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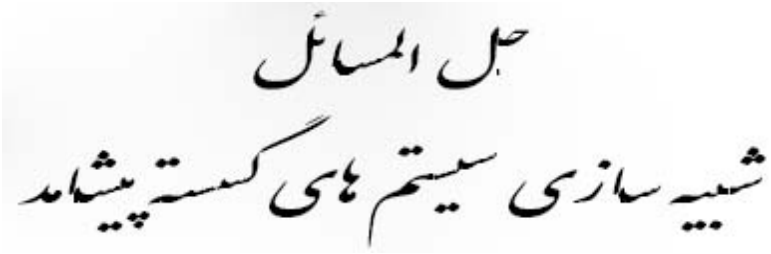

Third Edition

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Foreword

There are approximately three hundred exercises for solution in the text. These exercises emphasize principles of discrete-event simulation and provide practice in utilizing concepts found in the text.

Answers provided here are selective, in that not every problem in every chapter is solved. Answers in some instances are suggestive rather than complete. These two caveats hold particularly in chapters where building of computer simulation models is required. The solutions manual will give the instructor a basis for assisting the student and judging the student's progress. Some instructors may interpret an exercise differently than we do, or utilize an alternate solution method; they are at liberty to do so. We have provided solutions that our students have found to be understandable.

When computer solutions are provided they will be found on the text web site, `www.bcnn.net`, rather than here. We have invited simulation software vendors to submit solutions to a number of modeling and analysis problems; these solutions will also be found on the web site. Instructors are encouraged to submit solutions to the web site as well.

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Chapter 1

Introduction to Simulation

For additional solutions check the course web site at `www.bcnn.net`.

1. Solution to Exercise 1:

	SYSTEM	ENTITIES	ATTRIBUTES	ACTIVITIES	EVENTS	STATE VARIABLES
a.	Small appliance repair shop	Appliances	Type of appliance Age of appliance Nature of problem	Repairing the appliance	Arrival of a job Completion of a job	Number of appliances waiting to be repaired Status of repair person busy or idle
b.	Cafeteria	Diners	Size of appetite Entree preference	Selecting food Paying for food	Arrival at service line Departures from service line	Number of diners in waiting line Number of servers working
c.	Grocery store	Shoppers	Length of grocery list	Checking out	Arrival at checkout counters Departure from checkout counter	Number of shoppers in line Number of checkout lanes in operation
d.	Laundromat	Washing machine	Breakdown rate	Repairing a machine	Occurrence of breakdowns Completion of service	Number of machines running Number of machines in repair Number of Machines waiting for repair

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	SYSTEM	ENTITIES	ATTRIBUTES	ACTIVITIES	EVENTS	STATE VARIABLES
e.	Fast food restaurant	Customers	Size of order desired	Placing the order Paying for the order	Arrival at the counter Completion of purchase	Number of customers waiting Number of positions operating
f.	Hospital emergency room	Patients	Attention level required	Providing service required	Arrival of the patient Departure of the patient	Number of patients waiting Number of physicians working
g.	Taxicab company	Fares	Origination Destination	Traveling	Pick-up of fare Drop-off of fare	Number of busy taxi cabs Number of fares waiting to be picked up
h.	Automobile assembly line	Robot welders	Speed Breakdown rate	Spot welding	Breaking down	Availability of machines

3. Abbreviated solution to Exercise 3:

Iteration	Problem Formulation	Setting of Objectives and Overall Project Plan
1	Cars arriving at the intersection are controlled by a traffic light. The cars may go straight, turn left, or turn right.	How should the traffic light be sequenced? Criterion for evaluating effectiveness: average delay time of cars. Resources required: 2 people for 5 days for data collection, 1 person for 2 days for data analysis, 1 person for 3 days for model building, 1 person for 2 days for running the model, 1 person for 3 days for implementation.
2	Same as 1 above plus the following: Right on red is allowed after full stop provided no pedestrians are crossing and no vehicle is approaching the intersection.	How should the traffic light be sequenced? Criterion for evaluating effectiveness: average delay time of cars. Resources required: 2 people for 8 days for data collection, 1 person for 3 days for data analysis, 1 person for 4 days for model building, 1 person for 2 days for running the model, 1 person for 3 days for implementation.
3	Same as 2 above plus the following: Trucks arrive at the intersection. Vehicles break down in the intersection making one lane impassable. Accidents occur blocking traffic for varying amounts of time.	How should the traffic light be sequenced? Should the road be widened to 4 lanes? Method of evaluating effectiveness: average delay time of all vehicles. Resources required: 2 people for 10 days for data collection, 1 person for 5 days for data analysis, 1 person for 5 days for model building, 1 person for 3 days for running the model, 1 person for 4 days for implementation.

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4. Solution to Exercise 4:

Data Collection (step 4) - Storage of raw data in a file would allow rapid accessibility and a large memory at a very low cost. The data could be easily augmented as it is being collected. Analysis of the data could also be performed using currently available software.

Model Translation (step 5) - Many simulation languages are now available (see Chapter 4).

Validation (step 7) - Validation is partially a statistical exercise. Statistical packages are available for this purpose.

Experimental Design (step 3) - Same response as for step 7.

Production Runs (step 9) - See discussion of step 5 above.

Documentation and Reporting (step 11) - Software is available for documentation assistance and for report preparation.

5. Data Needed

Number of guests attending
Time required for boiling water
Time required to cook pasta
Time required to dice onions, bell peppers, mushrooms
Time required to saute onions, bell peppers, mushrooms, ground beef
Time required to add necessary condiments and spices
Time required to add tomato sauce, tomatoes, tomato paste
Time required to simmer sauce
Time required to set the table
Time required to drain pasta
Time required to dish out the pasta and sauce

Events

Begin cooking
Complete pasta cooking } Simultaneous
Complete sauce cooking }
Arrival of dinner guests
Begin eating

Activities

Boiling the water
Cooking the pasta
Cooking sauce
Serving the guests

State variables

Number of dinner guests
Status of the water (boiling or not boiling)
Status of the pasta (done or not done)
Status of the sauce (done or not done)

7. Event

Deposit
Withdrawal

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Activities
- Writing a check
- Cashing a check
- Making a deposit
- Verifying the account balance
- Reconciling the checkbook with the bank statement

Chapter 2

Simulation Examples

For additional solutions check the course web site at `www.bcnn.net`.

4. Solution to Exercise 4:

$\widehat{L} = \sum_{i=0}^{\infty} iT_i/T$ where

$\widehat{L}$ = time weighted average number of customers in the system

T_i = total time during $[0, T]$ in which the system contains exactly i customers

$\widehat{L} = \sum_{i=0}^{4} iT_i/86 = [0(18) + 1(32) + 2(20) + 3(14) + 4(2)]/86 = 1.419$ customers

$\widehat{L}_Q = \sum_{i=0}^{\infty} iT_i^Q/T$ where

$\widehat{L}_Q$ = time weighted average number of customers waiting during $[0, T]$

T_i^Q = Total time during $[0, T]$ in which exactly i customers are waiting in the queue

$\widehat{L}_Q = [0(50) + 1(20) + 2(14) + 3(2)]/86 = .628$ customers

6. Solution to Exercise 6:

New Service Distribution for Able

Service Time	Probability	Cumulative Probability	RD Assignment
3	.30	.30	01-30
4	.30	.60	31-60
5	.25	.85	61-85
6	.15	1.00	86-00

6a.

Number	RD for Arrival	Inter-Arrival Time	Arrival Clock Time	RD for Service	Able			Baker			Time in Queue
					Time Service Begins	Service Time	Time Service Ends	Time Service Begins	Service Time	Time Service Ends	
1	-	-	0	95				0	6	6	0
2	26	2	2	25	2	3	5				0
3	98	4	6	51				6	4	10	0
4	90	4	10	92				10	6	16	0
5	26	2	12	89	12	6	18				0
6	42	2	14	38				16	4	20	2
7	74	3	17	13	18	3	21				1
8	80	3	20	61				20	5	25	0
.											
.											
.											
25	16	1	55	87	6	63					2

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Typical results of a simulation:

Able serves only 12 cars rather than 16 as in the previous simulation.
Average time in queue = 1.5 minutes.

6b. Simulation for Able, Baker and Charlie using some random digits.

					Able			Baker			Charlie			
Number	RD for Arrival	Inter-Arrival Time	Arrival Clock Time	RD for Service	Time Service Begins	Service Time	Time Service Ends	Time Service Begins	Service Time	Time Service Ends	Time Service Begins	Service Time	Time Service Ends	Time in Queue
1	-	-	0	95				0	6	6			0	
2	26	2	2	25	2	3	5						0	
3	98	4	6	51				6	4	10				0
4	90	4	10	92				10	6	16				0
5	26	2	12	89	12	6	18				14	4	18	0
6	42	2	14	38						20				2
7	74	3	17	13				17	3					0
8	80	3	20	61				20	5	25				0
.														
.														
.														
25	16	1	25	55							55	6	61	0
26	74	4	59	47				59	4	63				

Typical results of a simulation:

Baker still has first shot at cars and thus has the most, or 12.
Able serves 8 cars, and Charlie gets the leftovers, or 6 cars.
There is no waiting time in the queue.

10. Profit = Revenue from retail sales - Cost of bagels made + Revenue from grocery store sales - Lost profit.

Let Q = number of dozens baked/day
$S = \sum_i 0_i$, where 0_i = Order quantity in dozens for the ith customer
$Q - S$ = grocery store sales in dozens, $Q > S$
$S - Q$ = dozens of excess demand, $S > Q$

$$\text{Profit} = \$5.40 \min(S, Q) - \$3.80Q + \$2.70(Q - S) - \$1.60(S - Q)$$

Number of Customers	Probability	Cumulative Probability	RD Assignment
8	.35	.35	01-35
10	.30	.65	36-65
12	.25	.90	66-90
14	.10	1.00	91-100

Dozens Ordered	Probability	Cumulative Probability	RD Assignment
1	.4	.4	1-4
2	.3	.7	5-7
3	.2	.9	8-9
4	.1	1.0	0

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Pre-analysis

$$\begin{aligned} E(\text{Number of Customers}) &= .35(8) + .30(10) + .25(12) + .10(14) \\ &= 10.20 \end{aligned}$$

$$E(\text{Dozens ordered}) = .4(1) + .3(2) + .2(3) + .1(4) = 2$$

$$E(\text{Dozens sold}) = \bar{S} = (10.20)(2) = 20.4$$

$$\begin{aligned} E(\text{Profit}) &= \$5.40\text{Min}(\bar{S}, Q) - \$3.80Q + \$2.70(Q - \bar{S}) - \$1.60(\bar{S} - Q) \\ &= \$5.40\text{Min}(20.4, Q) - \$3.80Q + \$2.70(Q - 20.4) \\ &\quad -\$0.67(20.4 - Q) \end{aligned}$$

$$E(\text{Profit}|Q = 0) = 0 - 0 + \$1.60(20.4) = -\$32.64$$

$$\begin{aligned} E(\text{Profit}|Q = 10) &= \$5.40(10) - \$3.80(10) + 0 - \$1.60(20.4 - 10) \\ &= -\$0.64 \end{aligned}$$

$$\begin{aligned} E(\text{Profit}|Q = 20) &= \$5.40(20) - \$3.80(20) + 0 - \$1.60(20.4 - 20) \\ &= \$15.36 \end{aligned}$$

$$\begin{aligned} E(\text{Profit}|Q = 30) &= \$5.40(20.4) - \$3.80(30) + \$2.70(30 - 20.4) - 0 \\ &= \$22.08 \end{aligned}$$

$$\begin{aligned} E(\text{Profit}|Q = 40) &= \$5.40(20.4) - \$3.80(40) + \$2.70(40 - 20.4) - 0 \\ &= \$11.08 \end{aligned}$$

The pre-analysis, based on expectation only, indicates that simulation of the policies $Q = 20, 30$, and 40 should be sufficient to determine the policy. The simulation should begin with $Q = 30$, then proceed to $Q = 40$, then, most likely to $Q = 20$.

Initially, conduct a simulation for $Q = 20, 30$ and 40. If the profit is maximized when $Q = 30$, it will become the policy recommendation.

The problem requests that the simulation for each policy should run for 5 days. This is a very short run length to make a policy decision.

$Q = 30$

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Day	RD for Customer	Number of Customers	RD for Demand	Dozens Ordered	Revenue from Retail $	Lost Profit $
1	44	10	8	3	16.20	0
			2	1	5.40	0
			4	1	5.40	0
			8	3	16.20	0
			1	1	5.40	0
			6	2	10.80	0
			3	1	5.40	0
			0	4	21.60	0
			2	1	5.40	0
			0	4	21.60	0
				21	113.40	0

For Day 1,

Profit $= \$113.40 - \$152.00 + \$24.30 - 0 = \$14.30$
Days 2, 3, 4 and 5 are now analyzed and the five day total profit is determined.

11. Solution to Exercise 11:

Daily Demand	Probability	Cumulative Probability	RD Assignment
0	.33	.33	01-33
1	.25	.58	34-58
2	.20	.78	59-78
3	.12	.90	79-90
4	.10	1.00	91-00

Lead Time	Probability	Cumulative Probability	RD Assignment
1	.3	.3	1-3
2	.5	.8	4-8
3	.2	1.0	9-0

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Cycle	Day	Beginning Inventory	RD for Demand	Demand	Ending Inventory	Shortage Quantity	Order Quantity	RD for Lead Time	Days Until Order Arrives
1	1	12	56	1	11	0			
	2	11	30	0	11	0			
	3	11	79	3	8	0			
	4	8	84	3	5	0			
	5	5	20	0	5	0			
	6	5	10	0	5	0			
	7	5	83	3	2	0	10	2	1
2	1	2	62	2	0	0			0
	2	10	58	1	9	0			
	3	9	32	0	9	0			
	4	9	42	1	8	0			
	5	8	87	3	5	0			
	6	5	88	3	2	0			
	7	2	00	4	0	2	10	7	2
⋮									
6	1	0	71	2	0	2			1
	2	10	34	1	7	0			0
	3	7	14	0	7	0			
	4	7	46	1	6	0			
	5	6	84	3	3	0			
	6	3	09	0	3	0			
	7	3	65	2	1	0	10	2	1

Typical results from simulation of current system:

Probability of shortage = 0.25

Average ending inventory = 3.5 units

Effect on Shortages Caused by Policy Variable Changes

	Policy Variable		
Change	Review Period	Reorder Quantity	Reorder Point
Increase	Increase	Decrease	No effect in this case since all values were below current reorder point.
Decrease	Decrease	Increase	Decrease would have to be drastic, say to a reorder point of < 2 units. Such a change would increase shortages.

12. Solution to Exercise 12:

Daily Demand	Probability	Cumulative Probability	RD Assignment
0	.18	.18	01-18
1	.39	.57	19-57
2	.29	.86	58-86
3	.09	.95	87-95
4	.05	1.00	96-00

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Lead Time	Probability	Cumulative Probability	RD Assignment
0	.135	.135	001-135
1	.223	.358	136-358
2	.288	.646	359-646
3	.213	.859	647-859
4	.118	.977	860-977
5	.023	1.000	978-000

Cycle	RD for Lead Time	Lead Time	RD for Demand	Demand	Lead Time Demand
1	024	0	-	-	0
2	330	1	14	0	0
3	288	1	53	1	1
4	073	0	-	-	0
5	197	1	24	1	1
6	924	4	53	1	
			81	2	
			70	2	
			18	0	5

Narrow histogram intervals (say 1 time unit) seem to be more descriptive and less blocky than larger intervals. For a realistic determination many more cycles would need to be simulated. With a large number of cycles, narrow histogram intervals will probably be favored.

