15. ■ A well-known example of an ill-conditioned matrix is the *Hilbert matrix*:

$$\mathbf{A} = \begin{bmatrix} 1 & 1/2 & 1/3 & \cdots \\ 1/2 & 1/3 & 1/4 & \cdots \\ 1/3 & 1/4 & 1/5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Write a program that specializes in solving the equations $\mathbf{Ax} = \mathbf{b}$ by Doolittle's decomposition method, where $\mathbf{A}$ is the Hilbert matrix of arbitrary size $n \times n$, and

$$b_i = \sum_{j=1}^{n} A_{ij}$$

The program should have no input apart from n. By running the program, determine the largest n for which the solution is within six significant figures of the exact solution

$$\mathbf{x} = \begin{bmatrix} 1 & 1 & 1 & \cdots \end{bmatrix}^T$$

16. Derive the forward and back substitution algorithms for the solution phase of Choleski's method. Compare them with the function `choleskiSol`.
17. ■ Determine the coefficients of the polynomial $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ that passes through the points (0, 10), (1, 35), (3, 31), and (4, 2).
18. ■ Determine the fourth-degree polynomial $y(x)$ that passes through the points (0, −1), (1, 1), (3, 3), (5, 2), and (6, −2).
19. ■ Find the fourth-degree polynomial $y(x)$ that passes through the points (0, 1), (0.75, −0.25), and (1, 1) and has zero curvature at (0, 1) and (1, 1).
20. ■ Solve the equations $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 3.50 & 2.77 & -0.76 & 1.80 \\ -1.80 & 2.68 & 3.44 & -0.09 \\ 0.27 & 5.07 & 6.90 & 1.61 \\ 1.71 & 5.45 & 2.68 & 1.71 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 7.31 \\ 4.23 \\ 13.85 \\ 11.55 \end{bmatrix}$$

By computing $|\mathbf{A}|$ and $\mathbf{Ax}$ comment on the accuracy of the solution.

21. Compute the condition number of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

based on (a) the euclidean norm and (b) the infinity norm. You may use the function `inv(A)` in `numpy.linalg` to determine the inverse of $\mathbf{A}$.