Chapter 1: Systems of Linear Equations

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Contents

1. Introduction to Systems of Linear Equations

Definition 1.1 (**Linear Equation in n Variables**). *A **linear equation in n variables** $x_1, x_2, \ldots, x_n$ has the form*

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b.$$

*The **coefficients** $a_1, a_2, \ldots, a_n$ are real numbers, and the **constant term** b is a real number. The number a_1 is the **leading coefficient**, and x_1 is the **leading variable**.*

The word "linear" has a geometric meaning in the 2-dimensional space or 3-dimensional space:

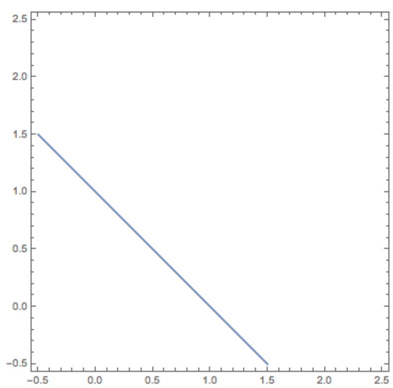


Figure 1. 2-dimensional line

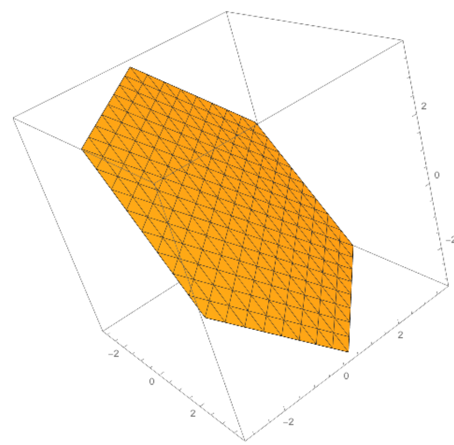


Figure 2. 3-dimensional plane

Example 1.1. *Each equation is linear*

$$\text{(a) } 3x + 2y = 7 \text{ (b) } \frac{1}{2}x + y - \pi z = \sqrt{2} \text{ (c) } (\sin\pi)x_1 - 4x_2 = e^2$$

Each equation is not linear.

$$\text{(a) } xy + z = 2 \text{ (b) } e^x - 2y = 4 \text{ (c) } \sin x_1 + 2x_2 - 3x_3 = 0$$

A **system of m linear equations in n variables** (or a **linear system**) is a set of m equations, each of which is linear in the same n variables:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = b_2 \\ & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = b_m. \end{cases}$$

For example,

$$\begin{array}{ccccccc} 2x_1 & - & x_2 & + & 1.5x_3 & = & 8 \\ x_1 & & & - & 4x_3 & = & -7 \end{array}$$

is a system of 2 equations in 3 variables.

Remark 1.1. *Notice that a linear equation is a special case of a linear system (with just one equation). So when we say a linear system, it may imply a linear equation.*

A **solution** of a linear system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, \ldots, s_n$ are substituted for $x_1, \ldots, x_n$, respectively. For instance, the system

$$3x_1 + 2x_2 = 3$$

$$-x_1 + x_2 = 4$$

has $(-1, 3)$ as a solution because $x_1 = -1$ and $x_2 = 3$ satisfy both equations. On the other hand, $x_1 = 1$ and $x_2 = 0$ is not a solution of the system because these values satisfy only the first equation in the system.

The set of all possible solutions is called the **solution set** of the linear system. Two linear system are called **equivalent** if they have the same solution set.

The goal of this chapter is to find a systematic method (or an algorithm) to solve a system of linear equations. Finding the solution set of a system of two linear equations in two variables is intuitive, it amounts to finding the intersection of two lines. An example:

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

The graphs of these equations are lines, which we denote by l_1 and l_2. See Figure 3. The intersection $(3, 2)$ is the solution to the system.

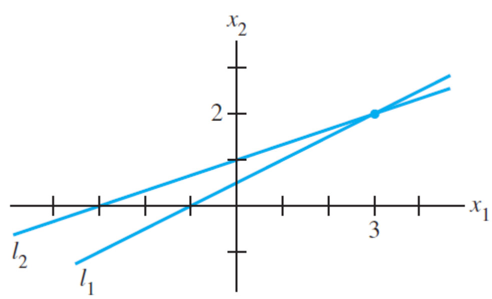


FIGURE 3. Exactly one solution

Example 1.2. *Solve and graph each system of linear equation.*

a. $\begin{cases} x + y = 3 \\ x - y = -1 \end{cases}$ *b.* $\begin{cases} x + y = 3 \\ 2x + 2y = 6 \end{cases}$ *c.* $\begin{cases} x + y = 3 \\ x + y = 1 \end{cases}$

Fact 1.1 (Number of Solutions of a System of Linear Equations). *For a system of linear equation, precisely one of the statements below is true.*

(1) The system has exactly one solution.
(2) The system has infinitely many solutions.
(3) The system has no solution.

