HW3 due 11:30a Mon Oct 24

1. Inner product on $\mathbb{C}^n$

Given $x, y \in \mathbb{C}^n$, define the inner product $\langle x, y\rangle$ by the formula

$$\langle x, y\rangle = \sum_{\ell=1}^{n} \bar{x}_\ell \cdot y_\ell$$

where $\bar{x}_\ell \cdot y_\ell$ denotes the complex scalar multiplication between the complex conjugate $\bar{x}_\ell$ of x_ℓ and the complex number y_ℓ.

a. $\langle \cdot, \cdot\rangle$ is a function that takes two complex n-vectors as arguments; what is its codomain?

We claim that the codomain of $\langle \cdot, \cdot\rangle$ is $\mathbb{C}$. To show this, let n be an arbitrary natural number and select arbitrarily $x, y \in \mathbb{C}^n$. Then, observe that x, y can be represented as

$$x = (a_1 + b_1 j, a_2 + b_2 j, \ldots, a_n + b_n j)$$
$$y = (c_1 + d_1 j, c_2 + d_2 j, \ldots, c_n + d_n j)$$

where $a_i, b_i, c_i, d_i \in \mathbb{R}$.

Now consider

$$\langle x, y\rangle = \sum_{\ell=1}^{n} \bar{x}_\ell \cdot y_\ell = \sum_{\ell=1}^{n} (a_\ell - b_\ell j)(c_\ell + d_\ell j) = \sum_{\ell=1}^{n} (a_\ell c_\ell + b_\ell d_\ell) + j \sum_{\ell=1}^{n} (a_\ell d_\ell - b_\ell c_\ell)$$

Which is a complex number, as desired.

b. Is $\langle \cdot, \cdot\rangle$ linear in its second argument? In other words, does the following equality hold?

$$\forall x, y, z \in \mathbb{C}^n, \zeta \in \mathbb{C} : \langle x, y + \zeta z\rangle = \langle x, y\rangle + \zeta\langle x, z\rangle$$

We claim that yes, $\langle \cdot, \cdot\rangle$ is linear in its second argument. To show this directly, let n be a natural number, ζ a complex number, and let $x, y, z \in \mathbb{C}^n$ be arbitrary elements. Then consider that

$$x = (a_1 + b_1 j, a_2 + b_2 j, \ldots, a_n + b_n j)$$
$$y = (c_1 + d_1 j, c_2 + d_2 j, \ldots, c_n + d_n j)$$
$$z = (p_1 + q_1 j, p_2 + q_2 j, \ldots, p_n + q_n j)$$
$$\zeta = u + vj$$

where $a_i, b_i, c_i, d_i, p_i, q_i, u, v \in \mathbb{R}$. Then elements of $y + \zeta z$ look like:

$$(c_i + up_i - vq_i) + (d_i + vp_i + uq_i)j$$

for $i \in (1, 2, \ldots, n)$. Then we have that

$$\langle x, y + \zeta z\rangle = \sum_{\ell=1}^{n} \left[a_\ell(c_\ell + up_\ell - vq_\ell) + b_\ell(d_\ell + vp_\ell + uq_\ell))\right] + j$$
$$\sum_{\ell=1}^{n} \left[a_\ell(d_\ell + vp_\ell + uq_\ell) - b_\ell(c_\ell + up_\ell - vq_\ell)\right]$$

Recall that

$$\langle x, y\rangle = \sum_{\ell=1}^{n} (a_\ell c_\ell + b_\ell d_\ell) + j \sum_{\ell=1}^{n} (a_\ell d_\ell - b_\ell c_\ell)$$

And observe that

$$
\begin{aligned}
\zeta\langle x, z\rangle &= (u+vj)\left[\sum_{\ell=1}^{n}(a_\ell p_\ell + b_\ell q_\ell) + j\sum_{\ell=1}^{n}(a_\ell q_\ell - b_\ell p_\ell)\right] \\
&= \sum_{\ell=1}^{n} u(a_\ell p_\ell + b_\ell q_\ell) - v(a_\ell q_\ell - b_\ell p_\ell) + j\sum_{\ell=1}^{n} v(a_\ell p_\ell + b_\ell q_\ell) + u(a_\ell q_\ell - b_\ell p_\ell)
\end{aligned}
$$

But this means that $\langle x, y\rangle + \zeta\langle x, z\rangle = \langle x, y + \zeta z\rangle$, as desired.

c. Is $\langle\cdot,\cdot\rangle$ symmetric? In other words, does the following equality hold?

$$\forall x, y \in \mathbb{C}^n : \langle x, y\rangle = \langle y, x\rangle$$

If not, how is $\langle x, y\rangle$ related to $\langle y, x\rangle$?