15. Solution to Exercise 15:

Time Between Calls	Probability	Cumulative Probability	RD Assignment
15	.14	.14	01-14
20	.22	.36	15-36
25	.43	.79	37-79
30	.17	.96	80-96
35	.04	1.00	97-00

Service Time	Probability	Cumulative Probability	RD Assignment
5	.12	.12	01-12
15	.35	.47	13-47
25	.43	.90	48-90
35	.06	.96	91-96
45	.04	1.00	97-00

First, simulate for one taxi for 5 days.
Then, simulate for two taxis for 5 days. } Shown on simulation tables

Comparison

Smalltown Taxi would have to decide which is more important—paying for about 43 hours of idle time in a five day period with no customers having to wait, or paying for around 4 hours of idle time in a five day period, but having a probability of waiting equal to 0.59 with an average waiting time for those who wait of around 20 minutes.

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One Taxi

Day	Call	RD for Time between Calls	Time between Calls	Call Time	RD for Service Time	Service Time	Time Service Begins	Time Customer Waits	Time Service Ends	Time Customer in System	Idle Time of Taxi
1	1	15	-	0	01	5	0	0	5	5	0
	2	01	20	20	53	25	20	0	55	25	0
	3	14	15	35	62	25	55	20	80	45	0
	4	65	25	60	55	25	80	20	105	45	0
	5	73	25	85	95	35	105	20	140	55	0
	6	48	25	110	22	15	140	30	155	45	0
	⋮										
	20	77	25	444	63	25	470	25	495	50	0
2											
⋮											

Typical results for a 5 day simulation:

Total idle time = 265 minutes = 4.4 hours
Average idle time per call = 2.7 minutes
Proportion of idle time = .11
Total time customers wait = 1230 minutes
Average waiting time per customer = 11.9 minutes
Number of customers that wait = 61 (of 103 customers)
Probability that a customer has to wait = .59
Average waiting time of customers that wait = 20.2 minutes

Two taxis (using common RDs for time between calls and service time)

					Taxi 1			Taxi 2						
Day	Call	Time between Calls	Call Time	Service Time	Time Service Begins	Service Time	Time Service Ends	Time Service Begins	Service Time	Time Service Ends	Time Customer Waits	Time Customer in System	Idle Time Taxi 1	Idle Time Taxi 2
1	1	-	0	5	0	5	5				0	5		
	2	20	20	25	20	25	45				0	25		
	3	15	35	25				35	25	60	0	25		35
	4	25	60	25	60	25	85				0	25	15	
	5	25	85	35	80	35	120				0	35		
	6	25	110	15				110	15	125	0	15		50
	⋮													
	20	20	480	25	480	25	505				0	25	10	
2														
⋮														

Typical results for a 5 day simulation:

Idle time of Taxi 1 = 685 minutes
Idle time of Taxi 2 = 1915 minutes
Total idle time = 2600 minutes = 43 hours
Average idle time per call = 25.7 minutes
Proportion of idle time = .54
Total time customers wait = 0 minutes
Number of customers that wait = 0

17. Solution to Exercise 17:

$$\begin{aligned} X &= 100 + 10RNN_x \\ Y &= 300 + 15RNN_y \\ Z &= 40 + 8RNN_z \end{aligned}$$

Typical results...

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	RNN_x	X	RNN_y	Y	RNN_z	Z	W
1	-.137	98.63	.577	308.7	-.568	35.46	11.49
2	.918	109.18	.303	304.55	-.384	36.93	11.20
3	1.692	116.92	-.383	294.26	-.198	38.42	10.70
4	-.199	98.01	1.033	315.50	.031	40.25	10.27
5	-.411	95.89	.633	309.50	.397	43.18	9.39
⋮							

19. Solution to Exercise 19:

$$\begin{aligned} T &= \text{Lead Time} \\ T &\sim N(7, 2^2) \\ T &= 7 + 2(RNN)(\text{Rounded to nearest integer}) \end{aligned}$$

Daily Demand	Probability	Cumulative Probability	RD Assignment
0	0.367	0.367	001-367
1	0.368	0.735	368-735
2	0.184	0.919	736-919
3	0.062	0.981	920-981
4	0.019	1.000	982-000

Cycle	RNN for Lead Time	Lead Time	Day	RD for Demand	Demand	Lead Time Demand
1	-.82	5	1	127	0	
			2	313	0	
			3	818	2	
			4	259	0	
			5	064	0	2
2	-.45	6	1	912	2	
			2	651	1	
			3	139	0	
			4	288	0	
			5	524	1	
			6	772	2	6
⋮						

21. Solution to Exercise 21:

Lead Time (Days)	Probability	Cumulative Probability	RD Assignment
0	.166	.166	001-166
1	.166	.332	167-332
2	.166	.498	333-498
3	.166	.664	499-664
4	.166	.830	665-830
5	.166	.996	831-996
			996-000 (discard)

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Assume 5-day work weeks.

$$D = \text{Demand}$$
$$D = 5 + 1.5(RNN)(\text{ Rounded to nearest integer})$$

Week	Day	Beginning Inventory	RNN for Demands	Demand	Ending Inventory	Order Quantity	RD for Lead Time	Lead Time	Lost Sales
1	1	18	-1.40	3	15				0
	2	15	-.35	4	11				0
	3	11	-.38	4	7	13	691	4	0
	4	7	.05	5	2				0
	5	2	.36	6	0				4
2	6	0	.00	5	0				5
	7	0	-.83	4	0				4
	8	13	-1.83	2	11				0
	9	11	-.73	4	7	13	273	1	0
	10	7	-.89	4	3				0
⋮									

Typical results
Average number of lost sales/week = 24/5 = 4.8 units/weeks

22. Solution to Exercise 22:

Material A (200kg/box)

Interarrival Time	Probability	Cumulative Probability	RD Assignment
3	.2	.2	1-2
4	.2	.4	3-4
5	.2	.6	5-6
6	.2	.8	7-8
7	.2	1.0	9-0

Box	RD for Interarrival Time	Interarrival Time	Clock Time
1	1	3	3
2	4	4	7
3	8	6	13
4	3	4	17
⋮			
14	4	4	60

Material B (100kg/box)

Box	1	2	3	⋯	10
Clock Time	6	12	18	⋯	60

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Material C (50kg/box)

Interarrival Time	Probability	Cumulative Probability	RD Assignment
2	.33	.33	01-33
3	.67	1.00	34-00

Box	RD for Interarrival Time	Interarrival Time	Clock Time
1	58	5	3
2	92	3	6
3	87	3	9
4	31	2	11
⋮	⋮	⋮	⋮
22	62	3	60

Clock Time	A Arrival	B Arrival	C Arrival
3	1		1
6		1	2
7	2		
9			3
11			4
12		2	
⋮			

Simulation table shown below.
Typical results:
Average transit time for box A ($\bar{t}_A$)

$$\begin{aligned}\bar{t}_A &= \frac{\text{Total waiting time of } A + \text{ (No. of boxes of } A)(1 \text{ minute up to unload})}{\text{No. of boxes of } A} \\ &= \frac{28 + 12(1)}{12} = 3.33 \text{ minutes}\end{aligned}$$

Average waiting time for box B ($\bar{w}_B$)

$$\bar{w}_B = \frac{(\text{Total time } B \text{ in Queue})}{\text{No. of boxes of } B} = \frac{10}{10} = 1 \text{ minute/box of } B$$

Total boxes of C shipped = Value of C Counter = 22 boxes

Clock Time	No. of A in Queue	No. of B in Queue	No. of C in Queue	Queue Weight	Time Service Begins	Time Service Ends	Time A in Queue	Time B in Queue	A Counter	B Counter	C Counter
3	1	0	1	250							
6	0	0	0	0	6	10	3	0	1	1	2
7	1	0	0	200							
9	1	0	1	250							
11	1	0	2	300							
12	0	0	0	350	12	16	5	0	2	2	4
⋮											

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25. Solution can be obtained from observing those clearance values in Exercise 24 that are greater than 0.006.

26. Degrees $=360(RD/100)$

Replication 1

RD	Degrees
57	205.2
45	162.0
22	79.2

Range $= 205.2^0 - 79.2^0 = 126^0$ (on the same semicircle).
Continue this process for 5 replications and estimate the desired probability.

27. Solution to Exercise 27:

$$V = 1.02^2 + (-.72)^2 + .28^2 = 1.7204$$

$$T = \frac{^{-}.18}{\sqrt{\frac{1.7204}{3}}} = -.2377$$

28. Solution to Exercise 28:

Cust.	RD for Arrival	IAT	AT	RD for Service	Serv. Time	No. in Queue	TimeServ. Begins	Time Serv. Ends	Go Into Bank?
1	30	2	2	27	2	1	-		√
2	46	2	4	26	2	0	4	6	
3	39	2	6	99	4	0	6	10	
4	86	4	10	72	3	0	10	13	
5	63	3	13	12	1	0	13	14	
6	83	4	17	17	1	0	17	18	
7	07	0	17	78	3	1	18	21	
8	37	2	19	91	4	1	-		√
9	69	3	22	82	3	0	22	25	
10	78	4	26	62	3	0	26	29	

Chapter 3

General Principles

For solutions check the course web site at `www.bcnn.net`.

Chapter 4

Simulation Software

For solutions check the course web site at `www.bcnn.net`.

Chapter 5

Statistical Models in Simulation

1. Let X be defined as the number of defectives in the sample. Then X is binomial $(n = 100, p = .01)$ with the probability mass function

$$p(x) = \binom{100}{x} (.01)^x (.99)^{100-x}, x = 0, 1, \ldots, 100$$

The probability of returning the shipment is

$$\begin{aligned} P(X > 2) &= 1 - P(X \leq 2) \\ &= 1 - \binom{100}{0} (.99)^{100} - \binom{100}{1} (.01)(.99)^{99} \\ &\quad - \binom{100}{2} (.01)^2 (.99)^{98} = .0794 \end{aligned}$$

2. Let X be defined as the number of calls received until an order is placed. Then, X is geometric $(p = .48)$ with the probability mass function

$$p(x) = (.52)^{x-1}(.48), x = 0, 1, 2 \ldots$$

(a) The probability that the first order will come on the fourth call is

$$p(4) = .0675$$

(b) The number of orders, Y, in eight calls is binomial $(n = 8, p = .48)$ with the probability mass function

$$p(y) = \binom{8}{y} (.48)^y (.52)^{8-y}, y = 0, 1, \ldots, 8$$

The probability of receiving exactly six orders in eight calls is

$$p(6) = .0926$$

(c) The number of orders, X, in four calls is binomial $(n = 4, p = .48)$ with probability mass function

$$p(x) = \binom{4}{x} (.48)^x (.52)^{8-x}, x = 0, 1, 2, 3, 4$$

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The probability of receiving one or fewer orders in four calls is

$$\begin{aligned} P(X \leq 1) &= \binom{4}{0}(.52)^4 + \binom{4}{1}(.48)(.52)^3 \\ &= .3431 \end{aligned}$$

3. Let X be defined as the number of women in the sample never married

$$\begin{aligned} P(2 \leq X \leq 3) &= p(2) + p(3) \\ &= \binom{20}{2}(.18)^2(.82)^{18} + \binom{20}{3}(.18)^3(.82)^{17} \\ &= .173 + .228 = .401 \end{aligned}$$

4. Let X be defined as the number of games won in the next two weeks. The random variable X is described by the binomial distribution:

$$p(x) = \binom{5}{x}(.55)^x(.45)^{5-x}$$

$$P(3 \leq X \leq 5) = p(3) + p(4) + p(5)$$

$$= \binom{5}{3}(.55)^3(.45)^2 + \binom{5}{4}(.55)^4(.45) + \binom{5}{5}.55^5$$

$$= .337 + .206 + .050 = .593$$

5. Solution to Exercise 5:
(a) Using the geometric probability distribution, the desired probability is given by

$$p(.4) = (.6)^3(.4) = .0864$$

(b) Using the binomial distribution, the desired probability is given by

$$\begin{aligned} P(X \leq 2) &= \sum_{i=0}^{5}\binom{5}{i}(.4)^i(.6)^{5-i} \\ &= .07776 + .2592 + .3456 \\ &= .68256 \end{aligned}$$

6. $X = X_1 + X_2 \sim$ Erlang with $K\theta = 1$. Since $K = 2, \theta = 1/2$

$$F(2) = 1 - \sum_{i=0}^{1} e^{-2}2^i/i! = 0.406$$

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$$P(X_1 + X_2 > 2) = 1 - F(2) = .594$$

7. The geometric distribution is memoryless if

$$P(X > s + t | X > s) = P(X > t)$$

where s and t are integers and X is a geometrically distributed random variable. The probability of a failure is denoted by q and

$$\begin{aligned}
&P(X > s) = \sum_{j=s+1}^{\infty} q^{j-1}p = q^s, \\
&P(X > t) = q^t, \text{and} \\
&P(X > s + t) = q^{s+t}; \text{so,} \\
&P[(X > s + t)|X > s] = \left(q^{s+t}/q^s\right) = q^t
\end{aligned}$$

which is equal to $P(X > t)$.

8. The number of hurricanes per year, X, is Poisson ($\alpha = 0.8$) with the probability mass function

$$p(x) = e^{-0.8}(0.8)^x/x!, x = 0, 1, \ldots$$

(a) The probability of more than two hurricanes in one year is

$$\begin{aligned}
P(X > 2) &= 1 - P(X \leq 2) \\
&= 1 - e^{-0.8} - e^{-0.8}(0.8) - e^{-0.8}(0.8^2/2) \\
&= .0474
\end{aligned}$$

(b) The probability of exactly one hurricane in one year is

$$p(1) = .3595$$

9. The number of arrivals at a bank teller's cage, X, is Poisson ($\alpha = 1.2$) with the probability mass function

$$p(x) = e^{-1.2}(1.2)^x/x!, x = 0, 1, 2, \ldots$$

(a) The probability of zero arrivals during the next minute is

$$p(0) = .3012$$

(b) The probability of zero arrivals during the next two minutes ($\alpha = 2.4$) is $p(0) = 0.0907$.

