Homework #7 (SOLUTION KEY) Name:

1. (Prob. 15.8 in text) Find the Laplace transform, F(s), given that f(t) is:
   a. 2t·u(t-4) (u is the unit step function)
   b. 4cos(t)δ(t-2) (δ is the Dirac delta function)
   c. e^{-t}·u(t-τ)
   d. sin(2t) ·u(t- τ)

(a) 2t=2(t-4) + 8

f(t) = 2tu(t-4) = 2(t-4)u(t-4) + 8u(t-4)

$$F(s) = \frac{2}{s^2}e^{-4s} + \frac{8}{s}e^{-4s} = \underline{\left(\frac{2}{s^2} + \frac{8}{s}\right)e^{-4s}}$$

(b) $$F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty 5\cos t\,\delta(t-2)e^{-st}dt = 5\cos t e^{-st}\Big|_{t=2} = \underline{\mathbf{5cos(2)e^{-2s}}}$$

(c) $$e^{-t} = e^{-(t-\tau)}e^{-\tau}$$

$$f(t) = e^{-\tau}e^{-(t-\tau)}u(t-\tau)$$

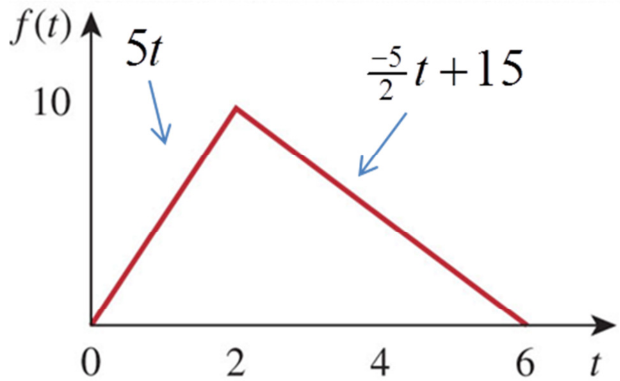
$$F(s) = e^{-\tau}e^{-\tau s}\frac{1}{s+1} = \underline{\frac{e^{-\tau(s+1)}}{s+1}}$$

(d) $$\sin 2t = \sin[2(t-\tau)+2\tau] = \sin 2(t-\tau)\cos 2\tau + \cos 2(t-\tau)\sin 2\tau$$

$$f(t) = \cos 2\tau \sin 2(t-\tau)u(t-\tau) + \sin 2\tau \cos 2(t-\tau)u(t-\tau)$$

$$F(s) = \underline{\cos 2\tau e^{-\tau s}\frac{2}{s^2+4} + \sin 2\tau e^{-\tau s}\frac{s}{s^2+4}}$$

2. (Prob. 15.14 from Text) Find the Laplace transform of the signal in the figure below:



$$f(t)=\begin{cases} 5t & 0\le t<2 \\ \frac{-5}{2}t+15 & 2\le t<6 \\ 0 & 6\le t<\infty \end{cases}$$

$$F(s)=\mathcal{L}[f(t)]=\int_0^2 5te^{-st}dt+\int_2^6\left(\tfrac{-5}{2}t+15\right)e^{-st}dt$$

$$=5\int_0^2 te^{-st}dt+\tfrac{-5}{2}\int_2^6 te^{-st}dt+15\int_2^6 e^{-st}dt$$

Make use of the following integral identities:

$$\int_a^b te^{-st}dt=\frac{1}{s^2}e^{-st}(-st-1)\Big|_a^b=\frac{-1}{s^2}e^{-bs}(bs+1)+\frac{1}{s^2}e^{-as}(as+1)$$

$$\int_a^b te^{-st}dt=\left(\frac{a}{s}+\frac{1}{s^2}\right)e^{-as}-\left(\frac{b}{s}+\frac{1}{s^2}\right)e^{-bs}$$

$$\int_a^b e^{-st}dt=\frac{-1}{s}\left[e^{-bs}-e^{-as}\right]=\frac{1}{s}e^{-as}-\frac{1}{s}e^{-bs}$$

