CS 240 – Data Structures and Data Management

Module 6: Dictionaries for special keys

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Based on lecture notes by many previous cs240 instructors

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Outline

6. Dictionaries for special keys
   - Lower bound
   - Interpolation Search
   - Tries
     - Standard Tries
     - Variations of Tries
     - Compressed Tries

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Lower bound for search

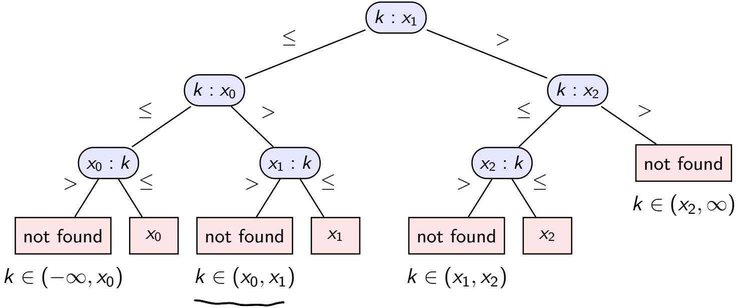
The fastest realizations of *ADT Dictionary* require $\Theta(\log n)$ time to search among n items. Is this the best possible?

Lower bound for search

The fastest realizations of *ADT Dictionary* require $\Theta(\log n)$ time to search among n items. Is this the best possible?

Theorem: In the comparison model (on the keys), $\Omega(\log n)$ comparisons are required to search a size-n dictionary.

Proof: via decision tree for items $x_0, \ldots, x_{n-1}$



But can we beat the lower bound for special keys?

Assume we are searching for k in items $a_1, \dots, a_n$

We only use comparisons $\rightsquigarrow$ Can draw decision tree!

leaves correspond to answers returned by the algorithm.

we have at least $n+1$ possible answers.

$\Rightarrow$ $\geq n+1$ leaves in decision tree.

The number of leaves of depth h is at most 2^h.

$$n+1 \leq \# \text{ of leaves} \leq 2^h$$

$$\Rightarrow n+1 \leq 2^h \longrightarrow \lg(n+1) \leq h$$

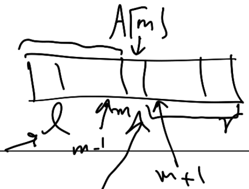
$$\Rightarrow h \in \Omega(\lg n)$$

Outline

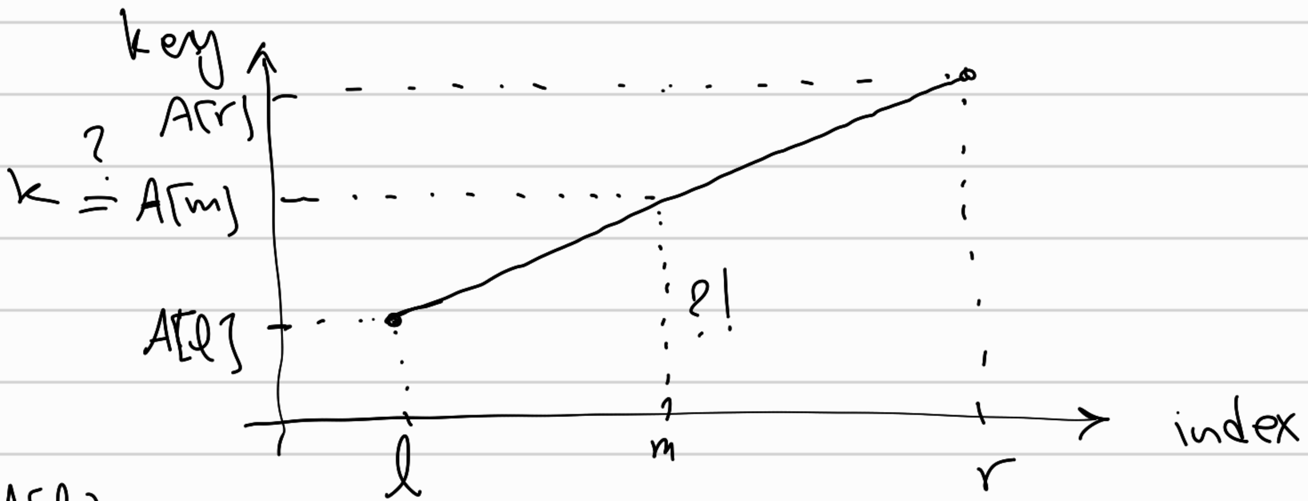
Binary Search

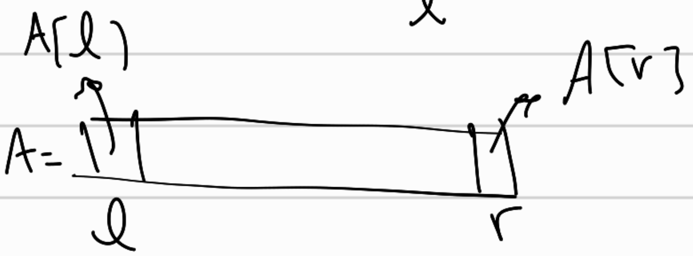
Recall the run-times in a *sorted array*:

