Chapter 4: Vector Spaces

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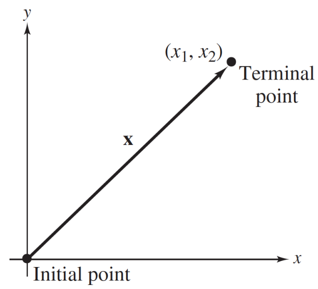
Contents

1. Vectors in $\mathbb{R}^n$

First we look at vectors in $\mathbb{R}^2$ (the plane). Geometrically, a vector in the plane is represented by a directed line segment with its initial point at the origin and its terminal point at (x_1, x_2), as shown below.



The same ordered pair used to represent its terminal point also represents the vector. That is, $\vec{x} = (x_1, x_2)$ or $\mathbf{x} = (x_1, x_2)$. Sometimes we write vectors in columns instead of rows, for example, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. The coordinates x_1 and x_2 are the components of the vector $\vec{x}$. Two vectors $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$ are **equal** if and only if

$$u_1 = v_1 \text{ and } u_2 = v_2.$$

Example 1.1. *Use directed line segments to represent the following vectors in the plane:* $\mathbf{u} = (2, 3)$ *and* $\vec{v} = (-1, 2)$.

A quick review of the vectors (or matrices) addition and scalar multiplication. Given two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$, their **sum** is the vector $\mathbf{u}+\mathbf{v}$ obtained by adding corresponding entries of two vectors. Given a vector $\mathbf{u}$ and a real number c, the **scalar multiple** of $\mathbf{u}$ by c is the vector $c\mathbf{u}$ obtained by multiplying each entry in the vector by c.

The sum of two vectors has a useful geometric representation.

Theorem 1.1 (Parallelogram Rule for Addition)**.** *If* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^2$ *are represented as points in the plane, then* $\mathbf{u}+\mathbf{v}$ *corresponds to the fourth vertex of the parallelogram whose other vertices are* $\mathbf{u}, \mathbf{0}$ *and* $\mathbf{v}$*. See Figure 1.*

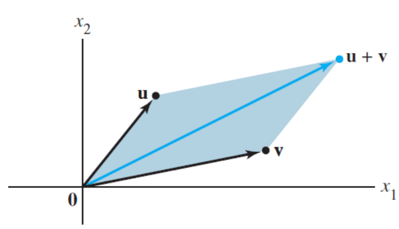


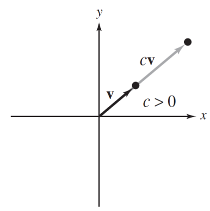
Figure 1. The parallelogram rule.

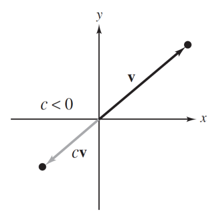
As before, we use $\vec{0}$ (or $\mathbf{0}$) to represent the zero vector $(0, 0)$ (or $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$).

Example 1.2. *Find each vector sum* $\mathbf{u}+\mathbf{v}$ *using the Parallelogram Rule.*

(1) $\vec{u} = (1, 4), \vec{v} = (2, -2)$.
(2) $\vec{u} = (3, -2), \vec{v} = (-3, 2)$.
(3) $\vec{u} = (2, 1), \vec{0} = \vec{0}$.

Just like addition, there is a geometric representation for scalar multiplication. For instance, when a vector $\vec{v}$ is multiplied by 2, the resulting vector $2\vec{v}$ is a vector having the same direction as $\vec{v}$ and twice the length. In general, for a scalar c, the vector $c\mathbf{v}$ will be $|c|$ times as long as $\mathbf{v}$. If c is positive, then $c\mathbf{v}$ and $\mathbf{v}$ have the same direction, and if c is negative, then $c\mathbf{v}$ and $\mathbf{v}$ have opposite directions (see the figure below).





The product of a vector $\mathbf{v}$ and scalar -1 is denoted by

$$-\mathbf{v} = (-1)\mathbf{v}.$$

The vector $-\mathbf{v}$ is the negative of $\mathbf{v}$. The **difference** of $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}).$$

So we can also represent the subtraction of vectors using the Parallelogram Rule. First draw the vector $\mathbf{u}$ and the "negated" vector $-\mathbf{v}$ on the plane. They will form the two adjacent sides of a parallelogram, then $\mathbf{u} - \mathbf{v}$ is the main diagonal of the parallelogram.

Example 1.3. *Let $\vec{v} = (-2, 5)$ and $\vec{u} = (3, 4)$. Perform each vector operation.*

$$\textit{(1)}\ \frac{1}{2}\vec{v} \qquad \textit{(2)}\ \vec{u} - \vec{v} \qquad \textit{(3)}\ \frac{1}{2}\vec{v} + \vec{u}$$

Vector addition and scalar multiplication share many properties with matrix addition and scalar multiplication.