Using these identities for each integral as part of F(s):

$$5\int_0^2 te^{-st}dt=5\left[\left(\frac{0}{s}+\frac{1}{s^2}\right)e^{-0s}-\left(\frac{2}{s}+\frac{1}{s^2}\right)e^{-2s}\right]=\frac{5}{s^2}-\frac{10}{s}e^{-2s}-\frac{5}{s^2}e^{-2s}$$

$$\frac{-5}{2}\int_2^6 te^{-st}dt=\frac{-5}{2}\left[\left(\frac{2}{s}+\frac{1}{s^2}\right)e^{-2s}-\left(\frac{6}{s}+\frac{1}{s^2}\right)e^{-6s}\right]=\left(\frac{-5}{s}+\frac{-2.5}{s^2}\right)e^{-2s}+\left(\frac{15}{s}+\frac{2.5}{s^2}\right)e^{-6s}$$

$$15\int_2^6 e^{-st}dt=\frac{-15}{s}\left[e^{-6s}-e^{-2s}\right]=\frac{15}{s}e^{-2s}-\frac{15}{s}e^{-6s}$$

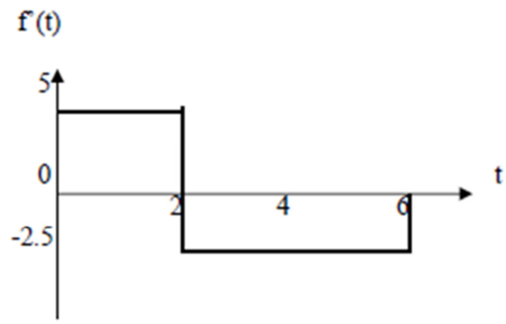
Combining the results of each integral:

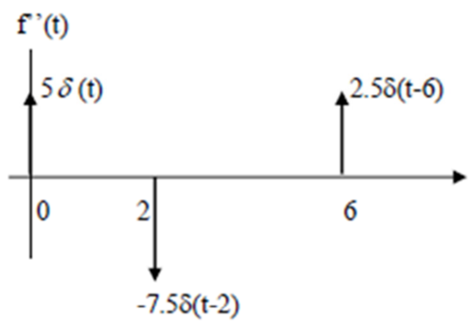
$$F(s)=\frac{5}{s^2}-\cancel{\frac{10}{s}}e^{-2s}-\frac{5}{s^2}e^{-2s}+\left(\cancel{\frac{-5}{s}}+\frac{-2.5}{s^2}\right)e^{-2s}+\left(\cancel{\frac{15}{s}}+\frac{2.5}{s^2}\right)e^{-6s}+\cancel{\frac{15}{s}}e^{-2s}-\cancel{\frac{15}{s}}e^{-6s}$$

$$F(s)=\frac{5}{s^2}-\frac{5}{s^2}e^{-2s}-\frac{2.5}{s^2}e^{-2s}+\frac{2.5}{s^2}e^{-6s}=\boxed{\frac{5}{s^2}-\frac{7.5}{s^2}e^{-2s}+\frac{2.5}{s^2}e^{-6s}}$$

Author's solution making use of time differentiation property:

Taking the derivative of f(t) twice, we obtain the figures below.





$$f'' = 5\delta(t) - 7.5\delta(t-2) + 2.5\delta(t-6)$$

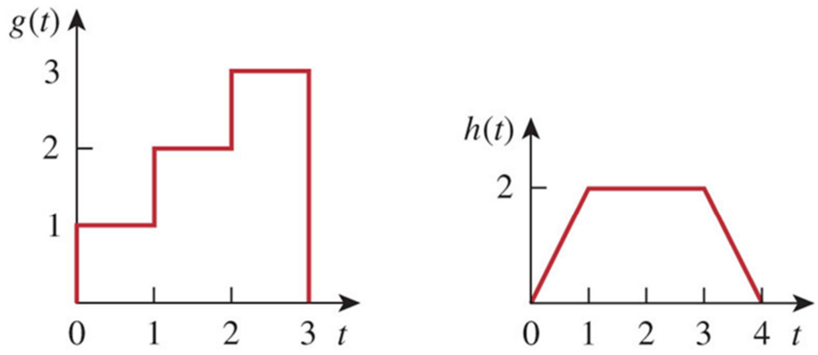
Taking the Laplace transform of each term,

$$s^2F(s) = 5 - 7.5e^{-2s} + 2.5e^{-6s} \text{ or } \mathbf{F(s)} = \underline{\frac{5}{s} - 7.5\frac{e^{-2s}}{s^2} + 2.5\frac{e^{-6s}}{s^2}}$$

Please note that we can obtain the same answer by representing the function as,

$$f(t) = 5tu(t) - 7.5u(t-2) + 2.5u(t-6).$$

3. (Prob. 15.18 from Text) Find the Laplace transform of the signals in the figures a) and b) below:



(a) (b)

