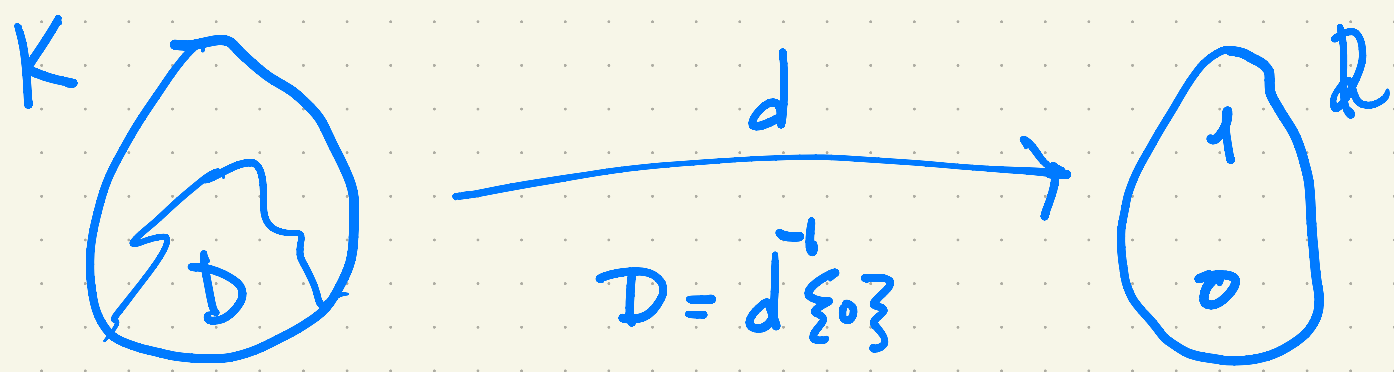
Kripke Models of λ-Calculus

CMU – SS'23

For the propositional logic PPC we had semantics in CC posets, of which one special kind was the downsets in any poset K,

$$\mathrm{Down}(K) \cong \mathrm{Pos}(K, 2) \cong 2^K$$



This gave rise to Kripke Semantics for PPC by:

$$\frac{[\![-]\!] : \mathrm{PPC} \longrightarrow 2^K}{\Vdash\, : \mathrm{PPC} \times K \Longrightarrow 2}$$

where:

$$[\![\varphi]\!]_k = 0 \quad \in 2$$

$$\iff$$

$$k \Vdash \varphi \,.$$

- Then functoriality & ccc of the model
$$[\![-]\!] : \mathrm{PPC} \longrightarrow 2^K$$
gave rise to the Kripke conditions:

  - $j \leq k \Vdash \varphi \rightarrow j \Vdash \varphi$
  - $k \Vdash \varphi \Rightarrow \psi$ iff $\forall_{k \leq l}.\ l \Vdash \varphi$ implies $l \Vdash \psi$
  - etc.

- For completeness we took:
$$K = \mathcal{C}_{\mathrm{PPC}}^{op}$$
$$[\![-]\!] = \downarrow : \mathcal{C}_{\mathrm{PPC}} \longrightarrow 2^{\mathcal{C}_{\mathrm{PPC}}^{op}}$$

which was just the Yoneda embedding, factored through $2 \hookrightarrow \mathrm{Set}$:

$$\begin{tikzcd}
\mathcal{C}_{\mathrm{PPC}} \arrow[r, "\downarrow"] \arrow[dr, bend right, "y"'] & 2^{\mathcal{C}_{\mathrm{PPC}}^{op}} \arrow[d, hook] & 2 \arrow[d, hook] \\
& \mathrm{Set}^{\mathcal{C}_{\mathrm{PPC}}^{op}} & \mathrm{Set}
\end{tikzcd}$$

Then we generalized from Propositions to Types.

$$\begin{array}{cc} \mathrm{STT} & 1, \times, \to \\ & (0, +) \\ | & \\ \mathrm{PL} & \top, \wedge, \Rightarrow \\ & (\bot, \vee) \end{array}$$

The (syntactic) classifying category $\mathcal{C}_{\mathbb{T}}$ of a λ-theory $\mathbb{T}$ is a CCC, and a model of $\mathbb{T}$ in any CCC $\mathcal{C}$ is a CC functor,

$$M : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{C} .$$

We have a completeness theorem w/resp. to variable models, i.e. those of the form

$$M : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathrm{Set}^{\mathbb{C}} .$$

as a common generalization of :

- functorial semantics of algebraic theories,
- Kripke semantics of propositional logic.