10. Using the Poisson approximation with the mean, α, given by

$$\alpha = np = 200(.018) = 3.6$$

The probability that $0 \leq x \leq 3$ students will drop out of school is given by

$$F(3) = \sum_{x=o}^{3} \frac{e^{\alpha}\alpha^x}{x!} = .5148$$

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11. Let X be the number of calls received. The variance and mean are equal. Thus,

$$\sigma^2 = \alpha = 4$$

and the standard deviation is

$$\sigma = 2$$

Then using the Poisson distribution

$$P(X > \ 6) = 1 - .889 = .111$$

12. Let X be defined as the lead time demand. Then, X is Poisson ($\alpha = 6$) with cumulative distribution function

$$F(x) = \sum_{i=0}^{x} e^{-6}(6)^i/i!$$

The order size at various protection levels is given by:

Order Size	Protection(%)	$F(x)$
6	50	.606
8	80	.847
9	90	.916
10	95	.957
11	97	.979
11	97.5	.979
12	99	.991
13	99.5	.996
15	99.9	.999

13. A random variable, X, has a discrete uniform distribution if its probability mass function is

$$p(x) = 1/(n+1) \quad R_X = \{0, 1, 2, \ldots n\}$$

(a) The mean and variance are found by using

$$\sum_{i=0}^{n} i = [n(n+1)]/2 \text{ and}$$

$$\sum_{i=0}^{n} i^2 = [n(n+1)(2n+1)]/6$$

$$\begin{aligned} E(X) &= \sum_{i=0}^{n} x_i p(x_i) = \sum_{i=0}^{n} ip(i) \\ &= [1/(n+1)] \sum_{i=0}^{n} i = n/2 \\ V(X) &= E(X^2) - [E(X)]^2 \\ &= \sum_{i=0}^{n} x_i^2 p(x_i) - (n/2)^2 = (n^2 + 2n)/12 \end{aligned}$$

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(b) If $R_X = \{a, a+1, a+2, \ldots, b\}$, the mean and variance are

$$E(X) = a + (b-a)/2 = (a+b)/2$$
$$V(X) = [(b-a)^2 + 2(b-a)]/12$$

14. Let X be defined as the lifetime of the satellite. Then, X is exponential ($\lambda = .4$) with cumulative distribution function

$$F(x) = 1 - e^{-.4x}, x \geq 0$$

(a) The probability of the satellite lasting at least five years is

$$P(X \geq 5) = 1 - F(5) = .1353$$

(b) The probability that the satellite dies between three and six years is

$$P(3 \leq X \leq 6) = F(6) - F(3) = .2105$$

15. Let X be the number of hours until a crash occurs. Using the exponential distribution, the desired probability is given by

$$F(48) - F(24) = [1 - e^{-\frac{1}{36}(48)}] - [1 - e^{-\frac{1}{36}(24)}]$$

$$= e^{-2/3} - e^{-4/3} = .513 - .264 = .249$$

16. Let X be defined as the number of ball bearings with defects in a random sample of 4000 bearings. Then, X is binomial ($n = 4000, p = 1/800$) with probability mass function

$$p(x) = \binom{4000}{x} (1/800)^x (1 - (1/800))^{n-x}, \quad x = 0, 1, 2, \ldots, 4000$$

The probability that the random sample yields three or fewer ball bearings with defects is

$$\begin{aligned} P(X \leq 3) &= p(0) + p(1) + p(2) + p(3) \\ &= .2649 \end{aligned}$$

Also, X can be approximated as Poisson ($\lambda = 4000/800$) with a probability mass function

$$p(x) = e^{-5}(5)^x/x!, \quad x = 0, 1, 2, \ldots$$

The probability that the random sample yields three or fewer ball bearings with defects is

$$\begin{aligned} P(X \leq 3) &= p(0) + p(1) + p(2) + p(3) \\ &= .2650 \end{aligned}$$

17. An exponentially distributed random variable, X, that satisfies

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$$P(X \leq 3) = .9P(X \leq 4),$$

can be specified by letting

$$1 - e^{-3\lambda} = .9(1 - e^{-4\lambda})$$

By letting $z = e^{-\lambda}$,

$$0 = z^3 - .90z^4 - .10, \text{ or}$$
$$z = .6005 \text{ and } \lambda = .51$$

18. Let X be the number of accidents occuring in one week. The mean is given by

$$\alpha = 1$$

The probability of no accidents in one week is given by

$$p(0) = \frac{e^{-1}\alpha^0}{0!} = .368$$

The probability of no accidents in three successive weeks is given by

$$[p(0)]^3 = .368^3 = .05$$

19. Let X be defined as the lifetime of the component. Then X is exponential ($\lambda = 1/10,000$ hours) with cumulative distribution function

$$F(x) = 1 - e^{-x/10000}, \quad x > 0$$

Given that the component has not failed for $s = 10,000$ or $s = 15,000$ hours, the probability that it lasts 5000 more hours is

$$P(X \geq 5000 + s | X > s) = P(X \geq 5000) = .6065$$

In both cases, this is due to the memoryless property of the exponential distribution.

20. Let X be defined as the lifetime of the battery. Then, X is exponential ($\lambda = 1/48$) with cumulative distribution function

$$F(x) = 1 - e^{-x/48}, \quad x > 0$$

(a) The probability that the battery will fail within the next twelve months, given that it has operated for sixty months is

$$\begin{aligned} P(X \leq 72 | X > 60) &= P(X \leq 12) \\ &= F(12) = .2212 \end{aligned}$$

due to the memoryless property.

(b) Let Y be defined as the year in which the battery fails, Then,

$$P(Y = \text{odd year}) = (1 - e^{-.25}) + (e^{-.50}e^{-.75}) + \ldots$$

$$P(Y = \text{even year}) = (1 - e^{-.50}) + (e^{-.75} - e^{-1}) + \ldots$$

So,

$$P(Y = \text{even year}) = e^{-.25}P(Y = \text{odd year}),$$
$$P(Y = \text{even year}) + P(Y = \text{odd year}) = 1, \text{ and}$$
$$e^{-.25}P(Y = \text{odd year}) = 1 - P(Y = \text{odd year})$$

The probability that the battery fails during an odd year is

$$P(Y = \text{odd year}) = 1/(1 + e^{-.25}) = .5622$$

(c) Due to the memoryless property of the exponential distribution, the remaining expected lifetime is 48 months.

21. Service time, X_i, is exponential ($\lambda = 1/50$) with cumulative distribution function

$$F(x) = 1 - e^{-x/50}, x > 0$$

(a) The probability that two customers are each served within one minute is

$$P(X_1 \le 60, X_2 \le 60) = [F(60)]^2 = (.6988)^2 = .4883$$

(b) The total service time, $X_1 + X_2$, of two customers has an Erlang distribution (assuming independence) with cumulative distribution function

$$F(x) = 1 - \sum_{i=0}^{1}[e^{-x/50}(x/50)^i/i!], x > 0$$

The probability that the two customers are served within two minutes is

$$P(X_1 + X_2 \le 120) = F(120) = .6916$$

22. A random variable, X, has a triangular distribution with probability density function

$$f(x) = \begin{cases} [2(x-a)]/[(b-a)(c-a)], & a \le x \le b \\ [2(c-x)]/[(c-b)(c-a)], & b \le x \le c \end{cases}$$

The variance is

$$\begin{aligned} V(X) &= E(X^2) - [E(X)]^2 \\ E(X) &= (a+b+c)/3 \\ E(X^2) &= \left(\frac{2}{(b-a)(c-a)}\right)\int_a^b x^2(x-a)dx \\ &\quad + \left(\frac{2}{(c-b)(c-a)}\right)\int_b^c x^2(c-x)dx \\ &= [1/6(c-a)][c(c^2+cb+b^2) - a(b^2+ab+a^2)] \\ V(X) &= [(a+b+c)^2/18] - [(ab+ac+bc)/6] \end{aligned}$$

23. The daily use of water, X, is Erlang ($k = 2, \theta = .25$) with a cumulative distribution function

$$F(x) = 1 - \sum_{i=0}^{2-1}[e^{-x/2}(x/2)^i/i!], \quad x > 0$$

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The probability that demand exceeds 4000 liters is

$$P(X > 4) = 1 - F(4) = .4060$$

24. Let X_i be defined as the lifetime of the ith battery and $X = X_1 + X_2 + X_3$. Then X is Erlang $(k = 3, \theta = 1/36)$ with cumulative distribution function

$$F(x) = 1 - \sum_{i=0}^{3-1} [e^{-x/12}(x/12)^i/i!], \quad x > 0$$

The probability that three batteries are sufficient is

$$P(X > 30) = 1 - F(30) = .5438$$

25. Let X represent the time between dial up connections. The desired probability is Erlang distributed with

$$K\theta = 1/15 \text{ and } X = 30$$

The probability that the third connection occurs within 30 seconds is given by

$$\begin{aligned} F(30) &= 1 - \sum_{i=0}^{2} \frac{e^{-\frac{1}{15}(30)}[\frac{1}{15}30]^i}{i!} \\ &= .323 \end{aligned}$$

and its complement gives the desired probability, or $1 - .323 = .677$.

26. Let X represent the life of a single braking system. Using the Erlang distribution, the probability of a crash within 5,000 hours is given by

$$\begin{aligned} F(5,000) &= 1 - \frac{\sum_{i=0}^{1} e^{-2(8,000)(5,000)}[2(1/8,000)(5,000)]^i}{i!} \\ &= i - e^{-5/4} - e^{-5/4}(5/4) \\ &= 1 - .2865 - .3581 = .3554 \end{aligned}$$

The complement gives the desired probability, or,

$$p(\text{no crash}) = .6446$$

27. Let X represent the time until a car arrives. Using the Erlang distribution with

$$K\theta = 4 \text{ and } X = 1$$

the desired probability is given by

$$F(1) = 1 - \sum_{i=o}^{2} \frac{e^{-4(1)}[4(1)]^i}{i!} = .762$$

28. Let X be defined as the number of arrivals during the next five minutes. Then X is Poisson $(\alpha = 2.5)$ with cumulative distribution function

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$$F(x) = \sum_{i=0}^{x} e^{-2.5}(2.5)^i/i!, \quad x = 0, 1, \ldots$$

The probability that two or more customers will arrive in the next five minutes is

$$P(X \geq 2) = 1 - F(1) = .7127$$

29. Let X be defined as the grading time of all six problems. Then X is Erlang ($k = 6, \theta = 1/180$) with cumulative distribution function

$$F(x) = 1 - \sum_{i=0}^{6-1}[e^{-x/30}(x/30)^i/i!], \quad x > 0$$

(a) The probability that grading is finished in 150 minutes or less is

$$P(X \leq 150) = F(150) = .3840$$

(b) The most likely grading time is the mode $= (k-1)/k\theta = 150$ minutes.

(c) The expected grading time is

$$E(X) = 1/\theta = 180 \text{ minutes}$$

30. Let X be defined as the life of a dual hydraulic system consisting of two sequentially activated hydraulic systems each with a life, Y, which is exponentially distributed ($\lambda = 2000$ hours). Then X is Erlang ($k = 2, \theta = 1/4000$) with cumulative distribution function

$$F(x) = 1 \sum_{i=0}^{2-1}[e^{-x/2000}(x/2000)^i/i!], \quad x > 0$$

(a) The probability that the system will fail within 2500 hours is

$$P(X \leq 2500) = F(2500) = .3554$$

(b) The probability of failure within 3000 hours is

$$P(X \leq 3000) = F(3000) = .4424$$

If inspection is moved from 2500 to 3000 hours, the probability that the system will fail increases by .087.

32. Letting X represent the lead time in 100's of units, the Erlang distribution with

$$\beta = K = 3, \ \theta = 1, \text{ and } X = 2$$

will provide the probability that the lead time is less than 2 with

$$F(2) = 1 - \sum_{i=o}^{2} \frac{e^{-6}6^i}{i!} = .938$$

The complement gives the desired probability, or

$$P(\text{Lead Time} \geq 2) = 1 - .938 = .062$$

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33. Let X be the lifetime of the card in months. The Erlang distribution gives the desired probability where

$$\beta = K = 4,\ K\theta = 4(1/16) = \frac{1}{4},\ \text{and } X = 24$$

Then

$$F(24) = 1 - \sum_{i=o}^{3} \frac{e^6 6^i}{i!} = 1 - .151 = .849$$

The complement gives the desired probability, or

$$P(X \geq 2 \text{ years}) = 1 - .849 = .151$$

34. Let X be defined as the number on a license tag. Then X is discrete uniform $(a = 100, b = 999)$ with cumulative distribution function

$$F(x) = (x - 99)/900, \quad x = 100, 101, \ldots, 999$$

(a) The probability that two tag numbers are 500 or higher is

$$[P(X \geq 500)]^2 = [1 - F(499)]^2 = .5556^2 = .3086$$

(b) Let Y be defined as the sum of two license tag numbers. Then Y is discrete triangular which can be approximated by

$$F(y) = \begin{cases} (y-a)^2/[(b-a)(c-a)], & a \leq y \leq b \\ 1 - [(c-y)^2/[(c-a)(c-b)]], & b \leq y \leq c \end{cases}$$

where $a = 2(100) = 200$, $c = 2(999) = 1998$, and $b = (1998 + 200)/2 = 1099$.

The probability that the sum of the next two tags is 1000 or higher is

$$P(Y \geq 1000) = 1 - F(999) = .6050$$

35. A normally distributed random variable, X, with a mean of 10, a variance of 4, and the following properties

$$P(a < X < b) = .90 \text{ and } |\mu - a| = |\mu - b|$$

exists as follows

$$\begin{aligned} P(X < b) = P(X > a) &= .95 \text{ due to symmetry} \\ \Phi[(b-10)/2] &= .95\ b = 13.3 \\ 1 - \Phi[(a-10)/2] &= .95\ a = 6.7 \end{aligned}$$

36. Solution to Exercise 36:

Normal (10, 4)

$$\begin{aligned} F(8) - F(6) &= F\left(\frac{8-10}{2}\right) - F\left(\frac{6-10}{2}\right) \\ &= F(-1) - F(-2) = (1 - .84134) - (1 - .97725) \\ &= .13591 \end{aligned}$$

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Triangular (4, 10, 16)

$$\begin{aligned} F(8) - F(6) &= \frac{(8-4)^2}{(10-4)(16-4)} - \frac{(6-4)^2}{(10-4)(16-4)} \\ &= 1/6 = .1667 \end{aligned}$$

Uniform (4, 16)

$$\begin{aligned} F(8) - F(6) &= \frac{(8-4)}{16-4} - \frac{(6-4)}{16-4} \\ &= 1/6 = .1667 \end{aligned}$$

37. Letting X be the random variable

$$\begin{aligned} Z &= \frac{x-u}{\sigma} \\ 2.33 &= \frac{x-20}{2} \\ x &= 24.66 \quad (1\%) \\ 1.645 &= \frac{x-20}{2} \\ x &= 23.29 \quad (5\%) \\ 1.283 &= \frac{x-20}{2} \\ x &= 22.57 \quad (10\%) \end{aligned}$$

38. Let X be defined as I.Q. scores. Then X is normally distributed ($\mu = 100, \sigma = 15$).

(a) The probability that a score is 140 or greater is

$$P(X \geq 140) = 1 - \Phi[140 - 100)/15] = .00383$$

(b) The probability that a score is between 135 and 140 is

$$\begin{aligned} P(135 \leq X \leq 140) &= \Phi[(140-100)/15] - \Phi[(135-100)/15] \\ &= .00598 \end{aligned}$$

(c) The probability that a score is less than 110 is

$$P(X < 110) = \Phi[(110 - 100)/15] = .7475$$

39. Let X be defined as the length of the ith shaft, and Y as the linkage formed by i shafts. Then X_i is normally distributed.

