Chapter 3: Determinants

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1. The Determinant of a Matrix

In the last chapter, we talked about the invertibility of a matrix A. One of the applications is to solve a linear system $A\vec{x} = \vec{b}$ by finding the multiplicative inverse of the coefficient matrix A provided it exists.

How do we know if a square matrix A is invertible or non-invertible? We will introduce a new notion, called the determinant of a matrix.

First consider the easiest case.

Definition 1.1. *The **determinant** of the matrix*

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

denoted by $\det(A)$ *or* $|A|$, *is* $a_{11}a_{22} - a_{21}a_{12}$.

Once again, both the notations $\det(\cdot)$ and $|\cdot|$ represent the determinant.

Example 1.1. *Compute:*

$$\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix}, \quad \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix}, \quad \det\left(\begin{bmatrix} 0 & \frac{3}{2} \\ 2 & 4 \end{bmatrix}\right)$$

Next we will try to define the determinant of a more complex square matrix inductively. First, some more definitions.

Definition 1.2 (**Minors and Cofactors of a Square Matrix**)**.** *If A is a square matrix, then the **minor** M_{ij} of the entry a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A. The **cofactor** C_{ij} of the entry a_{ij} is $C_{ij} = (-1)^{i+j}M_{ij}$.*

The minors and cofactors of a matrix can differ only in sign.

Example 1.2. *Find the minors and cofactors of all the entries in the second row.*

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

Now we're ready to define the determinant of an $n \times n$ matrix when $n \geq 2$.

Definition 1.3 (**Determinant of a Square Matrix**)**.** *If A is a square matrix of order $n \geq 2$, then the determinant of A is the sum of the entries in the first row of A multiplied by their respective cofactors. That is,*

$$\det(A) = |A| = \sum_{j=1}^{n} a_{1j}C_{1j} = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.$$

Example 1.3. *Find the determinant of*

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}.$$

Remark 1.1. *Although the determinant is defined as an expansion by the cofactors in the first row, it can be shown that the determinant can be evaluated by expanding in any row or column.*

Theorem 1.1. *Let A be a square matrix of order n. Then the determinant of A is*

$$\det(A) = \sum_{j=1}^{n} a_{ij}C_{ij}$$

or

$$\det(A) = \sum_{i=1}^{n} a_{ij}C_{ij}$$

Because of the previous theorem, when we try to evaluate the determinant of a matrix, we use the row (or column) containing the most zeros, as we do not need to find cofactors of zero entries. The next example demonstrates this idea.

Example 1.4. *Find the determinant of*

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & -2 \end{bmatrix}$$

1.1. **Triangular Matrices.**

A square matrix is **upper triangular** when it has all zero entries below its main diagonal, and **lower triangular** when it has all zero entries above it's main diagonal.

The determinant of a triangular matrix is relatively easy to find, as stated in the next theorem.

Theorem 1.2. *If A is a triangular matrix of order n (i.e., an $n \times n$ matrix), then its determinant is the product of the entries on the main diagonal. That is,*

$$\det(A) = |A| = a_{11}a_{22}\cdots a_{nn}.$$

Example 1.5. *The determinant of the lower triangular matrix*

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ -5 & 6 & 1 & 0 \\ 1 & 5 & 3 & 3 \end{bmatrix}$$

is $|A| = (2)(-2)(1)(3) = -12.$

2. Determinants and Elementary Operations

Using the definition to find the determinant of a matrix is usually tedious. In this section, we will learn how to find it in a more practical way.

First consider the following determinants

$$|A| = \begin{vmatrix} 1 & -2 & 3 & 1 \\ 4 & -6 & 3 & 2 \\ -2 & 4 & -9 & -3 \\ 3 & -6 & 9 & 2 \end{vmatrix}, \text{ and } |B| = \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 2 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

Clearly, the second determinant $|B|$ is easier to calculate, as it's a triangular matrix. In fact, B is an echelon form of A. We know that B is obtained from A by performing elementary row operations, so how does an elementary row operation change the determinant of a matrix?