Remark 1.2. *Each system of linear equation must fit into one of the three previous categories. A system of linear equations is said to be* ***consistent*** *if it has either one solution or infinitely many solutions; a system is* ***inconsistent*** *if it has no solution.*

When a linear system has infinitely many solutions, we want to describe the solution set using a parametric representation. The following two examples illustrate this concept.

Example 1.3. *Find the solution set of the linear equation* $x_1 + 2x_2 = 4$.

Example 1.4. *Solve the linear equation* $3x + 2y - z = 3$.

1.1. **Solving a System of Linear Equations.**

Compare the following equivalent linear systems:

$$\begin{array}{rrcr} x - 2y & +3z & = & 9 \\ -x + 3y & & = & -4 \\ 2x - 5y & +5z & = & 17 \end{array} \qquad \begin{array}{rcr} x - 2y + 3z & = & 9 \\ y + 3z & = & 5 \\ z & = & 2 \end{array}$$

The system on the right is clearly easier to solve. This system is in **row-echelon form**, which means that it has a "stair-step" pattern with leading coefficients of 1. To solve such a system, use **back-substitution**:

- Substitute 2 for z in the second equation: $y + 3 \cdot 2 = 5 \Rightarrow y = -1$.
- Substitute 2 for z and -1 for y in the first equation: $x - 2(-1) + 3(2) = 9 \Rightarrow x = 1$.
- Therefore, the solution is $(1, -1, 2)$.

To solve a linear system that is not in row-echelon form, first rewrite it as an equivalent system that is in row-echelon form using the following operations.

Fact 1.2 (Operations That Produce Equivalent Systems). *Each of these operations on a system of linear equations produces an* equivalent *system.*

(1) Interchange two equations.
(2) Multiply an equation by a nonzero constant.
(3) Add a multiply of an equation to another equation.

Rewriting a system of linear equations in row-echelon form usually involves a chain of equivalent systems, using one of the three basic operations to obtain each system. This process is called **Gaussian elimination**.

Example 1.5. *Solve the system*

$$\begin{array}{rrcr} x - 2y & +3z & = & 9 \\ -x + 3y & & = & -4 \\ 2x - 5y & +5z & = & 17 \end{array}$$

Example 1.6. *Solve the system.*

$$x_1 - 3x_2 + x_3 = 1$$
$$2x_1 - x_2 - 2x_3 = 2$$
$$x_1 + 2x_2 - 3x_3 = -1$$

Example 1.7. *Solve the system*

$$\begin{array}{rrrcr} & x_2 & -x_3 & = & 0 \\ x_1 & & -3x_3 & = & -1 \\ -x_1 & +3x_2 & & = & 1 \end{array}$$

2. Gaussian Elimination and Gauss-Jordan Elimination

Definition 2.1. *If m and n are positive integers, then an $m \times n$ (read "m by n") matrix is a rectangular array*

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

in which each ***entry****, a_{ij}, of the matrix is a number. An $m \times n$ matrix has m rows and n columns. Matrices are usually denoted by capital letters.*

The **size** of a matrix tells how many rows and columns it has. If m and n are positive integers, an $m \times n$ **matrix** is a rectangular array of numbers with m rows and n columns. When $m = n$, the matrix is a **square** and $a_{11}, a_{22}, \ldots, a_{n,n}$ are the **main diagonal** entries.

Example 2.1. *Identify the size of the matrices.*

$$a.\ [2] \quad b.\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad c.\ \begin{bmatrix} e & 2 & -7 \\ \pi & \sqrt{2} & 4 \end{bmatrix}$$

We can extract the essential information of a linear system and write it in a more compact way. The essential information of a system can be recorded compactly in a matrix. Given the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

with the coefficients of each variable aligned in columns, the matrix

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$$

is called the **coefficient matrix** (or **matrix of coefficients**) of the system, and

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

is called the **augmented matrix** of the system. An augmented matrix of a system consists of the coefficient matrix with an added column containing the constants from the right sides of the equations.

