Overview

I. Propositional Logic

1) CPC & Boolean Algebras
2) IPC & Heyting Algebras
3) Embedding & Completeness Thm.s

II. Simple Type Theory

1) λ-calculus & CCCs
2) Kripke Semantics
3) Embedding & Completness Thm.s

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"Proof Irrelevant"
Posets

Categories
"Proof Relevant"

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"Proof Irrelevant"

$$
\begin{tikzcd}
\text{Posets} \arrow[d, squiggly, shift right=2, "\text{"categorification"}"'] \\
\text{Categories} \arrow[u, squiggly, shift right=2, "\text{truncation}"']
\end{tikzcd}
$$

"Proof Relevant"

II. Simple Type Theory

1) λ-calculus & CCCs
2) Kripke Semantics
3) Embedding & Completness Thm.s

I. Propositional Logic

1) CPC & Boolean Algebras

- Deductive system of Classical Propositional Calculus

$$\phi \vdash \psi$$

- Equational reasoning in Boolean Algebra

$$\vdash_{eq} \varphi = \psi$$

- Semantic entailment

$$\phi \models \psi$$

based on truth values.

Goal: Show that TFAEq:

$$\vdash \varphi \qquad \vdash_{eq} \varphi = T \qquad \models \varphi$$

Review of Propositional Logic : Syntax

- Formulas :

$$\varphi ::= p_1, p_2, \ldots$$
$$\top, \bot$$
$$\neg\varphi$$
$$\varphi \wedge \psi, \varphi \vee \psi$$
$$\varphi \rightarrow \psi, \varphi \leftrightarrow \psi$$

$$\begin{aligned} \text{Form} &= \text{Set of all fmlas} \\ \cup\!\mid \quad & \\ \text{Form}(n) &= \text{fmlas in } \{p_1, \ldots, p_n\} \end{aligned}$$

- Fact :

Form is generated inductively by the operations $\top, \neg, \wedge$, etc. from the set $\underset{\{p_1, \ldots, p_n, \ldots\}}{\underset{\shortparallel}{Var}} \subseteq$ Form.

Review of Propositional Logic: Semantics

Def. A *valuation* is a function

$$[\![\cdot]\!]^v : \mathrm{Form} \longrightarrow 2 = \{0,1\}$$

determined by recursion from an arbitrary function

$$v : \mathrm{Var} \longrightarrow 2$$

where

$$[\![p]\!]^v = v(p)$$

$$[\![\top]\!] = 1$$

$$[\![\bot]\!] = 0$$

$$[\![\neg\varphi]\!] = 1 - [\![\varphi]\!]$$

$$[\![\varphi \wedge \psi]\!] = \min([\![\varphi]\!], [\![\psi]\!])$$

$$[\![\varphi \vee \psi]\!] = \max([\![\varphi]\!], [\![\psi]\!])$$

$$[\![\varphi \to \psi]\!] = 1 \text{ iff } [\![\varphi]\!] \leq [\![\psi]\!]$$

$$[\![\varphi \leftrightarrow \psi]\!] = 1 \text{ iff } [\![\varphi]\!] = [\![\psi]\!]$$

Review of Propositional Logic: Semantics

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$0 \leq 1$

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Review of Propositional Logic: Semantics

Def. φ fmla, $\Phi \subseteq$ Form,

- $\models \varphi$ valid $:= \quad [\![\varphi]\!]^v = 1$, f. every v.

- $\Phi \models \psi$ entails $:=$ f. every v:
  if $[\![\varphi]\!]^v = 1$ f. all $\varphi \in \Phi$,
  then $[\![\psi]\!]^v = 1$,

- a truth table computes the value $[\![\varphi]\!]^v$ f. each v

p	q	r	p	$\vee$	$\neg$	q	$\to$	r
1	1	1	1	1	0	1	1	1
1	1	0	1	1	0	1	0	0
1	0	1	1	1	1	0	1	1
1	0	0	1	1	1	0	0	0
0	1	1	0	0	0	1	0	1
0	0	0	0	0	0	1	1	0
0	1	1	0	1	1	0	1	1
0	0	0	0	1	1	0	0	0

