

# Kripke-Joyal Forcing for Martin-Löf Type Theory

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# Motivation

- Martin L f type theory (MLTT) is common generalization of first-order logic (FOL) and the simply-typed lambda calculus, and is a powerful and expressive system of formal logic.
- It serves as the basis of Homotopy Type Theory, as well as several computer proof systems such as Agda, Coq, and Lean.
- It is a challenging problem to give semantics for MLTT that are both precise enough to strictly model the syntax and yet flexible enough to admit basic mathematical constructions.
- Kripke-Joyal forcing provides such semantics for both FOL and HOL and is here generalized to MLTT.

# Kripke-Joyal forcing for FOL

Let  $\mathbb{C}$  be a small category. For the topos of presheaves, write

$$\widehat{\mathbb{C}} = [\mathbb{C}^{\text{op}}, \text{Set}].$$

We interpret a FOL formula  $x:X \mid \phi$  over  $X \in \widehat{\mathbb{C}}$  as a subobject,

$$\{x:X \mid \phi\} \rightarrowtail X.$$

**Definition.** Let  $x:yc \rightarrow X$ . We say that  $x$  **forces  $\phi$  at stage  $c$** , if there is a factorization as on the right below.

$$c \Vdash \phi(x) \qquad \begin{array}{ccc} & & \{x:X \mid \phi\} \\ & \nearrow \text{dotted} & \downarrow \\ yc & \xrightarrow{x} & X \end{array}$$

# Kripke-Joyal forcing for FOL

## Remark

- If  $c \Vdash \phi(x)$  for *all* elements  $x : yc \rightarrow X$  we then have

$$\{x : X \mid \phi\} \cong X,$$

- If  $\phi$  is *closed* we then have

$$\{\phi\} \cong 1.$$

- We then say that  $\phi$  **holds** on  $\mathbb{C}$  and write

$$\mathbb{C} \Vdash \phi.$$

# Kripke-Joyal forcing for FOL

**Key fact:** We can recursively unwind the condition  $c \Vdash \phi(x)$  according to the structure of  $\phi$ ,

$c \Vdash \phi(x) \vee \psi(x)$       iff     $c \Vdash \phi(x)$  or  $c \Vdash \psi(x)$

$c \Vdash \phi(x) \wedge \psi(x)$       iff     $c \Vdash \phi(x)$  and  $c \Vdash \psi(x)$

$c \Vdash \phi(x) \Rightarrow \psi(x)$       iff     $d \Vdash \phi(xf)$  implies  $d \Vdash \psi(xf)$ , for all  $f : d \rightarrow c$

$c \Vdash \exists y. \vartheta(x, y)$       iff     $c \Vdash \vartheta(x, y)$  for some  $y : yc \rightarrow Y$

$c \Vdash \forall y. \vartheta(x, y)$       iff     $d \Vdash \vartheta(xf, y)$  for all  $f : d \rightarrow c$  and  $y : yd \rightarrow Y$

This provides a quasi-mechanical procedure for determining whether a formula holds in a model.

# Kripke-Joyal forcing for MLTT

For MLTT we instead need to force a *dependent type*.

$$x:X \vdash A$$

This is interpreted as a map  $A \rightarrow X$  (an indexed family  $A_x$ ), rather than a mere subobject  $\{x:X \mid \phi\} \rightarrowtail X$ .

We will also need to *force a term*.

$$x:X \vdash t:A$$

This is interpreted as a section.

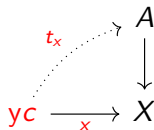


The diagram shows a vertical arrow pointing downwards from the object  $A$  to the object  $X$ . To the left of this arrow, there is a curved arrow pointing upwards from  $X$  to  $A$ . This curved arrow is labeled with a red  $t$  and is drawn with a dotted line.

