Introduction to Categorical Logic

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Contents

Chapter 3

Cartesian Closed Categories and the λ-Calculus

3.1 Categorification and the Curry-Howard correspondence

Consider the following natural deduction proof in propositional calculus.

$$\dfrac{\dfrac{\dfrac{\dfrac{[(A \wedge B) \wedge (A \Rightarrow B)]^1}{A \wedge B}}{A} \qquad \dfrac{[(A \wedge B) \wedge (A \Rightarrow B)]^1}{A \Rightarrow B}}{B}}{(A \wedge B) \wedge (A \Rightarrow B) \Rightarrow B}\,{\scriptstyle(1)}$$

This deduction shows that

$$\vdash (A \wedge B) \wedge (A \Rightarrow B) \Rightarrow B.$$

But so does the following:

$$\dfrac{\dfrac{\dfrac{[(A \wedge B) \wedge (A \Rightarrow B)]^1}{A \Rightarrow B} \qquad \dfrac{\dfrac{[(A \wedge B) \wedge (A \Rightarrow B)]^1}{A \wedge B}}{A}}{B}}{(A \wedge B) \wedge (A \Rightarrow B) \Rightarrow B}\,{\scriptstyle(1)}$$

As does:

$$\dfrac{\dfrac{\dfrac{[(A \wedge B) \wedge (A \Rightarrow B)]^1}{A \wedge B}}{B}}{(A \wedge B) \wedge (A \Rightarrow B) \Rightarrow B}\,{\scriptstyle(1)}$$

There is a sense in which the first two proofs are "equivalent", but not the first and the third. The relation (or property) of *provability* in propositional calculus $\vdash \phi$ discards such differences in the proofs that witness it. According to the "proof-relevant" point of view, sometimes called *propositions as types*, one retains as relevant some information about the way in which a proposition is proved. This is effected by annotating the proofs with *proof-terms* as they are constructed, as follows:

$$\dfrac{\dfrac{\dfrac{[x : (A \wedge B) \wedge (A \Rightarrow B)]^1}{\pi_2(x) : A \Rightarrow B} \qquad \dfrac{\dfrac{[x : (A \wedge B) \wedge (A \Rightarrow B)]^1}{\pi_1(x) : A \wedge B}}{\pi_1(\pi_1(x)) : A}}{\pi_2(x)(\pi_1(\pi_1(x))) : B}}{\lambda x.\pi_2(x)(\pi_1(\pi_1(x))) : (A \wedge B) \wedge (A \Rightarrow B) \Rightarrow B}\ (1)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[x : (A \wedge B) \wedge (A \Rightarrow B)]^1}{\pi_1(x) : A \wedge B}}{\pi_1(\pi_1(x)) : A} \qquad \dfrac{[x : (A \wedge B) \wedge (A \Rightarrow B)]^1}{\pi_2(x) : A \Rightarrow B}}{\pi_2(x)(\pi_1(\pi_1(x))) : B}}{\lambda x.\pi_2(x)(\pi_1(\pi_1(x))) : (A \wedge B) \wedge (A \Rightarrow B) \Rightarrow B}\ (1)$$

$$\dfrac{\dfrac{\dfrac{[x : (A \wedge B) \wedge (A \Rightarrow B)]^1}{\pi_1(x) : A \wedge B}}{\pi_2(\pi_1(x)) : B}}{\lambda x.\pi_2(\pi_1(x)) : (A \wedge B) \wedge (A \Rightarrow B) \Rightarrow B}\ (1)$$

The proof terms for the first two proofs are the same, namely $\lambda x.\pi_2(x)(\pi_1(\pi_1(x)))$, but the term for the third one is $\lambda x.\pi_2(\pi_1(x))$, reflecting the difference in the proofs. The assignment works by labelling assumptions as variables, and then associating term-constructors to the different rules of inference: pairing and projection to conjunction introduction and elimination, function application and λ-abstraction to implication elimination (*modus ponens*) and introduction. The use of variable binding to represent cancellation of premisses is a particularly effective device.

From the categorical point of view, the relation of deducibility $\phi \vdash \psi$ is a mere preorder. The addition of proof terms $x : \phi \vdash t : \psi$ results in a *categorification* of this preorder, in the sense that it is a "proper" category, the preordered reflection of which is the deducibility preorder. And now the following remarkable fact emerges: it is hardly surprising that the deducibility preorder has, say, finite products $\phi \wedge \psi$ or even exponentials $\phi \Rightarrow \psi$; but it is *amazing* that the category with proof terms $x : \phi \vdash t : \psi$ as arrows, also turns out to be a cartesian closed category, and indeed a proper one, with distinct parallel arrows, such as

$$\begin{aligned} \pi_2(x)(\pi_1(\pi_1(x))) &: (A \wedge B) \wedge (A \Rightarrow B) \longrightarrow B, \\ \pi_2(\pi_1(x)) &: (A \wedge B) \wedge (A \Rightarrow B) \longrightarrow B. \end{aligned}$$

This *category of proofs* contains information about the "proof theory" of the propositional calculus, as opposed to its mere relation of deducibility. The calculus of proof terms can be presented formally in a system of *simple type theory*, with an alternate interpretation as a formal system of function application and abstraction. This dual interpretation—as the proof theory of propositional logic, and as a system of type theory for the specification of functions—is called the *Curry-Howard correspondence* []. From the categorical point of view, it expresses the structural equivalence between the cartesian closed categories of proofs in propositional logic and terms in simple type theory. Both of these can be seen as categorifications of their preorder reflection, the deducibility preorder of propositional logic (cf. [MH92]).

In the following sections, we shall consider this remarkable correspondence in detail, as well as some extensions of the basic case represented by cartesian closed categories: categories with coproducts, cocomplete categories, and categories equipped with modal operators. In the next chapter, it will be seen that this correspondence even extends to proofs in quantified predicate logic and terms in dependent type theory, and beyond.

3.2 Cartesian closed categories

Exponentials

We begin with the notion of an exponential B^A of two objects A, B in a category, motivated by a couple of important examples. Consider first the category Pos of posets and monotone functions. For posets P and Q the set $\mathsf{Hom}(P, Q)$ of all monotone functions between them is again a poset, with the pointwise order:

$$f \leq g \iff fx \leq gx \quad \text{for all } x \in P\,. \qquad\qquad (f, g : P \to Q)$$

Thus $\mathsf{Hom}(P, Q)$ is again an object of Pos, when equipped with a suitable order.

Similarly, given monoids $K, M \in \mathsf{Mon}$, there is a natural monoid structure on the set $\mathsf{Hom}(K, M)$, defined pointwise by

$$(f \cdot g)x = fx \cdot gx\,. \qquad\qquad (f, g : K \to M,\, x \in K)$$

Thus the category Mon also admits such "internal Hom"s. The same thing works in the category Group of groups and group homomophisms, where the set $\mathsf{Hom}(G, H)$ of all homomorphisms between groups G and H can be given a pointwise group structure.

These examples suggest a general notion of "internal Hom" in a category: an "object of morphisms $A \to B$" which corresponds to the hom-set $\mathsf{Hom}(A, B)$. The other ingredient needed is an "evaluation" operation $\epsilon : B^A \times A \to B$ which evaluates a morphism $f \in B^A$ at an argument $x \in A$ to give a value $\epsilon \circ \langle f, x\rangle \in B$. This is always going to be present for the underlying functions if we're starting from a set of functions $\mathsf{Hom}(A, B)$, but it needs to be an actual morphism in the category. Finally, we need an operation of "transposition", taking a morphism $f : C \times A \to B$ to one $\widetilde{f} : C \to A^B$. We shall see that this in fact separates the previous two examples.