We claim that $\langle\cdot,\cdot\rangle$ is not symmetric. To show this, let n be a natural number and $x, y \in \mathbb{C}^n$ be arbitrary. Then consider that

$$\langle x, y\rangle = \sum_{\ell=1}^{n}(a_\ell c_\ell + b_\ell d_\ell) + j\sum_{\ell=1}^{n}(a_\ell d_\ell - b_\ell c_\ell)$$

and

$$\langle y, x\rangle = \sum_{\ell=1}^{n}(a_\ell c_\ell + b_\ell d_\ell) + j\sum_{\ell=1}^{n}(b_\ell c_\ell - a_\ell d_\ell) = \sum_{\ell=1}^{n}(a_\ell c_\ell + b_\ell d_\ell) - j\sum_{\ell=1}^{n}(a_\ell d_\ell - b_\ell c_\ell)$$

Since x, y were arbitrary, we conclude that for any $x, y \in \mathbb{C}^n$, that $\langle x, y\rangle = \overline{\langle y, x\rangle}$.

d. Is $\langle\cdot,\cdot\rangle$ positive definite? In other words, is the following true?

$$\forall x \in \mathbb{C}^n : \langle x, x\rangle \geq 0,\ \langle x, x\rangle = 0 \iff x = 0$$

We wish to show that $\langle\cdot,\cdot\rangle$ is positive definite. To that end, we will demonstrate two things, $\forall x \in \mathbb{C}^n$:

1. $\langle x, x\rangle \geq 0$
2. $\langle x, x\rangle = 0 \iff x = 0$

Select arbitraily $x \in \mathbb{C}^n$ where $n \in \mathbb{N}$. Then we know that

$$x = (a_1 + b_1 j, a_2 + b_2 j, \ldots, a_n + b_n j)$$

where a_i, b_i are real numbers.

Now we start with the first claim and observe that from part a. we know that:

$$\langle x, x\rangle = \sum_{\ell=1}^{n}(a_\ell a_\ell + b_\ell b_\ell) + j\sum_{\ell=1}^{n}(a_\ell b_\ell - b_\ell a_\ell) = \sum_{\ell=1}^{n} a_\ell^2 + b_\ell^2 \geq 0$$

Now the second claim. Well, we just showed that for any $x \in \mathbb{C}^n$, that

$$\langle x, x\rangle \sum_{\ell=1}^{n} a_\ell^2 + b_\ell^2$$

But a_i, b_i are real numbers and we know that the square of any nonzero real number is positive. It follows that

$$\langle x, x\rangle \sum_{\ell=1}^{n} a_\ell^2 + b_\ell^2 = 0 \iff a_i = b_i = 0, i \in \{1, 2, \ldots, n\} \iff x = 0$$

.

We have thus shown that $\langle\cdot,\cdot\rangle$ is positive definite, as desired.

2. Linear functions

Let $(m \times n)$ denote the set

$$(m \times n) = \{(i,j) : i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}\}.$$

a. Show that the set $\mathcal{A} = \{A : (m \times n) \to \mathbb{F}\}$ of matrices with m rows and n columns is a vector space over the field $\mathbb{F}$. (You need to define vector addition $+ : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and scalar multiplication $\cdot : \mathbb{F} \times \mathcal{A} \to \mathcal{A}$ and show that they satisfy the commutative, associative, distributive, and zero element properties that define a vector space.)

We want to define vector addition and scalar multiplication set of matrices with m rows and n columns is a vector space over the field $\mathbb{F}$. To that end, define addition as follows. Let $M, N \in \mathcal{A}$ be two matrices with m rows and n columns. Arbitrarily select $x = (i,j) \in (m \times n)$. Then, since $M \in \mathcal{A}$, we have that

$$M(x) = f_{ij} \in \mathbb{F}$$

Similarly, let

$$N(x) = g_{ij} \in \mathbb{F}$$

Now define addition $M \oplus N$ as, for any $x = (i,j) \in (m \times n)$:

$$M \oplus N(x) = f_{ij} + g_{ij}$$

where addition of f_{ij}, g_{ij} is defined as in $\mathbb{F}$. We know then that $M \oplus N(x) \in \mathbb{F}$ since $\mathbb{F}$ is a field. Thus, $M \oplus N \in \mathcal{A}$, and $\mathcal{A}$ is closed under this addition.

Now define scalar multiplcation as follows. Let $k \in \mathbb{F}$ be arbitrary. Then let

$$k \cdot M(x) = kf_{ij}$$

with scalar multiplication of k and f_{ij} as defined in $\mathbb{F}$. But this means that $k \cdot M(x) \in \mathbb{F}$ for all $x \in (m \times n)$, and thus $\mathcal{A}$ is closed under this multiplication.