Theorem 2.1. *Let A and B be square matrices.*

(1) When B is obtained from A by interchanging two rows of A, $\det(B) = -\det(A)$.
(2) When B is obtained from A by adding a multiply of a row of A to another row of A, $\det(B) = \det(A)$.
(3) When B is obtained from A by multiplying a row of A by a nonzero constant c, $\det(A) = c\det(A)$.

With this theorem in hand, we can evaluate the determinant of a matrix in the following way. Use elementary row operations to obtain a triangular matrix B. For each step in the elimination process, use the previous theorem to determine the effect of the elementary row operation on the determinant. Finally, find the determinant of B by multiplying the entries on its main diagonal.

The next example illustrate this process.

Example 2.1. *Find the determinant of*

$$A = \begin{bmatrix} 0 & -7 & 14 \\ 1 & 2 & -2 \\ 0 & 3 & -8 \end{bmatrix}.$$

Although Theorem 2.1 is stated in terms of elementary row operations, the theorem remains valid when the word "column" replaced the word "row". Sometimes it is easier to work with columns rather than rows.

Example 2.2. *Find the determinant of* $A = \begin{bmatrix} -1 & 2 & 2 \\ 3 & -6 & 4 \\ 5 & -10 & -3 \end{bmatrix}$.

In the next theorem, we will look at some special cases, where the determinant of a matrix is equal to 0.

Theorem 2.2. *If A is a square matrix and any one of the conditions below is true, then* $\det(A) = 0$.

(1) An entire row (or an entire column) consists of zeros.
(2) Two rows (or columns) are equal.
(3) One row (or column) is a multiply of another row (or column).

We can't conclude from the previous theorem, that they are the only conditions that produce a zero determinant. On the other hand, when we work on the determinant of a matrix, we can try to reduce the matrix by elementary row (or column) operations to one of the three conditions. If succeeded, then we can claim that the determinant of the original matrix is zero. The next example illustrates this.

Example 2.3. *Find the determinant of*

$$A = \begin{bmatrix} 1 & 4 & 1 \\ 2 & -1 & 0 \\ 0 & 18 & 4 \end{bmatrix}.$$

In general, a square matrix has a determinant of zero if and only if it is row-(or column-) equivalent to a matrix that satisfies one of the conditions in Theorem 2.2.

When evaluating a determinant by hand, we sometimes save steps by using elementary row (or column) operations to create a row (or column) having zeros in all but one position and then using cofactor expansion to reduce the order of the matrix by 1. The next example illustrates this approach.

Example 2.4. *Find the determinant of*

$$A = \begin{bmatrix} -3 & 5 & 2 \\ 2 & -4 & -1 \\ -3 & 0 & 6 \end{bmatrix}.$$

3. Properties of Determinants

In this section, we will learn several important properties of determinants.

Theorem 3.1. *If A and B are square matrices of order n, then* $\det(AB) = \det(A)\det(B)$.

Example 3.1. *Find $|A|, |B|$, and $|AB|$ for the matrices*

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{bmatrix}.$$

Theorem 3.2. *If A is a square matrix of order n and c is a scalar, then the determinant of cA is*

$$\det(cA) = c^n \det(A).$$

Example 3.2. *Find the determinant of the matrix.*

$$A = \begin{bmatrix} 10 & -20 & 40 \\ 30 & 0 & 50 \\ -20 & -30 & 10 \end{bmatrix}$$

Remark 3.1. *In general, we don't have a formula for the determinant of the sum (or difference) of two matrices. For example, if*

$$A = \begin{bmatrix} 6 & 2 \\ 2 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 7 \\ 0 & -1 \end{bmatrix}$$

then $|A| = 2$ and $|B| = -3$, but $A + B = \begin{bmatrix} 9 & 9 \\ 2 & 0 \end{bmatrix}$ and $|A + B| = -18$.

At the beginging of this chapter, we asked the question that how can we tell whether or not a square matrix has a multiplicative inverse. This can be answered using determinants.

Theorem 3.3. *A square matrix A is invertible (non-singular) if and only if* $\det(A) \neq 0$.