- *insert, delete*: $\Theta(n)$
- *search*: $\Theta(\log n)$




```
binary-search(A, n, k)
A: Sorted array of size n, k: key
    ℓ ← 0, r ← n − 1
    while (ℓ ≤ r)
        m ← ⌊(ℓ+r)/2⌋
        if (A[m] == k) then return "found at A[m]"
        else if (A[m] < k) then ℓ ← m + 1
        else r ← m − 1
    return "not found, but would be between A[ℓ−1] and A[ℓ]"
```





search(k)

Find the equation of the line, using the first and last points.

slope: $\dfrac{A[r] - A[l]}{r-l}$

line $$y - A[l] = \frac{A[r]-A[l]}{r-l}(x-l)$$

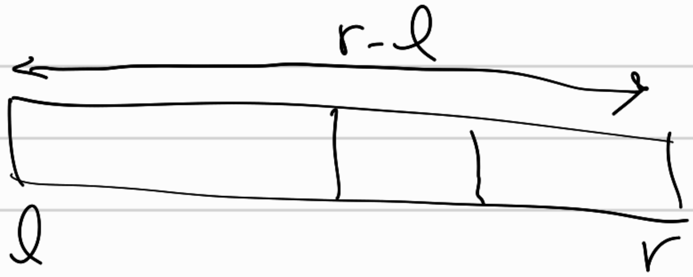
Our guess is that (m, k) is on the line:

$$k - A[l] = \frac{A[r]-A[l]}{r-l}(m-l)$$

$$k - A[l] = \frac{A[r] - A[l]}{r - l}(m - l)$$

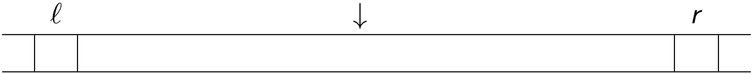
$$\frac{k - A[l]}{A[r] - A[l]}(r - l) = m - l$$

$$m = l + \frac{k - A[l]}{A[r] - A[l]}(r - l)$$



Interpolation Search: Motivation

binary-search$(A[\ell, r], k)$: Compare at index $\lfloor \frac{\ell+r}{2} \rfloor = \ell + \lfloor \frac{1}{2}(r - \ell) \rfloor$



Interpolation Search: Motivation

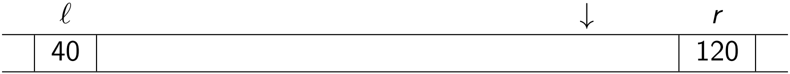
binary-search$(A[\ell, r], k)$: Compare at index $\lfloor \frac{\ell+r}{2} \rfloor = \ell + \lfloor \frac{1}{2}(r - \ell) \rfloor$

	ℓ		$\downarrow$	r	
	40			120	

Question: If keys are *numbers*, where would you expect key $k = 100$?

Interpolation Search: Motivation

binary-search$(A[\ell, r], k)$: Compare at index $\lfloor \frac{\ell+r}{2} \rfloor = \ell + \lfloor \frac{1}{2}(r - \ell) \rfloor$



Question: If keys are *numbers*, where would you expect key $k = 100$?

interpolation-search$(A[\ell, r], k)$: Compare at index $\ell + \left\lfloor \frac{k - A[\ell]}{A[r] - A[\ell]}(r - \ell) \right\rfloor$

Interpolation Search

- Code very similar to binary search, but compare at interpolated index
- Need a few extra tests to avoid crash during computation of m.

interpolation-search(A, n, k)
A: Sorted array of size n, k: key

$\ell \leftarrow 0$, $r \leftarrow n-1$
while $(\ell \leq r)$
  if $(k < A[\ell]$ or $k > A[r])$ **return** "not found"
  if $(k = A[r])$ **then return** "found at $A[r]$"
  $m \leftarrow \ell + \lfloor \frac{k - A[\ell])}{A[r]-A[\ell]} \cdot (r - \ell) \rfloor$
  if $(A[m] == k)$ **then return** "found at $A[m]$"
  else if $(A[m] < k)$ **then** $\ell \leftarrow m+1$
  else $r \leftarrow m-1$
// We always return from somewhere within while-loop

$A[\ell] \leq k \leq A[r]$

$=$

Interpolation Search Example

	0	1	2	3	4	5	6	7	8	9	10
A =	0	1	2	3	449	450	600	800	1000	1200	1500
	ℓ										r

interpolation-search(A[0..10],449):
ℓ r

Interpolation Search Example

0	1	2	3	4	5	6	7	8	9	10
0	1	2	3	449	450	600	800	1000	1200	1500
ℓ		$\uparrow$								r

interpolation-search(A[0..10],449):

- Initially $\ell = 0$, $r = n - 1 = 10$, $m = \ell + \lfloor \frac{449-0}{1500-0}(10-0) \rfloor = \overset{0}{\ell} + 2 = 2$

Interpolation Search Example

0	1	2	3	4	5	6	7	8	9	10
0	1	2	3	449	450	600	800	1000	1200	1500
			ℓ		$\uparrow$					r

interpolation-search(A[0..10],449):