Finally, we are asked to show commutativity, associativity, distributivity, and a zero element. Well recall that we've defined

$$M \oplus N(x) = f_{ij} + g_{ij}$$

and it follows that

$$N \oplus M(x) = g_{ij} + f_{ij} = f_{ij} + g_{ij} = M \oplus N(x)$$

since $\mathbb{F}$ is a field. Thus, our vector addition is commutative. Now let $O \in \mathcal{A}$. Then $O(x) = h_{ij} \in \mathbb{F}$. Consider

$$(M \oplus N)(x) \oplus O(x) = (f_{ij} + g_{ij}) + h_{ij}$$

And observe that

$$M(x) \oplus (N \oplus O)(x) = f_{ij} + (g_{ij} + h_{ij}) = (f_{ij} + g_{ij}) + h_{ij} = (M \oplus N)(x) \oplus O(x)$$

since $\mathbb{F}$ is a field. Thus, our vector addition is associative. Now let $\ell \in \mathbb{F}$ be another arbitary element in our field. Then consider

$$\ell \cdot (k \cdot M(x)) = \ell(kf_{ij})$$

and

$$(\ell k) \cdot M(x) = (\ell k)f_{ij} = \ell(kf_{ij}) = \ell \cdot (k \cdot M(x))$$

since $\mathbb{F}$ is a field. Thus, our scalar multiplication is associative.

Consider

$$(\ell + k) \cdot M(x) = (\ell + k) f_{ij}$$

and that

$$\ell \cdot M(x) + k \cdot M(x) = \ell f_{ij} + k f_{ij} = (\ell + k) f_{ij} = (\ell + k) \cdot M(x)$$

since $\ell, k, f_{ij} \in \mathbb{F}$ which is a field. Thus, scalar sums are distributive with our scalar multiplcation and vector addition.

Finally, observe that

$$\ell \cdot (M \oplus N)(x) = \ell(f_{ij} + g_{ij})$$

and that

$$\ell \cdot M(x) + \ell \cdot N(x) = \ell f_{ij} + \ell g_{ij} = \ell(f_{ij} + g_{ij}) = \ell \cdot (M \oplus N)(x)$$

since $\ell, k, f_{ij}, g_{ij} \in \mathbb{F}$ which is a field. Thus, vector sums are distributive with our scalar multiplcation and vector addition.

Now let $Z \in \mathcal{A}$ be defined as

$$Z(x) = \hat{0}$$

for all $x \in (m \times n)$ where $\hat{0}$ is the zero element or additive identity in $\mathbb{F}$. Then consider

$$Z \oplus M(x) = \hat{0} + f_{ij} = f_{ij}$$

Thus, Z is an additive identity or zero element of $\mathcal{A}$. Since the above analysis was done with arbitrary $x \in (m \times n)$ and $M, N, O \in \mathcal{A}$ and $\ell, k \in \mathbb{F}$, we have shown the properties generally, as desired.

b. Let V, W be vector spaces over the same field $\mathbb{F}$. Show that the set $\mathcal{L} = \{L : V \to W \mid L \text{ is linear}\}$ of linear maps from V to W is a vector space over the field $\mathbb{F}$ (You need to define vector addition $+ : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ and scalar multiplication $\cdot : \mathbb{F} \times \mathcal{L} \to \mathcal{L}$ and show that they satisfy the commutative, associative, distributive, and zero element properties that define a vector space.)

We start by selecting arbitrarily $F, G \in \mathcal{L}$ and $k, \ell, m \in \mathbb{F}$. Now let $v_1, v_2 \in V$ be arbitrary. Then, since F, G are linear, we know that:

$$kF(v_1) + \ell F(v_2) = F(kv_1 + \ell v_2)$$
$$kG(v_1) + \ell G(v_2) = G(kv_1 + \ell v_2)$$

Let $x \in V$ and define our vector addition such that

$$F \oplus G(x) = F(x) + G(x)$$

where the addition of $F(x), G(x)$ is as defined in W. Note first that if F, G are well defined, then so is $F \oplus G$ in this construction, since $F(x) + G(x)$ is precisely one element of W. $F \oplus G$ is also linear, since

$$\begin{aligned}
k(F \oplus G)(v_1) + \ell(F \oplus G)(v_2) &= k\left[F(v_1) + G(v_1)\right] + \ell\left[F(v_2) + G(v_2)\right] \\
&= \left[F(kv_1) + F(\ell v_2)\right] + \left[G(kv_1) + G(\ell v_2)\right] \\
&= F(kv_1 + \ell v_2) + G(kv_1 + \ell v_2) \\
&= (F \oplus G)(kv_1 + \ell v_2)
\end{aligned}$$

Thus $\mathcal{L}$ is closed under this addition. Now consider scalar multiplication, and suppose that $F(v_1) = w_1 \in W$. Then define with $m \in \mathbb{F}$ and $F \in \mathcal{L}$:

$$m \cdot F(v_1) = mw_1$$

where the multplication of m, w_1 is as defined in vector space W. Then for any scalar $m \in \mathbb{F}$ and any $v \in V$, we know that $m \cdot F(v)$ is an element of W, and is well defined since F is. Now we show that $m \cdot F$ is linear.

