Introduction to Categorical Logic

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Chapter 2

Propositional Logic

Propositional logic is the logic of propositional connectives like $p \wedge q$ and $p \Rightarrow q$. As was the case for algebraic theories, the general approach will be to determine suitable categorical structures to model the logical operations, and then use categories with such structure to represent (abstract) propositional theories. Adjoints will play a special role, as we will describe the basic logical operations as such. We again show that the semantics is "functorial", meaning that the models of a theory are functors that preserve the categorical structure. We will show that there are classifying categories for all propositional theories, as was the case for the algebraic theories that we have already met.

A more abstract, algebraic perspective will then relate the propositional case of syntax-semantics duality with classical Stone duality for Boolean algebras, and related results from lattice theory will provide an algebraic treatment of Kripke semantics for intuitionistic (and modal) propositional logic.

2.1 Propositional calculus

Before going into the details of the categorical approach, we first briefly review the propositional calculus from a conventional point of view, as we did for algebraic theories. We focus first on the *classical* propositional logic, before considering the intuitionistic case in Section 2.9.

In the style of Section **??**, we have the following (abstract) syntax for (propositional) formulas:

$$\begin{aligned} \text{Propositional variable } p &::= \mathtt{p}_1 \mid \mathtt{p}_2 \mid \mathtt{p}_3 \mid \cdots \\ \text{Propositional formula } \phi &::= p \mid \top \mid \bot \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid \phi_1 \Rightarrow \phi_2 \mid \phi_1 \Leftrightarrow \phi_2 \end{aligned}$$

An example of a formula is therefore $\big(\mathtt{p}_3 \Leftrightarrow ((((\neg\mathtt{p}_1) \vee (\mathtt{p}_2 \wedge \bot)) \vee \mathtt{p}_1) \Rightarrow \mathtt{p}_3)\big)$. We will make use of the usual conventions for parenthesis, with binding order $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$. Thus e.g. the foregoing may also be written unambiguously as $\mathtt{p}_3 \Leftrightarrow \neg\mathtt{p}_1 \vee \mathtt{p}_2 \wedge \bot \vee \mathtt{p}_1 \Rightarrow \mathtt{p}_3$.

2.1.1 Natural deduction

The system of *natural deduction* for propositional logic has one form of judgement

$$\mathtt{p}_1, \ldots, \mathtt{p}_n \mid \phi_1, \ldots, \phi_m \vdash \phi$$

where $\mathtt{p}_1, \ldots, \mathtt{p}_n$ is a *context* consisting of distinct propositional variables, the formulas $\phi_1, \ldots, \phi_m$ are the *hypotheses* and ϕ is the *conclusion*. The variables in the hypotheses and the conclusion must occur among those listed in the context. The hypotheses are regarded as a (finite) set; so they are unordered, have no repetitions, and may be empty. We may abbreviate the context of variables by Γ, and we often omit it.

Deductive entailment (or *derivability*) $\Phi \vdash \phi$ is thus a relation between finite sets of formulas Φ and single formulas ϕ. It is defined as the smallest such relation satisfying the following rules:

1. Hypothesis:
$$\frac{}{\Phi \vdash \phi} \ \text{if } \phi \text{ occurs in } \Phi$$

2. Truth:
$$\frac{}{\Phi \vdash \top}$$

3. Falsehood:
$$\frac{\Phi \vdash \bot}{\Phi \vdash \phi}$$

4. Conjunction:
$$\frac{\Phi \vdash \phi \qquad \Phi \vdash \psi}{\Phi \vdash \phi \wedge \psi} \qquad \frac{\Phi \vdash \phi \wedge \psi}{\Phi \vdash \phi} \qquad \frac{\Phi \vdash \phi \wedge \psi}{\Phi \vdash \psi}$$

5. Disjunction:
$$\frac{\Phi \vdash \phi}{\Phi \vdash \phi \vee \psi} \qquad \frac{\Phi \vdash \psi}{\Phi \vdash \phi \vee \psi} \qquad \frac{\Phi \vdash \phi \vee \psi \qquad \Phi, \phi \vdash \theta \qquad \Phi, \psi \vdash \theta}{\Phi \vdash \theta}$$

6. Implication:
$$\frac{\Phi, \phi \vdash \psi}{\Phi \vdash \phi \Rightarrow \psi} \qquad \frac{\Phi \vdash \phi \Rightarrow \psi \qquad \Phi \vdash \phi}{\Phi \vdash \psi}$$

For the purpose of deduction, we define $\neg\phi := \phi \Rightarrow \bot$ and $\phi \Leftrightarrow \psi := (\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi)$. To obtain *classical* logic we need only include one of the following additional rules.

7. Classical logic:
$$\frac{}{\Phi \vdash \phi \vee \neg\phi} \qquad \frac{\Phi \vdash \neg\neg\phi}{\Phi \vdash \phi}$$

A *proof* of a judgement $\Phi \vdash \phi$ is a *finite* tree built from the above inference rules whose root is $\Phi \vdash \phi$. For example, here is a proof of $\phi \vee \psi \vdash \psi \vee \phi$ using the disjunction rules:

$$\dfrac{\dfrac{}{\phi \vee \psi \vdash \phi \vee \psi} \qquad \dfrac{\dfrac{}{\phi \vee \psi, \phi \vdash \phi}}{\phi \vee \psi, \phi \vdash \psi \vee \phi} \qquad \dfrac{\dfrac{}{\phi \vee \psi, \psi \vdash \psi}}{\phi \vee \psi, \psi \vdash \psi \vee \phi}}{\phi \vee \psi \vdash \psi \vee \phi}$$

A judgment $\Phi \vdash \phi$ is *provable* if there exists a proof of it. Observe that every proof has at its leaves either the rule for $\top$ or an instance of the rule of hypothesis (or the Excluded Middle rule for classical logic).

Remark 2.1.1. An alternate form of presentation for proofs in natural deduction that is more, well, natural uses trees of formulas, rather than of judgements, with leaves labelled by *assumptions* ϑ that may also occur in *cancelled* form $[\vartheta]$. Thus for example the introduction and elimination rules for conjunction would be written in the form:

$$\dfrac{\begin{array}{c}\Phi \\ \vdots \\ \phi\end{array} \quad \begin{array}{c}\Phi \\ \vdots \\ \psi\end{array}}{\phi \wedge \psi} \qquad\qquad \dfrac{\begin{array}{c}\Phi \\ \vdots \\ \phi \wedge \psi\end{array}}{\phi} \qquad \dfrac{\begin{array}{c}\Phi \\ \vdots \\ \phi \wedge \psi\end{array}}{\psi}$$

An example of a proof tree with cancelled assumptions is the one for disjunction elimination:

$$\dfrac{\begin{array}{c}\Phi \\ \vdots \\ \phi \vee \psi\end{array} \qquad \begin{array}{c}\Phi, [\phi] \\ \vdots \\ \vartheta\end{array} \qquad \begin{array}{c}\Phi, [\psi] \\ \vdots \\ \vartheta\end{array}}{\vartheta}$$

And the above rule of implication introduction takes the form:

$$\dfrac{\begin{array}{c}\Phi, [\phi] \\ \vdots \\ \psi\end{array}}{\phi \Rightarrow \psi}$$

In these examples, the cancellation occurred at the last step. In order to continue such a proof, we need a device to indicate *when* a cancellation occurs, *i.e.* at which step of the proof. This can be done as follows:

$$\dfrac{\dfrac{\begin{array}{c}\Phi, [\alpha]^2 \\ \vdots \\ \phi \vee \psi\end{array} \qquad \begin{array}{c}\Phi, [\phi]^1 \\ \vdots \\ \vartheta\end{array} \qquad \begin{array}{c}\Phi, [\psi]^1 \\ \vdots \\ \vartheta\end{array}}{\vartheta}\,(1)}{\alpha \Rightarrow \vartheta}\,(2)$$

This proof tree represents a derivation of the judgement $\Phi \vdash \alpha \Rightarrow \vartheta$. A proof tree in which all the assumptions have been cancelled represents a derivation of an unconditional judgement such as $\vdash \phi$.

We will have a better way to record such proofs in Section **??**.

Exercise 2.1.2. Derive each of the two classical rules (2.1.1), called *Excluded Middle* and *Double Negation*, from the other.

2.2 Truth values

The idea of an axiomatic system of deductive, logical reasoning goes to back to Frege, who gave the first such system for propositional calculus (and more) in his *Begriffsschrift* of 1879. The question soon arose whether Frege's rules (or rather, their derivable consequences — it was clear that one could chose the primitive basis in different but equivalent ways) were correct, and if so, whether they were *all* the correct ones. An ingenious solution was proposed by Russell's student Wittgenstein, who came up with an entirely different way of singling out a set of "valid" propositional formulas in terms of assignments of truth values to the variables occurring in them. He interpreted this as showing that logical validity was really a matter of the logical structure of a proposition, rather than depending any particular system of derivations. The same idea seems to have been had independently by Post, who proved that the valid propositional formulas coincide with the ones derivable in Whitehead and Russell's *Principia Mathematica* (which is propositionally equivalent to Frege's system), a fact that we now refer to as the *soundness* and *completeness* of propositional logic.

In more detail, let a *valuation* v be an assignment of a "truth-value" $0, 1$ to each propositional variable, $v(\mathtt{p}_n) \in \{0, 1\}$. We can then extend the valuation to all propositional formulas $[\![\phi]\!]^v$ by the following recursion.

$$
\begin{aligned}
[\![\mathtt{p}_n]\!]^v &= v(\mathtt{p}_n)\\
[\![\top]\!]^v &= 1\\
[\![\bot]\!]^v &= 0\\
[\![\neg\phi]\!]^v &= 1 - [\![\phi]\!]^v\\
[\![\phi \wedge \psi]\!]^v &= \min([\![\phi]\!]^v, [\![\psi]\!]^v)\\
[\![\phi \vee \psi]\!]^v &= \max([\![\phi]\!]^v, [\![\psi]\!]^v)\\
[\![\phi \Rightarrow \psi]\!]^v &= 1 \text{ iff } [\![\phi]\!]^v \leq [\![\psi]\!]^v\\
[\![\phi \Leftrightarrow \psi]\!]^v &= 1 \text{ iff } [\![\phi]\!]^v = [\![\psi]\!]^v
\end{aligned}
$$

This is sometimes expressed using the "semantic consequence" notation $v \vDash \phi$ to mean that $[\![\phi]\!]^v = 1$. The above specification then takes the following form, in which the condition for the truth of a formula is given in terms of its informal "meaning":

$$
\begin{array}{rll}
v \vDash \top & \text{always} \\
v \vDash \bot & \text{never} \\
v \vDash \neg\phi & \text{iff} & \text{not } v \vDash \phi \\
v \vDash \phi \wedge \psi & \text{iff} & v \vDash \phi \text{ and } v \vDash \psi \\
v \vDash \phi \vee \psi & \text{iff} & v \vDash \phi \text{ or } v \vDash \psi \\
v \vDash \phi \Rightarrow \psi & \text{iff} & v \vDash \phi \text{ implies } v \vDash \psi \\
v \vDash \phi \Leftrightarrow \psi & \text{iff} & v \vDash \phi \text{ iff } v \vDash \psi
\end{array}
$$

Finally, ϕ is *valid*, written $\vDash \phi$, is defined by,

$$
\begin{array}{rll}
\vDash \phi & \text{iff} & v \vDash \phi \text{ for all } v \\
 & \text{iff} & [\![\phi]\!]^v = 1 \text{ for all } v\,.
\end{array}
$$

And, more generally, we define $\phi_1, ..., \phi_n$ *semantically entails* ϕ, written

$$\phi_1, ..., \phi_n \vDash \ \phi, \tag{2.1}$$

to mean that for all valuations v such that $v \vDash \phi_k$ for all k, also $v \vDash \phi$.

Given a formula in context $\Gamma \mid \phi$ and a valuation v for the variables in Γ, one can check whether $v \vDash \phi$ using a *truth table*, which is a systematic way of calculating the value of $[\![\phi]\!]^v$. For example, under the assignment $v(\mathtt{p}_1) = 1, v(\mathtt{p}_2) = 0, v(\mathtt{p}_3) = 1$ we can calculate $[\![\phi]\!]^v$ for $\phi = \big(\mathtt{p}_3 \Leftrightarrow ((((\neg\mathtt{p}_1) \vee (\mathtt{p}_2 \wedge \bot)) \vee \mathtt{p}_1) \Rightarrow \mathtt{p}_3)\big)$ as follows.

$\mathtt{p}_1$	$\mathtt{p}_2$	$\mathtt{p}_3$	$\mathtt{p}_3$	$\Leftrightarrow$	$\neg$	$\mathtt{p}_1$	$\vee$	$\mathtt{p}_2$	$\wedge$	$\bot$	$\vee$	$\mathtt{p}_1$	$\Rightarrow$	$\mathtt{p}_3$
1	0	1	1	1	0	1	0	0	0	0	1	1	1	1

The value of the formula ϕ under the valuation v is then the value in the column under the main connective, in this case $\Leftrightarrow$, and thus $[\![\phi]\!]^v = 1$.

Displaying all 2^3 valuations for the context $\Gamma = \mathtt{p}_1, \mathtt{p}_2, \mathtt{p}_3$, therefore results in a table that checks for validity of ϕ,

$\mathtt{p}_1$	$\mathtt{p}_2$	$\mathtt{p}_3$	$\mathtt{p}_3$	$\Leftrightarrow$	$\neg$	$\mathtt{p}_1$	$\vee$	$\mathtt{p}_2$	$\wedge$	$\bot$	$\vee$	$\mathtt{p}_1$	$\Rightarrow$	$\mathtt{p}_3$
1	1	1	.	1	…									
1	1	0	.	1		…								
1	0	1	1	1	0	1	0	0	0	0	1	1	1	1
1	0	0	.	1				…						
0	1	1	.	1					…					
0	1	0	.	1						…				
0	0	1	.	1							…			
0	0	0	.	1								…		

In this case, working out the other rows shows that ϕ is indeed valid, thus $\vDash \phi$.

Theorem 2.2.1 (Soundness and Completeness of Propositional Calculus)**.** *Let Φ be any set of formulas and ϕ any formula, then*

$$\Phi \vdash \phi \iff \Phi \vDash \phi.$$

In particular, for any propositional formula ϕ we have

$$\vdash \phi \iff \vDash \phi.$$

Thus derivability and validity coincide.

Proof. Let us sketch the usual proof, for later reference.

(*Soundness*:) First assume $\Phi \vdash \phi$ is provable, meaning there is a finite derivation of $\Phi \vdash \phi$ by the rules of inference. We show by induction on the set of derivations that $\Phi \vDash \phi$, meaning that for any valuation v such that $v \vDash \Phi$ also $v \vDash \phi$. For this, observe that in each individual rule of inference, if $\Psi \vDash \psi$ for all the premisses of the rule, then $\Phi \vDash \phi$ for the conclusion (the set of premisses may change from the premisses to the conclusion if the rule involves a cancellation).

(*Competeness*:) Suppose that $\Phi \nvdash \phi$, then $\Phi, \neg\phi \nvdash \bot$ (using double negation elimination). By Lemma 2.2.2 below, there is a valuation v such that $v \vDash \{\Phi, \neg\phi\}$. Thus in particular $v \vDash \Phi$ and $v \nvDash \phi$, therefore $\Phi \nvDash \phi$. □

The key lemma is this:

Lemma 2.2.2 (Model Existence)**.** *If a set Φ of formulas is* consistent*, in the sense that $\Phi \nvdash \bot$, then it has a* model*, i.e. a valuation v such that $v \vDash \Phi$.*

Proof. Let Φ be any consistent set of formulas. We extend $\Phi \subseteq \Psi$ to one that is *maximally consistent*, meaning Ψ is consistent, and if $\Psi \subseteq \Psi'$ and Ψ' is consistent, then $\Psi = \Psi'$. Enumerate the formulas $\phi_0, \phi_1, ...$, and let,

$$\begin{aligned}
\Phi_0 &= \Phi, \\
\Phi_{n+1} &= \Phi_n \cup \phi_n \text{ if consistent, else } \Phi_n, \\
\Psi &= \textstyle\bigcup_n \Phi_n.
\end{aligned}$$

One can then show that Ψ is indeed maximally consistent, and for every formula ψ, either $\psi \in \Psi$ or $\neg\psi \in \Psi$ and not both (exercise!). Now for each propositional variable $\mathtt{p}$, define $v_\Psi(\mathtt{p}) = 1$ just if $\mathtt{p} \in \Psi$. Finally, one shows that $[\![\phi]\!]^{v_\Psi} = 1$ just if $\phi \in \Psi$, and therefore $v_\Psi \vDash \Psi \supseteq \Phi$. □

Exercise 2.2.3. Show that for any maximally consistent set Ψ of formulas, either $\psi \in \Psi$ or $\neg\psi \in \Psi$ and not both. Conclude from this that for the valuation v_Ψ defined by $v_\Psi(\mathtt{p}) = 1$ just if $\mathtt{p} \in \Psi$, we indeed have $[\![\phi]\!]^{v_\Psi} = 1$ just if $\phi \in \Psi$, as claimed in the proof of the Model Existence Lemma 2.2.2.

2.3 Boolean algebra

There is of course another approach to propositional logic, which also goes back to the 19th century, namely that of Boolean algebra, which draws on the analogy between the propositional operations and the arithmetical ones.

Definition 2.3.1. A *Boolean algebra* is a set B equipped with the operations:

$$\begin{aligned} 0, 1 : 1 &\to B \\ \neg : B &\to B \\ \wedge, \vee : B \times B &\to B \end{aligned}$$

satisfying the following equations:

$$\begin{aligned}
x \vee x &= x & x \wedge x &= x \\
x \vee y &= y \vee x & x \wedge y &= y \wedge x \\
x \vee (y \vee z) &= (x \vee y) \vee z & x \wedge (y \wedge z) &= (x \wedge y) \wedge z \\
x \wedge (y \vee z) &= (x \wedge y) \vee (x \wedge z) & x \vee (y \wedge z) &= (x \vee y) \wedge (x \vee z) \\
0 \vee x &= x & 1 \wedge x &= x \\
1 \vee x &= 1 & 0 \wedge x &= 0 \\
\neg(x \vee y) &= \neg x \wedge \neg y & \neg(x \wedge y) &= \neg x \vee \neg y \\
x \vee \neg x &= 1 & x \wedge \neg x &= 0
\end{aligned}$$

This is of course an algebraic theory, like those considered in the previous chapter. Familiar examples of Boolean algebras are $2 = \{0, 1\}$, with the usual operations, and more generally, any powerset $\mathcal{P}X$, with the set-theoretic operations $A \vee B = A \cup B$, etc. (indeed, $2 = \mathcal{P}1$ is a special case.).