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v's

p	q	r	p	$\vee$	$\neg$	q	$\rightarrow$	r
1	1	1	1	1	0	1	1	1
1	1	0	1	1	0	1	0	0
1	0	1	1	1	1	0	1	1
1	0	0	1	1	1	0	0	0
0	1	1	0	0	0	1	0	1
0	0	0	0	0	0	1	1	0
0	1	1	0	1	1	0	1	1
0	0	0	0	1	1	0	0	0

φ

Review of Propositional Logic: Semantics

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v's

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1	1	0	1	1	0	1	0	0
1	0	1	1	1	1	0	1	1
1	0	0	1	1	1	0	0	0
0	1	1	0	0	0	1	0	1
0	0	0	0	0	0	1	1	0
0	1	1	0	1	1	0	1	1
0	0	0	0	1	1	0	0	0

So $\not\models p \vee \neg q \to r$!

Review of Propositional Logic: Deduction

Deduction We build derivations to define $\Phi \vdash \psi$ deducibility for $\Phi \subseteq \mathrm{Form}$, $\psi \in \mathrm{Form}$.

For example this derivation

$$\cfrac{\cfrac{\varphi \wedge \psi}{\varphi} \qquad \varphi \to \vartheta}{\vartheta}$$

shows that

$$\varphi \wedge \psi, \varphi \to \psi \vdash \vartheta .$$

It also shows that

$$\Phi, \varphi \wedge \psi, \varphi \to \psi \vdash \vartheta$$

f. any $\Phi \subseteq \mathrm{Form}$!

Review of Propositional Logic: Deduction

Deduction We build derivations to define $\Phi \vdash \psi$ deducibility for $\Phi \subseteq Form$, $\psi \in Form$.

For example this derivation

$$\dfrac{\dfrac{\varphi \wedge \psi}{\varphi} \qquad \varphi \to \vartheta}{\vartheta}$$

shows that

$$\{\varphi \wedge \psi, \varphi \to \psi\} \vdash \vartheta .$$

It also shows that

$$\Phi \cup \{\varphi \wedge \psi, \varphi \to \psi\} \vdash \vartheta$$

f. any $\Phi \subseteq Form$!

Review of Propositional Logic: Deduction

The rules for derivations are easier to write in sequent form:

$$\dfrac{\begin{array}{c}\vdots \\ \varphi\end{array} \quad \begin{array}{c}\vdots \\ \psi\end{array}}{\varphi \wedge \psi} \qquad \dfrac{\begin{array}{c}\vdots \\ \varphi \wedge \psi\end{array}}{\varphi} \qquad \dfrac{\begin{array}{c}\vdots \\ \varphi \wedge \psi\end{array}}{\psi}$$

$$\dfrac{\Phi \vdash \varphi, \ \Phi \vdash \psi}{\Phi \vdash \varphi \wedge \psi} \qquad \dfrac{\Phi \vdash \varphi \wedge \psi}{\Phi \vdash \varphi} \qquad \dfrac{\Phi \vdash \varphi \wedge \psi}{\Phi \vdash \psi}$$

Some rules cancel assumptions:

$$\dfrac{\begin{array}{c}\Phi, \ \text{---} \ \varphi \\ \vdots \\ \psi\end{array}}{\begin{array}{c}\Phi, \ \text{---} \ [\varphi] \\ \vdots \\ \dfrac{\psi}{\varphi \Rightarrow \psi}\end{array}}$$

Review of Propositional Logic: Deduction

The rules for derivations are easier to write in sequent form:

$$\dfrac{\begin{matrix}\vdots & & \vdots\\ \varphi & & \psi\end{matrix}}{\varphi \wedge \psi} \qquad \dfrac{\begin{matrix}\vdots\\ \varphi \wedge \psi\end{matrix}}{\varphi} \qquad \dfrac{\begin{matrix}\vdots\\ \varphi \wedge \psi\end{matrix}}{\psi}$$

$$\rightsquigarrow$$

$$\dfrac{\Phi \vdash \varphi \,,\; \Phi \vdash \psi}{\Phi \vdash \varphi \wedge \psi} \qquad \dfrac{\Phi \vdash \varphi \wedge \psi}{\Phi \vdash \varphi} \qquad \dfrac{\Phi \vdash \varphi \wedge \psi}{\Phi \vdash \psi}$$

