HW4 due 11:30a Mon Oct 31

1. Bases and matrix representations

Let $L : V \to W$ be a linear function where $\dim V = n < \infty$ and $\dim W = m < \infty$, and let $r = \operatorname{rank} L$.

a. Find bases for V and W with respect to which the matrix representation of L is:

$$\begin{bmatrix} I_{r\times r} & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix}$$

Let $\{v_\ell\}_{\ell=1}^n$ be a basis for V for which $\operatorname{null}(L) = \operatorname{span}\{v_\ell\}_{r+1}^n$, and set $w_\ell = L(v_\ell)$ for each $\ell \in \{1, \ldots, r\}$.

The collection $\{w_\ell\}_{\ell=1}^r$ are linearly independent (if this were not the case, it is straightforward to show using an argument we used several times in class that the vectors $\{v_\ell\}_{\ell=1}^r$ are linearly dependent, which is a contradiction).

Furthermore, it is straightforward to show that the collection $\{w_\ell\}_{\ell=1}^r$ span $\operatorname{range}(L)$ by construction.

Let $\{w_\ell\}_{\ell=r+1}^m$ be such that $\{w_\ell\}_{\ell=1}^m$ is a basis for W.

The matrix representation for L in the basis chosen for V and W is the desired matrix.

b. Find a vector space U and a linear function $\tilde{L} : U \to U$ such that, no matter which basis you choose for U, the matrix representation of $\tilde{L}$ does not have the form from (a.).

Let $U = \mathbb{R}$ and $\tilde{L} : U \to U$ be defined $\forall u \in U : \tilde{L}(u) = \alpha u$, $\alpha \notin \{-1, 0, 1\}$. Given any $\mu \neq 0$, the matrix representation for $\tilde{L}$ in the basis $\{\mu\}$ is $\alpha\mu^{-1}\mu = \alpha$, which does not have the form from (a.).

2. Eigenvalues, eigenvectors, eigenbases

Let $A \in \mathbb{R}^{n\times n}$ be a given matrix.

Suppose that, for each $\ell \in \{1, \ldots, k\}$, there exists $\lambda_\ell \in \mathbb{C}$ and $v_\ell \in \mathbb{R}^n$ such that $v_\ell \neq 0$ and $Av_\ell = \lambda_\ell v_\ell$ (i.e. λ_ℓ is an eigenvalue for A with eigenvector v_ℓ).

a. If the eigenvalues $\{\lambda_\ell\}_{\ell=1}^k$ are distinct (i.e. $\lambda_i = \lambda_j \iff i = j$), show that the eigenvectors $\{v_\ell\}_{\ell=1}^k$ are linearly independent. (*Hint: use induction.*)

We proved the base case in class, that two eigenvectors associated with distinct eigenvalues are linearly independent.

Now suppose for $m \in \mathbb{N}$ such that $1 \leq k < m$ that $\{v_\ell\}_{\ell=1}^m$ are linearly independent but $\{v_\ell\}_{\ell=1}^{m+1}$ is not linearly independent so that $\exists \alpha \in \mathbb{C}^{m+1}$, $\alpha \neq 0$, such that $\sum_{\ell=1}^{m+1} \alpha_\ell v_\ell = 0$.

Then $L\left(\sum_{\ell=1}^{m+1} \alpha_\ell v_\ell\right) = \sum_{\ell=1}^{m+1} \alpha_\ell \lambda_\ell v_\ell = 0$, but also $\sum_{\ell=1}^{m+1} \alpha_\ell \lambda_{m+1} v_\ell = 0$.

Subtracting these two equations, we conclude $\sum_{\ell=1}^{m+1} \alpha_\ell(\lambda_{m+1} - \lambda_\ell)v_\ell$. But since $\lambda_{m+1} \neq \lambda_\ell$ for any $\ell \in \{1, \ldots, m\}$, this contradicts linear independence of $\{v_\ell\}_{\ell=1}^k$.

We conclude that $\{v_\ell\}_{\ell=1}^{m+1}$ is linearly independent, so by induction we conclude that $\{v_\ell\}_{\ell=1}^k$ is linearly independent.

Now let $L : U \to U$ be linear and $\dim U = n$.

Suppose that $\lambda \in \mathbb{C}$ and $W = \{w_\ell\}_{\ell=1}^n$ is a basis for U such that $Lw_1 = \lambda w_1$ and $Lw_k = \lambda w_k + w_{k-1}$ for all $k \in \{2, \ldots, n\}$.

b. Obtain the matrix representation of L with respect to the basis W.

$$\begin{bmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \lambda & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \lambda \end{bmatrix}$$

3. Spectral mapping theorem

Let $\operatorname{spec} A = \{\lambda_1, \ldots, \lambda_n\}$ denote the spectrum of $A \in \mathbb{C}^{n \times n}$ (i.e. the set of eigenvalues of A).

Theorem If $f : \mathbb{C} \to \mathbb{C}$ is analytic, then $\operatorname{spec} f(A) = \{f(\lambda_1), \ldots, f(\lambda_n)\}$.

a. Prove or provide a counterexample: if $\lambda_1 \neq \lambda_2$, then $f(\lambda_1) \neq f(\lambda_2)$.

If f is not injective, then it can easily happen that $f(\lambda_1) = f(\lambda_2)$. Consider, for instance, the zero function: $\forall z \in \mathbb{C} : f(z) = 0$

b. Prove or provide a counterexample: if A is invertible, then $f(A)$ is invertible.

If f sends an eigenvalue of A to zero, then $f(A)$ will not be invertible. Consider, for instance, the zero function: $\forall z \in \mathbb{C} : f(z) = 0$