Introduction to Categorical Logic

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Steve Awodey Andrej Bauer

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Chapter 1

Algebraic Theories

Algebraic theories are descriptions of structures determined by operations and equations. There are familiar examples from elementary algebra, such as groups, but also many concepts that are not evidently algebraic, such as adjoint functors, can be given algebraic formulations. Thus the scope of algebraic theories is actually much greater than first appears. On the other hand, all such algebraic notions have in common some quite deep and general properties, from the existence of free algebras to Lawvere's duality theory. The most important of these are presented in this chapter. The development also serves as a first example and template for the scheme of "functorial semantics," to be applied to other logical notions in later chapters.

1.1 Syntax and semantics

We begin with a general approach to algebraic structures such as groups, rings, modules, and lattices. These are characterized by axiomatizations which involve only variables, constants, operations, and equations. It is important that the operations are defined everywhere, which excludes two important examples: fields because the inverse of 0 is undefined, and categories because composition is defined only for some pairs of morphisms.

Let us start with the quintessential algebraic theory—the theory of groups. A group can be described as a set G with a binary operation $\cdot : G \times G \to G$, satisfying the two axioms:

$$\forall\, x, y, z \in G \,.\, (x \cdot y) \cdot z = x \cdot (y \cdot z)$$
$$\exists\, e \in G \,.\, \forall\, x \in G \,.\, \exists\, y \in G \,.\, (e \cdot x = x \cdot e = x \wedge x \cdot y = y \cdot x = e)$$

Taking a closer look at the logical form of these axioms, we see that the second one, which expresses the existence of a unit and inverse elements, is somewhat unsatisfactory because it involves nested quantifiers. Not only does this complicate the interpretation, but it is not really necessary, since the unit element and inverse operation in a group are uniquely determined. Thus we can add them to the structure and reformulate as follows. We require the unit to be a distinguished *constant* $e \in G$ and the inverse to be an *operation* ${}^{-1} : G \to G$. We then obtain an equivalent formulation in which all axioms are now *equations*:

$$\begin{gathered} x \cdot (y \cdot z) = (x \cdot y) \cdot z \\ \begin{aligned} x \cdot e &= x \qquad & e \cdot x &= x \\ x \cdot x^{-1} &= e \qquad & x^{-1} \cdot x &= e \end{aligned} \end{gathered}$$

Notice that the universal quantifier $\forall\, x \in G$ is no longer needed in stating the axioms, since we interpret all variables as ranging over all elements of G. Nor do we really need to explicitly mention the particular set G in the specification. Finally, since the constant e can be regarded as a nullary operation, i.e., a function $e : \mathbf{1} \to G$, the specification of the group concept consists solely of operations and equations. This leads us to the general definition of an algebraic theory.

Definition 1.1.1. A *signature* Σ for an algebraic theory consists of a family of sets $\{\Sigma_k\}_{k \in \mathbb{N}}$. The elements of Σ_k are called the *k-ary operations*. In particular, the elements of Σ_0 are the *nullary operations* or *constants*.

The *terms* of a signature Σ are expressions constructed inductively by the following rules:

1. variables x, y, z, …, are terms,
2. if $t_1, \ldots, t_k$ are terms and $f \in \Sigma_k$ is a k-ary operation then $f(t_1, \ldots, t_k)$ is a term.

Definition 1.1.2 (cf. Definition 1.2.9)**.** An *algebraic theory* $\mathbb{T} = (\Sigma_{\mathbb{T}}, A_{\mathbb{T}})$ is given by a signature $\Sigma_{\mathbb{T}}$ and a set $A_{\mathbb{T}}$ of *axioms*, which are equations between terms (formally, pairs of terms).

Algebraic theories are also called *equational theories*.

Example 1.1.3. The theory of a commutative ring with unit is an algebraic theory. There are two nullary operations (constants) 0 and 1, a unary operation $-$, and two binary operations $+$ and $\cdot$. The equations are:

$$\begin{aligned} (x + y) + z &= x + (y + z) \qquad & (x \cdot y) \cdot z &= x \cdot (y \cdot z) \\ x + 0 &= x & x \cdot 1 &= x \\ 0 + x &= x & 1 \cdot x &= x \\ x + (-x) &= 0 & (x + y) \cdot z &= x \cdot z + y \cdot z \\ (-x) + x &= 0 & z \cdot (x + y) &= z \cdot x + z \cdot y \\ x + y &= y + x & x \cdot y &= y \cdot x \end{aligned}$$

Example 1.1.4. The "empty" theory with no operations and no equations is the theory of a set.

Example 1.1.5. The theory with one constant and no equations is the theory of a *pointed set*, cf. Example A.4.11.

Example 1.1.6. Let R be a ring. There is an algebraic theory of left R-modules. It has one constant 0, a unary operation $-$, a binary operation $+$, and for each $a \in R$ a unary operation $\overline{a}$, called *scalar multiplication by a*. The following equations hold:

$$\begin{aligned} (x+y)+z &= x+(y+z) \;, & x+y &= y+x \;, \\ x+0 &= x \;, & 0+x &= x \;, \\ x+(-x) &= 0 \;, & (-x)+x &= 0 \;. \end{aligned}$$

For every $a, b \in R$ we also have the equations

$$\overline{a}(x+y) = \overline{a}\,x + \overline{a}\,y \;, \qquad \overline{a}(\overline{b}\,x) = \overline{(ab)}\,x \;, \qquad \overline{(a+b)}\,x = \overline{a}\,x + \overline{b}\,x \;.$$

Scalar multiplication by a is usually written as $a \cdot x$ instead of $\overline{a}\,x$. If we replace the ring R by a field $\mathbb{F}$ we obtain the algebraic theory of a vector space over $\mathbb{F}$ (even though the theory of fields is not algebraic!).

Example 1.1.7. In computer science, inductive datatypes are examples of algebraic theories. For example, the datatype of binary trees with leaves labeled by integers might be defined as follows in a programming language:

```
type tree = Leaf of int | Node of tree * tree
```

This corresponds to the algebraic theory with a constant `Leaf` n for each integer n and a binary operation `Node`. There are no equations. Actually, when computer scientists define a datatype like this, they have in mind a particular model of the theory, namely the *free* one.

Example 1.1.8. An obvious non-example is the theory of posets, formulated with a binary relation symbol $x \leq y$ and the usual axioms of reflexivity, transitivity and anti-symmetry, namely:

$$\begin{gathered} x \leq x \\ x \leq y \wedge y \leq z \Rightarrow x \leq z \\ x \leq y \wedge y \leq x \Rightarrow x = x \end{gathered}$$

On the other hand, using an operation of greatest lower bound or "meet" $x \wedge y$, one can make the equational theory of "$\wedge$-semilattices":

$$\begin{gathered} x \wedge x = x \\ x \wedge y = y \wedge x \\ x \wedge (y \wedge z) = (x \wedge y) \wedge z \end{gathered}$$

Then, defining a partial ordering $x \leq y \iff x \wedge y = x$ we arrive at the notion of a "poset with meets", which *is* equational (of course, the same can be done with joins $x \vee y$ as well). We'll have a proof later (in section **??**) that there is no reformulation of the general theory of posets into an equivalent equational one however.

Exercise 1.1.9. Let G be a group. Formulate the notion of a (left) G-set (i.e. a functor $G \to \mathsf{Set}$) as an algebraic theory.

1.1.1 Models of algebraic theories

Let us now consider what a *model* of an algebraic theory is. In classical algebra, a group is given by a set G, an element $e \in G$, a function $m : G \times G \to G$ and a function $i : G \to G$, satisfying the group axioms:

$$
\begin{gathered}
m\,(x, m\,(y,z)) = m\,(m\,(x,y)\,, z) \\
m\,(x, i\,x) = m\,(i\,x, x) = e \\
m\,(x, e) = m\,(e, x) = x
\end{gathered}
$$

This notion can easily be generalized so that we can speak of models of group theory in categories other than Set. This is accomplished simply by translating the equations between certain elements into equations between the operations themselves: thus a group is given by an object $G \in \mathsf{Set}$ and three morphisms

$$e : \mathsf{1} \to G\,, \qquad m : G \times G \to G\,, \qquad i : G \to G\,.$$

Associativity of m is expressed by the commutativity of the following diagram:

$$
\begin{tikzcd}
G \times G \times G \arrow[r, "m \times \pi_2"] \arrow[d, "\pi_0 \times m"'] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"'] & G
\end{tikzcd}
\tag{1.1}
$$

Similarly, the axioms for the unit and the inverse are expressed by commutativity of the following diagrams:

$$
\begin{tikzcd}
G \times \mathsf{1} \arrow[r, "\mathsf{1}_G \times e"] \arrow[dr, "\pi_0"'] & G \times G \arrow[d, "m"] & \mathsf{1} \times G \arrow[l, "e \times \mathsf{1}_G"'] \arrow[dl, "\pi_1"] \\
& G &
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "\langle \mathsf{1}_G, i\rangle"] \arrow[d, "!_G"'] & G \times G \arrow[d, "m"] & G \arrow[l, "\langle i, \mathsf{1}_G\rangle"'] \arrow[d, "!_G"] \\
\mathsf{1} \arrow[r, "e"'] & G & \mathsf{1} \arrow[l, "e"]
\end{tikzcd}
\tag{1.2}
$$

Moreover, this formulation makes sense in any category $\mathcal{C}$ with finite products. So we can define a *group in* $\mathcal{C}$ to consist of an object G equipped with arrows:

$$
\begin{tikzcd}
G \times G \arrow[r, "m"] & G & G \arrow[l, "i"'] \\
& \mathsf{1} \arrow[u, "e"'] &
\end{tikzcd}
$$

such that the above diagrams (1.1) and (1.2) expressing the group equations commute.

There is also an obvious corresponding generalization of a group homomorphism in Set to homomorphisms of groups in $\mathcal{C}$. Namely, an arrow in $\mathcal{C}$ between groups $h : M \to N$ is a homomorphism if it commutes with the interpretations of the basic operations m, i, and e,

$$h \circ m^M = m^N \circ h^2 \qquad h \circ i^M = i^N \circ h \qquad h \circ e^M = e^N$$

as indicated in:

$$\begin{tikzcd} M^2 \arrow[r, "h^2"] \arrow[d, "m^M"'] & N^2 \arrow[d, "m^N"] \\ M \arrow[r, "h"'] & N \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "h"] \arrow[d, "i^M"'] & N \arrow[d, "i^N"] \\ M \arrow[r, "h"'] & N \end{tikzcd} \qquad \begin{tikzcd} \mathsf{1} \arrow[r, "="] \arrow[d, "e^M"'] & \mathsf{1} \arrow[d, "e^N"] \\ M \arrow[r, "h"'] & N \end{tikzcd}$$

Together with the evident composition and identity arrows inherited from $\mathcal{C}$, this gives a category of groups in $\mathcal{C}$ which we denote:

$$\mathsf{Group}(\mathcal{C})$$

In general, we define an *interpretation* I of a theory $\mathbb{T}$ in a category $\mathcal{C}$ with finite products to consist of an object $I \in \mathcal{C}$ and, for each basic operation f of arity k, a morphism $f^I : I^k \to I$. (More formally, I is the tuple consisting of an underlying set $|I|$ and the interpretations f^I, but we shall write simply I for $|I|$.) In particular, basic constants are interpreted as morphisms $\mathsf{1} \to I$. The interpretation can be extended to all terms as follows: a general term t is always interpreted together with a *context* of variables $x_1, \ldots, x_n$, where the variables appearing in t are among the variables appearing in the context. We write

$$x_1, \ldots, x_n \mid t \tag{1.3}$$

to indicate that the term t is to be understood in context $x_1, \ldots x_n$. The interpretation of a term in context (1.3) is a morphism $t^I : I^n \to I$, determined by the following specification:

1. The interpretation of a variable x_i is the i-th projection $\pi_i : I^n \to I$.

2. A term of the form $f(t_1, \ldots, t_k)$ is interpreted as the composite:

$$I^n \xrightarrow{\left({t_1}^I, \ldots, {t_k}^I\right)} I^k \xrightarrow{\quad f^I \quad} I$$

where ${t_i}^I : I^n \to I$ is the interpretation of the subterm t_i, for $i = 1, \ldots, k$, and f^I is the interpretation of the basic operation f.

It is clear that the interpretation of a term really depends on the context, and when necessary we shall write $t^I = [\, x_1, \ldots, x_n \mid t\,]^I$. For example, the term $f\,x_1$ is interpreted as a morphism $f^I : I \to I$ in context x_1, and as the morphism $f^I \circ \pi_1 : I^2 \to I$ in the context x_1, x_2.

Suppose u and v are terms in context $x_1, \ldots, x_n$. Then we say that the equation $u = v$ is *satisfied* by the interpretation I if u^I and v^I are the same morphism in $\mathcal{C}$. In particular, if $u = v$ is an axiom of the theory, and $x_1, \ldots, x_n$ are all the variables appearing in u and v, we say that I *satisfies the axiom* $u = v$ if $[x_1, \ldots, x_n \mid u]^I$ and $[x_1, \ldots, x_n \mid v]^I$ are the same morphism,

$$I^n \underset{[x_1, \ldots, x_n \mid v]^I}{\overset{[x_1, \ldots, x_n \mid u]^I}{\rightrightarrows}} I \tag{1.4}$$

which we also write as:

$$I \models u = v \qquad \Longleftrightarrow \qquad u^I = v^I.$$

Of course, we can now define as usual:

Definition 1.1.10 (cf. Definition 1.2.9)**.** A *model* M of an algebraic theory $\mathbb{T}$ in a category $\mathcal{C}$ with finite products is an interpretation I of the theory that satisfies the axioms of $\mathbb{T}$,

$$I \models u = v,$$

for all $(u = v) \in A_{\mathbb{T}}$.

A *homomorphism* of models $h : M \to N$ is an arrow in $\mathcal{C}$ that commutes with the interpretations of the basic operations,

$$h \circ f^M = f^N \circ h^k$$

for all $f \in \Sigma_{\mathbb{T}}$, as indicated in:

$$\begin{tikzcd} M^k \arrow[r, "h^k"] \arrow[d, "f^M"'] & N^k \arrow[d, "f^N"] \\ M \arrow[r, "h"'] & N \end{tikzcd}$$

The category of $\mathbb{T}$-models in $\mathcal{C}$ is written,

$$\mathsf{Mod}(\mathbb{T}, \mathcal{C}).$$

A model of the empty theory $\mathbb{T}_0$ in a category $\mathcal{C}$ with finite products is just an object $A \in \mathcal{C}$, and similarly for homomorphisms, so

$$\mathsf{Mod}(\mathbb{T}_0, \mathcal{C}) = \mathcal{C}.$$

A model of the theory $\mathbb{T}_{\mathsf{Group}}$ of groups in $\mathcal{C}$ is a group in $\mathcal{C}$, in the above sense, and similarly for homomorphisms, so:

$$\mathsf{Mod}(\mathbb{T}_{\mathsf{Group}}, \mathcal{C}) = \mathsf{Group}(\mathcal{C}).$$

In particular, a model in Set is just a group in the usual sense:

$$\mathsf{Mod}(\mathbb{T}_{\mathsf{Group}}, \mathsf{Set}) = \mathsf{Group}(\mathsf{Set}) = \mathsf{Group}.$$

An example of a new kind is provided the following.

Example 1.1.11. A model of the theory of groups in a functor category $\mathsf{Set}^{\mathbb{C}}$ is the same thing as a functor from $\mathbb{C}$ into groups,

$$\mathsf{Group}(\mathsf{Set}^{\mathbb{C}}) \cong \mathsf{Hom}(\mathbb{C}, \mathsf{Group}).$$

Indeed, for each object $C \in \mathbb{C}$ there is an evaluation functor,

$$\mathsf{eval}_C : \mathsf{Set}^{\mathbb{C}} \to \mathsf{Set}$$

with $\mathsf{eval}_C(F) = F(C)$, and evaluation preserves products since these are computed pointwise in the functor category. Moreover, every arrow $h : C \to D$ in $\mathbb{C}$ gives rise to an obvious natural transformation $h : \mathsf{eval}_C \to \mathsf{eval}_D$. Thus for any group G in $\mathsf{Set}^{\mathbb{C}}$, we have groups $\mathsf{eval}_C(G)$ for each $C \in \mathbb{C}$ and group homomorphisms $h_G : C(G) \to D(G)$, comprising a functor $G : \mathbb{C} \to \mathsf{Group}$. Conversely, it is clear that any such functor $H : \mathbb{C} \to \mathsf{Group}$ arises in this way from a group H in $\mathsf{Set}^{\mathbb{C}}$, at least up to isomorphism.

In this way, a group in a category of variable sets can be regarded as a *variable group*.

Exercise 1.1.12. Verify the details of the isomorphism of categories

$$\mathsf{Mod}(\mathbb{T}, \mathsf{Set}^{\mathbb{C}}) \cong \mathsf{Hom}(\mathbb{C}, \mathsf{Mod}(\mathbb{T}, \mathsf{Set}))$$

discussed in example 1.1.11 for arbitrary algebraic theories $\mathbb{T}$.

Exercise 1.1.13. Determine what a group is in the following categories: the category of finite sets $\mathsf{Set}_{\mathrm{fin}}$, the category of topological spaces Top, the category of graphs Graph, and the category of groups Group.

Hint: Only the last case is tricky. Before thinking about it, prove the following lemma [Bor94, Lemma 3.11.6]. Let G be a set provided with two binary operations $\cdot$ and $\star$ and a common unit e, so that $x \cdot e = e \cdot x = x \star e = e \star x = x$. Suppose the two operations commute, i.e., $(x \star y) \cdot (z \star w) = (x \cdot z) \star (y \cdot w)$. Then they coincide, are *commutative* and associative.

1.1.2 Theories as categories

The syntactically presented notion of an algebraic theory, say of groups, is a notational convenience, but as a specification of, say, the mathematical concept of a group it has some defects. We want to find a *presentation-free* notion that captures the group concept without tying it to a specific syntactic presentation. The notion we seek can be given by a category with a certain universal mapping property which determines it uniquely (up to equivalence). This also results in a reformulation of the usual conception of syntax and semantics — so distinctive of conventional logic — bringing it more in line with other fields of modern mathematics.

Let us consider group theory again. The algebraic axiomatization in terms of unit, multiplication and inverse is not the only possible one. For example, an alternative formulation uses the unit e and a binary operation $\odot$, called *double division*, along with a single axiom [McC93]:

$$(x \odot (((x \odot y) \odot z) \odot (y \odot e))) \odot (e \odot e) = z\ .$$

The usual group operations are related to double division as follows:

$$x \odot y = x^{-1} \cdot y^{-1} \,, \qquad x^{-1} = x \odot e \,, \qquad x \cdot y = (x \odot e) \odot (y \odot e) \,.$$

There may be various reasons why we prefer to work with one formulation of group theory rather than another, but this should not be reflected in the general idea of what a group is. We want to avoid particular choices of basic constants, operations, and axioms. This is akin to the situation where an algebra is presented by generators and relations: the algebra itself is regarded as independent of any particular choice of presentation. Similarly, one usually prefers a basis-free theory of vector spaces: it is better to formulate the idea of a vector space without speaking explicitly of vector bases, even though every vector space has one. Without a doubt, vector bases are important, but they really are an auxiliary concept.

As a first step, we could simply take *all* operations built from unit, multiplication, and inverse as basic, and *all* valid equations of group theory as axioms. But we can go a step further and collect all the operations into a category, thus forgetting about which ones were "basic" and which ones "derived", and which equalities were "axioms". We first describe this construction of a category $\mathcal{C}_{\mathbb{T}}$ for a general algebraic theory $\mathbb{T}$, and then determine another characterization of it.

As objects of $\mathcal{C}_{\mathbb{T}}$ we take *contexts*, i.e. sequences of distinct variables,

$$[x_1, \dots, x_n] \,. \qquad (n \geq 0)$$

Actually, it will be convenient to take equivalence classes under renaming of variables, so that $[x_1, x_3] = [x_2, x_1]$.

A morphism from $[x_1, \dots, x_m]$ to $[x_1, \dots, x_n]$ is an n-tuple $(t_1, \dots, t_n)$, where each t_k is a term in the context, $x_1, \dots, x_m \mid t_k$. Two such morphisms $(t_1, \dots, t_n)$ and $(s_1, \dots, s_n)$ are equal if, and only if, the axioms of the theory imply that $t_k = s_k$ for every $k = 1, \dots, n$,

$$\mathbb{T} \vdash t_k = s_k$$

Strictly speaking, morphisms are thus (tuples of) *equivalence classes* of terms in context

$$[x_1, \dots, x_m \mid t_1, \dots, t_n\,] : [x_1, \dots, x_m] \longrightarrow [x_1, \dots, x_n],$$

where two terms are equivalent when the theory proves them to be equal. Since it is rather cumbersome to work with equivalence classes, we shall work with the terms directly, but keeping in mind that equality between them is equivalence. Note also that the context of the morphism agrees with its domain, so we can omit it from the notation when the domain is clear. The composition of morphisms

$$\begin{aligned}(t_1, \dots, t_m) &: [x_1, \dots, x_k] \to [x_1, \dots, x_m] \\ (s_1, \dots, s_n) &: [x_1, \dots, x_m] \to [x_1, \dots, x_n]\end{aligned}$$

is the morphism $(r_1, \ldots, r_n)$ whose i-th component is obtained by simultaneously substituting in s_i the terms $t_1, \ldots, t_m$ for the variables $x_1, \ldots, x_m$:

$$r_i = s_i[t_1, \ldots, t_m/x_1, \ldots, x_m] \qquad (1 \leq i \leq n)$$

The identity morphism on $[x_1, \ldots, x_n]$ is $(x_1, \ldots, x_n)$. Using the usual rules of deduction for equational logic, it is easy to verify that these specifications are well-defined on equivalence classes and thus make $\mathcal{C}_{\mathbb{T}}$ a category.

Definition 1.1.14. The category $\mathcal{C}_{\mathbb{T}}$ just defined is called the *syntactic category* of the theory $\mathbb{T}$.

The syntactic category $\mathcal{C}_{\mathbb{T}}$ — which may be thought of as the "Lindenbaum-Tarski category" of $\mathbb{T}$ — contains the same "algebraic" information as the theory $\mathbb{T}$ from which it was built, but in a syntax-invariant way. Any two different presentations of $\mathbb{T}$ — like the ones for groups mentioned above — will give rise to essentially the same category $\mathcal{C}_{\mathbb{T}}$ (i.e. up to isomorphism). In this sense, the category $\mathcal{C}_{\mathbb{T}}$ is the abstract, algebraic gadget presented by the operations and equations of the theory $\mathbb{T}$, in just the way a group can be presented by generators and relations. But there is another, much more important, sense in which $\mathcal{C}_{\mathbb{T}}$ represents $\mathbb{T}$, as we next show.

Exercise 1.1.15. Show that the syntactic category $\mathcal{C}_{\mathbb{T}}$ has all finite products.

1.1.3 Models as functors

Having now represented an algebraic theory $\mathbb{T}$ as a special category $\mathcal{C}_{\mathbb{T}}$, the syntactic category constructed from $\mathbb{T}$, we next show that $\mathcal{C}_{\mathbb{T}}$ has the special property that models of $\mathbb{T}$ correspond uniquely to certain functors from $\mathcal{C}_{\mathbb{T}}$. More precisely, given any FP-category $\mathcal{C}$ there is a natural equivalence,

$$\frac{\mathcal{M} \in \mathsf{Mod}(\mathbb{T}, \mathcal{C})}{\overline{\overline{M : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}}}}$$

between models $\mathcal{M}$ of $\mathbb{T}$ in $\mathcal{C}$ and FP-functors $M : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}$. The equivalence is mediated by a "universal model" $\mathcal{U}$ in $\mathcal{C}_{\mathbb{T}}$, so that every model $\mathcal{M}$ arises as the functorial image $M(\mathcal{U}) \cong \mathcal{M}$ of $\mathcal{U}$ under an essentially unique FP-functor $M : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}$. The universal model $\mathcal{U}$ is of course the one corresponding to the identity functor $1_{\mathcal{C}_{\mathbb{T}}} : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}_{\mathbb{T}}$ in the above displayed correspondence. The possibility of such universal models (with their attendant property of being logically generic, as described in the next section 1.1.4 below) is a benefit of the generalized notion of a model in a category other than Set. Classical Set-valued models are almost never universal in this way.

To give the details of this correspondence, let $\mathbb{T}$ be an arbitrary algebraic theory and $\mathcal{C}_{\mathbb{T}}$ the syntactic category constructed from $\mathbb{T}$ as in the foregoing section. It is easy to show that the product in $\mathcal{C}_{\mathbb{T}}$ of two objects $[x_1, \ldots, x_n]$ and $[x_1, \ldots, x_m]$ is the object $[x_1, \ldots, x_{n+m}]$, and that $\mathcal{C}_{\mathbb{T}}$ has all finite products (including $1 = [-]$, the empty context). Moreover, there is a $\mathbb{T}$-model U in $\mathcal{C}_{\mathbb{T}}$ consisting of the language itself: The underlying object is the context $U = [x_1]$ of length one, and each operation symbol f of, say, arity k is interpreted as itself,

$$f^U = [\,x_1, \dots, x_k \mid f(x_1, \dots, x_k)\,] : U^k = [x_1, \dots, x_k] \longrightarrow [x_1] = U.$$

The axioms are of course all satisfied, since for any terms s, t:

$$U \models s = t \iff s^U = t^U \iff \mathbb{T} \vdash s = t. \tag{1.5}$$

This *syntactic model* U in $\mathcal{C}_{\mathbb{T}}$ is "universal" in the following sense: any model M in any category $\mathcal{C}$ with finite products is the image of U under an essentially unique, finite product preserving functor $\mathcal{C}_{\mathbb{T}} \to \mathcal{C}$. In a certain sense, then, $\mathcal{C}_{\mathbb{T}}$ is the "free finite product category with a model of $\mathbb{T}$". We now proceed to make this more precise.

First, observe that any FP-functor $F : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}$ takes the syntactic model U in $\mathcal{C}_{\mathbb{T}}$ to a model FU in $\mathcal{C}$, with interpretations

$$f^{FU} : FU^k \to FU \qquad \text{for each } f \in \Sigma_k.$$

Moreover, any natural transformation $\vartheta : F \to G$ between FP-functors determines a homomorphism of models $h = \vartheta_U : FU \to GU$. In more detail, suppose $f : U \times U \to U$ is a basic operation, then there is a commutative diagram,

$$\begin{tikzcd}
FU \times FU \arrow[r, "h \times h"] \arrow[d, "\cong"] \arrow[dd, bend right=60, "f^{FU}"'] & GU \times GU \arrow[d, "\cong"'] \arrow[dd, bend left=60, "f^{GU}"] \\
F(U \times U) \arrow[r, "\vartheta_{U \times U}"] \arrow[d, "Ff"] & G(U \times U) \arrow[d, "Gf"'] \\
FU \arrow[r, "h = \vartheta_U"'] & GU
\end{tikzcd}$$

where the upper square commutes by preservation of products, and the lower one by naturality. Thus the operation "evaluation at U" determines a functor,

$$\mathsf{eval}_U : \mathsf{Hom}_{\mathrm{FP}}(\mathcal{C}_{\mathbb{T}}, \mathcal{C}) \longrightarrow \mathsf{Mod}(\mathbb{T}, \mathcal{C}) \tag{1.6}$$

from the category of finite product preserving functors $\mathcal{C}_{\mathbb{T}} \to \mathcal{C}$, with natural transformations as arrows, into the category of $\mathbb{T}$-models in $\mathcal{C}$.

Proposition 1.1.16. *The functor* (1.6) *is an equivalence of categories, natural in* $\mathcal{C}$.

Proof. Let M be any model in an FP-category $\mathcal{C}$. Then the assignment $f \mapsto f^M$ given by the interpretation determines a functor $M^\sharp : \mathcal{C}_\mathbb{T} \to \mathcal{C}$, defined on objects by

$$M^\sharp[x_1, \dots, x_k] = M^k$$

and on morphisms by

$$M^\sharp(t_1, \dots, t_n) = \langle {t_1}^M, \dots, {t_n}^M \rangle.$$

In detail, $M^\sharp$ is defined on morphisms

$$[x_1, \dots, x_k \mid t\,] : [x_1, \dots, x_k] \to [x_1, \dots, x_n]$$

in $\mathcal{C}_\mathbb{T}$ by the following rules:

1. The morphism

   $$(x_i) : [x_1, \dots, x_k] \to [x_1]$$

   is mapped to the i-th projection

   $$\pi_i : M^k \to M.$$

2. The morphism

   $$(f\,(t_1, \dots, t_m)) : [x_1, \dots, x_k] \to [x_1]$$

   is mapped to the composite

   $$M^k \xrightarrow{\left(M^\sharp t_1, \dots, M^\sharp t_m\right)} M^m \xrightarrow{\quad M^\sharp f \quad} M$$

   where $M^\sharp t_i : M^k \to M$ is the value of $M^\sharp$ on the morphisms $(t_i) : [x_1, \dots, x_k] \to [x_1]$, for $i = 1, \dots, m$, and $M^\sharp f = f^M$ is the interpretation of the basic operation f.

3. The morphism

   $$(t_1, \dots, t_n) : [x_1, \dots, x_k] \to [x_1, \dots, x_n]$$

   is mapped to the morphism $\langle M^\sharp t_1, \dots, M^\sharp t_n \rangle$ where $M^\sharp t_i$ is the value of $M^\sharp$ on the morphism $(t_i) : [x_1, \dots, x_k] \to [x_1]$, and

   $$\langle M^\sharp t_1, \dots, M^\sharp t_n \rangle : M^k \longrightarrow M^n$$

   is the evident n-tuple in the FP-category $\mathcal{C}$.

That $M^\sharp : \mathcal{C}_\mathbb{T} \to \mathcal{C}$ really is a functor now follows from the assumption that the interpretation M is a model, which means that all the equations of the theory are satisfied by it, so that the above specification is well-defined on equivalence classes. Observe that the functor $M^\sharp$ is defined in such a way that it obviously preserves finite products, and that there is an isomorphism of models,

$$M^\sharp(U) \cong M.$$

Thus we have shown that the functor "evaluation at U",

$$\mathsf{eval}_U : \mathsf{Hom}_{\mathrm{FP}}(\mathcal{C}_{\mathbb{T}}, \mathcal{C}) \longrightarrow \mathsf{Mod}(\mathbb{T}, \mathcal{C}) \tag{1.7}$$

is essentially surjective on objects, since $\mathsf{eval}_U(M^\sharp) = M^\sharp(U) \cong M$.

We leave the verification that it is full and faithful as an easy exercise.

Exercise 1.1.17. Verify this.

Naturality in $\mathcal{C}$ means the following. Suppose M is a model of $\mathbb{T}$ in any category $\mathcal{C}$ with finite products ("FP-category"). Any finite product-preserving functor ("FP-functor") $F : \mathcal{C} \to \mathcal{D}$ to another FP-category $\mathcal{D}$ then takes M to a model $F(M)$ in $\mathcal{D}$. The interpretation is given by setting $f^{F(M)} = F(f^M)$ for the basic operations f (and composing with the canonical isos coming from preservation of products, $F(M) \times F(M) \cong F(M \times M)$, etc.). Since equations are described by commuting diagrams, F takes a model to a model, and the same is true for homomorphisms. Thus $F : \mathcal{C} \to \mathcal{D}$ induces a functor on $\mathbb{T}$-models,

$$\mathsf{Mod}(\mathbb{T}, F) : \mathsf{Mod}(\mathbb{T}, \mathcal{C}) \longrightarrow \mathsf{Mod}(\mathbb{T}, \mathcal{D}).$$

By naturality of (1.6) we mean that the following square commutes, up to natural isomorphism:

$$\begin{tikzcd}
\mathsf{Hom}_{\mathrm{FP}}(\mathcal{C}_{\mathbb{T}}, \mathcal{C}) \arrow[r, "\mathsf{eval}_U"] \arrow[d, "\mathsf{Hom}_{\mathrm{FP}}(\mathcal{C}_{\mathbb{T}}, F)"'] & \mathsf{Mod}(\mathbb{T}, \mathcal{C}) \arrow[d, "\mathsf{Mod}(\mathbb{T}, F)"] \\
\mathsf{Hom}_{\mathrm{FP}}(\mathcal{C}_{\mathbb{T}}, \mathcal{D}) \arrow[r, "\mathsf{eval}_U"'] & \mathsf{Mod}(\mathbb{T}, \mathcal{C})
\end{tikzcd} \tag{1.8}$$

But this is clear, since for any FP-functor $M : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}$ we have:

$$\begin{aligned}
\mathsf{eval}_U \circ \mathsf{Hom}_{\mathrm{FP}}(\mathcal{C}_{\mathbb{T}}, F)(M) &= (\mathsf{Hom}_{\mathrm{FP}}(\mathcal{C}_{\mathbb{T}}, F)(M))(U) \\
&= (F \circ M)(U) \\
&= F(M(U)) \\
&= F(\mathsf{eval}_U(M)) \\
&\cong \mathsf{Mod}(\mathbb{T}, F)(\mathsf{eval}_U(M)) \\
&= \mathsf{Mod}(\mathbb{T}, F) \circ \mathsf{eval}_U(M).
\end{aligned}$$

□

The equivalence of categories

$$\mathsf{Hom}_{\mathrm{FP}}(\mathcal{C}_{\mathbb{T}}, \mathcal{C}) \simeq \mathsf{Mod}(\mathbb{T}, \mathcal{C}) \tag{1.9}$$

actually determines $\mathcal{C}_{\mathbb{T}}$ and the universal model U uniquely, up to equivalence of categories and isomorphism of models. Indeed, to recover U, just put $\mathcal{C}_{\mathbb{T}}$ for $\mathcal{C}$ and the identity functor $1_{\mathcal{C}_{\mathbb{T}}}$ on the left, to get U in $\mathsf{Mod}(\mathbb{T}, \mathcal{C}_{\mathbb{T}})$ on the right! To see that $\mathcal{C}_{\mathbb{T}}$ itself is also determined, observe that (1.9) essentially says that the functor $\mathsf{Mod}(\mathbb{T}, \mathcal{C})$ is representable, with representing object $\mathcal{C}_{\mathbb{T}}$. As usual, this fact can also be formulated in elementary terms as a universal mapping property of $\mathcal{C}_{\mathbb{T}}$, as follows:

Definition 1.1.18. The *classifying category* of an algebraic theory $\mathbb{T}$ is an FP-category $\mathcal{C}_{\mathbb{T}}$ with a distinguished model U, called the *universal model*, such that:

(i) for any model M in any FP-category $\mathcal{C}$, there is an FP-functor

$$M^\sharp : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}$$

and an isomorphism of models $M \cong M^\sharp(U)$.

(ii) for any FP-functors $F, G : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}$ and model homomorphism $h : F(U) \to G(U)$, there is a unique natural transformation $\vartheta : F \to G$ with

$$\vartheta_U = h.$$

Observe that (i) says that the evaluation functor (1.6) is essentially surjective, and (ii) that it is full and faithful. The category $\mathcal{C}_{\mathbb{T}}$ is clearly determined up to equivalence by this universal mapping property in the usual way. Specifically, if $(\mathcal{C}, U)$ and $(\mathcal{D}, V)$ are both classifying categories for the same theory, then there are classifying functors,

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "V^\sharp"] & \mathcal{D} \arrow[l, bend left, "U^\sharp"] \end{tikzcd}$$

the composites of which are necessarily isomorphic to the respective identity functors, since e.g. $U^\sharp(V^\sharp(U)) \cong U^\sharp(V) \cong U$.

We have now shown not only that every algebraic theory has a classifying category, but also that the syntactic category is essentially determined by that distinguishing property. We record this as the following.

Theorem 1.1.19. *Every algebraic theory $\mathbb{T}$ has the syntactic category $\mathcal{C}_{\mathbb{T}}$ as a classifying category.*

Example 1.1.20. Let us see what the foregoing definitions give us in the case of group theory $\mathbb{G} = \mathbb{T}_{\mathsf{Group}}$. Recall that the category $\mathbb{G}$ consists of contexts $[x_1, \dots, x_n]$ and terms built from variables and the basic group operations. A finite product preserving functor $M : \mathbb{G} \to \mathsf{Set}$ is then determined up to natural isomorphism by its action on the context $[x_1]$ and the terms representing the basic operations. If we set

$$\begin{aligned} G &= M[x_1]\,, & e &= M(\cdot \mid e)\,, \\ i &= M(x_1 \mid {x_1}^{-1})\,, & m &= M(x_1, x_2 \mid x_1 \cdot x_2)\,, \end{aligned}$$

then (G, e, i, m) is a just a group with unit e, inverse i and multiplication m. That G satisfies the axioms for groups follows from functoriality of M. Conversely, any group (G, e, i, m) determines a finite product preserving functor $M_G : \mathbb{G} \to \mathsf{Set}$ defined by

$$\begin{aligned} M_G[x_1, \ldots, x_n] &= G^n \;, & M_G(\cdot \mid e) \;, \\ M_G(x_1 \mid {x_1}^{-1}) &= i \;, & M_G(x_1, x_2 \mid x_1 \cdot x_2) = m \;. \end{aligned}$$

This shows that $\mathsf{Mod}_{\mathsf{Set}}(\mathbb{G})$ is indeed equivalent to Group, provided both categories have the same notion of morphisms.

Suppose then that (G, e_G, i_G, m_G) and (H, e_H, i_H, m_H) are groups, and let $\phi : M_G \Longrightarrow M_H$ be a natural transformation between the corresponding functors. Then ϕ is already determined by its component at $[x_1]$ because by naturality the following diagram commutes, for $1 \leq k \leq n$:

$$\begin{tikzcd} G^n \arrow[r, "\phi_{[x_1,\ldots,x_n]}"] \arrow[d, "G\pi_k = \pi_k"'] & H^n \arrow[d, "H\pi_k = \pi_k"] \\ G \arrow[r, "\phi_{[x_1]}"'] & H \end{tikzcd}$$

If we write $\phi' = \phi_{[x_1]}$ then it follows that $\phi_{[x_1,\ldots,x_n]} = \phi' \times \cdots \times \phi'$. Again, by naturality of ϕ we see that the following diagram commutes:

$$\begin{tikzcd} G \times G \arrow[r, "\phi' \times \phi'"] \arrow[d, "m_G"'] & H \times H \arrow[d, "m_h"] \\ G \arrow[r, "\phi'"'] & H \end{tikzcd}$$

Similar commutative squares show that ϕ' preserves the unit and commutes with the inverse operation, therefore $\phi' : G \to H$ is indeed a group homomorphism. Conversely, a group homomorphism $\psi' : G \to H$ determines a natural transformation $\psi : G \Longrightarrow H$ whose component at $[x_1, \ldots, x_n]$ is the n-fold product $\psi' \times \cdots \psi' : G^n \to H^n$. This demonstrates that

$$\mathsf{Mod}_{\mathsf{Set}}(\mathbb{G}) \simeq \mathsf{Group} \;.$$

Example 1.1.21. Recall from 1.1.11 that a group G in the functor category $\mathsf{Set}^{\mathbb{C}}$ is essentially the same thing as a functor $G : \mathbb{C} \to \mathsf{Group}$. From the point of view of algebras as functors, this amounts to the observation that product-preserving functors $\mathbb{G} \to \mathsf{Hom}(\mathbb{C}, \mathsf{Set})$ correspond (by exponential transposition) to functors $\mathbb{C} \to \mathsf{Hom}_{\mathsf{FP}}(\mathbb{G}, \mathsf{Set})$, where the latter Hom-set consists just of product-preserving functors. Indeed, the correspondence extends to natural transformations to give the previously observed equivalance of categories,

$$\mathsf{Group}(\mathsf{Set}^{\mathbb{C}}) \simeq (\mathsf{Group}(\mathsf{Set}))^{\mathbb{C}} \simeq \mathsf{Group}^{\mathbb{C}}.$$

1.1.4 Completeness

Consider an algebraic theory $\mathbb{T}$ and an equation $s = t$ between terms of the theory. If the equation can be proved from the axioms of the theory, then every model of the theory satisfies the equation; this is just the *soundness* of the equational calculus with respect to models in categories. The converse statement is:

$$\text{“Every model of } \mathbb{T} \text{ satisfies } s = t.\text{”} \Rightarrow \text{“}\mathbb{T} \text{ proves equation } s = t.\text{”} .$$

This property is called *completeness*, and (together with soundness) it says that the calculus of equations suffices for proving all (and only) the ones that hold in the semantics. This holds in an especially strong sense for categorical semantics, as shown by the following.

Theorem 1.1.22 (Strong completeness)**.** *Suppose $\mathbb{T}$ is an algebraic theory.*

1. *For any equation $s = t$: every model M of $\mathbb{T}$ in every FP-category $\mathbb{C}$ satisfies $s = t$, i.e. $M \models s = t$, if and only if $\mathbb{T} \vdash s = t$.*
2. *Then there exists an FP-category $\mathbb{C}$ and a model $U \in \mathsf{Mod}_{\mathbb{C}}(\mathbb{T})$ with the property that, for every equation $s = t$ between terms of the theory $\mathbb{T}$,*

$$U \models s = t \iff \mathbb{T} \vdash s = t.$$

That is, satisfaction by U is equivalent to provability in $\mathbb{T}$.

We will say that the equational calculus of algebraic theories is strongly complete *with respect to general categorical semantics.*

Proof. The second statement follows from the syntactic construction of the classifying category 1.1.19 as follows: Let $\mathcal{C} = \mathcal{C}_{\mathbb{T}}$ be the classifying category and U the universal model. If $\mathbb{T} \vdash s = t$, then by the syntactic construction of $\mathcal{C}_{\mathbb{T}}$ we have $s^U = t^U$. Conversely, if $U \models s = t$, then $s^U = t^U$. But by the syntactic construction of $\mathcal{C}_{\mathbb{T}}$, it then must be the case that $\mathbb{T} \vdash s = t$.

For the first statement, any model M in an FP-category $\mathcal{C}$ has a classifying functor $M^\sharp : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}$, which preserves the interpretations of s and t in the sense that (up to canonical isomorphism):

$$M^\sharp(s^U) = s^{M^\sharp(U)} = s^M$$

and similarly for t. Thus from $s^U = t^U$ we can infer $s^M = t^M$, i.e. $M \models s = t$. Thus $M \models s = t$ for every model M if and only if this holds in the universal model U, which is equivalent to provability by part (2). $\square$

Definition 1.1.23. A single model with the property mentioned in the theorem, of satisfying all and only those equations that are provable from the theory, shall be said to be *logically generic*.

Thus, by the foregoing, the universal model is logically generic. Classically, it is seldom the case that there exists a single, logically generic model; instead, for classical completeness, we consider the range of all models in Set. Completeness with respect to such a restricted range of models is of course a stronger statement than completeness with respect to all models in all categories. Toward the classical result, we first consider completeness with respect to "variable models" in Set, i.e. models in presheaf categories $\mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$.

Proposition 1.1.24. *Let $\mathbb{T}$ be an algebraic theory. The Yoneda embedding*

$$\mathsf{y} : \mathcal{C}_{\mathbb{T}} \to \widehat{\mathcal{C}_{\mathbb{T}}}$$

is a generic model for $\mathbb{T}$.

Proof. The Yoneda embedding $\mathsf{y} : \mathcal{C}_{\mathbb{T}} \to \widehat{\mathcal{C}_{\mathbb{T}}}$ preserves limits, and in particular finite products, hence it corresponds to a model $U' = \mathsf{y}(U)$ of $\mathbb{T}$ in $\widehat{\mathcal{C}_{\mathbb{T}}}$. Simply because y is a functor, U' satisfies all equations that hold in U, but because it is faithful, U' does not validate any equations that do not already hold in U. Since U is logically generic, so is U'. $\square$

Example 1.1.25. We consider group theory one last time. As a presheaf on the theory of groups, the universal group satisfies every equation that is satisfied by all groups, and no others. Let us describe it explicitly as a variable set. Recall that the theory of groups is the category $\mathbb{G}$ whose objects are contexts $[x_1, \dots, x_n]$, $n \in \mathbb{N}$. The carrier U of the universal group is the presheaf represented by the context with one variable,

$$U = \mathsf{y}[x_1] = \mathbb{G}(-, [x_1]) .$$

This is a set parametrized by the objects of $\mathbb{G}$. For every $n \in \mathbb{N}$, we get the set $U_n = \mathbb{G}([x_1, \dots, x_n], [x_1])$ that consists of all terms built from n variables, modulo equations of group theory; but this is precisely the free group on n generators! Unit, inverse, and multiplication on U are defined at each stage U_n as the corresponding operations on the free group on n generators (the reader should verify this in detail).

To summarize, as a presheaf, the universal group is the free group on n-generators, where $n \in \mathbb{N}$ is a parameter.

Finally, we consider completeness with respect to Set-valued models $M : \mathcal{C}_{\mathbb{T}} \to \mathsf{Set}$, which of course correspond to classical models. We need the following:

Lemma 1.1.26. *For any small category $\mathcal{C}$, there is a jointly faithful set of FP-functors $E_i : \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}} \to \mathsf{Set}, i \in I$. That is, for any maps $f, g : A \to B$ in $\mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$, if $E_i(f) = E_i(g)$ for all $i \in I$, then $f = g$.*

Proof. Consider the evaluation functors $\mathrm{ev}_C : \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}} \to \mathsf{Set}$ for all $C \in \mathcal{C}$. These are clearly jointly faithful, and they preserve all limits and colimits, since these are constructed pointwise in presheaves. $\square$

Proposition 1.1.27. *Suppose $\mathbb{T}$ is an algebraic theory. For every equation $s = t$ between terms of the theory $\mathbb{T}$,*

$$M \models s = t \quad \textit{for all models } M \textit{ in } \mathsf{Set} \iff \mathbb{T} \vdash s = t.$$

Thus the equational calculus of algebraic theories is complete with respect to Set*-valued semantics.*

Proof. Combine the foregoing lemma with the fact, from Proposition 1.1.24, that the Yondea embedding is a generic model. □

Exercise 1.1.28. The universal group U is a functor $\mathbb{G}^{\mathsf{op}} \to \mathsf{Set}$. In the last example we described the object part of U. What is the action of U on morphisms? Also describe the group structure on U explicitly.

Exercise 1.1.29. Let s be a term of group theory with variables $x_1, \ldots, x_n$. On one hand we can think of s as an element of the free group U_n, and on the other we can consider the interpretation of s in the universal group U, namely a natural transformation $Us : U^n \Longrightarrow U$. Suppose t is another term of group theory with variables $x_1, \ldots, x_n$. Show that $Us = Ut$ if, and only if, $s = t$ in the free group U_n.

1.1.5 Functorial semantics

Let us now summarize our treatment of algebraic theories so far. We have reformulated the traditional *logical* notions in terms of "algebraic" or *categorical* ones. The traditional approach to logic may be described as involving four different parts:

Type theory
: There is an underlying type theory, which is a calculus of types and terms. For algebraic theories the calculus of types is trivial, since there is only one type which is not even explicitly mentioned. The terms are built from variables and basic operations.

Logic
: A variety of different kinds of logic can be considered. Algebraic theories have a very simple kind of logic that only involves equations between terms and equational reasoning.

Theory
: A theory is given by basic types, basic terms, and axioms. The types and the terms are expressed in the type theory of the system, and the axioms are be expressed in the logic of the system.

Interpretations and Models
: The type theory and logic of a logical system can be interpreted in any category of the appropriate kind. For algebraic theories we considered categories with finite products. The interpretation is *denotational*, in the sense that the types and terms

of the theory are assigned to objects and morphisms (which they "denote") by induction on the structure of types, terms, and logical formulas. An interpretation of a theory is a *model* if it satisfies all the axioms of the theory, where in the present case the notion of satisfaction just means that the arrows interpreting the terms occurring in the equations are actually equal.

The alternative approach developed here — called *functorial semantics* — may be summarized as follows:

Theories are categories

From a theory we can construct a category which expresses essentially the same information as the theory but is syntax-invariant, in the sense that it does not depend on a particular presentation by (basic) operations and axioms. The structure of the category reflects the underlying type theory and logic. For example, single sorted algebraic theories give rise to categories with finite products.

Models are functors

A model is a (structure-preserving) functor from a (category representing a) theory to a category with appropriate structure to interpret the logic. The requirement that all axioms of the theory must be satisfied by a model translates to the requirement that the model is a functor and that it preserves the structure of the theory. For models of algebraic theories we only required that they preserve finite products, whereas functoriality ensures that all valid equations of the theory are preserved, thus satisfying the axioms.

Homomorphisms are natural transformations

We obtain a notion of homomorphisms between models for free: since models are functors, homomorphisms are natural transformations between them. Homomorphisms between models of algebraic theories turned out to be the usual notion of morphisms that preserved the algebraic structure.

Universal model

By admitting models in categories other than Set, functorial semantics allows the possibility of *universal models*: a model U in the classifying category $\mathcal{C}_{\mathbb{T}}$, such that every model anywhere is a functorial image of U by an essentially unique, logic-preserving functor. Such a universal model is then "logically generic", in the sense that it has all and only those logical properties had by all models, since such properties are preserved by the functors in question.

Logical completeness

The construction of the classifying category from the logical syntax of the theory shows the *soundness and completeness* of the theory with respect to general categorical semantics. Completeness with respect to a special class of models (e.g. Set-valued ones) results from an embedding theorem for the classifying category.

1.2 Lawvere duality

The scheme of functorial semantics that we have developed also applies to a wide range of logics other than algebraic theories, and we shall consider some of these in later chapters. A further aspect of functorial semantics is not nearly as transparent in the general case as it is in that of algebraic theories, however; namely, a deep and fascinating duality relating syntax and semantics. We devote the rest of this chapter to its investigation.

1.2.1 Logical duality

There is a remarkable and far-reaching duality in logic of the form:

$$\text{Syntax} \quad \simeq \quad \text{Semantics}^{\mathsf{op}}$$

It was first presented by F.W. Lawvere in his thesis, and developed in some early papers [Law63a, Law63b, Law65], but it has hardly even been noticed by conventional logicians—probably because its recognition requires the tools of category theory.

We can see this duality quite clearly in the case of algebraic theories. Let $\mathcal{C}_{\mathbb{T}}$ be the classifying category for an equational theory $\mathbb{T}$, like the theory of groups, constructed syntactically as in section 1.1.2 above. So the objects of $\mathcal{C}_{\mathbb{T}}$ are contexts of variables $[x_1 \ldots, x_n]$, up to renaming, and the arrows are terms in context $[x_1 \ldots, x_n \mid t]$, up to $\mathbb{T}$-provable equality. We will see that $\mathcal{C}_{\mathbb{T}}$ is actually dual to a certain subcategory $\mathbb{M}$ of classical models of $\mathbb{T}$ (in Set). Specifically, there is a full subcategory, $\mathbb{M} \hookrightarrow \mathsf{Mod}(\mathbb{T})$ and an equivalence of categories,

$$\mathcal{C}_{\mathbb{T}} \simeq \mathbb{M}^{\mathsf{op}},$$

making the *syntactic* category $\mathcal{C}_{\mathbb{T}}$ dual to a subcategory of the *semantic* category $\mathsf{Mod}(\mathbb{T})$. Thus, in particular, there is an invariant representation of the syntax of the theory $\mathbb{T}$ "hidden" inside the category of models of $\mathbb{T}$. Indeed, it is quite easy to specify $\mathbb{M}$: it is the *full* subcategory on the finitely generated free models of $\mathbb{T}$,

$$\mathsf{Mod}_{\mathrm{fg}}(\mathbb{T})_0 = \big\{F(n) \bigm| F(n) \text{ free } \mathbb{T}\text{-model}, n \in \mathbb{N}\big\}\,.$$

Theorem 1.2.1. *Let $\mathbb{T}$ be an algebraic theory, and let*

$$\mathbb{M} = \mathsf{Mod}_{\mathrm{fg}}(\mathbb{T}) \hookrightarrow \mathsf{Mod}(\mathbb{T})$$

be the full subcategory of finitely generated, free models of $\mathbb{T}$. Then $\mathbb{M}^{\mathsf{op}}$ classifies $\mathbb{T}$ models. That is to say, for any FP-category $\mathcal{C}$, there is an equivalence of categories,

$$\mathsf{Hom}_{\mathrm{FP}}(\mathbb{M}^{\mathsf{op}}, \mathcal{C}) \simeq \mathsf{Mod}(\mathbb{T}, \mathcal{C}), \tag{1.10}$$

which is natural in $\mathcal{C}$.

Before giving the somewhat lengthy proof, let us observe that the claimed duality follows almost directly. Namely, there is an equivalence,

$$\mathcal{C}_{\mathbb{T}} \simeq \mathsf{Mod}_{\mathrm{fg}}(\mathbb{T})^{\mathsf{op}} \tag{1.11}$$

between the syntactic category $\mathcal{C}_{\mathbb{T}}$ and the opposite of the category $\mathsf{Mod}_{\mathrm{fg}}(\mathbb{T})$ of finitely generated, free models. This is because both objects $\mathcal{C}_{\mathbb{T}}$ and $\mathsf{Mod}_{\mathrm{fg}}(\mathbb{T})^{\mathsf{op}}$ represent the same functor, $\mathsf{Mod}(\mathbb{T}, \mathcal{C})$.

of theorem 1.2.1. First, observe that $\mathbb{M}^{\mathsf{op}}$ has all finite products, since $\mathbb{M}$ has all finite coproducts. Indeed, we have

$$\begin{aligned} F(n) + F(m) &\cong F(n+m), \\ 0 &\cong F(0), \end{aligned}$$

since the left adjoint F preserves all colimits.

To determine the universal $\mathbb{T}$-algebra in M^{op}, let,

$$U = F(1),$$

so that every object is a power of U in M^{op},

$$F(n) \cong U^n.$$

Next, we interpret the signature. For each basic operation f of $\mathbb{T}$, with arity n, there is an element of $F(n)$ built from f and the n generators $x_1, \ldots, x_n$, namely

$$f(x_1, \ldots, x_n) \in F(n).$$

E.g. in the theory of groups, there is the element $x \cdot y$ in the free group on the two generators x, y. By freeness of $F(1)$, each element $t \in F(n)$ determines a unique homomorphism $\bar{t} : F(1) \to F(n)$ in $\mathbb{M}$ taking the generator in $F(1)$ to t. Thus there is a homomorphism

$$\overline{f(x_1, \ldots, x_n)} : F(1) \to F(n) \qquad \text{in } \mathbb{M}$$

associated to $f(x_1, \ldots, x_n)$ in $F(n)$. This map is the interpretation of f,

$$f^U : U^n \to U \qquad \text{in } \mathbb{M}^{\mathsf{op}}.$$

Similarly, if $x_1 \ldots, x_n \mid t$ is any term in context, then the interpretation

$$[x_1 \ldots, x_n \mid t] : U^n \to U$$

is just the unique homomorphism corresponding to the element $t \in F(n)$ (proof by induction!).

It now follows that for every equation of $\mathbb{T}$,

$$s = t$$

we have $U \models s = t$. Indeed,

$$[x_1 \ldots, x_n \mid s] = [x_1 \ldots, x_n \mid t] : U^n \to U$$

if $\mathbb{T} \vdash s = t$, since clearly these terms must agree in the free algebra $F(n)$. For instance, $x \cdot y = y \cdot x$ for the two generators x, y of the free *abelian* group $F(2)$, but not in the free (non-abelian) group.

Thus we indeed have a model U of $\mathbb{T}$ in $\mathbb{M}^{\mathsf{op}}$, made from the free algebras. We show that this model has the required universal property, in three steps:

Step 1. Let A be any $\mathbb{T}$-algebra in Set. Then there is a product-preserving functor,

$$A^\sharp : \mathbb{M}^{\mathsf{op}} \to \mathsf{Set}$$

with $A^\sharp(U) \cong A$ (as $\mathbb{T}$-models), namely:

$$A^\sharp(-) = \mathsf{Hom}_{\mathsf{Mod}(\mathbb{T})}(-, A),$$

where we of course restrict the representable functor $\mathsf{Hom}_{\mathsf{Mod}(\mathbb{T})}(-, A) : \mathsf{Mod}(\mathbb{T})^{\mathsf{op}} \to \mathsf{Set}$ along the (full) inclusion

$$\mathbb{M} = \mathsf{Mod}_{\mathrm{fg}}(\mathbb{T}) \hookrightarrow \mathsf{Mod}(\mathbb{T})$$

of the finitely generated, free algebras. The functor

$$A^\sharp : \mathbb{M}^{\mathsf{op}} \to \mathsf{Set}$$

clearly preserves products: for each object $U^n \in \mathbb{M}^{\mathsf{op}}$, we have

$$A^\sharp(U^n) = \mathsf{Hom}_{\mathsf{Mod}(\mathbb{T})}(F(n), A) \cong = \mathsf{Hom}_{\mathsf{Set}}(n, V(A)) \cong |A|^n.$$

And in particular $A^\sharp(U) \cong |A|$.