Definition 3.2.1. In a category $\mathcal{C}$ with binary products, an *exponential* (B^A, ϵ) of objects A and B is an object B^A together with a morphism $\epsilon : B^A \times A \to B$, called the *evaluation* morphism, such that for every $f : C \times A \to B$ there exists a *unique* morphism $\widetilde{f} : C \to B^A$, called the *transpose*[1] of f, for which the following diagram commutes.

$$
\begin{tikzcd}
B^A & B^A \times A \arrow[r, "\epsilon"] & B \\
C \arrow[u, dotted, "\widetilde{f}"] & C \times A \arrow[u, "\widetilde{f} \times 1_A"] \arrow[ur, "f"'] &
\end{tikzcd}
$$

Commutativity of the diagram of course means that $f = \epsilon \circ (\widetilde{f} \times 1_A)$.

Definition 3.2.1 is called the *universal property of the exponential.* It is just the category-theoretic way of saying that a function $f : C \times A \to B$ of two variables can be viewed as a function $\widetilde{f} : C \to B^A$ of one variable that maps $z \in C$ to a function $\widetilde{f}z = f\langle z, -\rangle : A \to B$ that maps $x \in A$ to $f\langle z, x\rangle$. The relationship between f and $\widetilde{f}$ is then

$$f\langle z, x\rangle = (\widetilde{f}z)x \; .$$

That is all there is to it, really, except that variables and elements never need to be mentioned. The benefit of this is that the definition is applicable also in categories whose objects are not *sets* and whose morphisms are not *functions*—even though some of the basic examples are of that sort.

In Poset the exponential Q^P of posets P and Q is the set of all monotone maps $P \to Q$, ordered pointwise, as above. The evaluation map $\epsilon : Q^P \times P \to Q$ is just the usual evaluation of a function at an argument. The transpose of a monotone map $f : R \times P \to Q$ is the map $\widetilde{f} : R \to Q^P$, defined by, $(\widetilde{f}z)x = f\langle z, x\rangle$, i.e. the transposed *function.* We say that the category Pos *has all exponentials.*

Definition 3.2.2. Suppose $\mathcal{C}$ has all finite products. An object $A \in \mathcal{C}$ is *exponentiable* when the exponential B^A exists for every $B \in \mathcal{C}$. We say that $\mathcal{C}$ *has exponentials* if every object is exponentiable. A *cartesian closed category (ccc)* is a category that has all finite products and exponentials.

Example 3.2.3. Consider again the example of the set $\mathsf{Hom}(M, N)$ of homomorphisms between two monoids M, N, equipped with the pointwise monoid structure. To be a monoid homomorphism. the transpose $\widetilde{h} : 1 \to \mathsf{Hom}(M, N)$ of a homomorphism $h : 1 \times M \to N$ would have to take the unit element $u \in 1$ to the unit homomorphism $u : M \to N$, which is the constant function at the unit $u \in N$. Since $1 \times M \cong M$, that would mean that *all* homomorphisms $h : M \to N$ would have the same transpose $\widetilde{h} = u : 1 \to \mathsf{Hom}(M, N)$. So Mon cannot be cartesian closed. The same argument works in the category Group, and in many related ones. (But see **??** below on one way of embedding Group into a CCC.)

Exercise 3.2.4. Is the evaluation function $\mathsf{eval} : \mathsf{Hom}(M, N) \times M \to N$ a homomorphism of monoids?

[1] Also, f is called the transpose of $\widetilde{f}$, so that f and $\widetilde{f}$ are each other's transpose.

Two characterizations of CCCs

Proposition 3.2.5. *In a category $\mathcal{C}$ with binary products an object A is exponentiable if, and only if, the functor*

$$- \times A : \mathcal{C} \to \mathcal{C}$$

has a right adjoint

$$-^A : \mathcal{C} \to \mathcal{C} \ .$$

Proof. If such a right adjoint exists then the exponential of A and B is (B^A, ϵ_B), where $\epsilon : -^A \times A \Longrightarrow 1_\mathcal{C}$ is the counit of the adjunction. The universal property of the exponential is precisely the universal property of the counit ϵ.

Conversely, suppose for every B there is an exponential (B^A, ϵ_B). As the object part of the right adjoint we then take B^A. For the morphism part, given $g : B \to C$, we can define $g^A : B^A \to C^A$ to be the transpose of $g \circ \epsilon_B$,

$$g^A = (g \circ \epsilon_B)^\sim$$

as indicated below.

$$\begin{tikzcd} B^A \times A \arrow[r, "\epsilon_B"] \arrow[d, "g^A \times 1_A"'] & B \arrow[d, "g"] \\ C^A \times A \arrow[r, "\epsilon_C"'] & C \end{tikzcd} \tag{3.1}$$

The counit $\epsilon : -^A \times A \Longrightarrow 1_\mathcal{C}$ at B is then ϵ_B itself, and the naturality square for ϵ is then exactly (3.1), i.e. the defining property of $(f \circ \epsilon_B)^\sim$:

$$\epsilon_C \circ (g^A \times 1_A) = \epsilon_C \circ ((g \circ \epsilon_B)^\sim \times 1_A) = g \circ \epsilon_B \ .$$

The universal property of the counit ϵ is precisely the universal property of the exponential (B^A, ϵ_B) □

Note that because exponentials can be expressed as right adjoints to binary products, they are determined uniquely up to isomorphism. Moreover, the definition of a cartesian closed category can then be phrased entirely in terms of adjoint functors: we just need to require the existence of the terminal object, binary products, and exponentials.

Proposition 3.2.6. *A category $\mathcal{C}$ is cartesian closed if, and only if, the following functors have* right *adjoints:*

$$\begin{aligned} !_\mathcal{C} &: \mathcal{C} \to \mathsf{1} \ , \\ \Delta &: \mathcal{C} \to \mathcal{C} \times \mathcal{C} \ , \\ (- \times A) &: \mathcal{C} \to \mathcal{C} \ . \end{aligned} \qquad (A \in \mathcal{C})$$

Here $!_\mathcal{C}$ is the unique functor from $\mathcal{C}$ to the terminal category $\mathsf{1}$ and Δ is the diagonal functor $\Delta A = \langle A, A \rangle$, and the right adjoint of $- \times A$ is exponentiation by A.

$\square$

The significance of the adjoint formulation is that it implies the possibility of a purely *equational* specification (adjoint structure on a category is "equational" in a sense that can be made precise; see [**?**]). We can therefore give an explicit, equational formulation of cartesian closed categories.

Proposition 3.2.7 (Equational version of CCC)**.** *A category $\mathcal{C}$ is cartesian closed if, and only if, it has the following structure:*

1. *An object $1 \in \mathcal{C}$ and a morphism $!_A : A \to 1$ for every $A \in \mathcal{C}$.*
2. *An object $A \times B$ for all $A, B \in \mathcal{C}$ together with morphisms $\pi_0 : A \times B \to A$ and $\pi_1 : A \times B \to B$, and for every pair of morphisms $f : C \to A$, $g : C \to B$ a morphism $\langle f, g\rangle : C \to A \times B$.*
3. *An object B^A for all $A, B \in \mathcal{C}$ together with a morphism $\epsilon : B^A \times A \to B$, and a morphism $\widetilde{f} : C \to B^A$ for every morphism $f : C \times A \to B$.*

These new objects and morphisms are required to satisfy the following equations:

1. *For every $f : A \to 1$,*
$$f = !_A \, .$$
2. *For all $f : C \to A$, $g : C \to B$, $h : C \to A \times B$,*
$$\pi_0 \circ \langle f, g\rangle = f \, , \qquad \pi_1 \circ \langle f, g\rangle = g \, , \qquad \langle \pi_0 \circ h, \pi_1 \circ h\rangle = h \, .$$
3. *For all $f : C \times A \to B$, $g : C \to B^A$,*
$$\epsilon \circ (\widetilde{f} \times 1_A) = f \, , \qquad (\epsilon \circ (g \times 1_A))^{\sim} = g \, .$$
where for $e : E \to E'$ and $f : F \to F'$ we define
$$e \times f := \langle e\pi_0, f\pi_1\rangle : E \times F \to E' \times F'.$$

