# HW2 due 11:30a Mon Oct 10

# 1. Abstract vector spaces

Let $\rho(x) = \left\{ \sum_{\ell=1}^{n} c_\ell x^\ell \mid n \in \mathbb{N},\ c_1, \ldots, c_n \in \mathbb{R} \right\}$denote the set of polynomials in the variable $x$.

a. Provide an injection from $\rho(x)$ into $\mathbb{R}^{\mathbb{N}}$.

We need to provide an injection from $\rho(x)$ to $\mathbb{R}^{\mathbb{N}}$. To that end, select arbitrarily $p \in \rho(x)$. Then by the definition of $\rho(x)$, $p$ is of the form:

$$p = c_1 x + c_2 x^2 + \ldots + c_n x^n$$

where $c_i \in \mathbb{R}$ and $n \in \mathbb{N}$ and $n$ is the largest power of $x$ in $p$ whose coefficient is nonzero. Also, define a subset of $\mathbb{N}$, call it $N_p = \{1, 2, \ldots, n\}$.

Now consider the mapping $f : \rho(x) \to \mathbb{R}^{\mathbb{N}}$. This mapping takes our arbitrary element in $\rho(x)$, $p = c_1 x + c_2 x^2 + \ldots + c_n x^n$, and maps it to a function $g : \mathbb{N} \to \mathbb{R}$ where for $i \in \mathbb{N}$:

$$g(i) = \begin{cases} c_i & i \in N_p \\ 0 & \text{else} \end{cases}$$

This maps the power of any term in $p$ to its coefficient, and any other natural number to zero. Note that $g$ is a function since it maps any arbitrary element of domain $\mathbb{N}$ to precisely one element in $\mathbb{R}$, i.e. $g$ is well-defined, the domain of $g$ is $\mathbb{N}$, the codomain $\mathbb{R}$, and $(n, r_1), (n, r_2) \in g \implies r_1 = r_2$.

We claim that $f$ is an injection from $\rho(x)$ into $\mathbb{R}^{\mathbb{N}}$. First observe that $f$ is well-defined, since it maps an arbitrary element of $\rho(x)$ to prescisely one element in $\mathbb{R}^{\mathbb{N}}$, namely $g$ as defined above.

To show that $f$ is an injection, we need to show that $(p_1, \hat{g}), (p_2, \hat{g}) \in f \implies p_1 = p_2$. We proceed with a contradiction strategy and first assume the contrary, that $(p_1, \hat{g}), (p_2, \hat{g}) \in f$ and $p_1 \neq p_2$.

Consider that $p_1$ and $p_2$ can be expressed as:

$$p_1 = c_1 x + c_2 x^2 + \ldots. + c_q x^q$$
$$p_2 = k_1 x + k_2 x^2 + \ldots + k_r x^r$$

where $q, r$ are natural numbers again corresponding to the largest power of $x$ in $p_1, p_2$ respectively with nonzero coefficients, and $c_i, k_i$ are reals.

Then two cases arise:

1. $q = r$
2. $q \neq r$

We consider the former case, that $q = r$. Well then $N_{p_1} = N_{p_2}$, and since $p_1 \neq p_2$, it must be the case that $\exists i \in N_{p_1} = N_{p_2} \ni k_i \neq c_i$. But this means that both $(i, k_i)$ and $(i, c_i)$ are in $\hat{g}$. But we showed that $\hat{g}$ is a function and cannot map one element of the domain to multiple elements in the codomain. Our contradiction is thus established.

Now suppose the latter, that $q \neq r$. Then WLOG, let $q > r$. By the way we constructed $f$, $p_1$ is mapped to a function $\hat{g}$ such that $\hat{g}(q) = c_q \neq 0$. In the same way, $f$ maps $p_2$ to a function $\hat{g}$, which maps $q$ to $0$. But we showed that $\hat{g}$ is a function and cannot map one element of the domain to multiple elements in the codomain. Our contradiction is thus established.

Since we showed that assuming the contrary to our claim resulted in a contradiction, we have shown the original claim, that $f$ is an injection, as desired.

b. Endow $\rho(x)$ and $\mathbb{R}^{\mathbb{N}}$ with abstract vector space structures. (You need to define the operations of vector addition and scalar multiplication for the two sets, and show that the two sets are closed under their respective operations.)