Exercise 2.3.2. Show that the free Boolean algebra $B(n)$ on n-many generators is the double powerset $\mathcal{P}\mathcal{P}(n)$, and determine the free functor on finite sets.

One can use equational reasoning in Boolean algebra as an alternative to the deductive propositional calculus as follows. For a propositional formula in context $\Gamma \mid \phi$, let us say that ϕ is *equationally provable* if we can prove $\phi = 1$ by equational reasoning (Section **??**), from the laws of Boolean algebras above. More generally, for a set of formulas Φ and a formula ψ let us define the (*ad hoc*) relation of *equational provability*,

$$\Phi \vdash_{\mathsf{eq}} \psi \tag{2.2}$$

to mean that $\psi = 1$ can be proven equationally from (the Boolean equations and) the set of all equations $\phi = 1$, for $\phi \in \Phi$. Since we don't have any laws for the connectives $\Rightarrow$ or $\Leftrightarrow$, let us replace them with their Boolean equivalents, by adding the equations:

$$\begin{aligned} \phi \Rightarrow \psi &= \neg\phi \vee \psi \,, \\ \phi \Leftrightarrow \psi &= (\neg\phi \vee \psi) \wedge (\neg\psi \vee \phi) \,. \end{aligned}$$

Here for example is an equational proof of $(\phi \Rightarrow \psi) \vee (\psi \Rightarrow \phi)$.

$$\begin{aligned}(\phi \Rightarrow \psi) \vee (\psi \Rightarrow \phi) &= (\neg\phi \vee \psi) \vee (\neg\psi \vee \phi) \\ &= \neg\phi \vee (\psi \vee (\neg\psi \vee \phi)) \\ &= \neg\phi \vee ((\psi \vee \neg\psi) \vee \phi) \\ &= \neg\phi \vee (1 \vee \phi) \\ &= \neg\phi \vee 1 \\ &= 1 \vee \neg\phi \\ &= 1\end{aligned}$$

Thus we have

$$\vdash_{\mathsf{eq}} (\phi \Rightarrow \psi) \vee (\psi \Rightarrow \phi)\,.$$

We now ask: *How is equational provability $\Phi \vdash_{\mathsf{eq}} \phi$ related to deductive entailment $\Phi \vdash \phi$ and semantic entailment $\Phi \vDash \phi$?*

Exercise 2.3.3. Using equational reasoning, show that every propositional formula ϕ has both a *conjunctive* $\phi^\wedge$ and a *disjunctive* $\phi^\vee$ *Boolean normal form* such that:

1. The formula $\phi^\vee$ is an n-fold disjunction of m-fold conjunctions of *positive* $\mathtt{p_i}$ or *negative* $\neg\mathtt{p_j}$ propositional variables,

$$\phi^\vee \;=\; (\mathsf{q}_{11} \wedge \ldots \wedge \mathsf{q}_{1m_1}) \vee \ldots \vee (\mathsf{q}_{n1} \wedge \ldots \wedge \mathsf{q}_{nm_n})\,, \qquad \mathsf{q}_{ij} \in \{\mathsf{p}_{ij}, \neg\mathsf{p}_{ij}\}\,,$$

   and $\phi^\wedge$ is the same, but with the roles of $\vee$ and $\wedge$ reversed.

2. Both

$$\vdash_{\mathsf{eq}} \phi \Leftrightarrow \phi^\vee \qquad \text{and} \qquad \vdash_{\mathsf{eq}} \phi \Leftrightarrow \phi^\wedge\,.$$

(*Hint:* Rewrite the formula in terms of just conjunction, disjunction, and negation, and then do both normal forms at the same time, by structural induction on the formula.)

Remark 2.3.4. We can already use Exercise 2.3.3 to show that equational provability is equivalent to semantic validity,

$$\vdash_{\mathsf{eq}} \phi \iff \vDash \phi\,.$$

To show this, we first put the formula ϕ into conjunctive normal form, and then read off a truth valuation that falsifies it, just if there is one. Indeed, the CNF is valued as 1 just if each conjunct is, and that holds just if each conjunct contains a propositional letter $\mathtt{p}$ in both positive and negative $\neg\mathtt{p}$ form. And in that case, the CNF clearly reduces to 1 by an equational calculation. Conversely, if the CNF does not so reduce, it must have a conjunct that does not satisfy the condition just stated – and so we can read off a valuation making all propositional letters in that conjunct 0.

Exercise 2.3.5. A Boolean algebra can be partially ordered by defining $x \leq y$ as

$$x \leq y \iff x \vee y = y \qquad \text{or equivalently} \qquad x \leq y \iff x \wedge y = x\,.$$

Thus a Boolean algebra is a (poset) category. Show that as a category, a Boolean algebra has all finite limits and colimits and is cartesian closed, with $x \Rightarrow y := \neg x \vee y$ as the exponential of x and y. Moreover, a finitely complete and cocomplete cartesian closed poset is a Boolean algebra just if it satisfies $x = (x \Rightarrow 0) \Rightarrow 0$. Finally, show that homomorphisms of Boolean algebras $f : B \to B'$ are the same thing as functors (i.e. monotone maps) that preserve all finite limits and colimits.

2.4 Lawvere duality for Boolean algebras

Let us apply the machinery of algebraic theories from Chapter **??** to the algebraic theory of Boolean algebras and see what we get. The algebraic theory $\mathbb{B}$ of Boolean algebras is a finite product (FP) category with objects $1, B, B^2, ...$, containing a Boolean algebra $\mathsf{U}_{\mathbb{B}}$, with underlying object $|\mathsf{U}_{\mathbb{B}}| = B$. By Theorem **??**, $\mathbb{B}$ has the universal property that finite product preserving (FP) functors from $\mathbb{B}$ into any FP-category $\mathcal{C}$ correspond (pseudo-)naturally to Boolean algebras in $\mathcal{C}$,

$$\mathsf{Hom}_{\mathsf{FP}}(\mathbb{B}, \mathcal{C}) \simeq \mathsf{BA}(\mathcal{C})\,. \tag{2.3}$$

The correspondence is mediated by evaluating an FP functor $F : \mathbb{B} \to \mathcal{C}$ at (the underlying structure of) the Boolean algebra $\mathsf{U}_{\mathbb{B}}$ to get a Boolean algebra $F(\mathsf{U}_{\mathbb{B}})$ in $\mathcal{C}$:

$$\frac{F : \mathbb{B} \longrightarrow \mathcal{C} \qquad \mathsf{FP}}{\overline{F(\mathsf{U}_{\mathbb{B}}) \qquad\quad \mathsf{BA}(\mathcal{C})}}$$

We call $\mathsf{U}_{\mathbb{B}}$ the *universal Boolean algebra*. Given a Boolean algebra B in $\mathcal{C}$, we write

$$\mathsf{B}^{\sharp} : \mathbb{B} \longrightarrow \mathcal{C}$$

for the associated *classifying functor*. By the equivalence of categories (2.3), we have isos,

$$\mathsf{B}^{\sharp}(\mathsf{U}_{\mathbb{B}}) \cong \mathsf{B}, \qquad F(\mathsf{B})^{\sharp} \cong F\,.$$

And in particular, $\mathsf{B}^{\sharp} \cong 1_{\mathbb{B}} : \mathbb{B} \to \mathbb{B}$.

By (the logical form of) Lawvere duality, Corollary **??**, we know that $\mathbb{B}^{\mathsf{op}}$ can be identified with a full subcategory $\mathsf{mod}(\mathbb{B})$ of $\mathbb{B}$-models in Set (i.e. Boolean algebras),

$$\mathbb{B}^{\mathsf{op}} = \mathsf{mod}(\mathbb{B}) \hookrightarrow \mathsf{Mod}(\mathbb{B}) = \mathsf{BA}(\mathsf{Set})\,, \tag{2.4}$$

namely, that consisting of the finitely generated free Boolean algebras $F(n) = PP([n])$ for $[n]$ an n-element set. Composing (2.4) and (2.3), we have an embedding of $\mathbb{B}^{\mathsf{op}}$ into the functor category,

$$\mathbb{B}^{\mathsf{op}} \hookrightarrow \mathsf{BA}(\mathsf{Set}) \simeq \mathsf{Hom}_{\mathsf{FP}}(\mathbb{B}, \mathsf{Set}) \hookrightarrow \mathsf{Set}^{\mathbb{B}}\,, \tag{2.5}$$

which, up to isomorphism, is just the (contravariant) Yoneda embedding, taking $B^n \in \mathbb{B}$ to the covariant representable functor $\mathsf{y}^{\mathbb{B}}(B^n) = \mathsf{Hom}_{\mathbb{B}}(B^n, -)$ (cf. Theorem **??**).

Now let us consider provability of equations between terms $\phi : B^n \to B$ in the theory $\mathbb{B}$, which are essentially the same as propositional formulas in context $(\mathsf{p}_1, ..., \mathsf{p}_n \mid \phi)$ modulo $\mathbb{B}$-provable equality. The universal Boolean algebra $\mathsf{U}_{\mathbb{B}}$ is logically generic, in the sense that for any such formulas ϕ, ψ, we have $\mathsf{U}_{\mathbb{B}} \vDash \phi = \psi$ just if $\mathbb{B} \vdash \phi = \psi$ (Proposition **??**). The latter condition is equational provability from the axioms for Boolean algebras, which was used in the definition of $\vdash_{\mathsf{eq}} \phi$ (cf. 2.2). So we have:

$$\vdash_{\mathsf{eq}} \phi \iff \mathbb{B} \vdash \phi = 1 \iff \mathsf{U}_{\mathbb{B}} \vDash \phi = 1\,.$$

As we showed in Proposition **??**, the image of the universal model $\mathsf{U}_{\mathbb{B}}$ under the (FP) *covariant* Yoneda embedding,

$$\mathsf{y}_{\mathbb{B}} : \mathbb{B} \to \mathsf{Set}^{\mathbb{B}^{\mathsf{op}}}$$

is also a logically generic model, with underlying object $|\mathsf{y}_{\mathbb{B}}(\mathsf{U}_{\mathbb{B}})| = \mathsf{Hom}_{\mathbb{B}}(-, B)$. By Proposition **??** we can use that fact to restrict attention to Boolean algebras in Set, and in particular, to the finitely generated free ones $F(n)$, when testing for equational provability. Specifically, using the (FP) evaluation functors $\mathsf{eval}_{B^n} : \mathsf{Set}^{\mathbb{B}^{\mathsf{op}}} \to \mathsf{Set}$ for all objects $B^n \in \mathbb{B}$, we can continue the above reasoning as follows:

$$\begin{aligned}
\vdash_{\mathsf{eq}} \phi &\iff \mathbb{B} \vdash \phi = 1 \\
&\iff \mathsf{U}_{\mathbb{B}} \vDash \phi = 1 \\
&\iff \mathsf{y}_{\mathbb{B}}(\mathsf{U}_{\mathbb{B}}) \vDash \phi = 1 \\
&\iff \mathsf{eval}_{B^n}\mathsf{y}_{\mathbb{B}}(\mathsf{U}_{\mathbb{B}}) \vDash \phi = 1 \quad \text{for all } B^n \in \mathbb{B} \\
&\iff F(n) \vDash \phi = 1 \quad \text{for all } n.
\end{aligned}$$

The last step holds because the image of $\mathsf{y}_{\mathbb{B}}(\mathsf{U}_{\mathbb{B}})$ under eval_{B^n} is exactly the free Boolean algebra $\mathsf{eval}_{B^n}\mathsf{y}_{\mathbb{B}}(\mathsf{U}_{\mathbb{B}}) = F(n)$ (cf. Exercise **??**). Indeed, for the underlying objects we have

$$\mathsf{eval}_{B^n}\mathsf{y}_{\mathbb{B}}(\mathsf{U}_{\mathbb{B}}) \cong \mathsf{Hom}_{\mathbb{B}}(B^n, B) \cong \mathsf{Hom}_{\mathsf{BA}^{\mathsf{op}}}(F(n), F(1)) \cong \mathsf{Hom}_{\mathsf{BA}}(F(1), F(n)) \cong |F(n)|\,.$$

Thus to test for equational provability it suffices to check the equations in the free algebras $F(n)$ (which makes sense, since $F(n)$ is usually *defined* in terms of equational provability). We have therefore shown:

Lemma 2.4.1. *A formula in context $\mathsf{p}_1, ..., \mathsf{p}_k \mid \phi$ is equationally provable $\vdash_{\mathsf{eq}} \phi$ just in case it holds in every finitely generated free Boolean algebra $F(n)$,* i.e. $F(n) \vDash \phi = 1$.

Recall that the condition $F(n) \vDash \phi = 1$ means that the equation $\phi = 1$ holds *generally* in $F(n)$, i.e. for any elements $f_1, ..., f_k \in F(n)$, we have $\phi[f_1/\mathsf{p}_1, ..., f_k/\mathsf{p}_k] = 1$, where the expression $\phi[f_1/\mathsf{p}_1, ..., f_k/\mathsf{p}_k]$ denotes the element of $F(n)$ resulting from interpreting the propositional variables p_i as the elements f_i and evaluating the resulting expression using the Boolean operations of $F(n)$. But now observe that the recipe:

for any elements $f_1, ..., f_k \in F(n)$, let the expression

$$\phi[f_1/\mathtt{p}_1, ..., f_k/\mathtt{p}_k] \tag{2.6}$$

denote the element of $F(n)$ resulting from interpreting the propositional variables $\mathtt{p}_i$ as the elements f_i and evaluating the resulting expression using the Boolean operations of $F(n)$

just describes the unique Boolean homomorphism

$$F(1) \xrightarrow{\overline{\phi}} F(k) \xrightarrow{\overline{(f_1, ..., f_k)}} F(n)\,,$$

where $\overline{(f_1, ..., f_k)} : F(k) \to F(n)$ is determined by the elements $f_1, ..., f_k \in F(n)$, and $\overline{\phi} : F(1) \to F(k)$ by the corresponding element $(\mathtt{p}_1, ..., \mathtt{p}_k \mid \phi) \in F(k)$. It is therefore equivalent to check the case $k = n$ and $f_i = \mathtt{p}_i$, i.e. the "universal case"

$$(\mathtt{p}_1, ..., \mathtt{p}_k \mid \phi) = 1 \quad \text{in } F(k)\,. \tag{2.7}$$

Finally, then, we have:

Proposition 2.4.2 (Boolean-valued completeness of the equational propositional calculus)**.** *Equational propositional calculus is sound and complete with respect to* boolean-valued models *in* Set*, in the sense that a propositional formula* ϕ *is equationally provable from the laws of Boolean algebra,*

$$\vdash_{\mathsf{eq}} \phi\,,$$

just if it holds generally in any Boolean algebra (in Set*), which we may denote*

$$\vDash_{\mathsf{BA}} \phi\,.$$

Proof. By "holding generally" is meant that it holds for all elements of the Boolean algebra B, in the sense displayed after the Lemma. But, as above, this is equivalent to the condition that for all $b_1, ..., b_k \in \mathsf{B}$, for $\overline{(b_1, ..., b_k)} : F(k) \to \mathsf{B}$ we have $\overline{(b_1, ..., b_k)}(\phi) = 1$ in B, which in turn is clearly equivalent to the previously determined "universal" condition (2.7) that $\phi = 1$ in $F(k)$. $\square$

We leave the analogous statement for equational entailment $\Phi \vdash_{\mathsf{eq}} \phi$ and Boolean-valued entailment $\Phi \vDash_{\mathsf{BA}} \phi$ as an exercise.

Corollary 2.4.3. *Show that a propositional formula* $\mathtt{p}_1, ..., \mathtt{p}_k \mid \phi$ *is equationally provable* $\vdash_{\mathsf{eq}} \phi$*, just if it holds in the free Boolean algebra* $F(\omega)$ *on countably many generators* $\omega = \{\mathtt{p}_1, \mathtt{p}_2, ...\}$*, with the variables* $\mathtt{p}_1, ..., \mathtt{p}_k$ *interpreted as the corresponding generators of* $F(\omega)$*.*

Exercise 2.4.4. Prove this as an easy corollary of Proposition 2.4.2.

Let us summarize what we know so far. By Exercise **??**, we already knew that equational provability in Boolean algebra is equivalent to semantic validity,

$$\vdash_{\mathsf{eq}} \phi \iff \vDash \phi\,.$$

This was based on a certain *decision procedure* for validity in classical propositional logic, originally due to Bernays [**?**], restated in terms of Boolean algebra. Like the classical proof of the Completeness Theorem 2.2.1,

$$\vdash \phi \iff \vDash \phi\,,$$

we would like to analyze this result, too, in general categorical terms, in order to be able to extend and generalize it to other systems of logic.

Our algebraic approach via Lawvere duality resulted in Proposition 2.4.2, which says that equational provability is equivalent to what we have called *Boolean-valued validity*,

$$\vdash_{\mathsf{eq}} \phi \iff \vDash_{\mathsf{BA}} \phi \quad\iff\quad \mathsf{B} \vDash \phi \quad \text{for all } \mathsf{B}\,. \tag{2.8}$$

This is essentially the Boolean algebra case of our Proposition **??**, the completeness of equational reasoning with respect to algebras in Set, originally proved by Birkhoff.

It still remains to relate equational provability $\vdash_{\mathsf{eq}} \phi$ with deduction $\vdash \phi$, and Boolean-valued validity $\vDash_{\mathsf{BA}} \phi$ with semantic validity $\vDash \phi$, which is just the special case $\mathbb{2} \vDash_{\mathsf{BA}} \phi$. We shall consider deduction $\vdash \phi$ via a different approach in the following section, one that regards Boolean algebras as special finite product categories, rather than special Lawvere algebraic theories.