These equations ensure that certain diagrams commute and that the morphisms that are required to exist are unique. For example, let us prove that $(A \times B, \pi_0, \pi_1)$ is the product of A and B. For $f : C \to A$ and $g : C \to B$ there exists a morphism $\langle f, g\rangle : C \to A \times B$. Equations

$$\pi_0 \circ \langle f, g\rangle = f \qquad \textit{and} \qquad \pi_1 \circ \langle f, g\rangle = g$$

enforce the commutativity of the two triangles in the following diagram:

$$\begin{tikzcd}
& C \arrow[dl, "g"'] \arrow[d, "\langle f,g\rangle"] \arrow[dr, "f"] & \\
A & A \times B \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & B
\end{tikzcd}$$

Suppose $h : C \to A \times B$ is another morphism such that $f = \pi_0 \circ h$ and $g = \pi_1 \circ h$. Then by the third equation for products we get

$$h = \langle \pi_0 \circ h, \pi_1 \circ h \rangle = \langle f, g \rangle ,$$

and so $\langle f, g \rangle$ is unique.

Exercise 3.2.8. Use the equational characterization of CCCs, Proposition 3.2.7, to show that the category Pos of posets and monotone functions *is* cartesian closed, as claimed. Also verify that that Mon is not. Which parts of the definition fail in Mon?

3.3 Positive propositional calculus

We begin with the example of a cartesian closed poset and a first application to propositional logic.

Example 3.3.1. Consider the *positive propositional calculus* PPC with conjunction and implication, as in Section **??**. Recall that PPC is the set of all propositional formulas ϕ constructed from propositional variables $p_1, p_2, ...$, a constant $\top$ for truth, and binary connectives for conjunction $\phi \wedge \psi$, and implication $\phi \Rightarrow \psi$.

As a category, PPC is a preorder under the relation $\phi \vdash \psi$ of logical entailment, determined for instance by the natural deduction system **??** of section **??**. As usual, it will be convenient to pass to the poset reflection of the preorder, which we shall denote by

$$\mathcal{C}_{\mathsf{PPC}}$$

by identifying ϕ and ψ when $\phi \dashv\vdash \psi$. (This is just the usual *Lindenbaum-Tarski* algebra of the system of propositional logic, as in Section **??**.)

The conjunction $\phi \wedge \psi$ is a greatest lower bound of ϕ and ψ in $\mathcal{C}_{\mathsf{PPC}}$, because we have $\phi \wedge \psi \vdash \phi$ and $\phi \wedge \psi \vdash \psi$ and for all ϑ, if $\vartheta \vdash \phi$ and $\vartheta \vdash \psi$ then $\vartheta \vdash \phi \wedge \psi$. Since binary products in a poset are the same thing as greatest lower bounds, we see that $\mathcal{C}_{\mathsf{PPC}}$ has all binary products; and of course $\top$ is a terminal object.

We have already remarked that implication is right adjoint to conjunction in propositional calculus,

$$(-) \wedge \phi \;\dashv\; \phi \Rightarrow (-) . \tag{3.2}$$

Therefore $\phi \Rightarrow \psi$ is an exponential in $\mathcal{C}_{\mathsf{PPC}}$. The counit of the adjunction (the "evaluation" arrow) is the entailment

$$(\phi \Rightarrow \psi) \wedge \phi \vdash \psi ,$$

i.e. the familiar logical rule of *modus ponens*.

We have now shown:

Proposition 3.3.2. *The poset $\mathcal{C}_{\mathsf{PPC}}$ of positive propositional calculus is cartesian closed.*

Let us now use this fact to show that the positive propositional calculus is *deductively complete* with respect to the following notion of *Kripke semantics* [].

Definition 3.3.3 (Kripke model)**.** Let K be a poset. Suppose we have a relation

$$k \Vdash p$$

between elements $k \in K$ and propositional variables p, such that

$$j \leq k,\ k \Vdash p \quad \text{implies} \quad j \Vdash p. \tag{3.3}$$

Extend $\Vdash$ to all formulas ϕ in PPC by defining

$$\begin{array}{lll} k \Vdash \top & \text{always}, & \\ k \Vdash \phi \wedge \psi & \text{iff} & k \Vdash \phi \text{ and } k \Vdash \psi\,, \\ k \Vdash \phi \Rightarrow \psi & \text{iff} & \text{for all } j \leq k, \text{ if } j \Vdash \phi, \text{ then } j \Vdash \psi\,. \end{array} \tag{3.4}$$

Finally, say that ϕ *holds on* K, written

$$K \Vdash \phi$$

if $k \Vdash \phi$ for all $k \in K$ (for all such relations $\Vdash$).

Theorem 3.3.4 (Kripke completeness for PPC)**.** *A propositional formulas ϕ is provable from the rules of deduction for PPC if, and only if, $K \Vdash \phi$ for all posets K. Briefly:*

$$\mathsf{PPC} \vdash \phi \quad \textit{iff} \quad K \Vdash \phi \textit{ for all } K.$$

We will require the following (which extends the discussion in Section **??**).

Lemma 3.3.5. *For any poset P, the poset $\downarrow\! P$ of all downsets in P, ordered by inclusion, is cartesian closed. Moreover, the downset embedding,*

$$\downarrow\!(-) : P \to \downarrow\! P$$

preserves any CCC structure that exists in P.

Proof. The total downset P is obviously terminal, and for any downsets $S, T \in \downarrow\! P$, the intersection $S \cap T$ is also closed down, so we have the products $S \wedge T = S \cap T$. For the exponential, set

$$S \Rightarrow T = \{p \in P \mid \ \downarrow\!(p) \cap S \subseteq T\}.$$

Then for any downset Q we have

$$Q \subseteq S \Rightarrow T \quad \text{iff} \quad \downarrow\!(q) \cap S \subseteq T, \text{ for all } q \in Q. \tag{3.5}$$

But that means that

$$\textstyle\bigcup_{q \in Q}(\downarrow\!(q) \cap S) \subseteq T\,,$$

which is equivalent to $Q \cap S \subseteq T$, since $\bigcup_{q \in Q}(\downarrow(q) \cap S) = (\bigcup_{q \in Q} \downarrow(q)) \cap S = Q \cap S$.

The preservation of CCC structure by $\downarrow(-) : P \to \downarrow P$ follows from its preservation by the Yoneda embedding, of which $\downarrow(-)$ is a factor,

$$\begin{tikzcd} & \mathsf{Set}^{P^{\mathsf{op}}} \\ P \arrow[ur, hook, "\mathsf{y}"] \arrow[r, "\downarrow(-)"'] & \downarrow P \arrow[u, hook] \end{tikzcd}$$

But it is also easy enough to check directly: preservation of any limits 1, $p \wedge q$ that exist in P are clear. Suppose $p \Rightarrow q$ is an exponential; then for any downset D we have:

$$\begin{array}{lll} D \subseteq \downarrow(p \Rightarrow q) & \text{iff} & \downarrow(d) \subseteq \downarrow(p \Rightarrow q)\ ,\ \text{for all } d \in D \\ & \text{iff} & d \leq p \Rightarrow q\ ,\ \text{for all } d \in D \\ & \text{iff} & d \wedge p \leq q\ ,\ \text{for all } d \in D \\ & \text{iff} & \downarrow(d \wedge p) \subseteq \downarrow(q)\ ,\ \text{for all } d \in D \\ & \text{iff} & \downarrow(d) \cap \downarrow(p) \subseteq \downarrow(q)\ ,\ \text{for all } d \in D \\ & \text{iff} & D \subseteq \downarrow(p) \Rightarrow \downarrow(q) \end{array}$$

where the last line is by (3.5). (Note that in line (3) we assumed that $d \wedge p$ exists for all $d \in D$; this can be avoided by a slightly more complicated argument.) □

Proof. (of Theorem 3.3.4) The proof follows a now-familiar pattern, which we only sketch:

1. The syntactic category $\mathcal{C}_{\mathsf{PPC}}$ is a CCC, with $\top = 1$, $\phi \times \psi = \phi \wedge \psi$, and $\psi^{\phi} = \phi \Rightarrow \psi$. In fact, it is the free cartesian closed poset on the generating set $\mathsf{Var} = \{p_1, p_2, \dots\}$ of propositional variables.

