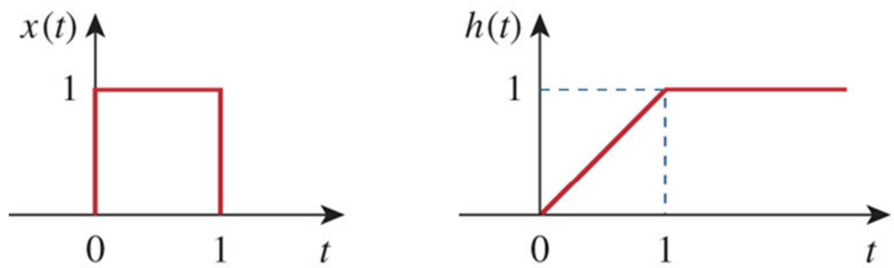
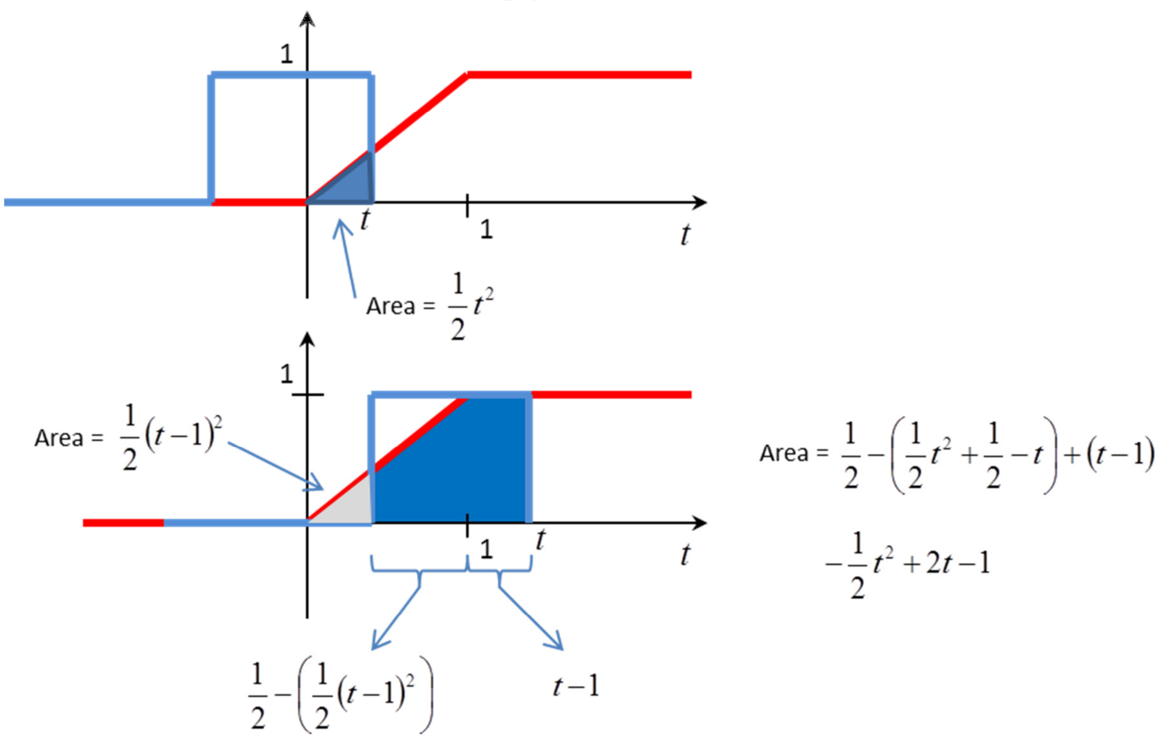
Homework #8 (SOLUTION KEY) Name:

1. (Prob. 15.43a in text) Find $y(t) = x(t) * h(t)$ (convolution of x(t) and h(t)) for $x(t)$ and $h(t)$ in the figure below:
   a. Solve using the graphical method (evaluate the integral to find the area under the curve).
   b. Solve by multiplying in the s-domain (use the Laplace Transform & Inverse Laplace transform):



a) Graphical Solution:

Fold x(t), shift, slide, multiply and find area under curve



$$y(t)=\begin{cases}\frac{1}{2}t^2 & 0\le t<1\\ -\frac{1}{2}t^2+2t-1 & 1\le t<2\\ 1 & 2\le t<\infty\end{cases}$$

b) Laplace Solution:

The signals $x(t)$ and $h(t)$ can be described as follows:

$$x(t) = u(t) - u(t-1)$$

$$h(t) = \underbrace{tu(t)}_{\text{+ ramp at t=0}} - \underbrace{(t-1)u(t-1)}_{\text{- ramp at t=1 (Cancels previous slope)}}$$

Converting to S-domain:

$$X(s) = \frac{1}{s} - \frac{1}{s}e^{-s} = \frac{1}{s}\left(1 - e^{-s}\right)$$

$$H(s) = \frac{1}{s^2} - \frac{1}{s^2}e^{-s} = \frac{1}{s^2}\left(1 - e^{-s}\right)$$

Multiplying together to get the output:

$$Y(s) = X(s)H(s) = \frac{1}{s^3}\left(1 - e^{-s}\right)\left(1 - e^{-s}\right)$$

$$Y(s) = X(s)H(s) = \frac{1}{s^3}\left(1 - 2e^{-s} + e^{-2s}\right)$$

From Laplace Table $\mathcal{L}^{-1}\left[\frac{1}{s^3}\right] = \frac{1}{2}t^2u(t)$

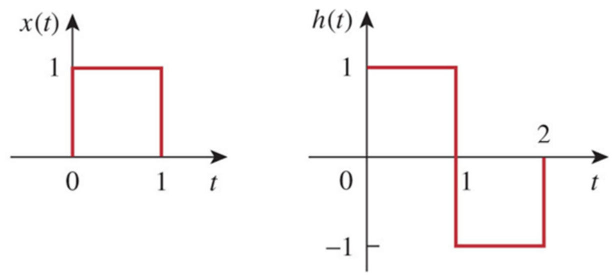
Time Delay Property $\mathcal{L}^{-1}\left[-\frac{1}{s^3}2e^{-s}\right] = \frac{1}{2}(-2)(t-1)^2u(t-1)$

$$y(t) = \frac{1}{2}t^2u(t) - (t-1)^2u(t-1) + \frac{1}{2}(t-2)^2u(t-2)$$

$$y(t) = \frac{1}{2}t^2u(t) - \left(t^2 - 2t + 1\right)u(t-1) + \frac{1}{2}\left(t^2 - 4t + 4\right)u(t-2)$$

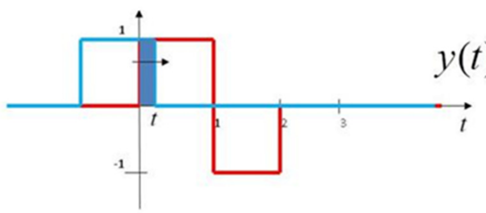
$$y(t) = \begin{cases} \frac{1}{2}t^2 & 0 \le t < 1 \\ -\frac{1}{2}t^2 + 2t - 1 & 1 \le t < 2 \\ 1 & 2 \le t < \infty \end{cases}$$

2. (Prob. 15.44a from Text) Find $y(t) = x(t) * h(t)$ (convolution of x(t) and h(t)) for $x(t)$ and $h(t)$ in the figure below:
   a. Solve using the graphical method (evaluate the integral to find the area under the curve).
   b. Solve by multiplying in the s-domain (use the Laplace Transform & Inverse Laplace transform):