Finally, let us show that for any basic operation f, we have $A^\sharp(f^U) = f^A$, up to isomorphism. Indeed, given any algebra A and operation $f^A : |A|^n \to |A|$, we have a commutative diagram,

$$\begin{tikzcd}
|A|^n \arrow[rr, "f^A"] \arrow[d, "\cong"'] & & |A| \arrow[d, "\cong"] \\
\mathsf{Hom}(F(n), A) \arrow[rr, "f^*"'] & & \mathsf{Hom}(F(1), A)
\end{tikzcd} \tag{1.12}$$

where f^* is precomposition with the homomorphism

$$F(n) \xleftarrow{\overline{f(x_1, \ldots, x_n)}} F(1)$$

To see that (1.12) commutes, take any $(a_1, \ldots, a_n) \in |A|^n$ with associated homomorphism $\overline{(a_1, \ldots, a_n)} : F(n) \to A$ and precompose with $\overline{f(x_1, \ldots, x_n)}$ to get a map $F(1) \to A$ picking out the element

$$\begin{aligned}
\overline{(a_1, \ldots, a_n)} \circ \overline{f(x_1, \ldots, x_n)}(x) &= \overline{(a_1, \ldots, a_n)}(f(x_1, \ldots, x_n)) \\
&= \overline{(a_1, \ldots, a_n)} \circ f^{F(n)}(x_1, \ldots, x_n) \\
&= f^A \circ \overline{(a_1, \ldots, a_n)}(x_1, \ldots, x_n) \\
&= f^A(a_1, \ldots, a_n)
\end{aligned}$$

where x is the generator of $F(1)$, and using the fact that $\overline{(a_1, \ldots, a_n)}$ is a homomorphism and therefore commutes with the respective interpretations of f.

But now note that

$$F(n) \xleftarrow{\overline{f(x_1, \ldots, x_n)}} F(1)$$

in $\mathbb{M}$ is

$$U^n \xrightarrow[f^U]{} U$$

in $\mathbb{M}^{\mathsf{op}}$, and that $\mathsf{Hom}(F(n), A) = A^\sharp(U^n)$ and $f^* = A^\sharp(f^U)$. Thus (1.12) shows that indeed $A^\sharp(f^U) = f^A$, up to isomorphism.

Thus, as algebras, $A^\sharp(U) \cong A$, as required.

We leave it to the reader to verify that any homomorphism $h : F(U) \to G(U)$ of $\mathbb{T}$-algebras $F(U), G(U)$ arising from FP-functors $F, G : \mathbb{M}^{\mathsf{op}} \to \mathsf{Set}$ is of the form $h = \vartheta_U$ for a unique natural transformation $\vartheta : F \to G$.

Exercise 1.2.2. Show this.

Step 2. Let $\mathbb{C}$ be any (locally small) category, and $\mathcal{A}$ a $\mathbb{T}$-algebra in $\mathsf{Set}^{\mathbb{C}}$. Since

$$\mathsf{Mod}(\mathbb{T}, \mathsf{Set}^{\mathbb{C}}) \cong \mathsf{Mod}(\mathbb{T})^{\mathbb{C}},$$

each $\mathcal{A}(C)$ is a $\mathbb{T}$-algebra (in Set), which by Step 1 has a classifying functor,

$$\mathcal{A}(C)^\sharp : \mathbb{M}^{\mathsf{op}} \to \mathsf{Set}.$$

Together, these determine a single functor $\mathcal{A}^\sharp : \mathbb{M}^{\mathsf{op}} \to \mathsf{Set}^{\mathbb{C}}$, defined on $U \in \mathbb{M}^{\mathsf{op}}$ by:

$$(\mathcal{A}^\sharp(U))(C) \cong \mathcal{A}(C) = \mathcal{A}(C)^\sharp(U),$$

and on U^n by

$$(\mathcal{A}^\sharp(U^n))(C) \cong \mathcal{A}(C)^n = \mathcal{A}(C)^\sharp(U^n).$$

The functor $\mathcal{A}^\sharp(U) : \mathbb{C} \to \mathsf{Set}$ acts on an arrow $g : C \to D$ in $\mathbb{C}$ as indicated in the diagram:

$$\begin{tikzcd}
\mathcal{A}^\sharp(U)(C) \arrow[r, "\mathcal{A}^\sharp(U)(g)"] \arrow[d, "\cong"'] & \mathcal{A}^\sharp(U)(C) \arrow[d, "\cong"] \\
\mathcal{A}(C) \arrow[r, "\mathcal{A}(g)"'] & \mathcal{A}(D).
\end{tikzcd} \tag{1.13}$$

The case of U^n is precisely analogous. Finally, the action of $\mathcal{A} : \mathbb{M}^{\mathsf{op}} \to \mathsf{Set}^{\mathbb{C}}$ on arrows $U^n \to U^m$ in $\mathbb{M}^{\mathsf{op}}$ is similarly determined pointwise, i.e. by the components

$$(\mathcal{A}^\sharp(U^n))(C) \cong \mathcal{A}(C)^\sharp(U^n) \to \mathcal{A}(C)^\sharp(U^m) = (\mathcal{A}^\sharp(U^m))(C),$$

for all $C \in \mathbb{C}$.

Step 3. For the general case, let $\mathcal{C}$ be any (locally small) FP-category, and A a $\mathbb{T}$-algebra in $\mathcal{C}$. Use the Yoneda embedding

$$\mathsf{y} : \mathcal{C} \hookrightarrow \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$$

to send A to an algebra $\mathcal{A} = \mathsf{y}(A)$ in $\mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$ (since y preserves finite products). Now apply Step 2 to get a classifying functor,

$$\mathcal{A}^\sharp : \mathbb{M}^{\mathsf{op}} \to \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}.$$

We claim that $\mathcal{A}^\sharp$ factors through the Yoneda embedding,

$$\begin{tikzcd}
 & \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}} \\
\mathbb{M}^{\mathsf{op}} \arrow[ur, "\mathcal{A}^\sharp"] \arrow[r, dotted, "A^\sharp"'] & \mathcal{C}. \arrow[u, hook, "\mathsf{y}"']
\end{tikzcd}$$

Indeed, we know that the objects of $\mathbb{M}^{\mathsf{op}}$ all have the form U^n, so their images

$$\mathcal{A}^\sharp(U^n) \cong \mathcal{A}^n \cong \mathsf{y}(A)^n \cong \mathsf{y}(A^n)$$

are all representable. Since y is full and faithful, the claim is established, and the resulting functor $A^\sharp : \mathbb{M}^{\mathsf{op}} \to \mathcal{C}$ preserves finite products because $\mathcal{A}^\sharp$ does so, and y creates them. Clearly,

$$A^\sharp(U) \cong A.$$

Naturality of the equivalence

$$\mathsf{Hom}_{\mathrm{FP}}(\mathbb{M}^{\mathsf{op}}, \mathcal{C}) \simeq \mathsf{Mod}(\mathbb{T}, \mathcal{C}),$$

in $\mathcal{C}$ is essentially automatic, using the fact that it is induced by evaluating an FP functor $F : \mathbb{M}^{\mathsf{op}} \to \mathcal{C}$ at the universal model U in $\mathbb{M}^{\mathsf{op}}$. □

As already mentioned, since the classifying category is uniquely determined, up to equivalence, by its universal property, combining the foregoing theorem with the syntactic construction of $\mathcal{C}_{\mathbb{T}}$ given in theorem 1.1.19 yields the following:

Corollary 1.2.3 (Logical duality for algebraic theories)**.** *For any algebraic theory* $\mathbb{T}$*, there is an equivalence,*

$$\mathcal{C}_{\mathbb{T}} \simeq \mathsf{Mod}_{\mathrm{fg}}(\mathbb{T})^{\mathsf{op}} \tag{1.14}$$

between the syntactic category $\mathcal{C}_{\mathbb{T}}$ *and the opposite of the category* $\mathsf{Mod}_{\mathrm{fg}}(\mathbb{T})$ *of finitely generated, free models.*

Thus the syntactic construction of the classifying category $\mathcal{C}_{\mathbb{T}}$, on the one hand, and the semantic construction of it as $\mathsf{Mod}_{\mathrm{fg}}(\mathbb{T})^{\mathsf{op}}$, taken together imply that there is an invariant representation of the syntax of $\mathbb{T}$ just sitting there, as it were, in the opposite of the semantics $\mathsf{Mod}(\mathbb{T})$. In section **??** below, we shall consider how to actually *recover* this category $\mathcal{C}_{\mathbb{T}}$ from the semantics $\mathsf{Mod}(\mathbb{T})$, by identifying the subcategory $\mathsf{Mod}_{\mathrm{fg}}(\mathbb{T})$ intrinsically.

Before doing so, however, let us examine the equivalence (1.14) explicitly in a very special case: the "empty" theory $\mathbb{T}_0$ with no basic operations or equations. A model of this theory in Set is just a set X, equipped with no operations, and satisfying no further conditions (and similarly in any other FP category). Thus $\mathbb{T}_0$ is the pure theory of equality on an object.

All $\mathbb{T}_0$-algebras are free, and the finitely generated ones are the finite sets, so

$$\mathsf{Mod}_{\mathrm{fg}}(\mathbb{T}_0) = \mathsf{Set}_{\mathrm{fin}},$$

is the category of finite sets (to be more specific, let us take one n-element set $[n]$ for each $n \in \mathbb{N}$). Our theorem 1.2.1 tells us that, for any FP category $\mathcal{C}$, there is an equivalence

$$\mathsf{Hom}_{\mathrm{FP}}(\mathsf{Set}_{\mathrm{fin}}^{\mathsf{op}}, \mathcal{C}) \simeq \mathsf{Mod}(\mathbb{T}_0, \mathcal{C}) \simeq \mathcal{C}.$$

This simply says that $\mathsf{Set}_{\mathrm{fin}}^{\mathsf{op}}$ is the free FP category on one object. Equivalently, $\mathsf{Set}_{\mathrm{fin}}$ is the free finite *coproduct* category on one object. This is indeed the case, as can easily be seen directly (the objects are $0, 1, 1+1, 1+1+1, \ldots$).

The duality of corollary 1.2.3 now tells us that the dual to the category of finite sets is the *syntactic category* of the theory of equality $\mathbb{T}_0$. The terms of this theory are simply tuples of variables $(x_1, \ldots, x_n)$, and the valid equations are those that are true of them as terms, like $(x_2, x_5) = (x_2, x_5)$. Our corollary tells us that this is the category of finite sets, if we read the contexts $[x_1, \ldots, x_n]$ as coproducts $1 + \cdots + 1$ and the tuples $(x_1, \ldots, x_n)$ as *cotuples* $[x_1, \ldots, x_n] : 1 + \cdots + 1 \to 1$, etc.

Example 1.2.4. For a less trivial example, consider the theory $\mathbb{T}_{\mathsf{Ab}}$ of abelian groups. Duality tells us that the syntactic category $\mathcal{C}_{\mathbb{T}_{\mathsf{Ab}}}$ is dual to the category of finitely generated, free abelian groups $\mathsf{Ab}_{\mathrm{fg}}$,

$$\mathcal{C}_{\mathbb{T}_{\mathsf{Ab}}} \simeq \mathsf{Ab}_{\mathrm{fg}}^{\mathsf{op}}.$$

This gives us a representation of the syntax of (abelian) group theory in the category of abelian groups, which is summarized as follows:

- the basic types of variables $[-] = 1,\ [x_1] = U,\ [x_1, x_2] = U \times U, \ldots$ are represented by the groups $\{0\},\ \mathbb{Z},\ \mathbb{Z} + \mathbb{Z} = \mathbb{Z}^2, \ldots,$
- the group unit $0 : 1 \to U$ is the (unique) homomorphism $0 : \mathbb{Z} \to \{0\}$,
- the inverse $i : U \to U$ is the (unique) homomorphism $- : \mathbb{Z} \to \mathbb{Z}$ taking 1 to -1,
- the group operation $U \times U \to U$ is the (unique) homomorphism $+ : \mathbb{Z} \to \mathbb{Z} + \mathbb{Z}$ taking 1 to $\langle 1, 1\rangle = \langle 1, 0\rangle + \langle 0, 1\rangle$ (using $\mathbb{Z} + \mathbb{Z} \cong \mathbb{Z} \times \mathbb{Z}$),
- the laws of abelian groups (and no further ones!) hold under this interpretation, because the group structure on any abelian group A is induced by precomposing with these "co-operations", as indicated in the following diagram for the sum $a + b$ of elements $a, b \in A$.

$$\begin{tikzcd}
\mathbb{Z} + \mathbb{Z} \arrow[r, "{(a,b)}"] & A \\
\mathbb{Z} \arrow[u, "+"] \arrow[ur, "a+b"'] &
\end{tikzcd}$$

Example 1.2.5. The category of *affine schemes* is by definition the dual of the category of commutative rings with unit,

$$\mathsf{Scheme}_{\mathrm{aff}} = \mathsf{Ring}^{\mathsf{op}}$$

There is therefore a ring object in affine schemes – called the *affine line* – based on the finitely generated free algebra $F(1) = \mathbb{Z}[x]$, the ring of polynomials in one variable x with integer coefficients. The "co-operations" of $+$ and $\cdot$ are given in rings by the homomorphisms $\mathbb{Z}[x] \to \mathbb{Z}[x, y]$ taking the generator x to the elements $x + y$ and $x \cdot y$.

Exercise 1.2.6. Prove directly that $\mathsf{Set}_{\mathrm{fin}}$ is the free finite coproduct category on one object.

Exercise 1.2.7. Show that for any algebraic theory $\mathbb{T}$, the forgetful functor $V : \mathsf{Mod}(\mathbb{T}) \to \mathsf{Set}$ is an algebra in the functor category $\mathsf{Set}^{\mathsf{Mod}(\mathbb{T})}$. In more detail, each n-ary operation f determines a natural transformation $f^V : V^n \to V$, since the homomorphisms in $\mathsf{Mod}(\mathbb{T})$ commute with the various operations interpreting f. Indeed, given any algebra A we have the underlying set $V(A) = |A|$ and an operation $f^A : |A|^n \to |A|$, and for every homomorphism $h : A \to B$ to another algebra B, there is a commutative square,

$$\begin{tikzcd}
|A|^n \arrow[r, "|h|^n"] \arrow[d, "f^A"'] & |B|^n \arrow[d, "f^B"] \\
|A| \arrow[r, "|h|"'] & |B|.
\end{tikzcd} \tag{1.15}$$

So we can set $(f^V)_A = f^A$ to get a natural transformation $f^V : V^n \to V$. Now check that this really is an algebra in $\mathsf{Set}^{\mathsf{Mod}(\mathbb{T})}$.

Exercise 1.2.8. * Show that the algebra described in the previous exercise is represented by the universal algebra $U = F(1)$ in $\mathbb{M}^{\mathsf{op}}$, in the sense of the (covariant) Yoneda embedding,

$$\mathsf{y} : \mathsf{Mod}(\mathbb{T})^{\mathsf{op}} \longrightarrow \mathsf{Set}^{\mathsf{Mod}(\mathbb{T})}.$$

1.2.2 Lawvere algebraic theories

Nothing in the foregoing account of duality for algebraic theories really depended on the primarily syntactic nature of such theories, i.e. their specification in terms of operations and equations. We can thus immediately generalize it to "abstract" algebraic theories, which can be regarded as providing a *presentation-free* notion of an algebraic theory.

Definition 1.2.9 (cf. Definition 1.1.2)**.** A *Lawvere algebraic theory* $\mathbb{A}$ is a small category with finite products whose objects form a sequence $A^0, A^1, A^2, \ldots$ such that $A^m \times A^n = A^{m+n}$ for all $m, n \in \mathbb{N}$. In particular, $\mathsf{1} = A^0$ is the terminal object and every object is a product of finitely many copies of $A = A^1$.

A *model* of a Lawvere algebraic theory $\mathbb{A}$ in any category $\mathcal{C}$ with finite products is a finite-product-preserving functor $M : \mathbb{A} \to \mathcal{C}$, and a *homomorphism of models* is a natural transformation $\vartheta : M \to M'$.

We could just as well have taken the natural numbers $0, 1, 2, \ldots$ themselves as the objects of a Lawvere algebraic theory $\mathbb{A}$, but the notation A^n is more suggestive. A Lawvere algebraic theory $\mathbb{A}$ in the sense of the above definition determines an algebraic theory in the sense of Definition 1.1.2 as follows. As basic operations with arity k we take all of the morphisms $A^k \to A$:

$$\Sigma(\mathbb{A})_k = \mathsf{Hom}_{\mathbb{A}}(A^k, A) \tag{1.16}$$

There is a canonical interpretation in $\mathbb{A}$ of terms built from variables and morphisms $A^k \to A$, namely each morphism is interpreted by itself, and variables are interpreted as product projects, as usual. An equation $u = v$ is taken as an axiom of the theory $\mathbb{A}$ if the canonical interpretations of u and v coincide. Of course, the conventional logical notions of model 1.1.10 and homomorphism of models then also correspond to the new, functorial ones in an obvious way.

This new, abstract view of algebraic theories immediately suggest some interesting examples.

Example 1.2.10. The algebraic theory $\mathcal{C}^\infty$ of smooth maps is the category whose objects are n-dimensional Euclidean spaces $\mathsf{1}$, $\mathbb{R}$, $\mathbb{R}^2$, $\ldots$, and whose morphisms are $\mathcal{C}^\infty$-maps between them. Recall that a $\mathcal{C}^\infty$-map $f : \mathbb{R}^n \to \mathbb{R}$ is a function which has all higher partial derivatives, and that a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is a $\mathcal{C}^\infty$-map when its compositions $\pi_k \circ f : \mathbb{R}^n \to \mathbb{R}$ with projections $\pi_k : \mathbb{R}^m \to \mathbb{R}$ are $\mathcal{C}^\infty$-maps.

A model of this theory in Set is a finite product preserving functor $A : \mathcal{C}^\infty \to \mathsf{Set}$. Up to natural isomorphism it can be described as follows. A $\mathcal{C}^\infty$-model is given by a set A and for every smooth map $f : \mathbb{R}^n \to \mathbb{R}$ a function $Af : A^n \to A$ such that if $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^m \to \mathbb{R}$, $i = 1, \ldots, n$, are smooth maps then, for all $a_1, \ldots, a_m \in A$,

$$Af\big((Ag_1)\langle a_1, \ldots, a_m\rangle, \ldots, (Ag_n)\langle a_1, \ldots, a_m\rangle\big) = A(f \circ \langle g_1, \ldots, g_n\rangle)\langle a_1, \ldots, a_m\rangle \ .$$

In particular, since multiplication and addition are smooth maps, A is a commutative ring with unit. Such structures are known as $\mathcal{C}^\infty$*-rings*. Therefore, the models in Set (cf. [MR91])

Example 1.2.11. Recall that a *(total) recursive function* $f : \mathbb{N}^m \to \mathbb{N}^n$ is one that can be computed by a Turing machine. This means that there exists a Turing machine which on input $\langle a_1, \ldots, a_m\rangle$ outputs the value of $f\langle a_1, \ldots, a_m\rangle$. The algebraic theory Rec of recursive functions is the category whose objects are finite powers of the natural numbers 1, $\mathbb{N}$, $\mathbb{N}^2$, $\ldots$, and whose morphisms are recursive functions between them. The models of this theory in a category $\mathcal{C}$ with finite products give a theory of computability in $\mathcal{C}$.

Indeed, let us consider the category of its set-theoretic models $\mathbb{R} = \mathsf{Mod}_{\mathsf{Set}}(\mathsf{Rec})$. First, there is the "identity" model $I \in \mathbb{R}$, defined by $I\mathbb{N}^k = \mathbb{N}^k$ and $If = f$. Given any model $S \in \mathbb{R}$, its object part is determined by $S_1 = S\mathbb{N}$ since $S\mathbb{N}^k = S_1^k$. For every $n \in \mathbb{N}$ there is a morphism $1 \to \mathbb{N}$ in Rec defined by $\star \mapsto n$. Thus we have for each $n \in \mathbb{N}$ an element $s_n = S(\star \mapsto n) : 1 \to S_1$. This defines a function $s : \mathbb{N} \to S_1$ which in turn determines a natural transformation $\sigma : I \Longrightarrow S$ whose component at $\mathbb{N}^k$ is $s \times \cdots \times s : \mathbb{N}^k \to S_1^k$.

Example 1.2.12. In a category $\mathcal{C}$ with finite products every object $A \in \mathcal{C}$ determines a full subcategory consisting of the finite powers $1, A, A^2, A^3, \ldots$ and all morphisms between them. This is the *total theory* $\mathbb{T}(A)$ of the object A in $\mathcal{C}$.

Free algebras

In order to extend the duality theory of the foregoing section to the abstract case, we will require the notion of a *free model* of an abstract algebraic theory. Of course, we already have the conventional notion of free models determined in terms of the associated conventional algebraic theory given by (1.16). But we can also determine free models directly in terms of the abstract theory, which then of course applies to the conventional case as well.

Let $\mathbb{A}$ be a Lawvere algebraic theory, with objects $1, A, A^2, \ldots$. We have the category of models,

$$\mathsf{Mod}(\mathbb{A}) = \mathsf{Hom}_{\mathsf{FP}}(\mathbb{A}, \mathsf{Set}).$$

Let us first define the *forgetful functor* by,

$$U : \mathsf{Mod}(\mathbb{A}) \to \mathsf{Set} \tag{1.17}$$

$$(M : \mathbb{A} \to \mathsf{Set}) \mapsto M(A). \tag{1.18}$$

We shall also write
$$|M| = U(M) = M(A).$$

Now for the (finitary) free functor $F_{\text{fin}} : \mathsf{Set}_{\text{fin}} \to \mathsf{Mod}(\mathbb{A})$, we set:

$$\begin{aligned} F(0) &= \mathsf{Hom}_{\mathbb{A}}(1,-) \\ F(1) &= \mathsf{Hom}_{\mathbb{A}}(A,-) \\ &\vdots \\ F(n) &= \mathsf{Hom}_{\mathbb{A}}(A^n,-). \end{aligned}$$

Note that this is a composite of the two (contravariant) functors,

$$\mathsf{Set}_{\text{fin}} \to \mathbb{A}^{\mathsf{op}}, \quad \mathbb{A}^{\mathsf{op}} \to \mathsf{Mod}(\mathbb{A}),$$

given by $n \mapsto A^n$ and $X \mapsto \mathsf{Hom}_{\mathbb{A}}(X,-)$, and is therefore functorial. Note also that the representables $\mathsf{Hom}_{\mathbb{A}}(A^n,-)$ do indeed preserve finite products, and are therefore in the full subcategory $\mathsf{Mod}(\mathbb{A}) \hookrightarrow \mathsf{Set}^{\mathbb{A}}$.

For adjointness we need to check that:

$$\mathsf{Hom}_{\mathsf{Mod}(\mathbb{A})}(F(n), M) \cong \mathsf{Hom}_{\mathsf{Set}}(n, |M|) \tag{1.19}$$

(naturally in both arguments, of course). The right-hand side is plainly just

$$\mathsf{Hom}_{\mathsf{Set}}(n, |M|) \cong |M|^n.$$

For the left-hand side we have:

$$\begin{aligned} \mathsf{Hom}_{\mathsf{Mod}(\mathbb{A})}(F(n), M) &= \mathsf{Hom}_{\mathsf{Mod}(\mathbb{A})}(\mathsf{Hom}_{\mathbb{A}}(A^n,-), M) \\ &= \mathsf{Hom}_{\mathsf{Set}^{\mathbb{A}}}(\mathsf{Hom}_{\mathbb{A}}(A^n,-), M) \\ &\cong M(A^n) && \text{(by Yoneda)} \\ &\cong M(A)^n && (M \text{ is FP}) \\ &= |M|^n && (1.17). \end{aligned}$$

The full definition of the free functor

$$F : \mathsf{Set} \to \mathsf{Mod}(\mathbb{A})$$

is then given by writing an arbitrary set X as a (filtered) colimit of its finite subsets $X_i \subseteq X$, and then setting $F(X) = \mathrm{colim}_i\, F(X_i)$ in the category $\mathsf{Set}^{\mathbb{A}}$. Since filtered colimits commute with finite products, these colimits taken in $\mathsf{Set}^{\mathbb{A}}$ and will remain in $\mathsf{Mod}(\mathbb{A})$.

Theorem 1.2.13. *For any set X with free algebra $F(X)$ as just defined, there is a natural isomorphism,*
$$\mathsf{Hom}_{\mathsf{Mod}(\mathbb{A})}(F(X), M) \cong \mathsf{Hom}_{\mathsf{Set}}(X, |M|)\,. \tag{1.20}$$

Proof. The proof is an easy exercise. □

By definition, the finitely generated free models $F(n)$ are just the representables $\mathsf{Hom}_{\mathbb{A}}(A^n, -)$; therefore as the "semantic dual" $\mathsf{Mod}_{\mathrm{fg}}(\mathbb{A}) \hookrightarrow \mathsf{Mod}(\mathbb{A})$ of the theory $\mathbb{A}$, in the sense of corollary 1.2.3, we simply have the full subcategory of $\mathsf{Hom}_{\mathsf{FP}}(\mathbb{A}, \mathsf{Set})$ on the image of the Yoneda embedding,

$$\begin{tikzcd}
\mathsf{Mod}_{\mathrm{fg}}(\mathbb{A}) \arrow[r, hook] & \mathsf{Mod}(\mathbb{A}) = \mathsf{Hom}_{\mathrm{FP}}(\mathbb{A}, \mathsf{Set}) \arrow[r, hook] & \mathsf{Set}^{\mathbb{A}} \\
\mathbb{A}^{\mathsf{op}} \arrow[u, "\cong"] \arrow[rr, "="'] & & \mathbb{A}^{\mathsf{op}}. \arrow[u, hook, "\mathsf{y}"']
\end{tikzcd}$$

In the abstract case, then, the logical duality

$$\mathbb{A} \simeq \mathsf{Mod}_{\mathrm{fg}}(\mathbb{A})^{\mathsf{op}}$$

comes down to the fact that the (contravariant) Yoneda embedding

$$\mathbb{A}^{\mathsf{op}} \hookrightarrow \mathsf{Set}^{\mathbb{A}}$$

presents $\mathbb{A}$ as (the dual of) a full subcategory of (product-preserving!) functors. Thus we have shown:

Theorem 1.2.14. *For any Lawvere algebraic theory* $\mathbb{A}$*, there is an equivalence,*

$$\mathbb{A} \simeq \mathsf{Mod}_{\mathrm{fg}}(\mathbb{A})^{\mathsf{op}}$$

between $\mathbb{A}$ *and the full subcategory of finitely generated free models.*

Exercise 1.2.15. Prove theorem 1.2.13.

1.2.3 Algebraic categories

Given an *arbitrary* category $\mathcal{A}$, we may ask, when is $\mathcal{A}$ the category of models for some algebraic theory? Such categories are often called *varieties*, at least in universal algebra, and there are well-known "recognition theorems" such the famous "HSP-theorem" of Birkhoff, which says that a collection of algebas for some fixed signature are exactly those satisfyig a set of equations if the collection is closed under Products, Subalgebras, and Homomorphic images (i.e. quotient algebras). Toward the goal of "recognizing" a *category* of algebras (without even being given the signature), let us define:

Definition 1.2.16. An *algebraic category* $\mathcal{A}$ is a (locally small) category equivalent to one of the form

$$\mathsf{Hom}_{\mathsf{FP}}(\mathbb{A}, \mathsf{Set}) \hookrightarrow \mathsf{Set}^{\mathbb{A}}$$

where $\mathbb{A}$ is any small finite product category and $\mathsf{Hom}_{\mathsf{FP}}(\mathbb{A}, \mathsf{Set})$ is the full subcategory of finite product preserving, set-valued functors and natural transformations. If $\mathbb{A}$ is a Lawvere algebraic theory (i.e. the objects are generated under finite products by a single object), then we will say that $\mathcal{A}$ is a *Lawvere algebraic category.*

If $\mathcal{A}$ is the category of models of a Lawvere algebraic theory $\mathbb{A}$, with generating object A, then in particular there will be a forgetful functor

$$U : \mathcal{A} \to \mathsf{Set},$$

determined by evaluation at A, and we know moreover that U preserves all limits, and one can show without too much difficulty that it also preserves all filtered colimits (cf. exercise 1.2.21). We require only one further condition to "recognize" $\mathcal{A}$ as algebraic, namely creation of "U-absolute coequalizers".

Definition 1.2.17. In any category $\mathcal{C}$, a coequalizer $c : Y \to Z$ of maps $a, b : X \rightrightarrows Y$ is *absolute* if, for every category $\mathcal{D}$ and functor $F : \mathcal{C} \to \mathcal{D}$, the image $Fc : FY \to FZ$ is a coequalizer of the images $Fa, Fb : FX \rightrightarrows FY$. A functor $F : \mathcal{C} \to \mathcal{D}$ *creates F-absolute coequalizers* if for every parallel pair of maps $a, b : X \rightrightarrows Y$ in $\mathcal{C}$ and absolute coequalizer $q : FY \to Q$ for $Fa, Fb : FX \rightrightarrows FY$ in $\mathcal{D}$, there is a unique object Z and map $c : Y \to Z$ in $\mathcal{C}$ with $FZ = Q$ and $Fc = q$, which, moreover, is a coequalizer in $\mathcal{C}$.

$$\begin{tikzcd} \mathcal{C} \arrow[d, "F"'] & X \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & Y \arrow[r, dotted, "c"] & Z \\ \mathcal{D} & FX \arrow[r, shift left, "Fa"] \arrow[r, shift right, "Fb"'] & FY \arrow[r, "q"'] & Q \end{tikzcd} \tag{1.21}$$

Theorem 1.2.18. *Given a category $\mathcal{A}$, equipped with a functor $U : \mathcal{A} \to \mathsf{Set}$, the following conditions are equivalent.*

1. *$\mathcal{A}$ is a Lawvere algebraic category; i.e. there is a Lawvere algebraic theory $\mathbb{A}$, and an equivalence,*
$$\mathcal{A} \simeq \mathsf{Hom}_{\mathsf{FP}}(\mathbb{A}, \mathsf{Set}) \hookrightarrow \mathsf{Set}^{\mathbb{A}}$$
between $\mathcal{A}$ and the full subcategory of finite product preserving functors on $\mathbb{A}$, associating $U : \mathcal{A} \to \mathsf{Set}$ to the evaluation at the generating object of $\mathbb{A}$.

2. *$U : \mathcal{A} \to \mathsf{Set}$ has a left adjoint $F : \mathsf{Set} \to \mathcal{A}$, preserves all (small) filtered colimits, and creates U-absolute coequalizers.*

3. *$\mathcal{A}$ is monadic over Set (via $U : \mathcal{A} \to \mathsf{Set}$),*
$$\mathcal{A} \simeq \mathsf{Set}^T$$
for a finitary monad $T : \mathsf{Set} \to \mathsf{Set}$.

Proof. (1⇒2) Suppose first that $\mathcal{A}$ is (Lawvere) algebraic, so

$$\mathcal{A} \simeq \mathsf{Mod}(\mathbb{A}) = \mathsf{Hom}_{\mathsf{FP}}(\mathbb{A}, \mathsf{Set}) \hookrightarrow \mathsf{Set}^{\mathbb{A}}$$

for an algebraic theory $\mathbb{A}$. First, observe that U preserves all limits: since these are computed pointwise in $\mathsf{Set}^{\mathbb{A}}$ and U is evaluation at the generating object of $\mathbb{A}$, it suffices to show that the limit of a diagram of FP functors, calculated in $\mathsf{Set}^{\mathbb{A}}$, is again an FP functor. But this is true because limits commute with finite products. The same argument works for filtered colimits in place of limits, since these also commute with finite products. Thus U preserves both limits and filtered colimits.

We also know by theorem 1.2.13 that we can construct a free algebra $F(X)$ for any set X, so (by the adjoint functor theorem), U has a left adjoint $F : \mathsf{Set} \to \mathcal{A}$ (we could also have used the fact that U has a left adjoint to infer that it preserves limits).

For creation of U-absolute coequalizers, suppose we have maps $f, g : A \rightrightarrows B$ in $\mathcal{A}$ and an absolute coequalizer $c : UB \to C$ for $Uf, Ug : UA \rightrightarrows UB$ in Set; we want to put an algebra structure on C making c a homomorphism $c : B \to C$ in $\mathcal{A}$ and a coequalizer of f and g.

$$\begin{tikzcd}
UA^n \arrow[r, shift left, "Uf^n"] \arrow[r, shift right, "Ug^n"'] \arrow[d, "\sigma^A"'] & UB^n \arrow[r, "c^n"] \arrow[d, "\sigma^B"] & C^n \arrow[d, dotted, "\sigma^C"] \\
UA \arrow[r, shift left, "Uf"] \arrow[r, shift right, "Ug"'] & UB \arrow[r, "c"'] & C
\end{tikzcd} \tag{1.22}$$

For each function symbol $\sigma \in \Sigma$ we have commutative squares as on the left in the above diagram, because f and g are homomorphisms. It follows by a simple diagram chase that $c \circ \sigma^B$ coequalizes the pair $Uf^n, Ug^n : UA^n \rightrightarrows UB^n$. Since $c : UB \to C$ is absolute, it is preserved by the functor $(-)^n$, and therefore $c^n : UB^n \to C^n$ is a coequalizer of Uf^n, Ug^n. There is therefore a unique map $\sigma^C : C^n \to C$ as indicated, making the right hand square commute. Doing this for each $\sigma \in \Sigma$ gives an interpretation of Σ on C. This is seen to be an algebra structure because the maps c^n are surjections. Thus $c : B \to C$ is a homomorphism, which is easily seen to be a coequalizer in $\mathcal{A}$.

(2⇒3) Taking the standard monad (T, η, μ) on Set with underlying functor $T = U \circ F$, we want to show that the canonical comparison map

$$\mathcal{A} \to \mathsf{Set}^T$$

to the category of T-algebras is an isomorphism. This follows from the condition that U creates absolute coequalizers by Beck's theorem; see [Lan71, VI.7]. Moreover, T preserves filtered colimits (i.e. is "finitary") because each of F and U do so.

(3⇒1) Let (T, η, μ) be a finitary monad on Set and $U : \mathsf{Set}^T \to \mathsf{Set}$ the forgetful functor from the category of T-algebras. We want an algebraic theory $\mathbb{A}$ and an equivalence

$$\mathsf{Set}^T \simeq \mathsf{Mod}(\mathbb{A})$$

over U and evaluation at the generator of $\mathbb{A}$, where recall $\mathsf{Mod}(\mathbb{A}) = \mathsf{Hom}_{\mathsf{FP}}(\mathbb{A}, \mathsf{Set})$. Let

$$\mathbb{A} = \mathsf{FGF}(\mathsf{Set}^T)^{\mathsf{op}} \tag{1.23}$$

be the dual of the full subcategory of finitely generated free T-algebras. The objects of $\mathbb{A}$ are of the form $T_0, T_1, T_2, \ldots$ where $T_n = T(n)$, equipped with the multiplication $\mu_n : T^2(n) \to T(n)$ as algebra structure map. Since, as free algebras, $T(n+m) \cong T(n) + T(m)$ we indeed have $T_n \times T_m \cong T_{n+m}$ as objects of $\mathbb{A}$, and T_1 as the generating object.
By the first two steps of this proof, we know that the algebraic category $\mathsf{Mod}(\mathbb{A})$ is also (finitary) monadic,

$$\mathsf{Mod}(\mathbb{A}) \simeq \mathsf{Set}^M,$$

with monad $M = U_M \circ F_M$, where $F_M \dashv U_M$ is the free-forgetful adjunction for $\mathsf{Mod}(\mathbb{A}) = \mathsf{Hom}_{\mathsf{FP}}(\mathbb{A}, \mathsf{Set})$, and $U_M \cong \mathsf{eval}_{T_1}$. Thus it will suffice to show that $M \cong T$, as monads, in order to conclude that

$$\mathsf{Mod}(\mathbb{A}) \simeq \mathsf{Set}^M \simeq \mathsf{Set}^T.$$

Moreover, since both M and T are finitary, it suffices to show that their respective restrictions to the dense subcategory $\mathsf{Set}_{\mathrm{fin}} \hookrightarrow \mathsf{Set}$ are isomorphic. By (1.19), we know that the finite free functor $F_M(n)$ has the form

$$F_M(n) = \mathsf{Hom}_{\mathbb{A}}(T_n, -) = \mathsf{Hom}_{\mathsf{FGF}(\mathsf{Set}^T)}(-, \langle T(n), \mu_n \rangle)$$

thus using the fact that $U_M \cong \mathsf{eval}_{T_1}$ we see that

$$\begin{aligned} M(n) = U_M(F_M(n)) &= U_M\big(\mathsf{Hom}_{\mathsf{FGF}(\mathsf{Set}^T)}(-, \langle T(n), \mu_n \rangle)\big) \\ &\cong \mathsf{Hom}_{\mathsf{FGF}(\mathsf{Set}^T)}(\langle T(1), \mu_1 \rangle, \langle T(n), \mu_n \rangle) \\ &\cong \mathsf{Hom}_{\mathsf{Set}}(1, T(n)) \cong T(n). \end{aligned}$$

□

Remark 1.2.19. Another "recognition theorem" that can be found in [Bor94] is the following:

Theorem (Borceux II.3.9)**.** *Given a category $\mathcal{A}$, equipped with a functor $U : \mathcal{A} \to \mathsf{Set}$, the following conditions are equivalent.*

1. *$\mathcal{A}$ is equivalent to the category of models of some algebraic theory $\mathbb{T}$,*

   $$\mathcal{A} \simeq \mathsf{Mod}(\mathbb{T})$$

   with $U : \mathcal{A} \to \mathsf{Set}$ the corresponding forgetful functor.

2. *$\mathcal{A}$ has coequalizers and kernel pairs, and $U : \mathcal{A} \to \mathsf{Set}$ has a left adjoint $F : \mathsf{Set} \to \mathcal{A}$, preserves all (small) filtered colimits and regular epimorphisms, and reflects isomorphisms.*

Condition (1) is of course equivalent to condition (1) or our theorem 1.2.18, by theorem 1.2.1.

Exercise 1.2.20. A *split coequalizer* for maps $f, g : A \rightrightarrows B$ is a map $e : B \to C$ together with s and t as indicated below,

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "e"] \arrow[l, bend left=60, "t"] & C \arrow[l, bend left=60, "s"] \end{tikzcd} \tag{1.24}$$

satisfying the equations

$$ef = eg, \quad ft = 1_B, \quad gt = se, \quad es = 1_C.$$

Show that a split coequalizer is an absolute coequalizer.

Exercise 1.2.21. A filtered colimit of algebras can be described directly as follows: First consider the case of sets. Let the index category $\mathbb{J}$ be filtered and $D : \mathbb{J} \to \mathsf{Set}$ a diagram. The colimiting set $\mathrm{colim}_j\, D_j$ can be described as the quotient of the coproduct $(\coprod_j Dj)/\sim$, where the equivalence relation $\sim$ is defined by:

$$(d_i \in D_i) \sim (d_j \in D_j) \;\Leftrightarrow\; t_{ik}(d_i) = t_{jk}(d_j) \text{ for some } t_{ik} : i \to k \text{ and } t_{jk} : j \to k \text{ in } \mathbb{J}.$$

1. Show that this is an equivalence relation using the filteredness of $\mathbb{J}$.
2. Now assume that the D_j all have an algebra structure and that all the transition maps $t_{ik} : D_i \to D_k$ are homomorphisms. Show that the colimit set $D_\infty = \mathrm{colim}_j\, D_j$ is also an algebra of the same kind by defining each of the operations $\sigma_\infty : D_\infty \times \ldots \times D_\infty \to D_\infty$ on equivalence classes as

   $$\sigma_\infty\langle [d_i], \ldots, [d'_j]\rangle = [\sigma_k\langle t_{ik}(d_i), \ldots, t_{jk}(d'_j)\rangle]$$

   for suitable k. Show that this is well-defined, and that D_∞, so equipped, also satisfies the equations satisfied by the D_j.

Example 1.2.22. A field is a ring in which every non-zero element has a multiplicative inverse. The theory of fields is (apparently) not algebraic, because the axiom

$$x \neq 0 \;\Rightarrow\; \exists y(x \cdot y = 1)$$

is not simply an equation, but in principle there could be an equivalent algebraic formulation of the theory which would somehow circumvent this problem. We can show that this is not the case by proving that the category Field of fields and field homomorphisms is not algebraic.

First observe that a category of models $\mathsf{Mod}(\mathbb{A})$ always has a terminal object because Set has a terminal object 1, and the constant functor $\Delta_1 : \mathbb{A} \to \mathsf{Set}$ which maps everything to 1 is a model. The functor Δ_1 is the terminal object in $\mathsf{Mod}(\mathbb{A})$ because it is the terminal functor in the functor category $\mathsf{Set}^{\mathbb{A}}$. Now in order to see that Field is not algebraic it suffices to show that there is no terminal field.

Exercise 1.2.23. Show that the category Field does not have a terminal object. (Hint: suppose that T is the terminal field and use the unique homomorphism $\mathbb{Z}_2 \to T$ to see that $1 + 1 = 0$ in T, then reason similarly using the unique homomorphism $\mathbb{Z}_3 \to T$.)

1.2.4 Algebraic functors

A *syntactic translation* of one algebraic theory into another is an assignment of types to types and terms to terms, and thus can be represented as a finite product preserving functor,

$$T : \mathbb{A} \to \mathbb{B}$$

between the associated Lawvere algebraic theories. Every such translation induces a "definable" functor on the semantics:

$$T^*(M) = M \circ T.$$

$$\mathsf{Mod}(\mathbb{A}) \xleftarrow{T^*} \mathsf{Mod}(\mathbb{B}) \tag{1.25}$$

$$\begin{tikzcd} \mathbb{A} \arrow[r, "T"] \arrow[dr, "T^*M"'] & \mathbb{B} \arrow[d, "M"] \\ & \mathsf{Set} \end{tikzcd}$$

For instance, let $\mathbb{F} = (\mathsf{Set}_{\mathrm{fin}})^{\mathsf{op}}$, the "empty" theory of an object, so that $\mathsf{Mod}(\mathbb{F}) = \mathsf{Set}$. Then the object underlying the universal model $U \in \mathbb{A}$ has a classifying functor

$$U : \mathbb{F} \to \mathbb{A}$$

which induces the forgetful functor by precomposition,

$$\mathsf{Set} = \mathsf{Mod}(\mathbb{F}) \xleftarrow{U^*} \mathsf{Mod}(\mathbb{A})$$

$$\begin{tikzcd} \mathbb{F} \arrow[r, "U"] \arrow[dr, "U^*M"'] & \mathbb{A} \arrow[d, "M"] \\ & \mathsf{Set} \end{tikzcd}$$

More generally, every translation corresponds to a "model in syntax":

$$\frac{T : \mathbb{A} \to \mathbb{B}}{\hat{T} \in \mathsf{Mod}(\mathbb{A}, \mathbb{B})}$$

by the universal property of $\mathbb{A}$. For instance, since every ring R has an underlying group $\mathsf{Grp}(R)$, the universal ring $U_{\mathbb{R}}$ in $\mathbb{R}$ has one $\mathsf{Grp}(U_{\mathbb{R}})$, which is classified by a unique functor from the theory of groups,

$$\mathsf{Grp}(U_{\mathbb{R}}) : \mathbb{G} \to \mathbb{R}.$$

This translation induces a functor on the corresponding categories of models,

$$\mathsf{Grp}(U_{\mathbb{R}})^* : \mathsf{Group} \leftarrow \mathsf{Ring}, \tag{1.26}$$

which of course is just the forgetful functor $\mathsf{Grp} : \mathsf{Ring} \to \mathsf{Group}$.

Now let us ask, which functors $f : \mathsf{Mod}(\mathbb{B}) \to \mathsf{Mod}(\mathbb{A})$ between algebraic categories are of the form $f = T^*$ for a translation $T : \mathbb{A} \to \mathbb{B}$ of theories? Let us call these *algebraic functors*. We consider first the case where T takes the generator A_1 of $\mathbb{A}$ to the generator B_1 of $\mathbb{B}$,

$$T(A_1) \cong B_1.$$

Then T^* commutes with the forgetful functors, which, recall, are evaluation at the generators, $U_{\mathbb{A}}(M) = M(A_1)$, etc.:

$$\begin{tikzcd} \mathsf{Mod}(\mathbb{B}) \arrow[rr, "T^*"] \arrow[dr, "U_{\mathbb{B}}"'] & & \mathsf{Mod}(\mathbb{A}) \arrow[dl, "U_{\mathbb{A}}"] \\ & \mathsf{Set} & \end{tikzcd}$$

because:

$$(U_{\mathbb{A}} \circ T^*)(M) = U_{\mathbb{A}}(M \circ T) = (M \circ T)(A_1) \cong M(T(A_1)) \cong M(B_1) = U_{\mathbb{B}}(M).$$

In fact, this condition is already sufficient!

Proposition 1.2.24. *Given any functor* $f : \mathsf{Mod}(\mathbb{B}) \to \mathsf{Mod}(\mathbb{A})$ *with*

$$U_{\mathbb{B}} \cong U_{\mathbb{A}} \circ f,$$

there is a unique (up to iso) translation $T : \mathbb{A} \to \mathbb{B}$ *such that*

$$f \cong T^*.$$

Proof. (Cf. [Bor94, 3.9.2]) Consider the diagram:

$$\begin{tikzcd} \mathsf{Mod}(\mathbb{B}) \arrow[rr, "f"] \arrow[dr, shift left, "U_{\mathbb{B}}"] & & \mathsf{Mod}(\mathbb{A}) \arrow[dl, shift right, "U_{\mathbb{A}}"'] \\ & \mathsf{Set} \arrow[ul, shift left, "F_{\mathbb{B}}"] \arrow[ur, shift right, "F_{\mathbb{A}}"'] & \end{tikzcd}$$

where in each pair $F \dashv U$. We seek an FP-functor $T : \mathbb{A} \to \mathbb{B}$; thus, using the explicit description of $\mathbb{A}$ as the subcategory of finitely generated free models $\mathsf{Mod}_{\mathrm{fg}}(\mathbb{A})^{\mathsf{op}}$ (and the same for $\mathbb{B}$), it suffices to give a functor T as indicated in:

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "T"] \arrow[d, "\cong"'] & \mathbb{B} \arrow[d, "\cong"'] \\
\mathsf{Mod}_{\mathrm{fg}}(\mathbb{A})^{\mathsf{op}} \arrow[r, dotted] & \mathsf{Mod}_{\mathrm{fg}}(\mathbb{B})^{\mathsf{op}}
\end{tikzcd}$$

So we need a functor T with:

$$T(F_{\mathbb{A}}(n)) \cong F_{\mathbb{B}}(n).$$

Consider the following construction: beginning with the unit $\eta_n : n \to U_{\mathbb{B}}F_{\mathbb{B}}(n)$, we have a natural bijection

$$\frac{\eta_n : n \to U_{\mathbb{B}}F_{\mathbb{B}}(n) = U_{\mathbb{A}}fU_{\mathbb{B}}}{h_n : F_{\mathbb{A}}(n) \to fF_{\mathbb{B}}(n)}$$

The pair $(F_{\mathbb{A}}(n), h_n)$ has the following universal property: for any $M \in \mathsf{Mod}(\mathbb{B})$ and $k : F_{\mathbb{A}}(n) \to fM$, there is a unique "lift" $\bar{k} : F_{\mathbb{B}}(n) \to M$ with $k = f(\bar{k}) \circ h_n$, as indicated in the following.

$$\begin{tikzcd}
\mathsf{Mod}(\mathbb{B}) \arrow[d, "f"'] & F_{\mathbb{B}}(n) \arrow[r, "\bar{k}"] & M \\
\mathsf{Mod}(\mathbb{A}) & fF_{\mathbb{B}}(n) \arrow[r, "f\bar{k}"] & fM \\
& F_{\mathbb{A}}(n) \arrow[u, "h_n"] \arrow[ur, "k"'] &
\end{tikzcd}$$

Indeed, we have natural bijections:

$$\begin{array}{c}
k : F_{\mathbb{A}}(n) \longrightarrow fM \\
\hline\hline
k' : n \longrightarrow U_{\mathbb{A}}fM \cong U_{\mathbb{B}}M \\
\hline\hline
\bar{k} : F_{\mathbb{B}}(n) \longrightarrow M
\end{array} \tag{1.27}$$

Now, to define T, take any $\alpha : F_{\mathbb{A}}(n) \to F_{\mathbb{A}}(m)$ and use

$$\begin{tikzcd}
& fF_{\mathbb{B}}(m) \\
F_{\mathbb{A}}(n) \arrow[r, "\alpha"'] \arrow[ur, "h_m\alpha"] & F_{\mathbb{A}}(m) \arrow[u, "h_m"']
\end{tikzcd}$$

to get the lift:

$$T(\alpha) := \overline{h_m\alpha} : F_{\mathbb{B}}(n) \longrightarrow F_{\mathbb{B}}(m).$$

This assignment is easily seen to be functorial. Finally, to show that it is induced by precomposing with $T : \mathbb{A} \to \mathbb{B}$, we need the following to commute up to natural isomorphism.

$$\begin{tikzcd}
\mathsf{Mod}(\mathbb{B}) \arrow[r, "f"] \arrow[d, "\cong"'] & \mathsf{Mod}(\mathbb{A}) \arrow[d, "\cong"'] \\
\mathsf{FP}(\mathbb{B}, \mathsf{Set}) \arrow[r, "T^*"'] & \mathsf{FP}(\mathbb{A}, \mathsf{Set})
\end{tikzcd}$$

Identifying $\mathbb{B} = \mathsf{Mod}_{\mathrm{fg}}(\mathbb{B})^{\mathsf{op}}$ and $\mathbb{A} = \mathsf{Mod}_{\mathrm{fg}}(\mathbb{A})^{\mathsf{op}}$, going aroung the Southwest, we have

$$M \mapsto \mathsf{Hom}_{\mathsf{Mod}(\mathbb{B})}(-, M) \mapsto \mathsf{Hom}_{\mathsf{Mod}(\mathbb{A})}(T(-), M).$$

Going around the Northeast gives

$$M \mapsto fM \mapsto \mathsf{Hom}(-, fM).$$

But by (1.27), we have a natural bijection:

$$\frac{F_{\mathbb{A}}(n) \to fM}{\overline{T(F_{\mathbb{A}}(n)) = F_{\mathbb{B}}(n) \to M}}$$

□

Corollary 1.2.25. *For a functor $f : \mathsf{Mod}(\mathbb{B}) \to \mathsf{Mod}(\mathbb{A})$ between algebraic categories, the following are equivalent.*

1. *f is algebraic, $f = T^*$, for some FP functor $T : \mathbb{A} \to \mathbb{B}$ which, moreover, preserves the generator, $T(A_1) \cong B_1$*
2. *f commutes with the forgetful functors, $U_{\mathbb{A}} \circ f \cong U_(B)$*

□

Exercise 1.2.26. Show that for any algebraic theory $\mathbb{A}$, the full inclusion $\mathsf{Mod}(\mathbb{A}) \hookrightarrow \mathsf{Set}^{\mathbb{A}}$ has a left adjoint. (Hint: use the Adjoint Functor Theorem.)

Exercise 1.2.27. Assuming the result of the previous exercise, show that the precomposition functor $T^* : \mathsf{Mod}(\mathbb{B}) \longrightarrow \mathsf{Mod}(\mathbb{A})$ induced by any translation $T : \mathbb{A} \to \mathbb{B}$ (not necessarily preserving the generator) always has a left adjoint $T_! : \mathsf{Mod}(\mathbb{A}) \longrightarrow \mathsf{Mod}(\mathbb{B})$.

Exercise 1.2.28. Assuming the results of the previous two exercises, show that a functor $f : \mathsf{Mod}(\mathbb{B}) \longrightarrow \mathsf{Mod}(\mathbb{A})$ is induced by a translation $T : \mathbb{A} \to \mathbb{B}$ as $f = T^*$ iff the following conditions hold:

1. f has a left adjoint $f_! : \mathsf{Mod}(\mathbb{A}) \longrightarrow \mathsf{Mod}(\mathbb{B})$.
2. $f_!(U_{\mathbb{A}}) \cong (U_{\mathbb{B}})^n$ for some $0 \leq n$, where as usual we identify $\mathbb{A} = \mathsf{Mod}_{\mathrm{fgf}}(\mathbb{A})^{\mathsf{op}}$ and similarly for $\mathbb{B}$.

Remark 1.2.29. The general question, when is a "semantic functor"

$$f : \mathsf{Mod}(\mathbb{B}) \longrightarrow \mathsf{Mod}(\mathbb{A})$$

between (Lawvere) algebraic categories induced by a "syntactic translation" $T : \mathbb{A} \to \mathbb{B}$ of the Lawvere algebraic theories (not necessarily preserving the generating objects) can also be answered in the abstract, providing a definition of an *algebraic functor*: it is one that preserves all limits, filtered colimits, and regular epimorphisms. However, this characterization requires a slight modification in the notion of algebraic theory to include closure under retracts — so-called *Cauchy completeness*. In that setting, there is a neat duality theory between the "syntax category" of theories and translations and the "semantics category" of algebraic categories and algebraic functors, which is developed in detail in [ALR03].

Chapter 2

First-Order Logic

Having considered equational theories, we now move on to first-order logic. This is the usual predicate logic with propositional connectives like $\wedge$ and $\Rightarrow$, and quantifiers $\forall$ and $\exists$. The general approach to studying logic via category theory is to determine categorical structures that model the first-order logical operations, or a suitable fragment of it, and then consider categories with these structures and functors that preserve them. Here adjoint functors play an imporant role, as the basic logical operations are recognized as adjoints. We again show that the semantics is "functorial", meaning that models of a theory are functors that preserve suitable categorical structure. We again construct classifying categories representing theories, which are the counterparts of the algebraic theories that we have already met.

Let us demonstrate our approach informally with an example. In section 1.1.1 we considered models of algebraic theories in categories with finite products. Recall that e.g. a group is a structure of the form:

$$e : 1 \to G\,, \qquad m : G \times G \to G\,, \qquad i : G \to G\,.$$

for which, moreover, certain diagrams built from these basic arrows must commute. We can express some properties of groups in terms of further equations, for example commutativity

$$x \cdot y = y \cdot x\,.$$

As we saw, such equations can be interpreted in any category with finite products. This provides a large scope for categorical semantics of algebraic theories.

However, there are also many significant properties of algebraic structures which cannot be expressed with equations. Consider the statement that a group (G, e, m, i) has no non-trivial square roots of unity,

$$\forall\, x : G\,.\,(x \cdot x = e \Rightarrow x = e)\ . \tag{2.1}$$

This is a first-order logical statement which cannot be rewritten as a system of equations (proof!). To see what its categorical interpretation ought to be, we look at its usual set-theoretic interpretation. Each of subformulas, $x \cdot x = e$ and $x = e$, determines a subset of G:

$$\begin{tikzcd}
\{x \in G \mid x \cdot x = e\} \arrow[rr, tail, dotted] \arrow[dr, tail, "i"'] & & \{x \in G \mid x = e\} \arrow[dl, tail, "j"] \\
& G &
\end{tikzcd}$$

The implication $x \cdot x = e \Rightarrow x = e$ holds when $\{x \in G \mid x \cdot x = e\}$ is contained in $\{x \in G \mid x = e\}$. In categorical language we can say that the inclusion i factors through the inclusion j. Observe also that such a factorization is necessarily a mono and is unique, if it exists. The defining formulas of the subsets $\{x \in G \mid x \cdot x = e\}$ and $\{x \in G \mid x = e\}$ are equations, and so the subsets themselves can be constructed as equalizers (as above, interpreting $\cdot$ as m):

$$\begin{tikzcd}
\{x \in G \mid x \cdot x = e\} \arrow[r, tail] & G \arrow[r, "\langle 1_G, 1_G \rangle"] \arrow[rr, bend right, "e \circ !_G"'] & G \times G \arrow[r, "m"] & G
\end{tikzcd}$$

$$\begin{tikzcd}
\{x \in G \mid x = e\} \arrow[r, tail] & G \arrow[r, shift left, "1_G"] \arrow[r, shift right, "e \circ !_G"'] & G
\end{tikzcd}$$

In sum, we can interpret condition (2.1) in any category with products and equalizers, i.e. in any category with finite limits.[1] This allows us to define the notion of a group without square roots of unity in any category $\mathcal{C}$ with finite limits as an object G with morphisms $e : \mathbf{1} \to G$, $m : G \times G \to G$ and $i : G \to G$ such that (G, e, m, i) is a group in $\mathcal{C}$, and the equalizer of $m \circ \langle 1_G, 1_G \rangle$ and $e \circ !_G$ factors through $e : \mathbf{1} \to G$.

