

# Introduction to Categorical Logic

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# Chapter 3

## First-Order Logic

Having considered equational and propositional logic, we now move on to first-order logic, which is the usual predicate logic with propositional connectives like  $\wedge$  and  $\Rightarrow$  and the quantifiers  $\forall$  and  $\exists$ . This logic can be seen as propositional logic indexed over an equational theory, in a sense that will become clear; quantification will then be seen to result from completeness with respect to the indexing.

We pursue the same general approach to studying logic via category theory as in the previous chapters, determining categorical structures that model the logical operations, and regarding (certain) categories with these structures as theories, and functors that preserve them as models. We again construct the classifying category for a theory from the syntax of a deductive system and establish its universal property, leading again to functorial semantics. We then establish basic completeness theorems by embedding such classifying categories in particular semantic categories of interest, such as presheaves.

### 3.1 Predicate logic

Let us first demonstrate our general approach informally with an example. In Chapter ?? we considered models of algebraic theories in categories with finite products. Recall that e.g. a group is a structure of the form:

$$m : G \times G \rightarrow G , \quad i : G \rightarrow G , \quad e : 1 \rightarrow G ,$$

for which, moreover, certain diagrams built from these basic arrows must commute. We can express some properties of groups in terms of further equations, for example commutativity

$$x \cdot y = y \cdot x ,$$

which is expressed by the diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\sim} & G \times G \\ m \downarrow & & \downarrow m \\ G & \xlongequal{\quad} & G \end{array}$$



1. A language  $\mathcal{L}$  for a first-order theory consists, as usual, of some basic relation, function, and constant symbols, say  $\mathcal{L} = (R, f, c)$ .
2. An  $\mathcal{L}$ -structure in a category  $\mathcal{C}$  with finite limits is an interpretation of  $\mathcal{L}$  in  $\mathcal{C}$  as an object  $M$  equipped with corresponding relations (subobjects) and operations (morphisms) of appropriate arities,

$$\begin{aligned} R^M &\rhd M \times \cdots \times M \\ f^M &: M \times \cdots \times M \longrightarrow M \\ c^M &: 1 \rightarrow M. \end{aligned}$$

3. Formulas  $\varphi$  in first-order logic will be interpreted as subobjects,

$$\llbracket \varphi \rrbracket \rhd M \times \cdots \times M.$$

The interpretation makes use of categorical operations in  $\mathcal{C}$  corresponding to the logical ones appearing in the formula  $\varphi$ .

4. A theory  $\mathbb{T}$  in first-order logic consists of a set of (binary) sequents,

$$\varphi \vdash \psi.$$

5. A model of  $\mathbb{T}$  is then an interpretation  $M$  in which the corresponding subobjects “satisfy” all the sequents of  $\mathbb{T}$ , in the sense that

$$\llbracket \varphi \rrbracket \leq \llbracket \psi \rrbracket \quad \text{in } \mathbf{Sub}(M^n).$$

6. We shall give a deductive calculus for such sequents  $\varphi \vdash \psi$ , prove that it is sound with respect to categorical models, and then use it to construct a classifying category  $\mathcal{C}_{\mathbb{T}}$  with the expected universal property: models of  $\mathbb{T}$  in any suitably-structured category  $\mathcal{C}$  correspond uniquely to structure-preserving functors  $\mathcal{C}_{\mathbb{T}} \rightarrow \mathcal{C}$ .
7. Completeness of the calculus with respect to general models follows from classification, while completeness with respect to special models, such as “Kripke-models”  $\mathbf{Set}^K$ , follows from embedding  $\mathcal{C}_{\mathbb{T}}$  in such special categories.

Not only does having such categorical semantics permit us to prove things about different systems of logic (such as consistency of formal systems and independence of axioms), it also allows us to *use* the systems of logic to reason formally about logical structures in categories of various kinds.

### 3.1.1 Theories

A *first-order theory*  $\mathbb{T}$  consists of an underlying *type theory* and a set of formulas in a *fragment of first-order logic*. Anticipating Chapter ??, the type theory is given by a set of basic types, a set of basic constants together with their types, rules for forming types, and rules and axioms for deriving typing judgments,

$$x_1 : A_1, \dots, x_n : A_n \mid t : B ,$$

expressing that term  $t$  has type  $B$  in typing context  $x_1 : A_1, \dots, x_n : A_n$ . There is also a set of axioms and rules of inference which tell us which equations between terms,

$$x_1 : A_1, \dots, x_n : A_n \mid t = u : B ,$$

are assumed to hold. This part of the theory  $\mathbb{T}$  may be regarded as providing the underlying structure, on top of which the logical formulas are defined. For first-order logic, the underlying type theory is essentially the same as the equational logic that we already met in Chapter ??.

A fragment of first-order logic is then given by a set of *basic relation symbols*, together with a specification of which first-order operations are to be used in building formulas. Each basic relation symbol has a *signature*  $(A_1, \dots, A_n)$ , which specifies the types of its arguments. The *arity* of a relation symbol is the number of arguments it takes. The judgment<sup>2</sup>

$$x_1 : A_1, \dots, x_n : A_n \mid \phi \text{ pred}$$

states that  $\phi$  is a well-formed formula in typing context  $x_1 : A_1, \dots, x_n : A_n$ . For each basic relation symbol  $R$  with signature  $(A_1, \dots, A_n)$  there is an inference rule

$$\frac{\Gamma \mid t_1 : A_1 \quad \dots \quad \Gamma \mid t_n : A_n}{\Gamma \mid R(t_1, \dots, t_n) \text{ pred}}$$

which says that the atomic formula  $R(t_1, \dots, t_n)$  is well formed in context  $\Gamma$ . Depending on what fragment of first-order logic is involved, there may be other rules for forming logical formulas. For example, if equality is present as a formula, then for each type  $A$  there is a rule:

$$\frac{\Gamma \mid t : A \quad \Gamma \mid u : A}{\Gamma \mid t =_A u \text{ pred}}$$

And if conjunction is present, then there is a rule:

$$\frac{\Gamma \mid \varphi \text{ pred} \quad \Gamma \mid \psi \text{ pred}}{\Gamma \mid \varphi \wedge \psi \text{ pred}}$$

Other such rules will be given when we come to the study of particular logical operations.

<sup>2</sup>We follow type-theoretic practice here by adding the tag **pred** to turn what would otherwise be an exhibited formula in context into a judgement concerning the formula.



The basic logical judgment of a first-order theory is *entailment* between formulas,

$$x_1 : A_1, \dots, x_n : A_n \mid \varphi_1, \dots, \varphi_m \vdash \psi ,$$

which states that in the typing context  $x_1 : A_1, \dots, x_n : A_n$ , the assumptions  $\varphi_1, \dots, \varphi_m$  entail  $\psi$ . It is understood that the terms appearing in the formulas are well-typed in the typing context, and that the formulas  $\varphi_1, \dots, \varphi_m, \psi$  are part of the fragment of the logic of  $\mathbb{T}$ . When the fragment contains conjunction  $\wedge$  it is convenient to restrict attention to *binary* sequents,

$$x_1 : A_1, \dots, x_n : A_n \mid \varphi \vdash \psi,$$

by replacing  $\varphi_1, \dots, \varphi_m$  with  $\varphi_1 \wedge \dots \wedge \varphi_m$ . When the fragment contains equality, we may replace the type-theoretic equality judgments

$$x_1 : A_1, \dots, x_n : A_n \mid t = u : B$$

with the entailments

$$x_1 : A_1, \dots, x_n : A_n \mid \cdot \vdash t =_B u .$$

The subscript at the equality sign indicates the type at which the equality is taken. In a theory  $\mathbb{T}$  there are basic entailments, or axioms, which together with the inference rules for the operations involved can be used for deriving judgments, as usual.

We shall consider several standard fragments of first-order logic, determined by selecting a subset of logical connectives and quantifiers. These are as follows:

1. *Full first-order logic* consists of formulas built from the logical operations

$$= \top \perp \neg \wedge \vee \Rightarrow \forall \exists .$$

2. *Cartesian logic* is the fragment

$$= \top \wedge .$$

3. *Regular logic* is the fragment

$$= \top \wedge \exists .$$

4. *Coherent logic* is the fragment built from

$$= \top \wedge \exists \perp \vee .$$

5. *Geometric logic* consists of formulas of the form

$$\forall x : A . (\varphi \Rightarrow \psi) ,$$

where  $\varphi$  and  $\psi$  are coherent formulas.<sup>3</sup>

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<sup>3</sup>There is also *infinitary geometric logic*, in which  $\varphi$  and  $\psi$  may contain disjunctions  $\bigvee_i \vartheta_i$  of infinitely many formulas  $\vartheta_i$ .

The names for these fragments come from the names of the various kinds of categories in which they are interpreted. We shall also consider both *Heyting* and *Boolean* theories in full first-order logic, which differ according to their assumed rules of inference and their intended interpretations.

The well-formed terms and formulas of a first-order theory  $\mathbb{T}$  constitute its *language*. It may seem that we are doing things backwards, because we should have spoken of first-order languages before we spoke of first-order theories. While this is possible for simple theories, it becomes difficult to do when we consider more complicated theories in which types and logical formulas are intertwined. In such cases the typing judgments and entailments may be given by a mutual recursive definition. In order to find out whether a given term is well-formed, we might have to prove a logical statement. In everyday mathematics this occurs all the time, for example, to show that the term  $\int_0^\infty f$  denotes a real number, it may be necessary to prove that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is an integrable function and that the integral has a finite value. This is why it does not always make sense to strictly differentiate a language from a theory.<sup>4</sup>

In order to focus on the logical part of first-order theories, we will limit attention to only two very simple kinds of type theory. A *single-sorted* first-order theory has as its underlying type theory a single type  $A$ , and for each  $k \in \mathbb{N}$  a set of basic  $k$ -ary function symbols. The rules for typing judgments are:

1. Variables in contexts:

$$\frac{}{x_1 : A, \dots, x_n : A \mid x_i : A}$$

2. For each basic function symbol  $f$  of arity  $k$ , there is an inference rule

$$\frac{\Gamma \mid t_1 : A \quad \dots \quad \Gamma \mid t_n : A}{\Gamma \mid f(t_1, \dots, t_n) : A}$$

This much is essentially an algebraic theory. In addition, however, a single-sorted first-order theory may contain relation symbols, formulas, axioms, and rules of inference which an algebraic theory does not.

A slight generalization of a single-sorted theory is a *many-sorted* one. Its underlying type theory is given by a set of types, and a set of basic function symbols. Each function symbol  $f$  has a *signature*  $(A_1, \dots, A_n; B)$ , where  $n$  is the arity of  $f$  and  $A_1, \dots, A_n, B$  are types. The rules for typing judgments are:

1. Variables in contexts:

$$\frac{}{x_1 : A_1, \dots, x_n : A_n \mid x_i : A_i}$$

2. For each basic function symbol  $f$  with signature  $(A_1, \dots, A_n; B)$ , there is an inference rule

$$\frac{\Gamma \mid t_1 : A_1 \quad \dots \quad \Gamma \mid t_n : A_n}{\Gamma \mid f(t_1, \dots, t_n) : B}$$

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<sup>4</sup>However, it *does* make sense to distinguish *syntax* from theories. Rules of substitution and the behaviour of free and bound variables are syntactic considerations, for example.

