Introduction to Categorical Logic

[DRAFT: January 22, 2023]

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Introduction

Once upon a time, there was logic, and there was category theory. Traditional logic once consisted of:

- Propositional calculus, first-order logic, formal systems of deduction, Tarski-style semantics, Gödel's completeness and incompleteness theorems.
- On that basis were erected model theory, set theory, computability theory, and proof theory.
- Logic was considered the study of the foundations of mathematics, but it was largely unrealted to other branches of mathematics.

And category theory originally consisted of:

- Homological algebra, homotopy theory, the study of various kinds of limits,
- Universal constructions like free algebras and tensor products,
- Duality theories such as that of Gelfand and Stone,
- Grothendieck's algebraic geometry and sheaf theory,
- The theory of monads and universal algebra, like Birkhoff's theorems.

Then along came F.W. Lawvere and noticed how the basic framework of Stone duality could be applied to algebraic theories, inventing *functorial semantics*. From this, the basic ideas of categorical logic followed:

- An equational theory is represented as a category $\mathbb{T}$ with finite products that's "freely generated as such by the signature"; a model of the theory, or $\mathbb{T}$-algebra, is then a finite product preserving functor $A : \mathbb{T} \to \mathsf{Set}$. The completeness of equational reasoning (one of Birkhoff's theorems) is then the fact that we have a contravariant embedding,
$$\mathbb{T}^{\mathsf{op}} \hookrightarrow \mathrm{Mod}(\mathbb{T}) = \mathrm{FP}(\mathbb{T}, \mathsf{Set}),$$
so that the "syntax" is a (dual) subcategory of the "semantics".

- Following Rasiowa-Sikorski, propositional logic can be treated as Boolean algebra: formal deduction is a way to specify a free algebra, truth-table semantics is a description of the Boolean homomorphisms into $\{0, 1\}$, and Stone's representation theorem is the completenes theorem for propositional logic.

- First-order logic can be understoood as a Boolean algebra indexed over an algebraic theory, with the quantifiers as adjoints to the reindexing functors (Lawvere's hyperdoctrines). More generally, one can define the notion of a "Boolean category" as a solution to the analogy: "propositional logic is to Boolean algebra as first-order logic is to X", generalizing from posets to (proper) categories. Gödel completeness can be formulated as a (sheaf) representation theorem for Boolean (or Heyting) categories.

- The same ideas also apply to various fragments of first-order logic to relate diffferent kinds of logical theories (syntax) and their categories of models (semantics) via the general framework of functorial semantics.

- Finally, topos theory subsumes and generalizes logical duality, unifying the "algebraic" (syntactic) and "geometric" (semantic) aspects in the single category of Grothendieck toposes and geometric morphisms. A topos can also be seen a forcing model of set theory, Kripke model of intuitionistic or modal logic, a model of infinitary first-order logic, a model of higher-order (predicate) logic, or even a realizability model of computability.

There is also another, "constructive" tradition in logic, more closely related to proof theory and influenced by theoretical computer science.

- The *Curry-Howard correspondence* is a somewhat mysterious connection between propositional logic and type theory, according to which the "meaning" of a propositional formula is not just a truth-value, but rather the collection of its proofs. Propositions-as-Types, Proofs-as-Terms (or -Programs) is a proof-theoretic (or computational) alterative to Tarskian, truth-value semantics. It also extends to first-order logic and dependent type theory.

- Associated to this perspective, one also has categorical semantics of type theories like the λ-calculus in (locally) cartesian closed categories, like the category of Scott domains, rather than in Boolean and Heyting algebras (for propositional logic) and (pre)toposes (for first-order and higher-order predicate logic).

- The algebra used for the truth-value semantics of propositional or predicate logic (e.g. the Boolean algebra $\{0, 1\}$) is then seen to be the poset reflection of a proper category (e.g. Set) modeling the type theory that is the "proof-relevant" version of the logic. The general scheme can be represented as follows, with the righthand side the proof-relevant version of the left:

Logic	Algebra	Type Theory	Category
Propositional	Boolean algebra	Simple	CCC
Predicate	Boolean category	Dependent	LCCC

- The relationship between *validity* and *provability* classically described by that between logic and type theory, is described categorically by the relations of generalization and "poset reflection" between (structured) posets and categories. In this way, the Curry-Howard correspondence relates to the idea of "categorification": a structured category whose poset reflection is a given structured poset. For example, the categorification of a $\wedge$-semilattice is a category with finite products, and the categorification of the Boolean algebra $\{0,1\}$ is the category Set.

Such was the state of Categorical Logic when these notes were begun, around the turn of the century. In the meantime, some new ideas have shifted the focus: the Curry-Howard paradigm relating truth-value semantics (model theory) and type-theoretic syntax (proof theory)—viewed as an instance of categorification—has turned out to capture only the first two levels of an infinite hierarchy of levels of structure, related by inclusion, truncation, (co-)reflection, and other operations. The importance of "proof-relevance" that underlies the Propositions-as-Types idea is essnetially just a special case of the coherence issue that arises everywhere in higher category theory. And the once-bold replacement of both *truth-values* and *sets* by *types* in constructive logic and the foundations of computation parallels the replacement of discrete structures (sheaves) by "higher" ones (stacks) in algebra and geometry, except that we have now learned that the gap between the levels is not just a single step, but rather an infinite hierarchy of levels of structure, each just as significant as the first step. These insights are reflected in current categorical logic in the recent extension from algebraic logic (level 0) and topos theory (level 1) to higher topos theory and homotopy type theory (level ∞). The latter are the focus of much current research, but the unification of the various earlier topics that has been achieved already shows how much we have learned about what happens in passing from 0 to 1, by passing from the finite to the infinite.

For instance, the dualities of Stone, Lawvere, and Makkai that spurred the early development of categorical logic now fall more neatly into place, from the modern standpoint shown in table 1, that focuses on typing, variance, and h-level, rather than the traditional distinction between syntax and semantics.

	0-types	1-types	n-types
Simple types	Positive PL	Alg. Theories	HITs
Dependent types	Coherent FOL	Gen. Alg. Theories	W-types

Table 1: Covariant fragments with duality

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