- Initially $\ell = 0$, $r = n - 1 = 10$, $m = \ell + \lfloor \frac{449-0}{1500-0}(10-0) \rfloor = \ell + 2 = 2$
- $\ell = 3$, $r = 10$, $m = \ell + \lfloor \frac{449-3}{1500-3}(10-3) \rfloor = \underset{3}{\ell} + 2 = 5$

Interpolation Search Example

0	1	2	3	4	5	6	7	8	9	10
0	1	2	3	449	450	600	800	1000	1200	1500
			ℓ	$\uparrow, r$						

interpolation-search(A[0..10],449):

- Initially $\ell = 0$, $r = n - 1 = 10$, $m = \ell + \lfloor \frac{449-0}{1500-0}(10-0) \rfloor = \ell + 2 = 2$
- $\ell = 3$, $r = 10$, $m = \ell + \lfloor \frac{449-3}{1500-3}(10-3) \rfloor = \ell + 2 = 5$
- $\ell = 3$, $\underline{r = 4}$, $\underline{\text{found at } A[4]}$

Interpolation Search Example

0	1	2	3	4	5	6	7	8	9	10
0	1	2	3	449	450	600	800	1000	1200	1500

interpolation-search(A[0..10],449):

- Initially $\ell = 0$, $r = n-1 = 10$, $m = \ell + \lfloor \frac{449-0}{1500-0}(10-0) \rfloor = \ell + 2 = 2$
- $\ell = 3$, $r = 10$, $m = \ell + \lfloor \frac{449-3}{1500-3}(10-3) \rfloor = \ell + 2 = 5$
- $\ell = 3$, $r = 4$, found at $A[4]$

Works well if keys are *uniformly* distributed:

- Can show: Recurrence relation is $T^{(\mathrm{avg})}(n) = T^{(\mathrm{avg})}(\sqrt{n}) + \Theta(1)$.
- This resolves to $T^{(\mathrm{avg})}(n) \in \Theta(\log\log n)$.

But: Worst case performance $\Theta(n)$ $\longleftarrow$

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- Tries
  - Standard Tries
  - Variations of Tries
  - Compressed Tries

Tries: Introduction

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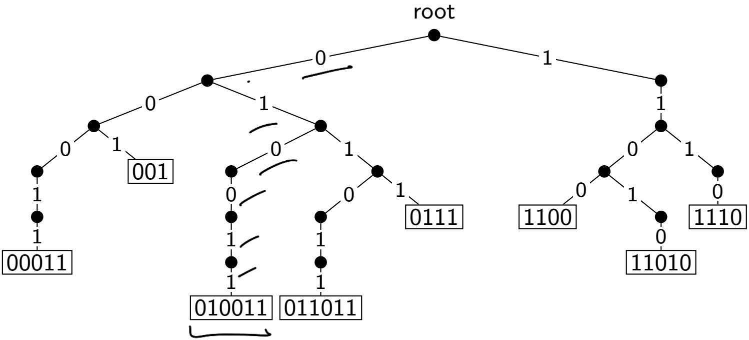
Trie (also know as **radix tree**): A dictionary for bitstrings.
(Should know: string, word, $|w|$, alphabet, prefix, suffix, comparing words,....)

- Comes from retrieval, but pronounced "try"
- A tree based on *bitwise comparisons*: Edge labelled with corresponding bit
- Similar to *radix sort*: use individual bits, not the whole key

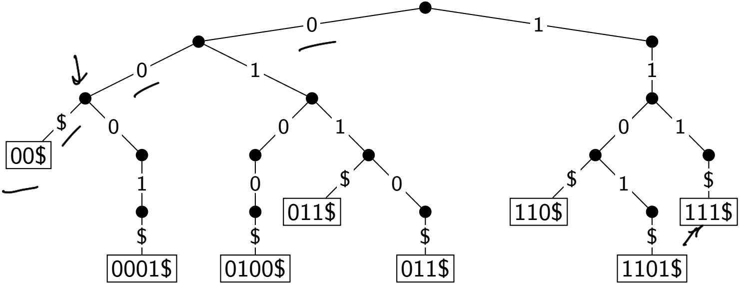


More on tries

Assumption: Dictionary is **prefix-free**: no string is a prefix of another

- Assumption satisfied if all strings have the same length.
- Assumption satisfied if all strings end with 'end-of-word' character \$.