We may write suggestively  $f : A_1 \times \cdots \times A_n \rightarrow B$  to indicate that  $(A_1, \dots, A_n; B)$  is the signature of  $f$ . However, this does not mean that  $A_1 \times \cdots \times A_n \rightarrow B$  is a type! A many-sorted first-order theory does *not* have any type forming operations, such as  $\times$  and  $\rightarrow$ . We shall consider type theories with such operations in Chapter ??.

### 3.1.2 Subobjects

Formulas of first-order logic will be interpreted as “generalized subsets”, i.e. subobjects. We therefore need to review some of the basic theory of these.

Let  $A$  be an object in a category  $\mathcal{C}$ . If  $i : I \rightarrowtail A$  and  $j : J \rightarrowtail A$  are monos into  $A$ , we say that  $i$  is smaller than  $j$ , and write  $i \leq j$ , when there exists a morphism  $k : I \rightarrow J$  such that the following diagram commutes:

$$\begin{array}{ccc} I & \xrightarrow{k} & J \\ & \searrow i & \swarrow j \\ & A & \end{array}$$

If such a  $k$  exists then it, too, is monic, since  $i$  is, and it is unique, since  $j$  is monic. The class  $\mathbf{Mono}(A)$  of all monos into  $A$  is thus preordered by the relation  $\leq$ . It is the same as the slice category  $\mathbf{Mono}(\mathcal{C})/A$  consisting of all monos with codomain  $A$  and commutative triangles between them. Let  $\mathbf{Sub}(A)$  be the poset reflection of the preorder  $\mathbf{Mono}(A)$ . Thus the elements of  $\mathbf{Sub}(A)$  are equivalence classes of monos into  $A$ , where  $i : I \rightarrowtail A$  and  $j : J \rightarrowtail A$  are equivalent when  $i \leq j$  and  $j \leq i$  (note that then  $I \cong J$ ). The induced relation  $\leq$  on  $\mathbf{Sub}(A)$  is then a partial order.

We have to be a bit careful with the formation of  $\mathbf{Sub}(A)$ , since it is defined as a quotient of a *class*  $\mathbf{Mono}(A)$ . In many particular cases the general construction by quotients can be avoided. If we can demonstrate that the preorder  $\mathbf{Mono}(A)$  is equivalent, as a category, to a poset  $P$  then we can simply take  $\mathbf{Sub}(A) = P$ . We will usually simply require that  $\mathbf{Sub}(A)$  is small.

**Definition 3.1.1.** A category  $\mathcal{C}$  is *well-powered* when, for all  $A \in \mathcal{C}$ , there is at most a *set* of subobjects of  $A$ , so that the category  $\mathbf{Mono}(A)$  is equivalent to a (small) poset. In other words,  $\mathbf{Sub}(A)$  is a small category for every  $A \in \mathcal{C}$ .

We shall often speak of subobjects as if they were monos rather than equivalence classes of monos. It is then understood that we mean the subobjects represented by monos and not the monos themselves. Sometimes we refer to a mono  $i : I \rightarrowtail A$  by its domain  $I$  only, even though the object  $I$  itself does not determine the morphism  $i$ . Hopefully this will not cause confusion, as it is always going to be clear which mono is meant to go along with the object  $I$ .

In a category  $\mathcal{C}$  with finite limits the assignment  $A \mapsto \mathbf{Sub}(A)$  is the object part of the contravariant *subobject functor*,

$$\mathbf{Sub} : \mathcal{C}^{\mathrm{op}} \rightarrow \mathbf{Poset} .$$

The morphism part of **Sub** is given by pullback; in detail, given any  $f : A \rightarrow B$ , let  $\mathbf{Sub}(f) = f^* : \mathbf{Sub}(B) \rightarrow \mathbf{Sub}(A)$  be the monotone map that takes the subobject (represented by)  $i : I \rightarrow B$  to the subobject (represented by)  $f^*i : f^*I \rightarrow A$ , where  $f^*i : f^*I \rightarrow A$  is a pullback of  $i$  along  $f$ :

$$\begin{array}{ccc} f^*I & \xrightarrow{\quad} & I \\ \downarrow f^*i & \lrcorner & \downarrow i \\ A & \xrightarrow{\quad f \quad} & B \end{array}$$

Recall that a pullback of a mono is again mono, so this definition makes sense. We need to verify (1) that if two monos  $i : I \rightarrow A$  and  $j : J \rightarrow A$  are equivalent, then their pullbacks are so as well; and (2) that  $\mathbf{Sub}(1_A) = 1_{\mathbf{Sub}(A)}$  and  $\mathbf{Sub}(g \circ f) = \mathbf{Sub}(f) \circ \mathbf{Sub}(g)$ . These all follow easily from the fact that pullback is a functor  $\mathcal{C}/B \rightarrow \mathcal{C}/A$ , which reduces to the familiar “2-pullbacks” lemma:

**Lemma 3.1.2.** *Suppose both squares in the following diagram are pullbacks:*

$$\begin{array}{ccccc} \cdot & \xrightarrow{\quad} & \cdot & \xrightarrow{\quad} & \cdot \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ \cdot & \xrightarrow{\quad} & \cdot & \xrightarrow{\quad} & \cdot \end{array}$$

*Then the outer rectangle is a pullback diagram as well. Moreover, if the outer rectangle and the right square are pullbacks, then so is the left square.*

*Proof.* This is left as an exercise in diagram chasing. □

Of course, pullbacks are really only determined up to isomorphism, but this does not cause any problems because isomorphic monos represent the same subobject.

In the semantics to be given below, a formula

$$x : A \mid \varphi \text{ pred}$$

will be interpreted as a subobject

$$\llbracket x : A \mid \varphi \rrbracket \twoheadrightarrow \llbracket A \rrbracket.$$

Thus  $\mathbf{Sub}(A)$  can be regarded as the poset of “predicates” on  $A$ , generalizing the powerset of a set  $A$ . Logical operations on formulas then correspond to operations on  $\mathbf{Sub}(A)$ . The structure of  $\mathbf{Sub}(A)$  therefore determines which logical connectives can be interpreted. If  $\mathbf{Sub}(A)$  is a Heyting algebra, then we can interpret the (propositional part of) the full intuitionistic propositional calculus (cf. Subsection ??), but if  $\mathbf{Sub}(A)$  only has binary meets, then all that can be interpreted are  $\top$  and  $\wedge$ . We will work out details of different operations in the following sections, but one common aspect that we require is the “stability” of the interpretation of the logical operations, in a sense that we now make clear.

### Substitution and stability

Let us consider the interpretation of substitution of terms for variables. There are two kinds of substitution, into a term, and into a formula. We may substitute a term  $x : A \mid t : B$  for a variable  $y$  in a term  $y : B \mid u : C$  to obtain a new term  $x : A \mid u[t/y] : C$ . If  $t$  and  $u$  are interpreted as morphisms

$$\llbracket A \rrbracket \xrightarrow{\llbracket t \rrbracket} \llbracket B \rrbracket \xrightarrow{\llbracket u \rrbracket} \llbracket C \rrbracket$$

then  $u[t/y]$  is interpreted as their composition:

$$\llbracket x : A \mid u[t/y] : C \rrbracket = \llbracket y : B \mid u : C \rrbracket \circ \llbracket x : A \mid t : B \rrbracket .$$

Thus, *substitution into a term is composition*.

The second kind of substitution occurs when we substitute a term  $x : A \mid t : B$  for a variable  $y$  in a formula  $y : B \mid \varphi$  to obtain a new formula  $x : A \mid \varphi[t/y]$ . If  $t$  is interpreted as a morphism  $\llbracket t \rrbracket : \llbracket A \rrbracket \rightarrow \llbracket B \rrbracket$  and  $\varphi$  is interpreted as a subobject  $\llbracket \varphi \rrbracket \rightarrow \llbracket B \rrbracket$  then the interpretation of  $\varphi[t/y]$  is the pullback of  $\llbracket \varphi \rrbracket$  along  $\llbracket t \rrbracket$ :

$$\begin{array}{ccc} \llbracket \varphi[t/y] \rrbracket = \llbracket t \rrbracket^* \llbracket \varphi \rrbracket & \xrightarrow{\quad} & \llbracket \varphi \rrbracket \\ \downarrow \lrcorner & & \downarrow \\ \llbracket A \rrbracket & \xrightarrow{\llbracket t \rrbracket} & \llbracket B \rrbracket \end{array}$$

Thus, *substitution into a formula is pullback*,

$$\llbracket x : A \mid \varphi[t/y] \rrbracket = \llbracket x : A \mid t : B \rrbracket^* \llbracket y : B \mid \varphi \rrbracket .$$

Now, because substitution respects the syntactical, logical operations, e.g.

$$(\varphi \wedge \psi)[t/x] = \varphi[t/x] \wedge \psi[t/x],$$

the categorical structures used to interpret the various logical operations such as  $\wedge$  must also behave well with respect to the interpretation of substitution, i.e. pullback. We say that a categorical property or structure is *stable (under pullbacks)* if it is preserved by pullbacks, so that e.g.

$$\begin{aligned} \llbracket t \rrbracket^* \llbracket (\varphi \wedge \psi) \rrbracket &= \llbracket (\varphi \wedge \psi)[t/x] \rrbracket = \llbracket \varphi[t/x] \wedge \psi[t/x] \rrbracket \\ &= \llbracket \varphi[t/x] \rrbracket \wedge \llbracket \psi[t/x] \rrbracket = \llbracket t \rrbracket^* \llbracket \varphi \rrbracket \wedge \llbracket t \rrbracket^* \llbracket \psi \rrbracket . \end{aligned}$$

In more detail, say that a category  $\mathcal{C}$  has *stable meets* if each poset  $\mathbf{Sub}(A)$  has binary meets, and the pullback of a meet  $I \wedge J \rightarrow A$  along any map  $f : B \rightarrow A$  is the meet  $f^* I \wedge f^* J \rightarrow A$  of the respective pullbacks,

$$f^*(I \wedge J) = f^* I \wedge f^* J.$$

This means that the meet operation,

$$\wedge : \text{Sub}(A) \times \text{Sub}(A) \longrightarrow \text{Sub}(A)$$

is natural in  $A$ , in the sense that for any map  $f : B \rightarrow A$  the following diagram commutes.

$$\begin{array}{ccc} \text{Sub}(A) \times \text{Sub}(A) & \xrightarrow{\wedge_A} & \text{Sub}(A) \\ f^* \times f^* \downarrow & & \downarrow f^* \\ \text{Sub}(B) \times \text{Sub}(B) & \xrightarrow{\wedge_B} & \text{Sub}(B) \end{array}$$

**Exercise 3.1.3.** Show that any category  $\mathcal{C}$  with finite limits has stable meets in the foregoing sense: each poset  $\text{Sub}(A)$  has all finite meets (i.e. including the “empty meet”  $\mathbf{1}$ ), and these are stable under pullbacks. Conclude that for any finite limit category  $\mathcal{C}$ , the subobject functor  $\text{Sub} : \mathcal{C}^{\text{op}} \rightarrow \text{Pos}$  therefore factors through the subcategory of  $\wedge$ -semilattices.