# Kripke-Joyal forcing for MLTT

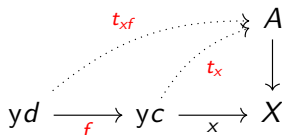
To force terms **in stages** we use **partial** sections.

$$c \Vdash t_x : A_x$$



Changing stages along  $f : d \rightarrow c$  results in a **coherence condition**.

$$\frac{c \Vdash t_x : A_x}{d \Vdash t_{xf} : A_{xf}}$$



# Kripke-Joyal forcing for MLTT

This requires a stricter interpretation than in the propositional case:

$$\frac{c \Vdash \phi(x)}{d \Vdash \phi(xf)}$$

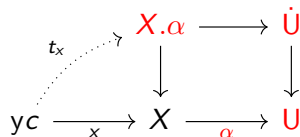
The diagram illustrates the forcing relation for the function symbol  $f$ . It shows a node  $yd$  on the left, which has a solid arrow labeled  $f$  pointing to a node  $yc$ . From  $yc$ , there is a solid arrow labeled  $x$  pointing to a node  $X$ . Above the node  $X$  is a set notation  $\{x:X \mid \phi\}$ , with a solid arrow pointing down to  $X$ . Two dotted curved arrows originate from the node  $yd$ : one points to the set notation  $\{x:X \mid \phi\}$ , and the other points to the node  $yc$ .



# Kripke-Joyal forcing for MLTT

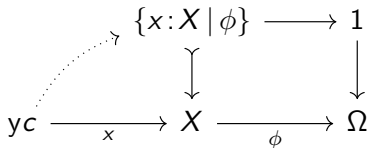
We will use a *universe* to ensure coherence.

$$c \Vdash t_x : \alpha(x)$$



This is like using the *subobject classifier* to interpret FOL.

$$c \Vdash \phi(x)$$



# Kripke-Joyal forcing for MLTT

## Proposition (Forcing terms)

*For any type in context  $X \vdash \alpha$  the following are equivalent.*

- *there is a term  $t$  such that*

$$X \vdash t : \alpha$$

- *for all  $x : yc \rightarrow X$  there is given coherently  $t_x$  such that*

$$c \Vdash t_x : \alpha(x).$$

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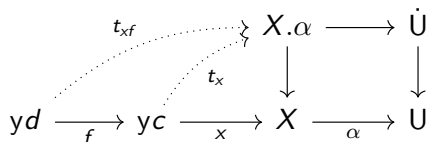
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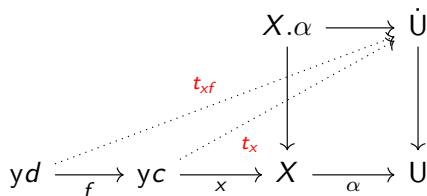
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# Kripke-Joyal forcing for MLTT

**Proof.** Coherence means that  $t_{xf} = t_x \circ f$ .

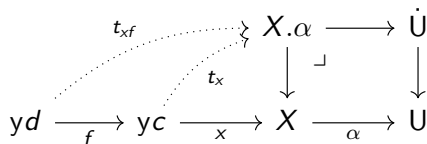


But these partial sections correspond to partial lifts of  $\alpha$ ,

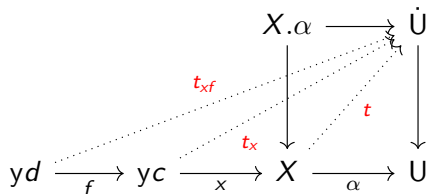


# Kripke-Joyal forcing for MLTT

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But these partial sections correspond to partial lifts of  $\alpha$ ,



So the proof that  $X \vdash t : \alpha$  is complete by Yoneda.

# Outline

- 1 The universe  $\dot{U} \rightarrow U$
- 2 The natural model of MLTT
- 3 The Kripke-Joyal forcing rules
- 4 The completeness theorem

# 1. The universe $\dot{U} \rightarrow U$

For  $\kappa$  sufficiently large, define small categories:

$$\mathrm{Set}_{\kappa} \hookrightarrow \mathrm{Set} \qquad \textit{small sets}$$

$$\dot{\mathrm{Set}}_{\kappa} \hookrightarrow \dot{\mathrm{Set}} \qquad \textit{small pointed sets}$$

These can be lifted to presheaves:

$$\dot{U} = \mathrm{Cat}(\mathbb{C}/_{-}^{\mathrm{op}}, \dot{\mathrm{Set}}_{\kappa})$$

$$U = \mathrm{Cat}(\mathbb{C}/_{-}^{\mathrm{op}}, \mathrm{Set}_{\kappa})$$

with the evident natural map  $\dot{U} \rightarrow U$  that “forgets the point”.

