

2. Propositional Calculus

- Consider the positive propositional calculus:

$$\text{PPC} := P \mid T \mid \varphi \wedge \psi \mid \varphi \Rightarrow \psi$$

- The rules of inference are as usual:

$$\frac{\cdot}{\varphi \vdash T} \qquad \frac{\vartheta \vdash \varphi, \vartheta \vdash \psi}{\vartheta \vdash \varphi \wedge \psi} \qquad \frac{\vartheta, \varphi \vdash \psi}{\vartheta \vdash \varphi \Rightarrow \psi}$$

- A Kripke model (K, \models) is a poset K , with a relation $\models \models_P$ s.t.
 $i \leq j, j \models_P \varphi \Rightarrow i \models_P \varphi$.
- Extend \models to all $\varphi \in \text{PPC}$ by:
 - $j \models T$ always,
 - $j \models \varphi \wedge \psi$ if $j \models \varphi$ & $j \models \psi$,
 - $j \models \varphi \Rightarrow \psi$ iff $i \models \varphi$ implies $i \models \psi$, f.a. $i \leq j$
- Let $K \Vdash \varphi$ if $j \models \varphi$ f.a. $j \in K$, and $\Vdash \varphi$ if $K \Vdash \varphi$ f.a. (K, \models) .
" φ is Kripke valid"

Prop (Kripke completeness of PPC)

$$\vdash \varphi \quad \text{iff} \quad \Vdash \varphi .$$

Pf. (i) Order (the formulas of) PPC by $\varphi \vdash \psi$,
 & identify $\varphi = \psi$ iff $\varphi \dashv \vdash \psi$,
 Call the resulting poset \mathcal{C}_{PPC} .

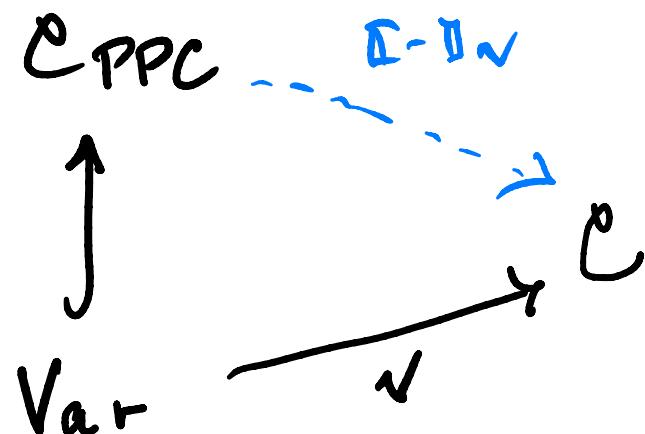
(ii) \mathcal{C}_{PPC} has:

- a terminal object T
 - products $\varphi \wedge \psi$
 - exponentials $\varphi \Rightarrow \psi$
- } a cartesian
closed poset
=: CCP

(iii) C_{PPC} is the free CCP on the set

$$\{P, q, \dots\} := \text{Var},$$

meaning:



If C is any CCP & ν is any function,
then there's a unique CCP map $\Gamma - \Pi_\nu$ s.th.

$$[\Gamma_P] = \nu P$$

$$[\Gamma_T] = T$$

$$[\Gamma_{\varphi \wedge \psi}] = [\Gamma_\varphi] \wedge [\Gamma_\psi]$$

$$[\Gamma_{\varphi \Rightarrow \psi}] = [\Gamma_\varphi] \Rightarrow [\Gamma_\psi],$$

briefly:

$$CCP(C_{PPC}, C) \cong \text{Set}(\text{VAR}, C).$$

Next, we need the following ...