2. A (Kripke) model $(K, \Vdash)$ is the same thing as a CCC functor $\mathcal{C}_{\mathsf{PPC}} \to \downarrow K$, which by Step 1 is just an arbitrary map $\mathsf{Var} \to \downarrow K$, as in (3.3). To see this, observe that we have a bijective correspondence between CCC functors $[\![-]\!]$ and Kripke relations $\Vdash$; indeed, by the exponential adjunction in the cartesian closed category Pos, there is a natural bijection,
$$\frac{[\![-]\!] : \mathcal{C}_{\mathsf{PPC}} \longrightarrow \downarrow K \cong 2^{K^{\mathsf{op}}}}{\Vdash : K^{\mathsf{op}} \times \mathcal{C}_{\mathsf{PPC}} \longrightarrow 2}$$
where we use the poset 2 to classify downsets in a poset P (via upsets in P^{op}),
$$\downarrow P \cong 2^{P^{\mathsf{op}}} \cong \mathsf{Pos}(P^{\mathsf{op}}, 2)\,,$$
by taking the 1-kernel $f^{-1}(1) \subseteq P$ of a monotone map $f : P^{\mathsf{op}} \to 2$. (The contravariance will be convenient in Step 3). Note that the monotonicity of $\Vdash$ yields the conditions
$$p \leq q\,,\ q \Vdash \phi \implies p \Vdash \phi$$

and

$$p \Vdash \phi\,,\ \phi \vdash \psi \implies p \Vdash \psi\,.$$

and the CCC preservation of the transpose $[\![-]\!]$ yields the Kripke forcing conditions (3.4) (exercise!).

3. For any model $(K, \Vdash)$, by the adjunction in (2) we then have $K \Vdash \phi$ iff $[\![\phi]\!] = K$, the total downset.

4. Because the downset/Yoneda embedding $\downarrow$ preserves the CCC structure (by Lemma 3.3.5), $\mathcal{C}_{\mathsf{PPC}}$ has a *canonical model*, namely $(\mathcal{C}_{\mathsf{PPC}}, \Vdash)$, where:

$$\frac{\downarrow(-)\ :\ \mathcal{C}_{\mathsf{PPC}} \longrightarrow {\downarrow}\mathcal{C}_{\mathsf{PPC}} \cong 2^{\mathcal{C}_{\mathsf{PPC}}{}^{\mathsf{op}}} \hookrightarrow \mathsf{Set}^{\mathcal{C}_{\mathsf{PPC}}{}^{\mathsf{op}}}}{\Vdash\ :\ \mathcal{C}_{\mathsf{PPC}}{}^{\mathsf{op}} \times \mathcal{C}_{\mathsf{PPC}} \longrightarrow 2 \hookrightarrow \mathsf{Set}}$$

5. Now note that for the Kripke relation $\Vdash$ in (4), we have $\Vdash\, = \,\vdash$, since it's essentially the transpose of the Yoneda embedding. Thus the model is logically generic, in the sense that $\mathcal{C}_{\mathsf{PPC}} \Vdash \phi$ iff $\mathsf{PPC} \vdash \phi$.

$\square$

Exercise 3.3.6. Verify the claim that CCC preservation of the transpose $[\![-]\!]$ of $\Vdash$ yields the Kripke forcing conditions (3.4).

Exercise 3.3.7. Give a countermodel to show that $\mathsf{PPC} \nvdash (\phi \Rightarrow \psi) \Rightarrow \phi$

3.4 Heyting algebras

We now extend the positive propositional calculus to the full intuitionistic propositional calculus. This involves adding the finite coproducts 0 and $p \vee q$ to notion of a cartesian closed poset, to arrive at the general notion of a Heyting algebra. Heyting algebras are to intuitionistic logic as Boolean algebras are to classical logic: each is an algebraic description of the corresponding logical calculus. We shall review both the algebraic and the logical points of view; as we shall see, many aspects of the theory of Boolean algebras carry over to Heyting algebras. For instance, in order to prove the Kripke completeness of the full system of intuitionistic propositional calculus, we will need an alternative to Lemma 3.3.5, because the Yoneda embedding does not in general preserve coproducts. For that we will again use a version of the Stone representation theorem, this time in a generalized form due to Joyal.

Distributive lattices

Recall first that a (bounded) *lattice* is a poset that has finite limits and colimits. In other words, a lattice $(L, \leq, \wedge, \vee, 1, 0)$ is a poset $(L, \leq)$ with distinguished elements $1, 0 \in L$, and binary operations meet $\wedge$ and join $\vee$, satisfying for all $x, y, z \in L$,

$$0 \leq x \leq 1 \qquad \frac{z \leq x \qquad z \leq y}{\overline{z \leq x \wedge y}} \qquad \frac{x \leq z \qquad x \leq y}{\overline{x \vee y \leq z}}$$

A *lattice homomorphism* is a function $f : L \to K$ between lattices which preserves finite limits and colimits, i.e., $f0 = 0$, $f1 = 1$, $f(x \wedge y) = fx \wedge fy$, and $f(x \vee y) = fx \vee fy$. The category of lattices and lattice homomorphisms is denoted by Lat.

A lattice can be axiomatized equationally as a set with two distinguished elements 0 and 1 and two binary operations $\wedge$ and $\vee$, satisfying the following equations:

$$\begin{aligned} (x \wedge y) \wedge z &= x \wedge (y \wedge z)\ , & (x \vee y) \vee z &= x \vee (y \vee z)\ , \\ x \wedge y &= y \wedge x\ , & x \vee y &= y \vee x\ , \\ x \wedge x &= x\ , & x \vee x &= x\ , \\ 1 \wedge x &= x\ , & 0 \vee x &= x\ , \\ & x \wedge (y \vee x) = x = (x \wedge y) \vee x\ . \end{aligned} \tag{3.6}$$

The partial order on L is then determined by

$$x \leq y \iff x \wedge y = x\ .$$

Exercise 3.4.1. Show that in a lattice $x \leq y$ if, and only if, $x \wedge y = x$ if, and only if, $x \vee y = y$.

A lattice is *distributive* if the following distributive laws hold in it:

$$\begin{aligned} (x \vee y) \wedge z &= (x \wedge z) \vee (y \wedge z)\ , \\ (x \wedge y) \vee z &= (x \vee z) \wedge (y \vee z)\ . \end{aligned} \tag{3.7}$$

It turns out that if one distributive law holds then so does the other [Joh82, I.1.5].

A *Heyting algebra* is a cartesian closed lattice H. This means that it has an operation $\Rightarrow$, satisfying for all $x, y, z \in H$

$$\frac{z \wedge x \leq y}{\overline{z \leq x \Rightarrow y}}$$

A *Heyting algebra homomorphism* is a lattice homomorphism $f : K \to H$ between Heyting algebras that preserves implication, i.e., $f(x \Rightarrow y) = (fx \Rightarrow fy)$. The category of Heyting algebras and their homomorphisms is denoted by Heyt.