Some rules cancel assumptions:

$$\dfrac{\begin{matrix}\Phi, \; --- \; \varphi\\ \ddots \quad \cdot^{\cdot^{\cdot}}\\ \psi\end{matrix}}{\dfrac{\begin{matrix}\Phi, \; --- \; [\varphi]\\ \ddots \quad \cdot^{\cdot^{\cdot}}\\ \psi\end{matrix}}{\varphi \Rightarrow \psi}} \quad \rightsquigarrow \quad \dfrac{\Phi, \varphi \vdash \psi}{\Phi \vdash \varphi \Rightarrow \psi}$$

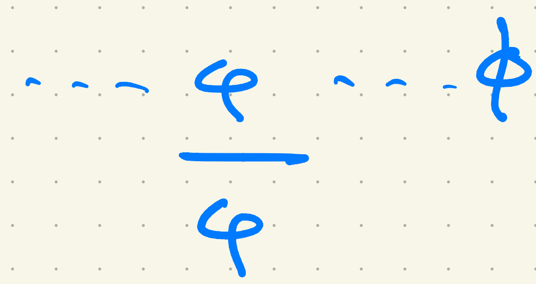
Rules of Deduction

- $\Phi \vdash \varphi$ if $\varphi \in \Phi$

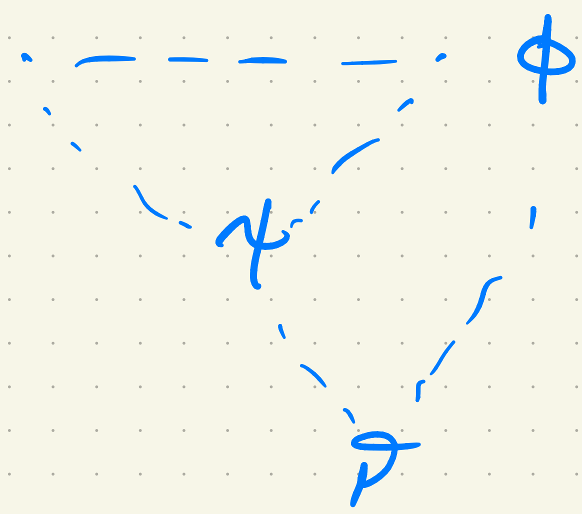
- $$\frac{\Phi \vdash \psi \qquad \Phi, \psi \vdash \vartheta}{\Phi \vdash \vartheta}$$

Rules of Deduction

- $\Phi \vdash \varphi$ if $\varphi \in \Phi$

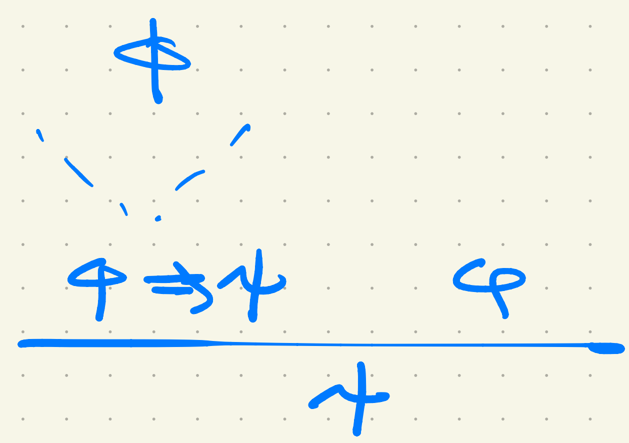


- $$\frac{\Phi \vdash \psi \qquad \Phi, \psi \vdash \vartheta}{\Phi \vdash \vartheta}$$