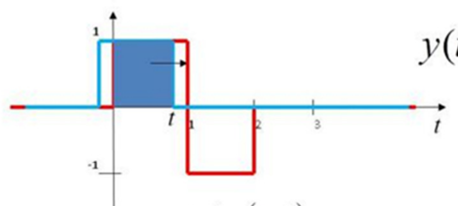


a.) Graphical Solution

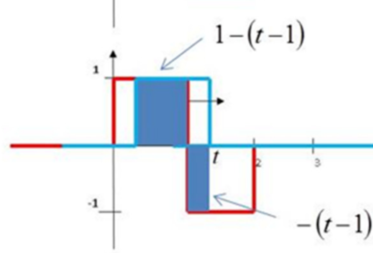
Fold x(t), shift, slide, multiply and find area under curve



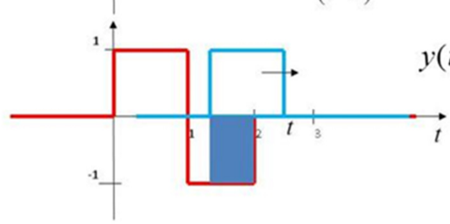
$$y(t) = t \quad 0 \le t < 1$$



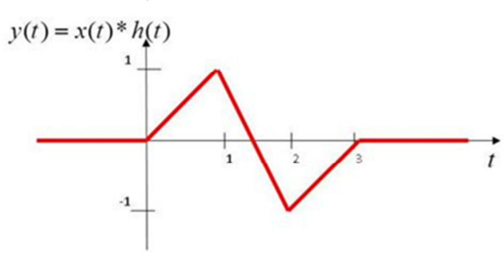
$$y(t) = t \quad 0 \le t < 1$$



$$y(t) = \{1-(t-1)\} - \{t-1\} = 3-2t \quad 1 \le t < 2$$



$$y(t) = -1 \cdot (2-(t-1)) = t-3 \quad 2 \le t < 3$$



$$y(t) = \begin{cases} t & 0 \le t < 1 \\ 3-2t & 1 \le t < 2 \\ t-3 & 2 \le t < 3 \\ 0 & 3 \le t \end{cases}$$

b.) Laplace Solution:

The signals $x(t)$ and $h(t)$ can be described as follows:

$$x(t) = u(t) - u(t-1)$$

$$h(t) = u(t) - 2(t-1)u(t-1) + u(t-2)$$

Converting to S-domain:

$$X(s) = \frac{1}{s} - \frac{1}{s}e^{-s} = \frac{1}{s}\left(1 - e^{-s}\right)$$

$$H(s) = \frac{1}{s} - \frac{2}{s}e^{-s} + \frac{1}{s}e^{-2s} = \frac{1}{s}\left(1 - 2e^{-s} + e^{-2s}\right)$$

Multiplying together to get the output:

$$Y(s) = X(s)H(s) = \frac{1}{s^2}\left(1 - e^{-s}\right)\left(1 - 2e^{-s} + e^{-2s}\right)$$

$$Y(s) = \frac{1}{s^2}\left(1 - 2e^{-s} + e^{-2s} - e^{-s} + 2e^{-2s} - e^{-3s}\right)$$

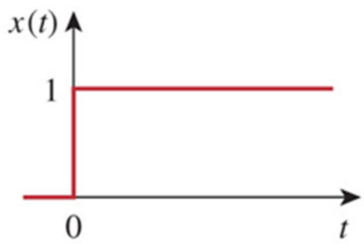
$$Y(s) = \frac{1}{s^2}\left(1 - 3e^{-s} + 3e^{-2s} - e^{-3s}\right)$$

