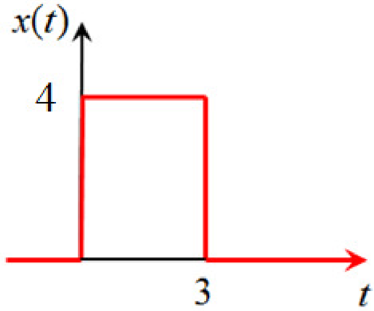
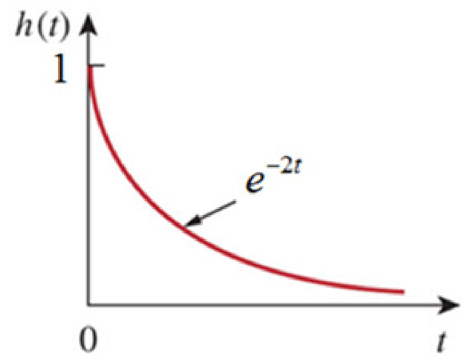
IUPUI ECE 202 Spring 2015:

Exam #3 (SOLUTION KEY) Name:

1. (25 points) Find $\boldsymbol{y}(t) = \boldsymbol{x}(t) * \boldsymbol{h}(t)$ for the paired $\boldsymbol{x}(t)$ and $\boldsymbol{h}(t)$ below using two methods:
   a) Perform the **convolution** (either graphical method or by direct use of the convolution integral).
   b) Solve by converting to the **s-domain** then back to time domain to find $\boldsymbol{y}(t)$. (*The Laplace Transform Pairs tables and Laplace Transform Properties tables are provided at the back of this exam*).





a) for $0 \le t < 3$

$$y(t) = \int_0^t (4)e^{-2\lambda} d\lambda = \left(-\frac{4}{2}\right)e^{-2\lambda}\Big|_0^t = (-2)\left(e^{-2t}-1\right) = 2\left(1-e^{-2t}\right)$$

for $3 \le t < \infty$

$$y(t) = \int_{t-3}^t (4)e^{-2\lambda} d\lambda = \left(-\frac{4}{2}\right)\left(e^{-2t}-e^{-2(t-3)}\right) = 2e^{-2t}\left(e^{6}-1\right)$$

$$\boxed{y(t) = \begin{cases} 0 & t \le 0 \\ 2\left(1-e^{-2t}\right) & 0 < t \le 3 \\ 2e^{-2t}\left(e^{6}-1\right) & 3 < t \end{cases}}$$

b) $x(t) = 4\left(u(t)-u(t-3)\right) \qquad X(s) = \dfrac{4}{s} - \dfrac{4}{s}e^{-3s} \qquad H(s) = \dfrac{1}{s+2}$

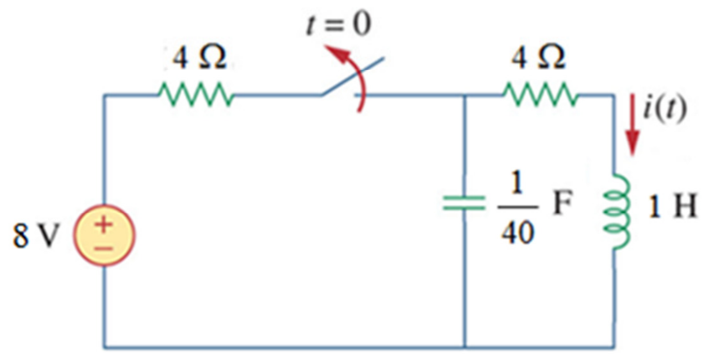
$$Y(s) = X(s)H(s) = \frac{4}{s(s+2)} - \frac{4}{s(s+2)}e^{-3s} = F(s) - F(s)e^{-3s}$$

$$F(s) = \frac{4}{s(s+2)} = \frac{k_0}{s} + \frac{k_1}{s+2} = \frac{2}{s} - \frac{2}{s+2} \quad \Rightarrow \quad f(t) = 2\left(1-e^{-2t}\right)u(t)$$

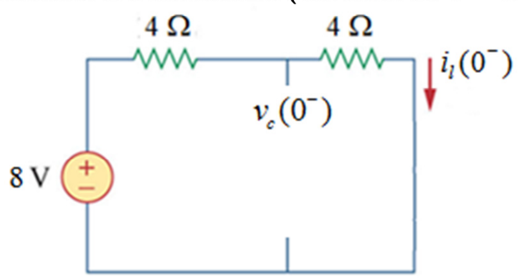
$$y(t) = \mathcal{L}^{-1}\left[F(s)-F(s)e^{-3s}\right] = f(t)u(t) - f(t-3)u(t-3) \qquad \text{Time Delay Property}$$

$$\boxed{y(t) = 2\left(1-e^{-2t}\right)u(t) - 2\left(1-e^{-2(t-3)}\right)u(t-3) = 2\left(1-e^{-2t}\right)u(t) - 2\left(1-e^{-2t}e^{6}\right)u(t-3)}$$