The aim of this chapter is to analyze how such examples can be treated in general. We want to relate first-order logic and fragments of it to categorical structures that are suitable for the interpretation of the logic. The general outline will be as follows:

1. A language $\mathcal{L}$ for a first-order theory consists, as usual, of some basic relation, function, and constant symbols, say $\mathcal{L} = (R, f, c)$.

2. An $\mathcal{L}$-structure in a category $\mathcal{C}$ with finite limits is an interpretation of $\mathcal{L}$ in $\mathcal{C}$ as an object M equipped with corresponding relations and operations (of appropriate arities), e.g.

$$R^M \rightarrowtail M \times \cdots \times M$$
$$f^M : M \times \cdots \times M \to M$$
$$c^M : 1 \to M.$$

[1] We are *not* claiming that finite limits suffice for an interpretation of arbitrary formulas built from universal quantifiers and implications. The formula at hand has a very special form $\forall x . (\varphi(x) \Longrightarrow \psi(x))$, where $\varphi(x)$ and $\psi(x)$ do not contain further $\forall$ or $\Longrightarrow$.

3. Formulas $\varphi(x_1, \ldots, x_n)$ in (some fragment of) first-order logic will be interpreted as "generalized subsets", i.e. subobjects,

$$[\![\varphi(x_1, \ldots, x_n)]\!] \rightarrowtail M \times \cdots \times M.$$

The interpretation makes use of categorical operations in $\mathcal{C}$ corresponding to the logical ones appearing in the formula $\varphi(x_1, \ldots, x_n)$.

4. A theory $\mathbb{T}$ in (a fragment of) first-order logic will consist of a set of (binary) sequents,

$$\varphi(x_1, \ldots, x_n) \vdash \psi(x_1, \ldots, x_n).$$

5. A model of the theory is then an interpretation M in which the corresponding subobjects satisfy all the sequents of $\mathbb{T}$, in the sense that

$$[\![\varphi(x_1, \ldots, x_n)]\!] \leq [\![\psi(x_1, \ldots, x_n)]\!] \qquad \text{in } \mathsf{Sub}(M^n).$$

6. We shall give a deductive calculus for such sequents, prove that it is sound with respect to categorical models, and then use it to construct a classifying category $\mathcal{C}_{\mathbb{T}}$, with the expected universal property: models of $\mathbb{T}$ correspond to (structure-preserving) functors on $\mathcal{C}_{\mathbb{T}}$.

7. Completeness of the calculus in general follows from classification, and more specialized completeness results from embedding theorems applied to the classifying category.

2.1 Theories

A *first-order theory* $\mathbb{T}$ consists of an underlying *type theory* and a set of formulas in a *fragment of first-order logic*. Anticipating Chapter **??**, the type theory is given by a set of basic types, a set of basic constants together with their types, rules for forming types, and rules and axioms for deriving typing judgments

$$x_1 : A_1, \ldots, x_n : A_n \mid t : B\ ,$$

expressing that term t has type B in typing context $x_1 : A_1,\ \ldots,\ x_n : A_n$, and a set of axioms and rules of inference which tell us which equations between terms

$$x_1 : A_1, \ldots, x_n : A_n \mid t = u : B\ ,$$

are valid. This part of the theory $\mathbb{T}$ may be regarded as providing the underlying structure, on top of which the logical formulas are defined. For first-order logic, the underlying type theory will be essentially the same as the equational logic that we already met in Chapter 1.

A fragment of first-order logic is then given by a set of *basic relation symbols* together with a specification of which first-order operations are being considered in building formulas. Each basic relation symbol has a *signature* $(A_1, \ldots, A_n)$, which specifies the types of its arguments. The *arity* of a relation symbol is the number of arguments it takes. The judgment[2]

$$x_1 : A_1, \ldots, x_n : A_n \mid \phi \ \mathtt{pred}$$

states that ϕ is a well-formed formula in typing context $x_1 : A_1, \ldots, x_n : A_n$. For each basic relation symbol R with signature $(A_1, \ldots, A_n)$ there is an inference rule

$$\frac{\Gamma \mid t_1 : A_1 \qquad \cdots \qquad \Gamma \mid t_n : A_n}{\Gamma \mid R(t_1, \ldots, t_n) \ \mathtt{pred}}$$

Depending on what fragment of first-order logic is involved, there may be other rules for forming logical formulas. For example, if equality is present, then for each type A there is a rule

$$\frac{\Gamma \mid t : A \qquad \Gamma \mid u : A}{\Gamma \mid t =_A u \ \mathtt{pred}}$$

and if conjunction is present, then there is a rule

$$\frac{\Gamma \mid \varphi \ \mathtt{pred} \qquad \Gamma \mid \psi \ \mathtt{pred}}{\Gamma \mid \varphi \wedge \psi \ \mathtt{pred}}$$

Other such rules will be given when we come to the study of particular logical operations.

The basic logical judgment of a first-order theory is *logical entailment* between formulas,

$$x_1 : A_1, \ldots, x_n : A_n \mid \varphi_1, \ldots, \varphi_m \vdash \psi$$

which states that in the typing context $x_1 : A_1, \ldots, x_n : A_n$, the hypotheses $\varphi_1, \ldots, \varphi_m$ entail ψ. It is understood that the terms appearing in the formulas are well-typed in the typing context, and that formulas $\varphi_1, \ldots, \varphi_m, \psi$ are part of the fragment of the logic of $\mathbb{T}$. When the fragment contains conjunction $\wedge$ it is convenient to restrict attention to *binary* sequents,

$$x_1 : A_1, \ldots, x_n : A_n \mid \varphi \vdash \psi,$$

by replacing $\varphi_1, \ldots, \varphi_m$ with $\varphi_1 \wedge \ldots \wedge \varphi_m$. When the fragment contains equality, we may replace the type-theoretic equality judgments

$$x_1 : A_1, \ldots, x_n : A_n \mid t = u : B$$

with the logical statements

$$x_1 : A_1, \ldots, x_n : A_n \mid \cdot \vdash t =_B u \ .$$

[2] We follow type-theoretic practice here by adding the tag pred to turn what would otherwise be an exhibited formula in context into a judgement concerning the formula.

The subscript at the equality sign indicates the type at which the equality is taken. In a theory $\mathbb{T}$ there are basic entailments, or axioms, which together with the inference rules for the opertations involved can be used for deriving valid judgments, as usual.

We shall consider several standard fragments of first-order logic, determined by selecting a subset of logical connectives and quantifiers. These are as follows:

1. *Full first-order logic* is built from logical operations
$$= \quad \top \quad \bot \quad \neg \quad \wedge \quad \vee \quad \Rightarrow \quad \forall \quad \exists\,.$$
2. *Cartesian logic* is the fragment built from
$$= \quad \top \quad \wedge\,.$$
3. *Regular logic* is the fragment built from
$$= \quad \top \quad \wedge \quad \exists\,.$$
4. *Coherent logic* is the fragment built from
$$= \quad \top \quad \wedge \quad \exists \quad \bot \quad \vee\,.$$
5. A *geometric formula* is a formula of the form
$$\forall\, x : A\,.\,(\varphi \Longrightarrow \psi)\ ,$$
where φ and ψ are coherent formulas.

The names for these fragments come from the names of various categorical structures in which they are interpreted.

The well-formed terms and formulas of a first-order theory $\mathbb{T}$ constitute its *language*. It may seem that we are doing things backwards, because we should have spoken of first-order languages before we spoke of first-order theories. While this is possible for simple theories, it becomes difficult to do when we consider more complicated theories in which types and logical formulas are intertwined. In such cases the typing judgments and logical entailments may be given by a mutual recursive definition. In order to find out whether a given term is well-formed, we might have to prove a logical statement. In everyday mathematics this occurs all the time, for example, to show that the term $\int_0^\infty f$ denotes a real number, it may be necessary to prove that $f : \mathbb{R} \to \mathbb{R}$ is an integrable function and that the integral has a finite value. This is why it does not always make sense to strictly differentiate a language from a theory.[3]

In order to focus on the logical part of first-order theories, we are going to limit attention to only two very simple kinds of type theory. A *single-sorted* first-order theory has as its underlying type theory a single type A, and for each $k \in \mathbb{N}$ a set of basic k-ary function symbols. The rules for typing judgments are:

[3] However, it *does* make sense to distinguish syntax from theory. Rules of substitution and the behaviour of free and bound variables are syntactic considerations, for example.

1. Variables in contexts:
$$\frac{}{x_1 : A, \dots, x_n : A \mid x_i : A}$$
2. For each basic function symbol f of arity k, there is an inference rule
$$\frac{\Gamma \mid t_1 : A \qquad \cdots \qquad \Gamma \mid t_n : A}{\Gamma \mid f(t_1, \dots, t_n) : A}$$

This much is essentially an algebraic theory. In addition, however, a single-sorted first-order theory may contain relation symbols, formulas, axioms, and rules of inference which an algebraic theory does not.

A slight generalization of a single-sorted theory is a *multi-sorted* one. Its underlying type theory is given by a set of types, and a set of basic function symbols. Each function symbol f has a *signature* $(A_1, \dots, A_n; B)$, where n is the arity of f and $A_1, \dots, A_n, B$ are types. The rules for typing judgments are:

1. Variables in contexts:
$$\frac{}{x_1 : A_1, \dots, x_n : A_n \mid x_i : A_i}$$
2. For each basic function symbol f with signature $(A_1, \dots, A_n; B)$, there is an inference rule
$$\frac{\Gamma \mid t_1 : A_1 \qquad \cdots \qquad \Gamma \mid t_n : A_n}{\Gamma \mid f(t_1, \dots, t_n) : B}$$

We often write suggestively $f : A_1 \times \cdots \times A_n \to B$ to indicate that $(A_1, \dots, A_n; B)$ is the signature of f. However, this does not mean that $A_1 \times \cdots \times A_n \to B$ is a type! A multi-sorted first-order theory does *not* have any type forming operations, such as $\times$ and $\to$.

2.2 Predicates as subobjects

Formulas of first-order logic will be interpreted as "generalized subsets", i.e. subobjects. We therefore need to review some of the basic theory of these.

Let A be an object in a category $\mathcal{C}$. If $i : I \rightarrowtail A$ and $j : J \rightarrowtail A$ are monos into A, we say that i is smaller than j, and write $i \leq j$, when there exists a morphism $k : I \to J$ such that the following diagram commutes:

$$\begin{tikzcd} I \arrow[rr, dotted, "k"] \arrow[dr, "i"'] & & J \arrow[dl, "j"] \\ & A & \end{tikzcd}$$

If such a k exists then it, too, is monic, since i is, and it is unique, since j is monic. The class $\mathsf{Mono}(A)$ of all monos into A is this preordered by this relation $\leq$, it is the same as the slice category $\mathsf{Mono}(\mathcal{C})/A$ of all monos in $\mathcal{C}$, sliced over the object A. Let $\mathsf{Sub}(A)$ be the poset reflection of this preorder. Thus the elements of $\mathsf{Sub}(A)$ are equivalence classes of monos into A, where monos $i : I \rightarrowtail A$ and $j : J \rightarrowtail A$ are equivalent when $i \leq j$ and $j \leq i$ (note that then $I \cong J$). The induced relation $\leq$ on $\mathsf{Sub}(A)$ is then a partial order.

We have to be a bit careful with the formation of $\mathsf{Sub}(A)$, since it is defined as a quotient of a *class* $\mathsf{Mono}(A)$. In many particular cases the general construction by quotients can be avoided. If we can demonstrate that the preorder $\mathsf{Mono}(A)$ is equivalent, as a category, to a poset P then we can simply take $\mathsf{Sub}(A) = P$. At any rate, we usually require that $\mathsf{Sub}(A)$ is small.

Definition 2.2.1. A category $\mathcal{C}$ is *well-powered* when, for all $A \in \mathcal{C}$, there is at most a *set* of subobjects of A, so that the category $\mathsf{Mono}(A)$ is equivalent to a small poset. In other words, for every $A \in \mathcal{C}$, $\mathsf{Sub}(A)$ is a small category.

We shall often speak of subobjects as if they were monos rather than equivalence classes of monos. It is understood that we mean the subobjects represented by monos and not the monos themselves. Sometimes we refer to a mono $i : I \rightarrowtail A$ by its domain I only, even though the object I itself does not determine the morphism i. Hopefully this will not cause confusion, as it is always going to be clear which mono is meant to go along with the object I.

In a category $\mathcal{C}$ with finite limits the assignment $A \mapsto \mathsf{Sub}(A)$ is the object part of the *subobject functor*

$$\mathsf{Sub} : \mathcal{C}^{\mathsf{op}} \to \mathsf{Poset}\ .$$

The morphism part of Sub is pullback. More precisely, given a morphism $f : A \to B$, let $\mathsf{Sub}(f) = f^* : \mathsf{Sub}(B) \to \mathsf{Sub}(A)$ be the monotone map which maps the subobject $[i : I \rightarrowtail B]$ to the subobject $[f^*i : f^*I \rightarrowtail A]$, where $f^*i : f^*I \rightarrowtail A$ is a pullback of i along f:

$$\begin{tikzcd}
f^*I \arrow[r] \arrow[d, tail, "f^*i"'] \arrow[dr, phantom, "\lrcorner", very near start] & I \arrow[d, tail, "i"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

Recall that a pullback of a mono is again mono, so this definition makes sense. We need to verify (1) that if two monos $i : I \rightarrowtail A$ and $j : J \rightarrowtail A$ are equivalent, then their pullbacks are so as well; and (2) that $\mathsf{Sub}(1_A) = 1_{\mathsf{Sub}(A)}$ and $\mathsf{Sub}(g \circ f) = \mathsf{Sub}(f) \circ \mathsf{Sub}(g)$. These all follow easily from the fact that pullback is a functor $\mathcal{C}/B \to \mathcal{C}/A$, which reduces to the familiar "2-pullbacks" lemma:

Lemma 2.2.2. *Suppose both squares in the following diagram are pullbacks:*

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

Then the outer rectangle is a pullback diagram as well. Moreover, if the outer rectangle and the right square are pullbacks, then so is the left square.

Proof. This is left as an exercise in diagram chasing. □

Of course, pullbacks are really only determined up to isomorphism, but this does not cause any problems because isomorphic monos represent the same subobject.

In the semantics to be given below, a formula

$$x : A \mid \varphi \ \texttt{pred}$$

will be interpreted as a subobject

$$[\![x : A \mid \varphi]\!] \rightarrowtail [\![A]\!].$$

Thus $\mathsf{Sub}(A)$ can be regarded as the poset of "predicates" on A, generalizing the powerset of a set A. Logical operations on formulas then correspond to operations on $\mathsf{Sub}(A)$. The structure of $\mathsf{Sub}(A)$ therefore determines which logical connectives can be interpreted. If $\mathsf{Sub}(A)$ is a Heyting algebra, then we can interpret the full intuitionistic propositional calculus (cf. Subsection **??**), but if $\mathsf{Sub}(A)$ only has binary meets then all that can be interpreted are $\top$ and $\wedge$. We will work out details of different operations in the following sections, but one common aspect that we require is the "stability" of the interpretation of the logical operations, in a sense that we now make clear.

Substitution and stability

Let us consider the interpretation of substitution of terms for variables. There are two kinds of substitution, into a term, and into a formula. We may substitute a term $x : A \mid t : B$ for a variable y in a term $y : B \mid u : C$ to obtain a new term $x : A \mid u[t/y] : C$. If t and u are interpreted as morphisms

$$[\![A]\!] \xrightarrow{[\![t]\!]} [\![B]\!] \xrightarrow{[\![u]\!]} [\![C]\!]$$

then $u[t/y]$ is interpreted as their composition:

$$[\![x : A \mid u[t/y] : C]\!] = [\![y : B \mid u : C]\!] \circ [\![x : A \mid t : B]\!] \ .$$

Thus, *substitution into a term is composition.*

The second kind of substitution occurs when we substitute a term $x : A \mid t : B$ for a variable y in a formula $y : B \mid \varphi$ to obtain a new formula $x : A \mid \varphi[t/y]$. If t is interpreted as a morphism $[\![t]\!] : [\![A]\!] \to [\![B]\!]$ and φ is interpreted as a subobject $[\![\varphi]\!] \rightarrowtail [\![B]\!]$ then the interpretation of $\varphi[t/y]$ is the pullback of $[\![\varphi]\!]$ along $[\![t]\!]$:

$$\begin{tikzcd}
{[\![\varphi[t/y]]\!] = [\![t]\!]^*[\![\varphi]\!]} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & {[\![\varphi]\!]} \arrow[d, tail] \\
{[\![A]\!]} \arrow[r, "{[\![t]\!]}"'] & {[\![B]\!]}
\end{tikzcd}$$

Thus, *substitution into a formula is pullback*,

$$[\![x : A \mid \varphi[t/y]\,]\!] = [\![x : A \mid t : B]\!]^* [\![y : B \mid \varphi]\!].$$

Now, because substitution respects the syntactical, logical operations, e.g.

$$(\varphi \wedge \psi)[t/x] = \varphi[t/x] \wedge \psi[t/x],$$

the categorical structures used to interpret the various logical operations such as $\wedge$ must also behave well with respect to the interpretation of substitution, i.e. pullback. We say that a categorical property or structure is *stable (under pullbacks)* if it is preserved by pullbacks.

For example, a category $\mathcal{C}$ has *stable meets* if each poset $\mathsf{Sub}(A)$ has binary meets, and the pullback of a meet $I \wedge J \rightarrowtail A$ along any map $f : B \to A$ is the meet $f^*I \wedge f^*J \rightarrowtail A$ of the respective pullbacks,

$$f^*(I \wedge J) = f^*I \wedge f^*J.$$

This means that the meet operation,

$$\wedge : \mathsf{Sub}(A) \times \mathsf{Sub}(A) \longrightarrow \mathsf{Sub}(A)$$

is natural in A, in the sense that for any map $f : B \to A$ the following diagram commutes.

$$\begin{tikzcd}
\mathsf{Sub}(A) \times \mathsf{Sub}(A) \arrow[r, "\wedge_A"] \arrow[d, "f^* \times f^*"'] & \mathsf{Sub}(A) \arrow[d, "f^*"] \\
\mathsf{Sub}(B) \times \mathsf{Sub}(B) \arrow[r, "\wedge_B"'] & \mathsf{Sub}(B)
\end{tikzcd}$$

Exercise 2.2.3. Show that any category $\mathcal{C}$ with finite limits has stable meets in the foregoing sense: each poset $\mathsf{Sub}(A)$ has all finite meets (i.e. including the "empty meet" 1), and these are stable under pullbacks. Conclude that $\mathsf{Sub} : \mathcal{C}^{\mathsf{op}} \longrightarrow \mathsf{Posets}$ factors through the subcategory of $\wedge$-semi-lattices.

Generalized elements

In any category, we sometimes consider arbitrary arrows $x : X \to C$ as *generalized elements* of C, thinking thereby of variable elements or parameters. With respect to a subobject $S \rightarrowtail C$, such an element is said to be *in the subobject*, writtten

$$x \in_C S,$$

if it factors through (any mono representing) the subobject,

$$\begin{tikzcd}
 & S \arrow[d, tail] \\
X \arrow[ur, dotted] \arrow[r, "x"'] & C
\end{tikzcd}$$

which, observe, it then does uniquely. The following “generalized element semantics” can be a useful technique for “externalizing” the operations on subobjects into statements about generalized elements.

Proposition 2.2.4. *Let C be any object in a category $\mathcal{C}$ with finite limits.*

1. *for the top element $1 \in \mathsf{Sub}(C)$, and for all $x : X \to C$,*

$$x \in_C 1.$$

2. *for any $S, T \in \mathsf{Sub}(C)$,*

$$S \leq T \iff x \in_C S \text{ implies } x \in_C T, \text{ for all } x : X \to C.$$

3. *for any $S, T \in \mathsf{Sub}(C)$, and for all $x : X \to C$,*

$$x \in_C S \wedge T \iff x \in_C S \text{ and } x \in_C T.$$

4. *for the subobject $\Delta = [\langle 1_C, 1_C \rangle] \in \mathsf{Sub}(C \times C)$, and for all $x, y : X \to C$,*

$$\langle x, y \rangle \in \Delta \iff x = y.$$

5. *for the equalizer $E_{(f,g)} \rightarrowtail A$ of a pair of arrows $f, g : A \rightrightarrows B$, and for all $x : X \to A$,*

$$x \in_A E_{(f,g)} \iff fx = gx.$$

6. *for the pullback $f^*S \rightarrowtail A$ of a subobject $S \rightarrowtail B$ along any arrow $f : A \to B$, and for all $x : X \to A$,*

$$x \in_A f^*S \iff fx \in_B S.$$

Exercise 2.2.5. Prove the proposition.

2.3 Cartesian logic

As a first example we look at the logic of *cartesian categories*, which are categories with finite limits, to be called *cartesian logic*. This is a logic of formulas over a multi-sorted type theory with unit type `1`. (See section **??** for multi-sorted type theories and the axioms for the unit type). The logical operations are $=$, $\top$, and $\wedge$.

Formation rules for cartesian logic

Given a basic language consisting of a stock of relation and function symbols (with arities), the terms are built up as usual from the basic function symbols and variables (we take "constants" to be 0-ary function symbols). The rules for constructing formulas are as follows:

1. The 0-ary relation symbol $\top$ is a formula:
$$\frac{}{\Gamma \mid \top \ \mathtt{pred}}$$
2. For each basic relation symbol R with signature $(A_1, \ldots, A_n)$ there is a rule
$$\frac{\Gamma \mid t_1 : A_1 \qquad \cdots \qquad \Gamma \mid t_n : A_n}{\Gamma \mid R(t_1, \ldots, t_n) \ \mathtt{pred}}$$
3. For each type A, there is a rule
$$\frac{\Gamma \mid s : A \qquad \Gamma \mid t : A}{\Gamma \mid s =_A t \ \mathtt{pred}}$$
4. Conjunction:
$$\frac{\Gamma \mid \varphi \ \mathtt{pred} \qquad \Gamma \mid \psi \ \mathtt{pred}}{\Gamma \mid \varphi \wedge \psi \ \mathtt{pred}}$$
5. Weakening:
$$\frac{\Gamma \mid \varphi \ \mathtt{pred}}{\Gamma, x : A \mid \varphi \ \mathtt{pred}}$$

Observe that, as usual, there is then a derived operation of substitution of terms for variables into formulas, defined by structural recursion on the above specification of formulas:

Substitution:
$$\frac{\Gamma \mid t : A \qquad \Gamma, x : A \mid \varphi \ \mathtt{pred}}{\Gamma \mid \varphi[t/x] \ \mathtt{pred}}$$

Inference rules for cartesian logic

Although we do not yet need them, we state the rules of inference here, too, for the convenience of having the entire specification of cartesian logic in one place. As already mentioned, we can conveniently state this deductive calculus entirely in terms of *binary* sequents,

$$\Gamma \mid \psi \vdash \varphi.$$

We omit mention of the context Γ when it is the same in the premisses and conclusion of a rule.

1. Weakening:
$$\frac{\Gamma \mid \psi \vdash \varphi}{\Gamma, x : A \mid \psi \vdash \varphi}$$
2. Substitution:
$$\frac{\Gamma \mid t : A \qquad \Gamma, x : A \mid \psi \vdash \varphi}{\Gamma \mid \psi[t/x] \vdash \varphi[t/x]}$$
3. Identity:
$$\frac{}{\varphi \vdash \varphi}$$
4. Cut:
$$\frac{\psi \vdash \theta \qquad \theta \vdash \varphi}{\psi \vdash \varphi}$$
5. Equality:
$$\frac{}{\psi \vdash t =_A t} \qquad \frac{\psi \vdash t =_A u \qquad \psi \vdash \varphi[t/z]}{\psi \vdash \varphi[u/z]}$$
6. Truth:
$$\frac{}{\psi \vdash \top}$$
7. Conjunction:
$$\frac{\vartheta \vdash \varphi \qquad \vartheta \vdash \psi}{\vartheta \vdash \varphi \wedge \psi} \qquad \frac{\vartheta \vdash \varphi \wedge \psi}{\vartheta \vdash \psi} \qquad \frac{\vartheta \vdash \varphi \wedge \psi}{\vartheta \vdash \varphi}$$

Exercise 2.3.1. Derive symmetry and transitivity of equality:
$$\frac{\Gamma \mid \psi \vdash t =_A u}{\Gamma \mid \psi \vdash u =_A t} \qquad \frac{\Gamma \mid \psi \vdash t =_A u \qquad \Gamma \mid \psi \vdash u =_A v}{\Gamma \mid \psi \vdash t =_A v}$$

Example 2.3.2. The theory of a poset is a cartesian theory. There is one basic sort $\mathtt{P}$ and one binary relation symbol $\leq$ with signature $(\mathtt{P}, \mathtt{P})$. The axioms are the familiar axioms for reflexivity, transitivity, and antisymmetry:
$$x : \mathtt{P} \mid \cdot \vdash x \leq x$$
$$x : \mathtt{P}, y : \mathtt{P}, z : \mathtt{P} \mid x \leq y,\ y \leq z \vdash x \leq z$$
$$x : \mathtt{P}, y : \mathtt{P} \mid x \leq y,\ y \leq x \vdash x =_{\mathtt{P}} y$$

There are also many examples, such as ordered groups, ordered fields, etc., that are posets with further algebraic structure.

Example 2.3.3. An *equivalence relation* in a cartesian category is a model of the corresponding theory with one basic sort $\mathtt{A}$ and one binary relation symbol $\sim$ with signature $(\mathtt{A}, \mathtt{A})$. The axioms are the familiar axioms for reflexivity, symmetry, and transitivity:
$$x : \mathtt{A} \mid \cdot \vdash x \sim x$$
$$x : \mathtt{A}, y : \mathtt{A} \mid x \sim y \vdash y \sim x$$
$$x : \mathtt{A}, y : \mathtt{A}, z : \mathtt{A} \mid x \sim y \wedge y \sim z \vdash x \sim z$$

Semantics for cartesian logic

In order to give the semantics of cartesian logic, we require a couple of useful propositions regarding cartesian categories.

Proposition 2.3.4. *If a category $\mathcal{C}$ has pullbacks then, for every $A \in \mathcal{C}$, $\mathsf{Sub}(A)$ has finite limits. Moreover, these are stable under pullback.*

Proof. The poset $\mathsf{Sub}(A)$ has finite limits if it has a top object and binary meets. The top object of $\mathsf{Sub}(A)$ is the subobject $[1_A : A \to A]$. The meet of subobjects $i : I \rightarrowtail A$ and $j : J \rightarrowtail A$ is the subobject $i \wedge j = i \circ (i^* j) = j \circ (j^* i) : I \wedge J \rightarrowtail A$ obtained by pullback, as in the following diagram:

$$\begin{tikzcd}
I \wedge J \arrow[r, tail, "j^* i"] \arrow[d, tail, "i^* j"'] \arrow[dr, phantom, "\lrcorner", very near start] & J \arrow[d, tail, "j"] \\
I \arrow[r, tail, "i"'] & A
\end{tikzcd}$$

It is easy to verify that $I \wedge J$ is the infimum of I and J. Finally, stability follows from a familiar diagram chase on a cube of pullbacks. $\square$

Proposition 2.3.5. *If a category has finite products and pullbacks of monos along monos then it has all finite limits.*

Proof. It is sufficient to show that the category has equalizers. To construct the equalizer of parallel arrows $f : A \to B$ and $g : A \to B$, first observe that the arrows

$$A \xrightarrow{\langle 1_A, f \rangle} A \times B \qquad\qquad A \xrightarrow{\langle 1_A, g \rangle} A \times B$$

are monos because the projection $\pi_0 : A \times B \to A$ is their left inverse. Therefore, we may construct the pullback

$$\begin{tikzcd}
P \arrow[r, tail, "p"] \arrow[d, tail, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, tail, "\langle 1_A, f \rangle"] \\
A \arrow[r, tail, "\langle 1_A, g \rangle"'] & A \times B
\end{tikzcd}$$

The morphisms p and q coincide because $\langle 1_A, f \rangle$ and $\langle 1_A, g \rangle$ have a common left inverse π_0:

$$p = 1_A \circ p = \pi_0 \circ \langle 1_A, f \rangle \circ p = \pi_0 \circ \langle 1_A, f \rangle \circ q = 1_A \circ q = q \;.$$

Let us show that $p : P \to A$ is the equalizer of f and g. First, p equalizes f and g,

$$f \circ p = \pi_1 \circ \langle 1_A, f \rangle \circ p = \pi_1 \circ \langle 1_A, g \rangle \circ q = g \circ q = g \circ p \;.$$

If $k : K \to A$ also equalizes f and g then

$$\langle 1_A, f\rangle \circ k = \langle k, f \circ k\rangle = \langle k, g \circ k\rangle = \langle 1_A, g\rangle \circ k\ ,$$

therefore by the universal property of the constructed pullback there exists a unique factorization $\overline{k} : K \to P$ such that $k = p \circ \overline{k}$, as required. □

We now explain how cartesian logic is interpreted in a cartesian category $\mathcal{C}$ (i.e. $\mathcal{C}$ is finitely complete). Let $\mathbb{T}$ be a multi-sorted cartesian theory. Recall that the type theory of $\mathbb{T}$ is specified by a set of sorts (types) $A, \ldots$ and a set of basic function symbols $f, \ldots$ together with their signatures, while the logic is given by a set of basic relation symbols $R, \ldots$ with their signatures, and a set of axioms in the form of logical entailments between formulas in context,

$$\Gamma \mid \psi \vdash \varphi.$$

Definition 2.3.6. An *interpretation* of $\mathbb{T}$ in $\mathcal{C}$ is given by the following data, where Γ stands for a typing context $x_1 : A_1, \ldots, x_n : A_n$, and ψ and φ are formulas:

1. Each sort A is interpreted as an object $[\![A]\!]$, with the unit sort $\mathtt{1}$ being interpreted as the terminal object 1.
2. A typing context $x_1 : A_1, \ldots, x_n : A_n$ is interpreted as the product $[\![A_1]\!] \times \cdots \times [\![A_n]\!]$. The empty context is interpreted as the terminal object 1.
3. A basic function symbol f with signature $(A_1, \ldots, A_m; B)$ is interpreted as a morphism $[\![f]\!] : [\![A_1]\!] \times \cdots [\![A_m]\!] \to [\![B]\!]$.
4. A basic relation symbol R with signature $(A_1, \ldots, A_n)$ is interpreted as a subobject $[\![R]\!] \in \mathsf{Sub}([\![A_1]\!] \times \cdots \times [\![A_n]\!])$.

We then extend the interpretation to all terms and formulas as follows:

1. A term in context $\Gamma \mid t : B$ is interpreted as a morphism

$$[\![\Gamma \mid t : B]\!] : [\![\Gamma]\!] \to [\![B]\!]$$

according to the following specification.

- A variable $x_0 : A_1, \ldots, x_n : A_n \mid x_i : A_i$ is interpreted as the i-th projection $\pi_i : [\![A_1]\!] \times \cdots \times [\![A_n]\!] \to [\![A_i]\!]$.
- The interpretation of $\Gamma \mid * : \mathtt{1}$ is the unique morphism $!_{[\![\Gamma]\!]} : [\![\Gamma]\!] \to 1$.
- A composite term $\Gamma \mid f(t_1, \ldots, t_m) : B$, where f is a basic function symbol with signature $(A_1, \ldots, A_m; B)$, is interpreted as the composition

$$[\![\Gamma]\!] \xrightarrow{\langle [\![t_1]\!], \ldots, [\![t_m]\!]\rangle} [\![A_1]\!] \times \cdots \times [\![A_m]\!] \xrightarrow{[\![f]\!]} [\![B]\!]$$

Here $[\![t_i]\!]$ is shorthand for $[\![\Gamma \mid t_i : A_i]\!]$.

2. A formula in a context $\Gamma \mid \varphi$ is interpreted as a subobject $[\![\Gamma \mid \varphi]\!] \in \mathsf{Sub}([\![\Gamma]\!])$ according to the following specification.

   - The logical constant $\top$ is interpreted as the maximal subobject, represented by the identity arrow:
   $$[\![\Gamma \mid \top]\!] = [\, \mathbf{1}_{[\![\Gamma]\!]} : [\![\Gamma]\!] \to [\![\Gamma]\!] \,]$$
   - An atomic formula $\Gamma \mid R(t_1, \ldots, t_m)$, where R is a basic relation symbol with signature $(A_1, \ldots, A_m)$ is interpreted as the left-hand side of the pullback:
   $$\begin{tikzcd} {[\![\Gamma \mid R(t_1, \ldots, t_m)]\!]} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & {[\![R]\!]} \arrow[d, tail] \\ {[\![\Gamma]\!]} \arrow[r, "{\langle [\![t_1]\!], \ldots, [\![t_m]\!] \rangle}"'] & {[\![A_1]\!] \times \cdots \times [\![A_m]\!]} \end{tikzcd}$$
   - An equation $\Gamma \mid t =_A u$ `pred` is interpreted as the subobject represented by the equalizer of $[\![\Gamma \mid t : A]\!]$ and $[\![\Gamma \mid u : A]\!]$:
   $$\begin{tikzcd} {[\![\Gamma \mid t =_A u]\!]} \arrow[r, tail] & {[\![\Gamma]\!]} \arrow[r, shift left, "{[\![t]\!]}"] \arrow[r, shift right, "{[\![u]\!]}"'] & {[\![A]\!]} \end{tikzcd}$$
   - By Proposition 2.3.4, each $\mathsf{Sub}(A)$ is a poset with binary meets. Thus we interpret a conjunction $\Gamma \mid \varphi \wedge \psi$ `pred` as the meet of subobjects
   $$[\![\Gamma \mid \varphi \wedge \psi]\!] = [\![\Gamma \mid \varphi]\!] \wedge [\![\Gamma \mid \psi]\!] .$$
   - A formula formed by weakening is interpreted as pullback along a projection:
   $$\begin{tikzcd} {[\![\Gamma, x : A \mid \varphi]\!]} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & {[\![\Gamma \mid \varphi]\!]} \arrow[d, tail, "i"] \\ {[\![\Gamma]\!] \times [\![A]\!]} \arrow[r, "\pi"'] & {[\![\Gamma]\!]} \end{tikzcd}$$
   Computing this pullback one sees that the interpretation of $[\![\Gamma, x : A \mid \varphi]\!]$ turns out to be the subobject
   $$\begin{tikzcd} {[\![\Gamma \mid \varphi]\!] \times [\![A]\!]} \arrow[r, tail, "i \times \mathbf{1}_A"] & {[\![\Gamma]\!] \times [\![A]\!]} \end{tikzcd}$$

This concludes the definition of an interpretation of a cartesian theory $\mathbb{T}$ in a cartesian category $\mathcal{C}$.

As was explained in the previous section, the operation of substitution of terms into formulas is interpreted as pullback:

Lemma 2.3.7. *Let the formula* $\Gamma, x : A \mid \varphi$ *and the term* $\Gamma \mid t : A$ *be given. Then the substituted formula* $\Gamma \mid \varphi[t/x]$ *is interpreted as the pullback indicated in the following diagram:*

$$\begin{tikzcd}
\llbracket \Gamma \mid \varphi[t/x] \rrbracket \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \llbracket \Gamma, x : A \mid \varphi \rrbracket \arrow[d, tail] \\
\llbracket \Gamma \rrbracket \arrow[r, "\langle 1_{\llbracket \Gamma \rrbracket}, \llbracket t \rrbracket \rangle"'] & \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket
\end{tikzcd}$$

Proof. A simple induction on the structure of φ. We do the case where φ is an atomic formula $R(t_1, \ldots, t_m)$. Let $\Gamma = x_1 : A_1, \ldots, x_n : A_n$ and $\Gamma, x : A \mid t_i : B_i$ for $i = 1, \ldots, m$, where $(B_1, \ldots, B_m)$ is the signature of R. For the interpretation of $\Gamma, x : A \mid R(t_1, \ldots, t_m)$, by Definition 2.3.6 we have a pullback diagram:

$$\begin{tikzcd}
\llbracket \Gamma \mid R(t_1, \ldots, t_m) \rrbracket \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \llbracket R \rrbracket \arrow[d, tail] \\
\llbracket \Gamma, x : A \rrbracket \arrow[r, "\langle \llbracket t_1 \rrbracket, \ldots, \llbracket t_m \rrbracket \rangle"'] & \llbracket B_1 \rrbracket \times \cdots \times \llbracket B_m \rrbracket
\end{tikzcd}$$

Now suppose $\Gamma \mid t : A$, and consider the substitution

$$\Gamma \mid R(t_1, \ldots, t_m)[t/x] = \Gamma \mid R(t_1[t/x], \ldots, t_m[t/x])$$

For the interpretations of the substituted terms $t_i[t/x]$ we have the composites

$$\llbracket t_i[t/x] \rrbracket = \llbracket t_i \rrbracket \circ \langle 1_{\llbracket \Gamma \rrbracket}, \llbracket t \rrbracket \rangle : \llbracket \Gamma \rrbracket \longrightarrow \llbracket \Gamma, x : A \rrbracket \longrightarrow \llbracket B_i \rrbracket$$

by (associativity of composition and) the definition of the interpretation of terms. Thus for the interpretation of $\Gamma \mid R(t_1, \ldots, t_m)[t/x]$ we have the outer pullback rectangle below.

$$\begin{tikzcd}
\llbracket \Gamma \mid R(t_1, \ldots, t_m)[t/x] \rrbracket \arrow[r, dotted] \arrow[d, tail] \arrow[rr, bend left] & \llbracket \Gamma, x : A \mid R(t_1, \ldots, t_m) \rrbracket \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \llbracket R \rrbracket \arrow[d, tail] \\
\llbracket \Gamma \rrbracket \arrow[r, "\langle 1_{\llbracket \Gamma \rrbracket}, \llbracket t \rrbracket \rangle"'] \arrow[rr, bend right, "\langle \llbracket t_1[t/x] \rrbracket, \ldots, \llbracket t_m[t/x] \rrbracket \rangle"'] & \llbracket \Gamma, x : A \rrbracket \arrow[r, "\langle \llbracket t_1 \rrbracket, \ldots, \llbracket t_m \rrbracket \rangle"'] & \llbracket B_1 \rrbracket \times \cdots \times \llbracket B_m \rrbracket
\end{tikzcd}$$

But since the righthand square is a pullback, there is a unique dotted arrow as indicated. By the 2-pullbacks lemma, the lefthand square is then also a pullback, as required. □

Exercise 2.3.8. Complete the proof.

When we deal with several different interpretations at once we may name them M, N, ..., and subscript the semantic brackets accordingly, $[\![\Gamma]\!]_M$, $[\![\Gamma]\!]_N$, ...

Definition 2.3.9. If $\Gamma \mid \psi \vdash \psi$ is one of the logical entailment axioms of $\mathbb{T}$ and

$$[\![\Gamma \mid \psi]\!]_M \leq [\![\Gamma \mid \varphi]\!]_M$$

holds in an interpretation M, then we say that M *satisfies* or *models* $\Gamma \mid \psi \vdash \varphi$ and write

$$M \models \ \Gamma \mid \psi \vdash \varphi \ .$$

An interpretation M is a *model* of $\mathbb{T}$ if it satisfies all the axioms of $\mathbb{T}$.

Theorem 2.3.10 (Soundness of cartesian logic)**.** *If a cartesian theory* $\mathbb{T}$ *proves an entailment*

$$\Gamma \mid \psi \vdash \varphi$$

then every model M *of* $\mathbb{T}$ *satisfies the entailment:*

$$M \models \ \Gamma \mid \psi \vdash \varphi \ .$$

Proof. The proof proceeds by induction on the proof of the entailment. In the following we often omit the typing context Γ to simplify notation, and all inequalities are interpreted in $\mathsf{Sub}([\![\Gamma]\!])$. We consider all possible last steps in the proof of the entailment:

1. Weakening: if $[\![\Gamma \mid \psi]\!] \leq [\![\Gamma \mid \varphi]\!]$ in $\mathsf{Sub}([\![\Gamma]\!])$ then

$$[\![\Gamma, x : A \mid \psi]\!] = [\![\Gamma \mid \psi]\!] \times A \leq [\![\Gamma \mid \varphi]\!] \times A = [\![\Gamma, x : A \mid \varphi]\!] \quad \text{in } \mathsf{Sub}([\![\Gamma, x : A]\!]).$$

2. Substitution: by lemma 2.3.7, substitution is interpreted by pullback so that $[\![\varphi[t/x]]\!] = \langle 1_{[\![\psi]\!]}, [\![t]\!]\rangle^* [\![\varphi]\!]$ and $[\![\psi[t/x]]\!] = \langle 1_{[\![\psi]\!]}, [\![t]\!]\rangle^* [\![\psi]\!]$. Because

$$\langle 1_{[\![\psi]\!]}, [\![t]\!]\rangle^* : \mathsf{Sub}([\![\psi]\!]) \to \mathsf{Sub}([\![\psi]\!] \times [\![A]\!])$$

is a functor it is a monotone map, therefore $[\![\psi]\!] \leq [\![\varphi]\!]$ implies

$$\langle 1_{[\![\psi]\!]}, [\![t]\!]\rangle^* [\![\psi]\!] \leq \langle 1_{[\![\psi]\!]}, [\![t]\!]\rangle^* [\![\varphi]\!] \ .$$

3. Identity: trivially

$$[\![\varphi]\!] \leq [\![\varphi]\!] \ .$$

4. Cut: if $[\![\psi]\!] \leq [\![\theta]\!]$ and $[\![\theta]\!] \leq [\![\varphi]\!]$ then clearly $[\![\psi]\!] \leq [\![\varphi]\!]$, since $\mathsf{Sub}([\![\Gamma, x : A]\!])$ is a poset.

5. Truth: trivially $\llbracket \psi \rrbracket \leq \llbracket \top \rrbracket$.

6. The rules for conjunction clearly hold because by the definition of infimum $\llbracket \vartheta \rrbracket \leq \llbracket \varphi \wedge \psi \rrbracket$ if, and only if, $\llbracket \vartheta \rrbracket \leq \llbracket \varphi \rrbracket$ and $\llbracket \vartheta \rrbracket \leq \llbracket \psi \rrbracket$.

7. Equality: the axiom $t =_A t$ is satisfied because an equalizer of $\llbracket t \rrbracket$ with itself is the maximal subobject:
$$\llbracket \psi \rrbracket \leq [1_{\llbracket \Gamma \rrbracket} : \llbracket \Gamma \rrbracket \to \llbracket \Gamma \rrbracket] = \llbracket t =_A t \rrbracket \,.$$
For the other axiom, suppose $\llbracket \psi \rrbracket \leq \llbracket t =_A u \rrbracket$ and $\llbracket \psi \rrbracket \leq \llbracket \varphi[t/z] \rrbracket$. It suffices to show $\llbracket t =_A u \rrbracket \wedge \llbracket \varphi[t/z] \rrbracket \leq \llbracket \varphi[u/z] \rrbracket$ for then
$$\llbracket \psi \rrbracket \leq \llbracket t =_A u \rrbracket \wedge \llbracket \varphi[t/z] \rrbracket \leq \llbracket \varphi[u/z] \rrbracket \,.$$
The interpretation of $P = \llbracket t =_A u \rrbracket \wedge \llbracket \varphi[t/z] \rrbracket$ is obtained by two successive pullbacks, as in the following diagram:
$$\begin{tikzcd}
P \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \llbracket \varphi[t/z] \rrbracket \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \llbracket \varphi \rrbracket \arrow[d, tail] \\
\llbracket t =_A u \rrbracket \arrow[r, tail, "e"'] & \llbracket \Gamma \rrbracket \arrow[r, "\langle 1_\Gamma, \llbracket t \rrbracket \rangle"'] & \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket
\end{tikzcd}$$
Here e is the equalizer of $\llbracket u \rrbracket$ and $\llbracket t \rrbracket$. Observe that e equalizes $\langle 1_{\llbracket \Gamma \rrbracket}, \llbracket t \rrbracket \rangle$ and $\langle 1_{\llbracket \Gamma \rrbracket}, \llbracket u \rrbracket \rangle$ as well:
$$\langle 1_{\llbracket \Gamma \rrbracket}, \llbracket t \rrbracket \rangle \circ e = \langle e, \llbracket t \rrbracket \circ e \rangle = \langle e, \llbracket u \rrbracket \circ e \rangle = \langle 1_{\llbracket \Gamma \rrbracket}, \llbracket u \rrbracket \rangle \circ e \,.$$
Therefore, if we replace $\langle 1_{\llbracket \Gamma \rrbracket}, \llbracket t \rrbracket \rangle$ with $\langle 1_{\llbracket \Gamma \rrbracket}, \llbracket u \rrbracket \rangle$ in the above diagram, the outer rectangle still commutes. By the universal property of the pullback
$$\begin{tikzcd}
\llbracket \varphi[u/z] \rrbracket \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \llbracket \varphi \rrbracket \arrow[d, tail] \\
\llbracket \Gamma \rrbracket \arrow[r, "\langle 1_\Gamma, \llbracket u \rrbracket \rangle"'] & \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket
\end{tikzcd}$$
it follows that P also factors through $\llbracket \varphi[u/z] \rrbracket$, as required.

□

Example 2.3.11. Recall the cartesian theory of posets (example 2.3.2). There is one basic sort `P` and one binary relation symbol $\leq$ with signature $(\mathtt{P}, \mathtt{P})$ and the axioms of reflexivity, transitivity, and antisymmetry. A poset in a cartesian category $\mathcal{C}$ is thus given by an object P, which is the interpretation of the sort `P`, and a subobject $r : R \rightarrowtail P \times P$, which the interpretation of $\leq$, such that the axioms are satisfied. As an example we spell out when the reflexivity axiom is satisfied. The interpretation of $x : P \mid x \leq x$ is obtained by the following pullback:

$$\begin{tikzcd}
[\![x \leq x]\!] \arrow[r] \arrow[d, tail] & R \arrow[d, tail, "r"] \\
P \arrow[r, "\Delta"'] \arrow[ur, dotted, "\rho"] & P \times P
\end{tikzcd}$$

where $\Delta = \langle 1_P, 1_P \rangle$ is the diagonal. The first axiom is satisfied when $[\![x \leq x]\!] = 1_P$, which happens if, and only if, Δ factors through r, as indicated. Therefore, reflexivity can be expressed as follows: there exists a "reflexivity" morphism $\rho : P \to R$ such that $r \circ \rho = \Delta$. Equivalently, the morphisms $\pi_0 \circ r$ and $\pi_1 \circ r$ have a common right inverse ρ.

As an example, of a poset in a cartesian category other than Set, observe that since the definition is stated entirely in terms of finite limits, and these are computed pointwise in functor categories $\mathsf{Set}^{\mathbb{C}}$, it follows that a poset P in $\mathsf{Set}^{\mathbb{C}}$ is the same thing as a functor $P : \mathbb{C} \to \mathsf{Poset}$. Indeed, as was the case for algebraic theories, we have an equivalence (an isomorphism, actually) of categories,

$$\mathsf{Poset}(\mathsf{Set}^{\mathbb{C}}) \;\cong\; \mathsf{Poset}(\mathsf{Set})^{\mathbb{C}} \;\cong\; \mathsf{Poset}^{\mathbb{C}}.$$

2.3.1 Subtypes

Let us consider whether the theory of a category is a cartesian theory. We begin by expressing the definition of a category so that it can be interpreted in any cartesian category $\mathcal{C}$. An *internal category* in $\mathcal{C}$ consists of an *object of morphisms* C_1, an *object of objects* C_0, and *domain*, *codomain*, and *identity* morphisms,

$$\mathsf{dom} : C_1 \to C_0 \;, \qquad \mathsf{cod} : C_1 \to C_0 \;, \qquad \mathsf{id} : C_0 \to C_1 \;.$$

There is also a *composition* morphism $c : C_2 \to C_1$, where C_2 is obtained by the pullback

$$\begin{tikzcd}
C_2 \arrow[r, "p_1"] \arrow[d, "p_0"'] & C_1 \arrow[d, "\mathsf{dom}"] \\
C_1 \arrow[r, "\mathsf{cod}"'] & C_0
\end{tikzcd}$$

The following equations must hold:

$$\begin{gathered}
\mathsf{dom} \circ i = 1_{C_0} = \mathsf{cod} \circ i \;, \\
\mathsf{cod} \circ p_1 = \mathsf{cod} \circ c \;, \qquad \mathsf{dom} \circ p_0 = \mathsf{dom} \circ c \;. \\
c \circ \langle 1_{C_1}, i \circ \mathsf{dom} \rangle = 1_{C_1} = c \circ \langle i \circ \mathsf{cod}, 1_{C_1} \rangle \;,
\end{gathered}$$

The first two equations state that the domain and codomain of an identity morphism 1_A are both A. The second equation states that $\mathsf{cod}\,(f \circ g) = \mathsf{cod}\, f$ and the third one that

$\mathsf{dom}\,(f \circ g) = \mathsf{dom}\, g$. The fourth equation states that $f \circ 1_{\mathsf{dom}\, f} = f = 1_{\mathsf{cod}\, f} \circ f$. It remains to express associativity of composition. For this purpose we construct the pullback

$$\begin{tikzcd} C_3 \arrow[r, "q_2"] \arrow[d, "q_{01}"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \arrow[d, "\mathsf{dom}"] \\ C_2 \arrow[r, "\mathsf{cod} \circ p_1"'] & C_0 \end{tikzcd}$$

The object C_3 can be thought of as the set of triples of morphisms (f, g, h) such that $\mathsf{cod}\, f = \mathsf{dom}\, g$ and $\mathsf{cod}\, g = \mathsf{dom}\, h$. We denote $q_0 = p_0 \circ q_{01}$ and $q_1 = p_1 \circ q_{01}$. The morphisms $q_0, q_1, q_2 : C_3 \to C_1$ are like three projections which select the first, second, and third element of a triple, respectively. With this notation we can write $q_{01} = \langle q_0, q_1 \rangle_{C_2}$ because q_{01} is the unique morphism such that $p_0 \circ q_{01} = q_0$ and $p_1 \circ q_{01} = q_1$. The subscript C_2 reminds us that the "pair" $\langle q_0, q_1 \rangle_{C_2}$ is obtained by the universal property of the pullback C_2.

Morphisms $c \circ q_{01} : C_3 \to C_1$ and $q_2 : C_3 \to C_1$ factor through the pullback C_2 because

$$\mathsf{cod} \circ c \circ q_{01} = \mathsf{cod} \circ p_1 \circ q_0 = \mathsf{dom} \circ q_2 \;.$$

Thus let $r : C_3 \to C_2$ be the unique factorization for which $p_0 \circ r = c \circ q_{01}$ and $p_1 \circ r = q_2$. Because p_0 and p_1 are like projections from C_2 to C_1, morphism r can be thought of as a pair of morphisms, so we write $r = \langle c \circ q_{01}, q_2 \rangle_{C_2}$. Morphism $c \circ \langle c \circ q_{01}, q_2 \rangle_{C_2} : C_3 \to C_1$ corresponds to the operations $\langle f, g, h \rangle \mapsto (f, g) \circ h$, whereas the morphism corresponding to $\langle f, g, h \rangle \mapsto f \circ (g \circ h)$ is obtained in a similar way and is equal to

$$c \circ \langle q_0, c \circ \langle q_1, q_2 \rangle_{C_2} \rangle_{C_2} : C_3 \to C_1 \;.$$

Thus associativity is expressed by the equation

$$c \circ \langle c \circ \langle q_0, q_1 \rangle_{C_2}, q_2 \rangle_{C_2} = c \circ \langle q_0, c \circ \langle q_1, q_2 \rangle_{C_2} \rangle_{C_2} \;.$$

Example 2.3.12. An internal category in Set is a small category.

Example 2.3.13. An internal category in $\mathsf{Set}^{\mathbb{C}}$ is a functor $\mathbb{C} \to \mathsf{Cat}$. Indeed, as in previous examples of cartesian theories we have an equivalence of categories,

$$\mathsf{Cat}(\mathsf{Set}^{\mathbb{C}}) \;\cong\; \mathsf{Cat}(\mathsf{Set})^{\mathbb{C}} \;\cong\; \mathsf{Cat}^{\mathbb{C}}.$$

We have successfully formulated the theory of a category so that it makes sense in any cartesian category. In fact, the definition of an internal category refers only to certain pullbacks, hence the notion of an internal category makes sense in any category with pullbacks. However, if we try to formulate it as a multi-sorted cartesian theory, there is a problem. Obviously, there ought to be a basic sort of objects $\mathtt{C}_0$ and a basic sort of morphisms $\mathtt{C}_1$. There are also basic function symbols with signatures

$$\mathtt{dom} : (\mathtt{C}_1; \mathtt{C}_0) \qquad \mathtt{cod} : (\mathtt{C}_1; \mathtt{C}_0) \qquad \mathtt{id} : (\mathtt{C}_0, \mathtt{C}_1)\ .$$

However, it is not clear what the signature for composition should be. It is not $(\mathtt{C}_1, \mathtt{C}_1; \mathtt{C}_1)$ because composition is undefined for non-composable pairs of morphisms. We might be tempted to postulate another basic sort $\mathtt{C}_2$ but then we would have no way of stating that $\mathtt{C}_2$ is the pullback of $\mathtt{dom}$ and $\mathtt{cod}$. And even if we somehow axiomatized the fact that $\mathtt{C}_2$ is a pullback, we would then still have to formalize the object $\mathtt{C}_3$ of composable triples, $\mathtt{C}_4$ of composable quadruples, and so on. What we lack is the ability to define the type $\mathtt{C}_2$ as a *subtype* of $\mathtt{C}_1 \times \mathtt{C}_1$.

One way to remedy the situation is to use a richer underlying type theory; in Chapter **??** we will consider the system of *dependent type theory*, which provides the means to capture such notions as the theory of categories (and much more). Here we consider a small step in that direction, namely *simple subtypes*. The formation rule for simple subtypes is

$$\frac{x : A \mid \varphi\ \mathtt{pred}}{\{x : A \mid \varphi\}\ \mathtt{type}}$$

We can think of $\{x : A \mid \varphi\}$ as the subobject of all those $x : A$ that satisfy φ. Note that we did not allow an arbitrary context Γ to be present. This means that we cannot define subtypes that depend on parameters, which why they are called "simple".

Inference rules for subtypes are as follows:

$$\frac{\Gamma \mid t : \{x : A \mid \varphi\}}{\Gamma \mid \mathtt{in}_\varphi\, t : A} \qquad \frac{\Gamma \mid t : \{x : A \mid \varphi\}}{\Gamma \mid \cdot \vdash \varphi[\mathtt{in}_\varphi\, t/x]} \qquad \frac{\Gamma \mid t : A \qquad \Gamma \mid \cdot \vdash \varphi[t/x]}{\Gamma \mid \mathtt{rs}_\varphi\, t : \{x : A \mid \varphi\}}$$

$$\frac{\Gamma, x : A \mid \varphi, \psi \vdash \theta}{\overline{\Gamma, y : \{x : A \mid \varphi\} \mid \psi[\mathtt{in}_\varphi\, y/x] \vdash \theta[\mathtt{in}_\varphi\, y/x]}}$$

The first rule states that a term t of subtype $\{x : A \mid \varphi\}$ can be converted to a term $\mathtt{in}_\varphi\, t$ of type A. We can think of the constant $\mathtt{in}_\varphi$ as the *inclusion* $\mathtt{in}_\varphi : \{x : A \mid \varphi\} \to A$. The second rule states that every term of a subtype $\{x : A \mid \varphi\}$ satisfies the defining predicate φ. The third rule states that a term t of type A which satisfies φ can be converted to a term $\mathtt{rs}_\varphi\, t$ of type $\{x : A \mid \varphi\}$. A good way to think of the constant $\mathtt{rs}_\varphi$ is as a partially defined *restriction*, or a type-casting operations, $\mathtt{rs}_\varphi : A \rightharpoonup \{x : A \mid \varphi\}$.[4] The last rule tells us how to replace a variable x of type A and an assumption φ about it with a variable y of type $\{x : A \mid \varphi\}$ and remove the assumption. Note that this is a two-way rule.