Exercise 2.4.5. For a formula in context $\mathsf{p}_1, ..., \mathsf{p}_k \mid \vartheta$ and a Boolean algebra B, let the expression $\vartheta[b_1/\mathsf{p}_1, ..., b_k/\mathsf{p}_k]$ denote the element of B resulting from interpreting the propositional variables p_i in the context as the elements b_i of B, and evaluating the resulting expression using the Boolean operations of B. For any *finite* set of propositional formulas Φ and any formula ψ, let $\Gamma = \mathsf{p}_1, ..., \mathsf{p}_k$ be a context for (the formulas in) $\Phi \cup \{\psi\}$. Finally, recall that $\Phi \vdash_{\mathsf{eq}} \psi$ means that $\psi = 1$ is equationally provable from the set of equations $\{\phi = 1 \mid \phi \in \Phi\}$. Show that $\Phi \vdash_{\mathsf{eq}} \psi$ just if for all finitely generated free Boolean algebras $F(n)$, the following condition holds:

> For any elements $f_1, ..., f_k \in F(n)$, if $\phi[f_1/\mathsf{p}_1, ..., f_k/\mathsf{p}_k] = 1$ for all $\phi \in \Phi$, then $\psi[f_1/\mathsf{p}_1, ..., f_k/\mathsf{p}_k] = 1$.

Is it sufficient to just take $F(k)$ and its generators $\mathsf{p}_1, ..., \mathsf{p}_k$ as the $f_1, ..., f_k$? Is it equivalent to take all Boolean algebras B, rather than the finitely generated free ones $F(n)$? Determine a condition that is equivalent to $\Phi \vdash_{\mathsf{eq}} \psi$ for not necessarily finite sets Φ.

2.5 Functorial semantics for propositional logic

Considering the algebraic theory of Boolean algebras suggested the idea of a Boolean valuation of propositional logic, generalizing the truth valuations of section 2.2. This can be seen as applying the framework of functorial semantics to a different system of logic than that of equational theories, represented as finite product categories, namely that represented categorically by *poset* categories with finite products $\wedge$ *and coproducts* $\vee$ (each of these cases could, of course, also be considered separately, giving $\wedge$-semi-lattices and categories with finite products $\times$ and coproducts $+$, respectively). Thus we are moving from the top right corner to the bottom center position in the following Hasse diagram of structured categories:

$$\begin{tikzcd}[arrows=dash]
+ \arrow[dr] \arrow[dd] & & \times \arrow[dl] \arrow[dd] & \mathsf{Cat} \\
& +\,,\times \arrow[dd] & & \\
\vee \arrow[dr] & & \wedge \arrow[dl] & \mathsf{Pos} \\
& \vee\,,\wedge & &
\end{tikzcd}$$

In Chapter **??** we shall see how first-order logic results categorically from these two cases by "indexing the lower one over the upper one", and in Chapters **??** and **??** we shall consider simple and dependent type theory as "categorified" versions of propositional and first-order logic. It is for this reason (rather than a dogmatic commitment to categorical methods!) that we continue our reformulation of the basic results of classical propositional logic in functorial terms.

Definition 2.5.1. A *propositional theory* $\mathbb{T}$ consists of a set $V_{\mathbb{T}}$ of propositional variables, called the *basic* or *atomic propositions*, and a set $A_{\mathbb{T}}$ of propositional formulas (over $V_{\mathbb{T}}$), called the *axioms*. The *consequences* $\Phi \vdash_{\mathbb{T}} \phi$ are those judgements that are derivable by natural deduction (as in Section 2.1.1), from the axioms $A_{\mathbb{T}}$.

Definition 2.5.2. Let $\mathbb{T} = (V_{\mathbb{T}}, A_{\mathbb{T}})$ be a propositional theory and $\mathcal{B}$ a Boolean algebra. A *model* of $\mathbb{T}$ in $\mathcal{B}$, also called a *Boolean valuation of* $\mathbb{T}$ is an *interpretation function* $v : V_{\mathbb{T}} \to |\mathcal{B}|$ such that, for every $\alpha \in A_{\mathbb{T}}$, we have $[\![\alpha]\!]^v = 1_{\mathcal{B}}$ in $\mathcal{B}$, where the extension

$[\![-]\!]^v$ of v from $V_{\mathbb{T}}$ to all formulas (over $V_{\mathbb{T}}$) is defined in the expected way, namely:

$$\begin{aligned}
[\![\mathsf{p}]\!]^v &= v(\mathsf{p}), \quad \mathsf{p} \in V_{\mathbb{T}} \\
[\![\top]\!]^v &= 1_{\mathcal{B}} \\
[\![\bot]\!]^v &= 0_{\mathcal{B}} \\
[\![\neg\phi]\!]^v &= \neg_{\mathcal{B}}[\![\phi]\!]^v \\
[\![\phi \wedge \psi]\!]^v &= [\![\phi]\!]^v \wedge_{\mathcal{B}} [\![\psi]\!]^v \\
[\![\phi \vee \psi]\!]^v &= [\![\phi]\!]^v \vee_{\mathcal{B}} [\![\psi]\!]^v \\
[\![\phi \Rightarrow \psi]\!]^v &= \neg_{\mathcal{B}}[\![\phi]\!]^v \vee_{\mathcal{B}} [\![\psi]\!]^v
\end{aligned}$$

Finally, let $\mathsf{Mod}(\mathbb{T}, \mathcal{B})$ be the set of all $\mathbb{T}$-models in $\mathcal{B}$. Given a Boolean homomorphism $f : \mathcal{B} \to \mathcal{B}'$, there is an induced mapping $\mathsf{Mod}(\mathbb{T}, f) : \mathsf{Mod}(\mathbb{T}, \mathcal{B}) \to \mathsf{Mod}(\mathbb{T}, \mathcal{B}')$, determined by setting $\mathsf{Mod}(\mathbb{T}, f)(v) = f \circ v$, which is clearly functorial.

Theorem 2.5.3. *The functor* $\mathsf{Mod}(\mathbb{T}) : \mathsf{BA} \to \mathsf{Set}$ *is representable, with representing Boolean algebra* $\mathcal{B}_{\mathbb{T}}$*, the* classifying Boolean algebra *of* $\mathbb{T}$.

The classifying Boolean algebra $\mathcal{B}_{\mathbb{T}}$ is closely related to the conventional *Lindenbaum-Tarski algebra* of $\mathbb{T}$.

Proof. We construct $\mathcal{B}_{\mathbb{T}}$ from the "syntax of $\mathbb{T}$" in two steps:

Step 1: Suppose first that $A_{\mathbb{T}}$ is empty, so $\mathbb{T}$ is just a set V of propositional variables. Then define the classifying Boolean algebra $\mathcal{B}[V]$ by

$$\mathcal{B}[V] \;=\; \{\phi \mid \phi \text{ is a formula in context } V\}/\!\sim$$

where the equivalence relation $\sim$ is *(deductively) provable bi-implication*,

$$\phi \sim \psi \iff \vdash \psi \Leftrightarrow \psi.$$

The operations are (well-)defined on equivalence classes by setting,

$$[\phi] \wedge [\psi] \;=\; [\phi \wedge \psi],$$

and so on. (The reader who has not seen this construction before should fill in the details!)

Step 2: In the general case $\mathbb{T} = (V_{\mathbb{T}}, A_{\mathbb{T}})$, let

$$\mathcal{B}_{\mathbb{T}} = \mathcal{B}[V_{\mathbb{T}}]/\!\sim_{\mathbb{T}},$$

where the equivalence relation $\sim_{\mathbb{T}}$ is now $A_{\mathbb{T}}$*-provable bi-implication*,

$$\phi \sim_{\mathbb{T}} \psi \iff A_{\mathbb{T}} \vdash \psi \Leftrightarrow \psi.$$

The operations are defined as before, but now on equivalence classes $[\phi]$ modulo $A_{\mathbb{T}}$.

Observe that the construction of $\mathcal{B}_{\mathbb{T}}$ is a variation on that of the *syntactic category* construction $\mathcal{C}_{\mathbb{T}} = \mathsf{Syn}(\mathbb{T})$ of the classifying category of an algebraic theory $\mathbb{T}$, in the sense of the previous chapter. Indeed, the statement of the theorem is just the universal property of $\mathcal{B}_{\mathbb{T}}$ as the classifying category of $\mathbb{T}$-models, namely

$$\mathsf{Hom}_{\mathsf{BA}}(\mathcal{B}_{\mathbb{T}}, \mathcal{B}) \;\cong\; \mathsf{Mod}(\mathbb{T}, \mathcal{B})\,, \tag{2.9}$$

naturally in $\mathcal{B}$. (Since $\mathsf{Mod}(\mathbb{T}, \mathcal{B})$ is now a *set* rather than a category, we can classify it up to *isomorphism* rather than equivalence of categories.) The proof of this fact is a variation on the proof of the corresponding theorem **??** from Chapter 1. Further details are given in the following Remark 2.5.5 for the interested reader. □

Remark 2.5.4. The Lindenbaum-Tarski algebra of a propositional theory is usually defined in semantic terms using (truth) valuations. Our definition of $\mathcal{B}_{\mathbb{T}}$ in terms of *provability* is more useful in the present setting, as it parallels that of the syntactic category $\mathcal{C}_{\mathbb{T}}$ of an algebraic theory, and will allow us to prove Theorem 2.2.1 by analogy to Theorem **??** for algebraic theories.

Remark 2.5.5 (Adjoint Rules for Propositional Calculus)**.** For the construction of the classifying algebra $\mathcal{B}_{\mathbb{T}}$, it is convenient to reformulate the rules of inference for the propositional calculus in the following equivalent *adjoint form*: Contexts Γ may be omitted, since the rules leave them unchanged (there is no variable binding). We may also omit hypotheses that remain unchanged. Thus e.g. the *hypothesis* rule may be written in any of the following equivalent ways.

$$\frac{}{\Gamma \mid \phi_1, \ldots, \phi_m \vdash \phi_i} \qquad\qquad \frac{}{\phi_1, \ldots, \phi_m \vdash \phi_i} \qquad\qquad \frac{}{\phi \vdash \phi}$$

The structural rules can then be stated as follows:

$$\frac{}{\phi \vdash \phi} \qquad\qquad \frac{\phi \vdash \psi \qquad \psi \vdash \vartheta}{\phi \vdash \vartheta}$$

$$\frac{\phi \vdash \vartheta}{\psi, \phi \vdash \vartheta} \qquad\qquad \frac{\phi, \phi \vdash \vartheta}{\phi \vdash \vartheta} \qquad\qquad \frac{\phi, \psi \vdash \vartheta}{\psi, \phi \vdash \vartheta}$$

The rules for the propositional connectives can be given in the following adjoint form, where the double line indicates a two-way rule (with the obvious two instances when there are two conclusions, in going from bottom to top).

$$\frac{}{\phi \vdash \top} \qquad\qquad \frac{}{\bot \vdash \phi}$$

$$\genfrac{}{}{1.5pt}{}{\vartheta \vdash \phi \quad \vartheta \vdash \psi}{\vartheta \vdash \phi \wedge \psi} \qquad\qquad \genfrac{}{}{1.5pt}{}{\phi \vdash \vartheta \quad \psi \vdash \vartheta}{\phi \vee \psi \vdash \vartheta} \qquad\qquad \genfrac{}{}{1.5pt}{}{\vartheta, \phi \vdash \psi}{\vartheta \vdash \phi \Rightarrow \psi}$$

For the purpose of deduction, negation $\neg\phi$ is again treated as defined by $\phi \Rightarrow \bot$ and bi-implication $\phi \Leftrightarrow \psi$ by $(\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi)$. For *classical* logic we also include the rule of *double negation*:

$$\frac{}{\neg\neg\phi \vdash \phi} \tag{2.10}$$

It is now obvious that the set of formulas is preordered by $\phi \vdash \psi$, and that the poset reflection agrees with the deducibility equivalence relation,

$$\phi \dashv\vdash \psi \iff \phi \sim \psi\,.$$

Moreover, $\mathcal{B}_{\mathbb{T}}$ clearly has all finite limits $\top, \wedge$ and colimits $\bot, \vee$, is cartesian closed $\wedge \dashv \Rightarrow$, and is therefore a *Heyting algebra* (see Section **??** below). The rule of double negation then makes it a Boolean algebra.

The proof of the universal property of $\mathcal{B}_{\mathbb{T}}$ is essentially the same as that for $\mathcal{C}_{\mathbb{T}}$.

Exercise 2.5.6. Fill in the details of the proof that $\mathcal{B}_{\mathbb{T}}$ is a well-defined Boolean algebra, with the universal property stated in (2.9). (*Hint:* The well-definedness of the operations $[\phi] \wedge [\psi]$, etc., just requires a few deductions, but the well-definedness of the Boolean homomorphism $v^\sharp : \mathcal{B}_{\mathbb{T}} \to \mathcal{B}$ classifying a model $v : V_{\mathbb{T}} \to |\mathcal{B}|$ requires the *soundness* of deduction with respect to Boolean-valued semantics. Just state this precisely and sketch a proof of it.)

Just as for the case of algebraic theories and FP categories, we now have the following corollary of the classifying theorem 2.5.3, which again follows from the fact that the classifying Boolean algebra $\mathcal{B}_{\mathbb{T}}$ is *logically generic*, in virtue of its syntactic construction.

Corollary 2.5.7. *For any formula ϕ, derivability from the axioms $A_{\mathbb{T}} \vdash \phi$ is equivalent to validity under all Boolean-valued models of $\mathbb{T}$,*

$$A_{\mathbb{T}} \vdash \phi \iff A_{\mathbb{T}} \vDash_{\mathsf{BA}} \phi\,.$$

Proof. We have

$$A_{\mathbb{T}} \vdash \phi \iff \mathcal{B}_{\mathbb{T}} \vDash_{\mathsf{BA}} \phi\,,$$

essentially by definition, where on the righthand side it suffices to check the canonical model $u : V_{\mathbb{T}} \to |\mathcal{B}_{\mathbb{T}}|$ associated to the identity $\mathcal{B}_{\mathbb{T}} \to \mathcal{B}_{\mathbb{T}}$. But if $u \vDash_{\mathsf{BA}} \phi$, then also $v \vDash_{\mathsf{BA}} \phi$ for any $v : V_{\mathbb{T}} \to |\mathcal{B}|$, since $v = v^\sharp u$, and the homomorphism $v^\sharp : \mathcal{B}_{\mathbb{T}} \to \mathcal{B}$ preserves models. Thus $\mathcal{B}_{\mathbb{T}} \vDash_{\mathsf{BA}} \phi \;\Rightarrow\; A_{\mathbb{T}} \vDash_{\mathsf{BA}} \phi$. The converse is immediate. □

Note that the recipe displayed at (2.6) for a Boolean valuation in $F(n)$ of a formula in context $\mathsf{p}_1, ..., \mathsf{p}_k \mid \phi$ is exactly the (canonical) *model* in $F(n)$, with underlying valuation $\{\mathsf{p}_1, ..., \mathsf{p}_k\} \to F(n)$, of the theory $\mathbb{T} = \{\mathsf{p}_1, ..., \mathsf{p}_k\}$. So

$$F(n) \vDash_{\mathsf{BA}} \phi \iff [\![\phi]\!] = 1 \text{ in } F(n)\,.$$

Inspecting the universal property (2.9) of $\mathcal{B}_{\mathbb{T}}$ for the case $\mathbb{T} = \{\mathsf{p}_1, ..., \mathsf{p}_n\}$, we obtain:

Corollary 2.5.8. *The classifying Boolean algebra for the theory* $\{\mathtt{p}_1, ..., \mathtt{p}_n\}$ *is the finitely generated, free Boolean algebra,*

$$\mathcal{B}[\mathtt{p}_1, ..., \mathtt{p}_n] \cong F(n)\,,$$

(which, recall, is the double powerset $PP[n]$*). And generally,* $\mathcal{B}[V]$ *is the free Boolean algebra on the set* V*, for any set* V*.*

Indeed, for any valuation (= arbitrary function) $v : \{\mathtt{p}_1, ..., \mathtt{p}_n\} \to |\mathcal{B}|$ we have a unique extension $[\![-]\!]^v : \mathcal{B}[\mathtt{p}_1, ..., \mathtt{p}_n] \to \mathcal{B}$, which upon inspection of Definition 2.5.2 we recognize as exactly a Boolean homomorphism.

$$\begin{tikzcd}
\mathcal{B}[\mathtt{p}_1, ..., \mathtt{p}_n] \arrow[r, "{[\![-]\!]^v}"] & \mathcal{B} \\
\{\mathtt{p}_1, ..., \mathtt{p}_n\} \arrow[u] \arrow[ur, "v"'] &
\end{tikzcd}$$

The isomorphism $\mathcal{B}[\mathtt{p}_1, ..., \mathtt{p}_n] \cong F(n)$ of Corollary 2.5.7 expresses the fact that the relations of derivability by natural deduction $\Phi \vdash \phi$ and equational provability $\Phi \vdash_{\mathsf{eq}} \phi$ agree,

$$\Phi \vdash \phi \iff \Phi \vdash_{\mathsf{eq}} \phi\,, \tag{2.11}$$

answering one of the two questions from the end of Section 2.4.

Toward answering the other question of the relation between Boolean-valued validity $\Phi \vDash_{\mathsf{BA}} \phi$ and truth-valued validity $\Phi \vDash \phi$, consider the *finitely presented* Boolean algebras, which can be described as those of the form

$$\mathcal{B}_{\mathbb{T}} = \mathcal{B}[\mathtt{p}_1, ..., \mathtt{p}_n]/\alpha\,,$$

for a finite theory $\mathbb{T} = (\mathtt{p}_1, ..., \mathtt{p}_n; \alpha_1, ..., \alpha_m)$, where the slice category of a Boolean algebra $\mathcal{B}$ over an element $\beta \in \mathcal{B}$ is the *downset* (or *principal ideal*)

$$\mathcal{B}/\beta = \downarrow(\beta) = \{b \in \mathcal{B} \mid b \leq \beta\}\,.$$

To see this, given $\mathbb{T} = (V_{\mathbb{T}}, A_{\mathbb{T}})$, if $A_{\mathbb{T}}$ is finite, then let

$$\alpha_{\mathbb{T}} := \bigwedge_{\alpha \in A_{\mathbb{T}}} \alpha\,,$$

so we clearly have

$$\mathcal{B}_{\mathbb{T}} = \mathcal{B}[V_{\mathbb{T}}]/\alpha_{\mathbb{T}}\,.$$

If $V_{\mathbb{T}} = \{\mathtt{p}_1, ..., \mathtt{p}_n\}$ is also finite, then we have

$$\mathcal{B}_{\mathbb{T}} \cong \mathcal{B}[\mathtt{p}_1, ..., \mathtt{p}_n]/\alpha_{\mathbb{T}}\,.$$

It is now easy to show that the finitely presented objects in the category of Boolean algebras are exactly those of the form $\mathcal{B}[\mathsf{p}_1, ..., \mathsf{p}_n]/\alpha_{\mathbb{T}}$, using the fact that a (Boolean) algebra A is finitely presented if and only if it has a presentation (by n-many generators and m-many equations) as a coequalizer of finitely generated free algebras,

$$F(m) \rightrightarrows F(n) \longrightarrow A\,. \tag{2.12}$$

Exercise 2.5.9. Show that the classifying Boolean algebras $\mathcal{B}_{\mathbb{T}}$, for *finite sets* $V_{\mathbb{T}}$ of variables and $A_{\mathbb{T}}$ of formulas, are exactly the *finitely presented* ones in the sense stated in (2.12). In general algebraic categories $\mathcal{A}$ such coequalizers of finitely generated free algebras are exactly those for which the representable functor $\mathsf{Hom}(A, -) : \mathcal{A} \to \mathsf{Set}$ preserves all filtered colimits. Show that the finitely presented Boolean algebras in the sense of (2.12) do indeed have this property.

