

Algebraic Type Theory

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1. Natural Models of DTT

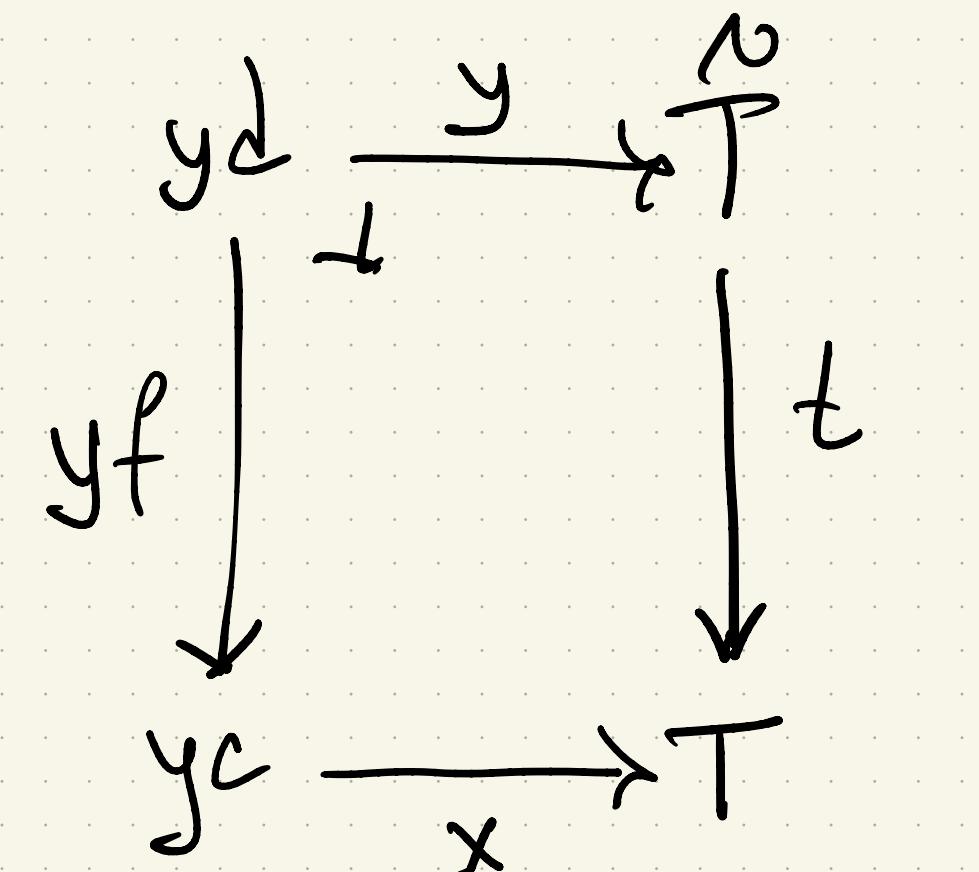
Def. (A.2012) A natural model consists of :

- a cat C
- presheaves T, \tilde{T}
- a natural transf. $t: \tilde{T} \rightarrow T$

that is representable:

$\forall c \in C \quad \forall x \in T_c :$

$\exists f: d \rightarrow c \quad \exists y \in \tilde{T}_d :$



Remarks

(1) • C cat of ctx's

• T presheaf of types

• \hat{T} presheaf of terms

(2) Representability is ctx-extension:

(3) This is equivalent to CwF.

(4) (A.2012) gives conditions on t equivalent to the CwF having

$1, \Sigma, \Pi, Eq, Id$

$C \vdash a : A$

\iff

$$\begin{array}{ccc} & a & \vdash t \\ & \swarrow & \downarrow \\ yC & \xrightarrow[A]{} & T \end{array}$$

$$\begin{array}{ccc} yC, A & \xrightarrow{q_A} & \hat{T} \\ \downarrow P_A & \downarrow \perp & \downarrow t \\ yC & \xrightarrow[A]{} & T \end{array}$$

(5) Namely, e.g.:

$$\begin{array}{ccc} 1 & \xrightarrow{\quad} & \tilde{T} \\ \downarrow ! & \downarrow t & \downarrow \\ 1 & \xrightarrow{\quad} & T \end{array}$$

unit type

$$\begin{array}{ccc} \tilde{T}^2 & \xrightarrow{\quad} & \tilde{T} \\ \downarrow t^2 & \downarrow (\Sigma) & \downarrow t \\ T^2 & \xrightarrow{\quad} & T \end{array}$$

dependent sum

$$\begin{array}{ccc} \tilde{T}^* & \xrightarrow{\quad} & \tilde{T} \\ \downarrow t^* & \downarrow (\pi) & \downarrow t \\ \tilde{T}^* & \xrightarrow{\quad} & T \end{array}$$

dependent product

(6) We shall abstract this structure to form
that of a "Martin-Löf algebra".

2. Polynomial Functors

Let \mathcal{E} be LCCC.

