HW4 due 11:30a Mon Oct 31

1. Bases and matrix representations

Let $L : V \to W$ be a linear function where $\dim V = n < \infty$ and $\dim W = m < \infty$, and let $r = \operatorname{rank} L$.

a. Prove that there are bases $\{v_j\}_{j=1}^n \subset V$ and $\{w_i\}_{i=1}^m \subset W$ for V and W, respectively, with respect to which the matrix representation of L is:

$$\begin{bmatrix} I_{r\times r} & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix}$$

b. Prove that (a.) is not true if $V = W$ (i.e. you are forced to use the same basis for the domain and range of L when constructing its matrix representation).

2. Eigenvalues, eigenvectors, eigenbases

Let $A \in \mathbb{R}^{n\times n}$ be a given matrix.

Suppose that, for each $\ell \in \{1, \ldots, k\}$, there exists $\lambda_\ell \in \mathbb{C}$ and $v_\ell \in \mathbb{R}^n$ such that $v_\ell \neq 0$ and $Av_\ell = \lambda_\ell$ (i.e. λ_ℓ is an eigenvalue for A with eigenvector v_ℓ).

a. If the eigenvalues $\{\lambda_\ell\}_{\ell=1}^k$ are distinct (i.e. $\lambda_i = \lambda_j \iff i = j$), show that the eigenvectors $\{v_\ell\}_{\ell=1}^k$ are linearly independent. (*Hint: use induction.*)

Now let $L : U \to U$ be linear and $\dim U = n$.

Suppose that $\lambda \in \mathbb{C}$ and $W = \{w_\ell\}_{\ell=1}^n$ is a basis for U such that $Lw_1 = \lambda w_1$ and $Lw_k = \lambda w_k + w_{k-1}$ for all $k \in \{2, \ldots, n\}$.

b. Obtain the matrix representation of L with respect to the basis W.

3. Spectral mapping theorem

Let $\operatorname{spec} A = \{\lambda_1, \ldots, \lambda_n\}$ denote the spectrum of $A \in \mathbb{C}^{n\times n}$ (i.e. the set of eigenvalues of A).

Theorem If $f : \mathbb{C} \to \mathbb{C}$ is analytic, then $\operatorname{spec} f(A) = \{f(\lambda_1), \ldots, f(\lambda_n)\}$.

a. Prove or provide a counterexample: if $\lambda_1 \neq \lambda_2$, then $f(\lambda_1) \neq f(\lambda_2)$.

b. Prove or provide a counterexample: if A is invertible, then $f(A)$ is invertible.