Chapter 16

Applications of the Laplace Transform

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- 16.2 Circuit Element Models
- 16.3 Circuit Analysis
- 16.4 Transfer Functions

16.2 Circuit Element Models (1)

Steps in Applying the Laplace Transform:

1. Transform the circuit from the time domain to the s-domain
2. Solve the circuit using nodal analysis, mesh analysis, source transformation, superposition, or any circuit analysis technique with which we are familiar
3. Take the inverse transform of the solution and thus obtain the solution in the time domain.

16.2 Circuit Element Models (2)

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- Remember the time differentiation property from Chapter 15.

$$\boxed{\mathcal{L}\left[f'(t)\right] = sF(s) - f(0^-)}$$

- We can apply this to the definitions of the inductor and capacitor.

Inductor Equation

$$\boxed{v = L\frac{d\,i}{d\,t}}$$

$$\mathcal{L}\{v(t)\} = \mathcal{L}\{L\frac{di(t)}{dt}\}$$

$$\boxed{V(s) = L\left[sI(s) - i(0^-)\right]}$$

$$\boxed{I(s) = \frac{V(s)}{sL} + \frac{i(0^-)}{s}}$$

Capacitor Equation

$$\boxed{i = C\frac{d\,v}{d\,t}}$$

$$\mathcal{L}\{i(t)\} = \mathcal{L}\{C\frac{dv(t)}{dt}\}$$

$$\boxed{I(s) = C\left[sV(s) - v(0^-)\right]}$$

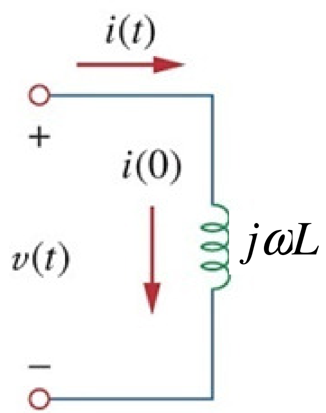
$$\boxed{V(s) = \frac{I(s)}{sC} + \frac{v(0^-)}{s}}$$

16.2 Circuit Element Models (3)

Inductor Model

- We can model the inductor equations with an equivalent circuit as follows:

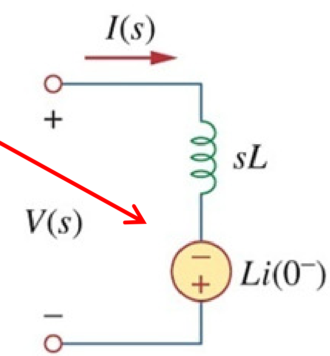
Time Domain



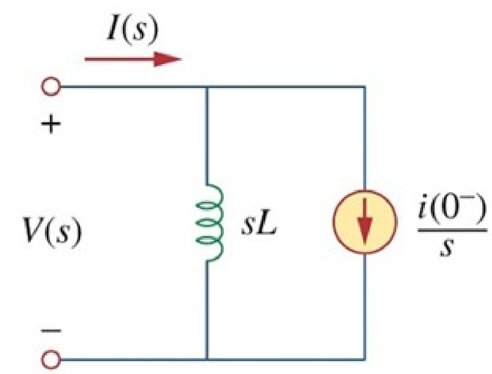
Initial current through inductor

$$V(s) = sLI(s) - Li(0^-)$$

S - Domain



$$I(s) = \frac{V(s)}{sL} + \frac{i(0^-)}{s}$$



16.2 Circuit Element Models (4)

Capacitor Model

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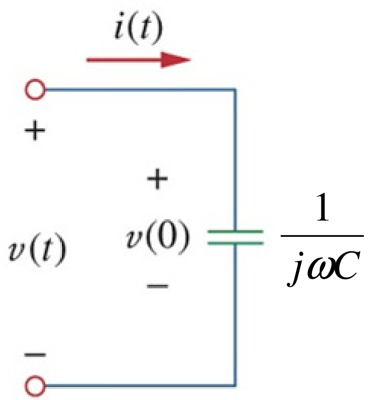
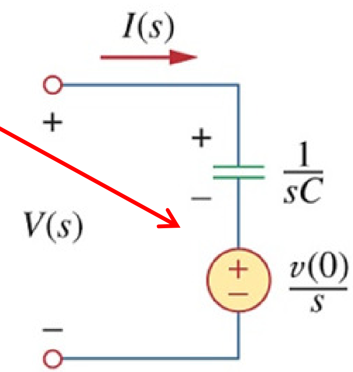
- Likewise for the capacitor we can model an equivalent circuit as follows:

Time Domain **S - Domain**

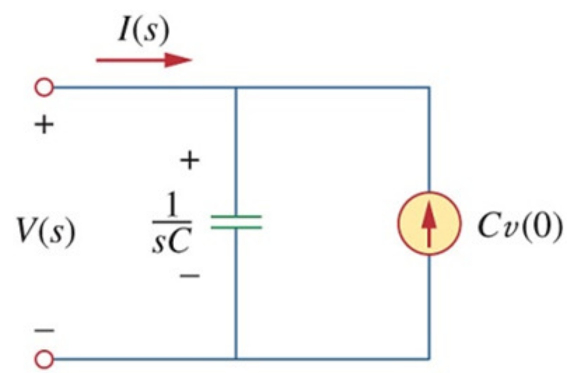


Initial voltage across capacitor

$$V(s)=\frac{I(s)}{sC}+\frac{v(0^-)}{s}$$



$$I(s)=sCV(s)-vC(0^-)$$



16.2 Circuit Element Models (5)

Resistor Model

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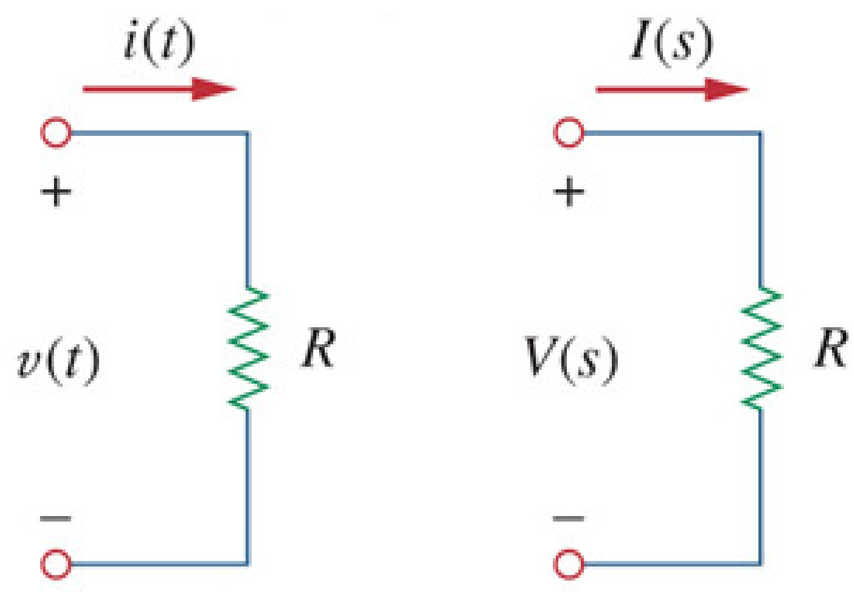
- Laplace transform does not affect the resistor (no time variation).
- Resistor in time domain is same in S-domain.