# 1. The universe $\dot{U} \rightarrow U$

## Definition (Small presheaves)

A presheaf  $A$  is *small* if all its values are small.

A map  $A \rightarrow X$  is *small* if all its fibers  $A_x$  are small.

$$\begin{array}{ccc} A_x & \longrightarrow & A \\ \downarrow & & \downarrow \\ yx & \xrightarrow{x} & X \end{array}$$

## Lemma ( $\dot{U} \rightarrow U$ classifies small maps)

For small  $A \rightarrow X$  there is an  $\alpha : X \rightarrow U$  and a pullback

$$\begin{array}{ccc} A & \longrightarrow & \dot{U} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\alpha} & U \end{array}$$



# 1. The universe $\dot{U} \rightarrow U$

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## Lemma ( $\dot{U} \rightarrow U$ classifies small maps)

For small  $A \rightarrow X$  there is a *canonical*  $\alpha : X \rightarrow U$  and a *chosen* pullback

$$\begin{array}{ccccc} A & \xrightarrow{\cong} & X.\alpha & \longrightarrow & \dot{U} \\ \downarrow & & \downarrow \lrcorner & & \downarrow \\ X & \xrightarrow{=} & X & \xrightarrow{\alpha} & U \end{array}$$

## 2. The natural model of MLTT

Let  $f : Y \rightarrow X$  and consider the two-pullbacks diagram arising from substitution.

$$\frac{X \vdash \alpha}{Y \vdash \alpha f}$$

$$\begin{array}{ccccc} Y.\alpha f & \longrightarrow & X.\alpha & \longrightarrow & \dot{U} \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ Y & \xrightarrow{f} & X & \xrightarrow{\alpha} & U \end{array}$$

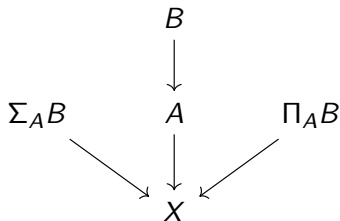
The pullback functor  $f^*$  is modeled by precomposition of classifying maps into  $U$ .

$$\begin{array}{ccccc} Y & & \text{Hom}(Y, U) & \xrightarrow{\sim} & \mathcal{S}/_Y \hookrightarrow \mathcal{E}/_Y \\ \downarrow f & & \uparrow - \circ f & & \uparrow f^* \\ X & \xrightarrow{\alpha} & U & & \text{Hom}(X, U) \xrightarrow{\sim} \mathcal{S}/_X \hookrightarrow \mathcal{E}/_X \end{array}$$

## 2. The natural model of MLTT

For small  $A \rightarrow X$  the adjoint functors

$$\Sigma_A B \dashv A^* \dashv \Pi_A B$$



all preserve the small maps,

$$\begin{array}{ccc}
 \mathcal{S}/_A & \hookrightarrow & \mathcal{E}/_A \\
 \Sigma_A \left( \begin{array}{c} \uparrow \\ \downarrow \end{array} \right) A^* \Pi_A & & \Sigma_A \left( \begin{array}{c} \uparrow \\ \downarrow \end{array} \right) A^* \Pi_A \\
 \mathcal{S}/_X & \hookrightarrow & \mathcal{E}/_X
 \end{array}$$

## 2. The natural model of MLTT

The type formers  $\Sigma, \Pi$  are then induced by structure on  $\dot{U} \rightarrow U$ , just as the quantifiers  $\exists, \forall$  are induced by structure on  $1 \rightarrow \Omega$ .

