Sheaf representations and duality in logic

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1 Outline

1. Gelfand Duality
2. Grothendieck duality for commutative rings
3. Sheaf representation for toposes 1: Lambek and Moerdijk
4. Sheaf representation for toposes 2: Awodey
5. Stone duality for Boolean algebras
6. Lawvere duality and others
7. Sheaf representation for pretoposes
8. Stone-type dualities for pretoposes

2 Gelfand duality

Let X be a space, and consider the ring of real-valued functions,

$$\mathcal{C}(X) \;=\; \mathsf{Top}(X, \mathbb{R}).$$

This is a (contravariant) functor from "geometry" to "algebra":

$$\mathcal{C} : \mathsf{Top}^{\mathsf{op}} \longrightarrow \mathsf{CRng}.$$

It is full and faithful if we restrict to compact Hausdorff spaces and bounded continuous functions $\mathcal{C}^*(X)$:

$$\mathcal{C}^* : \mathsf{KHaus}^{\mathsf{op}} \hookrightarrow \mathsf{CRng}.$$

Theorem 1 (Gelfand duality)**.** KHaus *is dual to the category of all commutative rings of the form* $\mathcal{C}^*(X)$ *and ring homomorphisms between them.*

It then requires some work to determine *which* rings are of the form $\mathcal{C}^*(X)$! They are called *C^*-algebras*.

When can we recover the space X from its ring of functions $\mathcal{C}^*(X)$?

- The points $x \in X$ determine maximal ideals in the ring $\mathcal{C}^*(X)$,

$$M_x = \{ f : X \longrightarrow \mathbb{R} \mid f(x) = 0 \}$$

- The (Zariski) topology on the set $\mathsf{MaxIdl}(A)$ in any ring A has a basis of open sets of the form:

$$B_a \;=\; \{ M \in X \mid a \notin M \}, \qquad a \in A.$$

- If A is a C^*-*algebra*, then this specification will determine a compact Hausdorff space $X = \mathsf{MaxIdl}(A)$ such that $A \cong \mathcal{C}(X)$.

Theorem 2 (Gelfand-Stone-Naimark)**.** KHaus *is dual to the category of* C^**-algebras,*

$$\mathsf{KHaus}^{\mathsf{op}} \;\simeq\; C^*\mathsf{Alg}.$$

3 Grothendieck's sheaf representation for commutative rings

Grothendieck extended this duality from C^*-algebras to *all* commutative rings, by generalizing on the "geometric" side from spaces to *(affine) schemes*,

$$\mathsf{Scheme}_{\mathsf{aff}}^{\mathsf{op}} \;\simeq\; \mathsf{CRng}.$$

The essential change was to generalize the "ring of values" from the constant ring $\mathbb{R}$ to a ring $\mathcal{R}$ that "varies continuously over the space X", i.e. a *sheaf of rings*.

The various rings $\mathcal{R}_x$ generalize the *local rings* of real-valued functions that vanish at $x \in X$.

This allows *every* commutative ring A to be seen as a ring of continuous functions on a suitable space X_A, with values in a suitable sheaf of rings $\mathcal{R}$ on X_A.

Definition 3. A ring (commutative, with unit $1 \neq 0$) is called *local* if it has a unique maximal ideal. Equivalently:

$$x + y \text{ is a unit} \quad \text{implies} \quad x \text{ is a unit or } y \text{ is a unit.}$$

Theorem 4 (Grothendieck)**.** *Let A be a ring. There is a space X with a sheaf of rings $\mathcal{R}$ such that:*

1. *for every $p \in X$, the stalk $\mathcal{R}_p$ is a local ring,*
2. *for the ring of global sections, we have:* $\Gamma(\mathcal{R}) \cong A$.

Thus every ring is isomorphic to the ring of global sections of a sheaf of local rings.