The completeness theorem is again proved "by Yoneda":

$$\mathcal{C} = \mathcal{C}_{\mathbb{T}}^{op}$$

$$\mathcal{U} = y : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathrm{Set}^{\mathcal{C}_{\mathbb{T}}^{op}}.$$

Then we obtain:

1) y is a model, since it's cc.

2) $\mathbb{T} \vdash (\cdot \vdash a : A)$ "inhabitation"

$\Longleftrightarrow \quad \mathcal{U} \Vdash a : A$ "PAT provability"

$\Longleftrightarrow \quad 1 \longrightarrow [\![A]\!]^{M}$ "pointedness" in all M (and universality)

by fullness of y

3) $\mathbb{T} \vdash u = v : A$ provably

$\Longleftrightarrow \quad \mathcal{U} \models u = v : A$ in $\mathcal{U}$

$\Longleftrightarrow \quad M \models u = v : A$ in all M (and universality)

by faithfullness of y

Thm (Scott 1980) (Presheaf Completeness)

For any λ-theory $\mathbb{T}$ we have:

(1) $\mathbb{T} \vdash t : T \iff \mathcal{M} \models T$
f. all $\mathbb{C}$ and all $\mathbb{T}$ models $\mathcal{M}$ in $\hat{\mathbb{C}}$

(2) $\mathbb{T} \vdash s = t : E \iff \mathcal{M} \models s = t$
f. all $\mathbb{C}$ and all $\mathbb{T}$ models $\mathcal{M}$ in $\hat{\mathbb{C}}$.

Pf:

1. Build the syntactic CCC $\mathcal{C}_{\mathbb{T}}$, consisting of types & terms, mod equ's.
2. $\mathcal{C}_{\mathbb{T}}$ has a canonical model $\mathcal{U}$, consisting of the basic types & terms.
3. $\mathcal{U}$ is generic, in the sense:

$$\mathbb{T} \vdash t : T \iff \mathcal{U} \models T$$

$$\mathbb{T} \vdash s = t : E \iff \mathcal{U} \models s = t$$

4. $\mathcal{C}_{\mathbb{T}}$ is the free CCC on a $\mathbb{T}$-model:

$$\forall \mathcal{M}\ \exists! \mathcal{M}^{\#} : \qquad \begin{array}{ccc} \mathcal{C}_{\mathbb{T}} & \xrightarrow{\mathcal{M}^{\#}} & \mathcal{C} \\ \mathcal{U} & \longmapsto & \mathcal{M} \end{array} \qquad \text{CCC}$$

Next we need the following generalization of the main lemma from PL for $\downarrow : P \to \hat{P}$.

Lemma For any small cat $\mathbb{C}$, the cat

$$\hat{\mathbb{C}} = \mathrm{Set}^{\mathbb{C}^{op}}$$

of presheaves on $\mathbb{C}$ is cartesian closed, and the Yoneda embedding

$$y : \mathbb{C} \hookrightarrow \hat{\mathbb{C}}$$

preserves any CCC structure in $\mathbb{C}$.

pf. For $P, Q \in \hat{\mathbb{C}}$ what should Q^P be?

$$\begin{aligned} (Q^P)c &\cong \hat{\mathbb{C}}(yc, Q^P) && \text{Yoneda} \\ &\cong \hat{\mathbb{C}}(yc \times P, Q) && \text{CCC} \end{aligned}$$

So let $Q^P := \hat{\mathbb{C}}(y(-) \times P, Q)$. ✓

Given $c, d \in \hat{\mathbb{C}}$,

$$\begin{aligned} y(d^c) &= \mathbb{C}(-, d^c) && \text{def} \\ &= \mathbb{C}(- \times c, d) && \text{CCC} \\ &\cong \hat{\mathbb{C}}(y(- \times c), yd) && \text{Yoneda} \\ &\cong \hat{\mathbb{C}}(y(-) \times yc, yd) && \text{UMP of } \times \\ &= (yd)^{yc}. \ ✓ && \text{def} \end{aligned}$$

To finish the proof of the thm:

if $\mathbb{T} \vdash t : T$, then $\mathcal{U} \models T$, namely

$$(*) \qquad [\![t]\!] : 1 \longrightarrow [\![T]\!] \quad \text{in } \mathcal{C}_{\mathbb{T}} .$$

