Introduction to Categorical Logic

[DRAFT: March 29, 2022]

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Contents

Chapter 3

Cartesian Closed Categories and the λ-Calculus

3.1 Categorification and the Curry-Howard correspondence

Consider the following natural deduction proof in propositional calculus.

$$\dfrac{\dfrac{\dfrac{\dfrac{[(A \wedge B) \wedge (A \Rightarrow B)]^1}{A \wedge B}}{A} \qquad \dfrac{[(A \wedge B) \wedge (A \Rightarrow B)]^1}{A \Rightarrow B}}{B}}{(A \wedge B) \wedge (A \Rightarrow B) \Rightarrow B}\ (1)$$

This deduction shows that

$$\vdash (A \wedge B) \wedge (A \Rightarrow B) \Rightarrow B.$$

But so does the following:

$$\dfrac{\dfrac{\dfrac{[(A \wedge B) \wedge (A \Rightarrow B)]^1}{A \Rightarrow B} \qquad \dfrac{\dfrac{[(A \wedge B) \wedge (A \Rightarrow B)]^1}{A \wedge B}}{A}}{B}}{(A \wedge B) \wedge (A \Rightarrow B) \Rightarrow B}\ (1)$$

As does:

$$\dfrac{\dfrac{\dfrac{[(A \wedge B) \wedge (A \Rightarrow B)]^1}{A \wedge B}}{B}}{(A \wedge B) \wedge (A \Rightarrow B) \Rightarrow B}\ (1)$$

There is a sense in which the first two proofs are "equivalent", but not the first and the third. The relation (or property) of *provability* in propositional calculus $\vdash \phi$ discards such differences in the proofs that witness it. According to the "proof-relevant" point of view, sometimes called *propositions as types*, one retains as relevant some information about the way in which a proposition is proved. This is effected by annotating the proofs with *proof-terms* as they are constructed, as follows:

$$\dfrac{\dfrac{\dfrac{[x:(A\wedge B)\wedge(A\Rightarrow B)]^1}{\pi_2(x):A\Rightarrow B} \qquad \dfrac{\dfrac{[x:(A\wedge B)\wedge(A\Rightarrow B)]^1}{\pi_1(x):A\wedge B}}{\pi_1(\pi_1(x)):A}}{\pi_2(x)(\pi_1(\pi_1(x))):B}}{\lambda x.\pi_2(x)(\pi_1(\pi_1(x))):(A\wedge B)\wedge(A\Rightarrow B)\Rightarrow B}\,(1)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[x:(A\wedge B)\wedge(A\Rightarrow B)]^1}{\pi_1(x):A\wedge B}}{\pi_1(\pi_1(x)):A} \qquad \dfrac{[x:(A\wedge B)\wedge(A\Rightarrow B)]^1}{\pi_2(x):A\Rightarrow B}}{\pi_2(x)(\pi_1(\pi_1(x))):B}}{\lambda x.\pi_2(x)(\pi_1(\pi_1(x))):(A\wedge B)\wedge(A\Rightarrow B)\Rightarrow B}\,(1)$$

$$\dfrac{\dfrac{\dfrac{[x:(A\wedge B)\wedge(A\Rightarrow B)]^1}{\pi_1(x):A\wedge B}}{\pi_2(\pi_1(x)):B}}{\lambda x.\pi_2(\pi_1(x)):(A\wedge B)\wedge(A\Rightarrow B)\Rightarrow B}\,(1)$$

The proof terms for the first two proofs are the same, namely $\lambda x.\pi_2(x)(\pi_1(\pi_1(x)))$, but the term for the third one is $\lambda x.\pi_2(\pi_1(x))$, reflecting the difference in the proofs. The assignment works by labelling assumptions as variables, and then associating term-constructors to the different rules of inference: pairing and projection to conjunction introduction and elimination, function application and λ-abstraction to implication elimination (*modus ponens*) and introduction. The use of variable binding to represent cancellation of premisses is a particularly effective device.

