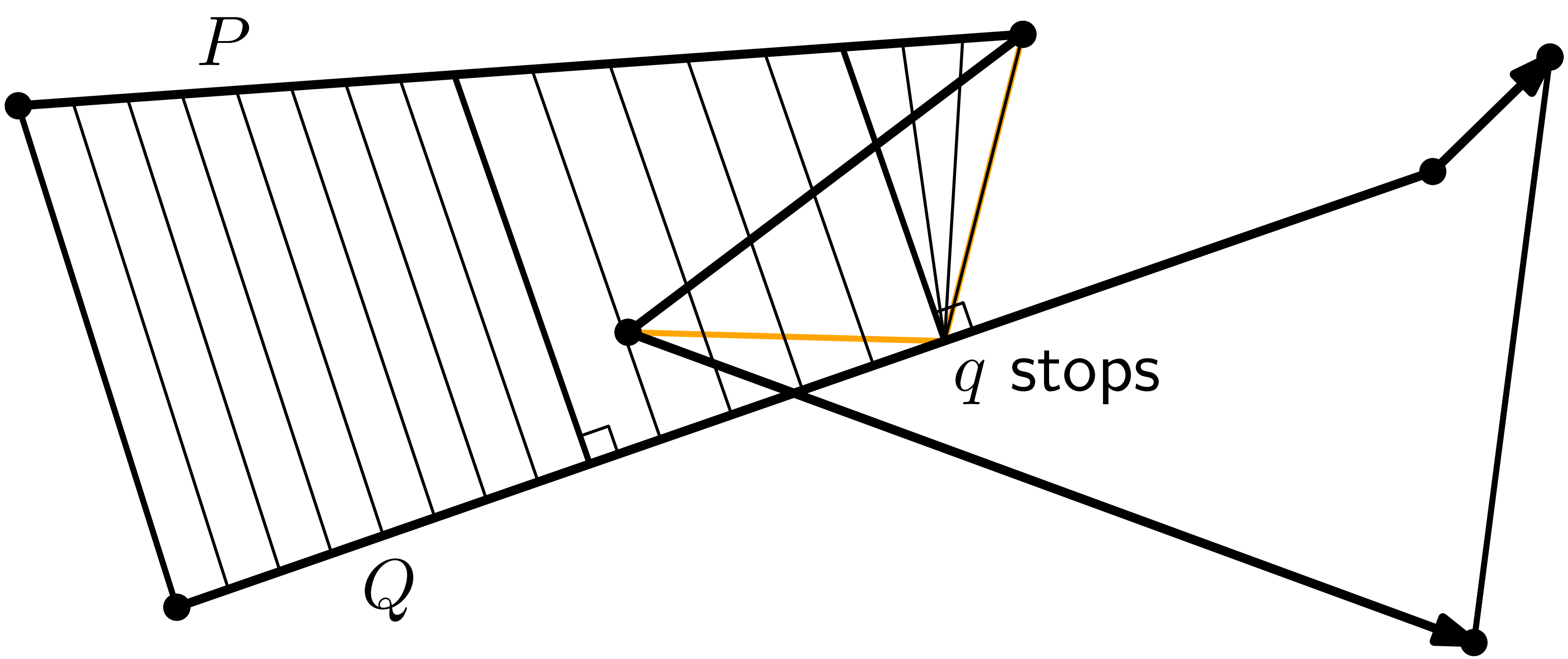

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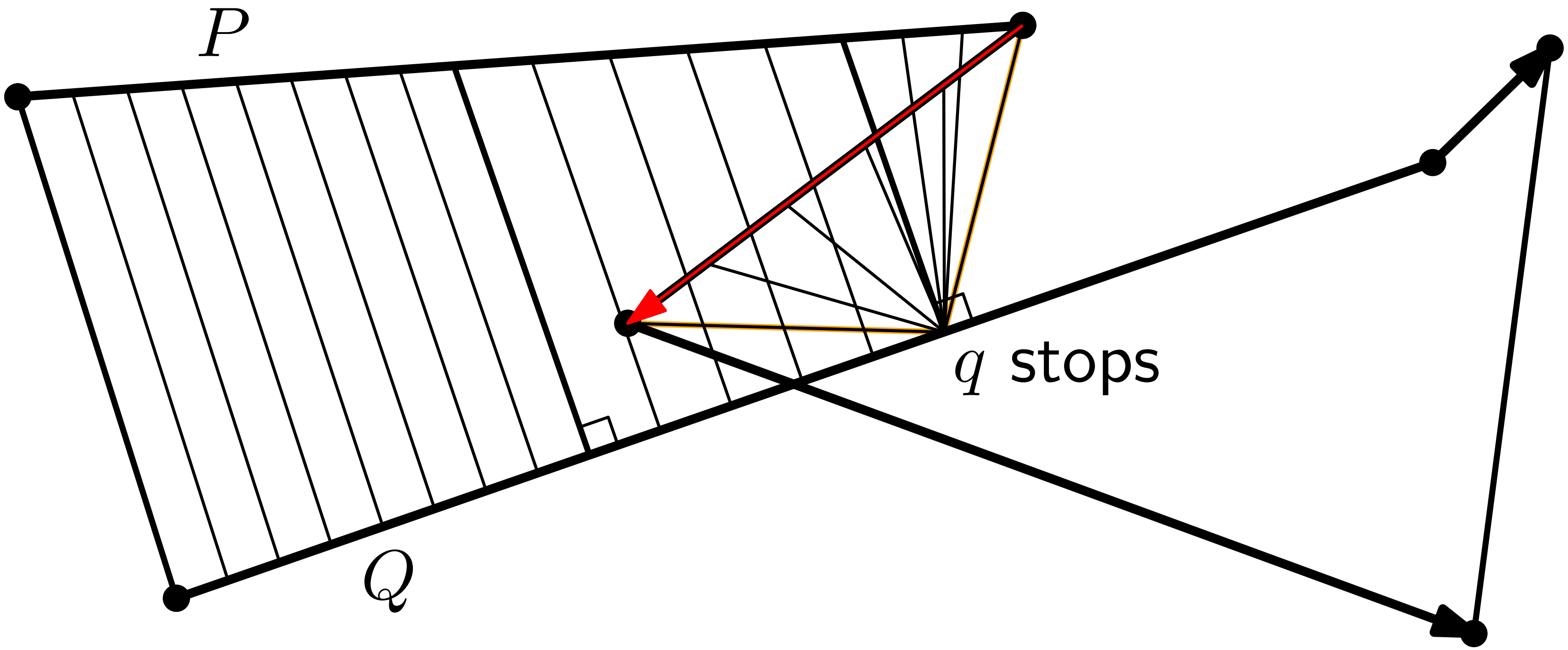

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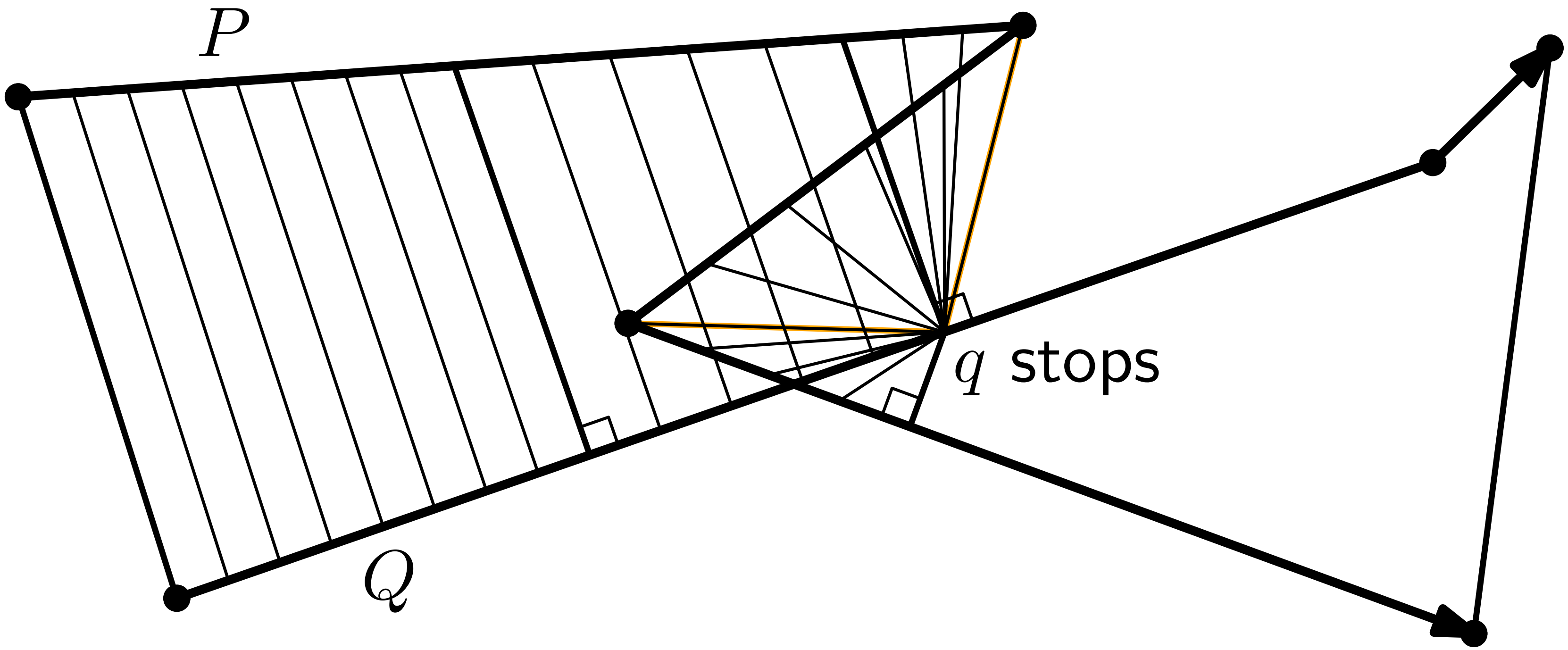

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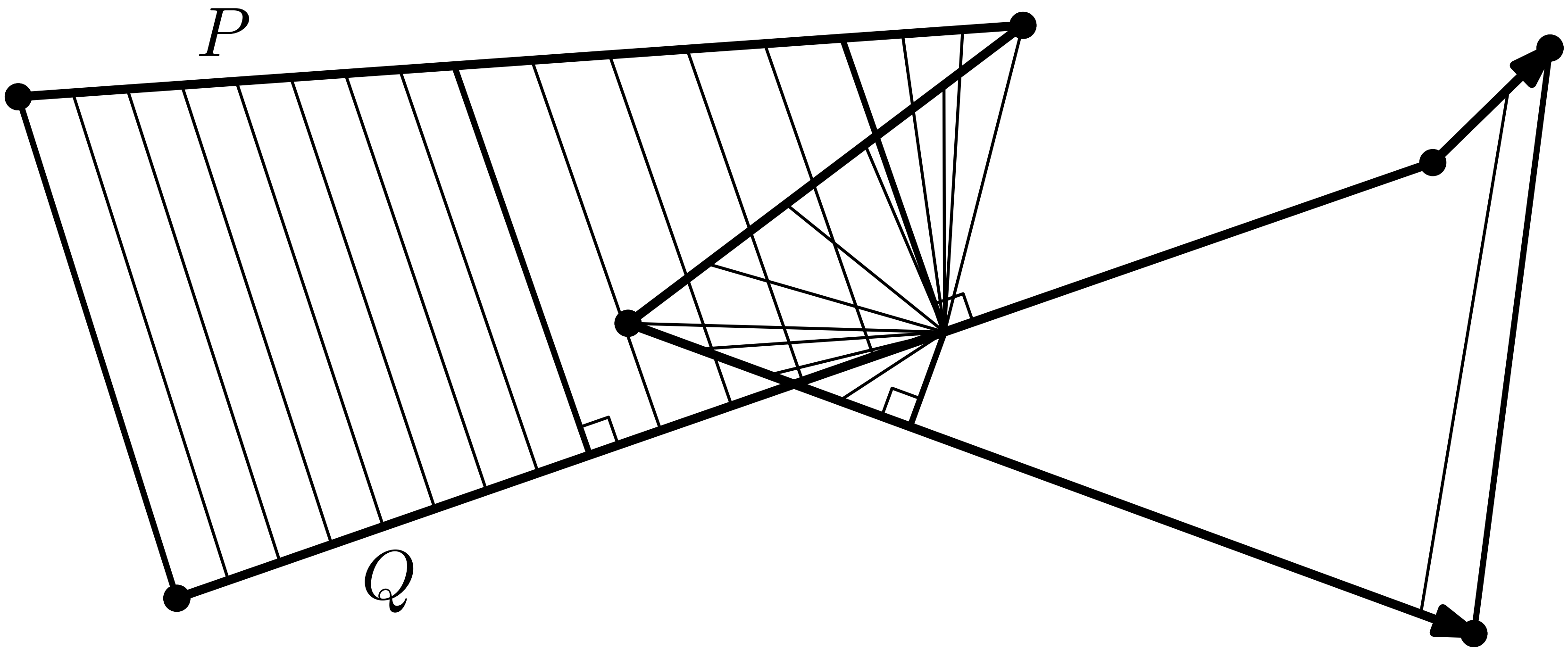

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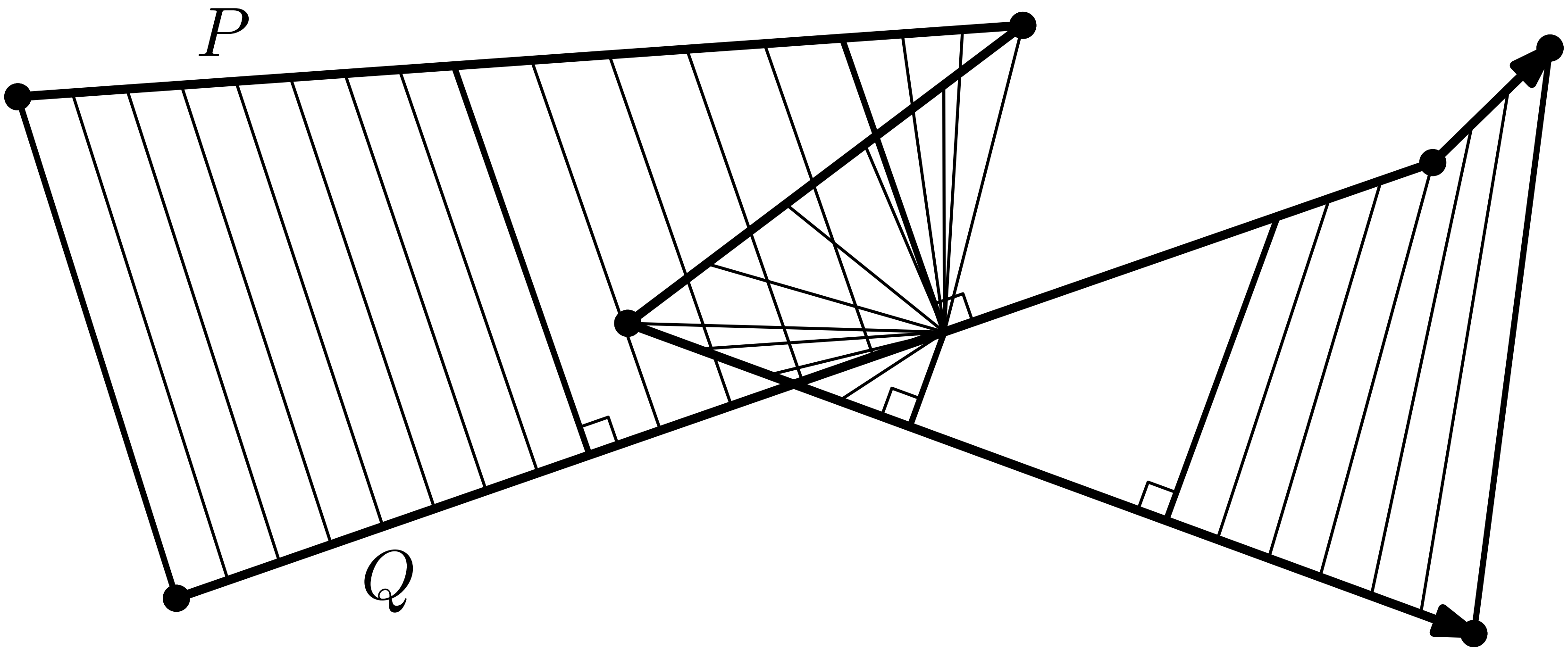