Given any $\mathbb{T}$ model $\mathcal{M}$ in any $\hat{\mathbb{C}}$,
since $\mathcal{C}_{\mathbb{T}}$ is free on $\mathcal{U}$ we have:

$$\begin{aligned} \mathcal{C}_{\mathbb{T}} &\xrightarrow{\mathcal{M}^{\#}} \mathcal{C} \\ \mathcal{U} &\longmapsto \mathcal{M} . \end{aligned}$$

So from $(*)$ we obtain:

$$\begin{tikzcd} \mathcal{M}^{\#} 1 \arrow[r, "\mathcal{M}^{\#} [\![t]\!]"] \arrow[d, phantom, "\cong" sloped] & \mathcal{M}^{\#} [\![T]\!] \arrow[d, phantom, "\cong" sloped] \\ 1 \arrow[r] & [\![T]\!]^{\mathcal{M}} \end{tikzcd}$$

where the $\cong$'s are because $\mathcal{M}^{\#}$ is ccc.

Thus indeed:
$$\mathcal{M} \models T .$$

Conversely, if $\mathcal{M} \models T$ for all $\mathcal{M}$,
then in particular $\mathcal{U} \models T$.

Whence:
$$\mathbb{T} \vdash T ,$$

since $\mathcal{U}$ is generic.

(1) ✓

(2) %.

The result can now be specialized from general cats $\mathbb{C}$ to posets K:

Thm (Kripke Completeness of λ-Calculus)
For any λ-theory $\mathbb{T}$ we have:

(1) $\mathbb{T} \vdash t : T \iff \mathcal{M} \models T$
f. all posets K and $\mathbb{T}$-models $\mathcal{M}$ in $\hat{K}$

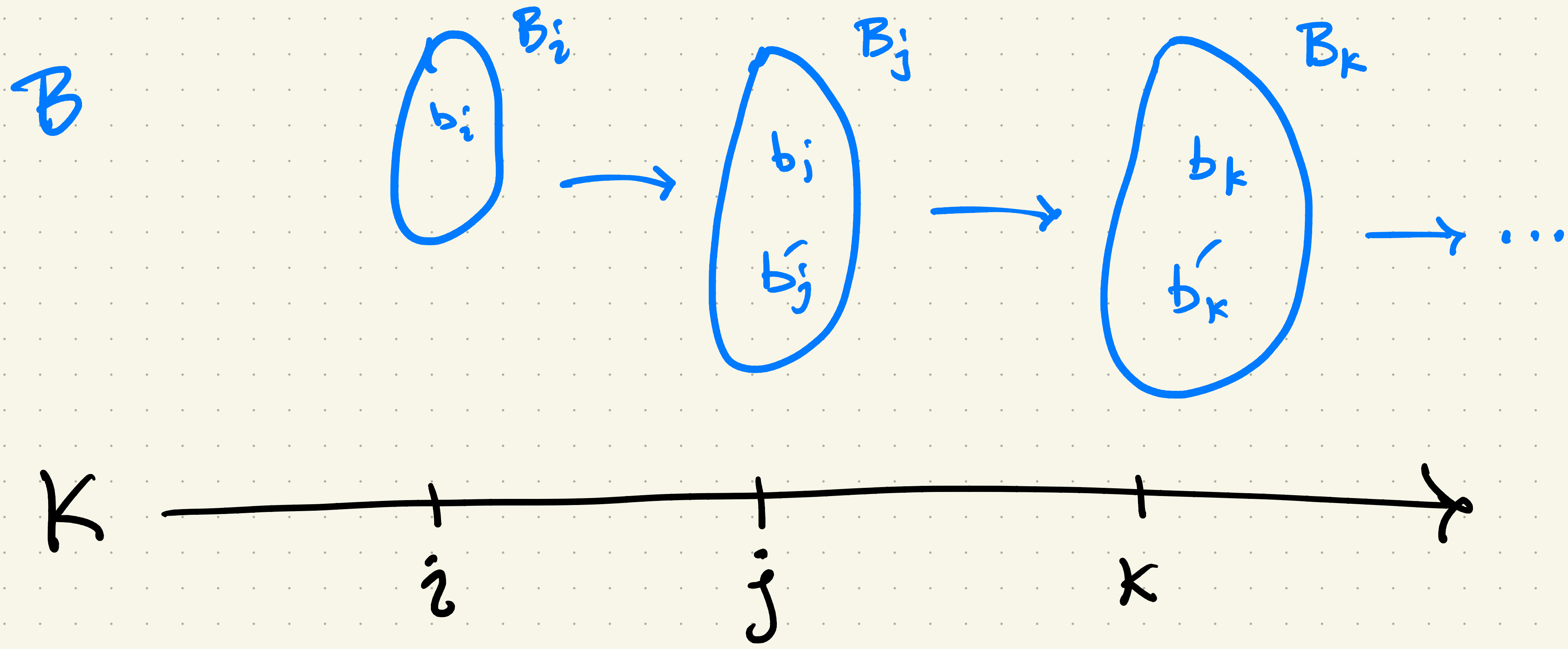
(2) $\mathbb{T} \vdash s = t : E \iff \mathcal{M} \models s = t$
f. all posets K and $\mathbb{T}$-models $\mathcal{M}$ in $\hat{K}$.