(4)

Lemma 1: A Kripke model (K, H) of PPC

is the same thing as a CCP map

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CCP

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is the same thing as a CCP map

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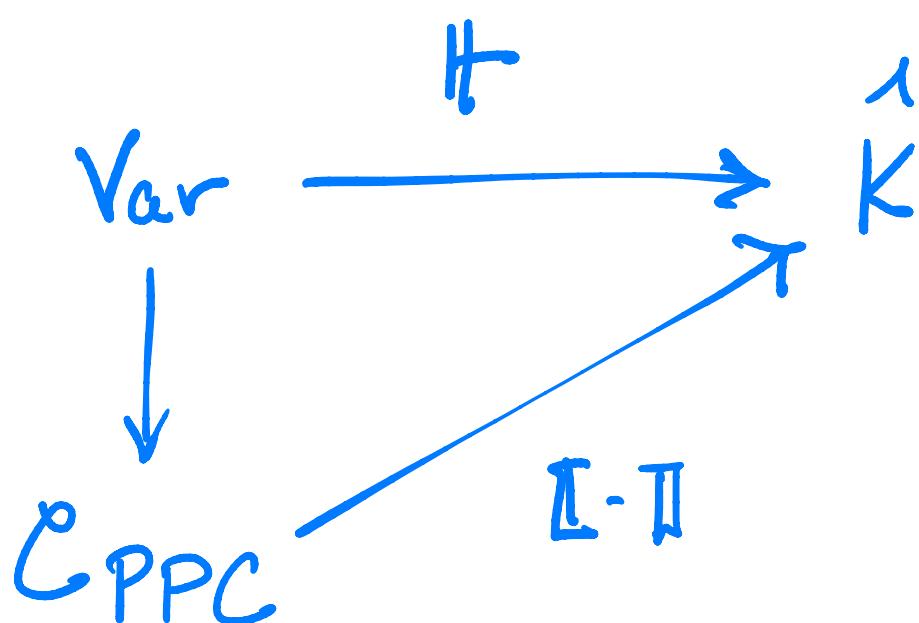
Pf: There's a bijection:

$$\mathbb{H} \subseteq K \times \text{Var} \quad \text{w/ } i \leq j \# p \Rightarrow i \# p$$

$$K \times \text{Var} \longrightarrow \mathbb{Z} = (0 \leq 1) \quad \text{in Pos}$$

$$\text{Var} \longrightarrow \mathbb{Z}^K = \hat{K} \quad \text{in Pos}$$

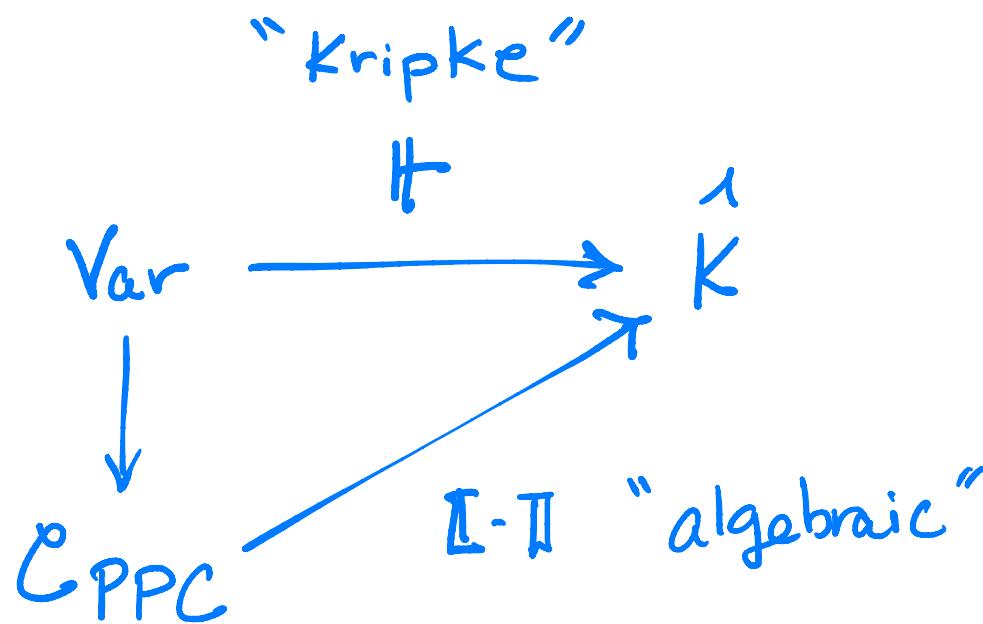
$$\mathbb{I}-\mathbb{J} : \mathcal{C}_{PPC} \longrightarrow \hat{K} \quad \text{in CCP}$$