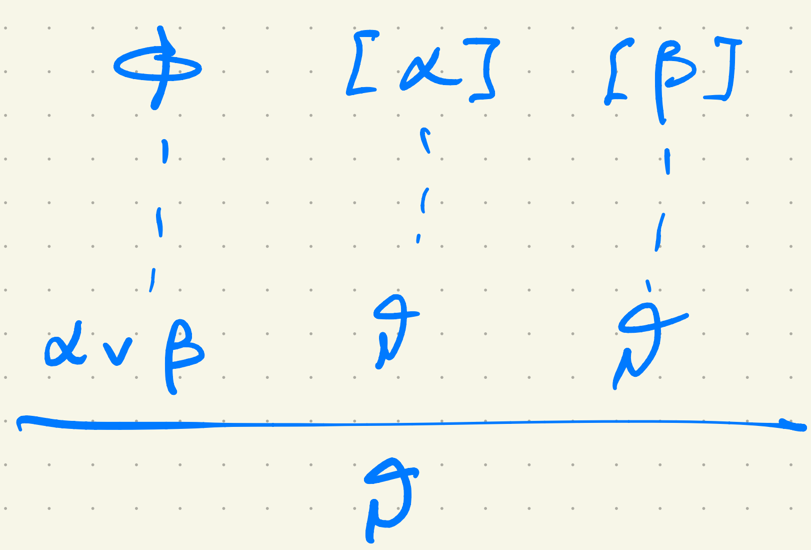


- $$\frac{\Phi \vdash \varphi \qquad \Phi \vdash \psi}{\Phi \vdash \varphi \wedge \psi}$$ (double line)

- $$\frac{\Phi, \varphi \vdash \psi}{\Phi \vdash \varphi \Rightarrow \psi}$$ (double line)



- $$\frac{\begin{array}{c}\Phi, \alpha \vdash \vartheta \\ \Phi, \beta \vdash \vartheta\end{array}}{\Phi, \alpha \vee \beta \vdash \vartheta}$$



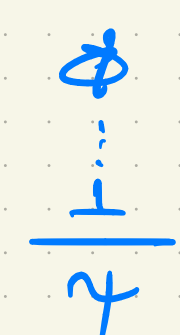
- $\Phi, \alpha \vdash \alpha \vee \beta$

  $\Phi, \beta \vdash \alpha \vee \beta$

$$\frac{\Phi \quad \alpha}{\alpha \vee \beta} \qquad \frac{\Phi \quad \beta}{\alpha \vee \beta}$$

- $\Phi \vdash \bot$

- $$\frac{\Phi \vdash \bot}{\Phi \vdash \psi}$$



Rules of Deduction

- Define negation $\neg\varphi := \varphi \Rightarrow \bot$ for deduction, so that

(1)
$$\frac{\varphi \qquad \neg\varphi}{\bot} \quad (\varphi \Rightarrow \bot)$$

(2)
$$\begin{array}{c} \varphi \\ \vdots \\ \hline \bot \end{array}$$

Rules of Deduction

- Define <u>negation</u> $\neg\varphi := \varphi \Rightarrow \bot$ for deduction, so that

(1)
$$\frac{\varphi \qquad \neg\varphi \quad (\varphi \Rightarrow \bot)}{\bot}$$

(2)
$$\frac{\begin{array}{c}[\varphi] \\ \vdots \\ \hline \bot\end{array}}{\neg\varphi \quad (\varphi \Rightarrow \bot)}$$

- For <u>classical logic</u>

$$\frac{\neg\neg\varphi}{\varphi}$$ Double Negation Elim

- In <u>sequent</u> style:

$$\varphi, \neg\varphi \vdash \bot \quad , \quad \frac{\Phi, \varphi \vdash \bot}{\Phi \vdash \neg\varphi}$$

$$\Phi, \neg\neg\varphi \vdash \varphi$$

Poset of Formulas

Now order the set Form by the relation of deducibility

$$\varphi \vdash \psi \quad .$$

Indeed we have

$$\varphi \vdash \varphi$$

$$\frac{\varphi \vdash \psi \ , \ \psi \vdash \vartheta}{\varphi \vdash \vartheta}$$

Poset of Formulas

Now order the set Form by the relation of deducibility

$$\varphi \vdash \psi .$$

Indeed we have

$$\varphi \vdash \varphi \qquad \begin{array}{c} \varphi \\ \vdots \\ \varphi \end{array}$$

$$\frac{\varphi \vdash \psi \,,\; \psi \vdash \vartheta}{\varphi \vdash \vartheta} \qquad \begin{array}{c} \varphi \\ \vdots \\ \psi \\ \vdots \\ \vartheta \end{array}$$

Later we'll take the category of proofs, but for now we just take the poset

$$\mathrm{Form} / \dashv\vdash \,.$$

So identify

$$\varphi = \psi \quad \text{iff} \quad \varphi \dashv\vdash \psi \,.$$

NB: we don't always bother to write $[\varphi] = [\psi]$, etc.

Poset of Formulas

<u>Def.</u> $\mathrm{Syn}^{CPC} = \mathrm{Form}/\dashv\vdash$

is the syntactic poset of formulas in CPC, and

$$\mathrm{Syn}^{CPC}_n = \mathrm{Form}(n)/\dashv\vdash$$

the same for fmlas in $p_1 \cdots p_n$.

