Chapter 15
Introduction to the Laplace Transform

- 15.1 Introduction
- 15.2 Definition of the Laplace Transform
- 15.3 Properties of Laplace Transform
- 15.4 The Inverse Laplace Transform
- 15.5 The Convolution Integral

15.1 Introduction (1)

- The Laplace transform method involves changing differential equations into algebraic equations to simplify the solution process.

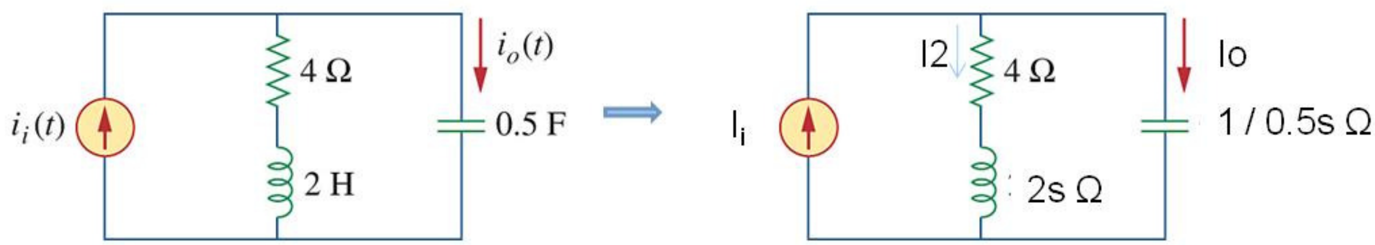
Example:

$$\frac{d^2v(t)}{dt^2} + 6\frac{dv(t)}{dt} + 8v(t) = 2u(t)$$

$$\Downarrow$$

$$\left[s^2V(s) - sv(0) - v'(0)\right] + 6\times\left[sV(s) - v(0)\right] + 8V(s) = \frac{2}{s}$$

- The Laplace transform method follows the same method we used with the phasor analysis of circuits.



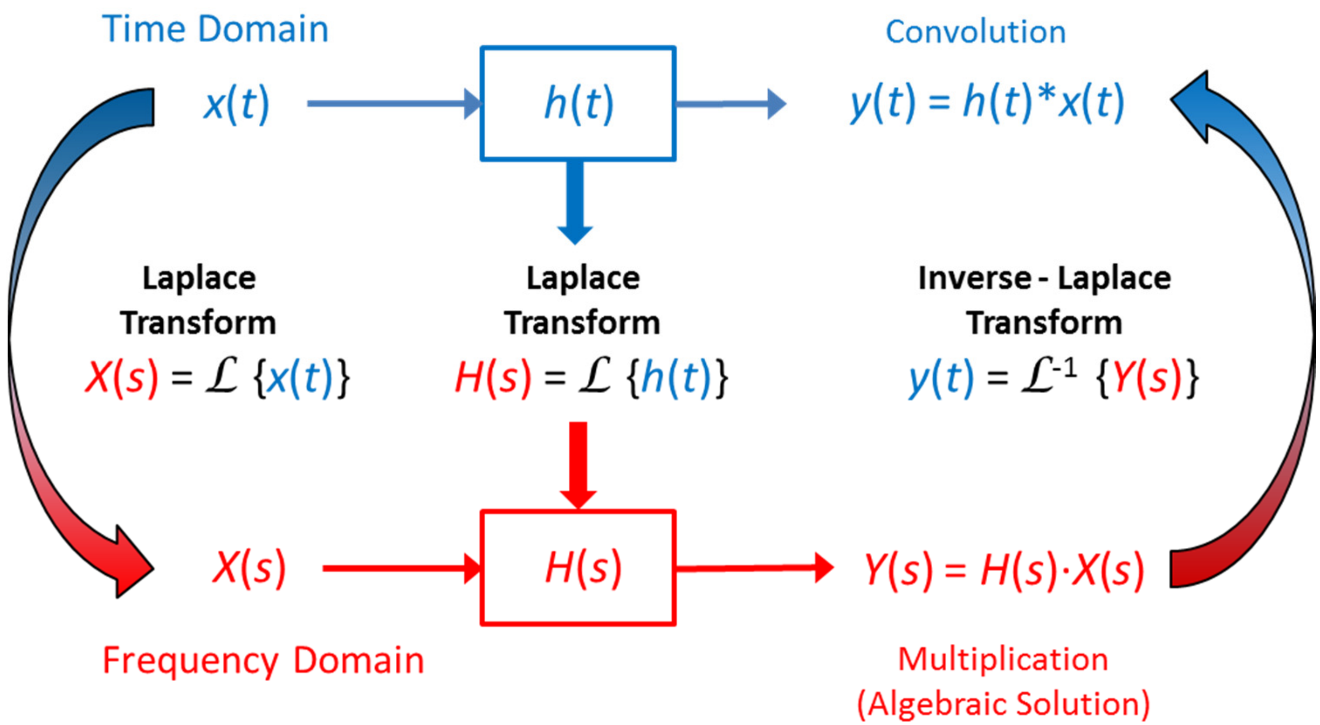
Substitute s for jω

- We saw a “sneak peak” of this with the transfer function last chapter

15.1 Introduction (2)

Linear Time Invariant Systems - Laplace Transform

- The Laplace transform allows us to transform Linear Time Invariant (LTI) systems from the Time domain to the Frequency domain



- We can then solve algebraically in the frequency domain, then apply the inverse Laplace transform to change it back to the time domain.

15.2 Definition of Laplace Transform (1)

- The Laplace transform of a function *f(t)* is defined by the following equation:

$$\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt; \qquad s = \sigma + j\omega$$

- Note: This one-sided Laplace is used for circuits.
- Since the term -st is dimensionless and t is time in seconds, therefore s is frequency in Hz.
- In order for f(t) to have a Laplace transform, $\int_{0-}^{\infty} f(t)e^{-st}dt$ **must converge** to a finite value.
- For example, periodic functions in general do not converge, so no Laplace transform is available (needs Fourier analysis, which is similar).
- Fortunately, all functions of interest in circuit analysis satisfy the convergence criteria and have Laplace transforms.

15.2 Definition of Laplace Transform (2)

- Inverse Laplace transform is defined as follows:

$$\mathcal{L}^{-1}\left[F(s)\right] = f(t) = 1/\, j2\pi \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} ds$$

- Where the integration is performed in the region of convergence.
- We will be using a look-up table to determine Laplace and inverse Laplace transforms.
- The functions f(t) and F(s) are regarded as a Laplace transform pair.

$$f(t) \Leftrightarrow F(s)$$

- There is a one to one correspondence between f(t) and F(s).

Origin of the Laplace Transform

Aside (relation to the power series)

- Interesting lecture about where the Laplace Transform comes from:
  - (Part 1) http://www.youtube.com/watch?v=zvbdoSeGAgI
  - (Part 2) http://www.youtube.com/watch?v=hqOboV2jgVo
- Synopsis:
  - Let A(x) be a power series from 0 to infinity

$$A(x) = \sum_{n=0}^{\infty} a(n) x^n$$

  - The function a(n) defines the coefficients of each polynomial term.
  - Change this from a discrete function to a continuous function

$$F(x) = \int_0^{\infty} f(t) x^t \, dt$$

  - Make the following substitution in x → $x^t = e^{\ln(x)t}$

$$F(e^{\ln(x)}) = \int_0^{\infty} f(t) e^{\ln(x)t} \, dt$$

  - Substitute $\ln(x) = -s \quad ; \quad 0 < s < \infty$

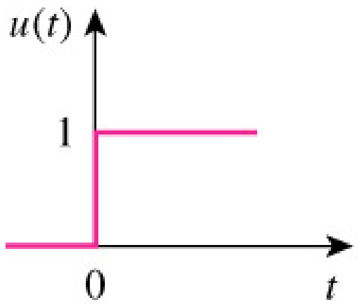
$$F(e^{-s}) = F(s) = \int_0^{\infty} f(t) e^{-st} \, dt$$ Laplace Transform

15.2 Definition of Laplace Transform (3)

Direct Calculation of Laplace Transform

Example 15.1

Determine the Laplace transform of each of the following functions

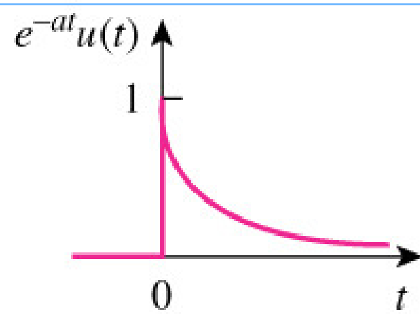


$$\mathcal{L}[u(t)] = \int_0^\infty 1e^{-st}\,dt$$

$$= -\frac{1}{s}e^{-st}\Big|_0^\infty$$

$$= -\frac{1}{s}\{\cancelto{0}{e^{-s\infty}} - \cancelto{1}{e^{-0}}\}$$

$$= \frac{1}{s}$$



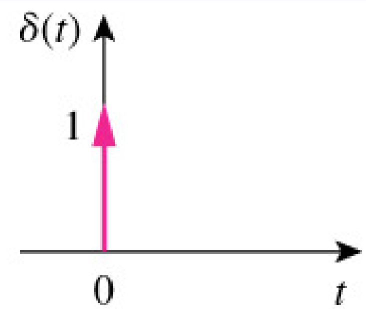
$$\mathcal{L}[e^{-\alpha t}u(t)] = \int_0^\infty e^{-\alpha t}e^{-st}\,dt$$

$$= \int_0^\infty e^{-(s+\alpha)t}\,dt$$

$$= -\frac{1}{s+\alpha}e^{-(s+\alpha)t}\Big|_0^\infty$$

$$= -\frac{1}{s+\alpha}\{e^{-\infty} - e^{0}\}$$

$$= \frac{1}{s+\alpha}$$



$$\mathcal{L}[\delta(t)] = \int_0^\infty \delta(t)e^{-st}\,dt$$

$$= 1$$

15.2 Definition of Laplace Transform (4)

Laplace Transform Pairs (Table 15.2 in text)

- For many "common" functions we can develop a lookup table of transform "Pairs":

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\dfrac{1}{s}$
e^{-at}	$\dfrac{1}{s+a}$
t	$\dfrac{1}{s^2}$
t^n	$\dfrac{n!}{s^{n+1}}$
te^{-at}	$\dfrac{1}{(s+a)^2}$
$t^n e^{-at}$	$\dfrac{n!}{(s+a)^{n+1}}$

$f(t)$	$F(s)$
$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
$\sin(\omega t+\theta)$	$\dfrac{s\sin\theta+\omega\cos\theta}{s^2+\omega^2}$
$\cos(\omega t+\theta)$	$\dfrac{s\cos\theta-\omega\sin\theta}{s^2+\omega^2}$
$e^{-at}\sin \omega t$	$\dfrac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos \omega t$	$\dfrac{s+a}{(s+a)^2+\omega^2}$

$\boxed{\text{*Defined for } t \geq 0;\ f(t) = 0, \text{ for } t < 0.}$

Essentially all these functions are $f(t)\cdot u(t)$ *!*

15.2 Definition of Laplace Transform (5)

Examples

Find the Laplace Transform for the following Function:

Use the Transform Pairs Table!

$$f(t) = (4 + 3e^{-2t})u(t)$$

$$f(t) = 4u(t) + 3e^{-2t}u(t)$$

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$

$a = 2$

$$F(s) = \frac{4}{s} + 3\frac{1}{s+2}$$

Note: The Laplace Transform Pairs table is defined for t ≥ 0. For t < 0 it is assumed f(t) = 0. Therefore, "all pairs" are essentially for f(t)•u(t).

$$f(t) = 4u(t) + 3e^{-2t}u(t)$$

$$F(s) = \frac{4}{s} + 3\frac{1}{s+2} \cdot \cancel{\frac{1}{s}}$$

Don't Make this mistake!
This would be WRONG.