Example: A trie for $\{00\$, 0001\$, 0100\$, 011\$, 0110\$, 110\$, 1101\$, 111\$\}$



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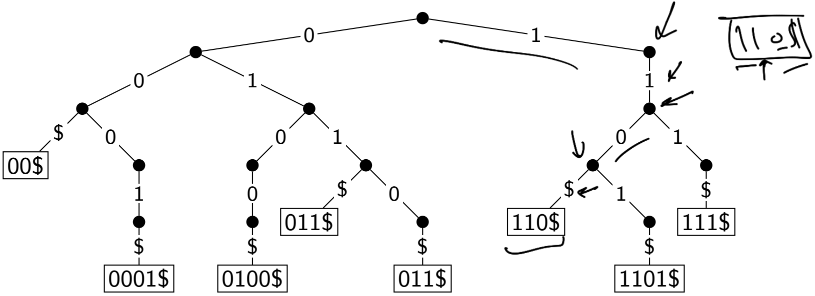


Then items (keys) are stored *only* in the leaf nodes

Tries: Search

- start from the root and the most significant bit of x
- follow the link that corresponds to the current bit in x; return failure if the link is missing
- return success if we reach a leaf (it must store x)
- else recurse on the new node and the next bit of x

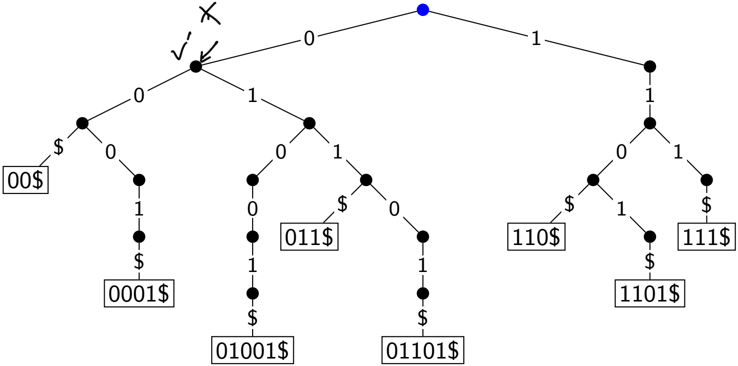
Trie::search$(v \leftarrow \text{root}, d \leftarrow 0, x)$
v: node of trie; d: level of v, x: word stored as array of chars

    if v is a leaf
        return v
    else
        let v' be child of v labelled with $x[d]$
        if there is no such child
            return "not found"
        else *Trie::search*$(v', d+1, x)$

Tries: Search Example

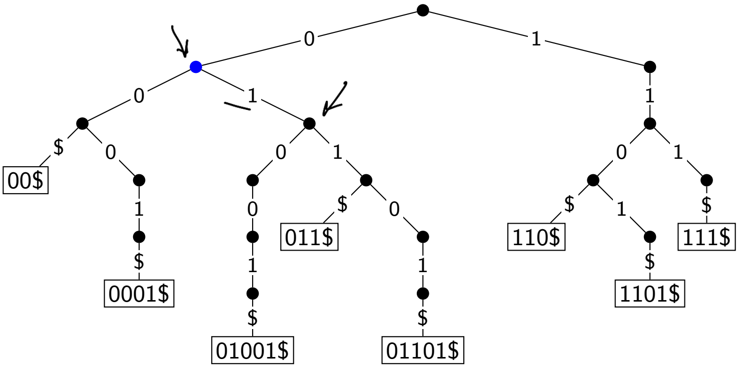
Example: Trie::search(011$)

✓ ✗



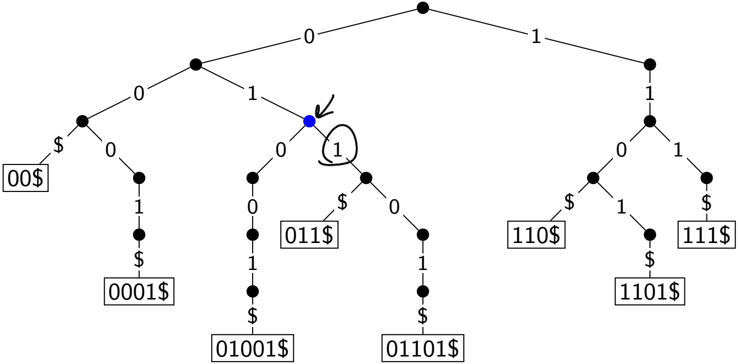
Tries: Search Example

Example: Trie::search(011$)



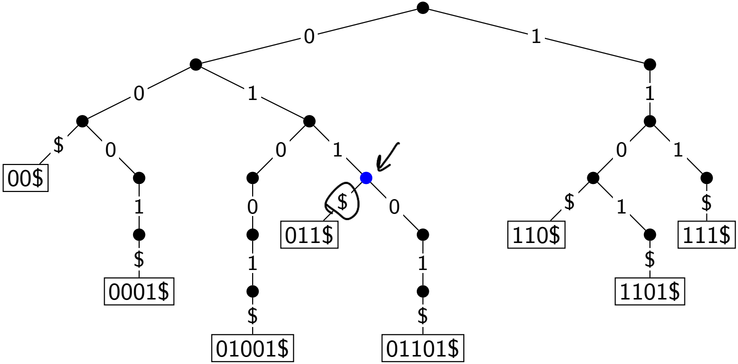
Tries: Search Example

Example: Trie::search(011$)