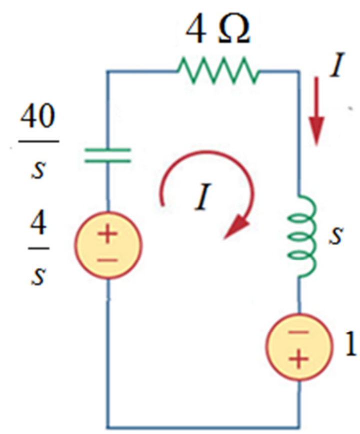
2. (30 points) Assume the source is connected to the circuit shown below for a very long time prior to the switch being opened at $t = 0$. For the circuit shown:
   a) Find the initial voltage $v_c(0^-)$ across the capacitor and initial current $i(0^-)$ through the inductor prior to the opening of the switch.
   b) Find $I(s)$ at $t > 0$ (after the switch has opened).
   c) Find $i(t)$ at $t > 0$.



Initial Conditions (circuit at t < 0)



a)
$$\boxed{\begin{aligned} v_c(0^-) &= \frac{4}{4+4}(8) = 4 \\ i_l(0^-) &= \frac{(8)}{4+4} = 1 \end{aligned}}$$



Find circuit at t > 0, translate to s-domain

b)
$$I\left(\frac{40}{s} + 4 + s\right) = 1 + \frac{4}{s}$$

$$I\left(s^2 + 4s + 40\right) = s + 4$$

$$\boxed{I = \frac{s+4}{\left(s^2+4s+40\right)} = \frac{s+4}{(s+2)^2+6^2}}$$

Use "completing the square" method and numerator "trick" we discussed in lecture notes

$$s^2 + 4s + 40 = s^2 + 2as + \left(a^2 + \omega^2\right)$$

$$a = 2$$

$$40 = 2^2 + \omega^2 \quad \Rightarrow \quad \omega = 6$$

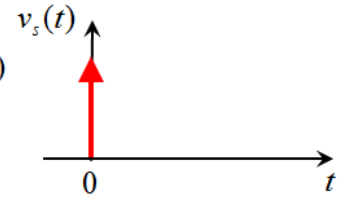
c)
$$I = \frac{(s+2)+2}{(s+2)^2+6^2} = \frac{(s+2)}{(s+2)^2+6^2} + \left(\frac{2}{6}\right)\frac{6}{(s+2)^2+6^2}$$

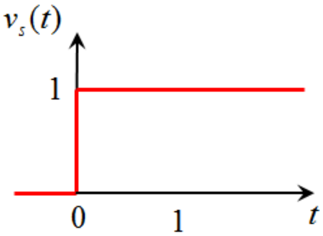
$$\boxed{i(t) = e^{-2t}\cos 6t + \frac{1}{3}e^{-2t}\sin 6t \text{ A} \quad \text{for } t > 0}$$

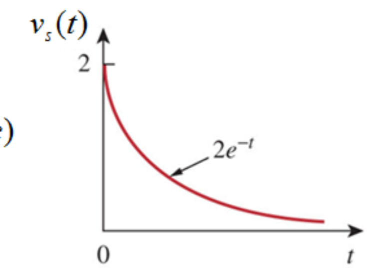
$f(t)$	$F(s)$
$e^{-at}\sin \omega t$	$\dfrac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos \omega t$	$\dfrac{s+a}{(s+a)^2+\omega^2}$

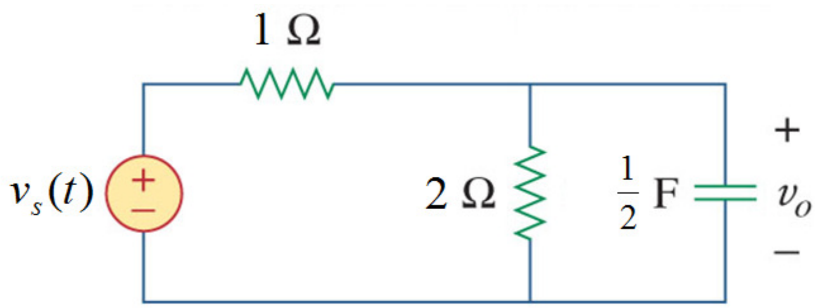
3. (25 points) For the circuit shown below find the transfer function **H**(s) = **V_o**(s) / **V_s**(s) and use to find the following outputs (assume the circuit has no initial energy):

a) Find the output $v_o(t)$ given the input $v_s(t)$ is an implulse $\delta(t)$
b) Find the output $v_o(t)$ given the input $v_s(t)$ is a unit step $u(t)$
c) Find the output $v_o(t)$ given the input $v_s(t)$ is $2e^{-t}\,u(t)$

a)


b)


c)