Every $f: A \rightarrow B$ determines a polynomial functor

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\quad P_f \quad} & \mathcal{E} \\ A^* \downarrow & & \nearrow B! \\ \mathcal{E}/A & \xrightarrow{f^*} & \mathcal{E}/B \end{array}$$

$$\begin{array}{ccc} X & \xleftarrow{X \times A} & P_f X \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$

2. Polynomial Functors

(1) In the internal DTT of \mathcal{E} :

$$Pf(X) = B! f_+ A^*(X) = B! f_+ f^* B^*(X) = \sum_{b:B} X^{A_b}.$$

(2) UMP of $Pf X$ is: $(b,x): \mathbb{Z} \xrightarrow{\quad} Pf X$

$$\begin{array}{ccc} X & \xleftarrow{x} & A_b \\ & & \downarrow \\ \mathbb{Z} & \xrightarrow[b]{} & B \end{array} .$$

(3) The assignment $f \mapsto Pf$ is functorial on pullbacks:

$$\begin{array}{ccc} A & \xrightarrow{\quad} & C \\ f \downarrow \perp & & \downarrow g \\ B & \xrightarrow{\quad} & D \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} Pf & \xrightarrow{\quad} & Pg \\ \varepsilon \downarrow \perp & \nearrow \cdot \perp & \downarrow \varepsilon \\ \varepsilon & \xrightarrow{\quad} & \varepsilon \end{array}$$

$$\varepsilon^I \supset \varepsilon^I \xrightarrow{\quad P \quad} End^c(\varepsilon) \subset [\varepsilon, \varepsilon]$$

$\xrightarrow{\quad f \otimes f \quad}$

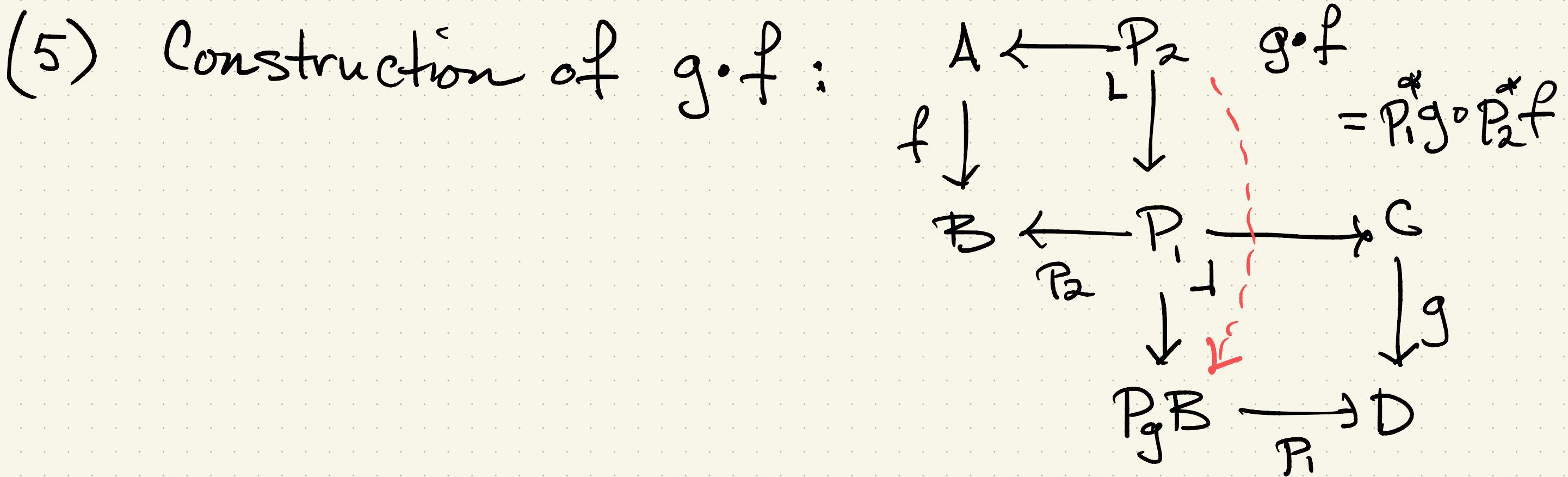
(4) The composite of polynomial functors is polynomial:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\quad P_{g \circ f} \quad} & \mathcal{E} \\ & \searrow P_f & \swarrow P_g \\ & \mathcal{E} & \end{array}$$

$$\begin{array}{ccccc} A & & C & & E \\ f \downarrow & & g \downarrow & \rightsquigarrow & \downarrow g \circ f \\ B & & D & & F \end{array}$$

And $P(! \downarrow^1) = 1_{\mathcal{E}}$, so there is an equivalence of monoids:

$$(\mathcal{E}^I, \cdot, !) \cong (\text{Poly}(\mathcal{E}), \circ, 1_{\mathcal{E}}).$$