The proof* uses a theorem from topos theory (due to Joyal & Tierney) to move from $\hat{\mathbb{C}}$ to $\hat{K}_{\mathbb{C}}$ for a poset $K_{\mathbb{C}}$.

Remark: One can also go between
"Scott style" $j \in [\![A]\!]$ and
"Kripke style" $j \Vdash A$ for λ-theories (see AGH 2021).

* In (AR 2011).

What is a Kripke model of the λ-calculus?



$k \Vdash b : B$ f.a. $k \in K$

"pointwise inhabitation"

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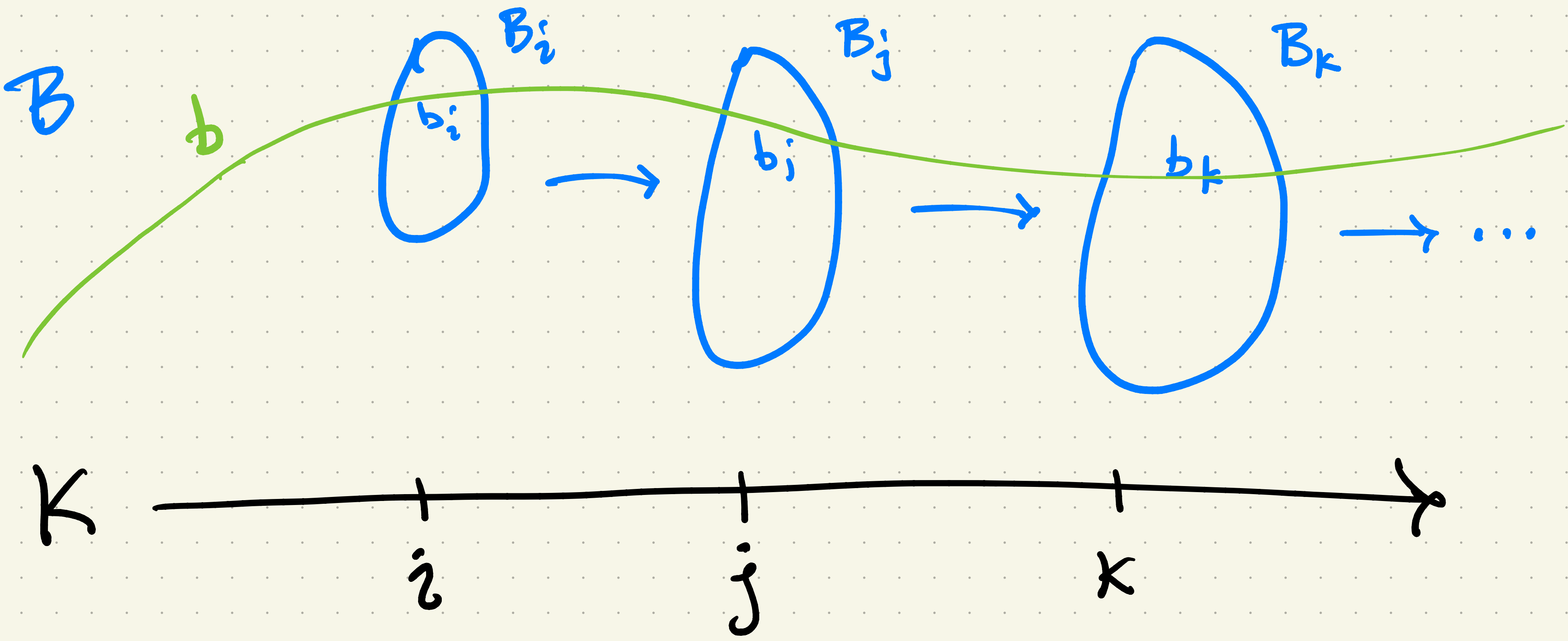