The following is a special case of the universal property of the slice category

$$X^* : \mathbb{C} \to \mathbb{C}/_X\,,$$

for any $\mathbb{C}$ with finite limits. The reader not already familiar with this fact should definitely do the exercise!

Exercise 2.5.10. For any Boolean algebra $\mathcal{B}$ and any $\beta \in \mathcal{B}$, consider the map

$$\beta^* : \mathcal{B} \to \mathcal{B}/\beta\,,$$

with $\beta^*(x) = \beta \wedge x$.

(i) Show that $\mathcal{B}/\beta \cong \downarrow(\beta)$ is a Boolean algebra, and that β^* is a Boolean homomorphism with $\beta^*(\beta) = 1 \in \mathcal{B}/\beta$.

(ii) If $h : \mathcal{B} \to \mathcal{B}'$ is any homomorphism, then $h(\beta) = 1 \in \mathcal{B}'$ if and only if there is a factorization

$$\begin{tikzcd} \mathcal{B} \arrow[r, "h"] \arrow[d, "\beta^*"'] & \mathcal{B}' \\ \mathcal{B}/\beta \arrow[ur, "\overline{h}"'] & \end{tikzcd} \tag{2.13}$$

of h through β^*, and then $\overline{h}$ is unique with $\overline{h} \circ \beta^* = h$.

(iii) Show that if $\mathcal{B}_{\mathbb{T}} = \mathcal{B}[\mathsf{p}_1, ..., \mathsf{p}_n]/\alpha$ classifies (models of) the theory $\mathbb{T} = (\mathsf{p}_1...\mathsf{p}_n, \alpha)$ and $\mathsf{p}_1, ..., \mathsf{p}_n \mid \beta$, then $\mathcal{B}_{\mathbb{T}}/\beta$ classifies models of the extended theory $\mathbb{T}' = (\mathsf{p}_1...\mathsf{p}_n, \alpha, \beta)$.

Lemma 2.5.11. *Let* $\mathcal{B}[\mathsf{p}_1, ..., \mathsf{p}_n]/\alpha$ *be a* finitely presented *Boolean algebra in which* $0 \neq 1$. *Then there is a Boolean homomorphism*

$$h : \mathcal{B}[\mathsf{p}_1, ..., \mathsf{p}_n]/\alpha \to \mathbb{2}\,.$$

Proof. By Exercise 2.5.9, we can assume that $\mathcal{B}[\mathtt{p}_1, ..., \mathtt{p}_n]/\alpha = \mathcal{B}_{\mathbb{T}}$ classifies (models of) $\mathbb{T} = (\mathtt{p}_1 ... \mathtt{p}_n, \alpha)$. By the assumption that $0 \neq 1$ in $\mathcal{B}[\mathtt{p}_1, ..., \mathtt{p}_n]/\alpha$, we must have $\alpha \neq 0$ in the free Boolean algebra $\mathcal{B}[\mathtt{p}_1, ..., \mathtt{p}_n]$. It then suffices to give a valuation $v : \{\mathtt{p}_1, ..., \mathtt{p}_n\} \to 2$ such that $[\![\alpha]\!]^v = 1$, for then (by Exercise 2.5.10) we will have a factorization,

$$\begin{tikzcd} \mathcal{B}[\mathtt{p}_1, ..., \mathtt{p}_n] \arrow[r, "m_v"] \arrow[d, "\alpha^*"'] & \mathbb{2} \\ \mathcal{B}[\mathtt{p}_1, ..., \mathtt{p}_n]/\alpha \arrow[ur, "\overline{m_v}"'] & \end{tikzcd} \tag{2.14}$$

where $m_v = [\![-]\!]^v$ is the "model" associated to the valuation $v : \{\mathtt{p}_1, ..., \mathtt{p}_n\} \to 2$, and $\alpha \wedge - : \mathcal{B}[\mathtt{p}_1, ..., \mathtt{p}_n] \to \mathcal{B}[\mathtt{p}_1, ..., \mathtt{p}_n]/\alpha$ is the canonical Boolean projection to the "quotient" Boolean algebra given by the slice category, and $\overline{m_v}$ is the extension of m_v along α^* resulting from the universal property of slicing a category with finite products. (Informally, α has a truth table with 2^n rows, corresponding to the valuations $v : \{\mathtt{p}_1, ..., \mathtt{p}_n\} \to 2$, and we know that the main column for α is not all 0's, so we can find a row in which it is 1 and read off the corresponding valuation.) More formally, as in Remark 2.3.4, we can put α into a disjunctive normal form $\alpha = \alpha_1 \vee ... \vee \alpha_k$ and one of the disjuncts α_i must then also be non-zero. Since $\alpha_i = q_1 \wedge ... \wedge q_m$ with each q_j either positive $\mathtt{p}$ or negative $\neg\mathtt{p}$, if both $\mathtt{p}$ and $\neg\mathtt{p}$ occur, then $\alpha_i = 0$, so the $\mathtt{p}$ in each q_j must occur only once in α_i. We can then define v accordingly, with $v(\mathtt{p}) = 1$ iff $\mathtt{p}$ occurs positively in α_i, and we will have $[\![\alpha_i]\!]^v = 1$. This valuation $v : \{\mathtt{p}_1, ..., \mathtt{p}_n\} \to \mathbb{2}$ then determines a Boolean homomorphism $[\![-]\!]^v : \mathcal{B}[\mathtt{p}_1 ... \mathtt{p}_n] \to \mathbb{2}$ with $[\![\alpha]\!]^v = 1$, as required for a homomorphism

$$\mathcal{B}[\mathtt{p}_1 ... \mathtt{p}_n]/\alpha \to \mathbb{2}\,.$$

□

Proposition 2.5.12. *For any formula ϕ, Boolean-valued validity and truth-valued validity are equivalent,*

$$\vDash_{\mathsf{BA}} \phi \iff \vDash \phi\,. \tag{2.15}$$

Proof. Since $\vDash_{\mathsf{BA}} \phi$ means that $\mathcal{B} \vDash_{\mathsf{BA}} \phi$ for all Boolean algebras $\mathcal{B}$, and $\vDash \phi$ means the same for valuations in $\mathbb{2}$, the implication from left to right is trivial. For the converse, let $(\mathtt{p}_1, ..., \mathtt{p}_n \,|\, \phi)$, and consider $\phi \in \mathcal{B}[\mathtt{p}_1, ..., \mathtt{p}_n]$. If $h(\phi) = 1$ for all homomorphisms $h : \mathcal{B}[\mathtt{p}_1, ..., \mathtt{p}_n] \to \mathbb{2}$, then $\mathcal{B}[\mathtt{p}_1, ..., \mathtt{p}_n]/\neg\phi$ can have no homomorphism $\overline{h} : \mathcal{B}[\mathtt{p}_1, ..., \mathtt{p}_n]/\neg\phi \to \mathbb{2}$ (else $\overline{h}(\neg\phi) = 1$ would give $h(\neg\phi) = 1$ and so $h(\phi) = 0$). Therefore $0 = 1$ in $\mathcal{B}[\mathtt{p}_1, ..., \mathtt{p}_n]/\neg\phi$ by Lemma 2.5.11. But then $0 = \neg\phi \wedge 1 = \neg\phi$ in $\mathcal{B}[\mathtt{p}_1, ..., \mathtt{p}_n]$, whence $\phi = \neg\neg\phi = \neg 0 = 1$, so $h(\phi) = 1 \in \mathcal{B}$ for all $h : \mathcal{B}[\mathtt{p}_1, ..., \mathtt{p}_n] \to \mathcal{B}$. □

Exercise 2.5.13. Extend Proposition 2.5.14 to entailment, for any finite set Φ,

$$\Phi \vDash_{\mathsf{BA}} \phi \iff \Phi \vDash \phi\,.$$

Combining this last result (2.15) with the previous one (2.11) and (2.8) from the last section, we arrive finally at our desired reconstruction of the classical completeness theorem:

Proposition 2.5.14. *For any formula ϕ, provability by deduction and truth-valued validity are equivalent,*

$$\vdash \phi \iff \vDash \phi\,. \tag{2.16}$$

And the same holds relative to a set Φ of premises.

Let us now unwind the foregoing "reproof" into a direct argument, from the present point of view: A formula ϕ in context $\mathsf{p}_1, ..., \mathsf{p}_n \mid \phi$ determines an element in the free Boolean algebra $\mathcal{B}[\mathsf{p}_1, ..., \mathsf{p}_n]$. If $\vdash \phi$ then $\phi = 1$ in $\mathcal{B}[\mathsf{p}_1, ..., \mathsf{p}_n]$, so clearly $h(\phi) = 1$ for every $h : \mathcal{B}[\mathsf{p}_1, ..., \mathsf{p}_n] \to \mathbb{2}$, which means exactly $\vDash \phi$. Conversely, if $\vDash \phi$ then $h(\phi) = 1$ for every $h : \mathcal{B}[\mathsf{p}_1, ..., \mathsf{p}_n] \to \mathbb{2}$, so $\neg\phi$ can have no model in $\mathbb{2}$. Thus $\mathcal{B}[\mathsf{p}_1, ..., \mathsf{p}_n]/\neg\phi$ must be degenerate, with $0 = 1$. So $[\bot] = [\neg\phi]$ and therefore $\neg\phi \vdash \bot$, so $\vdash \neg\neg\phi$, so $\vdash \phi$.

The main fact used here is that the finitely generated, free Boolean algebras $\mathcal{B}(n) = \mathcal{B}[\mathsf{p}_1, ..., \mathsf{p}_n]$ have enough Boolean homomorphisms $h : \mathcal{B}(n) \to \mathbb{2}$ to separate any non-zero element $\phi \neq 0$, in the sense that if $h(\phi) = 0$ for all such h then $\phi = 0$. In other words, the canonical homomorphism

$$\mathcal{B}(n) \longrightarrow \Pi_{h \in \mathcal{B}(n)^*} \mathbb{2}\,, \tag{2.17}$$

is injective, for $\mathcal{B}(n)^* = \mathsf{BA}(\mathcal{B}(n), \mathbb{2})$. This is reminiscent of the proof of completeness for algebraic theories, which also used an embedding of the syntactic category $\mathcal{C}_{\mathbb{T}}$ into a power of Set by a "sufficient" set of models $\mathcal{C}_{\mathbb{T}} \to \mathsf{Set}$,

$$\mathcal{C}_{\mathbb{T}} \hookrightarrow \mathsf{Set}^{\mathsf{mod}(\mathbb{T})}$$

namely those of the form $\mathcal{C}_{\mathbb{T}}(U^n, -) \cong \mathsf{mod}(\mathbb{T})(-, F(n)) : \mathcal{C}_{\mathbb{T}} \to \mathsf{Set}$. For Boolean algebras, the embedding (2.17) is the main point of the Stone Representation Theorem.

2.6 Stone representation

Regarding a Boolean algebra $\mathcal{B}$ as a category with finite products, consider its Yoneda embedding $\mathsf{y} : \mathcal{B} \hookrightarrow \mathsf{Set}^{\mathcal{B}^{\mathsf{op}}}$. Since the hom-set $\mathcal{B}(x, y)$ is always $\mathbb{2}$-valued, we have a factorization,

$$\mathsf{y} : \mathcal{B} \hookrightarrow \mathbb{2}^{\mathcal{B}^{\mathsf{op}}} \hookrightarrow \mathsf{Set}^{\mathcal{B}^{\mathsf{op}}} \tag{2.18}$$

in which each factor still preserves the finite products (note that the products in $\mathbb{2}$ are preserved by the inclusion $\mathbb{2} \hookrightarrow \mathsf{Set}$, and the products in the functor categories $\mathbb{2}^{\mathcal{B}^{\mathsf{op}}}$ and $\mathsf{Set}^{\mathcal{B}^{\mathsf{op}}}$ are taken pointwise). Indeed, this is an instance of a general fact. In the category $\mathsf{Cat}_\times$ of finite product categories (and $\times$-preserving functors), the inclusion of the full subcategory of posets with $\wedge$ (the $\wedge$-semilattices) has a *right adjoint* R, in addition to the left adjoint L of poset reflection.

$$\begin{tikzcd} \mathsf{Cat}_\times \arrow[d, bend right=60, "L"'] \arrow[d, bend left=60, "R"] \\ \mathsf{Pos}_\wedge \arrow[u, hook, "i"] \end{tikzcd}$$

For a finite product category $\mathbb{C}$, the poset $R\mathbb{C}$ is the subcategory $\mathsf{Sub}(1) \hookrightarrow \mathbb{C}$ of subobjects of the terminal object 1 (equivalently, the category of monos $m : M \rightarrowtail 1$). The reason for this is that a $\times$-preserving functor $f : A \to \mathbb{C}$ from a poset A with meets takes every object $a \in A$ to a mono $f(a) \rightarrowtail 1$ in $\mathbb{C}$, since $a = a \wedge a$ implies the following is a product diagram in A.

$$\begin{tikzcd} a \arrow[r] & 1 \\ a \arrow[u] \arrow[r] & a \arrow[u] \end{tikzcd}$$

Exercise 2.6.1. Prove this, and use it to verify that $R = \mathsf{Sub}(1)$ is indeed a right adjoint to the inclusion of $\wedge$-semilattices into finite-product categories.

Now the functor category $\mathbb{2}^{\mathcal{B}^{\mathsf{op}}} = \mathsf{Pos}(\mathcal{B}^{\mathsf{op}}, \mathbb{2})$ occurring in (2.18), consists of all *contravariant*, monotone maps $\mathcal{B}^{\mathsf{op}} \to \mathbb{2}$ (which indeed is $\mathsf{Sub}(1) \hookrightarrow \mathsf{Set}^{\mathcal{B}^{\mathsf{op}}}$), and is easily seen to be isomorphic to the poset $\mathsf{Down}(\mathcal{B})$ of all downsets (or "*sieves*") in $\mathcal{B}$: subsets $S \subseteq \mathcal{B}$ that are downward closed, $x \leq y \in S \Rightarrow x \in S$, ordered by subset inclusion $S \subseteq T$. Explicitly, the isomorphism

$$\mathsf{Pos}(\mathcal{B}^{\mathsf{op}}, \mathbb{2}) \cong \mathsf{Down}(\mathcal{B}) \tag{2.19}$$

is given by taking $f : \mathcal{B}^{\mathsf{op}} \to \mathbb{2}$ to $f^{-1}(1)$ and $S \subseteq \mathcal{B}$ to the function $f_S : \mathcal{B}^{\mathsf{op}} \to \mathbb{2}$ with $f_S(b) = 1 \Leftrightarrow b \in S$. Under this isomorphism, the Yoneda embedding takes an element $b \in \mathcal{B}$ *covariantly* to the *principal downset* $\downarrow b \subseteq \mathcal{B}$ of all $x \leq b$.

Exercise 2.6.2. Show that (2.19) is indeed an isomorphism of posets, and that it sends the Yoneda embedding to the principal sieve mapping, as claimed.

For algebraic theories $\mathbb{A}$, we used the Yoneda embedding to give a completeness theorem for equational logic with respect to Set-valued models, by composing the (faithful) functor $\mathsf{y} : \mathbb{A} \hookrightarrow \mathsf{Set}^{\mathbb{A}^{\mathsf{op}}}$ with the (jointly faithful) evaluation functors $\mathsf{eval}_A : \mathsf{Set}^{\mathbb{A}^{\mathsf{op}}} \to \mathsf{Set}$, for all objects $A \in \mathbb{A}$. This amounts to considering all *covariant* representables $\mathsf{eval}_A \circ \mathsf{y} = \mathbb{A}(A, -) : \mathbb{A} \to \mathsf{Set}$, and observing that these are then (both $\times$-preserving and) jointly faithful.

We can do exactly the same thing for a Boolean algebra $\mathcal{B}$ (which is, after all, a finite product category) to get a jointly faithful family of $\times$-preserving, monotone maps $\mathcal{B}(b, -) : \mathcal{B} \to \mathbb{2}$, i.e. $\wedge$-semilattice homomorphisms. By taking the preimages of $1 \in \mathbb{2}$, such homomorphisms correspond to *filters* in $\mathcal{B}$: "upsets" that are also closed under $\wedge$ (and non-empty!).

$$\mathsf{Pos}_{\wedge}(\mathcal{B}, \mathbb{2}) \cong \mathsf{Filters}(\mathcal{B}) \tag{2.20}$$

The representables $\mathcal{B}(b, -)$ now correspond to the *principal filters* $\uparrow b \subseteq \mathcal{B}$.

The *problem* with using this approach for a completeness theorem for *propositional* logic, however, is that such $\wedge$-homomorphisms $\mathcal{B} \to \mathbb{2}$ are not *models*, because they need not preserve the joins $\phi \vee \psi$ (nor the complements $\neg\phi$).

Lemma 2.6.3. *Let $\mathcal{B}, \mathcal{B}'$ be Boolean algebras and $f : \mathcal{B} \to \mathcal{B}'$ a distributive lattice homomorphism. Then f preserves negation, and so is Boolean. The category* BA *of Boolean algebras is thus a* full *subcategory of the category* DLat *of distributive lattices.*

Proof. The complement $\neg b$ is the unique element of $\mathcal{B}$ such that both $b \vee \neg b = 1$ and $b \wedge \neg b = 0$. □

This suggests representing a Boolean algebra $\mathcal{B}$, not by its filters, but by its *prime* filters, which correspond bijectively to distributive lattice homomorphisms $\mathcal{B} \to 2$.