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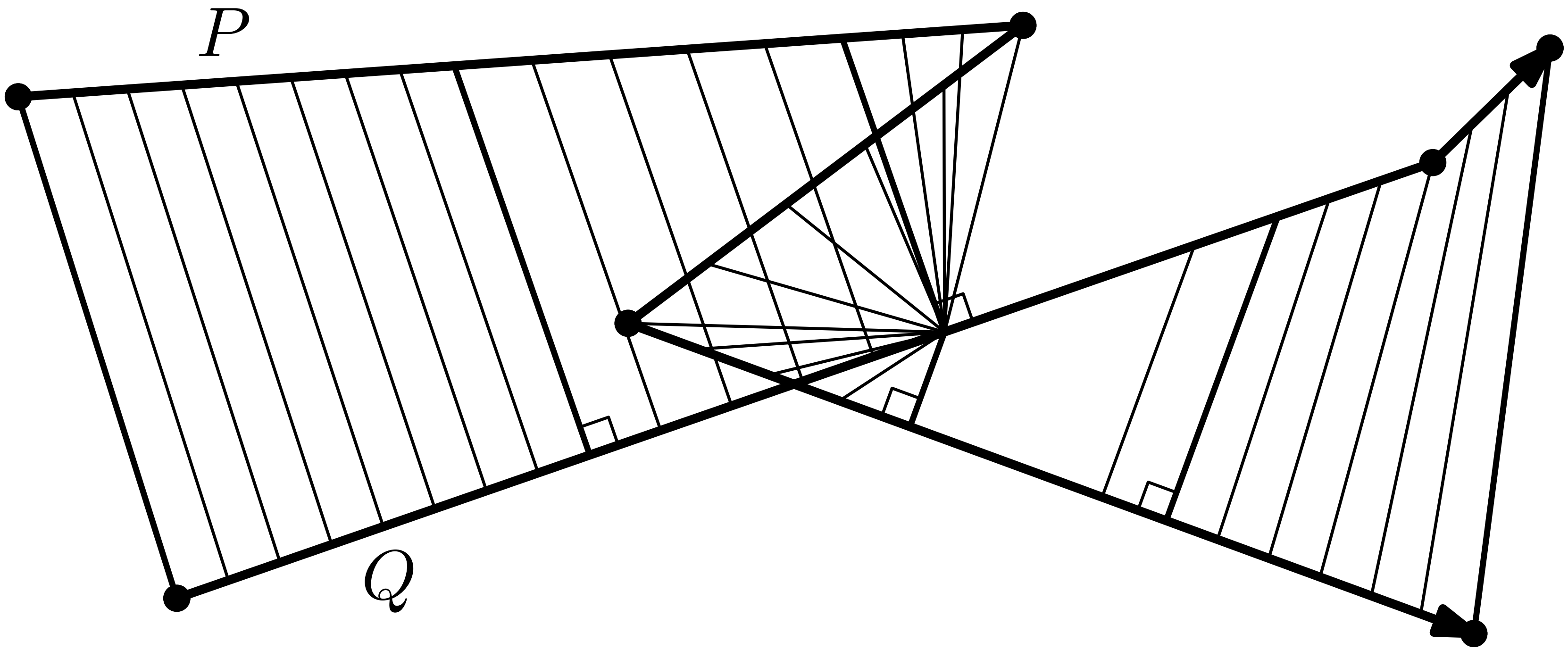

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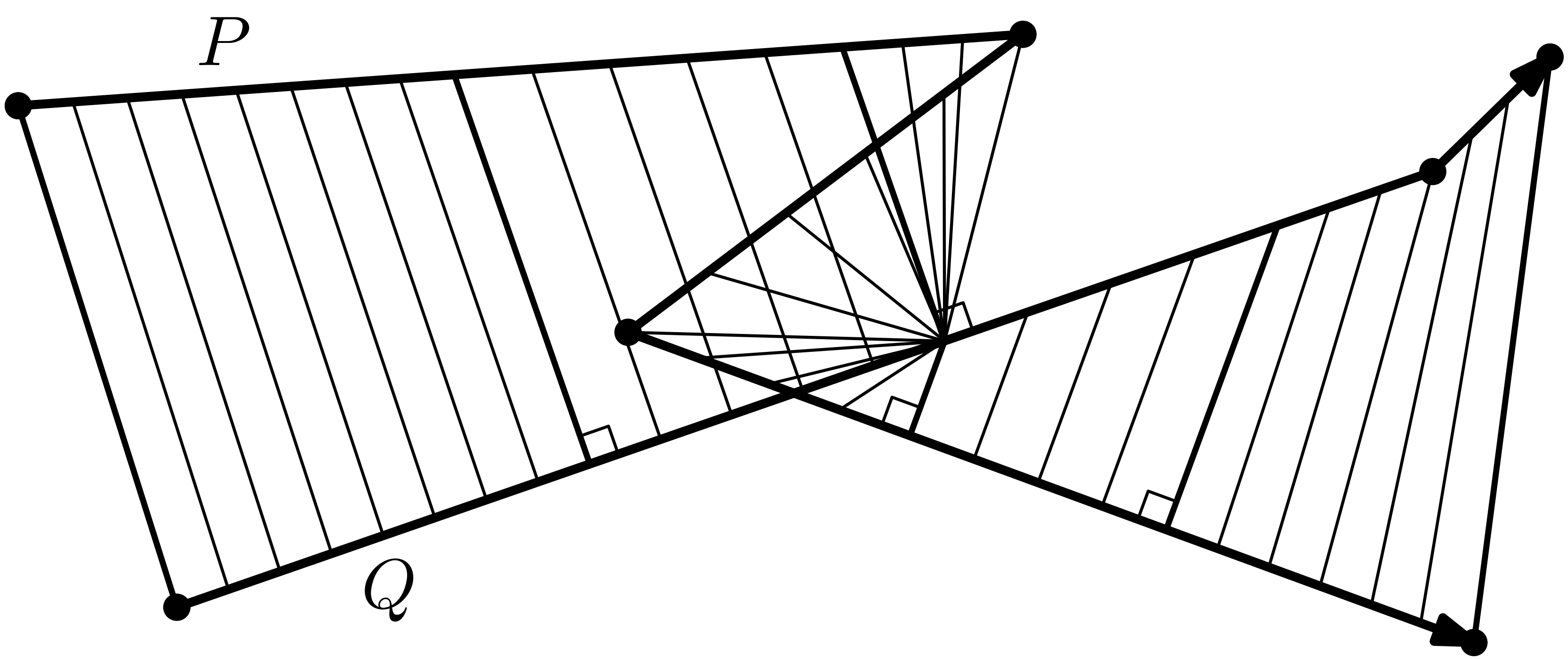
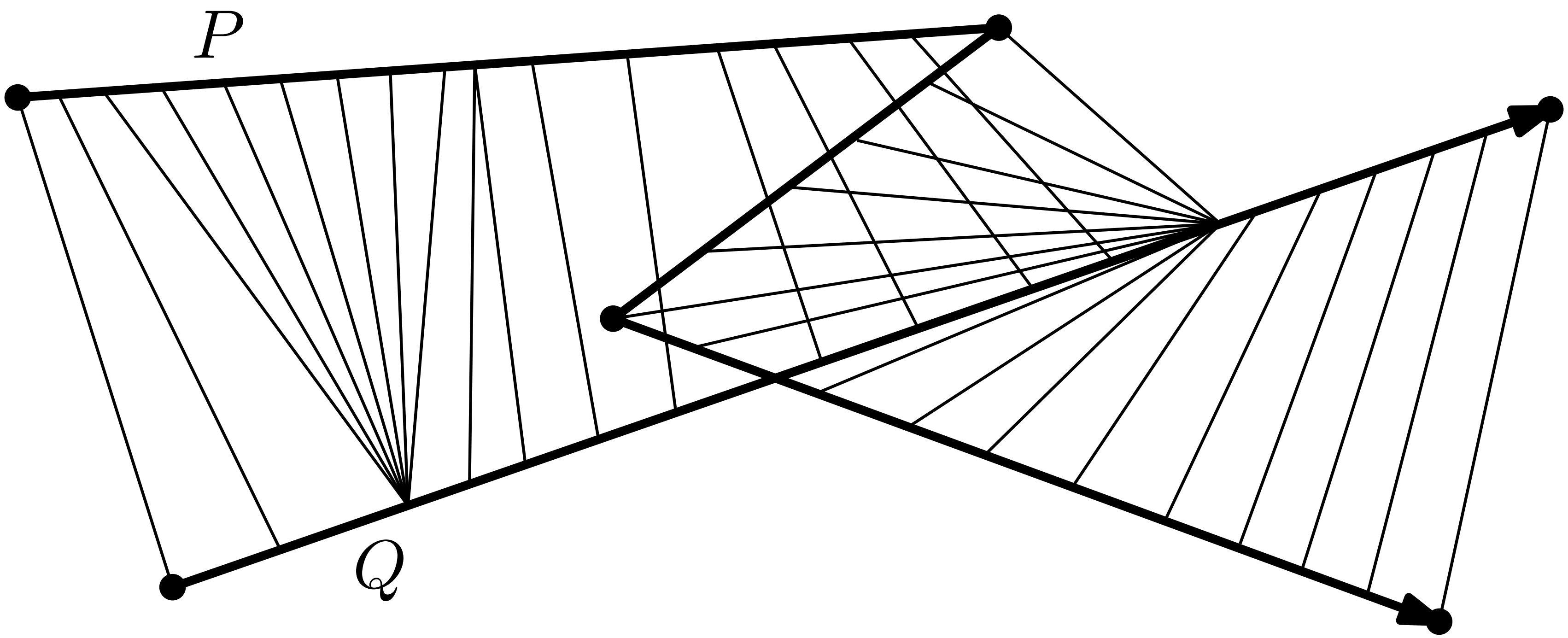


Locally Correct Fréchet Matching

Related Work: [Buchin, Buchin, Meulemans, Speckmann 2012]

The maximum distance between any two matched subcurves must be the Fréchet distance between these two curves.



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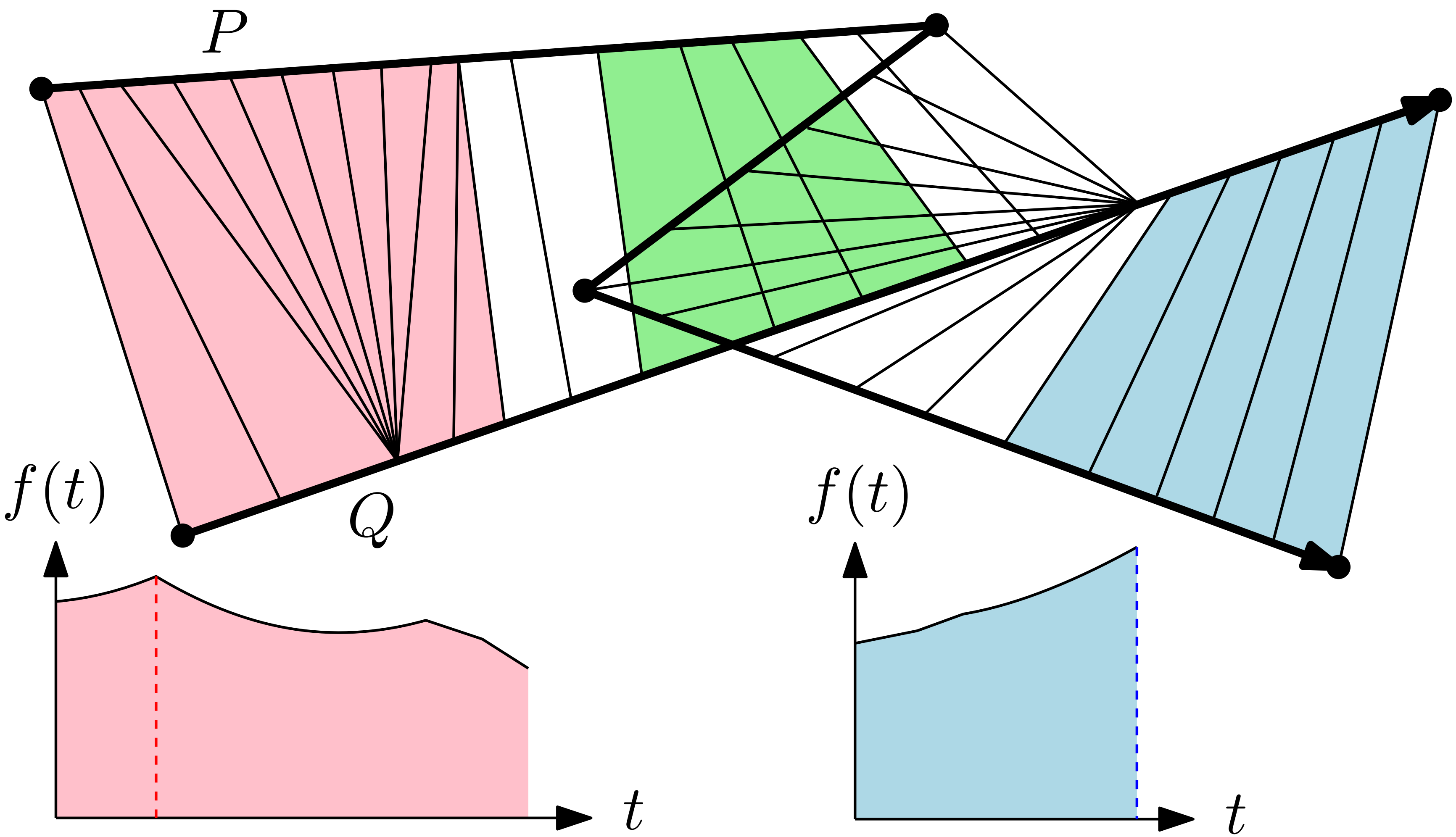

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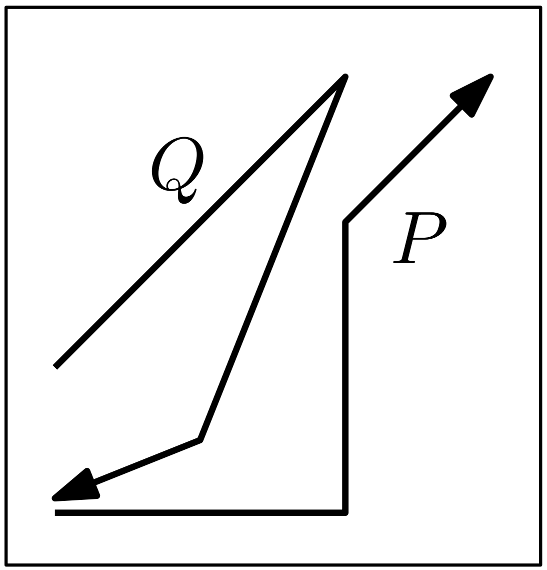