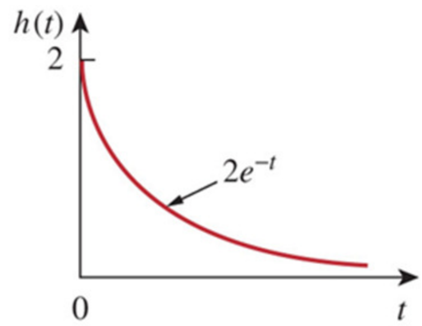
Inverse Laplace Transform:

$$y(t) = tu(t) - 3(t-1)u(t-1) + 3(t-2)u(t-2) - (t-3)u(t-3)$$

$$y(t) = \begin{cases} t & 0 \le t < 1 \\ 3 - 2t & 1 \le t < 2 \\ t - 3 & 2 \le t < 3 \\ 0 & 3 \le t \end{cases}$$

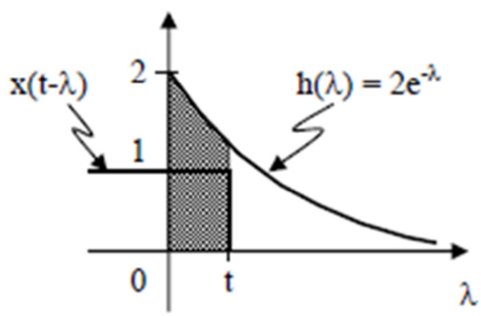
3. (Prob. 15.43b from Text) Find $y(t) = x(t) * h(t)$ for the paired $x(t)$ and $h(t)$ below using two methods:
   a. Solve using the graphical method (evaluate the integral to find the area under the curve).
   b. Solve by multiplying in the s-domain (use the Laplace Transform & Inverse Laplace transform):





(a) For $t > 0$, the two functions overlap as shown in Fig. (c).

$$y(t) = x(t) * h(t) = \int_0^t (1)\,2e^{-\lambda}\,d\lambda = -2e^{-\lambda}\Big|_0^t$$



(c)

Therefore

$$\boxed{y(t) = 2(1 - e^{-t}), \quad t > 0}$$

(b)

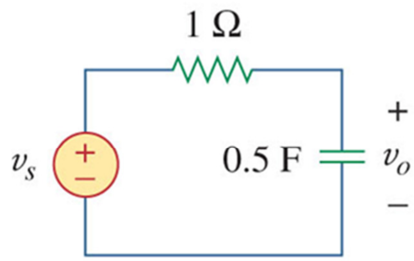
$$x(t) = u(t) \qquad h(t) = 2e^{-t}u(t)$$

$$X(s) = \frac{1}{s} \qquad H(s) = \frac{2}{s+1}$$

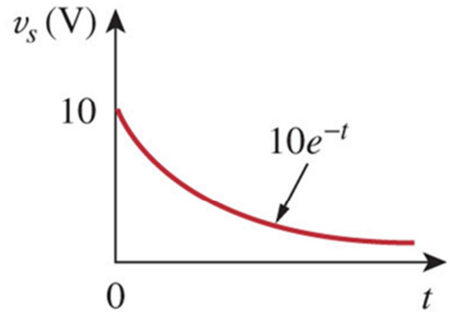
$$Y(s) = X(s)H(x) = \frac{2}{s(s+1)} = \frac{k_0}{s} + \frac{k_1}{(s+1)} = \frac{2}{s} - \frac{2}{(s+1)}$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{2}{s} - \frac{2}{(s+1)}\right] = \boxed{2\left(1 - e^{-t}\right) \quad t > 0}$$

4. (Practice problem 15.14 from Text) Use convolution to find $v_o(t)$ in the circuit below in figure (a) when the excitation is the signal shown in figure (b). Verify your answer by performing the equivalent operation in the s-domain.



