

Toward a higher realizability topos (Notes)

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October 7, 2024

Here are some conventions:

- For a small category \mathbb{C} let $\widehat{\mathbb{C}} = [\mathbb{C}^{\text{op}}, \mathbf{Set}]$ be the category of presheaves, and

$$y : \mathbb{C} \hookrightarrow \widehat{\mathbb{C}}$$

the Yoneda embedding.

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1 PCAs

We review the basic definitions of partial combinatory algebras and applicative morphisms, which can be found in [?, ?]. For a partial function $f : X \rightharpoonup Y$ the notation $fx \downarrow$ means that f is defined for the argument $x \in X$. For a partial binary operation $x \cdot y$, the expression $x \cdot y \doteq u \cdot v$ means that if one side is defined then so is the other and they are equal.

Definition 1. A *partial combinatory algebra (PCA)* (A, \cdot, K, S) is a set A together with a partial binary operation

$$\cdot : A \times A \rightharpoonup A$$

called *application*, and distinguished elements $K, S \in A$, such that for all $x, y, z \in A$,

$$Kxy \doteq x, \quad Sxy \downarrow, \quad Sxyz \doteq (xz)(yz),$$

where we write xy instead of $x \cdot y$, and associate application to the left. We only consider non-trivial PCAs, satisfying $K \neq S$.

Example 2. The *first Kleene Algebra* \mathbb{N} is the set of natural numbers \mathbb{N} equipped with *Kleene application* $n \cdot m = \{n\}m$ which applies the n -th partial recursive function $\{n\}$ to m . The existence of \mathbf{K} and \mathbf{S} is a consequence of the s-m-n theorem [?].

2 Assemblies

Definition 3. An *assembly* (X, α) over a PCA \mathbb{A} is a set X together with a map $\alpha : X \rightarrow \mathcal{P}A$ such that $\alpha(x) \neq \emptyset$ for all $x \in X$. One says that the elements $a \in \alpha(x) \subseteq A$ *realize* the element x . Thus every element in the carrier X of the assembly is realized by (at least one) $a \in A$.

The assembly is called *partitioned* if the subsets $\alpha(x) \subseteq A$ are all singletons, so that each $x \in X$ has exactly one realizer. In that case, we may regard α as a map $\alpha : X \rightarrow A$.

A *morphism of assemblies* $f : (X, \alpha) \rightarrow (Y, \beta)$ is a function $f : X \rightarrow Y$ for which there is an element $\phi \in A$ such that, for all $x \in X$ and all $a \in \alpha(x)$,

$$\phi \cdot a \in \beta(f(x)).$$

One says that “ ϕ tracks f ”.

For *partitioned* assemblies, this condition reduces to

$$\phi \cdot \alpha(x) \doteq \beta(f(x)),$$

which means that (the left action of) ϕ fits into a commutative diagram as follows.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \alpha \downarrow & & \downarrow \beta \\ A & \xrightarrow{\phi \cdot (-)} & A \end{array}$$

Proposition 4. *The category \mathbf{PAsm} of partitioned assemblies has all finite limits. The category \mathbf{Asm} of assemblies is regular and locally cartesian closed.*

3 Completions

Proposition 5. *For any partial combinatory algebra \mathbb{A} ,*

1. *the category of assemblies is the regular completion of the category of partitioned assemblies,*

$$\mathbf{Asm} = \mathbf{PAsm}_{\text{reg/lex}} ,$$

2. *the realizability topos $\mathbf{RT}(\mathbb{A})$ is the exact completion of the lex category $\mathbf{PAsm}(\mathbb{A})$, which can therefore be constructed as the exact completion of the regular category $\mathbf{Asm}(\mathbb{A})$,*

$$\mathbf{RT}(\mathbb{A}) = \mathbf{Asm}(\mathbb{A})_{\text{ex/reg}} = \mathbf{PAsm}_{\text{ex/lex}}(\mathbb{A}) .$$

The effective topos $\mathcal{E}ff$ is the realizability topos of the first Kleene Algebra,

$$\mathcal{E}ff = \mathbf{RT}(\mathbb{N}) .$$

In more detail,