The **space** X is the *prime spectrum* $\mathsf{Spec}(A)$:

1. points $p \in \mathsf{Spec}(A)$ are prime ideals $p \subseteq A$,
2. the topology has basic opens of the following form, for all $f \in A$:

$$B_f = \{p \in \mathsf{Spec}(A) \mid f \notin p\}.$$

The **structure sheaf** $\mathcal{R}$ is determined by "localizing" A at f,

$$\mathcal{R}(B_f) = [f]^{-1}A$$

where $A \to [f]^{-1}A$ freely inverts all of the elements $f, f^2, f^3, \dots$.

The **stalk** $\mathcal{R}_p$ is then the localization of A at p,

$$\mathcal{R}_p = S^{-1}A,$$

where $S = A \setminus p$.

The **affine scheme** $(\mathsf{Spec}(A), \mathcal{O})$ represents A as a "ring of continuous functions"

$$f : \mathsf{Spec}(A) \longrightarrow \mathcal{R}\,,$$

except that the ring $\mathcal{R}$ is itself "varying continuously over the space $\mathsf{Spec}(A)$" (i.e. it is a sheaf).

The local ring $\mathcal{R}_p$ has a **unique maximal ideal**, consisting of "those functions $f : \mathsf{Spec}(A) \longrightarrow \mathcal{R}$ that vanish at p".

It is a "representation" of A because there is always an injective homomorphism

$$A \cong \Gamma(\mathcal{R}) \rightarrowtail \prod_p \mathcal{R}_p\,.$$

Corollary 5 (Sub-direct-product representation)**.** *Every ring A is isomorphic to a **sub**ring of a **direct product** of local rings.*

4 Lambek-Moerdijk sheaf representation for toposes

Definition 6. A (small, elementary) topos is called *sublocal* if its subterminal lattice $\mathsf{Sub}(1)$ has a unique maximal ideal. Equivalently, for $x, y \in \mathsf{Sub}(1)$:

$$x \vee y = 1 \quad \text{implies} \quad x = 1 \text{ or } y = 1\,.$$

Theorem 7 (Lambek-Moerdijk 1982)**.** *Let $\mathcal{E}$ be a topos. There is a space X with a sheaf of toposes $\tilde{\mathcal{E}}$ such that:*

1. *for every $p \in X$, the stalk $\tilde{\mathcal{E}}_p$ is a sublocal topos,*
2. *for the topos of global sections, we have: $\Gamma(\tilde{\mathcal{E}}) \cong \mathcal{E}$.*

Thus every topos is isomorphic to the topos of global sections of a sheaf of sublocal toposes.

The **space** X is the so-called *(sub)spectrum of the topos*, $\mathsf{Spec}(\mathcal{E})$.

It is the prime spectrum of the distributive lattice $\mathsf{Sub}(1)$:

1. the points $P \in \mathsf{Spec}(\mathcal{E})$ are prime ideals $P \subseteq \mathsf{Sub}(1)$,
2. the basic opens have the following form, for all $q \in \mathsf{Sub}(1)$:

$$B_q = \{P \in \mathsf{Spec}(\mathcal{E}) \mid q \notin P\}.$$

The lattice of all open sets of $\mathsf{Spec}(\mathcal{E})$ is isomorphic to the ideal completion of $\mathsf{Sub}(1)$,

$$O(\mathsf{Spec}(\mathcal{E})) = \mathsf{Idl}(\mathsf{Sub}(1))\,.$$

The **structure sheaf** $\tilde{\mathcal{E}}$ is determined by "slicing" $\mathcal{E}$ at $q \in \mathsf{Sub}(1)$,

$$\tilde{\mathcal{E}}(B_q) = \mathcal{E}/q\,.$$

This takes the place of localization. Note that it also "inverts" all those elements $p \in \mathsf{Sub}(1)$ with $q \leq p$.