From the categorical point of view, the relation of deducibility $\phi \vdash \psi$ is a mere preorder. The addition of proof terms $x:\phi \vdash t:\psi$ results in a *categorification* of this preorder, in the sense that it is a "proper" category, the preordered reflection of which is the deducibility preorder. And now the following remarkable fact emerges: it is hardly surprising that the deducibility preorder has, say, finite products $\phi \wedge \psi$ or even exponentials $\phi \Rightarrow \psi$; but it is *amazing* that the category with proof terms $x:\phi \vdash t:\psi$ as arrows, also turns out to be a cartesian closed category, and indeed a proper one, with distinct parallel arrows, such as

$$\begin{aligned}\pi_2(x)(\pi_1(\pi_1(x))):(A\wedge B)\wedge(A\Rightarrow B) &\longrightarrow B,\\ \pi_2(\pi_1(x)):(A\wedge B)\wedge(A\Rightarrow B) &\longrightarrow B.\end{aligned}$$

This *category of proofs* contains information about the "proof theory" of the propositional calculus, as opposed to its mere relation of deducibility. The calculus of proof terms can be presented formally in a system of *simple type theory*, with an alternate interpretation as a formal system of function application and abstraction. This dual interpretation—as the proof theory of propositional logic, and as a system of type theory for the specification of functions—is called the *Curry-Howard correspondence* []. From the categorical point of view, it expresses the structural equivalence between the cartesian closed categories of proofs in propositional logic and terms in simple type theory. Both of these can be seen as categorifications of their preorder reflection, the deducibility preorder of propositional logic.

In the following sections, we shall consider this remarkable correspondence in detail, as well as some extensions of the basic case represented by cartesian closed categories: categories with coproducts, cocomplete categories, and categories equipped with modal operators. In the next chapter, it will be seen that this correspondence even extends to proofs in quantified predicate logic and terms in dependent type theory, and beyond.

3.2 Cartesian closed categories

Exponentials

We begin with the notion of an exponential B^A of two objects A, B in a category, motivated by a couple of important examples. Consider first the category Pos of posets and monotone functions. For posets P and Q the set $\mathsf{Hom}(P, Q)$ of all monotone functions between them is again a poset, with the pointwise order:

$$f \leq g \iff fx \leq gx \quad \text{for all } x \in P \, . \qquad (f, g : P \to Q)$$

Thus $\mathsf{Hom}(P, Q)$ is again an object of Pos, when equipped with a suitable order.

Similarly, given monoids $K, M \in \mathsf{Mon}$, there is a natural monoid structure on the set $\mathsf{Hom}(K, M)$, defined pointwise by

$$(f \cdot g)x = fx \cdot gx \, . \qquad (f, g : K \to M, \, x \in K)$$

Thus the category Mon also admits such "internal Hom"s. The same thing works in the category Group of groups and group homomophisms, where the set $\mathsf{Hom}(G, H)$ of all homomorphisms between groups G and H can be given a pointwise group structure.

These examples suggest a general notion of "internal Hom" in a category: an "object of morphisms $A \to B$" which corresponds to the hom-set $\mathsf{Hom}(A, B)$. The other ingredient needed is an "evaluation" operation $\epsilon : B^A \times A \to B$ which evaluates a morphism $f \in B^A$ at an argument $x \in A$ to give a value $\epsilon \circ \langle f, x \rangle \in B$. This is always going to be present for the underlying functions if we're starting from a set of functions $\mathsf{Hom}(A, B)$, but it needs to be an actual morphism in the category. Finally, we need an operation of "transposition", taking a morphism $f : C \times A \to B$ to one $\widetilde{f} : C \to A^B$. We shall see that this in fact separates the previous two examples.

Definition 3.2.1. In a category $\mathcal{C}$ with binary products, an *exponential* (B^A, ϵ) of objects A and B is an object B^A together with a morphism $\epsilon : B^A \times A \to B$, called the *evaluation* morphism, such that for every $f : C \times A \to B$ there exists a *unique* morphism $\widetilde{f} : C \to B^A$, called the *transpose*[1] of f, for which the following diagram commutes.

$$\begin{tikzcd} B^A & B^A \times A \arrow[r, "\epsilon"] & B \\ C \arrow[u, dotted, "\widetilde{f}"] & C \times A \arrow[u, "\widetilde{f} \times 1_A"] \arrow[ur, "f"'] & \end{tikzcd}$$

Commutativity of the diagram of course means that $f = \epsilon \circ (\widetilde{f} \times 1_A)$.