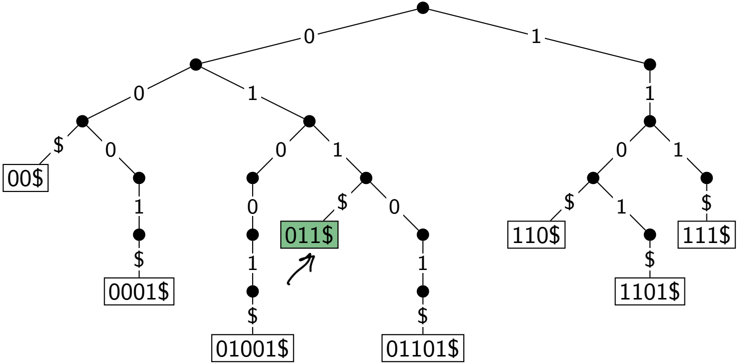
Tries: Search Example

Example: Trie::search(011$)



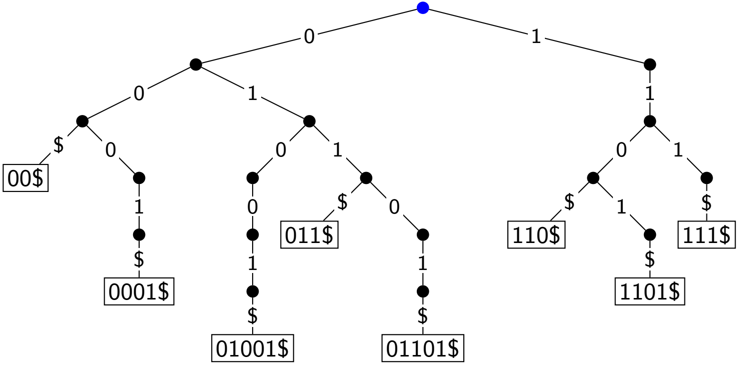
Tries: Search Example

Example: Trie::search(011$) successful



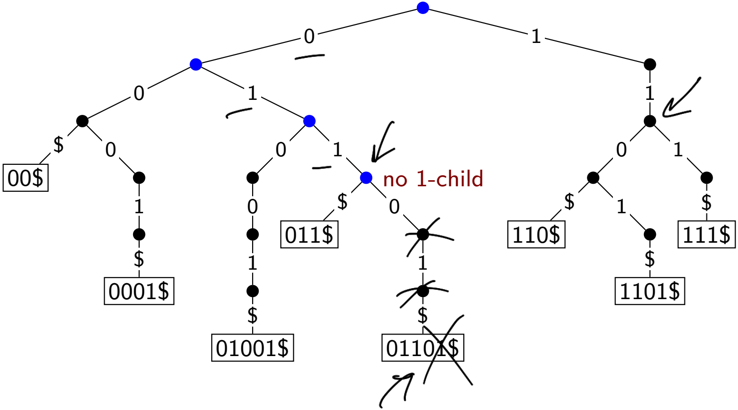
Tries: Search Example

Example: Trie::search(0111$)



Tries: Search Example

Example: Trie::search(0111$) unsuccessful

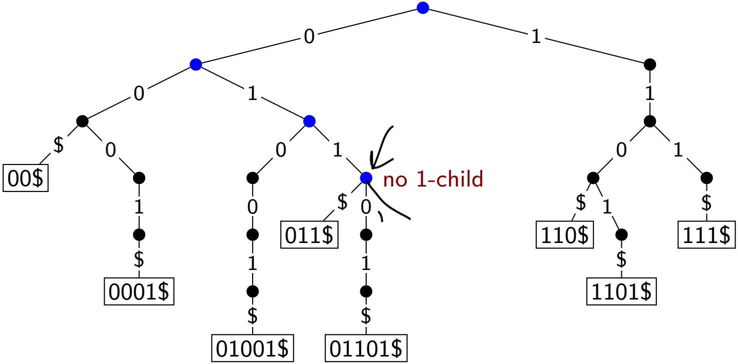


Tries: Insert & Delete

- *Trie::insert(x)*
  - Search for x, this should be unsuccessful
  - Suppose we finish at a node v that is missing a suitable child. Note: x has extra bits left.
  - Expand the trie from the node v by adding necessary nodes that correspond to extra bits of x.
- *Trie::delete*(x)
  - Search for x
  - let v be the leaf where x is found
  - delete v and all ancestors of v until we reach an ancestor that has two children.
- **Time Complexity** of all operations: $\boxed{\Theta(|x|)}$
  $|x|$: length of binary string x, i.e., the number of bits in x

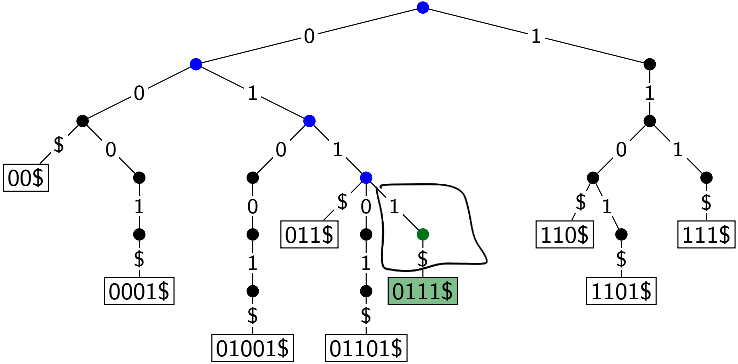
Tries: Insert Example

Example: *Trie::insert*(0111\$)