Time Domain **S - Domain**

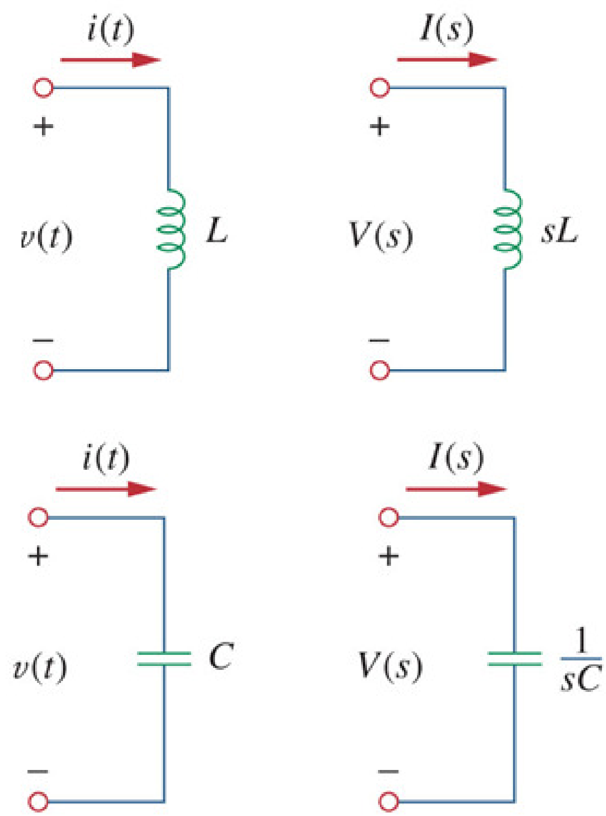


16.2 Circuit Element Models (6)

Zero Initial Condition Model

- If there are zero initial conditions, this results in the simple approach of just replacing **jω** with **s**.

Time Domain **S - Domain**



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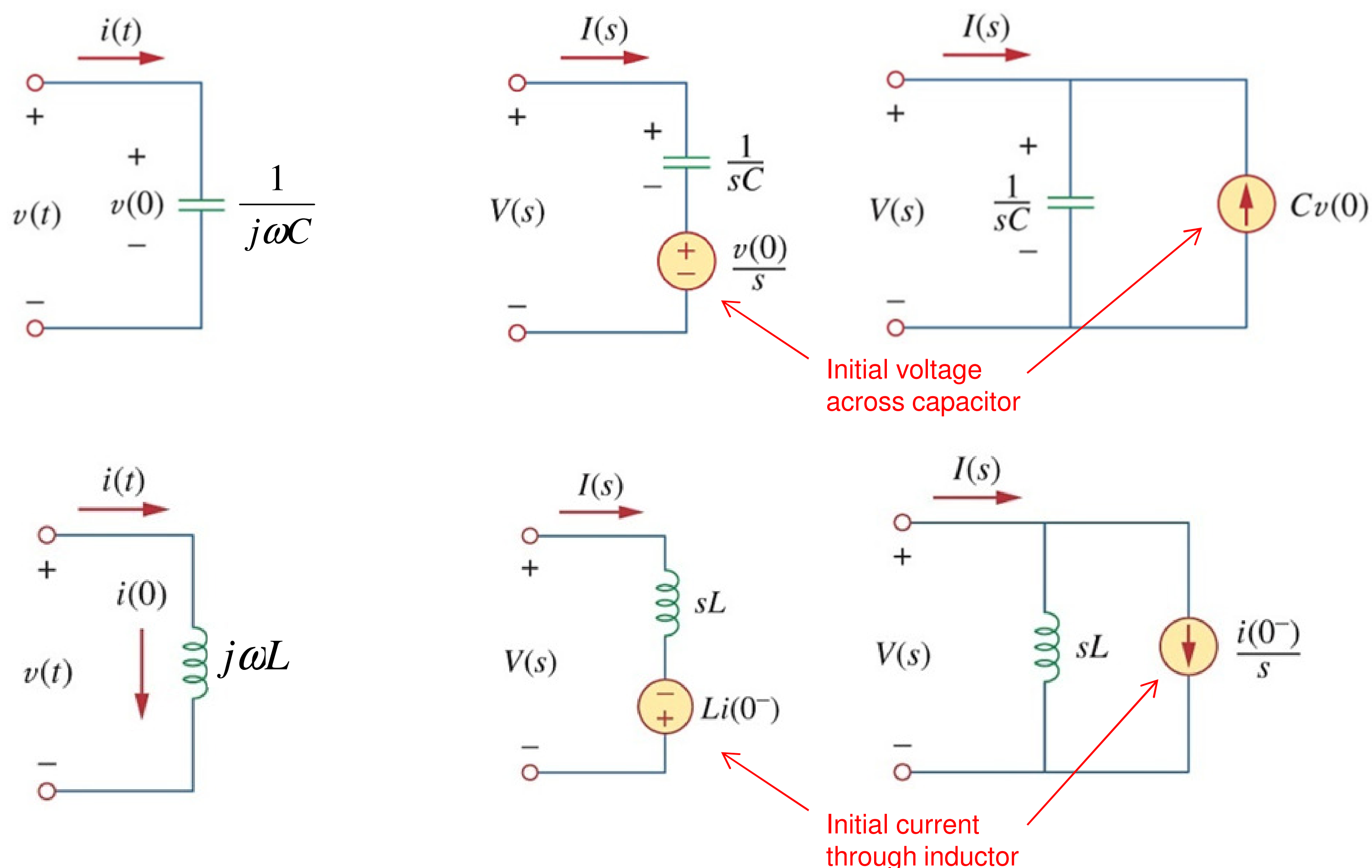
- With initial conditions, the following models are used:

Time Domain **S - Domain**



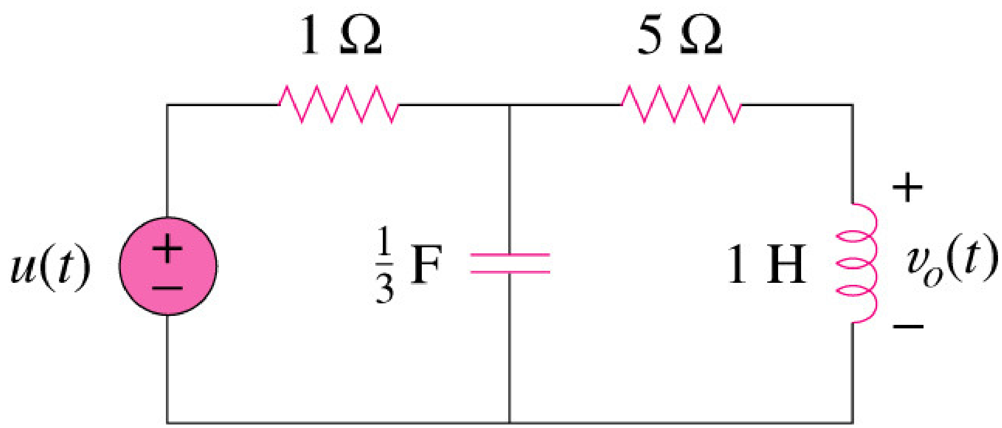
16.2 Circuit Element Models (8)

- The Laplace transform can readily be used to solve first and second order circuits.
- From the previous equations for R, L and C, observe that the initial conditions are part of the transformation which is an advantage of using the Laplace transform in the circuit analysis.
- Another advantage is that a complete response, including transient and steady state, is obtained.
- Also observe the duality of the inductor and capacitor s-domain equations.
- The use of the Laplace transform in the circuit analysis enables the use of various signal sources such as impulse, step, ramp, exponential and sinusoidal.

16.2 Circuit Element Models (9)

Example 16.1

Find $v_0(t)$ in the circuit shown below, assuming zero initial conditions.