15.2 Definition of Laplace Transform (6)

Examples

Find the Laplace Transform for the following function:

$$f(t) = (e^{-2t}\cosh(4t))u(t)$$

Hyperbolic cosine identity

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2}$$

$$f(t) = \left(e^{-2t}\left(\frac{e^{4t} + e^{-4t}}{2}\right)\right)u(t)$$

$$f(t) = \left(\frac{e^{2t} + e^{-6t}}{2}\right)u(t)$$

Transform Pairs Table

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$

$a = -2$

$a = 6$

$$F(s) = \frac{1}{2}\left(\frac{1}{s-2} + \frac{1}{s+6}\right)$$

$$F(s) = \frac{1}{2}\left(\frac{s+6}{(s-2)(s+6)} + \frac{s-2}{(s-2)(s+6)}\right) = \frac{1}{2}\left(\frac{s+4}{s^2 + 4s - 12}\right)$$

15.3 Properties of Laplace Transform (1)

Table 15.1 text

- Understanding the properties of the Laplace Transform allows us to obtain transform pairs without directly solving the integral equation
- Table 15.1 in the text lists several key properties
- We will discuss several of these next

TABLE 15.1

Properties of the Laplace transform.

Property	$f(t)$	$F(s)$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
Scaling	$f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
Time shift	$f(t-a)u(t-a)$	$e^{-as} F(s)$
Frequency shift	$e^{-at} f(t)$	$F(s+a)$
Time differentiation	$\frac{df}{dt}$	$sF(s) - f(0^-)$
	$\frac{d^2f}{dt^2}$	$s^2F(s) - sf(0^-) - f'(0^-)$
	$\frac{d^3f}{dt^3}$	$s^3F(s) - s^2 f(0^-) - sf'(0^-) - f''(0^-)$
	$\frac{d^nf}{dt^n}$	$s^nF(s) - s^{n-1}f(0^-) - s^{n-2}f'(0^-) - \cdots - f^{(n-1)}(0^-)$
Time integration	$\int_0^t f(x)dx$	$\frac{1}{s} F(s)$
Frequency differentiation	$tf(t)$	$-\frac{d}{ds} F(s)$
Frequency integration	$\frac{f(t)}{t}$	$\int_s^\infty F(s) ds$
Time periodicity	$f(t) = f(t + nT)$	$\frac{F_1(s)}{1 - e^{-sT}}$
Initial value	$f(0)$	$\lim_{s\to\infty} sF(s)$
Final value	$f(\infty)$	$\lim_{s\to 0} sF(s)$
Convolution	$f_1(t) * f_2(t)$	$F_1(s)F_2(s)$

Linearity:

The Laplace Transform is linear. If $F_1(s)$ and $F_2(s)$ are, respectively, the Laplace Transforms of $f_1(t)$ and $f_2(t)$, then:

$$\boxed{\mathcal{L}[a_1 f_1(t) + a_2 f_2(t)] = a_1 F_1(s) + a_2 F_2(s)}$$

Example:

$$\mathcal{L}[\cos(\omega t)u(t)] = \mathcal{L}\left[\frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)u(t)\right]$$

$$= \mathcal{L}\left[\frac{1}{2}\left(e^{j\omega t}\right)u(t)\right] + \mathcal{L}\left[\frac{1}{2}\left(e^{-j\omega t}\right)u(t)\right]$$

$$= \frac{1}{2(s+j\omega)} + \frac{1}{2(s-j\omega)} = \frac{s}{s^2+\omega^2}$$

Cosine identity

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

IUPUI SCHOOL OF ENGINEERING AND TECHNOLOGY
A PURDUE UNIVERSITY SCHOOL
Indianapolis

Scaling:

If *F (s)* is the Laplace Transforms of *f (t),* then

$$\boxed{\mathcal{L}[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)}$$

Example:

$a = 2$

$f(t)$	$F(s)$
$\sin \omega t$	$\dfrac{\omega}{s^2 + \omega^2}$

$$\boxed{\mathcal{L}[\sin(2\omega t)u(t)] = \frac{1}{2}\frac{\omega}{\left(\frac{s}{2}\right)^2 + \omega^2} = \frac{0.5\omega}{\frac{s^2}{4} + \omega^2} = \frac{2\omega}{s^2 + 4\omega^2}}$$

Time Shift:

If *F (s)* is the Laplace Transforms of *f (t),* then

$$\boxed{\mathcal{L}\left[f(t-a)u(t-a)\right]=e^{-as}F(s)}$$

If a function is delayed in time by a, the result in the s-domain is found by multiplying the Laplace transform by e^{-as} .

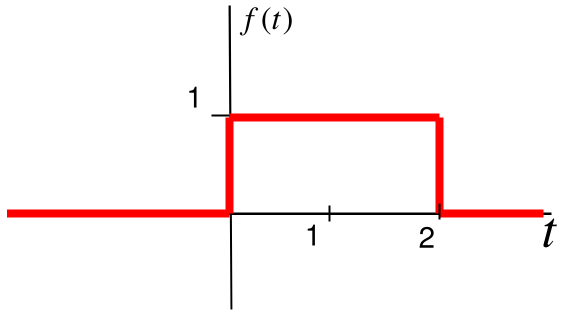
Example:

$$\boxed{\mathcal{L}\left[\cos(\omega(t-a))u(t-a)\right]=e^{-as}\frac{s}{s^2+\omega^2}}$$

IUPUI | SCHOOL OF ENGINEERING AND TECHNOLOGY
A PURDUE UNIVERSITY SCHOOL
Indianapolis

Time Shift Example:

Find the Laplace Transform of the following Function:



$$f(t) = u(t) - u(t-2)$$

$$\mathcal{L}[f(t)] = \mathcal{L}[u(t) - u(t-2)]$$

Linearity Property $\Rightarrow$ $\mathcal{L}[f(t)] = \mathcal{L}[u(t)] - \mathcal{L}[u(t-2)]$

Time Shift Property $\Rightarrow$ $\mathcal{L}[f(t)] = \mathcal{L}[u(t)] - e^{-2t}\mathcal{L}[u(t)]$

Transform Pairs Table $\Rightarrow$ $\boxed{\mathcal{L}[f(t)] = \frac{1}{s} - e^{-2t}\frac{1}{s}}$

Frequency Shift:

If *F (s)* is the Laplace Transforms of *f (t)*, then

$$\boxed{\mathcal{L}\left[e^{-at} f(t)u(t)\right] = F(s+a)}$$

The Laplace transform of e^{-at} f(t) can be obtained from the Laplace transform of f(t) by replacing s with s+a.

Example:

$$\boxed{\mathcal{L}\left[e^{-at}\cos(\omega t)u(t)\right] = \frac{s+a}{(s+a)^2+\omega^2}}$$

Time Differentiation:

If *F (s)* is the Laplace Transforms of *f (t)*, then the Laplace Transform of its derivative is

$$\boxed{\mathcal{L}\left[f'(t)\right] = sF(s) - f(0^-)}$$

Example:

$$\frac{d}{dt}\cos(\omega t) = -\omega \sin(\omega t)$$

$$\frac{-1}{\omega}\frac{d}{dt}\cos(\omega t) = \sin(\omega t)$$

$$\begin{aligned}
\mathcal{L}\left[\sin(\omega t)u(t)\right] &= \mathcal{L}\left[\frac{-1}{\omega}\frac{d}{dt}\cos(\omega t)u(t)\right] \\
&= \frac{-1}{\omega} L\left[\frac{d}{dt}\cos(\omega t)u(t)\right] = \frac{-1}{\omega}\left[sL\left[\cos(\omega t)u(t)\right] - \cos(0^-)\right] \\
&= \frac{-1}{\omega}\left(s \times \frac{s}{s^2+\omega^2}\right) - 1 = \frac{\omega}{s^2+\omega^2}
\end{aligned}$$

$L\left[\cos(\omega t)\right]$

15.3 Properties of Laplace Transform (8)

Time Differentiation (Chapter 16 preview)

- Remember the following relationships:

$$\boxed{v = L\frac{di}{dt}} \qquad \boxed{i = C\frac{dv}{dt}}$$

- We can use "Time Differentiation" property to find the Laplace Transforms of these:

$$\mathcal{L}\{v(t)\} = \mathcal{L}\{L\frac{di(t)}{dt}\} \qquad \mathcal{L}\{i(t)\} = \mathcal{L}\{C\frac{dv(t)}{dt}\}$$

$$V(s) = L\left[sI(s) - i(0^-)\right] \qquad I(s) = C\left[sV(s) - v(0^-)\right]$$

$$I(s) = \frac{V(s)}{sL} + \frac{i(0^-)}{s} \qquad V(s) = \frac{I(s)}{sC} + \frac{v(0^-)}{s}$$

- We will revisit this in chapter 16

IUPUI SCHOOL OF ENGINEERING AND TECHNOLOGY
A PURDUE UNIVERSITY SCHOOL
Indianapolis

Time Integration:

If *F (s)* is the Laplace Transforms of *f (t),* then the Laplace Transform of its integral is

$$\boxed{\mathcal{L}\left[\int_0^t f(t)dt\right] = \frac{1}{s}F(s)}$$

Examples:

$$\boxed{\mathcal{L}\left[t^2\right] = \frac{2}{s^3}} \qquad \boxed{\mathcal{L}\left[t^n\right] = \frac{n!}{s^{n+1}}}$$

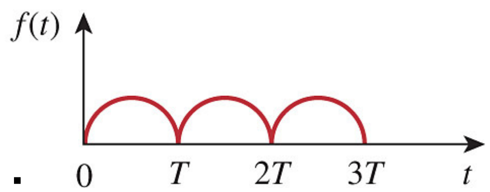
Frequency Differentiation:

If *F(s)* is the Laplace Transforms of *f (t),* then the derivative with respect to s, is

$$\boxed{\mathcal{L}\left[tf(t)\right] = -\frac{dF(s)}{ds}}$$

Example:

$$\boxed{\mathcal{L}\left[te^{-at}u(t)\right] = \frac{1}{(s+a)^2}}$$



Time Periodicity:

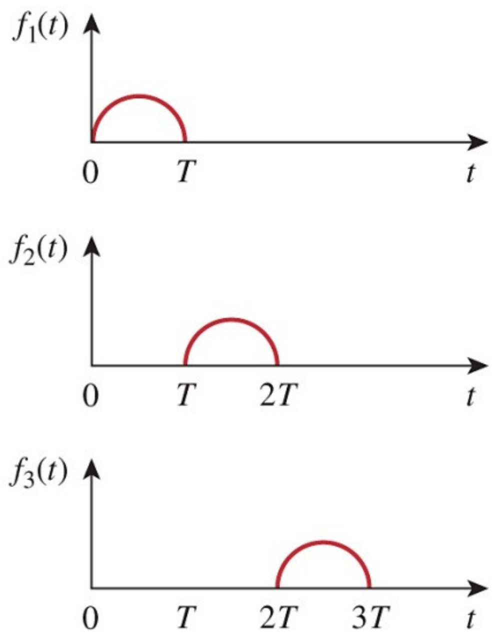
$f(t)$=f_1(t) + f_1(t-T)u(t-T)+ f_1(t-2T)u(t-2T)…

$$F(s) = F_1(s) + F_1(s)e^{-Ts} + F_1(s)e^{-2Ts} + F_1(s)e^{-3Ts} \ldots$$

$$F(s) = F_1(s)(1 + e^{-Ts} + e^{-2Ts} + e^{-3Ts} \ldots)$$

However: $(1 + x + x^2 + x^3 \ldots) = 1/(1 - x)$

Therefore: $F(s) = F_1(s)/(1 - e^{-Ts})$



Initial and Final Values:

The initial-value and final-value properties allow us to find the initial value *f(0)* and *f(∞)* of *f(t)* directly from its Laplace transform *F(s)*.

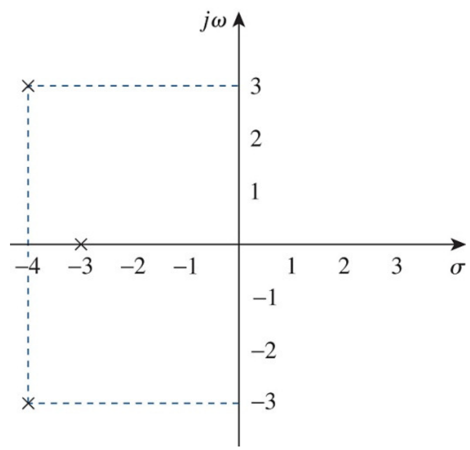
$$\boxed{f(0) = \lim_{s\to\infty} sF(s)}$$

Initial-value theorem

$$\boxed{f(\infty) = \lim_{s\to 0} sF(s)}$$

Final-value theorem, but must check that poles are all left of jω axis in s-plane. except for simple pole at s=0.

Example 1. $e^{(-2t)}\sin(5t)u(t)$
Example 2. $\sin(t)u(t)$



Example 15.7:

Find initial and final values of:

$$F(s)=\frac{20}{(s+3)(s^2+8s+25)}$$

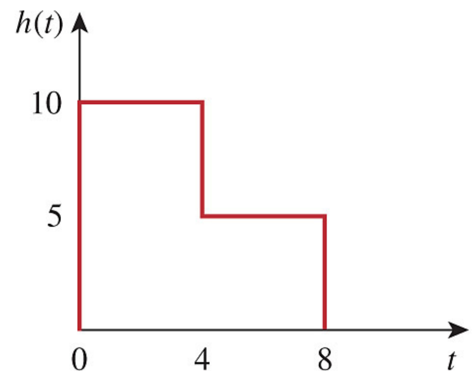
Initial-value theorem

$$f(0)=\lim_{s\to\infty} sF(s)=\lim_{s\to\infty}\frac{s20/s^3}{(s+3)(s^2+8s+25)/s^3}=\lim_{s\to\infty}\frac{20/s^2}{(1+3/s)(1+8/s+25/s^2)}=0$$

Final-value theorem, but must check that poles are all left of jω axis in s-plane.

$$f(\infty)=\lim_{s\to 0} sF(s)=\lim_{s\to 0}\frac{s20}{(s+3)(s^2+8s+25)}=\lim_{s\to 0}\frac{0}{(0+3)(0+0+25)}=0$$

- Find transform for:



$$\mathbf{h(t)} = 10\left[\mathbf{u(t)} - \mathbf{u(t}-4)\right] + 5\left[\mathbf{u(t}-4) - \mathbf{u(t}-8)\right]$$

Time Shift Property Used

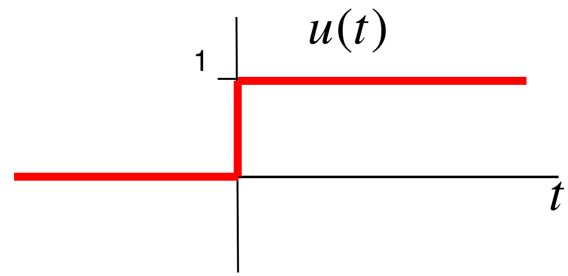
$$\mathbf{H(s)} = 10\left(\frac{1}{\mathbf{s}} - \frac{\mathbf{e}^{-4s}}{\mathbf{s}}\right) + 5\left(\frac{\mathbf{e}^{-4s}}{\mathbf{s}} - \frac{\mathbf{e}^{-8s}}{\mathbf{s}}\right)$$

$$\mathrm{H(s)} = \frac{5}{\mathbf{s}}\left(2 - \mathbf{e}^{-4s} - \mathbf{e}^{-8s}\right)$$

IUPUI SCHOOL OF ENGINEERING AND TECHNOLOGY
A PURDUE UNIVERSITY SCHOOL
Indianapolis

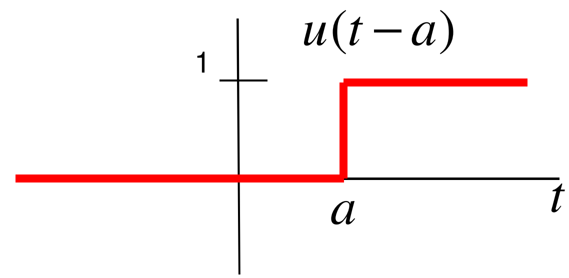
- Properties of unit step function:
  - Definition

$$u(t)=\begin{cases}0, & t<0\\ 1, & t>0\end{cases}$$



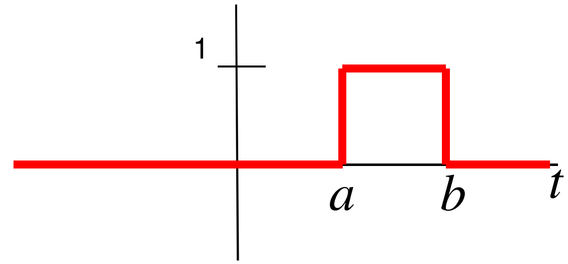
  - Time Delay

$$u(t-a)=\begin{cases}0, & t<a\\ 1, & t>a\end{cases}$$



  - Pulse

$$u(t-a)-u(t-b)=\begin{cases}0, & t<a\\ 1, & a<t<b\\ 0, & t>b\end{cases}$$

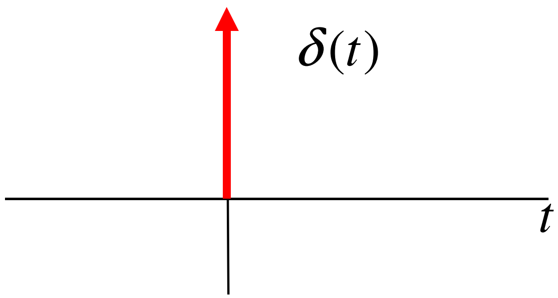


15.3 Properties of Laplace Transform (16)

Properties of impulse function (review)

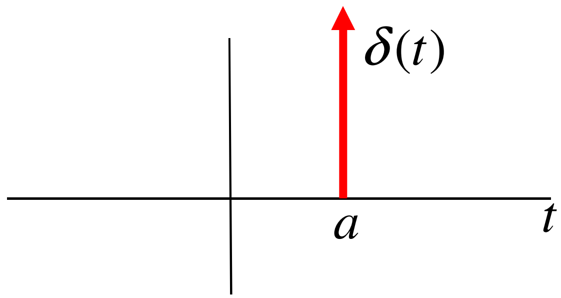
- Properties of impulse function:
  - Definition

$$\delta(t) = \begin{cases} 0 & t < 0 \\ undefined & t = 0 \\ 0 & t > 0 \end{cases}$$



  - Delay

$$\delta(t-a) = \begin{cases} 0 & t < a \\ undefined & t = a \\ 0 & t > a \end{cases}$$



  - Integration

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

  - Sifting property

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$$

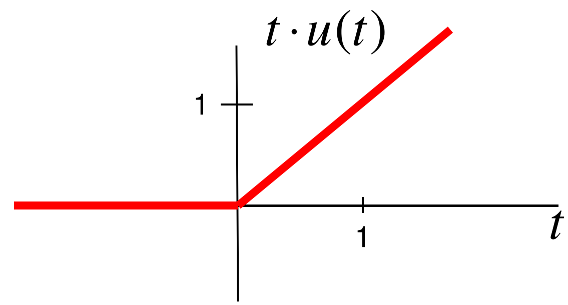
$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$$

15.3 Properties of Laplace Transform (17)

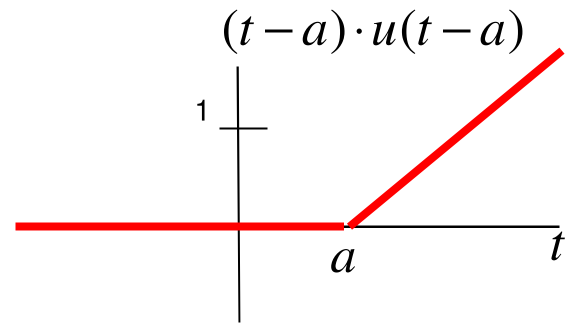
Properties of some ramp functions (review)