Find the impedance of the resistor and capacitor in parallel Z_L

$$Z_L = \frac{2\left(\frac{2}{s}\right)}{2+\left(\frac{2}{s}\right)} = \frac{4}{2s+2} = \frac{2}{(s+1)}$$

Transfer function H(s) = $V_o(s)$ / $V_s(s)$ is just a voltage divider equation:

$$H(s) = \frac{V_s}{V_o} = \frac{Z_L}{1+Z_L} = \frac{2/(s+1)}{1+2/(s+1)} = \frac{2}{(s+1)+2} = \boxed{\frac{2}{(s+3)}}$$

a) $$V_o(s) = V_s(s)H(s) = \mathcal{L}[\delta(t)]\frac{2}{(s+3)} = (1)\frac{2}{(s+3)} = \frac{2}{(s+3)}$$

$$\boxed{v_o(t) = \mathcal{L}^{-1}\left[\frac{2}{(s+3)}\right] = 2e^{-3t}u(t)}$$

b) $$V_o(s) = \mathcal{L}[u(t)]\frac{2}{(s+3)} = \left(\frac{1}{s}\right)\frac{2}{(s+3)} = \frac{k_0}{s} + \frac{k_1}{(s+3)} = \frac{2/3}{s} - \frac{2/3}{(s+3)}$$

$$\boxed{v_o(t) = \mathcal{L}^{-1}\left[\frac{2/3}{s} - \frac{2/3}{(s+3)}\right] = \frac{2}{3}\left(1-e^{-3t}\right)u(t)}$$

c) $$V_o(s) = \mathcal{L}[2e^{-t}]\left(\frac{2}{s+3}\right) = \left(\frac{2}{s+1}\right)\left(\frac{2}{s+3}\right) = \frac{k_0}{(s+1)} + \frac{k_1}{(s+3)} = \frac{2}{(s+1)} - \frac{2}{(s+3)}$$

$$\boxed{v_o(t) = \mathcal{L}^{-1}\left[\frac{2}{(s+1)} - \frac{2}{(s+3)}\right] = 2\left(e^{-t} - e^{-3t}\right)u(t)}$$

4. (20 points) Given a function $\boldsymbol{Y}(s)$:

$$Y(s) = \frac{5(s+1)}{s(s+2)(s+3)}$$

a) Use the initial value theorem to find $\boldsymbol{y}(0)$
b) Use the final value theorem to find $\boldsymbol{y}(\infty)$
c) Find the inverse Laplace Transform $\boldsymbol{y}(t)$ and **verify** the answers from parts **a** & **b**.

$$a)\quad y(0) = \lim_{s\to\infty} sY(s) = \lim_{s\to\infty} \cancel{s}\frac{5(s+1)}{\cancel{s}(s+2)(s+3)} = \lim_{s\to\infty}\frac{5(s+1)}{(s+2)(s+3)} = \boxed{0}$$

$$b)\quad y(\infty) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \cancel{s}\frac{5(s+1)}{\cancel{s}(s+2)(s+3)} = \frac{5(0+1)}{(0+2)(0+3)} = \boxed{\frac{5}{6}}$$

$$c)\quad \frac{5(s+1)}{s(s+2)(s+3)} = \frac{k_0}{s} + \frac{k_1}{s+2} + \frac{k_2}{s+3}$$

$$k_0 = \frac{5(0+1)}{(0+2)(0+3)} = \frac{5}{6}$$

$$k_1 = \frac{5((-2)+1)}{(-2)((-2)+3)} = \frac{5(-1)}{(-2)(1)} = \frac{5}{2}$$

$$k_3 = \frac{5((-3)+1)}{(-3)((-3)+2)} = \frac{5(-2)}{(-3)(-1)} = \frac{-10}{3}$$

$$Y(s) = \left(\frac{5}{6}\right)\frac{1}{s} + \left(\frac{5}{2}\right)\frac{1}{s+2} + \left(\frac{-10}{3}\right)\frac{1}{s+3}$$

$$\boxed{y(t) = \left(\left(\frac{5}{6}\right) + \left(\frac{5}{2}\right)e^{-2t} + \left(\frac{-10}{3}\right)e^{-3t}\right)u(t) = \left(\frac{5}{6}\right)\left(1 + 3e^{-2t} - 4e^{-3t}\right)u(t)}$$

$$y(0) = \left(\frac{5}{6}\right) + \left(\frac{5}{2}\right)e^{0} + \left(\frac{-10}{3}\right)e^{0} = \frac{5}{6} + \frac{5}{2} + \frac{-10}{3} = \frac{5}{6} + \frac{15}{6} + \frac{-20}{6} = 0$$

$$y(\infty) = \lim_{t\to\infty} y(t) = \left(\frac{5}{6}\right) + \left(\frac{5}{2}\right)\cancelto{0}{e^{-\infty}} + \left(\frac{-10}{3}\right)\cancelto{0}{e^{-\infty}} = \frac{5}{6} + 0 + 0 = \frac{5}{6}$$