16.2 Circuit Element Models (10)

Example 16.1

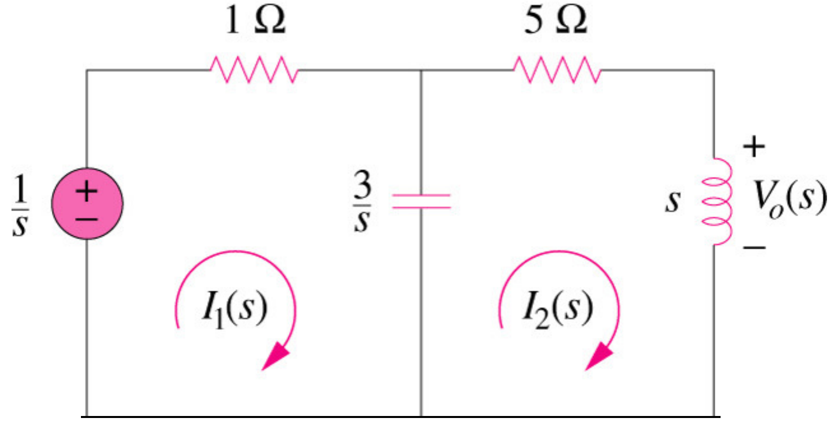
Solution:

Transform the circuit from the time domain to the s-domain, we have

$$u(t) \Rightarrow \frac{1}{s}$$

$$1\,\mathrm{H} \Rightarrow sL = s$$

$$\frac{1}{3}\mathrm{F} \Rightarrow \frac{1}{sC} = \frac{3}{s}$$



Solve by performing Mesh analysis for I_1 and I_2. This will result in:

$$I_2 = \frac{3}{s^3 + 8s^2 + 18s} \qquad V_o = sI_2 = \frac{3}{s^2 + 8s + 18}$$

Not best form for finding Inverse Laplace Transform
We can use the method "Completing the square" to solve (next slide)

16.2 Circuit Element Models (11)

Example 16.1 – "Completing the Square"

We will use the method known as "Completing the Square" to put the denominator into a better from for the inverse Laplace Transform.

This is what we have

$$V_o = \frac{3}{s^2 + 8s + 18}$$

This is the form we want

$$\frac{\omega}{(s+a)^2 + \omega^2} = \mathcal{L}^{-1}\left[e^{-at} \sin \omega t\right]$$

Expand out to find values of a and ω that will work

$$s^2 + 8s + 18 = s^2 + 2as + \left(a^2 + \omega^2\right)$$

$$a = 4$$

$$18 = 4^2 + \omega^2 \quad \Rightarrow \quad \omega = \sqrt{2}$$

Substituting back into the original equation gives:

$$V_o = \frac{3}{(s+4)^2 + \left(\sqrt{2}\right)^2} = \frac{3}{\sqrt{2}} \cdot \frac{\sqrt{2}}{(s+4)^2 + \left(\sqrt{2}\right)^2}$$

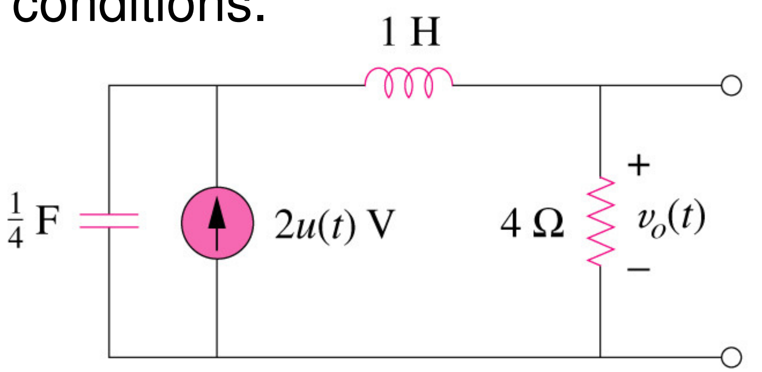
We now have a form that we can easily find the inverse Laplace Transform for

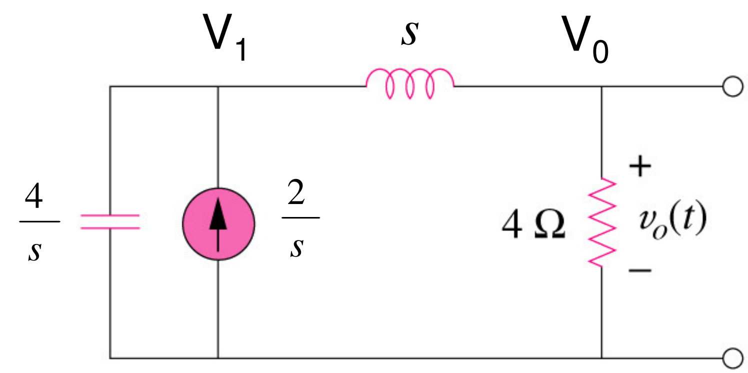
$$\boxed{v_0(t) = \frac{3}{\sqrt{2}} e^{-4t} \sin(\sqrt{2}t) \ \text{V}, \quad t \geq 0}$$

16.2 Circuit Element Models (12)

Example (Similar to 16.1)

Determine $v_0(t)$ in the circuit shown below, assuming zero initial conditions.





Solve by performing Nodal analysis for V_1 and V_0

Node V_1:

$$\frac{sV_1}{4}+\frac{V_1-V_0}{s}=\frac{2}{s}$$

Node V_1:

$$\frac{V_0}{4}+\frac{V_0-V_1}{s}=0$$

Solving for V_0 gives:

$$V_0=\frac{2}{s}\cdot\frac{16}{s^2+4s+4}$$

$$V_0=\frac{32}{s(s+2)^2}$$

16.2 Circuit Element Models (13)

Example (Similar to 16.1)

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Partial Fraction Decomposition:

$f(t)$	$F(s)$
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
te^{-at}	$\frac{1}{(s+a)^2}$

$$V_0 = \frac{32}{s(s+2)^2} = \frac{k_0}{s} + \frac{k_1}{(s+2)^2} + \frac{k_2}{(s+2)}$$

$$k_0 = s\frac{32}{s(s+2)^2}\bigg|_{s=0} = \frac{32}{(0+2)^2} = 8$$

$$k_1 = (s+2)^2\frac{32}{s(s+2)^2}\bigg|_{s=-2} = \frac{32}{-2} = -16$$

Residue Method

To avoid doing a derivative, substitute the values for k_0 and k_1 into the original equation and pick a value for s that makes solving for k_0 simple (i.e. 2).

$$\frac{32}{2(2+2)^2} = \frac{8}{2} + \frac{-16}{(2+2)^2} + \frac{k_2}{(2+2)}$$

$$1 = 4 - 1 + \frac{k_2}{4} \quad \Rightarrow \quad k_2 = -8$$

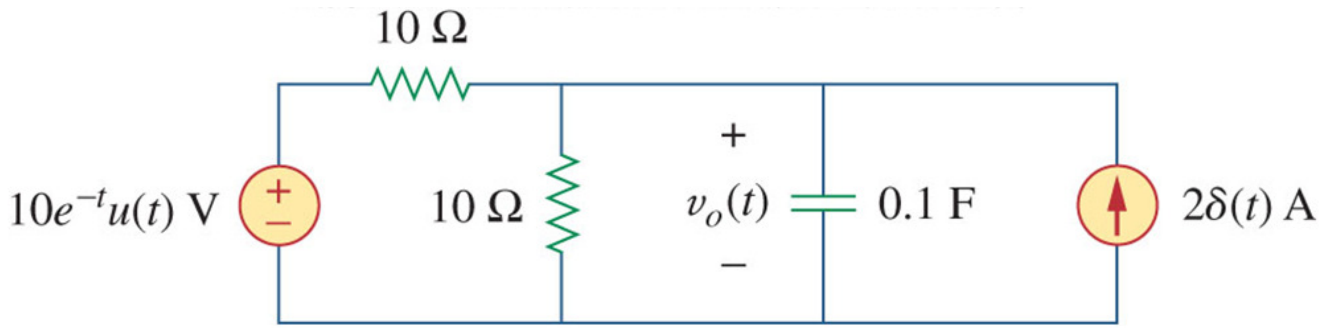
Can now take inverse Laplace to get

$$V_0 = \frac{8}{s} - \frac{16}{(s+2)^2} - \frac{8}{(s+2)} \Longrightarrow \boxed{v(t) = 8(1 - e^{-2t} - 2te^{-2t})u(t)}$$