### Generalized elements

In any category, we can regard arbitrary arrows  $x : X \rightarrow C$  as *generalized elements* of  $C$ , thinking thereby of variable elements or parameters. With respect to a subobject  $S \rightarrowtail C$ , such an element is said to be *in the subobject*, written

$$x \in_C S,$$

if it factors through (any mono representing) the subobject,

$$\begin{array}{ccc} & & S \\ & \nearrow & \downarrow \\ X & \xrightarrow{x} & C \end{array}$$

which, observe, it then does uniquely. The following “generalized element semantics” can be a useful technique for “externalizing” the operations on subobjects into statements about generalized elements.

**Proposition 3.1.4.** *Let  $C$  be any object in a category  $\mathcal{C}$  with finite limits.*

1. *for the top element  $\mathbf{1} \in \text{Sub}(C)$ , and for all  $x : X \rightarrow C$ ,*

$$x \in_C \mathbf{1}.$$

2. *for any  $S, T \in \text{Sub}(C)$ ,*

$$S \leq T \iff x \in_C S \text{ implies } x \in_C T, \text{ for all } x : X \rightarrow C.$$

3. for any  $S, T \in \mathbf{Sub}(C)$ , and for all  $x : X \rightarrow C$ ,

$$x \in_C S \wedge T \iff x \in_C S \text{ and } x \in_C T.$$

4. for the subobject  $\Delta = [\langle 1_C, 1_C \rangle] \in \mathbf{Sub}(C \times C)$ , and for all  $x, y : X \rightarrow C$ ,

$$\langle x, y \rangle \in \Delta \iff x = y.$$

5. for the equalizer  $E_{(f,g)} \rightarrowtail A$  of a pair of arrows  $f, g : A \rightrightarrows B$ , and for all  $x : X \rightarrow A$ ,

$$x \in_A E_{(f,g)} \iff fx = gx.$$

6. for the pullback  $f^*S \rightarrowtail A$  of a subobject  $S \rightarrowtail B$  along any arrow  $f : A \rightarrow B$ , and for all  $x : X \rightarrow A$ ,

$$x \in_A f^*S \iff fx \in_B S.$$

**Exercise 3.1.5.** Prove the proposition.

### 3.1.3 Cartesian logic

We begin with a basic system of logic for categories with finite limits, also called *cartesian categories*, which we therefore call *cartesian logic*. This is a logic of formulas built from the logical operations  $=$ ,  $\top$ , and  $\wedge$ , over a multi-sorted type theory with unit type  $1$ . (See section ?? for multi-sorted type theories and the axioms for the unit type. In a dependently-typed formulation as in Chapter ?? one would also include equality types.).

#### Formation rules for cartesian logic

Given a basic language consisting of a stock of relation and function symbols (with arities), the terms are built up as explained in Section 3.1.1 from the basic function symbols and variables (we take “constants” to be 0-ary function symbols). The rules for constructing formulas are as follows:

1. The 0-ary relation symbol  $\top$  is a formula:

$$\overline{\Gamma \mid \top \text{ pred}}$$

2. For each basic relation symbol  $R$  with signature  $(A_1, \dots, A_n)$  there is a rule

$$\frac{\Gamma \mid t_1 : A_1 \quad \dots \quad \Gamma \mid t_n : A_n}{\Gamma \mid R(t_1, \dots, t_n) \text{ pred}}$$

3. For each type  $A$ , there is a rule

$$\frac{\Gamma \mid s : A \quad \Gamma \mid t : A}{\Gamma \mid s =_A t \text{ pred}}$$

4. Conjunction:

$$\frac{\Gamma \mid \varphi \text{ pred} \quad \Gamma \mid \psi \text{ pred}}{\Gamma \mid \varphi \wedge \psi \text{ pred}}$$

5. Weakening:

$$\frac{\Gamma \mid \varphi \text{ pred}}{\Gamma, x : A \mid \varphi \text{ pred}}$$

Observe that, as usual, there is then a derived operation of substitution of terms for variables into formulas, defined by structural recursion on the above specification of formulas:

Substitution:

$$\frac{\Gamma \mid t : A \quad \Gamma, x : A \mid \varphi \text{ pred}}{\Gamma \mid \varphi[t/x] \text{ pred}}$$

### Inference rules for cartesian logic

Although we do not yet need them, we state the rules of inference here, too, for the convenience of having the entire specification of cartesian logic in one place. As already mentioned, we can conveniently state this deductive calculus using only *binary* sequents,

$$\Gamma \mid \psi \vdash \varphi.$$

We omit mention of the context  $\Gamma$  when it is the same in the premisses and conclusion of a rule.

1. Weakening:

$$\frac{\Gamma \mid \psi \vdash \varphi}{\Gamma, x : A \mid \psi \vdash \varphi}$$

2. Substitution:

$$\frac{\Gamma \mid t : A \quad \Gamma, x : A \mid \psi \vdash \varphi}{\Gamma \mid \psi[t/x] \vdash \varphi[t/x]}$$

3. Identity:

$$\overline{\varphi \vdash \varphi}$$

4. Cut:

$$\frac{\psi \vdash \theta \quad \theta \vdash \varphi}{\psi \vdash \varphi}$$

5. Equality:

$$\frac{}{\psi \vdash t =_A t} \quad \frac{\psi \vdash t =_A u \quad \psi \vdash \varphi[t/z]}{\psi \vdash \varphi[u/z]}$$

6. Truth:

$$\overline{\psi \vdash \top}$$



7. Conjunction:

$$\frac{\vartheta \vdash \varphi \quad \vartheta \vdash \psi}{\vartheta \vdash \varphi \wedge \psi} \quad \frac{\vartheta \vdash \varphi \wedge \psi}{\vartheta \vdash \varphi} \quad \frac{\vartheta \vdash \varphi \wedge \psi}{\vartheta \vdash \psi}$$

**Exercise 3.1.6.** Derive symmetry and transitivity of equality:

$$\frac{\Gamma \mid \psi \vdash t =_A u}{\Gamma \mid \psi \vdash u =_A t} \quad \frac{\Gamma \mid \psi \vdash t =_A u \quad \Gamma \mid \psi \vdash u =_A v}{\Gamma \mid \psi \vdash t =_A v}$$

**Example 3.1.7.** The theory of a poset is a cartesian theory. There is one basic sort  $\mathbf{P}$  and one binary relation symbol  $\leq$  with signature  $(\mathbf{P}, \mathbf{P})$ . The axioms are the familiar axioms for reflexivity, transitivity, and antisymmetry:

$$\begin{aligned} x : \mathbf{P} \mid \cdot \vdash x &\leq x \\ x : \mathbf{P}, y : \mathbf{P}, z : \mathbf{P} \mid x &\leq y \wedge y \leq z \vdash x \leq z \\ x : \mathbf{P}, y : \mathbf{P} \mid x &\leq y \wedge y \leq x \vdash x =_{\mathbf{P}} y \end{aligned}$$

There are also many examples, such as ordered groups, ordered fields, etc., that extend the theory of posets with further algebraic operations and equations.

**Example 3.1.8.** An *equivalence relation* in a cartesian category is a model of the corresponding theory with one basic sort  $\mathbf{A}$  and one binary relation symbol  $\sim$  with signature  $(\mathbf{A}, \mathbf{A})$ . The axioms are the familiar axioms for reflexivity, symmetry, and transitivity:

$$\begin{aligned} x : \mathbf{A} \mid \cdot \vdash x &\sim x \\ x : \mathbf{A}, y : \mathbf{A} \mid x &\sim y \vdash y \sim x \\ x : \mathbf{A}, y : \mathbf{A}, z : \mathbf{A} \mid x &\sim y \wedge y \sim z \vdash x \sim z \end{aligned}$$

## Semantics of cartesian logic

In order to give the semantics of cartesian logic, we require a couple of useful propositions regarding cartesian categories.

**Proposition 3.1.9.** *If a category  $\mathcal{C}$  has pullbacks then, for every  $A \in \mathcal{C}$ , the poset  $\mathbf{Sub}(A)$  has finite limits. Moreover, these are stable under pullback.*

*Proof.* The poset  $\mathbf{Sub}(A)$  has finite limits if it has a top object and binary meets. The top object of  $\mathbf{Sub}(A)$  is the subobject  $[1_A : A \rightarrow A]$ . The meet of subobjects  $i : I \rightarrowtail A$  and  $j : J \rightarrowtail A$  is the subobject  $i \wedge j = i \circ (i^*j) = j \circ (j^*i) : I \wedge J \rightarrowtail A$  obtained by pullback, as in the following diagram:

$$\begin{array}{ccc} I \wedge J & \xrightarrow{j^*i} & J \\ \downarrow i^*j & \lrcorner & \downarrow j \\ I & \xrightarrow{i} & A \end{array}$$

It is easy to verify that  $I \wedge J$  is the infimum of  $I$  and  $J$ . Finally, stability follows from a familiar diagram chase on a cube of pullbacks.  $\square$

**Proposition 3.1.10.** *A category has all finite limits just if it has all finite products and pullbacks of monos along monos.*

*Proof.* It is sufficient to show that the category has equalizers. To construct the equalizer of parallel arrows  $f : A \rightarrow B$  and  $g : A \rightarrow B$ , first observe that the arrows

$$A \xrightarrow{\langle 1_A, f \rangle} A \times B \qquad A \xrightarrow{\langle 1_A, g \rangle} A \times B$$

are monos because the projection  $\pi_0 : A \times B \rightarrow A$  is their left inverse. Therefore, we may construct the pullback

$$\begin{array}{ccc} P & \xrightarrow{p} & A \\ \downarrow q & \lrcorner & \downarrow \langle 1_A, f \rangle \\ A & \xrightarrow{\langle 1_A, g \rangle} & A \times B \end{array}$$

The morphisms  $p$  and  $q$  coincide because  $\langle 1_A, f \rangle$  and  $\langle 1_A, g \rangle$  have a common left inverse  $\pi_0$ :

$$p = 1_A \circ p = \pi_0 \circ \langle 1_A, f \rangle \circ p = \pi_0 \circ \langle 1_A, f \rangle \circ q = 1_A \circ q = q .$$

Let us show that  $p : P \rightarrow A$  is the equalizer of  $f$  and  $g$ . First,  $p$  equalizes  $f$  and  $g$ ,

$$f \circ p = \pi_1 \circ \langle 1_A, f \rangle \circ p = \pi_1 \circ \langle 1_A, g \rangle \circ q = g \circ q = g \circ p .$$

If  $k : K \rightarrow A$  also equalizes  $f$  and  $g$  then

$$\langle 1_A, f \rangle \circ k = \langle k, f \circ k \rangle = \langle k, g \circ k \rangle = \langle 1_A, g \rangle \circ k ,$$

therefore by the universal property of the constructed pullback there exists a unique factorization  $\bar{k} : K \rightarrow P$  such that  $k = p \circ \bar{k}$ , as required.  $\square$

We now explain how cartesian logic is interpreted in a cartesian category  $\mathcal{C}$  (i.e.  $\mathcal{C}$  is finitely complete). Let  $\mathbb{T}$  be a multi-sorted cartesian theory. Recall that the type theory of  $\mathbb{T}$  is specified by a set of sorts (types)  $A, \dots$  and a set of basic function symbols  $f, \dots$  together with their signatures, while the logic is given by a set of basic relation symbols  $R, \dots$  with their signatures, and a set of axioms in the form of logical entailments between formulas in context,

$$\Gamma \mid \psi \vdash \varphi .$$

**Definition 3.1.11.** An *interpretation* of  $\mathbb{T}$  in  $\mathcal{C}$  is given by the following data, where  $\Gamma$  stands for a typing context  $x_1 : A_1, \dots, x_n : A_n$ , and  $\psi$  and  $\varphi$  are formulas:

1. Each sort  $A$  is interpreted as an object  $\llbracket A \rrbracket$ , with the unit sort  $1$  being interpreted as the terminal object  $1$ .