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(a) The linkage, Y, formed by the three shafts is distributed as

$$Y \sim N\left(\sum_{i=1}^{3}\mu_i, \sum_{i=1}^{3}\sigma_i^2\right)$$

$$Y \sim N(150, .25)$$

(b) The probability that the linkage is larger than 150.2 is

$$P(Y > 150.2) = 1 - \Phi[(150.2 - 150)/.5] = .3446$$

(c) The probability that the linkage is within tolerance is

$$P(149.83 \leq Y \leq 150.21) = \Phi[(150.21 - 150)/.5] - \Phi[(149.83 - 150)/.5]$$

$$= .2958$$

40. Let X be defined as the circumference of battery posts. Then X is Weibull ($\gamma = 3.25, \beta = 1/3, \alpha = .005$) with cumulative distribution function

$$F(x) = 1 - \exp[-((x - 3.25)/.005)^{1/3}] \ , \ x \geq 3.25$$

(a) The probability of a post having a circumference greater than 3.40 is

$$P(X > 3.40) = 1 - F(3.40) = .0447$$

(b) The probability of a post not meeting tolerance is

$$1 - P(3.3 < X < 3.5) = 1 - F(3.5) + F(3.3) = .9091$$

41. Let X be defined as the time to failure of a battery. Then X is Weibull ($\gamma = 0, \beta = 1/4, \alpha = 1/2$) with cumulative distribution function

$$F(x) = 1 - \exp[-(2x)^{1/4}], x \geq 0$$

(a) The probability that a battery will fail within 1.5 years is

$$P(X < 1.5) = F(1.5) = .7318$$

(b) The mean life of a battery is

$$E(X) = (1/2)\Gamma(4 + 1) = 12 \text{ years}$$

The probability of a battery lasting longer than twelve years is

$$P(X > 12) = 1 - F(12) = .1093$$

(c) The probability that a battery will last from between 1.5 and 2.5 years is

$$P(1.5 \leq X \leq 2.5) = F(2.5) - F(1.5) = .0440$$

42. Let X be the demand for electricity. Suppose

$$1000 = a < \text{median} = 1425 \leq b = \text{Mode}$$

so that the probability that the demand is less than or equal to 1425 kwh is given by

$$F(1425) = 0.5 = \frac{(1425 - 1000)^2}{(b - 1000)(1800 - 1000)} = \frac{425^2}{(b - 1000(800)}$$

implying $b = 1451.56$ kwh. Since $1451.56 \geq 1425$ we have Mode = 1451.56.

43. Letting X represent the time to failure

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(a) $E(X) = 100\Gamma(1+2) = 1000\Gamma(3) = 2000$ hours

(b) $F(3000) = 1 - \exp\left[-\left(\frac{3000}{1000}\right)^{\frac{1}{2}}\right]$

$F(3000) = 1 - e^{-1.732} = .823$

44. Let X be defined as the gross weight of three axle trucks. Then X is Weibull ($\gamma = 6.8, \beta = 1.5, \alpha = 1/2$) with cumulative distribution function

$$F(x) = 1 - \exp[-((x-6.8)/.5)^{1.5}], x \geq 6.8$$

The weight limit, a, such that .01 of the trucks are considered overweight is

$$\begin{aligned} P(X > a) &= 1 - F(a) = .01 \\ \exp[-((a-6.8)/.5)^{1.5}] &= .01 \\ a &= 8.184 \text{ tons} \end{aligned}$$

45. Let X be defined as the car's gas mileage. Then X is triangular ($a = 0, c = 50$) with an expected value, $E(X)$, equal to 25.3 miles per gallon.

The median can be determined by first finding the mode, b, by setting

$$E(X) = (a+b+c)/3 = 25.3$$

$$b = 25.9 \text{ miles per gallon,}$$

then, determining which interval of the distribution contains the median by setting

$$F(b) = (x-a)^2/[(b-a)(c-a)], a \leq x \leq b$$

to compute $F(25.9) = .518$, so the median is in the interval (0,25.9). The median is then computed by finding x such that $F(x) = .50$, or median $= 25.45$ miles per gallon.

46. Let T represent the time to complete the route. Then $T \sim N(\mu_T, \sigma_T^2)$

(a) $\mu_T = \sum_i \mu_i = 38 + 99 + 85 + 73 + 52 + 90 + 10 + 15 + 30 = 492$ minutes

(b) $\sigma_T^2 = \sum_i \sigma_i^2 = 16 + 29 + 25 + 20 + 12 + 25 + 4 + 4 + 9 = 144$ minutes^2 and $\sigma_T = 12$ minutes

$\Phi(z) = \Phi\left(\frac{x-\mu}{\sigma_T}\right) = \Phi\left(\frac{480-492}{12}\right) = \Phi(-1) = .3413$

$P(X > 480) = 1 - .3413 = .6587$

(c) $P(X > 2) = 1 - P(X < 2) = 1 - \sum_{x=0}^{2} \binom{6}{x} (.6587)^x (.3413)^{6-x}$
$= 1 - .108 = .892$

(d) $P(456 < X < 504) = F(504) - F(456)$

$= \Phi\left(\frac{504-496}{12}\right) - \Phi\left(\frac{456-496}{12}\right)$

$= \Phi(2/3) - \Phi(-3\ 1/3) = .7476 - .0001 = .7475$

47. $1 - F(600) = \exp[-(600/400)^{1/2}] = e^{-(1.5)1/2} = e^{-1.22} = .295$

48. $R(x) = 1 - F(x) = \sum_{i=0}^{2} e^{-.0001(32,000)} \frac{[(.001)(32,000)]^i}{i!} = .2364$

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49. Solution to Exercise 49.
(a)

$$E(X^2) = \int_{85}^{92} \frac{x^2(2)(x-85)}{119}dx + \int_{95}^{102} \frac{x^2(2)(102-x)}{170}dx = 3311.75 + 5349.41 = 8661.16$$

$E(X) = (a+b+c)/3 = (85+92+102)/3 = 93$

$V(X) = E(X^2) - [E(X)]^2 = 8661.16 - (93)^2 = 12.16° F^2$

(b)

$$\begin{aligned} 0.5 &= 1 - \frac{(102-x)^2}{170} \\ (102-x)^2 &= 85 \\ x &= 92.8° \ F \end{aligned}$$

(c) Mode $= b = 92° \ F$

50. (a) $E(X) = 1.8 + 1/3 \ \Gamma(2+1) = 1.8 + 1/3(2) = 2.47 \times 10^3$ hours

$$F(2.47) = 1 - \exp\left[-\left(\frac{2.47-1.80}{.33}\right)^{1/2}\right] = 1 - \exp[-(2)^{1/2}] = .757$$

$$P(X > 2.47) = 1 - .757 = .243$$

(b)

$$.5 = 1 - \exp\left[-\left(\frac{x-1.8}{.33}\right)^{1/2}\right], \text{ where } x = \text{ median}$$

$$.5 = \exp\left[-\left(\frac{x-1.8}{.33}\right)\right]^{1/2}$$

$$\ell n \ .5 = -\left(\frac{x-1.8}{.33}\right)^{1/2}$$

$$x = 1.96 \times 10^3 \text{ hours}$$

51.

$$\begin{aligned} F(4) &= 1 - \sum_{i=0}^{1} e^{-2(1/4)4}\left[\frac{[2(1/4)(4)]^i}{i!}\right] \\ &= 1 - \sum_{i=0}^{1} \frac{e^{-2}2^i}{i!} = .594 \\ P(X>4) &= 1 - .594 = .406 \end{aligned}$$

Chapter 6

Queueing Models

For Maple procedures that help in evaluating queueing models see the course web site at `www.bcnn.net`.

1. The tool crib is modeled by an M/M/c queue ($\lambda = 1/4, \mu = 1/3, c = 1$ or 2). Given that attendants are paid \$6 per hour and mechanics are paid \$10 per hour,

$$\text{Mean cost per hour } = \$10c + \$15L$$

assuming that mechanics impose cost on the system while in the queue and in service.

CASE 1: one attendant - M/M/1 ($c = 1, \rho = \lambda/\mu = .75$)

$L = \rho/(1-\rho) = 3$ mechanics

Mean cost per hour = \$10(1) + \$15(3) = \$55 per hour.

CASE 2: two attendants - M/M/2 ($c = 2, \rho = \lambda/c\mu = .375$)

$$L = c\rho + \left[(c\rho)^{c+1}P_0\right] / \left[c(c!)(1-\rho)^2\right] = .8727,$$

where

$$P_0 = \left\{ \left[\sum_{n=0}^{c-1} (c\rho)^n/n! \right] + [(c\rho)^c(1/c!)(1/(1-\rho))] \right\}^{-1} = .4545$$

Mean cost per hour = \$10(2) + \$15(.8727) = \$33.09 per hour

It would be advisable to have a second attendant because long run costs are reduced by \$21.91 per hour.

2. A single landing strip airport is modeled by an M/M/1 queue ($\mu = 2/3$). The maximum arrival rate, λ, such that the average wait, w_Q, does not exceed three minutes is computed as follows:

$$w_Q = \lambda/[\mu(\mu - \lambda)] \leq 3$$

or

$$\lambda = \mu/[1/\mu w_Q + 1] \leq .4444 \text{ airplanes per minute.}$$

Therefore, $\lambda_{\max} = .4444$ airplanes per minute.

3. The Port of Trop is modeled by an M/M/1/4 queue ($\lambda = 7, \mu = 8,\ a = 7/8, N = 4$). The expected number of ships waiting or in service, L, is

$$L = \frac{a[1-(N+1)a^N + Na^{N+1}]}{(1-a^{N+1})(1-a)} = 1.735 \text{ ships}$$

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since $\lambda \neq \mu$ and system capacity is $N = 4$ ships.

4. String pulling at City Hall is modeled by an M/M/2 queue ($\lambda = 1/10, \mu = 1/15, \rho = .75$).

(a) The probability that there are no strings to be pulled is

$$P_0 = \left\{ \left[\sum_{n=0}^{c-1} (c\rho)^n / n! \right] + [(c\rho)^c (1/c!)/(1-\rho)] \right\}^{-1} = .1429$$

(b) The expected number of strings waiting to be pulled is

$$L_Q = \left[(c\rho)^{c+1} P_0\right] / \left[c(c!)(1-\rho)^2\right] = 1.929 \text{ strings}$$

(c) The probability that both string pullers are busy is

$$P(L(\infty) \geq 2) = \left[(c\rho)^2 P_0\right] / \left[c!(1-\rho)\right] = .643$$

(d) If a third string puller is added to the system, (M/M/3 queue, $c = 3, \rho = .50$), the measures of performance become

$$P_0 = .2105, L_Q = .2368, P(L(\infty) \geq 3) = .2368$$

5. The bakery is modeled by an M/G/1 queue ($\mu = 4, \sigma^2 = 0$). The maximum arrival rate, λ, such that the mean length of the queue, L_Q, does not exceed five cakes is

$$L_Q = [\lambda^2 / 2\mu^2 (1 - \lambda/\mu)] \leq 5 \text{ cakes}$$
$$\lambda^2 + 40\lambda - 160 \leq 0$$
$$\lambda \leq 3.6643 \text{ cakes per hour.}$$

6. The physical examination is modeled as an $M/G/1$ queue. The arrival rate is $\lambda = 1/60$ patient per minute. The mean service time is $15 + 15 + 15 = 45$ minutes, so the service rate is $\mu = 1/45$ patient per minute. Thus, $\rho = \lambda/\mu = 3/4$. The variance of the service time is $\sigma^2 = 15^2 + 15^2 + 15^2 = 675$ minutes, the sum of the variance of three exponentially distributed random variables, each with mean 15. Applying the formula for L_Q for the $M/G/1$ queue we obtain

$$L_Q = \frac{\rho^2(1 + \sigma^2\mu^2)}{2(1-\rho)} = 1\frac{1}{2} \text{ patients.}$$

7. The tool crib is modeled as an $M/G/1$ queue with arrival rate $\lambda = 10$ per hour, service rate $\mu = 60/4 = 15$ per hour, and service-time variance $\sigma^2 = (2/60)^2 = (1/30)^2$ hours. Thus, $\rho = \lambda/\mu = 2/3$. The wages for non-productive waiting in line amounts to $15w_Q$ per mechanic's visit to the tool crib. Since there are $\lambda = 10$ visits per hour on average, the average cost per hour of having mechanics delayed is $\lambda(\$15w_Q) = \$15L_Q$, using $L_Q = \lambda w_Q$. Applying the formula for L_Q for the $M/G/1$ queue we obtain

$$L_Q = \frac{\rho^2(1 + \sigma^2\mu^2)}{2(1-\rho)} = 0.833 \text{ mechanics.}$$

Thus, the average cost per hour is $\$15L_Q = \$12.50$.

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8. The airport is modeled as an $M/G/1$ queue with arrival rate $\lambda = 30/60 = 0.5$ per minute, service rate $\mu = 60/90 = 2/3$ per minute, and service-time variance $\sigma^2 = 0$. The runway utilization is $\rho = \lambda/\mu = 3/4$. Applying the formulas for the $M/G/1$ queue we obtain

$$\begin{aligned} L_Q &= \frac{\rho^2(1+\sigma^2\mu^2)}{2(1-\rho)} = 1.125 \text{ aircraft} \\ w_Q &= \frac{L_Q}{\lambda} = 2.25 \text{ minutes} \\ w &= w_Q + \frac{1}{\mu} = 3.75 \text{ minutes} \\ L &= \frac{\lambda}{\mu} + L_Q = 1.875 \text{ aircraft.} \end{aligned}$$

9. The machine shop is modeled by an M/G/1 queue ($\lambda = 12/40 = .3$/hour, $\mu = 1/2.5 = .4$/hour, $\rho = .75, \sigma^2 = 1$).

(a) The expected number of working hours that a motor spends at the machine shop is

$$w = \mu^{-1} + [\lambda(\mu^{-2} + \sigma^2)]/[2(1-\rho)] = 6.85 \text{ hours}$$

(b) The variance that will reduce the expected number of working hours, w, that a motor spends in the shop to 6.5 hours is calculated by solving the equation in (a) for σ^2:

$$\begin{aligned} \sigma^2 &= [(w - \mu^{-1})(2(1-\rho))]/\lambda - \mu^{-2} \\ \sigma^2 &= .4167 \text{ hours}^2. \end{aligned}$$

10. The self-service gasoline pump is modeled by an M/G/1 queue with ($\lambda = 12$/hour, $\mu = 15$/hour, $\rho = .8, \sigma^2 = 1.333^2\text{min}^2 = .0222^2$ hour^2. The expected number of vehicles in the system is

$$L = \rho + [\rho^2(1+\sigma^2\mu^2)]/[2(1-\rho)] = 2.5778 \text{ vehicles.}$$

11. The car wash is modeled by an M/G/1 queue ($\lambda = 1/45, \mu = 1/36$, $\rho = .8, \sigma^2 = 324$).

(a) The average time a car waits to be served is

$$w_Q = 90 \text{ minutes}$$

(b) The average number of cars in the system is

$$L = 2.8 \text{ cars}$$

(c) The average time required to wash a car is

$$1/\mu = 36 \text{ minutes.}$$

12. The cotton spinning room is modeled by an M/M/c/10/10 queue with ($\lambda = 1/40, \mu = 1/10, N = K = 10$). Given that operators are paid \$10 per hour, and idle looms cost \$40 per hour, the mean cost per hour of the system is

$$\text{Mean cost per hour } = \$10c + \$40L$$

The table below is generated for various levels of c.