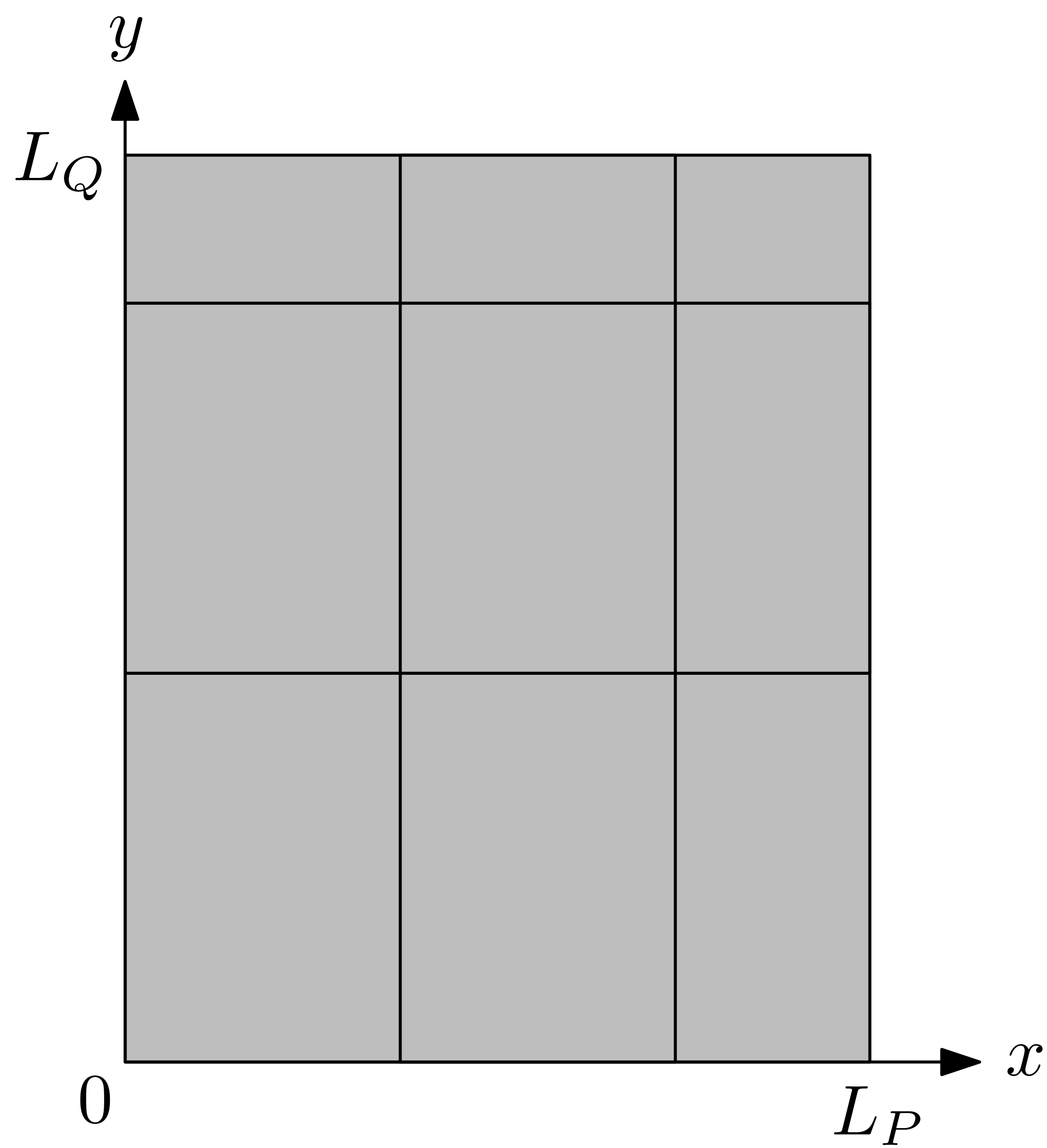

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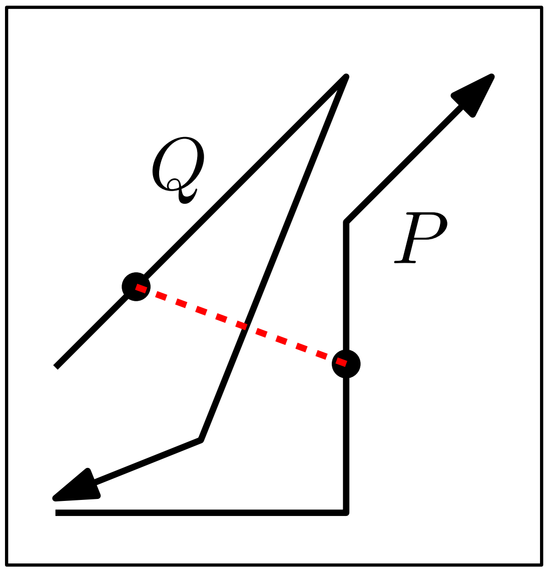

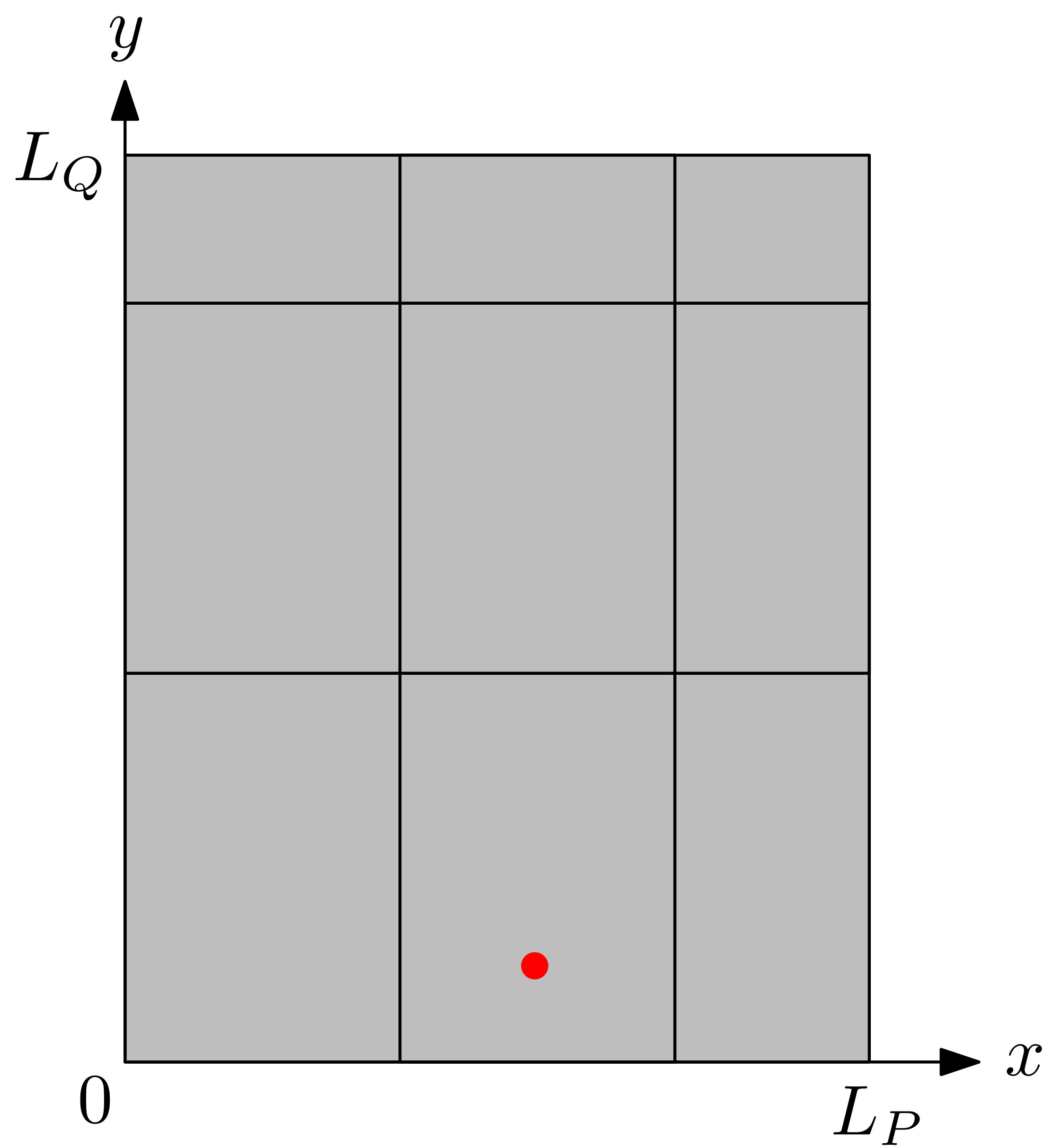

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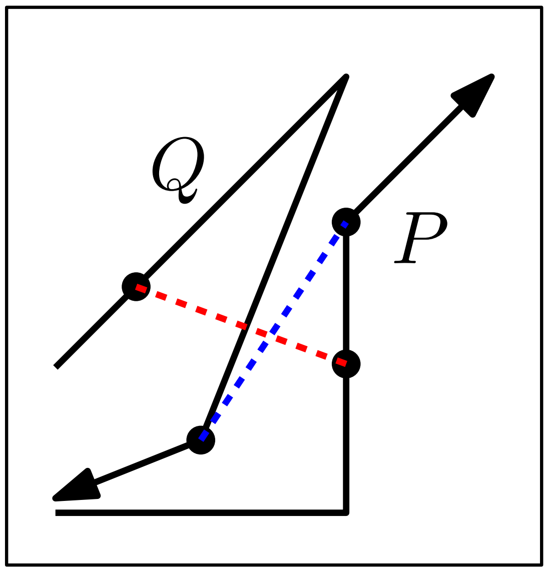

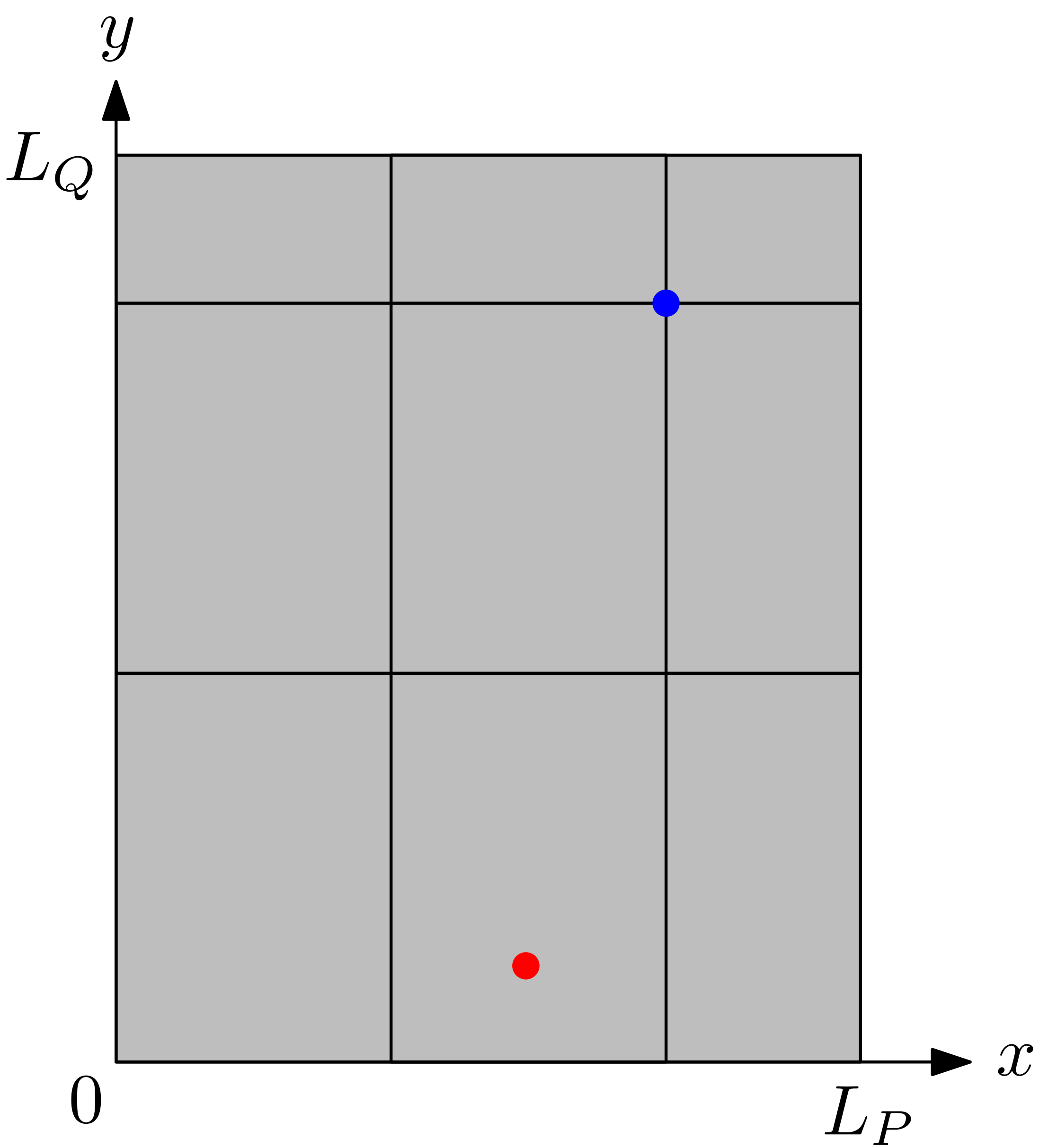

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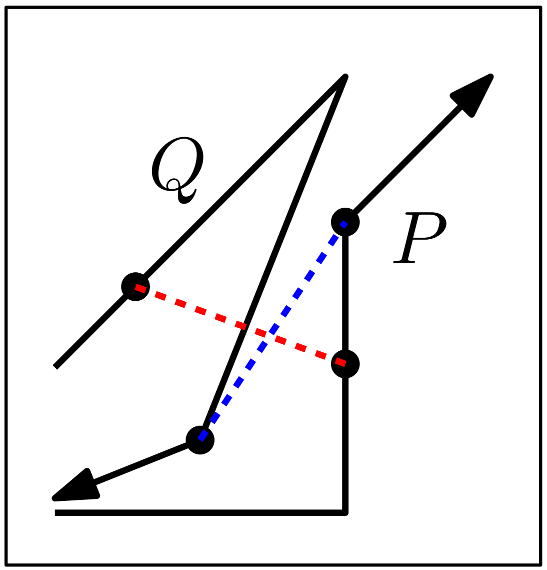

