

The Effective 2-Topos

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jww/

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The Effective 1·Topos

Recall the Effective topos $\mathcal{E}ff$:

$PAsm \rightarrow Asm \rightarrow \mathcal{E}ff$

Lex

Regular

Exact

$Asm = \frac{\text{reg}}{\text{lex}}(PAsm)$ free completion

$\mathcal{E}ff = \frac{\text{ex}}{\text{reg}}(Asm)$ "

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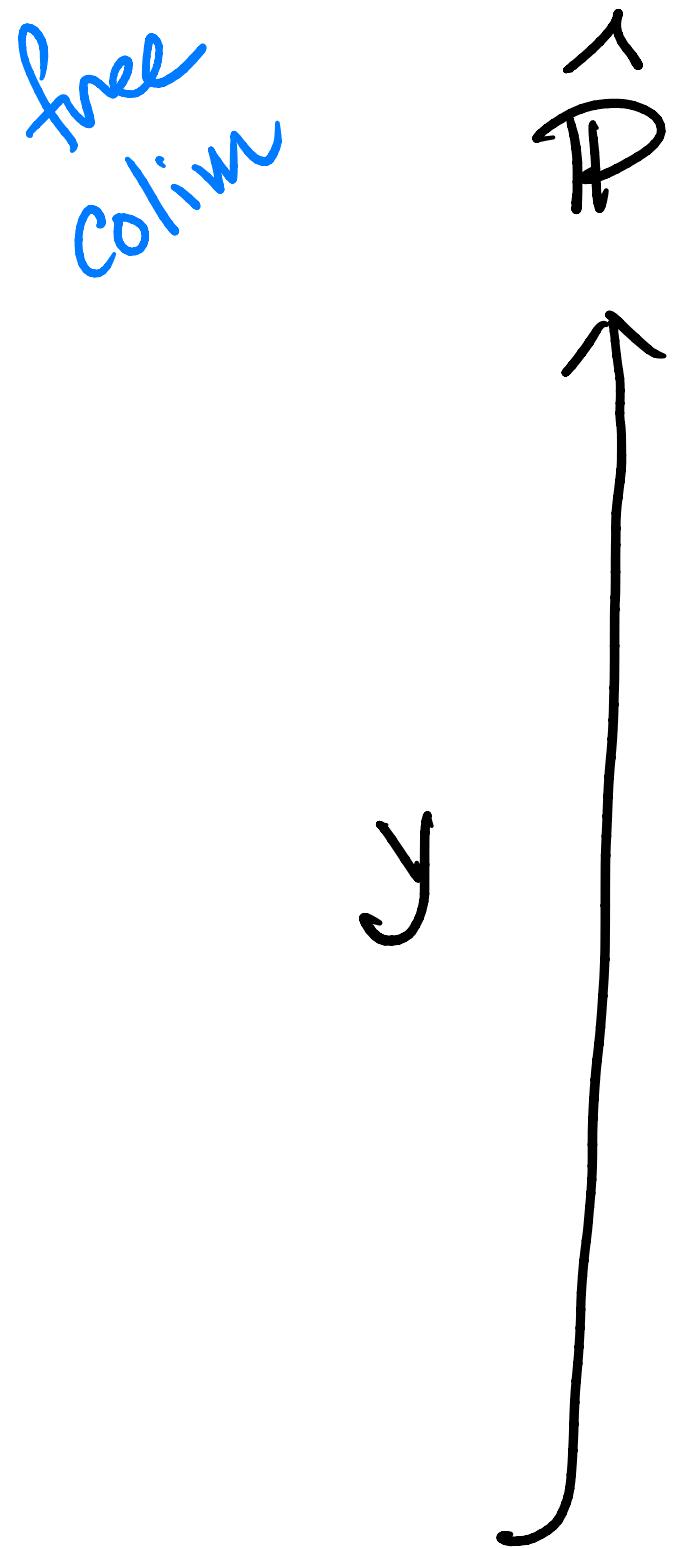
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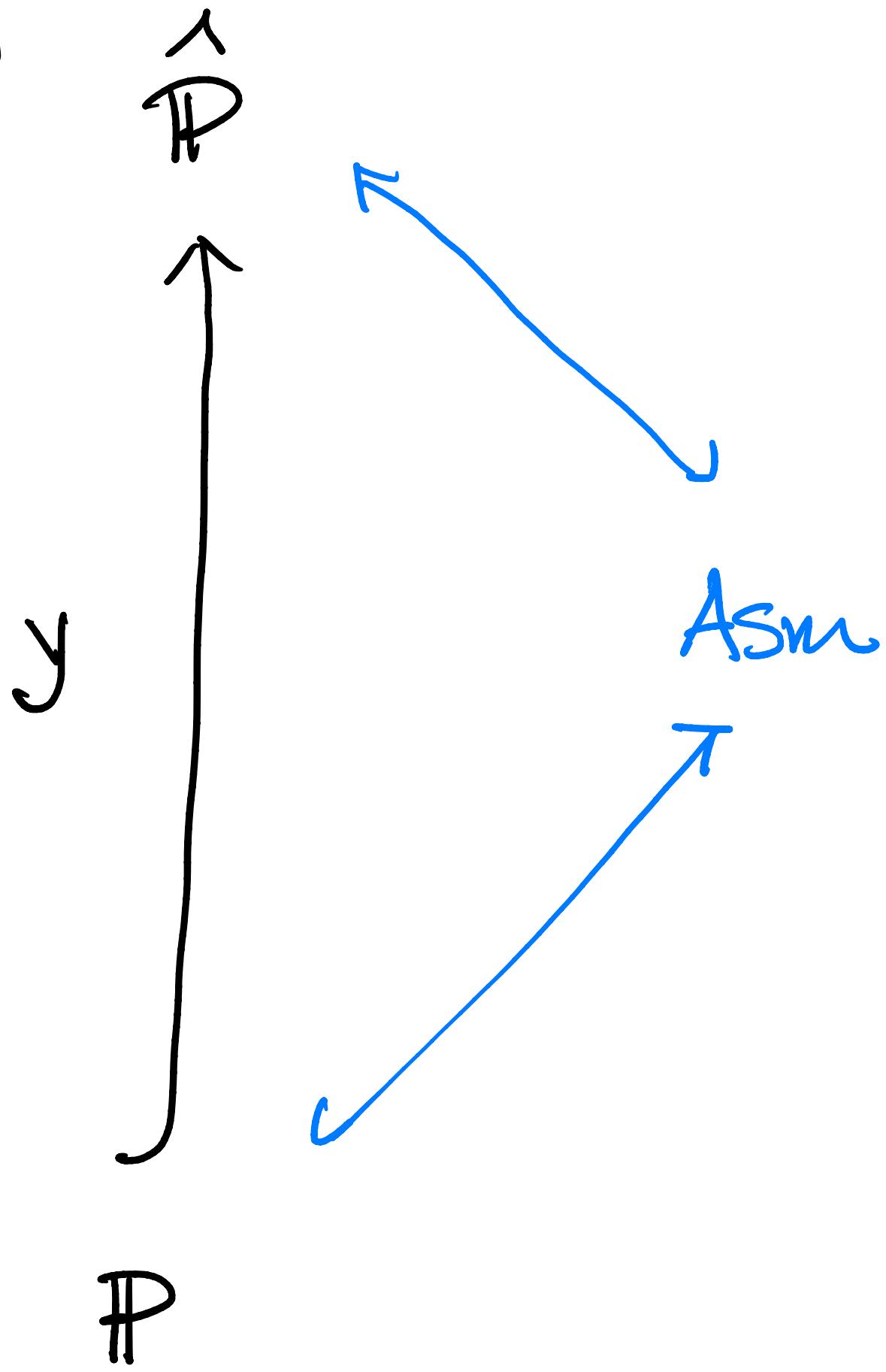
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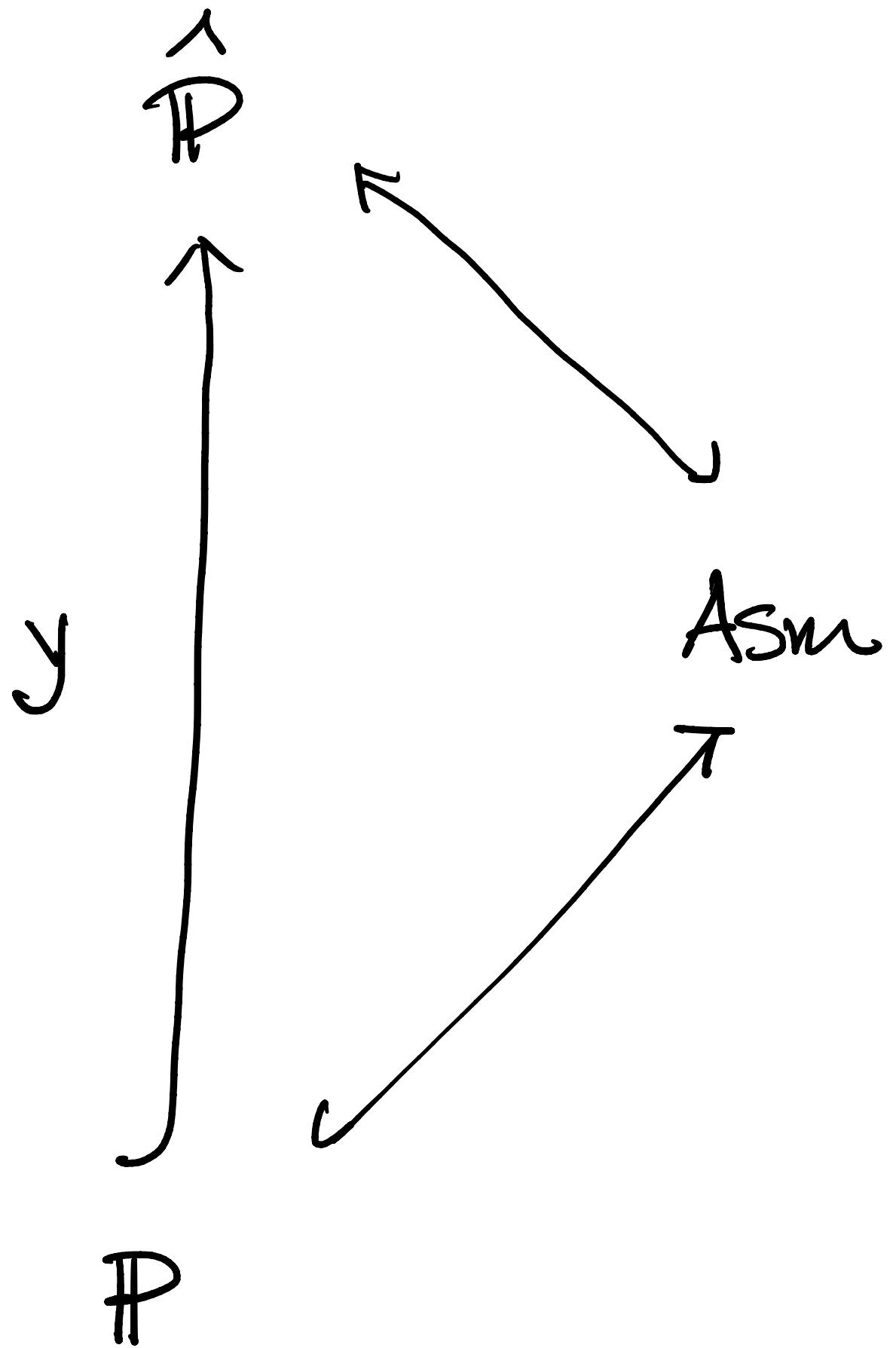
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$$PAsm = \hat{P}$$