<u>Prop.</u> The syntactic poset

$$\mathrm{Syn}^{CPC}_n$$

is the free Boolean Algebra $F(n)$ on the set $\{p_1 \cdots p_n\}$, and

$$\mathrm{Syn}^{CPC} = F(\omega)$$

is the free BA on

$$\omega = \{p_0, p_1, \cdots\}.$$

Poset of Formulas

Explicitly, this means that if

- B is any BA
- $b_1 \dots b_n \in B$

then there is a unique Boolean homom.

$$\tilde{b} : \mathrm{Syn}_n^{CPC} \to B$$

with $\tilde{b}(p_i) = b_i \qquad i = 1, \dots, n$.

NB : The Lindenbaum-Tarski algebra LT^{CPC} is a similar construction using instead semantic equivalence

$$LT^{CPC} = \mathrm{Form} / \models \varphi \Leftrightarrow \psi .$$

We don't yet know that

$$\mathrm{Syn}^{CPC} \overset{?}{\cong} LT^{CPC} .$$

Boolean Algebras

Recall that a Boolean Algebra is a set B with the structure:

$$0, 1 \in B$$

$$\neg : B \longrightarrow B$$

$$\wedge, \vee : B \times B \longrightarrow B$$

satisfying the equations

...

Laws of BA

$$x \wedge x = x \qquad x \vee x = x$$

$$x \wedge y = y \wedge x \qquad x \vee y = y \vee x$$

$$x \wedge (y \wedge z) = (x \wedge y) \wedge z \qquad x \vee (y \vee z) = (x \vee y) \vee z$$

$$x \wedge 0 = 0 \qquad x \vee 1 = 1$$

$$x \wedge 1 = x \qquad x \vee 0 = x$$

$$x \wedge \neg x = 0 \qquad x \vee \neg x = 1$$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

$$\neg (x \wedge y) = \neg x \vee \neg y$$

$$\neg (x \vee y) = \neg x \wedge \neg y$$

$$\neg 1 = 0$$

$$\neg 0 = 1$$

$$\neg\neg x = x$$

NB: These are <u>not</u> independent.

Boolean Algebras

Some facts:

- Any BA can be ordered by

$$x \leq y \quad \text{iff} \quad x \wedge y = x,$$
$$\text{iff} \quad x = x \vee y.$$

- then $0 \leq x \leq 1$ f.all x

- $x \leq a \wedge b$ iff $x \leq a$ & $x \leq b$

- $a \vee b \leq x$ iff $a \leq x$ & $b \leq x$

- $a \wedge b \leq x$ iff $a \leq b \Rightarrow x$

  where $b \Rightarrow x := \neg b \vee x$

- So it's a CC poset w/ $0, +, \neg$.

- Basic examples are powersets

$$P(n) \cong 2^n.$$

Free Boolean Algebra

We construct the free BA as

$$F(n) = \mathrm{Form}(n)/\sim$$

where

$$\varphi \sim \psi \quad \text{iff} \quad \vdash_{eq} \varphi = \psi \qquad ,$$

i.e. equational provability from the laws of BA!

Prop. $F(n)$ is the free BA on $\{p_1 \ldots p_n\}$.

pf (1) It's a BA. ✓

Since $\vdash_{eq}$ includes the laws of BA's and substitution of $=$'s for $=$'s.

(2) It's the free one. ✓

Given $b_1 \dots b_n \in B$, let

$$\tilde{b} : F(n) \longrightarrow B$$

by

$$\tilde{b}(p_i) := b_i$$

and extend to all terms by structural induction:

$$\tilde{b}(\top) := 1$$

$$\tilde{b}(\bot) := 0$$

$$\tilde{b}(\neg\varphi) := 1 - \tilde{b}(\varphi)$$

$$\tilde{b}(\varphi \wedge \psi) := \tilde{b}(\varphi) \wedge \tilde{b}(\psi)$$

$$\tilde{b}(\varphi \vee \psi) := \tilde{b}(\varphi) \vee \tilde{b}(\psi).$$

NTS: $\tilde{b}$ is well-defined on eq-classes:

$$\varphi \sim \psi \Rightarrow \tilde{b}(\varphi) = \tilde{b}(\psi).$$

But since $\varphi \sim \psi$ means

$$\vdash_{eq} \varphi = \psi$$

and B is a BA, it follows that

$$\tilde{b}(\varphi) = \tilde{b}(\psi) \quad \text{in } B.$$

NB: this is Birkhoff's completeness thm for $\vdash_{eq} \varphi = \psi$.