There are two more axioms that relate inclusions and restrictions:

$$\frac{\Gamma \mid t : \{x : A \mid \varphi\}}{\Gamma \mid \cdot \vdash \mathtt{rs}_\varphi\, (\mathtt{in}_\varphi\, t) = t} \qquad \frac{\Gamma \mid t : A \qquad \Gamma \mid \cdot \vdash \varphi[t/x]}{\Gamma \mid \cdot \vdash \mathtt{in}_\varphi\, (\mathtt{rs}_\varphi\, t) = t\ .}$$

[4]Inclusions and restrictions are like type-casting operations in some programming languages. For example in Java, an inclusion corresponds to an (implicit) type cast from a class to its superclass, whereas a restriction corresponds to a type cast from a class to a subclass. Must I write that Java is a registered trademark of Sun Microsystems?

In an informal discussion it is customary for the inclusions and restrictions to be omitted, or at least for the subscript φ to be missing.[5]

Exercise 2.3.14. Suppose $x : A \mid \psi$ and $x : A \mid \varphi$ are formulas. Show that

$$x : A \mid \psi \vdash \varphi$$

is provable if, and only if, $\{x : A \mid \psi\}$ factors through $\{x : A \mid \varphi\}$, which means that there exists a term k,

$$y : \{x : A \mid \psi\} \mid k : \{x : A \mid \varphi\} \ ,$$

such that

$$y : \{x : A \mid \psi\} \mid \cdot \vdash \mathtt{in}_\psi\, y =_A \mathtt{in}_\varphi\, k$$

is provable. Show also that k is determined uniquely up to provable equality.

Example 2.3.15. We are now able to formulate the theory of a category as a cartesian theory whose underlying type theory has product types and subset types. The basic types are the type of objects $\mathtt{C}_0$ and the type of morphisms $\mathtt{C}_1$. We define the type $\mathtt{C}_2$ to be

$$\mathtt{C}_2 \equiv \{p : C_1 \times C_1 \mid \mathtt{cod}(\mathtt{fst}\, p) = \mathtt{dom}(\mathtt{snd}\, p)\} \ .$$

The basic function symbols and their signatures are:

$$\mathtt{dom} : \mathtt{C}_1 \to \mathtt{C}_0 \ , \qquad \mathtt{cod} : \mathtt{C}_1 \to \mathtt{C}_0 \ , \qquad \mathtt{id} : \mathtt{C}_0 \to \mathtt{C}_1 \ , \qquad \mathtt{c} : \mathtt{C}_2 \to \mathtt{C}_1 \ .$$

The axioms are:

$$\begin{aligned}
a : \mathtt{C}_0 \mid \cdot &\vdash \mathtt{dom}(\mathtt{id}(a)) = a \\
a : \mathtt{C}_0 \mid \cdot &\vdash \mathtt{cod}(\mathtt{id}(a)) = a \\
f : \mathtt{C}_1, g : \mathtt{C}_1 \mid \mathtt{cod}(f) = \mathtt{dom}(g) &\vdash \mathtt{dom}(\mathtt{c}(\mathtt{rs}\,\langle f, g\rangle)) = f \\
f : \mathtt{C}_1, g : \mathtt{C}_1 \mid \mathtt{cod}(f) = \mathtt{dom}(g) &\vdash \mathtt{cod}(\mathtt{c}(\mathtt{rs}\,\langle f, g\rangle)) = g \\
f : \mathtt{C}_1 \mid \cdot &\vdash \mathtt{c}(\mathtt{rs}\,\langle \mathtt{id}(\mathtt{dom}(f)), f\rangle) = f \\
f : \mathtt{C}_1 \mid \cdot &\vdash \mathtt{c}(\mathtt{rs}\,\langle f, \mathtt{id}(\mathtt{cod}(f))\rangle) = f
\end{aligned}$$

Lastly, the associativity axiom is

$$\begin{aligned}
&f : \mathtt{C}_1, g : \mathtt{C}_1, h : \mathtt{C}_1 \mid \mathtt{cod}(f) = \mathtt{dom}(g), \mathtt{cod}(g) = \mathtt{dom}(h) \vdash \\
&\qquad\qquad \mathtt{c}(\mathtt{rs}\,\langle \mathtt{c}(\mathtt{rs}\,\langle f, g\rangle), h\rangle) = \mathtt{c}(\mathtt{rs}\,\langle f, \mathtt{c}(\mathtt{rs}\,\langle g, h\rangle)\rangle) \ .
\end{aligned}$$

This notation is quite unreadable. If we write $g \circ f$ instead of $\mathtt{c}(\mathtt{rs}\,\langle f, g\rangle)$ then the axioms take on a more familiar form. For example, associativity is just $h \circ (g \circ f) = (h \circ g) \circ f$. However, we need to remember that we may form the term $g \circ f$ only if we first prove $\mathtt{dom}(g) = \mathtt{cod}(f)$.

[5] Strictly speaking, even the notation $\mathtt{in}_\varphi\, t$ is imprecise because it does not indiciate that ϕ stands in the context $x : A$. The correct notation would be $\mathtt{in}_{(x:A\mid\varphi)}\, t$, where x is bound in the subscript. A similar remark holds for $\mathtt{rs}_\varphi\, t$.

A subtype $\{x : A \mid \varphi\}$ is interpreted as the domain of a monomorphism representing $x : A \mid \varphi$:

$$[\![\{x : A \mid \varphi\}]\!] \rightarrowtail^{[\![x : A \mid \varphi]\!]} [\![A]\!]$$

Some care must be taken here because monos representing a given subobject are only determined up to isomorphism. We assume that a suitable canonical choice of monos can be made.

An inclusion $\Gamma \mid \mathtt{in}_\varphi\, t : A$ is interpreted as the composition

$$[\![\Gamma]\!] \xrightarrow{[\![t]\!]} [\![\{x : A \mid \varphi\}]\!] \rightarrowtail^{[\![x : A \mid \varphi]\!]} [\![A]\!]$$

A restriction $\Gamma \mid \mathtt{rs}_\varphi\, t : \{x : A \mid \varphi\}$ is interpreted as the unique $\overline{[\![t]\!]}$ which makes the following diagram commute:

$$\begin{tikzcd} {[\![\Gamma]\!]} \arrow[r, "\overline{[\![t]\!]}"] \arrow[dr, "{[\![t]\!]}"'] & {[\![x : A \mid \varphi]\!]} \arrow[d, tail] \\ & {[\![A]\!]} \end{tikzcd}$$

Exercise 2.3.16. Formulate and prove a soundness theorem for subtypes. Pay attention to the interpretation of restrictions, where you need to show unique existence of $\overline{[\![t]\!]}$.

Remark 2.3.17. Another approach to the logic of cartesian categories that captures the theory of categories and related notions involving partial operations is that of *essentially algebraic theories*, due to P. Freyd; see [Fre72, PV07]. A third approach is that of *dependent type theory* to be developed in **??** below. Finally, we will see in Section 2.5.3 that the theory of categories can be formulated as a *regular theory*.

2.4 Quantifiers as adjoints

The categorical semantics of quantification is one of the central features of the subject, and quite possibly one of the nicest contributions of categorical logic to the field of logic. You might expect that the quantifiers $\forall$ and $\exists$ are "just a big conjunction and disjunction", respectively. In fact the Polish school of algebraic logicians worked to realize this point of view—but categorical logic shows how quantifiers are treated algebraically as adjoint functors to give a much more satisfactory theory. The original treatment can be found in the classic paper [Law69].

Let us first recall the rules of inference for quantifiers. The formation rules are:

$$\frac{\Gamma, x : A \mid \varphi\ \mathtt{pred}}{\Gamma \mid (\forall\, x : A\,.\,\varphi)\ \mathtt{pred}} \qquad \frac{\Gamma, x : A \mid \varphi\ \mathtt{pred}}{\Gamma \mid (\exists\, x : A\,.\,\varphi)\ \mathtt{pred}}$$

The variable x is bound in $\forall\, x : A \,.\, \varphi$ and $\exists\, x : A \,.\, \varphi$. If x and y are distinct variables and x does not occur freely in the term t then substitution of t for y commutes with quantification over x:

$$\begin{aligned}(\exists\, x : A \,.\, \varphi)[t/y] &= \exists\, x : A \,.\, (\varphi[t/y]) \ , \\ (\forall\, x : A \,.\, \varphi)[t/y] &= \forall\, x : A \,.\, (\varphi[t/y]) \ .\end{aligned}$$

For each quantifier we have a two-way rule of inference:

$$\frac{\Gamma, x : A \mid \psi \vdash \varphi}{\Gamma \mid \psi \vdash \forall\, x : A \,.\, \varphi} \qquad\qquad \frac{\Gamma, x : A \mid \varphi \vdash \vartheta}{\Gamma \mid (\exists\, x : A \,.\, \varphi) \vdash \vartheta}$$

Note that these rules implicitly impose the usual condition that x must not occur freely in ψ and ϑ, because ψ and ϑ are supposed to be well formed in context Γ, which does not contain x.

Exercise 2.4.1. A common way of stating the inference rules for quantifiers is as follows. For the universal quantifier, the introduction and elimination rules are

$$\frac{\Gamma, x : A \mid \psi \vdash \varphi}{\Gamma \mid \psi \vdash \forall\, x : A \,.\, \varphi} \qquad\qquad \frac{\Gamma \mid t : A \qquad \Gamma \mid \psi \vdash \forall\, x : A \,.\, \varphi}{\Gamma \mid \psi \vdash \varphi[t/x]}$$

The introduction rule for existential quantifier is

$$\frac{\Gamma \mid t : A \qquad \Gamma \mid \psi \vdash \varphi[t/x]}{\Gamma \mid \psi \vdash \exists\, x : A \,.\, \varphi}$$

and the elimination rule is

$$\frac{\Gamma \mid \psi \vdash \exists\, x : A \,.\, \varphi \qquad \Gamma, x : A \mid \varphi \vdash \vartheta}{\Gamma \mid \psi \vdash \vartheta}$$

Note that these rules implicitly impose a requirement that x does not occur in Γ and that it does not occur freely in ψ because the context $\Gamma, x : A$ must be well formed and the hypotheses ψ must be well formed in context Γ. Show that these rules can be derived from the ones above, and vice versa. Of course, you may also use the inference rules for cartesian logic, cf. page 55.

In order to discover what the semantics of existential quantifier ought to be, we look at the following instance of the two-way rule for quantifiers:

$$\frac{y : B, x : A \mid \varphi \vdash \vartheta}{y : B \mid \exists\, x : A \,.\, \varphi \vdash \vartheta} \tag{2.2}$$

First observe that this rule implicitly requires

$$y : B, x : A \mid \varphi \ \mathtt{pred} \qquad\qquad y : B \mid \vartheta \ \mathtt{pred} \qquad\qquad y : B \mid (\exists\, x : A \,.\, \varphi) \ \mathtt{pred}$$

This is required for the entailments to be well-formed. The fourth judgement

$$y : B, x : A \mid \vartheta \ \texttt{pred}$$

follows from the second one above by weakening,

$$\frac{y : B \mid \vartheta \ \texttt{pred}}{y : B, x : A \mid \vartheta \ \texttt{pred}}$$

The interpretations of φ, ϑ, and $\exists\, x : A \,.\, \varphi$ are therefore subobjects

$$\begin{aligned}
[\![y : B, x : A \mid \varphi]\!] &\in \mathsf{Sub}([\![B]\!] \times [\![A]\!]) \,, \\
[\![y : B \mid \vartheta]\!] &\in \mathsf{Sub}([\![B]\!]) \,, \\
[\![y : B \mid \exists\, x : A \,.\, \varphi]\!] &\in \mathsf{Sub}([\![B]\!]) \,.
\end{aligned}$$

And the weakened instance of ϑ in the context $y : B, x : A$ is interpreted by pullback along a projection, cf. page 59, as in the following pullback diagram:

$$\begin{tikzcd}
{[\![y : B, x : A \mid \vartheta]\!]} \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[\![B]\!] \times [\![A]\!]} \arrow[d, "\pi"] \\
{[\![y : B \mid \vartheta]\!]} \arrow[r, tail] & {[\![B]\!]}
\end{tikzcd}$$

Thus we have

$$[\![y : B, x : A \mid \vartheta]\!] = \pi^* [\![y : B \mid \vartheta]\!] \,,$$

with weakening interpreted as the pullback functor

$$\pi^* : \mathsf{Sub}([\![B]\!]) \to \mathsf{Sub}([\![B]\!] \times [\![A]\!]) \,.$$

We will interpret existential quantification $\exists x : A$ as a suitable functor

$$\exists_A : \mathsf{Sub}([\![B]\!] \times [\![A]\!]) \to \mathsf{Sub}([\![B]\!])$$

so that

$$[\![y : B \mid \exists\, x : A \,.\, \varphi]\!] = \exists_A [\![y : B, x : A \mid \varphi]\!] \,.$$

The interpretation of the two-way rule (2.2) then becomes a two-way inequality rule

$$\frac{[\![y : B, x : A \mid \varphi]\!] \leq \pi^* [\![y : B \mid \vartheta]\!]}{\exists_A [\![y : B, x : A \mid \varphi]\!] \leq [\![y : B \mid \vartheta]\!]}$$

Replacing the interpretations of φ and ϑ by general subobjects $S \in \mathsf{Sub}([\![B]\!] \times [\![A]\!])$ and $T \in \mathsf{Sub}([\![B]\!])$, we obtain the more suggestive formulation

$$\frac{S \leq \pi^* T}{\exists_A S \leq T} \tag{2.3}$$

This is nothing but an adjunction between $\exists_A$ and π^*! Indeed, the operations $\exists_A$ and π^* are functors on the subjects posets $\mathsf{Sub}(\llbracket B \rrbracket \times \llbracket A \rrbracket)$ and $\mathsf{Sub}(\llbracket B \rrbracket)$, and the bijection of hom-sets (2.3) is exactly the statement of an adjunction between them. Thus *existential quantification is left-adjoint to weakening*:

$$\exists_A \dashv \pi^*$$

An exactly dual argument shows that *universal quantification is right-adjoint to weakening*:

$$\pi^* \dashv \forall_A$$

Thus, in sum, we have shown that the rules of inference require the quantifiers to be interpreted as operations that are adjoints to the interpretation of weakening, i.e. pullback π^* along the projection $\pi : \llbracket B \rrbracket \times \llbracket A \rrbracket \to \llbracket B \rrbracket$.

$$\begin{tikzcd}
\mathsf{Sub}(\llbracket B \rrbracket \times \llbracket A \rrbracket) \arrow[d, bend right=70, "\exists"'] \arrow[d, bend left=70, "\forall"] \\
\mathsf{Sub}(\llbracket B \rrbracket) \arrow[u, "\dashv \quad \pi^* \quad \dashv"{description}]
\end{tikzcd}$$

Note that the familiar side-conditions on the conventional rules for the quantifiers, to the effect that "x cannot occur freely in ψ", etc., which may seem like tiresome bookkeeping, are actually of the essence, since they express the weakening operation to which the quantifiers themselves are adjoints.

Let us see how this works for the usual interpretation in Set. A predicate $y : B, x : A \mid \varphi$ corresponds to a subset $\Phi \subseteq B \times A$, and $y : B \mid \vartheta$ corresponds to a subset $\Theta \subseteq B$. Weakening of Θ is the subset $\pi^*\Theta = \Theta \times A \subseteq B \times A$. Then we have

$$\begin{aligned}
\exists_A \Phi &= \big\{ y \in B \bigm| \exists\, x : A \,.\, \langle x, y \rangle \in \Phi \big\} \subseteq B \;, \\
\forall_A \Phi &= \big\{ y \in B \bigm| \forall\, x : A \,.\, \langle x, y \rangle \in \Phi \big\} \subseteq B \;.
\end{aligned}$$

A moment's thought convinces us that with this interpretation we do indeed have

$$\frac{\Phi \subseteq \Theta \times A}{\exists_A \Phi \subseteq \Theta} \qquad\qquad \frac{\Theta \times A \subseteq \Phi}{\Theta \subseteq \forall_A \Phi}$$

The unit of the adjunction $\exists_A \dashv \pi^*$ amounts to the inequality

$$\Phi \subseteq (\exists_A \Phi) \times A \;, \tag{2.4}$$

and the universal property of the unit says that $\exists_A \Phi$ is the smallest set satisfying (2.4). Similarly, the counit of the adjunction $\pi^* \dashv \forall_A$ is just the inequality

$$(\forall_A \Phi) \times A \subseteq \Phi \;, \tag{2.5}$$

and the universal property of the counit says that $\forall_A \Phi$ is the largest set satisfying (2.5). Figure 2.1 shows the geometric meaning of existential and universal quantification.

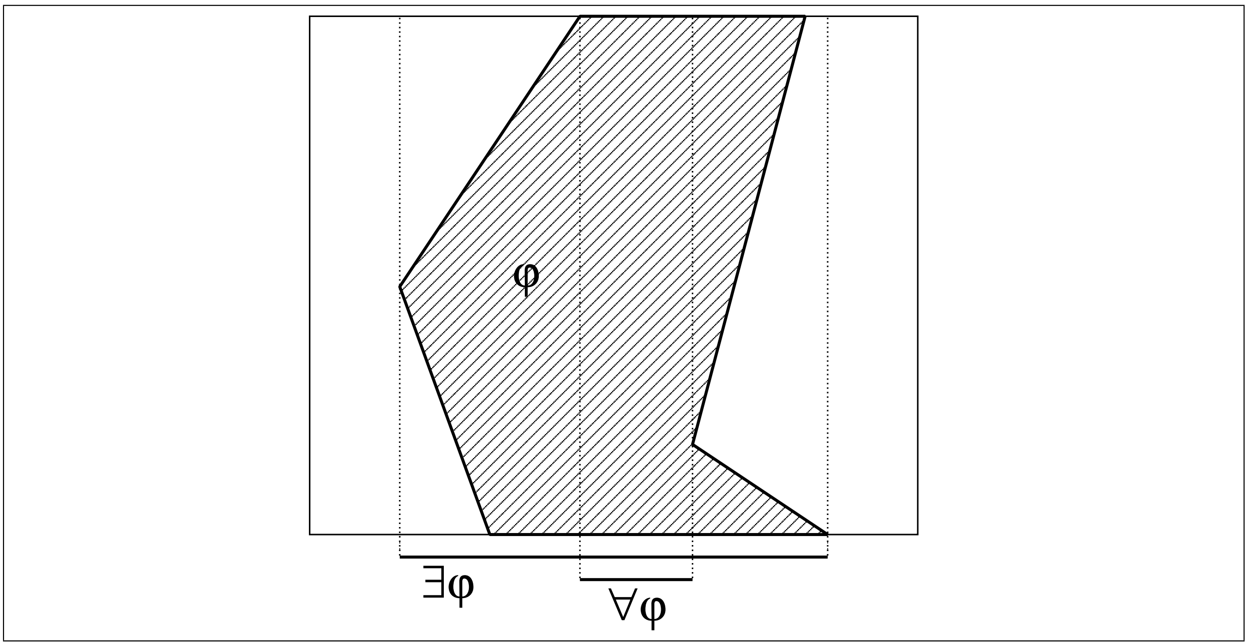

Figure 2.1: $\exists\varphi$ and $\forall\varphi$

Exercise 2.4.2. What do the universal properties of the counit of $\exists_A \dashv \pi^*$ and the unit of $\pi^* \dashv \forall_A$ say?

The weakening functor π^* is a special case of a pullback functor $f^* : \mathsf{Sub}(B) \to \mathsf{Sub}(A)$ for a morphism $f : B \to A$. This gives us the idea that we may regard the left and the right adjoint to f^* as a kind of generalized existential and universal quantifier.

We may also be tempted to *define* quantifiers as left and right adjoints to pullback functors. However there is a bit more to quantifiers than that—we are still missing the important *Beck-Chevalley condition.*

2.4.1 The Beck-Chevalley condition

Recall that quantification commutes with substitution, as long as no variables are captured by the quantifier. Thus if $\Gamma \mid t : B$ and $\Gamma, y : B, x : A \mid \varphi$ `pred` then

$$(\exists\, x : A \,.\, \varphi)[t/y] = \exists\, x : A \,.\, (\varphi[t/y]) \;.$$
$$(\forall\, x : A \,.\, \varphi)[t/y] = \forall\, x : A \,.\, (\varphi[t/y]) \;.$$

If semantics of quantifiers is to be sound, the interpretation of these equations must be valid. Because substitution of a term in a formula is interpreted as pullback this means that quantifiers must be *stable* under pullbacks. This is known as the *Beck-Chevalley condition*.

Definition 2.4.3. A family of functors $F_f : \mathsf{Sub}(A) \to \mathsf{Sub}(B)$ parametrized by morphisms $f : A \to B$ is said to satisfy the *Beck-Chevalley condition* when for every pullback on the left-hand side, the right-hand square commutes:

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[d, "k"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ D \arrow[r, "g"'] & B \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Sub}(C) \arrow[d, "F_k"'] & \mathsf{Sub}(A) \arrow[l, "h^*"'] \arrow[d, "F_f"] \\ \mathsf{Sub}(D) & \mathsf{Sub}(B) \arrow[l, "g^*"] \end{tikzcd}$$

To convince ourselves that Beck-Chevalley condition is what we want, we spell it out explicitely in the case of a substitution into an existentially quantified formula. In order to keep the notation simple we omit the semantic brackets $[\![-]\!]$. Suppose we have a term $\Gamma \mid t : B$ and a formula $\Gamma, y : B, x : A \mid \varphi$ `pred`. The diagram

$$\begin{tikzcd} \Gamma \times A \arrow[r, "\langle \pi_0, t \circ \pi_0, \pi_1 \rangle"] \arrow[d, "\pi_0^{\Gamma,A}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma \times B \times A \arrow[d, "\pi_0^{\Gamma,B,A}"] \\ \Gamma \arrow[r, "\langle 1_\Gamma, t \rangle"'] & \Gamma \times B \end{tikzcd}$$

is a pullback. By Beck-Chevalley condition for $\exists$, the following square commutes:

$$\begin{tikzcd} \mathsf{Sub}(\Gamma \times A) \arrow[d, "\exists_A^{\Gamma,A}"'] & \mathsf{Sub}(\Gamma \times B \times A) \arrow[l, "\langle \pi_0, t \circ \pi_0, \pi_1 \rangle^*"'] \arrow[d, "\exists_A^{\Gamma,B,A}"] \\ \mathsf{Sub}(\Gamma) & \mathsf{Sub}(\Gamma \times B) \arrow[l, "\langle 1_\Gamma, t \rangle^*"] \end{tikzcd}$$

Therefore, for $\Gamma, y : B, x : A \mid \varphi$ `pred`,

$$[\![(\exists\, x : A \,.\, \varphi)[t/y]]\!] = \langle 1_\Gamma, t\rangle^*(\exists_A^{\Gamma,B,A}[\![\varphi]\!]) = \\ \exists_A^{\Gamma,A}(\langle \pi_0, t \circ \pi_0, \pi_1\rangle^*[\![\varphi]\!]) = [\![\exists\, x : A \,.\, (\varphi[t/y])]\!] \;.$$

This is precisely the equation we wanted. The Beck-Chevalley condition says that (interpretations of) the quantifiers commute with pullbacks, in just the way that the syntactic operations of applying quantifiers to formulas commute with substitutions of terms, which are interpreted as pullbacks.

Definition 2.4.4. A cartesian category $\mathcal{C}$ has *existential quantifiers* if, for every $f : A \to B$, the left adjoint $\exists_f \dashv f^*$ exists and it satisfies the Beck-Chevalley condition. Similarly, $\mathcal{C}$ has *universal quantifiers* if the right adjoints $f^* \dashv \forall_f$ exist and they satisfy the Beck-Chevalley condition.

Given both adjoints $\exists_f \dashv f^* \dashv \forall_f$, it actually suffices to have the Beck-Chevalley condition for either one in order to infer it for both:

Proposition 2.4.5. *If for every $f : A \to B$, both the left and right adjoints exist*

$$\exists_f \dashv f^* \dashv \forall_f$$

then the left adjoint satisfies the Beck-Chevalley condition iff the right adjoint does.

Proof. Suppose we have the Beck-Chevalley condition for the left adjoints $\exists$, and that we are given a pullback square as on the left below. We want to check the Beck-Chevalley square for the right adjoints $\forall$, as indicated on the right below.

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[d, "k"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ D \arrow[r, "g"'] & B \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Sub}(C) \arrow[d, "\forall_k"'] & \mathsf{Sub}(A) \arrow[l, "h^*"'] \arrow[d, "\forall_f"] \\ \mathsf{Sub}(D) & \mathsf{Sub}(B) \arrow[l, "g^*"] \end{tikzcd}$$

Swapping all the functors in the righthand diagram for their left adjoints we obtain the following.

$$\begin{tikzcd} \mathsf{Sub}(C) \arrow[r, "\exists_h"] & \mathsf{Sub}(A) \\ \mathsf{Sub}(D) \arrow[u, "k^*"] \arrow[r, "\exists_g"'] & \mathsf{Sub}(B) \arrow[u, "f^*"'] \end{tikzcd}$$

But this is a Beck-Chevalley square for (the "transpose" of) the original pullback diagram, and therefore commutes by the Beck-Chevalley condition for the left adjoints $\exists$. The original diagram of right adjoints therefore also commutes, by uniqueness of adjoints.

The argument for the dual case is, well, dual. $\square$

Exercise 2.4.6. In Set we can identify $\mathsf{Sub}(-)$ with powersets because $\mathsf{Sub}(X) \cong \mathcal{P}X$. Then quantifiers along a function $f : A \to B$ are functions

$$\exists_f : \mathcal{P}A \to \mathcal{P}B\ , \qquad\qquad \forall_f : \mathcal{P}A \to \mathcal{P}B\ .$$

Verify that

$$\exists_f U = \big\{ b \in B \;\big|\; \exists\, a : A \,.\, (fa = b \wedge a \in U) \big\}\ ,$$
$$\forall_f U = \big\{ b \in B \;\big|\; \forall\, a : A \,.\, (fa = b \Rightarrow a \in U) \big\}\ .$$

Thus $\exists_f U$ is just the usual direct image of U by f, sometimes written $f_!(U)$, or simply $f(U)$. But have you seen $\forall_f U$ before? It can also be written as $\forall_f U = \{b \in B \mid f^* \{b\} \subseteq U\}$. What is the meaning of $\exists_q$ and $\forall_q$ when $q : A \to A/{\sim}$ is a canonical quotient map that maps an element $x \in A$ to its equivalence class $qx = [x]$ under an equivalence relation $\sim$ on A?

2.5 Regular logic

We next consider the question of when a cartesian category has existential quantifiers. It turns out that this is closely related to the notion of a *regular category*, a concept which first arose in the context of abelian categories and axiomatic homology theory, quite independently of categorical logic. We will see for instance that all algebraic categories, in the sense of Chapter 1, are regular.

2.5.1 Regular categories

Throughout this section we work in a cartesian category $\mathcal{C}$. We begin with some general definitions. The *kernel pair* of a morphism $f : A \to B$ is the pair of morphisms $k_1, k_2 : K \rightrightarrows A$ obtained as in the following pullback

$$\begin{tikzcd} K \arrow[r, "k_2"] \arrow[d, "k_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Note that a kernel pair determines an equivalence relation $\langle k_1, k_2 \rangle : K \rightarrowtail A \times A$, in the sense that the map $\langle k_1, k_2 \rangle$ is a mono that satisfies the reflexivity, symmetry and transitivity conditions. In Set the mono $\langle k_1, k_2 \rangle : K \rightarrowtail A \times A$ is the equivalence relation $\sim$ on A defined by

$$x \sim y \iff fx = fy \,.$$

Indeed, a kernel pair in a general cartesian category is a model of the cartesian theory of an equivalence relation, in the sense of example 2.3.3.

Exercise 2.5.1. Prove this.

In general, the *quotient* by the equivalence relation determined by the kernel pair k_1, k_2 is their coequalizer $q : A \to Q$, if it exists,

$$\begin{tikzcd} K \arrow[r, shift left, "k_1"] \arrow[r, shift right, "k_2"'] & A \arrow[r, "q"] & Q \end{tikzcd}$$

Such a coequalizer is called a *kernel quotient*.

Because $f \circ k_1 = f \circ k_2$, we see that f factors through q by a unique morphism $m : Q \to A$,

$$\begin{tikzcd}
K \arrow[r, shift left, "k_1"] \arrow[r, shift right, "k_2"'] & A \arrow[rr, "f"] \arrow[dr, "q"'] & & B \\
& & Q \arrow[ur, "m"'] &
\end{tikzcd} \tag{2.6}$$

As a coequalizer, $q : A \to Q$ is always epic; indeed, epis that are coequalizers will be called *regular epimorphisms* and will be denoted by arrows with triangular heads:

$$e : A \longrightarrow\!\!\triangleright B$$

It is of some interest to know when the second factor $m : Q \to B$ in (2.6) is guaranteed to be a mono. For example, in Set the function $m : Q \to B$ is defined by $m[x] = fx$, where $Q = A/{\sim}$ as above. In this case m is indeed injective, because $m[x] = m[y]$ implies $fx = fy$, hence $x \sim y$ and $[x] = [y]$.

Definition 2.5.2. A category with finite limits is *regular* when it has kernel quotients, and regular epis are stable under pullback. Thus, in detail:

1. the kernel pair of any map has a coequalizer, and
2. any pullback of a regular epi is a regular epi.

Exercise 2.5.3. Suppose $e : A \longrightarrow\!\!\triangleright B$ is a regular epi. Prove that it is the coequalizer of its own kernel pair.

Let us return to (2.6) and show that m is monic in any regular category. Consider the following diagram, in which h_1, h_2 are constructed as the kernel pair of m, and the other three squares are constructed as pullbacks:

$$\begin{tikzcd}
K \arrow[r, -{Triangle[open]}, "p_2"] \arrow[d, -{Triangle[open]}, "p_1"'] \arrow[dr, -{Triangle[open]}, "r"] & \cdot \arrow[r] \arrow[d, -{Triangle[open]}, "s_2"] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, -{Triangle[open]}, "q"] \\
\cdot \arrow[r, -{Triangle[open]}, "s_1"'] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & H \arrow[r, "h_2"] \arrow[d, "h_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Q \arrow[d, "m"] \\
A \arrow[r, -{Triangle[open]}, "q"'] & Q \arrow[r, "m"'] & B
\end{tikzcd}$$

Because all the smaller squares are pullbacks the large square is a pullback as well, therefore the left-hand vertical morphism is $k_1 : K \to A$, and the morphism across the top is $k_2 : K \to A$, and we have the kernel pair $k_1, k_2 : K \rightrightarrows A$ of $f = m \circ q$. The morphisms s_1,

s_2, p_1, and p_2 are all regular epis because they are pullbacks of the regular epi q. The morphism $r = s_2 \circ p_2 = s_1 \circ p_1$ is epic because it is a composition of regular epis. Observe that

$$h_1 \circ r = q \circ k_1 = q \circ k_2 = h_2 \circ r \ ,$$

and so, because r is epic, $h_1 = h_2$. But this means that m is monic, since the maps in its kernel pair are equal; indeed, given any $u, v : U \to Q$ with $m \circ u = m \circ v$, there exists a $w : U \to H$ such that $u = w \circ h_1 = w \circ h_2 = v$.

Proposition 2.5.4. *In a regular category every morphism $f : A \to B$ factors as a composition of a regular epi q followed by a mono m,*

$$\begin{tikzcd} A \arrow[r, twoheadrightarrow, "q"'] \arrow[rr, bend left, "f"] & Q \arrow[r, rightarrowtail, "m"'] & B \end{tikzcd}$$

The factorization is unique up to isomorphism.

Proof. By uniqueness of the factorization we mean that if

$$\begin{tikzcd} A \arrow[r, twoheadrightarrow, "q'"'] \arrow[rr, bend left, "f"] & Q' \arrow[r, rightarrowtail, "m'"'] & B \end{tikzcd}$$

is another such factorization, then there exists an isomorphism $i : Q \to Q'$ such that $q' = i \circ q$ and $m = m' \circ i$.

$$\begin{tikzcd} A \arrow[r, twoheadrightarrow, "q'"] \arrow[d, twoheadrightarrow, "q"'] & Q' \arrow[d, rightarrowtail, "m'"] \\ Q \arrow[r, rightarrowtail, "m"'] \arrow[ur, dotted, "i"'] & B \end{tikzcd}$$

As the factorization of f we take the one constructed in (2.6). Then q is a regular epi by construction, and we have just shown that m is monic. So it only remains to show that the factorization is unique. Suppose f also factors as $f = m' \circ q'$ where q' is a regular epi and m' is monic. Consider the following diagram, in which k_1, k_2 is the kernel pair of f, q is the coequalizer of k_1 and k_2, and h_1, h_2 is the kernel pair of q' so that q' is the coequalizer of h_1 and h_2:

$$\begin{tikzcd}
 & H \arrow[d, shift right, "h_1"'] \arrow[d, shift left, "h_2"] & \\
K \arrow[r, shift left, "k_1"] \arrow[r, shift right, "k_2"'] & A \arrow[r, two heads, "q"] \arrow[d, two heads, "q'"'] & Q \arrow[d, tail, "m"] \arrow[dl, dotted, shift right, "i"'] \\
 & Q' \arrow[r, tail, "m'"'] \arrow[ur, dotted, shift right, "j"'] & B
\end{tikzcd}$$

Because $m' \circ q' \circ k_1 = m \circ q \circ k_1 = m \circ q \circ k_2 = m' \circ q' \circ k_2$ and m' is monic, $q' \circ k_1 = q' \circ k_2$. So there exists a unique $i : Q \to Q'$ such that $q' = i \circ q$. But then $m' \circ i \circ q = m' \circ q' = f = m \circ q$ and because q is epi, $m' \circ i = m$.

We prove that i is iso by constructing its inverse j. Because $m \circ q \circ h_1 = m' \circ q \circ h_1 = m' \circ q \circ h_2 = m \circ q \circ h_2$ and m is monic, $q \circ h_1 = q \circ h_2$. So there exists a unique $j : Q' \to Q$ such that $q = j \circ q'$. Now we have $i \circ j \circ q' = i \circ q = 1_{Q'} \circ q'$, from which we conclude that $i \circ j = 1_{Q'}$ because q' is epi. Similarly, $j \circ i \circ q = j \circ q' = 1_Q \circ q$, therefore $j \circ i = 1_Q$. □

Corollary 2.5.5. *A map $f : A \to B$ that is both a regular epi and a mono is an iso.*

Proof. Consider the following outer square, regarded as two different reg-epi/mono factorizations.

$$\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[d, "f"'] & A \arrow[d, "f"] \\
B \arrow[r, "1_B"'] \arrow[ur, dotted, "d"'] & B
\end{tikzcd}$$

A diagonal d is then an inverse of f. □

A factorization $f = m \circ q$ as in Proposition 2.5.4 determines a subobject

$$\mathsf{im}(f) = [m : Q \rightarrowtail B] \in \mathsf{Sub}(B) \ ,$$

called the *image of f*. It is characterized as the least subobject of B through which f factors.

Proposition 2.5.6. *For a morphism $f : A \to B$ in a regular category $\mathcal{C}$, the image $\mathsf{im}(f) \rightarrowtail B$ is the least subobject $U \rightarrowtail B$ of B through which f factors.*

Proof. Suppose f factors through $v : V \rightarrowtail B$ as

$$\begin{tikzcd}
A \arrow[r, "g"'] \arrow[rr, bend left, "f"] & V \arrow[r, tail, "v"'] & B
\end{tikzcd}$$

and consider the factorization of f, as in (2.6). Since $v \circ g \circ k_1 = f \circ k_1 = f \circ k_2 = v \circ g \circ k_2$ and v is mono, $g \circ k_1 = g \circ k_2$, therefore there exists a unique $\overline{g} : Q \to V$ such that $g = \overline{g} \circ q$. Now $v \circ \overline{g} \circ q = v \circ g = f = m \circ q$ and because q is epic, $v \circ \overline{g} = m$ as required. (The reader should draw the corresponding diagram.) □

Definition 2.5.7. A functor $F : \mathcal{C} \to \mathcal{D}$ is *regular* if it preserves finite limits and regular epis. It follows that F preserves image factorizations. The category of regular functors $\mathcal{C} \to \mathcal{D}$ and natural transformations is denoted by $\mathsf{Reg}(\mathcal{C}, \mathcal{D})$.

Examples of regular categories

Let us consider some examples of regular categories.

1. The category Set is regular. It is complete and cocomplete, so it has in particular all finite limits and coequalizers. To show that the pullback of a regular epi is again a regular epi, note that in Set the epis are exactly the surjections, and a surjection is a quotient of its kernel pair, and thus a regular epi. It therefore it suffices to show that the pullback of a surjection is a surjection, which is easy.

2. More generally, any presheaf category $\widehat{\mathcal{C}}$ is also regular, because it is complete and cocomplete, with (co)limits computed pointwise. Thus, again, every epi is regular, and epis are stable under pullbacks.

3. ("Fuzzy logic") Let H be a complete Heyting algebra; thus H is a cartesian closed poset with all small joins $\bigvee_i p_i$. The category of *H-presets* has as objects all pairs $(X, e_X : X \to H)$ where X is a set and e_X is a function, called the *existence predicate of X*. For $x \in X$, $e_X(x)$ can be thought of as "the amount by which x exists". A *morphism of presets* is a function $f : X \to Y$ satisfying, for all $x \in X$,

$$e_X(x) \leq e_Y(fx).$$

   This is a regular category, with the following structure.

   - the terminal object is $\top : 1 \to H$,
   - the product of $e_A : A \to H$ and $e_B : B \to H$ is

     $$e_A \wedge e_B : A \times B \to H,$$

     where $(e_a \wedge e_B)(a, b) = e_A(a) \wedge e_B(b)$,
   - the equalizer of two maps $f, g : A \to B$ is their equalizer as functions, $A' = \{a \mid f(a) = g(a)\} \hookrightarrow A$, with the restriction of $e_A : A \to H$ to $A' \subseteq A$.
   - a map $f : A \to B$ is a regular epi if and only if it is a surjective function and for all $b \in B$:

     $$e_B(b) = \bigvee_{f(a)=b} e_A(a)$$

Exercise 2.5.8. Verify that H-presets form a regular category, and compute the regular epi-mono factorization of a map.

The next example deserves to be a proposition.

Proposition 2.5.9. *The category* $\mathsf{Mod}(\mathbb{A}, \mathsf{Set})$ *of set-theoretic models of an algebraic theory* $\mathbb{A}$ *is regular.*

Proof. We sketch a proof, for details see [Bor94, Theorem 3.5.4]. Recall that the objects of $\mathsf{Mod}(\mathbb{A}) = \mathsf{Mod}(\mathbb{A}, \mathsf{Set})$ are $\mathbb{A}$-algebras, which are structures $A = (|A|, f_1, f_2, \ldots)$ where $|A|$ is the underlying set and $f_1, f_2, \ldots$ are the basic operations on $|A|$. Every such $\mathbb{A}$-algebra is also required to satisfy the equational axioms of $\mathbb{A}$. A morphism $h : A \to B$ is a function $h : |A| \to |B|$ that preserves the basic operations.

The category $\mathsf{Mod}(\mathbb{A})$ of $\mathbb{A}$-algebras has small limits, which are created by the forgetful functor $U : \mathsf{Mod}(\mathbb{A}) \to \mathsf{Set}$. Thus the product of $\mathbb{A}$-algebras A and B has as its underlying set $|A \times B| = |A| \times |B|$, and the basic operations of $A \times B$ are computed separately on each factor, and similarly for products of arbitrary (small) families $\prod_i A_i$. An equalizer of morphisms $g, h : A \to B$ has as its underlying set the equalizer of $g, h : |A| \to |B|$, and the basic operations inherited from A.

To see that coequalizers of kernel pairs exist, consider a morphism $h : A \to B$. We can form the quotient $\mathbb{A}$-algebra Q whose underlying set is $|Q| = |A|/{\sim}$, where $\sim$ is the relation defined by

$$x \sim y \iff hx = hy ,$$

which is just the kernel quotient of the underlying function h. A basic operation $f_Q : |Q|^k \to |Q|$ is induced by the basic operation $f_A : |A|^k \to |A|$ by

$$f_Q\langle [x_1], \ldots, [x_k] \rangle = [f_A\langle x_1, \ldots, x_k \rangle] .$$

It is easily verified that this is well-defined, that Q is an $\mathbb{A}$-algebra, and that the canonical quotient map $q : A \to Q$ is the coequalizer of the kernel pair of h.

Lastly regular epis in $\mathsf{Mod}(\mathbb{A})$ are stable because pullbacks and kernel pairs are computed as in Set, and a morphism $h : A \to B$ is a regular epi in $\mathsf{Mod}(\mathbb{A})$ if, and only if, the underlying function $h : |A| \to |B|$ is a regular epi in Set, which is therefore stable under pullback. $\square$

We now know that categories of groups, rings, modules, $\mathcal{C}^\infty$-rings and other algebraic categories are regular. The preceding proposition is useful also for showing that certain structures can*not* be axiomatized by algebraic theories. The category of posets is an example of a category that is not regular; therefore the theory of partial orders cannot be axiomatized solely by equations.

Exercise 2.5.10. Show that Poset is not regular. (Hint: find a regular epi that is not stable under pullback.) Conclude that there is no purely equational reformulation of the cartesian theory of posets.

Exercise* 2.5.11. Is Top regular? Hint: is there is a topological quotient map $q : X \twoheadrightarrow X'$ and a space Y such that $q \times 1_Z : X \times Y \twoheadrightarrow X' \times Y$ is not a quotient map?

Remark 2.5.12 (Exactness)**.** A regular category $\mathcal{C}$ is said to be *exact* [?] if *every* equivalence relation (not just those arising as kernel pairs) has a quotient. It can be shown fairly easily that categories of algebras are not just regular but also exact: an equivalence relation in such a category is a congruence relation with respect to the algebraic operations, and its (underlying set) quotient is then necessarily also a homomorphism, and thus a coequalizer of algebras.

Exercise 2.5.13. Prove that the regular epis and monos in a regular category $\mathcal{C}$ form the two classes $(\mathcal{L}, \mathcal{R})$, respectively, of an *orthogonal factorization system* in the following sense:

1. every arrow $f : A \to B$ factors as $f = r \circ l$ with $l \in \mathcal{L}$ and $r \in \mathcal{R}$,

2. $\mathcal{L}$ is the class of all arrows left-orthogonal to all maps in $\mathcal{R}$, and $\mathcal{R}$ is the class of all arrows right-orthogonal to all maps in $\mathcal{L}$, where $l : A \to B$ is said to be *left-orthogonal* to $r : X \to Y$, and r is said to be *right-orthogonal* to l, if for every commutative square as on the outside below,

$$\begin{tikzcd} A \arrow[r] \arrow[d, "l"'] & X \arrow[d, "r"] \\ B \arrow[r] \arrow[ur, dotted, "d"'] & Y, \end{tikzcd}$$

there is a unique diagonal arrow d as indicated making both triangles commute.

2.5.2 Images and existential quantifiers

Recall that the poset $\mathsf{Sub}(A)$ is equivalent to the preordered category $\mathsf{Mono}(A)$ of monos into A. If we compose an equivalence functor $\mathsf{Sub}(A) \to \mathsf{Mono}(A)$ with the inclusion $\mathsf{Mono}(A) \to \mathcal{C}/A$ we obtain a (full and faithful) inclusion functor

$$I : \mathsf{Sub}(A) \hookrightarrow \mathcal{C}/A\,. \tag{2.7}$$

In the other direction we have the "image functor" $\mathsf{im} : \mathcal{C}/A \to \mathsf{Sub}(A)$, which maps an object $f : B \to A$ in $\mathcal{C}/A$ to the subobject $\mathsf{im}(f) \rightarrowtail A$.

Exercise 2.5.14. In order to show that im is in fact a functor, prove that $f = g \circ h$ implies $\mathsf{im}(f) \leq \mathsf{im}(g)$.

Proposition 2.5.6 says that the image functor is left adjoint to the inclusion functor (2.7),

$$\mathsf{im} \dashv I\,.$$

Furthermore, images are stable in the sense that the following diagram commutes for all $f : A \to B$ (as does the corresponding one with the inclusion I in place of im).

$$\begin{tikzcd} \mathcal{C}/A \arrow[d, "\mathsf{im}_A"'] & \mathcal{C}/B \arrow[l, "f^*"'] \arrow[d, "\mathsf{im}_B"] \\ \mathsf{Sub}(A) & \mathsf{Sub}(B) \arrow[l, "f^*"] \end{tikzcd} \tag{2.8}$$

The functor f^* on the top is the "change of base" functor given by pullback of an arbitrary map, and the functor f^* on the bottom is the pullback functor acting on subjects. To see that (2.8) commutes, consider $g : C \to B$ and the following diagram:

$$\begin{tikzcd} f^*C \arrow[r] \arrow[d, two heads] \arrow[dd, bend right=60, "f^*g"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, two heads] \arrow[dd, bend left=60, "g"] \\ \cdot \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{im}(g) \arrow[d, tail] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

On the right-hand side we have the factorization of g, which is then pulled back along f. Because monos and regular epis are both stable, this gives a factorization of the pullback f^*g, hence (by the uniqueness of factorizations, Proposition 2.5.4) the claimed equality

$$\mathsf{im}(f^*g) = f^*(\mathsf{im}(g)) \; .$$

Proposition 2.5.15. *A regular category has existential quantifiers. The existential quantifier along $f : A \to B$,*

$$\exists_f : \mathsf{Sub}(A) \longrightarrow \mathsf{Sub}(B),$$

is given by

$$\exists_f[m : M \rightarrowtail A] = \mathsf{im}(f \circ m) \; ,$$

as indicated below.

$$\begin{tikzcd} M \arrow[r, two heads] \arrow[d, tail, "m"'] & \mathsf{im}(f \circ m) \arrow[d, tail] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Proof. Recall that composition

$$\Sigma_f : \mathcal{C}/A \longrightarrow \mathcal{C}/B$$

by a map $f : A \to B$ is left adjoint to pullback f^* along f. Thus we are defining $\exists_f = \mathsf{im} \circ \Sigma_f \circ I$ as shown below.

$$\begin{tikzcd}
\mathsf{Sub}(A) \arrow[r, dotted, "\exists_f"] \arrow[d, hook, "I"'] & \mathsf{Sub}(B) \\
\mathcal{C}/A \arrow[r, "\Sigma_f"'] & \mathcal{C}/B \arrow[u, "\mathsf{im}"']
\end{tikzcd}$$

First we verify that $\exists_f \dashv f^*$ on subobjects. For $U \rightarrowtail A$ and $V \rightarrowtail B$:

$$\begin{array}{lr}
\exists_f U \leq V & \text{in } \mathsf{Sub}(B) \\
\hline\hline
\mathsf{im} \circ \Sigma_f \circ I(U) \leq V & \text{in } \mathsf{Sub}(B) \\
\hline\hline
\Sigma_f \circ I(U) \leq I(V) & \text{in } \mathcal{C}/B \\
\hline\hline
I(U) \to f^* I(V) & \text{in } \mathcal{C}/A \\
\hline\hline
I(U) \to I(f^* V) & \text{in } \mathcal{C}/A \\
\hline\hline
U \leq f^* V & \text{in } \mathsf{Sub}(A)
\end{array}$$

In the second step in the above derivation we used the adjunction between $\mathsf{im} : \mathcal{C}/B \to \mathsf{Sub}(B)$ and the inclusion $\mathsf{Sub}(B) \to \mathcal{C}/B$.

The Beck-Chevalley condition follows from stability of image factorizations. Indeed, given a pullback

$$\begin{tikzcd}
D \arrow[r, "h"] \arrow[d, "k"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

and a subobject $U \rightarrowtail C$, (2.8) gives

$$\begin{aligned}
f^*(\exists_g U) = f^* \circ \mathsf{im} \circ \Sigma_g \circ I(U) = \mathsf{im} \circ f^* \circ \Sigma_g \circ I(U) &= \mathsf{im} \circ \Sigma_k \circ h^* \circ I(U) \\
&= \mathsf{im} \circ \Sigma_k \circ I \circ h^*(U) = \exists_k(h^* U)
\end{aligned}$$

$$\begin{tikzcd}
 & f^*U \arrow[r] \arrow[dl, two heads] \arrow[d, tail] \arrow[dr, phantom, very near start, "\lrcorner"] & U \arrow[d, tail] \arrow[dr, two heads] & \\
\exists_k f^*U \arrow[dr, tail] & D \arrow[r, "h"] \arrow[d, "k"'] \arrow[dr, phantom, very near start, "\lrcorner"] & C \arrow[d, "g"] & \exists_g U \arrow[dl, tail] \\
 & A \arrow[r, "f"'] & B &
\end{tikzcd}$$

as required. □

Summarizing the results of this section, we have the following.

Proposition 2.5.16. *In any regular category, for every map $f : A \to B$ we have the following situation, where f^* is pullback:*

$$\begin{tikzcd}
\mathsf{Sub}(A) \arrow[r, shift right, "\exists_f"'] \arrow[d, hook, shift left, "I"] & \mathsf{Sub}(B) \arrow[l, shift right, "f^*"'] \arrow[d, hook, shift left, "I"] \\
\mathcal{C}/A \arrow[u, shift left, "\mathsf{im}"] \arrow[r, shift right, "\Sigma_f"'] & \mathcal{C}/B \arrow[l, shift right, "f^*"'] \arrow[u, shift left, "\mathsf{im}"]
\end{tikzcd}$$

with adjunctions

$$\exists_f \dashv f^*, \quad \mathsf{im} \dashv I, \quad \Sigma_f \dashv f^*$$

and natural isos

$$f^* \circ \mathsf{im} \cong \mathsf{im} \circ f^*, \quad f^* \circ I \cong I \circ f^*.$$

Note, moreover, that

$$\exists_f \circ \mathsf{im} \cong \mathsf{im} \circ \Sigma_f$$

then follows.

Finally, we call attention to the following special fact.

Proposition 2.5.17 (Frobenius Reciprocity)**.** *Given a map $f : A \to B$ and subobjects $U \leq A$ and $V \leq B$, the following equation holds in* $\mathsf{Sub}(B)$*.*

$$\exists_f(U \wedge f^*V) = \exists_f U \wedge V$$

Exercise 2.5.18. Prove Frobenius reciprocity, using the following diagram.

$$
\begin{tikzcd}
& U \wedge f^*V \arrow[rr] \arrow[d] \arrow[ddl] & & \exists_f U \wedge V \arrow[d] \arrow[ddl] \\
& f^*V \arrow[rr] \arrow[ddl] & & V \arrow[ddl, tail] \\
U \arrow[rr] \arrow[d, tail] & & \exists_f U \arrow[d] & \\
A \arrow[rr, "f"'] & & B &
\end{tikzcd}
$$

2.5.3 Regular theories

A regular category has finite limits and image factorizations, therefore it allows us to interpret a type theory with the terminal type and binary products, and a logic with equality, conjunction, and existential quantifiers. This system is called *regular logic*.

Definition 2.5.19. A (many-sorted) *regular theory* $\mathbb{T}$ is a (many-sorted) type theory together with a set of axioms expressed in the fragment of logic built from $=$, $\top$, $\wedge$, and $\exists$.

In more detail, a regular theory consists of the following data, extending the notion of cartesian theory from section 2.3.

- basic type symbols $A_1, \ldots, A_k$,
- basic function symbols $f, \ldots$ (with signature) $(A_1, \cdots, A_m; B)$,
- basic relation symbols $R, \ldots$ (with signature) $(A_1, \cdots, A_n)$.

We then define by induction the set of terms in context,

$$\Gamma \mid t : A\,,$$

as well as the formulas in context,

$$\Gamma \mid \varphi \ \mathtt{pred}\,.$$

Here is the first place where things differ from cartesian logic; we extend the formation rules for cartesian formulas (section 2.3) by the further clause:

6. Existential Quantifier:

$$\frac{\Gamma, x : A \mid \varphi \ \mathtt{pred}}{\Gamma \mid \exists x : A.\, \varphi \ \mathtt{pred}}$$

(We also add the evident additional clause for sustitution of terms into existentially quantified formulas, namely $(\exists x : A.\, \varphi)[t/y] = \exists x : A.\, (\varphi[t/y])$.) This defines the notion of a *regular formula*, i.e. ones built from the atomic formulas $s = t$ and $R(t_1, \dots, t_n)$ using the logical operations $\top$, $\wedge$, and $\exists$.

A regular theory then includes, finally, a set of axioms of the form

$$\Gamma \mid \varphi \vdash \psi$$

where φ, ψ are *regular formulas.*

Example 2.5.20. 1. A ring A (with unit 1) is called *von Neumann regular* if for every element a there is at least one element x for which $a = a \cdot x \cdot a$. Such an x may be thought of as a "weak inverse" of a. The theory of *von Neumann regular rings* is thus an extension of the usual theory of rings with unit by adding the single axiom

$$a : A \mid \top \vdash \exists x : A \,.\, a = a \cdot x \cdot a$$

2. A perhaps more familiar example is the theory of categories, with two basic types A, O for arrows and objects, 3 basic function symbols $\mathrm{dom}, \mathrm{cod} : (A; O)$ and $\mathrm{id} : (O; A)$ and one basic relation symbol $C : (A, A, A)$, where the latter is for the relation $C(x, y, z) =$ "z is the composite of x and y". The axioms for C are as follows (with abbreviated notation for the context):

$$\begin{aligned} x, y, z : A \mid C(x, y, z) &\vdash \mathrm{cod}(x) = \mathrm{dom}(y) \wedge \mathrm{dom}(z) = \mathrm{dom}(x) \wedge \mathrm{cod}(z) = \mathrm{cod}(y) \\ x, y : A \mid \mathrm{cod}(x) = \mathrm{dom}(y) &\vdash \exists z.\, C(x, y, z) \\ x, y, z, z' : A \mid C(x, y, z) \wedge C(x, y, z') &\vdash z = z' \end{aligned}$$

Recall the previous versions of the theory of categories as cartesian theories in 2.3.17. Are the homomorphisms of categories, as models of a regular theory, the same thing as functors?

3. The theory of an *inhabited object* has a single type A, no function or relation symbols, and the single axiom:

$$\cdot \mid \top \vdash \exists x : A.\, x = x$$

A model is an object that is "inhabited" by at least one (unnamed) element, but the homomorphisms need not preserve anything – in this sense being inhabited is a *property*, not a *structure.*

The *rules of inference* of regular logic are those of cartesian logic (section 2.3), with an additional rule for the existential quantifier:

8. Existential Quantifier:

$$\frac{y : B, x : A \mid \varphi \vdash \vartheta}{\overline{\;y : B \mid \exists x : A \,.\, \varphi \vdash \vartheta\;}}$$

Note that the lower judgement is well-formed only if $x : A$ does not occur freely in ϑ.

We also add a rule coresponding to Frobenius reciprocity, Proposition 2.5.17, in the form

9. Frobenius:
$$x : A \mid (\exists y : B.\varphi) \wedge \psi \;\vdash\; \exists y : B.(\varphi \wedge \psi)$$
provided the variable $y : B$ does not occur freely in ψ.

Note that the converse of Frobenius is easily derivable, so we have the interderivability of $(\exists y : B.\varphi) \wedge \psi$ and $\exists y : B.(\varphi \wedge \psi)$ when $y : B$ is not free in ψ. The Frobenius rule will be derivable in the extended system of Heyting logic (see Proposition 2.6.15), and could be made derivable in a suitably formulated system of regular logic using multi-sequents $\Gamma \mid \varphi_1, \ldots, \varphi_n \vdash \psi$.