Theorem 1.2. *Let $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ be vectors in the plane, and let c and d be scalars.*
(1) $\mathbf{u} + \mathbf{v}$ *is a vector in the plane.*
(2) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
(3) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
(4) $\mathbf{u} + \mathbf{0} = \mathbf{u}$
(5) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
(6) $c\mathbf{u}$ *is a vector in the plane.*
(7) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
(8) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(9) $c(d\mathbf{u}) = (cd)\mathbf{u}$
(10) $1(\mathbf{u}) = \mathbf{u}$

The discussion of vectors in the plane can be extended to a discussion of vectors in n-space. An **ordered n-tuple** represents a vector in n-space. For instance, an ordered triple has the form (x_1, x_2, x_3), an ordered quadruple has the form (x_1, x_2, x_3, x_4), and a general ordered n-tuple has the form $(x_1, x_2, x_3, \ldots, x_n)$. The set of all n-tuples is n-space and is denoted by $\mathbb{R}^n$.

An n-tuple $(x_1, x_2, x_3, \ldots, x_n)$ can be viewed as a **point** in $\mathbb{R}^n$ with the x_i's as its coordinates, or as a **vector** $\vec{x} = (x_1, x_2, x_3, \ldots, x_n)$ with the x_i's as its components. The equality between two vectors in $\mathbb{R}^n$ is defined similarly, say being component-wisely equal.

The sum and the scalar multiplication are defined below (which is identical to the matrix addition and scalar multiplication)

Definition 1.1. *Let $\vec{u} = (u_1, u_2, \ldots, u_n)$ and $\vec{v} = (v_1, v_2, \ldots, v_n)$ be vectors in $\mathbb{R}^n$ and let c be a real number. the sum of $\vec{u}$ and $\vec{v}$ is the vector*

$$\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$$

and the scalar multiple of $\vec{u}$ by c is the vector

$$c\vec{u} = (cu_1, cu_2, \ldots, cu_n).$$

The negation $-\vec{u}$ and the difference $\vec{u} - \vec{v}$ are defined similarly as in the $\mathbb{R}^2$. And the zero vector in $\mathbb{R}^n$ is also denoted by $\vec{0}$ (or $\mathbf{0}$).

Example 1.4. *Let $\vec{u} = (-1, 0, 1)$ and $\vec{v} = (2, -1, 5)$ in $\mathbb{R}^3$. Perform each vector operation.*

(1) $\vec{u} + \vec{v}$
(2) $2\vec{u}$
(3) $\vec{v} - 2\vec{u}$

For $\mathbb{R}^n$, there is an identical version of Theorem 1.2, we will not list them here. Actually, in the future sections, we will revisit these properties and make use of them in defining a vector space.

The zero vector $\vec{0}$ in $\mathbb{R}^n$ is the **additive identity** in $\mathbb{R}^n$. Similarly, the vector $-\vec{v}$ is the **additive inverse** of $\vec{v}$. The next theorem summarizes several important properties of the additive identity and additive inverse in $\mathbb{R}^n$.

Theorem 1.3. *Let $\vec{v}$ be a vector in $\mathbb{R}^n$, and let c be a scalar. Then the properties below are true.*

(1) The additive identity is unique. That is, if $\vec{v} + \vec{u} = \vec{v}$, then $\vec{u} = \vec{0}$.
(2) The additive inverse of $\vec{v}$ is unique. That is, if $\vec{v} + \vec{u} = \vec{0}$, then $\vec{u} = -\vec{v}$.
(3) $0\vec{v} = \vec{0}$
(4) $c\vec{0} = \vec{0}$
(5) If $c\vec{v} = \vec{0}$, then $c = 0$ or $\vec{v} = \vec{0}$.
(6) $-(-\vec{v}) = \vec{v}$

Finally, let us recall the definition of linear combinations. Suppose $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, and $\vec{x}$ are vectors in $\mathbb{R}^n$, we say that $\vec{x}$ is a linear combination of $\vec{v}_1, \ldots, \vec{v}_n$ if there exists scalars $c_1, c_2, \ldots, c_n$, such that

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n.$$

In the previous chapters, we've seen linear combinations in terms of column vectors, starting from this chapter, we can also refer linear combinations in terms of row vectors.

Example 1.5. *Let* $\vec{x} = (-1, -2, -2)$*,* $\vec{u} = (0, 1, 4), \vec{v} = (-1, 1, 2)$*, and* $\vec{w} = (3, 1, 2)$ *in* $\mathbb{R}^3$*. Find scalars* $a, b,$ *and* c *such that*

$$\vec{x} = a\vec{u} + b\vec{v} + c\vec{w}.$$

2. Vector Spaces

Theorem 1.2 lists ten properties of vector addition and scalar multiplication in $\mathbb{R}^n$. Suitable definitions of addition and scalar multiplication reveal that many other mathematical quantities (such as matrices, polynomials, and functions) also share these ten properties. Any set that satisfies these properties (or **axioms**) is called a **vector space**, and the objects in the set are **vectors**.