Remark

A Kripke model (K, \mathbb{H}) thus corresponds to an algebraic model $\mathbb{I}-\mathbb{J}$ in \hat{K} via

$$j \models \varphi \quad \text{iff} \quad j \in \llbracket \varphi \rrbracket .$$



The Kripke conditions correspond to algebraic ones:

$$i \leq j \models P \Rightarrow i \models P$$

$$\llbracket P \rrbracket \in \hat{K}$$

$$j \models T \quad \text{f.a. } j$$

$$\llbracket T \rrbracket = k$$

$$j \models \varphi \wedge \psi \quad \text{iff} \quad j \models \varphi \wedge j \models \psi$$

$$\llbracket \varphi \wedge \psi \rrbracket = \llbracket \varphi \rrbracket \wedge \llbracket \psi \rrbracket$$

$$j \models \varphi \Rightarrow \psi \quad \text{iff f.a. } i \leq j ,$$

$$\llbracket \varphi \Rightarrow \psi \rrbracket = \llbracket \varphi \rrbracket \Rightarrow \llbracket \psi \rrbracket$$

$$i \models \varphi \text{ implies } i \models \psi$$

(iv) The syntactic CCP \mathcal{C}_{PPC} has a
canonical algebraic model, namely

$$\downarrow : \mathcal{C}_{CCP} \rightarrow \mathcal{C}_{CCP}^*,$$

it corresponds to a canonical Kripke model

$$(\mathcal{C}_{PPC}, \vdash),$$

with

$$\varphi \vdash \psi \text{ iff } \varphi \in \llbracket \psi \rrbracket \text{ iff } \varphi \vdash \psi.$$

So we have:

$$(*) \quad \mathcal{C}_{PPC} \vdash \varphi \text{ iff } \varphi \vdash \varphi \text{ f.a. } \varphi \in \mathcal{C}_{PPC}$$

$$\text{ iff } \vdash \varphi.$$

(v) Thus we have :

$$\begin{aligned} \vdash \varphi &= K \vdash \varphi \text{ f.a. } (K, \vdash) \\ &\Rightarrow \mathcal{C}_{PPC} \vdash \varphi \\ &\Leftrightarrow \vdash \varphi. \text{ Completeness } \checkmark \end{aligned}$$

Conversely :

$$\begin{aligned} \vdash \varphi &\stackrel{*}{\Leftrightarrow} \mathcal{C}_{PPC} \vdash \varphi \text{ Soundness} \\ &\Rightarrow K \vdash \varphi \text{ f.a. } (K, \vdash) \end{aligned}$$

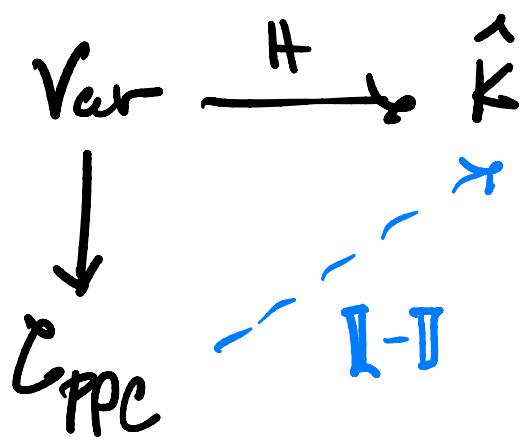
by the following ...

Lemma 2: Given $\varphi \in \text{PPC}$,

$\mathcal{L}_{\text{PPC}} \Vdash \varphi$ implies $K \Vdash \varphi$ f.a. (K, \Vdash) .

pf. $\mathcal{L}_{\text{PPC}} \Vdash \varphi \Rightarrow T \Vdash \varphi$
 $\Rightarrow T \models \varphi$
 $\Rightarrow T \leq \varphi$ in \mathcal{L}_{PPS} .