2. A typing context  $x_1 : A_1, \dots, x_n : A_n$  is interpreted as the product  $\llbracket A_1 \rrbracket \times \dots \times \llbracket A_n \rrbracket$ . The empty context is interpreted as the terminal object  $1$ .
3. A basic function symbol  $f$  with signature  $(A_1, \dots, A_m; B)$  is interpreted as a morphism  $\llbracket f \rrbracket : \llbracket A_1 \rrbracket \times \dots \times \llbracket A_m \rrbracket \rightarrow \llbracket B \rrbracket$ .
4. A basic relation symbol  $R$  with signature  $(A_1, \dots, A_n)$  is interpreted as a subobject  $\llbracket R \rrbracket \in \mathbf{Sub}(\llbracket A_1 \rrbracket \times \dots \times \llbracket A_n \rrbracket)$ .

We then extend the interpretation to all terms and formulas as follows:

1. A term in context  $\Gamma \mid t : B$  is interpreted as a morphism

$$\llbracket \Gamma \mid t : B \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket B \rrbracket$$

according to the following specification.

- A variable  $x_0 : A_1, \dots, x_n : A_n \mid x_i : A_i$  is interpreted as the  $i$ -th projection  $\pi_i : \llbracket A_1 \rrbracket \times \dots \times \llbracket A_n \rrbracket \rightarrow \llbracket A_i \rrbracket$ .
- The interpretation of  $\Gamma \mid * : 1$  is the unique morphism  $!_{\llbracket \Gamma \rrbracket} : \llbracket \Gamma \rrbracket \rightarrow 1$ .
- A composite term  $\Gamma \mid f(t_1, \dots, t_m) : B$ , where  $f$  is a basic function symbol with signature  $(A_1, \dots, A_m; B)$ , is interpreted as the composition

$$\llbracket \Gamma \rrbracket \xrightarrow{\langle \llbracket t_1 \rrbracket, \dots, \llbracket t_m \rrbracket \rangle} \llbracket A_1 \rrbracket \times \dots \times \llbracket A_m \rrbracket \xrightarrow{\llbracket f \rrbracket} \llbracket B \rrbracket$$

Here  $\llbracket t_i \rrbracket$  is shorthand for  $\llbracket \Gamma \mid t_i : A_i \rrbracket$ .

2. A formula in a context  $\Gamma \mid \varphi$  is interpreted as a subobject  $\llbracket \Gamma \mid \varphi \rrbracket \in \mathbf{Sub}(\llbracket \Gamma \rrbracket)$  according to the following specification.

- The logical constant  $\top$  is interpreted as the maximal subobject, represented by the identity arrow:

$$\llbracket \Gamma \mid \top \rrbracket = [1_{\llbracket \Gamma \rrbracket} : \llbracket \Gamma \rrbracket \rightarrow \llbracket \Gamma \rrbracket]$$

- An atomic formula  $\Gamma \mid R(t_1, \dots, t_m)$ , where  $R$  is a basic relation symbol with signature  $(A_1, \dots, A_m)$  is interpreted as the left vertical arrow in the following pullback square:

$$\begin{array}{ccc} \llbracket \Gamma \mid R(t_1, \dots, t_m) \rrbracket & \xrightarrow{\quad} & \llbracket R \rrbracket \\ \downarrow & & \downarrow \\ \llbracket \Gamma \rrbracket & \xrightarrow{\langle \llbracket t_1 \rrbracket, \dots, \llbracket t_m \rrbracket \rangle} & \llbracket A_1 \rrbracket \times \dots \times \llbracket A_m \rrbracket \end{array}$$

- An equation  $\Gamma \mid t =_A u$  **pred** is interpreted as the subobject represented by the equalizer of  $\llbracket \Gamma \mid t : A \rrbracket$  and  $\llbracket \Gamma \mid u : A \rrbracket$ :

$$\llbracket \Gamma \mid t =_A u \rrbracket \rightrightarrows \llbracket \Gamma \rrbracket \begin{array}{c} \xrightarrow{\llbracket t \rrbracket} \\ \xleftarrow{\llbracket u \rrbracket} \end{array} \llbracket A \rrbracket$$

- By Proposition 3.1.9, each  $\mathbf{Sub}(A)$  is a poset with binary meets. Thus we interpret a conjunction  $\Gamma \mid \varphi \wedge \psi$  **pred** as the meet of subobjects

$$\llbracket \Gamma \mid \varphi \wedge \psi \rrbracket = \llbracket \Gamma \mid \varphi \rrbracket \wedge \llbracket \Gamma \mid \psi \rrbracket .$$

- A formula formed by weakening is interpreted as pullback along a projection:

$$\begin{array}{ccc} \llbracket \Gamma, x : A \mid \varphi \rrbracket & \longrightarrow & \llbracket \Gamma \mid \varphi \rrbracket \\ \downarrow \lrcorner & & \downarrow i \\ \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket & \xrightarrow{\pi} & \llbracket \Gamma \rrbracket \end{array}$$

Computing this pullback one sees that the interpretation of  $\llbracket \Gamma, x : A \mid \varphi \rrbracket$  turns out to be the subobject

$$\llbracket \Gamma \mid \varphi \rrbracket \times \llbracket A \rrbracket \xrightarrow{i \times 1_A} \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket$$

This concludes the definition of an interpretation of a cartesian theory  $\mathbb{T}$  in a cartesian category  $\mathcal{C}$ .

As was explained in the previous section, the operation of substitution of terms into formulas is interpreted as pullback:

**Lemma 3.1.12.** *Let the formula  $\Gamma, x : A \mid \varphi$  and the term  $\Gamma \mid t : A$  be given. Then the substituted formula  $\Gamma \mid \varphi[t/x]$  is interpreted as the pullback indicated in the following diagram:*

$$\begin{array}{ccc} \llbracket \Gamma \mid \varphi[t/x] \rrbracket & \longrightarrow & \llbracket \Gamma, x : A \mid \varphi \rrbracket \\ \downarrow \lrcorner & & \downarrow \\ \llbracket \Gamma \rrbracket & \xrightarrow{\langle 1_{\llbracket \Gamma \rrbracket}, \llbracket t \rrbracket \rangle} & \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket \end{array}$$

*Proof.* A simple induction on the structure of  $\varphi$ . We do the case where  $\varphi$  is an atomic formula  $R(t_1, \dots, t_m)$ . Let  $\Gamma = x_1 : A_1, \dots, x_n : A_n$  and  $\Gamma, x : A \mid t_i : B_i$  for  $i = 1, \dots, m$ ,

where  $(B_1, \dots, B_m)$  is the signature of  $R$ . For the interpretation of  $\Gamma, x : A \mid R(t_1, \dots, t_m)$ , by Definition 3.1.11 we have a pullback diagram:

$$\begin{array}{ccc} \llbracket \Gamma \mid R(t_1, \dots, t_m) \rrbracket & \xrightarrow{\quad} & \llbracket R \rrbracket \\ \downarrow & & \downarrow \\ \llbracket \Gamma, x : A \rrbracket & \xrightarrow{\langle \llbracket t_1 \rrbracket, \dots, \llbracket t_m \rrbracket \rangle} & \llbracket B_1 \rrbracket \times \dots \times \llbracket B_m \rrbracket \end{array}$$

Now suppose  $\Gamma \mid t : A$ , and consider the substitution

$$\Gamma \mid R(t_1, \dots, t_m)[t/x] = \Gamma \mid R(t_1[t/x], \dots, t_m[t/x])$$

For the interpretations of the substituted terms  $t_i[t/x]$  we have the composites

$$\llbracket t_i[t/x] \rrbracket = \llbracket t_i \rrbracket \circ \langle \mathbf{1}_{\llbracket \Gamma \rrbracket}, \llbracket t \rrbracket \rangle : \llbracket \Gamma \rrbracket \longrightarrow \llbracket \Gamma, x : A \rrbracket \longrightarrow \llbracket B_i \rrbracket$$

by (associativity of composition and) the definition of the interpretation of terms. Thus for the interpretation of  $\Gamma \mid R(t_1, \dots, t_m)[t/x]$  we have the outer pullback rectangle below.

$$\begin{array}{ccccc} & & \xrightarrow{\quad} & & \\ & \searrow & & \searrow & \\ \llbracket \Gamma \mid R(t_1, \dots, t_m)[t/x] \rrbracket & \xrightarrow{\quad} & \llbracket \Gamma, x : A \mid R(t_1, \dots, t_m) \rrbracket & \xrightarrow{\quad} & \llbracket R \rrbracket \\ & \searrow & \downarrow & \searrow & \downarrow \\ \llbracket \Gamma \rrbracket & \xrightarrow{\langle \mathbf{1}_{\llbracket \Gamma \rrbracket}, \llbracket t \rrbracket \rangle} & \llbracket \Gamma, x : A \rrbracket & \xrightarrow{\langle \llbracket t_1 \rrbracket, \dots, \llbracket t_m \rrbracket \rangle} & \llbracket B_1 \rrbracket \times \dots \times \llbracket B_m \rrbracket \\ & \searrow & \downarrow & \searrow & \downarrow \\ & & \langle \llbracket t_1[t/x] \rrbracket, \dots, \llbracket t_m[t/x] \rrbracket \rangle & & \end{array}$$

But since the righthand square is a pullback, there is a unique dotted arrow as indicated. By the 2-pullbacks lemma, the lefthand square is then also a pullback, as required.  $\square$

**Exercise 3.1.13.** Complete the proof.

When we deal with several different interpretations at once we may name them  $M, N, \dots$ , and superscript the semantic brackets accordingly,  $\llbracket \Gamma \rrbracket^M, \llbracket \Gamma \rrbracket^N, \dots$

**Definition 3.1.14.** If  $\Gamma \mid \psi \vdash \varphi$  is one of the logical entailment axioms of  $\mathbb{T}$  and

$$\llbracket \Gamma \mid \psi \rrbracket^M \leq \llbracket \Gamma \mid \varphi \rrbracket^M$$

holds in an interpretation  $M$ , then we say that  $M$  *satisfies* or *models*  $\Gamma \mid \psi \vdash \varphi$ , which we may write as

$$M \models (\Gamma \mid \psi \vdash \varphi).$$

An interpretation  $M$  is a *model* of  $\mathbb{T}$  if it satisfies all the axioms of  $\mathbb{T}$ .