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c	L_Q	L	w_Q(min)	$K-L$	Cost
1	5.03	6.02	50.60	3.98	\$250.80
2	1.46	3.17	8.55	6.83	146.80
3	0.32	2.26	1.65	7.74	120.40
4	0.06	2.05	0.30	7.95	122.00
5	0.01	2.01	0.05	7.99	130.40

(a) The number of operators that should be employed to minimize the total cost of the room is three, resulting in a total cost of \$120.40.

(b) Four operators should be employed to ensure that, on the average, no loom should wait for more than one minute for an operator (i.e., to ensure $w_Q \leq 1$ min.). In this case, a loom will only have to wait an average of $w_q = 0.3$ min. $= 18$ seconds for a cost of \$122.00.

(c) Three operators should be employed to ensure that an average of at least 7.5 looms are running at all times (i.e., to ensure $K - L \geq 7.5$ looms)

13. Given an M/M/2/10/10 queue ($\lambda = 1/82, \mu = 1/15, c = 2,\ K = 10, N = 10$), the average number of customers in the queue is $L_Q = 0.72$. The average waiting time of a customer in the queue is

$$W_Q = L_Q/\lambda_e = 0.72/0.09567 = 7.526 \text{ time units.}$$

The value of λ such that $L_Q = L/2$ is found by trial and error to be

$$\lambda = 0.0196$$

14. Assuming Figure 6.6 represents a single-server LIFO system, the time in system, W_i, of the ith customer can be found to be $W_1 = 2, W_2 = 5, W_3 = 9, W_4 = 3,\ W_5 = 4$, so $\sum_{i=1}^{N} W_i = 23$.

Also, $\widehat{\lambda} = N/T = 5/20 = 0.25$

$$\widehat{w} = \sum_{i=1}^{N} w_i/N = 4.6 \text{ time units}$$

$$\widehat{L} = (1/T)\sum_{i=0}^{\infty} iT_i = 1.15 \text{ customers}$$

Note that: $\widehat{L} = 1.15 = (.25)(4.6) = \widehat{\lambda}\widehat{w}$

Allowing $T \longrightarrow \infty$, and $N \longrightarrow \infty$, implies that $\widehat{L} \longrightarrow L$, $\widehat{\lambda} \longrightarrow \lambda$, and $\widehat{w} \longrightarrow w$, and

$$\widehat{L} = \widehat{\lambda}\widehat{w} \text{ becomes } L = \lambda w$$

The total area under the $L(t)$ function can be written as:

$$\int_0^T L(t)dt = \sum_{i=1}^{N} W_i$$

Note that LIFO did not change the equations.

15. (a) Assume Figure 6.6 is for a FIFO system with $c = 2$ servers. As before, $N = 5$ and $T = 20$, so $\widehat{\lambda} = N/T = 0.25$ customer/time unit. The solution for this system is given by Figure 6.8. Hence,

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$W_1 = 2, W_2 = 8 - 3 = 5$, $W_3 = 10 - 5 = 5, W_4 = 14 - 7 = 7$, and $W_5 = 20 - 16 = 4$. To show $\widehat{L} = \widehat{\lambda}\widehat{w}$, one proceeds as in Exercise 14.

(b) Assume Figure 6.6 is for LIFO system with $c = 2$ servers. The solution is identical to that of Exercise 11.

16. (d) The values of μ_1, μ_2, and p needed to achieve a distribution with mean $E(X) = 1$ and coefficient of variation cv $= 2$ can be determined as follows:

Note that

$$E(X) = p/\mu_1 + (1-p)/\mu_2$$

and

$$(\text{cv})^2 = [2p(1-p)(1/\mu_1 - 1/\mu_2)^2]/[E(X)]^2 + 1$$

By choosing $p = 1/4$ arbitrarily, the following equations can be simultaneously solved

$$1/(4\mu_1) + 3/(4\mu_2) = 1 \text{ and } 3/8(1/\mu_1 - 1/\mu_2)^2 + 1 = 4$$

Solving the left equation for μ_1 yields

$$\mu_1 = \mu_2/(4\mu_2 - 3)$$

Substituting μ_1 into the right equation and solving for μ_2 yields

$$\mu_2 = 1/(1 - \sqrt{2/2}) = 3.4142$$
$$\mu_1 = 3.4142/[4(3.4142 - 3)] = .3204$$

17. In Example 6.18, the milling machine station is modeled by M/M/c/K/K queue ($\lambda = 1.20, \mu = 1/5, K = 10$). A table comparing the relevant parameters of the system for $c = 1, 2$, and 3 is given below:

	$c=1$	$c=2$	$c=3$
L_Q	5.03	1.46	0.32
$L - L_Q$	0.994	1.708	1.936
ρ	0.994	0.854	0.645

As more servers are hired, the average server utilization, ρ, decreases; but the average queue length, L_Q, also decreases.

18. Modeling the system as an $M/M/c/12/12$ queue we need λ_e to obtain $\rho = \lambda_e/(c\mu)$, where $\lambda = 1/20$ and $\mu = 1/5$. Results are given in the table below:

c	λ_e	ρ
1	0.200	0.999
2	0.374	0.934
3	0.451	0.752

19. The lumber yard is modeled by a M/M/c/N/K queue ($\lambda = 1/3, \mu = 1, N = K = 10$).

(a) Assume that unloading time is exponentially distributed with mean $1/\mu = 1$ hour. Also assume that travel time to get the next load of logs and return is exponentially distributed with mean $1/\lambda = 3$ hours. The exponential distribution is highly variable (mean=std.dev.) and therefore it may be reasonable for travel times provided the trucks travel varying distances and/or run into congested traffic conditions. On the other hand, actual unloading times are probably less variable than the exponential distribution.

(b) With one crane to unload trucks, $c = 1$.

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The average number of trucks waiting to be unloaded is

$$L_Q = 6 \text{ trucks.}$$

The average number of trucks arriving at the yard each hour is

$$\lambda_e = 1.0 \text{ trucks/hour.}$$

The fraction of trucks finding the crane busy upon arrival is

$$1 - P_0 = .997 = 99.7\%$$

The long run proportion of time the crane is busy is

$$\rho = 1.0$$

(c) With two cranes to unload trucks, $c = 2$.
A table comparing one crane and two cranes follows:

	one crane	two cranes
c	1	2
L_Q	6.0	2.47
λ_e	1.0	1.88
busy	0.997	0.844
ρ	1.0	0.94

(d) The value of a truckload is \$200 and the cost of a crane is \$50 per hour independent of utilization. The cost per hour is \$50 (number of cranes) - \$200 (number of arrivals per hour), or cost per hour = $\$50c - \$200\lambda_e$.

c	λ_e	Cost (\$) per hour Exercise 19(d)	Cost (\$) per hour Exercise 19(e)
1	1.000	-150.00	90.00
2	1.883	-276.60	-177.80
3	2.323	-314.60	-286.20
4	2.458	-291.60	-284.80
5	2.493	-248.60	-247.40

Three cranes should be used because the value of logs received per hour is \$314.60 more than the cost of three cranes, and is higher than with any other option.

(e) In addition to the above costs, the cost of an idle truck and driver is \$40 per hour. Then,

$$\text{cost } = \$50c + \$40L_Q - \$200\lambda_e$$

and three cranes should be installed as shown in the table above, since the value of the logs is \$286.20 more than the combined cost of three cranes and $L_Q = .71$ idle trucks and drivers on the average.

20. The tool crib is modeled by an M/M/c/N/K queue ($\lambda = 1.20$, $\mu = 1.3, N = K = 10, c = 1$ or 2). As in Exercise 1,

$$\text{mean cost per hour } = \$6c + \$10L$$

Case 1: one attendant ($c = 1$)

$L_Q = 2.82$
$\lambda_e = 0.311$

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$L = 3.75$
Mean cost per hour = \$6(1) + \$10(3.75) = \$43.50

Case 2: two attendants ($c = 2$)

$L_Q = 0.42$
$L = 1.66$
Mean cost per hour = \$6(2) + \$10(1.66) = \$28.60

A second attendant reduces mean costs per hour by \$43.50 - \$28.60 = \$14.90.

21. For an M/G/∞ queue with $\lambda = 1000$/hour and $1/\mu = 3$ hours,

$$P_n = \frac{e^{-\lambda/\mu}(\lambda/\mu)^n}{n!}$$

If c is the number of parking spaces, the probability we need more than c spaces is

$$\sum_{n=c+1}^{\infty} P_n = 1 - \sum_{n=0}^{c} P_n$$

By trial and error we find that $c = 3169$ spaces makes this probability < 0.001.

22. If the overall arrival rate increases to $\lambda = 160$/hour, then $\lambda_1 = .4\lambda = 64, \lambda_2 = .6\lambda = 96$, and $\lambda_3 = \lambda_1 + \lambda_2 = 160$. The offered load at service center 2 is $\lambda_2/\mu_2 = 96/20 = 4.8$, so we need at least $c = 5$ clerks. At service center 3, $\lambda_3/\mu_3 = 160/90 = 1.8$, so we need at least $c = 2$ clerks.

23. The system can be approximated as an M/M/c queue with arrival rate $\lambda = 24$ per hour and service rate $\mu = 1/2$ per minute = 30 per hour. Currently $c = 1$ server (copy machine), but the proposal is for $c = 2$ servers. The steady-state probability that the line reaches outside the store is

$$p = \sum_{n=5}^{\infty} P_n = 1 - \sum_{n=0}^{4} P_n$$

For the M/M/1 queue $p \approx 0.33$, while for the M/M/2 queue $p \approx 0.01$. Thus, adding another copier substantially reduces the likelihood of having a line reach outside the store.

24. The system can be approximated as an M/M/c/N queue. In both system designs the capacity is $N = 7$ cars. Currently there are $c = 4$ servers (stalls), and the proposal is to change to $c = 5$ stalls. The arrival rate is $\lambda = 34$ cars per hour, so the rate at which cars are lost is λP_7.

The expected service time is

$$3(0.2) + 7(0.7) + 12(0.1) = 6.7 \text{ minutes per car}$$

implying a service rate of approximately $\mu = 9$ cars per hour. Clearly the service time is not exponentially distributed, but we are approximating it as exponentially distributed with the same mean.

When $c = 4$ we have $\lambda P_7 \approx (34)(0.14) = 4.8$ cars per hour lost, but when $c = 5$ we have $\lambda P_7 \approx (34)(0.08) = 2.7$ cars per hour lost.

Chapter 7

Random-Number Generation

1. Place 10 slips of paper into a hat, where each slip has one of the integers $0, 1, 2, \ldots, 9$ written on it. Draw two slips of paper (one-at-a-time, with replacement), and let the resulting numbers be F, S. Then set

$$R = 0.FS$$

This procedure generates random numbers on the interval $[0, 0.99]$.

2. Video gambling games, military draft, assigning subjects to treatments in a pharmaceutical experiment, state lotteries and pairing teams in a sports tournament.

3. Let $X = -11 + 28R$.

4. Solution to Exercise 4:

$X_0 = 27,\ a = 8,\ c = 47,\ m = 100$

$X_1 = (8 \times 27 + 47) \text{mod } 100 = 63, \quad R_1 = 63/100 = .63$

$X_2 = (8 \times 63 + 47) \text{mod } 100 = 51, \quad R_2 = 51/100 = .51$

$X_3 = (8 \times 51 + 47) \text{mod } 100 = 55, \quad R_3 = 55/100 = .55$

5. None. A problem would occur only if $c = 0$ also.

6. Solution to Exercise 6:

$X_0 = 117, a = 43, m = 1,000$

$X_1 = [43(117)] \text{mod } 1,000 = 31$

$X_2 = [43(31)] \text{mod } 1,000 = 333$

$X_3 = [43(333)] \text{mod } 1,000 = 319$

$X_4 = [43(319)] \text{mod } 1,000 = 717$

7. Solution to Exercise 7:

$R_{(i)}$	.11	.54	.68	.73	.98
i/N	.20	.40	.60	.80	1.0
$i/N - R_{(i)}$	.09	–	–	.07	.02
$R_{(i)} - (i-1)/N$	.11	.34	.28	.13	.18

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$D^+ = \max_{1 \le i \le N}(i/N - R_{(i)}) = .09$

$D^- = \max_{1 \le i \le N}(R_{(i)} - (i-1)/N) = .34$

$D = \max(D^+, D^-) = .34$

The critical value, D_α, obtained from Table A.8 is

$$D_{.05} = .565$$

since $D < D_{.05}$, the hypothesis that there is no difference between the true distribution of $\{R_1, R_2, \ldots, R_5\}$ and the uniform distribution on $[0, 1]$ cannot be rejected on the basis of this test.

8. Let ten intervals be defined each from $(10i - 9)$ to $(10i)$ where $i = 1, 2, \ldots, 10$. By counting the numbers that fall within each interval and comparing this to the expected value for each interval, $E_i = 10$, the following table is generated:

Interval	O_i	$(O_i - E_i)^2/E_i$
(01-10)	9	0.1
(11-20)	9	0.1
(21-30)	9	0.1
(31-40)	6	1.6
(41-50)	17	4.9
(51-60)	5	2.5
(61-70)	10	0.0
(71-80)	12	0.4
(81-90)	7	0.9
(91-00)	16	3.6
	100	14.2= χ_0^2

From Table A.6, $\chi^2_{.05,9} = 16.9$. Since $\chi_0^2 < \chi_{.05,9}$, then the null hypothesis of no difference between the sample distribution and the uniform distribution is not rejected.

9. The numbers are given a "+" or a "−" depending on whether they are followed by a larger or smaller number:

$$+ - + - - - + + + + - + - + + + - - + + + - - - -$$

$$+ + + - + - + - - + - + - + - - - + + - - + +$$

There are $a = 27$ runs in this sequence.

For $N = 50$,

$$\begin{aligned} \mu_a &= (2N-1)/3 = 33, \text{ and} \\ \sigma_a^2 &= (16N - 29)/90 = 8.5667 \\ Z_0 &= (a - \mu_a)/\sigma_a = -2.05 \\ z_{\alpha/2} &= z_{.025} = 1.96 \end{aligned}$$

Since $Z_0 < -z_{.025}$, the null hypothesis of independence can be rejected.

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10. A "+" sign is used to denote an observation above the mean (.495) and a "−" sign will denote an observation below the mean.

$$+-+--+-------++-+-+-+-+-+-++-$$

$$++++--+--+--++--++++-+--+$$

$$\begin{aligned} n_1 &= 24, n_2 = 26, \text{ and } b = 31 \\ \mu_b &= [(2n_1n_2)/N] + 1/2 = 25.46 \\ \sigma_b^2 &= [2n_1n_2(2n_1n_2 - N)]/[N^2(N-1)] = 12.205 \\ Z_0 &= (b - \mu_b)/\sigma_b = 1.586 \\ z_{\alpha/2} &= z_{.025} = 1.96 \end{aligned}$$

Since $-z_{.025} < Z_0 < z_{.025}$, the null hypothesis of independence cannot be rejected.