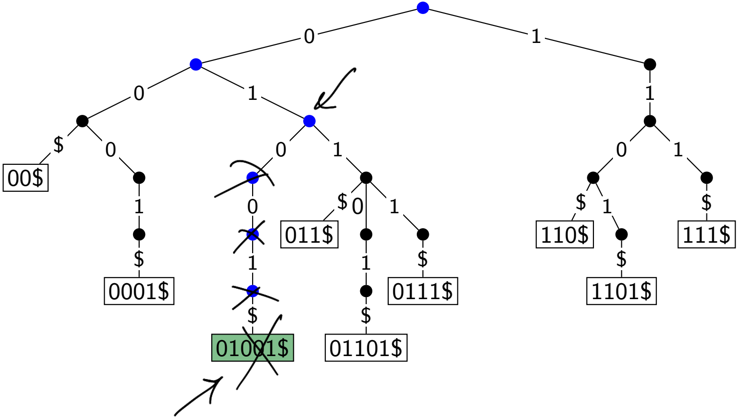
Tries: Insert Example

Example: *Trie::insert*(0111$)



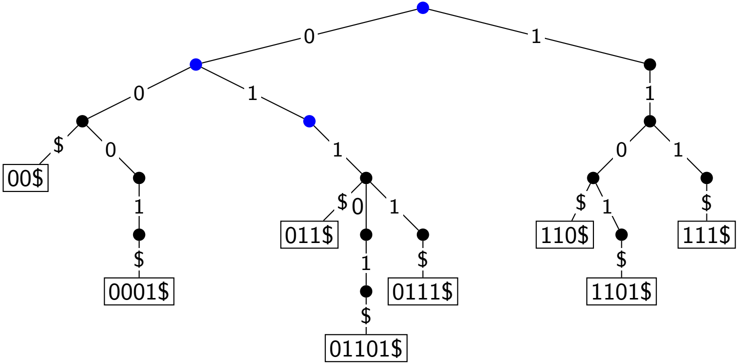
Tries: Delete Example

Example: *Trie::delete*(01001$)



Tries: Delete Example

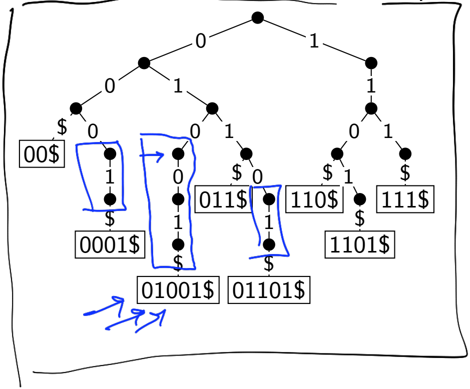
Example: *Trie::delete*(01001\$)

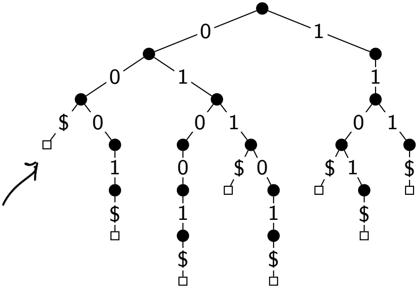


Variation 1 of Tries: No leaf labels

Do not store actual keys at the leaves.

- The key is stored implicitly through the characters along the path to the leaf. It therefore need not be stored again.
- This halves the amount of space needed.

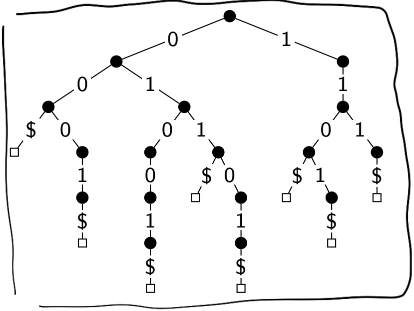


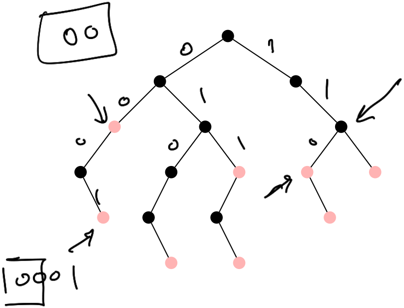


Variation 2 of Tries: Allow Proper Prefixes

Allow prefixes to be in dictionary.

- Internal nodes may now also represent keys.
  Use a *flag* to indicate such nodes.
- No need for end-of-word character $
- Now a trie of bitstrings is a binary tree. Can express 0-child and 1-child implicitly via left and right child.
- More space-efficient.

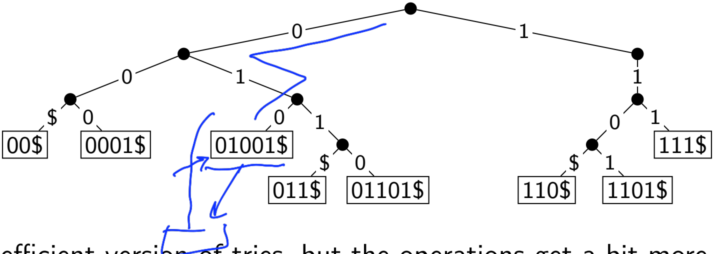




Variations 3 of Tries

Pruned Trie: Stop adding nodes to trie as soon as the key is unique.