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Versus

$K \Vdash b : B$

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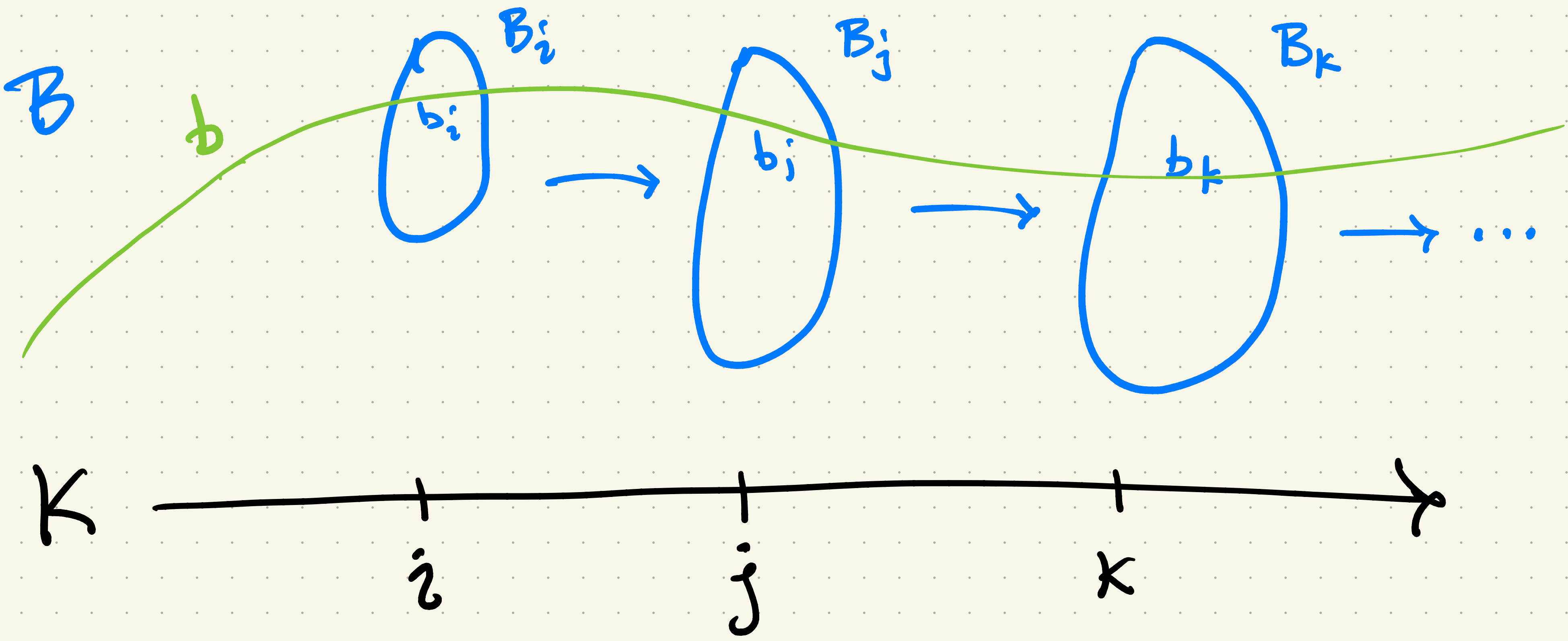

$k \Vdash b : B$ f.a. $k \in K$

"pointwise inhabitation"

Versus

$K \Vdash b : B$

"global inhabitation"