free
Colim

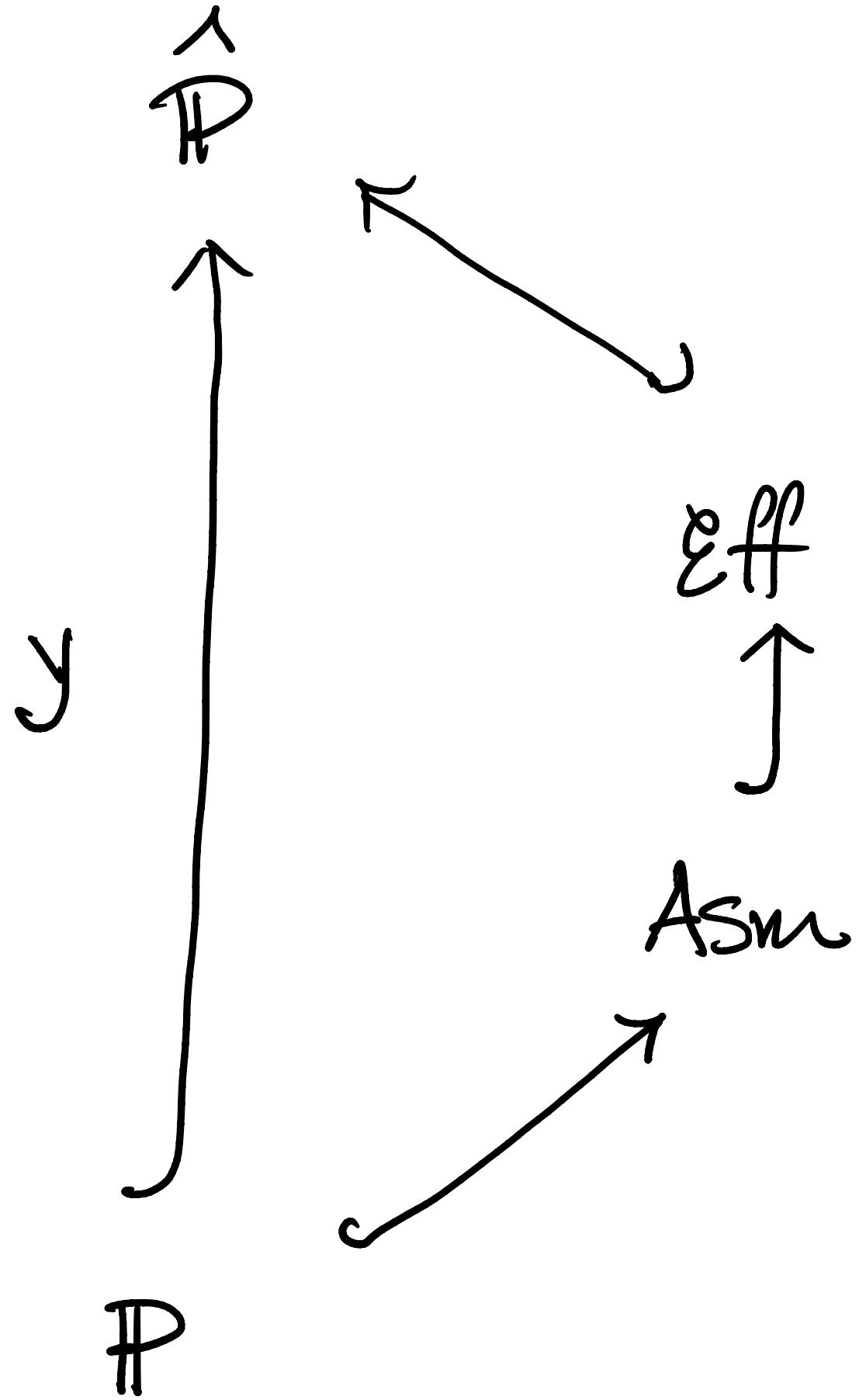




free kernel quotients

$$k \rightarrowtail P \xrightarrow{\quad} Q$$

$$\downarrow \rightarrowtail P/K \rightarrowtail Q$$



free exact quotients

$$E \rightarrowtail P \twoheadrightarrow P/E$$

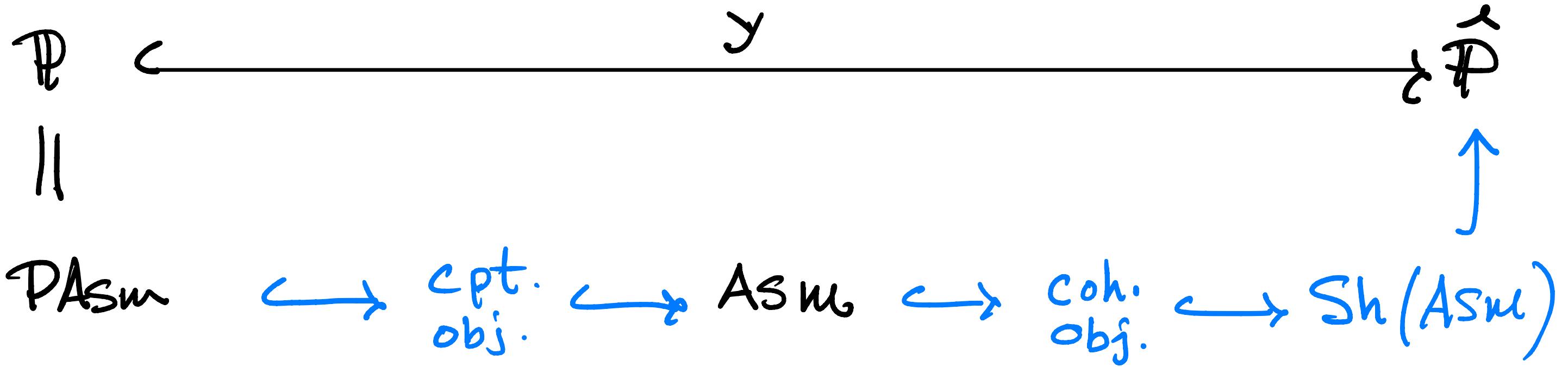
free Kernel quotients

$$K \rightarrowtail P \longrightarrow Q$$