**Theorem 3.1.15** (Soundness of cartesian logic). *If a cartesian theory  $\mathbb{T}$  proves an entailment*

$$\Gamma \mid \psi \vdash \varphi$$

*then every model  $M$  of  $\mathbb{T}$  satisfies the entailment:*

$$M \models (\Gamma \mid \psi \vdash \varphi) .$$

*Proof.* The proof proceeds by induction on the proof of the entailment. In the following we often omit the typing context  $\Gamma$  to simplify the notation, and all inequalities are interpreted in  $\mathbf{Sub}(\llbracket \Gamma \rrbracket)$ . We consider all possible last steps in the proof of the entailment:

1. Weakening: if  $\llbracket \Gamma \mid \psi \rrbracket \leq \llbracket \Gamma \mid \varphi \rrbracket$  in  $\mathbf{Sub}(\llbracket \Gamma \rrbracket)$  then

$$\llbracket \Gamma, x : A \mid \psi \rrbracket = \llbracket \Gamma \mid \psi \rrbracket \times A \leq \llbracket \Gamma \mid \varphi \rrbracket \times A = \llbracket \Gamma, x : A \mid \varphi \rrbracket \quad \text{in } \mathbf{Sub}(\llbracket \Gamma, x : A \rrbracket).$$

2. Substitution: by lemma 3.1.12, substitution is interpreted by pullback so that  $\llbracket \varphi[t/x] \rrbracket = \langle \mathbf{1}_{\llbracket \psi \rrbracket}, \llbracket t \rrbracket \rangle^* \llbracket \varphi \rrbracket$  and  $\llbracket \psi[t/x] \rrbracket = \langle \mathbf{1}_{\llbracket \psi \rrbracket}, \llbracket t \rrbracket \rangle^* \llbracket \psi \rrbracket$ . Because

$$\langle \mathbf{1}_{\llbracket \psi \rrbracket}, \llbracket t \rrbracket \rangle^* : \mathbf{Sub}(\llbracket \psi \rrbracket) \rightarrow \mathbf{Sub}(\llbracket \psi \rrbracket \times \llbracket A \rrbracket)$$

is a functor it is a monotone map, therefore  $\llbracket \psi \rrbracket \leq \llbracket \varphi \rrbracket$  implies

$$\langle \mathbf{1}_{\llbracket \psi \rrbracket}, \llbracket t \rrbracket \rangle^* \llbracket \psi \rrbracket \leq \langle \mathbf{1}_{\llbracket \psi \rrbracket}, \llbracket t \rrbracket \rangle^* \llbracket \varphi \rrbracket .$$

3. Identity: trivially

$$\llbracket \varphi \rrbracket \leq \llbracket \varphi \rrbracket .$$

4. Cut: if  $\llbracket \psi \rrbracket \leq \llbracket \theta \rrbracket$  and  $\llbracket \theta \rrbracket \leq \llbracket \varphi \rrbracket$  then clearly  $\llbracket \psi \rrbracket \leq \llbracket \varphi \rrbracket$ , since  $\mathbf{Sub}(\llbracket \Gamma, x : A \rrbracket)$  is a poset.

5. Truth: trivially  $\llbracket \psi \rrbracket \leq \llbracket \top \rrbracket$ .

6. The rules for conjunction clearly hold because by the definition of infimum  $\llbracket \vartheta \rrbracket \leq \llbracket \varphi \wedge \psi \rrbracket$  if, and only if,  $\llbracket \vartheta \rrbracket \leq \llbracket \varphi \rrbracket$  and  $\llbracket \vartheta \rrbracket \leq \llbracket \psi \rrbracket$ .

7. Equality: the axiom  $t =_A t$  is satisfied because an equalizer of  $\llbracket t \rrbracket$  with itself is the maximal subobject:

$$\llbracket \psi \rrbracket \leq [\mathbf{1}_{\llbracket \Gamma \rrbracket} : \llbracket \Gamma \rrbracket \rightarrow \llbracket \Gamma \rrbracket] = \llbracket t =_A t \rrbracket .$$

For the other axiom, suppose  $\llbracket \psi \rrbracket \leq \llbracket t =_A u \rrbracket$  and  $\llbracket \psi \rrbracket \leq \llbracket \varphi[t/z] \rrbracket$ . It suffices to show  $\llbracket t =_A u \rrbracket \wedge \llbracket \varphi[t/z] \rrbracket \leq \llbracket \varphi[u/z] \rrbracket$  for then

$$\llbracket \psi \rrbracket \leq \llbracket t =_A u \rrbracket \wedge \llbracket \varphi[t/z] \rrbracket \leq \llbracket \varphi[u/z] \rrbracket .$$

The interpretation of  $P = \llbracket t =_A u \rrbracket \wedge \llbracket \varphi[t/z] \rrbracket$  is obtained by two successive pullbacks, as in the following diagram:

$$\begin{array}{ccccc}
 P & \xrightarrow{\quad} & \llbracket \varphi[t/z] \rrbracket & \xrightarrow{\quad} & \llbracket \varphi \rrbracket \\
 \downarrow \lrcorner & & \downarrow \lrcorner & & \downarrow \\
 \llbracket t =_A u \rrbracket & \xrightarrow{\quad e \quad} & \llbracket \Gamma \rrbracket & \xrightarrow{\langle 1_\Gamma, \llbracket t \rrbracket \rangle} & \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket
 \end{array}$$

Here  $e$  is the equalizer of  $\llbracket u \rrbracket$  and  $\llbracket t \rrbracket$ . Observe that  $e$  equalizes  $\langle 1_\Gamma, \llbracket t \rrbracket \rangle$  and  $\langle 1_\Gamma, \llbracket u \rrbracket \rangle$  as well:

$$\langle 1_\Gamma, \llbracket t \rrbracket \rangle \circ e = \langle e, \llbracket t \rrbracket \circ e \rangle = \langle e, \llbracket u \rrbracket \circ e \rangle = \langle 1_\Gamma, \llbracket u \rrbracket \rangle \circ e.$$

Therefore, if we replace  $\langle 1_\Gamma, \llbracket t \rrbracket \rangle$  with  $\langle 1_\Gamma, \llbracket u \rrbracket \rangle$  in the above diagram, the outer rectangle still commutes. By the universal property of the pullback

$$\begin{array}{ccc}
 \llbracket \varphi[u/z] \rrbracket & \xrightarrow{\quad} & \llbracket \varphi \rrbracket \\
 \downarrow \lrcorner & & \downarrow \\
 \llbracket \Gamma \rrbracket & \xrightarrow{\langle 1_\Gamma, \llbracket u \rrbracket \rangle} & \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket
 \end{array}$$

it follows that  $P$  also factors through  $\llbracket \varphi[u/z] \rrbracket$ , as required.  $\square$

**Example 3.1.16.** Recall the cartesian theory of posets (example 3.1.7). There is one basic sort  $\mathbf{P}$  and one binary relation symbol  $\leq$  with signature  $(\mathbf{P}, \mathbf{P})$  and the axioms of reflexivity, transitivity, and antisymmetry. A poset in a cartesian category  $\mathcal{C}$  is thus given by an object  $P$ , which is the interpretation of the sort  $\mathbf{P}$ , and a subobject  $r : R \hookrightarrow P \times P$ , which is the interpretation of  $\leq$ , such that the axioms are satisfied. As an example we spell out when the reflexivity axiom is satisfied. The interpretation of  $x : P \mid x \leq x$  is obtained by the following pullback:

$$\begin{array}{ccc}
 \llbracket x \leq x \rrbracket & \xrightarrow{\quad} & R \\
 \downarrow \lrcorner & \nearrow \rho & \downarrow r \\
 P & \xrightarrow{\Delta} & P \times P
 \end{array}$$

where  $\Delta = \langle 1_P, 1_P \rangle$  is the diagonal. The first axiom is satisfied when  $\llbracket x \leq x \rrbracket = 1_P$ , which happens if, and only if,  $\Delta$  factors through  $r$ , as indicated. Therefore, reflexivity can be expressed as follows: there exists a “reflexivity” morphism  $\rho : P \rightarrow R$  such that  $r \circ \rho = \Delta$ . Equivalently, the morphisms  $\pi_0 \circ r$  and  $\pi_1 \circ r$  have a common right inverse  $\rho$ .

As an example, of a poset in a cartesian category other than **Set**, observe that since the definition is stated entirely in terms of finite limits, and these are computed pointwise

in functor categories  $\mathbf{Set}^{\mathbb{C}}$ , it follows that a poset  $P$  in  $\mathbf{Set}^{\mathbb{C}}$  is the same thing as a functor  $P : \mathbb{C} \rightarrow \mathbf{Poset}$ . Indeed, as was the case for algebraic theories, we have an equivalence (an isomorphism, actually) of categories,

$$\mathbf{Poset}(\mathbf{Set}^{\mathbb{C}}) \cong \mathbf{Poset}(\mathbf{Set})^{\mathbb{C}} \cong \mathbf{Poset}^{\mathbb{C}}.$$

**Exercise 3.1.17.** An *ordered group* is a group  $(G, \cdot, i, e)$  equipped with a partial ordering  $x \leq y$  that is compatible with the group multiplication, in the sense that  $x \leq y$  implies  $x \cdot z \leq y \cdot z$  and  $z \cdot x \leq z \cdot y$ . Is this the same thing as a group in the category of posets? A poset in the category of groups?

### Subtypes

Let us consider whether the theory of a category is a cartesian theory. We begin by expressing the definition of a category so that it can be interpreted in any cartesian category  $\mathcal{C}$ . An *internal category* in  $\mathcal{C}$  consists of an *object of morphisms*  $C_1$ , an *object of objects*  $C_0$ , and *domain*, *codomain*, and *identity* morphisms,

$$\text{dom} : C_1 \rightarrow C_0, \quad \text{cod} : C_1 \rightarrow C_0, \quad \text{id} : C_0 \rightarrow C_1.$$

There is also a *composition* morphism  $c : C_2 \rightarrow C_1$ , where  $C_2$  is obtained by the pullback

$$\begin{array}{ccc} C_2 & \xrightarrow{p_1} & C_1 \\ p_0 \downarrow & \lrcorner & \downarrow \text{dom} \\ C_1 & \xrightarrow{\text{cod}} & C_0 \end{array}$$

The following equations must hold:

$$\begin{aligned} \text{dom} \circ i &= 1_{C_0} = \text{cod} \circ i, \\ \text{cod} \circ p_1 &= \text{cod} \circ c, \quad \text{dom} \circ p_0 = \text{dom} \circ c, \\ c \circ \langle 1_{C_1}, i \circ \text{dom} \rangle &= 1_{C_1} = c \circ \langle i \circ \text{cod}, 1_{C_1} \rangle, \end{aligned}$$

The first two equations state that the domain and codomain of an identity morphism  $1_A$  are both  $A$ . The second equation states that  $\text{cod}(f \circ g) = \text{cod } f$  and the third one that  $\text{dom}(f \circ g) = \text{dom } g$ . The fourth equation states that  $f \circ 1_{\text{dom } f} = f = 1_{\text{cod } f} \circ f$ . It remains to express associativity of composition. For this purpose we construct the pullback

$$\begin{array}{ccc} C_3 & \xrightarrow{q_2} & C_1 \\ q_{01} \downarrow & \lrcorner & \downarrow \text{dom} \\ C_2 & \xrightarrow{\text{cod} \circ p_1} & C_0 \end{array}$$



The object  $C_3$  can be thought of as the set of triples of morphisms  $(f, g, h)$  such that  $\text{cod } f = \text{dom } g$  and  $\text{cod } g = \text{dom } h$ . We denote  $q_0 = p_0 \circ q_{01}$  and  $q_1 = p_1 \circ q_{01}$ . The morphisms  $q_0, q_1, q_2 : C_3 \rightarrow C_1$  are like three projections which select the first, second, and third element of a triple, respectively. With this notation we can write  $q_{01} = \langle q_0, q_1 \rangle_{C_2}$  because  $q_{01}$  is the unique morphism such that  $p_0 \circ q_{01} = q_0$  and  $p_1 \circ q_{01} = q_1$ . The subscript  $C_2$  reminds us that the “pair”  $\langle q_0, q_1 \rangle_{C_2}$  is obtained by the universal property of the pullback  $C_2$ .