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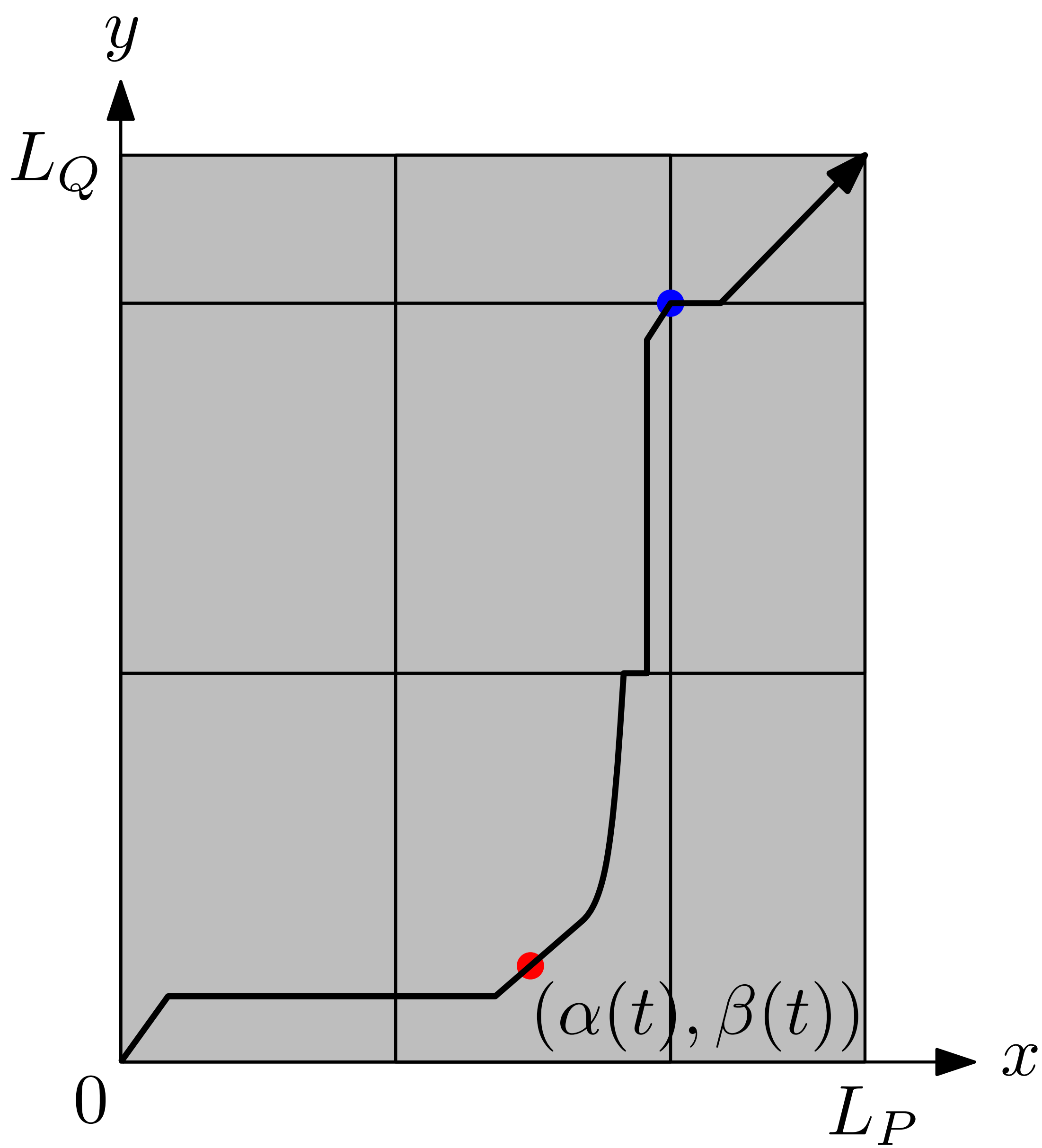

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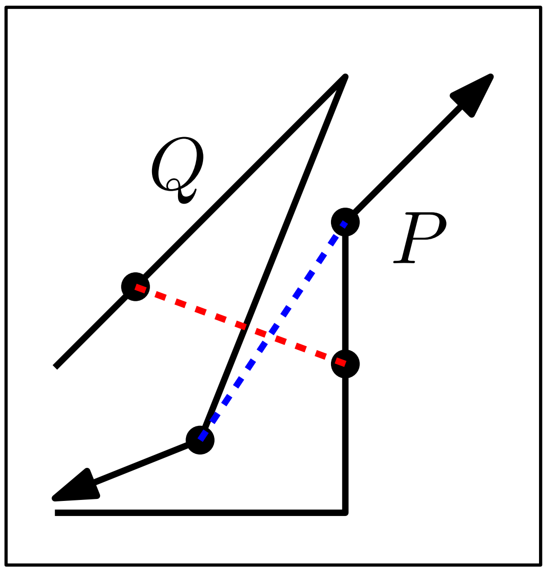

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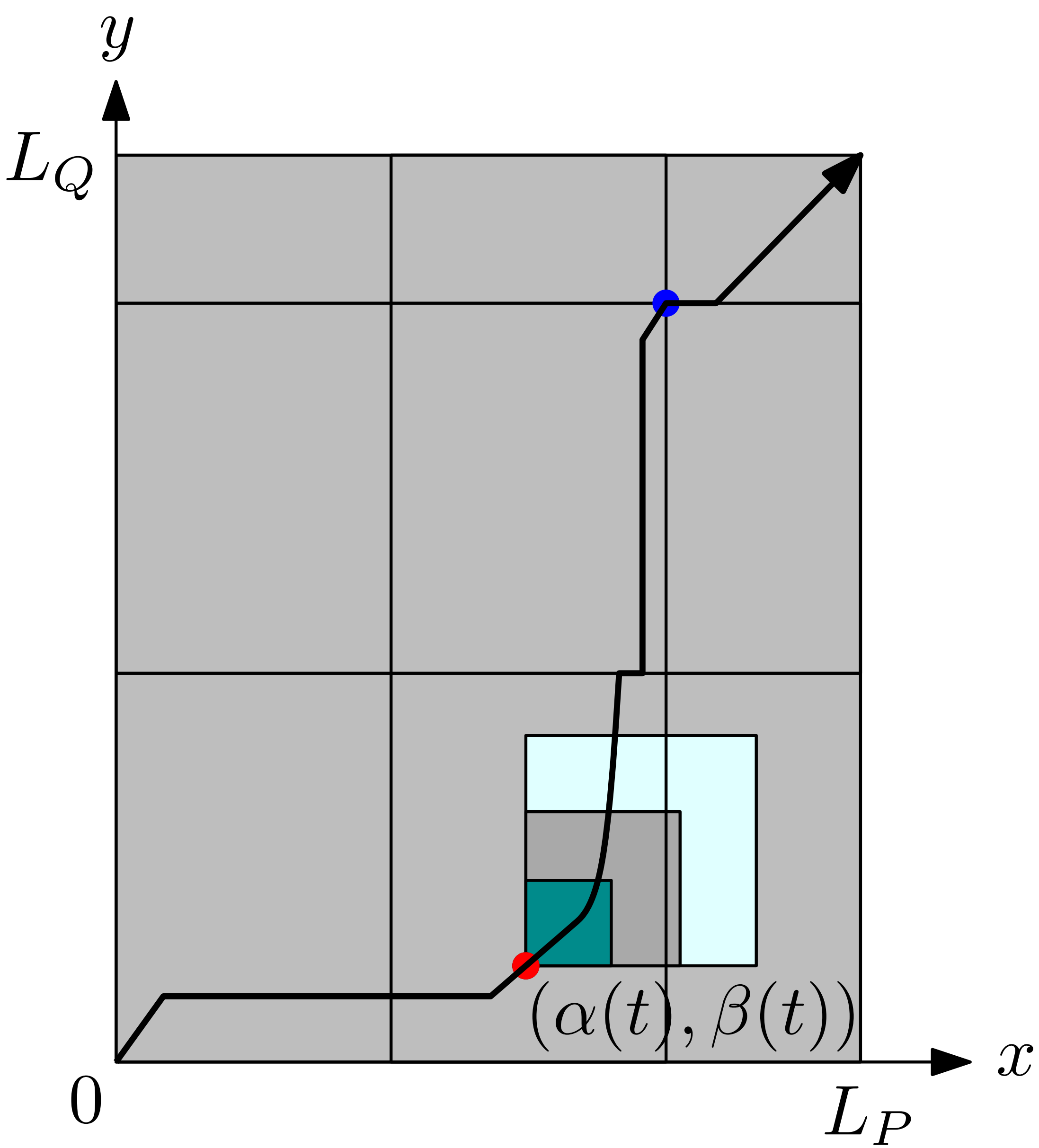

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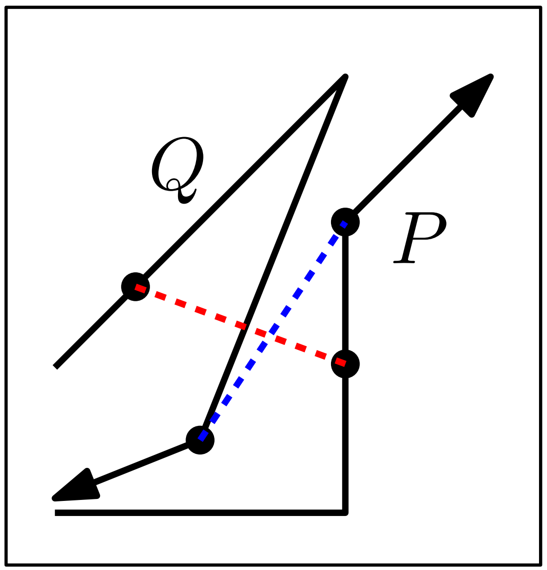

