Chapter 7: Eigenvalues and Eigenvectors

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1. Eigenvalues and Eigenvectors

Definition 1.1. *Let A be an $n \times n$ matrix. The scalar λ is an* ***eigenvalue*** *of A when there is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$. The vector $\vec{x}$ is an* ***eigenvector*** *of A corresponding to λ.*

Example 1.1. *For the matrix*

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

verify that $\vec{x}_1 = (1, 0)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 2$, and that $\vec{x}_2 = (0, 1)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_2 = -1$.

Example 1.2. *For the matrix*

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

verify that

$$\vec{x}_1 = (-3, -1, 1) \text{ and } \vec{x}_2 = (1, 0, 0)$$

are eigenvectors of A and find their corresponding eigenvalues.

If A is an $n \times n$ matrix with an eigenvalue λ and a corresponding eigenvector $\vec{x}$, then every nonzero scalar multiple of $\vec{x}$ is also an eigenvector of A. It is also true that if $\vec{x}_1$ and $\vec{x}_2$ are eigenvectors corresponding to the same eigenvalue λ, then their sum is also a eigenvector corresponding to λ. In other words, the set of all eigenvectors of an eigenvalue λ, together with the zero vector, is a subspace of $\mathbb{R}^n$.

Theorem 1.1. *If A is an $n \times n$ matrix with an eigenvalue λ, then the set of all eigenvectors of λ, together with the zero vector*

$$\{\vec{x} : \vec{x} \text{ is an eigenvector of } \lambda\} \cup \{\vec{0}\}$$

is a subspace of $\mathbb{R}^n$. This subspace is the ***eigenspace*** *of λ.*

Example 1.3. *Find the eigenvalues and corresponding eigenspaces of $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.*

To find the eigenvalues and eigenvectors of an $n \times n$ matrix A, let I be the $n \times n$ identity matrix. Suppose λ is an eigenvalue and $\vec{x}$ is a corresponding eigenvector, then they must satisfy the equation

$$A\vec{x} = \lambda\vec{x} \text{ or } A\vec{x} = \lambda I\vec{x}$$

rearranging gives

$$(\lambda I - A)\vec{x} = \vec{0}.$$

This homogeneous system of equations has nonzero solutions (by the definition of eigenvectors, $\vec{x}$ must be nonzero) if and only if the coefficient matrix $(\lambda I - A)$ is not invertible—that is, if and only if its determinant is zero.

Theorem 1.2. *Let A be an $n \times n$ matrix.*

(1) *An eigenvalue of A is a scalar λ such that $\det(\lambda I - A) = 0$.*
(2) *The eigenvectors of A corresponding to λ are the nonzero solutions of $(\lambda I - A)\vec{x} = \vec{0}$.*

The equation $\det(\lambda I - A) = 0$ is the **characteristic equation** of A. Moreover, when expanded to polynomial form, the polynomial $|\lambda I - A|$ is the **characteristic polynomial** of A.

Example 1.4. *Find the eigenvalues and corresponding eigenvectors of* $A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$.

A summary of the steps used to find the eigenvalues and corresponding eigenvectors of a matrix is below.

Procedure 1 (Finding Eigenvalues and Eigenvectors)**.** *Let A be an $n \times n$ matrix.*

(1) Form the characteristic equation $|\lambda I - A| = 0$. It will be a polynomial equation of degree n in the variable λ.
(2) Find the real roots of the characteristic equation. These are the eigenvalues of A.
(3) For each eigenvalue λ_i, find the eigenvectors corresponding to λ_i by solving the homogeneous system $(\lambda_i I - A)\vec{x} = \vec{0}$. This can require row reducing an $n \times n$ matrix. The reduced row-echelon form must have at least one row of zeros.

Example 1.5. *Find the eigenvalues and corresponding eigenvectors of*

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

What is the dimension of the eigenspace of each eigenvalue?

If an eigenvalue λ_i occurs as a multiple root (k times) of the characteristic polynomial, then λ_i has **multiplicity** k. In general, the multiplicity of an eigenvalue is greater than or equal to the dimension of its eigenspace.