- Properties of ramp functions:
  - Definition

$$r(t) = t \cdot u(t) = \begin{cases} 0, & t < 0 \\ t, & t > 0 \end{cases}$$



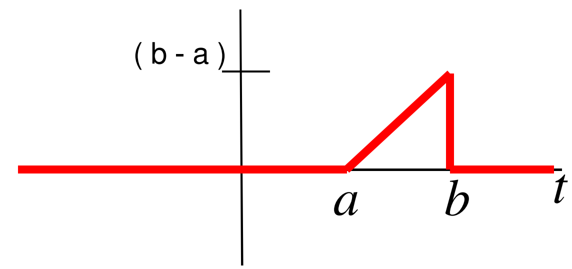
  - Time delay



$$r(t-a) = (t-a) \cdot u(t-a) = \begin{cases} 0, & t < a \\ (t-a), & t > a \end{cases}$$

  - Saw tooth

$$(t-a) \cdot \left(u(t-a) - u(t-b)\right) = \begin{cases} 0, & t < a \\ t-a, & a < t < b \\ 0, & t > b \end{cases}$$



Homework #7 (part 1)

Due in class Monday, March 23

- 15.8
- 15.14
- 15.18

More Problems to be assigned on Wednesday

15.4 The Inverse Laplace Transform (1)

Suppose F(s) has the following general form:

$$F(s) = \frac{N(s)}{D(s)} \begin{array}{l} \longleftarrow \text{Numerator Polynomial } (a_0 + a_1 s + a_2 s^2 + a_3 s^3 \ldots) \\ \hline \longleftarrow \text{Denominator Polynomial } (b_0 + b_1 s + b_2 s^2 + b_3 s^3 \ldots) \end{array}$$

The finding the inverse Laplace transform of *F(s)* involves two steps:

1. Decompose F(s) into simple terms using partial fraction expansion.
2. Find the inverse of each term by matching entries in Laplace Transform Table.

15.4 The Inverse Laplace Transform (2)

Example: 15.8

Find the inverse Laplace transform of

$$F(s) = \frac{3}{s} - \frac{5}{s+1} + \frac{6}{s^2+4}$$

Solution:

$f(t)$	$F(s)$
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$

Use table to perform Inverse Transform

$$f(t) = \mathcal{L}^{-1}\left(\frac{3}{s}\right) - \mathcal{L}^{-1}\left(\frac{5}{s+1}\right) + \mathcal{L}^{-1}\left(\frac{6}{s^2+4}\right)$$

$$\omega = 2$$

$$= (\ 3 - 5e^{-t} + 3\sin(2t)\ )u(t), \quad t \geq 0$$

15.4 The Inverse Laplace Transform (3)

- The Laplace transform F(s) can take three possible forms:
  - Only simple poles (i.e. first-order poles)
  - Repeated poles
  - Complex poles (simple or repeated)
- Methods we will use to solve:
  - Residue method
  - Algebra method
  - “Completing the Square” for complex poles
  - Substitution to solve single coefficient

15.4 The Inverse Laplace Transform (4)

Simple Poles Form

- If F(s) has only simple poles, then it can be expressed as:

$$F(s)=\frac{N(s)}{D(s)}=\frac{N(s)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

- This can be expanded by partial fraction expansion as follows:

$$F(s)=\frac{k_1}{(s+p_1)}+\frac{k_2}{(s+p_2)}+\cdots+\frac{k_n}{(s+p_n)}$$

Key Assumption

Assuming degree of **N(s)** is less than the degree of **D(S)** !!!

- We can use the residue method to find each term k_n

1) Multiply F(s) by each "pole"

$$(s+p_1)F(s)=k_1+\frac{k_2(s+p_1)}{(s+p_2)}+\cdots+\frac{k_n(s+p_1)}{(s+p_n)}$$

2) Set s = -p_i to find k_i

$$\left.(s+p_1)F(s)\right|_{s=-p_1}=k_1+\cancelto{0}{\frac{k_2(s+p_1)}{(s+p_2)}}+\cdots+\cancelto{0}{\frac{k_n(s+p_1)}{(s+p_n)}}=k_1$$

15.4 The Inverse Laplace Transform (5)

Simple Poles Form

- Residue method general form:
  - 1) Multiply the transfer function by $(s+p_i)$
  - 2) Set $s = p_i$ to find k_i

$$\boxed{(s+p_i)F(s)\Big|_{s=-p_i} = k_i}$$

- We can also note by looking at the transform pairs table, that if the transfer function is in this form, the Laplace will be in the following form:

$$F(s) = \frac{k_1}{(s+p_1)} + \frac{k_2}{(s+p_2)} + \cdots + \frac{k_n}{(s+p_n)}$$

$f(t)$	$F(s)$
e^{-at}	$\frac{1}{s+a}$

$$f(t) = k_1 e^{-p_1 t} + k_2 e^{-p_2 t} + \cdots + k_n e^{-p_n t}$$

For t > 0

15.4 The Inverse Laplace Transform (6)

Simple Poles Form

- What about a simple pole at origin?
- Same principle:

$$\boxed{(s)F(s)\big|_{s=0} = k_0}$$

$f(t)$	$F(s)$
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$

$$F(s) = \frac{N(s)}{s(s+p_1)\cdots(s+p_n)} = \frac{k_0}{s} + \frac{k_1}{(s+p_1)} + \cdots + \frac{k_n}{(s+p_n)}$$

$$f(t) = k_0 u(t) + k_1 e^{-p_1 t} + \cdots + k_n e^{-p_n t}$$ For t > 0

Note: We could have also wrote this as follows:

$$f(t) = \left(k_0 + k_1 e^{-p_1 t} + \cdots + k_n e^{-p_n t}\right) u(t)$$

Example: 15.9

Find the inverse Laplace transform of

$$F(s)=\frac{s^2+12}{s(s+2)(s+3)}$$

Note: Order of N(s) is 2, Order of D(s) is 3

Solution:

$$F(s)=\frac{s^2+12}{s(s+2)(s+3)}=\frac{k_0}{s}+\frac{k_1}{(s+2)}+\frac{k_2}{(s+3)}$$

$$k_0=sF(s)\big|_{s=0}=\cancel{s}\frac{s^2+12}{\cancel{s}(s+2)(s+3)}\bigg|_{s=0}=\frac{0^2+12}{(0+2)(0+3)}=2$$

$$k_1=(s+2)F(s)\big|_{s=-2}=\cancel{(s+2)}\frac{s^2+12}{s\cancel{(s+2)}(s+3)}\bigg|_{s=-2}=\frac{(-2)^2+12}{-2(-2+3)}=-8$$

$$k_2=(s+3)F(s)\big|_{s=-3}=\cancel{(s+3)}\frac{s^2+12}{s(s+2)\cancel{(s+3)}}\bigg|_{s=-3}=\frac{(-3)^2+12}{-3(-3+2)}=7$$

15.4 The Inverse Laplace Transform (8)

Simple Poles Form – Example 15.9

Solution (continued)

$f(t)$	$F(s)$
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$

$$F(s)=\frac{s^2+12}{s(s+2)(s+3)}=\frac{2}{s}-\frac{8}{(s+2)}+\frac{7}{(s+3)}$$

$$f(t)=2u(t)-8e^{-2t}+7e^{-3t}$$ For t > 0

Same

or

$$f(t)=\left(2-8e^{-2t}+7e^{-3t}\right)u(t)$$

15.4 The Inverse Laplace Transform (9)

Repeated Poles Form

IUPUI | SCHOOL OF ENGINEERING AND TECHNOLOGY
A PURDUE UNIVERSITY SCHOOL
Indianapolis

- Suppose F(s) has n repeated poles at s = -p, we can then express as follows:

$$F(s)=\frac{k_n}{(s+p)^n}+\frac{k_{n-1}}{(s+p)^{n-1}}+\cdots+\frac{k_2}{(s+p)^2}+\frac{k_1}{(s+p)}+F_1(s)$$

- We find the coefficients by taking derivatives

$$k_n=(s+p)^n F(s)\Big|_{s=-p}$$

$$k_{n-1}=\frac{d}{ds}(s+p)^n F(s)\Big|_{s=-p}$$

$$k_{n-2}=\frac{1}{2!}\frac{d^2}{ds^2}(s+p)^n F(s)\Big|_{s=-p}$$

$$k_{n-m}=\frac{1}{m!}\frac{d^m}{ds^m}(s+p)^n F(s)\Big|_{s=-p}$$

Example: 15.10

Find the inverse Laplace transform of $F(s)=\dfrac{10s^2+4}{s(s+1)(s+2)^2}$

From repeated pole

Solution: $$F(s)=\frac{10s^2+4}{s(s+1)(s+2)^2}=\frac{k_0}{s}+\frac{k_1}{(s+1)}+\boxed{\frac{k_2}{(s+2)^2}+\frac{k_3}{(s+2)}}$$

Start with repeated pole

Keep decrementing till order = 1

$$k_0=sF(s)\big|_{s=0}=s\frac{10s^2+4}{s(s+1)(s+2)^2}\bigg|_{s=0}=\frac{0^2+4}{(0+1)(0+2)^2}=1$$

$$k_1=(s+1)F(s)\big|_{s=-1}=(s+1)\frac{10s^2+4}{s(s+1)(s+2)^2}\bigg|_{s=-1}=\frac{10(-1)^2+4}{-1(-1+2)^2}=-14$$

$$k_2=(s+2)^2F(s)\big|_{s=-2}=(s+2)^2\frac{10s^2+4}{s(s+1)(s+2)^2}\bigg|_{s=-2}=\frac{10(-2)^2+4}{-2(-2+1)}=22$$

$$k_3=\frac{d}{ds}(s+2)^2F(s)\big|_{s=-2}=\frac{d}{ds}\frac{10s^2+4}{s(s+1)}\bigg|_{s=-2}=(\textit{ see text})=13$$

15.4 The Inverse Laplace Transform (11)

Repeated Poles – Example 15.10

$$F(s)=\frac{1}{s}-\frac{14}{(s+1)}+\frac{13}{(s+2)}+\frac{22}{(s+2)^2}$$

$$f(t)=u(t)-14e^{-t}+13e^{-2t}+22te^{-2t}$$ For t > 0

or

$$f(t)=\left(1-14e^{-t}+13e^{-2t}+22te^{-2t}\right)u(t)$$

$f(t)$	$F(s)$
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
te^{-at}	$\frac{1}{(s+a)^2}$

15.4 The Inverse Laplace Transform (12)

Algebraic Method

- The "Algebraic Method" multiplies both sides by the denominator of F(s) and collects the terms of powers of s:

Example 15.10:

$$F(s)=\frac{10s^2+4}{s(s+1)(s+2)^2}=\frac{k_0}{s}+\frac{k_1}{(s+1)}+\frac{k_2}{(s+2)^2}+\frac{k_3}{(s+2)}$$