(6) Polynomials preserve pullbacks, so they lift to \mathcal{E}^I :

$$\begin{matrix} \mathcal{E}^I & \xrightarrow{P_f^I} & \mathcal{E}^I \\ \downarrow & & \downarrow \\ \mathcal{E} & \xrightarrow{P_f} & \mathcal{E} \end{matrix}$$

$$\begin{matrix} C & \xrightarrow{P_f^I} & P_f C \\ g \downarrow & \nearrow & \downarrow \\ D & & P_f D \end{matrix}$$

$P_f g := f^* g$

3. M-L Algebras

Def. A M-L algebra in a LCCC \mathcal{E} is a map

$$t: \tilde{T} \rightarrow T$$

with structure:

$$\begin{array}{ccc} ! & \downarrow & \tilde{T} \\ \downarrow & \downarrow & \downarrow t \\ - & \rightarrow & T \end{array}$$

$$\begin{array}{ccc} \tilde{T}^2 & \xrightarrow{+} & \tilde{T} \\ \downarrow t \cdot t & & \downarrow t \\ T^2 & \xrightarrow{\quad} & T \end{array}$$

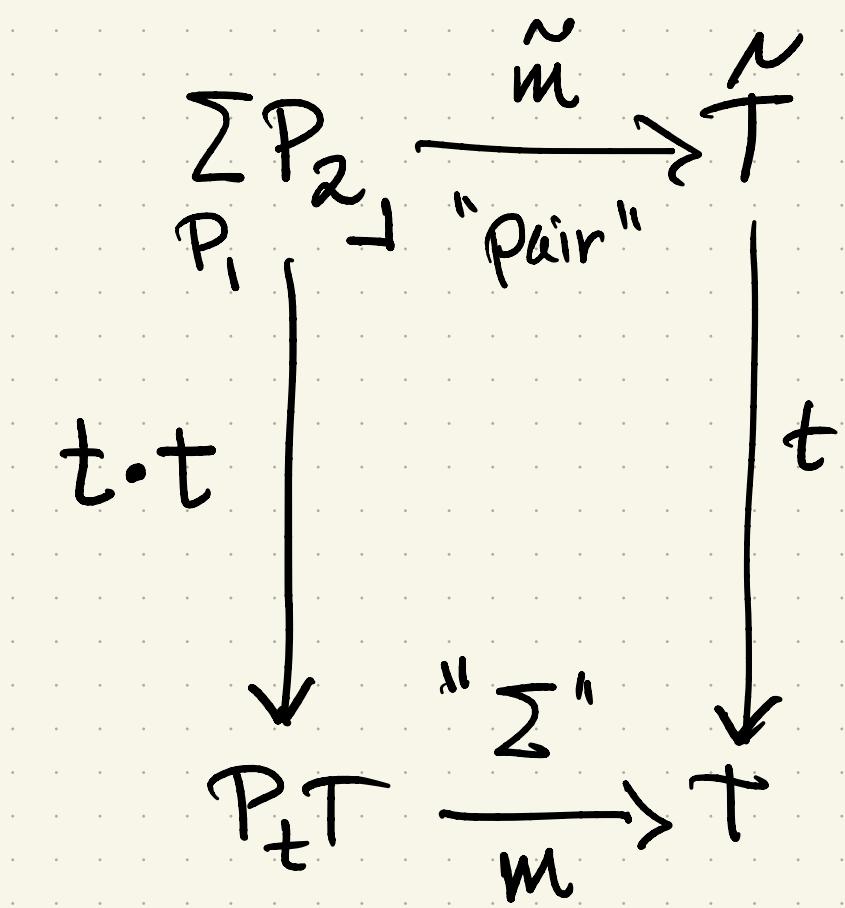
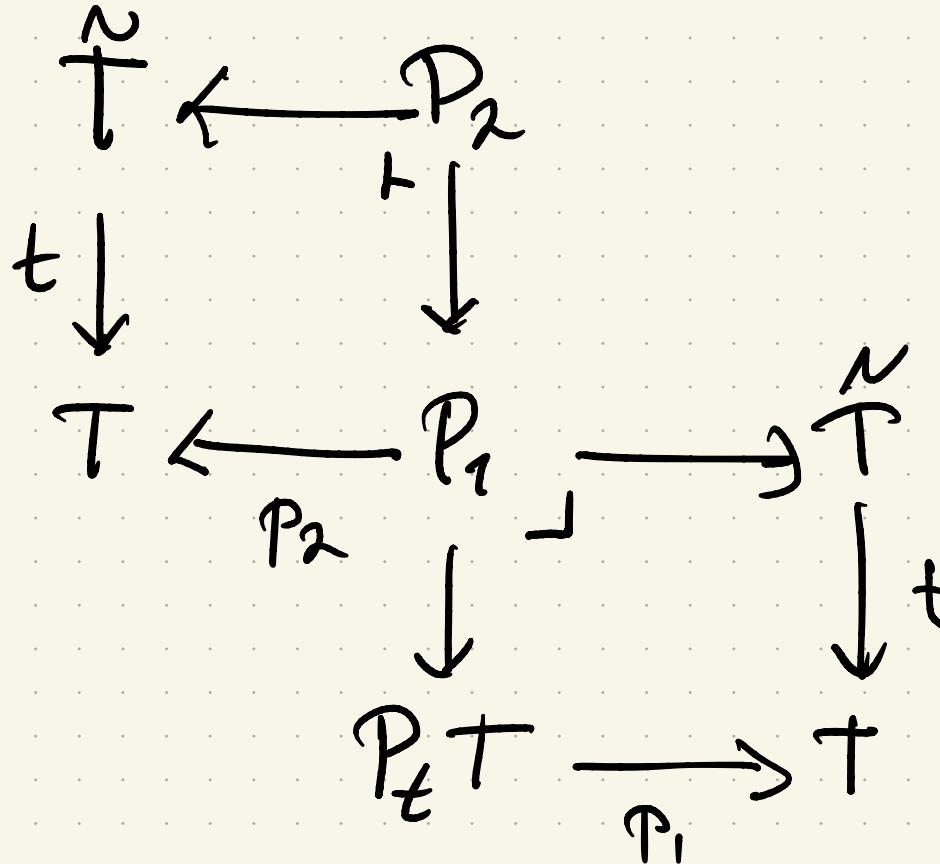
$$\begin{array}{ccc} \tilde{T}^* & \xrightarrow{t} & \tilde{T} \\ \downarrow t^*t & & \downarrow t \\ T^* & \xrightarrow{\quad} & T \end{array}$$