Definition 2.6.4. A filter $F \subseteq \mathcal{D}$ in a distributive lattice $\mathcal{D}$ is *prime* if $0 \notin F$ and $b \vee b' \in F$ implies $b \in F$ or $b' \in F$. Equivalently, just if the corresponding $\wedge$-semilattice homomorphism $f_F : \mathcal{B} \to 2$ is a lattice homomorphism.

Now if $\mathcal{B}$ is Boolean, it follows from Lemma 2.6.3 that prime filters $F \subseteq \mathcal{B}$ are in bijection with Boolean homomorphisms $\mathcal{B} \to 2$, via the assignment $F \mapsto f_F : \mathcal{B} \to 2$ with $f_F(b) = 1 \Leftrightarrow b \in F$ and $(f : \mathcal{B} \to 2) \mapsto F_f := f^{-1}(1) \subseteq \mathcal{B}$,

$$\mathsf{BA}(\mathcal{B}, 2) \cong \mathsf{PrFilters}(\mathcal{B}) . \tag{2.21}$$

The homomorphism $f_F : \mathcal{B} \to 2$ may be called the *classifying map* of the prime filter $F \subseteq \mathcal{B}$. The prime filter F_f may be called the *(filter)-kernel* (or 1-*kernel*) of the homomorphism $f : \mathcal{B} \to 2$.

Proposition 2.6.5. *In a Boolean algebra $\mathcal{B}$, the following conditions on a filter $F \subseteq \mathcal{B}$ are equivalent.*

1. *F is prime,*
2. *the complement $\mathcal{B} \backslash F$ is a* prime ideal *(defined as a prime filter in $\mathcal{B}^{\mathsf{op}}$),*
3. *the complement $\mathcal{B} \backslash F$ is an* ideal *(defined as a filter in $\mathcal{B}^{\mathsf{op}}$),*
4. *for each $b \in \mathcal{B}$, either $b \in F$ or $\neg b \in \mathcal{F}$ and not both,*
5. *F is* maximal*: if $F \subseteq G$ and G is a filter, then $F = G$ (also called an* ultrafilter*),*
6. *the map $f_F : \mathcal{B} \to 2$ given by $f_F(b) = 1 \Leftrightarrow b \in F$ (as in (2.19)) is a Boolean homomorphism.*

Proof. Exercise! □

The following lemma is sometimes referred to as the *(Boolean) prime ideal theorem.*

Lemma 2.6.6. *Let $\mathcal{B}$ be a Boolean algebra, $I \subseteq \mathcal{B}$ an ideal, and $F \subseteq \mathcal{B}$ a filter, with $I \cap F = \emptyset$. There is a prime filter $P \supseteq F$ with $I \cap P = \emptyset$.*

Proof. Suppose first that $I = \{0\}$ is the trivial ideal, and that $\mathcal{B}$ is countable, with $b_0, b_1, ...$ an enumeration of its elements. As in the proof of the Model Existence Lemma, we build an increasing sequence of filters $F_0 \subseteq F_1 \subseteq \dots$ as follows:

$$F_0 = F$$
$$F_{n+1} = \begin{cases} F_n & \text{if } \neg b_n \in F_n \\ \{f \wedge b \mid f \in F_n,\ b_n \leq b\} & \text{otherwise} \end{cases}$$
$$P = \bigcup_n F_n$$

One then shows that each F_n is a filter, that $I \cap F_n = \emptyset$ for all n and so $I \cap P = \emptyset$, and that for each b_n, either $b_n \in P$ or $\neg b_n \in P$, whence P is prime.

For $I \subseteq \mathcal{B}$ a nontrivial ideal we take the quotient Boolean algebra $\mathcal{B} \twoheadrightarrow \mathcal{B}/I$, defined as the algebra of equivalence classes $[b]$ where $a \sim_I b \Leftrightarrow a \vee i = b \vee j$ for some $i, j \in I$. One shows that this is indeed a Boolean algebra and that the projection onto equivalence classes $\pi_I : \mathcal{B} \twoheadrightarrow \mathcal{B}/I$ is a Boolean homomorphism with (ideal) kernel $\pi^{-1}([0]) = I$. Now apply the foregoing argument to obtain a prime filter $P : \mathcal{B}/I \to \mathbb{2}$. The composite $p_I = P \circ \pi_I : \mathcal{B} \to \mathbb{2}$ is then a Boolean homomorphism with (filter) kernel $p_I^{-1}(1)$ which is prime, contains F and is disjoint from I.

The case where $\mathcal{B}$ is uncountable is left as an exercise. $\square$

Exercise 2.6.7. Finish the proof of Lemma 2.6.6 by (i) verifying the construction of the quotient Boolean algebra $\mathcal{B} \twoheadrightarrow \mathcal{B}/I$, and (ii) considering the case where $\mathcal{B}$ is uncountable (*Hint*: either use Zorn's lemma, or well-order $\mathcal{B}$.)

Theorem 2.6.8 (Stone representation theorem)**.** *Let $\mathcal{B}$ be a Boolean algebra. There is an injective Boolean homomorphism $\mathcal{B} \rightarrowtail \mathcal{P}X_{\mathcal{B}}$ into a powerset.*

Proof. We take $X_{\mathcal{B}} = \mathsf{PrFilters}(\mathcal{B})$, the set of prime filters in $\mathcal{B}$, and consider the map $h : \mathcal{B} \to \mathcal{P}X_{\mathcal{B}}$ given by $h(b) = \{F \mid b \in F\}$. Clearly $h(0) = \emptyset$ and $h(1) = X$. Moreover, for *any* filter F, we have $b \in F$ and $b' \in F$ if and only if $b \wedge b' \in F$, so $h(b \wedge b') = h(b) \cap h(b')$. If F is prime, then $b \in F$ or $b' \in F$ if and only if $b \vee b' \in F$, so $h(b \vee b') = h(b) \cup h(b')$. Thus h is a Boolean homomorphism. Let $a \neq b \in \mathcal{B}$, and we want to show that $h(a) \neq h(b)$. It suffices to assume that $a < b$ (otherwise, consider $a \wedge b$, for which we cannot have both $a \wedge b = a$ and $a \wedge b = b$). We seek a prime filter $P \subseteq \mathcal{B}$ with $b \in P$ but $a \notin P$. Apply Lemma 2.6.6 to the ideal $\downarrow a$ and the filter $\uparrow b$. $\square$

2.7 Stone duality

Note that in the Stone representation $\mathcal{B} \rightarrowtail \mathcal{P}(X_{\mathcal{B}})$ with $X_{\mathcal{B}}$ set of prime filters in B, the powerset Boolean algebra

$$\mathcal{P}(X_{\mathcal{B}}) \cong \mathsf{Set}\big(\mathsf{BA}(\mathcal{B}, \mathbb{2}), 2\big)$$

is evidently (covariantly) functorial in $\mathcal{B}$, and has an apparent "double-dual" form $\mathcal{B}^{**}$. This suggests a possible duality between the categories BA and Set,

$$\begin{tikzcd} \mathsf{BA}^{\mathsf{op}} \arrow[r, bend left, "*"] & \mathsf{Set} \arrow[l, bend left, "*"] \end{tikzcd} \tag{2.22}$$

with contravariant functors

$$\mathcal{B}^* = \mathsf{BA}(\mathcal{B}, \mathbb{2}),$$

the set of prime filters of the Boolean algebra $\mathcal{B}$, and

$$S^* = \mathsf{Set}(S, 2),$$

the powerset Boolean algebra of the set S. This indeed gives a contravariant adjunction "on the right",

$$\frac{\mathcal{B} \to \mathcal{P}S \qquad \mathsf{BA}}{S \to X_{\mathcal{B}} \qquad \mathsf{Set}} \tag{2.23}$$

by applying the corresponding contravariant functors

$$\begin{aligned} \mathcal{P}S &= \mathsf{Set}(S, 2), \\ X_{\mathcal{B}} &= \mathsf{BA}(\mathcal{B}, \mathbb{2}), \end{aligned}$$

and then precomposing with the respective "evaluation" natural transformations,

$$\begin{aligned} \eta_{\mathcal{B}} : \mathcal{B} &\longrightarrow \mathcal{P}(X_{\mathcal{B}}) \cong \mathsf{Set}\big(\mathsf{BA}(\mathcal{B}, \mathbb{2}), 2\big), \\ \varepsilon_S : S &\longrightarrow X_{\mathcal{P}S} \cong \mathsf{BA}\big(\mathsf{Set}(S, 2), \mathbb{2}\big). \end{aligned}$$

The homomorphism $\eta_{\mathcal{B}}$ takes an element $b \in \mathcal{B}$ to (the characteristic function of) the set of (characteristic functions of) prime filters that contain it, and the function ε_S takes an element $s \in S$ to the (...) principal filter $\uparrow\{s\} \subseteq \mathcal{P}S$, which is prime since the singleton set $\{s\}$ is an *atom* in $\mathcal{P}S$, i.e., a minimal, non-zero element.

Exercise 2.7.1. Verify the adjunction (2.22).

The adjunction (2.22) is not an equivalence of categories, however, because neither of the units $\eta_{\mathcal{B}}$ nor ε_S is in general an isomorphism. (Recall that a right adjoint is full and faithful just if the counit is an iso, and an equivalence if both the unit and the counit are isos.) We can improve the adjunction ref(2.22) by *topologizing* the set $X_{\mathcal{B}}$ of prime filters, in order to be able to cut down the powerset $\mathcal{P}(X_{\mathcal{B}}) \cong \mathsf{Set}(X_{\mathcal{B}}, 2)$ from *all* functions to just the *continuous* functions into the discrete space 2, which will then correspond to the *clopen* sets in $X_{\mathcal{B}}$.

To do this, we take as *basic open sets* of $X_{\mathcal{B}}$ all those subsets of the form:

$$B_b = \eta_{\mathcal{B}}(b) = \{P \in X_{\mathcal{B}} \mid b \in P\}, \qquad b \in \mathcal{B}. \tag{2.24}$$

These sets are closed under finite intersections, because $B_a \cap B_b = B_{a \wedge b}$. Indeed, if $P \in B_a \cap B_b$ then $a \in P$ and $b \in P$, whence $a \wedge b \in P$, and conversely (after all, $\eta_{\mathcal{B}}$ is a Boolean homomorphism!). Thus the family $(B_b)_{b \in \mathcal{B}}$ is a basis of open sets for a topology on $X_{\mathcal{B}}$.

Definition 2.7.2. For any Boolean algebra $\mathcal{B}$, the *prime spectrum* of $\mathcal{B}$ is a topological space $X_{\mathcal{B}}$ with the prime filters $P \subseteq \mathcal{B}$ as points, and the sets B_b of (2.24), for all $b \in \mathcal{B}$, as basic open sets. The prime spectrum $X_{\mathcal{B}}$ is also called the *Stone space* of $\mathcal{B}$.

Proposition 2.7.3. *The open sets $\mathcal{O}(X_{\mathcal{B}})$ of the Stone space are in order-preserving, bijective correspondence with the ideals $I \subseteq \mathcal{B}$ of the Boolean algebra, with the principal ideals $\downarrow b$ corresponding exactly to the clopen sets B_b.*

Proof. Exercise! □

We now have an improved adjunction

$$\begin{tikzcd} \mathsf{BA}^{\mathsf{op}} \arrow[r, bend left, "\mathsf{Spec}"] & \mathsf{Top} \arrow[l, bend left, "\mathsf{Clop}"] \end{tikzcd} \tag{2.25}$$

$$\begin{aligned} \mathsf{Spec}(\mathcal{B}) &= (X_{\mathcal{B}}, \mathcal{O}(X_{\mathcal{B}})) \\ \mathsf{Clop}(X) &= \mathsf{Top}(X, 2), \end{aligned}$$

for which, up to isomorphism, the space $\mathsf{Spec}(\mathcal{B})$ has the underlying set $\mathsf{BA}(\mathcal{B}, \mathbb{2})$ given by "homming" into the Boolean algebra $\mathbb{2}$, and the Boolean algebra $\mathsf{Clop}(X) = \mathsf{Top}(X, 2)$ is similarly determined by mapping into the "topological Boolean algebra" given by the discrete space 2. Such an adjunction is said to be induced by a *dualizing object*: an object that can be regarded as "living in two different categories". Here the dualizing object 2 is acting both as a space and as a Boolean algebra. In the Lawvere duality of Chapter 1, the role of dualizing object was played by the category Set of all sets!

Now if $\eta_{\mathcal{B}} : \mathcal{B} \cong \mathsf{ClopSpec}(\mathcal{B})$, it would follow that the functor $\mathsf{Spec} : \mathsf{BA}^{\mathsf{op}} \to \mathsf{Top}$ is full and faithful. So if we then cut down the improved adjunction (2.25) to just the spaces in the image of Spec, we will obtain a "duality" (a contraviant equivalence). Toward that end, observe first that the Stone space $X_{\mathcal{B}}$ of a Boolean algebra $\mathcal{B}$ is a subspace of a product of finite discrete spaces,

$$X_{\mathcal{B}} \cong \mathsf{BA}(\mathcal{B}, \mathbb{2}) \hookrightarrow \prod_{|\mathcal{B}|} 2,$$

and is therefore a compact Hausdorff space, by Tychonoff's theorem. Indeed, the basis (2.24) is just the subspace topology on $X_{\mathcal{B}}$ with respect to the product topology on $\prod_{|\mathcal{B}|} 2$. The latter space is moreover *totally disconnected*, meaning that it has a subbasis of clopen subsets, namely all those of the form $f^{-1}(\delta) \subseteq |\mathcal{B}|$ for $f : |\mathcal{B}| \to 2$ and $\delta = 0, 1$.

Lemma 2.7.4. *The prime spectrum $X_{\mathcal{B}}$ of a Boolean algebra $\mathcal{B}$ is a totally disconnected, compact, Hausdorff space.*

Proof. Since $\prod_{|\mathcal{B}|} 2$ has just been shown to be a totally disconnected, compact Hausdorff space, we need only see that the subspace $X_{\mathcal{B}}$ is closed. Consider the subspaces

$$2^{|\mathcal{B}|}_{\wedge},\ 2^{|\mathcal{B}|}_{\vee},\ 2^{|\mathcal{B}|}_{1},\ 2^{|\mathcal{B}|}_{0} \subseteq 2^{|\mathcal{B}|}$$

consisting of the functions $f : |\mathcal{B}| \to 2$ that preserve $\wedge, \vee, 1, 0$ respectively. Since each of these is closed, so is their intersection $X_{\mathcal{B}}$. In more detail, the set of maps $f : |\mathcal{B}| \to 2$ that preserve e.g. $\wedge$ can be described as an equalizer

$$\begin{tikzcd} 2^{|\mathcal{B}|}_{\wedge} \arrow[r, tail] & 2^{|\mathcal{B}|} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & 2^{|\mathcal{B}|\times|\mathcal{B}|} \end{tikzcd}$$

where the maps s, t take an arrow $f : |\mathcal{B}| \to 2$ to the two different composites around the square

$$\begin{tikzcd} {|\mathcal{B}| \times |\mathcal{B}|} \arrow[r, "\wedge"] \arrow[d, "f \times f"'] & {|\mathcal{B}|} \arrow[d, "f"] \\ 2 \times 2 \arrow[r, "\wedge"'] & 2. \end{tikzcd}$$

But the equalizer $2^{|\mathcal{B}|}_{\wedge} \rightarrowtail 2^{|\mathcal{B}|}$ is the pullback of the diagonal on $2^{|\mathcal{B}|\times|\mathcal{B}|}$, which is closed since $2^{|\mathcal{B}|\times|\mathcal{B}|}$ is Hausdorff. The other cases are analogous. □

Definition 2.7.5. A topological space is called *Stone* if it is totally disconnected, compact, and Hausdorff. Let $\mathsf{Stone} \hookrightarrow \mathsf{Top}$ be the full subcategory of topological spaces consisting of Stone spaces and continuous functions between them.

Now in order to cut down the adjunction (2.25) to a duality, we can restrict it on the topological side to just the Stone spaces, since we know this subcategory will contain the image of the functor Spec. In fact, up to isomorphism, this is exactly the image:

Theorem 2.7.6. *There is a contravariant equivalence of categories between* BA *and* Stone*,*

$$\begin{tikzcd} \mathsf{BA}^{\mathsf{op}} \arrow[r, bend left, "*"] & \mathsf{Stone}\,, \arrow[l, bend left, "*"] \end{tikzcd}$$

with contravariant functors $\mathcal{B}^* = X_{\mathcal{B}}$ the Stone space of a Boolean algebra $\mathcal{B}$, as in Definition 2.7.2, and $X^* = \mathsf{clopen}(X)$, the Boolean algebra of all clopen sets in the Stone space X.

Proof. We just need to show that the two units of the adjunction

$$\eta_{\mathcal{B}} : \mathcal{B} \to \mathsf{Top}\big(\mathsf{BA}(\mathcal{B}, \mathbb{2}), 2\big),$$
$$\varepsilon_S : S \to \mathsf{BA}\big(\mathsf{Top}(S, 2), \mathbb{2}\big).$$

are isomorphisms, the second assuming S is a Stone space.

We know by the Stone representation theorem 2.6.8 that $\eta_{\mathcal{B}}$ is an injective Boolean homomorphism, so its image, say

$$\mathcal{B}' \subseteq \mathsf{Top}\big(\mathsf{BA}(\mathcal{B}, 2), 2\big) \cong \mathsf{Clop}(X_{\mathcal{B}}),$$

is a sub-Boolean algebra of the clopen sets of $X_{\mathcal{B}}$. It suffices to show that every clopen set of $X_{\mathcal{B}}$ is in $\mathcal{B}'$. Thus let $K \subseteq X_{\mathcal{B}}$ be clopen, and take $K = \bigcup_i B_i$ a cover by basic opens B_i, all of which, note, are of the form (2.24), and so are in $\mathcal{B}'$. Since K is closed and $X_{\mathcal{B}}$ compact, K is also compact, so there is a finite subcover, each element of which is in $\mathcal{B}'$. Thus their finite union K is also in $\mathcal{B}'$.