(a) $$g(t) = u(t) + u(t-1) + u(t-2) - 3u(t-3)$$

Use Laplace Transform Pairs table and Time Shift property

$$G(s) = \frac{1}{s} + \frac{1}{s}e^{-s} + \frac{1}{s}e^{-2s} - \frac{3}{s}e^{-3s}$$

(b) $$h(t) = \begin{cases} 2t & 0 \le t < 1 \\ 2 & 1 \le t < 3 \\ 8-2t & 3 \le t < 4 \end{cases}$$

$$H(s) = \mathcal{L}[h(t)] = 2\int_0^1 te^{-st}dt + 2\int_1^3 e^{-st}dt + 8\int_3^4 e^{-st}dt - 2\int_3^4 te^{-st}dt$$

Make use of the following integral identities:

$$\int_a^b te^{-st}dt = \left(\frac{a}{s} + \frac{1}{s^2}\right)e^{-as} - \left(\frac{b}{s} + \frac{1}{s^2}\right)e^{-bs}$$

$$\int_a^b e^{-st}dt = \frac{-1}{s}\left[e^{-bs} - e^{-as}\right] = \frac{1}{s}e^{-as} - \frac{1}{s}e^{-bs}$$

(b) $$H(s)=\mathcal{L}\left[h(t)\right]=2\int_0^1 te^{-st}dt+2\int_1^3 e^{-st}dt+8\int_3^4 e^{-st}dt-2\int_3^4 te^{-st}dt$$

$$2\int_0^1 te^{-st}dt=2\left[\left(\frac{0}{s}+\frac{1}{s^2}\right)e^{-0s}-\left(\frac{1}{s}+\frac{1}{s^2}\right)e^{-1s}\right]=\frac{2}{s^2}-\frac{2}{s}e^{-s}-\frac{2}{s^2}e^{-s}$$

$$2\int_1^3 e^{-st}dt=\frac{2}{s}e^{-s}-\frac{2}{s}e^{-3s}$$

$$8\int_3^4 e^{-st}dt=\frac{8}{s}e^{-3s}-\frac{8}{s}e^{-4s}$$

$$-2\int_3^4 te^{-st}dt=-2\left[\left(\frac{3}{s}+\frac{1}{s^2}\right)e^{-3s}-\left(\frac{4}{s}+\frac{1}{s^2}\right)e^{-4s}\right]=\frac{-6}{s}e^{-3s}+\frac{-2}{s^2}e^{-3s}+\frac{8}{s}e^{-4s}+\frac{2}{s^2}e^{-4s}$$

$$H(s)=\frac{2}{s^2}-\cancel{\frac{2}{s}e^{-s}}-\frac{2}{s^2}e^{-s}+\cancel{\frac{2}{s}e^{-s}}-\cancel{\frac{2}{s}e^{-3s}}+\cancel{\frac{8}{s}e^{-3s}}-\cancel{\frac{8}{s}e^{-4s}}+\cancel{\frac{-6}{s}e^{-3s}}+\frac{-2}{s^2}e^{-3s}+\cancel{\frac{8}{s}e^{-4s}}+\frac{2}{s^2}e^{-4s}$$

$$\boxed{H(s)=\frac{2}{s^2}-\frac{2}{s^2}e^{-s}+\frac{-2}{s^2}e^{-3s}+\frac{2}{s^2}e^{-4s}}$$

4. (Prob. 15.25 from Text) For the given transfer function F(s):

$$F(s) = \frac{5(s+1)}{(s+2)(s+3)}$$

a. Use the initial and final value theorems to find $f(0)$ and $f(\infty)$
b. Verify your answer in part (a) by finding $f(t)$, using partial fractions

(a) $f(0) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{5s(s+1)}{(s+2)(s+3)} = \lim_{s\to\infty} \frac{5(1+1/s)}{(1+2/s)(1+3/s)} = \underline{5}$

$f(\infty) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{5s(s+1)}{(s+2)(s+3)} = \underline{0}$

(b) $F(s) = \frac{5(s+1)}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$

$$A = \frac{5(-1)}{1} = -5, \qquad B = \frac{5(-2)}{-1} = 10$$

$$F(s) = \frac{-5}{s+2} + \frac{10}{s+3} \longrightarrow f(t) = -5e^{-2t} + 10e^{-3t}$$

f(0) = -5 + 10 = **5**
f(∞)= -0 + 0 = **0**

5. (Prob. 15.27 from Text) Determine the inverse Laplace transform of the following functions:

(a) $F(s) = \frac{1}{s} + \frac{2}{s+1}$

(b) $G(s) = \frac{3s+1}{s+4}$

(c) $H(s) = \frac{4}{(s+1)(s+3)}$

(d) $J(s) = \frac{12}{(s+2)^2(s+4)}$

(a) $\mathbf{f(t) = u(t) + 2e^{-t}u(t)}$

(b) $G(s) = \frac{3(s+4)-11}{s+4} = 3 - \frac{11}{s+4}$

$\mathbf{g(t) = 3\delta(t) - 11e^{-4t}u(t)}$

(c) $H(s) = \frac{4}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}$

$A = 2, \qquad B = -2$

$H(s) = \frac{2}{s+1} - \frac{2}{s+3}$

$\mathbf{h(t) = [2e^{-t} - 2e^{-3t}]u(t)}$

(d) $J(s) = \frac{12}{(s+2)^2(s+4)} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s+4}$

$B = \frac{12}{2} = 6, \quad C = \frac{12}{(-2)^2} = 3$

$12 = A(s+2)(s+4) + B(s+4) + C(s+2)^2$

Equating coefficients :

$s^2: \quad 0 = A + C \longrightarrow A = -C = -3$

$s^1: \quad 0 = 6A + B + 4C = 2A + B \longrightarrow B = -2A = 6$

$s^0: \quad 12 = 8A + 4B + 4C = -24 + 24 + 12 = 12$

$J(s) = \frac{-3}{s+2} + \frac{6}{(s+2)^2} + \frac{3}{s+4}$

$\mathbf{j(t) = [3e^{-4t} - 3e^{-2t} + 6te^{-2t}]u(t)}$

6. (Prob. 15.37 from Text) Determine the inverse Laplace transform of the following functions:

(a) $H(s) = \dfrac{s+4}{s(s+2)}$

(b) $G(s) = \dfrac{s^2+4s+5}{(s+3)(s^2+2s+2)}$

(c) $F(s) = \dfrac{e^{-4s}}{s+2}$

(d) $D(s) = \dfrac{10s}{(s^2+1)(s^2+4)}$

(a) $$H(s) = \frac{s+4}{s(s+2)} = \frac{k_0}{s} + \frac{k_1}{(s+2)}$$

$$k_0 = sH(s)\big|_{s=0} = \frac{0+4}{(0+2)} = 2$$

$$k_1 = (s+2)H(s)\big|_{s=-2} = \frac{-2+4}{-2} = -1$$

$$H(s) = \frac{s+4}{s(s+2)} = \frac{2}{s} + \frac{-1}{(s+2)}$$

$$h(t) = \mathcal{L}^{-1}[H(s)] = 2u(t) - e^{-2t}u(t)$$

(b) $$G(s) = \frac{A}{s+3} + \frac{Bs+C}{s^2+2s+2}$$

$$s^2+4s+5 = (Bs+C)(s+3) + A(s^2+2s+2)$$

Equating coefficients,

s^2:	$1 = B + A$	(1)
s:	$4 = 3B + C + 2A$	(2)
Constant:	$5 = 3C + 2A$	(3)

Solving (1) to (3) gives

$$A = \frac{2}{5}, \quad B = \frac{3}{5}, \quad C = \frac{7}{5}$$

$$G(s) = \frac{0.4}{s+3} + \frac{0.6s+1.4}{s^2+2s+2} = \frac{0.4}{s+3} + \frac{0.6(s+1)+0.8}{(s+1)^2+1}$$

$$g(t) = \underline{0.4e^{-3t} + 0.6e^{-t}\cos t + 0.8e^{-t}\sin t}\ u(t)$$

(c) $f(t) = \underline{e^{-2(t-4)}u(t-4)}$

(d) $D(s) = \dfrac{10s}{(s^2+1)(s^2+4)} = \dfrac{As+B}{s^2+1} + \dfrac{Cs+D}{s^2+4}$

$$10s = (s^2+4)(As+B) + (s^2+1)(Cs+D)$$

Equating coefficients,

s^3 : $0 = A + C$

s^2 : $0 = B + D$

s: $10 = 4A + C$

constant: $0 = 4B+D$

Solving these leads to

A = -10/3, B = 0, C = -10/3, D = 0

$$D(s) == \frac{10s/3}{s^2+1} - \frac{10s/3}{s^2+4}$$

$$d(t) = \underline{\frac{10}{3}\cos t - \frac{10}{3}\cos 2t\,\mathrm{u(t)}}$$