unit

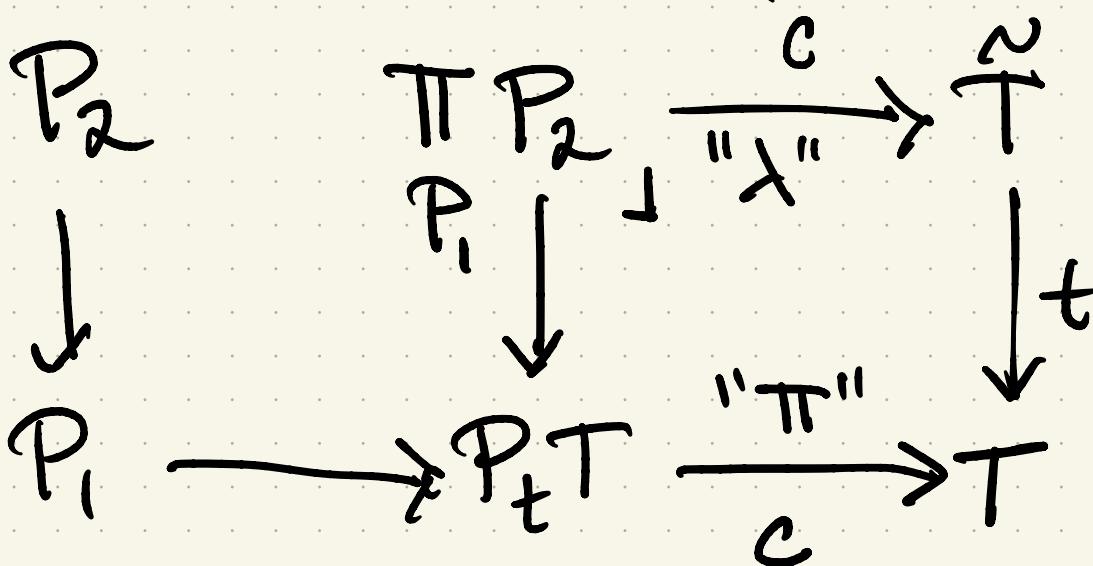
dominance multiplication

closure

The general pattern :



Note :



$$\frac{\pi P_2}{P_1} = \frac{P_t T}{P_t T}$$

$$P_t T = t * t$$

- The unit determines a cart. nat. trans.

$$u: 1_{\mathcal{E}} \rightarrow P_t$$

- The mult. determines

$$m: P_t \circ P_t \rightarrow P_t$$

- The closure determines an algebra structure

$$c: P_t(t) \rightarrow t$$

- In terms of the monoidal cat $(\mathcal{E}^I, \cdot, !)$:

- a monoid structure $! \rightarrow t \leftarrow t \cdot t$
- a module structure $t \otimes t \rightarrow t$

Basic Example: A CwF $(\mathcal{C}, t: \hat{T} \rightarrow T)$ is a

ML-algebra in $\hat{\mathcal{C}}$ iff it has $1, \Sigma, \Pi$.

Conversely:

Thm Let $t: \hat{T} \rightarrow T$ be a ML-algebra in \mathcal{E} .