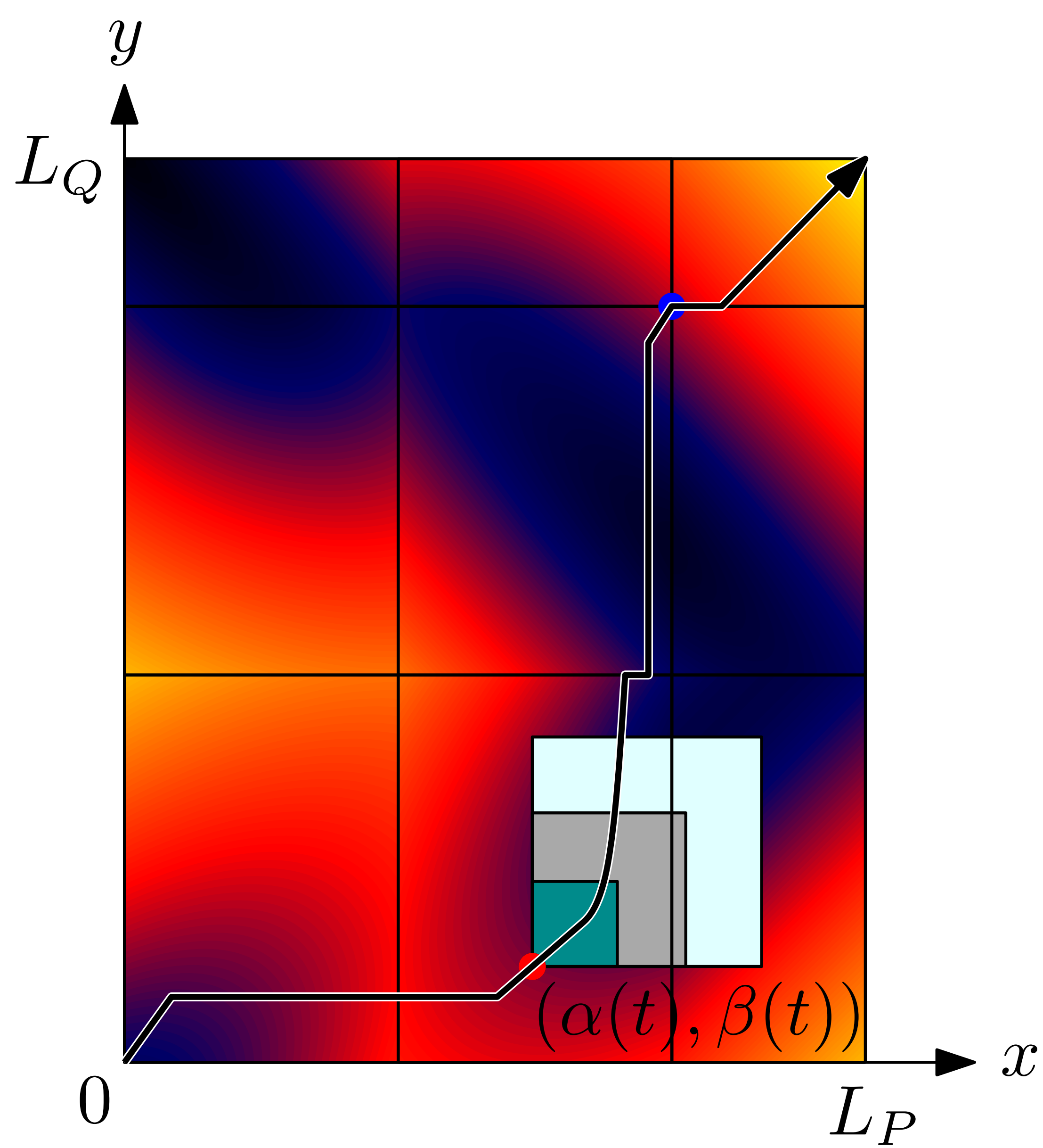

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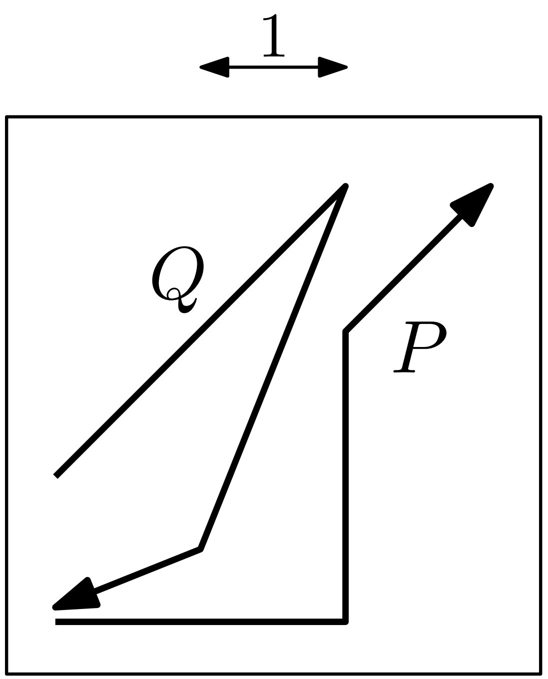
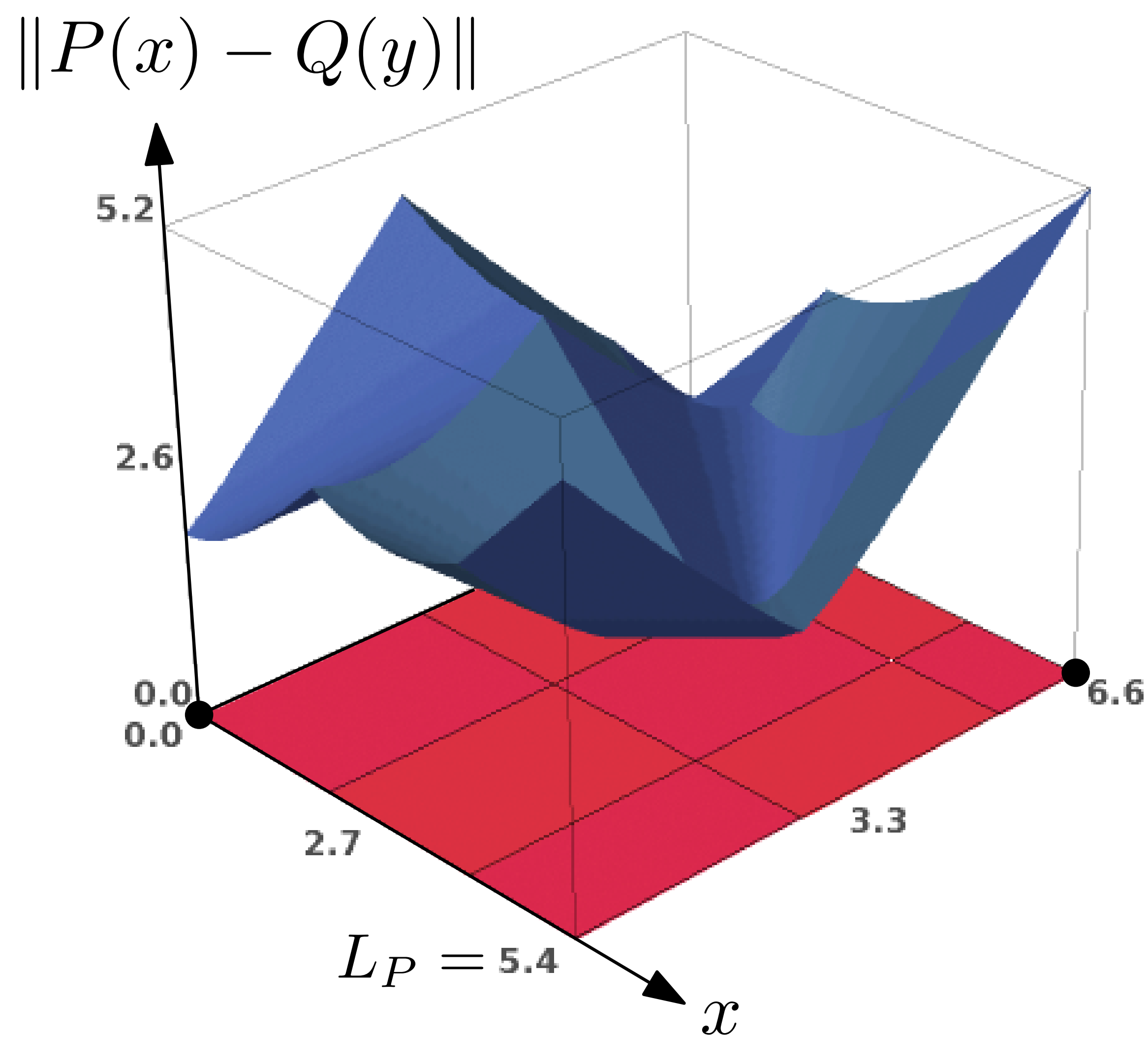
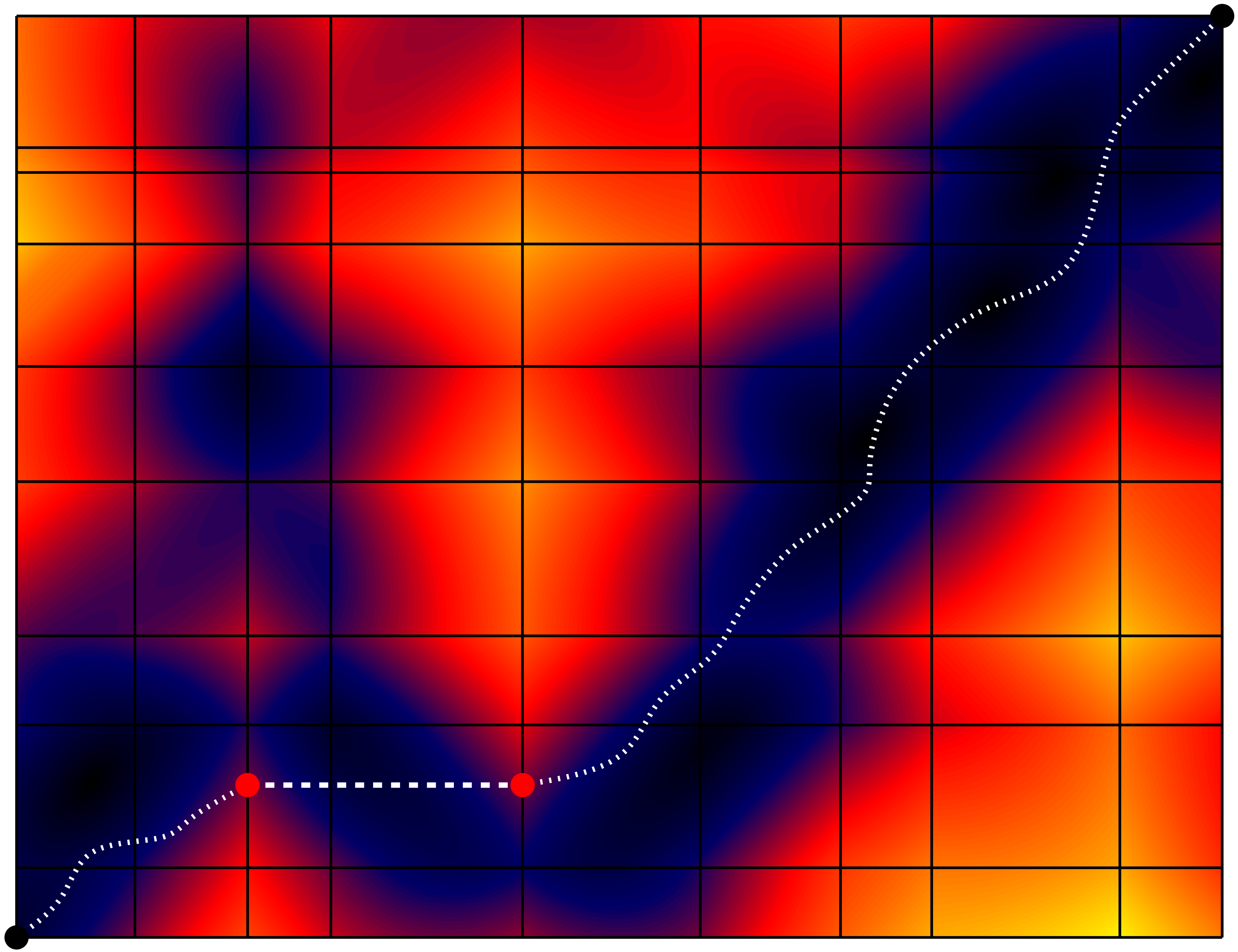
Avoid high values of $\|P(x) - Q(y)\|$

The Distance Landscape





A Critical Passage



ASSUME: There is a UNIQUE critical passage.

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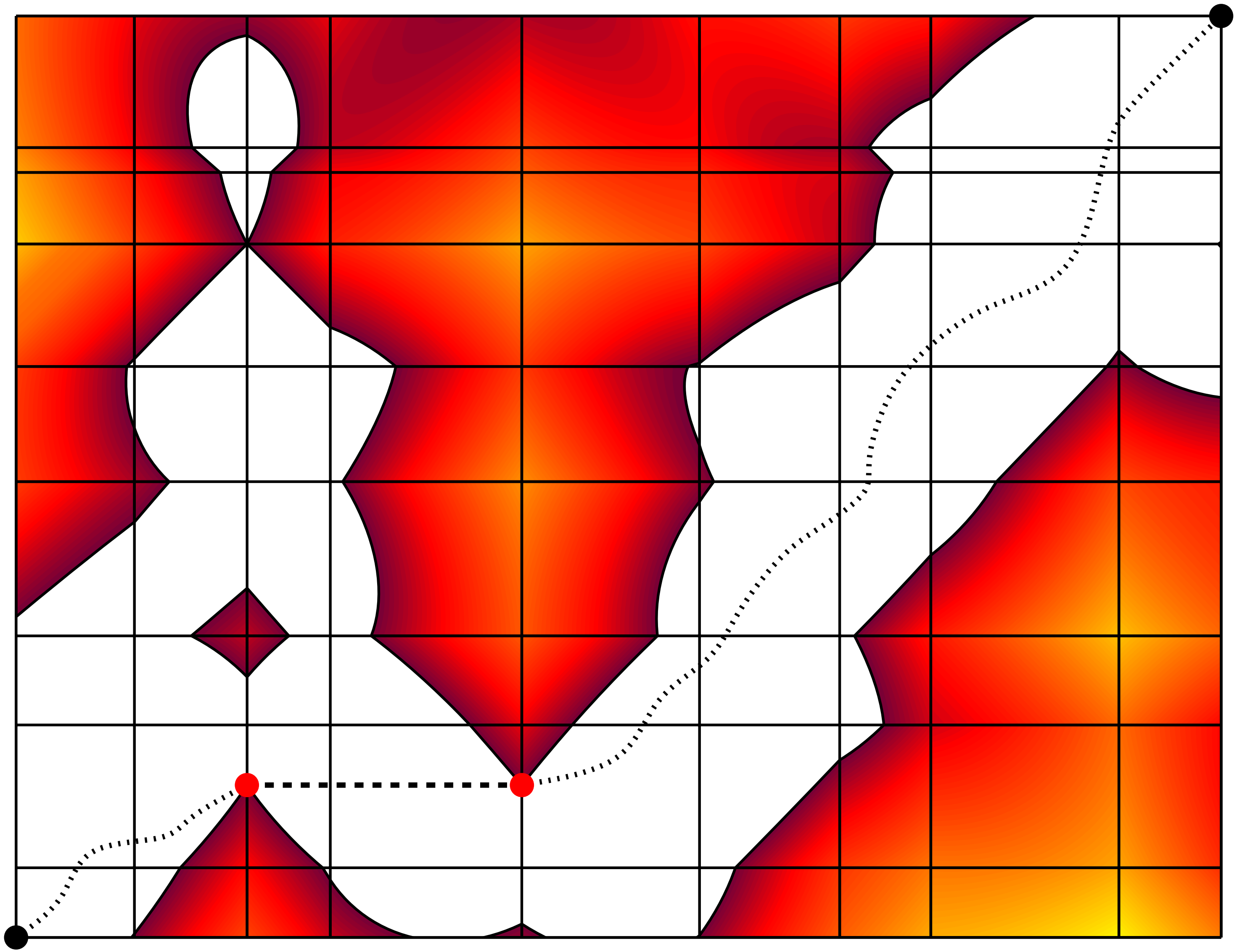

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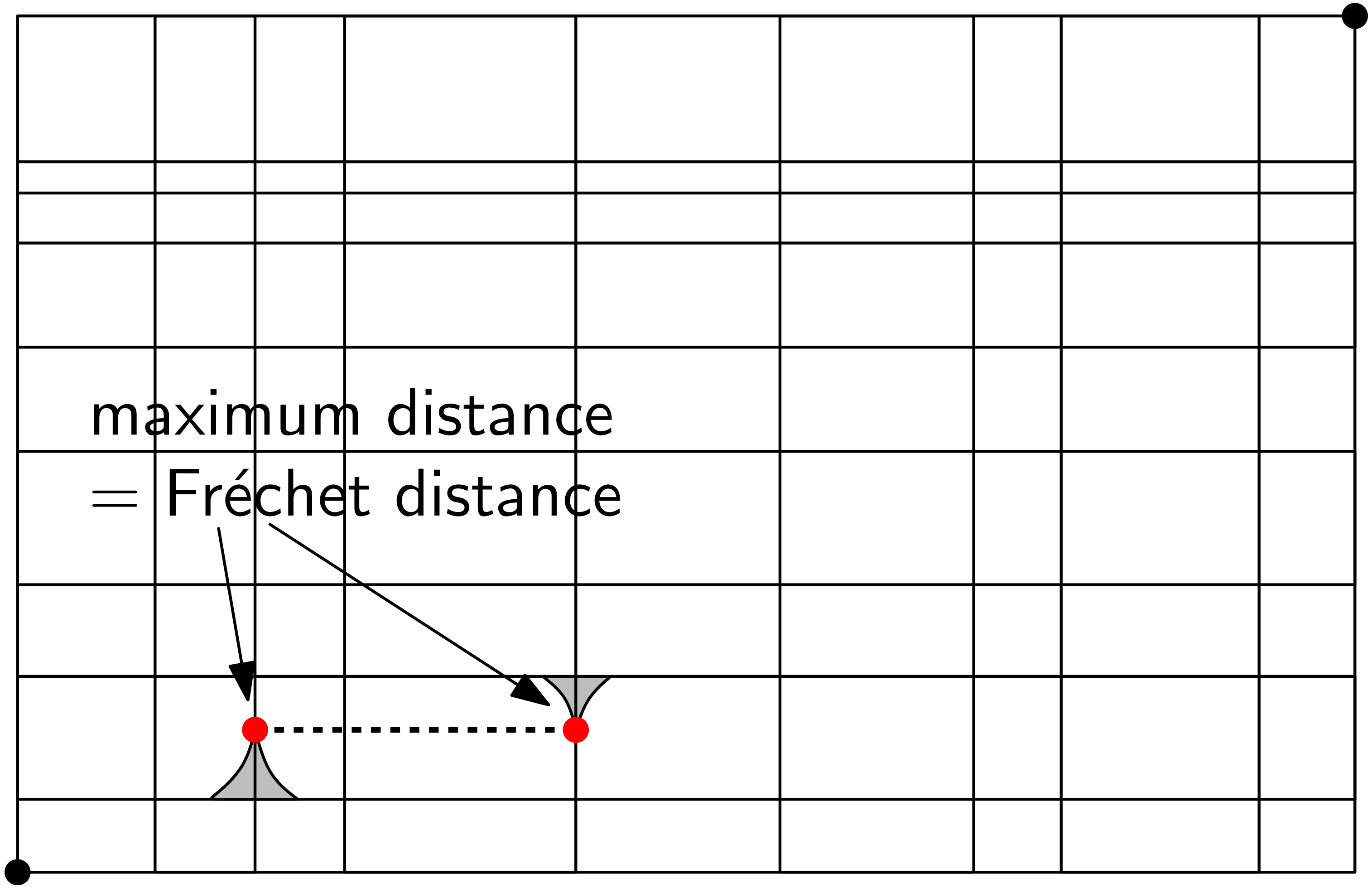