Semantics of regular theories

Turning to semantics, an *interpretation* of a regular theory $\mathbb{T}$ in a regular category $\mathcal{C}$ extends the notion for cartesian logic (section 2.3), and is given by the following data:

1. Each basic sort A is interpreted as an object $[\![A]\!]$.
2. Each basic constant f with signature $(A_1, \ldots, A_n; B)$ is interpreted as a morphism $[\![f]\!] : [\![A_1]\!] \times \cdots \times [\![A_n]\!] \to [\![B]\!]$.
3. Each basic relation symbol R with signature $(A_1, \ldots, A_n)$ is interpreted as a subobject $[\![R]\!] \in \mathsf{Sub}([\![A_1]\!] \times \cdots \times [\![A_1]\!])$.

This is the same as for cartesian logic, as is the extension of the interpretation to all terms,
$$[\![\Gamma \mid t : A]\!] : [\![\Gamma]\!] \longrightarrow [\![A]\!]$$
For the formulas, we extended the interpretation to cartesian formulas as before (section **??**),
$$[\![\Gamma \mid \varphi]\!] \rightarrowtail [\![\Gamma]\!] \,.$$
Finally, existential formulas $\exists\, x : A \,.\, \varphi$ are interpreted by the existential quantifiers in the regular category,
$$[\![\Gamma \mid \exists\, x : A \,.\, \varphi]\!] = \exists_A [\![\Gamma, x : A \bigm| \varphi]\!] \,,$$
where
$$\exists_A = \exists_\pi : \mathsf{Sub}([\![\Gamma]\!] \times [\![A]\!]) \to \mathsf{Sub}([\![\Gamma]\!])$$
is the existential quantifier along the projection $\pi : [\![\Gamma]\!] \times [\![A]\!] \to [\![\Gamma]\!]$.

The following is immediate from these definitions, and the considerations in section 2.4.

Proposition 2.5.21. *The rules of regular logic are sound with respect to the interpretation in regular categories.*

Exercise 2.5.22. Prove this.

If all the axioms of $\mathbb{T}$ hold in a given interpretation, then we again say that the interpretation is a *model* of the theory $\mathbb{T}$. Morphisms of models are just morphisms of the underlying cartesian structures. Thus for any regular theory $\mathbb{T}$ and regular category $\mathcal{C}$, there is a *category of models*,

$$\mathsf{Mod}(\mathbb{T}, \mathcal{C})\,.$$

Moreover, this semantic category is functorial in $\mathcal{C}$ with respect to regular functors $\mathcal{C} \to \mathcal{D}$, which, recall, preserve finite limits and regular epis. Indeed, if $F : \mathcal{C} \to \mathcal{D}$ is regular then given a model M in $\mathcal{C}$ with underlying cartesian structure $[\![A]\!]_M, [\![f]\!]_M, [\![R]\!]_M$, etc., we can determine an interpretation FM in $\mathcal{D}$ by setting:

$$[\![A]\!]_{FM} = F([\![A]\!]_M),\ [\![f]\!]_{FM} = F([\![f]\!]_M),\ [\![R]\!]_{FM} = F([\![f]\!]_M)$$

etc., and these will have the correct types (up to isomorphism). To show that FM is a $\mathbb{T}$-model, if M is one and F is regular, consider an axiom of $\mathbb{T}$ of the form $\Gamma \mid \varphi \vdash \psi$. Satisfaction by M means that $[\![\Gamma \mid \varphi]\!]_M \leq [\![\Gamma \mid \psi]\!]_M$ in $\mathsf{Sub}([\![\Gamma]\!]_M)$, which in turn means that there is a (necessarily unique) factorization,

$$\begin{tikzcd}
{[\![\Gamma \mid \varphi]\!]_M} \arrow[rr, dotted] \arrow[dr, hook] & & {[\![\Gamma \mid \psi]\!]_M} \arrow[dl, hook] \\
& {[\![\Gamma]\!]_M} &
\end{tikzcd},$$

Applying the cartesian functor F will result in an inclusion of subobjects $F[\![\Gamma \mid \varphi]\!]_M \leq F[\![\Gamma \mid \psi]\!]_M$ in $\mathsf{Sub}(F[\![\Gamma]\!]_M) = \mathsf{Sub}([\![\Gamma]\!]_{FM})$. Thus is clearly suffices to show that for any regular formula φ,

$$F[\![\Gamma \mid \varphi]\!]_M = [\![\Gamma \mid \varphi]\!]_{FM}\,.$$

This is an easy induction on φ, using the regularity of F.

Proposition 2.5.23. *Given a regular functor $F : \mathcal{C} \to \mathcal{D}$, taking images determines a functor*

$$F_* : \mathsf{Mod}(\mathbb{T}, \mathcal{C}) \longrightarrow \mathsf{Mod}(\mathbb{T}, \mathcal{D})\,.$$

Proof. It only remains show the effect of F_* on morphisms of models. But these are just homomorphisms of the underlying cartesian structure, so they are clearly preserved by the cartesian functor F. □

An associated result, which we will need, is the following.

Proposition 2.5.24. *Given regular categories $\mathcal{C}$ and $\mathcal{D}$ and a model M in $\mathcal{C}$, evaluation at M determines a functor*

$$\mathsf{eval}_M : \mathsf{Reg}(\mathcal{C}, \mathcal{D}) \longrightarrow \mathsf{Mod}(\mathbb{T}, \mathcal{D})\,,$$

which is natural in $\mathcal{D}$.

The proof is straightforward and can be left as an exercise. The naturality means that for any a regular functor $G : \mathcal{D} \longrightarrow \mathcal{D}'$, the following commutes (up to natural isomorphism, as usual):

$$\begin{tikzcd}
\mathsf{Reg}(\mathcal{C}, \mathcal{D}) \arrow[r, "\mathsf{eval}_M"] \arrow[d, "\mathsf{Reg}(\mathcal{C}, G)"'] & \mathsf{Mod}(\mathbb{T}, \mathcal{D}) \arrow[d, "G_*"] \\
\mathsf{Reg}(\mathcal{C}, \mathcal{D}') \arrow[r, "\mathsf{eval}_M"'] & \mathsf{Mod}(\mathbb{T}, \mathcal{D}')
\end{tikzcd}$$

Exercise 2.5.25. Prove this.

Exercise 2.5.26. Show that for any small category $\mathbb{C}$ and regular theory $\mathbb{T}$, there is an equivalence between models in the functor category and functors into the category of models,

$$\mathsf{Mod}(\mathbb{T}, \mathsf{Set}^{\mathbb{C}}) \;\simeq\; \mathsf{Mod}(\mathbb{T})^{\mathbb{C}} .$$

Hint: this is just as for the algebraic and cartesian cases.

2.5.4 Classifying category of a regular theory

We will next show that the framework of *functorial semantics* applies to regular logic and regular categories: there is a *classifying category* $\mathcal{C}_{\mathbb{T}}$ for $\mathbb{T}$-models, for which there is an equivalence, natural in $\mathcal{C}$,

$$\mathsf{Mod}(\mathbb{T}, \mathcal{C}) \;\simeq\; \mathsf{Reg}(\mathcal{C}_{\mathbb{T}}, \mathcal{C}) ,$$

where $\mathsf{Reg}(-,-)$ is the category of regular functors and natural transformations.

Remark 2.5.27. The construction of $\mathcal{C}_{\mathbb{T}}$, and the corollary completeness theorem, are analogous to a perhaps familiar way of proving the completeness theorem for classical propositional logic: one first constructs the *Lindenbaum-Tarski algebra* of propositional logic with respect to a propositional theory $\mathbb{T}$ (a set of formulas) as the set $\mathsf{PL} = \{\varphi \mid \varphi \text{ a propositional formula}\}$, quotiented by $\mathbb{T}$-provable logical equivalence, $\varphi \sim_{\mathbb{T}} \psi$ iff $\mathbb{T} \vdash \varphi \leftrightarrow \psi$,

$$\mathcal{L}_{\mathbb{T}} = \mathsf{PL}/\!\sim_{\mathbb{T}} \; .$$

The quotient set $\mathcal{L}_{\mathbb{T}}$ becomes a Boolean algebra by defining the Boolean operations in terms of the expected propositional logical analogues,

$$[\varphi] \wedge [\psi] = [\varphi \wedge \psi] , \quad \neg[\varphi] = [\neg\varphi] , \quad [\top] = 1 , \quad \text{etc.} .$$

One then has a Boolean-valuation of PL in $\mathcal{L}_{\mathbb{T}}$, namely $[-]$, for which

$$[\varphi] = [\psi] \quad \text{iff} \quad \mathbb{T} \vdash \varphi \leftrightarrow \psi .$$

In particular, then, $[\varphi] = 1$ iff $\mathbb{T} \vdash \varphi$. Classical completeness with respect to valuations in the Boolean algebra $\mathbf{2} = \{1, 0\}$ then follows e.g. from Stone's representation theorem, which embeds the Boolean algebra $\mathcal{L}_{\mathbb{T}}$ into a powerset $\mathcal{P}(X) \cong \mathbf{2}^X$, where X is the set of prime ideals in $\mathcal{L}_{\mathbb{T}}$, corresponding to Boolean valuations $\mathcal{L}_{\mathbb{T}} \to \mathbf{2}$ ("rows of a truth table").

Our syntactic construction of the classifying category $\mathcal{C}_{\mathbb{T}}$ can be regarded as a generalization of this method, with $\mathcal{C}_{\mathbb{T}}$ as the "Lindenbaum-Tarski category" of the (regular) theory $\mathbb{T}$. This will give a completeness theorem with respect to models in regular categories, which can in turn be specialized to Set-valued completeness by embedding $\mathcal{C}_{\mathbb{T}}$ into a "power of Set", i.e. Set^X for a set X of classical Set-valued models, i.e. regular functors $\mathcal{C}_{\mathbb{T}} \to \mathsf{Set}$. See Section **??** below for the second step.

We first sketch the construction of the classifying category $\mathcal{C}_{\mathbb{T}}$ of an arbitrary regular theory $\mathbb{T}$ (a more detailed account can be found in [But98, Joh03]). An object of $\mathcal{C}_{\mathbb{T}}$ is represented by a formula in context,

$$[\Gamma \mid \varphi],$$

where $\Gamma \mid \varphi$ `pred`. Two such objects $[\Gamma \mid \varphi]$ and $[\Gamma \mid \psi]$ are equal if $\mathbb{T}$ proves both

$$\Gamma \mid \varphi \vdash \psi \,, \qquad\qquad \Gamma \mid \psi \vdash \varphi \,.$$

Objects which differ only in the names of free variables are also considered equal:

$$[x : A \mid \varphi] = [y : A \mid \varphi[y/x]] \qquad (\text{no } y \text{ in } \varphi)$$

A morphism

$$[x : A \mid \varphi] \xrightarrow{\rho} [y : B \mid \psi]$$

is represented by a formula $x : A, y : B \mid \rho$ such that $\mathbb{T}$ proves that ρ is a *functional relation* from φ to ψ:

$$\begin{aligned} x : A \mid \varphi &\vdash \exists\, y : B \,.\, \rho && \text{(total)} \\ x : A, y : B, z : B \mid \rho \wedge \rho[z/y] &\vdash y = z && \text{(single-valued)} \\ x : A, y : B \mid \rho &\vdash \varphi \wedge \psi && \text{(well-typed)} \end{aligned}$$

Two functional relations ρ and σ represent the same morphism if $\mathbb{T}$ proves both

$$x : A, y : B \mid \rho \vdash \sigma \,, \qquad\qquad x : A, y : B \mid \sigma \vdash \rho \,.$$

Relations which only differ in the names of free variables are also considered equal.

(Strictly speaking, a morphism

$$[x : A, y : B \mid \rho] : [x : A \mid \varphi] \to [y : B \mid \psi]$$

should be taken to be the triple

$$\big([x : A, y : B \mid \rho],\ [x : A \mid \varphi],\ [y : B \mid \psi]\big)$$

so that one knows what the domain and codomain are, but we shall often write simply

$$\rho : [x : A \mid \varphi] \to [y : B \mid \psi]$$

since the rest can be recovered from that much data.)

The identity morphism on $[x : A \mid \varphi]$ is

$$1_{[x:A|\varphi]} = [x : A, x' : A \mid (x = x') \wedge \varphi] : [x : A \mid \varphi] \to [x' : A \mid \varphi[x'/x]] \ .$$

Note that we used the variable substitution $\varphi[x'/x]$ and the identification $[x : A \mid \varphi] = [x' : A \mid \varphi[x'/x]]$ in order to make this definition.

Composition of morphisms

$$[x : A \mid \varphi] \xrightarrow{\ \rho\ } [y : B \mid \psi] \xrightarrow{\ \tau\ } [z : C \mid \theta]$$

is given by the relational product,

$$\tau \circ \rho = (\exists\, y : B \,.\, (\rho \wedge \tau)) \ .$$

Of course, one needs to check that this *is* a morphism from φ to ϑ, i.e. that it is total, single-valued, and well-typed. We leave the detailed proof that $\mathcal{C}_{\mathbb{T}}$ is a category as an exercise; let us just show how to prove that composition of morphisms is associative. Given morphisms

$$[x : A \mid \varphi] \xrightarrow{\ \rho\ } [y : B \mid \psi] \xrightarrow{\ \tau\ } [z : C \mid \theta] \xrightarrow{\ \sigma\ } [u : D \mid \zeta]$$

we need to derive in context $x : A, u : D$

$$\exists\, z : C \,.\, ((\exists\, y : B \,.\, (\rho \wedge \tau)) \wedge \sigma) \ \dashv\vdash\ \exists\, y : B \,.\, (\rho \wedge (\exists\, z : C \,.\, (\tau \wedge \sigma)))$$

This follows easily with repeated application of the Frobenius rule (Section 2.5.3).

Exercise 2.5.28. Extend the definition of $\mathcal{C}_{\mathbb{T}}$ to morphisms between objects with arbitrary contexts,

$$[\Gamma \mid \varphi] \xrightarrow{\ \rho\ } [\Delta \mid \psi]$$

(use relations $\Gamma, \Delta \mid \rho$), and provide a proof that $\mathcal{C}_{\mathbb{T}}$ is a category.

Proposition 2.5.29. *The category $\mathcal{C}_{\mathbb{T}}$ is regular.*

Proof. We sketch the constructions required for regularity.

- The terminal object is $[\cdot \mid \top]$.
- The product of $[x : A \mid \varphi]$ and $[y : B \mid \psi]$, where x and y are distinct variables, is the object

  $$[x : A, y : B \mid \varphi \wedge \psi] \ .$$

  The first projection from the product is

  $$x : A, y : B, x' : A \mid x = x' \wedge \varphi \wedge \psi \,,$$

  and the second projection is

  $$x : A, y : B, y' : B \mid y = y' \wedge \varphi \wedge \psi \,,$$

  where we rename the codomains of the projections $[x : A \mid \varphi] = [x' : A \mid \varphi[x'/x]]$, etc., to make the context variables distinct.

- An equalizer of morphisms

$$[x : A \mid \varphi] \underset{\tau}{\overset{\rho}{\rightrightarrows}} [y : B \mid \psi]$$

is

$$[x : A \mid \exists\, y : B \,.\, (\rho \wedge \tau)] \xrightarrow{\quad \varepsilon \quad} [x' : A \mid \varphi[x'/x]]$$

where ε is the morphism

$$x : A, x' : A \mid (x = x') \wedge \exists\, y : B \,.\, (\rho \wedge \tau) \ .$$

- Finally, let us consider coequalizers of kernel pairs. The kernel pair of a map

$$\rho : [x : A \mid \varphi] \longrightarrow [y : B \mid \psi]$$

is

$$K \underset{\kappa_2}{\overset{\kappa_1}{\rightrightarrows}} [x : A \mid \varphi]$$

where K is the object

$$[u : A, v : A \mid \exists\, y : B \,.\, (\rho[u/x] \wedge \rho[v/x])] \ ,$$

the morphism κ_1 is

$$u : A, v : A, x : A \mid (u = x) \wedge \exists\, y : B \,.\, (\rho[u/x] \wedge \rho[v/x]) \ ,$$

and κ_2 is

$$u : A, v : A, x : A \mid (v = x) \wedge \exists\, y : B \,.\, (\rho[u/x] \wedge \rho[v/x]) \ .$$

Now the coequalizer of κ_1 and κ_2 can be shown to be the morphism

$$[x : A \mid \varphi] \xrightarrow{\quad \rho \quad} [y : B \mid \exists x : A.\, \rho] \ ,$$

where $[y : B \mid \exists\, x : A \,.\, \rho]$ is the image of ρ, as a subobject of $[y : B \mid \psi]$.

The following lemma shows that regular epis are stable under pullback. $\square$

Lemma 2.5.30. *1. A map $\rho : [x : A \mid \varphi] \longrightarrow [y : B \mid \psi]$ is a regular epi if and only if*

$$y : B \mid \psi \vdash \exists x : A.\, \rho$$

2. Regular epis are stable under pullback in $\mathcal{C}_{\mathbb{T}}$.

Proof. For (1), suppose $\rho : [x : A \mid \varphi] \to [y : B \mid \psi]$ is a regular epi. We claim first that if ρ factors through some subobject $U \rightarrowtail [y : B \mid \psi]$ then $U = [y : B \mid \psi]$ is the maximal suboject. Indeed, since ρ is regular epi it is a coequalizer of its kernel pair. But if ρ factors through a subobject $U \rightarrowtail [y : B \mid \psi]$, say by $r : [x : A \mid \varphi] \to U$, then r is also a coequalizer of the kernel pair of ρ, as one can easily check. Thus $U \rightarrowtail [y : B \mid \psi]$ must be iso.

Now, up to iso, every $U \rightarrowtail [y : B \mid \psi]$ is of the form $U = [y : B \mid \vartheta]$ with $y \mid \vartheta \vdash \psi$, and ρ factors through $[y : B \mid \vartheta]$ iff

$$y : B \mid \exists x : A \,.\, \rho \vdash \vartheta \,.$$

Thus for all ϑ we have that:

$$(y : B \mid \exists x : A \,.\, \rho \vdash \vartheta) \;\Rightarrow\; (y : B \mid \psi \vdash \vartheta) \,.$$

Whence $y : B \mid \psi \vdash \exists x : A \,.\, \rho$. The convere is immediate from the specification of the kernel quotient above.

For (2), suppose we have a pullback diagram, which has the form indicated below.

$$\begin{tikzcd}
{[x : A, y : B \mid \varphi \wedge \psi \wedge \exists z : C.\, (\sigma \wedge \rho)]} \arrow[r, "\rho^*\sigma"] \arrow[d, "\sigma^*\rho"'] & {[y : B \mid \psi]} \arrow[d, "\rho"] \\
{[x : A \mid \varphi]} \arrow[r, "\sigma"'] & {[z : C \mid \vartheta]}
\end{tikzcd}$$

The maps $\sigma^*\rho$ and $\rho^*\sigma$ are represented by the relations:

$$\sigma^*\rho = \big(x : A,\ y : B,\ x' : A \mid x = x' \wedge \varphi \wedge \psi \wedge \exists z : C.\, (\sigma \wedge \rho)\big)$$
$$\rho^*\sigma = \big(x : A,\ y : B,\ y' : B \mid y = y' \wedge \varphi \wedge \psi \wedge \exists z : C.\, (\sigma \wedge \rho)\big)$$

If ρ is regular epi, then by (1) we have

$$z : C \mid \vartheta \vdash \exists y : B.\, \rho \,. \tag{2.9}$$

To show that the pullback $\sigma^*\rho$ is regular epi, again by (1) we need to show

$$x' : A \mid \varphi[x'/x] \vdash \exists x : A\, \exists y : B.\, \big(x = x' \wedge \varphi \wedge \psi \wedge \exists z : C.\, (\sigma \wedge \rho)\big) \,. \tag{2.10}$$

We can make use thereby of the functionality of σ and ρ, specifically we have

$$x : A,\ z : C \mid \sigma \vdash \varphi \wedge \vartheta \qquad \text{and} \qquad x : A \mid \varphi \vdash \exists z : C.\, \sigma \,. \tag{2.11}$$

The result now follows by a simple deduction. $\square$

Exercise 2.5.31. Show that in $\mathcal{C}_{\mathbb{T}}$ the regular-epi mono factorization of a morphism $\rho : [x : A \mid \varphi] \to [y : B \mid \psi]$ is given by

$$[x : A \mid \varphi] \xrightarrow{\rho} [y : B \mid \exists\, x : A \,.\, \rho] \xrightarrow{\iota} [z : B \mid \psi[z/y]]$$

where ι is the morphism

$$y : B, z : B \mid (y = z) \wedge (\exists\, x : A \,.\, \rho) \ .$$

Theorem 2.5.32 (Functorial semantics for regular logic)**.** *For any regular theory $\mathbb{T}$, the syntactic category $\mathcal{C}_{\mathbb{T}}$ classifies $\mathbb{T}$-models in regular categories. Specifically, for any regular category $\mathcal{C}$, there is an equivalence of categories*

$$\mathsf{Reg}(\mathcal{C}_{\mathbb{T}}, \mathcal{C}) \simeq \mathsf{Mod}(\mathbb{T}, \mathcal{C}) \tag{2.12}$$

which is natural in $\mathcal{C}$. In particular, there is a universal model U in $\mathcal{C}_{\mathbb{T}}$.

Proof. We have just constructed $\mathcal{C}_{\mathbb{T}}$ and shown that it is regular.

The universal model U, corresponding to the identity functor $\mathcal{C}_{\mathbb{T}} \to \mathcal{C}_{\mathbb{T}}$ under (2.12), is determined as follows:

- Each sort A is interpreted by the object $[x : A \mid \top]$
- A basic constant f with signature $(A_1, \dots, A_n; B)$ is interpreted by the formula

  $$x_1 : A_1, \dots, x_n : A_n, y : B \mid f(x_1, \dots, x_n) = y \ .$$

  which is plainly a functional relation and thus a morphism $[\![A_1]\!] \times \cdots \times [\![A_n]\!] \longrightarrow [\![B]\!]$.
- A relation symbol R with signature $(A_1, \dots, A_n)$ is interpreted by the subobject represented by the morphism

  $$\rho : [x_1 : A_1, \dots, x_n : A_n \mid R(x_1, \dots, x_n)] \longrightarrow [y_1 : A_1, \dots, y_n : A_n \mid \top]$$

  where ρ is the formula

  $$x_1 : A_1, \dots, x_n : A_n, y_1 : A_1, \dots, y_n : A_n \mid R(x_1, \dots, x_n) \wedge x_1 = y_1 \wedge \cdots \wedge x_n = y_n \ ,$$

  which is easily shown to be monic.

It is now straightforward to show that with respect to this structure, a formula $\Gamma \mid \varphi$ is interpreted as (the subobject determined by) the map

$$\iota : [\Gamma \mid \varphi] \longrightarrow [\Gamma \mid \top]$$

where ι is the formula

$$\Gamma, \Gamma' \mid \Gamma = \Gamma' \wedge \varphi \ ,$$

(with obvious abbreviations) which, again, is easily shown to be monic. Moreover, for any formulas $\Gamma \mid \varphi$ and $\Gamma \mid \psi$ we then have

$$U \models \Gamma \mid \varphi \vdash \psi \quad \iff \quad \mathbb{T} \text{ proves } \Gamma \mid \varphi \vdash \psi \,.$$

Thus in particular U is indeed a $\mathbb{T}$-model.

We next construct a functor $\mathsf{Reg}(\mathcal{C}_{\mathbb{T}}, \mathcal{C}) \to \mathsf{Mod}(\mathbb{T}, \mathcal{C})$. Suppose $\mathcal{C}$ is regular and $F : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{C}$ a regular functor, then by Proposition 2.5.24, applying F to U determines a model FU in $\mathcal{C}$ with

$$[\![A]\!]_{FU} = F([\![A]\!]_U)\,,$$

and similarly for the other parts of the structure f, R, etc. Satisfaction of an entailment $\Gamma \mid \varphi \vdash \psi$ is preserved, because the interpretation of the logical operations is determined by the regular structure: pullbacks, images, etc., so that $[\![\varphi]\!]_U \leq [\![\psi]\!]_U$ in $\mathsf{Sub}([\![\Gamma]\!])$ implies

$$[\![\varphi]\!]_{FU} = F([\![\varphi]\!]_U) \leq F([\![\psi]\!]_U) = [\![\psi]\!]_{FU}$$

in $\mathsf{Sub}([\![\Gamma]\!]_{FU})$.

Moreover, just as for algebraic structures, every natural transformation between regular functors $\vartheta : F \Rightarrow G$ determines a homomorphism of the evaluated models by taking components $\vartheta_U : FU \to GU$. In this way, as in Proposition 2.5.24, evaluation at U is a functor

$$\mathsf{eval}_U : \mathsf{Reg}(\mathcal{C}_{\mathbb{T}}, \mathcal{C}) \longrightarrow \mathsf{Mod}(\mathbb{T}, \mathcal{C})\,.$$

We claim that this functor, which is the one mentioned in (2.12), is full and faithful and essentially surjective. The naturality in $\mathcal{C}$ of the equivalence then follows directly from its determination by evaluation at U and Proposition 2.5.24.

To see that eval_U is essentially surjective, let M be a model in $\mathcal{C}$. We will define a regular functor

$$M^\sharp : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{C}$$

with $M^\sharp(U) \cong M$. Since M is a model, there are objects $[\![A]\!]_M$ interpreting each type A, as well as interpretations

$$[\![\Gamma \mid \varphi]\!] \rightarrowtail [\![\Gamma]\!]$$

for all formulas and

$$[\![\Gamma \mid t : B]\!] : [\![\Gamma]\!] \longrightarrow [\![B]\!]$$

for all terms. Using these, we determine the functor $M^\sharp : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}$ by taking an object $[\Gamma \mid \varphi]$ to $[\![\Gamma \mid \varphi]\!]_M$, i.e. the domain of a mono representing the subobject $[\![\Gamma \mid \varphi]\!]_M \rightarrowtail [\![\Gamma]\!]_M$. Thus, for the record,

$$M^\sharp[\Gamma \mid \varphi] \;=\; [\![\Gamma \mid \varphi]\!]_M\,.$$

In the verification that those formulas in context $[\Gamma \mid \varphi]$ that are identified in $\mathcal{C}_{\mathbb{T}}$ are also identified in $\mathcal{C}$, we use the fact that the rules of inference for regular logic are sound in the regular category $\mathcal{C}$. Note in particular that for each basic type A, we then have

$$M^\sharp([\![A]\!]_U) = M^\sharp([x : A \mid \top]) \cong [\![x : A \mid \top]\!]_M \cong [\![A]\!]_M\,,$$

so that $M^\sharp(U) \cong M$ as required.

Functional relations in $\mathcal{C}_\mathbb{T}$ determine functional relations in $\mathcal{C}$, again by soundness, which determines the action of $M^\sharp$ on arrows, as well as the functoriality of these assignments.

Finally, to show that eval_U is full and faithful, let $F, G : \mathcal{C}_\mathbb{T} \longrightarrow \mathcal{C}$ be regular functors classifying models FU and GU, and let $h : FU \to GU$ be a model homomorphism. We then have maps

$$h_{[x:A \mid \top]} : F([x : A \mid \top]) \longrightarrow G([x : A \mid \top])$$

for all basic types A, and these commute with the interpretations of the function symbols f, and preserve the basic relations R, in the obvious sense, because h is a homomorphism. It only remains to determine the components

$$h_{[\Gamma \mid \varphi]} : F([\Gamma \mid \varphi]) \to G([\Gamma \mid \varphi]), \tag{2.13}$$

and to show that they commute with all maps $\rho : [\Gamma \mid \varphi] \to [\Delta \mid \psi]$. Define

$$h_{[\Gamma \mid \varphi]} : F[\Gamma \mid \varphi] = [\![\Gamma \mid \varphi]\!]_{FU} \longrightarrow [\![\Gamma \mid \varphi]\!]_{GU} = G[\Gamma \mid \varphi]$$

by induction on the structure of φ. The base cases involving the primitive relations $R, \ldots$ and equality of terms are given by the assumption that $h : FU \to GU$ is a model homomorphism, so we just need to check that for every definable subobject

$$[\![\Gamma \mid \varphi]\!]_{FU} \rightarrowtail [\![\Gamma \mid \top]\!]_{FU}$$

the following diagram can be filled in as indicated.

$$\begin{tikzcd}
[\![\Gamma \mid \varphi]\!]_{FU} \arrow[r, tail] \arrow[d, dotted, "h_{[\Gamma \mid \varphi]}"'] & [\![\Gamma \mid \top]\!]_{FU} \arrow[d, "h_{[\Gamma \mid \top]}"] \\
[\![\Gamma \mid \varphi]\!]_{GU} \arrow[r, tail] & [\![\Gamma \mid \top]\!]_{GU}
\end{tikzcd} \tag{2.14}$$

Suppose we have e.g. $\varphi = \exists x : A.\, \psi$, and we have already determined

$$h_{[\Gamma, x:A \mid \psi]} : [\![\Gamma, x : A \mid \psi]\!]_{FU} \longrightarrow [\![\Gamma, x : A \mid \psi]\!]_{GU}\,.$$

An easy diagram chase shows that there is a unique $h_{[\Gamma \mid \exists x:A.\, \psi]}$ determined by the image factorizations indicated below.

$$\begin{tikzcd}
[\![\Gamma, x : A \mid \psi]\!]_{FU} \arrow[r, two heads] \arrow[d, "h_{[\Gamma, x:A \mid \psi]}"'] & [\![\Gamma \mid \varphi]\!]_{FU} \arrow[r, tail] \arrow[d, dotted, "h_{[\Gamma \mid \exists x:A.\, \psi]}"'] & [\![\Gamma \mid \top]\!]_{FU} \arrow[d, "h_{[\Gamma \mid \top]}"] \\
[\![\Gamma, x : A \mid \psi]\!]_{GU} \arrow[r, two heads] & [\![\Gamma \mid \varphi]\!]_{GU} \arrow[r, tail] & [\![\Gamma \mid \top]\!]_{GU}
\end{tikzcd}$$

The other cases are even more direct. Thus we have defined the components (2.13); we leave the required naturality with respect to all maps $\rho : [\Gamma \mid \varphi] \to [\Delta \mid \psi]$ as an exercise. $\square$

Exercise 2.5.33. Prove the naturality of the maps (2.13), using the following trick. In any category with finite products, suppose we have objects and arrows

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "g"'] & D \end{tikzcd} \tag{2.15}$$

Let $\hat{f} = \langle 1_A, f\rangle : A \rightarrowtail A \times B$ be the graph of f, and similarly for $\hat{g} : C \rightarrowtail C \times D$. Then the diagram (2.15) commutes iff the following one does.

$$\begin{tikzcd} A \arrow[r, tail, "\hat{f}"] \arrow[d, "\alpha"'] & A \times B \arrow[d, "\alpha \times \beta"] \\ C \arrow[r, tail, "\hat{g}"'] & C \times D \end{tikzcd}$$

Corollary 2.5.34. *The rules of regular logic are sound and complete with respect to semantics in regular categories: a regular theory $\mathbb{T}$ proves an entailment*

$$\Gamma \mid \varphi \vdash \psi \tag{2.16}$$

if, and only if, every model of $\mathbb{T}$ satisfies it.

Proof. As for algebraic logic, soundness follows from classification (although we have of course already proved it separately in Proposition 2.6.7, and made use of it in the proof of the theorem!): if (2.16) is provable from $\mathbb{T}$, then it holds in the universal model U in $\mathcal{C}_{\mathbb{T}}$ by the construction of U,

$$U \models \ \Gamma \mid \varphi \vdash \psi\,.$$

But since regular functors preserve the interpretations of regular formulas $[\![\Gamma \mid \varphi]\!], [\![\Gamma \mid \psi]\!]$ (as well as entailments between them), the entailment (2.16) then holds also in any model M in any regular $\mathcal{C}$, since there is a classifying functor $M^\sharp : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}$ taking U to M, for which

$$M^\sharp([\![\Gamma \mid \varphi]\!]_U) \cong [\![\Gamma \mid \varphi]\!]_M\,.$$

Completeness follows from the syntactic construction of the universal model U in $\mathcal{C}_{\mathbb{T}}$. The model U is logically generic, in the sense that

$$U \models \ \Gamma \mid \varphi \vdash \psi \quad \Longleftrightarrow \quad \mathbb{T} \text{ proves } \Gamma \mid \varphi \vdash \psi\,.$$

Thus if $\Gamma \mid \varphi \vdash \psi$ holds in all models, then it holds in particular in U, and is therefore provable. □

2.5.5 Coherent logic

A regular category is coherent if all the subobject posets are distributive lattices, and that structure is stable under pullback. We add rules to regular logic to describe this further structure, show that the rules are sound in coherent categories, and extend the results on functorial semantics of the previous section to the coherent case, including the completeness theorem.

Definition 2.5.35. A cartesian category $\mathcal{C}$ is *coherent* if:

1. $\mathcal{C}$ is regular, i.e. it has coequalizers of kernel pairs, and regular epimorphisms are stable under pullback,
2. each subobject poset $\mathsf{Sub}(A)$ has all finite joins, in particular 0 and $U \vee V$,
3. for each map $f : A \to B$, the pullback functor $f^* : \mathsf{Sub}(B) \longrightarrow \mathsf{Sub}(A)$ preserves the joins:
$$f^* 0_B = 0_A, \qquad f^*(U \vee V) = f^*U \vee f^*V\,.$$

Note that since joins are stable under pullback in a coherent category, the meets distribute over the joins,
$$U \wedge (V \vee W) = (U \wedge V) \vee (U \wedge W)\,, \tag{2.17}$$
so that the posets $\mathsf{Sub}(A)$ are distributive lattices. Indeed, this follows from the fact that $U \wedge V$ may be written as
$$U \wedge V = \Sigma_U \circ U^*(V) \tag{2.18}$$
where $\Sigma_U : \mathsf{Sub}(U) \to \mathsf{Sub}(A)$ is the left adjoint (composition) of the pullback functor $U^* : \mathsf{Sub}(A) \to \mathsf{Sub}(U)$ along the inclusion $U \rightarrowtail A$. Since left adjoints preserve colimits, and thus joins, we therefore have
$$U \wedge (V \vee W) = \Sigma_U \circ U^*(V \vee W) = \Sigma_U \circ U^*(V) \vee \Sigma_U \circ U^*(W) = (U \wedge V) \vee (U \wedge W)\,.$$

A category is said to have have *stable sums* if it has all finite coproducts, in particular an initial object 0 and binary coproducts $A+B$, and these are stable under pullback, in the expected sense. The following simple observation provides plenty of examples of coherent categories.

Proposition 2.5.36. *Regular categories with stable sums are coherent.*

Proof. Given subobjects $U, V \rightarrowtail A$, let $U \vee V$ be the image of the canonical map $U+V \to A$ as indicated below.

$$\begin{tikzcd}
 & U+V \arrow[d, two heads] & \\
U \arrow[ur] \arrow[dr, tail] & U \vee V \arrow[d, tail] & V \arrow[ul] \arrow[dl, tail] \\
 & A &
\end{tikzcd}$$

This is easily seem to be the supremum of U and V in $\mathsf{Sub}(A)$. Since the unique map $0 \to A$ is always monic, it determines the subobject $0 \rightarrowtail A$. Thus $\mathsf{Sub}(A)$ has all finite joins, and they are stable by stability of the coproducts and image factorizations. □

As examples of coherent categories we thus have Set and $\mathsf{Set}_{\text{fin}}$, as well as all functor categories $\mathsf{Set}^{\mathbb{C}}$ since limits and colimits (and thus image factorizations) there are computed pointwise.

Exercise 2.5.37. Is the category of H-presets for a heyting algebra H from Section 2.5.1 coherent?

Coherent logic is the extension of regular logic by adding rules corresponding to joins.

Definition 2.5.38. A *coherent theory* $\mathbb{T}$ is (a type theory together with) a set of axioms expressed in the fragment of logic built from $=$, $\top$, $\bot$, $\wedge$, $\vee$, and $\exists$.

We thus extend the formation rules for formulas in context by two additional clauses:

7. The 0-ary relation symbol $\bot$ (pronounced "false") is a formula :

$$\frac{\cdot}{\Gamma \mid \bot \ \texttt{pred}}$$

8. Disjunction:

$$\frac{\Gamma \mid \varphi \ \texttt{pred} \qquad \Gamma \mid \psi \ \texttt{pred}}{\Gamma \mid \varphi \vee \psi \ \texttt{pred}}$$

(We also again add the evident additional clauses for substitution of terms into formulas.)

A *coherent theory* then consists of axioms of the form

$$\Gamma \mid \varphi \vdash \psi$$

where φ, ψ are *coherent formulas*. Coherent logic not only allows for disjunctions $\varphi \vee \psi$ on both side of the $\vdash$, but the presence of the symbol $\bot$ allows for a certain amount of negation, in the form $\varphi \vdash \bot$, as the following classical example illustrates.

Example 2.5.39.
1. A ring A (with unit 1) is called *local* if it has a unique maximal ideal. This can be captured with two coherent axioms of the form $0 = 1 \vdash \bot$ (to ensure that $0 \neq 1$), and

$$x : A, y : A \mid \exists z : A.\, z(x + y) = 1 \vdash (\exists z : A.\, zx = 1) \vee (\exists z : A.\, zy = 1)$$

2. Another example is the theory of *fields*, which can be axiomatized by again adding to the theory of rings the law $0 = 1 \vdash \bot$, together with the following:

$$x : A \mid \top \vdash x = 0 \vee (\exists y : A.\, xy = 1)$$

which is a clever way of saying that every non-zero element has a multiplicative inverse.

3. An order example is the notion of a *linear order*, which adds to the cartesian theory of posets the *totality* axiom:
$$x : P, y : P \mid x \leq y \vee y \leq q\,.$$

4. For another example of how we can make use of the constant *false* $\bot$ to get the effect of negation, at least for entire axioms, even though the coherent fragment does not include negation, consider the theory of graphs, with two basic sorts E for edges and V for verticies, and two operations $s, t : (E; V)$ for source and target. A graph $G = (E_G, V_G, s_G, t_G)$ is *acyclic* if it satisfies all the finitely many axioms
$$\exists e_1 \ldots e_n : E.\, \big(t(e_1) = s(e_2) \wedge \ldots \wedge t(e_n) = s(e_1)\big) \vdash \bot\,.$$

The *rules of inference of coherent logic* are those of regular logic (Section 2.5.3), with additional rules for falshood the disjunctions:

10. Falsehood:
$$\frac{}{\bot \vdash \psi}$$

11. Disjunction:
$$\frac{\varphi \vdash \vartheta \qquad \psi \vdash \vartheta}{\varphi \vee \psi \vdash \vartheta} \qquad \frac{\varphi \vee \psi \vdash \vartheta}{\varphi \vdash \vartheta} \qquad \frac{\varphi \vee \psi \vdash \vartheta}{\psi \vdash \vartheta}$$

12. Distributivity:
$$\varphi \wedge (\psi \vee \vartheta) \vdash\ (\varphi \wedge \psi) \vee (\varphi \wedge \vartheta)$$

The latter of course coresponds to the distributive law (2.17); note that the converse can be derived. Like the Frobenius rule, this will be derivable in the extended system of Heyting logic (see Proposition 2.6.14), and could also be made derivable in a suitably formulated system of coherent logic using multi-sequents $\Gamma \mid \varphi_1, \ldots, \varphi_n \vdash \psi$.

The *semantics for coherent logic* extends that for regular logic in the expected way: the disjunctive formulas are interpreted as the corresponding joins in the subobject lattices,
$$[\![\Gamma \mid \bot]\!] = 0\,, \qquad\qquad [\![\Gamma \mid \varphi \vee \psi]\!] = [\![\Gamma \mid \varphi]\!] \vee [\![\Gamma \mid \psi]\!]\,.$$

The additional clauses in the proof of soundness are routine. We can then extend the syntactic construction of the regular classifying category $\mathcal{C}_{\mathbb{T}}$ to include all coherent formulas and prove the following extended functorial semantics theorem for models in coherent categories and *coherent functors*, which are defined to be regular functors that preserve all finite joins of subobjects.

Theorem 2.5.40 (Functorial semantics for coherent logic)**.** *For any coherent theory $\mathbb{T}$, the syntactic category $\mathcal{C}_{\mathbb{T}}$ classifies $\mathbb{T}$-models in coherent categories. Specifically, for any coherent category $\mathcal{C}$, there is an equivalence of categories, natural in $\mathcal{C}$,*
$$\mathsf{Coh}(\mathcal{C}_{\mathbb{T}}, \mathcal{C}) \simeq \mathsf{Mod}(\mathbb{T}, \mathcal{C})\,, \tag{2.19}$$
where $\mathsf{Coh}(\mathcal{C}_{\mathbb{T}}, \mathcal{C})$ is the category of coherent functors and natural transformations. In particular, there is a universal model U in $\mathcal{C}_{\mathbb{T}}$.

The corresponding completeness theorem 2.5.34 then holds as well. We leave the routine details to the reader.

Exercise 2.5.41. Extend the functorial semantics theorem 2.5.32 from regular to coherent logic. Specifically, one must determine the components (2.13) of a natural transformation for the extended language of coherent logic.

2.6 Heyting categories

In this section we consider coherent categories that also model the universal quantifier $\forall$, in the sense of Section 2.4; such categories will be seen to model full first-order logic. One could also consider *cartesian* categories modeling $\forall$, without being coherent, and thus modeling the fragment of logic consisting of $u = v, \top, \wedge, \forall$, but we will not do so separately.

Definition 2.6.1. A *Heyting category* is a coherent category with universal quantifiers in the sense of Section 2.4. Thus for every map $f : A \to B$, the pullback functor $f^* : \mathsf{Sub}(B) \to \mathsf{Sub}(A)$ has a right adjoint,

$$\forall_f : \mathsf{Sub}(A) \to \mathsf{Sub}(B)\,,$$

in addition to the left adjoint $\exists_f : \mathsf{Sub}(A) \to \mathsf{Sub}(B)$ given by taking images.

Note that in a Heyting category, one therefore has both adjoints to pullback along any map $f : A \to B$,

$$\begin{tikzcd} \mathsf{Sub}(A) \arrow[r, shift left=2, "\exists_f"] \arrow[r, shift right=2, "\forall_f"'] & \mathsf{Sub}(B) \arrow[l, "f^*" description] \end{tikzcd} \qquad \exists_f \dashv f^* \dashv \forall_f\,. \tag{2.20}$$

Moreover, the Beck-Chevalley conditions from Section 2.4.1 are satisfied for both $\exists_f$ (by Proposition 2.5.15) and $\forall_f$ (by Proposition 2.4.5).

One way to get a Heyting structure on a category $\mathcal{C}$ is when the operations on the subobject lattices $\mathsf{Sub}(A)$ are inherited from related ones on the slice categories $\mathcal{C}/A$; this happens when $\mathcal{C}$ is *locally cartesian closed.* Recall that a cartesian closed category is a category that has products and exponentials. A category of locally cartesian closed when every slice is cartesian closed.

Definition 2.6.2. A category $\mathcal{C}$ is *locally cartesian closed (lccc)* when it has a terminal object and every slice $\mathcal{C}/A$ is cartesian closed.

Note that every slice category $\mathcal{C}/A$ has a terminal object, namely the identity morphism $1_A : A \to A$, and all $\mathcal{C}/A$ have binary products if, and only if, $\mathcal{C}$ has pullbacks. Thus a locally cartesian closed category has all finite limits because it has a terminal object and pullbacks. In addition, a locally cartesian closed category is cartesian closed because $\mathcal{C} \cong \mathcal{C}/1$. We describe how exponentials in a slice $\mathcal{C}/A$ can be computed in terms of *change* *of base functors* and *dependent products*. Given a morphism $f : A \to B$ in $\mathcal{C}$, the "change of base along f" is the pullback functor

$$f^* : \mathcal{C}/B \to \mathcal{C}/A .$$

A right adjoint to f^*, when it exists, is called a *dependent product along* f, denoted

$$\Pi_f : \mathcal{C}/A \to \mathcal{C}/B .$$

Now an exponential of $b : B \to A$ and $c : C \to A$ in $\mathcal{C}/A$ can be computed in terms of Π_b and b^*. For any $d : D \to A$, we have $b \times_A d = (b^* d) \circ b = \Sigma_b(b^* d)$, hence

$$\frac{\dfrac{\dfrac{b \times_A d \to c}{\Sigma_b(b^* d) \to c}}{b^* d \to b^* c}}{d \to \Pi_b(b^* c)}$$

Therefore, $c^b = \Pi_b(b^* c)$.

We have proved that if a cartesian category $\mathcal{C}$ has dependent product $\Pi_f : \mathcal{C}/A \to \mathcal{C}/B$ along every morphism $f : A \to B$ then it is locally cartesian closed. The converse holds as well, that is every lccc has dependent products. For a proof see Section **??** or [Awo10, 9.20].

Proposition 2.6.3. *A category $\mathcal{C}$ with a terminal object is locally cartesian closed if, and only if, for any $f : A \to B$ the change of base functor $f^* : \mathcal{C}/B \to \mathcal{C}/A$ has a right adjoint $\Pi_f : \mathcal{C}/A \to \mathcal{C}/B$.*

Proposition 2.6.4. *In an lccc $\mathcal{C}$, for any $f : A \to B$ the change of base functor $f^* : \mathcal{C}/B \to \mathcal{C}/A$ preserves the ccc structure.*

Proof. We need to show that f^* preserves terminal objects, binary products, and exponentials in slices. Because f^* is a right adjoint it preserves limits, hence it preserves terminal objects and binary products. To see that it preserves exponentials we first show that $f^* \circ \Pi_g \cong \Pi_{f^* g} \circ (g^* f)^*$ for $g : C \to B$. Given any $d : D \to C$, and $e : E \to A$:

$$\begin{array}{c} e \to f^*(\Pi_g d) \\ \hline\hline \Sigma_f e \to \Pi_g d \\ \hline\hline g^*(\Sigma_f e) \to d \\ \hline\hline g^*(f \circ e) \to d \\ \hline\hline (g^* f) \circ ((f^* g)^* e) \to d \\ \hline\hline (f^* g)^* e \to (g^* f)^* d \\ \hline\hline e \to \Pi_{f^* g}((g^* f)^* d) \end{array}$$

By the Yoneda Lemma it follows that $f^*(\Pi_g d) \cong \Pi_{f^*g}((g^*f)^*d)$. Now we have, for any $c : C \to B$ and $d : D \to B$,

$$f^*c^d = f^*(\Pi_d(d^*c)) = \Pi_{f^*d}((d^*f)^*(d^*c)) = \Pi_{f^*d}((f^*d)^*(f^*c)) = (f^*c)^{(f^*d)} .$$

□

Exercise 2.6.5. In the preceding proof we used the fact that $(d^*f)^*(d^*c) \cong (f^*d)^*(f^*c)$ and $g^*(f \circ e) \cong (g^*f) \circ ((f^*g)^*e)$. Prove that this is really so.

Locally cartesian closed categories are an important example of categories with universal quantifiers.

Proposition 2.6.6. *A locally cartesian closed category has universal quantifiers.*

Proof. Suppose $\mathcal{C}$ is locally cartesian closed. First observe that a morphism $m : M \to A$ is mono if, and only if, the morphism

$$\begin{tikzcd} M \arrow[rr, "m"] \arrow[dr, "m"'] & & A \arrow[dl, "\mathsf{1}_A"] \\ & A & \end{tikzcd}$$

is mono in $\mathcal{C}/A$. Because right adjoints preserve monos, $\Pi_f : \mathcal{C}/A \to \mathcal{C}/B$ preserve monos for any $f : A \to B$, that is, if $m : M \rightarrowtail A$ is mono then $\Pi_f m : \Pi_f M \to B$ is mono in $\mathcal{C}$. Therefore, we may define $\forall_f$ as the restriction of Π_f to $\mathsf{Sub}(A)$. To be more precise, a subobject $[m : M \rightarrowtail A]$ is mapped by $\forall_f$ to the subobject $[\Pi_f m : \Pi_f M \rightarrowtail B]$. This works because for any monos $m : M \rightarrowtail A$ and $n : N \rightarrowtail B$ we have

$$\begin{array}{cr} f^*[m : M \rightarrowtail A] \leq [n : N \rightarrowtail B] & \text{in } \mathsf{Sub}(B) \\ \hline\hline f^*m \to n & \text{in } \mathcal{C}/B \\ \hline\hline m \to \Pi_f n & \text{in } \mathcal{C}/A \\ \hline\hline [m] \leq \forall_f[n] & \text{in } \mathsf{Sub}(A) \end{array}$$

The Beck-Chevalley condition for $\forall_f$ follows from Proposition 2.6.4. Indeed, if $g : C \to B$ and $m : M \rightarrowtail C$ then

$$f^*(\Pi_g m) \cong \Pi_{f^*g}((g^*f)^*m) ,$$

therefore

$$f^*(\forall_g[m : M \rightarrowtail C]) = \forall_{f^*g}((g^*f)^*[m : M \rightarrowtail C]) ,$$

as required. □

Summarizing, diagram (2.21), which may be called *Lawvere's hyperdoctrine diagram*, displays the relation between the quantifiers and the change of base functors.

$$
\begin{tikzcd}
\mathcal{C}/A \arrow[rr, shift left=6, "\Sigma_f"] \arrow[rr, shift right=6, "\Pi_f"'] \arrow[d, shift right=3, "\mathsf{im}"'] & & \mathcal{C}/B \arrow[ll, "f^*" description] \arrow[d, shift right=3, "\mathsf{im}"'] \\
\mathsf{Sub}(A) \arrow[rr, shift left=6, "\exists_f"] \arrow[rr, shift right=6, "\forall_f"'] \arrow[u, hook, shift right=3, "I"'] & & \mathsf{Sub}(B) \arrow[ll, "f^*" description] \arrow[u, hook, shift right=3, "I"']
\end{tikzcd}
\tag{2.21}
$$

In Section 2.6.3 below we shall see that all presheaf categories $\mathsf{Set}^{\mathbb{C}^{\mathsf{op}}}$ are Heyting, and therefore have universal quantifiers, which we will compute explicitly (they are *not* pointwise!).

2.6.1 Heyting logic

We can now extend the *formation rules* for the logical language to include universally quantified formulas in the expected way:

$$\frac{\Gamma, x : A \mid \varphi \ \texttt{pred}}{\Gamma \mid \forall x : A.\, \varphi \ \texttt{pred}}$$

The corresponding additional *rule of inference* for the universal quantifier is:

$$\frac{y : B, x : A \mid \vartheta \vdash \varphi}{y : B \mid \vartheta \vdash \forall x : A.\, \varphi}$$

Note that the lower judgement is well-formed only if $x : A$ does not occur freely in ϑ.

Finally, we extend the *interpretation* from coherent formulas from (Section 2.5.5) to formulas including universal quantifiers by the additional clause for $\forall x : A.\, \varphi$ using the universal quantifiers in the Heyting category,

$$[\![\Gamma \mid \forall x : A.\, \varphi]\!] = \forall_A [\![\Gamma, x : A \mid \varphi]\!] \ ,$$

where

$$\forall_A = \forall_\pi : \mathsf{Sub}([\![\Gamma]\!] \times [\![A]\!]) \to \mathsf{Sub}([\![\Gamma]\!])$$

is the universal quantifier along the projection $\pi : [\![\Gamma]\!] \times [\![A]\!] \to [\![\Gamma]\!]$.

The following is then immediate from the results of section 2.4.

Proposition 2.6.7. *The rules for the universal quantifier are sound with respect to the interpretation in Heyting categories.*

Implication

Recall that the rules of inference for implication state that $\Rightarrow$ is right adjoint to $\wedge$:

$$\frac{\Gamma \mid \vartheta \ \mathtt{pred} \qquad \Gamma \mid \varphi \ \mathtt{pred}}{\Gamma \mid (\vartheta \Rightarrow \varphi) \ \mathtt{pred}} \qquad\qquad \frac{\Gamma \mid \psi \wedge \vartheta \vdash \varphi}{\Gamma \mid \psi \vdash \vartheta \Rightarrow \varphi}$$

Exercise 2.6.8. Show that the above two-way rule can be replaced by the following introduction and elimination rules:

$$\frac{\Gamma \mid \psi \wedge \vartheta \vdash \varphi}{\Gamma \mid \psi \vdash \vartheta \Rightarrow \varphi} \qquad\qquad \frac{\Gamma \mid \psi \vdash \vartheta \Rightarrow \varphi \qquad \Gamma \mid \psi \vdash \vartheta}{\Gamma \mid \psi \vdash \varphi}$$

We expect that in order to interpret implication in a cartesian category $\mathcal{C}$ we require $\mathsf{Sub}(A)$ to be a Heyting algebra for every $A \in \mathcal{C}$. However, we must not forget that implication interacts with substitution by the rule

$$(\vartheta \Rightarrow \varphi)[t/x] \ = \ \vartheta[t/x] \Rightarrow \varphi[t/x] \ .$$

Semantically this means that implication is *stable* under pullbacks.

Definition 2.6.9. A cartesian category $\mathcal{C}$ has *implications* when, for every $A \in \mathcal{C}$, the poset $\mathsf{Sub}(A)$ is a Heyting algebra with stable implication $\Rightarrow$. This means that for $U, V \in \mathsf{Sub}(A)$ and $f : B \to A$,

$$f^*(U \Rightarrow V) = (f^*U \Rightarrow f^*V) \ .$$

Proposition 2.6.10. *If a cartesian category has universal quantifiers then it has implications.*

Proof. Let $[u : U \rightarrowtail A]$ and $[v : V \rightarrowtail A]$ be subobjects of A. Define

$$([u] \Rightarrow [v]) = \forall_u (u^*[v]) \ ,$$

as indicated below

$$\begin{tikzcd}
u^*([v]) \arrow[r, tail] \arrow[d, tail] & V \arrow[d, tail, "v"] & \forall_u u^*([v]) \arrow[dl, tail] \\
U \arrow[r, tail, "u"'] & A &
\end{tikzcd}$$

Then for any subobject $[w : W \rightarrowtail A]$ we have:

$$\begin{array}{ll}
[w] \leq [u] \Rightarrow [v] & \text{in } \mathsf{Sub}(A) \\ \hline\hline
[w] \leq \forall_u (u^*[v]) & \text{in } \mathsf{Sub}(A) \\ \hline\hline
u^*[w] \leq u^*[v] & \text{in } \mathsf{Sub}(U) \\ \hline\hline
\exists_u (u^* w) \leq v & \text{in } \mathsf{Sub}(A) \\ \hline\hline
[u] \wedge [w] \leq [v] & \text{in } \mathsf{Sub}(A)
\end{array}$$

Note that we used the decomposition of $[u] \wedge [w]$ as $\exists_u (u^* w)$ from (2.18).

Finally, stability of $\Rightarrow$ follows from Beck-Chevalley condition for $\forall$. $\square$

Exercise 2.6.11. Prove the last claim of the proof.

Corollary 2.6.12. *Any LCCC has universal quantifiers and implications.*

Negation

Now that we have Heyting implication $U \Rightarrow V$ making each $\mathsf{Sub}(A)$ a Heyting algebra, we can also define *negation* $\neg U$ as usual in a Heyting algebra, namely:

$$\neg U \;=\; (U \Rightarrow 0)\,, \tag{2.22}$$

where 0 is the bottom element $[0 \rightarrowtail A]$ of $\mathsf{Sub}(A)$. These negations are stable under pullback because the Heyting implications and the bottom element 0 are stable.

We can therefore add *formulas* with negation to the logical language, along with the evident two-way *rule of inference*:

$$\frac{\Gamma \mid \varphi \ \mathtt{pred}}{\Gamma \mid \neg\varphi \ \mathtt{pred}} \qquad\qquad \frac{\Gamma \mid \vartheta \vdash \neg\varphi}{\Gamma \mid \vartheta \wedge \varphi \vdash \bot}$$

We give negated formulas the obvious *interpretation*: given $[\![\varphi]\!]$ in $\mathsf{Sub}(A)$, we set

$$[\![\neg\varphi]\!] \;=\; \neg[\![\varphi]\!] \;=\; [\![\varphi]\!] \Rightarrow 0\,.$$

using the Heyting implication $\Rightarrow$ and bottom element 0 in $\mathsf{Sub}(A)$. The following is then immediate.

Proposition 2.6.13. *The rules for negation are sound in any Heyting category.*

Given Heyting implication, we can prove the distributivity rule from Section 2.5.5 for conjunction and disjunction.