Define a CwF \hat{t} on \mathcal{E} by just "mapping in":

$$\begin{array}{ccc} \text{tm}(-) & = & \text{Hom}_{\mathcal{E}}(-, \hat{T}) \\ \hat{t} \downarrow & & \downarrow \text{Hom}_{\mathcal{E}}(-, t) \\ \text{ty}(-) & = & \text{Hom}_{\mathcal{E}}(-, T) \end{array}$$

Then $\hat{\mathcal{E}}$ has $1, \Sigma, \Pi$ as a CwF. (pf: Yoneda preserves ML alg.s.)

4. Examples

(i) Display maps Take any map $t: \tilde{T} \rightarrow T$ in \mathcal{E} and define display maps $\mathcal{D}_t \subseteq \mathcal{C}_1$ by:

$$d \downarrow \in \mathcal{D}_t \Leftrightarrow \begin{array}{ccc} D & \xrightarrow{\quad} & \tilde{T} \\ d \downarrow & \lrcorner & \downarrow t \\ E & \xrightarrow{\quad} & T \end{array}$$

Then \mathcal{D}_t is closed under pullbacks, and:

- under isos & composition if t is a dominance,
- under pushforwards if t is closed.

So $(\mathcal{C}, \mathcal{D}_t)$ is a IT-clan* (Joyal) if t is a ML algebra.

Conversely:

Thm (A.2012) Given a display map cat $(\mathcal{C}, \mathcal{D})$,
there's a $d_{\mathcal{D}}: \tilde{\mathcal{D}} \rightarrow D$ in $\overset{\sim}{\mathcal{C}}$ that's a ML-algebra
if $(\mathcal{C}, \mathcal{D})$ is closed under isos, composition,
and pushforwards, i.e. if $(\mathcal{C}, \mathcal{D})$ is a Π -clan.

In fact:

$$\begin{array}{ccc} \tilde{\mathcal{D}} & & \amalg y_{\text{dom}(d)} \\ \downarrow d_{\mathcal{D}} := \amalg_{d \in \mathcal{D}} y_d & & \downarrow \\ D & & \amalg y_{\text{cod}(d)} \end{array}$$

$$\text{So } \mathcal{D}_{d_{\mathcal{D}}} = \mathcal{D} .$$

(ii) Finite sets In $\mathcal{E} = \text{Set}$, let

$$\begin{array}{c} \mathbb{N}^2 \\ \downarrow \text{nat} \\ \mathbb{N} \end{array} = \sum_{n: \mathbb{N}} \begin{array}{c} \mathbb{N} \\ \downarrow \\ \mathbb{N} \end{array} = \mathbb{N} \times \mathbb{N} \xrightarrow{\quad P_2 \quad} \mathbb{N}$$

Polynomial functor $P_{\text{nat}}: \text{Set} \rightarrow \text{Set}$ is then

$$\begin{aligned} P_{\text{nat}}(X) &= \sum_{n: \mathbb{N}} X^n \\ &= 1 + X + X^2 + \dots \end{aligned}$$

• Unit : $u_x: X \rightarrow 1 + X + \dots$ (+ · inclusion)

• multiplication :

$$\begin{array}{ccc} P_{\text{nat}}^2 & \longrightarrow & P_{\text{nat}} \\ P_{\text{nat}} \times P_{\text{nat}} & \xrightarrow{\quad \text{nat}^2 \quad} & X \end{array}$$

$$\begin{array}{ccccc} P_2 & \xrightarrow{\tilde{m}} & \mathbb{N}^N & & m_n: \mathbb{N}^n \longrightarrow \mathbb{N} \\ \downarrow \text{nat}^2 & \perp & \downarrow & & (k_1, \dots, k_n) \mapsto k_1 + \dots + k_n \\ \sum_n \mathbb{N}^n = P_{\text{nat}} \mathbb{N} & \xrightarrow{m} & \mathbb{N} & & \end{array}$$

• Closure : $c_n(k_1, \dots, k_n) = k_1 \cdot \dots \cdot k_n$

(iii) Bool

$$\begin{array}{ccccc} T & \downarrow & 2^2 & \downarrow & \sum \delta \\ 1 & = & & \{\emptyset\} & \delta:2 \\ 2 & & & \{\emptyset, \{1\}\} & \downarrow \\ & & & & 2 \end{array}$$

$$P_T(x) = \sum_{\delta:2} x^\delta = 1 + x.$$

• Unit : $x \rightarrow 1 + x$ t-incl.

