Introduction to Categorical Logic

[DRAFT: MARCH 24, 2022]

Steve Awodey   Andrej Bauer

Contents

Chapter 3

Cartesian Closed Categories and the λ-Calculus

3.1 Categorification and the Curry-Howard correspondence

Consider the following natural deduction proof in propositional calculus.

$$\dfrac{\dfrac{\dfrac{\dfrac{[(A \wedge B) \wedge (A \Rightarrow B)]^1}{A \wedge B}}{A} \qquad \dfrac{[(A \wedge B) \wedge (A \Rightarrow B)]^1}{A \Rightarrow B}}{B}}{(A \wedge B) \wedge (A \Rightarrow B) \Rightarrow B}\ (1)$$

This deduction shows that

$$\vdash (A \wedge B) \wedge (A \Rightarrow B) \Rightarrow B.$$

But so does the following:

$$\dfrac{\dfrac{\dfrac{[(A \wedge B) \wedge (A \Rightarrow B)]^1}{A \Rightarrow B} \qquad \dfrac{\dfrac{[(A \wedge B) \wedge (A \Rightarrow B)]^1}{A \wedge B}}{A}}{B}}{(A \wedge B) \wedge (A \Rightarrow B) \Rightarrow B}\ (1)$$

As does:

$$\dfrac{\dfrac{\dfrac{[(A \wedge B) \wedge (A \Rightarrow B)]^1}{A \wedge B}}{B}}{(A \wedge B) \wedge (A \Rightarrow B) \Rightarrow B}\ (1)$$

There is a sense in which the first two proofs are "equivalent", but not the first and the third. The relation (or property) of *provability* in propositional calculus $\vdash \phi$ discards such differences in the proofs that witness it. According to the "proof-relevant" point of view, sometimes called *propositions as types*, one retains as relevant some information about the way in which a proposition is proved. This is effected by annotating the proofs with *proof-terms* as they are constructed, as follows:

$$\dfrac{\dfrac{\dfrac{[x:(A\wedge B)\wedge(A\Rightarrow B)]^1}{\pi_2(x):A\Rightarrow B} \qquad \dfrac{\dfrac{[x:(A\wedge B)\wedge(A\Rightarrow B)]^1}{\pi_1(x):A\wedge B}}{\pi_1(\pi_1(x)):A}}{\pi_2(x)(\pi_1(\pi_1(x))):B}}{\lambda x.\pi_2(x)(\pi_1(\pi_1(x))):(A\wedge B)\wedge(A\Rightarrow B)\Rightarrow B}\;(1)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[x:(A\wedge B)\wedge(A\Rightarrow B)]^1}{\pi_1(x):A\wedge B}}{\pi_1(\pi_1(x)):A} \qquad \dfrac{[x:(A\wedge B)\wedge(A\Rightarrow B)]^1}{\pi_2(x):A\Rightarrow B}}{\pi_2(x)(\pi_1(\pi_1(x))):B}}{\lambda x.\pi_2(x)(\pi_1(\pi_1(x))):(A\wedge B)\wedge(A\Rightarrow B)\Rightarrow B}\;(1)$$

$$\dfrac{\dfrac{\dfrac{[x:(A\wedge B)\wedge(A\Rightarrow B)]^1}{\pi_1(x):A\wedge B}}{\pi_2(\pi_1(x)):B}}{\lambda x.\pi_2(\pi_1(x)):(A\wedge B)\wedge(A\Rightarrow B)\Rightarrow B}\;(1)$$

The proof terms for the first two proofs are the same, namely $\lambda x.\pi_2(x)(\pi_1(\pi_1(x)))$, but the term for the third one is $\lambda x.\pi_2(\pi_1(x))$, reflecting the difference in the proofs. The assignment works by labelling assumptions as variables, and then associating term-constructors to the different rules of inference: pairing and projection to conjunction introduction and elimination, function application and λ-abstraction to implication elimination (*modus ponens*) and introduction. The use of variable binding to represent cancellation of premisses is a particularly effective device.

From the categorical point of view, the relation of deducibility $\phi \vdash \psi$ is a mere preorder. The addition of proof terms $x : \phi \vdash t : \psi$ results in a *categorification* of this preorder, in the sense that it is a "proper" category, the preordered reflection of which is the deducibility preorder. And now the following remarkable fact emerges: it is hardly surprising that the deducibility preorder has, say, finite products $\phi \wedge \psi$ or even exponentials $\phi \Rightarrow \psi$; but it is *amazing* that the category with proof terms $x : \phi \vdash t : \psi$ as arrows, also turns out to be a cartesian closed category, and indeed a proper one, with distinct parallel arrows, such as

$$\begin{aligned} \pi_2(x)(\pi_1(\pi_1(x))) &: (A\wedge B)\wedge(A\Rightarrow B) \longrightarrow B,\\ \pi_2(\pi_1(x)) &: (A\wedge B)\wedge(A\Rightarrow B) \longrightarrow B. \end{aligned}$$

This *category of proofs* contains information about the "proof theory" of the propositional calculus, as opposed to its mere relation of deducibility. The calculus of proof terms can be presented formally in a system of *simple type theory*, with an alternate interpretation as a formal system of function application and abstraction. This dual interpretation—as the proof theory of propositional logic, and as a system of type theory for the specification of functions—is called the *Curry-Howard correspondence* []. From the categorical point of view, it expresses the structural equivalence between the cartesian closed categories of proofs in propositional logic and terms in simple type theory. Both of these can be seen as categorifications of their preorder reflection, the deducibility preorder of propositional logic.

In the following sections, we shall consider this remarkable correspondence in detail, as well as some extensions of the basic case represented by cartesian closed categories: categories with coproducts, cocomplete categories, and categories equipped with modal operators. In the next chapter, it will be seen that this correspondence even extends to proofs in quantified predicate logic and terms in dependent type theory, and beyond.

Bibliography

[Awo21] Steve Awodey. Sheaf representations and duality in logic. In C. Casadio and P.J. Scott, editors, *Joachim Lambek: The Interplay of Mathematics, Logic, and Linguistics*. Springer, 2021. arXiv:2001.09195.

[Joh82] P.T. Johnstone. *Stone Spaces*. Number 3 in Cambridge studies in advanced mathematics. Cambridge University Press, 1982.

[MR95] Michael Makkai and Gonzalo Reyes. Completeness results for intuitionistic and modal logic in a categorical setting. *Annals of Pure and Applied Logic*, 72:25–101, 1995.