We begin with $\rho(x)$. Now let $\alpha$ be a scalar in the field of real numbers, $\mathbb{R}$. Then we propose that, given arbitrary $p \in \rho(x)$ (i.e. $p = c_1 x + c_2 x^2 + .... + c_n x^n$, where $n \in \mathbb{N}$ and $c_i \in \mathbb{R}$) that scalar multiplication be defined as:

$$\alpha p = \alpha c_1 x + \alpha c_2 x^2 + .... + \alpha c_n x^n$$

We want to show that $\rho(x)$ is closed under this operation. Observe that $\alpha c_1, \alpha c_2, ..., \alpha c_n$ are also real numbers. Denote these real numbers as $\alpha c_i = d_i$. Then $\alpha p = d_1 x + d_2 x^2 + ... + d_n x^n = \sum_{\ell=1}^{n} d_\ell x^\ell$ where $n \in \mathbb{N}$, $d_1, \dots, d_n \in \mathbb{R}$, and is thus an element of $\rho(x)$. Thus, $\rho(x)$ over the field of reals is closed under our multiplication.

Next, we want to define addition. To that end, let $p_1, p_2$ be arbitrary elements of $\rho(x)$. Now consider that $p_1$ and $p_2$ can be expressed as:

$$p_1 = c_1 x + c_2 x^2 + .... + c_q x^q$$
$$p_2 = k_1 x + k_2 x^2 + ... + k_r x^r$$

where $q, r$ are natural numbers again corresponding to the largest power of $x$ in $p_1, p_2$ respectively with nonzero coefficients, and $c_i, k_i$ are reals.

Then two cases arise:

1. $q = r$
2. $q \neq r$

We consider the former case, that $q = r$. Well then $N_{p_1} = N_{p_2}$. Then define the addition as follows: \begin{equation} p*1+p_2 = (c_1+k_1)x+ (c_2+k_2)x^2+...+(c_q+k_q)x^q \end{equation} Since $c_i, k_i$ are all real numbers, so is their sum. Thus, $p_1+p_2 = \sum*{\ell=1}^q (c\ell+k\ell) x^\ell where q\in\mathbb{N},\ c_1+k_1,\dots,c_q+k_q \in\mathbb{R} and is an element of \rho(x)$.

Now suppose that $q \neq r$. Then WLOG, let $q > r$. Then define the addition as follows: \begin{equation} p*1+p_2 = (c_1+k_1)x+ (c_2+k_2)x^2+...+(c_r+k_r)x^r+c*{r+1}x^{r+1}+...+c*qx^q \end{equation} Now let $s_i = c_i + k_i$ for $i \in \{1,2,\dots,r\}$, and let $s_i = c_i$ for $i \in \{r+1, r+2, \dots, q\}$. Then the set of all $s_i$ is a set of real numbers, and $p_1+p_2 = \sum*{\ell=1}^q s_\ell x^\ell where q\in\mathbb{N}, s_1\dots,s_q\in\mathbb{R}, and is thus an element of \rho(x)$.

We have thus provided a scalar multiplication and addition for $\rho(x)$ and demonstrated closure.

Now consider $\mathbb{R}^{\mathbb{N}}$. Let $f$ be an element of $\mathbb{R}^{\mathbb{N}}$. Then $f \subseteq \mathbb{N} \times \mathbb{R}$. Then $f$ can be represented as $\{(1, r_1), (2, r_2), \dots\}$ where $r_i$ is a real number. Let $\alpha$ be a scalar in the field of reals, $\mathbb{R}$. Then define scalar multiplication $\alpha f$ such that $\alpha f$ is the set $\{(1, \alpha r_1), (2, \alpha r_2), \dots\}$. Note that the $\alpha r_i$ are real numbers. Observe that $\alpha f$ is clearly also a function (we don't lose well-defined) from the naturals to the reals, and is thus an element of $\mathbb{R}^{\mathbb{N}}$.

Now let $f, g$ be arbitrary elements of $\mathbb{R}^{\mathbb{N}}$. Then $f$ can be represented as $\{(1, r_1), (2, r_2), \dots\}$ and $g$ can be represented as $\{(1, u_1), (2, u_2), \dots\}$ where $r_i, u_i$ are real numbers. Then let $f + g = \{(1, r_1 + u_1), (2, r_2 + u_2), \dots\}$. Observe that $r_i + u_i$ is a real number. We have that $g + f$ is also a function from the naturals to the reals, and is thus an element of $\mathbb{R}^{\mathbb{N}}$.