Example 1.6. *Find the eigenvalues of*

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

and find a basis for each of the corresponding eigenspaces.

Theorem 1.3. *If A is an $n \times n$ triangular matrix, then its eigenvalues are the entries on its main diagonal.*

Example 1.7. *Find the eigenvalues of each matrix.*

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 3 & -3 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

1.1. **Eigenvalues and Eigenvectors of Linear Transformations.**

Eigenvalues and eigenvectors can also be defined in terms of linear transformations. A number λ is an **eigenvalue** of a linear transformation $T : V \to V$ when there is a nonzero vector $\vec{x}$ such that $T(\vec{x}) = \lambda\vec{x}$. The vector $\vec{x}$ is an **eigenvector** of T corresponding to λ, and the set of all eigenvectors of λ (with the zero vector) is the **eigenspace** of λ.

Example 1.8. *Find the eigenvalues and a basis for each corresponding eigenspace of*

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

2. Diagonalization

In this section, we will look at another classic problem in linear algebra called the diagonalization problem. The problem is "for a square matrix A, does there exsit an invertible matrix P such that $P^{-1}AP$ is a diagonal?

Definition 2.1. *An $n \times n$ matrix A is **diagonalizable** when A is similar to a diagonal matrix. That is, A is diagonalizable when there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.*

With this definition, the diagonalization problem can be stated as "which square matrices are diagonalizable?"

Example 2.1. *The matrix*

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

is diagonalizable because

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

has the property that

$$P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

The eigenvalue problem is related closely to the diagonalization problem. The next two theorems shed some light on this relationship.

Theorem 2.1. *If A are B are similar $n \times n$ matrices, then they have the same eigenvalues.*

Example 2.2. *The matrices A and D are similar.*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & -2 & 4 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Find the eigenvalues of A.

Theorem 2.2. *An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.*

Procedure 2 (Steps for Diagonalizing a Square Matrix)**.** *Let A be an $n \times n$ matrix.*

(1) *Find n linearly independent eigenvectors $\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n$ for A (if possible) with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. If n linearly independent eigenvectors do not exist, then A is not diagonalizable.*
(2) *Let P be the $n \times n$ matrix whose columns consists of these eigenvectors. That is, $P = [\vec{p}_1\, \vec{p}_2\, \cdots\, \vec{p}_n]$.*
(3) *The diagonal matrix $D = P^{-1}AP$ will have the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ on its main diagonal. Note that the order of the eigenvectors used to form P will determine the order in which the eigenvalues appear on the main diagonal of D.*

Example 2.3. *Show that the matrix A is diagonalizable.*

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

Then find a matrix P such that $P^{-1}AP$ is diagonal.

Example 2.4. *Show that the matrix A is diagonalizable.*

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

Then find a matrix P such that $P^{-1}AP$ is diagonal.

Example 2.5. *Show that the matrix A is not diagonalizable.*

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

For a square matrix A of order n to be diagonalizable, the sum of the dimensions of the eigenspaces must be equal to n. This can happen when A has n distinct eigenvalues.

Theorem 2.3. *If an $n \times n$ matrix A has n distinct eigenvalues, then the corresponding eigenvectors are linearly independent and A is diagonalizable.*

Example 2.6. *Determine whether the matrix A is diagonalizable.*

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

2.1. **Diagonalization and Linear Transformations.**

In terms of linear transformations, the diagonalization problem can be stated as: For a linear transformation

$$T : V \to V$$

does there exist a basis B for V such that the matrix for T relative to B is diagonal? The answer is "yes" when the standard matrix for T is diagonalizable.

Example 2.7. *Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation represented by*

$$T(x_1, x_2, x_3) = (x_1 - x_2 - x_3, x_1 + 3x_2 + x_3, -3x_1 + x_2 - x_3).$$

If possible, find a basis B for $\mathbb{R}^3$ such that the matrix for T relative to B is diagonal.