Proposition 2.6.14. *The distributivity rule is provable in Heyting logic:*

$$\varphi \wedge (\psi \vee \vartheta) \;\vdash\; (\varphi \wedge \psi) \vee (\varphi \wedge \vartheta)$$

Proof.

$$\begin{array}{c}
(\varphi \wedge \psi) \vee (\varphi \wedge \vartheta) \vdash \zeta \\
\hline\hline
(\varphi \wedge \psi) \vdash \zeta \qquad (\varphi \wedge \vartheta) \vdash \zeta \\
\hline\hline
\psi \vdash \varphi \Rightarrow \zeta \qquad \vartheta \vdash \varphi \Rightarrow \zeta \\
\hline\hline
\psi \vee \vartheta \vdash \varphi \Rightarrow \zeta \\
\hline\hline
\varphi \wedge (\psi \vee \vartheta) \vdash \zeta
\end{array}$$

Thus, in fact,

$$\varphi \wedge (\psi \vee \vartheta) \;\dashv\vdash\; (\varphi \wedge \psi) \vee (\varphi \wedge \vartheta).$$

□

Perhaps more surprisingly, given universal quantifiers, we can actually prove the Frobenius rule from Section 2.5.3 for existential quantifiers.

Proposition 2.6.15. *The Frobenius rule is provable in Heyting logic:*

$$(\exists y : B.\, \varphi) \wedge \psi \;\vdash\; \exists y : B.\, (\varphi \wedge \psi)$$

provided the variable $y : B$ does not occur freely in ψ.

Proof.

$$\begin{array}{c} \exists y : B.\, (\varphi \wedge \psi) \vdash \zeta \\ \hline\hline y : B \mid \varphi \wedge \psi \vdash \zeta \\ \hline\hline y : B \mid \varphi \vdash \psi \Rightarrow \zeta \\ \hline\hline (\exists y : B.\, \varphi) \vdash \psi \Rightarrow \zeta \\ \hline\hline (\exists y : B.\, \varphi) \wedge \psi \;\vdash \zeta \end{array}$$

Thus, in fact,

$$(\exists y : B.\varphi) \wedge \psi \;\dashv\vdash\; \exists y : B.(\varphi \wedge \psi).$$

□

Exercise 2.6.16. In classical logic, one has the *de Morgen laws* for negation,

$$\neg(\varphi \wedge \psi) \dashv\vdash \neg\varphi \vee \neg\psi$$
$$\neg(\varphi \vee \psi) \dashv\vdash \neg\varphi \wedge \neg\psi$$

Which of these four entailments can you prove in Heyting logic?

Adjoint rules of Heyting logic

Figure 2.2 collects the rules of inference for Heyting logic, also known as *intuitionistic first-order logic*. These are stated as two-way rules to emphasize the respective underlying adjunctions. The rules for disjunction and conjunction in the bottom-up direction are, of course, to be understood a two separate rules, left and right. The contexts are omitted where there is no change between the top and bottom, thus e.g. the rule for existential quantifier can be stated in full as:

$$\begin{array}{c} \Gamma, x : A \mid \varphi \vdash \vartheta \\ \hline\hline \Gamma \mid \exists x : A\,.\, \varphi \vdash \vartheta \end{array}$$

Negation $\neg\,\varphi$ is treated as a defined by

$$\neg\,\varphi \;:=\; \varphi \Rightarrow \bot\,.$$

$$\frac{}{\bot \vdash \varphi} \qquad\qquad \frac{}{\varphi \vdash \top}$$

$$\frac{\varphi \vdash \vartheta \qquad \psi \vdash \vartheta}{\varphi \vee \psi \vdash \vartheta} \qquad\qquad \frac{\vartheta \vdash \varphi \qquad \vartheta \vdash \psi}{\vartheta \vdash \varphi \wedge \psi}$$

$$\frac{\vartheta \wedge \varphi \vdash \psi}{\vartheta \vdash \varphi \Rightarrow \psi}$$

$$\frac{x : A \mid \varphi \vdash \vartheta}{\exists\, x : A \,.\, \varphi \vdash \vartheta} \qquad\qquad \frac{x : A \mid \vartheta \vdash \varphi}{\vartheta \vdash \forall\, x : A \,.\, \varphi}$$

Figure 2.2: Adjoint rules of inference for Heyting logic

It therefore satisfies the derived rule:

$$\frac{\vartheta \wedge \varphi \vdash \bot}{\vartheta \vdash \neg \varphi}$$

The rules for *equality*, recall from Section 2.3, were:

$$\frac{}{\psi \vdash t =_A t} \qquad\qquad \frac{\psi \vdash t =_A u \qquad \psi \vdash \varphi[t/z]}{\psi \vdash \varphi[u/z]} \tag{2.23}$$

Lawvere [Law70] observed that equality can also be seen as an adjoint, namely to the operation of pullback along the diagonal $\Delta : A \to A \times A$ in any cartesian category. Indeed, we have an adjunction

$$\begin{tikzcd} \mathsf{Sub}(A) \arrow[d, shift right, "\exists_\Delta"'] \\ \mathsf{Sub}(A \times A) \arrow[u, shift right, "\Delta^*"'] \end{tikzcd} \qquad\qquad \frac{x : A \mid \vartheta(x) \vdash \varphi(x, x)}{x : A, y : A \mid (x = y) \wedge \vartheta(x) \vdash \varphi(x, y)} \tag{2.24}$$

where we have displayed the variables in the style $\varphi(x, y)$ in order to emphasize the effect of Δ^* as a "contraction of variables",

$$\Delta^*(\varphi(x, y)) = \varphi(x, x)\,.$$

The effect of the left adjoint $\exists_\Delta$ (which is simply composition with Δ, because it is monic) is given by

$$\exists_\Delta(\vartheta(x)) \;=\; \big(x = y \wedge \vartheta(x)\big)\,.$$

The adjoint rule (2.24) may be called *Lawvere's Law.* It is equivalent to the standard rules (2.23).

Exercise 2.6.17. Prove the equivalence of (2.23) and (2.24).

We state the following for the record as a summary of the foregoing discussion.

Proposition 2.6.18 (Soundness)**.** *The adjoint rules of inference for Heyting logic, as stated in Figure 2.2 and including Lawvere's Law* (2.24)*, are sound in any Heyting category.*

We will show in Section **??** that these rules are also complete.

2.6.2 Intuitionistic first-order logic

Heyting logic with equality is often called *intuitionistic first-order logic*. It lacks the classical laws of excluded middle $\varphi \vee \neg\varphi$ and double negation elimination $\neg\neg\varphi \Rightarrow \varphi$, but adding either one of these implies the other (proof!), and gives a system equivalent to standard first-order logic – with one exception: one still cannot prove the classical law

$$\forall x : A.\, \varphi \vdash \exists x : A\,.\varphi\,. \tag{2.25}$$

The latter law, which is satisfied only in non-empty domains, is considered by many to be a defect of the conventional formulation of first-order logic. It would follow if we were to forget about the contexts, essentially permitting inferences of the form

$$\frac{x : A \mid \varphi \vdash \psi}{\cdot \mid \varphi \vdash \psi} \tag{2.26}$$

when $x : A$ does not occur freely in φ or ψ.

Exercise 2.6.19. Assume the rule (2.26) and prove the entailment (2.25).

Any conventional first-order theory can be formulated in Heyting logic, often in more than one way, since classical logic may collapse differences between concepts that are intuitionistically distinct (like, most simply, φ and $\neg\neg\varphi$). Our interest in intuitionistic logic does not arise from any philosophical scruples about the validity of the classical laws of excluded middle or double negation, but rather the fact that the logic of variable structures is naturally intuitionistic, as we will see in Section **??**.

Example 2.6.20. An example of a first-order theory that is not (immediately) coherent is the theory of dense linear orders. In addition to the poset axioms, and the totality axiom $x, y : P \mid \top \vdash (x \leq y \vee y \leq x)$, one adds density e.g. in the form

$$x, y : P \mid (x \leq y \wedge x \neq y) \vdash (\exists z : P.\, x \leq z \wedge x \neq z \wedge z \leq y \wedge z \neq y)\,.$$

Classifying category for a Heyting theory

Given a theory in first-order intuitionistic logic $\mathbb{T}$, we can build the syntactic category $\mathcal{C}_{\mathbb{T}}$ from the formulas over $\mathbb{T}$, as was done for coherent logic in Section 2.5.4. The objects again have the form $[\Gamma \mid \varphi]$, but now using the Heyting formulas φ, including the logical operations $\forall$, and $\Rightarrow$. The result will then be a coherent category with universal quantifiers, and thus a Heyting category in the sense of Definition 2.6.1. Given another Heyting category $\mathcal{C}$ with a $\mathbb{T}$-model $M \in \mathsf{Mod}(\mathbb{T}, \mathcal{C})$, the interpretation $[\![-]\!]_M$ associated to the model M determines a Heyting functor,

$$M^\sharp : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{C} \tag{2.27}$$

$$[\Gamma \mid \varphi] \longmapsto [\![\Gamma \mid \varphi]\!]_M \tag{2.28}$$

We would like to show that $\mathcal{C}_{\mathbb{T}}$ classifies $\mathbb{T}$-models, in the sense that this assignment determines an equivalence of categories, associating homomorphisms of $\mathbb{T}$-models $h : M \to N$ in the category $\mathsf{Mod}(\mathbb{T}, \mathcal{C})$, and natural transformations of the associated classifying Heyting functors $M^\sharp \to N^\sharp$ in $\mathcal{C}_{\mathbb{T}} \to \mathcal{C}$.

However, there is a problem. Reviewing the proof of Theorem 2.5.32, we needed to show that definable subobjects are natural in model homomorphisms, in the following sense: let $F, G : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{C}$ be functors classifying models FU and GU, and let $h : FU \to GU$ be a model homomorphism. We have maps $h_A : F(A) \longrightarrow G(A)$ for all basic types $A = [x : A \mid \top]$, commuting with the interpretations of the function symbols f and the basic relations R. For each object $[x : A \mid \varphi]$, say, the components

$$h_{[x:A|\varphi]} : F[x : A \mid \varphi] = [\![x : A \mid \varphi]\!]_{FU} \longrightarrow [\![x : A \mid \varphi]\!]_{GU} = G[x : A \mid \varphi]$$

were then defined on definable subobject $[\![x : A \mid \varphi]\!]_{FU} \rightarrowtail [\![A]\!]_{FU} = FA$, in such a way that the following diagram commutes as indicated.

$$\begin{tikzcd}
{[\![x : A \mid \varphi]\!]_{FU}} \arrow[r, tail] \arrow[d, dotted, "h_{[x:A|\varphi]}"'] & {[\![A]\!]_{FU}} \arrow[d, "h_A"] \\
{[\![x : A \mid \varphi]\!]_{GU}} \arrow[r, tail] & {[\![A]\!]_{GU}}
\end{tikzcd} \tag{2.29}$$

This we could do for all *coherent* formulas φ, as was shown by induction on the structure of φ. However, this is no longer possible when φ is Heyting. Most simply, if $\varphi = \neg\psi$ for coherent ψ, there is no need for the following to commute on the left.

$$\begin{tikzcd}
{[\![x : A \mid \neg\psi]\!]_{FU}} \arrow[r, tail] \arrow[d, dotted, "h_{[x:A|\neg\psi]}"'] & {[\![A]\!]_{FU}} \arrow[d, "h_A"] & {[\![x : A \mid \psi]\!]_{FU}} \arrow[l, tail] \arrow[d, "h_{[x:A|\psi]}"] \\
{[\![x : A \mid \neg\psi]\!]_{GU}} \arrow[r, tail] & {[\![A]\!]_{GU}} & {[\![x : A \mid \psi]\!]_{GU}} \arrow[l, tail]
\end{tikzcd} \tag{2.30}$$

Very concretely, let $\mathbb{T}$ be the theory of groups, FU and GU groups in Set and $h_A : [\![A]\!]_{FU} \to [\![A]\!]_{GU}$ the trivial homomorphism that takes everything $a \in [\![A]\!]_{FU}$ to the unit $e_{GU} \in [\![A]\!]_{GU}$, and ψ the formula $x : A \mid x = e$. Then $[\![x : A \mid \psi]\!]_{GU} = \{e_{GU}\}$ and so $[\![x : A \mid \neg\psi]\!]_{GU} = \{y \in [\![A]\!]_{GU} \mid y \neq e_{GU}\}$, so there is a factorization $h_{[x:A|\neg\psi]} : [\![x : A \mid \neg\psi]\!]_{FU} \to [\![x : A \mid \neg\psi]\!]_{GU}$ only if FU is trivial.

The same holds, of course, for subobjects defined by the other Heyting operations, such as $[x : A \mid \vartheta \Rightarrow \psi]$ and $[x : A \mid \forall y : B.\psi]$; there need not be any factorizations $h_{[x:A|\varphi]}$ as indicted in (2.29).

Our solution (although not the only possible one) is to consider only *isomophisms of models* $h : M \cong N$ and *natural isomorphisms* between the classifying functors.

Lemma 2.6.21. *In the situation of diagram* (2.29)*, if the model homomorphism* $h : FU \to GU$ *is an isomorphism, then for any Heyting formula* $[\Gamma \mid \varphi]$ *there is a unique factorization*

$$h_{[\Gamma|\varphi]} : F[\Gamma \mid \varphi] = [\![x : A \mid \varphi]\!]_{FU} \longrightarrow [\![x : A \mid \varphi]\!]_{GU} = G[x : A \mid \varphi]$$

making the corresponding diagram (2.29) *commute.*

Proof. Induction on φ. □

Now for every Heyting category $\mathcal{C}$, let us define $\mathsf{Mod}(\mathbb{T}, \mathcal{C})^i$ to be the category of $\mathbb{T}$-models in $\mathcal{C}$, and their isomorphisms; thus $\mathsf{Mod}(\mathbb{T}, \mathcal{C})^i$ is a groupoid. Accordingly we let $\mathsf{Heyt}(\mathcal{C}_{\mathbb{T}}, \mathcal{C})^i$ to be the category of all Heyting functors $\mathcal{C}_{\mathbb{T}} \to \mathcal{C}$ and natural *iso*morphisms between them – thus also a groupoid. Then just as in previous cases we can show:

Theorem 2.6.22 (Functorial semantics for intuitionistic first-order logic)**.** *For any theory* $\mathbb{T}$ *in (intuitionistic) first-order logic, the syntactic category* $\mathcal{C}_{\mathbb{T}}$ *classifies* $\mathbb{T}$*-models in Heyting categories. Specifically, for any Heyting category* $\mathcal{C}$*, there is an equivalence of categories, natural in* $\mathcal{C}$*,*

$$\mathsf{Heyt}(\mathcal{C}_{\mathbb{T}}, \mathcal{C})^i \simeq \mathsf{Mod}(\mathbb{T}, \mathcal{C})^i \,, \tag{2.31}$$

where $\mathsf{Heyt}(\mathcal{C}_{\mathbb{T}}, \mathcal{C})^i$ *is the groupoid of Heyting functors and natural isomorphisms, and* $\mathsf{Mod}(\mathbb{T}, \mathcal{C})^i$ *is the groupoid of* $\mathbb{T}$*-models in* $\mathcal{C}$*. In particular, there is a universal model* U *in* $\mathcal{C}_{\mathbb{T}}$*.*

The corresponding completeness theorem 2.5.34 for intuitionistic first-order logic with respect to models in Heyting categories then holds as well. We leave the routine details to the reader.

Boolean categories

A Boolean category may be defined as a coherent category in which every subobject $U \rightarrowtail A$ is *complemented*, in the sense that it there is some (necessarily unique) $V \rightarrowtail A$ such that $U \wedge V \leq 0$ and $1 \leq U \vee V$ in $\mathsf{Sub}(A)$. One can then introduce the Boolean negation $\neg U = V$, and show that each $\mathsf{Sub}(A)$ is a Boolean algebra. Indeed one can then show that every Boolean category is Heyting, using the familiar definitions $\forall\varphi = \neg\exists\neg\varphi$ and $\varphi \Rightarrow \psi = \neg\varphi \vee \psi$.

This definition, however, leads to the wrong notion of a "Boolean classifying category", for the reasons just discussed with respect to Heyting categories: although every coherent functor between Boolean categories is Boollean, the natural transformations between classifying functors will not be simply the homomorphisms. (They will be something interesting, namely elementary embeddings, but we shall not pursue this further here; see [**?**].) Thus it seems preferable for our purposed to define a Boolean category to be a Heyting category with complemented subobjects:

Definition 2.6.23. A Heyting catgeory $\mathcal{C}$ is *Boolean* if every subobject lattice $\mathsf{Sub}(A)$ is a Booean algebra. Thus for all subobjects $U \rightarrowtail A$, the Heyting complement $\neg U$ satisfies $U \vee \neg U = 1$ in $\mathsf{Sub}(A)$.

Of course, the category Set is Boolean. A presheaf category $\mathsf{Set}^{\mathbb{C}}$ is in general *not* Boolean, but an important special case always is, namely when $\mathbb{C}$ is a groupoid. (Set^G is called the *category of G-sets.*)

Exercise 2.6.24. Regard a group G as a category with one object. Show that in the functor category Set^G, every subobject lattice $\mathsf{Sub}(A)$ is a Boolean algebra.

The classifying category theorem 2.6.22 for Heyting categories, and indeed the entire framework of functorial semantics, applies *mutatis mutandis* to classical first-order logic and Boolean categories. We will not spell out the details, which do not differ in any unexpected way from the more general Heyting case.

Exercise 2.6.25. Assume that $\mathcal{C}$ is coherent and has complemented subobjects in the sense just defined. Prove that then each $\mathsf{Sub}(A)$ is a Boolean algebra, and that $\mathcal{C}$ is a Heyting category.

Exercise 2.6.26. Show that a Heyting category $\mathcal{C}$ is Boolean if, and only if, in each $\mathsf{Sub}(A)$ the Heyting complement $\neg U$ always satisfies $\neg\neg U = U$.

2.6.3 Examples of Heyting categories

Sets. The category Set is of course complete and cocomplete. It is cartesian closed, with function sets $B^A = \{f : A \to B\}$ as exponentials. It is also locally cartesian closed, because the slice category Set/I is equivalent to the category Set^I of I-indexed families of sets $(A_i)_{i\in I}$, for which the exponentials can be computed pointwise: for $A = (A_i)_{i\in I}$ and $B = (B_i)_{i\in I}$ we can set $B^A = ({B_i}^{A_i})_{i\in I}$. Since pullback is therefore a left adjoint, regular epis are stable and so Set is coherent. It is then Heyting by Proposition 2.6.6.

In order to compute the Heyting structure explicitly, consider any map $f : A \to B$ and the resulting adjunctions from (2.20),

$$\begin{tikzcd} \mathsf{Sub}(A) \arrow[r, bend left, "\exists_f"] \arrow[r, bend right, "\forall_f"'] & \mathsf{Sub}(B) \arrow[l, "f^*"'] \end{tikzcd} \qquad \exists_f \dashv f^* \dashv \forall_f\,.$$

For $U \in \mathsf{Sub}(A)$ and $V \in \mathsf{Sub}(B)$ we then have:

$$\begin{aligned} f^*(V) &= f^{-1}(V) = \{a \in A \mid f(a) \in V\} \\ \exists_f(U) &= \{b \in B \mid \text{for some } a \in f^{-1}\{b\}, a \in U\} \\ \forall_f(U) &= \{b \in B \mid \text{for all } a \in f^{-1}\{b\}, a \in U\} \end{aligned} \tag{2.32}$$

It follows that in Set the implications $U \Rightarrow V$ for $U, V \in \mathsf{Sub}(A)$ have the form

$$\begin{aligned} (U \Rightarrow V) &= \{a \in A \mid a \in U \text{ implies } a \in V\} \\ &= (A - U) \cup V\,. \end{aligned}$$

For negation, we then have

$$\begin{aligned} \neg U &= \{a \in A \mid a \notin U\} \\ &= (A - U)\,, \end{aligned}$$

as expected. Of course, Set is Boolean.

Exercise 2.6.27. In Set consider the dependent sum and product along the unique function $I \to 1$. Show that for $a : A \to I$ the set $\Pi_I A$ is the set of right inverses of a:

$$\Pi_I A = \left\{ s : I \to A \;\middle|\; a \circ s = 1_I \right\}\;.$$

If $(A_i)_{i \in I}$ is a family of sets indexed by I and we take

$$A = \textstyle\coprod_{i \in I} A_i = \left\{ \langle i, x \rangle \in I \times \bigcup_{i \in I} A_i \;\middle|\; i \in I \;\&\; x \in A_i \right\}$$

with $a = \pi_0 : \langle i, x \rangle \mapsto i$ then $\Pi_{!_I} A$ is precisely the cartesian product $\Pi_{i \in I} A_i$. Calculate what Π_f is in Set for a general $f : J \to I$, and conclude that Set is locally cartesian closed.

Presheaves. For a small category $\mathbb{C}$, the presheaf category $\widehat{\mathbb{C}} = \mathsf{Set}^{\mathbb{C}^{\mathsf{op}}}$ has pointwise limits and colimits and is cartesian closed with the exponential of presheaves P, Q calculated using Yoneda as,

$$Q^P(C) \cong \mathsf{Hom}(\mathsf{y}C, Q^P) \cong \mathsf{Hom}(\mathsf{y}C \times P, Q)\,, \qquad \text{for } C \in \mathbb{C}.$$

But then $\widehat{\mathbb{C}}$ is also LCC, because for any presheaf P, the slice category $\widehat{\mathbb{C}}/P$ is equivalent to presheaves on the *category of elements* $\int_{\mathbb{C}} P$,

$$\widehat{\mathbb{C}}/P \;=\; (\mathsf{Set}^{\mathbb{C}^{\mathsf{op}}})/P \;\simeq\; \mathsf{Set}^{(\int_{\mathbb{C}} P)^{\mathsf{op}}}\,.$$

See [Awo10, 9.23].

We first consider the poset $\mathsf{Sub}(P)$ for any presheaf P on $\mathbb{C}$. Let $U \rightarrowtail P$ be any subobject, then since monos in are pointwise in $\widehat{\mathbb{C}}$, and they are represented by subsets in Set, we can represent U by a family $UC \subseteq PC$ of subsets. If $f : P \to Q$ is a natural transformation, the inverse image of $V \rightarrowtail Q$ can then be calculated pointwise from $f_C : PC \to QC$ as

$$f^*(V)(C) = f_C^{-1}(VC) = \{x \in PC \mid f_C(x) \in VC\}\,.$$

The image $\exists_f(U)$, as a coequalizer, is also pointwise, therefore

$$\exists_f(U)(C) = \{y \in QC \mid \text{for some } x \in f_C^{-1}\{y\},\, x \in UC\}\,.$$

The direct image $\forall_f(U)$ is however *not pointwise*, so we must determine it directly. The problem with the obvious attempt

$$\forall_f(U)(C) \overset{?}{=} \{y \in QC \mid \text{for all } x \in f_C^{-1}\{y\},\, x \in UC\}\,.$$

is that it is not functorial in C! In order to correct this, have to modify it by taking instead

$$\forall_f(U)(C) = \{y \in QC \mid \text{for all } h : D \to C, \text{for all } x \in f_D^{-1}\{y.h\},\, x \in UD\}\,, \tag{2.33}$$

where we have written $y.h$ for the action of Q on $y \in QC$, i.e. $Q(h)(y) \in QD$.

Lemma 2.6.28. *The specification* (2.33) *is the universal quantifier* $\forall_f$ *in presheaves.*

Proof. Consider the diagram

$$\begin{tikzcd}
\mathsf{y}C \times_Q P \arrow[rr, dotted] \arrow[dr, "y'"] \arrow[ddd] & & U \arrow[dl, tail] \\
& P \arrow[d, "f"] & \\
& Q & \\
\mathsf{y}C \arrow[ur, "y"] \arrow[rr, dotted] & & \forall_f U \arrow[ul, tail]
\end{tikzcd}$$

For all $y \in QC$, we have $y \in \forall_f U$ iff the pullback $y' = f^*y$ factors through $U \rightarrowtail P$, as indicated. Replacing the pullback $\mathsf{y}C \times_Q P$ by its generalized elements, the latter condition is equivalent to saying that for all $\mathsf{y}D$ and $\mathsf{y}h : \mathsf{y}D \to \mathsf{y}C$ and $x \in PD$, if $f \circ x = y \circ \mathsf{y}h$, then $x \in UD$, as shown below.

$$\begin{tikzcd}
\mathsf{y}D \arrow[rrrd, dotted, bend left] \arrow[rrdd, bend left, "x"] \arrow[rdddd, bend right, "\mathsf{y}h"'] & & & \\
& \mathsf{y}C \times_Q P \arrow[rr, dotted] \arrow[rd, "y'"'] \arrow[ddd] & & U \arrow[ld, tail] \\
& & P \arrow[d, "f"] & \\
& & Q & \\
& \mathsf{y}C \arrow[ru, "y"'] \arrow[rr, dotted] & & \forall_f U \arrow[lu, tail]
\end{tikzcd}$$

But the last condition is equivalent to saying for all D and all $h : D \to C$ and all $x \in PD$, if $x \in f_D^{-1}\{y.h\}$, then $x \in UD$, which is the righthand side of (2.33). $\square$

Proposition 2.6.29. *For any natural transformation $f : P \to Q$, there are adjoints*

$$\begin{tikzcd}
\mathsf{Sub}(P) \arrow[r, bend left, "\exists_f"] \arrow[r, bend right, "\forall_f"'] & \mathsf{Sub}(Q) \arrow[l, "f^*"']
\end{tikzcd} \qquad \exists_f \dashv f^* \dashv \forall_f \,.$$

These are determined by the following formulas, where $U \rightarrowtail P$ and $V \rightarrowtail Q$ and $C \in \mathbb{C}$:

$$\begin{aligned}
&f^*(V)(C) = \{x \in PC \mid f_C(x) \in VC\} \qquad (2.34)\\
&\exists_f(U)(C) = \{y \in QC \mid \text{for some } x \in PC,\ f_C(x) = y \ \&\ x \in UC\}\\
&\forall_f(U)(C) = \{y \in QC \mid \text{for all } h : D \to C, \text{ for all } x \in PD,\ f_D(x) = y.h \text{ implies } x \in UD\}
\end{aligned}$$

$\square$

The implication $U \Rightarrow V$ for $U, V \in \mathsf{Sub}(P)$ therefore has the form, for each $C \in \mathbb{C}$,

$$(U \Rightarrow V)(C) \ = \ \{x \in PC \mid \text{for all } h : D \to C,\ x.h \in UD \text{ implies } x.h \in VD\}\,.$$

And the negation $\neg U \in \mathsf{Sub}(P)$ is then, for each $C \in \mathbb{C}$,

$$(\neg U)(C) \ = \ \{x \in PC \mid \text{for all } h : D \to C,\ x.h \notin UD\}\,.$$

Exercise 2.6.30. Prove the last two statements, computing $U \Rightarrow V$ and $\neg U$.

Sets through time. For presheaves on a poset K, the foregoing description of the Heyting structure becomes a bit simpler. Let us consider "covariant presheaves", i.e. functors $A : K \to \mathsf{Set}$. We can regard such a functor as a "set developing through (branching) time", with each later time $i \leq j$ giving rise to a transition map $A_i \to A_j$, which we may denote by

$$A_i \ni a \longmapsto a_j \in A_j\,.$$

For any map $f : A \to B$ (a family of functions $f_i : A_i \to B_i$ compatible with the development over time), we again have the adjunctions

$$\begin{tikzcd} \mathsf{Sub}(A) \arrow[r, bend left, "\exists_f"] \arrow[r, bend right, "\forall_f"'] & \mathsf{Sub}(B) \arrow[l, "f^*"'] \end{tikzcd} \qquad \exists_f \dashv f^* \dashv \forall_f\,.$$

These can now be described by the following formulas, where $U \in \mathsf{Sub}(A)$ and $V \in \mathsf{Sub}(B)$ and $i \in K$:

$$\begin{aligned} f^*(V)_i &= \{x \in A_i \mid f_i(x) \in V_i\} \\ \exists_f(U)_i &= \{y \in B_i \mid \text{for some } x \in A_i,\ f_i(x) = y \ \&\ x \in U_i\} \\ \forall_f(U)_i &= \{y \in B_i \mid \text{for all } j \geq i, \text{ for all } x \in A_j,\ f_j(x) = y_j \text{ implies } x \in U_j\} \end{aligned} \tag{2.35}$$

The implication $U \Rightarrow V$ for $U, V \in \mathsf{Sub}(A)$ then has the form, for each $i \in K$,

$$(U \Rightarrow V)_i \;=\; \{x \in A_i \mid \text{for all } j \geq i,\ x_j \in U_j \text{ implies } x_j \in V_j\}\,.$$

And the negation $\neg U \in \mathsf{Sub}(A)$ is then, for each $i \in K$,

$$(\neg U)_i \;=\; \{x \in A_i \mid \text{for all } j \geq i,\ x_j \notin U_j\}\,.$$

Exercise 2.6.31. Show that for the arrow category $\mathbf{2} = \cdot \to \cdot$ the functor category $\mathsf{Set}^{\to}$ is *not* Boolean.

Remark 2.6.32 (Bi-Heyting categories)**.** We know by Proposition 2.6.29 that in presheaf categories $\mathsf{Set}^{\mathbb{C}^{\mathsf{op}}}$, each subobject lattice $\mathsf{Sub}(P)$ is a Heyting algebra. Define a *bi-Heyting category* to be a Heyting category in which each $\mathsf{Sub}(P)$ is a *bi-Heyting algebra*, meaning that both $\mathsf{Sub}(P)$ and its opposite $\mathsf{Sub}(P)^{\mathsf{op}}$ are Heyting algebras. One can show that any presheaf category is also bi-Heyting (this follows from the fact that limits and colimits in presheaves are computed pointwise, but see also Exercise 2.6.33 below). See [Law91, MR95, GER96] for more on bi-Heyting categories.

Exercise 2.6.33. Complete the following sketch to show that any presheaf category $\mathsf{Set}^{\mathbb{C}^{\mathsf{op}}}$ is bi-Heyting.

1. Every presheaf P is covered by a coproduct of representables,

$$\coprod_{C \in \mathbb{C},\, x \in PC} \mathsf{y}C \twoheadrightarrow P\,.$$

2. There is therefore an injective lattice homomorphism
$$\mathsf{Sub}(P) \rightarrowtail \prod_{C \in \mathbb{C}, x \in PC} \mathsf{Sub}(\mathsf{y}C)\,.$$
3. It thus suffices to show that all $\mathsf{Sub}(\mathsf{y}C)$ are bi-Heyting.
4. The poset $\mathsf{Sub}(\mathsf{y}C)$ is isomorphic to the poset of *sieves* on C in $\mathbb{C}$: sets S of arrows with codomain C, closed under precomposition by arbitrary arrows, i.e. $(s : C' \to C) \in S$ and $t : C'' \to C'$ implies $s \circ t \in S$.
5. Writing $|\mathbb{D}|$ for the poset reflection of an arbitrary category $\mathbb{D}$, the sieves on C are the same as lower sets in the poset reflection of the slice category $|\mathbb{C}/C|$, thus $\mathsf{Sub}(\mathsf{y}C) \cong {\downarrow}|\mathbb{C}/C|$.
6. For *any* poset P, the poset of lower sets ${\downarrow} P$, ordered by inclusion, form a Heyting algebra.
7. The opposite category of ${\downarrow} P$ is isomorphic to the upper sets ${\uparrow} P$.
8. But since ${\uparrow} P = {\downarrow}(P^{\mathsf{op}})$, by (6) the poset $({\downarrow} P)^{\mathsf{op}}$ is also a Heyting algebra.
9. Thus $\mathsf{Sub}(\mathsf{y}C)$ is a bi-Heyting algebra.

2.7 Kripke-Joyal semantics

In section 2.2, we introduced the idea of using "generalized elements" $z : Z \to C$ as a way of externalizing the interpretation of the logical language. With respect to a subobject $S \rightarrowtail C$, such an element is said to be *in the subobject*, writtten $z \in_C S$, if it factors through $S \rightarrowtail C$.

$$\begin{tikzcd} & S \arrow[d, tail] \\ Z \arrow[ur, dotted] \arrow[r, "z"'] & C \end{tikzcd}$$

Generalized elements provide a way of *testing for satisfaction* of a formula $(x : A \mid \varphi)$ by a model M, as follows. Let A_M be the interpretion of the type A in the model M, so that the formula determines a subobject $[\![x : A \mid \varphi]\!]_M \rightarrowtail A_M$. Note that in Heyting logic, with $\forall$ and $\Rightarrow$, we can consider satisfaction of individual formulas $(x : A \mid \varphi)$ rather than entailments $(x : A \mid \varphi \vdash \psi)$, by turning the latter into $(\top \vdash \forall x : A.\, \varphi \Rightarrow \psi)$.

Definition 2.7.1. For a theory $\mathbb{T}$ in first-order logic we say that a model M *satisfies* a formula $(x : A \mid \varphi)$, written $M \models (x : A \mid \varphi)$, if the subobject $[\![x : A \mid \varphi]\!]_M \rightarrowtail A_M$ is the maximal one 1_{A_M}.

Note that this notion of satisfaction of a formula agrees with our previous notion of satisfaction for the entailment $x : A \mid \top \vdash \varphi$,

$$\begin{aligned} M \models (x : A \mid \varphi) \quad &\text{iff} \quad [\![x : A \mid \varphi]\!]_M = 1_{A_M} \\ &\text{iff} \quad M \models (x : A \mid \top \vdash \varphi)\,. \end{aligned} \tag{2.36}$$

Now observe that the condition $[\![x : A \mid \varphi]\!]_M = 1_{A_M}$ holds just in case every element $z : Z \to A_M$ factors through the subobject $[\![x : A \mid \varphi]\!]_M \rightarrowtail A_M$. It is convenient to use the *forcing* notation $\Vdash$ for this condition, writing

$$Z \Vdash \varphi(z) \quad \text{for} \quad z \in_{A_M} [\![x : A \mid \varphi]\!]_M\,.$$

We can then use forcing to test for satisfaction, by asking whether *all* generalized elements $z : Z \to A_M$ factor through $[\![x : A \mid \varphi]\!]_M \rightarrowtail A_M$, and thus "force" the formula $(x : A \mid \varphi)$:

$$M \models (x : A \mid \varphi) \qquad \text{iff} \qquad \text{for all } z : Z \to A_M,\ Z \Vdash \varphi(z)\,.$$

We summarize these conventions in the following Definition and Lemma.

Definition 2.7.2 (Kripke-Joyal Forcing)**.** In any Heyting category $\mathcal{C}$, define the *forcing relation* $\Vdash$ as follows: for a formula $(x : A \mid \varphi)$ in the langage of a theory $\mathbb{T}$, and a $\mathbb{T}$-model M, let A_M interpret the type symbol A; then for any $z : Z \to A_M$, we define the relation "z forces φ" by

$$\begin{aligned} Z \Vdash \varphi(z) \quad &\text{iff} \quad z \in_{A_M} [\![x : A \mid \varphi]\!]_M \\ &\text{iff} \quad z : Z \to A_M \text{ factors as} \quad \begin{tikzcd} & {[\![x : A \mid \varphi]\!]_M} \arrow[d, tail] \\ Z \arrow[ur, dotted] \arrow[r, "z"'] & A_M \end{tikzcd}\,. \end{aligned} \tag{2.37}$$

Lemma 2.7.3. *For any model M, we have:*

$$M \models (x : A \mid \varphi) \qquad \textit{iff} \qquad \textit{for all } z : Z \to A_M,\ Z \Vdash \varphi(z)\,. \tag{2.38}$$

□

Of course, we also define forcing for formulas with a context of variables $\Gamma = x_1 : A_1, \ldots x_n : A_n$, and then we have

$$M \models (\Gamma \mid \varphi) \qquad \text{iff} \qquad \text{for all } z : Z \to \Gamma_M,\ Z \Vdash \varphi(z)\,.$$

where $\Gamma_M = (A_1)_M \times \ldots \times (A_1)_M$, and $\varphi(z) = \varphi(z_1, \ldots, z_n)$ where $z_i = \pi_i z : Z \to \Gamma_M \to (A_i)_M$. In the extremal case, we have a formula $\cdot \mid \varphi$ with no free variables (a *closed* *formula* or *sentence*), for which the interpretation $[\![\cdot \mid \varphi]\!] \rightarrowtail 1$ is in $\mathsf{Sub}(1)$. For such a closed formula, we have

$$\begin{aligned} M \models (\cdot \mid \varphi) \qquad &\text{iff} \qquad \text{for all } z : Z \to 1,\ Z \Vdash \varphi \\ &\text{iff} \qquad [\![\cdot \mid \varphi]\!] = 1\,. \end{aligned} \tag{2.39}$$

In this sense, the Heyting algebra $\mathsf{Sub}(1)$ contains the *truth-values* of statements $(\cdot \mid \varphi)$ in the internal logic, which hold if and only if $[\![\cdot \mid \varphi]\!] = 1$.

The forcing relation $Z \Vdash \varphi(z)$ defined in (2.37) allows us to turn an internal statement $[\![x : A \mid \varphi]\!]_M$, i.e. a formula interpreted as an object of $\mathcal{C}$, into an external one, i.e. an ordinary statement that makes reference to objects an arrows of $\mathcal{C}$. We first restrict attention to categories of presheaves $\widehat{\mathbb{C}} = \mathsf{Set}^{\mathbb{C}^{\mathsf{op}}}$, for the sake of simplicity (but see Remark 2.7.5 below.) In this case, we can restrict to generalized elements $z : Z \to A_M$ of the special form $c : \mathsf{y}C \to A_M$, i.e. with representable domains, because Lemma 2.7.3 clearly still holds when so restricted: $M \models (x : A \mid \varphi)$ iff for all $c : \mathsf{y}C \to A_M$, we have $\mathsf{y}C \Vdash \varphi(c)$. Moreover, we then write simply $C \Vdash \varphi(c)$ for $\mathsf{y}C \Vdash \varphi(c)$. Observe that because (by Yoneda) $c : \mathsf{y}C \to A_M$ corresponds to $c \in A_M(C)$ in Set, with subset $([\![x : A \mid \varphi]\!]_M)(C) \subseteq A_M(C)$, we have, finally, the equivalence

$$C \Vdash \varphi(c) \quad \text{iff} \quad c \in [\![x : A \mid \varphi]\!]_M(C)\,. \tag{2.40}$$

Theorem 2.7.4 (Kripke-Joyal Semantics)**.** *For any presheaf category $\widehat{\mathbb{C}}$ and model M of a theory $\mathbb{T}$ in first-order logic, let $(x : A \mid \varphi)$, $(x : A \mid \psi)$, and $(x : A, y : B \mid \vartheta)$ be formulas in the language of $\mathbb{T}$, and let $C \in \mathbb{C}$ and $c, c_1, c_2 : \mathsf{y}C \to A_M$ be any maps. Then we have*

1. $C \Vdash \top(c)$ *always.*
2. $C \Vdash \bot(c)$ *never.*
3. $C \Vdash c_1 = c_2$ *iff* $c_1 = c_2$ *as arrows* $\mathsf{y}C \to A_M$*.*
4. $C \Vdash \varphi(c) \wedge \psi(c)$ *iff* $C \Vdash \varphi(c)$ *and* $C \Vdash \psi(c)$*.*
5. $C \Vdash \varphi(c) \vee \psi(c)$ *iff* $C \Vdash \varphi(c)$ *or* $C \Vdash \psi(c)$*.*
6. $C \Vdash \varphi(c) \Rightarrow \psi(c)$ *iff for all* $d : D \to C$, $D \Vdash \varphi(c.d)$ *implies* $D \Vdash \psi(c.d)$*.*
7. $C \Vdash \neg\varphi(c)$ *iff for no* $d : D \to C$, $D \Vdash \varphi(c.d)$*.*
8. $C \Vdash \exists y : B.\, \vartheta(c, y)$ *iff for some* $c' : C \to B_M$, $C \Vdash \vartheta(c, c')$*.*
9. $C \Vdash \forall y : B.\, \vartheta(c, y)$ *iff for all* $d : D \to C$*, for all* $d' : D \to B_M$, $D \Vdash \vartheta(c.d, d')$*.*

Proof. We just do a few cases and leave the rest to the reader.

...

Use (2.34) for the non-obvious cases. □

Examples: LEM, DN, a map is epic, monic, iso. Constant domains.

Remark 2.7.5. There are several variations on Kripke-Joyal semantics for various special kinds of categories: presheaves on a poset P, sheaves on a topological space or a complete Heyting algebra, G-sets for a group or groupoid G, sheaves on a Grothendieck site (i.e. a Grothendieck topos), as well as a general case for arbitrary Heyting categories. Many of these are discussed in [MM92]. In the case of sheaves, the clauses for falsehood $\bot$, disjunction $\vee$, and the existential quantifier $\exists$ typically become more involved. The result is then akin to what is known in constructive logic as Beth semantics.

We next consider another case that is even simpler than presheaves, namely covariant Set-valued functors on a poset P, which may be called "Kripke models".

Exercise 2.7.6. Show that for a group G, regarded as a category with one object, the functor category Set^G is Boolean.

Exercise 2.7.7. Prove Lemma 2.7.3 in the restricted case of presheaves and generalized elements with representable domains, $a : \mathsf{y}C \to A_M$.

2.7.1 Kripke models

As already mentioned, we can regard covariant functors $A : K \to \mathsf{Set}$ on a *poset* K as "sets developing through time". A model in such a category Set^K is a parametrized family of models, $(M_i)_{i \in I}$, or a *variable model*, which can be thought of as changing through space or (non-linearly ordered) time, represented by K. The satisfaction of a formula by such a variable structure can be tested by forcing, as a special case of Theorem 2.7.4. The result becomes simplified somewhat in the clauses for $\forall$ and $\Rightarrow$, in a way that agrees with the original semantics of Kripke [**?**].

Theorem 2.7.8 (Kripke Semantics). *For any first-order theory $\mathbb{T}$ and poset K and model M in the functor category Set^K, let $(x : A \mid \varphi)$, $(x : A \mid \psi)$, and $(x : A, y : B \mid \vartheta)$ be formulas in the language of $\mathbb{T}$, and let $i \in K$ and $a, a_1, a_2 : \mathsf{y}i \to A_M$ be any maps (respectively elements $a, a_1, a_2 \in (A_M)_i$. Then for each $i \in K$ we write $i \Vdash \varphi(a)$ for the relation $a \in (\llbracket x : A \mid \varphi \rrbracket_M)_i$. We can then calculate:*

1. *$i \Vdash \top(a)$ always.*
2. *$i \Vdash \bot(a)$ never.*
3. *$i \Vdash a_1 = a_2$ iff $a_1 = a_2$ as elements of the set $(A_M)_i$.*
4. *$i \Vdash \varphi(a) \wedge \psi(a)$ iff $i \Vdash \varphi(a)$ and $i \Vdash \psi(a)$.*
5. *$i \Vdash \varphi(a) \vee \psi(a)$ iff $i \Vdash \varphi(a)$ or $i \Vdash \psi(a)$.*
6. *$i \Vdash \varphi(a) \Rightarrow \psi(a)$ iff for all $j \geq i$, $j \Vdash \varphi(a_j)$ implies $j \Vdash \psi(a_j)$.*
7. *$i \Vdash \neg\varphi(a)$ iff for no $j \geq i$, $j \Vdash \varphi(a_j)$.*

8. $i \Vdash \exists y : B.\, \vartheta(a, y)$ *iff for some* $b : \mathsf{y}i \to B_M$*,* $i \Vdash \vartheta(a, b)$*.*

9. $i \Vdash \forall y : B.\, \vartheta(a, y)$ *iff for all* $j \geq i$*, for all* $b : \mathsf{y}j \to B_M$*,* $j \Vdash \vartheta(a_j, b)$*.*

Proof. Use (2.35) for the non-obvious cases. □

Examples: LEM, DN, a map is epic, monic, iso. Constant domain, increasing domain, individuals and trans-world identity. Presheaf of real-valued functions on a space is an ordered ring.

2.7.2 Completeness

We know by Theorem 2.6.22 that intuitionstic first-order logic is complete with respect to models in Heyting categories, and moreover, that for every theory $\mathbb{T}$, there is a *generic model*, namely the universal one U in the classifying category $\mathcal{C}_{\mathbb{T}}$. The model U is logically generic in the sense that, for any formula $(x : A \mid \varphi)$, there is an equivalence

$$U \models (x : A \mid \varphi) \quad \text{iff} \quad \mathbb{T} \vdash (x : A \mid \varphi)\,.$$

(The symbol $\vdash$ is once again available for provability from a set of formulas, the axioms of $\mathbb{T}$, now that we can consider single formulas rather than entailments $\varphi \vdash \psi$; see Definition 2.7.1.)

Lemma 2.7.9. *A functor* $F : \mathcal{C} \to \mathcal{D}$ *is said to be* conservative *if it reflects isomorphisms. A conservative Heyting functor between Heyting categories induces an injective homomorphism on the Heyting algebras* $\mathsf{Sub}(A)$ *for all* $A \in \mathcal{C}$*. Such a functor is always faithful.*

Proof. Let $F : \mathcal{C} \to \mathcal{D}$ be Heyting and conservative. The induced functor $\mathsf{Sub}(F) : \mathsf{Sub}(A) \to \mathsf{Sub}(FA)$, taking $U \rightarrowtail A$ to $FU \rightarrowtail FA$, is easily seen to preserve the Heyting operations, because F is Heyting. As for groups, a homomorphism of Heyting algebras is injective iff it has a trivial kernel $\mathsf{Sub}(F)^{-1}(1)$. Let $U \rightarrowtail A$ be in the kernel, i.e. $FU \rightarrowtail FA$ is iso. Then $U \rightarrowtail A$ is iso since F is conservative. To see that F is faithful consider the equalizer of a parallel pair of maps. □

By the foregoing lemma, in order to show completeness of first-order intuitionistic logic with respect to the Kripke-Joyal semantics of Theorem 2.7.4, it will suffice if we can embed $\mathcal{C}_{\mathbb{T}}$ by a conservative Heyting functor into a functor category $\widehat{\mathbb{C}} = \mathsf{Set}^{\mathbb{C}^{\mathsf{op}}}$ for some suitable (small) category $\mathbb{C}$,

$$F : \mathcal{C}_{\mathbb{T}} \rightarrowtail \widehat{\mathbb{C}}\,.$$

For then, if $FU \models (x : A \mid \varphi)$ in $\widehat{\mathbb{C}}$, then $U \models (x : A \mid \varphi)$ in $\mathcal{C}_{\mathbb{T}}$, since

$$\begin{aligned} FU \models (x : A \mid \varphi) \text{ iff } & 1 = [\![x : A \mid \varphi]\!]_{FU} = F([\![x : A \mid \varphi]\!]_U) \\ \text{iff } & 1 = [\![x : A \mid \varphi]\!]_U \\ \text{iff } & U \models (x : A \mid \varphi)\,. \end{aligned}$$

Such an embedding suffices, therefore, to prove completeness with respect to models in categories of the form $\widehat{\mathbb{C}}$, for which we have Kripke-Joyal semantics. The following representation theorem from [MR95] is originally due to Joyal.

Theorem 2.7.10 (Joyal). *For any small Heyting category $\mathcal{H}$ there is a regular category $\mathcal{R}$ and a conservative Heyting functor*

$$\mathcal{H} \rightarrowtail \mathsf{Set}^{\mathcal{R}}. \tag{2.41}$$

The proof of Joyal's theorem is beyond the scope of these notes, but we will mention that the category $\mathcal{R}$ is itself a subcategory of a functor category $\mathcal{R} \hookrightarrow \mathsf{Reg}(\mathcal{H}, \mathsf{Set})$ where $\mathsf{Reg}(\mathcal{H}, \mathsf{Set})$ is the category of all *regular* (not Heyting!) functors $\mathcal{H} \to \mathsf{Set}$. Here we obtain a glimpse of a generalization of Lawvere duality (as well as Stone duality, as emphasized in [MR95]) to regular categories, as developed by Makkai in [**?**]. The remarkable fact here is that the “double dual” embedding 2.41 is not just regular, but Heyting.

Theorem 2.7.11. *Intuitionistic first-order logic is sound and complete with respect to the Kripke-Joyal semantics of 2.7.4. Specifically, for every theory $\mathbb{T}$, there is a model M in a presheaf category $\widehat{\mathbb{C}}$ with the property that, for every closed formula φ,*

$$\mathbb{T} \vdash \varphi \quad \textit{iff} \quad M \models \varphi \quad \textit{iff} \quad \mathbb{C} \Vdash \varphi\,,$$

where by $\mathbb{C} \Vdash \varphi$ we mean $C \Vdash \varphi$ for all $C \in \mathbb{C}$.

Finally, in order to specialize even further to the case of a Kripke model Set^K for a poset K, we can use the following “covering theorem”.

Theorem 2.7.12 (Diaconescu). *For any small category $\mathbb{C}$ there is a poset K and a conservative Heyting functor*

$$\mathsf{Set}^{\mathbb{C}} \rightarrowtail \mathsf{Set}^{K}. \tag{2.42}$$

For a sketch of the proof (see [MM92, IX.9] for details), the poset K may be taken to be $\mathsf{String}(\mathbb{C})$, consisting of finite strings of arrows in $\mathbb{C}$,

$$s = (C_n \xrightarrow{s_n} C_{n-1} \longrightarrow \dots \longrightarrow C_1 \xrightarrow{s_1} C_0)$$

ordered by $t \leq s$ iff t extends s to the left, i.e. $s_i = t_i$ for all s_i in the string s. There is an evident functor

$$\pi : \mathsf{String}(\mathbb{C}) \longrightarrow \mathbb{C}$$

taking $s = (s_0, \dots, s_n)$ to the “first” object C_n and $t \leq s$ to the evident composite of the extra initial t's. The functor π induces one on the functor categories by precomposition

$$\pi^* : \mathsf{Set}^{\mathbb{C}} \longrightarrow \mathsf{Set}^{\mathsf{String}(\mathbb{C})}.$$

One can show by a direct calculation that π^* is Heyting and that it is conservative, using the fact that π is surjective on both arrows and objects.

Corollary 2.7.13. *Intuitionistic first-order logic is sound and complete with respect to the Kripke semantics of Theorem 2.7.8. Specifically, for every theory $\mathbb{T}$, there is a poset K and a model M in Set^K with the property that, for every closed formula φ,*

$$\mathbb{T} \vdash \varphi \quad \textit{iff} \quad M \models \varphi \quad \textit{iff} \quad K \Vdash \varphi\,,$$

where by $K \Vdash \varphi$ we mean $k \Vdash \varphi$ for all $k \in K$.

Appendix A

Category Theory

A.1 Categories

Definition A.1.1. A *category* $\mathcal{C}$ consists of classes

$\mathcal{C}_0$ of objects A, B, C, ...
$\mathcal{C}_1$ of morphisms f, g, h, ...

such that:

- Each morphism f has uniquely determined *domain* $\mathsf{dom}\, f$ and *codomain* $\mathsf{cod}\, f$, which are objects. This is written as

$$f : \mathsf{dom}\, f \to \mathsf{cod}\, f$$

- For any morphisms $f : A \to B$ and $g : B \to C$ there exists a uniquely determined *composition* $g \circ f : A \to C$. Composition is associative:

$$h \circ (g \circ f) = (h \circ g) \circ f \ ,$$

where domains are codomains are as follows:

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

- For every object A there exists the *identity* morphism $\mathsf{1}_A : A \to A$ which is a unit for composition:

$$\mathsf{1}_A \circ f = f \ , \qquad\qquad g \circ \mathsf{1}_A = g \ ,$$

where $f : B \to A$ and $g : A \to C$.

Morphisms are also called *arrows* or *maps*. Note that morphisms do not actually have to be functions, and objects need not be sets or spaces of any sort. We often write $\mathcal{C}$ instead of $\mathcal{C}_0$.

Definition A.1.2. A category $\mathcal{C}$ is *small* when the objects $\mathcal{C}_0$ and the morphisms $\mathcal{C}_1$ are sets (as opposed to proper classes). A category is *locally small* when for all objects $A, B \in \mathcal{C}_0$ the class of morphisms with domain A and codomain B is a set.

We normally restrict attention to locally small categories, so unless we specify otherwise all categories are taken to be locally small. Next we consider several examples of categories.

The empty category $\mathsf{0}$ The empty category has no objects and no arrows.

The unit category $\mathsf{1}$ The unit category, also called the terminal category, has one object $\star$ and one arrow $1_\star$:

$$\begin{tikzcd} \star \arrow[loop right, "1_\star"] \end{tikzcd}$$

Other finite categories There are other finite categories, for example the category with two objects and one (non-identity) arrow, and the category with two parallel arrows:

$$\begin{tikzcd} \star \arrow[r] & \bullet \end{tikzcd} \qquad\qquad \begin{tikzcd} \star \arrow[r, bend left] \arrow[r, bend right] & \bullet \end{tikzcd}$$

Groups as categories Every group $(G, \cdot)$, is a category with a single object $\star$ and each element of G as a morphism:

$$\begin{tikzcd} \star \arrow[loop above, "b"] \arrow[loop right, "a"] \arrow[loop below, "c"] \end{tikzcd} \qquad\qquad a, b, c, \ldots \in G$$

The composition of arrows is given by the group operation:

$$a \circ b = a \cdot b$$

The identity arrow is the group unit e. This is indeed a category because the group operation is associative and the group unit is the unit for the composition. In order to get a category, we do not actually need to know that every element in G has an inverse. It suffices to take a *monoid*, also known as *semigroup*, which is an algebraic structure with an associative operation and a unit.

We can turn things around and *define* a monoid to be a category with a single object. A group is then a category with a single object in which every arrow is an *isomorphism* (in the sense of definition A.1.5 below).

Posets as categories Recall that a *partially ordered set*, or a *poset* $(P, \leq)$, is a set with a reflexive, transitive, and antisymmetric relation:

$$x \leq x \tag{reflexive}$$

$$x \leq y \wedge y \leq z \Rightarrow x \leq z \tag{transitive}$$

$$x \leq y \wedge y \leq z \Rightarrow x = y \tag{antisymmetric}$$

Each poset is a category whose objects are the elements of P, and there is a single arrow $p \to q$ between $p, q \in P$ if, and only if, $p \leq q$. Composition of $p \to q$ and $q \to r$ is the unique arrow $p \to r$, which exists by transitivity of $\leq$. The identity arrow on p is the unique arrow $p \to p$, which exists by reflexivity of $\leq$.

Antisymmetry tells us that any two isomorphic objects in P are equal.[1] We do not need antisymmetry in order to obtain a category, i.e., a *preorder* would suffice.

Again, we may *define* a preorder to be a category in which there is at most one arrow between any two objects. A poset is a skeletal preorder. We allow for the possibility that a preorder or a poset is a proper class rather than a set.

A particularly important example of a poset category is the poset of open sets $\mathcal{O}X$ of a topological space X, ordered by inclusion.

Sets as categories Any set S is a category whose objects are the elements of S and the only arrows are identity arrows. A category in which the only arrows are the identity arrows is a *discrete category*.

A.1.1 Structures as categories

In general, structures like groups, topological spaces, posets, etc., determine categories in which composition is composition of functions and identity morphisms are identity functions:

- Group is the category whose objects are groups and whose morphisms are group homomorphisms.
- Top is the category whose objects are topological spaces and whose morphisms are continuous maps.
- Set is the category whose objects are sets and whose morphisms are functions.[2]
- Graph is the category of (directed) graphs an graph homomorphisms.
- Poset is the category of posets and monotone maps.

Such categories of structures are generally *large*, but locally small.

Exercise A.1.3. The *category of relations* Rel has as objects all sets $A, B, C, \ldots$ and as arrows $A \to B$ the relations $R \subseteq A \times B$. The composite of $R \subseteq A \times B$ and $S \subseteq B \times C$, and the identity arrow on A, are defined by:

$$\begin{aligned} S \circ R &= \left\{ \langle x, z \rangle \in A \times C \;\middle|\; \exists y \in B \,.\, xRy \;\&\; ySz \right\}, \\ 1_A &= \left\{ \langle x, x \rangle \;\middle|\; x \in A \right\}. \end{aligned}$$

Show that this is indeed a category!

[1] A category in which isomorphic object are equal is a *skeletal* category.