For subobjects:

$$\begin{array}{ccc}
 A & \Omega^A & \text{Hom}(1, \Omega^A) \xrightarrow{\cong} \text{Sub}(A) \\
 \downarrow & \left( \begin{array}{c} \uparrow \\ \exists_A \quad \quad \forall_A \\ \downarrow \end{array} \right) \begin{array}{c} * \\ \downarrow \end{array} & \left( \begin{array}{c} \uparrow \\ \exists_A \quad \quad \forall_A \\ \downarrow \end{array} \right) \\
 1 & \Omega & \text{Hom}(1, \Omega) \xrightarrow{\cong} \text{Sub}(1)
 \end{array}$$

For types we require some preliminaries.

## 2. The natural model of MLTT

The polynomial object

$$PU = \sum_{A:U} U^{[A]}$$

classifies *types in context*:

$$\frac{(A, B) : \Gamma \longrightarrow PU}{\Gamma.A \vdash B}$$

Similarly, the object

$$P\dot{U} = \sum_{A:U} \dot{U}^{[A]}$$

classifies *terms in context*  $\Gamma.A \vdash \textcolor{red}{b} : B$ .

## 2. The natural model of MLTT

### Proposition

*The universe  $\dot{U} \rightarrow U$  models the rules for products just if there are maps  $\lambda, \Pi$  making a pullback diagram.*

$$\begin{array}{ccc} P\dot{U} & \xrightarrow{\lambda} & \dot{U} \\ \downarrow & \lrcorner & \downarrow \\ PU & \xrightarrow{\Pi} & U \end{array}$$

## 2. The natural model of MLTT

The right adjoint  $A^* \dashv \Pi_A B$  is then induced by composing classifying maps with  $\Pi : PU \rightarrow U$ .

$$\begin{array}{ccc}
 B & & \text{Hom}(X, PU) \longrightarrow \mathcal{S}^2/X \\
 \downarrow & & \uparrow \quad \downarrow \quad \Pi \\
 A & \swarrow \Pi_A B & * \\
 \downarrow & & \text{Hom}(X, U) \longrightarrow \mathcal{S}/X \\
 X & & \uparrow \quad \downarrow \quad \Pi \\
 & & *
 \end{array}$$

## 2. The natural model of MLTT

There is a similar structure  $\Sigma : PU \rightarrow U$  inducing the left adjoint  $\Sigma_A \dashv A^*$ .

$$\begin{array}{c} B \\ \downarrow \\ A \\ \downarrow \\ X \end{array} \quad \begin{array}{c} \Sigma_A B \\ \searrow \end{array}$$

$$\begin{array}{ccc} \mathrm{Hom}(X, PU) & \longrightarrow & \mathcal{S}^2/X \\ \Sigma \uparrow \quad * & & \Sigma \uparrow \quad * \\ \mathrm{Hom}(X, U) & \longrightarrow & \mathcal{S}/X \end{array}$$



## 2. The natural model of MLTT

### Proposition

*The natural model structure on the universe provides a strict interpretation of MLTT.*

$$\begin{array}{ccc}
 & PU & \\
 \Sigma \left( \begin{array}{c} \uparrow \\ * \\ \downarrow \end{array} \right) \Pi & & \\
 & U &
 \end{array}
 \qquad
 \begin{array}{ccc}
 \text{Hom}(X, PU) & \xrightarrow{\cong} & \mathcal{S}^2/X \\
 \Sigma \left( \begin{array}{c} \uparrow \\ * \\ \downarrow \end{array} \right) \Pi & & \Sigma \left( \begin{array}{c} \uparrow \\ * \\ \downarrow \end{array} \right) \Pi \\
 \text{Hom}(X, U) & \xrightarrow[\cong]{} & \mathcal{S}/X
 \end{array}$$

We use this structure to give forcing conditions for  $\Sigma$  and  $\Pi$  at  $x : y \rightarrow X$ , as in

$$c \Vdash t : \Sigma_{y:\alpha(x)} \beta(x, y)$$

$$c \Vdash t : \Pi_{y:\alpha(x)} \beta(x, y)$$

### 3. The Kripke-Joyal forcing rules

#### Theorem (A-Gambino-Hazratpour)

Let  $X \in \widehat{\mathbb{C}}$  and  $\alpha : X \rightarrow \mathbf{U}$  and  $\beta : X.\alpha \rightarrow \mathbf{U}$ .