Now let S be a Stone space and consider the continuous function

$$\varepsilon_S : S \to \mathsf{BA}\big(\mathsf{Top}(S, 2), 2\big) \cong X_{\mathsf{Clop}(S)}$$

which takes $s \in S$ to the prime filter $F_s = \{K \in \mathsf{Clop}(S) \mid s \in K\}$ of all clopen sets containing it. Since S is Hausdorff, ε_S is a bijection on points, and it is continuous by construction. To see that it is open, let $K \subseteq S$ be a basic clopen set. The complement $S - K$ is therefore closed, and thus compact, and so is its image $\varepsilon_S(S - K)$, which is therefore closed. But since ε_S is a bijection, $\varepsilon_S(S - K)$ is the complement of $\varepsilon_S(K)$, which is therefore open. □

Remark 2.7.7. Another way to cut down the adjunction (2.22),

$$\begin{tikzcd} \mathsf{BA}^{\mathsf{op}} \arrow[r, bend left, "*"] & \mathsf{Set} \arrow[l, bend left, "*"] \end{tikzcd}$$

to an equivalence is to restrict the Boolean algebra side to the *complete, atomic* Boolean algebras $\mathsf{BA}_{\mathrm{ca}}$ and continuous (i.e. $\bigvee$-preserving) homomorphisms between them. One then obtains a duality

$$\mathsf{BA}_{\mathrm{ca}}^{\mathsf{op}} \simeq \mathsf{Set},$$

between complete, atomic Boolean algebras and sets (see Johnstone [Joh82]).

Remark 2.7.8. See Johnstone [Joh82] for a more detailed presentation of the material in this section (and much more). Also see [MR95] for a generalization to distributive lattices and Heyting algebras, as well as to "Boolean algebras with operators", i.e. algebraic models of modal logic. For more on logical duality see [Awo21]

2.8 Cartesian closed posets

We can relax the Boolean condition $\neg\neg b = b$ in order to generalize some of our results to other systems of propositional logic, represented by structured poset categories. This will be useful when we consider the "proof-relevant" versions of these as proper (*i.e.* non-poset) categories arising from systems of type theory. We begin with a basic system without the coproducts $\bot$ or $\phi \vee \psi$, and thus also without negation $\neg\phi$, which we shall therefore call the *positive propositional calculus* (a non-standard designation).

Positive propositional calculus Classically, implication $\phi \Rightarrow \psi$ can be defined by $\neg\phi \vee \psi$, but in categorical logic we prefer to consider $\phi \Rightarrow \psi$ as an *exponential*, of ψ by ϕ, defined as right adjoint to the conjunction $(-) \wedge \phi$. Since this makes sense without negation $\neg\phi$ or joins $\phi \vee \psi$, we can study just the cartesian closed fragment separately, and then add those other operations later. The same approach will be used for type theory in Chapter **??**.

Definition 2.8.1. The *positive propositional calculus* PPC is the subsystem of the propositional calculus of Section 2.1 containing just (finite) conjunction and implication. So PPC is the set of all propositional formulas ϕ constructed from propositional variables $p_1, p_2, ...$, a constant $\top$ for truth, and binary connectives for conjunction $\phi \wedge \psi$, and implication $\phi \Rightarrow \psi$.

As a category, PPC is a preorder under the relation $\phi \vdash \psi$ of logical entailment, determined, say, by the natural deduction system of section 2.1 . As usual, it will be convenient to pass to the poset reflection of the preorder, which we shall denote by

$$\mathcal{C}_{\mathsf{PPC}}$$

by identifying ϕ and ψ when $\phi \dashv\vdash \psi$. (This is the (syntactic) *Lindenbaum-Tarski* algebra of the system PPC of positive propositional logic, as in Section 2.5.)

The conjunction $\phi \wedge \psi$ is a greatest lower bound of ϕ and ψ in $\mathcal{C}_{\mathsf{PPC}}$, because $\phi \wedge \psi \vdash \phi$ and $\phi \wedge \psi \vdash \psi$, and for all ϑ, if $\vartheta \vdash \phi$ and $\vartheta \vdash \psi$ then $\vartheta \vdash \phi \wedge \psi$. Since binary products in a poset are the same thing as greatest lower bounds, we see that $\mathcal{C}_{\mathsf{PPC}}$ has all binary products; and of course $\top$ is a terminal object, so $\mathcal{C}_{\mathsf{PPC}}$ is a $\wedge$-semilattice.

We have already remarked that implication is right adjoint to conjunction in the sense that for any ϕ,

$$(-) \wedge \phi \ \dashv \ \phi \Rightarrow (-)\ . \tag{2.26}$$

Therefore $\phi \Rightarrow \psi$ is an exponential in $\mathcal{C}_{\mathsf{PPC}}$. The counit of the adjunction (the "evaluation" arrow) is the entailment

$$(\phi \Rightarrow \psi) \wedge \phi \vdash \psi\ ,$$

i.e. the familiar logical rule of *modus ponens.*

We therefore have the following:

Proposition 2.8.2. *The poset $\mathcal{C}_{\mathsf{PPC}}$ of positive propositional calculus is cartesian closed.*

We can use this fact to show that the positive propositional calculus is *deductively complete* with respect to the following notion of *Kripke semantics* [**?**].

Definition 2.8.3 (Kripke semantics)**.** 1. A *Kripke model* is a poset K (the *worlds*) equipped with a relation

$$k \Vdash p$$

between elements $k \in K$ and propositional variables p, such that for all $j \in K$,

$$j \leq k,\ k \Vdash p \quad \text{implies} \quad j \Vdash p\,. \tag{2.27}$$

2. Given a Kripke model $(K, \Vdash)$, extend the relation $\Vdash$ to all formulas ϕ in PPC by defining the relation of *holding in a world* $k \in K$ inductively by the following conditions:

$$\begin{array}{lll} k \Vdash \top & \text{always,} & \\ k \Vdash \phi \wedge \psi & \text{iff} & k \Vdash \phi \text{ and } k \Vdash \psi\,, \\ k \Vdash \phi \Rightarrow \psi & \text{iff} & \text{for all } j \leq k, \text{ if } j \Vdash \phi, \text{ then } j \Vdash \psi\,. \end{array} \tag{2.28}$$

3. Finally, say that ϕ *holds in the Kripke model* $(K, \Vdash)$, written

$$K \Vdash \phi$$

if $k \Vdash \phi$ for all $k \in K$. (One sometimes also says that ϕ *holds on* the poset K if $K \Vdash \phi$ for all such Kripke relations $\Vdash$ on K.)

Theorem 2.8.4 (Kripke completeness for PPC)**.** *A propositional formula ϕ is provable from the rules of deduction for PPC if, and only if, $K \Vdash \phi$ for all Kripke models $(K, \Vdash)$,*

$$\mathsf{PPC} \vdash \phi \qquad \textit{iff} \qquad K \Vdash \phi \quad \textit{for all } (K, \Vdash).$$

For the proof, we first require the following, which generalizes the discussion around (2.19) in Section 2.6.

Lemma 2.8.5. *For any poset P, the poset $\mathsf{Down}(P)$ of all downsets in P, ordered by inclusion, is cartesian closed. Moreover, the downset embedding,*

$$\downarrow(-) : P \longrightarrow \mathsf{Down}(P)$$

preserves any CCC structure that exists in P.

Proof. The total downset P is obviously terminal, and for any downsets $S, T \in \mathsf{Down}(P)$, the intersection $S \cap T$ is also closed down, so we have the products $S \wedge T = S \cap T$. For the exponential, set

$$S \Rightarrow T = \{p \in P \mid \downarrow(p) \cap S \subseteq T\}. \tag{2.29}$$

Then for any downset Q we have

$$\begin{array}{rll} Q \subseteq S \Rightarrow T & \text{iff} & \text{for all } q \in Q, \quad q \in S \Rightarrow T\,, \\ & \text{iff} & \text{for all } q \in Q, \quad \downarrow(q) \cap S \subseteq T\,, \\ & \text{iff} & \bigcup_{q \in Q} (\downarrow(q) \cap S) \subseteq T\,, \\ & \text{iff} & (\bigcup_{q \in Q} \downarrow(q)) \cap S \subseteq T\,, \\ & \text{iff} & Q \cap S \subseteq T\,. \end{array}$$

The preservation of CCC structure by $\downarrow(-) : P \longrightarrow \mathsf{Down}(P)$ follows from its preservation by the Yoneda embedding, of which we know $\downarrow(-)$ to be a factor,

$$\begin{tikzcd}
 & \mathsf{Set}^{P^{\mathsf{op}}} \\
P \arrow[ur, hook, "\mathsf{y}"] \arrow[r, "\downarrow(-)"'] & \mathsf{Down}(P) \arrow[u, hook]
\end{tikzcd}$$

Indeed, we can identify $\mathsf{Down}(P)$ with the subcategory $\mathsf{Sub}(1)$ of subobjects of 1 in $\mathsf{Set}^{P^{\mathsf{op}}}$ and the result follows by using the left adjoint left inverse sup of the inclusion

$$\mathsf{sup} \dashv i : \mathsf{Sub}(1) \hookrightarrow \mathsf{Set}^{P^{\mathsf{op}}} ,$$

to be considered later **??**.

But it is also easy enough to check it directly: Preservation of any limits 1, $p \wedge q$ that exist in P are clear, since these are pointwise. Then suppose $p \Rightarrow q$ is an exponential; so for any downset D we have:

$$\begin{array}{rll}
D \subseteq \downarrow(p \Rightarrow q) & \text{iff} & d \in \downarrow(p \Rightarrow q) \text{ , for all } d \in D \\
& \text{iff} & d \leq p \Rightarrow q \text{ , for all } d \in D \\
& \text{iff} & d \wedge p \leq q \text{ , for all } d \in D \\
& \text{iff} & \downarrow(d \wedge p) \subseteq \downarrow(q) \text{ , for all } d \in D \\
& \text{iff} & \downarrow(d) \cap \downarrow(p) \subseteq \downarrow(q) \text{ , for all } d \in D \\
& \text{iff} & D \subseteq \downarrow(p) \Rightarrow \downarrow(q)
\end{array}$$

where the last line is by (2.29). Now take D to be $\downarrow(p \Rightarrow q)$ and $\downarrow(p) \Rightarrow \downarrow(q)$ respectively (or just apply Yoneda!). (Note that in line (3) we assumed that $d \wedge p$ exists for all $d \in D$; this can be avoided by a slightly more complicated argument.) □

Proof. (of Theorem 2.8.4) The proof follows a now-familiar pattern, which we only sketch:

1. The syntactic category $\mathcal{C}_{\mathsf{PPC}}$ is a CCC, with $\top = 1$, $\phi \times \psi = \phi \wedge \psi$, and $\psi^{\phi} = \phi \Rightarrow \psi$. In fact, it is the free cartesian closed poset on the generating set $\mathsf{Var} = \{p_1, p_2, \dots\}$ of propositional variables.

2. A (Kripke) model $(K, \Vdash)$ is the same thing as a CCC functor $\mathcal{C}_{\mathsf{PPC}} \to \mathsf{Down}(K)$, which by Step 1 is just an arbitrary map $\mathsf{Var} \to \mathsf{Down}(K)$, as in (2.27). To see this, observe that we have a bijective correspondence between CCC functors $[\![-]\!]$ and Kripke relations $\Vdash$; indeed, by the exponential adjunction in the cartesian closed category Pos, there is a natural bijection,

$$\frac{\Vdash : K^{\mathsf{op}} \times \mathcal{C}_{\mathsf{PPC}} \longrightarrow 2}{[\![-]\!] : \mathcal{C}_{\mathsf{PPC}} \longrightarrow 2^{K^{\mathsf{op}}} \cong \mathsf{Down}(K)}$$

where we use the poset $\mathbb{2}$ to classify downsets in a poset K via upsets in K^{op},

$$\mathbb{2}^{P^{\mathsf{op}}} \cong \mathsf{Pos}(K^{\mathsf{op}}, \mathbb{2}) \cong \mathsf{Down}(K)\,,$$

by taking the 1-kernel $f^{-1}(1) \subseteq K$ of a monotone map $f : K^{\mathsf{op}} \to \mathbb{2}$. (The contravariance will be convenient in Step 3). Note that the monotonicity of $\Vdash$ yields the conditions

$$j \leq k\,,\ k \Vdash \phi \implies j \Vdash \phi$$

and

$$k \Vdash \phi\,,\ \phi \vdash \psi \implies k \Vdash \psi\,.$$

And the CCC preservation of the transpose $[\![-]\!]$ yields the Kripke forcing conditions (2.28) (exercise!).

3. For any model $(K, \Vdash)$, by the adjunction in (2) we then have

$$K \Vdash \phi \iff [\![\phi]\!] = K\,,$$

with $K \subseteq K$ the maximal downset.

4. Because the downset/Yoneda embedding $\downarrow$ preserves the CCC structure (by Lemma 2.8.5), $\mathcal{C}_{\mathsf{PPC}}$ has a *canonical model*, namely the special case of (2) with $K = \mathcal{C}_{\mathsf{PPC}}$ and $\Vdash$ resulting from the trasposition:

$$\dfrac{\downarrow(-)\ :\ \mathcal{C}_{\mathsf{PPC}} \longrightarrow \mathsf{Down}(\mathcal{C}_{\mathsf{PPC}}) \cong \mathbb{2}^{\mathcal{C}^{\mathsf{op}}_{\mathsf{PPC}}}}{\Vdash\ :\ \mathcal{C}^{\mathsf{op}}_{\mathsf{PPC}} \times \mathcal{C}_{\mathsf{PPC}} \longrightarrow \mathbb{2}}$$

5. Now note that for the Kripke relation $\Vdash$ in (4), we have $\Vdash\ =\ \vdash$ since it's just the transpose of the Yoneda embedding, and the poset $\mathcal{C}_{\mathsf{PPC}}$ is ordered by $\phi \vdash \psi$. So the canonical model is *logically generic*, in the sense that

$$\phi \Vdash \psi \quad \iff \quad \phi \vdash \psi\,,$$

and so in particular,

$$\mathcal{C}_{\mathsf{PPC}} \Vdash \phi \quad \iff \quad \mathsf{PPC} \vdash \phi\,.$$

□

Exercise 2.8.6. Verify the claim in (2) that CCC preservation of the transpose $[\![-]\!]$ of $\Vdash$ yields the Kripke forcing conditions (2.28).

Exercise 2.8.7. Give a Kripke countermodel to show that $\mathsf{PPC} \nvdash (\phi \Rightarrow \psi) \Rightarrow \phi$.

2.9 Heyting algebras

Let us now extend the positive propositional calculus to the full intuitionistic propositional calculus. This involves adding the finite coproducts 0 and $p \vee q$ to the notion of a cartesian closed poset, to arrive at the general notion of a Heyting algebra. Heyting algebras are to intuitionistic logic as Boolean algebras are to classical logic: each is an algebraic description of the corresponding logical calculus. We shall review both the algebraic and the logical points of view; as we shall see, many aspects of the theory of Boolean algebras carry over to Heyting algebras. For instance, in order to prove the Kripke completeness of the full system of intuitionistic propositional calculus, we will need an alternative to Lemma 2.8.5, because the Yoneda embedding does not in general preserve coproducts. For that we will again use a version of the Stone representation theorem, this time in a generalized form due to Joyal.

Distributive lattices

Recall first that a (bounded) *lattice* is a poset that has finite limits and colimits. In other words, a lattice $(L, \leq, \wedge, \vee, 1, 0)$ is a poset $(L, \leq)$ with distinguished elements $1, 0 \in L$, and binary operations of meet $\wedge$ and join $\vee$, satisfying for all $x, y, z \in L$,

$$0 \leq x \leq 1 \qquad \frac{z \leq x \qquad z \leq y}{\overline{z \leq x \wedge y}} \qquad \frac{x \leq z \qquad y \leq z}{\overline{x \vee y \leq z}}$$

A *lattice homomorphism* is a function $f : L \to K$ between lattices which preserves finite limits and colimits, i.e., $f0 = 0$, $f1 = 1$, $f(x \wedge y) = fx \wedge fy$, and $f(x \vee y) = fx \vee fy$. The category of lattices and lattice homomorphisms is denoted by Lat.

Lattices are an algebraic theory, and can be axiomatized equationally in a signature with two distinguished elements 0 and 1 and two binary operations $\wedge$ and $\vee$, satisfying the following equations:

$$\begin{aligned} (x \wedge y) \wedge z &= x \wedge (y \wedge z) \;, & (x \vee y) \vee z &= x \vee (y \vee z) \;, \\ x \wedge y &= y \wedge x \;, & x \vee y &= y \vee x \;, \\ x \wedge x &= x \;, & x \vee x &= x \;, \\ 1 \wedge x &= x \;, & 0 \vee x &= x \;, \\ & x \wedge (y \vee x) = x = (x \wedge y) \vee x \;. \end{aligned} \tag{2.30}$$

The partial order on L is then determined by

$$x \leq y \iff x = x \wedge y \;.$$

Exercise 2.9.1. Show that in a lattice we also have $x \leq y$ if and only if $x \vee y = y$.

A lattice is *distributive* if the following distributive laws hold:

$$
\begin{aligned}
(x \vee y) \wedge z &= (x \wedge z) \vee (y \wedge z) , \\
(x \wedge y) \vee z &= (x \vee z) \wedge (y \vee z) .
\end{aligned}
\tag{2.31}
$$

It turns out that if one distributive law holds then so does the other [Joh82, I.1.5].

Definition 2.9.2. A *Heyting algebra* is a cartesian closed lattice. This means that a Heyting algebra $\mathcal{H}$ has a binary operation of *implication* $x \Rightarrow y$, satisfying the following condition, for all $x, y, z \in \mathcal{H}$:

$$
\frac{z \leq x \Rightarrow y}{\overline{z \wedge x \leq y}}
$$

A *Heyting algebra homomorphism* is a lattice homomorphism $f : \mathcal{K} \to \mathcal{H}$ between Heyting algebras that preserves implication, i.e., $f(x \Rightarrow y) = (fx \Rightarrow fy)$. The category of Heyting algebras and their homomorphisms is denoted by Heyt. (*Caution*: unlike Boolean algebras, the subcategory of lattices consisting of Heyting algebras and their homomorphisms is *not full.*)

Heyting algebras can be axiomatized equationally as a set H with two distinguished elements 0 and 1 and three binary operations $\wedge$, $\vee$ and $\Rightarrow$. The equations for a Heyting algebra are the ones listed in (2.30), as well as the following ones for $\Rightarrow$.

$$
\begin{aligned}
(x \Rightarrow x) &= 1 , \\
x \wedge (x \Rightarrow y) &= x \wedge y , \\
y \wedge (x \Rightarrow y) &= y , \\
(x \Rightarrow (y \wedge z)) &= (x \Rightarrow y) \wedge (x \Rightarrow z) .
\end{aligned}
\tag{2.32}
$$

For a proof, see [Joh82, I.1], where one can also find a proof that every Heyting algebra is distributive (exercise!).