Consider

$$\begin{aligned} m \cdot F(kv_1 + \ell v_2) &= m(kF(v_1) + \ell F(v_2)) \\ &= mkF(v_1) + m\ell F(v_2) \\ &= k(m \cdot F(v_1)) + \ell(m \cdot F(v_2)) \end{aligned}$$

and thus $m \cdot F$ is linear. Note we could do some of the operation above since F is linear.

Finally, we are asked to show commutativity, associativity, distributivity, and a zero element. Well recall that we've defined

$$F \oplus G(x) = F(x) + G(x)$$

and it follows that

$$F \oplus G(x) = F(x) + G(x) = G(x) + F(x) = G \oplus F(x)$$

since W is a vector space. Thus, $\oplus$ is commutative in $\mathcal{L}$. Now suppose $H \in \mathcal{L}$. Then

$$F \oplus (G \oplus H)(x) = F \oplus (G(x) + H(x)) = (F(x) + G(x)) + H(x) = (F \oplus G) \oplus H(x)$$

since W is a vector space and addition is associative. Then, our vector addition $\oplus$ is associative. Now consider $F(v_1) = w_1 \in W$. Again, let k, ℓ be random in the field. Then

$$k \cdot (\ell \cdot F(v_1)) = k(\ell w_1)$$

And further that

$$(k\ell) \cdot F(v_1) = k\ell(w_1) = k(\ell w_1) = k \cdot (\ell \cdot F(v_1))$$

and thus our scalar multiplication is associative. Now consider

$$(\ell + k) \cdot F(v_1) = (\ell + k)w_1$$

and that

$$\ell \cdot F(v_1) + k \cdot F(v_1) = \ell w_1 + kw_1 = (\ell + k)w_1 = (\ell + k) \cdot F(v_1)$$

since $\ell, k, w_1 \in \mathbb{F}$ which is a field. Thus, scalar sums are distributive with our scalar multiplcation and vector addition.

Finally, let $G(v_1) = w_2 \in W$. Observe that

$$\ell \cdot (F \oplus G)(v_1) = \ell(w_1 + w_2)$$

and that

$$\ell \cdot F(v_1) + \ell \cdot G(v_1) = \ell w_1 + \ell w_2 = \ell(w_1 + w_2) = \ell \cdot (F \oplus G)(v_1)$$

since $\ell, k, w_1, w_2 \in \mathbb{F}$ which is a field. Thus, vector sums are distributive with our scalar multiplcation and vector addition.

Now let Z be a function that maps any element in V to the zero element, call it $\hat{0}$, in W. This function is clearly linear, and is thus an element of $\mathcal{L}$. Then, consider arbitrary $v_1 \in V$, and observe that

$$Z \oplus F(v_1) = \hat{0} + w_1 = w_1 = F(v_1)$$

Thus, Z is an additive identity or zero element of $\mathcal{L}$. Since the above analysis was done with arbitrary $x, v_1, v_2 \in V$ and $F, G, H \in \mathcal{A}$ and $\ell, k, m \in \mathbb{F}$, we have shown the properties generally, as desired.

3. Project control system

This problem is repeated from HW2 to provide the opportunity for you to revise your project system to address any issues that arose during the self-assessment or feedback from the TAs. If you already addressed each point below in your HW2 submission and dot wish to make any changes, simply include a statement to that effect in your submission for HW3.

Select a control system for your Project; refer to Canvas/Pages/Project for ideas and requirements.

a. What is the system state? Indicate any parameters (i.e. "states" that don't change in time). (≥ 3 dimensions)

b. What are the inputs to the system? Explain the inputs in physical terms, i.e. what physical device or mechanism actuates the input. (≥ 2 inputs; create one if needed)

c. What are the outputs from the system? Explain the outputs in physical terms, i.e. what physical device or mechanism measures the output. (≥ 2 outputs; create one if needed)

d. Write an ODE control system model for your system's dynamics in the form $\dot{x} = f(x, u),\ y = h(x, u)$. Be sure to specify the domain and codomain of f, h.

e. Is the control system linear or nonlinear? Show algebraically or graphically the source of nonlinarity. (must be nonlinear)

f. What disturbances could affect the system's dynamics? Specify what elements of f and/or h the disturbance would affect. (≥ 1 disturbance that affects f, ≥ 1 disturbance that affects h)

g. Why is your Project system synergistic with your education, research, and/or professional interests?

h. Add your Project system title and a link to ≥ 1 relevant paper / preprint / technical report in Canvas/Collaboration/Projects; upload the paper .pdf with your hw2 Assigment on Canvas.