For the global sections Γ, we have:

$$\Gamma(\tilde{\mathcal{E}}) \cong \tilde{\mathcal{E}}(B_\top) = \mathcal{E}/1 \cong \mathcal{E}\,.$$

So the topos of global sections of $\tilde{\mathcal{E}}$ is indeed isomorphic to $\mathcal{E}$.

The **stalk** $\tilde{\mathcal{E}}_P$ at a prime ideal $P \in \mathsf{Spec}(\mathcal{E})$ is the filter-quotient topos,

$$\tilde{\mathcal{E}}_P = \varinjlim_{q \notin P} \mathcal{E}/q,$$

at the prime **filter** $\mathsf{Sub}(1)\backslash P$.

One then has:

$$\mathsf{Sub}_{\tilde{\mathcal{E}}_P}(1) \cong P\,,$$

so the stalk topos $\tilde{\mathcal{E}}_P$ is indeed sublocal.

Again, there is always an injection from the global sections into the product of the stalks,

$$\mathcal{E} \cong \Gamma(\tilde{\mathcal{E}}) \rightarrowtail \prod_{P \in X} \tilde{\mathcal{E}}_P\,.$$

Corollary 8 (Sub-direct-product representation for toposes)**.** *Every topos $\mathcal{E}$ is isomorphic to a **sub**topos of a **direct product** of sublocal toposes.*

We have the following **logical interpretation** of the sheaf representation:

- A topos $\mathcal{E}$ is (the term model of) a theory in Intuitionistic Higher-Order Logic.
- A sublocal topos $\mathcal{S}$ is one that has the *disjunction property*:

  $$\mathcal{S} \vdash p \vee q \qquad \text{iff} \qquad \mathcal{S} \vdash p \ \text{ or } \ \mathcal{S} \vdash q\,,$$

  for all "propositions" p, q.
- The subdirect-product embedding is a logical completeness theorem with respect to such "semantic" toposes $\mathcal{S}$.
- The sheaf representation is a Kripke-style completeness theorem for IHOL, with $\tilde{\mathcal{E}}$ as a "sheaf of possible worlds".

4.1 Lambek's modified sheaf representation for toposes

But this result is **not entirely satisfactory**, because we would like the "semantic worlds" $\mathcal{S}$ to also have the *existence property*:

$$\mathcal{S} \vdash (\exists x : A)\varphi(x) \qquad \text{iff} \qquad \mathcal{S} \vdash \varphi(a) \text{ for some closed } a : A\,,$$

(we know that we can prove completeness with respect to such).

Definition 9. A topos $\mathcal{S}$ is called **local** if the terminal object 1 is indecomposable and projective, i.e. the global sections functor

$$\Gamma = \mathrm{Hom}_{\mathcal{S}}(1, -) : \mathcal{S} \longrightarrow \mathsf{Set}$$

preserves coproducts and epimorphisms.

Note that a local topos has **both** the disjunction and existence properties.

Lambek gave the following improvement over the sublocal sheaf representation:

Theorem 10 (Lambek 1989)**.** *Let $\mathcal{E}$ be a topos. There is a faithful logical functor $\mathcal{E} \rightarrowtail \mathcal{F}$ and a space X with a sheaf of toposes $\tilde{\mathcal{F}}$ such that:*

1. *for every $p \in X$, the stalk $\tilde{\mathcal{F}}_p$ is a **local** topos,*
2. *for the topos of global sections, we have: $\Gamma(\tilde{\mathcal{F}}) \cong \mathcal{F}$.*

*Thus every topos is a **subtopos** of one that is isomorphic to the topos of global sections of a sheaf of **local** toposes.*

This suffices for a *sub-direct-product representation* into **local** toposes, and therefore gives the desired *logical completeness* with respect to **local** toposes.

But conceptually it is still not entirely satisfactory.