• Mult : $1 + (1 + x) \rightarrow 1 + x$ \triangleright

• Closure : $\begin{array}{ccc} 1+1 & \rightarrow & 1 \\ \downarrow & + & \downarrow \\ 1+2 & \rightarrow & 2 \end{array}$ Soc

(iv) Groth. Universe

Take any cardinal α & do "the same thing":

$$\begin{array}{c} \sim \\ \tilde{S}_\alpha \\ \downarrow \\ S_\alpha \end{array} \quad \begin{array}{l} \text{"}\sum a\text{"} \\ a \in S_\alpha \end{array}$$

ML-algebra
if α is inaccessible

$$S_\alpha \quad \text{"Sets of size } \prec \alpha\text{"}$$

(v) Syntactic CWF of DTT w/ 1, Σ, Π

$$\begin{array}{ll} \text{In Ctx:} & \text{TermS}(-) = \{ - + a : A \} \\ & \downarrow \\ & \text{Types}(-) = \{ - + A \text{ type} \} \end{array} \quad \begin{array}{l} \text{this should be} \\ \text{the } \underline{\text{initial}} \\ \text{ML-algebra!} \end{array}$$

(vi) Hofmann-Streicher Universe

Given a cardinal $\alpha \in \text{Set}$, for any \mathcal{C} ,
we have the HS universe in $\widehat{\mathcal{C}}$:

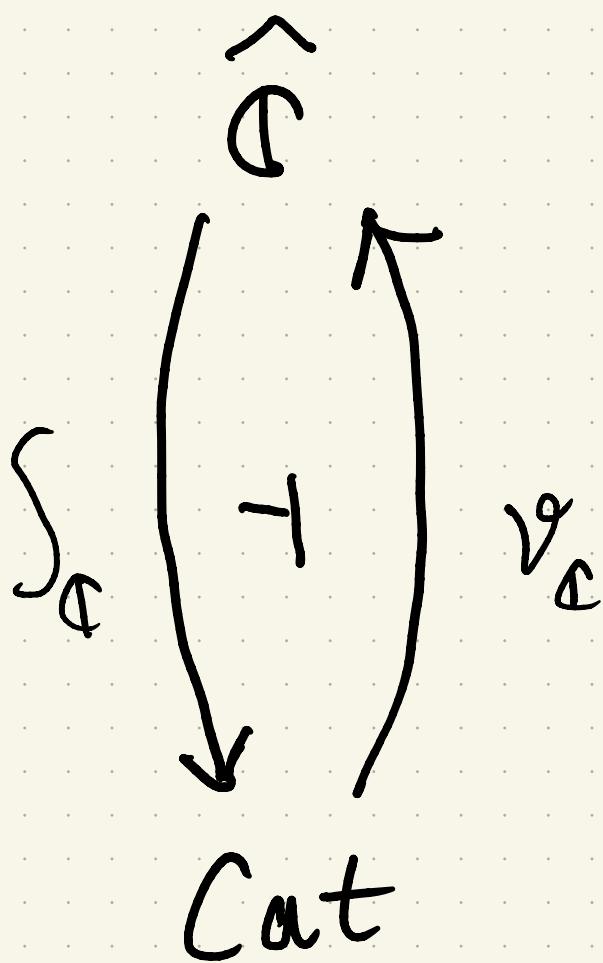
$$\begin{array}{ccc} U : \mathcal{P}^\alpha & \longrightarrow & \text{Set}_\alpha \\ E : \mathcal{S}\mathcal{U}_C^\alpha & \longrightarrow & \text{Set}_\alpha \\ & & \downarrow \\ & & \widetilde{U}_\alpha \\ & & \downarrow \\ & & U_\alpha \end{array}$$

Prop. $\widetilde{U}_\alpha \rightarrow U_\alpha$ is a ML-algebra (for suitable α).

This follows directly from the 3 facts...

Fact 1 :

$\tilde{U}_\alpha \rightarrow U_\alpha$ is the "nerve" of the universal α -small discrete fibration $\text{Set}_\alpha^{\text{op}} \rightarrow \text{Set}_\alpha^\alpha$ in Cat :



$$\begin{array}{ccc} \tilde{U}_\alpha & \xrightarrow{\nu_C} & \text{Set}_\alpha^\alpha \\ \downarrow & = & \downarrow \\ U_\alpha & \xrightarrow{\nu_C} & \text{Set}_\alpha^{\text{op}} \end{array}$$
$$\begin{array}{ccc} S_D & \xrightarrow{\cong} & D \\ \downarrow & \perp & \downarrow \\ C & \longrightarrow & \text{Set}_\alpha^\alpha \\ D & \longrightarrow & \text{Set}_\alpha^{\text{op}} \end{array}$$

Fact 2: The nerve $\mathcal{N}_{\mathcal{C}}: \text{Cat} \longrightarrow \widehat{\mathcal{C}}$ preserves
ML-algebras.

Fact 3: $\text{Set}_2^{\alpha} \rightarrow \text{Set}_2^{\alpha}$ is a ML-algebra in Cat^*
(for suitable α).

So indeed:

Prop. $\widetilde{U}_2 \rightarrow U_2$ is a ML-algebra in $\widehat{\mathcal{C}}$
(for suitable α).

Fun Corollary : the Soc $1 \rightarrow \Omega$ in $\widehat{\mathcal{C}}$
is also an MG-algebra.

Because it's the nerve of $T: 1 \rightarrow 2$,

$$\begin{array}{ccc} 1 & \xrightarrow{\nu_1} & \\ t \downarrow = & & \downarrow \nu_T \\ \Omega & \xrightarrow{\nu_2} & \end{array}$$

5. Free Completions

Let $(\mathcal{C}, t: \tilde{\mathcal{T}} \rightarrow \mathcal{T})$ be a CwF in $\mathcal{E} = \hat{\mathbb{C}}$.