Exercise 2.9.3. Show that every Heyting algebra is indeed a distributive lattice.

Example 2.9.4. We know from Lemma 2.8.5 that for any poset P, the poset $\mathsf{Down}(P)$ of all downsets in P, ordered by inclusion, is cartesian closed. Moreover, we know that

$$
\mathsf{Down}(P) \cong \mathbb{2}^{P^{\mathsf{op}}} \cong \mathsf{Pos}(P^{\mathsf{op}}, \mathbb{2}) ,
$$

the latter regarded as a poset with the pointwise ordering on the monotone maps $P^{\mathsf{op}} \to \mathbb{2}$ (*i.e.* the natural transformations). The assignment takes a map $f : P^{\mathsf{op}} \to \mathbb{2}$ to the filter-kernel $f^{-1}(1) \subseteq P^{\mathsf{op}}$, which is therefore a downset in P. Indeed, if $f \leq g$ then $p \in f^{-1}(1) \iff fp = 1$ which implies $gp = 1 \iff p \in g^{-1}(1)$, so $f^{-1}(1) \subseteq g^{-1}(1)$, and these upsets in P^{op} are downsets in P.

Since $\mathbb{2}$ is a lattice, we can take joins $f \vee g$ in $\mathsf{Pos}(P^{\mathsf{op}}, \mathbb{2})$ pointwise, in order to get joins in $\mathsf{Down}(P) \cong \mathsf{Pos}(P^{\mathsf{op}}, \mathbb{2})$, which then correspond to (set theoretic) unions of the corresponding downsets $f^{-1}(1) \cup g^{-1}(1)$. Thus for any poset P, the lattice $\mathsf{Down}(P)$ is a

Heyting algebra, with the downsets ordered by inclusion, and the (contravariant) classifying maps $P^{\mathsf{op}} \to \mathbb{2}$ ordered pointwise.

Of course, one can compose the classifying maps with the negation iso $\neg : \mathbb{2} \xrightarrow{\sim} \mathbb{2}$ to get $\mathsf{Down}(P) \cong \mathsf{Pos}(P, \mathbb{2})$, with *covariant* classifying maps $P \to \mathbb{2}$ for the downsets, using the ideal-kernels $f^{-1}(0) \subseteq P$ instead; but then the ordering on $\mathsf{Pos}(P, \mathbb{2})$ will be the *reverse pointwise ordering* of maps $f : P \to \mathbb{2}$.

Intuitionistic propositional calculus

There is an obvious forgetful functor $U : \mathsf{Heyt} \to \mathsf{Set}$ mapping a Heyting algebra to its underlying set, and a homomorphism of Heyting algebras to the underlying function. Because Heyting algebras are also models of an equational theory, there is a left adjoint $H \dashv U$, which is the usual "free" construction for algebras, mapping a set S to the free Heyting algebra $H(S)$ generated by it. As for all algebraic structures, the construction of $H(S)$ can be performed in two steps: first, define a set $H[S]$ of formal expressions in the signature, and then quotient it by an equivalence relation generated by the equations.

In more detail, let $H[S]$ be the set of formal expressions generated inductively by the following rules:

1. Generators: if $x \in S$ then $x \in H[S]$.

2. Constants: $\bot, \top \in H[S]$.

3. Connectives: if $\phi, \psi \in H[S]$ then $(\phi \wedge \psi), (\phi \vee \psi), (\phi \Rightarrow \psi) \in H[S]$.

We then impose an equivalence relation $\sim$ on $H[S]$, defined as the smallest equivalence relation containing all instances of the axioms (2.30) and (2.32) and closed under substitution of equals for equals (sometimes called the smallest *congruence*). This then forces the quotient

$$H(S) = H[S]/_{\sim}$$

to be a Heyting algebra, as is easily checked.

We define the action of the functor H on morphisms as usual: a function $f : S \to T$ is mapped to the Heyting algebra homomorphism $H(f) : H(S) \to H(T)$ (well-)defined (on equivalence classes) by

$$H(f)\bot = \bot\,, \qquad H(f)\bot = \bot\,, \qquad H(f)x = fx\,,$$
$$H(f)(\phi * \psi) = (H(f)\phi) * (H(f)\psi)\,,$$

where $*$ stands for $\wedge$, $\vee$ or $\Rightarrow$.

The inclusion of generators $\eta_S : S \to UH(S)$ into the underlying set of the free Heyting algebra $H(S)$ is then the component at S of a natural transformation $\eta : 1_{\mathsf{Set}} \Longrightarrow U \circ H$, which is of course the unit of the adjunction $H \dashv U$. To see this, consider a Heyting algebra $\mathcal{K}$ and an arbitrary function $f : S \to U\mathcal{K}$. Then the Heyting algebra homomorphism

$\overline{f} : H(S) \to \mathcal{K}$ is defined in the evident way, by

$$\overline{f}\bot = \bot\,, \qquad \overline{f}\bot = \bot\,, \qquad \overline{f}x = fx\,,$$
$$\overline{f}(\phi * \psi) = (\overline{f}\phi) * (\overline{f}\psi)\,,$$

where, again, $*$ stands for $\wedge$, $\vee$ or $\Rightarrow$. The map $\overline{f}$ then makes the following triangle in Set commute:

$$\begin{tikzcd} S \arrow[r, "\eta_S"] \arrow[dr, "f"'] & UH(S) \arrow[d, "U\overline{f}"] \\ & U\mathcal{K} \end{tikzcd}$$

The homomorphism $\overline{f} : H(S) \to \mathcal{K}$ is the unique one with this property, because any two homomorphisms from $H(S)$ that agree on generators must clearly be equal (formally, this can be proved by induction on the structure of the expressions in $H[S]$).

We can now define the *intuitionistic propositional calculus* IPC to be the free Heyting algebra $H(p_0, p_1, \dots)$ on countably many generators $\{p_0, p_1, \dots\}$, called *atomic propositions* or *propositional variables*. This is a somewhat unorthodox definition from a logical point of view—normally we would start from a *deductive calculus* consisting of a formal language, entailment judgements, and rules of inference. But of course, by now, we realize that the two approaches are essentially equivalent.

Having said that, let us also briefly describe IPC in the conventional way: The formulas are all those given in Section 2.1, and the rules of inference are those of the system of natural deduction from Section 2.1.1, but *without the classical rules*.

Then let $\mathcal{C}_{\mathsf{IPC}}$ be the poset reflection of the formulas of IPC, preordered by entailment $\phi \vdash \psi$. The elements of $\mathcal{C}_{\mathsf{IPC}}$ are thus equivalence classes $[\phi]$ of formulas, where two formulas ϕ and ψ are equivalent if both $\phi \vdash \psi$ and $\psi \vdash \phi$ are provable in natural deduction, without the classical rules,

$$[\phi] = [\psi] \iff \phi \dashv\vdash \psi\,.$$

This syntactic category $\mathcal{C}_{\mathsf{IPC}}$ is then easily seen to be the free Heyting algebra on countably many generators $\{p_0, p_1, \dots\}$,

$$\mathcal{C}_{\mathsf{IPC}} \cong H(p_0, p_1, \dots)\,,$$

just as the corresponding “Lindenbaum-Tarski” Boolean algebra $\mathcal{B}[p_0, p_1, \dots]$ was seen to be the free Boolean algebra on the propositional variables as generators.

Classical propositional calculus redux

Let us have another look back at the theory of classical propositional logic from the current point of view, *i.e.* as a special kind of Heyting algebra. An element $x \in L$ of a lattice L is said to be *complemented* when there exists $y \in L$ such that

$$x \wedge y = 0\,, \qquad\qquad x \vee y = 1\,.$$

We say that y is the *complement* of x. In a distributive lattice, the complement of x is unique if it exists. Indeed, if both y and z are complements of x then

$$y \wedge z = (y \wedge z) \vee 0 = (y \wedge z) \vee (y \wedge x) = y \wedge (z \vee x) = y \wedge 1 = y \ ,$$

hence $y \leq z$. A symmetric argument shows that $z \leq y$, therefore $y = z$. The complement of x, if it exists, is denoted by $\neg x$.

A *Boolean algebra* can be defined as a distributive lattice in which every element is complemented. In other words, a Boolean algebra B has a *complementation operation* $\neg : B \to B$ which satisfies, for all $x \in B$,

$$x \wedge \neg x = 0 \ , \qquad\qquad x \vee \neg x = 1 \ . \tag{2.33}$$

The full subcategory of Lat consisting of Boolean algebras is denoted by BA.

Exercise 2.9.5. Prove that every Boolean algebra is a Heyting algebra. (*Hint*: how is implication encoded in terms of negation and disjunction in classical logic?)

In a Heyting algebra, not every element is complemented. However, we can still define a *pseudo complement* or *negation* operation $\neg$ by

$$\neg x = (x \Rightarrow 0) \ ,$$

Then $\neg x$ is the largest element for which $x \wedge \neg x = 0$. While in a Boolean algebra $\neg\neg x = x$, in a Heyting algebra we only have $x \leq \neg\neg x$ in general. An element x of a Heyting algebra for which $x = \neg\neg x$ is called *regular*.

Exercise 2.9.6. Derive the following properties of negation in a *Heyting* algebra:

$$\begin{gathered}
x \leq \neg\neg x \ , \\
\neg x = \neg\neg\neg x \ , \\
x \leq y \Rightarrow \neg y \leq \neg x \ , \\
\neg\neg(x \wedge y) = \neg\neg x \wedge \neg\neg y \ .
\end{gathered}$$

Exercise 2.9.7. Prove that the topology $\mathcal{O}X$ of any topological space X is a Heyting algebra. Describe in topological language the implication $U \Rightarrow V$, the negation $\neg U$, and the regular elements $U = \neg\neg U$ in $\mathcal{O}X$.

Exercise 2.9.8. Show that for a Heyting algebra H, the regular elements of H form a Boolean algebra $H_{\neg\neg} = \{x \in H \mid x = \neg\neg x\}$. Here $H_{\neg\neg}$ is viewed as a subposet of H. Hint: negation $\neg'$, conjunction $\wedge'$, and disjunction $\vee'$ in $H_{\neg\neg}$ are expressed as follows in terms of negation, conjunction and disjunction in H, for $x, y \in H_{\neg\neg}$:

$$\neg' x = \neg x \ , \qquad x \wedge' y = \neg\neg(x \wedge y) \ , \qquad x \vee' y = \neg\neg(x \vee y) \ .$$

From logical point of view, the *classical propositional calculus* CPC is obtained from the intuitionistic propositional calculus by the addition of the logical law known as *tertium non datur*, or the *law of excluded middle*:

$$\frac{}{\Gamma \vdash \phi \vee \neg\phi}$$

Alternatively, we could add the rule of *reductio ad absurdum*, or *proof by contradiction*:

$$\frac{\Gamma \vdash \neg\neg\phi}{\Gamma \vdash \phi} .$$

Identifying logically equivalent formulas of CPC, we obtain a poset $\mathcal{C}_{\mathsf{CPC}}$ ordered by logical entailment. This poset is, of course, the *free Boolean algebra* on the countably many generators $\{p_0, p_1, \dots\}$. The free Boolean algebra can be constructed just as the free Heyting algebra above, either equationally, or in terms of deduction. The equational axioms for a Boolean algebra are the axioms for a lattice (2.30), the distributive laws (2.31), and the complement laws (2.33).

Exercise* 2.9.9. Is $\mathcal{C}_{\mathsf{CPC}}$ isomorphic to the Boolean algebra $\mathcal{C}_{\mathsf{IPC}_{\neg\neg}}$ of the regular elements of $\mathcal{C}_{\mathsf{IPC}}$?

Exercise 2.9.10. Show that in a Heyting algebra H, one has $\neg\neg x = x$ for all $x \in H$ if, and only if, $y \vee \neg y = 1$ for all $y \in H$. *Hint*: half of the equivalence is easy. For the other half, observe that the assumption $\neg\neg x = x$ means that negation is an order-reversing bijection $H \to H$. It therefore transforms joins into meets and vice versa, and so the *De Morgan laws* hold:

$$\neg(x \wedge y) = \neg x \vee \neg y , \qquad\qquad \neg(x \vee y) = \neg x \wedge \neg y .$$

Together with $y \wedge \neg y = 0$, the De Morgan laws easily imply $y \vee \neg y = 1$. See [Joh82, I.1.11].

Kripke semantics for IPC

Let us now prove the Kripke completeness of IPC, extending Theorem 2.8.4, namely:

Theorem 2.9.11 (Kripke completeness for IPC)**.** *Let* $(K, \Vdash)$ *be a Kripke model,* i.e. *a poset* K *equipped with a* forcing relation $k \Vdash p$ *between elements* $k \in K$ *and propositional variables* p*, satisfying*

$$j \leq k,\ k \Vdash p \quad \textit{implies} \quad j \Vdash p. \tag{2.34}$$

Extend $\Vdash$ *to all formulas* ϕ *in* IPC *by defining*

$$\begin{array}{lll}
k \Vdash \top & \textit{always}, & \\
k \Vdash \bot & \textit{never}, & \\
k \Vdash \phi \wedge \psi & \textit{iff} & k \Vdash \phi \textit{ and } k \Vdash \psi , \qquad (2.35)\\
k \Vdash \phi \vee \psi & \textit{iff} & k \Vdash \phi \textit{ or } k \Vdash \psi , \qquad (2.36)\\
k \Vdash \phi \Rightarrow \psi & \textit{iff} & \textit{for all } j \leq k, \textit{ if } j \Vdash \phi, \textit{ then } j \Vdash \psi .
\end{array}$$

Finally, write $K \Vdash \phi$ if $k \Vdash \phi$ for all $k \in K$.

A propositional formula ϕ is then provable from the rules of deduction for IPC *if, and only if, $K \Vdash \phi$ for all Kripke models $(K, \Vdash)$. Briefly:*

$$\mathsf{IPC} \vdash \phi \quad \textit{iff} \quad K \Vdash \phi \textit{ for all } (K, \Vdash).$$

Let us first see that we cannot simply reuse the proof from Theorem 2.8.4 for the positive fragment PPC, because the downset (Yoneda) embedding that we used there

$$\downarrow\, : \mathcal{C}_{\mathsf{PPC}} \hookrightarrow \mathsf{Down}(\mathcal{C}_{\mathsf{PPC}}) \tag{2.37}$$

would not preserve the coproducts $\perp$ and $\phi \vee \psi$. Indeed, $\downarrow(\perp) \neq \emptyset$, because it contains $\perp$ itself! And in general $\downarrow(\phi \vee \psi) \neq \downarrow(\phi) \cup \downarrow(\psi)$, because the righthand side need not contain, e.g., $\phi \vee \psi$.

Instead, we will generalize the Stone Representation theorem 2.6.8 from Boolean algebras to Heyting algebras, using a theorem due to A. Joyal (cf. [MR95, MH92]). First, recall that the Stone representation provided, for any Boolean algebra $\mathcal{B}$, an injective Boolean homomorphism into a powerset,

$$\mathcal{B} \rightarrowtail \mathcal{P}X\,.$$

For X we took the set of prime filters, which we identified with the homset of Boolean homomorphisms $\mathsf{BA}(\mathcal{B}, \mathbb{2})$ by taking the filter-kernel $f^{-1}(1) \subseteq \mathcal{B}$ of a homomorphism $h : \mathcal{B} \to \mathbb{2}$. The injective homomorphism $\eta : \mathcal{B} \rightarrowtail \mathcal{P}(\mathsf{BA}(\mathcal{B}, \mathbb{2}))$ was then given by:

$$\eta(b) = \{F \mid b \in F\} = \{h : \mathcal{B} \to \mathbb{2} \mid f(b) = 1\}\,.$$

Now, the set $\mathsf{BA}(\mathcal{B}, \mathbb{2})$ can be regarded as a (discrete) poset, and since the inclusion $\mathsf{Set} \hookrightarrow \mathsf{Pos}$ as discrete posets is left adjoint to the forgetful functor $|-| : \mathsf{Pos} \to \mathsf{Set}$, for the powerset $\mathcal{P}(\mathsf{BA}(\mathcal{B}, \mathbb{2}))$ we have

$$\mathcal{P}(\mathsf{BA}(\mathcal{B}, \mathbb{2})) \cong \mathsf{Set}(\mathsf{BA}(\mathcal{B}, \mathbb{2}), 2) \cong \mathsf{Pos}(\mathsf{BA}(\mathcal{B}, \mathbb{2}), \mathbb{2}) \cong \mathbb{2}^{\mathsf{BA}(\mathcal{B}, \mathbb{2})}$$

where the latter is the exponential in the cartesian closed category Pos. Transposing the composite of this iso with the Stone representation $\eta : \mathcal{B} \rightarrowtail \mathcal{P}X$ in Pos,

$$\frac{\eta : \mathcal{B} \rightarrowtail \mathcal{P}(\mathsf{BA}(\mathcal{B}, \mathbb{2})) \cong \mathbb{2}^{\mathsf{BA}(\mathcal{B}, \mathbb{2})}}{\tilde{\eta} : \mathsf{BA}(\mathcal{B}, \mathbb{2}) \times \mathcal{B} \to \mathbb{2}}$$

we arrive at the (monotone) evaluation map

$$\tilde{\eta} = \mathsf{eval} : \mathsf{BA}(\mathcal{B}, \mathbb{2}) \times \mathcal{B} \to \mathbb{2}. \tag{2.38}$$

Finally, recall that the category of Boolean algebras is full in the category DLat of distributive lattices, so that

$$\mathsf{BA}(\mathcal{B}, \mathbb{2}) = \mathsf{DLat}(\mathcal{B}, \mathbb{2})\,.$$

Now for any *Heyting algebra* $\mathcal{H}$ (or indeed any distributive lattice), the homset $\mathsf{DLat}(\mathcal{H}, \mathbb{2})$, ordered pointwise, is isomorphic to the *poset* of all prime filters in $\mathcal{H}$ ordered by inclusion, again by taking $h : \mathcal{H} \to \mathbb{2}$ to its (filter) kernel $h^{-1}\{1\} \subseteq \mathcal{H}$. In particular, when $\mathcal{H}$ is not Boolean, the poset $\mathsf{DLat}(\mathcal{H}, \mathbb{2})$ is no longer discrete, since prime filters in a Heyting algebra need not be maximal. Indeed, recall that Proposition 2.6.5 described the prime filters in a Boolean algebra $\mathcal{B}$ as those with a classifying map $h : \mathcal{B} \to \mathbb{2}$ that is a lattice homomorphism and therefore those with a complement $h^{-1}(0) \subseteq \mathcal{B}$ that is a (prime) ideal. In the Boolean case, these were also the *maximal* filters, because the preservation of Boolean negation $\neg b$ allowed us to deduce that for every $b \in \mathcal{B}$, exactly one of b or $\neg b$ must be in such filter F. In a Heyting algebra, however, the last condition need not obtain; and indeed prime filters in a Heyting algebra need not be maximal.