5 Local sheaf representation for toposes

In my thesis, I proved:

Theorem 11 (A. 1998)**.** *Let $\mathcal{E}$ be a topos. There is a space X with a sheaf of toposes $\tilde{\mathcal{E}}$ such that:*

1. *for every $p \in X$, the stalk $\tilde{\mathcal{E}}_p$ is a **local** topos,*
2. *for the topos of global sections, we have: $\Gamma(\tilde{\mathcal{E}}) \cong \mathcal{E}$.*

*Thus every topos is isomorphic to the global sections of a sheaf of **local** toposes.*

As before, this gives a *sub-direct-product representation*,

$$\mathcal{E} \rightarrowtail \prod_p \mathcal{S}_p$$

into a product of local toposes $\mathcal{S}_p$, and therefore implies the desired *logical completeness* of IHOL with respect to local toposes.

The stronger result also gives better "Kripke semantics" for IHOL, since the "sheaf of possible worlds" now has **local** stalks.

For **classical** higher-order logic, more can be said:

Lemma 12. *Every local* ***boolean*** *topos is well-pointed, i.e. the global sections functor,*

$$\Gamma = \mathrm{Hom}_{\mathcal{S}}(1, -) : \mathcal{S} \longrightarrow \mathsf{Set}$$

is faithful.

A well-pointed topos is essentially a model of set theory.

Corollary 13. *Every boolean topos is isomorphic to the global sections of a sheaf of* ***well-pointed*** *toposes.*

For boolean toposes, we therefore have the representation,

$$\mathcal{B} \rightarrowtail \prod_p \mathcal{S}_p$$

as sub-direct-product of *well-pointed* toposes $\mathcal{S}_p$, along with its logical counterpart:

Corollary 14. *Classical HOL is complete with respect to models in well-pointed toposes.*

These are **standard** models of classical HOL, taken in varying ("non-standard") models of set theory.

Taking the global sections $\Gamma : \mathcal{S}_p \rightarrowtail \mathsf{Set}$ of each such well-pointed model then embeds any boolean topos $\mathcal{B}$ into a power of Set:

$$\mathcal{B} \rightarrowtail \prod_p \mathcal{S}_p \rightarrowtail \prod_p \mathsf{Set}_p \cong \mathsf{Set}^X ,$$

The various composites $\mathcal{B} \to \mathcal{S}_p \rightarrowtail \mathsf{Set}$ are Henkin style, "non-standard" models of HOL in Set.

Corollary 15. *Classical HOL is complete with respect to Henkin models in* Set.

These Henkin models can be taken as the points of the space $X_{\mathcal{E}}$ for the sheaf representation.

To define the **space** $X_{\mathcal{E}}$ **of models**:

In the **sublocal** case, the points were *prime ideals* $p \subseteq \mathsf{Sub}(1)$. These correspond exactly to *lattice homomorphisms*

$$p : \mathsf{Sub}_{\mathcal{E}}(1) \longrightarrow \mathbf{2}\,.$$

For the **local** case, we instead take *coherent functors*

$$P : \mathcal{E} \longrightarrow \mathsf{Set}\,.$$

These correspond exactly to Henkin models of (the theory represented by) $\mathcal{E}$.

The **topology** is given (roughly speaking) by basic open sets of the following form, for all formulas φ:

$$V_\varphi = \{P \mid P \models \varphi\}$$

The **structure sheaf** $\tilde{\mathcal{E}}$ is first defined as a **stack** on $\mathcal{E}$ by "slicing",

$$\tilde{\mathcal{E}}(A) \;=\; \mathcal{E}/A\,.$$

The stack is first strictified to a **sheaf**, and then transferred from $\mathcal{E}$ to the space $X_{\mathcal{E}}$ of models using a topos-theoretic covering theorem due to Butz and Moerdijk.

For the **global sections** Γ, we then have:

$$\Gamma(\tilde{\mathcal{E}}) \;\simeq\; \mathcal{E}/1 \;\cong\; \mathcal{E}\,.$$

And for the **stalks** $\tilde{\mathcal{E}}_P$ we have the colimit,

$$\tilde{\mathcal{E}}_P \;=\; \varinjlim_{A \in \int P} \mathcal{E}/A,$$

where the (filtered!) category of elements $\int P$ of the Henkin model P takes the place of the prime filter.