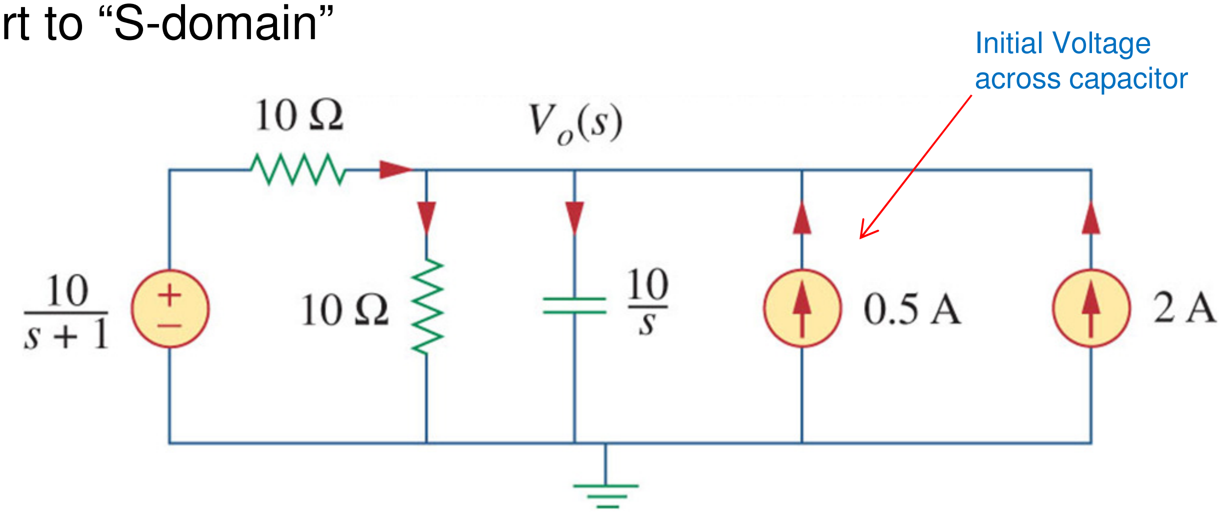
16.2 Circuit Element Models (14)

Example 16.2

Find $v_0(t)$ in the circuit shown below. Assume $v_0(0)=5V$.



Convert to "S-domain"



16.2 Circuit Element Models (15)

Example 16.2

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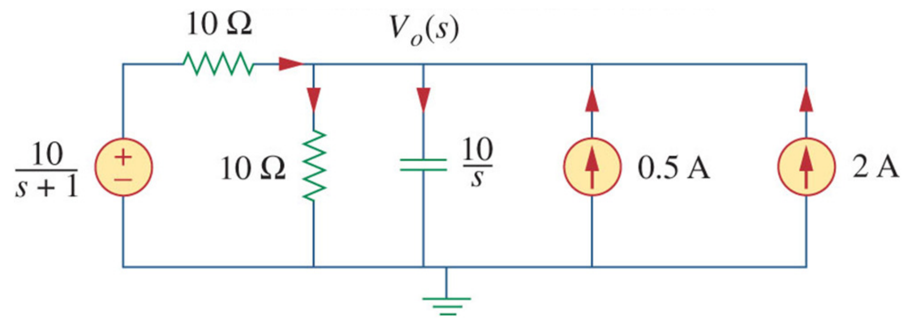


Nodal Analysis for V_o gives: $$\frac{10/(s-1)-V_0}{10}+2+0.5=\frac{V_0}{10}+\frac{V_0}{10/s}$$

Solving for V_o gives: $$V_0=\frac{25s+35}{(s+1)(s+2)}=\frac{k_0}{(s+1)}+\frac{k_1}{(s+2)}$$

Residue Method

$$\left\{\begin{aligned} k_0&=(s+1)\left.\frac{25s+35}{(s+1)(s+2)}\right|_{s=-1}=\frac{25(-1)+35}{(-1+2)}=10\\ k_1&=(s+2)\left.\frac{25s+35}{(s+1)(s+2)}\right|_{s=-2}=\frac{25(-2)+35}{(-2+1)}=15\end{aligned}\right.$$

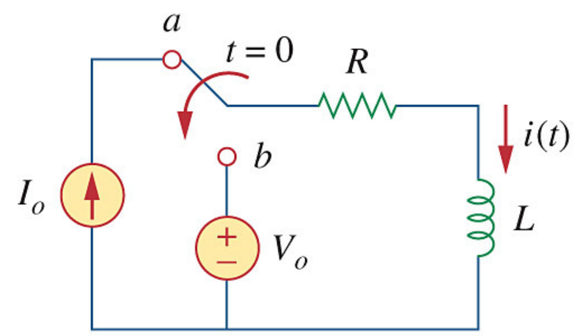
$$V_0=\frac{10}{(s+1)}+\frac{15}{(s+2)}$$

$$\boxed{\text{Answer}:\quad v_0(t)=(10e^{-t}+15e^{-2t})u(t)\quad \text{V}}$$

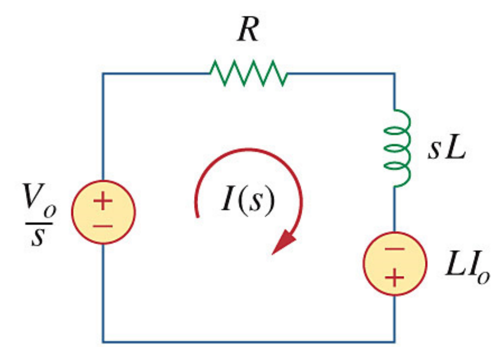
16.2 Circuit Element Models (16)

Example 16.3

Find i(t), initial condition $i(0)=I_0$



(a)



(b)

Mesh Analysis gives:

$$I(s)\cdot(R+sL)-LI_o-\frac{V_o}{s}=0$$

Solving for I(s) give:

$$I(s)=\frac{LI_o}{(R+sL)}+\frac{V_o}{s(R+sL)}=\frac{I_o}{(s+R/L)}+\frac{V_o/L}{s(s+R/L)}$$

Partial Fraction Expansion gives:

$$I(s)=\frac{I_o}{(s+R/L)}+\frac{V_o/R}{s}-\frac{V_o/R}{(s+R/L)}$$

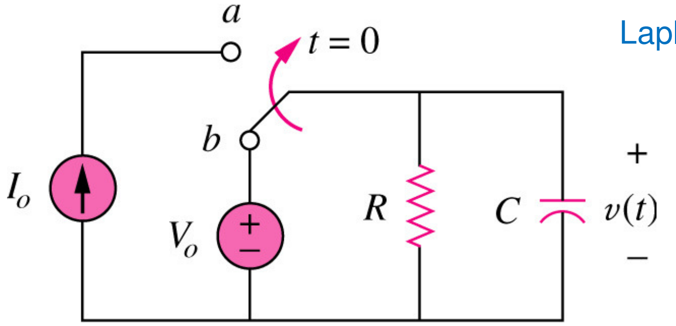
Inverse Laplace Transform gives:

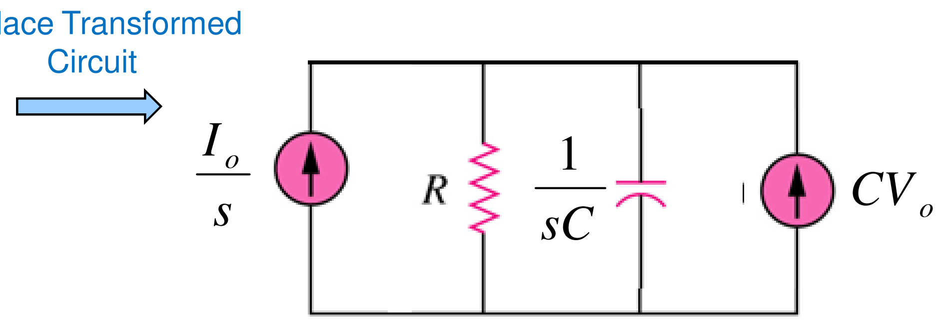
$$i(t)=L[I(s)]=I_oe^{-t/\tau}+\frac{V_o}{R}-\frac{V_o}{R}e^{-t/\tau} \qquad for \quad t\ge 0 \qquad \tau=\frac{R}{L}$$

16.2 Circuit Element Models (17)

Practice Problem 16.3

The switch shown below has been in position *b* for a long time. It is moved to position *a* at *t=0*. Determine *v(t)* for t > 0.





$$\text{Answer : } \quad v(t) = (V_0 - I_0 R)e^{-t/\tau} + I_0 R, \ \ t > 0, \text{ where } \tau = RC$$

16.3 Circuit Analysis (1)

- Circuit analysis is relatively easy to do in the s-domain.
- Transforms complicated sets of mathematical relationships (derivatives and integrals) in the time domain into simple algebraic equations (multipliers of s and 1/s) in the s-domain.
- All the circuit theorems and relationships developed for DC circuits are perfectly valid in the s-domain.

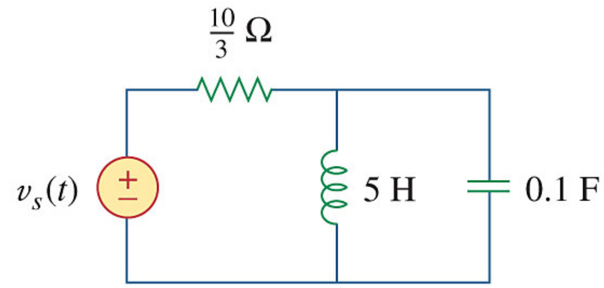
16.3 Circuit Analysis (2)

Example 16.4

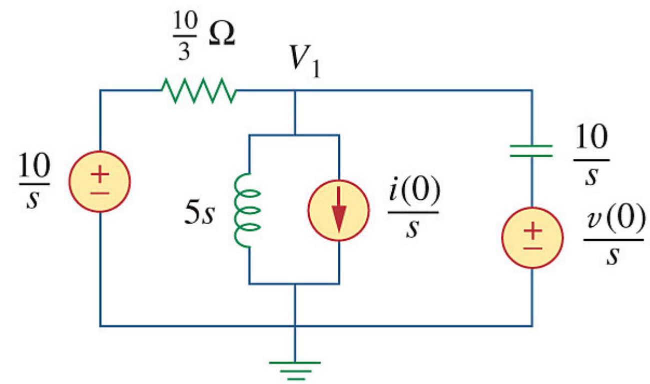
Consider the circuit in Fig (a).

Find the value of the voltage across the capacitor assuming that the value of $v_s(t)=10u(t)$ V

Assume that at t=0, -1 A flows through the inductor and +5 V is initially across the capacitor.



(a)



(b)

16.3 Circuit Analysis (3)

Example 16.4

Solution:

Transform the circuit from time-domain (a) into s-domain (b) using Laplace Transform. On rearranging the terms, we have

$$\boxed{V_1 = \frac{35}{s+1} - \frac{30}{s+2}}$$

By taking the inverse transform, we get

$$\boxed{v_1(t) = (35e^{-t} - 30e^{-2t})u(t) \quad \text{V}}$$

16.3 Circuit Analysis (4)

Practice Problem 16.6

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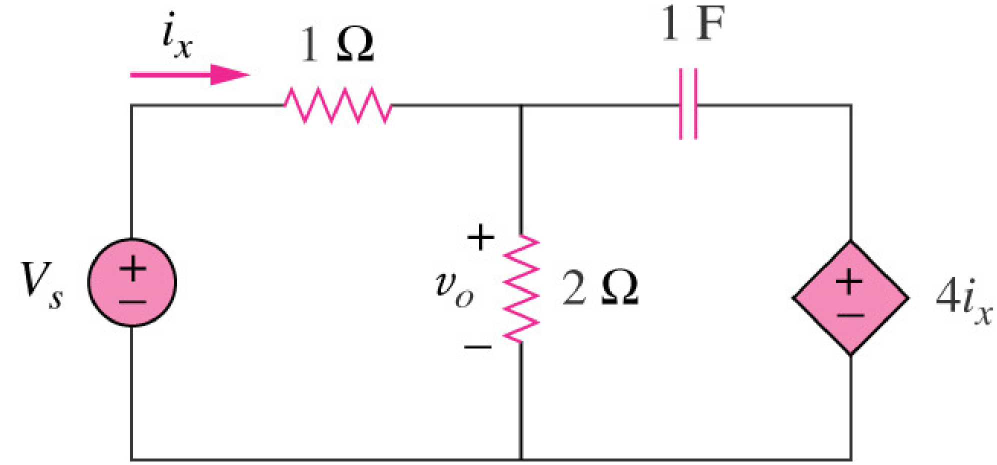
The initial energy in the circuit below is zero at *t=0*. Assume that $v_s=5u(t)$ V. (a) Find $V_0(s)$ using the thevenin theorem. (b) Apply the initial- and final-value theorem to find $v_0(0)$ and $v_0(\infty)$. (c) Obtain $v_0(t)$.



Answer: (a) $V_0(s) = 4(s+0.25)/(s(s+0.3))$,

(b) $v_0(0) = 4V$, $v_0(\infty)=3.33V$,

(c) $v_0(t)=(3.33+0.67e^{-0.3t})u(t)$ V.

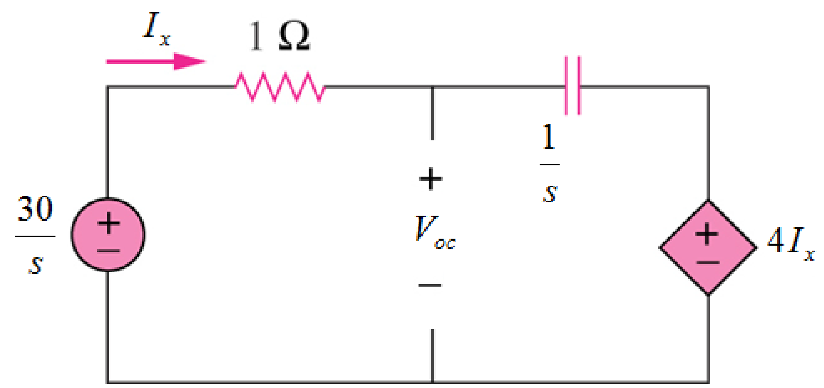
16.3 Circuit Analysis (5)

Practice Problem 16.6 (Continued)

Solution:

Find Open-Circuit Voltage by:

- Removing the resistor
- Find current I_x
- Find V_{oc}



Finding Current I_x

$$I_x(1)+I_x\left(\frac{1}{s}\right)+4I_x=\frac{30}{s}$$

$$I_x\left(5+\frac{1}{s}\right)=\frac{30}{s}$$

$$I_x(5s+1)=30$$

$$I_x=\frac{30}{(5s+1)}=\frac{6}{(s+0.2)}$$

Finding V_{oc}

$$I_x=\frac{V_s-V_{oc}}{1}\Rightarrow V_{oc}=V_s-I_x(1)$$

$$V_{oc}=\frac{30}{s}-\frac{6}{(s+0.2)}$$

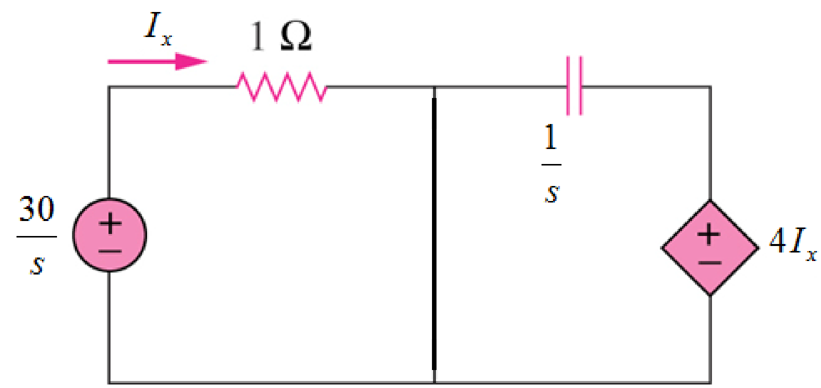
$$V_{oc}=\frac{30(s+0.2)}{s(s+0.2)}-\frac{6s}{s(s+0.2)}$$

$$V_{oc}=\frac{24s+6}{s(s+0.2)}=\boxed{\frac{24(s+0.25)}{s(s+0.2)}}$$

16.3 Circuit Analysis (6)

Practice Problem 16.6

Solution:



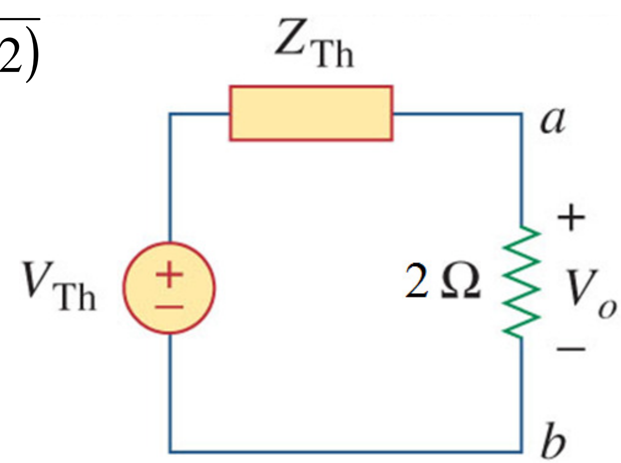
Next:

- Find Short-Circuit I_{sc}
- Find Z_{th}
- Find V_{oc}

$$I_{sc} = \frac{30}{s} + 4\left(\frac{30}{s}\right)\left(\frac{s}{1}\right) = 30\left(4 + \frac{1}{s}\right) = 30\left(\frac{4s+1}{s}\right) = 120\left(\frac{s+0.25}{s}\right)$$

$$Z_{th} = \frac{V_{oc}}{I_{sc}} = \frac{24(s+0.25)}{s(s+0.2)} \cdot \frac{1}{120}\left(\frac{s}{s+0.25}\right) = \frac{1}{5(s+0.2)}$$

V_{oc} can be found from Voltage Divider equation:



$$V_o = \frac{2}{2+Z_{th}} V_{th} = \left(\frac{2}{2+\left(5(s+0.2)\right)^{-1}}\right)\frac{24(s+0.25)}{s(s+0.2)}$$

$$V_o = \left(\frac{10(s+0.2)}{10(s+0.2)+1}\right)\frac{24(s+0.25)}{s(s+0.2)} = \frac{24(s+0.25)}{s(s+0.3)}$$

16.3 Circuit Analysis (4)

Practice Problem 16.6

Solution:

Next:

- Use Initial Value Theorem to find $v_o(0)$
- Use Final Value Theorem to find $v_o(\infty)$
- Find $v_o(t)$

Initial Value Theorem

$$v_o(0) = \lim_{s \to \infty}\left[sV_o(s)\right] = \lim_{s \to \infty} \frac{24(s+0.25)}{(s+0.3)} = 24$$

Final Value Theorem

$$v_o(\infty) = \lim_{s \to 0}\left[sV_o(s)\right] = \lim_{s \to 0} \frac{24(s+0.25)}{(s+0.3)} = \frac{24(0.25)}{(0.3)} = 20$$

$$V_o(s) = \frac{24(s+0.25)}{s(s+0.3)} = \frac{k_0}{s} + \frac{k_1}{(s+0.3)} = \frac{20}{s} + \frac{4}{(s+0.3)}$$

$$v_o(t) = \left(20 + 4e^{-0.3t}\right)u(t)$$

16.4 Transfer Functions (1)

- The transfer function of a network describes how the output behaves with respect to the input.
- The transfer function is a key concept in signal processing because it indicates how the signal is processed as it passes through a network.
- The transfer function H(s) is the ratio of the output response Y(s) to the input excitation X(s), assuming all the initial conditions are zero.

$$\boxed{H(s) = \frac{Y(s)}{X(s)}}$$ **, h(t) is the unit impulse response.**

- Since the input and output can be either current or voltage at any place in the circuit, there are four possible transfer functions:

1. H(s) = voltage gain = $V_0(s)/V_i(s)$
2. H(s) = Current gain = $I_0(s)/I_i(s)$
3. H(s) = Impedance = V(s)/I(s)
4. H(s) = Admittance = I(s)/V(s)

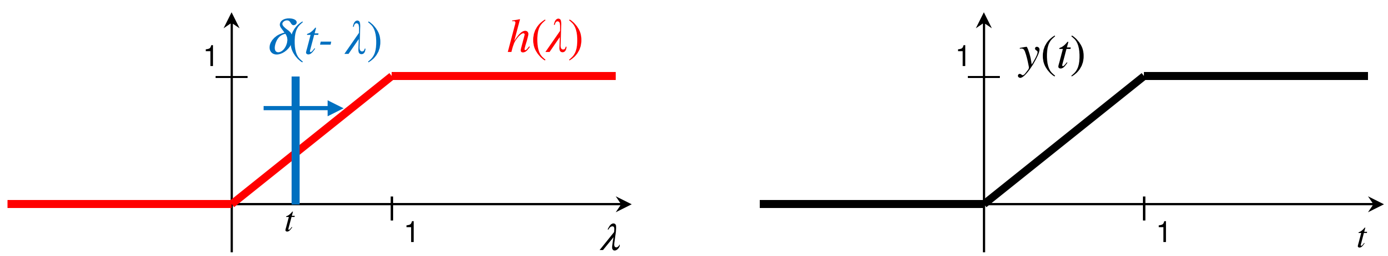
16.4 Transfer Functions (2)

Impulse Response

- The Transfer function is the response of a system to an impulse in the time domain $\delta(t)$.
- To see this, look at the convolution integral

$$y(t) = \delta(t) * h(t) = \int_{-\infty}^{\infty} \delta(t-\lambda)h(\lambda)d\lambda = h(t)$$

- This is the "sifting property" of the impulse function to the convolution integral.
- Think about this graphically, if we flipped and slid an impulse and multiplied it with the function it would just map out the function itself.