- A node has a child only if it has at least two descendants.
- Note that now we *must* store the full keys (why?)
- Saves space if there are only few bitstrings that are long.
- Could even store infinite bitstrings (e.g. real numbers)

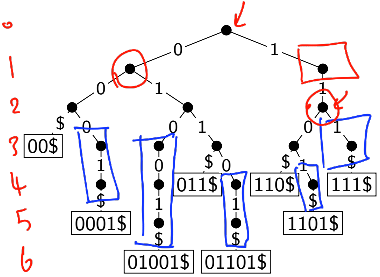


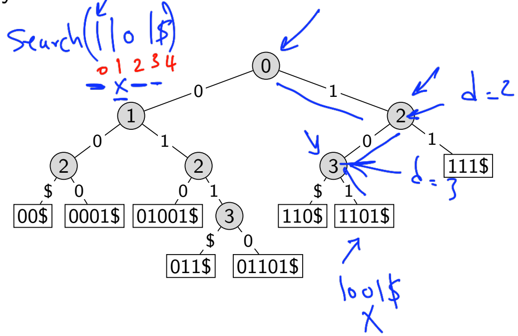
A more efficient version of tries, but the operations get a bit more complicated.

Variation 4 of Tries

Compressed Trie: compress paths of nodes with only one child

- Each node stores an *index*, corresponding to the depth in the uncompressed trie.
  - This gives the next bit to be tested during a search
- A compressed trie with n keys has at most $n-1$ internal nodes





Also known as **Patricia-Tries**:
Practical Algorithm to Retrieve Information Coded in Alphanumeric

Compressed Tries: Search

- start from the root and the bit indicated at that node
- follow the link that corresponds to the current bit in x; return failure if the link is missing
- if we reach a leaf, expicitly check whether word stored at leaf is x
- else recurse on the new node and the next bit of x

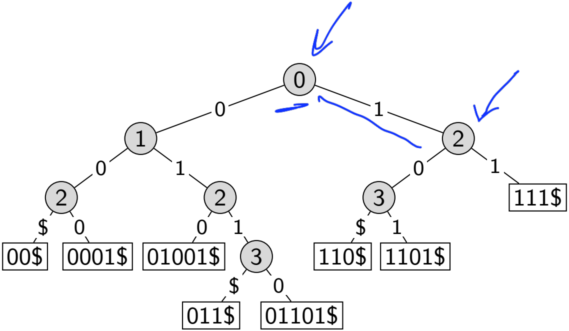
$|x|$

```
CompressedTrie::search(v ← root, x)
v: node of trie; x: word
    if v is a leaf
        return strcmp(x, v.key)
    d ← index stored at v
    if x has at most d bits
        return "not found"
    v′ ← child of v labelled with x[d]
    if there is no such child
        return "not found"
    CompressedTrie::search(v′, x)
```

Compressed Tries: Search Example

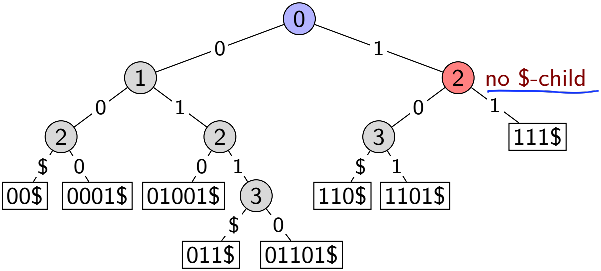
Example: CompressedTrie::search(10$)

0 1 2



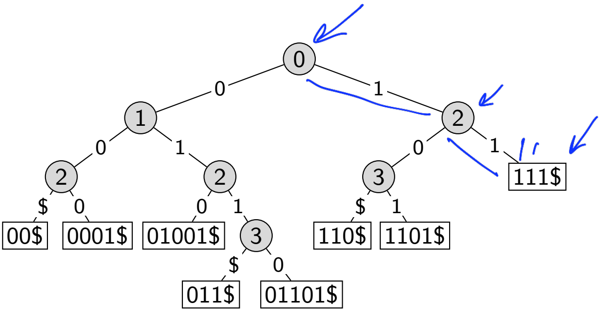
Compressed Tries: Search Example

Example: CompressedTrie::search(10$) unsuccessful



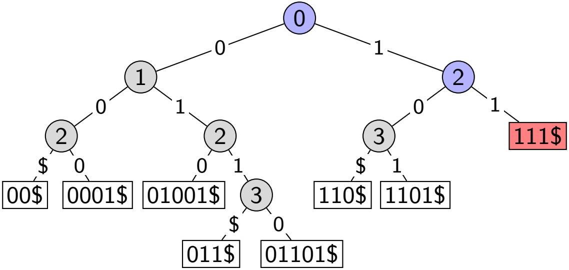
Compressed Tries: Search Example

Example: CompressedTrie::search(101$)