11. The lengths of runs up and down are

$$1,1,1,3,4,1,1,1,3,2,3,4,3,1,1,2,1,2,1,1,1,1,1,3,2,2,2$$

$$\begin{aligned} E(Y_i) &= [2/(i+3)!][N(i^2+3i+1) - (i^3+3i^2-i-4), i \le N-2 \\ E(Y_i) &= 2/N!,\ i = N-1 \\ E(Y_1) &= (2/24)[50(5) - (-1)] = 20.917 \\ E(Y_2) &= (2/120)[50(11) - 14] = 8.933 \\ E(Y_3) &= (2/720)[50(19) - 48] = 2.506 \\ E(Y_{\ge 4}) &= \mu_a - E(Y_1) - E(Y_2) - E(Y_3) \\ &= (2n-1)/3 - (20.917 + 8.933 + 2.506) = 0.644 \end{aligned}$$

Run Length (i)	Observed Runs (O_i)	Expected Runs $E(Y_i)$	$\frac{[O_i - E(Y_i)]^2}{E(Y_i)}$
1	14	20.917	2.2874
2	6	8.933	
3	5	2.506	0.0696
≥ 4	2	0.644	
			$\chi_0^2 = 2.3570$

$$\chi^2_{.05,1} = 3.84$$

since $\chi_0^2 < \chi^2_{.05,1}$, the null hypothesis of independence cannot be rejected. Notice that we grouped run lengths $i = 2, 3, \ge 4$ together into a single cell with $O_i = 13$ and $E(Y_i) = 12.083$.

12. Solution to Exercise 12:

The sequence is as follows:

$$+-+--+-------++-+-+-+-+-+-++-$$

$$++++--+--+--++--++++-+--+$$

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Run length, i	1	2	3	4
Observed Runs, O_i	19	8	2	2

$n_1 = 24$ and $n_2 = 26$
$w_1 = 2(24/50)(26/50) = .4992$
$w_2 = (24/50)^2(26/50) + (24/50)(26/50)^2 = .2496$
$w_3 = (24/50)^3(26/50) + (24/50)(26/50)^3 = .1250$
$E(I) = 24/26 + 26/24 = 2.00$
$E(A) = 50/2.00 = 25$
$E(Y_1) = 50(.4992)/2.00 = 12.48$
$E(Y_2) = 50(.2496)/2.00 = 6.24$
$E(Y_3) = 50(.1250)/2.00 = 3.125$

Run Length (i)	Observed Runs (O_i)	Expected Runs $E(Y_i)$	$\frac{[O_i - E(Y_i)]^2}{E(Y_i)}$
1	19	12.48	3.41
2	8	6.24	.50
3	2	3.125	
≥ 4	2	3.155	.83
			4.74

$$\chi^2_{.05,2} = 5.99$$

Therefore, do not reject the hypothesis of independence on the basis of this test. Notice that we grouped run lengths $i = 3, \geq 4$ together into a single cell with $O_i = 4$ and $E(Y_i) = 6.28$.

13. Solution to Exercise 13:

$\widehat{\rho}_{14} = (1/8)[(.48)(.61) + (.61)(.37) + (.37)(.37) + (.37)(.99) + (.99)(.09)$
$+(.09)(.55) + (.55)(.60) + (.60)(.19)] - .25 = -.0495$

$\sigma_{\widehat{\rho}_{14}} = .1030$

$Z_0 = -.0495/.1030 = -0.48$

Since $-z_{.025} < Z_0 < z_{.025}$, the null hypothesis of independence cannot be rejected on the basis of significant autocorrelation.

14. Solution to Exercise 14:

Gap Length Classes	Frequency	Relative Frequency	$S_N(x)$	$F(x)$	$\lvert F(X) - S_N(x)\rvert$
0-3	33	.3000	.3000	.3439	.0439
4-7	23	.2091	.5091	.5695	.0604
8-11	23	.2091	.7182	.7176	.0006
12-15	15	.1364	.8546	.8146	.0400
16-19	7	.0636	.9182	.8784	.0398
20-23	5	.0455	.9637	.9202	.0435
24-27	1	.0091	.9728	.9497	.0231
28-31	0	0	.9728	.9657	.0071
32-35	2	.0182	.9910	.9775	.0135
36-39	1	.0091	1.0	.9852	.0148
	110				

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$D = \max |F(x) - S_N(x)| = .0604$, and

$D_\alpha = D_{.05} = .136$. Since $D < D_{.05}$, the null hypothesis of independence cannot be rejected on the basis of this test.

15. Solution to Exercise 15:

(a)

$P(\text{4 different digits}) = (.9)(.8)(.7) = .5040$
$P(\text{exactly one pair}) = \binom{4}{2}(.1)(.9)(.8) = .4320$
$P(\text{two pairs}) = \binom{3}{2}(.1)(.9)(.1) = .0270$
$P(\text{triplet}) = \binom{4}{3}(.1)(.1)(.9) = .0360$
$P(\text{4 like digits}) = (.1)(.1)(.1) = .0010$

(b)
$P(\text{5 different digits}) = (.9)(.8)(.7)(.6) = .3024$
$P(\text{exactly one pair}) = \binom{5}{2}(.1)(.9)(.8)(.7) = .5040$
$P(\text{2 different pairs}) = 15(.1)(.9)(.1)(.8) = .1080$
$P(\text{triplet and pair}) = \binom{5}{3}(.1)(.9)(.1) = .0090$
$P(\text{exactly one triplet}) = \binom{5}{3}(.1)(.1)(.9)(.8) = .0720$
$P(\text{4 like digits}) = \binom{5}{4}(.1)(.1)(.1)(.9) = .0045$
$P(\text{5 like digits}) = (.1)(.1)(.1)(.1) = .0001$

16. Solution to Exercise 16:

Combination i	Observed O_i	Expected E_i	$\frac{(O_i - E_i)^2}{E_i}$
4 different digits	565	504	7.3829
1 pair	392	432	3.7037
2 pairs	17	27	3.7037
3 like digits	24	36	
4 like digits	2	1	3.2703
Totals	1000	1000	18.0606

$\chi^2_{.05,3} = 7.81 < \chi^2_0 = 18.0806$

Reject the null hypothesis of independence based on this test. Notice that we grouped "3 like digits" and "4 like digits" into a single cell with $O_i = 26$ and $E_i = 37$.

17. Solution to Exercise 17(c): $a = 1 + 4k \longrightarrow k = 1237.5$ which is not an integer. Therefore, maximum period cannot be achieved.

18. Solution to Exercise 18:

$X_1 = [7 \times 37 + 29] \bmod 100 = 88$
$R_1 = .88$
$X_2 = [7 \times 88 + 29] \bmod 100 = 45$
$R_2 = .45$
$X_3 = [7 \times 45 + 29] \bmod 100 = 44$
$R_3 = .44$

19. Use $m = 25$

$X_1 = [9 \times 13 + 35] \bmod 25 = 2$

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$X_2 = [9 \times 2 + 35] \bmod 25 = 3$
$X_3 = [9 \times 3 + 35] \bmod 25 = 12$

21. Solution to Exercise 21:

$X_1 = [4951 \times 3579] \bmod 256 = 77$
$R_1 = 77/256 = .3008$

23. Solution to Exercise 23:

	Case (a)	Case (b)	Case (c)	Case (d)
i	X_i	X_i	X_i	X_i
0	7	8	7	8
1	13	8	1	8
2	15		7	8
3	5			
4	7			

Inferences:
Maximum period, $p = 4$, occurs when X_0 is odd and $a = 3 + 8k$ where $k = 1$. Even seeds have the minimal possible period regardless of a.

24. $X_{1,0} = 100,\ X_{2,0} = 300,\ X_{3,0} = 500$
The generator is

$$\begin{aligned} X_{1,j+1} &= 157\ X_{1,j} \bmod 32363 \\ X_{2,j+1} &= 146\ X_{2,j} \bmod 31727 \\ X_{3,j+1} &= 142\ X_{3,j} \bmod 31657 \\ X_{j+1} &= (X_{1,j+1} - X_{2,j+1} + X_{3,j+1}) \bmod 32362 \\ R_{j+1} &= \begin{cases} \frac{X_{j+1}}{32363} & , \text{ if } X_{j+1} > 0 \\ \frac{32362}{32363} = 0.999 & , \text{ if } X_{j+1} = 0 \end{cases} \end{aligned}$$

The first 5 random numbers are
$X_{1,1} = [157 \times 100] \bmod 32363 = 15700$
$X_{2,1} = [146 \times 300] \bmod 31727 = 12073$
$X_{3,1} = [142 \times 500] \bmod 31657 = 7686$
$X_1 = [15700 - 12073 + 7686] \bmod 32362 = 11313$
$R_1 = 11313/32363 = 0.3496$

$X_{1,2} = 5312$
$X_{2,2} = 17673$
$X_{3,2} = 15074$
$X_2 = 2713$
$R_2 = 0.0838$

$X_{1,3} = 24909$
$X_{2,3} = 10371$
$X_{3,3} = 19489$
$X_3 = 1665$
$R_3 = 0.0515$

$X_{1,4} = 27153$
$X_{2,4} = 22997$

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$X_{3,4} = 13279$
$X_4 = 17435$
$R_4 = 0.5387$

$X_{1,5} = 23468$
$X_{2,5} = 26227$
$X_{3,5} = 17855$
$X_5 = 15096$
$R_5 = 0.4665$

29. Two results that are useful to solve this problem are

$$(c + d) \bmod m = c \bmod m + d \bmod m$$

and that if $g = h \bmod m$, then we can write $g = h - km$ for some integer $k \geq 0$. The last result is true because, by definition, g is the remainder after subtracting the largest integer multiple of m that is $\leq h$.

(a) Notice that

$$\begin{aligned} X_{i+2} &= aX_{i+1} \bmod m \\ &= a[aX_i \bmod m] \bmod m \\ &= a[aX_i - km] \bmod m \qquad \text{(for some integer } k \geq 0) \\ &= a^2X_i \bmod m - akm \bmod m \\ &= a^2X_i \bmod m \qquad \text{(since } akm \bmod m = 0). \end{aligned}$$

(b) Notice that

$$\begin{aligned} (a^nX_i) \bmod m &= \{(a^n \bmod m) + [a^n - (a^n \bmod m)]\} X_i \bmod m \\ &= \{(a^n \bmod m)X_i \bmod m\} + \{[a^n - (a^n \bmod m)]X_i \bmod m\} \\ &= \{(a^n \bmod m)X_i \bmod m\} + \{kmX_i \bmod m\} \qquad \text{(for some integer } k \geq 0) \\ &= (a^n \bmod m)X_i \bmod m. \end{aligned}$$

(c) In this generator $a = 19$, $m = 100$ and $X_0 = 63$. Therefore, $a^5 \bmod 100 = 19^5 \bmod 100 = 99$. Thus, $X_5 = (99)(63) \bmod 100 = 37$.

Chapter 8

Random-Variate Generation

1. Solution to Exercise 1:
   Step 1.

$$\text{cdf } = F(x) = \begin{cases} e^{2x}/2, & -\infty < x \le 0 \\ 1 - e^{-2x}, & 0 < x < \infty \end{cases}$$

   Step 2. Set $F(X) = R$ on $-\infty < X < \infty$

   Step 3. Solve for X to obtain

$$X = \begin{cases} 1/2 \ln 2R & 0 < R \le 1/2 \\ -1/2 \ln(2 - 2R) & 1/2 < R < 1 \end{cases}$$

2. Solution to Exercise 2:
   Step 1.

$$\text{cdf} = F(x) = \begin{cases} 1 - x + x^2/4, & 2 \le x < 3 \\ x - x^2/12 - 2, & 3 < x \le 6 \end{cases}$$

   Step 2. Set $F(X) = R$ on $2 \le X \le 6$

   Step 3. Solve for X to obtain

$$X = \begin{cases} 2 + 2\sqrt{2} & 0 \le R \le 1/4 \\ 6 - 2\sqrt{3 - 3R} & 1/4 < R \le 1 \end{cases}$$

The true mean is $(a + b + c)/3 = (2 + 3 + 6)/3 = 11/3$.

3. Triangular distribution with $a = 1, b = 4, c = 10$. Total area $= 1 =$ base $\times$ height$/2 = 9h/2$, so $h = 2/9$

   Step 1: Find cdf $F(x) =$ total area from 1 to x.

   For $1 \le x \le 4, f(x)/h = (x - 1)/(4 - 1)$ by similar triangles so

$$F(x) = (x - 1)f(x)/2 = (x - 1)^2/27$$

   For $4 < x \le 10, f(x)/h = (10 - x)/(10 - 4)$ by similar triangles so

$$F(x) = 1 - (10 - x)f(x)/2 = 1 - (10 - x)^2/54.$$

   Step 2: Set $F(X) = R$ on $1 \le X \le 10$.

   Step 3: Solve for X.

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$$X = \begin{cases} 1 + \sqrt{27R}, & 0 \le R \le 9/27 \\ 10 - \sqrt{54(1-R)}, & 9/27 < R \le 1 \end{cases}$$

4. Triangular distribution with $a = 1, c = 10$ and $E(X) = 4$. Since $(a + b + c)/3 = E(X)$, the mode is at $b = 1$. Thus, the height of the triangular pdf is $h = 2/9$. (See solution to previous problem. Note that the triangle here is a right triangle.)

Step 1: Find cdf $F(x)$ = total area from 1 to x.

$$= 1 - \text{ (total area from } x \text{ to } 10).$$

By similar triangles, $f(x)/h = (10 - x)/(10 - 1)$, so

$$F(x) = 1 - (10 - x)f(x)/2 = 1 - (10 - x)^2/81, \quad 1 \le x \le 10.$$

Step 2: Set $F(X) = R$ on $1 \le X \le 10$.

Step 3: $X = 10 - \sqrt{81(1 - R)}, \quad 0 \le R \le 1$

5. Solution to Exercise 5:

$$X = \begin{cases} 6(R - 1/2) & 0 \le R \le 1/2 \\ \sqrt{32(R - 1/2)} & 1/2 \le R \le 1 \end{cases}$$

6. $X = 2R^{1/4}, \quad 0 \le R \le 1$

7. Solution to Exercise 7:
$F(x) = x^3/27, \quad 0 \le x \le 3$
$X = 3R^{1/3}, \quad 0 \le R \le 1$

8. Solution to Exercise 8:
Step 1:

$$F(x) = \begin{cases} x/3, & 0 \le x \le 2 \\ 2/3 + (x - 2)/24, & 2 < x \le 10 \end{cases}$$

Step 2: Set $F(X) = R$ on $0 \le X \le 10$.