Definition 2.1 (**Vector Space**). *Let V be a set on which two operations (**vector addition** and **scalar multiplication**) are defined. If the listed axioms are satisfied for every $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ in V and every scalar (real number) c and d, then V is a **vector space**.*

(1) $\mathbf{u}+\mathbf{v}$ *is in* V.
(2) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
(3) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$
(4) *V has a **zero vector** $\mathbf{0}$ such that for every $\mathbf{u}$ in V, $\mathbf{u}+\mathbf{0}=\mathbf{u}$.*
(5) *For every $\mathbf{u}$ in V, there is a vector in V denoted by $-\mathbf{u}$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.*
(6) $c\mathbf{u}$ *is in* V.
(7) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$
(8) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$
(9) $c(d\mathbf{u})=(cd)\mathbf{u}$
(10) $1(\mathbf{u})=\mathbf{u}$

It is important to realize that a vector space consists of four entities: a set of vectors, a set of scalars, and two operations. When you refer to a vector space V, be sure that all four entities are clearly stated or understood.

Example 2.1. *The set of all ordered pairs of real numbers $\mathbb{R}^2$ with the standard operations is a vector space. The set of all ordered n-tuples of real numbers $\mathbb{R}^n$ with the standard operations is a vector space.*

In the next three examples, we will describe the set V (vector space) and defines the two vector operations. To show that the set is a vector space, we must verify all ten axioms.

Example 2.2. *Show that the set of all 2×3 matrices with the operations of matrix addition and scalar multiplication is a vector space.*

Example 2.3. *Let P_2 be the set of all polynomials of the form $p(x)=a_0+a_1x+a_2x^2$, where a_0, a_1, and a_2 are real numbers. The sum of two polynomials $p(x)=a_0+a_1x+a_2x^2$ and $q(x)=b_0+b_1x+b_2x^2$ is defined in the usual way,*

$$p(x)+q(x)=(a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2$$

and the scalar multiple of $p(x)$ by the scalar c is defined by

$$cp(x)=ca_0+ca_1x+ca_2x^2.$$

Show that P_2 is a vector space.

Remark 2.1. *P_n is defined as the set of all polynomials of degree n or less. The procedure used to verify that P_2 is a vector space can be extended to show that P_n is also a vector space.*

Example 2.4. *Let $C(-\infty, \infty)$ be the set of all real-valued continuous functions defined on the entire real line. This set consists of all polynomial functions and all other continuous functions on the entire real line.*

Addition is defined by

$$(f + g)(x) = f(x) + g(x).$$

Scalar multiplication is defined by

$$(cf)(x) = c[f(x)]$$

Show that $C(-\infty, \infty)$ is a vector space.

We've seen the versatility of the concept of a vector space. For instance, a vector an be a real number, an n-tuple, a matrix, a polynomial, or a continuous function, and so on. But what is the purpose of this abstraction, and why bother to define it?

The textbook mentioned one important reason (on page 164, first paragraph), here is another one, the coordinate systems. For example, in the xy-plane, or $\mathbb{R}^2$, each point (or vector) is represented by a pair of numbers (called coordinates), and we can operate among the vectors using basic arithmetic for real numbers.

For a more general vector space, consider the space of all polynomials of order less than or equal to 2, i.e. P_2 as in the previous example. We want to represent these polynomials using just triples (x_1, x_2, x_3) from $\mathbb{R}^3$, so that each polynomial is associated with a unique triple and then we can deal with arithmetic of real numbers instead of operations among functions. There are infinitely many such polynomials, and how do we know that the set of triples will be sufficient to differentiate all of them? Why is the set of pairs from $\mathbb{R}^2$ not enough, and why the set of quadruples from $\mathbb{R}^4$ will be redundant? We can answer these questions in the future.

In the next theorem, we will see the first theorem about a general vector space.

Theorem 2.1. *Let $\mathbf{v}$ be any element of a vector space V, and let c be any scalar. Then the properties below are true.*

(1) $0\mathbf{v} = \mathbf{0}$
(2) $c\mathbf{0} = \mathbf{0}$
(3) If $c\mathbf{v} = \mathbf{0}$, then $c = 0$ or $\mathbf{v} = \mathbf{0}$.
(4) $(-1)\mathbf{v} = -\mathbf{v}$

In the next two examples, we demonstrate some spaces that are not vector spaces.