Two matrices are **row-equivalent** when one can be obtained from the other by a finite sequence of elementary row operations.

Definition 2.2 (Elementary Row Operations)**.**

(1) *(Replacement) Replace one row by the sum of itself and a multiple of another row.*
(2) *(Interchange) Interchange two rows.*
(3) *(Scaling) Multiply all entries in a row by a nonzero constant.*

Remark 2.1. *It is important to note that row operations are* reversible. *If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.*

In the last section, we used Gaussian elimination with back-substitution to solve a system of linear equations. In the next example, we look at the matrix version of Gaussian elimination. The two methods are essentially the same. The basic difference is that with matrices you do not need to keep writing the variables.

Example 2.2. *Solve the system*

$$\begin{aligned} x - 2y \quad +3z &= \quad 9 \\ -x + 3y \qquad\quad &= \ -4 \\ 2x - 5y \quad +5z &= \quad 17 \end{aligned}$$

by applying elementary row operations to the associated augmented matrix.

Definition 2.3. *A matrix is in* ***echelon form*** *(or* ***row-echelon form****) if it has the following three properties:*

1. *All nonzero rows are above any rows of all zeros.*
2. *Each leading entry of a row is in a column to the right of the leading entry of the row above it.*
3. *All entries in a column below a leading entry are zeros.*

If a matrix in echelon form satisfies the following additional conditions, then it is in ***reduced echelon form*** *(or* ***reduced row-echelon form****):*

4. *The leading entry in each nonzero row is* 1*.*
5. *Each leading* 1 *is the only nonzero entry in its column.*

Remark 2.2. *Property 2 says that the leading entries form an* echelon *("steplike") pattern that moves down and to the right through the matrix. Property 3 is a simple consequence of property 2, but we include it for emphasis.*

An **echelon matrix** (respectively, **reduced echelon matrix**) is one that is in echelon form (respectively, reduced echelon form). The "triangular" matrices, such as

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

are in echelon form. In fact, the second matrix is in reduced echelon form.

Example 2.3. *Determine whether each matrix is in row-echelon form. If it is, determine whether the matrix is also in reduced row-echelon form.*

$$a. \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix} \qquad b. \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

$$c. \begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad d. \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem 2.1. *Each matrix is row-equivalent to one and only one reduced echelon matrix.*

Remark 2.3. *Any nonzero matrix may be* ***row reduced*** *into more than one matrix in echelon form, using different sequences of row operations. If a matrix A is row equivalent to an echelon matrix U, we call U* ***an echelon form of*** *A; if U is in reduced echelon form, we call U* ***the reduced echelon form of*** *A.*

Example 2.4. *Solve the system*

$$\begin{array}{rcrcrl} x_2 & + & x_3 & - & 2x_4 = & -3 \\ x_1 + 2x_2 & - & x_3 & & = & 2 \\ 2x_1 + 4x_2 & + & x_3 & - & 3x_4 = & -2 \\ x_1 - 4x_2 & - & 7x_3 & - & x_4 = & -19 \end{array}$$

Example 2.5. *Solve the system.*

$$\begin{array}{rcrcrl} x_1 & - & x_2 & + & 2x_3 & = 4 \\ x_1 & & & + & x_3 & = 6 \\ 2x_1 & - & 3x_2 & + & 5x_3 & = 4 \\ 3x_1 & + & 2x_2 & - & x_3 & = 1 \end{array}$$

With Gaussian elimination, you apply elementary row operations to a matrix to obtain a row-echelon form. A second method of elimination, called **Gauss-Jordan elimination** continues the reduction process until the reduced row-echelon form is obtained.

Example 2.6. *Use Gauss-Jordan elimination to solve the system.*

$$\begin{array}{rrrl} x & -2y & +3z = & 9 \\ -x & +3y & = & -4 \\ 2x & -5y & +5z = & 17 \end{array}$$

Example 2.7. *Solve the system of linear equations.*

$$\begin{array}{rrrl} 2x_1+ & 4x_2- & 2x_3 & = 0 \\ 3x_1+ & 5x_2 & & = 1 \end{array}$$

2.1. **Homogeneous System of Linear Equations.**

Systems of linear equation in which each of the constant terms is zero are called **homogeneous**. A homogeneous system must have at least one solution. Specifically, if all variables in a homogeneous system have the value zero, then each of the equations is satisfied. Such a solution is **trivial**.