Step 3:

$$X = \begin{cases} 3R, & 0 \le R \le 2/3 \\ 2 + 24(R - 2/3) = 24R - 14, & 2/3 < R \le 1 \end{cases}$$

9. Use Inequality (8.14) to conclude that, for R given, X will assume the value x in $R_X = \{1, 2, 3, 4\}$ provided

$$F(x - 1) = \frac{(x - 1)x(2x - 1)}{180} < R \le \frac{x(x + 1)(2x + 1)}{180} = F(x)$$

By direct computation, $F(1) = 6/180 = .033$, $F(2) = 30/180 = .167$, $F(3) = 42/180 = .233$, $F(4) = 1$. Thus, X can be generated by the table look-up procedure using the following table:

x	1	2	3	4
$F(x)$	.033	.167	.233	1

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$$\begin{aligned} R_1 &= 0.83 \longrightarrow X = 4 \\ R_2 &= 0.24 \longrightarrow X = 4 \\ R_3 &= 0.57 \longrightarrow X = 4 \end{aligned}$$

10. Weibull with $\beta = 2, \alpha = 10$. By Equation (9.6)

$$X = 10[-\ln(1-R)]^{0.5}$$

11. The table look-up method for service times:

i	Input r_i	Output x_i	Slope a_i
1	0	15	244.89
2	.0667	30	112.53
3	.2000	45	89.98
4	.3667	60	128.59
5	.6000	90	150.00
6	.8000	120	450.11
7	.9333	180	1799.10
8	1.0000	300	—

12. The table look-up method for fire crew response times, assuming $0.25 \le X \le 3$:

i	Input r_i	Output x_i	Slope a_i
1	0	.25	3.29
2	.167	.80	2.65
3	.333	1.24	1.26
4	.500	1.45	2.28
5	.667	1.83	5.60
6	.833	2.76	1.44
7	1.000	3.00	—

13. By Example 8.5, $\lceil 17R \rceil$ generates uniform random variates on $\{1, 2, \ldots, 17\}$, thus

$$X = 7 + \lceil 17R \rceil$$

generates uniform random variates on $\{8, 9, \ldots, 24\}$.

15. The mean is $(1/p) - 1 = 2.5$, so $p = 2/7$. By Equation (9.21),

$$X = \lceil -2.97 \ln(1-R) - 1 \rceil$$

17. Use $X = -3.7 \ln R$.

19. Generate $X = 8[-\ln R]^{4/3}$

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If $X \leq 5$, set $Y = X$.

Otherwise, set $Y = 5$.

(Note: for Equation 8.6, it is permissible to replace $1 - R$ by R.)

20.

Method 1: Generate $X_1 \sim U(0, 8)$ and $X_2 \sim U(0, 8)$.

Set $Y = \min(X_1, X_2)$.

Method 2: The cdf of Y is

$$\begin{aligned} F(y) = P(Y \leq y) &= 1 - P(Y > y) \\ &= 1 - P(X_1 > y, X_2 > y) \\ &= 1 - (1 - y/8)^2, 0 \leq y \leq 8 \end{aligned}$$

by independence of X_1 and X_2.

$$F(Y) = 1 - (1 - Y/8)^2 = R$$

implies

$$Y = 8 - 8\sqrt{1 - R}, \quad 0 \leq R \leq 1.$$

21. Assume X_i is exponentially distributed with mean $1/\lambda_i$, where $1/\lambda_1 = 2$ hours and $1/\lambda_2 = 6$ hours. Method 1 is similar to that in Exercise 20.

Method 2: The cdf of Y is

$$\begin{aligned} F(y) = P(Y \leq y) &= 1 - P(Y > y) \\ &= 1 - P(X_1 > y, X_2 > y) \\ &= 1 - e^{-\lambda_1 y} e^{-\lambda_2 y} \\ &= 1 - e^{-(\lambda_1 + \lambda_2) y} \end{aligned}$$

Therefore Y is exponential with parameter $\lambda_1 + \lambda_2 = 1/2 + 1/6 = 2/3$.

Generate $Y = -1.5 \ln R$.

Clearly, method 2 is twice as efficient as method 1.

22. Generate $R_1, R_2, \ldots R_n$.

$$\text{Set } X_i = \begin{cases} 1 & \text{if } R_i \leq p \\ 0 & \text{if } R_i > p. \end{cases}$$

Compute $X = \sum_{i=1}^{n} X_i$

23. Solution to Exercise 23:

Step 1: Set $n = 0$

Step 2: Generate R

Step 3: If $R \leq p$, set $X = n$, and go to step 4.

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If $R > p$, increment n by 1 and return to step 2.

Step 4: If more geometric variates are needed, return to step 1.

28. Recall that one can obtain exponentially distributed variates with mean 1 using the inverse cdf transformation

$$X = F^{-1}(1 - R) = -\ln(1 - R).$$

The reverse transformation (known as the probability-integral transformation) also works: If X is exponentially distributed with mean 1, then

$$R = F(X) = 1 - e^{-X}$$

is uniform $(0, 1)$. This gets us from X to R; we then use the inverse cdf for the triangular distribution to go from R to a triangularly distributed variate.

Chapter 9

Input Modeling

12. Solution to Exercise 12:
$\ln \bar{X} - 1.255787$
$\sum_{i=1}^{20} \ln X_i = 21.35591$
$1/M = 5.319392$
$\theta = 0.3848516$
$\beta = 2.815$

13. Solution to Exercise 13:

j	$\widehat{\beta}_j$	$\sum_{i=1}^{20} X_i^{\widehat{\beta}j}$	$\sum_{i=1}^{20} X_i^{\widehat{\beta}} j \ln X_i$	$\sum_{i=1}^{20} X_i^{\widehat{\beta}} j(\ln X_i)^2$	$f(\widehat{\beta}_j)$	$f'(\widehat{\beta}_j)$	$\widehat{\beta}_{j+1}$
0	2.539	1359.088	2442.221	4488.722	1.473	-4.577	2.861
1	2.861	2432.557	4425.376	8208.658	.141	-3.742	2.899
2	2.899	2605.816	4746.920	8813.966	.002	-3.660	2.899
3	2.899	2607.844	4750.684	8821.054	.000	-3.699	2.899

$\widehat{\beta} = 2.899$

$\widehat{\alpha} = 5.366$

14. H_0: Data are uniformly distributed

$R_{(i)}$	.0600	.0700	$\cdots$	.4070	$\cdots$	.8720	$\cdots$	.9970
$1/3$	.0333	.0667	$\cdots$	.4333	$\cdots$	.7333	$\cdots$	1.0000
$1/3 - R_{(i)}$	—	—	$\cdots$	.0653	$\cdots$	—	$\cdots$	.0030
$R_{(i)} - (i-1)/30$	.0600	.0367	$\cdots$	.0070	$\cdots$	.1720	$\cdots$	.0303

$D^+ = .0653$

$D^- = .1720$

$D = \max(.0653, .1720) = .1720$

$D_{.05,30} = .24 > D = .1720$

Therefore, do not reject H_0

16. Solution to Exercise 16:

(a) $\alpha = \bar{X} = 1.11$

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x_i	O_i	p_i	E_i	$\frac{(O_i-E_i)^2}{E_i}$
0	35	.3296	32.96	.126
1	40	.3658	36.58	.320
2	13	.2030	20.30	2.625
3	6	.0751	7.51	
4	4	.0209	2.09	
5	1	.0046	.46	
$\geq$6	1	.0010	.10	.333
Totals	100	1.0000	100	$3.404 = \chi_0^2$

$\chi^2_{.05,2} = 5.99$

Therefore, do not reject H_0. Notice that we have grouped cells $i = 3, 4, 5 \geq 6$ together into a single cell with $O_i = 12$ and $E_i = 10.16$.

(b) $\alpha = 1$

x_i	O_i	p_i	E_i	$\frac{(O_i-E_i)^2}{E_i}$
0	35	.3679	36.79	.087
1	40	.3679	36.79	.280
2	13	.1839	18.39	1.580
3	6	.0613	6.13	
4	4	.0153	1.53	
5	1	.0031	.31	
$\geq$6	1	.0006	.06	1.963
Totals	100	1.0000	100	$3.910 = \chi_0^2$

$\chi^2_{.05,3} = 7.81$

Therefore, do not reject H_0. Notice that we have grouped cells $3, 4, 5 \geq 6$ into a single cell with $O_i = 12$ and $E_i = 8.03$.

17. Solution to Exercise 17:

H_0 = Data are exponentially distributed

$\widehat{\lambda} = \bar{X} = 1.206$

$S = 1.267$

i	O_i	$\frac{(O_i-E_i)^2}{E_i}$
1	8	.013
2	11	.853
3	9	.053
4	5	1.333
5	10	.333
6	7	.213
Totals	50	$2.798 = \chi_0^2$

$\chi^2_{.05,4} = 9.49$

Therefore, do not reject H_0

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18. Using the Arena Input Analyzer, the Kolmogorov-Smirnov statistic for normality is 0.0985, which corresponds to a p-value greater than 0.15. The chi-square test statistic with 5 intervals (yielding 2 degrees of freedom) is 4.85, which corresponds to a p-value of 0.09. With 7 intervals (yielding 4 degrees of freedom), the chi-square statistic is 5.98, corresponding to a p-value of 0.21. These statistics show no strong evidence against the hypothesis of normality, although the chi-square statistic with 2 degrees of freedom could be interpreted as rejecting the hypothesis of normality.

19. H_0 = Data are normally distributed

$\widehat{\mu} = \bar{X} = 99.222$

$\widehat{\sigma}^2 = S^2 = 103.41$

Number of Cells (k)	χ_0^2	$\chi^2_{.05,k-3}$	Decision
10	3.2	14.1	Do not reject H_0
8	1.2	11.1	Do not reject H_0
5	1.0	5.99	Do not reject H_0

20. H_0: Data are normally distributed

$\widehat{\mu} = \bar{X} = 4.641$

$\widehat{\sigma}^2 = S^2 = 2.595$

Number of Cells (k)	χ_0^2	$\chi^2_{.05,k-3}$	Decisions
10	5.6	14.1	Do not reject H_0
8	1.52	11.1	Do not reject H_0
5	.6	5.99	Do not reject H_0

21. H_0: Data are exponentially distributed

$\widehat{\lambda} = 1/\bar{X} = 1/9.459 = .106$

i	O_i	$\frac{(O_i - E_i)^2}{E_i}$
1	7	.8
2	3	.8
3	5	0.0
4	5	0.0
5	5	0.0
6	6	.2
7	5	0.0
8	7	.8
9	4	.2
10	3	.8
Totals	50	$3.6 = \chi_0^2$

$\chi^2_{.05,8} = 15.5$

Therefore, do not reject H_0

22. H_0: Data are Poisson distributed

$\alpha = \bar{X} = .48$

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x_i	O_i	p_i	E_i	$\frac{(O_i-E_i)^2}{E_i}$
0	31	.6188	30.94	.0001
1	15	.2970	14.85	.0015
2	3	.0713	3.565	
≥ 3	1	.0129	.645	.0140
Totals	50	1.0000	50.00	$.0120 = \chi_0^2$

$\chi^2_{.05,1} = 3.84$

Therefore, do not reject H_0. Notice that we grouped cells $i = 2, 3$ into a single cell with $O_i = 4$ and $E_i = 4.21$.

Note: In Section 9.4.1 it was stated that there is no general agreement regarding the minimum size of E_i and that values of 3, 4 and 5 have been widely used. We prefer $E_i > 5$. If we follow our suggestion in this case, the degrees of freedom will equal zero, which results in an undefined tabular value of χ^2. The concern is that a very small E_i will result in an undue contribution to χ_0^2. With $E_i = 4.21$ this is certainly not a cause for concern. Thus, combining cells as shown is appropriate.

23. Solution to Exercise 23:

a) The data seem positively dependent.

b) The sample correlation is $\widehat{\rho} = 0.9560$.

c) To fit a bivariate normal distribution we need the sample means, sample variances, and sample correlation.

	Sample mean $\widehat{\mu}$	Sample Variance $\widehat{\sigma}^2$
Milling Time	17.7	$(6.7)^2$
Planning Time	13.1	$(3.6)^2$

Obtain $\widehat{\rho}$ from part (b).

26. For an AR(1) process
$\widehat{\mu} = \overline{X} = 20$
$\widehat{\phi} = \widehat{\rho} = 0.48$
$\widehat{\sigma}_\varepsilon^2 = \widehat{\sigma}^2 = (1 - \widehat{\phi}^2)(3.93)^2(1 - 0.48^2) = 11.89$

For an EAR(1) process
$\widehat{\lambda} = 1/\overline{X} = 0.05$
$\widehat{\phi} = \widehat{\rho} = 0.48$

A histogram and *q-q* plot suggest that AR(1) is a better fit since the distribution appears more normal than exponential.

27. Both exponential and lognormal models look feasible for this data (the Arena Input Analyzer gives *p*-values > 0.15 for the Kolmogorov-Smirnov test in both cases). Since many transactions in a bank are routine and brief, but there are occasional very long transaction times, an exponential model can be justified.

Chapter 10

Verification and Validation of Simulation Models

1. Solution to Exercise 1:

(a) System: $\mu_0 = 22.5$
Model:

$$\begin{aligned} \bar{Y} &= (18.9 + 22.0 + \ldots + 20.2)/7 = 20.614 \\ S_Y &= 1.36 \end{aligned}$$

Test for significance ($H_0 : E(Y) = \mu_0$)

$$t_0 = (20.614 - 22.5)/(1.36/\sqrt{7}) = -3.67$$

For $\alpha = 0.05,\ t_{6,0.025} = 2.45$
Since $|t_0| > 2.45$, reject null hypothesis

(b) Power of the test
$\delta = 2/1.36 = 1.47$
For $\alpha = 0.05$ and $n = 7,\ \delta(1.47) = 0.10$
Power $= 1 - 0.10 = 0.90$
Sample size needed for $\beta \leq 0.20$
Assume that $\sigma = 1.36$
Then for $\alpha = 0.05$ and $\delta = 1.47, n = 6$ observations

2. Solution to Exercise 2:

(a) System: $\mu_0 = 4$
Model:

$$\begin{aligned} \bar{Y} &= (3.70 + 4.21 + \ldots + 4.05)/7 = 4.084 \\ S_y &= 0.2441 \end{aligned}$$

Test for significance ($H_0 : E(Y) = \mu_0$)

$$t_0 = (4.084 - 4)/(0.2441/\sqrt{7}) = 0.91$$

For $\alpha = 0.01,\ t_{6,0.005} = 3.71$

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Since $|t_0| < 3.71$, do not reject null hypothesis

(b) Sample size needed for $\beta \leq 0.10$
$\delta = 0.5/0.2441 = 2.05$
for $\alpha = 0.01$ and $\delta = 2.05, n = 7$ observations.
Then, assuming that the population standard deviation is 0.2441, the current power of the test is 0.90.

3. Solution to Exercise 3:

(a) Test for significance ($H_0 : \mu_d = 0$)
Letting $d_i = y_i - z_i,\ \bar{d} = 3.35, S_d = 1.526$

$$t_0 = 3.35/(1.526/\sqrt{4}) = 4.39$$

For $\alpha = 0.05, t_{3,0.025} = 3.18$
Since $|t_0| > 3.18$, reject the null hypothesis.

(b) Sample size needed for $\beta \leq 0.20$
$\delta = 2/1.526 = 1.31$
For $\alpha = 0.05,\ \beta \leq 0.20$ and $\delta = 1.31$
$n = 8$ observations.

Chapter 11

Output Analysis for a Single Model

For additional solutions check the course web site at `www.bcnn.net`.

3. The 95% confidence interval based on only 5 replications is $[1.02, 16.93]$, which is much wider than the interval based on all 10 replications. From the ensemble averages across five replications, and upper and lower confidence limits, it is not possible to detect a trend in the data.