Heyting algebras can be axiomatized equationally as a set H with two distinguished elements 0 and 1 and three binary operations $\wedge$, $\vee$ and $\Rightarrow$. The equations for a Heyting algebra are the ones listed in (3.6), as well as the following ones for $\Rightarrow$.

$$\begin{aligned}(x \Rightarrow x) &= 1 \ ,\\ x \wedge (x \Rightarrow y) &= x \wedge y \ ,\\ y \wedge (x \Rightarrow y) &= y \ ,\\ (x \Rightarrow (y \wedge z)) &= (x \Rightarrow y) \wedge (x \Rightarrow z) \ .\end{aligned} \tag{3.8}$$

For a proof, see [Joh82, I.1], where one can also find a proof that every Heyting algebra is distributive (exercise!).

Example 3.4.2. We know from Lemma 3.3.5 that for any poset P, the poset $\downarrow P$ of all downsets in P, ordered by inclusion, is cartesian closed. Moreover, we know that $\downarrow P \cong \mathbb{2}^{P^{\mathsf{op}}}$, as a poset, with the reverse pointwise ordering on monotone maps $P^{\mathsf{op}} \to \mathbb{2}$, or equivalently, $\downarrow P \cong \mathbb{2}^{P}$, with the functions ordered pointwise. Since $\mathbb{2}$ is a lattice, we can also take joins $f \vee g$ pointwise, in order to get joins in $\mathbb{2}^P$, which then correspond to finite unions of the corresponding downsets $f^{-1}\{0\} \cup g^{-1}\{0\}$. Thus, in sum, for any poset P, the lattice $\downarrow P \cong \mathbb{2}^P$ is a Heyting algebra, with the downsets ordered by inclusion, and the functions ordered pointwise.

Intuitionistic propositional calculus

There is a forgetful functor $U : \mathsf{Heyt} \to \mathsf{Set}$ which maps a Heyting algebra to its underlying set, and a homomorphism of Heyting algebras to the underlying function. Because Heyting algebras are models of an equational theory, there is a left adjoint $H \dashv U$, which is the usual "free" construction mapping a set S to the free Heyting algebra HS generated by it. As for all algebraic strictures, the construction of HS can be performed in two steps: first, define a set HS of formal expressions, and then quotient it by an equivalence relation generated by the axioms for Heyting algebras.

Thus let HS be the set of formal expressions generated inductively by the following rules:

1. Generators: if $x \in S$ then $x \in HS$.

2. Constants: $\bot, \top \in HS$.

3. Connectives: if $\phi, \psi \in HS$ then $(\phi \wedge \psi), (\phi \vee \psi), (\phi \Rightarrow \psi) \in HS$.

We impose an equivalence relation on HS, which we write as equality $=$ and think of as such; it is defined as the smallest equivalence relation satisfying axioms (3.6) and (3.8). This forces HS to be a Heyting algebra. We define the action of the functor H on morphisms as usual: a function $f : S \to T$ is mapped to the Heyting algebra morphism $Hf : HS \to HT$ defined by

$$(Hf)\bot = \bot \ , \qquad (Hf)\bot = \bot \ , \qquad (Hf)x = fx \ ,$$
$$(Hf)(\phi \star \psi) = ((Hf)\phi) \star ((Hf)\psi) \ ,$$

where $\star$ stands for $\wedge$, $\vee$ or $\Rightarrow$.

The inclusion $\eta_S : S \to U(HS)$ of generators into the underlying set of the free Heyting algebra HS is then the component at S of a natural transformation $\eta : 1_{\mathsf{Set}} \Longrightarrow U \circ H$, which is of course the unit of the adjunction $H \dashv U$. To see this, consider a Heyting algebra K and an arbitrary function $f : S \to UK$. Then the Heyting algebra homomorphism $\overline{f} : HS \to K$ defined by

$$\overline{f}\bot = \bot\ , \qquad \overline{f}\bot = \bot\ , \qquad \overline{f}x = fx\ ,$$
$$\overline{f}(\phi \star \psi) = (\overline{f}\phi) \star (\overline{f}\psi)\ ,$$

where $\star$ stands for $\wedge$, $\vee$ or $\Rightarrow$, makes the following triangle commute:

$$\begin{tikzcd} S \arrow[r, "\eta_S"] \arrow[dr, "f"'] & U(HS) \arrow[d, "U\overline{f}"] \\ & K \end{tikzcd}$$

It is the unique such morphism because any two homomorphisms from HS which agree on generators must be equal. This is proved by induction on the structure of the formal expressions in HS.

We may now define the *intuitionistic propositional calculus* IPC to be the free Heyting algebra IPC on countably many generators p_0, p_1, …, called *atomic propositions* or *propositional variables*. This is a somewhat unorthodox definition from a logical point of view—normally we would start from a *calculus* consisting of a formal language, judgements, and rules of inference—but of course, by now, we realize that the two approaches are essentially equivalent.

Having said that, let us also describe IPC in the conventional way. The formulas of IPC are built inductively from propositional variables p_0, p_1, …, constants falsehood $\bot$ and truth $\top$, and binary operations conjunction $\wedge$, disjunction $\vee$ and implication $\Rightarrow$. The basic judgment of IPC is *logical entailment*

$$u_1 : A_1, \ldots, u_k : A_k \vdash B$$

which means "hypotheses A_1, …, A_k entail proposition B". The hypotheses are labeled with distinct labels u_1, …, u_k so that we can distinguish them, which is important when the same hypothesis appears more than once. Because the hypotheses are labeled it is irrelevant in what order they are listed, as long as the labels are not getting mixed up. Thus, the hypotheses $u_1 : A \vee B, u_2 : B$ are the same as the hypotheses $u_2 : B, u_1 : A \vee B$, but different from the hypotheses $u_1 : B, u_2 : A \vee B$. Sometimes we do not bother to label the hypotheses.

The left-hand side of a logical entailment is called the *context* and the right-hand side is the *conclusion*. Thus logical entailment is a relation between contexts and conclusions.

The context may be empty. If Γ is a context, u is a label which does not occur in Γ, and A is a formula, then we write $\Gamma, u : A$ for the context Γ extended by the hypothesis $u : A$. Logical entailment is the smallest relation satisfying the following rules:

1. Conclusion from a hypothesis:
$$\frac{}{\Gamma \vdash A} \text{ if } u : A \text{ occurs in } \Gamma$$

2. Truth:
$$\frac{}{\Gamma \vdash \top}$$

3. Falsehood:
$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A}$$

4. Conjunction:
$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \land B} \qquad \frac{\Gamma \vdash A \land B}{\Gamma \vdash A} \qquad \frac{\Gamma \vdash A \land B}{\Gamma \vdash B}$$

5. Disjunction:
$$\frac{\Gamma \vdash A}{\Gamma \vdash A \lor B} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \lor B} \qquad \frac{\Gamma \vdash A \lor B \qquad \Gamma, u : A \vdash C \qquad \Gamma, v : B \vdash C}{\Gamma \vdash C}$$

6. Implication:
$$\frac{\Gamma, u : A \vdash B}{\Gamma \vdash A \Rightarrow B} \qquad \frac{\Gamma \vdash A \Rightarrow B \qquad \Gamma \vdash A}{\Gamma \vdash B}$$

A *proof* of $\Gamma \vdash A$ is a *finite* tree built from the above inference rules whose root is $\Gamma \vdash A$. For example, here is a proof of $A \lor B \vdash B \lor A$: We did not bother to label the hypotheses. A judgment $\Gamma \vdash A$ is *provable* if there exists a proof of it. Observe that every proof has at its leaves either the rule for $\top$ or a conclusion from a hypothesis.