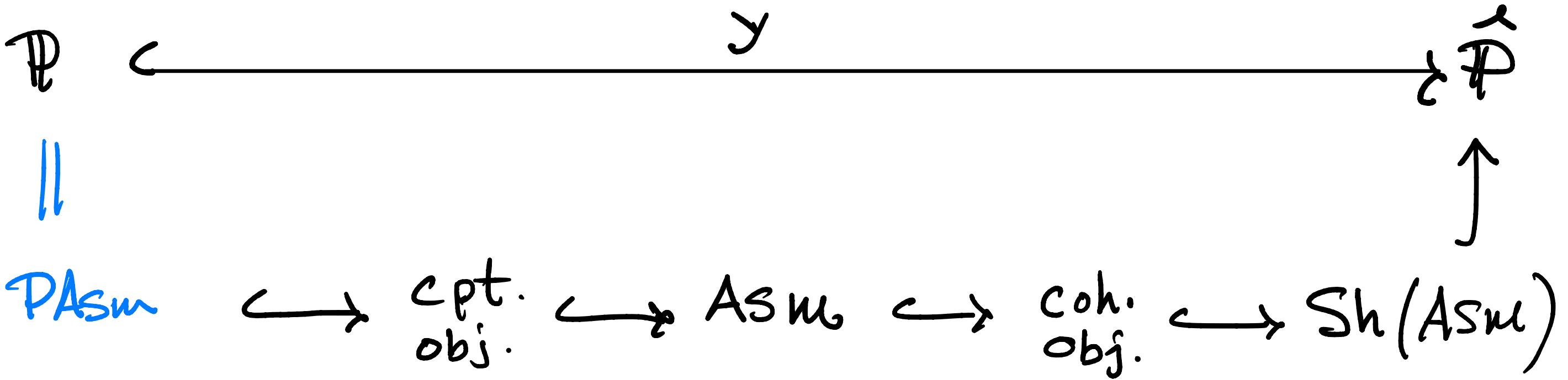
$$\downarrow \quad \downarrow$$

$$P/K \quad Q$$

Factorization of Yoneda



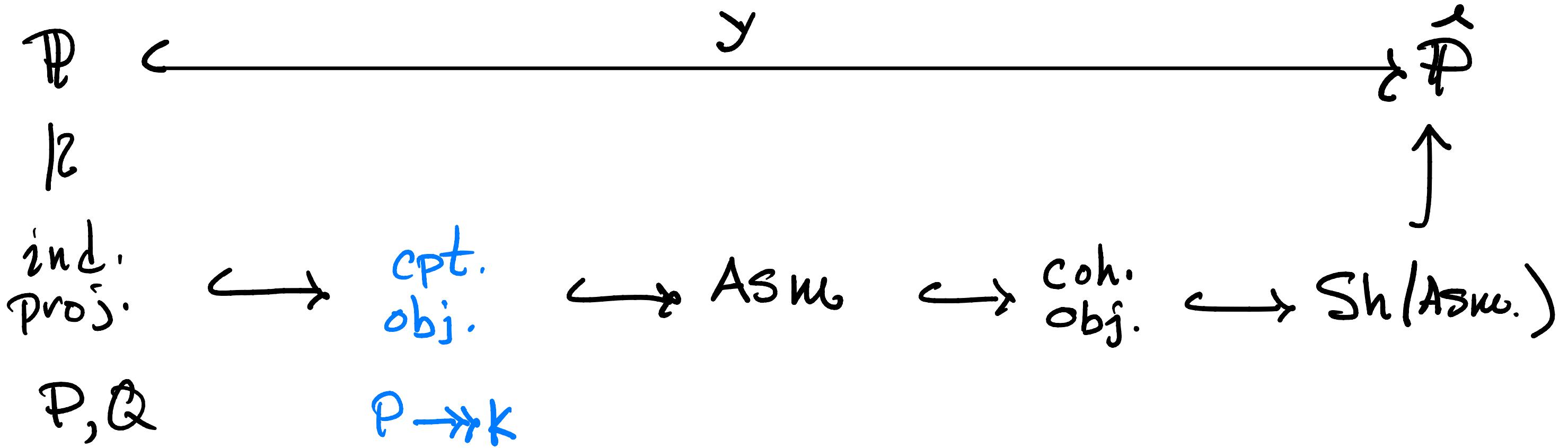
Factorization of Yoneda



indecomp. Proj. :

- $I = X + Y \Rightarrow I = X$
 $\sim I = Y$ } \Leftrightarrow $P = YP$
 $P \xrightarrow{\sim E} X$
- f. some $P \in \mathcal{P}$.

