HW01 Solution

1. Mathematical notation and reasoning

Given a set X, let $P(X) = \{S : S \subset X\}$ denote the *power set* (i.e. "the set of all subsets") of X.

Note: these solutions are verbose; your solutions (and our future solutions) may be more terse.

a. If $|X| < \infty$, compute $|P(X)|$ (i.e. the number of elements in the power set of X).

Given a subset $S \subset X$ (so $S \in P(X)$), define $f_S : X \to \{0, 1\}$ by the rule
$$\forall x \in X : (x \in S \Rightarrow f(x) = 1) \wedge (x \notin S \Rightarrow f(x) = 0),$$
i.e. $f(x)$ equals 1 if and only if $x \in S$. This injection assigns a unique binary-valued function to every $S \in P(X)$.

Given a binary function $f : X \to \{0, 1\}$, define $S_f \subset X$ by the rule
$$S_f = \{x \in X : f(x) = 1\},$$
or equivalently $S_f = f^{-1}(1)$. This injection assigns a unique subset to every binary function $f : X \to \{0, 1\}$.

Let $B(X) = \{f : X \to \{0, 1\}\}$ denote the set of binary-valued functions over X.

Using the cardinality definition in class, we conclude $|P(X)| = |B(X)|$.

We have reduced the problem of computing $|P(X)|$ to the problem of determining how many binary-valued functions there are over a set X for which $|X| < \infty$?

Since each element in X may be assigned either 0 or 1 independently from all other elements, we conclude $|P(X)| = |B(X)| = 2^{|X|}$.

Now let A, B be arbitrary sets.

b. Prove that there exists an injection $f : A \to B$ if and only if there exists a surjection $g : B \to A$ (therefore the two definitions we proposed for $|A| \leq |B|$ are equivalent).

We must show that for arbitrary sets A, B there exists an injection from A to B if and only if there exists a surjection from B to A. We proceed directly and must show two things:

1. If there exists an injection from A to B, then there exists a surjection from B to A.
2. If there exists a surjection from B to A, then there exists an injection from A to B.

We begin with the former and let $f : A \to B$ be the injection. Then by definition, for any $f(a_1), f(a_2) \in B$ where $a_1, a_2 \in A, f(a_1) = f(a_2) \implies a_1 = a_2$. We propose that a surjection exists from B to A, $g : B \to A$. We construct g as follows:

Select arbitrarily an element in domain B, call it b. Two cases arise:

b is in the image of f.

b is not in the image of f.

Start with the former. If b is in the image of f, then $b = f(\hat{a})$ for exactly one $\hat{a} \in A$ since f was injective. Let g map b precisely to its preimage in f, i.e. $g(b) = \hat{a}$.

Now if b is not in the image of f, let $g(b)$ map to some arbitrary element in A.

We now have that g is a function (it maps any element of B to precisely one element in A since f was injective). To show $g : B \to A$ is a surjection, select arbitrarily $a \in A$. Then there exists an element in B, namely $f(a) \in B$, such that $g(f(a)) = g \circ f(a) = a$. We have shown by construction that given injection $f : A \to B$ that there exists surjection $g : B \to A$. We have thus shown one direction.

Now consider the other direction of the coimplication. Recall that we need to show that for arbitrary sets A, B, if there exists a surjection from B to A, then there exists an injection from A to B. To that end, let $g : B \to A$ be a surjection from B to A. Then $\forall a \in A \exists b \in B \ni g(b) = a$. We propose that an injection exists from A to $B, f : A \to B$. We construct f as follows:

Select arbitrarily $a \in A$. Then the preimage of a under g is nonempty, since g was a surjection. Now for every $a \in A$, let us choose precisely one element in its preimage under g (we can do so by the Axiom of Choice). Then let us define f as mapping a to this chosen element. We now have constructed a mapping f which takes any element from A and maps it to exactly one element in B. Moreover, we know that this mapping is injective since g was a function. We have shown by construction that given surjection $g : B \to A$ that there exists injection $f : A \to B$.

Since we have shown both directions, we have proven the coimplication as desired.

c. If you are given a subset $S \subset A \times B$, how can you tell if it is the graph of some function? (That is, what are the set theoretic properties of a graph?)