- We saw that TFAE:

ML-Alg	CwF
$! \rightarrow t$	unit type $* : 1$
$t \cdot t \rightarrow t$	sum type $\Sigma_A B$
$t * t \rightarrow t$	product type $\prod_A B$

- Given any CwF t , we can freely add these structures to make the free ML-algebra on t .

Step 1 Using $P_f + P_g = P_{f+g}$ in $\text{Poly}(\mathcal{E}) \cong \mathcal{E}^I$, we have

in $\text{Poly}(\mathcal{E})$

$$1_{\mathcal{E}} \rightarrow 1_{\mathcal{E}} + P_t \leftarrow P_t ,$$

in \mathcal{E}^I

$$\begin{array}{ccccc} 1 & \xrightarrow{\quad + \quad} & 1 + \hat{t} & \xleftarrow{\quad \hat{t} \quad} & \hat{t} \\ ! \downarrow & & \downarrow ! + t & & \downarrow t \\ 1 & \xrightarrow{\quad + \quad} & 1 + t & \xleftarrow{\quad T \quad} & T \end{array} .$$

Since $1_{\mathcal{E}} + P_t = P_{!+t}$ is the free pointed ends functor on P_t ,
 the map $t \mapsto ! + t$ freely adds a unit type to the CwF t .

Step I: We seek U in $\mathcal{E}^{\mathcal{I}}$ with

$$t \mapsto U \leftarrow U \circ t \quad (\text{universal})$$

- As in the theory of endofunctors as "data types", this is given by solving the "domain equation"

$$U \cong 1 + U \circ t .$$

This also adds a point $1 \rightarrow U$.

- The solution is the lim of the sequence

$$0 \rightarrow T_0 \rightarrow T^2_0 \rightarrow \dots ,$$

for the endofunctor

$$T(x) = ! + x \circ t$$

on $\mathcal{E}^{\mathcal{I}} \cong \text{Poly}(\mathcal{E})$.

the colimit in \mathcal{E}^I is seen to be the object

$$\begin{aligned} u &= ! + t + t \cdot t + t^{3 \circ} + \dots \\ &= \sum_n t^n \end{aligned}$$

For this, one uses the following important

Fact: If $t: \tilde{T} \rightarrow T$ in $\hat{\mathcal{C}}$ is representable,
then the polynomial functor $P_t: \hat{\mathcal{C}} \rightarrow \hat{\mathcal{C}}$
has a right adjoint, and therefore preserves
all colimits. So in particular:

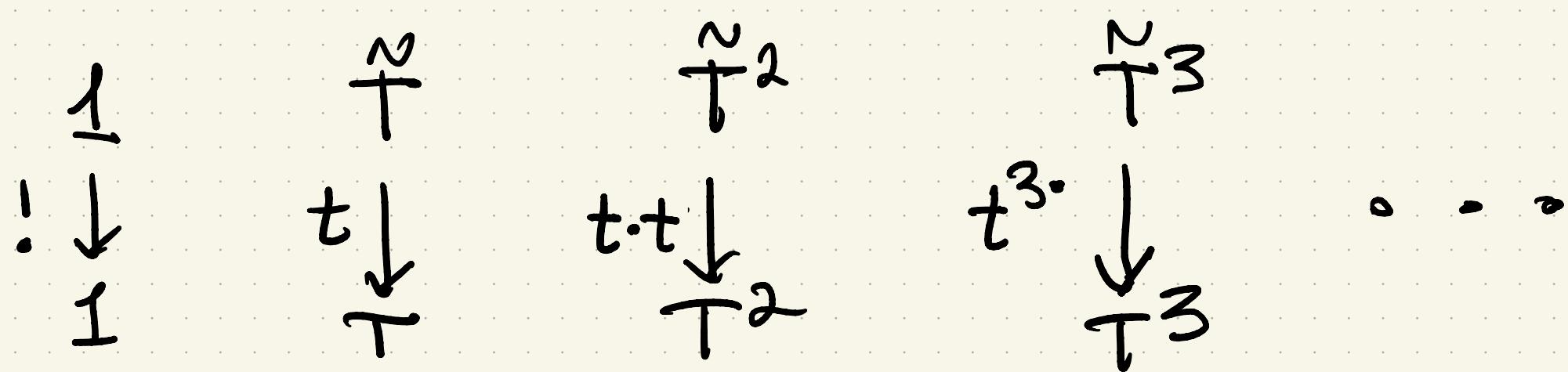
$$t \cdot (a+b) \cong t \cdot a + t \cdot b, \quad a, b \in \mathcal{E}^I.$$

- Let us consider this free completion under Σ -types in terms of the "type theory" $t: \tilde{T} \rightarrow T$.
- We have the objects in \mathcal{E}^I :

