Chapter 6: Linear Transformations

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Contents

1. Introduction to Linear Transformations

Recall that a real-valued function is a rule that assigns each number in the domain to a unique number in the codomain. In this section, we will look at a very important category of functions (or transformations), the domain and the codomain of these functions are vector spaces.

Definition 1.1. *Let V and W be vector spaces. The function*

$$T : V \to W$$

*is a **linear transformation** of V into W when the two properties below are true for all $\vec{u}$ and $\vec{v}$ in V and for any scalar c.*

(1) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$
(2) $T(c\vec{u}) = cT(\vec{u})$

Example 1.1. *Show that the function below is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$.*

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$$

Some functions are not linear transformations, as demonstrated in the next example.

Example 1.2. *Consider the following functions from $\mathbb{R}$ to $\mathbb{R}$.*

(1) $f(x) = \sin x$ *is not a linear transformation.*
(2) $f(x) = x^2$ *is not a linear transformation.*
(3) $f(x) = x + 1$ *is not a linear transformation.*

Two simple linear transformations are the **zero transformation** and the **identity transformation**, which are defined below.

(1) $T(\vec{v}) = \vec{0}$, for all $\vec{v}$
(2) $T(\vec{v}) = \vec{v}$, for all $\vec{v}$

Next, we look at some properties about linear transformations.

Theorem 1.1. *Let T be a linear transformation from V into W, where $\vec{u}$ and $\vec{v}$ are in V. Then the properties listed below are true.*

(1) $T(\vec{0}) = \vec{0}$
(2) $T(-\vec{v}) = -T(\vec{v})$
(3) $T(\vec{u} - \vec{v}) = T(\vec{u}) - T(\vec{v})$
(4) *If* $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$, *then*

$$T(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1T(\vec{v}_1) + \cdots + c_nT(\vec{v}_n).$$

In the next two examples, we explore the relationship between linear transformations among $\mathbb{R}^n$ and the linear transformations defined by matrix multiplications.

Example 1.3. *Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that*

$$T(1,0,0) = (2,-1,4)$$

$$T(0,1,0) = (1,5,-2)$$

$$T(0,0,1) = (0,3,1).$$

Find $T(2,3,-2)$.

Example 1.4. *Define the function* $T : \mathbb{R}^2 \to \mathbb{R}^3$ *as*

$$T(\vec{v}) = A\vec{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

(1) Find $T(\vec{v})$ *when* $\vec{v} = (2, -1)$.
(2) Show that T *is a linear transformation from* $\mathbb{R}^2$ *into* $\mathbb{R}^3$.

It turns out that all the linear transformations from $\mathbb{R}^n$ into $\mathbb{R}^m$ are given by matrices.

Theorem 1.2. *Let* A *be an* $m \times n$ *matrix. The function* T *defined by*

$$T(\vec{v}) = A\vec{v}$$

is a linear transformation from $\mathbb{R}^n$ *into* $\mathbb{R}^m$. *In order to conform to matrix multiplication with an* $m \times n$ *matrix,* $n \times 1$ *matrices represent the vectors in* $\mathbb{R}^n$ *and* $m \times 1$ *matrices represent the vectors in* $\mathbb{R}^m$.

Finally, we look at some interesting examples about linear transformations.

Example 1.5. *Show that the linear transformation* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *represented by the matrix*

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

has the property that it rotates every vector in $\mathbb{R}^2$ *counterclockwise about the origin through the angle* θ.

Example 1.6. *The linear transformation* $T : \mathbb{R}^3 \to \mathbb{R}^3$ *represented by*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is a projection in $\mathbb{R}^3$. *If* $\vec{v} = (x, y, z)$ *is a vector in* $\mathbb{R}^3$, *then* $T(\vec{v}) = (x, y, 0)$. *In other words,* T *maps every vector in* $\mathbb{R}^3$ *orthogonal projection in the xy-plane.*

Example 1.7. *Consider* $T : P \to \mathbb{R}$ *defined by*

$$T(p) = \int_a^b p(x)\,dx$$

where p *is a polynomial function. Show that* T *is a linear transformation from* P, *the vector space of polynomial functions, into* $\mathbb{R}$, *the vector space of real numbers.*

2. The Kernel and Range of a Linear Transformation

2.1. **The Kernel of a Linear Transformation.**

Definition 2.1. *Let $T : V \to W$ be a linear transformation. Then the set of all vectors $\vec{v}$ in V that satisfy $T(\vec{v}) = \vec{0}$ is the* ***kernel*** *of T and is denoted by* $\ker(T)$.