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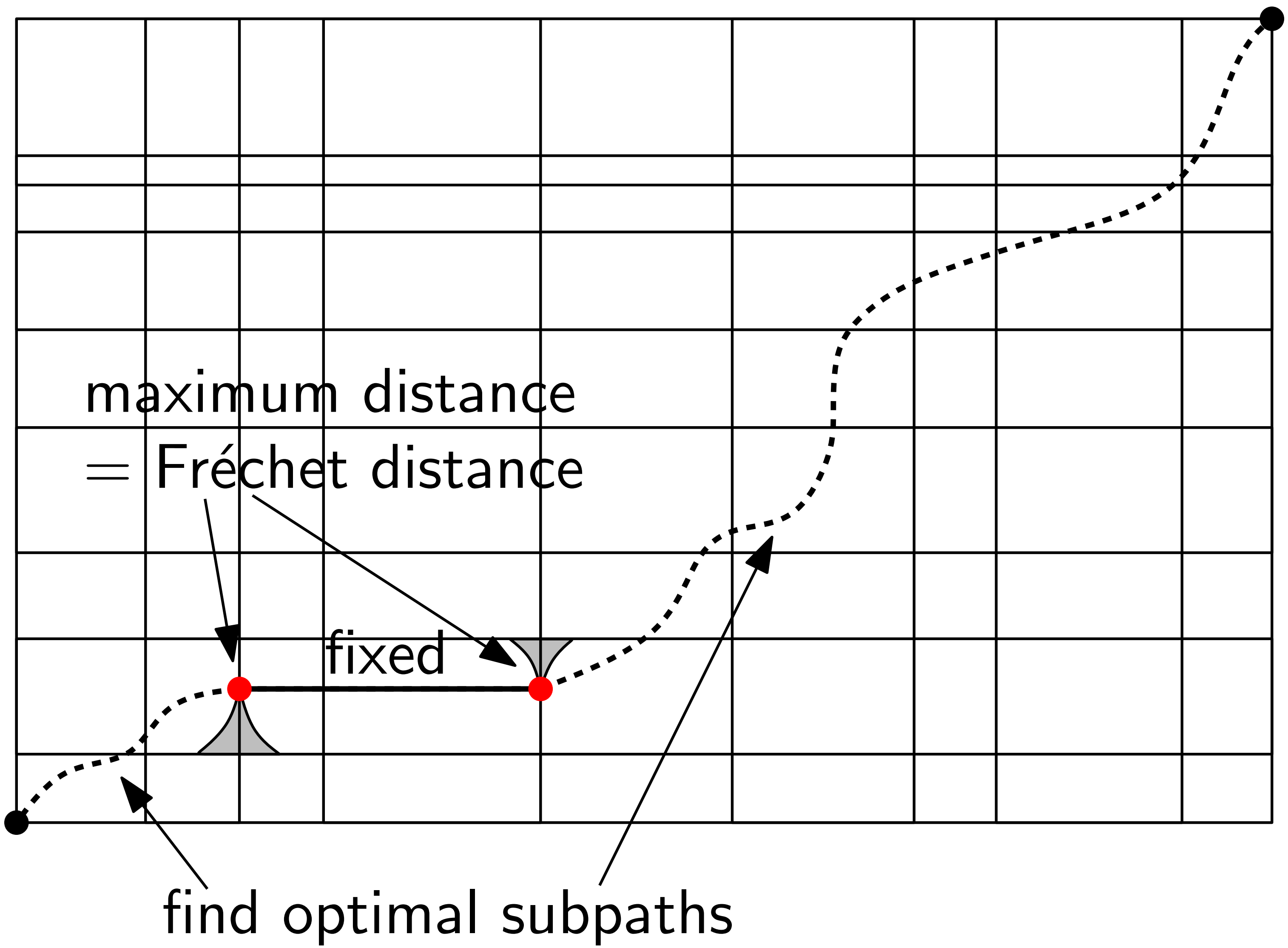

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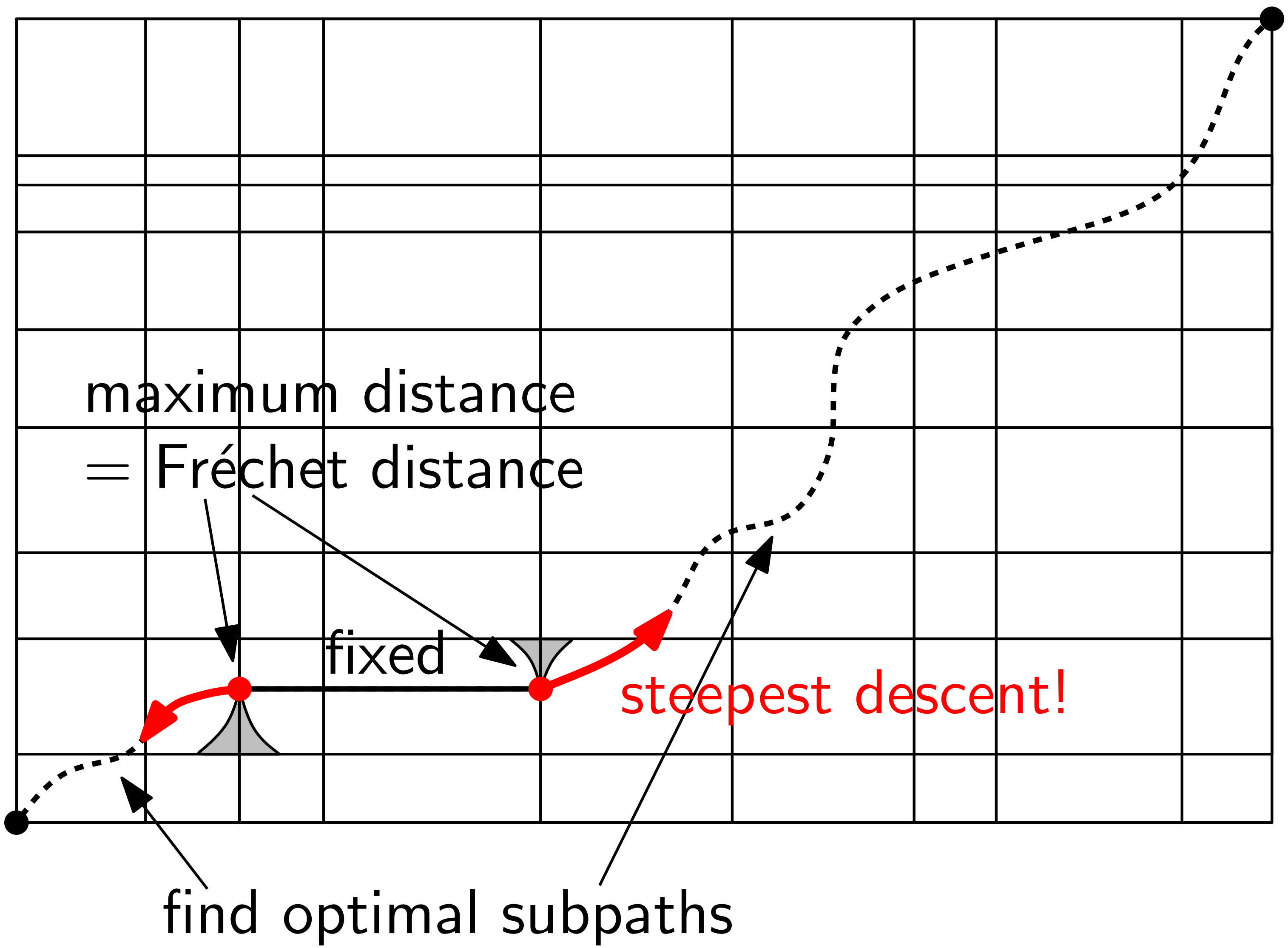


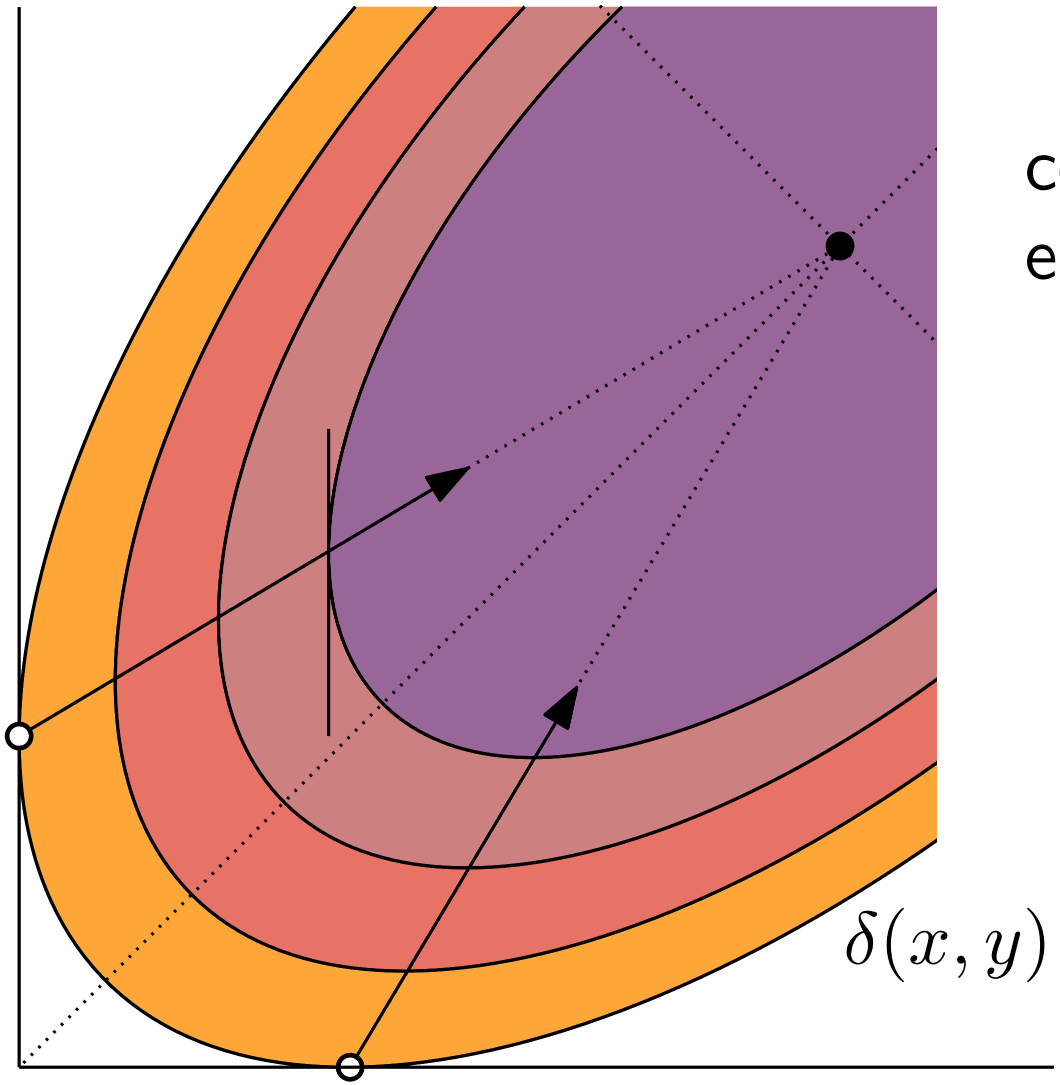
ASSUME: There is a UNIQUE critical passage.

Steepest Descent

Inside one cell: $\delta(x,y) := \|P(x) - Q(y)\|$

$$= \sqrt{(x-a)^2 + (y-b)^2 + \lambda(x-a)(y-b) + c}$$

$$(-2 \le \lambda \le 2,\ c \ge 0)$$



concentric homothetic ellipses with 45° axes

$\delta(x,y) =$ const

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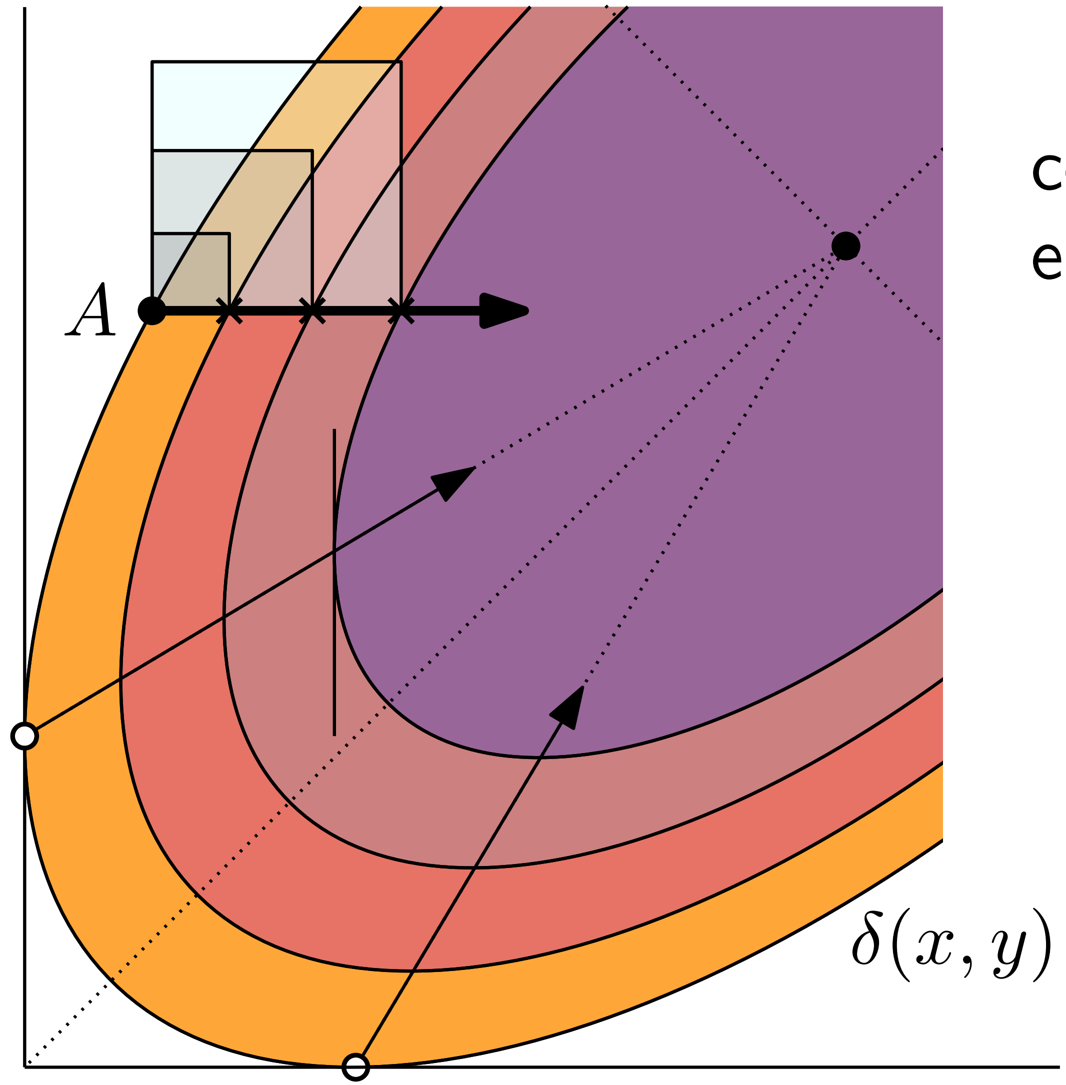