6 Toward logical duality

The results for toposes suggest an analogous treatment for **pretoposes** which would be somewhat better, because the models involved would all be **standard** ones, rather than Henkin style, non-standard models.

We then have the possibility of a **logical duality theory** analogous to Grothendieck's duality for schemes and commutative rings, with the sheaf representation playing the role of a **logical structure sheaf**.

This can be seen as a generalization of classical Stone duality for Boolean algebras (= Boolean rings): from a logical point of view, we have a **Stone duality for first-order logic**, with the classical theory for Boolean algebras appearing as the propositional case.

7 Boolean algebras and Stone duality

Recall that for a boolean algebra B we have the Stone space $\mathsf{Spec}(B)$, defined exactly as for the subterminal lattice $\mathsf{Sub}_{\mathcal{E}}(1)$ of a topos $\mathcal{E}$ (i.e. the prime spectrum). We can represent the **points** $p \in \mathsf{Spec}(B)$ as boolean homomorphisms,

$$p : B \longrightarrow \mathbf{2}\,.$$

We can recover B from the space $\mathsf{Spec}(B)$ as the **clopen subsets**, which are represented by continuous maps,

$$f : \mathsf{Spec}(B) \longrightarrow \mathbf{2}\,,$$

where $\mathbf{2}$ is given the discrete topology. (This is just a **constant** sheaf representation!)

There is a contravariant equivalence of categories,

$$\begin{tikzcd} \mathbf{Bool} \arrow[r, bend left, "\mathsf{Spec}"] & \mathbf{Stone}^{\mathsf{op}} \arrow[l, bend left, "\mathsf{Clop}"] \end{tikzcd}$$

The functors are given just by homming into $\mathbf{2}$.

Logically, a Boolean algebra is (the Lindenbaum-Tarski algebra of) a **theory in propositional logic**, and a boolean homomorphism $B \longrightarrow \mathbf{2}$ is a **model**, i.e. a truth-valuation.

We shall generalize this situation by replacing Boolean algebras with (Boolean) pretoposes, representing **first-order** logical theories, and replacing $\mathbf{2}$-valued models with **Set**-valued models.

8 Lawvere duality

Consider first the simpler case of equational logic, rather than full first-order logic:

- In place of a Boolean algebra representing a "propositional theory", we have a category $\mathbb{C}_T$ with finite products, representing an algebraic theory T (such as groups).
- $\mathbb{C}_T$ may be taken to be the *dual* of the category $T\mathsf{Alg}_{fg}$ of all finitely generated free algebras and algebra homomorphisms between them.
- The general T-algebras then correspond to FP-functors $\mathbb{C}_T \longrightarrow \mathsf{Set}$, where the category Set now plays the role of the "ring of values", in place of the Boolean algebra $\{0 \leq 1\}$ of "truth values".

Theorem 16 (Lawvere 1963)**.** *There is an equivalence of categories,*

$$T\mathsf{Alg} \;\simeq\; FP(T\mathsf{Alg}_{fg}^{\mathsf{op}}, \mathsf{Set})\,.$$

9 Lawvere duality and others

The following dualities are determined in essentially the same way:

Lawvere	equational logic operations and equations	finite product categories
Gabriel-Ulmer	relational logic operations, =, relations	finite limit categories
Makkai	regular logic operations, =, relations, $\exists$	regular categories

- In each case the "algebraic" side consists of a structured category representing the logical theory, and the "spatial" side consists of structure-preserving functors into Set, which are the models.

- Recovering the "algebra" from the "space" (the theory from the models) requires a Stone-like represetation/completeness theorem.