$\iff$ f. all $k : K$, there's a $b_k : B_k$

$$k \Vdash b_k : B_k$$

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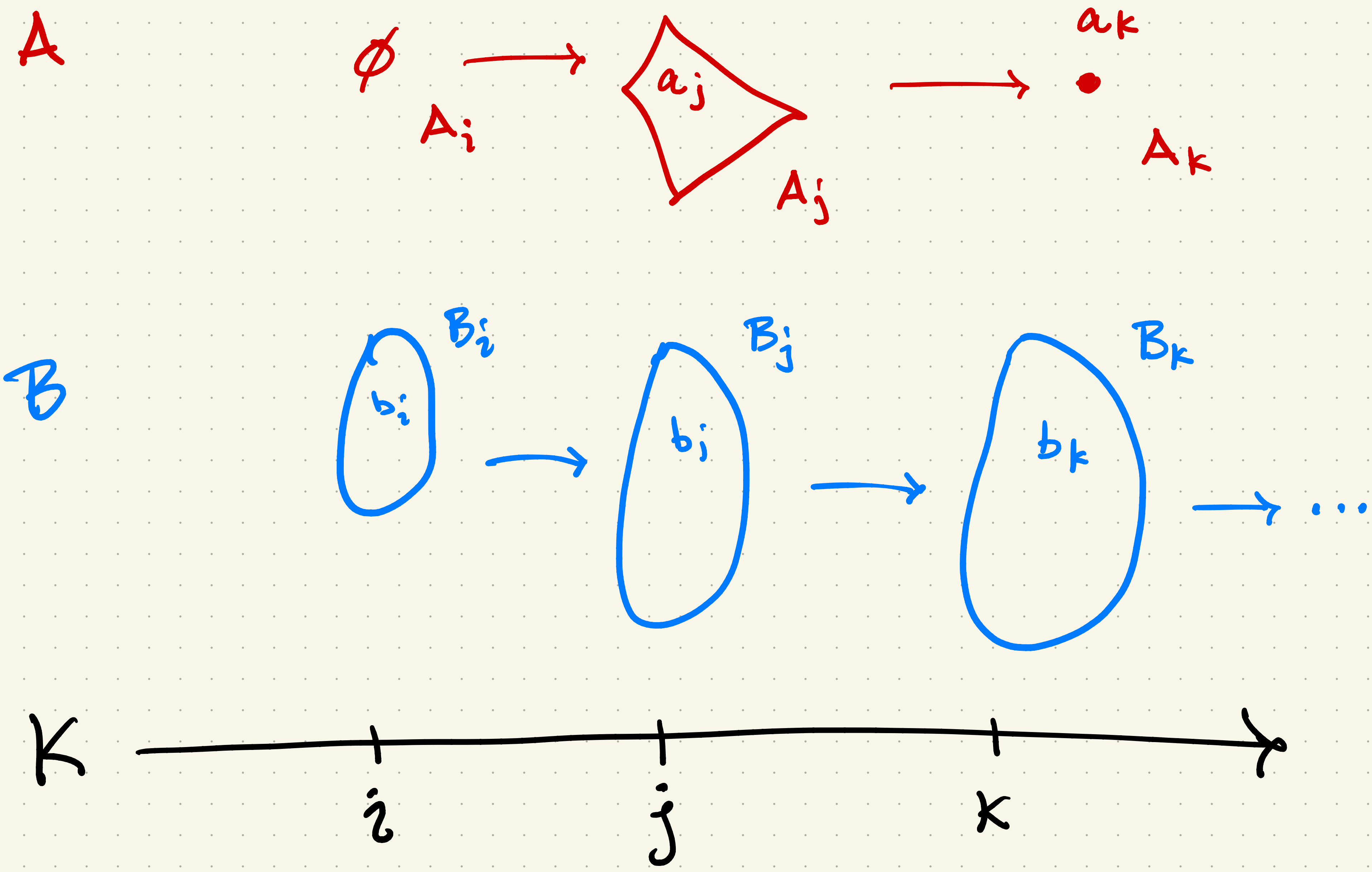