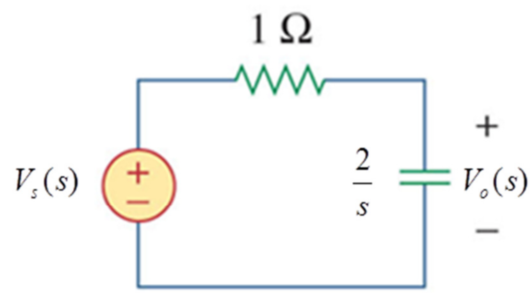
(a)



(b)

First convert circuit to s-domain and find the transfer function (Voltage Divider):

$$V_o(s) = \frac{2/s}{1+2/s}V_s(s) = \frac{2}{s+2}V_s(s) \quad \Rightarrow \quad H(s) = \frac{2}{s+2}$$



Convert the transfer function to the time domain

$$h(t) = \mathcal{L}^{-1}\left[\frac{2}{s+2}\right] = 2e^{-2t}u(t)$$

Perform the convolution using the integral equation

$$v_o(t) = v_s(t) * h(t) = \int_0^t v_s(t-\lambda)h(\lambda)d\lambda = \int_0^t \left(10e^{-(t-\lambda)}\right)\left(2e^{-2\lambda}\right)d\lambda = 20\int_0^t e^{-t}e^{-\lambda}d\lambda = 20e^{-t}\left(-e^{-\lambda}\right)\Big|_0^t$$

$$v_o(t) = 20e^{-t}\left(1-e^{-t}\right) = \boxed{20\left(e^{-t}-e^{-2t}\right) \quad \mathbf{t > 0}}$$

Verify by performing in the s-domain:

$$H(s) = \frac{2}{s+2} \qquad V_s(s) = \mathcal{L}\left[10e^{-t}\right] = \frac{10}{s+1}$$

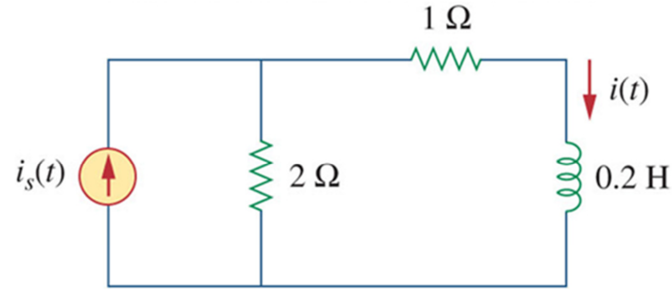
$$V_o(s) = V_s(s)H(s) = \left(\frac{10}{s+1}\right)\left(\frac{2}{s+2}\right) = \frac{20}{(s+1)(s+2)} = \frac{k_0}{(s+1)} + \frac{k_1}{(s+2)} = \frac{20}{(s+1)} - \frac{20}{(s+2)}$$

$$v_o(t) = 20\,\mathcal{L}^{-1}\left[\frac{1}{(s+1)} - \frac{1}{(s+2)}\right] = \boxed{20\left(e^{-t}-e^{-2t}\right) \quad \mathbf{t > 0}}$$

5. (Prob. 16.14 from Text) Find $i(t)$ for $t > 0$ for the circuit shown below:

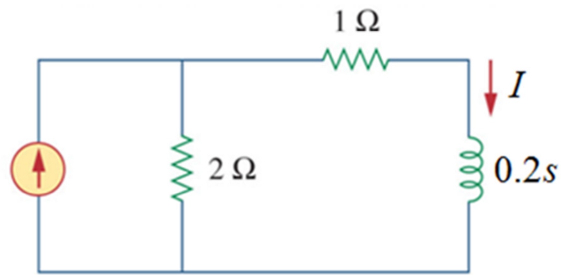
Assume:

$$i_s(t) = [4u(t) + 2\delta(t)] \ \text{mA}$$



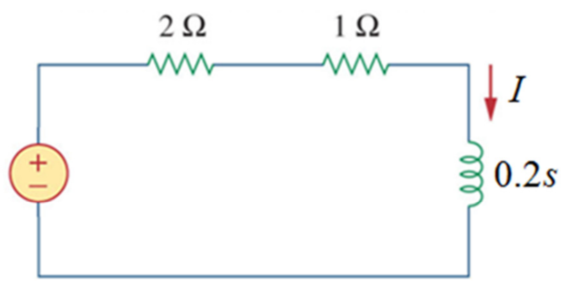
First convert circuit to s-domain:

$$I_s(s) = \mathcal{L}[4u(t) + 2\delta(t)] = \left(\frac{4}{s} + 2\right)$$



Use source transformation to convert current source to a voltage source:

$$\left(\frac{4}{s} + 2\right) \times 2$$



Solve circuit for I:

$$I = \frac{\left(\frac{8}{s} + 4\right)}{2 + 1 + 0.2s} = \frac{8 + 4s}{s(3 + 0.2s)} = \frac{20s + 40}{s(s + 15)} = \frac{k_0}{s} + \frac{k_1}{(s + 15)}$$

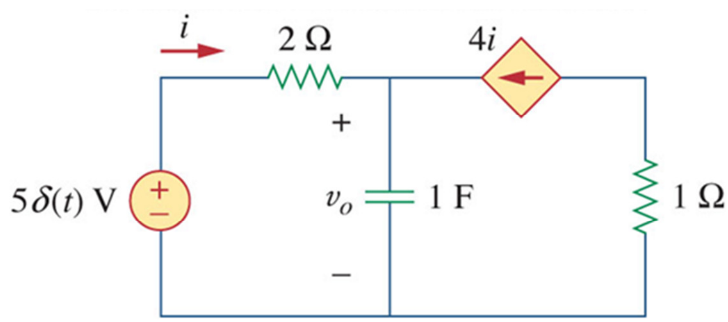
$$I = \left(\frac{8}{3}\right)\frac{1}{s} + \left(\frac{52}{3}\right)\frac{1}{(s + 15)}$$

$$k_0 = \frac{20(0) + 40}{(0 + 15)} = \frac{40}{15} = \frac{8}{3}$$

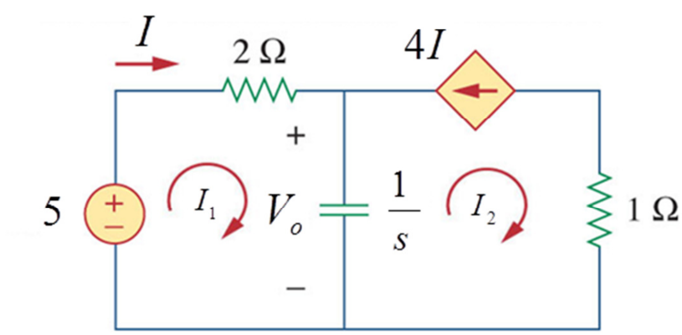
$$k_1 = \frac{20(-15) + 40}{-15} = \frac{-260}{-15} = \frac{52}{3}$$

$$i(t) = L^{-1}\left[\left(\frac{8}{3}\right)\frac{1}{s} + \left(\frac{52}{3}\right)\frac{1}{(s + 15)}\right] = \boxed{\left(\frac{8}{3}\right)u(t) + \left(\frac{52}{3}\right)e^{-15t}u(t) \ \text{mA} \quad t > 0}$$

6. (Prob. 16.16 from Text) The capacitor in the circuit below is initially uncharged. Find $v_o(t)$ for $t > 0$:



First convert circuit to s-domain:



Mesh for I_1:

$$I_1(2)+\frac{1}{s}(I_1-I_2)-5=0$$

However $I_2 = -4\ I_1$

$$I_1(2)+\frac{5}{s}(I_1)=5$$

$$I_1=\left(\frac{5}{2+\frac{5}{s}}\right)=\frac{5s}{2s+5}=\frac{2.5s}{s+2.5}$$

Solve for Voltage across capacitor

$$V_0=\frac{1}{s}(I_1-I_2)=\frac{5}{s}I_1=\frac{5}{s}\left(\frac{2.5s}{s+2.5}\right)=\frac{12.5}{s+2.5}$$

$$\boxed{v_o(t)=L^{-1}\left[\frac{12.5}{s+2.5}\right]=12.5e^{-2.5t} \quad \mathbf{t>0}}$$