Compressed Tries: Search Example

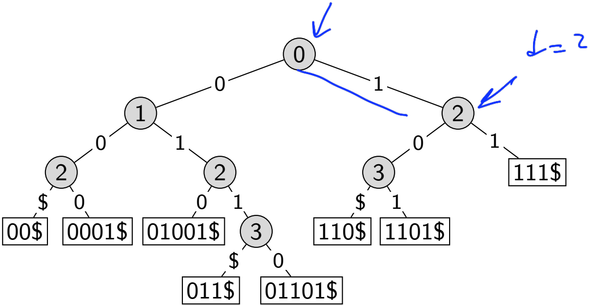
Example: CompressedTrie::search(101$) unsuccessful



Compressed Tries: Search Example

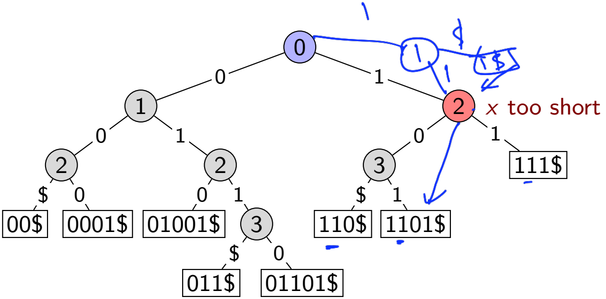
Example: CompressedTrie::search(1$)

0 1



Compressed Tries: Search Example

Example: CompressedTrie::search(1$) unsuccessful



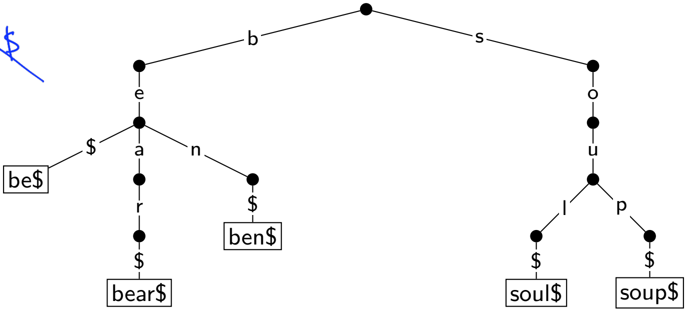
Compressed Tries: Insert & Delete

- *CompressedTrie::delete*(x): $O(|x|)$
  - Perform *search*(x)
  - Remove the node v that stored x
  - Compress along path to v whenever possible.
- *CompressedTrie::insert*(x):
  - Perform *search*(x)
  - Let v be the node where the search ended.
  - Conceptually simplest approach:
    - ★ Uncompress path from root to v.
    - ★ Insert x as in an uncompressed trie.
    - ★ Compress paths from root to v and from root to x.
  - But it can also be done by only adding those nodes that are needed.
  - Requires **leaf-links**: Every node stores a link to a leaf that is a descendant.
- All operations take $O(|x|)$ time.

Much more complicated, but space-savings are worth it if words are unevenly distributed.

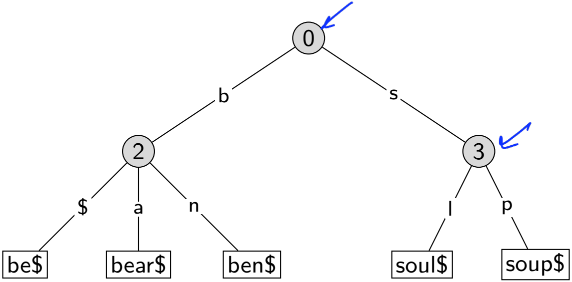
Multiway Tries: Larger Alphabet

- To represent *strings* over any *fixed alphabet* Σ
- Any node will have at most $|\Sigma| + 1$ children (one child for the end-of-word character \$)
- Example: A trie holding strings {bear\$, ben\$, be\$, soul\$, soup\$}



Compressed Multiway Tries

- **Variation**: Compressed multi-way tries: compress paths as before
- Example: A compressed trie holding strings {bear$, ben$, be$, soul$, soup$}



Multiway Tries: Summary

- Operations *search*(x), *insert*(x) and *delete*(x) are exactly as for tries for bitstrings.
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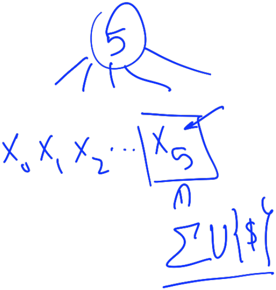

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$\Sigma = \{a, b, c, d\}$

Solution 1: Array of size $|\Sigma| + 1$ for each node.
Complexity: $O(1)$ time to find child, $O(|\Sigma|)$ space per node.

Solution 2: List of children for each node.
Complexity: $O(|\Sigma|)$ time to find child, $O(\#\text{children})$ space per node.

Solution 3: Dictionary (AVL-tree?) of children for each node.
Complexity: $O(\log(\#\text{children}))$ time, $O(\#\text{children})$ space per node.
Best in theory, but not worth it in practice unless $|\Sigma|$ is huge.

In practice, use *hashing* (keys are in (typically small) range Σ).