Morphisms  $c \circ q_{01} : C_3 \rightarrow C_1$  and  $q_2 : C_3 \rightarrow C_1$  factor through the pullback  $C_2$  because

$$\text{cod} \circ c \circ q_{01} = \text{cod} \circ p_1 \circ q_0 = \text{dom} \circ q_2 .$$

Thus let  $r : C_3 \rightarrow C_2$  be the unique factorization for which  $p_0 \circ r = c \circ q_{01}$  and  $p_1 \circ r = q_2$ . Because  $p_0$  and  $p_1$  are like projections from  $C_2$  to  $C_1$ , morphism  $r$  can be thought of as a pair of morphisms, so we write  $r = \langle c \circ q_{01}, q_2 \rangle_{C_2}$ . Morphism  $c \circ \langle c \circ q_{01}, q_2 \rangle_{C_2} : C_3 \rightarrow C_1$  corresponds to the operations  $\langle f, g, h \rangle \mapsto (f, g) \circ h$ , whereas the morphism corresponding to  $\langle f, g, h \rangle \mapsto f \circ (g \circ h)$  is obtained in a similar way and is equal to

$$c \circ \langle q_0, c \circ \langle q_1, q_2 \rangle_{C_2} \rangle_{C_2} : C_3 \rightarrow C_1 .$$

Thus associativity is expressed by the equation

$$c \circ \langle c \circ \langle q_0, q_1 \rangle_{C_2}, q_2 \rangle_{C_2} = c \circ \langle q_0, c \circ \langle q_1, q_2 \rangle_{C_2} \rangle_{C_2} .$$

**Example 3.1.18.** An internal category in **Set** is a small category.

**Example 3.1.19.** An internal category in  $\mathbf{Set}^{\mathbb{C}}$  is a functor  $\mathbb{C} \rightarrow \mathbf{Cat}$ . Indeed, as in previous examples of cartesian theories we have an equivalence of categories,

$$\mathbf{Cat}(\mathbf{Set}^{\mathbb{C}}) \cong \mathbf{Cat}(\mathbf{Set})^{\mathbb{C}} \cong \mathbf{Cat}^{\mathbb{C}} .$$

We have successfully formulated the theory of a category so that it makes sense in any cartesian category. In fact, the definition of an internal category refers only to certain pullbacks, hence the notion of an internal category makes sense in any category with pullbacks. However, if we try to formulate it as a multi-sorted cartesian theory, there is a problem. Obviously, there ought to be a basic sort of objects  $\mathbf{C}_0$  and a basic sort of morphisms  $\mathbf{C}_1$ . There are also basic function symbols with signatures

$$\text{dom} : (\mathbf{C}_1; \mathbf{C}_0) \qquad \text{cod} : (\mathbf{C}_1; \mathbf{C}_0) \qquad \text{id} : (\mathbf{C}_0, \mathbf{C}_1) .$$

However, it is not clear what the signature for composition should be. It is not  $(\mathbf{C}_1, \mathbf{C}_1; \mathbf{C}_1)$  because composition is undefined for non-composable pairs of morphisms. We might be tempted to postulate another basic sort  $\mathbf{C}_2$  but then we would have no way of stating that  $\mathbf{C}_2$  is the pullback of  $\text{dom}$  and  $\text{cod}$ . And even if we somehow axiomatized the fact that  $\mathbf{C}_2$  is a pullback, we would then still have to formalize the object  $\mathbf{C}_3$  of composable triples,  $\mathbf{C}_4$

of composable quadruples, and so on. What we lack is the ability to define the type  $\mathbf{C}_2$  as a *subtype* of  $\mathbf{C}_1 \times \mathbf{C}_1$ .

One way to remedy the situation is to use a richer underlying type theory; in Chapter ?? we will consider the system of *dependent type theory*, which provides the means to capture such notions as the theory of categories (and much more). Here we consider a small step in that direction, namely *simple subtypes*. The formation rule for simple subtypes is

$$\frac{x : A \mid \varphi \text{ pred}}{\{x : A \mid \varphi\} \text{ type}}$$

We can think of  $\{x : A \mid \varphi\}$  as the subobject of all those  $x : A$  that satisfy  $\varphi$ . Note that we did not allow an arbitrary context  $\Gamma$  to be present. This means that we cannot define subtypes that depend on parameters, which is why they are called “simple”.

Inference rules for subtypes are as follows:

$$\frac{\Gamma \mid t : \{x : A \mid \varphi\}}{\Gamma \mid \text{in}_\varphi t : A} \quad \frac{\Gamma \mid t : \{x : A \mid \varphi\}}{\Gamma \mid \cdot \vdash \varphi[\text{in}_\varphi t/x]} \quad \frac{\Gamma \mid t : A \quad \Gamma \mid \cdot \vdash \varphi[t/x]}{\Gamma \mid \text{rs}_\varphi t : \{x : A \mid \varphi\}}$$

$$\frac{\Gamma, x : A \mid \varphi, \psi \vdash \theta}{\Gamma, y : \{x : A \mid \varphi\} \mid \psi[\text{in}_\varphi y/x] \vdash \theta[\text{in}_\varphi y/x]}$$

The first rule states that a term  $t$  of subtype  $\{x : A \mid \varphi\}$  can be converted to a term  $\text{in}_\varphi t$  of type  $A$ . We can think of the constant  $\text{in}_\varphi$  as the *inclusion*  $\text{in}_\varphi : \{x : A \mid \varphi\} \rightarrow A$ . The second rule states that every term of a subtype  $\{x : A \mid \varphi\}$  satisfies the defining predicate  $\varphi$ . The third rule states that a term  $t$  of type  $A$  which satisfies  $\varphi$  can be converted to a term  $\text{rs}_\varphi t$  of type  $\{x : A \mid \varphi\}$ . A good way to think of the constant  $\text{rs}_\varphi$  is as a partially defined *restriction*, or a type-casting operations,  $\text{rs}_\varphi : A \rightarrow \{x : A \mid \varphi\}$ .<sup>5</sup> The last rule tells us how to replace a variable  $x$  of type  $A$  and an assumption  $\varphi$  about it with a variable  $y$  of type  $\{x : A \mid \varphi\}$  and remove the assumption. Note that this is a two-way rule.

There are two more axioms that relate inclusions and restrictions:

$$\frac{\Gamma \mid t : \{x : A \mid \varphi\}}{\Gamma \mid \cdot \vdash \text{rs}_\varphi(\text{in}_\varphi t) = t} \quad \frac{\Gamma \mid t : A \quad \Gamma \mid \cdot \vdash \varphi[t/x]}{\Gamma \mid \cdot \vdash \text{in}_\varphi(\text{rs}_\varphi t) = t}.$$

In an informal discussion it is customary for the inclusions and restrictions to be omitted, or at least for the subscript  $\varphi$  to be missing.<sup>6</sup>

**Exercise 3.1.20.** Suppose  $x : A \mid \psi$  and  $x : A \mid \varphi$  are formulas. Show that

$$x : A \mid \psi \vdash \varphi$$

<sup>5</sup>Inclusions and restrictions are like type-casting operations in some programming languages. For example in Java, an inclusion corresponds to an (implicit) type cast from a class to its superclass, whereas a restriction corresponds to a type cast from a class to a subclass. Must I write that Java is a registered trademark of Sun Microsystems?

<sup>6</sup>Strictly speaking, even the notation  $\text{in}_\varphi t$  is imprecise because it does not indicate that  $\phi$  stands in the context  $x : A$ . The correct notation would be  $\text{in}_{(x:A \mid \varphi)} t$ , where  $x$  is bound in the subscript. A similar remark holds for  $\text{rs}_\varphi t$ .

is provable if, and only if,  $\{x : A \mid \psi\}$  factors through  $\{x : A \mid \varphi\}$ , which means that there exists a term  $k$ ,

$$y : \{x : A \mid \psi\} \mid k : \{x : A \mid \varphi\} ,$$

such that

$$y : \{x : A \mid \psi\} \mid \cdot \vdash \text{in}_\psi y =_A \text{in}_\varphi k$$

is provable. Show also that  $k$  is determined uniquely up to provable equality.

**Example 3.1.21.** We are now able to formulate the theory of a category as a cartesian theory whose underlying type theory has product types and subset types. The basic types are the type of objects  $\mathbf{C}_0$  and the type of morphisms  $\mathbf{C}_1$ . We define the type  $\mathbf{C}_2$  to be

$$\mathbf{C}_2 \equiv \{p : \mathbf{C}_1 \times \mathbf{C}_1 \mid \text{cod}(\text{fst } p) = \text{dom}(\text{snd } p)\} .$$

The basic function symbols and their signatures are:

$$\text{dom} : \mathbf{C}_1 \rightarrow \mathbf{C}_0 , \quad \text{cod} : \mathbf{C}_1 \rightarrow \mathbf{C}_0 , \quad \text{id} : \mathbf{C}_0 \rightarrow \mathbf{C}_1 , \quad \text{c} : \mathbf{C}_2 \rightarrow \mathbf{C}_1 .$$

The axioms are:

$$\begin{aligned} a : \mathbf{C}_0 \mid \cdot \vdash \text{dom}(\text{id}(a)) &= a \\ a : \mathbf{C}_0 \mid \cdot \vdash \text{cod}(\text{id}(a)) &= a \\ f : \mathbf{C}_1, g : \mathbf{C}_1 \mid \text{cod}(f) = \text{dom}(g) \vdash \text{dom}(\text{c}(\text{rs } \langle f, g \rangle)) &= f \\ f : \mathbf{C}_1, g : \mathbf{C}_1 \mid \text{cod}(f) = \text{dom}(g) \vdash \text{cod}(\text{c}(\text{rs } \langle f, g \rangle)) &= g \\ f : \mathbf{C}_1 \mid \cdot \vdash \text{c}(\text{rs } \langle \text{id}(\text{dom}(f)), f \rangle) &= f \\ f : \mathbf{C}_1 \mid \cdot \vdash \text{c}(\text{rs } \langle f, \text{id}(\text{cod}(f)) \rangle) &= f \end{aligned}$$

Lastly, the associativity axiom is

$$\begin{aligned} f : \mathbf{C}_1, g : \mathbf{C}_1, h : \mathbf{C}_1 \mid \text{cod}(f) = \text{dom}(g), \text{cod}(g) = \text{dom}(h) \vdash \\ \text{c}(\text{rs } \langle \text{c}(\text{rs } \langle f, g \rangle), h \rangle) = \text{c}(\text{rs } \langle f, \text{c}(\text{rs } \langle g, h \rangle) \rangle) . \end{aligned}$$

This notation is quite unreadable. If we write  $g \circ f$  instead of  $\text{c}(\text{rs } \langle f, g \rangle)$  then the axioms take on a more familiar form. For example, associativity is just  $h \circ (g \circ f) = (h \circ g) \circ f$ . However, we need to remember that we may form the term  $g \circ f$  only if we first prove  $\text{dom}(g) = \text{cod}(f)$ .