Definition 3.2.1 is called the *universal property of the exponential.* It is just the category-theoretic way of saying that a function $f : C \times A \to B$ of two variables can be viewed as a function $\widetilde{f} : C \to B^A$ of one variable that maps $z \in C$ to a function $\widetilde{f}z = f\langle z, -\rangle : A \to B$ that maps $x \in A$ to $f\langle z, x\rangle$. The relationship between f and $\widetilde{f}$ is then

$$f\langle z, x\rangle = (\widetilde{f}z)x \; .$$

That is all there is to it, really, except that variables and elements never need to be mentioned. The benefit of this is that the definition is applicable also in categories whose objects are not *sets* and whose morphisms are not *functions*—even though some of the basic examples are of that sort.

In Poset the exponential Q^P of posets P and Q is the set of all monotone maps $P \to Q$, ordered pointwise, as above. The evaluation map $\epsilon : Q^P \times P \to Q$ is just the usual evaluation of a function at an argument. The transpose of a monotone map $f : R \times P \to Q$ is the map $\widetilde{f} : R \to Q^P$, defined by, $(\widetilde{f}z)x = f\langle z, x\rangle$, i.e. the transposed *function.* We say that the category Pos *has all exponentials.*

Definition 3.2.2. Suppose $\mathcal{C}$ has all finite products. An object $A \in \mathcal{C}$ is *exponentiable* when the exponential B^A exists for every $B \in \mathcal{C}$. We say that $\mathcal{C}$ *has exponentials* if every object is exponentiable. A *cartesian closed category (ccc)* is a category that has all finite products and exponentials.

Example 3.2.3. Consider again the example of the set $\mathsf{Hom}(M, N)$ of homomorphisms between two monoids M, N, equipped with the pointwise monoid structure. To be a monoid homomorphism. the transpose $\widetilde{h} : 1 \to \mathsf{Hom}(M, N)$ of a homomorphism $h : 1 \times M \to N$ would have to take the unit element $u \in 1$ to the unit homomorphism $u : M \to N$, which is the constant function at the unit $u \in N$. Since $1 \times M \cong M$, that would mean that *all* homomorphisms $h : M \to N$ would have the same transpose $\widetilde{h} = u : 1 \to \mathsf{Hom}(M, N)$. So Mon cannot be cartesian closed. The same argument works in the category Group, and in many related ones. (But see **??** below on one way of embedding Group into a CCC.)

Exercise 3.2.4. Is the evaluation function $\mathsf{eval} : \mathsf{Hom}(M, N) \times M \to N$ a homomorphism of monoids?

[1] Also, f is called the transpose of $\widetilde{f}$, so that f and $\widetilde{f}$ are each other's transpose.

Two more definitions of CCCs

Proposition 3.2.5. *In a category $\mathcal{C}$ with binary products an object A is exponentiable if, and only if, the functor*

$$- \times A : \mathcal{C} \to \mathcal{C}$$

has a right adjoint

$$-^A : \mathcal{C} \to \mathcal{C} \; .$$

Proof. If such a right adjoint exists then the exponential of A and B is (B^A, ϵ_B), where $\epsilon : -^A \times A \Longrightarrow 1_{\mathcal{C}}$ is the counit of the adjunction. The universal property of the exponential is precisely the universal property of the counit ϵ.

Conversely, suppose for every B there is an exponential (B^A, ϵ_B). As the object part of the right adjoint we then take B^A. For the morphism part, given $g : B \to C$, we can define $g^A : B^A \to C^A$ to be the transpose of $g \circ \epsilon_B$,

$$g^A = (g \circ \epsilon_B)^{\sim}$$

as indicated below.

$$\begin{tikzcd} B^A \times A \arrow[r, "\epsilon_B"] \arrow[d, "g^A \times 1_A"'] & B \arrow[d, "g"] \\ C^A \times A \arrow[r, "\epsilon_C"'] & C \end{tikzcd} \tag{3.1}$$