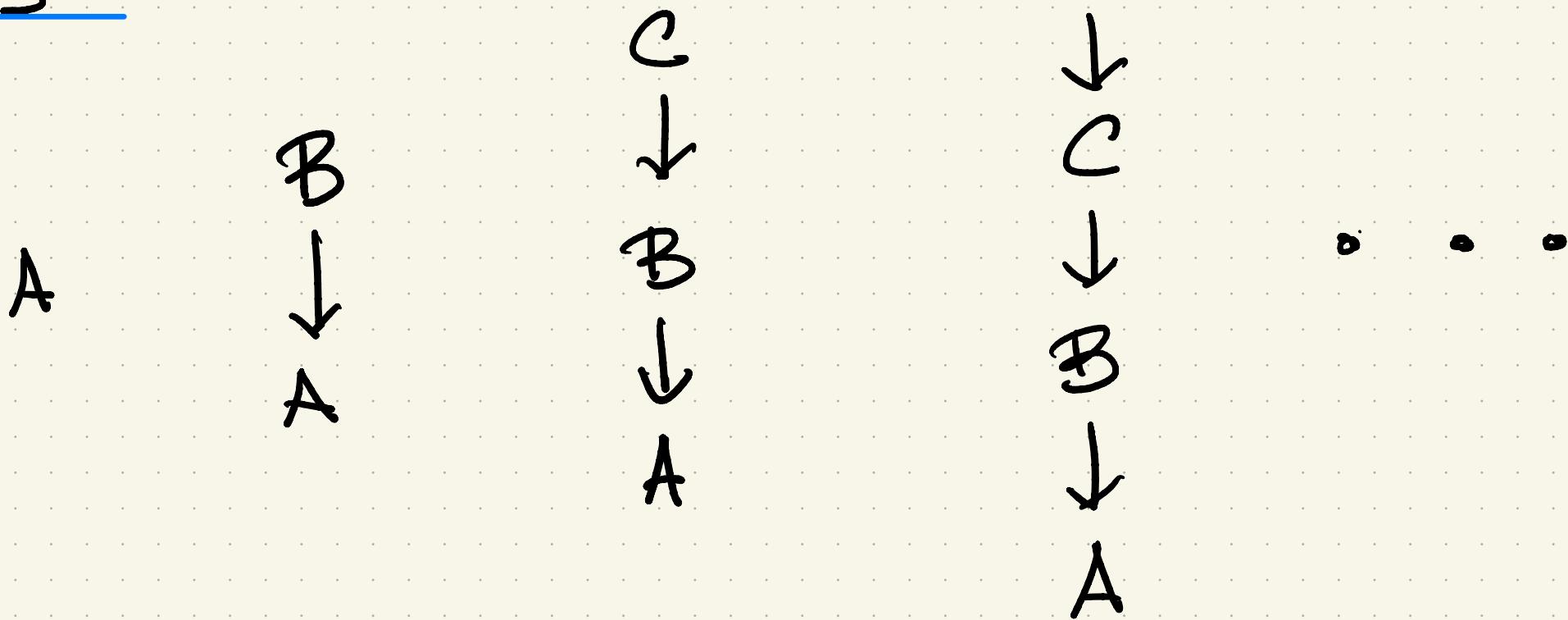
$$\begin{array}{ccccccc}
 & \frac{!}{1} & \frac{\vdash}{T} & \frac{\tilde{T}^2}{t \cdot t} & \frac{\tilde{T}^3}{t^3} & & \\
 & \downarrow & \downarrow & \downarrow & \downarrow & & \\
 1 & & T & T^2 & T^3 & & \dots
 \end{array}$$

- Mapping into these classifies ...

Objects:



classify:



ctx's:

A $A.B$ $A.B.C$ $A.B.C.D$ \dots

- Thus $u: \tilde{U} \rightarrow U$ classifies the ctx's of t :
 \Downarrow
 $(1+t+t^2+\dots)$

$$\begin{array}{ccc}
 G_u & \xrightarrow{\quad} & \tilde{U} \\
 \downarrow & \lrcorner & \downarrow u \\
 C_0, C_1, \dots, C_n : & \vdots & \\
 \downarrow & & \\
 C_0 & \xrightarrow{\quad} & U
 \end{array} .$$

- This agrees with the idea that the cat. of ctx's of a theory "freely adds Σ -types".
- Can be used to give a "base change" of CwFs :

$$(C, t) \longrightarrow (C^\Sigma, t^\Sigma) .$$

Work in Progress

- Π -types

$$u = t + u \otimes u ?$$

- Eq

$$\Delta u \rightarrow u$$

$$\begin{array}{ccc} \tilde{u} & \xrightarrow{\quad} & \tilde{u} \\ \downarrow & & \downarrow \\ \Delta u & & u \\ \tilde{u} \times \tilde{u} & \xrightarrow{\quad} & u \end{array}$$

- Id

$$u^I \xrightarrow{\quad} u$$

- u

?