Given any (K, \Vdash) we get a PPC map $\llbracket - \rrbracket$:



Thus:

$$\llbracket T \rrbracket \leq \llbracket \varphi \rrbracket$$

so:

$$K \Vdash \varphi .$$

Kripke completeness for PPC ✓

Next we can extend this result to :

(1) $\text{IPC} = \text{PPC} \wedge \perp, \vee$ (Kripke)

(2) Topological semantics (McKinsey
-Tarski)

(3) A translation (Gödel)

$$\text{IPC} \hookrightarrow \text{CPC} \wedge \#$$

(Discuss each one briefly)

1. Extension from PPC to IPC

(9)

Def. A Heyting algebra is a CCP with all finite joins: $\perp, P \vee q, \neg\neg$.

Equivalently, a bdd lattice w/ $a \Rightarrow b$.

Examples

i. any Boolean algebra B is a HA,

$$P \Rightarrow q := \neg P \vee q$$

ii. for any topological space (X, \mathcal{O}_X) , the open sets form a HA,

$$U \Rightarrow V = \bigcup_{W \cap U \subseteq V} W$$

iii. any V -complete distributive lattice,
e.g. \hat{P} f. any poset P .

iv. any complete linear order,

e.g. $[0, 1] \subseteq \mathbb{R}$. "fuzzy logic"

By (iii) the canonical model of PPC

$$\downarrow : \mathcal{E}_{PPC} \longrightarrow \widehat{\mathcal{E}}_{PPC}$$

is therefore valued in a HA !

Since being a HA is algebraic, there's a unique extension to the V-Completion, from CCPs to HAs:

$$\begin{array}{ccc} \mathcal{E}_{PPC} & \xrightarrow{\quad \downarrow \quad} & \widehat{\mathcal{E}}_{PPC} \\ & \searrow & \nearrow \tau \\ & \mathcal{E}_{PPC}^V & \end{array}$$

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$$\begin{array}{ccc} \mathcal{E}_{PPC} & \xrightarrow{\quad \downarrow \quad} & \widehat{\mathcal{E}}_{PPC} \\ & \searrow & \nearrow \\ & \mathcal{E}_{PPC}^V = \mathcal{E}_{IPC} & \end{array}$$

Now:

syntactic poset of IPC

And for φ, ψ in IPC, we again have:

$$\varphi \vdash_{IPC} \psi \Leftrightarrow \downarrow \varphi \leq \downarrow \psi \text{ in } \widehat{\mathcal{E}}_{IPC}.$$

But unfortunately the CCP embedding

$$\downarrow : \mathcal{E}_{IPC} \longrightarrow \widehat{\mathcal{E}}_{IPC}$$

is not Heyting (it does not preserve \perp, \vee).

Instead, we use the following :

Thm (Joyal) For H any HA the map

$$j : H \longrightarrow \widehat{H^*},$$

where $H^* = \text{Prime}(H)$

and $j(h) = \{p \mid h \in p\}$

is both Heyting and conservative.

$$jx \leq jy \Rightarrow x \leq y$$

Generalizes Stone Representation

Theorem from BAs to HAs.

Pf: Uses Birkhoff's prime ideal theorem.

Now we proceed as for $\text{PPC} \xleftrightarrow{\downarrow} \overset{\wedge}{\text{PPC}}$,
 but using $\text{IPC} \xleftrightarrow{j} \overset{\wedge}{\text{IPC}}^*$ instead: (12)

Theorem. (Kripke completeness of IPC)

Let $\text{IPC} = \text{PPC}$ extended by:

$$\frac{\bullet}{\perp \vdash \varphi} \quad \frac{\varphi \vdash \delta \quad \psi \vdash \delta}{\varphi \vee \psi \vdash \delta}$$

Then:

$\vdash \varphi$ iff $K \Vdash \varphi$ f.a. Kripke
 models (K, \Vdash)

Here K is a poset and again

$$\Vdash \subseteq \hat{K} \times \text{Var},$$

extended to $\text{IPC} \supseteq \text{Var}$ by:

- $j \Vdash T$ f.a. j
- $j \not\Vdash \perp$ f.a. j
- $j \Vdash \varphi \wedge \psi$ iff $j \Vdash \varphi$ & $j \Vdash \psi$
- $j \Vdash \varphi \vee \psi$ iff $j \Vdash \varphi$ or $j \Vdash \psi$
- $j \Vdash \varphi \Rightarrow \psi$ iff $i \Vdash \varphi$ implies $i \Vdash \psi$ for $i \leq j$.