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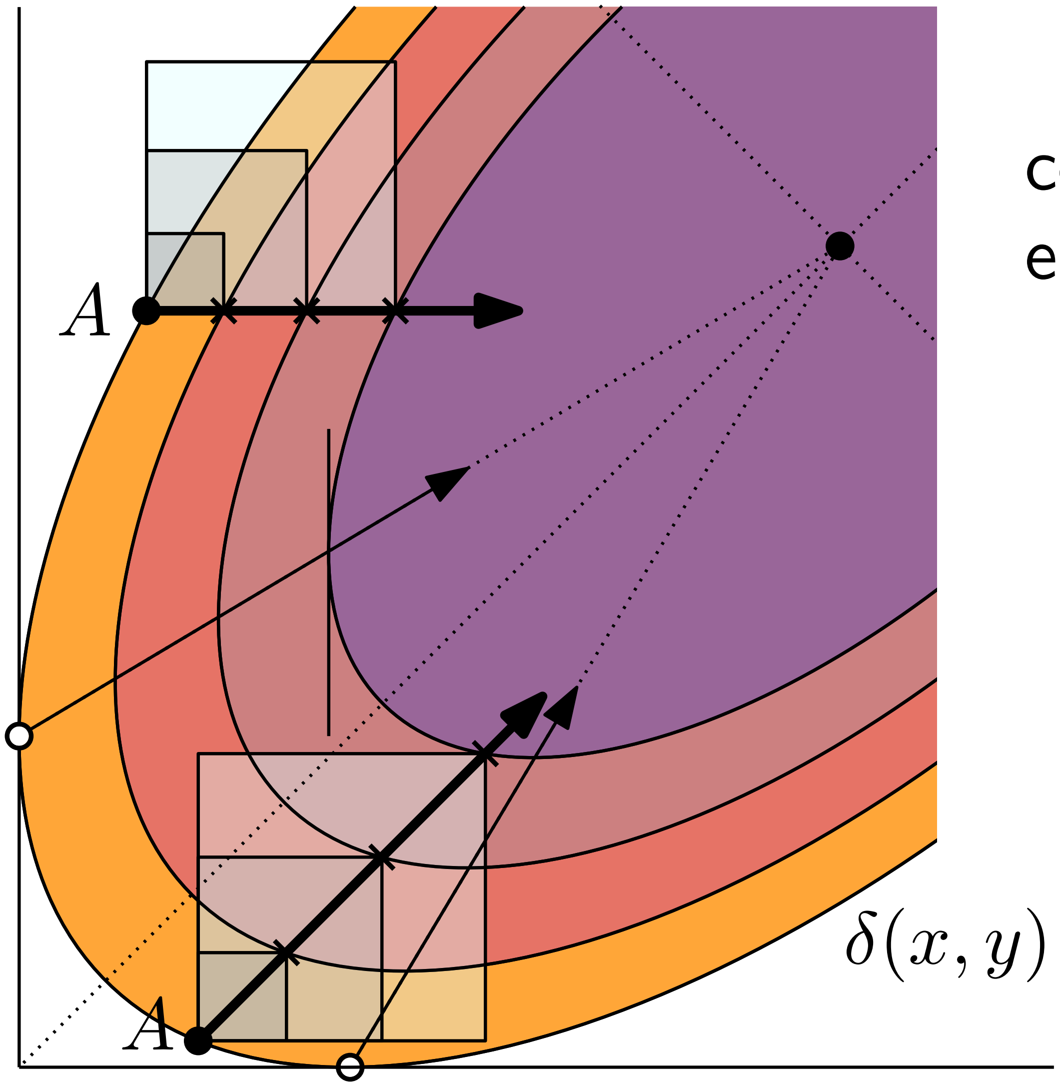

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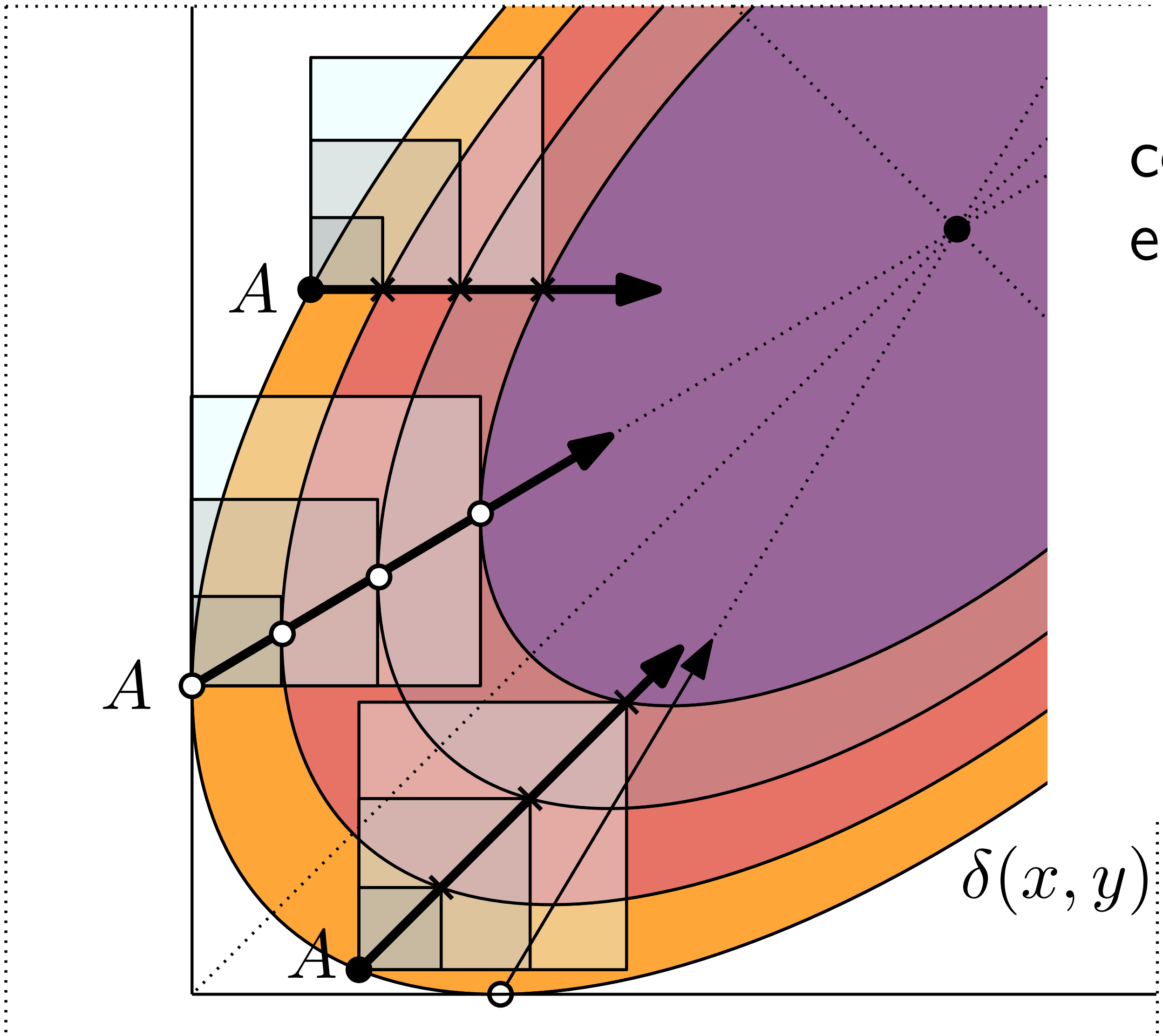

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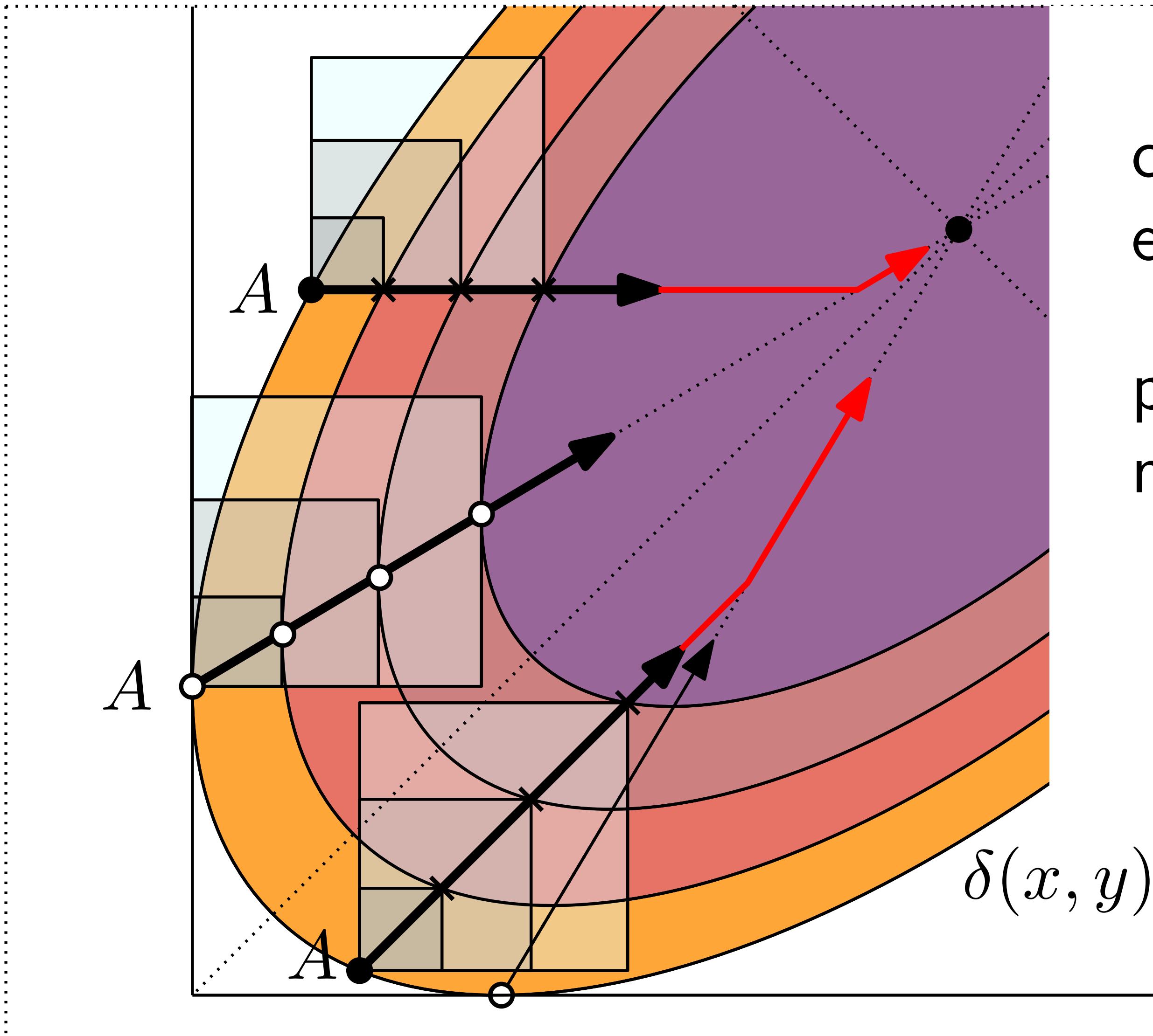
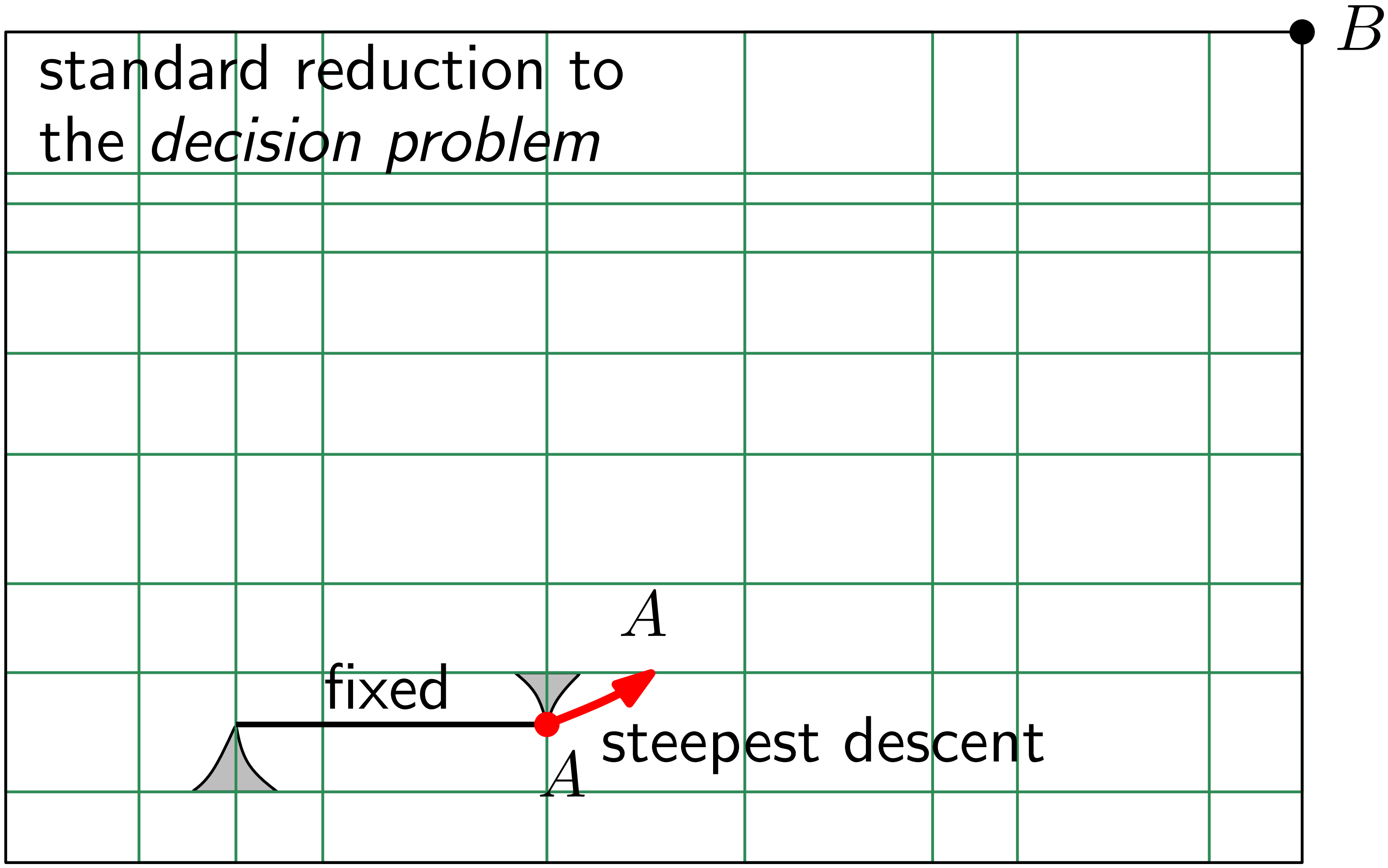


concentric homothetic ellipses with 45° axes

piecewise linear with at most two pieces

$\delta(x,y) = \text{const}$

Searching among Events



A follows steepest descent path: decrease height ε
while $\exists$ monotone path from A to B with height $\leq \varepsilon$