[2] A function between sets A and B is a relation $f \subseteq A \times B$ such that for every $x \in A$ there exists a unique $y \in B$ for which $\langle x, y \rangle \in f$. A morphism in Set is a triple $\langle A, f, B \rangle$ such that $f \subseteq A \times B$ is a function.

A.1.2 Further definitions

We recall some further basic notions in category theory.

Definition A.1.4. A *subcategory* $\mathcal{C}'$ of a category $\mathcal{C}$ is given by a subclass of objects $\mathcal{C}'_0 \subseteq \mathcal{C}_0$ and a subclass of morphisms $\mathcal{C}'_1 \subseteq \mathcal{C}_1$ such that $f \in \mathcal{C}'_1$ implies $\mathsf{dom}\, f, \mathsf{cod}\, f \in \mathcal{C}'_0$, $1_A \in \mathcal{C}'_1$ for every $A \in \mathcal{C}'_0$, and $g \circ f \in \mathcal{C}'_1$ whenever $f, g \in \mathcal{C}'_1$ are composable.

A subcategory $\mathcal{C}'$ of $\mathcal{C}$ is *full* if for all $A, B \in \mathcal{C}'_0$, every $f : A \to B$ in $\mathcal{C}_1$ is also in $\mathcal{C}'_1$.

Definition A.1.5. An *inverse* of a morphism $f : A \to B$ is a morphism $f^{-1} : B \to A$ such that

$$f \circ f^{-1} = 1_B \qquad \text{and} \qquad f^{-1} \circ f = 1_A \ .$$

A morphism that has an inverse is an *isomorphism*, or an *iso*. If there exists a pair of inverse morphisms $f : A \to B$ and $f^{-1} : B \to A$ we say that the objects A and B are *isomorphic*, written $A \cong B$.

The notation f^{-1} is justified because an inverse, if it exists, is unique. A *left inverse* is a morphism $g : B \to A$ such that $g \circ f = 1_A$, and a *right inverse* is a morphism $g : B \to A$ such that $f \circ g = 1_B$. A left inverse is also called a *retraction*, whereas a right inverse is called a *section*.

Definition A.1.6. A *monomorphism*, or *mono*, is a morphism $f : A \to B$ that can be cancelled on the left: for all $g : C \to A$, $h : C \to A$,

$$f \circ g = f \circ h \Rightarrow g = h \ .$$

An *epimorphism*, or *epi*, is a morphism $f : A \to B$ that can be cancelled on the right: for all $g : B \to C$, $h : B \to A$,

$$g \circ f = h \circ f \Rightarrow g = h \ .$$

In Set monomorphisms are the injective functions and epimorphisms are the surjective functions. Isomorphisms in Set are the bijective functions. Thus, in Set a morphism is iso if, and only if, it is both mono and epi. However, this example is misleading! In general, a morphism can be mono and epi without being an iso. For example, the non-identity morphism in the category consisting of two objects and one morphism between them is both epi and mono, but it has no inverse. (See examples in the next section.)

A more realistic example of morphisms that are both epi and mono but are not iso occurs in the category Top of topological spaces and continuous maps, because not every continuous bijection is a homeomorphism.

A *diagram* of objects and morphisms is a directed graph whose vertices are objects of a category and edges are morphisms between them, for example:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[r, "h"] & C \arrow[dl, "j"] \\
D \arrow[ur, "m"] \arrow[r, "k"'] & E &
\end{tikzcd}$$

Such a diagram is said to *commute* when the composition of morphisms along any two paths with the same beginning and end gives equal morphisms. Commutativity of the above diagram is equivalent to the following two equations:

$$f = m \circ g \,, \qquad\qquad j \circ h \circ m = k \,.$$

From these we can derive $k \circ g = j \circ h \circ f$ by a *diagram chase*.

A.2 Functors

Definition A.2.1. A *functor* $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ consists of functions

$$F_0 : \mathcal{C}_0 \to \mathcal{D}_0 \qquad \text{and} \qquad F_1 : \mathcal{C}_1 \to \mathcal{D}_1$$

such that, for all $f : A \to B$ and $g : B \to C$ in $\mathcal{C}$:

$$\begin{aligned} F_1 f &: F_0 A \to F_0 B \,, \\ F_1(g \circ f) &= (F_1 g) \circ (F_1 f) \,, \\ F_1(\mathsf{1}_A) &= \mathsf{1}_{F_0 A} \,. \end{aligned}$$

We usually write F for both F_0 and F_1.

A functor maps commutative diagrams to commutative diagrams because it preserves composition.

We may form the "category of categories" Cat whose objects are small categories and whose morphisms are functors. Composition of functors is composition of the corresponding functions, and the identity functor is one that is identity on objects and on morphisms. The category Cat is large and locally small.

Definition A.2.2. A functor $F : \mathcal{C} \to \mathcal{D}$ is *faithful* when it is "locally injective on morphisms", in the sense that for all $f, g : A \to B$, if $Ff = Fg$ then $f = g$.

A functor $F : \mathcal{C} \to \mathcal{D}$ is *full* when it is "locally surjective on morphisms": for every $g : FA \to FB$ there exists $f : A \to B$ such that $g = Ff$.

We consider several examples of functors.

A.2.1 Functors between sets, monoids and posets

When sets, monoids, groups, and posets are regarded as categories, the functors turn out to be the *usual morphisms*, for example:

- A functor between sets S and T is a function from S to T.
- A functor between groups G and H is a group homomorphism from G to H.
- A functor between posets P and Q is a monotone function from P to Q.

Exercise A.2.3. Verify that the above claims are correct.

A.2.2 Forgetful functors

For categories of structures Group, Top, Graphs, Poset, ..., there is a *forgetful* functor U which maps an object to the underlying set and a morphism to the underlying function. For example, the forgetful functor $U : \mathsf{Group} \to \mathsf{Set}$ maps a group $(G, \cdot)$ to the set G and a group homomorphism $f : (G, \cdot) \to (H, \star)$ to the function $f : G \to H$.

There are also forgetful functors that forget only part of the structure, for example the forgetful functor $U : \mathsf{Ring} \to \mathsf{Group}$ which maps a ring $(R, +, \times)$ to the additive group $(R, +)$ and a ring homomorphism $f : (R, +_R, \cdot_S) \to (S, +_S, \cdot_S)$ to the group homomorphism $f : (R, +_R) \to (S, +_S)$. Note that there is another forgetful functor $U' : \mathsf{Ring} \to \mathsf{Mon}$ from rings to monoids.

Exercise A.2.4. Show that taking the graph $\Gamma(f) = \big\{\langle x, f(x)\rangle \bigm| x \in A\big\}$ of a function $f : A \to B$ determines a functor $\Gamma : \mathsf{Set} \to \mathsf{Rel}$, from sets and functions to sets and relations, which is the identity on objects. Is this a forgetful functor?

A.3 Constructions of Categories and Functors

A.3.1 Product of categories

Given categories $\mathcal{C}$ and $\mathcal{D}$, we form the *product category* $\mathcal{C} \times \mathcal{D}$ whose objects are pairs of objects $\langle C, D\rangle$ with $C \in \mathcal{C}$ and $D \in \mathcal{D}$, and whose morphisms are pairs of morphisms $\langle f, g\rangle : \langle C, D\rangle \to \langle C', D'\rangle$ with $f : C \to C'$ in $\mathcal{C}$ and $g : D \to D'$ in $\mathcal{D}$. Composition is given by $\langle f, g\rangle \circ \langle f', g'\rangle = \langle f \circ f', g \circ g'\rangle$.

There are evident *projection* functors

$$
\begin{tikzcd}
& \mathcal{C} \times \mathcal{D} \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & \\
\mathcal{C} & & \mathcal{D}
\end{tikzcd}
$$

which act as indicated in the following diagrams:

$$
\begin{tikzcd}
& \langle C, D\rangle \arrow[dl, mapsto, "\pi_0"'] \arrow[dr, mapsto, "\pi_1"] & \\
C & & D
\end{tikzcd}
\qquad
\begin{tikzcd}
& \langle f, g\rangle \arrow[dl, mapsto, "\pi_0"'] \arrow[dr, mapsto, "\pi_1"] & \\
f & & g
\end{tikzcd}
$$

Exercise A.3.1. Show that, for any categories $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, there are distinguished isos:

$$
\begin{aligned}
\mathsf{1} \times \mathbb{C} &\cong \mathbb{C} \\
\mathbb{B} \times \mathbb{C} &\cong \mathbb{C} \times \mathbb{B} \\
\mathbb{A} \times (\mathbb{B} \times \mathbb{C}) &\cong (\mathbb{A} \times \mathbb{B}) \times \mathbb{C}
\end{aligned}
$$

Does this make Cat a (commutative) monoid?

A.3.2 Slice categories

Given a category $\mathcal{C}$ and an object $A \in \mathcal{C}$, the *slice* category $\mathcal{C}/A$ has as objects morphisms into A,

$$\begin{tikzcd} B \arrow[d, "f"] \\ A \end{tikzcd} \tag{A.1}$$

and as morphisms commutative diagrams over A,

$$\begin{tikzcd} B \arrow[rr, "g"] \arrow[dr, "f"'] & & B' \arrow[dl, "f'"] \\ & A & \end{tikzcd} \tag{A.2}$$

That is, a morphism from $f : B \to A$ to $f' : B' \to A$ is a morphism $g : B \to B'$ such that $f = f' \circ g$. Composition of morphisms in $\mathcal{C}/A$ is composition of morphisms in $\mathcal{C}$.

There is a forgetful functor $U_A : \mathcal{C}/A \to \mathcal{C}$ which maps an object (A.1) to its domain B, and a morphism (A.2) to the morphism $g : B \to B'$.

Furthermore, for each morphism $h : A \to A'$ in $\mathcal{C}$ there is a functor "composition by h",

$$\mathcal{C}/h : \mathcal{C}/A \to \mathcal{C}/A'$$

which maps an object (A.1) to the object $h \circ f : B \to A'$ and a morphisms (A.2) to the morphism

$$\begin{tikzcd} B \arrow[rr, "g"] \arrow[dr, "h \circ f"'] & & B' \arrow[dl, "h \circ f'"] \\ & A' & \end{tikzcd}$$

The construction of slice categories itself is a functor

$$\mathcal{C}/- : \mathcal{C} \to \mathsf{Cat}$$

provided that $\mathcal{C}$ is small. This functor maps each $A \in \mathcal{C}$ to the category $\mathcal{C}/A$ and each morphism $h : A \to A'$ to the functor $\mathcal{C}/h : \mathcal{C}/A \to \mathcal{C}/A'$.

Since Cat is itself a category, we may form the slice category $\mathsf{Cat}/\mathcal{C}$ for any small category $\mathcal{C}$. The slice functor $\mathcal{C}/-$ then factors through the forgetful functor $U_{\mathcal{C}} : \mathsf{Cat}/\mathcal{C} \to \mathsf{Cat}$ via a functor $\overline{\mathcal{C}} : \mathcal{C} \to \mathsf{Cat}/\mathcal{C}$,

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\overline{\mathcal{C}}"] \arrow[dr, "\mathcal{C}/-"'] & \mathsf{Cat}/\mathcal{C} \arrow[d, "U_{\mathcal{C}}"] \\ & \mathsf{Cat} \end{tikzcd}$$

where, for $A \in \mathcal{C}$, the object part $\overline{\mathcal{C}}A$ is

$$\begin{tikzcd} \mathcal{C}/A \arrow[d, "U_A"] \\ \mathcal{C} \end{tikzcd}$$

and, for $h : A \to A'$ in $\mathcal{C}$, the morphism part $\overline{\mathcal{C}}h$ is

$$\begin{tikzcd} \mathcal{C}/A \arrow[rr, "\mathcal{C}/h"] \arrow[dr, "U_A"'] & & \mathcal{C}/A' \arrow[dl, "U_{A'}"] \\ & \mathcal{C} & \end{tikzcd}$$

A.3.3 Arrow categories

Similar to the slice categories, an arrow category has arrows as objects, but without a fixed codomain. Given a category $\mathcal{C}$, the *arrow* category $\mathcal{C}^{\to}$ has as objects the morphisms of $\mathcal{C}$,

$$\begin{tikzcd} A \arrow[d, "f"] \\ B \end{tikzcd} \tag{A.3}$$

and as morphisms $f \to f'$ the commutative squares,

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\ B \arrow[r, "g'"'] & B'. \end{tikzcd} \tag{A.4}$$

That is, a morphism from $f : A \to B$ to $f' : A' \to B'$ is a pair of morphisms $g : A \to A'$ and $g' : B \to B'$ such that $g' \circ f = f' \circ g$. Composition of morphisms in $\mathcal{C}^{\to}$ is just componentwise composition of morphisms in $\mathcal{C}$.

There are two evident forgetful functors $U_1, U_2 : \mathcal{C}^{\to} \to \mathcal{C}$, given by the domain and codomain operations.

A.3.4 Opposite categories

For a category $\mathcal{C}$ the *opposite category* $\mathcal{C}^{\mathsf{op}}$ has the same objects as $\mathcal{C}$, but all the morphisms are turned around, that is, a morphism $f : A \to B$ in $\mathcal{C}^{\mathsf{op}}$ is a morphism $f : B \to A$ in $\mathcal{C}$. Composition and identity arrows in $\mathcal{C}^{\mathsf{op}}$ are the same as in $\mathcal{C}$. Clearly, the opposite of the opposite of a category is the original category.

A functor $F : \mathcal{C}^{\mathsf{op}} \to \mathcal{D}$ is sometimes called a *contravariant functor* (from $\mathcal{C}$ to $\mathcal{D}$), and a functor $F : \mathcal{C} \to \mathcal{D}$ is a *covariant* functor.

For example, the opposite category of a preorder $(P, \leq)$ is the preorder P turned upside down, $(P, \geq)$.

Exercise A.3.2. Given a functor $F : \mathcal{C} \to \mathcal{D}$, can you define a functor $F^{\mathsf{op}} : \mathcal{C}^{\mathsf{op}} \to \mathcal{D}^{\mathsf{op}}$ in such a way that $-^{\mathsf{op}}$ itself becomes a functor? On what category is it a functor?

A.3.5 Representable functors

Let $\mathcal{C}$ be a locally small category. Then for each pair of objects $A, B \in \mathcal{C}$ the collection of all morphisms $A \to B$ forms a set, written $\mathsf{Hom}_{\mathcal{C}}(A, B)$, $\mathsf{Hom}(A, B)$ or $\mathcal{C}(A, B)$. For every $A \in \mathcal{C}$ there is a functor

$$\mathcal{C}(A, -) : \mathcal{C} \to \mathsf{Set}$$

defined by

$$\begin{aligned} &\mathcal{C}(A, B) = \left\{ f \in \mathcal{C}_1 \;\middle|\; f : A \to B \right\} \\ &\mathcal{C}(A, g) : f \mapsto g \circ f \end{aligned}$$

where $B \in \mathcal{C}$ and $g : B \to C$. In words, $\mathcal{C}(A, g)$ is composition by g. This is indeed a functor because, for any morphisms

$$A \xrightarrow{\;f\;} B \xrightarrow{\;g\;} C \xrightarrow{\;h\;} D \tag{A.5}$$

we have

$$\mathcal{C}(A, h \circ g)f = (h \circ g) \circ f = h \circ (g \circ f) = \mathcal{C}(A, h)(\mathcal{C}(A, g)f) \ ,$$

and $\mathcal{C}(A, 1_B)f = 1_A \circ f = f = 1_{\mathcal{C}(A,B)}f$.

We may also ask whether $\mathcal{C}(-, B)$ is a functor. If we define its action on morphisms to be precomposition,

$$\mathcal{C}(f, B) : g \mapsto g \circ f \ ,$$

it becomes a *contravariant* functor,

$$\mathcal{C}(-, B) : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set} \ .$$

The contravariance is a consequence of precomposition; for morphisms (A.5) we have

$$\mathcal{C}(g \circ f, D)h = h \circ (g \circ f) = (h \circ g) \circ f = \mathcal{C}(f, D)(\mathcal{C}(g, D)h) \ .$$

A functor of the form $\mathcal{C}(A, -)$ is a *(covariant) representable functor*, and a functor of the form $\mathcal{C}(-, B)$ is a *(contravariant) representable functor*.

It follows that the hom-set is a functor

$$\mathcal{C}(-, -) : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Set}$$

which maps a pair of objects $A, B \in \mathcal{C}$ to the set $\mathcal{C}(A, B)$ of morphisms from A to B, and it maps a pair of morphisms $f : A' \to A$, $g : B \to B'$ in $\mathcal{C}$ to the function

$$\mathcal{C}(f, g) : \mathcal{C}(A, B) \to \mathcal{C}(A', B')$$

defined by

$$\mathcal{C}(f, g) : h \mapsto g \circ h \circ f \ .$$

A.3.6 Group actions

A group $(G, \cdot)$ is a category with one object $\star$ and elements of G as the morphisms. Thus, a functor $F : G \to \mathsf{Set}$ is given by a set $F\star = S$ and for each $a \in G$ a function $Fa : S \to S$ such that, for all $x \in S$, $a, b \in G$,

$$(Fe)x = x \ , \qquad\qquad (F(a \cdot b))x = (Fa)((Fb)x) \ .$$

Here e is the unit element of G. If we write $a \cdot x$ instead of $(Fa)x$, the above two equations become the familiar requirements for a *left group action on the set* S:

$$e \cdot x = x \ , \qquad\qquad (a \cdot b) \cdot x = a \cdot (b \cdot x) \ .$$

Exercise A.3.3. A *right group action* by a group $(G, \cdot)$ on a set S is an operation $\cdot : S \times G \to S$ that satisfies, for all $x \in S$, $a, b \in G$,

$$x \cdot e = x \ , \qquad\qquad x \cdot (a \cdot b) = (x \cdot a) \cdot b \ .$$

Exhibit right group actions as functors.

A.4 Natural Transformations and Functor Categories

Definition A.4.1. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\eta : F \Longrightarrow G$ from F to G is a map $\eta : \mathcal{C}_0 \to \mathcal{D}_1$ which assigns to every object $A \in \mathcal{C}$ a morphism $\eta_A : FA \to GA$, called the *component of* η *at* A, such that for every $f : A \to B$ in $\mathcal{C}$ we have $\eta_B \circ Ff = Gf \circ \eta_A$, i.e., the following diagram in $\mathcal{D}$ commutes:

$$\begin{tikzcd}
FA \arrow[r, "\eta_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\
FB \arrow[r, "\eta_B"'] & GB
\end{tikzcd}$$

A simple example is given by the "twist" isomorphism $t : A \times B \to B \times A$ (in Set). Given any maps $f : A \to A'$ and $g : B \to B'$, there is a commutative square:

$$\begin{tikzcd}
A \times B \arrow[r, "t_{A,B}"] \arrow[d, "f \times g"'] & B \times A \arrow[d, "g \times f"] \\
A' \times B' \arrow[r, "t_{A',B'}"'] & B' \times A'
\end{tikzcd}$$

Thus naturality means that the two *functors* $F(X,Y) = X \times Y$ and $G(X,Y) = Y \times X$ are related to each other (by $t : F \to G$), and not simply their individual values $A \times B$ and $B \times A$. As a further example of a natural transformation, consider groups G and H as categories and two homomorphisms $f, g : G \to H$ as functors between them. A natural transformation $\eta : f \Longrightarrow g$ is given by a single element $\eta_\star = b \in H$ such that, for every $a \in G$, the following diagram commutes:

$$\begin{tikzcd}
\star \arrow[r, "b"] \arrow[d, "fa"'] & \star \arrow[d, "ga"] \\
\star \arrow[r, "b"'] & \star
\end{tikzcd}$$

This means that $b \cdot fa = (ga) \cdot b$, that is $ga = b \cdot (fa) \cdot b^{-1}$. In other words, a natural transformation $f \Longrightarrow g$ is a *conjugation* operation $b^{-1} \cdot - \cdot b$ which transforms f into g.

For every functor $F : \mathcal{C} \to \mathcal{D}$ there exists the *identity transformation* $1_F : F \Longrightarrow F$ defined by $(1_F)_A = 1_A$. If $\eta : F \Longrightarrow G$ and $\theta : G \Longrightarrow H$ are natural transformations, then their composition $\theta \circ \eta : F \Longrightarrow H$, defined by $(\theta \circ \eta)_A = \theta_A \circ \eta_A$ is also a natural transformation. Composition of natural transformations is associative because it is composition in the codomain category $\mathcal{D}$. This leads to the definition of functor categories.

Definition A.4.2. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The *functor category* $\mathcal{D}^{\mathcal{C}}$ is the category whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$ and whose morphisms are natural transformations between them.

A functor category may be quite large, too large in fact. In order to avoid problems with size we normally require $\mathcal{C}$ to be a locally small category. The "hom-class" of all natural transformations $F \Longrightarrow G$ is usually written as

$$\mathsf{Nat}(F, G)$$

instead of the more awkward $\mathsf{Hom}_{\mathcal{D}^{\mathcal{C}}}(F, G)$.

Suppose we have functors F, G, and H with a natural transformation $\theta : G \Longrightarrow H$, as in the following diagram:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "G"{name=U}] \arrow[r, bend right, "H"'{name=L}] & \mathbb{E}
\arrow[from=U, to=L, phantom, "\Downarrow\theta"]
\end{tikzcd}$$

Then we can form a natural transformation $\theta \circ F : G \circ F \Longrightarrow H \circ F$ whose component at $A \in \mathcal{C}$ is $(\theta \circ F)_A = \theta_{FA}$.

Similarly, if we have functors and a natural transformation

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "G"{name=U}] \arrow[r, bend right, "H"'{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow\theta"] & \mathcal{D} \arrow[r, "F"] & \mathbb{E} \end{tikzcd}$$

we can form a natural transformation $(F \circ \theta) : F \circ G \Longrightarrow F \circ H$ whose component at $A \in \mathcal{C}$ is $(F \circ \theta)_A = F\theta_A$. These operations are known as *whiskering*.

A *natural isomorphism* is an isomorphism in a functor category. Thus, if $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ are two functors, a natural isomorphism between them is a natural transformation $\eta : F \Longrightarrow G$ whose components are isomorphisms. In this case, the inverse natural transformation $\eta^{-1} : G \Longrightarrow F$ is given by $(\eta^{-1})_A = (\eta_A)^{-1}$. We write $F \cong G$ when F and G are naturally isomorphic.

The definition of natural transformations is motivated in part by the fact that, for any small categories $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, we have

$$\mathsf{Cat}(\mathbb{A} \times \mathbb{B}, \mathbb{C}) \cong \mathsf{Cat}(\mathbb{A}, \mathbb{C}^{\mathbb{B}}) \ . \tag{A.6}$$

The isomorphism takes a functor $F : \mathbb{A} \times \mathbb{B} \to \mathbb{C}$ to the functor $\widetilde{F} : \mathbb{A} \to \mathbb{C}^{\mathbb{B}}$ defined on objects $A \in \mathbb{A}$, $B \in \mathbb{B}$ by

$$(\widetilde{F}A)B = F\langle A, B\rangle$$

and on a morphism $f : A \to A'$ by

$$(\widetilde{F}f)_B = F\langle f, 1_B\rangle \ .$$

The functor $\widetilde{F}$ is called the *transpose* of F.

The inverse isomorphism takes a functor $G : \mathbb{A} \to \mathbb{C}^{\mathbb{B}}$ to the functor $\widetilde{G} : \mathbb{A} \times \mathbb{B} \to \mathbb{C}$, defined on objects by

$$\widetilde{G}\langle A, B\rangle = (GA)B$$

and on a morphism $\langle f, g\rangle : A \times B \to A' \times B'$ by

$$\widetilde{G}\langle f, g\rangle = (Gf)_{B'} \circ (GA)g = (GA')g \circ (Gf)_B \ ,$$

where the last equation holds by naturality of Gf:

$$\begin{tikzcd} (GA)B \arrow[r, "(Gf)_B"] \arrow[d, "(GA)g"'] & (GA')B \arrow[d, "(GA')g"] \\ (GA)B' \arrow[r, "(Gf)_{B'}"'] & (GA')B' \end{tikzcd}$$

A.4.1 Directed graphs as a functor category

Recall that a *directed graph* G is given by a set of vertices G_V and a set of edges G_E. Each edge $e \in G_E$ has a uniquely determined *source* $\mathsf{src}_G\, e \in G_V$ and *target* $\mathsf{trg}_G\, e \in G_V$. We write $e : a \to b$ when a is the source and b is the target of e. A *graph homomorphism* $\phi : G \to H$ is a pair of functions $\phi_0 : G_V \to H_V$ and $\phi_1 : G_E \to H_E$, where we usually write ϕ for both ϕ_0 and ϕ_1, such that whenever $e : a \to b$ then $\phi_1 e : \phi_0 a \to \phi_0 b$. The category of directed graphs and graph homomorphisms is denoted by Graph.

Now let $\cdot \rightrightarrows \cdot$ be the category with two objects and two parallel morphisms, depicted by the following "sketch":

$$\begin{tikzcd} E \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & V \end{tikzcd}$$

An object of the functor category $\mathsf{Set}^{\cdot \rightrightarrows \cdot}$ is a functor $G : (\cdot \rightrightarrows \cdot) \to \mathsf{Set}$, which consists of two sets GE and GV and two functions $Gs : GE \to GV$ and $Gt : GE \to GV$. But this is precisely a directed graph whose vertices are GV, the edges are GE, the source of $e \in GE$ is $(Gs)e$ and the target is $(Gt)e$. Conversely, any directed graph G is a functor $G : (\cdot \rightrightarrows \cdot) \to \mathsf{Set}$, defined by

$$GE = G_E\ , \qquad GV = G_V\ , \qquad Gs = \mathsf{src}_G\ , \qquad Gt = \mathsf{trg}_G\ .$$

Now category theory begins to show its worth, for the morphisms in $\mathsf{Set}^{\cdot \rightrightarrows \cdot}$ are precisely the graph homomorphisms. Indeed, a natural transformation $\phi : G \Longrightarrow H$ between graphs is a pair of functions,

$$\phi_E : G_E \to H_E \qquad \text{and} \qquad \phi_V : G_V \to H_V$$

whose naturality is expressed by the commutativity of the following two diagrams:

$$\begin{tikzcd} G_E \arrow[r, "\phi_E"] \arrow[d, "\mathsf{src}_G"'] & H_E \arrow[d, "\mathsf{src}_H"] \\ G_V \arrow[r, "\phi_V"'] & H_V \end{tikzcd} \qquad \begin{tikzcd} G_E \arrow[r, "\phi_E"] \arrow[d, "\mathsf{trg}_G"'] & H_E \arrow[d, "\mathsf{trg}_H"] \\ G_V \arrow[r, "\phi_V"'] & H_V \end{tikzcd}$$

This is precisely the requirement that $e : a \to b$ implies $\phi_E e : \phi_V a \to \phi_V b$. Thus, in sum, we have,

$$\mathsf{Graph} = \mathsf{Set}^{\cdot \rightrightarrows \cdot}.$$

Exercise A.4.3. Exhibit the arrow category $\mathcal{C}^{\to}$ and the category of group actions $\mathsf{Set}(G)$ as functor categories.

A.4.2 The Yoneda Embedding

The example $\mathsf{Graph} = \mathsf{Set}^{\cdot \rightrightarrows \cdot}$ leads one to wonder which categories $\mathcal{C}$ can be represented as functor categories $\mathsf{Set}^{\mathcal{D}}$ for a suitably chosen $\mathcal{D}$ or, when that is not possible, at least as full subcategories of $\mathsf{Set}^{\mathcal{D}}$.

For a locally small category $\mathcal{C}$, there is the hom-set functor

$$\mathcal{C}(-,-) : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Set} \ .$$

By transposing as in (A.6) we obtain the functor

$$\mathsf{y} : \mathcal{C} \to \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$$

which maps an object $A \in \mathcal{C}$ to the functor

$$\mathsf{y}A = \mathcal{C}(-, A) : B \mapsto \mathcal{C}(B, A)$$

and a morphism $f : A \to A'$ in $\mathcal{C}$ to the natural transformation $\mathsf{y}f : \mathsf{y}A \Longrightarrow \mathsf{y}A'$ whose component at B is

$$(\mathsf{y}f)_B = \mathcal{C}(B, f) : g \mapsto f \circ g \ .$$

This functor is called the *Yoneda embedding.*

Exercise A.4.4. Show that this *is* a functor.

Theorem A.4.5 (Yoneda embedding)**.** *For any locally small category $\mathcal{C}$ the Yoneda embedding*

$$\mathsf{y} : \mathcal{C} \to \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$$

is full and faithful, and injective on objects. Therefore, $\mathcal{C}$ is a full subcategory of $\mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$.

The proof of the theorem uses the famous Yoneda Lemma.

Lemma A.4.6 (Yoneda)**.** *Every functor $F : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ is naturally isomorphic to the functor $\mathsf{Nat}(\mathsf{y}-, F)$. That is, for every $A \in \mathcal{C}$,*

$$\mathsf{Nat}(\mathsf{y}A, F) \cong FA \ ,$$

and this isomorphism is natural in A.

Indeed, the displayed isomorphism is also natural in F.

Proof. The desired natural isomorphism θ_A maps a natural transformation $\eta \in \mathsf{Nat}(\mathsf{y}A, F)$ to $\eta_A 1_A$. The inverse ${\theta_A}^{-1}$ maps an element $x \in FA$ to the natural transformation $({\theta_A}^{-1}x)$ whose component at B maps $f \in \mathcal{C}(B, A)$ to $(Ff)x$. To summarize, for $\eta : \mathcal{C}(-, A) \Longrightarrow F$, $x \in FA$ and $f \in \mathcal{C}(B, A)$, we have

$$\begin{aligned} &\theta_A : \mathsf{Nat}(\mathsf{y}A, F) \to FA \ , && {\theta_A}^{-1} : FA \to \mathsf{Nat}(\mathsf{y}A, F) \ , \\ &\theta_A \eta = \eta_A 1_A \ , && ({\theta_A}^{-1}x)_B f = (Ff)x \ . \end{aligned}$$

To see that θ_A and ${\theta_A}^{-1}$ really are inverses of each other, observe that

$$\theta_A({\theta_A}^{-1}x) = ({\theta_A}^{-1}x)_A 1_A = (F1_A)x = 1_{FA}x = x \ ,$$

and also

$$({\theta_A}^{-1}(\theta_A\eta))_B f = (Ff)(\theta_A\eta) = (Ff)(\eta_A 1_A) = \eta_B(1_A \circ f) = \eta_B f \ ,$$

where the third equality holds by the following naturality square for η:

$$\begin{tikzcd}
\mathcal{C}(A,A) \arrow[r, "\eta_A"] \arrow[d, "\mathcal{C}(f,A)"'] & FA \arrow[d, "Ff"] \\
\mathcal{C}(B,A) \arrow[r, "\eta_B"'] & FB
\end{tikzcd}$$

It remains to check that θ is natural, which amounts to establishing the commutativity of the following diagram, with $g : A \to A'$:

$$\begin{tikzcd}
\mathsf{Nat}(\mathsf{y}A, F) \arrow[r, "\theta_A"] & FA \\
\mathsf{Nat}(\mathsf{y}A', F) \arrow[r, "\theta_{A'}"'] \arrow[u, "\mathsf{Nat}(\mathsf{y}g, F)"] & FA' \arrow[u, "Fg"']
\end{tikzcd}$$

The diagram is commutative because, for any $\eta : \mathsf{y}A' \Longrightarrow F$,

$$\begin{aligned}(Fg)(\theta_{A'}\eta) = (Fg)(\eta_{A'}1_{A'}) = \eta_A(1_{A'} \circ g) = \\ \eta_A(g \circ 1_A) = (\mathsf{Nat}(\mathsf{y}g, F)\eta)_A 1_A = \theta_A(\mathsf{Nat}(\mathsf{y}g, F)\eta) \ ,\end{aligned}$$

where the second equality is justified by naturality of η. □

Proof of Theorem A.4.5. That the Yoneda embedding is full and faithful means that for all $A, B \in \mathcal{C}$ the map

$$\mathsf{y} : \mathcal{C}(A, B) \to \mathsf{Nat}(\mathsf{y}A, \mathsf{y}B)$$

which maps $f : A \to B$ to $\mathsf{y}f : \mathsf{y}A \Longrightarrow \mathsf{y}B$ is an isomorphism. But this is just the Yoneda Lemma applied to the case $F = \mathsf{y}B$. Indeed, with notation as in the proof of the Yoneda Lemma and $g : C \to A$, we see that the isomorphism

$$\theta_A^{-1} : \mathcal{C}(A, B) = (\mathsf{y}B)A \to \mathsf{Nat}(\mathsf{y}A, \mathsf{y}B)$$

is in fact y:

$$({\theta_A}^{-1}f)_C g = ((\mathsf{y}A)g)f = f \circ g = (\mathsf{y}f)_C g \ .$$

Furthermore, if $\mathsf{y}A = \mathsf{y}B$ then $1_A \in \mathcal{C}(A, A) = (\mathsf{y}A)A = (\mathsf{y}B)A = \mathcal{C}(B, A)$ which can only happen if $A = B$. Therefore, y is injective on objects. □

The following corollary is often useful.

Corollary A.4.7. *For $A, B \in \mathcal{C}$, $A \cong B$ if, and only if, $\mathsf{y}A \cong \mathsf{y}B$ in $\mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$.*

Proof. Every functor preserves isomorphisms, and a full and faithful one also reflects them. (A functor $F : \mathcal{C} \to \mathcal{D}$ is said to *reflect* isomorphisms when $Ff : FA \to FB$ being an isomorphisms implies that $f : A \to B$ is an isomorphism.) □

Exercise A.4.8. Prove that a full and faithful functor reflects isomorphisms.

Functor categories $\mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$ are important enough to deserve a name. They are called *presheaf categories*, and a functor $F : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ is a *presheaf* on $\mathcal{C}$. We also use the notation $\widehat{\mathcal{C}} = \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$.

A.4.3 Equivalence of Categories

An isomorphism of categories $\mathcal{C}$ and $\mathcal{D}$ in Cat consists of functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

such that $G \circ F = 1_{\mathcal{C}}$ and $F \circ G = 1_{\mathcal{D}}$. This is often too restrictive a notion. A more general notion which replaces the above identities with natural isomorphisms is required.

Definition A.4.9. An *equivalence of categories* is a pair of functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

such that

$$G \circ F \cong 1_{\mathcal{C}} \qquad \text{and} \qquad F \circ G \cong 1_{\mathcal{D}} .$$

We say that $\mathcal{C}$ and $\mathcal{D}$ are *equivalent categories* and write $\mathcal{C} \simeq \mathcal{D}$.

A functor $F : \mathcal{C} \to \mathcal{D}$ is called an *equivalence functor* if there exists $G : \mathcal{D} \to \mathcal{C}$ such that F and G form an equivalence.

The point of equivalence of categories is that it preserves almost all categorical properties, but ignores those concepts that are not of interest from a categorical point of view, such as identity of objects.

The following proposition requires the Axiom of Choice as stated in general form. However, in many specific cases a canonical choice can be made without appeal to the Axiom of Choice.

Proposition A.4.10. *A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence functor if, and only if, F is full and faithful, and* essentially surjective *on objects, which means that for every $B \in \mathcal{D}$ there exists $A \in \mathcal{C}$ such that $FA \cong B$.*

Proof. It is easily seen that the conditions are necessary, so we only show they are sufficient. Suppose $F : \mathcal{C} \to \mathcal{D}$ is full and faithful, and essentially surjective on objects. For each $B \in \mathcal{D}$, *choose* an object $GB \in \mathcal{C}$ and an isomorphism $\eta_B : F(GB) \to B$. If $f : B \to C$ is a morphism in $\mathcal{D}$, let $Gf : GB \to GC$ be the unique morphism in $\mathcal{C}$ for which

$$F(Gf) = {\eta_C}^{-1} \circ f \circ \eta_B \ . \tag{A.7}$$

Such a unique morphism exists because F is full and faithful. This defines a functor $G : \mathcal{D} \to \mathcal{C}$, as can be easily checked. In addition, (A.7) ensures that η is a natural isomorphism $F \circ G \Longrightarrow 1_{\mathcal{D}}$.

It remains to show that $G \circ F \cong 1_{\mathcal{C}}$. For $A \in \mathcal{C}$, let $\theta_A : G(FA) \to A$ be the unique morphism such that $F\theta_A = \eta_{FA}$. Naturality of θ_A follows from functoriality of F and naturality of η. Because F reflects isomorphisms, θ_A is an isomorphism for every A. $\square$

Example A.4.11. As an example of equivalence of categories we consider the category of sets and partial functions and the category of pointed sets.

A *partial function* $f : A \rightharpoonup B$ is a function defined on a subset $\mathsf{supp}\, f \subseteq A$, called the *support*[3] of f, and taking values in B. Composition of partial functions $f : A \rightharpoonup B$ and $g : B \rightharpoonup C$ is the partial function $g \circ f : A \rightharpoonup C$ defined by

$$\begin{aligned} \mathsf{supp}\,(g \circ f) &= \left\{x \in A \;\middle|\; x \in \mathsf{supp}\, f \wedge fx \in \mathsf{supp}\, g\right\} \\ (g \circ f)x &= g(fx) \quad \text{for } x \in \mathsf{supp}\,(g \circ f) \end{aligned}$$

Composition of partial functions is associative. This way we obtain a category Par of sets and partial functions.

A *pointed set* (A, a) is a set A together with an element $a \in A$. A *pointed function* $f : (A, a) \to (B, b)$ between pointed sets is a function $f : A \to B$ such that $fa = b$. The category $\mathsf{Set}_\bullet$ consists of pointed sets and pointed functions.

The categories Par and $\mathsf{Set}_\bullet$ are equivalent. The equivalence functor $F : \mathsf{Set}_\bullet \to \mathsf{Par}$ maps a pointed set (A, a) to the set $F(A, a) = A \setminus \{a\}$, and a pointed function $f : (A, a) \to (B, b)$ to the partial function $Ff : F(A, a) \rightharpoonup F(B, b)$ defined by

$$\mathsf{supp}\,(Ff) = \left\{x \in A \;\middle|\; fx \neq b\right\} \ , \qquad\qquad (Ff)x = fx \ .$$

The inverse equivalence functor $G : \mathsf{Par} \to \mathsf{Set}_\bullet$ maps a set $A \in \mathsf{Par}$ to the pointed set $GA = (A + \{\bot_A\}, \bot_A)$, where $\bot_A$ is an element that does not belong to A. A partial function $f : A \rightharpoonup B$ is mapped to the pointed function $Gf : GA \to GB$ defined by

$$(Gf)x = \begin{cases} fx & \text{if } x \in \mathsf{supp}\, f \\ \bot_B & \text{otherwise} \ . \end{cases}$$

[3] The support of a partial function $f : A \rightharpoonup B$ is usually called its *domain*, but this terminology conflicts with A being the domain of f as a morphism.

A good way to think about the "bottom" point $\bot_A$ is as a special "undefined value". Let us look at the composition of F and G on objects:

$$G(F(A,a)) = G(A \setminus \{a\}) = ((A \setminus \{a\}) + \bot_A, \bot_A) \cong (A,a) \; .$$
$$F(GA) = F(A + \{\bot_A\}, \bot_A) = (A + \{\bot_A\}) \setminus \{\bot_A\} = A \; .$$

The isomorphism $G(F(A,a)) \cong (A,a)$ is easily seen to be natural.

Example A.4.12. Another example of an equivalence of categories arises when we take the poset reflection of a preorder. Let $(P, \leq)$ be a preorder, If we think of P as a category, then $a, b \in P$ are isomorphic, when $a \leq b$ and $b \leq a$. Isomorphism $\cong$ is an equivalence relation, therefore we may form the quotient set $P/{\cong}$. The set $P/{\cong}$ is a poset for the order relation $\sqsubseteq$ defined by

$$[a] \sqsubseteq [b] \iff a \leq b \; .$$

Here $[a]$ denotes the equivalence class of a. We call $(P/{\cong}, \sqsubseteq)$ the *poset reflection* of P. The quotient map $q : P \to P/{\cong}$ is a functor when P and $P/{\cong}$ are viewed as categories. By Proposition A.4.10, q is an equivalence functor. Trivially, it is faithful and surjective on objects. It is also full because $qa \sqsubseteq qb$ in $P/{\cong}$ implies $a \leq b$ in P.

A.5 Adjoint Functors

The notion of adjunction is arguably the most important concept revealed by category theory. It is a fundamental logical and mathematical concept that occurs everywhere and often marks an important and interesting connection between two constructions of interest. In logic, adjoint functors are pervasive, although this is only recognizable from the category-theoretic approach.

A.5.1 Adjoint maps between preorders

Let us begin with a simple situation. We have already seen that a preorder $(P, \leq)$ is a category in which there is at most one morphism between any two objects. A functor between preorders is a monotone map. Suppose we have preorders P and Q with two monotone maps between them,

$$\begin{tikzcd} P \arrow[r, bend left, "f"] & Q \arrow[l, bend left, "g"] \end{tikzcd}$$

We say that f and g are *adjoint*, and write $f \dashv g$, when for all $x \in P$, $y \in Q$,

$$fx \leq y \iff x \leq gy \; . \tag{A.8}$$

Note that adjointness is *not* a symmetric relation. The map f is the *left adjoint* and g is the *right adjoint*.[4]

[4]Remember it like this: the left adjoint stands on the *left* side of $\leq$, the right adjoint stands on the *right* side of $\leq$.

Equivalence (A.8) is more conveniently displayed as

$$\frac{fx \leq y}{\overline{x \leq gy}}$$

The double line indicates the fact that this is a two-way rule: the top line implies the bottom line, and vice versa.

Let us consider two examples.

Conjunction is adjoint to implication Consider a propositional calculus whose only logical operations are conjunction $\land$ and implication $\Rightarrow$.[5] The formulas of this calculus are built from variables x_0, x_1, x_2, ..., the truth values $\bot$ and $\top$, and the logical connectives $\land$ and $\Rightarrow$. The logical rules are given in natural deduction style:

$$\frac{}{\top} \qquad \frac{\bot}{A} \qquad \frac{A \quad B}{A \land B} \qquad \frac{A \land B}{A} \qquad \frac{A \land B}{B}$$

$$\frac{A \Rightarrow B \qquad A}{B} \qquad \begin{array}{c} [u : A] \\ \vdots \\ \dfrac{B}{A \Rightarrow B}\, u \end{array}$$

For example, we read the last two inference rules as "from $A \Rightarrow B$ and A we infer B" and "if from assumption A we infer B, then (without any assumptions) we infer $A \Rightarrow B$", respectively. We indicate discharged assumptions by enclosing them in brackets. The symbol u in $[u : A]$ is a label for the assumption, which we write to the right of the inference rule that discharges it, as above.

Logical entailment $\vdash$ between formulas of the propositional calculus is the relation $A \vdash B$ which holds if, and only if, from assuming A we can prove B (by using only the inference rules of the calculus). It is trivially the case that $A \vdash A$, and also

$$\text{if } A \vdash B \text{ and } B \vdash C \text{ then } A \vdash C\ .$$

In other words, $\vdash$ is a reflexive and transitive relation on the set P of all propositional formulas, so that $(\mathsf{P}, \vdash)$ is a preorder.

Let A be a propositional formula. Define $f : \mathsf{P} \to \mathsf{P}$ and $g : \mathsf{P} \to \mathsf{P}$ to be the maps

$$fB = (A \land B)\ , \qquad\qquad gB = (A \Rightarrow B)\ .$$

The maps f and g are functors because they respect entailment. Indeed, if $B \vdash B'$ then

[5] Nothing changes if we consider a calculus with more connectives.

$A \wedge B \vdash A \wedge B'$ and $A \Rightarrow B \vdash A \Rightarrow B'$ by the following two derivations:

$$\dfrac{\dfrac{A \wedge B}{A} \qquad \begin{matrix} \dfrac{A \wedge B}{B} \\ \vdots \\ B' \end{matrix}}{A \wedge B'} \qquad\qquad \dfrac{\begin{matrix} \dfrac{A \Rightarrow B \qquad [u : A]}{B} \\ \vdots \\ B' \end{matrix}}{A \Rightarrow B'}\, u$$

We claim that $f \dashv g$. For this we need to prove that $A \wedge B \vdash C$ if, and only if, $B \vdash A \Rightarrow C$. The following two derivations establish the equivalence:

$$\dfrac{\begin{matrix} \dfrac{[u : A] \qquad B}{A \wedge B} \\ \vdots \\ C \end{matrix}}{A \Rightarrow C}\, u \qquad\qquad \dfrac{\begin{matrix} \dfrac{A \wedge B}{B} \\ \vdots \\ A \Rightarrow C \end{matrix} \qquad \dfrac{A \wedge B}{A}}{C}$$

Therefore, *conjunction is left adjoint to implication.*

Topological interior as an adjoint Recall that a *topological space* $(X, \mathcal{O}X)$ is a set X together with a family $\mathcal{O}X \subseteq \mathcal{P}X$ of subsets of X which contains $\emptyset$ and X, and is closed under finite intersections and arbitrary unions. The elements of $\mathcal{O}X$ are called the *open sets.*

The *topological interior* of a subset $S \subseteq X$ is the largest open set contained in S:

$$\mathsf{int}\, S = \bigcup \{U \in \mathcal{O}X \mid U \subseteq S\} \; .$$

Both $\mathcal{O}X$ and $\mathcal{P}X$ are posets ordered by subset inclusion. The inclusion $i : \mathcal{O}X \to \mathcal{P}X$ is a monotone map, and so is the interior $\mathsf{int} : \mathcal{P}X \to \mathcal{O}X$:

$$\begin{tikzcd} \mathcal{O}X \arrow[r, bend left, "i"] & \mathcal{P}X \arrow[l, bend left, "\mathsf{int}"] \end{tikzcd}$$

For $U \in \mathcal{O}X$ and $S \in \mathcal{P}X$ we have

$$\begin{array}{c} iU \subseteq S \\ \hline\hline U \subseteq \mathsf{int}\, S \end{array}$$

Therefore, *topological interior is a right adjoint* to the inclusion of $\mathcal{O}X$ into $\mathcal{P}X$.

A.5.2 Adjoint Functors

Let us now generalize the notion of adjoint monotone maps to the general situation

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

with arbitrary categories and functors. For monotone maps $f \dashv g$, the adjunction is a bijection

$$\frac{fx \to y}{\overline{x \to gy}}$$

between morphisms of the form $fx \to y$ and morphisms of the form $x \to gy$. This is the notion that generalizes the special case; for any $A \in \mathcal{C}$, $B \in \mathcal{D}$ we require a bijection between $\mathcal{D}(FA, B)$ and $\mathcal{C}(A, GB)$:

$$\frac{FA \to B}{\overline{A \to GB}}$$

Definition A.5.1. An *adjunction* $F \dashv G$ between the functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

is a natural isomorphism θ between functors

$$\mathcal{D}(F-, -) : \mathcal{C}^{\mathsf{op}} \times \mathcal{D} \to \mathsf{Set} \qquad \text{and} \qquad \mathcal{C}(-, G-) : \mathcal{C}^{\mathsf{op}} \times \mathcal{D} \to \mathsf{Set}\ .$$

This means that for every $A \in \mathcal{C}$ and $B \in \mathcal{D}$ there is a bijection

$$\theta_{A,B} : \mathcal{D}(FA, B) \cong \mathcal{C}(A, GB)\ ,$$

and naturality of θ means that for $f : A' \to A$ in $\mathcal{C}$ and $g : B \to B'$ in $\mathcal{D}$ the following diagram commutes:

$$\begin{tikzcd} \mathcal{D}(FA, B) \arrow[r, "\theta_{A,B}"] \arrow[d, "\mathcal{D}(Ff, g)"'] & \mathcal{D}(A, GB) \arrow[d, "\mathcal{C}(f, Gg)"] \\ \mathcal{D}(FA', B') \arrow[r, "\theta_{A',B'}"'] & \mathcal{C}(A', GB') \end{tikzcd}$$

Equivalently, for every $h : FA \to B$ in $\mathcal{D}$,

$$Gg \circ (\theta_{A,B} h) \circ f = \theta_{A',B'}(g \circ h \circ Ff)\ .$$

We say that F is a *left adjoint* and G is a *right adjoint*.

We have already seen examples of adjoint functors. For any category $\mathbb{B}$ we have functors $(-) \times \mathbb{B}$ and $(-)^{\mathbb{B}}$ from Cat to Cat. Recall the isomorphism (A.6),

$$\mathsf{Cat}(\mathbb{A} \times \mathbb{B}, \mathbb{C}) \cong \mathsf{Cat}(\mathbb{A}, \mathbb{C}^{\mathbb{B}}) .$$

This isomorphism is in fact natural, so that

$$(-) \times \mathbb{B} \dashv (-)^{\mathbb{B}} .$$

Similarly, for any set $B \in \mathsf{Set}$ there are functors

$$(-) \times B : \mathsf{Set} \to \mathsf{Set} , \qquad\qquad (-)^B : \mathsf{Set} \to \mathsf{Set} ,$$

where $A \times B$ is the cartesian product of A and B, and C^B is the set of all functions from B to C. For morphisms, $f \times B = f \times 1_B$ and $f^B = f \circ (-)$. Then we have, for all $A, C \in \mathsf{Set}$, a natural isomorphism

$$\mathsf{Set}(A \times B, C) \cong \mathsf{Set}(A, C^B) ,$$

which maps a function $f : A \times B \to C$ to the function $(\widetilde{f}x)y = f\langle x, y\rangle$. Therefore, $(-) \times B \dashv (-)^B$.

Exercise A.5.2. Verify that the definition (A.8) of adjoint monotone maps between preorders is a special case of Definition A.5.1.

For another example, consider the forgetful functor

$$U : \mathsf{Cat} \to \mathsf{Graph} ,$$

which maps a category to the underlying directed graph. It has a left adjoint $P \dashv U$. The functor P is the *free* construction of a category from a graph; it maps a graph G to the *category of paths* $P(G)$. The objects of $P(G)$ are the vertices of G. The morphisms of $P(G)$ are finite paths

$$v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} \cdots \xrightarrow{e_n} v_{n+1}$$

of edges in G, composition is concatenation of paths, and the identity morphism on a vertex v is the empty path starting and ending at v.

By using the Yoneda Lemma we can easily prove that adjoints are unique up to natural isomorphism.

Proposition A.5.3. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. If $F \dashv G$, $F \dashv G'$ and $F' \dashv G$ then $F \cong F'$ and $G \cong G'$.*

Proof. Suppose $F \dashv G$ and $F \dashv G'$. By the Yoneda embedding, $GB \cong G'B$ if, and only if, $\mathcal{C}(-, GB) \cong \mathcal{C}(-, G'B)$, which holds because, for any $A \in \mathcal{C}$,

$$\mathcal{C}(A, GB) \cong \mathcal{D}(FA, B) \cong \mathcal{C}(A, G'B) .$$

Therefore, $G \cong G'$. That $F \dashv G$ and $F' \dashv G$ implies $F \cong F'$ follows by duality. □

A.5.3 The Unit of an Adjunction

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be adjoint functors, $F \dashv G$, and let $\theta : \mathcal{D}(F-,-) \to \mathcal{C}(-,G-)$ be the natural isomorphism witnessing the adjunction. For any object $A \in \mathcal{C}$ there is a distinguished morphism $\eta_A = \theta_{A,FA} 1_{FA} : A \to G(FA)$,

$$\begin{array}{c} 1_{FA} : FA \to FA \\ \hline\hline \eta_A : A \to G(FA) \end{array}$$

The transformation $\eta : 1_{\mathcal{C}} \Longrightarrow G \circ F$ is natural. It is called the *unit of the adjunction* $F \dashv G$. In fact, we can recover θ from η as follows, for $f : FA \to B$:

$$\theta_{A,B} f = \theta_{A,B}(f \circ 1_{FA}) = Gf \circ \theta_{A,FA}(1_{FA}) = Gf \circ \eta_A \ ,$$

where we used naturality of θ in the second step. Schematically, given any $f : FA \to B$, the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "\theta_{A,B} f"'] & G(FA) \arrow[d, "Gf"] \\ & GB \end{tikzcd}$$

Since $\theta_{A,B}$ is a bijection, it follows that *every* morphism $g : A \to GB$ has the form $g = Gf \circ \eta_A$ for a *unique* $f : FA \to B$. We say that $\eta_A : A \to G(FA)$ is a *universal* morphism to G, or that η has the following *universal mapping property*: for every $A \in \mathcal{C}$, $B \in \mathcal{D}$, and $g : A \to GB$, there exists a *unique* $f : FA \to B$ such that $g = Gf \circ \eta_A$:

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "g"'] & G(FA) \arrow[d, "Gf"] & FA \arrow[d, dotted, "f"] \\ & GB & B \end{tikzcd}$$

This means that an adjunction can be given in terms of its unit. The isomorphism $\theta : \mathcal{D}(F-,-) \to \mathcal{C}(-,G-)$ is then recovered by

$$\theta_{A,B} f = Gf \circ \eta_A \ .$$

Proposition A.5.4. *A functor $F : \mathcal{C} \to \mathcal{D}$ is left adjoint to a functor $G : \mathcal{D} \to \mathcal{C}$ if, and only if, there exists a natural transformation*

$$\eta : 1_{\mathcal{C}} \Longrightarrow G \circ F \ ,$$

called the unit *of the adjunction, such that, for all $A \in \mathcal{C}$ and $B \in \mathcal{D}$ the map $\theta_{A,B} : \mathcal{D}(FA,B) \to \mathcal{C}(A,GB)$, defined by*

$$\theta_{A,B} f = Gf \circ \eta_A \ ,$$

is an isomorphism.

Let us demonstrate how the universal mapping property of the unit of an adjunction appears as a well known construction in algebra. Consider the forgetful functor from monoids to sets,

$$U : \mathsf{Mon} \to \mathsf{Set} .$$

Does it have a left adjoint $F : \mathsf{Set} \to \mathsf{Mon}$? In order to obtain one, we need a "most economical" way of making a monoid FX from a given set X. Such a construction readily suggests itself, namely the *free monoid* on X, consisting of finite sequences of elements of X,

$$FX = \left\{ x_1 \dots x_n \;\middle|\; n \geq 0 \land x_1, \dots, x_n \in X \right\} .$$

The monoid operation is concatenation of sequences

$$x_1 \dots x_m \cdot y_1 \dots y_n = x_1 \dots x_m y_1 \dots y_n ,$$

and the empty sequence is the unit of the monoid. In order for F to be a functor, it should also map morphisms to morphisms. If $f : X \to Y$ is a function, define $Ff : FX \to FY$ by

$$Ff : x_1 \dots x_n \mapsto (f x_1) \dots (f x_n) .$$

There is an inclusion $\eta_X : X \to U(FX)$ which maps every element $x \in X$ to the singleton sequence x. This gives a natural transformation $\eta : 1_{\mathsf{Set}} \Longrightarrow U \circ F$.

The free monoid FX is "free" in the sense that for every every monoid M and a function $f : X \to UM$ there exists a unique homomorphism $\overline{f} : FX \to M$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & U(FX) \arrow[d, "U\overline{f}"] \\ & UM \end{tikzcd}$$

This is precisely the condition required by Proposition A.5.4 for η to be the unit of the adjunction $F \dashv U$. In this case, the universal mapping property of η is just the usual characterization of free monoid FX generated by the set X: a homomorphism from FX is uniquely determined by its values on the generators.

A.5.4 The Counit of an Adjunction

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be adjoint functors, and let $\theta : \mathcal{D}(F-, -) \to \mathcal{C}(-, G-)$ be the natural isomorphism witnessing the adjunction. For any object $B \in \mathcal{D}$ there is a distinguished morphism $\varepsilon_B = \theta^{-1}_{GB,B} 1_{GB} : F(GB) \to B$,

$$\frac{1_{GB} : GB \to GB}{\varepsilon_B : F(GB) \to B}$$

The transformation $\varepsilon : F \circ G \Longrightarrow 1_{\mathcal{D}}$ is natural and is called the *counit* of the adjunction $F \dashv G$. It is the dual notion to the unit of an adjunction. We state briefly the basic properties of counit, which are easily obtained by "turning around" all morphisms in the previous section and exchanging the roles of the left and right adjoints.