Example 2.1. *The kernel of the zero transformation $T : V \to W$ consists of all of V because $T(\vec{v}) = \vec{0}$ for every $\vec{v}$ in V. That is,* $\ker(T) = V$.

The kernel of the identity transformation $T : V \to V$ consists of the single element $\vec{0}$. That is, $\ker(T) = \{\vec{v}\}$.

Example 2.2. *Find the kernel of the projection $T : \mathbb{R}^3 \to \mathbb{R}^3$ represented by $T(x, y, z) = (x, y, 0)$.*

Sometimes, the kernel of a linear transformation is not so obvious.

Example 2.3. *Find the kernel of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ represented by*

$$T(x_1, x_2) = (x_1 - 2x_2, 0, -x_1).$$

Example 2.4. *Find the kernel of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(\vec{x}) = A\vec{x}$, where*

$$A = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix}.$$

The next theorem states that the kernel of every linear transformation $T : V \to W$ is a subspace of V.

Theorem 2.1. *The kernel of a linear transformation $T : V \to W$ is a subspace of the domain V.*

As we mentioned in the last section, all the linear transformations among $\mathbb{R}^n$ are defined by matrices. In the next example, we shows how to find a basis for the kernel of a transformation defined by a matrix.

Example 2.5. *Define $T : \mathbb{R}^5 \to \mathbb{R}^4$ by $T(\vec{x}) = A\vec{x}$, where $\vec{x}$ is in $\mathbb{R}^5$ and*

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix}.$$

Find a basis for $\ker(T)$ *as a subspace of* $\mathbb{R}^5$.

Theorem 2.2. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation $T(\vec{x}) = A\vec{x}$. Then the kernel of T is equal to the solution space of $A\vec{x} = \vec{0}$.*

2.2. **The Range of a Linear Transformation.**

The kernel is one of the two critical subspaces associated with a linear transformation. The other is the range of T.

Theorem 2.3. *The range of a linear transformation $T : V \to W$ is a subspace of W.*

In the next theorem, we look at the range of a linear transformation defined by a matrix.

Theorem 2.4. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation $T(\vec{x}) = A\vec{x}$. Then the column space of A is equal to the range of T.*

The next definition gives the dimensions of the kernel and the range of a linear transformation.

Definition 2.2. *Let $T : V \to W$ be a linear transformation. The dimension of the kernel of T is called the **nullity** of T and is denoted by* $\text{nullity}(T)$*. The dimension of the range of T is called the **rank** of T and is denoted by* $\text{rank}(T)$.

Theorem 2.5. *Let $T : V \to W$ be a linear transformation from an n-dimensional vector space V into a vector space W. Then the sum of the dimensions of the range and kernel is equal to the dimension of the domain. That is,*

$$\text{rank}(T) + \text{nullity}(T) = n.$$

Example 2.6. *Find the rank and nullity of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by the matrix*

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Example 2.7. *Let $T : \mathbb{R}^5 \to \mathbb{R}^7$ be a linear transformation.*

(1) Find the dimension of the kernel of T when the dimension of the range is 2.

(2) Find the rank of T when the nullity of T is 4.

(3) Find the rank of T when $\ker(T) = \{\vec{0}\}$.

Theorem 2.6. *Let $T : V \to W$ be a linear transformation. Then T is one-to-one if and only if* $\ker(T) = \{\vec{0}\}$.

Theorem 2.7. *Let $T : V \to W$ be a linear transformation, where W is finite dimensional. Then T is onto if and only if the rank of T is equal to the dimension of W.*

Theorem 2.8. *Let $T : V \to W$ be a linear transformation with vector spaces V and W, both of dimension n. Then T is one-to-one if and only if it is onto.*

Example 2.8. *Consider the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ represented by $T(\vec{x}) = A\vec{x}$. Find the nullity and the rank of T, and determine whether T is one-to-one, onto, or neither.*

$$a.\ A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad b.\ A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$c.\ A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \qquad d.\ A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

2.3. **Isomorphisms of Vector Spaces.**

Distinct vector spaces such as $\mathbb{R}^3$ and $M_{3,1}$ can be thought of as being “essentially the same”—at least with respect to the operations of vector addition and scalar multiplication. Such spaces are isomorphic to each other.