Example 3.3. *Determine whether each matrix has an inverse.*

$$\begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

The next theorem provides a way to find the determinant of an inverse matrix.

Theorem 3.4. *If A is an $n \times n$ invertible matrix, then* $\det(A^{-1}) = \frac{1}{\det(A)}$.

Example 3.4. *Find $|A^{-1}|$ for the matrix*

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix}.$$

Let's make a summary of some equivalent conditions for an invertible (or non-singular) matrixk.

Theorem 3.5. *If A i s an $n \times n$ matrix, then the statements below are equivalent.*

(1) A is invertible.

(2) $A\vec{x} = \vec{b}$ *has a unique solution for every* $n \times 1$ *column matrix* $\vec{b}$*.*
(3) $A\vec{x} = \vec{0}$ *has only the trivial solution.*
(4) A *is row-equivalent to* I_n*.*
(5) $\det(A) \neq 0$.

Example 3.5. *Which of the systems has a unique solution?*

$$\begin{aligned} 2x_2 - x_3 &= -1 \\ 3x_1 - 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 - x_3 &= -4 \end{aligned} \qquad \begin{aligned} 2x_2 - x_3 &= -1 \\ 3x_1 - 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 + x_3 &= -4 \end{aligned}$$

The next theorem tells us that the determinant of the transpose of a square matrix is equal to the determinant of the original matrix.

Theorem 3.6. *If* A *is a square matrix, then*

$$\det(A) = \det(A^T).$$

4. Applications of Determinants

Recall that given a linear system $A\vec{x} = \vec{b}$ where A is a square matrix, we can determine whether or not the system has a unique solution by calculating the determinant of A, i.e. $\det(A)$. In this section, we learn a method to actually find that unique solution using the determinants.

Cramer's Rule, to be defined, uses determinants to solve a system of n linear equations in n variables. This rule applies only to systems with unique solutions. More specifically, given a linear system (assuming the system has a unique solution), the value of each variable is the quotient of two determinants. The denominator is the determinant of the coefficient matrix, and the numerator is the determinant of the matrix formed by replacing the column corresponding to the variable being solved for with the column representing the constants. For example, x_3 in the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

is

$$x_3 = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$

Theorem 4.1 (**Cramer's Rule**)**.** *If a system of n linear equations in n variables has a coefficient matrix A with a nonzero determinant $|A|$, then the solution of the system is*

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \ldots, \quad x_n = \frac{\det(A_n)}{\det(A)}$$

where the ith column of A_i is the column of constants in the system of equations.

Example 4.1. *Use Cramer's Rule to solve the system of linear equations for x.*

$$\begin{array}{rrcr} -x & +2y & - & 3z = 1 \\ 2x & & + & z = 0 \\ 3x & -4y & + & 4z = 2 \end{array}$$

Determinants have many applications in analytic geometry. One application is in finding the area of a triangle in the xy-plane.

Theorem 4.2. *The area of a triangle with vertices*

$$(x_1, y_1), (x_2, y_2), \text{ and } (x_3, y_3)$$

is

$$Area = \pm\frac{1}{2} \det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

where the sign ($\pm$) is chosen to give a positive area.

Example 4.2. *Find the area of the triangle whose vertices are*

$$(1, 1), (2, 2), \text{ and } (4, 3).$$

If the three points in the last example were on the same line, what would have happened when you applied the area formula? The answer is that the determinant would have been zero.

Theorem 4.3. *Three points* $(x_1, y_1), (x_2, y_2)$, *and* (x_3, y_3) *are collinear if and only if*

$$\det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 0.$$

The test for collinear points can be adapted to another use. That is, when we are given two points in the xy-plane, we can find an equation of the line passing through the two points, as shown below.

Theorem 4.4. *An equation of the line passing through the distinct points* (x_1, y_1) *and* (x_2, y_2) *is*

$$\det \begin{bmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{bmatrix} = 0.$$

Example 4.3. *Find an equation of the line passing through the points*

$$(2, 4) \text{ and } (-1, 3).$$