Toward a higher realizability topos (Notes)

Steve Awodey

October 6, 2024

Here are some conventions:

- For a small category $\mathbb{C}$ let $\widehat{\mathbb{C}} = [\mathbb{C}^{\mathsf{op}}, \mathsf{Set}]$ be the category of presheaves, and
$$\mathsf{y} : \mathbb{C} \hookrightarrow \widehat{\mathbb{C}}$$
the Yoneda embedding.
-

1 PCAs

We review the basic definitions of partial combinatory algebras and applicative morphisms, which can be found in [**?**, **?**]. For a partial function $f \colon X \rightharpoonup Y$ the notation $fx\downarrow$ means that f is defined for the argument $x \in X$. For a partial binary operation $x \cdot y$, the expression $x \cdot y \doteq u \cdot v$ means that if one side is defined then so is the other and they are equal.

Definition 1. A *partial combinatory algebra (PCA)* $(A, \cdot, \mathsf{K}, \mathsf{S})$ is a set A together with a partial binary operation

$$\cdot\ : A \times A \rightharpoonup A$$

called *application*, and distinguished elements $\mathsf{K}, \mathsf{S} \in A$, such that for all $x, y, z \in A$,

$$\mathsf{K}xy \doteq x\ , \qquad \mathsf{S}xy\downarrow\ , \qquad \mathsf{S}xyz \doteq (xz)(yz)\ ,$$

where write xy instead of $x \cdot y$, and associate application to the left. We only consider non-trivial PCAs, satisfying $\mathsf{K} \neq \mathsf{S}$.

Example 2. The *first Kleene Algebra* $\mathbb{N}$ is the set of natural numbers $\mathbb{N}$ equipped with *Kleene application* $n \cdot m = \{n\}m$ which applies the n-th partial recursive function $\{n\}$ to m. The existence of K and S is a consequence of the s-m-n theorem [**?**].

2 Assemblies

Definition 3. An *assembly* (X, α) over a PCA $\mathbb{A}$ is a set X together with a map $\alpha : X \to \mathcal{P}A$ such that $\alpha(x) \neq \emptyset$ for all $x \in X$. One says that the elements $a \in \alpha(x) \subseteq A$ *realize* the element x. Thus every element in the carrier X of the assembly is realized by (at least one) $a \in A$.

The assembly is called *partitioned* if the subsets $\alpha(x) \subseteq A$ are all singletons, so that each $x \in X$ has exactly one realizer. In that case, we may regard α as a map $\alpha : X \to A$.

A *morphism of assemblies* $f : (X, \alpha) \to (Y, \beta)$ is a function $f : X \to Y$ for which there is an element $\phi \in A$ such that, for all $x \in X$ and all $a \in \alpha(x)$,

$$\phi \cdot a \in \beta(f(x))\,.$$

One says that "ϕ *tracks* f".

For *partitioned* assemblies, this condition reduces to

$$\phi \cdot \alpha(x) \doteq \beta(f(x))\,,$$

which means that (the left action of) ϕ fits into a commutative diagram as follows.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"] \\ A \arrow[r, "\phi\cdot(-)"'] & A \end{tikzcd}$$

Proposition 4. *The category* PAsm *of partitioned assemblies has all finite limits. The category* Asm *of assemblies is regular and locally cartesian closed.*

3 Completions

Proposition 5. *For any partial combinatory algebra* $\mathbb{A}$*,*

1. *the category of assemblies is the regular completion of the category of partitioned assemblies,*

$$\mathsf{Asm} \;=\; \mathsf{PAsm}_{\mathsf{reg/lex}}\,,$$

2. *the* realizability topos $\mathsf{RT}(\mathbb{A})$ *is the exact completion of the lex category* $\mathsf{PAsm}(\mathbb{A})$*, which can therefore be constructed as the exact completion of the regular category* $\mathsf{Asm}(\mathbb{A})$*,*

$$\mathsf{RT}(\mathbb{A}) \;=\; \mathsf{Asm}(\mathbb{A})_{\mathsf{ex/reg}} \;=\; \mathsf{PAsm}_{\mathsf{ex/lex}}(\mathbb{A})\,.$$

The effective topos $\mathcal{E}ff$ *is the realizability topos of the first Kleene Algebra,*

$$\mathcal{E}ff = \mathsf{RT}(\mathbb{N})\,.$$

In more detail,