For all  $x : yc \rightarrow X$ , we have

$c \Vdash t : 0$	<i>iff</i> $t \neq t$
$c \Vdash t : 1$	<i>iff</i> $t = *$
$c \Vdash t : (\alpha + \beta)(x)$	<i>iff</i> $c \Vdash a : \alpha(x)$ or $c \Vdash b : \beta(x)$
$c \Vdash t : (\alpha \times \beta)(x)$	<i>iff</i> $c \Vdash a : \alpha(x)$ and $c \Vdash b : \beta(x)$
$c \Vdash t : (\Sigma_\alpha \beta)(x)$	<i>iff</i> $c \Vdash a : \alpha(x)$ and $c \Vdash b : \beta(x, a)$
$c \Vdash t : (\alpha \rightarrow \beta)(x)$	<i>iff</i> for all $f : d \rightarrow c$ and $d \Vdash a : \alpha(xf)$ there's $d \Vdash b_{f,a} : \beta(xf)$ coherently
$c \Vdash t : (\Pi_\alpha \beta)(x)$	<i>iff</i> for all $f : d \rightarrow c$ and $d \Vdash a : \alpha(xf)$ there's $d \Vdash b_{f,a} : \beta(xf, a)$ coherently

### 3. Kripke-Joyal forcing rules

Finally, we set:

**Definition.** Let  $X \in \widehat{\mathbb{C}}$  and  $\alpha : X \rightarrow \mathbf{U}$  a type over  $X$ .

We say that  $\mathbb{C}$  **forces a term of type**  $\alpha$ ,

$$\mathbb{C} \Vdash X \vdash t : \alpha$$

if for all  $c \in \mathbb{C}$  and all  $x : yc \rightarrow X$ , there is given coherently

$$c \Vdash t_x : \alpha(x).$$

## 4. The completeness theorem

### Theorem (A-Gambino-Hazratpour)

*Let  $C$  be a closed type in MLTT with the type forming operations*

$$0, 1, X, A + B, A \times B, A \rightarrow B, \Sigma_A B, \Pi_A B, s =_A t.$$

*There is a closed term  $\vdash t : C$  if, and only if, for all categories  $\mathbb{C}$  and all presheaves  $X$  on  $\mathbb{C}$ , one has  $\mathbb{C} \Vdash t : C$ . Briefly,*

$$\text{MLTT} \vdash t : C \quad \text{iff} \quad \mathbb{C} \Vdash t : C \quad \text{for all } \mathbb{C} \text{ and } X.$$

*Moreover, it suffices to assume that  $\mathbb{C}$  is a poset.*

## 4. The completeness theorem

**Proof.** Let  $\mathbb{T}$  be the classifying category of MLTT and  $p : \mathrm{Sh}(X_{\mathbb{T}}) \twoheadrightarrow \widehat{\mathbb{T}}$  the spatial cover. Take  $P = \mathcal{O}X_{\mathbb{T}}$ .

There are LCCC embeddings:

$$\mathbb{T} \xhookrightarrow{y} \widehat{\mathbb{T}} \xhookrightarrow{p^*} \mathrm{Sh}(X_{\mathbb{T}}) \hookrightarrow \widehat{\mathcal{O}X_{\mathbb{T}}}.$$

So we have:

$$\begin{array}{llll} \mathrm{MLTT} \vdash t : C & \iff & 1 \xrightarrow{t} C & \mathbb{T} \\ & \iff & 1 \cong y1 \xrightarrow{yt} yC \cong \llbracket C \rrbracket^{\mathbb{T}} & \widehat{\mathbb{T}} \\ & \iff & 1 \cong p^*y1 \xrightarrow{p^*yt} p^*yC \cong \llbracket C \rrbracket^{X_{\mathbb{T}}} & \mathrm{Sh}(X_{\mathbb{T}}) \\ & \iff & \mathcal{O}X_{\mathbb{T}} \Vdash t : C & \square \end{array}$$

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