Example 2.8. *Find the solution set of the system*

$$x_1 - 2x_2 + 3x_3 = 0$$

$$2x_1 + x_2 + 3x_3 = 0$$

Theorem 2.2. *Every homogeneous system of linear equations is consistent. Moreover, if the system has fewer equations than variables, then it must have infinitely many solutions.*

3. Applications of Systems of Linear Equations

3.1. **Polynomial Curve Fitting.**

Recall that the general form of a polynomial of degree $n-1$ is

$$p(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}$$

where $a_0, a_1, \ldots, a_{n-1}$ are constants. In order to completely determine a polynomial of degree $n-1$, say $p(x)$ as in the last equation, we would need to know at least n distinct points on the curve of the polynomial, this is because there are n unknown coefficients in $p(x)$, i.e. $a_0, a_1, \ldots, a_{n-1}$.

To solve for the n coefficients of $p(x)$, substitute each of the n points into the polynomial function and obtain n linear equations in n variables $a_0, a_1, \ldots, a_{n-1}$.

Example 3.1. *Determine the polynomial* $p(x) = a_0 + a_1 x + a_2 x^2$ *whose graph passes through the points* $(1, 4), (2, 0),$ *and* $(3, 12)$.

Example 3.2. *Find a polynomial that fits the points*

$$(-2, 3), (-1, 5), (0, 1), (1, 4), \text{ and } (2, 10).$$

Example 3.3. *Find a polynomial that relates the periods of the three planets that are closest to the Sun to their mean distances from the Sun, as shown in the table. Then use the polynomial to calculate the period of Mars, and compare it to the value shown in the table. (The mean distances are in astronomical units, and the periods are in years.)*

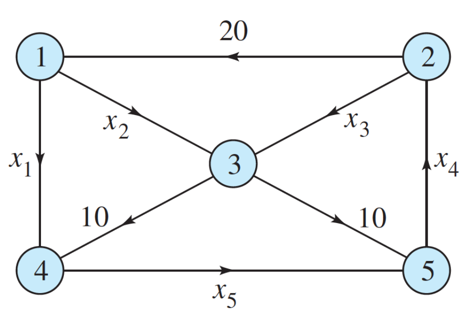
Planet	*Mercury*	*Venus*	*Earth*	*Mars*
Mean Distance	*0.387*	*0.723*	*1.000*	*1.524*
Period	*0.241*	*0.615*	*1.000*	*1.881*

3.2. **Network Analysis.**

Networks composed of branches and junctions are used as models in such such fields as traffic analysis, and electrical engineering.

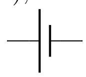
In a network model, you assume that the total flow into a junction is equal to the total flow out of the junction. Each junction in a network gives rise to a linear equation, so you can analyze the flow through a network composed of several junctions by solving a system of linear equations.

Example 3.4. *Set up a system of linear equations to represent the network shown in the figure below. Then solve the system.*

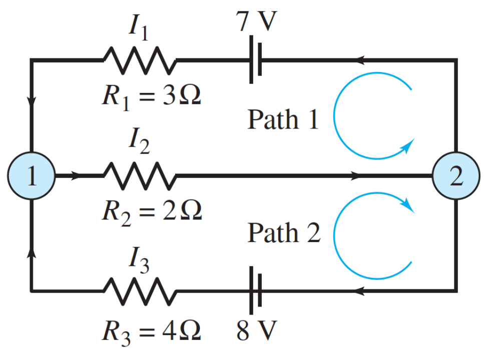


An electrical network is another type of network where analysis is commonly applied. We need the following properties (tools)

(1) All the current flowing into a junction must flow out of it.
(2) The sum of the products IR (I is the current and R is resistance) around a closed path is equal to the total voltage in the path.

In an electrical network, current is measured in amperes, or amps (A), resistance is measured in ohms (Ω), and the product of current and resistance is measured in volts (V). The symbol  represents a battery. The larger vertical bar denotes where the current flows out of the terminal. The symbol  denotes resistance. An arrow in the branch shows the direction of the current.

Example 3.5. *Determine the currents I_1, I_2, and I_3 for the electrical network shown in the figure.*



Example 3.6. *Determine the currents I_1, I_2, I_3, I_4, I_5, and I_6 for the electrical network shown below.*