A subtype  $\{x : A \mid \varphi\}$  is interpreted as the domain of a monomorphism representing  $x : A \mid \varphi$ :

$$\llbracket \{x : A \mid \varphi\} \rrbracket \xrightarrow{\llbracket x : A \mid \varphi \rrbracket} \llbracket A \rrbracket$$

Some care must be taken here because monos representing a given subobject are only determined up to isomorphism. We assume that a suitable canonical choice of monos can be made.

An inclusion  $\Gamma \mid \text{in}_\varphi t : A$  is interpreted as the composition

$$\llbracket \Gamma \rrbracket \xrightarrow{\llbracket t \rrbracket} \llbracket \{x : A \mid \varphi\} \rrbracket \xrightarrow{\llbracket x : A \mid \varphi \rrbracket} \llbracket A \rrbracket$$

A restriction  $\Gamma \mid \text{rs}_\varphi t : \{x : A \mid \varphi\}$  is interpreted as the unique  $\llbracket \overline{t} \rrbracket$  which makes the following diagram commute:

$$\begin{array}{ccc} \llbracket \Gamma \rrbracket & \xrightarrow{\llbracket \overline{t} \rrbracket} & \llbracket x : A \mid \varphi \rrbracket \\ & \searrow \llbracket t \rrbracket & \downarrow \\ & & \llbracket A \rrbracket \end{array}$$

**Exercise 3.1.22.** Formulate and prove a soundness theorem for subtypes. Pay attention to the interpretation of restrictions, where you need to show unique existence of  $\llbracket \overline{t} \rrbracket$ .

**Remark 3.1.23.** Another approach to the logic of cartesian categories that captures the theory of categories and related notions involving partial operations is that of *essentially algebraic theories*, due to P. Freyd; see [Fre72, PV07]. A third approach is that of *dependent type theory* to be developed in ?? below. Finally, we will see in Section ?? that the theory of categories can be formulated as a *regular theory*.

### 3.1.4 Quantifiers as adjoints

The categorical semantics of quantification is one of the central features of the subject, and quite possibly one of the nicest contributions of categorical logic to the field of logic. You might expect that the quantifiers  $\forall$  and  $\exists$  are “just a big conjunction and disjunction”, respectively. In fact the Polish school of algebraic logic worked to realize this point of view—but categorical logic shows how quantifiers can be treated algebraically as adjoint functors, giving a more satisfactory theory that generalizes to categories in which the subobject lattices are not (co)complete. The original treatment can be found in the classic paper [Law69].

Let us first recall the rules of inference for quantifiers. The formation rules are:

$$\frac{\Gamma, x : A \mid \varphi \text{ pred}}{\Gamma \mid (\exists x : A. \varphi) \text{ pred}} \qquad \frac{\Gamma, x : A \mid \varphi \text{ pred}}{\Gamma \mid (\forall x : A. \varphi) \text{ pred}}$$

The variable  $x$  is bound in  $\forall x : A. \varphi$  and  $\exists x : A. \varphi$ . If  $x$  and  $y$  are distinct variables and  $x$  does not occur freely in the term  $t$  then substitution of  $t$  for  $y$  commutes with quantification over  $x$ :

$$\begin{aligned} (\exists x : A. \varphi)[t/y] &= \exists x : A. (\varphi[t/y]) \ , \\ (\forall x : A. \varphi)[t/y] &= \forall x : A. (\varphi[t/y]) \ . \end{aligned} \tag{3.2}$$

For each quantifier we have a two-way rule of inference:

$$\frac{\Gamma, x : A \mid \varphi \vdash \vartheta}{\Gamma \mid (\exists x : A. \varphi) \vdash \vartheta} \qquad \frac{\Gamma, x : A \mid \psi \vdash \varphi}{\Gamma \mid \psi \vdash \forall x : A. \varphi}$$

Note that these rules implicitly impose the usual condition that  $x$  must not occur freely in  $\psi$  and  $\vartheta$ , because  $\psi$  and  $\vartheta$  are supposed to be well formed in context  $\Gamma$ , which does not contain  $x$ .

**Exercise 3.1.24.** A common way of stating the inference rules for quantifiers is as follows. For the universal quantifier, the introduction and elimination rules are

$$\frac{\Gamma, x : A \mid \psi \vdash \varphi}{\Gamma \mid \psi \vdash \forall x : A. \varphi} \qquad \frac{\Gamma \mid t : A \quad \Gamma \mid \psi \vdash \forall x : A. \varphi}{\Gamma \mid \psi \vdash \varphi[t/x]}$$

The introduction rule for existential quantifier is

$$\frac{\Gamma \mid t : A \quad \Gamma \mid \psi \vdash \varphi[t/x]}{\Gamma \mid \psi \vdash \exists x : A. \varphi}$$

and the elimination rule is

$$\frac{\Gamma \mid \psi \vdash \exists x : A. \varphi \quad \Gamma, x : A \mid \varphi \vdash \vartheta}{\Gamma \mid \psi \vdash \vartheta}$$

Note that these rules implicitly impose a requirement that  $x$  does not occur in  $\Gamma$  and that it does not occur freely in  $\psi$  because the context  $\Gamma, x : A$  must be well formed and the hypotheses  $\psi$  must be well formed in context  $\Gamma$ . Show that these rules can be derived from the ones above, and vice versa. Of course, you may also use the inference rules for cartesian logic, cf. page 16.

In order to discover what the semantics of existential quantifier ought to be, we look at the following instance of the two-way rule for quantifiers:

$$\frac{y : B, x : A \mid \varphi \vdash \vartheta}{y : B \mid \exists x : A. \varphi \vdash \vartheta} \tag{3.3}$$

First observe that this rule implicitly requires

$$y : B, x : A \mid \varphi \text{ pred} \qquad y : B \mid \vartheta \text{ pred} \qquad y : B \mid (\exists x : A. \varphi) \text{ pred}$$

This is required for the entailments to be well-formed. The fourth judgement

$$y : B, x : A \mid \vartheta \text{ pred}$$

follows from the second one above by weakening,

$$\frac{y : B \mid \vartheta \text{ pred}}{y : B, x : A \mid \vartheta \text{ pred}}$$

The interpretations of  $\varphi$ ,  $\vartheta$ , and  $\exists x : A . \varphi$  are therefore subobjects

$$\begin{aligned} \llbracket y : B, x : A \mid \varphi \rrbracket &\in \mathbf{Sub}(\llbracket B \rrbracket \times \llbracket A \rrbracket) , \\ \llbracket y : B \mid \vartheta \rrbracket &\in \mathbf{Sub}(\llbracket B \rrbracket) , \\ \llbracket y : B \mid \exists x : A . \varphi \rrbracket &\in \mathbf{Sub}(\llbracket B \rrbracket) . \end{aligned}$$

And the weakened instance of  $\vartheta$  in the context  $y : B, x : A$  is interpreted by pullback along a projection, cf. page 20, as in the following pullback diagram:

$$\begin{array}{ccc} \llbracket y : B, x : A \mid \vartheta \rrbracket & \xrightarrow{\quad} & \llbracket B \rrbracket \times \llbracket A \rrbracket \\ \downarrow \lrcorner & & \downarrow \pi \\ \llbracket y : B \mid \vartheta \rrbracket & \xrightarrow{\quad} & \llbracket B \rrbracket \end{array}$$

Thus we have

$$\llbracket y : B, x : A \mid \vartheta \rrbracket = \pi^* \llbracket y : B \mid \vartheta \rrbracket ,$$

with weakening interpreted as the pullback functor

$$\pi^* : \mathbf{Sub}(\llbracket B \rrbracket) \rightarrow \mathbf{Sub}(\llbracket B \rrbracket \times \llbracket A \rrbracket) .$$

We will interpret existential quantification  $\exists x : A$  as a suitable functor

$$\exists_A : \mathbf{Sub}(\llbracket B \rrbracket \times \llbracket A \rrbracket) \rightarrow \mathbf{Sub}(\llbracket B \rrbracket)$$

so that

$$\llbracket y : B \mid \exists x : A . \varphi \rrbracket = \exists_A \llbracket y : B, x : A \mid \varphi \rrbracket .$$

The interpretation of the two-way rule (3.3) then becomes a two-way inequality rule

$$\frac{\llbracket y : B, x : A \mid \varphi \rrbracket \leq \pi^* \llbracket y : B \mid \vartheta \rrbracket}{\exists_A \llbracket y : B, x : A \mid \varphi \rrbracket \leq \llbracket y : B \mid \vartheta \rrbracket}$$

Replacing the interpretations of  $\varphi$  and  $\vartheta$  by general subobjects  $S \in \mathbf{Sub}(\llbracket B \rrbracket \times \llbracket A \rrbracket)$  and  $T \in \mathbf{Sub}(\llbracket B \rrbracket)$ , we obtain the more suggestive formulation

$$\frac{S \leq \pi^* T}{\exists_A S \leq T} \tag{3.4}$$

This is of course nothing but an adjunction between  $\exists_A$  and  $\pi^*$ . Indeed, the operations  $\exists_A$  and  $\pi^*$  are functors on the posets of subjects  $\mathbf{Sub}(\llbracket B \rrbracket \times \llbracket A \rrbracket)$  and  $\mathbf{Sub}(\llbracket B \rrbracket)$ , and the bijection of hom-sets (3.4) is exactly the statement of an adjunction between them. Thus *existential quantification is left-adjoint to weakening*:

$$\exists_A \dashv \pi^*$$

An exactly dual argument shows that *universal quantification is right-adjoint to weakening*:

$$\pi^* \dashv \forall_A$$

Thus, in sum, we have that the rules of inference require that the quantifiers be interpreted as operations adjoint to the interpretation of weakening, i.e. pullback  $\pi^*$  along the projection  $\pi : \llbracket B \rrbracket \times \llbracket A \rrbracket \rightarrow \llbracket B \rrbracket$ .

$$\begin{array}{ccc} & \text{Sub}(\llbracket B \rrbracket \times \llbracket A \rrbracket) & \\ \exists \swarrow & \uparrow \pi^* & \searrow \forall \\ & \text{Sub}(\llbracket B \rrbracket) & \end{array}$$

Note that the familiar side-conditions on the conventional rules for the quantifiers, to the effect that “ $x$  cannot occur freely in  $\psi$ ”, etc., which may seem like tiresome book-keeping, are actually of the essence, since they actually express the weakening operation to which the quantifiers themselves are adjoints.