Definition 2.3. *A linear transformation $T : V \to W$ that is one-to-one and onto is called an **isomorphism**. Moreover, if V and W are vector spaces such that there exists and isomorphism from V to W, then V and W are **isomorphic** to each other.*

Theorem 2.9. *Two finite-dimensional vector spaces V and W are isomorphic if and only if they are of the same dimension.*

Example 2.9. *The vector spaces below are isomorphic to each other.*

(1) $\mathbb{R}^4$
(2) $M_{4,1}$
(3) $M_{2,2}$
(4) P_3
(5) $V = \{(x_1, x_2, x_3, x_4, 0) : x_i \text{ is a real number}\}$

3. Matrices for Linear Transformations

In this section, we will see that for linear transformation involving finite-dimensional vector spaces, matrix representation is always possible.

The key to representing a linear transformation $T : V \to W$ by a matrix is to determine how it acts on a basis for V.

Theorem 3.1. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation such that, for the standard basis vectors $\vec{e}_i$ of $\mathbb{R}^n$,*

$$T(\vec{e}_1) = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, T(\vec{e}_2) = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \dots, T(\vec{e}_n) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

Then the $m \times n$ matrix whose n columns correspond to $T(\vec{e}_i)$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

*is such that $T(\vec{v}) = A\vec{v}$ for every $\vec{v}$ in $\mathbb{R}^n$. A is called the **standard matrix** for T.*

Example 3.1. *Find the standard matrix for the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by*

$$T(x, y, z) = (x - 2y, 2x + y).$$

The next theorem emphasizes the usefulness of matrices for representing linear transformations.

Theorem 3.2. *Let $T_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $T_2 : \mathbb{R}^m \to \mathbb{R}^p$ be linear transformations with standard matrices A_1 and A_2, respectively. The **composition** $T : \mathbb{R}^n \to \mathbb{R}^p$, defined by $T(\vec{v}) = T_2(T_1(\vec{v}))$, is a linear transformation. Moreover, the standard matrix A for T is the matrix product*

$$A = A_2 A_1.$$

Example 3.2. *Let T_1 and T_2 be linear transformations from $\mathbb{R}^3$ into $\mathbb{R}^3$ such that*

$$T_1(x, y, z) = (2x + y, 0, x + z) \quad \text{and} \quad T_2(x, y, z) = (x - y, z, y).$$

Find the standard matrices for the compositions $T = T_2 \circ T_1$ and $T' = T_1 \circ T_2$.

Another benefit of matrix representation is that it can represent the inverse of a linear transformation.

Definition 3.1. *If $T_1 : \mathbb{R}^n \to \mathbb{R}^n$ and $T_2 : \mathbb{R}^n \to \mathbb{R}^n$ are linear transformations such that for every $\vec{v}$ in $\mathbb{R}^n$,*

$$T_2(T_1(\vec{v})) = \vec{v} \quad \text{and} \quad T_1(T_2(\vec{v})) = \vec{v}$$

*then T_2 is the **inverse** of T_1, and T_1 is said to be **invertible.***

Theorem 3.3. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with standard matrix A. Then the conditions listed below are equivalent.*

(1) T is invertible.
(2) T is an isomorphism.

(3) A is invertible.

If T is invertible with standard matrix A, then the standard matrix for T^{-1} is A^{-1}.

Example 3.3. *Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by*

$$T(x_1, x_2, x_3) = (2x_1 + 3x_2 + x_3, 3x_1 + 3x_2 + x_3, 2x_1 + 4x_2 + x_3).$$

Show that T is invertible, and find its inverse.

Finally, we will consider the more general problem of finding a matrix for a linear transformation $T : V \to W$, where B and B' are ordered bases for V and W, respectively.

Theorem 3.4. *Let V and W be finite-dimensional vector spaces with bases B and B', respectively, where*

$$B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}.$$

If $T : V \to W$ is a linear transformation such that

$$[T(\vec{v}_1)]_{B'} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, [T(\vec{v}_2)]_{B'} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \dots, [T(\vec{v}_n)]_{B'} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

Then the $m \times n$ matrix whose n columns correspond to $[T(\vec{v}_i)]_{B'}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is such that $[T(\vec{v})]_{B'} = A[\vec{v}]_B$ for every $\vec{v}$ in V.

Example 3.4. *Let $D_x : P_2 \to P_1$ be the differential operator that maps a polynomial p of degree 2 or less onto its derivative p'. Find the matrix for D_x using the bases*

$$B = \{1, x, x^2\} \quad \textit{and} \quad B' = \{1, x\}.$$

Example 3.5. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by $T(x_1, x_2) = (x_1 + x_2, 2x_1 - x_2)$. Find the matrix for T relative to the bases.*

$$B = \{(1, 2), (-1, 1)\} \quad \textit{and} \quad B' = \{(1, 0), (0, 1)\}.$$