Example 2.5. *The set of all integers (with the standard operations) does not form a vector space.*

Example 2.6. *The set of all second-degree polynomials is not a vector space.*

Example 2.7. *Let $V = \mathbb{R}^2$, the set of all ordered pairs of real numbers, with the standard operation of addition and the nonstandard definition of scalar multiplication listed below.*

$$c(x_1, x_2) = (cx_1, 0)$$

Show that V is not a vector space.

3. Subspaces of Vector Spaces

Sometimes we're interested in a subset of a vector space. We say that a nonempty subset of a vector space is a subspace when it is still a vector space with the same operations defined in the original vector space.

Definition 3.1. *A nonempty subset W of a vector space V is a* ***subspace*** *of V when W is a vector space under the operations of addition and scalar multiplication defined in V.*

Essentially, a subspace W of a vector space V is a vector space in its own right, with the two operations (addition and scalar multiplication) adopted from the vector space V. In the next example, we show that a subset of a vector space is a subspace by verifying all ten properties in the definition of vector spaces.

Example 3.1. *Show that the set $W = \{(x_1, 0, x_3) :\ x_1 \text{ and } x_3 \text{ are real numbers}\}$ is a subspace of $\mathbb{R}^3$ with the standard operations.*

From the last example, we see that all but two properties are intrinsically true for any subsect of a vector space, so the next theorem will shorten the process of showing some subset of a vector space is a subspace.

Theorem 3.1 (Test for Subspace). *If W is a nonempty subset of a vector space V, then W is a subspace of V if and only if the two closure conditions listed below hold.*

(1) If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\mathbf{u} + \mathbf{v}$ is in W.
(2) If $\mathbf{u}$ is in W and c is any scalar, then $c\mathbf{u}$ is in W.

Example 3.2. *Let W be the set of all 2×2 symmetric matrices. Show that W is a subspace of the vector space $M_{2,2}$, with the standard operations of matrix addition and scalar multiplication.*

Next we look at some examples of non-subspaces. In general, to show some set W is not a subspace, we either show that there exists two vectors $\mathbf{u}$ and $\mathbf{v}$ from W such that their sum $\mathbf{u} + \mathbf{v}$ is not in the set W anymore, i.e., the addition operation is not "closed"; or we can try to find a scalar c and a vector $\mathbf{u}$ such that $c\mathbf{u}$ is not in the set W, that is, the scalar multiplication is not "closed".

Example 3.3. *Let W be the set of singular matrices of order 2. Show that W is not a subspace of $M_{2,2}$ with the standard operations.*

Example 3.4. *Show that $W = \{(x_1, x_2) : x_1 \geq 0 \text{ and } x_2 \geq 0\}$, with the standard operations, is not a subspace of $\mathbb{R}^2$.*

Example 3.5. *Let W_5 be the vector space of all functions defined on $[0, 1]$, and let W_1, W_2, W_3, and W_4 be defined as shown below.*

- *W_1 = set of all polynomial functions that are defined on $[0, 1]$*
- *W_2 = set of all functions that are differentiable on $[0, 1]$*
- *W_3 = set of all functions that are continuous on $[0, 1]$*

Show that $W_1 \subset W_2 \subset W_3$ and that W_i is a subspace of W_j for $i \leq j$.

Given a vector space V, suppose we have two subspaces of V, say X and Y, can we find a subspace of V that is "smaller" than both of X and Y? Naturally, to find a set that is smaller than both X and Y, we can take the intersection $X \cap Y$, but do we still have a vector space by doing this? The next theorem confirms this.

Theorem 3.2. *If V and W are both subspaces of a vector space U, then the intersections of V and W (denoted by $V \cap W$) is also a subspace of U.*

3.1. Subspaces of $\mathbb{R}^n$.

Now we look at some subspaces and non-subspaces of $\mathbb{R}^n$.

Example 3.6. *Determine whether each subset is a subspace of $\mathbb{R}^2$.*

a. The set of points on the line $x + 2y = 0$
b. The set of points on the line $x + 2y = 1$

Notice that in the previous example, both equations are linear, however, the second equation is not a homogeneous equation. Can you give some examples of subspaces in $\mathbb{R}^3$? What about $\mathbb{R}^n$?

For the next example, it's a non-subspace. Pay attention to the equation, is that linear?

Example 3.7. *Show that the subset of $\mathbb{R}^2$ consisting of all points on $x^2 + y^2 = 1$ is not a subspace.*

Example 3.8. *Determine whether each subset is a subspace of $\mathbb{R}^3$.*

(1) $W = \{(x_1, x_2, 1) : x_1 \text{ and } x_2 \text{ are real numbers}\}$
(2) $W = \{(x_1, x_1 + x_3, x_3) : x_1 \text{ and } x_3 \text{ are real numbers}\}$