$S \subset A \times B$ is the graph of some function if and only if for all $a \in A$ there exists a unique $b_a \in B$ such that $(a, b_a) \in S$:

$$\forall a \in A : \exists! b_a \in B : (a, b_a) \in S.$$

If this condition is satisfied, then the function $f_S : A \to B$ is defined by $f(a) = b_a$ (where $b_a \in B$ is the unique element associated with the element $a \in A$).

d. Let $h : B \to C$ be a second function and consider the composition $h \circ f : A \to C$. With $A = B = C = [0, 1] \subset \mathbb{R}$, visualize $A \times B \times C \subset \mathbb{R}^3$ as the unit cube in 3--space. Show how the graphs of f, h, and $h \circ f$ can be represented on the face of the cube, and annotate representative $a \in A$, $b = f(a) \in B$, $c = h(b) = h \circ f(a) \in C$.

see hw1p1dsol.pdf on Canvas

2. Asteroids control system

Consider the classic arcade game ''Asteroids'' (if you're unfamiliar with this game, watch the video in Canvas/Files/watch/asteroids.mp4 or try your hand at the game at http://www.freeasteroids.org/ (http://www.freeasteroids.org/)).

a. What is the system state? (include the spaceship, asteroids, and missiles)

First we start with making some physical assumptions about our system. Let's assume that asteroids and missiles move with constant speed. Let's assume that pressing UP button on keyboard exerts a constant force u_1 on the spaceship. Let's also assume that pressing LEFT or RIGHT button on keyboard make the spaceship rotate with constant angular velocity u_2. Assuming the above, the information we need to uniquely determine the system's output will be horizontal x_1, x_1^a, x_1^m and vertical x_2, x_2^a, x_2^m coordinates of the spaceship, asteroids and missiles correspondingly, velocitites $v_1 = \dot{x_1}$ and $v_2 = \dot{x_2}$ of the spaceship, and the orientation θ of the spaceship.

b. What are the inputs to the system? (from the player's perspective)

Based on the assumptions above the inputs are the force u_1 exerted on the spaceship, and the angular velocity u_2.

c. What are the outputs from the system? (from the player's perspective)

The outputs are the x and y coordinates of the spaceship, and the orientation θ of the spaceship.

d. Write a DE control system model for the spaceship's dynamics in the form $\dot{x} = f(x, u),\ y = h(x)$. (specify the domain and range of the functions f, h)

$$\dot{x_1} = v_1$$
$$\dot{x_2} = v_2$$
$$\dot{v_1} = u_1 \cdot \cos\theta - k \cdot v_1$$
$$\dot{v_2} = u_1 \cdot \sin\theta - k \cdot v_2$$
$$\dot{\theta} = u_2$$
$$y_1 = x_1$$
$$y_2 = x_2$$
$$y_3 = \theta$$

Since $x = (x_1, x_2, v_1, v_2, \theta)' \in \mathbb{R}^5$, $u = (u_1, u_2)' \in \mathbb{R}^2$, $y = (y_1, y_2, y_3)' \in \mathbb{R}^3$, then

$f : \mathbb{R}^5 \times \mathbb{R}^2 \to \mathbb{R}^5$, and its domain and range are $\mathbb{R}^5 \times \mathbb{R}^2$ and $\mathbb{R}^5$ correspondingly.

$h : \mathbb{R}^5 \times \mathbb{R}^2 \to \mathbb{R}^3$, and its domain and range are $\mathbb{R}^5 \times \mathbb{R}^2$ and $\mathbb{R}^3$ correspondingly.

Notes:

- In the above ODE control system, the kv terms account for the drag force.
- Although (a.) asked for the states of the asteroids and missiles, this problem only asks for the dynamics of the spaceship.

e. Is the control system linear or nonlinear?

The system is nonlinear because of the $\sin\theta$ and $\cos\theta$ terms in $f(x, u)$.

Also, since θ is an angular variable it naturally resides in a non-vector space (S^1, the unit circle).

f. What disturbances could affect the system's dynamics; what elements of f and/or h would the disturbance affect? (≥ 2 disturbances; imagine this is a model of a real spacecraft)

Gusts of wind on earth, gravity in open space, broken rocket wings could be considered disturbances. These would primarily affect the state variable x in f and h.