You may wonder what happened to negation. In intuitionistic propositional calculus, negation is defined in terms of implication and falsehood as

$$\neg A \equiv A \Rightarrow \bot \, .$$

Properties of negation are then derived from the rules for implication and falsehood, see Exercise 3.4.6

Let P be the set of all formulas of IPC, preordered by the relation

$$A \vdash B \, , \qquad (A, B \in P)$$

where we did not bother to label the hypothesis A. Clearly, it is the case that $A \vdash A$. To see that $\vdash$ is transitive, suppose Π_1 is a proof of $A \vdash B$ and Π_2 is a proof of $B \vdash C$. Then we can obtain a proof of $A \vdash C$ from a proof Π_2 of $B \vdash C$ by replacing in it each use of the hypothesis B by the proof Π_1 of $A \vdash B$. This is worked out in detail in the next two exercises.

Exercise 3.4.3. Prove the following statement by induction on the structure of the proof Π: if Π is a proof of $\Gamma, u : A, v : A \vdash B$ then there is a proof of $\Gamma, u : A \vdash B$.

Exercise 3.4.4. Prove the following statement by induction on the structure of the proof Π_2: if Π_1 is a proof of $\Gamma \vdash A$ and Π_2 is a proof of $\Gamma, u : A \vdash B$, then there is a proof of $\Gamma \vdash B$.

Let IPC be the poset reflection of the preorder $(P, \vdash)$. The elements of IPC are equivalence classes $[A]$ of formulas, where two formulas A and B are equivalent if both $A \vdash B$ and $B \vdash A$ are provable. The poset IPC is just the free Heyting algebra on countably many generators p_0, p_1, ...

Classical propositional calculus

Another look:

An element $x \in L$ of a lattice L is said to be *complemented* when there exists $y \in L$ such that

$$x \vee y = 1 , \qquad\qquad x \wedge y = 0 .$$

We say that y is the *complement* of x.

In a distributive lattice, the complement of x is unique if it exists. Indeed, if both y and z are complements of x then

$$y \wedge z = (y \wedge z) \vee 0 = (y \wedge z) \vee (y \wedge x) = y \wedge (z \vee x) = y \wedge 1 = y ,$$

hence $y \leq z$. A symmetric argument shows that $z \leq y$, therefore $y = z$. The complement of x, if it exists, is denoted by $\neg x$.

A *Boolean algebra* is a distributive lattice in which every element is complemented. In other words, a Boolean algebra B has the *complementation* operation $\neg$ which satisfies, for all $x \in B$,

$$x \wedge \neg x = 0 , \qquad\qquad x \vee \neg x = 1 . \tag{3.9}$$

The full subcategory of Lat consisting of Boolean algebras is denoted by Bool.

Exercise 3.4.5. Prove that every Boolean algebra is a Heyting algebra. Hint: how is implication encoded in terms of negation and disjunction in classical logic?

In a Heyting algebra not every element is complemented. However, we can still define a *pseudo complement* or *negation* operation $\neg$ by

$$\neg x = (x \Rightarrow 0) ,$$

Then $\neg x$ is the largest element for which $x \wedge \neg x = 0$. While in a Boolean algebra $\neg\neg x = x$, in a Heyting algebra we only have $\neg\neg x \leq x$ in general. An element x of a Heyting algebra for which $\neg\neg x = x$ is called a *regular* element.

Exercise 3.4.6. Derive the following properties of negation in a *Heyting* algebra:

$$
\begin{aligned}
x &\le \neg\neg x \;, \\
\neg x &= \neg\neg\neg x \;, \\
x \le y &\Rightarrow \neg y \le \neg x \;, \\
\neg\neg(x \wedge y) &= \neg\neg x \wedge \neg\neg y \;.
\end{aligned}
$$

Exercise 3.4.7. Prove that the topology $\mathcal{O}X$ of any topological space X is a Heyting algebra. Describe in topological language the implication $U \Rightarrow V$, the negation $\neg U$, and the regular elements $U = \neg\neg U$ in $\mathcal{O}X$.

Exercise 3.4.8. Show that for a Heyting algebra H, the regular elements of H form a Boolean algebra $H_{\neg\neg} = \{x \in H \mid x = \neg\neg x\}$. Here $H_{\neg\neg}$ is viewed as a subposet of H. Hint: negation $\neg'$, conjunction $\wedge'$, and disjunction $\vee'$ in $H_{\neg\neg}$ are expressed as follows in terms of negation, conjunction and disjunction in H, for $x, y \in H_{\neg\neg}$:

$$
\neg' x = \neg x \;, \qquad x \wedge' y = \neg\neg(x \wedge y) \;, \qquad x \vee' y = \neg\neg(x \vee y) \;.
$$

The *classical propositional calculus (CPC)* is obtained from the intuitionistic propositional calculus by the addition of the logical rule known as *tertium non datur*, or the *law of excluded middle*:

$$
\frac{}{\Gamma \vdash A \vee \neg A}
$$

Alternatively, we could add the law known as *reductio ad absurdum*, or *proof by contradiction*:

$$
\frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A} \;.
$$

Identifying logically equivalent formulas of CPC, we obtain a poset CPC ordered by logical entailment. This poset is the *free Boolean algebra* on countably many generators. The construction of a free Boolean algebra can be performed just like described for the free Heyting algebra above. The equational axioms for a Boolean algebra are the axioms for a lattice (3.6), the distributive laws (3.7), and the complement laws (3.9).

Exercise* 3.4.9. Is CPC isomorphic to the Boolean algebra $\mathsf{IPC}_{\neg\neg}$ of the regular elements of IPC?

Exercise 3.4.10. Show that in a Heyting algebra H, one has $\neg\neg x = x$ for all $x \in H$ if, and only if, $y \vee \neg y = 1$ for all $y \in H$. Hint: half of the equivalence is easy. For the other half, observe that the assumption $\neg\neg x = x$ means that negation is an order-reversing bijection $H \to H$. It therefore transforms joins into meets and vice versa, and so the *De Morgan laws* hold:

$$
\neg(x \wedge y) = \neg x \vee \neg y \;, \qquad \neg(x \vee y) = \neg x \wedge \neg y \;.
$$

Together with $y \wedge \neg y = 0$, the De Morgan laws easily imply $y \vee \neg y = 1$. See [Joh82, I.1.11].

Kripke semantics for IPC

We now prove the Kripke completeness of IPC, extending Theorem 3.3.4. Let us first see that we cannot simply reuse the proof from that theorem, because the downset (Yoneda) embedding that we used there

$$\downarrow\, : \mathsf{IPC} \hookrightarrow\, \downarrow\!(\mathsf{IPC}) \tag{3.10}$$

would not preserve the coproducts $\perp$ and $\phi \vee \psi$. Indeed, $\downarrow(\perp) \neq \emptyset$, because it contains $\perp$ itself! And in general $\downarrow(\phi \vee \psi) \neq \downarrow(\phi) \cup \downarrow(\psi)$, because the righthand side need not contain, e.g., $\phi \vee \psi$.