Factorization of Yoneda

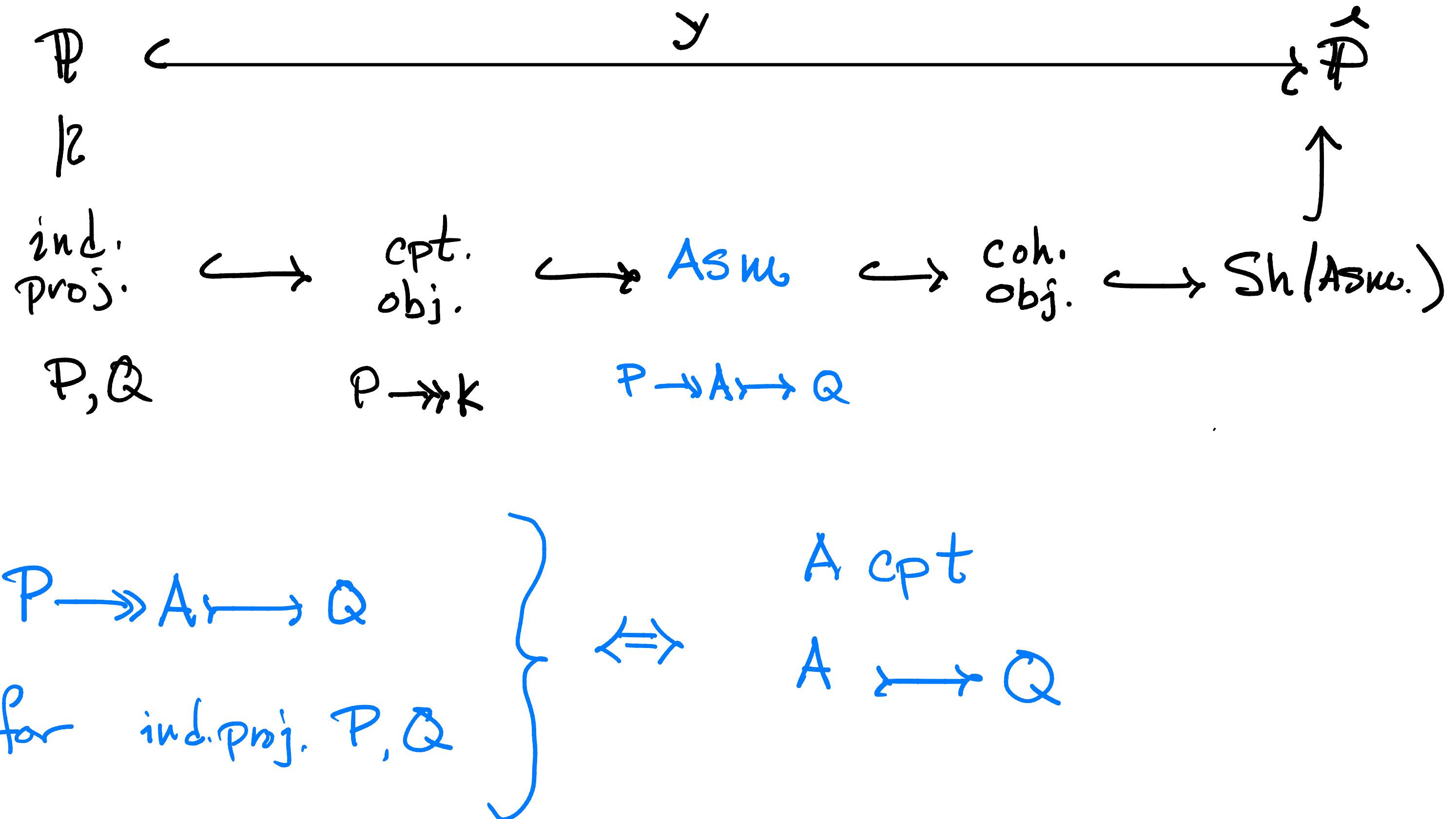


K cpt:

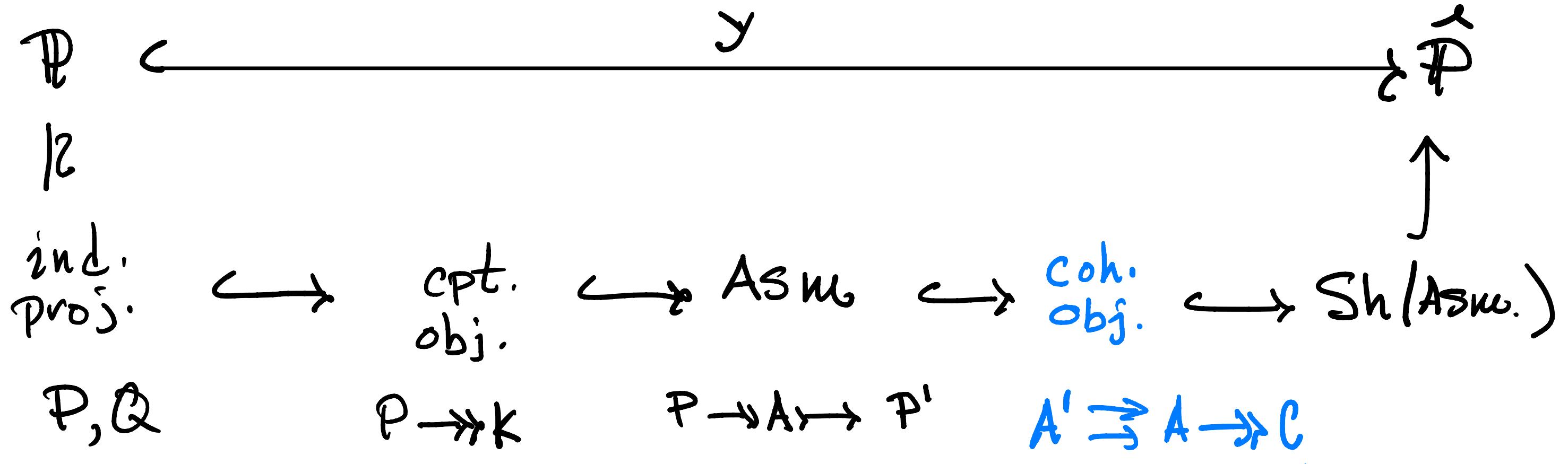
$(X_i \rightarrow K)_i$ covers $\Leftrightarrow X_k \rightarrow K$ f. some k

$P \rightarrow K$
f. some P

Factorization of Yoneda



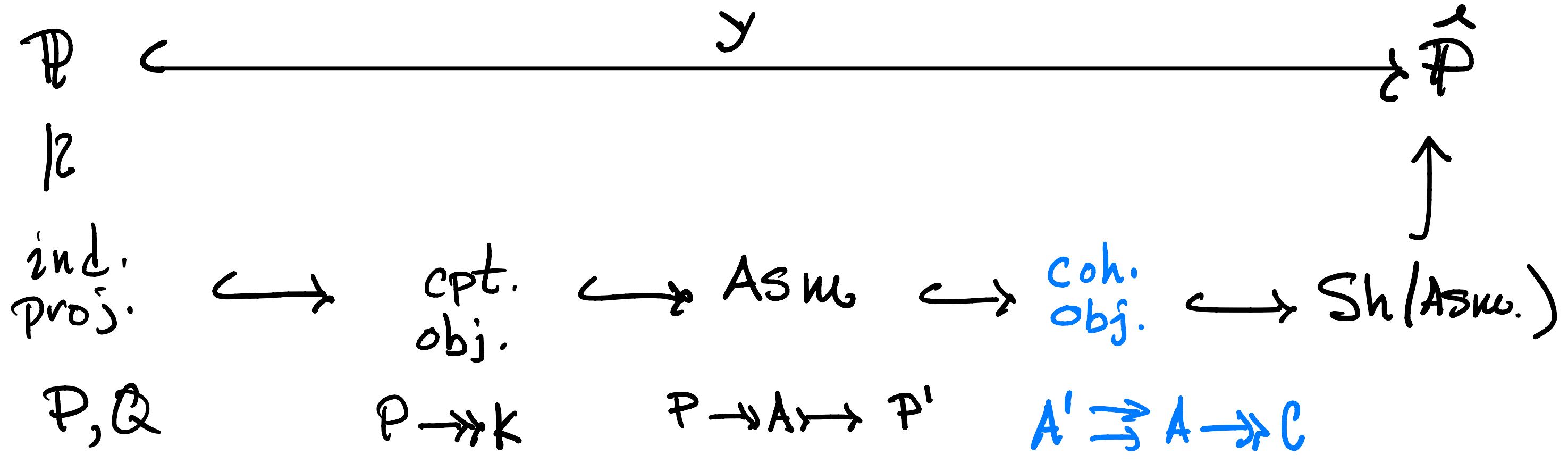
Factorization of Yoneda



C coherent :

- C is cpt.
- $C \xrightarrow{\Delta} C \times C$ cpt.