Multiply the left side by the denominator term:

$$F(s)s(s+1)(s+2)^2=10s^2+4$$

Multiply the right side by the denominator term:

$$10s^2+4=k_0(s+1)(s+2)^2+k_1s(s+2)^2+k_2s(s+1)+k_3s(s+1)(s+2)$$

Expand polynomials of s:

$$10s^2+4=k_0\left(s^3+5s^2+8s+4\right)+k_1\left(s^3+4s^2+4s\right)+k_2\left(s^2+s\right)+k_3\left(s^3+3s^2+2s\right)$$

15.4 The Inverse Laplace Transform (13)

Algebraic Method – Example 15.10

Next, collect the terms with the same powers of s:

$$10s^2+4=k_0\left(s^3+5s^2+8s+4\right)+k_1\left(s^3+4s^2+4s\right)+k_2\left(s^2+s\right)+k_3\left(s^3+3s^2+2s\right)$$

Constant:	4	$=$	$k_0(4)$
s	0	$=$	$k_0(8s)+k_1(4s)+k_2(s)+k_3(2s)$
s^2	$10s^2$	$=$	$k_0(5s^2)+k_1(4s^2)+k_2(s^2)+k_3(3s^2)$
s^3	0	$=$	$k_0(s^3)+k_1(s^3)+k_3(s^3)$

Results in n equations / n unknowns, solve set of linear equations. Can use substitution method or MATLAB to solve.

$$\begin{bmatrix}4&0&0&0\\8&4&1&2\\5&4&1&3\\1&1&0&1\end{bmatrix}\begin{bmatrix}k_0\\k_1\\k_2\\k_3\end{bmatrix}=\begin{bmatrix}4\\0\\10\\0\end{bmatrix}\Rightarrow\begin{bmatrix}k_0\\k_1\\k_2\\k_3\end{bmatrix}=\begin{bmatrix}1\\-14\\22\\13\end{bmatrix}$$

15.4 The Inverse Laplace Transform (14)

Complex Poles – (Form: $s^2 + as + b$)

- A pair of simple complex poles can result in a denominator of form $s^2 + as + b$
- We can expand by partial fractions as follows:

$$F(s) = F_1(s) + \frac{k_0 s + k_1}{\left(s^2 + as + b\right)}$$

- Were $F_1(s)$ is the expansion resulting from other poles

Example: (15.11)

$$F(s) = \frac{20}{(s+3)\left(s^2+8s+25\right)} = \frac{k_0}{(s+3)} + \frac{k_1 s + k_2}{\left(s^2+8s+25\right)}$$

- The coefficients can be found using the methods discussed previously.

IUPUI SCHOOL OF ENGINEERING AND TECHNOLOGY
A PURDUE UNIVERSITY SCHOOL
Indianapolis

Example: 15.11

Find the inverse Laplace transform of

$$F(s) = \frac{20}{(s+3)(s^2+8s+25)} = \frac{k_0}{(s+3)} + \frac{k_1 s + k_2}{(s^2+8s+25)}$$

Solution: First find k_o by residue method:

$$k_0 = (s+3)F(s)\big|_{s=-3} = \frac{20}{\left((-3)^2 + 8(-3) + 25\right)} = 2$$

Using residue method to find the other coefficients would involve complex algebra. Instead pick values of "s" that lead to one solution.

Note: This "picking a value of s" only works here because there is one solution for one coefficient.

Use known value of k_0 and try s=0 to solve for k_2

$$\frac{20}{(3)(25)} = \frac{2}{(3)} + \frac{k_2}{(25)} \Longrightarrow k_2 = 10$$

Now let s=1 to solve for k_1

$$\frac{20}{(1+3)(1+8+25)} = \frac{2}{(1+3)} + \frac{k_1 - 10}{(1+8+25)} \Longrightarrow k_1 = -2$$

15.4 The Inverse Laplace Transform (16)

Example 15.11 (Continued) – "Completing the Square"

So now we have: $F(s)=\dfrac{20}{(s+3)(s^2+8s+25)}=\dfrac{2}{(s+3)}-\dfrac{2s+10}{(s^2+8s+25)}$

How do we take the inverse Laplace Transform of the 2nd fraction?

We use a method called "Completing the Square":

This is what we have

$$\frac{2s+10}{(s^2+8s+25)}$$

This is the form we want

$$\frac{s+a}{(s+a)^2+\omega^2}=\mathcal{L}^{-1}\left[e^{-at}\cos\omega t\right]$$

Expand out to find values of a and ω that will work

$$s^2+8s+25=s^2+2as+(a^2+\omega^2)$$

$$a=4$$

$$25=4^2+\omega^2 \quad\Rightarrow\quad \omega=3$$

Substitute a and ω into original equation

Now we have the following form: $F(s)=\dfrac{2}{(s+3)}-\dfrac{2s+10}{(s+4)^2+3^2}$

Next we must fix the numerator

15.4 The Inverse Laplace Transform (17)

Example 15.11 (Continued) – "Completing the Square"

Closer, but not yet done. Still need to manipulate the numerator to look like something for which we can perform the inverse Laplace Transform

This is what we have

$$F(s) = \frac{2}{(s+3)} - \frac{2s+10}{(s+4)^2+3^2}$$

This is the form we want

$$\frac{s+a}{(s+a)^2+\omega^2} = \mathcal{L}^{-1}\left[e^{-at}\cos\omega t\right]$$

$$F(s) = \frac{2}{(s+3)} - 2\cdot\frac{s+4+1}{(s+4)^2+3^2}$$

This now matches the form we wanted

But what do we do about this?

$$F(s) = \frac{2}{(s+3)} - 2\cdot\frac{s+4}{(s+4)^2+3^2} - 2\cdot\frac{1}{(s+4)^2+3^2}$$

Manipulate to look like this

$$\frac{\omega}{(s+a)^2+\omega^2} = \mathcal{L}^{-1}\left[e^{-at}\sin\omega t\right]$$

$$F(s) = \frac{2}{(s+3)} - 2\cdot\frac{s+4}{(s+4)^2+3^2} - \frac{2}{3}\cdot\frac{3}{(s+4)^2+3^2}$$

We can now take the inverse Laplace

$$f(t) = \mathcal{L}^{-1}\left[F(s)\right] = 2e^{-3t} - 2e^{-4t}\cos(3t) - \tfrac{2}{3}e^{-4t}\sin(3t) \qquad for\quad t \geq 0$$

Homework #7

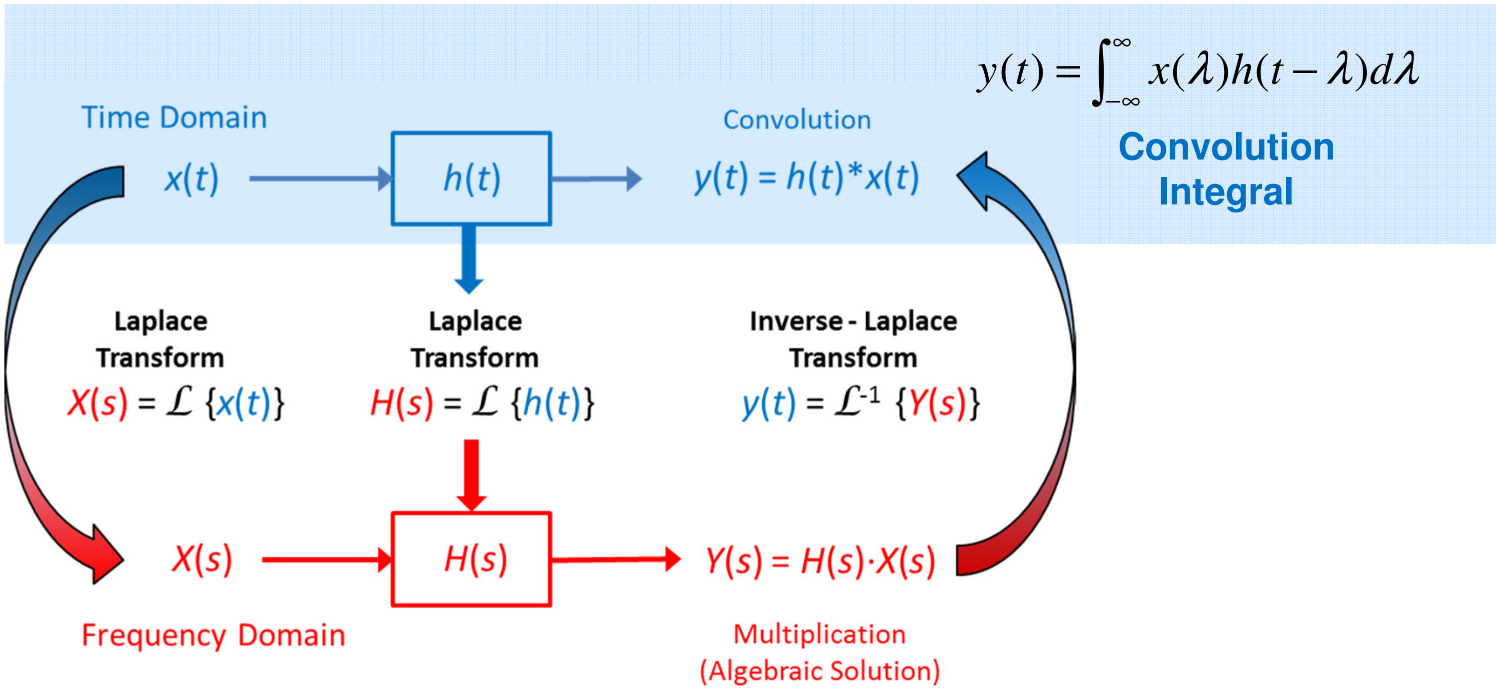
Due in class Monday, March 23

- 15.8
- 15.14
- 15.18
- 15.25
- 15.27
- 15.37

Problems due on Monday March 23th

15.5 The Convolution Integral (1)

- Up to this point we've worked in the frequency domain



- To obtain the output *y*(*t*) directly in the time domain without going through the Laplace Transform to the frequency domain we can use the convolution integral

15.5 The Convolution Integral (2)

- The notation we use for "convolution" is as follows:

$$y(t) = x(t) * h(t)$$

- This is NOT the same as multiplying the two signals! However, as a linear operator it abides by many of the same properties:

$$\begin{array}{lrcl}
\text{(Commutative)} & x(t) * h(t) & = & h(t) * x(t) \\
\text{(Distributive)} & f(t) * \big(x(t) + y(t)\big) & = & f(t) * x(t) + f(t) * y(t) \\
\text{(Associative)} & f(t) * \big(x(t) * y(t)\big) & = & \big(f(t) * x(t)\big) * y(t)
\end{array}$$