- Also, the Laplace Transform: $Y(s) = X(s)H(s) = \mathcal{L}[\delta(t)]H(s) = H(s)$

16.4 Transfer Functions (3)

Impulse Response

- Once we know the Transfer Function (or impulse response h(t)) of a network, we can obtain the response of the network to any input signal since: $y(t) = x(t) * h(t)$
- Example, Let: $\boxed{H(s) = \dfrac{1}{s+1}}$ $\quad Y(s) = X(s)H(s)$

$x(t)$	$X(s)$	$Y(s)$	$y(t)$
$\delta(t)$	1	$\dfrac{1}{s+1}$	$e^{-t}u(t)$
$u(t)$	$\dfrac{1}{s}$	$\dfrac{1}{s(s+1)} = \dfrac{1}{s} - \dfrac{1}{s+1}$	$\left(1 - e^{-t}\right)u(t)$
$tu(t)$	$\dfrac{1}{s^2}$	$\dfrac{1}{s^2(s+1)} = \dfrac{1}{s^2} - \dfrac{1}{s} + \dfrac{1}{s+1}$	$\left(t + e^{-t} - 1\right)u(t)$
$\sin(t)u(t)$	$\dfrac{1}{s^2+1}$	$\dfrac{1}{\left(s^2+1\right)\left(s+1\right)} = \dfrac{0.5}{(s+1)} + \dfrac{-1.5s+1}{\left(s^2+1\right)}$	$\left(0.5e^{-t} - 1.5\cos t + \sin t\right)u(t)$

16.4 Transfer Functions (4)

- The transfer function can be found in two ways:
  1. By Circuit Analysis (assume an input, find the output)
     - Assume a convenient input X(s) (such as impulse or step)
     - Use any circuit analysis technique to find Y(s)
     - Obtain the ratio of Y(s) and X(s)
  2. Ladder Method (assume an output, find the input)
     - Assume the output is 1V or 1A as appropriate
     - Use the basic circuit analysis technique to find the input
     - The transfer function is unity divided by the input
     - This approach may be more convenient to use when the circuit has many loops or nodes.

16.4 Transfer Function (5)

Example 16.7

The output of a linear system is $y(t)=10e^{-t}\cos 4t\, u(t)$ when the input is $x(t)=e^{-t}u(t)$. Find the transfer function of the system and its impulse response.

Solution:

Find the Laplace Transform of x(t) and y(t) to find H(s)

$$y(t)=10e^{-t}\cos 4t \longrightarrow \frac{10(s+1)}{(s+1)^2+4^2}=Y(s)$$

$$x(t)=e^{-t} \longrightarrow \frac{1}{(s+1)}=X(s)$$

$$\frac{Y(s)}{X(s)}=H(s)=\frac{10(s+1)^2}{(s+1)^2+4^2}$$

Rearrange the numerator into a convenient form:

$$H(s)=\frac{10\left[(s+1)^2+4^2\right]-160}{(s+1)^2+4^2}=10-40\frac{4}{(s+1)^2+4^2}$$

Study this numerator trick to get fraction into a convenient form

$$\boxed{h(t)=10\delta(t)-40e^{-t}\sin 4t}$$

$f(t)$	$F(s)$
$e^{-at}\sin\omega t$	$\dfrac{\omega}{(s+a)^2+\omega^2}$

16.4 Transfer Function (6)

Practice Problem 16.7

The transfer function of a linear system is

$$\boxed{H(s) = \frac{2s}{s+6}}$$

Find the output y(t) due to the input $5e^{-3t}u(t)$ and find its impulse response.

$$x(t) = 5e^{-3t} \longrightarrow \frac{5}{(s+3)} = X(s)$$

$$Y(s) = X(s)H(s) = \frac{5}{(s+3)}\frac{2s}{(s+6)} = \frac{10s}{(s+3)(s+6)} = \frac{-10}{(s+3)} + \frac{20}{(s+6)}$$

$$y(t) = -10e^{-3t} + 20e^{-6t}$$

For x(t) as an impulse function X(s) = 1

$$Y(s) = (1)H(s) = \frac{2s}{(s+6)} = \frac{2(s+6)-12}{(s+6)} = 2 - \frac{12}{(s+6)}$$

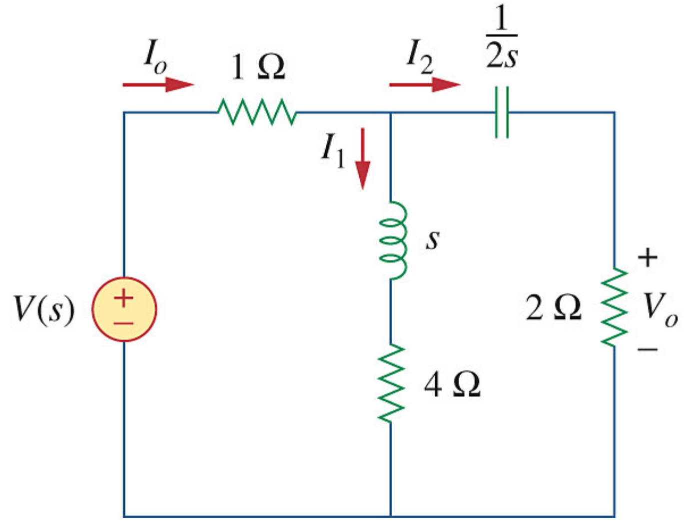
$$y(t) = 2\delta(t) - 12e^{-6t}$$

Here's that same trick again with the numerator

16.4 Transfer Function (7)

Example 16.8

Find H(s)=$V_0(s)/I_0(s)$



Method 1 (Circuit Analysis)

By current divider equation:

$$I_2 = \frac{(s+4)I_o}{s+4+2+1/2s}$$

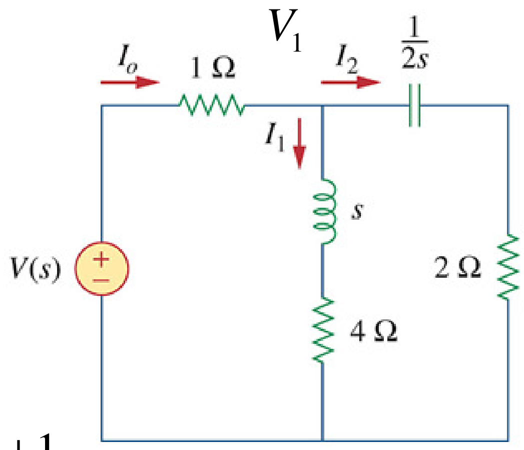
$$V_o = 2I_2 = \frac{2(s+4)I_o}{s+6+1/2s} = \frac{4s(s+4)I_o}{2s^2+12s+1}$$

$$\boxed{H(s) = \frac{V_o(s)}{I_o(s)} = \frac{4s(s+4)}{2s^2+12s+1}}$$

16.4 Transfer Function (8)

Example 16.8

Method 2 (Ladder Method)



Assume V_o = 1 Volt, then I_2 is:

$$I_2 = \frac{V_o}{2} = \frac{1}{2}$$

Working up the circuit, the voltage drop across the capacitor and resistor is:

$$V_1 = I_2\left(\frac{1}{2s} + 2\right) = \frac{1}{2}\left(\frac{1}{2s} + 2\right) = \frac{1}{4s} + 1 = \frac{4s+1}{4s}$$

We can now find I_1 by using V_1:

$$I_1 = \frac{V_1}{s+4} = \left(\frac{1}{s+4}\right)\frac{4s+1}{4s} = \frac{4s+1}{4s(s+4)}$$

I_o is just the sum of I_1 and I_2:

$$I_o = I_1 + I_2 = \frac{1}{2} + \frac{4s+1}{4s(s+4)} = \frac{2s(s+4)}{4s(s+4)} + \frac{4s+1}{4s(s+4)} = \frac{2s^2+12s+1}{4s(s+4)}$$