The counit $\epsilon : -^A \times A \Longrightarrow 1_{\mathcal{C}}$ at B is then ϵ_B itself, and the naturality square for ϵ is then exactly (3.1), i.e. the defining property of $(f \circ \epsilon_B)^{\sim}$:

$$\epsilon_C \circ (g^A \times 1_A) = \epsilon_C \circ ((g \circ \epsilon_B)^{\sim} \times 1_A) = g \circ \epsilon_B \; .$$

The universal property of the counit ϵ is precisely the universal property of the exponential (B^A, ϵ_B) □

Note that because exponentials can be expressed as right adjoints to binary products, they are determined uniquely up to isomorphism. Moreover, the definition of a cartesian closed category can then be phrased entirely in terms of adjoint functors: we just need to require the existence of the terminal object, binary products, and exponentials.

Proposition 3.2.6. *A category $\mathcal{C}$ is cartesian closed if, and only if, the following functors have* right *adjoints:*

$$\begin{aligned} !_{\mathcal{C}} &: \mathcal{C} \to \mathsf{1} \; , \\ \Delta &: \mathcal{C} \to \mathcal{C} \times \mathcal{C} \; , \\ (- \times A) &: \mathcal{C} \to \mathcal{C} \; . \end{aligned} \qquad (A \in \mathcal{C})$$

Here $!_{\mathcal{C}}$ is the unique functor from $\mathcal{C}$ to the terminal category $\mathsf{1}$ and Δ is the diagonal functor $\Delta A = \langle A, A \rangle$, and the right adjoint of $- \times A$ is exponentiation by A.

$\square$

The significance of the adjoint formulation is that it implies the possibility of a purely *equational* specification (adjoint structure on a category is “equational” in a sense that can be made precise; see [**?**]). We can therefore give an explicit, equational formulation of cartesian closed categories.

Definition 3.2.7 (Equational version of CCC)**.** A category $\mathcal{C}$ is cartesian closed if, and only if, it has the following structure:

1. An object $\mathbf{1} \in \mathcal{C}$ and a morphism $!_A : A \to \mathbf{1}$ for every $A \in \mathcal{C}$.
2. An object $A \times B$ for all $A, B \in \mathcal{C}$ together with morphisms $\pi_0 : A \times B \to A$ and $\pi_1 : A \times B \to B$, and for every pair of morphisms $f : C \to A$, $g : C \to B$ a morphism $\langle f, g\rangle : C \to A \times B$.
3. An object B^A for all $A, B \in \mathcal{C}$ together with a morphism $\epsilon : B^A \times A \to B$, and a morphism $\widetilde{f} : C \to B^A$ for every morphism $f : C \times A \to B$.

These new objects and morphisms are required to satisfy the following equations:

1. For every $f : A \to \mathbf{1}$,
$$f = {!_A} \; .$$
2. For all $f : C \to A$, $g : C \to B$, $h : C \to A \times B$,
$$\pi_0 \circ \langle f, g\rangle = f \; , \qquad \pi_1 \circ \langle f, g\rangle = g \; , \qquad \langle \pi_0 \circ h, \pi_1 \circ h\rangle = h \; .$$
3. For all $f : C \times A \to B$, $g : C \to B^A$,
$$\epsilon \circ (\widetilde{f} \times \mathbf{1}_A) = f \; , \qquad (\epsilon \circ (g \times \mathbf{1}_A))^{\sim} = g \; .$$

These equations ensure that certain diagrams commute and that the morphisms that are required to exist are unique. For example, let us prove that $(A \times B, \pi_0, \pi_1)$ is the product of A and B. For $f : C \to A$ and $g : C \to B$ there exists a morphism $\langle f, g\rangle : C \to A \times B$. Equations

$$\pi_0 \circ \langle f, g\rangle = f \qquad \text{and} \qquad \pi_1 \circ \langle f, g\rangle = g$$

enforce the commutativity of the two triangles in the following diagram:

$$\begin{tikzcd} & C \arrow[dl, "g"'] \arrow[d, "\langle f, g\rangle"] \arrow[dr, "f"] & \\ A & A \times B \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & B \end{tikzcd}$$

Suppose $h : C \to A \times B$ is another morphism such that $f = \pi_0 \circ h$ and $g = \pi_1 \circ h$. Then by the third equation for products we get

$$h = \langle \pi_0 \circ h, \pi_1 \circ h\rangle = \langle f, g\rangle \; ,$$

and so $\langle f, g\rangle$ is unique.