Factorization of Yoneda



\mathcal{C} coherent :

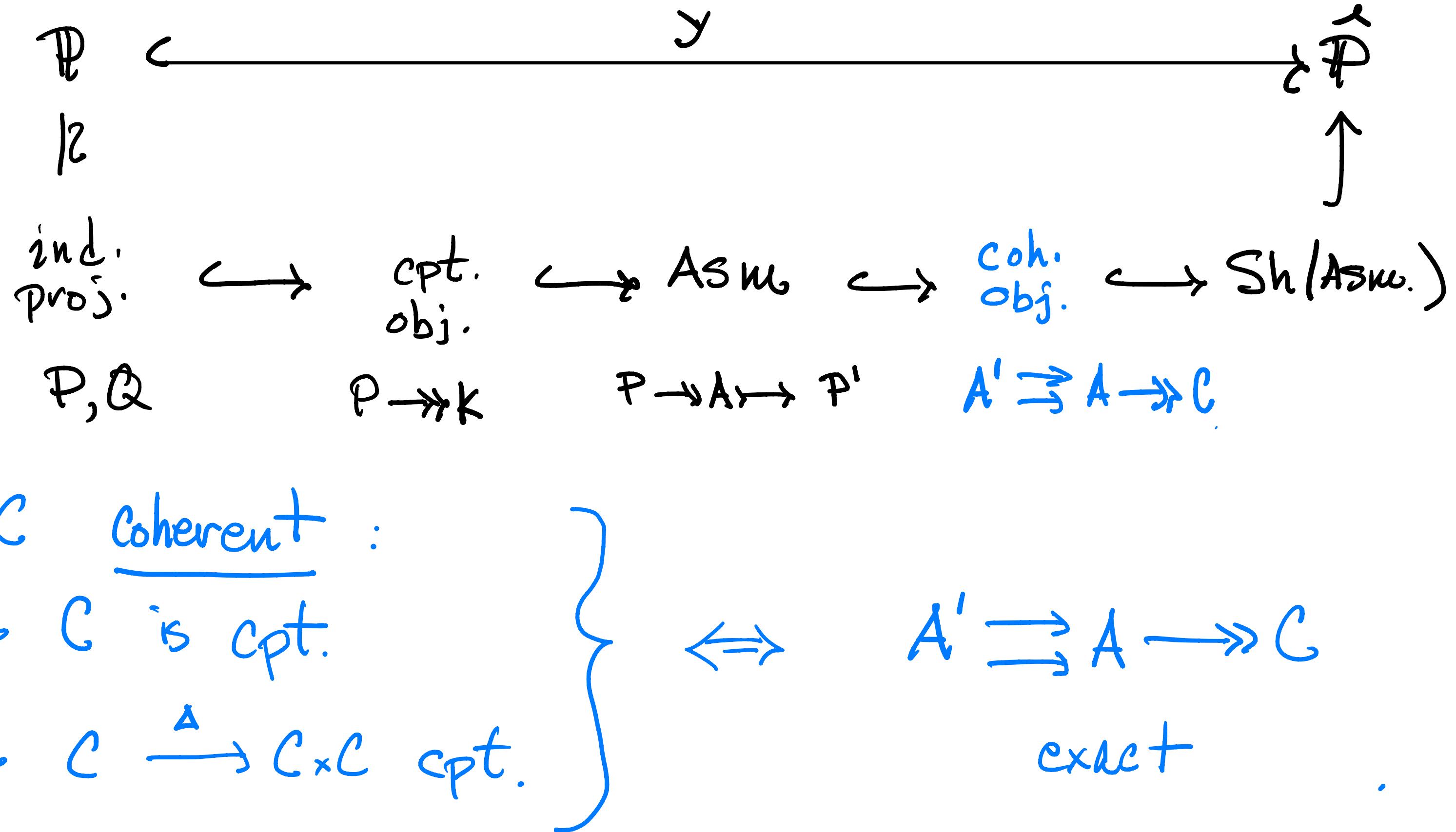
- \mathcal{C} is cpt.
- $\mathcal{C} \xrightarrow{\Delta} \mathcal{C} \times \mathcal{C}$ cpt.

where

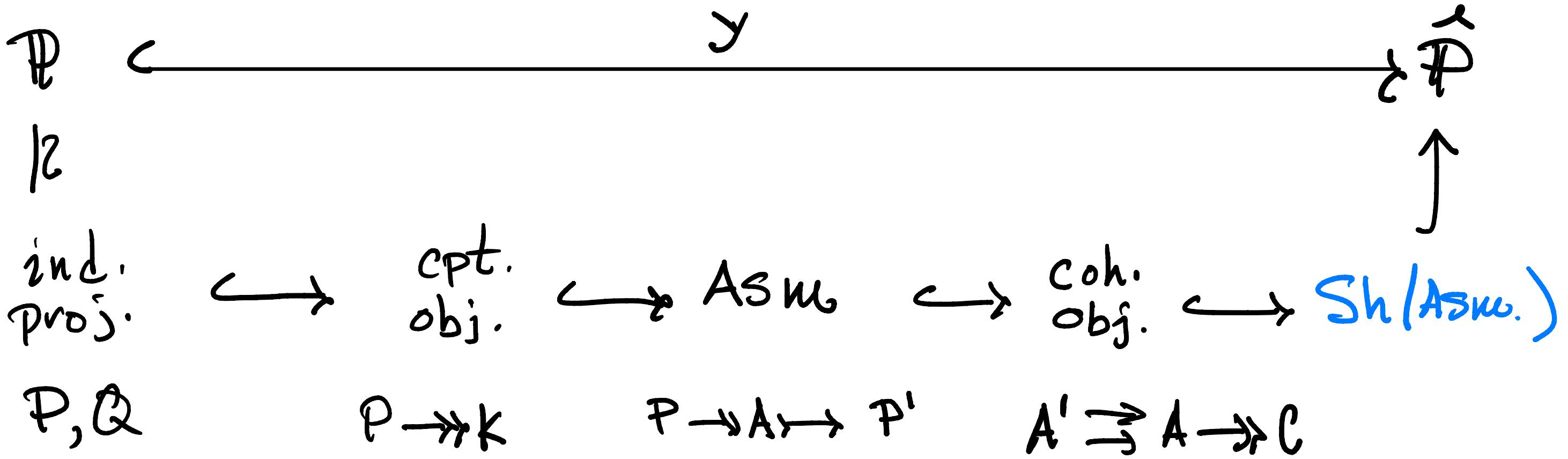
$X \rightarrowtail Y$ is cpt :

$$\begin{array}{ccc} \mathcal{K}' & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{K} & \longrightarrow & Y \end{array}$$