The bijection $\theta^{-1}_{A,B}$ can be recovered from the counit. For $g : A \to GB$ in $\mathcal{C}$, we have

$$\theta^{-1}_{A,B} g = \theta^{-1}_{A,B}(1_{GB} \circ g) = \theta^{-1}_{A,B} 1_{GB} \circ Fg = \varepsilon_B \circ Fg \; .$$

The universal mapping property of the counit is this: for every $A \in \mathcal{C}$, $B \in \mathcal{D}$, and $f : FA \to B$, there exists a *unique* $g : A \to GB$ such that $f = \varepsilon_B \circ Fg$:

$$\begin{tikzcd}
B & F(GB) \arrow[l, "\varepsilon_B"'] & & GB \\
& FA \arrow[ul, "f"] \arrow[u, "Fg"'] & & A \arrow[u, dotted, "g"']
\end{tikzcd}$$

The following is the dual of Proposition A.5.4.

Proposition A.5.5. *A functor $F : \mathcal{C} \to \mathcal{D}$ is left adjoint to a functor $G : \mathcal{D} \to \mathcal{C}$ if, and only if, there exists a natural transformation*

$$\varepsilon : F \circ G \Longrightarrow 1_{\mathcal{D}} \; ,$$

called the counit *of the adjunction, such that, for all $A \in \mathcal{C}$ and $B \in \mathcal{D}$ the map $\theta^{-1}_{A,B} : \mathcal{C}(A, GB) \to \mathcal{D}(FA, B)$, defined by*

$$\theta^{-1}_{A,B} g = \varepsilon_B \circ Fg \; ,$$

is an isomorphism.

Let us consider again the forgetful functor $U : \mathsf{Mon} \to \mathsf{Set}$ and its left adjoint $F : \mathsf{Set} \to \mathsf{Mon}$, the free monoid construction. For a monoid $(M, \star) \in \mathsf{Mon}$, the counit of the adjunction $F \dashv U$ is a monoid homomorphism $\varepsilon_M : F(UM) \to M$, defined by

$$\varepsilon_M(x_1 x_2 \dots x_n) = x_1 \star x_2 \star \cdots \star x_n \; .$$

It has the following universal mapping property: for $X \in \mathsf{Set}$, $(M, \star) \in \mathsf{Mon}$, and a homomorphism $f : FX \to M$ there exists a unique function $\overline{f} : X \to UM$ such that $f = \varepsilon_M \circ F\overline{f}$, namely

$$\overline{f} x = f x \; ,$$

where in the above definition $x \in X$ is viewed as an element of the set X on the left-hand side, and as an element of the free monoid FX on the right-hand side. To summarize, the universal mapping property of the counit ε is the familiar piece of wisdom that a homomorphism $f : FX \to M$ from a free monoid is already determined by its values on the generators.

A.6 Limits and Colimits

The follwing limits and colimits are all special cases of adjoint functors, as we shall see.

A.6.1 Binary products

In a category $\mathcal{C}$, the *(binary) product* of objects A and B is an object $A \times B$ together with *projections* $\pi_0 : A \times B \to A$ and $\pi_1 : A \times B \to B$ such that, for every object $C \in \mathcal{C}$ and all morphisms $f : C \to A$, $g : C \to B$ there exists a *unique* morphism $h : C \to A \times B$ for which the following diagram commutes:

$$\begin{tikzcd}
 & C \arrow[dl, "f"'] \arrow[d, dotted, "h"] \arrow[dr, "g"] & \\
A & A \times B \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & B
\end{tikzcd}$$

We normally refer to the product $(A\times B, \pi_0, \pi_1)$ just by its object $A\times B$, but you should keep in mind that a product is given by an object *and* two projections. The arrow $h : C \to A\times B$ is denoted by $\langle f, g\rangle$. The property

$$\forall\, C : \mathcal{C} \,.\, \forall\, f : C \to A \,.\, \forall\, g : C \to B \,.\, \exists!\, h : C \to A \times B \,.$$
$$(\pi_0 \circ h = f \wedge \pi_1 \circ h = g)$$

is the *universal mapping property* of the product $A \times B$. It characterizes the product of A and B uniquely up to isomorphism in the sense that if $(P, p_0 : P \to A, p_1 : P \to B)$ is another product of A and B then there exists a unique isomorphism $r : P \xrightarrow{\sim} A \times B$ such that $p_0 = \pi_0 \circ r$ and $p_1 = \pi_1 \circ r$.

If in a category $\mathcal{C}$ every two objects have a product, we can turn binary products into an operation[6] by *choosing* a product $A \times B$ for each pair of objects $A, B \in \mathcal{C}$. In general this requires the Axiom of Choice, but in many specific cases a particular choice of products can be made without appeal to the axiom of choice. When we view binary products as an operation, we say that "$\mathcal{C}$ has chosen products". The same holds for other specific and general instances of limits and colimits.

For example, in Set the usual cartesian product of sets is a product. In categories of structures, products are the usual construction: the product of topological spaces in Top is their topological product, the product of directed graphs in Graph is their cartesian product, the product of categories in Cat is their product category, and so on.

A.6.2 Terminal object

A *terminal object* in a category $\mathcal{C}$ is an object $\mathsf{1} \in \mathcal{C}$ such that for every $A \in \mathcal{C}$ there exists a *unique* morphism $!_A : A \to \mathsf{1}$.

[6] More precisely, binary product is a functor from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$, cf. Section A.6.11.

For example, in Set an object is terminal if, and only if, it is a singleton. The terminal object in Cat is the unit category $\mathsf{1}$ consisting of one object and one morphism.

Exercise A.6.1. Prove that if $\mathsf{1}$ and $\mathsf{1}'$ are terminal objects in a category then they are isomorphic.

Exercise A.6.2. Let Field be the category whose objects are fields and morphisms are field homomorphisms.[7] Does Field have a terminal object? What about the category Ring of rings?

A.6.3 Equalizers

Given objects and morphisms

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

we say that e *equalizes* f and g when $f \circ e = g \circ e$.[8] An *equalizer* of f and g is a *universal* equalizing morphism; thus $e : E \to A$ is an equalizer of f and g when it equalizes them and, for all $k : K \to A$, if $f \circ k = g \circ k$ then there exists a unique morphism $m : K \to E$ such that $k = e \circ m$:

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ K \arrow[u, dotted, "m"] \arrow[ur, "k"'] & & \end{tikzcd}$$

In Set the equalizer of parallel functions $f : A \to B$ and $g : A \to B$ is the set

$$E = \{x \in A \mid fx = gx\}$$

with $e : E \to A$ being the subset inclusion $E \subseteq A$, $ex = x$. In general, equalizers can be thought of as those subobjects (subsets, subgroups, subspaces, ...) that can be defined by a single equation.

Exercise A.6.3. Show that an equalizer is a monomorphism, i.e., if $e : E \to A$ is an equalizer of f and g, then, for all $r, s : C \to E$, $e \circ r = e \circ s$ implies $r = s$.

Definition A.6.4. A morphism is a *regular mono* if it is an equalizer.

The difference between monos and regular monos is best illustrated in the category Top: a continuous map $f : X \to Y$ is mono when it is injective, whereas it is a regular mono when it is a topological embedding.[9]

[7] A field $(F, +, \cdot, ^{-1}, 0, 1)$ is a ring with a unit in which all non-zero elements have inverses. We also require that $0 \neq 1$. A homomorphism of fields preserves addition and multiplication, and consequently also 0, 1 and inverses.

[8] Note that this does *not* mean the diagram involving f, g and e is commutative!

[9] A continuous map $f : X \to Y$ is a topological embedding when, for every $U \in \mathcal{O}X$, the image $f[U]$ is an open subset of the image $\mathsf{im}(f)$; this means that there exists $V \in \mathcal{O}Y$ such that $f[U] = V \cap \mathsf{im}(f)$.

A.6.4 Pullbacks

A *pullback* of $f : A \to C$ and $g : B \to C$ is an object P with morphisms $p_0 : P \to A$ and $p_1 : P \to B$ such that $f \circ p_0 = g \circ p_1$, and whenever $r_0 : R \to A$, $r_1 : R \to B$ are such that $f \circ r_0 = g \circ r_1$, then there exists a unique $h : R \to P$ such that $r_0 = p_0 \circ h$ and $r_1 = p_1 \circ h$:

$$
\begin{tikzcd}
R \arrow[dr, dotted, "h"] \arrow[drr, bend left, "r_1"] \arrow[ddr, bend right, "r_0"'] & & \\
& P \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}
$$

We indicate that P is a pullback by drawing a square corner next to it, as in the above diagram. Sometimes we denote the pullback P by $A \times_C B$, since it is indeed a product in the slice category over C.

In Set, the pullback of $f : A \to C$ and $g : B \to C$ is the set

$$P = \left\{ \langle x, y \rangle \in A \times B \;\middle|\; fx = gy \right\}$$

and the functions $p_0 : P \to A$, $p_1 : P \to B$ are the projections, $p_0\langle x, y\rangle = x$, $p_1\langle x, y\rangle = y$.

When we form the pullback of $f : A \to C$ and $g : B \to C$ we also say that we *pull back g along f* and draw the diagram

$$
\begin{tikzcd}
f^*B \arrow[r] \arrow[d, "f^*g"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}
$$

We think of $f^*g : f^*B \to A$ as the inverse image of B along f. This terminology is explained by looking at the pullback of a subset inclusion $u : U \hookrightarrow C$ along a function $f : A \to C$ in the category Set:

$$
\begin{tikzcd}
f^*U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d, hook, "u"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

In this case the pullback is $\left\{ \langle x, y \rangle \in A \times U \;\middle|\; fx = y \right\} \cong \left\{ x \in A \;\middle|\; fx \in U \right\} = f^*U$, the inverse image of U along f.

Exercise A.6.5. Prove that in a category $\mathcal{C}$, a morphism $f : A \to B$ is mono if, and only if, the following diagram is a pullback:

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "1_A"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

A.6.5 Limits

Let us now define the general notion of a limit.

A *diagram of shape* $\mathcal{I}$ in a category $\mathcal{C}$ is a functor $D : \mathcal{I} \to \mathcal{C}$, where the category $\mathcal{I}$ is called the *index category*. We use letters i, j, k, ... for objects of an index category $\mathcal{I}$, call them *indices*, and write D_i, D_j, D_k, ... instead of Di, Dj, Dk, ...

For example, if $\mathcal{I}$ is the category with three objects and three morphisms

$$\begin{tikzcd} & 1 \arrow[dl, "12"'] \arrow[d, "13"] \\ 2 \arrow[r, "23"'] & 3 \end{tikzcd}$$

where $13 = 23 \circ 12$ then a diagram of shape $\mathcal{I}$ is a commutative diagram

$$\begin{tikzcd} & D_1 \arrow[dl, "d_{12}"'] \arrow[d, "d_{13}"] \\ D_2 \arrow[r, "d_{23}"'] & D_3 \end{tikzcd} \tag{A.9}$$

Given an object $A \in \mathcal{C}$, there is a *constant diagram* of shape $\mathcal{I}$, which is the constant functor $\Delta_A : \mathcal{I} \to \mathcal{C}$ that maps every object to A and every morphism to 1_A.

Let $D : \mathcal{I} \to \mathcal{C}$ be a diagram of shape $\mathcal{I}$. A *cone* on D from an object $A \in \mathcal{C}$ is a natural transformation $\alpha : \Delta_A \Longrightarrow D$. This means that for every index $i \in \mathcal{I}$ there is a morphism $\alpha_i : A \to D_i$ such that whenever $u : i \to j$ in $\mathcal{I}$ then $\alpha_j = Du \circ \alpha_i$.

For a given diagram $D : \mathcal{I} \to \mathcal{C}$, we can collect all cones on D into a category $\mathsf{Cone}(D)$ whose objects are cones on D. A morphism between cones $f : (A, \alpha) \to (B, \beta)$ is a morphism $f : A \to B$ in $\mathcal{C}$ such that $\alpha_i = \beta_i \circ f$ for all $i \in \mathcal{I}$. Morphisms in $\mathsf{Cone}(D)$ are composed as morphisms in $\mathcal{C}$. A morphism $f : (A, \alpha) \to (B, \beta)$ is also called a factorization of the cone (A, α) through the cone (B, β).

A *limit* of a diagram $D : \mathcal{I} \to \mathcal{C}$ is a terminal object in $\mathsf{Cone}(D)$. Explicitly, a limit of D is given by a cone (L, λ) such that for every other cone (A, α) there exists a *unique* morphism $f : A \to L$ such that $\alpha_i = \lambda_i \circ f$ for all $i \in \mathcal{I}$. We denote a limit of D by one of the following:

$$\lim D \qquad \lim_{i\in\mathcal{I}} D_i \qquad \varprojlim_{i\in\mathcal{I}} D_i\ .$$

Limits are also called *projective limits*. We say that a category *has limits of shape* $\mathcal{I}$ when every diagram of shape $\mathcal{I}$ in $\mathcal{C}$ has a limit.

Products, terminal objects, equalizers, and pullbacks are all special cases of limits:

- a product $A \times B$ is the limit of the functor $D : \mathsf{2} \to \mathcal{C}$ where $\mathsf{2}$ is the discrete category on two objects 0 and 1, and $D_0 = A$, $D_1 = B$.
- a terminal object $\mathbf{1}$ is the limit of the (unique) functor $D : \mathsf{0} \to \mathcal{C}$ from the empty category.
- an equalizer of $f, g : A \to B$ is the limit of the functor $D : (\cdot \rightrightarrows \cdot) \to \mathcal{C}$ which maps one morphism to f and the other one to g.
- the pullback of $f : A \to C$ and $g : B \to C$ is the limit of the functor $D : \mathcal{I} \to \mathcal{C}$ where $\mathcal{I}$ is the category

$$\begin{tikzcd} & \bullet \arrow[d, "2"] \\ \bullet \arrow[r, "1"'] & \bullet \end{tikzcd}$$

  with $D1 = f$ and $D2 = g$.

It is clear how to define the product of an arbitrary family of objects

$$\{A_i \in \mathcal{C} \mid i \in I\}\ .$$

Such a family is a diagram of shape I, where I is viewed as a discrete category. A *product* $\prod_{i\in I} A_i$ is then given by an object $P \in \mathcal{C}$ and morphisms $\pi_i : P \to A_i$ such that, whenever we have a family of morphisms $\{f_i : B \to A_i \mid i \in I\}$ there exists a *unique* morphism $\langle f_i\rangle_{i\in I} : B \to P$ such that $f_i = \pi_i \circ f$ for all $i \in I$.

A *finite product* is a product of a finite family. As a special case we see that a terminal object is the product of an empty family. It is not hard to show that a category has finite products precisely when it has a terminal object and binary products.

A diagram $D : \mathcal{I} \to \mathcal{C}$ is *small* when $\mathcal{I}$ is a small category. A *small limit* is a limit of a small diagram. A *finite limit* is a limit of a diagram whose index category is finite.

Exercise A.6.6. Prove that a limit, when it exists, is unique up to isomorphism.

The following proposition and its proof tell us how to compute arbitrary limits from simpler ones. We omit detailed proofs as they can be found in any standard textbook on category theory.

Proposition A.6.7. *The following are equivalent for a category $\mathcal{C}$:*

1. *$\mathcal{C}$ has all pullbacks and a terminal object.*
2. *$\mathcal{C}$ has finite products and equalizers.*
3. *$\mathcal{C}$ has has finite limits.*

Proof. We only show how to get binary products from pullbacks and a terminal object. For objects A and B, let P be the pullback of $!_A$ and $!_B$:

$$\begin{tikzcd} P \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "!_B"] \\ A \arrow[r, "!_A"'] & 1 \end{tikzcd}$$

Then (P, π_0, π_1) is a product of A and B because, for all $f : X \to A$ and $g : X \to B$, it is trivially the case that $!_A \circ f = {!_B} \circ g$. □

Proposition A.6.8. *The following are equivalent for a category $\mathcal{C}$:*

1. *$\mathcal{C}$ has small products and equalizers.*
2. *$\mathcal{C}$ has small limits.*

Proof. We indicate how to construct an arbitrary limit from a product and an equalizer. Let $D : \mathcal{I} \to \mathcal{C}$ be a small diagram of an arbitrary shape $\mathcal{I}$. First form an $\mathcal{I}_0$-indexed product P and an $\mathcal{I}_1$-indexed product Q

$$P = \prod_{i \in \mathcal{I}_0} D_i \;, \qquad\qquad Q = \prod_{u \in \mathcal{I}_1} D_{\mathsf{cod}\, u} \;.$$

By the universal property of products, there are unique morphisms $f : P \to Q$ and $g : P \to Q$ such that, for all morphisms $u \in \mathcal{I}_1$,

$$\pi^Q_u \circ f = Du \circ \pi^P_{\mathsf{dom}\, u} \;, \qquad\qquad \pi^Q_u \circ g = \pi^P_{\mathsf{cod}\, u} \;.$$

Let E be the equalizer of f and g,

$$\begin{tikzcd} E \arrow[r, "e"] & P \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Q \end{tikzcd}$$

For every $i \in \mathcal{I}$ there is a morphism $\varepsilon_i : E \to D_i$, namely $\varepsilon_i = \pi^P_i \circ e$. We claim that (E, ε) is a limit of D. First, (E, ε) is a cone on D because, for all $u : i \to j$ in $\mathcal{I}$,

$$Du \circ \varepsilon_i = Du \circ \pi^P_i \circ e = \pi^Q_u \circ f \circ e = \pi^Q_u \circ g \circ e = \pi^P_j \circ e = \varepsilon_j \;.$$

If (A, α) is any cone on D there exists a unique $t : A \to P$ such that $\alpha_i = \pi_i^P \circ t$ for all $i \in \mathcal{I}$. For every $u : i \to j$ in $\mathcal{I}$ we have

$$\pi_u^Q \circ g \circ t = \pi_j^P \circ t = t_j = Du \circ t_i = Du \circ \pi_i^P \circ t = \pi_u^Q \circ f \circ t \ ,$$

therefore $g \circ t = f \circ t$. This implies that there is a unique factorization $k : A \to E$ such that $t = e \circ k$. Now for every $i \in \mathcal{I}$

$$\varepsilon_i \circ k = \pi_i^P \circ e \circ k = \pi_i^P \circ t = \alpha_i$$

so that $k : A \to E$ is the required factorization of the cone (A, α) through the cone (E, ε). To see that k is unique, suppose $m : A \to E$ is another factorization such that $\alpha_i = \varepsilon_i \circ m$ for all $i \in \mathcal{I}$. Since e is mono it suffices to show that $e \circ m = e \circ k$, which is equivalent to proving $\pi_i^P \circ e \circ m = \pi_i^P \circ e \circ k$ for all $i \in \mathcal{I}$. This last equality holds because

$$\pi_i^P \circ e \circ k = \pi_i^P \circ t = \alpha_i = \varepsilon_i \circ m = \pi_i^P \circ e \circ m \ .$$

□

A category is *(small) complete* when it has all small limits, and it is *finitely complete* or *lex* when it has finite limits.

Limits of presheaves Let $\mathcal{C}$ be a locally small category. Then the presheaf category $\widehat{\mathcal{C}} = \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$ has all small limits and they are computed pointwise, e.g., $(P \times Q)A = PA \times QA$ for $P, Q \in \widehat{\mathcal{C}}$, $A \in \mathcal{C}$. To see that this is really so, let $\mathcal{I}$ be a small index category and $D : \mathcal{I} \to \widehat{\mathcal{C}}$ a diagram of presheaves. Then for every $A \in \mathcal{C}$ the diagram D can be instantiated at A to give a diagram $DA : \mathcal{I} \to \mathsf{Set}$, $(DA)_i = D_i A$. Because Set is small complete, we can define a presheaf L by computing the limit of DA:

$$LA = \lim DA = \varprojlim_{i \in \mathcal{I}} D_i A \ .$$

We should keep in mind that $\lim DA$ is actually given by an object $(\lim DA)$ and a natural transformation $\delta A : \Delta_{(\lim DA)} \Longrightarrow DA$. The value of LA is supposed to be just the object part of $\lim DA$. From a morphism $f : A \to B$ we obtain for each $i \in \mathcal{I}$ a function $D_i f \circ (\delta A)_i : LA \to D_i B$, and thus a cone $(LA, Df \circ \delta A)$ on DB. Presheaf L maps the morphism $f : A \to B$ to the unique factorization $Lf : LA \Longrightarrow LB$ of the cone $(LA, Df \circ \delta A)$ on DB through the limit cone LB on DB.

For every $i \in \mathcal{I}$, there is a function $\Lambda_i = (\delta A)_i : LA \to D_i A$. The family $\{\Lambda_i\}_{i \in \mathcal{I}}$ is a natural transformation from Δ_{LA} to DA. This gives us a cone (L, Λ) on D, which is in fact a limit cone. Indeed, if (S, Σ) is another cone on D then for every $A \in \mathcal{C}$ there exists a unique function $\phi_A : SA \to LA$ because SA is a cone on DA and LA is a limit cone on DA. The family $\{\phi_A\}_{A \in \mathcal{C}}$ is the unique natural transformation $\phi : S \Longrightarrow L$ for which $\Sigma = \phi \circ \Lambda$.

A.6.6 Colimits

Colimits are the dual notion of limits. Thus, a *colimit* of a diagram $D : \mathcal{I} \to \mathcal{C}$ is a limit of the dual diagram $D^{\mathsf{op}} : \mathcal{I}^{\mathsf{op}} \to \mathcal{C}^{\mathsf{op}}$ in the dual category $\mathcal{C}^{\mathsf{op}}$:

$$\operatorname{colim}(D : \mathcal{I} \to \mathcal{C}) = \lim(D^{\mathsf{op}} : \mathcal{I}^{\mathsf{op}} \to \mathcal{C}^{\mathsf{op}}) .$$

Equivalently, the colimit of a diagram $D : \mathcal{I} \to \mathcal{C}$ is the initial object in the category of *cocones* $\mathsf{Cocone}(D)$ on D. A cocone (A, α) on D is a natural transformation $\alpha : D \Longrightarrow \Delta_A$. It is given by an object $A \in \mathcal{C}$ and, for each $i \in \mathcal{I}$, a morphism $\alpha_i : D_i \to A$, such that $\alpha_i = \alpha_j \circ Du$ whenever $u : i \to j$ in $\mathcal{I}$. A morphism between cocones $f : (A, \alpha) \to (B, \beta)$ is a morphism $f : A \to B$ in $\mathcal{C}$ such that $\beta_i = f \circ \alpha_i$ for all $i \in \mathcal{I}$.

Explicitly, a colimit of $D : \mathcal{I} \to \mathcal{C}$ is given by a cocone (C, ζ) on D such that, for every other cocone (A, α) on D there exists a unique morphism $f : C \to A$ such that $\alpha_i = f \circ \zeta_i$ for all $i \in D$. We denote a colimit of D by one of the following:

$$\operatorname{colim} D \qquad \operatorname{colim}_{i \in \mathcal{I}} D_i \qquad \varinjlim_{i \in \mathcal{I}} D_i \ .$$

Colimits are also called *inductive limits*.

Exercise A.6.9. Formulate the dual of Proposition A.6.7 and Proposition A.6.8 for colimits (coequalizers are defined in Section A.6.9).

A.6.7 Binary Coproducts

In a category $\mathcal{C}$, the *(binary) coproduct* of objects A and B is an object $A + B$ together with *injections* $\iota_0 : A \to A + B$ and $\iota_1 : B \to A + B$ such that, for every object $C \in \mathcal{C}$ and all morphisms $f : A \to C$, $g : B \to C$ there exists a *unique* morphism $h : A + B \to C$ for which the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "\iota_0"] \arrow[dr, "f"'] & A + B \arrow[d, dotted, "h"] & B \arrow[l, "\iota_1"'] \arrow[dl, "g"] \\
& C &
\end{tikzcd}$$

The arrow $h : A + B \to C$ is denoted by $[f, g]$.

The coproduct $A + B$ is the colimit of the diagram $D : 2 \to \mathcal{C}$, where $\mathcal{I}$ is the discrete category on two objects 0 and 1, and $D_0 = A$, $D_1 = B$.

In Set the coproduct is the disjoint union, defined by

$$X + Y = \big\{\langle 0, x\rangle \bigm| x \in X\big\} \cup \big\{\langle 1, y\rangle \bigm| x \in Y\big\} \ ,$$

where 0 and 1 are distinct sets, for example $\emptyset$ and $\{\emptyset\}$. Given functions $f : X \to Z$ and $g : Y \to Z$, the unique function $[f, g] : X + Y \to Z$ is the usual *definition by cases*:

$$[f, g]u = \begin{cases} fx & \text{if } u = \langle 0, x\rangle \\ gx & \text{if } u = \langle 1, x\rangle \ . \end{cases}$$

Exercise A.6.10. Show that the categories of posets and of topological spaces both have coproducts.

A.6.8 The initial object

An *initial object* in a category $\mathcal{C}$ is an object $\mathsf{0} \in \mathcal{C}$ such that for every $A \in \mathcal{C}$ there exists a *unique* morphism $\mathsf{o}_A : \mathsf{0} \to A$.

An initial object is the colimit of the empty diagram.

In Set, the initial object is the empty set.

Exercise A.6.11. What is the initial and what is the terminal object in the category of groups?

A *zero object* is an object that is both initial and terminal.

Exercise A.6.12. Show that in the category of Abelian[10] groups finite products and coproducts agree, that is $\mathsf{0} \cong \mathsf{1}$ and $A \times B \cong A + B$.

Exercise A.6.13. Suppose A and B are Abelian groups. Is there a difference between their coproduct in the category Group of groups, and their coproduct in the category $\mathsf{AbGroup}$ of Abelian groups?

A.6.9 Coequalizers

Given objects and morphisms

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "q"] & Q \end{tikzcd}$$

we say that q *coequalizes* f and g when $e \circ f = e \circ g$. A *coequalizer* of f and g is a *universal* coequalizing morphism; thus $q : B \to Q$ is a coequalizer of f and g when it coequalizes them and, for all $s : B \to S$, if $s \circ f = s \circ g$ then there exists a *unique* morphism $r : Q \to S$ such that $s = r \circ q$:

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "q"] \arrow[dr, "s"'] & Q \arrow[d, dotted, "r"] \\ & & S \end{tikzcd}$$

In Set the coequalizer of parallel functions $f : A \to B$ and $g : A \to B$ is the quotient set $Q = B/\sim$ where $\sim$ is the least equivalence relation on B satisfying

$$fx = gy \Rightarrow x \sim y\ .$$

[10] An Abelian group is one that satisfies the commutative law $x \cdot y = y \cdot x$.

The function $q : B \to Q$ is the canonical quotient map which assigns to each element $x \in B$ its equivalence class $[x] \in B/{\sim}$. In general, coequalizers can be thought of as quotients of those equivalence relations that that can be defined (generated) by a single equation.

Exercise A.6.14. Show that a coequalizer is an epimorphism, i.e., if $q : B \to Q$ is a coequalizer of f and g, then, for all $u, v : Q \to T$, $u \circ q = v \circ q$ implies $u = v$. [Hint: use the duality between limits and colimits and Exercise A.6.3.]

Definition A.6.15. A morphism is a *regular epi* if it is a coequalizer.

The difference between epis and regular epis is best illustrated in the category Top: a continuous map $f : X \to Y$ is epi when it is surjective, whereas it is a regular epi when it is a topological quotient map.[11]

A.6.10 Pushouts

A *pushout* of $f : A \to B$ and $g : A \to C$ is an object Q with morphisms $q_0 : B \to Q$ and $q_1 : C \to Q$ such that $q_0 \circ f = q_1 \circ g$, and whenever $r_0 : B \to R$, $r_1 : C \to R$ are such that $r_0 \circ f = r_1 \circ g$, then there exists a unique $s : Q \to R$ such that $r_0 = s \circ q_0$ and $r_1 = s \circ q_1$:

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "q_1"] \arrow[ddr, bend left, "r_1"] & \\
B \arrow[r, "q_0"'] \arrow[drr, bend right, "r_0"'] & Q \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dotted, "s"] & \\
 & & R
\end{tikzcd}$$

We indicate that Q is a pushout by drawing a square corner next to it, as in the above diagram. The above pushout Q is sometimes denotes by $B +_A C$.

A pushout, as in the above diagram, is the colimit of the diagram $D : \mathcal{I} \to \mathcal{C}$ where the index category $\mathcal{I}$ is

$$\begin{tikzcd}
\bullet \arrow[r, "2"] \arrow[d, "1"'] & \bullet \\
\bullet &
\end{tikzcd}$$

and $D1 = f$, $D2 = g$.

In Set, the pushout of $f : A \to C$ and $g : B \to C$ is the quotient set

$$Q = (B + C)/{\sim}$$

[11] A continuous map $f : X \to Y$ is a topological quotient map when it is surjective and, for every $U \subseteq Y$, U is open if, and only if, f^*U is open.

where $B + C$ is the disjoint union of B and C, and $\sim$ is the least equivalence relation on $B + C$ such that, for all $x \in A$,

$$fx \sim gx \ .$$

The functions $q_0 : B \to Q$, $q_1 : C \to Q$ are the injections, $q_0 x = [x]$, $q_1 y = [y]$, where $[x]$ is the equivalence class of x.

A.6.11 Limits and Colimits as Adjoints

Limits and colimits can be defined as adjoints to certain very simple functors.

First, observe that an object $A \in \mathcal{C}$ can be viewed as a functor from the terminal category $\mathsf{1}$ to $\mathcal{C}$, namely the functor which maps the only object $\star$ of $\mathsf{1}$ to A and the only morphism $1_\star$ to 1_A. Since $\mathsf{1}$ is the terminal object in Cat, there exists a unique functor $!_\mathcal{C} : \mathcal{C} \to \mathsf{1}$, which maps every object of $\mathcal{C}$ to $\star$.

Now we can ask whether this simple functor $!_\mathcal{C} : \mathcal{C} \to \mathsf{1}$ has any adjoints. Indeed, it has a right adjoint just if $\mathcal{C}$ has a terminal object $\mathsf{1}_\mathcal{C}$, for the corresponding functor $\mathsf{1}_\mathcal{C} : \mathsf{1} \to \mathcal{C}$ has the property that, for every $A \in \mathcal{C}$ we have a (trivially natural) bijective correspondence:

$$\frac{!_A : A \to \mathsf{1}_\mathcal{C}}{1_\star : !_\mathcal{C} A \to \star}$$

Similarly, an initial object is a left adjoint to $!_\mathcal{C}$:

$$\mathsf{0}_\mathcal{C} \dashv !_\mathcal{C} \dashv \mathsf{1}_\mathcal{C} \ .$$

Now consider the diagonal functor,

$$\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C},$$

defined by $\Delta A = \langle A, A \rangle$, $\Delta f = \langle f, f \rangle$. When does this have adjoints?

If $\mathcal{C}$ has all binary products, then they determine a functor

$$- \times - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

which maps $\langle A, B \rangle$ to $A \times B$ and a pair of morphisms $\langle f : A \to A', g : B \to B' \rangle$ to the unique morphism $f \times g : A \times B \to A' \times B'$ for which $\pi_0 \circ (f \times g) = f \circ \pi_0$ and $\pi_1 \circ (f \times g) = g \circ \pi_1$,

$$\begin{tikzcd}
A \arrow[d, "f"'] & A \times B \arrow[l, "\pi_0"'] \arrow[r, "\pi_1"] \arrow[d, "f \times g"] & B \arrow[d, "g"] \\
A' & A' \times B' \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & B'
\end{tikzcd}$$

The product functor $\times$ is right adjoint to the diagonal functor Δ. Indeed, there is a natural bijective correspondence:

$$\frac{\langle f, g \rangle : \langle A, A \rangle \to \langle B, C \rangle}{f \times g : A \to B \times C}$$

Similarly, binary coproducts are easily seen to be left adjoint to the diagonal functor,

$$+ \dashv \Delta \dashv \times \,.$$

Now in general, consider limits of shape $\mathcal{I}$ in a category $\mathcal{C}$. There is the constant diagram functor

$$\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{I}}$$

that maps $A \in \mathcal{C}$ to the constant diagram $\Delta_A : \mathcal{I} \to \mathcal{C}$. The limit construction is a functor

$$\varprojlim : \mathcal{C}^{\mathcal{I}} \to \mathcal{C}$$

that maps each diagram $D \in \mathcal{C}^{\mathcal{I}}$ to its limit $\varprojlim D$. These two are adjoint, $\Delta \dashv \varprojlim$, because there is a natural bijective correspondence between cones $\alpha : \Delta_A \Longrightarrow D$ on D, and their factorizations through the limit of D,

$$\frac{\alpha : \Delta_A \Longrightarrow D}{\overline{\overline{A \to \varprojlim D}}}$$

An analogous correspondence holds for colimits, so that we obtain a pair of adjunctions,

$$\varinjlim \dashv \Delta \dashv \varprojlim \,,$$

which, of course, subsume all the previously mentioned cases.

Exercise A.6.16. How are the functors $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{I}}$, $\lim : \mathcal{C}^{\mathcal{I}} \to \mathcal{C}$, and $\mathrm{colim} : \mathcal{C}^{\mathcal{I}} \to \mathcal{C}$ defined on morphisms?

A.6.12 Preservation of Limits and Colimits by Functors

We say that a functor $F : \mathcal{C} \to \mathcal{D}$ *preserves products* when, given a product

$$A \xleftarrow{\pi_0} A \times B \xrightarrow{\pi_1} B$$

its image in $\mathcal{D}$,

$$FA \xleftarrow{F\pi_0} F(A \times B) \xrightarrow{F\pi_1} FB$$

is a product of FA and FB. If $\mathcal{D}$ has chosen binary products, F preserves binary products if, and only if, the unique morphism $f : F(A \times B) \to FA \times FB$ which makes the following diagram commutative is an isomorphism: [12]

$$\begin{tikzcd}
& F(A \times B) \arrow[dl, "F\pi_0"'] \arrow[d, "f"] \arrow[dr, "F\pi_1"] & \\
FA & FA \times FB \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & FB
\end{tikzcd}$$

[12] Products are determined up to isomorphism only, so it would be too restrictive to require $F(A \times B) = FA \times FB$. When that is the case, however, we say that the functor F *strictly* preserves products.

In general, a functor $F : \mathcal{C} \to \mathcal{D}$ is said to *preserve limits* of shape $\mathcal{I}$ when it maps limit cones to limit cones: if (L, λ) is a limit of $D : \mathcal{I} \to \mathcal{C}$ then $(FL, F \circ \lambda)$ is a limit of $F \circ D : \mathcal{I} \to \mathcal{D}$.

Analogously, a functor $F : \mathcal{C} \to \mathcal{D}$ is said to *preserve colimits* of shape $\mathcal{I}$ when it maps colimit cocones to colimit cocones: if (C, ζ) is a colimit of $D : \mathcal{I} \to \mathcal{C}$ then $(FC, F \circ \zeta)$ is a colimit of $F \circ D : \mathcal{I} \to \mathcal{D}$.

Proposition A.6.17. *(a) A functor preserves finite (small) limits if, and only if, it preserves equalizers and finite (small) products. (b) A functor preserves finite (small) colimits if, and only if, it preserves coequalizers and finite (small) coproducts.*

Proof. This follows from the fact that limits are constructed from equalizers and products, cf. Proposition A.6.8, and that colimits are constructed from coequalizers and coproducts, cf. Exercise A.6.9. □

Proposition A.6.18. *For a locally small category $\mathcal{C}$, the Yoneda embedding $\mathsf{y} : \mathcal{C} \to \widehat{\mathcal{C}}$ preserves all limits that exist in $\mathcal{C}$.*

Proof. Suppose (L, λ) is a limit of $D : \mathcal{I} \to \mathcal{C}$. The Yoneda embedding maps D to the diagram $\mathsf{y} \circ D : \mathcal{I} \to \widehat{\mathcal{C}}$, defined by

$$(\mathsf{y} \circ D)_i = \mathsf{y}D_i = \mathcal{C}(-, D_i) \ .$$

and it maps the limit cone (L, λ) to the cone $(\mathsf{y}L, \mathsf{y} \circ \lambda)$ on $\mathsf{y} \circ D$, defined by

$$(\mathsf{y} \circ \lambda)_i = \mathsf{y}\lambda_i = \mathcal{C}(-, \lambda_i) \ .$$

To see that $(\mathsf{y}L, \mathsf{y} \circ \lambda)$ is a limit cone on $\mathsf{y} \circ D$, consider a cone (M, μ) on $\mathsf{y} \circ D$. Then $\mu : \Delta_M \Longrightarrow D$ consists of a family of functions, one for each $i \in \mathcal{I}$ and $A \in \mathcal{C}$,

$$(\mu_i)_A : MA \to \mathcal{C}(A, D_i) \ .$$

For every $A \in \mathcal{C}$ and $m \in MA$ we get a cone on D consisting of morphisms

$$(\mu_i)_A m : A \to D_i \ . \qquad (i \in \mathcal{I})$$

There exists a unique morphism $\phi_A m : A \to L$ such that $(\mu_i)_A m = \lambda_i \circ \phi_A m$. The family of functions

$$\phi_A : MA \to \mathcal{C}(A, L) = (\mathsf{y} \circ L)A \qquad (A \in \mathcal{C})$$

forms a factorization $\phi : M \Longrightarrow \mathsf{y}L$ of the cone (M, μ) through the cone (L, λ). This factorization is unique because each $\phi_A m$ is unique. □

In effect we showed that a covariant representable functor $\mathcal{C}(A, -) : \mathcal{C} \to \mathsf{Set}$ preserves existing limits,

$$\mathcal{C}(A, \varprojlim_{i \in \mathcal{I}} D_i) \cong \varprojlim_{i \in \mathcal{I}} \mathcal{C}(A, D_i) \ .$$

By duality, the contravariant representable functor $\mathcal{C}(-, A) : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ maps existing colimits to limits,

$$\mathcal{C}(\varinjlim_{i \in \mathcal{I}} D_i, A) \cong \varprojlim_{i \in \mathcal{I}} \mathcal{C}(D_i, A) \ .$$

Exercise A.6.19. Prove the above claim that a contravariant representable functor $\mathcal{C}(-, A) : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ maps existing colimits to limits. Use duality between limits and colimits. Does it also follow *by a simple duality argument* that a contravariant representable functor $\mathcal{C}(-, A)$ maps existing limits to colimits? How about a covariant representable functor $\mathcal{C}(A, -)$ mapping existing colimits to limits?

Exercise A.6.20. Prove that a functor $F : \mathcal{C} \to \mathcal{D}$ preserves monos if it preserves limits. In particular, the Yoneda embedding preserves monos. Hint: Exercise A.6.5.

Proposition A.6.21. *Right adjoints preserve limits, and left adjoints preserve colimits.*

Proof. Suppose we have adjoint functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

and a diagram $D : \mathcal{I} \to \mathcal{D}$ whose limit exists in $\mathcal{D}$. We would like to use the following slick application of Yoneda Lemma to show that G preserves limits: for every $A \in \mathcal{C}$,

$$\mathcal{C}(A, G(\varprojlim D)) \cong \mathcal{D}(FA, \varprojlim D) \cong \varprojlim_{i \in \mathcal{I}} \mathcal{D}(FA, D_i)$$
$$\cong \varprojlim_{i \in \mathcal{I}} \mathcal{C}(A, GD_i) \cong \mathcal{C}(A, \varprojlim (G \circ D)) \ .$$

Therefore $G(\lim D) \cong \lim(G \circ D)$. However, this argument only works if we already know that the limit of $G \circ D$ exists.

We can also prove the stronger claim that whenever the limit of $D : \mathcal{I} \to \mathcal{D}$ exists then the limit of $G \circ D$ exists in $\mathcal{C}$ and its limit is $G(\lim D)$. So suppose (L, λ) is a limit cone of D. Then $(GL, G \circ \lambda)$ is a cone on $G \circ D$. If (A, α) is another cone on $G \circ D$, we have by adjunction a cone (FA, γ) on D,

$$\frac{\alpha_i : A \to GD_i}{\gamma_i : FA \to D_i}$$

There exists a unique factorization $f : FA \to L$ of this cone through (L, λ). Again by adjunction, we obtain a unique factorization $g : A \to GL$ of the cone (A, α) through the cone $(GL, G \circ \lambda)$:

$$\frac{f : FA \to L}{g : A \to GL}$$

The factorization g is unique because γ is uniquely determined from α, f uniquely from α, and g uniquely from f.

By a dual argument, a left adjoint preserves colimits. $\square$

Appendix B

Logic

B.1 Concrete and Abstract Syntax

By *syntax* we generally mean manipulation of finite strings of symbols according to given *grammatical rules*. For instance, the strings "7)6 + /(8" and "$(6 + 8)/7$" both consist of the same symbols but you will recognize one as junk and the other as *well formed* because you have (implicitly) applied the grammatical rules for arithmetical expressions.

Grammatical rules are usually quite complicated, as they need to prescribe associativity of operators (does "$5 + 6 + 7$" mean "$(5 + 6) + 7$" or "$5 + (6 + 7)$"?) and their precedence (does "$6 + 8/7$" mean "$(6 + 8)/7$" or "$6 + (8/7)$"?), the role of *white space* (empty space between symbols and line breaks), rules for nesting and balancing parentheses, etc. It is not our intention to dwell on such details, but rather to focus on the mathematical nature of well-formed expressions, namely that they represent inductively generated finite trees.[1] Under this view the string "$(6+8)/7$" is just a concrete representation of the tree depicted in Figure B.1.

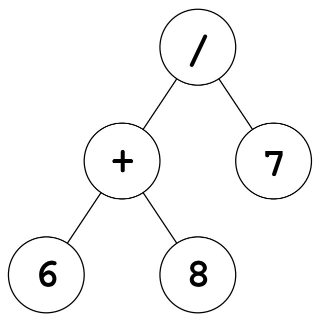

Figure B.1: The tree represented by $(6 + 8)/7$

Concrete representation of expressions as finite strings of symbols is called *concrete syntax*, while in *abstract syntax* we view expressions as finite trees. The passage from the

[1] We are limiting attention to the so-called *context-free* grammar, which are sufficient for our purposes. More complicated grammars are rarely used to describe formal languages in logic and computer science.

former to the latter is called *parsing* and is beyond the scope of this book. We will always specify only abstract syntax and assume that the corresponding concrete syntax follows the customary rules for parentheses, associativity and precedence of operators.

As an illustration we give rules for the (abstract) syntax of propositional calculus in *Backus-Naur* form:

$$\begin{aligned} &\text{Propositional variable } p ::= \mathtt{p}_1 \mid \mathtt{p}_2 \mid \mathtt{p}_3 \mid \cdots \\ &\text{Propositional formula } \phi ::= p \mid \bot \mid \top \mid \phi_1 \land \phi_2 \mid \phi_1 \lor \phi_2 \mid \phi_1 \Rightarrow \phi_2 \mid \neg\phi \end{aligned}$$

The vertical bars should be read as "or". The first rule says that a propositional variable is the constant $\mathtt{p}_1$, or the constant $\mathtt{p}_2$, or the constant $\mathtt{p}_3$, etc.[2] The second rule tells us that there are seven inductive rules for building a propositional formula:

- a propositional variable is a formula,
- the constants $\bot$ and $\top$ are formulas,
- if ϕ_1, ϕ_2, and ϕ are formulas, then so are $\phi_1 \land \phi_2$, $\phi_1 \lor \phi_2$, $\phi_1 \Rightarrow \phi_2$, and $\neg\phi$.

Even though abstract syntax rules say nothing about parentheses or operator associativity and precedence, we shall rely on established conventions for mathematical notation and write down concrete representations of propositional formulas, e.g., $\mathtt{p}_4 \land (\mathtt{p}_1 \lor \neg \mathtt{p}_1) \land \mathtt{p}_4 \lor \mathtt{p}_2$.

A word of warning: operator associativity in syntax is not to be confused with the usual notion of associativity in mathematics. We say that an operator $\star$ is *left associative* when an expression $x \star y \star z$ represents the left-hand tree in Figure B.2, and *right associative* when it represents the right-hand tree. Thus the usual operation of subtraction $-$ is left

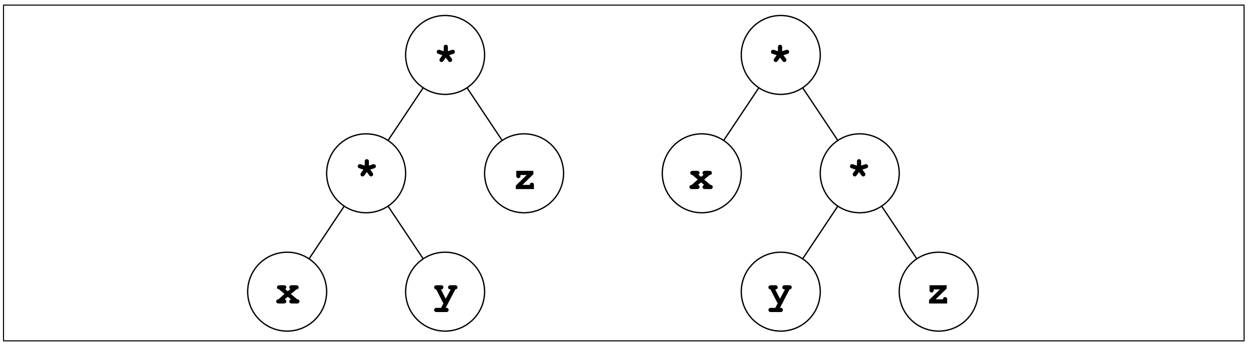

Figure B.2: Left and right associativity of $x \star y \star z$

associative, but is not associative in the usual mathematical sense.

[2] In an actual computer implementation we would allow arbitrary finite strings of letters as propositional variables. In logic we only care about the fact that we can never run out of fresh variables, i.e., that there are countably infinitely many of them.

B.2 Free and Bound Variables

Variables appearing in an expression may be *free* or *bound*. For example, in expressions

$$\int_0^1 \sin(a \cdot x)\,dx, \qquad x \mapsto ax^2 + bx + c, \qquad \forall\, x\,.\,(x < a \vee x > b)$$

the variables a, b and c are free, while x is bound by the integral operator $\int$, the function formation $\mapsto$, and the universal quantifier $\forall$, respectively. To be quite precise, it is an *occurrence* of a variable that is free or bound. For example, in expression $\phi(x) \vee \exists\, x\,.\,A\psi(x,x)$ the first occurrence of x is free and the remaining ones are bound.

In this book the following operators bind variables:

- quantifiers $\exists$ and $\forall$, cf. **??**,
- λ-abstraction, cf. **??**,
- search for others **??**.

When a variable is bound we may always rename it, provided the renaming does not confuse it with another variable. In the integral above we could rename x to y, but not to a because the binding operation would *capture* the free variable a to produce the unintended $\int_0^1 \sin(a^2)\,da$. Renaming of bound variables is called *α-renaming*.

We consider two expressions *equal* if they only differ in the names of bound variables, i.e., if one can be obtained from the other by α-renaming. Furthermore, we adhere to *Barendregt's variable convention [**?**, p. 2]*, which says that bound variables are always chosen so as to differ from free variables. Thus we would never write $\phi(x) \vee \exists\, x\,.\,A\psi(x,x)$ but rather $\phi(x) \vee \exists\, y\,.\,A\psi(y,y)$. By doing so we need not worry about capturing or otherwise confusing free and bound variables.

In logic we need to be more careful about variables than is customary in traditional mathematics. Specifically, we always specify which free variables may appear in an expression.[3] We write

$$x_1 : A_1, \ldots, x_n : A_n \mid t$$

to indicate that expression t may contain only free variables $x_1, \ldots, x_n$ of types $A_1, \ldots, A_n$. The list

$$x_1 : A_1, \ldots, x_n : A_n$$

is called a *context* in which t appears. To see why this is important consider the different meaning that the expression $x^2 + y^2 \leq 1$ recevieves in different contexts:

- $x : \mathbb{Z}, y : \mathbb{Z} \mid x^2 + y^2 \leq 1$ denotes the set of tuples $\{(-1,0),(0,1),(1,0),(0,-1)\}$,
- $x : \mathbb{R}, y : \mathbb{R} \mid x^2 + y^2 \leq 1$ denotes the closed unit disc in the plane, and

[3]This is akin to one of the guiding principles of good programming language design, namely, that all variables should be *declared* before they are used.

- $x : \mathbb{R}, y : \mathbb{R}, z : \mathbb{R} \mid x^2 + y^2 \leq 1$ denotes the infinite cylinder in space whose base is the closed unit disc.

In single-sorted theories there is only one type or sort A. In this case we abbreviate a context by listing just the variables, $x_1, \ldots, x_n$.

B.3 Substitution

Substitution is a basic syntactic operation which replaces (free occurrences of) distinct variables $x_1, \ldots, x_n$ in an expression t with expressions $t_1, \ldots, t_n$, which is written as

$$t[t_1/x_1, \ldots, t_n/x_n].$$

We sometimes abbreviate this as $t[\vec{t}/\vec{x}]$ where $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{t} = (t_1, \ldots, t_n)$. Here are several examples:

$$\begin{aligned}(x^2 + x + y)[(2+3)/x] &= (2+3)^2 + (2+3) + y \\ (x^2 + y)[y/x, x/y] &= y^2 + x \\ \left(\forall x \,.\, \left(x^2 < y + x^3\right)\right)[x+y/y] &= \forall z \,.\, \left(z^2 < (x+y) + z^3\right).\end{aligned}$$

Notice that in the third example we first renamed the bound variable x to z in order to avoid a capture by $\forall$.

Substitution is simple to explain in terms of trees. Assuming Barendregt's convention, the substitution $t[u/x]$ means that in the tree t we replace the leaves labeled x by copies of the tree u. Thus a substitution never changes the structure of the tree–it only "grows" new subtrees in places where the substituted variables occur as leaves.

Substitution satisfies the distributive law

$$(t[u/x])[v/y] = (t[v/y])[u[v/y]/x],$$

provided x and y are distinct variables. There is also a corresponding multivariate version which is written the same way with a slight abuse of vector notation:

$$(t[\vec{u}/\vec{x}])[\vec{v}/\vec{y}] = (t[\vec{v}/\vec{y}])[\vec{u}[\vec{v}/\vec{y}]/\vec{x}].$$

B.4 Judgments and deductive systems

A formal system, such as first-order logic or type theory, concerns itself with *judgments*. There are many kinds of judgments, such as:

- The most common judgments are equations and other logical statements. We distinguish a formula ϕ and the judgment "ϕ holds" by writing the latter as

  $$\vdash \phi \,.$$

  The symbol $\vdash$ is generally used to indicate judgments.

- Typing judgments
$$\vdash t : A$$
expressing the fact that a term t has type A. This is not to be confused with the set-theoretic statement $t \in u$ which says that individuals t and u (of type "set") are in relation "element of" $\in$.

- Judgments expressing the fact that a certain entity is well formed. A typical example is a judgment
$$\vdash x_1 : A_1, \ldots, x_n : A_n \quad \texttt{ctx}$$
which states that $x_1 : A_1, \ldots, x_n : A_n$ is a well-formed context. This means that $x_1, \ldots, x_n$ are distinct variables and that $A_1, \ldots, A_n$ are well-formed types. This kind of judgement is often omitted and it is tacitly assumed that whatever entities we deal with are in fact well-formed.

A *hypothetical judgement* has the form
$$H_1, \ldots, H_n \vdash C$$
and means that hypotheses $H_1, \ldots, H_n$ entail consequence C (with respect to a given decuctive system). We may also add a typing context to get a general form of judgment
$$x_1 : A_1, \ldots, x_n : A_n \mid H_1, \ldots, H_m \vdash C.$$
This should be read as: "if $x_1, \ldots, x_n$ are variables of types $A_1, \ldots, A_n$, respectively, then hypotheses $H_1, \ldots, H_m$ entail conclusion C." For our purposes such contexts will suffice, but you should not be surprised to see other kinds of judgments in logic.

A *deductive system* is a set of inference rules for deriving judgments. A typical inference rule has the form
$$\frac{J_1 \quad J_2 \quad \cdots \quad J_n}{J}\, C$$
This means that we can infer judgment J if we have already derived judgments $J_1, \ldots, J_n$, provided that the optional side-condition C is satisfied. An *axiom* is an inference rule of the form
$$\frac{}{J}$$
A *two-way rule*
$$\begin{array}{c} J_1 \quad J_2 \quad \cdots \quad J_n \\ \hline\hline K_1 \quad K_2 \quad \cdots \quad K_m \end{array}$$
is a combination of $n+m$ inference rules stating that we may infer each K_i from $J_1, \ldots, J_n$ and each J_i from $K_1, \ldots, K_m$.

A *derivation* of a judgment J is a finite tree whose root is J, the nodes are inference rules, and the leaves are axioms. An example is presented in the next subsection.

The set of all judgments that hold in a given deductive system is generated inductively by starting with the axioms and applying inference rules.

B.5 Example: Predicate calculus

We spell out the details of single-sorted predicate calculus and first-order theories. This is the most common deductive system taught in classical courses on logic.

The predicate calculus has the following syntax:

$$
\begin{aligned}
\text{Variable } v &::= x \mid y \mid z \mid \cdots \\
\text{Constant symbol } c &::= \mathtt{c}_1 \mid \mathtt{c}_2 \mid \cdots \\
\text{Function symbol}^4\ f^k &::= \mathtt{f}_1^{k_1} \mid \mathtt{f}_2^{k_2} \mid \cdots \\
\text{Term } t &::= v \mid c \mid f^k(t_1, \ldots, t_k) \\
\text{Relation symbol } R^m &::= \mathtt{R}_1^{m_1} \mid \mathtt{R}_2^{m_2} \mid \cdots \\
\text{Formula } \phi &::= \bot \mid \top \mid R^m(t_1, \ldots, t_m) \mid t_1 = t_2 \mid \\
&\qquad \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid \phi_1 \Rightarrow \phi_2 \mid \neg\phi \mid \forall x \,.\, \phi \mid \exists x \,.\, \phi.
\end{aligned}
$$

The variable x is bound in $\forall x \,.\, \phi$ and $\exists x \,.\, \phi$.

The predicate calculus has just one form of judgement

$$x_1, \ldots, x_n \mid \phi_1, \ldots, \phi_m \vdash \phi,$$

where $x_1, \ldots, x_n$ is a *context* consisting of distinct variables, $\phi_1, \ldots, \phi_m$ are *hypotheses* and ϕ is the *conclusion*. The free variables in the hypotheses and the conclusion must occur among the ones listed in the context. We abbreviate the context with Γ and Φ with hypotheses. Because most rules leave the context unchanged, we omit the context unless something interesting happens with it.

The following inference rules are given in the form of adjunctions. See Appendix **??** for the more usual formulation in terms of introduction an elimination rules.

$$\frac{}{\phi_1, \ldots, \phi_m \vdash \phi_i} \qquad \frac{}{\Phi \vdash \top} \qquad \frac{}{\Phi, \bot \vdash \phi}$$

$$\frac{\Phi \vdash \phi_1 \quad \Phi \vdash \phi_2}{\Phi \vdash \phi_1 \wedge \phi_2} \qquad \frac{\Phi, \phi_1 \vdash \psi \quad \Phi, \phi_2 \vdash \psi}{\Phi, \phi_1 \vee \phi_2 \vdash \psi} \qquad \frac{\Phi, \phi_1 \vdash \phi_2}{\Phi \vdash \phi_1 \Rightarrow \phi_2}$$

$$\frac{\Gamma, x, y \mid \Phi, x = y \vdash \phi}{\Gamma, x \mid \Phi \vdash \phi[x/y]} \qquad \frac{\Gamma, x \mid \Phi, \phi \vdash \psi}{\Gamma \mid \Phi, \exists x \,.\, \phi \vdash \psi} \qquad \frac{\Gamma, x \mid \Phi \vdash \phi}{\Gamma \mid \Phi \vdash \forall x \,.\, \phi}$$

The equality rule implicitly requires that y does not appear in Φ, and the quantifier rules implicitly require that x does not occur freely in Φ and ψ because the judgments below the lines are supposed to be well formed.

Negation $\neg\phi$ is defined to be $\phi \Rightarrow \bot$. To obtain *classical* logic we also need the law of excluded middle,

$$\frac{}{\Phi \vdash \phi \vee \neg\phi}$$

Comment on the fact that contraction and weakening are admissible.

Give an example of a derivation.

A *first-order theory* $\mathbb{T}$ consists of a set of constant, function and relation symbols with corresponding arities, and a set of formulas, called *axioms*.

Give examples of a first-order theories.

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