We have thus provided a scalar multiplication and addition for $\mathbb{R}^{\mathbb{N}}$ and demonstrated closure.

c. Is your injection from (a.) compatible with your vector space structure from (b.)? (In other words, is the result of applying the injection to a scaled sum in $\rho(x)$ the same as the corresponding scaled sum in $\mathbb{R}^{\mathbb{N}}$?)

We claim that our injection from (a.) is indeed compatible with the vector space structure from (b.). To that end, let $\alpha, \beta$ be real scalars, and let $p_1, p_2$ be arbitrary elements of $\rho(x)$.

Consider that $p_1$ and $p_2$ can be expressed as:

$$p_1 = c_1 x + c_2 x^2 + \ldots. + c_q x^q$$
$$p_2 = k_1 x + k_2 x^2 + \ldots + k_r x^r$$

where $q, r$ are natural numbers again corresponding to the largest power of $x$ in $p_1, p_2$ respectively with nonzero coefficients, and $c_i, k_i$ are reals.

Define $N_{p_1} = \{1, 2, \ldots, q\}$ and $N_{p_2} = \{1, 2, \ldots, r\}$.

Now consider the mapping $f : \rho(x) \to \mathbb{R}^{\mathbb{N}}$ from part (a.). $f(p_1)$ maps to a function $g_1$, where

$$g_1(i) = \begin{cases} c_i & i \in N_{p_1} \\ 0 & \text{else} \end{cases}$$

and $f(p_2)$ maps to a function $g_2$

$$g_2(i) = \begin{cases} k_i & i \in N_{p_2} \\ 0 & \text{else} \end{cases}$$

Recall $p_1, p_2 \in \rho(x)$, and consider $\alpha p_1 + \beta p_2$. Well, we know that

$$\alpha p_1 = \alpha c_1 x + \alpha c_2 x^2 + \ldots. + \alpha c_q x^q = d_1 x + d_2 x^2 + \ldots. + d_q x^q$$

where $d_i = \alpha c_i$. Similarly, we know that

$$\beta p_2 = \beta k_1 x + \beta k_2 x^2 + \ldots. + \beta k_r x^r = t_1 x + t_2 x^2 + \ldots. + t_r x^r$$

where $t_i = \beta k_i$.

Then two cases arise:

1. $q = r$
2. $q \neq r$

Begin with the former. If $q = r$, then $\alpha p_1 + \beta p_2 = (d_1 + t_1)x + (d_2 + t_2)x^2 + \ldots + (d_q + t_r)x^q$. Also, if $q = r$, then $N_{p_1} = N_{p_2}$.

Then $f(\alpha p_1 + \beta p_2)$ maps to a function, call it $g'$, such that

$$g'(i) = \begin{cases} d_i + t_i = \alpha c_i + \beta k_i & i \in N_{p_1} \\ 0 & \text{else} \end{cases}$$

Now consider $\alpha g_1 + \beta g_2$. Well,

$$\alpha g_1(i) = \begin{cases} \alpha c_i & i \in N_{p_1} \\ 0 & \text{else} \end{cases}$$

and

$$\beta g_2(i) = \begin{cases} \beta k_i & i \in N_{p_2} \\ 0 & \text{else} \end{cases}$$

But this means that

$$(\alpha g_1 + \beta g_2)(i) = \begin{cases} \alpha c_i + \beta k_i & i \in N_{p_1} = N_{p_2} \\ 0 & \text{else} \end{cases}$$

But this is precisely the result of $g' = f(\alpha p_1 + \beta p_2)$. Thus for this case we have shown that $f(\alpha p_1 + \beta p_2) = \alpha f(p_1) + \beta f(p_2)$.

Now consider the case where $q \neq r$. Then WLOG, let $q > r$. Then $\alpha p_1 + \beta p_2 = (d_1 + t_1)x + (d_2 + t_2)x^2 + \ldots + (d_r + t_r)x^r + d_{r+1}x^{r+1} + \ldots + d_q x^q$. Also observe that $N_{p_2} \subset N_{p_1}$. Then $f(\alpha p_1 + \beta p_2)$ maps to a function call if $g^*$.

$$g^*(i) = \begin{cases} d_i + t_i = \alpha c_i + \beta k_i & i \in N_{p_2} \\ d_i = \alpha c_i & i \in N_{p_1} \setminus N_{p_2} \\ 0 & \text{else} \end{cases}$$