Let us see how this works for the usual interpretation in **Set**. A predicate  $y : B, x : A \mid \varphi$  corresponds to a subset  $\Phi \subseteq B \times A$ , and  $y : B \mid \vartheta$  corresponds to a subset  $\Theta \subseteq B$ . Weakening of  $\Theta$  is the subset  $\pi^*\Theta = \Theta \times A \subseteq B \times A$ . Then we have

$$\begin{aligned} \exists_A \Phi &= \{y \in B \mid \exists x : A. \langle x, y \rangle \in \Phi\} \subseteq B, \\ \forall_A \Phi &= \{y \in B \mid \forall x : A. \langle x, y \rangle \in \Phi\} \subseteq B. \end{aligned}$$

A moment's thought convinces us that with this interpretation we do indeed have

$$\frac{\Phi \subseteq \Theta \times A}{\exists_A \Phi \subseteq \Theta} \qquad \frac{\Theta \times A \subseteq \Phi}{\Theta \subseteq \forall_A \Phi}$$

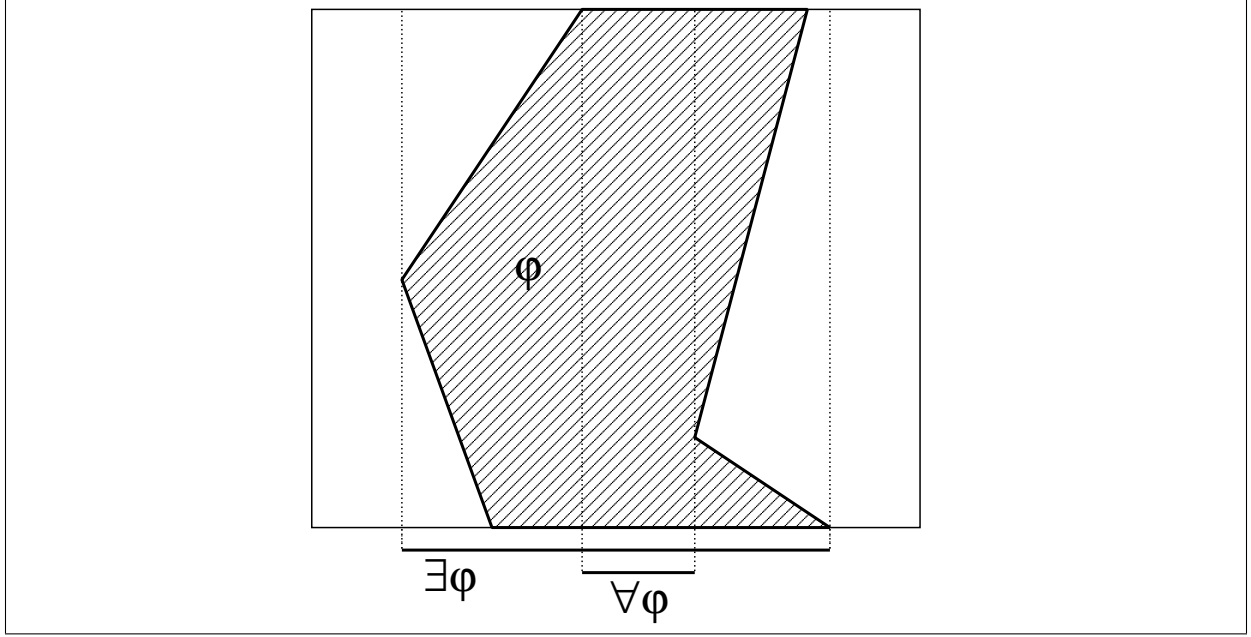
The unit of the adjunction  $\exists_A \dashv \pi^*$  amounts to the inequality

$$\Phi \subseteq (\exists_A \Phi) \times A, \tag{3.5}$$

and the universal property of the unit says that  $\exists_A \Phi$  is the smallest set satisfying (3.5). Similarly, the counit of the adjunction  $\pi^* \dashv \forall_A$  is just the inequality

$$(\forall_A \Phi) \times A \subseteq \Phi, \tag{3.6}$$

and the universal property of the counit says that  $\forall_A \Phi$  is the largest set satisfying (3.6). Figure 3.1 shows the geometric meaning of existential and universal quantification.

Figure 3.1:  $\exists\varphi$  and  $\forall\varphi$ 

**Exercise 3.1.25.** What do the universal properties of the counit of  $\exists_A \dashv \pi^*$  and the unit of  $\pi^* \dashv \forall_A$  say?

The weakening functor  $\pi^*$  is a special case of a pullback functor  $f^* : \mathbf{Sub}(B) \rightarrow \mathbf{Sub}(A)$  for a morphism  $f : B \rightarrow A$ . This suggests that we may regard the left and the right adjoint to  $f^*$  as a kind of generalized existential and universal quantifier. We may indeed be tempted to simply *define* the quantifiers as left and right adjoints to general pullback functors. However there is a bit more to quantifiers than that—we are still missing the important *Beck-Chevalley condition*.

### The Beck-Chevalley condition

Recall from (3.2) that quantification commutes with substitution, as long as no variables are captured by the quantifier. Thus if  $\Gamma \mid t : B$  and  $\Gamma, y : B, x : A \mid \varphi$  pred then

$$\begin{aligned} (\exists x : A. \varphi)[t/y] &= \exists x : A. (\varphi[t/y]) , \\ (\forall x : A. \varphi)[t/y] &= \forall x : A. (\varphi[t/y]) . \end{aligned}$$

If the semantics of quantification is to be sound, the interpretation of these equations must be valid. Because substitution of a term in a formula is interpreted as pullback, this means exactly that quantifiers must be *stable* under pullbacks. This is known as the *Beck-Chevalley condition*.

**Definition 3.1.26.** A family of functors  $F_f : \mathbf{Sub}(A) \rightarrow \mathbf{Sub}(B)$  parametrized by morphisms  $f : A \rightarrow B$  is said to satisfy the *Beck-Chevalley condition* when for every pullback



as on the left-hand side, the right-hand square commutes:

$$\begin{array}{ccc}
 C & \xrightarrow{h} & A \\
 \downarrow k & \lrcorner & \downarrow f \\
 D & \xrightarrow{g} & B
 \end{array}
 \qquad
 \begin{array}{ccc}
 \text{Sub}(C) & \xleftarrow{h^*} & \text{Sub}(A) \\
 \downarrow F_k & & \downarrow F_f \\
 \text{Sub}(D) & \xleftarrow{g^*} & \text{Sub}(B)
 \end{array}$$

To convince ourselves that Beck-Chevalley condition is what we want, we spell it out explicitly in the case of a substitution into an existentially quantified formula. In order to keep the notation simple we omit the semantic brackets  $\llbracket - \rrbracket$ . Suppose we have a term  $\Gamma \mid t : B$  and a formula  $\Gamma, y : B, x : A \mid \varphi$  **pred**. The diagram

$$\begin{array}{ccc}
 \Gamma \times A & \xrightarrow{\langle \pi_0, t \circ \pi_0, \pi_1 \rangle} & \Gamma \times B \times A \\
 \downarrow \pi_0^{\Gamma, A} & \lrcorner & \downarrow \pi_0^{\Gamma, B, A} \\
 \Gamma & \xrightarrow{\langle 1_\Gamma, t \rangle} & \Gamma \times B
 \end{array}$$

is a pullback. By the Beck-Chevalley condition for  $\exists$ , the following square commutes:

$$\begin{array}{ccc}
 \text{Sub}(\Gamma \times A) & \xleftarrow{\langle \pi_0, t \circ \pi_0, \pi_1 \rangle^*} & \text{Sub}(\Gamma \times B \times A) \\
 \downarrow \exists_A^{\Gamma, A} & & \downarrow \exists_A^{\Gamma, B, A} \\
 \text{Sub}(\Gamma) & \xleftarrow{\langle 1_\Gamma, t \rangle^*} & \text{Sub}(\Gamma \times B)
 \end{array}$$

Therefore, for  $\Gamma, y : B, x : A \mid \varphi$  **pred**, we have

$$\begin{aligned}
 \llbracket (\exists x : A. \varphi)[t/y] \rrbracket &= \langle 1_\Gamma, t \rangle^* (\exists_A^{\Gamma, B, A} \llbracket \varphi \rrbracket) = \\
 &= \exists_A^{\Gamma, A} (\langle \pi_0, t \circ \pi_0, \pi_1 \rangle^* \llbracket \varphi \rrbracket) = \llbracket \exists x : A. (\varphi[t/y]) \rrbracket .
 \end{aligned}$$

This is indeed precisely the equation we wanted. The Beck-Chevalley condition says that (the interpretations of) the quantifiers commute with pullbacks, in just the way that the syntactic operations of applying quantifiers to formulas commute with substitutions of terms (which are interpreted as pullbacks).

**Definition 3.1.27.** A cartesian category  $\mathcal{C}$  has *existential quantifiers* if, for every  $f : A \rightarrow B$ , the left adjoint  $\exists_f \dashv f^*$  exists and it satisfies the Beck-Chevalley condition. Similarly,  $\mathcal{C}$  has *universal quantifiers* if the right adjoints  $f^* \dashv \forall_f$  exist and they satisfy the Beck-Chevalley condition.

It is convenient to know that, if we have both adjoints  $\exists_f \dashv f^* \dashv \forall_f$ , it actually suffices to have the Beck-Chevalley condition for either one in order to infer it for both:

**Proposition 3.1.28.** *If for every  $f : A \rightarrow B$ , both the left and right adjoints exist*

$$\exists_f \dashv f^* \dashv \forall_f$$

*then the left adjoint satisfies the Beck-Chevalley condition iff the right adjoint does.*

*Proof.* Suppose we have the Beck-Chevalley condition for the left adjoints  $\exists$ , and that we are given a pullback square as on the left below. We want to check the Beck-Chevalley square for the right adjoints  $\forall$ , as indicated on the right below.

$$\begin{array}{ccc} C & \xrightarrow{h} & A \\ \downarrow k & \lrcorner & \downarrow f \\ D & \xrightarrow{g} & B \end{array} \qquad \begin{array}{ccc} \text{Sub}(C) & \xleftarrow{h^*} & \text{Sub}(A) \\ \downarrow \forall_k & & \downarrow \forall_f \\ \text{Sub}(D) & \xleftarrow{g^*} & \text{Sub}(B) \end{array}$$

Swapping all the functors in the righthand diagram for their left adjoints we obtain the following.

$$\begin{array}{ccc} \text{Sub}(C) & \xrightarrow{\exists_h} & \text{Sub}(A) \\ \uparrow k^* & & \uparrow f^* \\ \text{Sub}(D) & \xrightarrow{\exists_g} & \text{Sub}(B) \end{array}$$

But this is a Beck-Chevalley square for (the “transpose” of) the original pullback diagram, and therefore commutes by the Beck-Chevalley condition for the left adjoints  $\exists$ . The original diagram of right adjoints therefore also commutes, by uniqueness of adjoints.

The argument for the dual case is, well, dual.  $\square$

**Exercise 3.1.29.** In **Set** we can identify  $\text{Sub}(-)$  with powersets because  $\text{Sub}(X) \cong \mathcal{P}X$ . Then quantifiers along a function  $f : A \rightarrow B$  are functions

$$\exists_f : \mathcal{P}A \rightarrow \mathcal{P}B, \qquad \forall_f : \mathcal{P}A \rightarrow \mathcal{P}B.$$

Verify that

$$\begin{aligned} \exists_f U &= \{b \in B \mid \exists a : A. (fa = b \wedge a \in U)\}, \\ \forall_f U &= \{b \in B \mid \forall a : A. (fa = b \Rightarrow a \in U)\}. \end{aligned}$$

Thus  $\exists_f U$  is just the usual direct image of  $U$  by  $f$ , sometimes written  $f_!(U)$ , or simply  $f(U)$ . But have you seen  $\forall_f U$  before? It can also be written as  $\forall_f U = \{b \in B \mid f^* \{b\} \subseteq U\}$ . What is the meaning of  $\exists_q$  and  $\forall_q$  when  $q : A \rightarrow A/\sim$  is a canonical quotient map that maps an element  $x \in A$  to its equivalence class  $qx = [x]$  under an equivalence relation  $\sim$  on  $A$ ?

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