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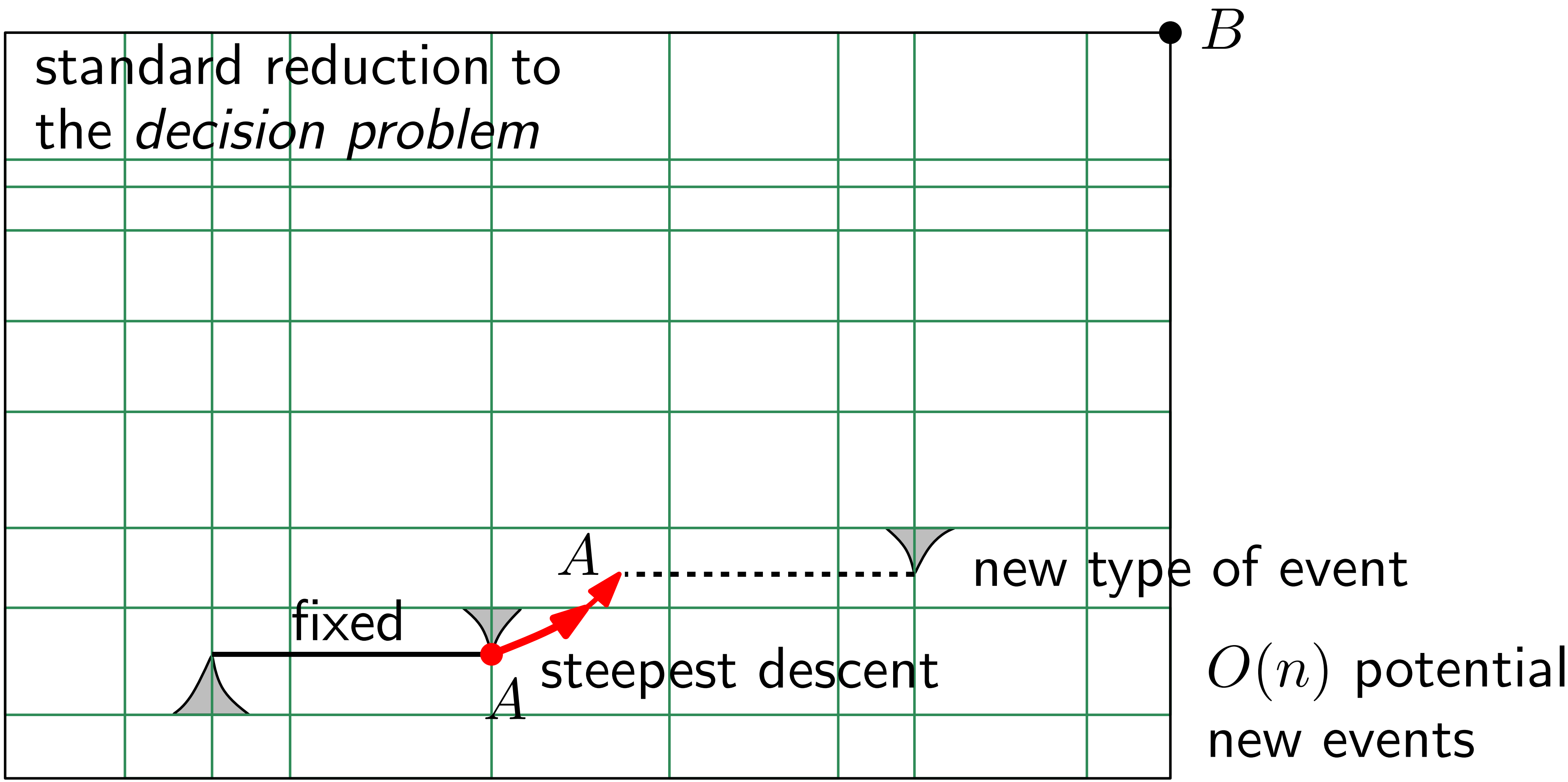

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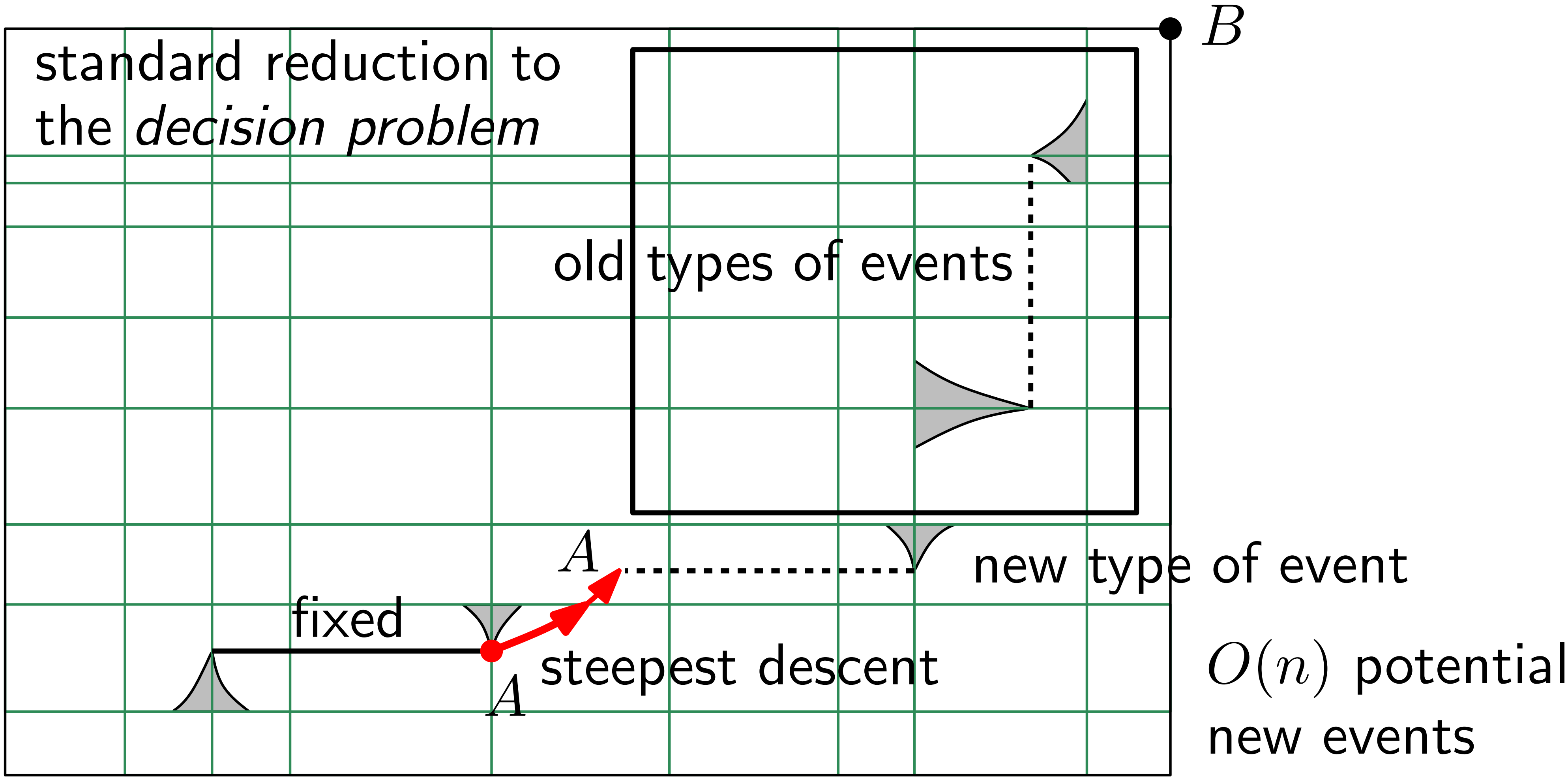

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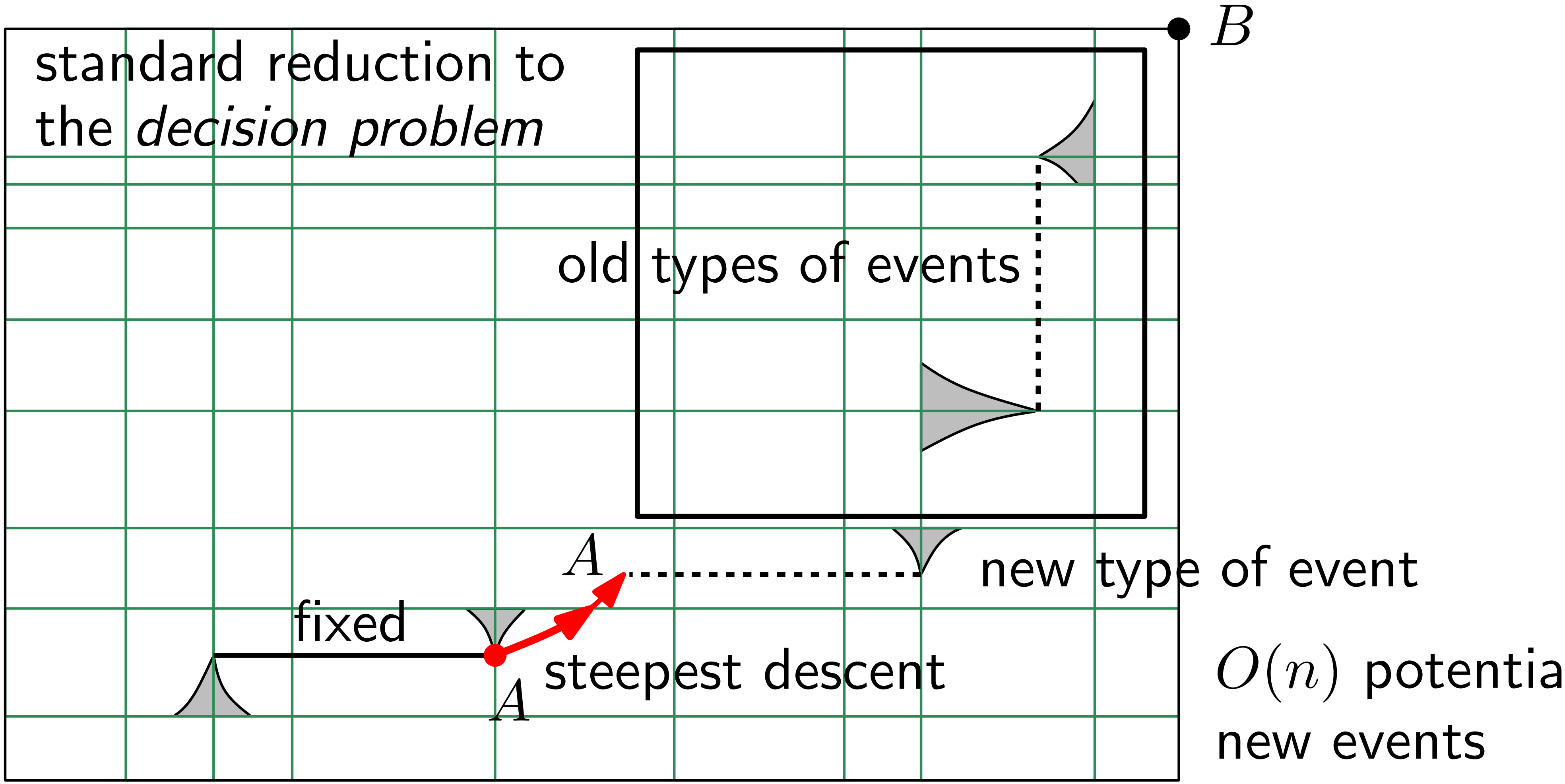


While A is in one cell (i.e., $O(n)$ times):

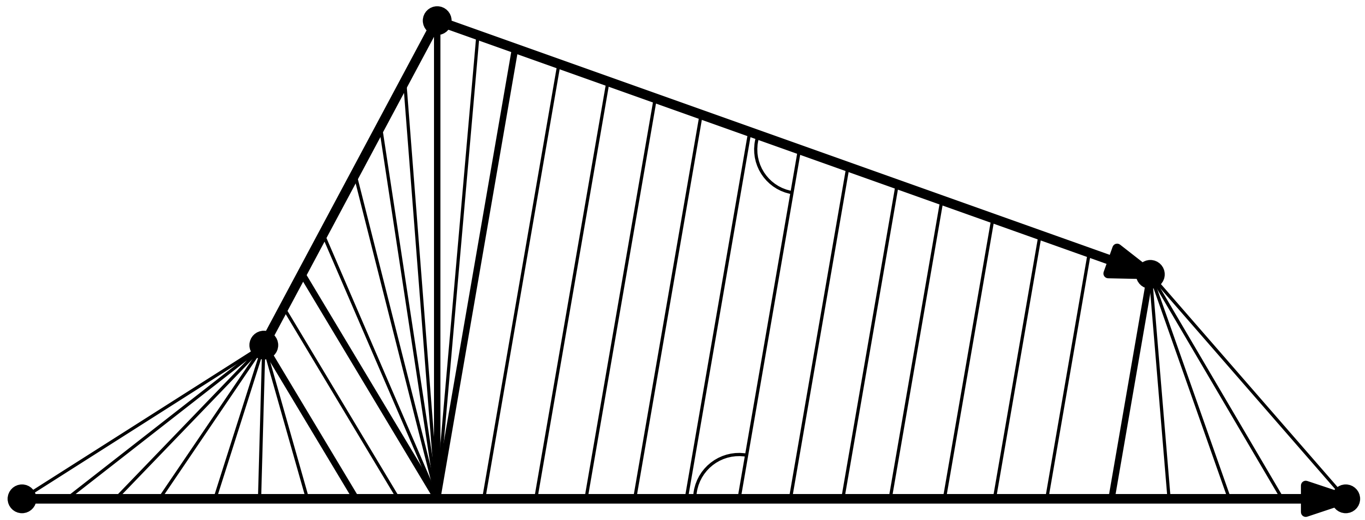
Search among *new* events: $O(\log n) \times$feasibility $= O(n^2 \log n)$

Search among *old* events: classical Fréchet $= O(n^2 \log n)$

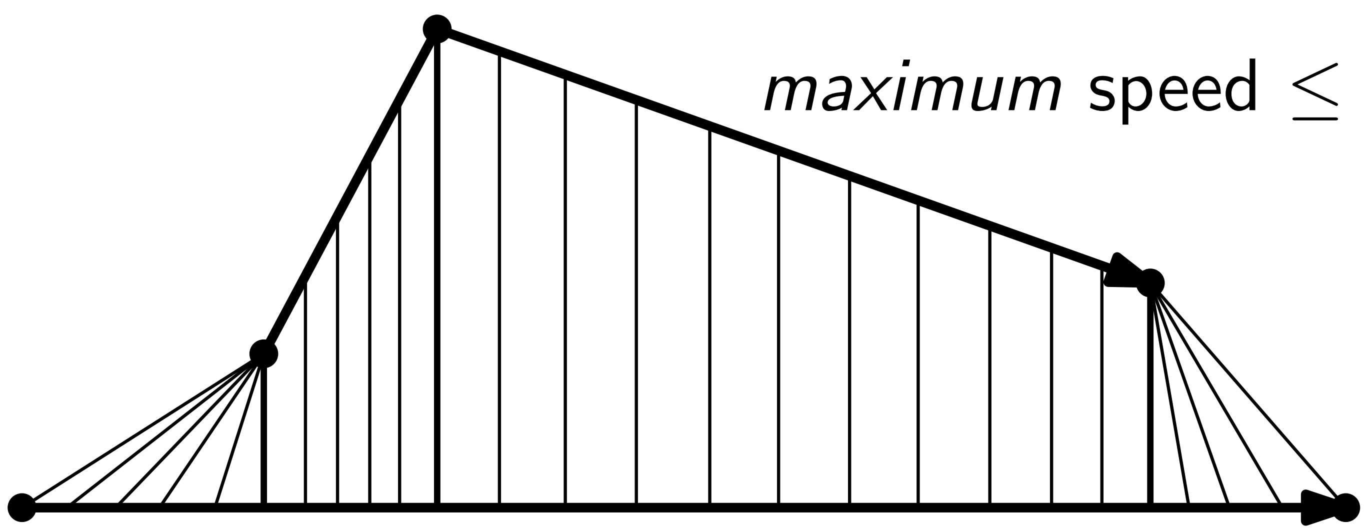
$\rightarrow$ Overall time $= O(n^3 \log n)$

Other Normalizations

The *sum* of the speeds is ≤ 1. (L_1-norm)



maximum speed ≤ 1.



An Unresolved Issue

several critical passages