The transpose in Pos of the evaluation map,

$$\mathsf{eval} : \mathsf{DLat}(\mathcal{H}, \mathbb{2}) \times \mathcal{H} \to \mathbb{2}. \tag{2.39}$$

is again a monotone map

$$\eta : \mathcal{H} \longrightarrow \mathbb{2}^{\mathsf{DLat}(\mathcal{H},\mathbb{2})}, \tag{2.40}$$

which takes $p \in \mathcal{H}$ to the "evaluation at p" map $h \mapsto h(p) \in \mathbb{2}$, i.e.,

$$\eta_p(h) = h(p) \qquad \text{for } p \in \mathcal{H} \text{ and } h : \mathcal{H} \to \mathbb{2}\,.$$

As before (cf. Example 2.9.4), the poset $\mathbb{2}^{\mathsf{DLat}(\mathcal{H},\mathbb{2})}$ (ordered pointwise) may be identified with the downsets in the poset $\mathsf{DLat}(\mathcal{H}, \mathbb{2})^{\mathsf{op}}$, ordered by inclusion, which recall from Example 2.9.4 is always a Heyting algebra. Thus, in sum, for any Heyting algebra $\mathcal{H}$, we have a monotone map,

$$\eta : \mathcal{H} \longrightarrow \mathsf{Down}(\mathsf{DLat}(\mathcal{H}, \mathbb{2})^{\mathsf{op}})\,, \tag{2.41}$$

generalizing the Stone representation from Boolean to Heyting algebras.

Theorem 2.9.12 (Joyal)**.** *Let $\mathcal{H}$ be a Heyting algebra. There is an injective homomorphism of Heyting algebras*

$$\mathcal{H} \rightarrowtail \mathsf{Down}(J)$$

into a Heyting algebra of downsets in a poset J.

Note that in this form, the theorem literally generalizes the Stone representation theorem: when $\mathcal{H}$ is Boolean we can take J to be discrete, and then $\mathsf{Down}(J) \cong \mathsf{Pos}(J, \mathbb{2}) \cong \mathsf{Set}(J, 2) \cong \mathcal{P}(J)$ is Boolean, whence the Heyting embedding is also Boolean.

The proof will again use the transposed evaluation map,

$$\eta : \mathcal{H} \longrightarrow \mathsf{Down}(\mathsf{DLat}(\mathcal{H}, \mathbb{2})^{\mathsf{op}}) \cong \mathbb{2}^{\mathsf{DLat}(\mathcal{H},\mathbb{2})}$$

which, as before, is injective, by the Prime Ideal Theorem (see Lemma 2.6.6). We will use it in the following form due to Birkhoff.

Lemma 2.9.13 (Prime Ideal Theorem)**.** *Let D be a distributive lattice, $I \subseteq D$ an ideal, and $x \in D$ with $x \notin I$. There is a prime ideal $I \subseteq P \subset D$ with $x \notin P$.*

Proof. As in the proof of Lemma 2.6.6, it suffice to prove it for the case $I = (0)$. This time, we use Zorn's Lemma: a poset in which every chain has an upper bound has maximal elements. Consider the poset $\mathcal{I}\backslash x$ of "ideals I without x", $x \notin I$, ordered by inclusion. The union of any chain $I_0 \subseteq I_1 \subseteq \ldots$ in $\mathcal{I}\backslash x$ is clearly also in $\mathcal{I}\backslash x$, so we have (at least one) maximal element $M \in \mathcal{I}\backslash x$. We claim that $M \subseteq D$ is prime. To that end, take $a, b \in D$ with $a \wedge b \in M$. If $a, b \notin M$, let $M[a] = \{n \leq m \vee a \mid m \in M\}$, the ideal join of M and $\downarrow(a)$, and similarly for $M[b]$. Since M is maximal without x, we therefore have $x \in M[a]$ and $x \in M[b]$. Thus let $x \leq m \vee a$ and $x \leq m' \vee b$ for some $m, m' \in M$. Then $x \vee m' \leq m \vee m' \vee a$ and $x \vee m \leq m \vee m' \vee b$, so taking meets on both sides gives

$$(x \vee m') \wedge (x \vee m) \leq (m \vee m' \vee a) \wedge (m \vee m' \vee b) = (m \vee m') \vee (a \wedge b).$$

Since the righthand side is in the ideal M, so is the left. But then $x \leq x \vee (m \wedge m')$ is also in M, contrary to our assumption that $M \in \mathcal{I}\backslash x$. □

Proof of Theorem 2.9.12. As in (2.41), let $J^{\mathsf{op}} = \mathsf{DLat}(\mathcal{H}, 2)$ be the poset of prime filters in $\mathcal{H}$, and consider the transposed evaluation map (2.41),

$$\eta : \mathcal{H} \longrightarrow \mathsf{Down}(\mathsf{DLat}(\mathcal{H}, 2)^{\mathsf{op}}) \cong 2^{\mathsf{DLat}(\mathcal{H},2)}$$

given by $\eta(p) = \{F \,|\, p \in F \text{ prime}\} \cong \{h : \mathcal{H} \to 2 \,|\, h(p) = 1\}$.

Clearly $\eta(0) = \emptyset$ and $\eta(1) = \mathsf{DLat}(\mathcal{H}, 2)$, and similarly for the other meets and joins, so η is a lattice homomorphism. Moreover, if $p \neq q \in \mathcal{H}$ then, as in the proof of 2.6.8, we have that $\eta(p) \neq \eta(q)$, by the Prime Ideal Theorem (Lemma 2.9.13). Thus it only remains to show that

$$\eta(p \Rightarrow q) \;=\; \eta(p) \Rightarrow \eta(q)\,.$$

Unwinding the definitions, this means that, for all $h \in \mathsf{DLat}(\mathcal{H}, 2)$,

$$h(p \Rightarrow q) = 1 \quad \text{iff} \quad \text{for all } g \geq h,\ g(p) = 1 \text{ implies } g(q) = 1. \tag{2.42}$$

Equivalently, for all prime filters $F \subseteq \mathcal{H}$,

$$p \Rightarrow q \in F \quad \text{iff} \quad \text{for all prime } G \supseteq F,\ p \in G \text{ implies } q \in G. \tag{2.43}$$

Now if $p \Rightarrow q \in F$, then for all (prime) filters $G \supseteq F$, also $p \Rightarrow q \in G$, and so $p \in G$ implies $q \in G$, since $(p \Rightarrow q) \wedge p \leq q$.

Conversely, suppose $p \Rightarrow q \notin F$, and we seek a prime filter $G \supseteq F$ with $p \in G$ but $q \notin G$. Consider the filter

$$F[p] = \{x \wedge p \leq h \in \mathcal{H} \mid x \in F\}\,,$$

which is the join of F and $\uparrow(p)$ in the poset of filters. If $q \in F[p]$, then $x \wedge p \leq q$ for some $x \in F$, whence $x \leq p \Rightarrow q$, and so $p \Rightarrow q \in F$, contrary to assumption; thus $q \notin F[p]$. By the Prime Ideal Theorem again (applied to the distributive lattice $\mathcal{H}^{\mathsf{op}}$) there is a prime filter $G \supseteq F[p]$ with $q \notin G$. □

Exercise 2.9.14. Give a Kripke countermodel to show that the Law of Excluded Middle $\phi \vee \neg\phi$ is not provable in IPC.

2.10 Frames and spaces

Recall that a *supremum* (least upper bound) of $S \subseteq P$ in a poset P is an element $\bigvee S \in P$ such that, for all $y \in S$,

$$\bigvee S \leq y \iff \forall x : S \, . \, x \leq y \, .$$

In particular, $\bigvee \emptyset$ is a least element of P and $\bigvee P$ is a greatest element of P.

A poset $(P, \leq)$ is said to be *complete* if it has suprema of *all* subsets. Viewed as a category, P is *both complete and cocomplete* when it is complete as a poset. This is so, first, because coequalizers in a poset always exist, and coproducts are exactly suprema, so a complete poset has all colimits. And moreover, it then also has infima (greatest lower bounds) of arbitrary subsets, and so it is also complete as a category. Indeed, an infimum of $S \subseteq P$ is an element $\bigwedge S \in P$ such that, for all $y \in S$,

$$y \leq \bigwedge S \iff \forall x : S \, . \, y \leq x \, .$$

Proposition 2.10.1. *A poset is complete if, and only if, it has infima $\bigwedge S$ for all subsets $S \subseteq P$.*

Proof. Infima and suprema are expressed in terms of each other as follows:

$$\bigwedge S = \bigvee \left\{ y \in P \;\middle|\; \forall x : S \, . \, y \leq x \right\} ,$$
$$\bigvee S = \bigwedge \left\{ y \in P \;\middle|\; \forall x : S \, . \, x \leq y \right\} .$$

□

The basic examples of complete posets are the powersets $\mathcal{P}X$, and these are Boolean algebras, and therefore also Heyting. When is a general complete poset P cartesian closed? Being complete, P has the terminal object, namely the greatest element $\bigvee P = 1 \in P$, and it has binary products which are binary infima. If P is cartesian closed then for all $x, y \in P$ there exists an exponential $(x \Rightarrow y) \in P$, which satisfies, for all $z \in P$,

$$\frac{z \wedge x \leq y}{\overline{z \leq x \Rightarrow y}} \, .$$

So, if it exists, we must have

$$(x \Rightarrow y) \;=\; \bigvee \{ \, z \mid z \leq (x \Rightarrow y) \, \} \;=\; \{ \, z \mid (z \wedge x) \leq y \, \} \, .$$

Indeed, observe that with the help of the adjunction, we can derive the *infinite distributive law*:

$$x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I} (x \wedge y_i) \tag{2.44}$$

for any family of elements $\{y_i \in P \mid i \in I\}$, as follows:

$$\frac{\frac{\frac{\frac{x \wedge \bigvee_{i\in I} y_i \leq z}{\bigvee_{i\in I} y_i \leq (x \Rightarrow z)}}{\forall i : I\,.\,(y_i \leq (x \Rightarrow z))}}{\forall i : I\,.\,(x \wedge y_i \leq z)}}{\bigvee_{i\in I}(x \wedge y_i) \leq z}$$

Now since $x \wedge \bigvee_{i\in I} y_i$ and $\bigvee_{i\in I}(x \wedge y_i)$ have the same upper bounds they must be equal.

Conversely, suppose the distributive law (2.44) holds, and as suggested above, let us *define* $x \Rightarrow y$ to be

$$(x \Rightarrow y) = \bigvee \{z \in P \mid x \wedge z \leq y\}\ . \tag{2.45}$$

Now check that this element $x \Rightarrow y$ has the universal property of an exponential (in a poset): for all $x, y \in P$,

$$x \wedge (x \Rightarrow y) \leq y\ , \tag{2.46}$$

and for $z \in P$,

$$x \wedge z \leq y \quad \text{implies} \quad z \leq x \Rightarrow y\ .$$

The implication follows directly from (2.45), and (2.46) follows from the distributive law:

$$x \wedge (x \Rightarrow y) = x \wedge \bigvee \{z \in P \mid x \wedge z \leq y\} = \bigvee \{x \wedge z \mid x \wedge z \leq y\} \leq y\ .$$

Such complete, cartesian closed posets are called *frames*.

Definition 2.10.2. A *frame* is a complete poset satisfying the (infinite) distributive law.

$$x \wedge \bigvee_{i\in I} y_i = \bigvee_{i\in I}(x \wedge y_i)\ .$$

Equivalently a frame is a complete Heyting algebra.

A *frame morphism* is a function $f : L \to M$ between frames that preserves finite infima $x \wedge y$ and arbitrary suprema $\bigvee_{i\in I} y_i$. The category of frames and frame morphisms is denoted by Frame.

Warning: just as we need to *require* the preservation of meets $x \wedge y$ although they are determined by the joins $\bigvee_{i\in I} y_i$, a frame morphism need not preserve exponentials $x \Rightarrow y$!

Example 2.10.3. Given a poset P, the downsets $\downarrow P$ form a complete lattice under the inclusion order $S \subseteq T$, and with the set theoretic operations $\bigcup$ and $\bigcap$ as $\bigvee$ and $\bigwedge$. Since $\mathsf{Down}(P)$ is already known to be a Heyting algebra (Example 2.9.4), it is therefore also a frame. (Alternately, we can show that it is a frame by noting that the operations $\bigcup$ and $\cap$ satisfy the infinite distributive law.)

Any monotone map $f : P \to Q$ between posets then gives rise to a frame map

$$\mathsf{Down}(f) :\ \mathsf{Down}(Q) \longrightarrow \mathsf{Down}(P)\ .$$

(Note the direction!) This is easily seen by recalling that $\mathsf{Down}(P) \cong \mathsf{Pos}(P, \mathbb{2})$ as posets.

Note, moreover, that $\mathsf{Pos}(f, \mathbb{2}) : \mathsf{Pos}(Q, \mathbb{2}) \longrightarrow \mathsf{Pos}(P, \mathbb{2})$ is a (co)limit preserving functor on complete posets, since the (co)limits are just set-theoretic unions and intersections. So $\mathsf{Pos}(f, \mathbb{2})$ therefore has both left and right adjoints. These functors are usually written $f_! \dashv f^* \dashv f_*$. Although it does not in general preserve Heyting implications $S \Rightarrow T$, the monotone map $f^* : \mathsf{Down}(Q) \longrightarrow \mathsf{Down}(P)$ is indeed a morphism of frames. We therefore have a contravariant functor

$$\mathsf{Down}(-) : \ \mathsf{Pos} \to \mathsf{Frame}^{\mathsf{op}}. \tag{2.47}$$

Example 2.10.4. The topology $\mathcal{O}X$ of a topological space X, ordered by inclusion, is a frame, because finite intersections and arbitrary unions of open sets are open. The distributive law holds because intersections distribute over unions. If $f : X \to Y$ is a continuous map between topological spaces, the inverse image map $f^* : \mathcal{O}Y \to \mathcal{O}X$ is a frame homomorphism. Thus, there is a functor

$$\mathcal{O} : \mathsf{Top} \to \mathsf{Frame}^{\mathsf{op}}$$

which maps a space X to its topology $\mathcal{O}X$ and a continuous map $f : X \to Y$ to the inverse image map $f^* : \mathcal{O}Y \to \mathcal{O}X$.

Motivated by the previous examples, we introduce the category of *locales* as the opposite of the category of frames:

$$\mathsf{Loc} = \mathsf{Frame}^{\mathsf{op}}\,.$$

We can think of a locale as a "generalized space".

Example 2.10.5. Let P be a poset and define a topology on the elements of P by defining the opens to be the downsets,

$$\mathcal{O}(P) = \mathsf{Down}(P) \cong \mathsf{Pos}(P, \mathbb{2}).$$

These open sets are not only closed under arbitrary unions and finite intersections, but also under *arbitrary* intersections. Such a topological space, in which the opens are closed under all intersections, is said to be an *Alexandroff space* (note that the opens could equivalently be defined to be the upsets). The associated locale is the one in Example 2.10.4, and the associated frame is the one in Example 2.10.3.

Exercise* 2.10.6. This exercise is meant for students with some knowledge of topology. For a topological space X and a point $x \in X$, let $N(x)$ be the neighborhood filter of x,

$$N(x) = \left\{ U \in \mathcal{O}X \;\middle|\; x \in U \right\}\ .$$

Recall that a T_0-*space* is a topological space X in which points are determined by their neighborhood filters,

$$N(x) = N(y) \Rightarrow x = y\ . \qquad (x, y \in X)$$

Let Top_0 be the full subcategory of Top on T_0-spaces. The functor $\mathcal{O} : \mathsf{Top} \to \mathsf{Loc}$ restricts to a functor $\mathcal{O} : \mathsf{Top}_0 \to \mathsf{Loc}$. Prove that $\mathcal{O} : \mathsf{Top}_0 \to \mathsf{Loc}$ is a faithful functor. Is it full?

Topological semantics for IPC

It should now be clear how to interpret IPC into a topological space X: each formula ϕ is assigned to an open set $[\![\phi]\!] \in \mathcal{O}X$ in such a way that $[\![-]\!]$ is a homomorphism of Heyting algebras.

Definition 2.10.7. A *topological model* of IPC consists of a space X and a function

$$[\![-]\!] : \mathsf{Var} \to \mathcal{O}(X)$$

from the propositional variables $\mathsf{Var} = \{p_0, p_1, \dots\}$ to open sets of X. The interpretation is then extended to all formulas,

$$[\![-]\!] : \mathsf{IPC} \to \mathcal{O}X\,,$$

by setting:

$$\begin{aligned}
[\![\top]\!] &= X \\
[\![\bot]\!] &= \emptyset \\
[\![\phi \wedge \psi]\!] &= [\![\phi]\!] \cap [\![\psi]\!] \\
[\![\phi \vee \psi]\!] &= [\![\phi]\!] \cup [\![\psi]\!] \\
[\![\phi \Rightarrow \psi]\!] &= [\![\phi]\!] \Rightarrow [\![\psi]\!]\,.
\end{aligned}$$

The Heyting implication $[\![\phi]\!] \Rightarrow [\![\psi]\!]$ in $\mathcal{O}X$, is defined as in (2.45) as

$$[\![\phi]\!] \Rightarrow [\![\psi]\!] \;=\; \bigcup \big\{ U \in \mathcal{O}X \;\big|\; U \wedge [\![\phi]\!] \leq [\![\psi]\!] \big\}\,.$$

Joyal's representation theorem 2.9.12 then easily implies that IPC is sound and complete with respect to topological semantics.

Corollary 2.10.8. *A formula ϕ is provable in* IPC *if, and only if, it holds in every topological interpretation $[\![-]\!]$ into a space X, briefly:*

$$\mathsf{IPC} \vdash \phi \qquad \textit{iff} \qquad [\![\phi]\!] = X \textit{ for all spaces } X\,.$$

Proof. Put the Alexandroff topology on the downsets of prime filters in the Heyting algebra IPC. □

Exercise 2.10.9. Give a topological countermodel to show that the Law of Double Negation $\neg\neg\phi \Rightarrow \phi$ is not provable in IPC.

Modal logic

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