- This notation " $*$ " for the "convolution" operation is equivalent to the following integral equation:

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\lambda) h(t-\lambda) d\lambda \quad \text{or} \quad \int_{-\infty}^{\infty} x(t-\lambda) h(\lambda) d\lambda}$$

- This is the "general" form. We can simplify further…

15.5 The Convolution Integral (3)

For Causal systems starting at t=0

- If we assume *x*(*t*) starts at t = 0. We can then simplify the convolution integral as follows:

$$y(t) = x(t) * h(t) = \int_0^{\infty} x(\lambda)h(t-\lambda)d\lambda$$

(Given x(t) = 0 for t < 0)

- Similarly we defined *h*(*t*) as only having values for *t* > 0.

$$\text{if} \quad h(t) = 0 \quad \text{for} \quad t < 0$$

$$\text{then} \quad h(t-\lambda) = 0 \quad \text{for} \quad t-\lambda < 0 \quad \text{or} \quad \lambda > t$$

- This modifies the “upper” limit of the integral as follows:

$$\boxed{y(t) = x(t) * h(t) = \int_0^{t} x(\lambda)h(t-\lambda)d\lambda}$$

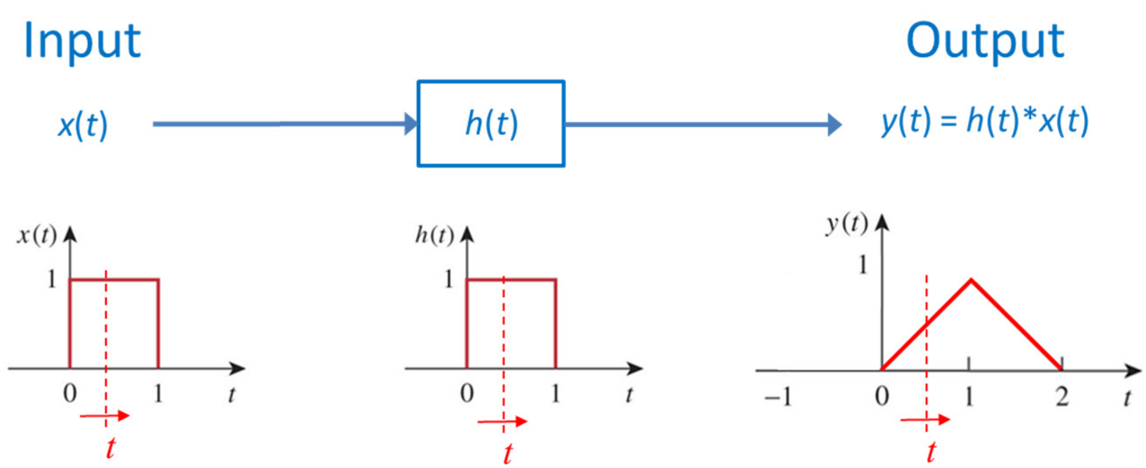
Given
x(t) = 0 for t < 0
h(t) = 0 for t < 0

- This definition of *h*(*t*) & *x*(*t*) indicates the system is “Causal”. That is the output depends on past and current inputs but not future inputs

15.5 The Convolution Integral (4)

Causal Example

- If the function $h(t)$ does not start before $x(t)$, the system is Causal.
- For a Causal system the output depends on past and current inputs but NOT future inputs



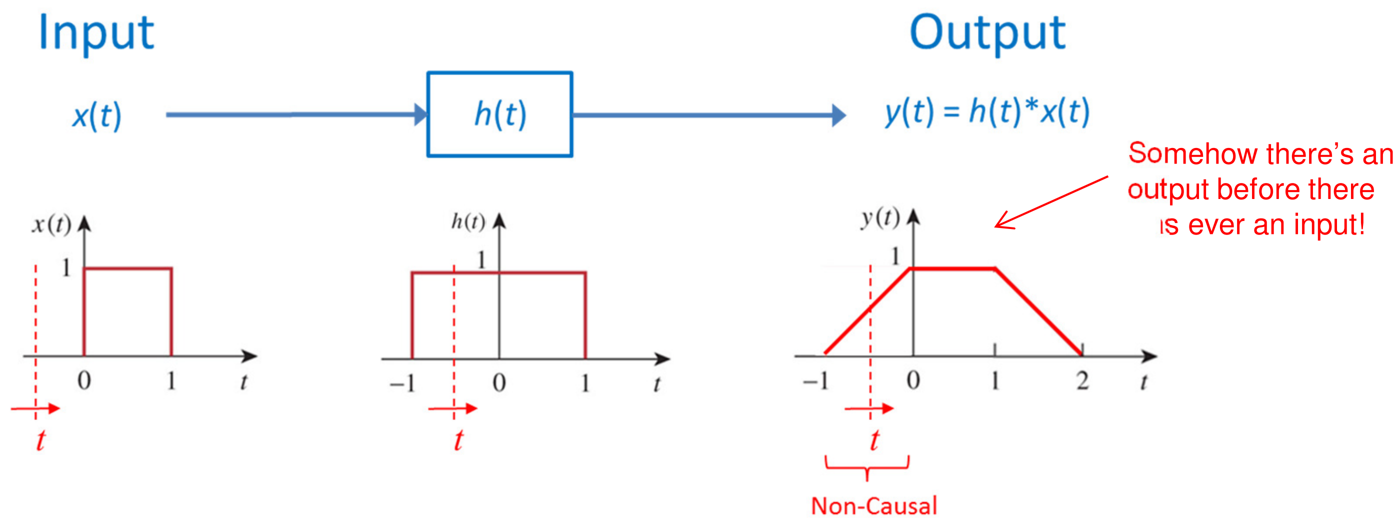
$$y(t) = x(t) * h(t) = \int_0^t x(\lambda)h(t-\lambda)d\lambda$$

Given
x(t) = 0 for t < 0
h(t) = 0 for t < 0

15.5 The Convolution Integral (5)

Non Causal Example

- If the function $h(t)$ has a value before the input $x(t)$, the system is said to be "Non-Causal".
- This implies something would output at $y(t)$ "before" anything was applied to the input. Somehow, the system would "predict" that an input is about to be applied!



- A "Non-Causal" output depends on past, present, AND future values of the input.
- We will only be dealing with "Causal" systems

15.5 The Convolution Integral (6)

Sifting property and the Impulse Response

- Let's see what happens when we apply an impulse function to the input:

$$y(t) = \delta(t) * h(t) = \int_{-\infty}^{\infty} \delta(\lambda) h(t-\lambda) d\lambda$$

- The impulse function "sifts out" all values of λ except for zero.
- The integral only has a value when λ = 0. At this point

$$y(t) = h(t-0) \int_{-\infty}^{\infty} \delta(\lambda) d\lambda = h(t)$$

- Because of the function *h*(*t*) is often called the "Impulse Response" of the system.

$$\boxed{\begin{array}{lll} h(t) & \rightarrow & \text{Impulse Response} \\ \text{H}(s) & \rightarrow & \text{Transfer Function} \end{array}}$$

15.5 Convolution Integral (7)

- Let's revisit the integral to describe what it's doing

$$y(t) = x(t) * h(t) = \int_0^t x(\lambda)h(t-\lambda)d\lambda$$

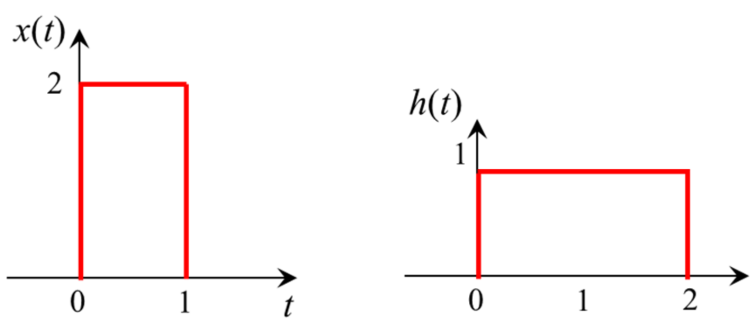
1.) First we "**FLIP**" $h(t)$. (Notice it's in terms of $-\lambda$)

2.) Next we "**SHIFT**" $h(-\lambda)$ by time "t" to get $h(t-\lambda)$.

3.) Then "**MULTIPLY**" $x(\lambda)\cdot h(t-\lambda)$

4.) Lastly, "**INTEGRATE**" to find the area under the curve

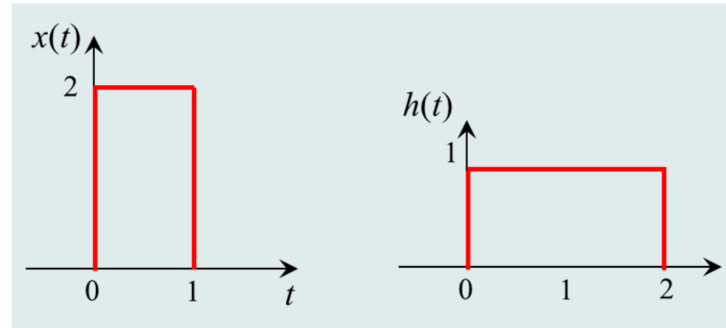
- Let's look at this process graphically with the following example:

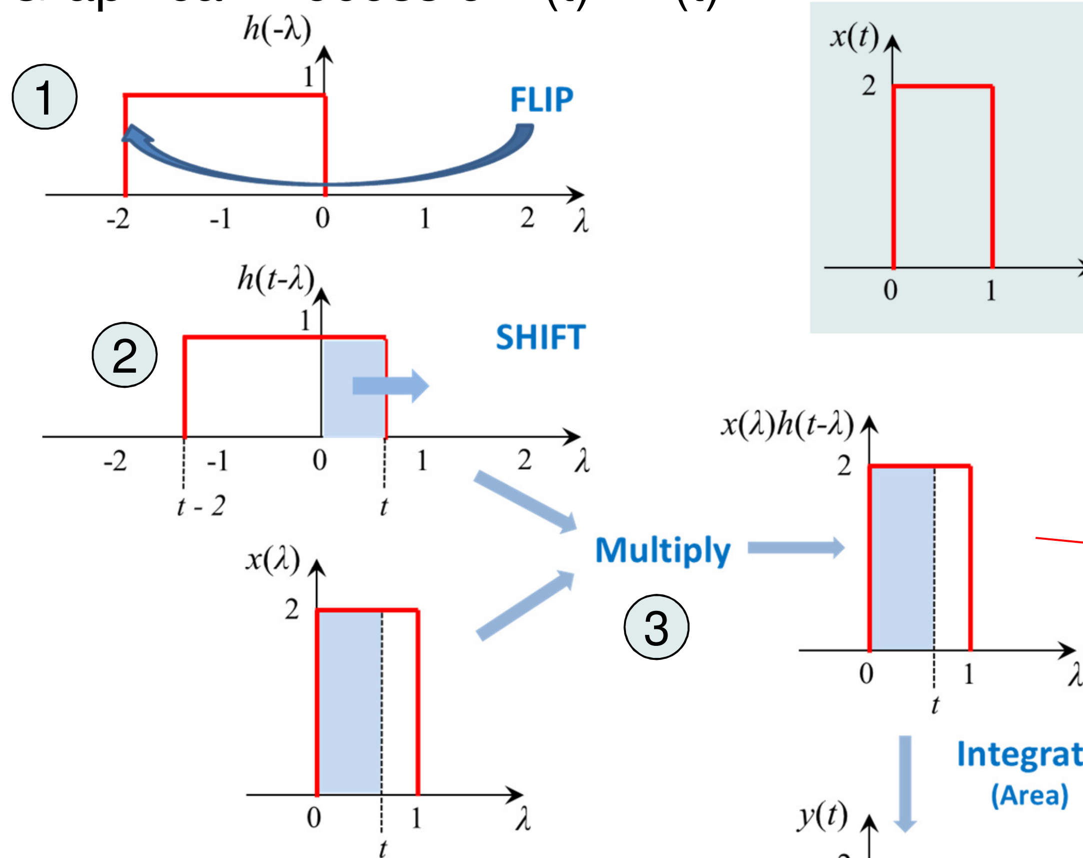


15.5 Convolution Integral (8)

Graphical Example

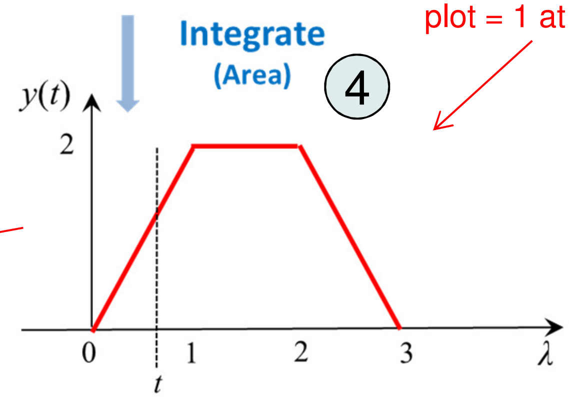
- Graphical Process of x(t) * h(t)





Notice that at t= 0.5, the area under curve is (2)(0.5) = 1.0.

Therefore, the output plot = 1 at t = 0.5.



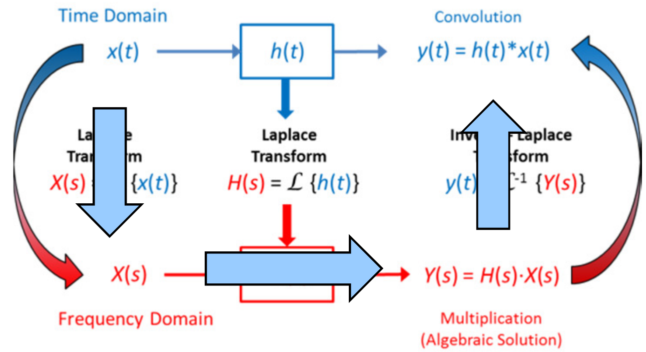
The output can be broken up into 3 "regions" we integrate over:

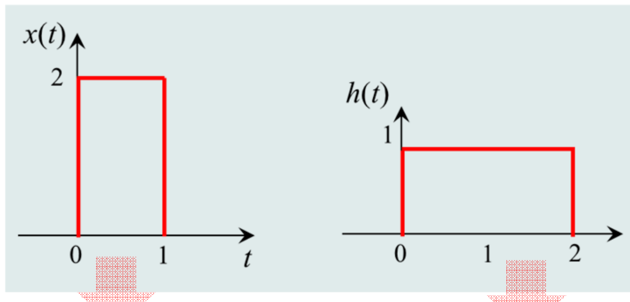
0 < t < 1	(Ramping up at rate 2t)
1 < t < 2	(2 x 1)
2 < t < 3	(Ramping down at rate -2t)
3 < t	(No overlay, y(t) = 0)

15.5 Convolution Integral (9)

Graphical Example (Continued)

- We can also obtain this result by performing the Laplace transform





$$x(t) = 2u(t) - 2u(t-1) \qquad h(t) = u(t) - u(t-2)$$

$$X(s) = \frac{2}{s} - \frac{2}{s}e^{-s} \qquad H(s) = \frac{1}{s} - \frac{1}{s}e^{-2s}$$

$$Y(s) = X(s)H(s) = \frac{2}{s}\left(1-e^{-s}\right)\frac{1}{s}\left(1-e^{-2s}\right) = \frac{2}{s^2}\left(1-e^{-s}-e^{-2s}+e^{-3s}\right)$$

Inverse Laplace Time Delay Property

$$y(t) = 2tu(t) - 2(t-1)u(t-1) - 2(t-2)u(t-2) + 2(t-3)u(t-3)$$

15.5 Convolution Integral (10)

Graphical Example (Continued)

$$y(t) = 2tu(t) - 2(t-1)u(t-1) - 2(t-2)u(t-2) + 2(t-3)u(t-3)$$

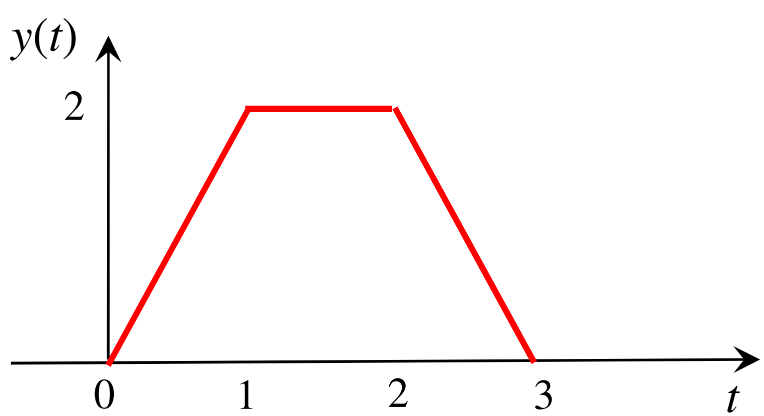
For $0 \le t \le 1$ $\quad y(t) = 2t$

For $1 \le t \le 2$ $\quad y(t) = 2t - 2(t-1) = 2$

For $2 \le t \le 3$ $\quad y(t) = 2t - 2(t-1) - 2(t-2) = 2 - 2t + 4 = 6 - 2t$

For $3 \le t$ $\quad y(t) = 2t - 2(t-1) - 2(t-2) + 2(t-3) = 0$

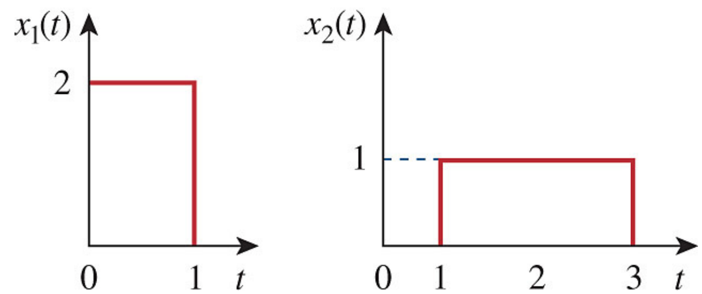
$$\boxed{y(t) = \begin{cases} 0 & t \le 0 \\ 2t & 0 \le t \le 1 \\ 2 & 1 \le t \le 2 \\ 6-2t & 2 \le t \le 3 \\ 0 & 3 \le t \end{cases}}$$



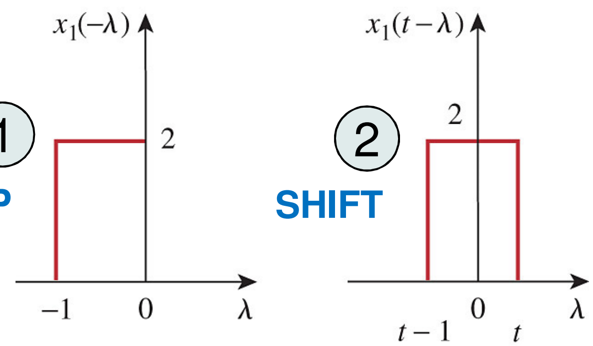
- Which gives the same result as the graphical approach !

15.5 The Convolution Integral (11)

- Another Example (from Text):



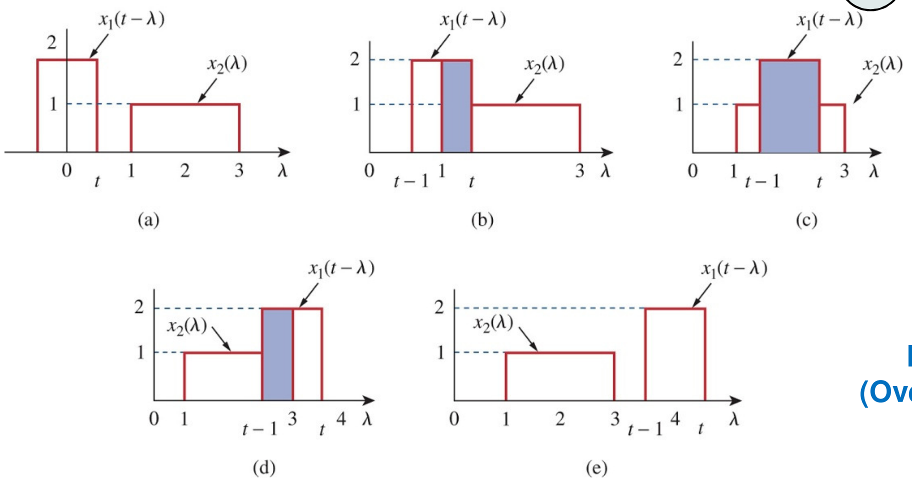
1 FLIP



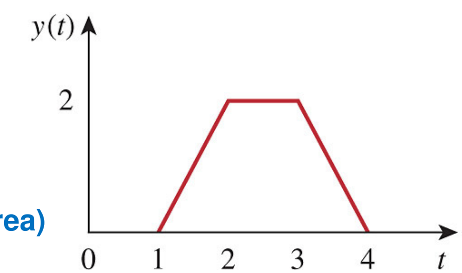
2 SHIFT

(a) (b)