Positive propositional calculus

We begin with the example of a cartesian closed poset and a first application to propositional logic.

Example 3.2.8. Consider the *positive propositional calculus* PPC with conjunction and implication, as in Subsection **??**. Recall that PPC is the set of all "propositions" ϕ constructed from propositional variables $p_1, p_2, ...$, a constant $\top$ for truth, conjunction $\phi \wedge \psi$, and implication $\phi \Rightarrow \psi$.

As a category, PPC is a preorder under the relation $\phi \vdash \psi$ of logical entailment, determined say by the natural deduction system **??** of section **??**. A conjunction $\phi \wedge \psi$ is a greatest lower bound of ϕ and ψ, because we have $\phi \wedge \psi \vdash \phi$ and $\phi \wedge \psi \vdash \psi$ and for all propositions ϑ,

$$\text{if } \vartheta \vdash \phi \text{ and } \vartheta \vdash \psi \text{ then } \vartheta \vdash \phi \wedge \psi.$$

Since in a preorder binary products are the same thing as greatest lower bounds, we see that conjunction is a binary product.

We have already remarked that, in general, implication is right adjoint to conjunction in propositional calculus,

$$(- \times \phi) \dashv (\phi \Rightarrow -)\ . \tag{3.2}$$

Therefore $\phi \Rightarrow \psi$ is an exponential in PPC. The counit of the adjunction, or equivalently, the "evaluation" morphism, is the entailment

$$(\phi \Rightarrow \psi) \wedge \phi \vdash \psi\ ,$$

i.e. the familiar logical rule of *modus ponens*. This is another example where the basic concepts of logic arise as adjoints.

Exercise 3.2.9. What is the unit of adjunction (3.2) in logical terms?

Let us now show that the positive propositional calculus PPC is *deductively complete* with respect to the following notion of *Kripke semantics* [].

Definition 3.2.10. Let K be a poset. Suppose we have a relation

$$k \Vdash p$$

between elements $k \in K$ and propositional variables p, such that

$$j \leq k,\ k \Vdash p \quad \text{implies} \quad j \Vdash p. \tag{3.3}$$

Then extend $\Vdash$ to all formulas ϕ by defining

$k \Vdash \top$	always	
$k \Vdash \phi \wedge \psi$	iff	$k \Vdash \phi$ and $k \Vdash \psi$
$k \Vdash \phi \Rightarrow \psi$	iff	for all $j \leq k$, if $j \Vdash \phi$, then $j \Vdash \psi$.

Finally, say that ϕ *holds on* K, written

$$K \Vdash \phi$$

if $k \Vdash \phi$ for all $k \in K$ (and all such relations $\Vdash$).

Proposition 3.2.11 (Kripke completeness for PPC)**.** *A propositional formulas ϕ is provable from the rules of deduction for* PPC *if, and only if, $K \Vdash \phi$ for all posets K. Briefly:*

$$\mathsf{PPC} \vdash \phi \quad \textit{iff} \quad K \Vdash \phi \textit{ for all } K.$$

Proof. The proof follows a now-familiar pattern, which we only sketch:

Step 1. Build the syntactic category $\mathcal{C}_{\mathsf{PPC}}$, consisting of formulas ϕ ordered by deductive entailment $\phi \vdash \psi$.

Step 2. $\mathcal{C}_{\mathsf{PPC}}$ is a ccc, with $\top = 1$, $\phi \times \psi = \phi \wedge \psi$, and $\psi^{\phi} = \phi \Rightarrow \psi$. It is logically generic, in the sense that $\mathsf{PPC} \vdash \phi$ iff $\phi = 1$.

Step 3. A model $\Vdash$ in a poset P, in the sense of (3.3), is a CCC functor $f : \mathcal{C}_{\mathsf{PPC}} \to \downarrow\!(\mathsf{P})$ into the poset of sieves on P, and $K \Vdash \phi$ means that $f\phi = 1$, the maximal sieve.

Step 4. The Yoneda embedding is a (conservative) such model of PPC.

$\square$