- But the situation with the further logical operations $\forall$, $\Rightarrow$, and $\neg$ is somewhat different, because they have a contravariant aspect that cannot be recovered from homomorphisms of models.

10 Stone duality for pretoposes (A.-Forssell)

The further generalization of Stone duality to Boolean pretoposes works like this:

Boolean algebra B propositional theory	Boolean pretopos $\mathcal{B}$ first-order theory
homomorphism $B \longrightarrow \mathbf{2}$ truth-valuation	pretopos functor $\mathcal{B} \longrightarrow \mathbf{Set}$ elementary model
topological space $\mathsf{Spec}(B)$ of all valuations	topological **groupoid** $\mathbf{Spec}(\mathcal{B})$ of all models and isos
continuous function $\mathsf{Spec}(B) \longrightarrow \mathbf{2}$ clopen set	coherent functor $\mathbf{Spec}(\mathcal{B}) \longrightarrow \mathbf{Set}$ coherent sheaf

Theorem 17 (A.-Forssell 2008)**.** *There is a contravariant* ***adjunction****,*

$$\begin{tikzcd} \mathbf{BPreTop} \arrow[rr, bend left, "\mathsf{Spec}"] & & \mathbf{StoneTopGpd}^{\mathsf{op}} \arrow[ll, bend left, "\mathsf{Coh}"] \end{tikzcd},$$

in which the functors are given by homming into $\mathbf{Set}$*.*

The spectrum $\mathbf{Spec}(\mathcal{B})$ of a Boolean pretopos $\mathcal{B}$ is the groupoid of models and isos, topologized by "satisfaction of formulas".

Recovering $\mathcal{B}$ from $\mathbf{Spec}(\mathcal{B})$ amounts to recovering an elementary theory from its models. This is done by taking the coherent, equivariant sheaves on $\mathbf{Spec}(\mathcal{B})$, using results from topos theory.

Makkai has found a related **equivalence**:

$$\begin{tikzcd} \mathbf{BPreTop} \arrow[rr, bend left] \arrow[rr, phantom, "\simeq"] & & \mathbf{UltraGpd}^{\mathsf{op}} \arrow[ll, bend left] \end{tikzcd}$$

We replace Makkai's **ultraproduct** structure on the groupoids of models by a Stone-Zariski type **logical topology**.

For us, however, the "semantic" functor,

$$\mathsf{Spec} : \mathbf{BPreTop} \longrightarrow \mathbf{StoneTopGpd}^{\mathsf{op}}$$

is **not full**: there are continuous functors between the groupoids of models that do not come from a "translation of theories".

Compare the case of commutative rings A, B: a continuous function

$$f : \mathsf{Spec}(B) \longrightarrow \mathsf{Spec}(A)$$

need not come from a ring homomorphism $h : A \longrightarrow B$.

11 Sheaf representation for pretoposes (A.-Breiner)

As for rings and affine schemes, we can equip the spectrum $\mathbf{Spec}(\mathcal{B})$ of the pretopos $\mathcal{B}$ with a "structure sheaf" $\tilde{\mathcal{B}}$,
defined as in the sheaf representation for toposes:

- start with the presheaf of categories $\tilde{\mathcal{B}} : \mathcal{B}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ with,

$$\tilde{\mathcal{B}}(X) \cong \mathcal{B}/X\,,$$

  for all $X \in \mathbb{B}$. This is a **stack** because $\mathcal{B}$ is a pretopos.

- Strictify to get a sheaf of categories on $\mathcal{B}$.

- Use the equivalence of toposes,

$$\mathbf{Sh}(\mathbb{B}) \;\simeq\; \mathbf{Sh}_{\mathsf{eq}}(\mathbf{Spec}(\mathcal{B}))$$

  $\mathbf{Spec}(\mathcal{B})$ is determined so that this equivalence holds.

- Move $\tilde{\mathcal{B}}$ along this equivalence in order to get an equivariant sheaf on $\mathbf{Spec}(\mathcal{B})$.