We can now find H(s): $\boxed{H(s) = \frac{V_o}{I_o} = \frac{1}{I_o} = \frac{4s(s+4)}{2s^2+12s+1}}$

16.4 Transfer Function (9)
Example 16.9

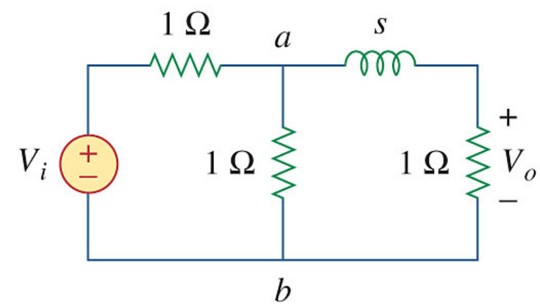
For the s-domain circuit shown, find:

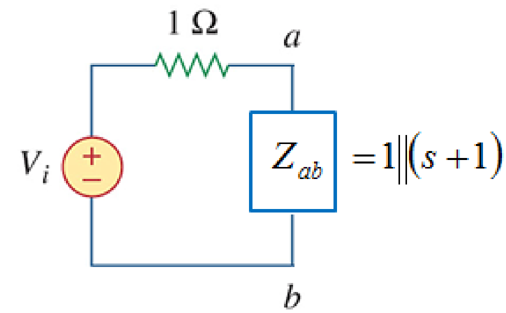
(a) The transfer function V_o/V_i

(b) The impulse response

(c) The response when $v_i(t)=u(t)$ Volts

(d) The response when $v_i(t)=8\cos 2t$ Volts





Using voltage divider equation:

$$V_o = \frac{1}{s+1} V_{ab}$$

$$V_{ab} = \frac{Z_{ab}}{1+Z_{ab}} V_i = \frac{1\|(s+1)}{1+1\|(s+1)} V_i = \frac{(s+1)/(s+2)}{1+(s+1)/(s+2)} V_i = \frac{s+1}{2s+3} V_i$$

Substituting V_{ab} into the equation above for V_o gives:

$$V_o = \frac{1}{2s+3} V_i \quad \Rightarrow \quad \boxed{H(s) = \frac{V_o}{V_i} = \frac{1}{2s+3} = \frac{1}{2\left(s+\frac{3}{2}\right)}} \qquad \boxed{h(t) = \frac{1}{2} e^{-\frac{3}{2}t} u(t)}$$

(a) (b)

16.4 Transfer Function (10)

Example 16.9

If the input is u(t), then:

$$Y(s) = X(s)H(s) = \left(\frac{1}{s}\right)\frac{1}{2\left(s+\frac{3}{2}\right)} = \frac{k_0}{s} + \frac{k_1}{\left(s+\frac{3}{2}\right)}$$

$$k_0 = sY(s)\Big|_{s=0} = \frac{1}{2\left(0+\frac{3}{2}\right)} = \frac{1}{3}$$

$$k_1 = \left(s+\tfrac{3}{2}\right)Y(s)\Big|_{s=-\frac{3}{2}} = \frac{1}{2\left(-\frac{3}{2}\right)} = -\frac{1}{3}$$

$$Y(s) = \frac{1}{3}\left(\frac{1}{s}\right) - \frac{1}{3}\frac{1}{\left(s+\frac{3}{2}\right)}$$

Inverse Laplace Transform gives:

$$\boxed{y(t) = \frac{1}{3}\left(1-e^{-\frac{3}{2}t}\right)u(t)}$$

(c)

16.4 Transfer Function (11)

Example 16.9

$f(t)$	$F(s)$
$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$

If the input is $8\cos(2t)$, then:

$$Y(s) = X(s)H(s) = \left(\frac{8s}{s^2+4}\right)\frac{1}{2\left(s+\frac{3}{2}\right)} = \frac{4s}{\left(s^2+4\right)\left(s+\frac{3}{2}\right)}$$

$$Y(s) = \frac{4s}{\left(s^2+4\right)\left(s+\frac{3}{2}\right)} = \frac{k_0}{\left(s+\frac{3}{2}\right)} + \frac{k_1 s + k_2}{\left(s^2+4\right)}$$

$$k_0 = \left(s+\tfrac{3}{2}\right)Y(s)\Big|_{s=-\frac{3}{2}} = \frac{4\left(-\frac{3}{2}\right)}{\left(-\frac{3}{2}^2+4\right)} = -\frac{24}{25}$$

Find k1 and k_2 by using the Algebraic method (equate powers of s):

$$4s = -\frac{24}{25}\left(s^2+4\right) + \left(k_1 s + k_2\right)\left(s+\tfrac{3}{2}\right)$$

$$k_1 = \frac{24}{25}$$

$$4s = \left(-\frac{24}{25}\right)s^2 - \left(\frac{(24)(4)}{25}\right) + k_1 s^2 + \left(\frac{3}{2}\right)k_1 s + k_2 s + \left(\frac{3}{2}\right)k_2$$

$$k_2 = 4 - \left(\frac{3}{2}\right)\frac{24}{25} = \frac{64}{25}$$

16.4 Transfer Function (12)

Example 16.9

$f(t)$	$F(s)$
$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$

Substituting in for k_0, k_1, and k_2 :

$$Y(s)=\frac{4s}{\left(s^2+4\right)\left(s+\frac{3}{2}\right)}=\left(-\frac{24}{25}\right)\frac{1}{\left(s+\frac{3}{2}\right)}+\left(\frac{1}{25}\right)\frac{24s+64}{\left(s^2+4\right)}$$

$$Y(s)=\left(-\frac{24}{25}\right)\frac{1}{\left(s+\frac{3}{2}\right)}+\left(\frac{24}{25}\right)\frac{s}{\left(s^2+4\right)}+\left(\frac{1}{25}\right)\frac{64}{\left(s^2+4\right)}$$

$$Y(s)=\left(-\frac{24}{25}\right)\frac{1}{\left(s+\frac{3}{2}\right)}+\left(\frac{24}{25}\right)\frac{s}{\left(s^2+2^2\right)}+\left(\frac{32}{25}\right)\frac{2}{\left(s^2+2^2\right)}$$

Now let's fix this numerator to look like the inverse transform of cosωt

Now we can take the inverse Laplace to get the final result:

$$y(t)=\left(-\frac{24}{25}\right)e^{-\frac{3}{2}t}+\left(\frac{24}{25}\right)\cos 2t+\left(\frac{32}{25}\right)\sin 2t \quad \text{for} \quad t>0$$

$$\boxed{y(t)=\frac{24}{25}\left(-e^{-\frac{3}{2}t}+\cos 2t+\left(\frac{4}{3}\right)\sin 2t\right)u(t)}$$

(d)

16.6 Applications (1)

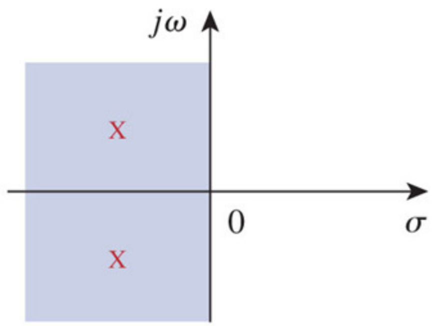
Network Stability

- A circuit is stable if the impulse response $h(t)$ is bounded as t→∞

$$\lim_{t \to \infty} |h(t)| = \text{finite}$$

- This imposes certain requirements on the transfer function H(s). Specifically:

$$H(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$



1) The degree of the numerator must be less than the denominator
2) The poles of H(s) must be in the left hand of the s plane (negative side)

$$h(t) = \left(k_1 e^{-p_1 t} + k_1 e^{-p_2 t} + \cdots + k_1 e^{-p_n t}\right) u(t)$$

If pole is negative, h(t) is unbounded