Factorization of Yoneda

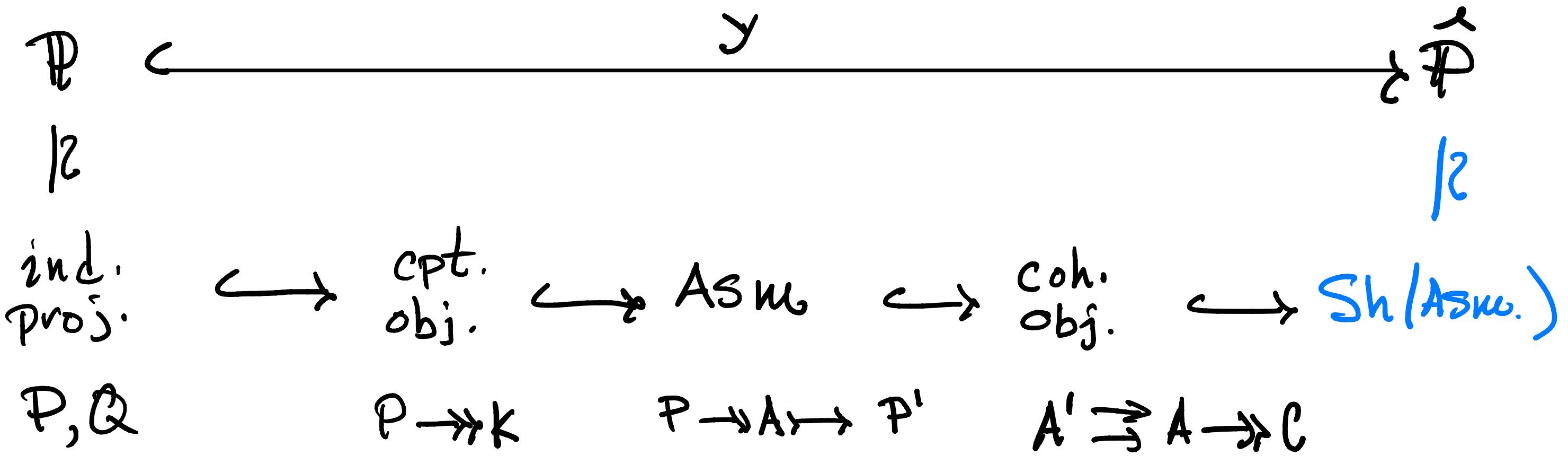


Factorization of Yoneda



F sheaf on Asm
 (for the reg. epi. top.) \iff $\begin{array}{ccc} A \times_A & \xrightarrow{\quad} & A \\ \downarrow B & & \downarrow B \\ \dots & \nearrow & \nearrow \end{array} F$

Factorization of Yoneda



F sheaf on Asm
 (for the reg. epis. top.) $\iff F \in \hat{P}$
 $= [P^{\text{op}}, \text{Set}]$.

Thm (Lack)

For a reg. cat. \mathcal{R} ,

$$\mathcal{R}_{\text{ex/reg}} \subseteq \text{Sh}(\mathcal{R}, \text{reg epi})$$

$$= \langle E \mid yA' \xrightarrow{\quad} yA \rightarrowtail E \rangle$$

f. $A, A' \in \mathcal{R}$

Thm (Lack)

For a reg. cat. R ,

$$R_{\text{ex/reg}} \subseteq \text{Sh}(R, \text{reg epi})$$

$$= \left\langle E \mid yA' \xrightarrow{\quad} yA \rightarrowtail E \right\rangle$$

f. $A, A' \in R$

Cor. $\text{Sh}(\text{Asm})_{\text{coh}} = \text{Asm}_{\text{ex/reg}}$.

Thm (Lack)

For a reg. cat. R ,

$$R_{\text{ex/reg}} \subseteq \text{Sh}(R, \text{reg epi}) \\ = \langle E \mid yA' \xrightarrow{\exists} yA \rightarrowtail E \rangle \\ \text{f. } A, A' \in R$$

Cor. $\text{Sh}(R)_{\text{coh}} = \text{Asm}_{\text{ex/reg}} = \text{Eff!}$

Summary

$\text{PAsm} \subset \text{Asm} \subset \text{Eff} \subset \overset{\text{1}}{\text{PAsm}}$

↓ ↓

$\text{CohSh}(\text{Asm}) \subset \text{Sh}(\text{Asm})$

Groth. topos

Summary

$$\text{PAsm} \subset \text{Asm} \subset \text{Eff} \subset \overset{1}{\text{PAsm}}$$

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Now in order to get a 2-topos we take
internal groupoids in the Groth. topos $\text{Sh}(\text{Asm})$,

$$\text{Gpd}(\text{Sh}(\text{Asm})) .$$

Summary

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Now in order to get a 2-topos we take
internal groupoids in the Groth. topos $\text{Sh}(\text{Asm})$.

$$\text{Gpd}(\text{Sh}(\text{Asm})) = \text{Gpd}([\mathbb{P}^{\text{op}}, \text{Set}]) = [\mathbb{P}^{\text{op}}, \text{Gpd}]$$

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Finally, we shall restrict to a subcat of
 "Coherent groupoids" in such a way that

$$\begin{array}{ccc} \mathrm{Coh}\mathrm{Gpd} & \hookrightarrow & \mathrm{Gpd} \\ \downarrow & \lrcorner & \uparrow \\ \mathrm{Eff} & \hookrightarrow & \mathrm{Gpd}_0 \end{array}$$

$$\begin{aligned} &\text{in } \mathrm{Sh}(\mathrm{Asm}) \\ &= [\mathbb{P}^{\mathrm{op}}, \mathrm{Gpd}] . \end{aligned}$$

QMS on $\text{Gpd}(\mathcal{E})$

There are many ways to put a QMS on $\text{Gpd}(\mathcal{E})$ for Groth. bpos \mathcal{E} . We choose one that:

- 1) is constructive,
- 2) works well with the groupoid model of HoTT,
- 3) gives \mathbf{Eff} as the coherent o-types:

$$\mathbf{Eff} = \text{Coh} \text{Gpd}(\mathcal{E})_o \subseteq \text{Gpd}(\mathcal{E})_o \subseteq \text{Gpd}(\mathcal{E}).$$

Using results of Everaert, Kieboom, vd Linden, we can show:

Thm. There's a model structure on $\text{Gpd}(\widehat{\mathcal{P}})$ with:

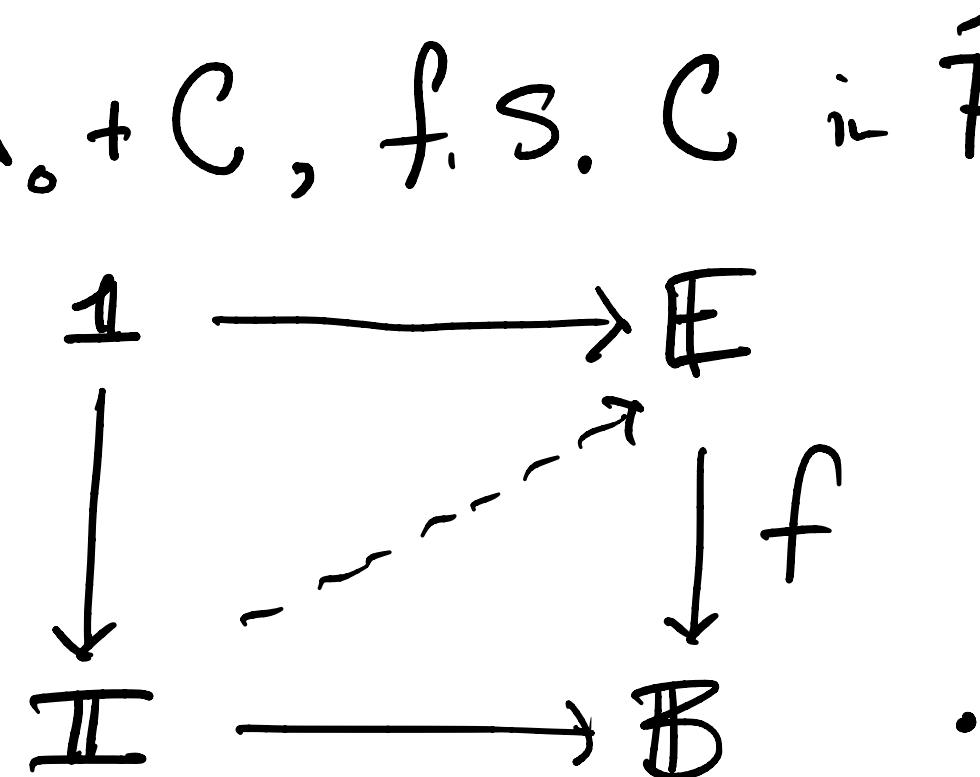
- Weak equivalences = equivalences of cats $\mathcal{G} \simeq \mathcal{H}$,

- Cofibrations = functors $c: A \rightarrow B$ s.t.

c_0 is a retract of $A_0 \rightarrow A_0 + C$, f.s. $C \in \widehat{\mathcal{P}}$,

- Fibrations =

f with "internal
path lifting":



Cohesive Groupoid

Def. A groupoid $G = (G, \rightrightarrows G_0)$ in $\hat{\mathcal{P}}$ is
Cohesive if :

- G_0 is cpt: $P \rightarrowtail G_0$ f. some $P \in \mathcal{P}$,
- $G_1 \rightarrow G_0 \times G_0$ is a cpt map.

Prop. Let $\mathbb{G} = (G_1, \Rightarrow G_0)$ be a coherent gpd

that is discrete, $\mathbb{G}^I \rightarrowtail \mathbb{G}$. Then

$\pi_0 \mathbb{G}$ is a coherent object.

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 then $\Pi_0 \mathbb{G}$ is a coherent object.

Pf. Since G_0 is cpt, there's a cover $P \rightarrow G_0$.

Take the p.b. K and its cover P' .

$$\begin{array}{ccccc}
 P' & \twoheadrightarrow & K & \longrightarrow & G_1 \\
 & \searrow & \downarrow & & \downarrow \\
 & & P_0 \times P_0 & \longrightarrow & G_0 \times G_0
 \end{array}$$

Prop. Let $\mathbb{G} = (G_1 \rightrightarrows G_0)$ be a coherent gpd that is discrete $\mathbb{G}^I \rightarrow \mathbb{G}$.
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so K is an assembly.

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So K is an assembly.

Write $\bar{\mathbb{G}} = G_0/G_1 = \pi_0 \mathbb{G}$.

We have $K \rightarrow P \rightarrow \bar{\mathbb{G}}$ exact.

So $\bar{\mathbb{G}} = \pi_0 \mathbb{G}$ is coherent. \square

$$\begin{array}{ccccccc}
 P' & \rightarrow & K & \rightarrow & G_1 & \rightarrow & \bar{\mathbb{G}} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 P \times P & \rightarrow & G_0 \times G_0 & \rightarrow & \bar{G} \times \bar{G} & &
 \end{array}$$

Then In $\text{Sh}(\text{As}_K) = \hat{P}$, we have

$$\begin{array}{ccc} \mathcal{E}\mathbf{ff} & \cong & \text{CohGpd}_0 \longrightarrow \text{Gpd}_0 \\ & & \downarrow \\ & & \text{CohGpd} \longrightarrow \text{Gpd} \end{array}$$

Then In $\text{Sh}(\text{As}_n) = \hat{\mathcal{P}}$, we have

$$\begin{array}{ccc} \mathcal{E}\mathbf{ff} & \cong & \text{CohGpd}_0 \longrightarrow \text{Gpd}_0 \\ & & \downarrow \\ \mathcal{E}\mathbf{ff}^2 := \text{CohGpd} & \longrightarrow & \text{Gpd} \end{array}$$

So we propose to take $\mathcal{E}\mathbf{ff}^2 = \text{CohGpd}(\text{Sh}(\text{As}_n))$,
a 2-topos with $\mathcal{E}\mathbf{ff}_0^2 = \mathcal{E}\mathbf{ff}$.

FINITOS !