Instead, we will generalize the Stone Representation theorem **??** from Boolean algebras to Heyting algebras, using a theorem due to Joyal (cf. [MR95, MH92]). First, recall that the Stone representation provided, for any Boolean algebra $\mathcal{B}$, an injective Boolean homomorphism into a powerset,

$$\mathcal{B} \rightarrowtail \mathcal{P}X\,.$$

For X we took the set of prime filters $\mathsf{Bool}(\mathcal{B}, 2)$, and the map $h : \mathcal{B} \rightarrowtail \mathcal{P}\mathsf{Bool}(\mathcal{B}, 2)$ was given by $h(b) = \{F \mid b \in F\}$. Transposing $\mathcal{P}\mathsf{Bool}(\mathcal{B}, 2) \cong 2^{\mathsf{Bool}(\mathcal{B},2)}$ in the cartesian closed category Pos, we arrive at the (monotone) evaluation map

$$\mathsf{eval} : \mathsf{Bool}(\mathcal{B}, 2) \times \mathcal{B} \to 2. \tag{3.11}$$

Now recall that the category of Boolean algebras is full in the category DLat of distributive lattices,

$$\mathsf{Bool}(\mathcal{B}, 2) = \mathsf{DLat}(\mathcal{B}, 2)\,.$$

For any Heyting algebra $\mathcal{H}$ (or indeed any distributive lattice), the Homset $\mathsf{DLat}(\mathcal{H}, 2)$, ordered pointwise, is isomorphic to the *poset* of all prime filters in $\mathcal{H}$ ordered by inclusion, by taking $f : \mathcal{H} \to 2$ to its (filter) kernel $f^{-1}\{1\} \subseteq \mathcal{H}$. In particular, the poset $\mathsf{DLat}(\mathcal{H}, 2)$ is no longer discrete when $\mathcal{H}$ is not Boolean, since a prime ideal in a Heyting algebra need not be maximal.

The transpose of the (monotone) evaluation map,

$$\mathsf{eval} : \mathsf{DLat}(\mathcal{H}, 2) \times \mathcal{H} \to 2. \tag{3.12}$$

will then be the (monotone) map

$$\epsilon\, : \mathcal{H} \longrightarrow 2^{\mathsf{DLat}(\mathcal{H},2)}, \tag{3.13}$$

which takes $p \in \mathcal{H}$ to the "evaluation at p" map $f \mapsto f(p) \in 2$, i.e.,

$$\epsilon_p(f) = f(p) \qquad \text{for } p \in \mathcal{H} \text{ and } f : \mathcal{H} \to 2\,.$$

As before, the poset $2^{\mathsf{DLat}(\mathcal{H},2)}$ (ordered pointwise) may be identified with the upsets in the poset $\mathsf{DLat}(\mathcal{H}, 2)$, ordered by inclusion, which recall from Example 3.4.2 is always a Heyting algebra. Thus, in sum, we have a monotone map,

$$\mathcal{H} \longrightarrow \uparrow\!\mathsf{DLat}(\mathcal{H}, 2)\,, \tag{3.14}$$

which generalizes the Stone representation from Boolean to Heyting algebras.

Theorem 3.4.11 (Joyal)**.** *Let $\mathcal{H}$ be a Heyting algebra. There is an injective Heyting homomorphism*

$$\mathcal{H} \rightarrowtail \uparrow J$$

into a Heyting algebra of upsets in a poset J.

For the proof, we again make use of the Prime Ideal Theorem (see Lemma **??**), now in the following form due to Birkhoff.

Lemma 3.4.12 (Birkhoff's Prime Ideal Theorem)**.** *Let D be a distributive lattice, $I \subseteq D$ an ideal, and $x \in D$ with $x \notin I$. There is a prime ideal $I \subseteq P \subset D$ with $x \notin P$.*

Proof. As in the proof of Lemma **??**, it suffice to prove it for the case $I = (0)$. This time, we use Zorn's Lemma: a poset in which every chain has an upper bound has maximal elements. Consider the poset $\mathcal{I}\backslash x$ of "ideals I without x", $x \notin I$, ordered by inclusion. The union of any chain $I_0 \subseteq I_1 \subseteq \ldots$ in $\mathcal{I}\backslash x$ is clearly also in $\mathcal{I}\backslash x$, so we have (at least one) maximal element $M \in \mathcal{I}\backslash x$. We claim that $M \subseteq D$ is prime. To that end, take $a, b \in D$ with $a \wedge b \in M$. If $a, b \notin M$, let $M_a = \{n \leq m \vee a \mid m \in M\}$, the ideal join of M and $\downarrow(a)$, and similarly for M_b. Since M is maximal without x, we therefore have $x \in M_a$ and $x \in M_b$. Thus let $x \leq m \vee a$ and $x \leq m' \vee b$ for some $m, m' \in M$. Then $x \vee m' \leq m \vee m' \vee a$ and $x \vee m \leq m \vee m' \vee b$, so taking meets on both sides gives

$$(x \vee m') \wedge (x \vee m) \leq (m \vee m' \vee a) \wedge (m \vee m' \vee b) = (m \vee m') \vee (a \wedge b).$$

Since the righthand side is in the ideal M, so is the left. But then $x \leq x \vee (m \wedge m')$ is also in M, contrary to our assumption that $M \in \mathcal{I}\backslash x$. □

Proof of Theorem 3.4.11. As in (3.14), let $J = \mathsf{DLat}(\mathcal{H}, \mathbb{2})$ be the poset of prime filters in $\mathcal{H}$, and consider the "evaluation" map (3.14),

$$\epsilon : \mathcal{H} \longrightarrow \mathbb{2}^{\mathsf{DLat}(\mathcal{H},\mathbb{2})} \cong \uparrow \mathsf{DLat}(\mathcal{H}, \mathbb{2})$$

given by $\epsilon(p) = \{F \,|\, p \in F \text{ prime}\}$.

Clearly $\epsilon(0) = \emptyset$ and $\epsilon(1) = \mathsf{DLat}(\mathcal{H}, \mathbb{2})$, and similarly for the other meets and joins, so ϵ is a lattice homomorphism. Moreover, if $p \neq q \in \mathcal{H}$ then, as in the proof of **??**, we have that $\epsilon(p) \neq \epsilon(q)$, by the Prime Ideal Theorem (Lemma 3.4.12). Thus it just remains to show that

$$\epsilon(p \Rightarrow q) \;=\; \epsilon(p) \Rightarrow \epsilon(q)\,.$$

Unwinding the definitions, it suffices to show that, for all $f \in \mathsf{DLat}(\mathcal{H}, \mathbb{2})$,

$$f(p \Rightarrow q) = 1 \quad \text{iff} \quad \text{for all } g \geq f,\ g(p) = 1 \text{ implies } g(q) = 1. \tag{3.15}$$

Equivalently, for all prime filters $F \subseteq \mathcal{H}$,

$$p \Rightarrow q \in F \quad \text{iff} \quad \text{for all prime } G \supseteq F,\ p \in G \text{ implies } q \in G. \tag{3.16}$$

Now if $p \Rightarrow q \in F$, then for all (prime) filters $G \supseteq F$, also $p \Rightarrow q \in G$, and so $p \in G$ implies $q \in G$, since $(p \Rightarrow q) \wedge p \leq q$.

Conversely, suppose $p \Rightarrow q \notin F$, and we seek prime $G \supseteq F$ with $p \in G$ but $q \notin G$. Consider the filter

$$F[p] = \{x \wedge p \leq h \in \mathcal{H} \mid x \in F\} ,$$

which is the join of F and $\uparrow(p)$ in the poset of filters. If $q \in F[p]$, then $x \wedge p \leq q$ for some $x \in F$, whence $x \leq p \Rightarrow q$, and so $p \Rightarrow q \in F$, contrary to assumption. Thus $q \notin F[p]$. By the Prime Ideal Theorem, again, there is a prime filter $G \supseteq F[p]$ with $q \notin G$. □

Frames and spaces

A poset $(P, \leq)$, viewed as a category, is *cocomplete* when it has suprema (least upper bounds) of arbitrary subsets. This is so because coequalizers in a poset always exist, and coproducts are precisely least upper bounds. Recall that the supremum of $S \subseteq P$ is an element $\bigvee S \in P$ such that, for all $y \in S$,

$$\bigvee S \leq y \iff \forall\, x : S \,.\, x \leq y \,.$$

In particular, $\bigvee \emptyset$ is the least element of P and $\bigvee P$ is the greatest element of P. Similarly, a poset is *complete* when it has infima (greatest lower bounds) of arbitrary subsets; the infimum of $S \subseteq P$ is an element $\bigwedge S \in P$ such that, for all $y \in S$,

$$y \leq \bigwedge S \iff \forall\, x : S \,.\, y \leq x \,.$$

Proposition 3.4.13. *A poset is complete if, and only if, it is cocomplete.*

Proof. Infima and suprema are expressed in terms of each other as follows:

$$\begin{aligned} \bigwedge S &= \bigvee \left\{ y \in P \mid \forall\, x : S \,.\, y \leq x \right\} , \\ \bigvee S &= \bigwedge \left\{ y \in P \mid \forall\, x : S \,.\, x \leq y \right\} . \end{aligned}$$

□

Thus, we usually speak of *complete* posets only, even when we work with arbitrary suprema.