$$j \Vdash a : A \;\Rightarrow\; k \Vdash a : A$$

$$K \nVdash a : A$$

A not inhabited

$$\Rightarrow \quad \mathbb{T} \nvdash A$$

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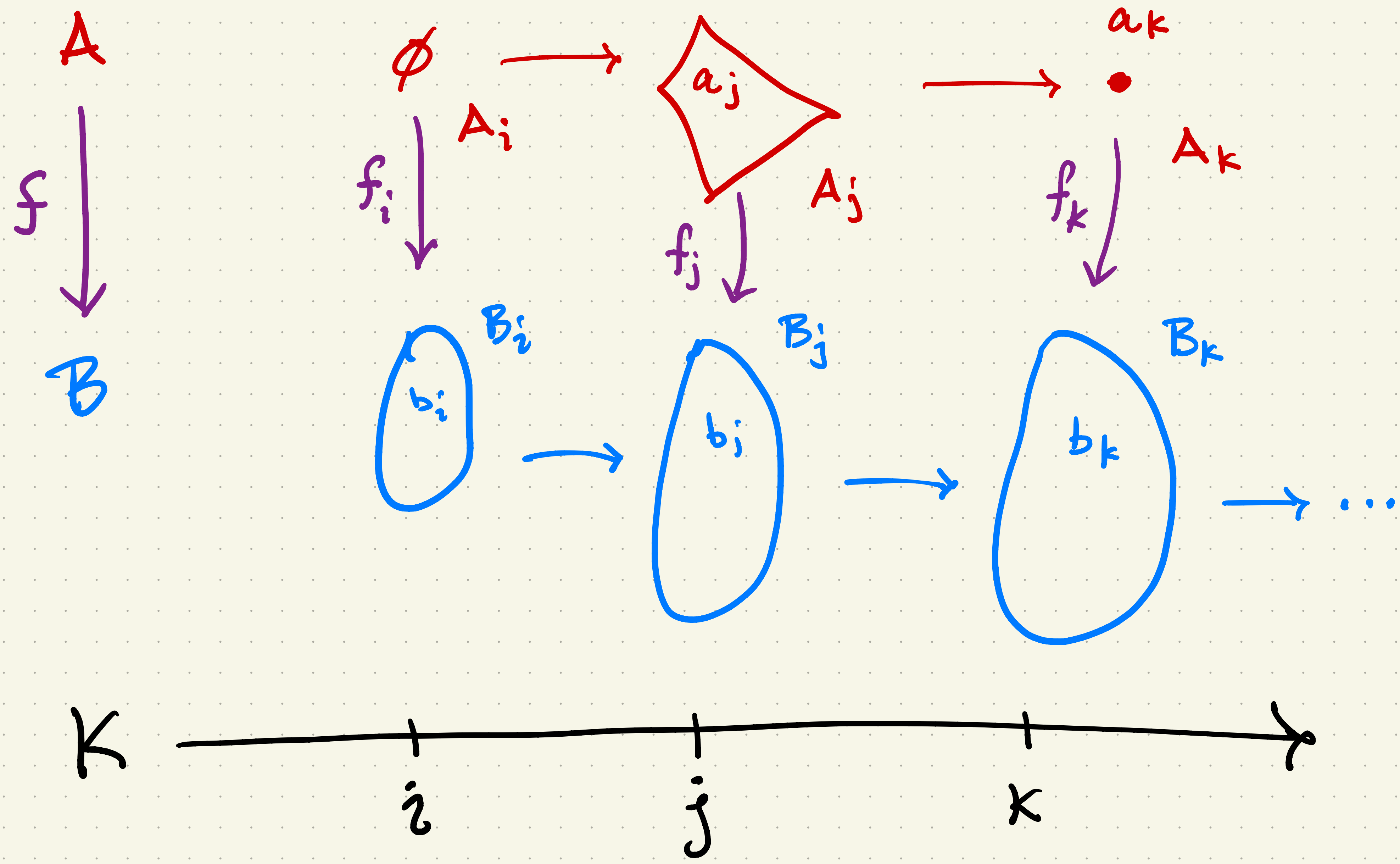
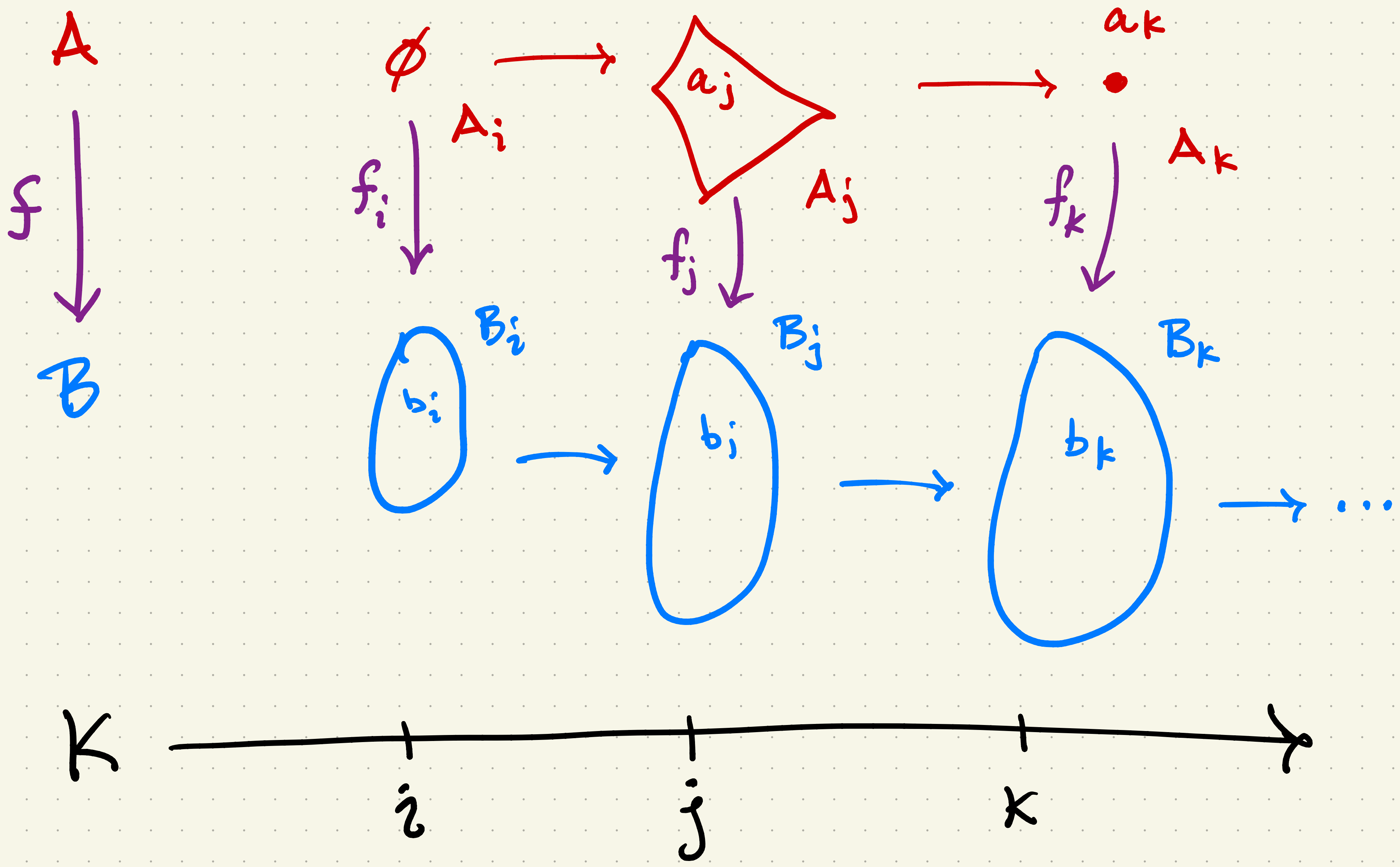


$$K \Vdash f : A \longrightarrow B$$

$$\iff \quad j \Vdash fa = b \quad \text{f.a. } j \leq k$$

So $\quad K\!\uparrow_j \Vdash f a = b$.

What is a Kripke model of the λ-calculus?



$$i \not\Vdash g : B \to A$$

$$\Rightarrow \quad K \not\Vdash g : B \to A$$

no map $B \to A$ in this model, so

$$\mathbb{T} \not\vdash * : B \to A$$

Open Problems

1) As in PPC, one should be able to add 0 & $A+B$ to the λ-calculus and still get the completeness theorems (both Scott $\hat{\mathbb{C}}$ and Kripke $\hat{K}$).

2) The use of Joyal-Tierney to get from $\hat{\mathbb{C}}$ to $\hat{K}$ is probably overkill. It actually produces a sheaf model over a space $X_{\mathbb{C}}$, and then

$$K \simeq \mathcal{O}X_{\mathbb{C}} \quad \text{(cf. A2000).}$$

Perhaps there is a more direct proof, following the idea of the PL case?

3) Can one add 0 & $A+B$ in the topological case?

4) Reformulate Kripke semantics in terms of

$$\mathrm{KSets} \hookrightarrow \mathrm{Pos}/K.$$

References

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