Introduction to Complexity and Computability: Homework 3

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Homework Problem 1

Consider the language $S = \{\langle M_1, M_2\rangle \mid L(M_1) \cap L(M_2) = \emptyset\}$. Decide whether the language S is a decidable language. If it is not decidable, decide if S or $\bar{S}$ is a partially decidable language.

By Rice's Theorem, language S is undecidable because the language is nontrivial.

Fix M_2 so that $L(M_2) = \{01\}$. Then it is nontrivial because for M_1 some Turing Machines accept languages that have an empty intersection with $\{01\}$ and others do not.

It is also clear that the property is a property of the language and not of the Turing Machines. If two TMs recognise the same language ($L(M_1') = L(M_1'')$), then either both have descriptions in S or neither do.

$\bar{S}$ is partially decidable. $\bar{S} = \{\langle M_1, M_2\rangle \mid L(M_1) \cap L(M_2) \neq \emptyset\}$.

For each x in Σ^* in lexicographic order, run M_1 on x for one move, and run M_2 on x for one move. If M_1 halts and accepts, add x to the set of strings of the language accepted by M_1. If M_2 halts and accepts, add x to the set of strings of the language accepted by M_2.

At each step of the computation, the Turing Machines run one more step of all previous strings, plus the first step of the next x. I.e. First, perform the first move of the TM for the first x, then perform the second step of the TM for the first x and the first step of the TM for the second x. Then perform the third step of the TM for the first x, the second step of the TM for the second x, and the first step of the TM for the third x, and so on. If there is a string that is accepted by both M_1 and M_2 (that is, string w has been added to both the set of strings of the language accepted by M_1 and the set of strings of the language accepted by M_2), then $\langle M_1, M_2\rangle \in \bar{S}$.

S is not partially decidable. By Post's Theorem, if both S and $\bar{S}$ are partially decidable, then S is decidable. But S is undecidable, and $\bar{S}$ is partially decidable, so S must not be partially decidable.

Homework Problem 2

Consider the language

$$S = \{\langle M\rangle \mid (\forall w \in \Sigma^*)(w \in L(M) \iff w^R \in L(M))\}.$$

Show that the following holds:

(a) $L_u \leq_m S$
Find

$$f(\langle M, x\rangle) = \langle M'\rangle$$

Given M and x, define M':
On input w:

- If $w = 10$, *accept*.
- If $w = 01$, run M on x. If M accepts, *accept*.
- Otherwise, *reject*.

$$L(M') = \begin{cases} \{01, 10\} & \text{if } x \in L(M), \\ \{10\} & \text{otherwise} \end{cases}$$

If M accepts x, $L(M') = \{01, 10\}$, so $\langle M'\rangle \in S$. If M does not accept $x, L(M') = \{10\}$, so there is a string $w \in L(M')$ for which $w^R \notin L(M')$, and $\langle M'\rangle \notin S$. So we have a total computable function $f(\langle M, x\rangle) = \langle M'\rangle$, and $L_u \leq_m S$.

(b) $L_u \leq_m \bar{S}$

$$\bar{S} = \{\langle M\rangle | (\exists w \in \Sigma^*)(w \in L(M) \iff w^R \notin L(M)\}$$

Given M and x, define M':
On input w:

- If $w = 01$, run M on x. If M accepts, *accept*.
- Otherwise, *reject*.

If M accepts x, 01 is in the language accepted by M' and its reverse (10) isn't. So $\langle M'\rangle \in \bar{S}$. If M does not accept x, the language is empty, so there is no string w for which the string is in the language and its reverse isn't. So $\langle M'\rangle \notin \bar{S}$, and we have a total computable function $f(\langle M, x\rangle) = \langle M'\rangle$. So $L_u \leq_m \bar{S}$.