Now consider $\alpha g_1 + \beta g_2$. Well,

$$\alpha g_1(i) = \begin{cases} \alpha c_i & i \in N_{p_1} \\ 0 & \text{else} \end{cases}$$

and

$$\beta g_2(i) = \begin{cases} \beta k_i & i \in N_{p_2} \\ 0 & \text{else} \end{cases}$$

Then by the way defined addition in $\mathbb{R}^{\mathbb{N}}$ and since $N_{p_2} \subset N_{p_1}$, we have that

$$(\alpha g_1 + \beta g_2)(i) = \begin{cases} \alpha c_i + \beta k_i & i \in N_{p_2} \\ \alpha c_i + 0 & i \in N_{p_1} \setminus N_{p_2} \\ 0 & \text{else} \end{cases}$$

but this is precisely $g^*$. Thus $f(\alpha p_1 + \beta p_2) = \alpha f(p_1) + \beta f(p_2)$.

We have thus shown that $f(\alpha p_1 + \beta p_2) = \alpha f(p_1) + \beta f(p_2)$ in all cases, as desired.

# 2. Project control system

Select a control system for your Project; refer to Canvas/Pages/Project for ideas and requirements.

*The project selected for this solution is a haptic device that consists of a linear motion controller and*

a. What is the system state? Indicate any parameters (i.e. "states" that don't change in time). ($\geq 3$ dimensions)

The states of the system are position and velocity of the gripper ($x_g$ and $v_g$) and the position and velocity of the controller ($x_c$ and $v_c$). Parameters are given in the ODE model below.

b. What are the inputs to the system? Explain the inputs in physical terms, i.e. what physical device or mechanism actuates the input. ($\geq 2$ inputs; create one if needed)

The inputs to the system are the force from the user's fingers ($u_1$) on the controller and an input (electric current) into the gripper's actuator ($u_2$).

c. What are the outputs from the system? Explain the outputs in physical terms, i.e. what physical device or mechanism measures the output. ($\geq 2$ outputs; create one if needed)

The outputs of the systems are positions of the gripper and controller ($x_c$ and $x_g$) measured by encoders on the two actuators.

d. Write an ODE control system model for your system's dynamics in the form $\dot{x} = f(x, u),\ y = h(x, u)$. Be sure to specify the domain and codomain of $f, h$.

$$
\begin{aligned}
\dot{x_c} &= v_c \\
\dot{v_c} &= (B_c/m_c) \cdot v_c + (1/m_c) \cdot u_1 \\
\dot{x_g} &= v_g \\
\dot{v_g} &= (B_g/m_g) \cdot v_g + (x_c - x_g)^3 + (1/m_g) \cdot u_2 \\
y_1 &= x_c \\
y_2 &= x_g
\end{aligned}
$$

$Note:\ m_c, m_g, B_c, B_g$ are parameters (masses and damping coefficients) of the system.

Since $x = (x_c, v_c, x_g, v_g)' \in \mathbb{R}^4$, $u = (u_1, u_2)' \in \mathbb{R}^2$, $y = (y_1, y_2)' \in \mathbb{R}^2$, then

$f : \mathbb{R}^4 \times \mathbb{R}^2 \to \mathbb{R}^4$, and its domain and range are $\mathbb{R}^4 \times \mathbb{R}^2$ and $\mathbb{R}^4$ correspondingly.

$h : \mathbb{R}^4 \times \mathbb{R}^2 \to \mathbb{R}^2$, and its domain and range are $\mathbb{R}^4 \times \mathbb{R}^2$ and $\mathbb{R}^2$ correspondingly.

e. Is the control system linear or nonlinear? Show algebraically or graphically the source of nonlinearity. (must be nonlinear)

Control system is non-linear due to the cubed terms in the ODE.

f. What disturbances could affect the system's dynamics? Specify what elements of $f$ and/or $h$ the disturbance would affect. ($\geq 1$ disturbance that affects $f$, $\geq 1$ disturbance that affects $h$)

Disturbances to the system would include sudden motion of the gripper (i.e. the gripper motion caused by outside source) and slippage of gears.

g. Why is your Project system synergistic with your education, research, and/or professional interests?

Project system is of special interest because it was used in a capstone haptics project.

h. Add your Project system title and a link to $\geq 1$ relevant paper / preprint / technical report in Canvas/Collaboration/Projects; upload the paper .pdf with your hw2 Assigment on Canvas.