The transported $\tilde{\mathcal{B}}$ is an equivariant sheaf of local, boolean pretoposes on the topological groupoid $\mathbf{Spec}(\mathcal{B})$.

- Logically, $\tilde{\mathcal{B}}$ is a sheaf of "local theories" on the groupoid of models, equipped with the logical topology.

- As before, $\tilde{\mathcal{B}}$ has global sections $\Gamma\tilde{\mathcal{B}} \simeq \mathcal{B}$. So the original "theory" $\mathcal{B}$ is the "theory of all models".

- The stalk $\tilde{\mathcal{B}}_P$ at a model $P : \mathcal{B} \longrightarrow \mathsf{Set}$ is a well-pointed pretopos: it is the "elementary diagram" of the model P.

- The global sections functors $\Gamma_P : \tilde{\mathcal{B}}_P \rightarrowtail \mathsf{Set}$ are faithful **pretopos** morphisms, i.e. "complete models".

In sum, we have the following:

Theorem 18 (A.-Breiner 2013)**.** *Let $\mathbb{B}$ be a boolean pretopos. There is a topological groupoid G with an equivariant sheaf of pretoposes $\tilde{\mathbb{B}}$ such that:*

1. *for every $x \in G$, the stalk $\tilde{\mathbb{B}}_x$ is a well-pointed pretopos,*
2. *for the pretopos of global sections, we have: $\Gamma(\tilde{\mathbb{B}}) \cong \mathbb{B}$.*

Thus every Boolean pretopos is isomorphic to the global sections of a sheaf of well-pointed pretoposes.

There is an analogous result for the non-Boolean case, with local pretoposes in place of well-pointed ones in the stalks.

The resulting sub-direct-product representation $\mathcal{B} \rightarrowtail \prod_x \tilde{\mathcal{B}}_x$ yields:

Corollary 19 (Gödel completeness theorem)**.** *There is a pretopos embedding,*

$$\mathcal{B} \rightarrowtail \prod_{x \in G} \tilde{\mathcal{B}}_x \rightarrowtail \prod_{x \in G} \mathsf{Set} \simeq \mathsf{Set}^{|G|},$$

with $|G|$ the set of points of the topological groupoid $G = \mathbf{Spec}(\mathcal{B})$ of models.

12 Logical schemes

For a Boolean pretopos $\mathbb{B}$, call the pair

$$(\mathbf{Spec}(\mathcal{B}), \tilde{\mathcal{B}})$$

an affine **logical scheme**.

A morphism of logical schemes

$$(h, \tilde{h}) : (\mathbf{Spec}(\mathcal{B}), \tilde{\mathcal{B}}) \longrightarrow (\mathbf{Spec}(\mathcal{A}), \tilde{\mathcal{A}})$$

consists of a continuous groupoid homomorphism

$$h : \mathbf{Spec}(\mathcal{B}) \longrightarrow \mathbf{Spec}(\mathcal{A}),$$

together with a pretopos functor

$$\tilde{h} : \tilde{\mathcal{A}} \longrightarrow h_* \tilde{\mathcal{B}}$$

over $\mathbf{Spec}(\mathcal{A})$.

Theorem 20 (A.-Breiner 2012)**.** *Every pretopos functor $\mathcal{A} \longrightarrow \mathcal{B}$ induces a morphism of the associated affine logical schemes. Moreover, the functor*

$$\mathsf{Spec} : \mathbf{BPreTop}^{\mathsf{op}} \longrightarrow \mathbf{LScheme}_{aff}$$

*is **full**: every map of schemes comes from a map of pretoposes.*

Corollary 21 (First-order logical duality)**.** *There is an equivalence,*

$$\mathbf{BPreTop}^{\mathsf{op}} \;\simeq\; \mathbf{LScheme}_{aff}\,.$$

Thus the category of Boolean pretoposes is dual to the category of affine logical schemes.

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