We still want to show

$$\mathrm{Syn}_n^{CPC} \cong F(n).$$

The main thing we need is

(*) $\vdash \varphi \Leftrightarrow \psi$ iff $\vdash_{eq} \varphi = \psi$

i.e. deductive equivalence

= equational equivalence.

In fact we have:

thm $\vdash \varphi \Leftrightarrow \psi$

iff $\vDash \varphi \Leftrightarrow \psi$ "semantic equivalence"

iff $\vdash_{eq} \varphi = \psi$,

where $\vDash \varphi$ is validity.

Completeness of BA

Prop. 1 $\quad \vdash_{eq} \varphi = \psi$

iff $\models \varphi \Leftrightarrow \psi$.

Completeness of BA

Prop. 1 $\vdash_{eq} \varphi = \psi$

iff $\models \varphi \Leftrightarrow \psi$.

Pf. Use Normal Form $\vartheta^\wedge$ to see that:

$$\models \vartheta^\wedge \quad \text{iff} \quad \vdash_{eq} \vartheta^\wedge = \top .$$

By example :

$$p \vee (q \Rightarrow r) \wedge \neg p$$

$$= p \vee ((\neg q \vee r) \wedge \neg p)$$

$$= (p \vee \neg q \vee r) \wedge (p \vee \neg p)$$

Not valid :

p	q	r
0	1	0

Completeness of CPC

Prop. 2 $\Phi \vdash \psi$ iff $\Phi \models \psi$.

Pf. If $\vdash$, then $\models$:

by str. ind. on the derivations.

Given

$$\frac{\Phi \vdash \psi}{\Phi \vdash \vartheta} \qquad \text{i.e.} \qquad \begin{array}{c} \varphi_1 \ \cdots\cdots \ \varphi_n \in \Phi \\ \vdots \\ \dfrac{\psi}{\vartheta} \end{array}$$

Assume $\Phi \models \psi$ & show $\Phi \models \vartheta$.

So take v s.th.

$$[\![\varphi]\!]^v = 1 \quad \text{f. all } \varphi \in \Phi$$

therefore $[\![\psi]\!]^v = 1$,

then show $[\![\vartheta]\!]^v = 1$.

E.g. if $\vartheta = \psi \vee \alpha$.

Completeness of CPC

Prop. 2 $\Phi \vdash \psi$ iff $\Phi \models \psi$.

Pf. If $\vdash$, then $\models$,

by str. ind. on the derivations.

For the converse, suppose

$$\Phi \nvdash \psi .$$

Then $\Phi \cup \{\neg\psi\} \nvdash \bot$,

else $\Phi \vdash \psi$ by DN.

Now use MEL to get a

valuation v s.th.

$$[\![\varphi]\!]^v = 1 \quad \text{f.a. } \varphi \in \Phi$$

$$\& \quad [\![\psi]\!]^v = 0 .$$

So $\Phi \not\models \psi$.

Model Existence Lemma

If $\Gamma \nvdash \bot$, then Γ has a model:

$$v : Var \longrightarrow 2$$

s.th. $[\![\gamma]\!]^v = 1$ f.a. $\gamma \in \Gamma$.

Pf 1) Extend $\Gamma \subseteq \Gamma'$ max. con.:

- $\Gamma' \nvdash \bot$
- $\Gamma' \subseteq \Delta \nvdash \bot \Rightarrow \Gamma' = \Delta$.

2) Show that for T' max. con.

- $T' \vdash \varphi \Rightarrow \varphi \in T'$
- $\gamma \in T' \Leftrightarrow \neg\gamma \notin T'$
- $\varphi \wedge \psi \in T' \Leftrightarrow \varphi \in T'$ & $\psi \in T'$

etc.

3) Define $v : Var \to 2$ by $v(p) = 1$ iff $p \in T'$.

4) Show $[\![\varphi]\!]^v = 1$ iff $\varphi \in T'$.

5) So v is a model of T, since $T \subseteq T'$.