6. It was assumed that orders could be partially fulfilled before backlogging occurred.

(a) For the (50,30) policy, the average monthly cost over 100 months, $\bar{Y}_{r\cdot}$, for replication r ($r = 1, 2, 3, 4$), is given by

$\bar{Y}_{1\cdot} = \$233.71, \bar{Y}_{2\cdot} = \$226.36, \bar{Y}_{3\cdot} = \$225.78, \bar{Y}_{4\cdot} = \$241.06.$

By Equation (12.39), the point estimate is

$\bar{Y}_{\cdot\cdot} = \$231.73$ and by Equation (12.40), $S^2 = (\$7.19)^2$.

An approximate 90% confidence interval is given by

$\$231.73 \pm t_{0.05,3}(\$7.19)/\sqrt{4}, (t_{0.05,3} = 2.353)$ or $[\$223.27, \$240.19]$

(b) The minimum number of replications is given by

$$R = \min\{R > R_0 : t_{\alpha/2,R-1} S_0/\sqrt{R} \leq \$5\} = 8$$

where $R_0 = 4, \alpha = 0.10, S_0 = \$7.19$ and $\epsilon = \$5$.

The calculation proceeds as follows:

$R \geq (z_{.05}S_0/\epsilon)^2 = [1.645(7.19)/5]^2 = 5.60$

R	6	7	8
$t_{.05,R-1}$	1.94	1.90	1.86
$t_{.05,R-1}S_0/\epsilon^2$	7.78	7.46	7.15

Thus, four additional replications are needed.

7. Solution to Exercise 7:

(a) The following estimates were obtained for the long-run monthly cost on each replication.

$$\bar{Y}_{1\cdot} = \$412.11, \bar{Y}_{2\cdot} = \$437.60, \bar{Y}_{3\cdot} = \$411.26, \bar{Y}_{4\cdot} = \$455.75, \bar{Y}_{\cdot\cdot} = \$429.18, S = \$21.52$$

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An approximate 90% c.i. for long-run mean monthly cost is given by

$\$429.18 \pm 2.353(\$21.52)/\sqrt{4}$, or

$[\$403.86, \$454.50]$

(b) With $R_0 = 4, \alpha = 0.10, S_0 = \$21.52$, and $\epsilon = \$25$ the number of replications needed is

$$\min\{R \geq R_0 : t_{\alpha/2,R-1} S/\sqrt{R} < \$25\} = 5$$

Thus, one additional replication is needed to achieve an accuracy of $\epsilon = \$25$.
To achieve an accuracy of $\epsilon = \$5$, the total number of replications needed is

$$\min\{R \geq R_0 : t_{.05,R-1} S_0/\sqrt{R} < 5\} = 53.$$

The calculations for $\epsilon = \$5$ are as follows:

$$R \geq [z_{.05} S_0/\epsilon]^2 = [1.645(21.52)/5]^2 = 50.12$$

R	51	52	53
$t_{.05,R-1}$	1.675	1.674	1.674
$[t_{.05,R-1} S_0/\epsilon]^2$	52.9	52.9	52.9

Therefore, for $\epsilon = \$5$, the number of additional replications is $53 - 4 = 49$.

10. Ten initial replications were made. The estimated profit is \$98.06 with a standard deviation of $S_0 =$ \$12.95.

For $\alpha = 0.10$ and absolute precision of $\epsilon = \$5.00$, the sample size is given by

$$\min\{R \geq 10 : t_{\alpha/2,R-1}(12.95)/\sqrt{R} < \$5\}$$

R	$t_{\alpha/2,R-1} S_0/\sqrt{R}$
19	5.15
20	5.01
21	4.87

Thus, 21 replications are needed. Based on 21 replications the estimated profit is:

$$\bar{Y} = \$96.38, S = \$13.16$$

and a 90% c.i. is given by

$$\$96.38 \pm t_{.05,20} S/\sqrt{21}$$

or $\$96.38 \pm \$4.94$.

If $\epsilon = \$0.50$ and $\alpha = 0.10$, then the sample size needed is approximately 1815.

13. The table below summarizes the results from each replication:

	Response Time (hrs.) for Job Type				Average Utilization at each Station			
Replications	1	2	3	4	1	2	3	4
1	146.6	88.82	82.81	42.53	0.509	0.533	0.724	0.516
2	146.4	89.79	80.45	46.48	0.517	0.537	0.772	0.569
3	144.4	88.40	81.59	45.01	0.468	0.516	0.692	0.491
4	144.3	88.00	82.13	47.17	0.486	0.489	0.673	0.496
5	144.9	88.29	82.53	43.26	0.471	0.473	0.627	0.461
$\bar{Y}_{..}$	145.3	88.66	81.90	44.89	0.465	0.510	0.698	0.507
S	1.103	.697	.932	1.998	0.022	0.028	0.054	0.049

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A 97.5% c.i. for utilization at each work station is given by

Station 1, $[.463, .518]$

Station 2, $[.475, .544]$

Station 3, $[.631, .765]$

Station 4, $[.457, .556]$

Note that by the Bonferroni inequality, Equation (12.20), the overall confidence level is 90% or greater.

A 95% c.i. for mean total response time (hrs.) of each job type is given by

Job type 1, $[143.6, 147.0]$

Job type 2, $[87.57, 89.75]$

Job type 3, $[80.44, 83.36]$

Job type 4, $[41.77, 48.01]$

Note that the overall confidence level is 80% or greater.

Chapter 12

Comparison and Evaluation of Alternative System Designs

For additional solutions check the course web site at `www.bcnn.net`.

2. Using common random numbers, the following results were obtained:

	Policy			
Rep.	(50,30)	(50,40)	(100,30)	(100,40)
1	\$233.71	\$226.21	\$257.73	\$261.90
2	\$226.36	\$232.12	\$252.58	\$257.89
3	\$225.78	\$221.02	\$266.48	\$258.16
4	\$241.06	\$243.95	\$270.61	\$270.51
$\bar{Y}_{\cdot i}$	\$231.73	\$230.83	\$261.85	\$262.12
S_i	\$7.19	\$9.86	\$8.19	\$5.89

To achieve an overall $\alpha_E = 0.10$, compute 97.5% confidence intervals (c.i.) for mean monthly cost for each policy by using

$\bar{Y}_{\cdot i} \pm t_{.0125,3} S_i/\sqrt{4}$, $\quad (t_{.0125,3} = 4.31$ by interpolation)

Policy	c.i.
(50,30)	\$231.73 ± \$15.49
(50,40)	\$230.83 ± \$21.25
(100,30)	\$261.85 ± \$17.65
(100,40)	\$262.12 ± \$12.69

The overall confidence level is at least 90%.

To obtain confidence intervals which do not overlap, policies (50,30) and (50,40) should be estimated with an accuracy $\epsilon = (\$231.73 - \$230.83)/2 = \$.45$, and policies (100,30) and (100,40) with $\epsilon = (\$262.12 - \$261.85)/2 = \$.135$.

An estimate for R is given by

$R > \left[\frac{z_{\alpha/2} S_i}{\epsilon}\right]^2$ with $z_{.0125} = 2.24$

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Policy	R (replications)
(50,30)	1281
(50,40)	2411
(100,30)	18,468
(100,40)	9551

The above number of replications might take excessive computer time and thus be too expensive to run. A better technique would be to compute c.i.'s for the differences.

At a 90% level, policies (50,30) and (50,40) appear to be better than the other two. A 90% c.i. for the difference between the (50,30) and (50,40) policies is given by

$\$.9025 \pm t_{.05,3} \times 6.250/\sqrt{4}$ or $[-\$6.451, \$8.256]$.

Since this interval includes zero, no significant difference is detected.

3. Using common random numbers, the following results were obtained for 4 replications:

	Policy				
Rep	(50,30)	(50,40)	(100,30)	(100,40)	D
1	\$412.11	\$405.69	\$419.57	\$398.78	\$6.91
2	\$437.60	\$409.54	\$429.82	\$410.60	-\$1.06
3	\$411.26	\$399.30	\$470.17	\$416.37	-\$17.07
4	\$455.75	\$418.01	\$466.55	\$438.95	-\$20.94
$\bar{Y}_i$	\$429.18	\$408.14	\$446.53	\$416.18	-\$8.04= $\bar{D}$
S_i	\$21.52	\$7.82	\$25.60	\$16.86	\$13.17 = S_D

It appears that the (50,40) policy dominates the other three policies. A 90% c.i. was computed for the mean difference in cost between the (50,40) and (100,40) policies. The differences, sample mean difference and sample standard deviation are given in the table above. It is clear that a 90% c.i. will contain zero. Thus, there is no significant difference between the 2 policies. The 90% c.i. is $-\$8.04 \pm \$15.47$. A complete analysis would compute c.i.'s for all differences, perhaps discard clearly inferior policies, and then replicate the remaining ones to determine the best policy.

6. Using common random numbers, 21 replications were made for different ordering sizes. The table below summarizes the results:

Q (cards)	Estimate of Mean Profit (\$)	Estimated Standard Deviation (\$)
250	85.05	51.17
300	96.38	13.16
350	101.4	20.89
356	101.8	20.92
357	101.9	20.88
360	101.9	21.00
375	101.5	21.71
400	99.91	22.83

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Based on Exercise 11.10, a 90% c.i. for mean total profit at $Q = 300$ was \$96.38 ± \$4.94. To obtain an accuracy of $\epsilon = \$5.00$ at $\alpha = 0.10$ additional replications should be made for Q in the range 350 to 400. Confidence intervals for differences could be computed to determine a range of Q significantly better than other Q.

9. Use $c_i > \lambda_i/\mu_i$ applied one station at a time.

Station 1

Station 1 receives types 1, 2 and 4 arrivals. Therefore,

Arrival rate $\lambda_1 = .4(.25) + .3(.25) + .1(.25) = .20$ per hour

Mean service time $\frac{1}{\mu_1} = \frac{.4}{.8}(20) + \frac{.3}{.8}(18) + \frac{.1}{.8}(30) = 20.5$ hours

$c_1 > \lambda_1/\mu_1 = .20(20.5) = 4.1, c_1 = 5$ servers.

Station 2

If station 1 is stable (i.e. has 5 or more servers), then departures occur at the same rate as arrivals. Station 2 receives type 1 arrivals from station 1 and type 3 arrivals from the outside. Therefore,

Arrival rate $\lambda_2 = .4(.25) + .2(.25) = .15$ per hour

Mean service time $\frac{1}{\mu_2} = \frac{.4}{.6}(30) + \frac{.2}{.6}(20) = 26.67$ hours

$c_2 > \lambda_2/\mu_2 = .15(26.67) = 4.00, c_2 = 5$ servers

Station 3

Station 3 receives types 1, 2, and 3 arrivals. Therefore,

Arrival rate $\lambda_3 = .4(.25) + .3(.25) + .2(.25) = .225$ per hour

Mean service time $\frac{1}{\mu_3} = \frac{.4}{.9}(75) + \frac{.3}{.9}(60) + \frac{.2}{.9}(50) = 64.44$ hours

$c_1 > \lambda_3/\mu_3 = .225(64.44) = 14.50, c_3 = 15$ servers

Station 4

Station 4 receives all arrivals. Therefore,

Arrival rate $\lambda_4 = .25$ per hour

Mean service times $\frac{1}{\mu_4} = .4(20) + .3(10) + .2(10) + .1(15) = 14.5$ hours

$c_4 > \lambda_4/\mu_4 = .25(14.5) = 3.63, c_4 = 4$ servers

For $c_1 = 5, c_2 = 5, c_3 = 15$, and $c_4 = 4$ the following results are obtained for one replication with $T_0 = 200$ hours and $T_E = 800$ hours.

Jobs	Average Response Time (hours)
Type 1	170.3
Type 2	106.8
Type 3	106.6
Type 4	56.44
All jobs	126.8

Station	Estimated Server Utilization
1	.754
2	.751
3	.828
4	.807

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Additional replications should be conducted and standard errors and confidence intervals computed. In addition, initialization bias should be investigated. Since $\lambda_4/c_4\mu_4$ was calculated to be $3.63/4 = .9075$ and $\widehat{\rho}_4 = .807$, it appears that significant bias may be present for $T_0 = 200$ hours and $T_E = 800$ hours.

13. Let S be the set-up time, which is exponentially distributed with mean 20. Let P_j be the time to process the jth application, which is normally distributed with mean 7 and standard deviation 2. For a particular design point, x, we generate n replications of total processing time as follows:

```
for i = 1 to n
do
   generate S
   for j = 1 to x
   do
     generate P_j
   enddo
   Y_i = S + P_1 + P_2 + ··· + P_x
enddo
```

15. Because the samples across design points are dependent, MS_E/S_{xx} is a biased estimator of the variance of $\widehat{\beta}_1$, and the degrees of freedom are not $n-2$.

18. Let m be the number of buffer spaces ($m = 50$ in this problem). Since $x_1 + x_2 + x_3 = m$, x_3 is determined once x_1 and x_2 are specified. Thus, what we really need are all assignments to x_1 and x_2 such that $x_1 + x_2 \le m$. Clearly there are $m+1$ possible assignments for x_1; specifically, $0, 1, 2, \ldots, m$. If x_1 is assigned value ℓ, then there are $m + 1 - \ell$ possible assignments for x_2; specifically, $0, 1, 2, \ldots, m - \ell$. If we sum over the possible assignments for x_1 we obtain

$$\sum_{\ell=0}^{m}(m+1-\ell) = \frac{(m+1)(m+2)}{2}$$

which is 1326 when $m = 50$.

The scheme we will develop for sampling (x_1, x_2, x_3) will first sample x_1, then x_2 given the value of x_1, and finally compute $x_3 = m - x_2 - x_1$.

Let $n = (m+1)(m+2)/2$, the number of possible outcomes for (x_1, x_2, x_3), all equally likely. The marginal probability that $x_1 = m$ is $1/n$, since $(m, 0, 0)$ is the only way it can happen. The marginal probability that $x_1 = m-1$ is $2/n$ since $(m-1, 1, 0)$ and $(m-1, 0, 1)$ are the only ways it can happen. Arguing this way we can show that

$$P(x_1 = j) = \frac{m-j+1}{n}$$

for $j = 0, 1, 2, \ldots, m$. Thus, we can use one of the general methods for sampling from discrete distributions to sample x_1.

Now given x_1, we can show that the marginal distribution of x_2 is discrete uniform on $\{0, 1, \ldots, m - x_1\}$, a distribution that is easy to sample. And finally, $x_3 = m - x_2 - x_1$.

19. For this problem the true optimal solution can be computed analytically: $x^* = 2.611$ years, giving an expected cost of \$11,586. This solution is obtained by minimizing the expected cost, which can be written as

$$2000x + \int_0^\infty 20000\, I(y \le 1)\frac{e^{-y/x}}{x}\, dx$$

where I is the indicator function.

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20. For this problem the true optimal solution can be computed analytically: $x^* = 2.611$ years, giving an expected cost of \$11,586. This solution is obtained by minimizing the expected cost, which can be written as

$$2000x + \int_0^\infty 20000\, I(y \leq 1)\frac{e^{-y/x}}{x}\, dx$$

where I is the indicator function.

21. There are two optimal solutions, $x^* = 9, 10$, with objective function value approximately 0.125.