Introduction to Computability and Complexity - Homework 4

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Determine the correct inclusion relationship between the following pairs of classes. That is, fill in one of the relation symbols $\subsetneq, \subseteq, =, \supseteq, \supsetneq$, or ? (if we cannot say based on our knowledge). Justify your answers.

(a) SPACE(n) ? TIME($2^{\log^3 n}$)

If we want to show any subset relationship from SPACE to TIME, we would have to go through Theorem 1.iii and the Corollary to Theorem 2. But NSPACE(n) $\not\subseteq$ TIME($2^{\log^3 n}$), as $n \notin o(\log^3 n)$.

If we want to show any subset relationship from TIME to SPACE, we would have to at least go through SPACE($2^{\log^3 n}$) (either by 1.i or by 1.ii and 1.iv), and then by Theorem 4 show that $2^{\log^3 n} \in o(n)$, which is false.

Therefore we cannot say the inclusion relationship.

(b) TIME($2^{\log^3 n}$) $\supsetneq$ NSPACE($\log^2 n$)

$$\text{NSPACE}(\log^2 n) \overset{(1)}{\subseteq} \text{TIME}(2^{\log^{2.5} n}) \overset{(2)}{\subsetneq} \text{TIME}(2^{\log^3 n})$$

(1) By Corollary. $\log^2 n \in o(\log^{2.5} n)$
(2) By Theorem 5.

$$2^{\log^{2.5} n} \in o\left(\frac{2^{\log^3 n}}{\log^3 n}\right) \iff \lim_{n\to\infty} \frac{2^{\log^{2.5} n} \cdot \log^3 n}{2^{\log^3 n}} = 0$$

$2^{\log^3 n}$ dominates, so at the limit the equation holds.

(c) NSPACE($\log^2 n$) $\subsetneq$ NTIME($2^{\log^3 n}$)

$$\text{NSPACE}(\log^2 n) \overset{(1)}{\subsetneq} \text{TIME}(2^{\log^3 n}) \overset{(2)}{\subseteq} \text{NTIME}(2^{\log^3 n})$$

(1) By Problem (b).
(2) By Theorem 1.ii.

(d) NTIME($2^{\log^3 n}$) ? NTIME($2^{n \log n}$)

To show an inclusion relationship from NTIME $\to$ NTIME, we'd have to go through NTIME $\to$ SPACE $\to$ NSPACE $\to$ TIME $\to$ NTIME. For NSPACE($f(n)$) $\subseteq$ TIME($2^{g(n)}$) such that $f(n) \in o(g(n))$ there is no possibility for either inclusion direction, so we cannot say based on our knowledge.

(e) $\text{NTIME}(2^{n\log n}) \supsetneq \text{SPACE}(n)$

$$\text{SPACE}(n) \subsetneq \text{SPACE}(n\log(\log n)) \tag{1}$$
$$\text{SPACE}(n\log(\log n)) \subseteq \text{NSPACE}(n\log(\log n)) \tag{2}$$
$$\text{NSPACE}(n\log(\log n)) \subseteq \text{TIME}(2^{n\log n}) \tag{3}$$
$$\text{TIME}(2^{n\log n}) \subseteq \text{NTIME}(2^{n\log n}) \tag{4}$$

(1) By Theorem 4. Clearly $n \in o(n\log(\log n))$
(2) By Theorem 1.iii.
(3) By Corollary.

$$\lim_{n\to\infty} \frac{n\log(\log n)}{n\log n} = \frac{\log(\log n)}{\log n} = 0$$

(4) By Theorem 1.ii.

(f) $\text{SPACE}(n) \supsetneq \text{NSPACE}(\log^2 n)$

$$\text{NSPACE}(\log^2 n) \overset{(1)}{\subseteq} \text{SPACE}(\log^4 n) \overset{(2)}{\subsetneq} \text{SPACE}(n)$$

(1) By Theorem 3.
(2) By Theorem 4. Clearly $\log^4 n \in o(n)$

(g) $\text{TIME}(2^{\log^3 n}) \subseteq \text{NTIME}(2^{\log^3 n})$

$$\text{TIME}(2^{\log^3 n}) \overset{(1)}{\subseteq} \text{NTIME}(2^{\log^3 n})$$

(1) By Theorem 1.ii.

Cannot be $\subsetneq$, as there would have to be a $g(n)$ greater than $2^{\log^3 n}$ so that Theorem 5 could apply, but also equal to $2^{\log^3 n}$ so that Theorem 1.ii can apply. Obviously this is not possible. Further, there cannot be an inclusion relationship from NTIME $\to$ TIME, as there would have to be an additional exponential factor for TIME, from the Corollary.

(h) $\text{NSPACE}(\log^2 n) \subsetneq \text{NTIME}(2^{n\log n})$

$$\text{NSPACE}(\log^2 n) \overset{(1)}{\subseteq} \text{TIME}(2^{\log^3 n}) \overset{(2)}{\subsetneq} \text{TIME}(2^{n\log n}) \overset{(3)}{\subseteq} \text{NTIME}(2^{n\log n})$$

(1) By Corollary. $\log^2 n \in o(\log^3 n)$
(2) By Theorem 5.

$$2^{\log^3 n} \in o\left(\frac{2^{n\log n}}{n\log n}\right) \iff \lim_{n\to\infty} \frac{2^{\log^3 n}\cdot n\log n}{2^{n\log n}} = 0$$

$2^{n\log n}$ dominates, so the limit holds.
(3) By Theorem 1.ii.

(i) NTIME($2^{\log^3 n}$) ? SPACE(n)

From SPACE $\rightarrow$ NTIME we would need to go from n to something that dominates n in the exponent (going through the Corollary), which $\log^3 n$ does not.

From NTIME $\rightarrow$ SPACE by Theorem 1.iv we get SPACE($2^{\log^3 n}$), but there is no way to get to SPACE(n) from here.

(j) NTIME($2^{n \log n}$) $\supsetneq$ TIME($2^{\log^3 n}$)

$$\mathrm{TIME}(2^{\log^3 n}) \overset{(1)}{\subsetneq} \mathrm{TIME}(2^{n \log n}) \overset{(2)}{\subseteq} \mathrm{NTIME}(2^{n \log n})$$

(1) By Theorem 5. (Same proof as (2) in Problem (h)).

(2) By Theorem 1.ii.