Suppose P is a complete poset. When is it cartesian closed? Being a complete poset, it has the terminal object, namely the greatest element $1 \in P$, and it has binary products which are binary infima. If P is cartesian closed then for all $x, y \in P$ there exists an exponential $(x \Rightarrow y) \in P$, which satisfies, for all $z \in P$,

$$\frac{z \wedge x \leq y}{z \leq x \Rightarrow y} \,.$$

With the help of this adjunction we derive the *infinite distributive law*, for an arbitrary family $\left\{ y_i \in P \mid i \in I \right\}$,

$$x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I} (x \wedge y_i) \tag{3.17}$$

as follows:

$$\cfrac{\cfrac{\cfrac{\cfrac{x \wedge \bigvee_{i\in I} y_i \leq z}{\bigvee_{i\in I} y_i \leq (x \Rightarrow z)}}{\forall\, i : I\,.\,(y_i \leq (x \Rightarrow z))}}{\forall\, i : I\,.\,(x \wedge y_i \leq z)}}{\bigvee_{i\in I}(x \wedge y_i) \leq z}$$

Now since $x \wedge \bigvee_{i\in I} y_i$ and $\bigvee_{i\in I}(x \wedge y_i)$ have the same upper bounds they must be equal.

Conversely, suppose the distributive law (3.17) holds. Then we can *define* $x \Rightarrow y$ to be

$$(x \Rightarrow y) = \bigvee \left\{ z \in P \;\middle|\; x \wedge z \leq y \right\} \;. \tag{3.18}$$

The best way to show that $x \Rightarrow y$ is the exponential of x and y is to use the characterization of adjoints by counit, as in Proposition **??**. In the case of $\wedge$ and $\Rightarrow$ this amounts to showing that, for all $x, y \in P$,

$$x \wedge (x \Rightarrow y) \leq y \;, \tag{3.19}$$

and that, for $z \in P$,

$$(x \wedge z \leq y) \Rightarrow (z \leq x \Rightarrow y) \;.$$

This implication follows directly from (3.4.18), and (3.19) follows from the distributive law:

$$x \wedge (x \Rightarrow y) = x \wedge \bigvee \left\{ z \in P \;\middle|\; x \wedge z \leq y \right\} = \bigvee \left\{ x \wedge z \;\middle|\; x \wedge z \leq y \right\} \leq y \;.$$

Complete cartesian closed posets are called *frames*.

Definition 3.4.14. A *frame* is a poset that is complete and cartesian closed, thus a frame is a complete Heyting algebra. Equivalently, a frame is a complete poset satisfying the (infinite) distributive law

$$x \wedge \bigvee_{i\in I} y_i = \bigvee_{i\in I}(x \wedge y_i) \;.$$

A *frame morphism* is a function $f : L \to M$ between frames that preserves finite infima and arbitrary suprema. The category of frames and frame morphisms is denoted by Frame.

Warning: a frame morphism need not preserve exponentials!

Example 3.4.15. The topology $\mathcal{O}X$ of a topological space X, ordered by inclusion, is a frame because finite intersections and arbitrary unions of open sets are open. The distributive law holds because intersections distribute over unions. If $f : X \to Y$ is a continuous map between topological spaces, the inverse image map $f^* : \mathcal{O}Y \to \mathcal{O}X$ is a frame homomorphism. Thus, there is a functor

$$\mathcal{O} : \mathsf{Top} \to \mathsf{Frame}^{\mathsf{op}}$$

which maps a space X to its topology $\mathcal{O}X$ and a continuous map $f : X \to Y$ to the inverse image map $f^* : \mathcal{O}Y \to \mathcal{O}X$.

The category $\mathsf{Frame}^{\mathsf{op}}$ is called the category of *locales* and is denoted by Loc. When we think of a frame as an object of Loc we call it a locale.

Example 3.4.16. Let P be a poset and define a topology on the elements of P by defining the opens to be the upsets,

$$\mathcal{O}P = {\uparrow} P \cong \mathsf{Pos}(P, \mathbb{2}).$$

These open sets are not only closed under arbitrary unions and finite intersections, but also under *arbitrary* intersections. Such a topological space is said to be an *Alexandrov space.*

Exercise* 3.4.17. This exercise is meant for students with some background in topology. For a topological space X and a point $x \in X$, let $N(x)$ be the neighborhood filter of x,

$$N(x) = \left\{ U \in \mathcal{O}X \;\middle|\; x \in U \right\} \ .$$

Recall that a T_0-*space* is a topological space X in which points are determined by their neighborhood filters,

$$N(x) = N(y) \Rightarrow x = y \ . \qquad (x, y \in X)$$

Let Top_0 be the full subcategory of Top on T_0-spaces. The functor $\mathcal{O} : \mathsf{Top} \to \mathsf{Loc}$ restricts to a functor $\mathcal{O} : \mathsf{Top}_0 \to \mathsf{Loc}$. Prove that $\mathcal{O} : \mathsf{Top}_0 \to \mathsf{Loc}$ is a faithful functor. Is it full?

Topological semantics for IPC

It should now be clear how to interpret IPC into a topological space X: each formula ϕ is assigned to an open set $[\![\phi]\!] \in \mathcal{O}X$ in such a way that $[\![-]\!]$ is a homomorphism of Heyting algebras.

Definition 3.4.18. A *topological model* of IPC is a space X and an interpretation of formulas,

$$[\![-]\!] : \mathsf{IPC} \to \mathcal{O}X \, ,$$

satisfying the conditions:

$$\begin{aligned} [\![\top]\!] &= X \\ [\![\bot]\!] &= \emptyset \\ [\![\phi \wedge \psi]\!] &= [\![\phi]\!] \cap [\![\psi]\!] \\ [\![\phi \vee \psi]\!] &= [\![\phi]\!] \cup [\![\psi]\!] \\ [\![\phi \Rightarrow \psi]\!] &= [\![\phi]\!] \Rightarrow [\![\psi]\!] \end{aligned}$$

The Heyting implication $[\![\phi]\!] \Rightarrow [\![\psi]\!]$ in $\mathcal{O}X$, is defined in (3.4.18) as

$$[\![\phi]\!] \Rightarrow [\![\psi]\!] \;=\; \bigcup \left\{ U \in \mathcal{O}X \;\middle|\; U \wedge [\![\phi]\!] \leq [\![\psi]\!] \right\} \ .$$

Joyal's representation theorem 3.4.11 easily implies that IPC is sound and complete with respect to topological semantics.

Corollary 3.4.19. *A formula ϕ is provable in* IPC *if, and only if, it holds in every topological interpretation $[\![-]\!]$ into a space X, briefly:*

$$\mathsf{IPC} \vdash \phi \qquad \textit{iff} \qquad [\![\phi]\!] = X \textit{ for all spaces } X\,.$$

Proof. Put the Alexandrov topology on the upsets of prime ideals in the Heyting algebra IPC $\square$

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