3 MULTIPLY



4 INTEGRATE (Overlapping Area)

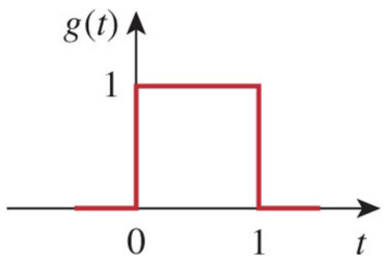


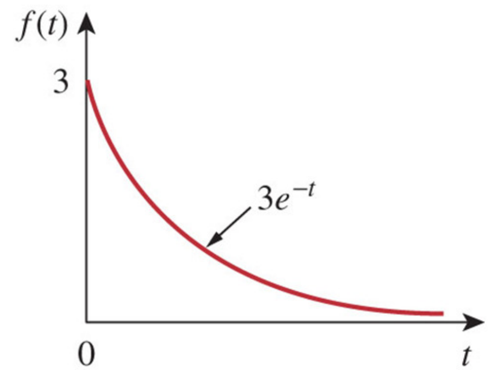
Practice Problem 15.13:

Find y*(t)=g(t)*f(t)* for the functions shown below.

a) Perform using y(t)=g(t)*f(t)

b) Perform using Y(s)=G(s)*F(s)





For 0 < t < 1 :

$$y(t) = \int_0^t (1)3e^{-\lambda}d\lambda = -3e^{-\lambda}\Big|_0^t = 3\left(1 - e^{-t}\right)$$

For 1 < t :

$$y(t) = \int_{t-1}^t (1)3e^{-\lambda}d\lambda = -3e^{-\lambda}\Big|_{t-1}^t = -e^{-t} + e^{-t+1}$$

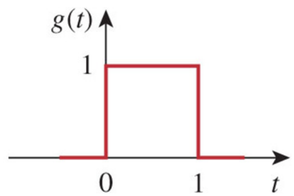
$$y(t) = \left(e^1 - 1\right)e^{-t}$$

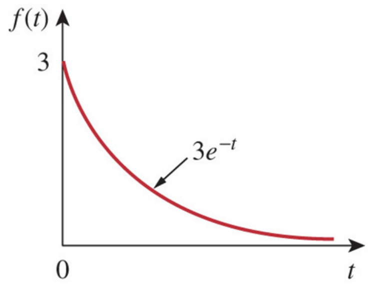
Solution:

$$y(t) = \begin{cases} 3\left(1 - e^{-t}\right) & 0 \le t \le 1 \\ 3(e-1)e^{-t} & 1 \le t \\ 0 & elseware \end{cases}$$

IUPUI SCHOOL OF ENGINEERING AND TECHNOLOGY
A PURDUE UNIVERSITY SCHOOL
Indianapolis

b.) Find Y(s)=G(s)*F(s)





$$g(t) = u(t) - u(t-1) \qquad f(t) = 3e^{-t}u(t)$$

$$\Downarrow \qquad \Downarrow$$

$$G(s) = \frac{1}{s} - \frac{1}{s}e^{-s} \qquad F(s) = \frac{3}{s+1}$$

$$Y(s) = G(s)F(s) = \left(\frac{1}{s} - \frac{1}{s}e^{-s}\right)\frac{3}{s+1}$$

$$Y(s) = \frac{3}{s(s+1)} - \frac{3}{s(s+1)}e^{-s} = Y'(s) - Y'(s)e^{-s}$$

$$Y'(s) = \frac{3}{s(s+1)} = \frac{k_0}{s} + \frac{k_1}{(s+1)}; \qquad k_0 = 3 \;\; ; \;\; k_1 = -3$$

$$Y'(s) = \frac{3}{s(s+1)} = \frac{3}{s} - \frac{3}{(s+1)}$$

$$y(t) = \left(3 - 3e^{-t}\right)u(t) - \left(3 - 3e^{-(t-1)}\right)u(t-1)$$

$$\boxed{y(t) = \begin{cases} 3\left(1 - e^{-t}\right) & 0 \le t \le 1 \\ 3(e-1)e^{-t} & 1 \le t \\ 0 & elseware \end{cases}}$$

15.5 The Convolution Integral (13)

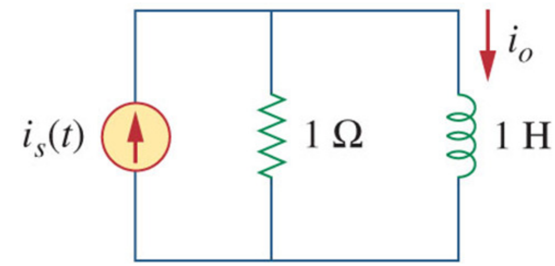
Example15.14

Example 15.14:

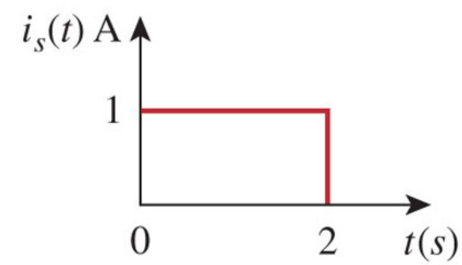
For the RL circuit shown in figure (a), use the convolution integral to find the response $i_o(t)$ due to the excitation show in figure (b)

a) Perform using $y(t)=i_s(t)*h(t)$

b) Perform using $Y(s)=I_s(s)*H(s)$



(a)



(b)

Solution (Part a):

Use Current Divider equation to find H(s):

$$I_o = \frac{1}{1+s} I_s \quad \text{(Current Divider)}$$

$$H(s) = \frac{I_o}{I_s} = \frac{1}{1+s} \quad \text{(Transfer Function)}$$

$$h(t) = \mathcal{L}^{-1}\left[\frac{1}{1+s}\right] = e^{-t}u(t) \quad \text{(Inverse Laplace)}$$

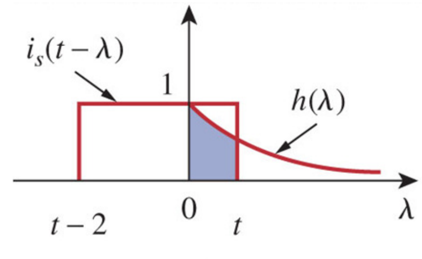
The input is described graphically by the following:

$$i_s(t) = u(t) - u(t-2)$$

Next, use Graphical method to determine ranges of integration

For < t < 0, there is no overlap, therefore:

$$i_o(t) = i_s(t) * h(t) = \boxed{0 \quad \text{for } t \le 0}$$

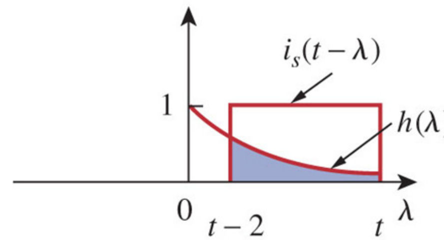


(a)

For 0 < t < 2, we can find the area under the curve shown in figure a:

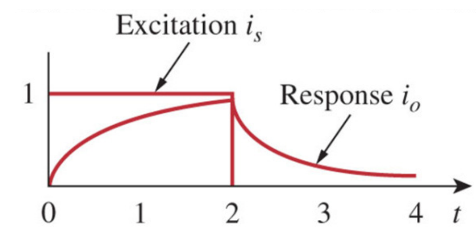
$$i_o(t) = i_s(t) * h(t)$$

$$i_o(t) = \int_0^t (1) e^{-\lambda} d\lambda = -e^{-\lambda}\Big|_0^t = \boxed{\left(1 - e^{-t}\right) \quad \text{for } 0 \le t \le 2}$$



(b)

For t > 2, we can find the area under the curve shown in figure b:



$$i_o(t) = i_s(t) * h(t)$$

$$i_o(t) = \int_{t-2}^t (1) e^{-\lambda} d\lambda = -e^{-\lambda}\Big|_{t-2}^t = -e^{-t} + e^{-t+2}$$

$$i_o(t) = \boxed{\left(e^2 - 1\right) e^{-t} \quad \text{for } 2 \le t}$$

Solution

$$\boxed{i_o(t) = \begin{cases} 1 - e^{-t} & 0 \le t \le 2 \\ \left(e^2 - 1\right) e^{-t} & 2 \le t \\ 0 & elseware \end{cases}}$$

IUPUI | SCHOOL OF ENGINEERING AND TECHNOLOGY
A PURDUE UNIVERSITY SCHOOL
Indianapolis

Solution (Part b), find y(t) using Laplace Transform to Y(s):

From earlier: $H(s) = \dfrac{1}{1+s}$ $\quad i_s(t) = u(t) - u(t-2)$

$$\Downarrow$$

$$I_s(s) = \frac{1}{s} - \frac{1}{s}e^{-2s}$$

Output Y(s) given by: $I_o(s) = I_s(s)H(s) = \left(\dfrac{1}{s} - \dfrac{1}{s}e^{-2s}\right)\dfrac{1}{s+1}$

$$I_o(s) = \frac{1}{s(s+1)} - \frac{1}{s(s+1)}e^{-2s} = I_o^{\prime}(s) - I_o^{\prime}(s)e^{-2s}$$

$$I_o^{\prime}(s) = \frac{1}{s(s+1)} = \frac{k_0}{s} + \frac{k_1}{(s+1)}; \quad k_0 = 1\,;\, k_1 = -1$$

$$I_o^{\prime}(s) = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{(s+1)}$$

Time Delay Property

$$i_o(t) = \left(1 - e^{-t}\right)u(t) - \left(1 - e^{-(t-2)}\right)u(t-2)$$

$$\boxed{i_o(t) = \begin{cases} 1 - e^{-t} & 0 \le t \le 2 \\ \left(e^{2} - 1\right)e^{-t} & 2 \le t \\ 0 & elseware \end{cases}}$$

15.5 The Convolution Integral (16)

Joy of Convolution – Other Resources

IUPUI SCHOOL OF ENGINEERING AND TECHNOLOGY
A PURDUE UNIVERSITY SCHOOL
Indianapolis

- http://www.jhu.edu/~signals/convolve/index.html
- Other examples of graphical convolution:
  - This example involves a rather complicated function however the concept is illustrated well:
    - http://www.youtube.com/watch?v=zoRJZDiPGds
  - This example is a bit simpler (two rectangular pulses)
    - http://www.youtube.com/watch?v=MEDjw6VcDTY
  - Flash video showing graphical convolution:
    - https://engineering.purdue.edu/VISE/ee438/demos/flash/convolution.html
  - Example involving a step and exponent (more realistic)
    - http://www.youtube.com/watch?annotation_id=annotation_671224&feature=iv&src_vid=SNdNf3mprrU&v=_UFyMWDoISk

Homework #8

Due in class Monday, March 30th

- 15.43 a) (Graphical & Laplace)
- 15.43 b) (Graphical & Laplace)
- 15.44 a) (Graphical & Laplace)
- Practice problem 15.14 (Graphical & Laplace)
- 16.14
- 16.16

Problems due on Monday, March 30th
(Chapter 16 material to be covered Wednesday)