Introduction to Categorical Logic

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Contents

Chapter 2

Propositional Logic

Propositional logic is the logic of propositional connectives like $p \wedge q$ and $p \Rightarrow q$. As was the case for algebraic theories, the general approach will be to determine suitable categorical structures to model the logical operations, and then use categories with such structure to represent (abstract) propositional theories. Adjoints will play a special role, as we will describe the basic logical operations as such. We again show that the semantics is "functorial", meaning that the models of a theory are functors that preserve the categorical structure. We will show that there are classifying categories for all propositional theories, as was the case for the algebraic theories that we have already met.

A more abstract, algebraic perspective will then relate the propositional case of syntax-semantics duality with classical Stone duality for Boolean algebras, and related results from lattice theory will provide an algebraic treatment of Kripke semantics for intuitionistic (and modal) propositional logic.

2.1 Propositional calculus

Before going into the details of the categorical approach, we first briefly review the propositional calculus from a conventional point of view, as we did for algebraic theories. We focus first on the *classical* propositional logic, before considering the intuitionisitc case in section **??**.

In the style of Section **??**, we have the following (abstract) syntax for (propositional) formulas:

$$\begin{array}{rcl} \text{Propositional variable } p & ::= & \mathtt{p}_1 \mid \mathtt{p}_2 \mid \mathtt{p}_3 \mid \cdots \\ \text{Propositional formula } \phi & ::= & p \mid \top \mid \bot \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid \phi_1 \Rightarrow \phi_2 \mid \phi_1 \Leftrightarrow \phi_2 \end{array}$$

An example of a formula is therefore $\big(\mathtt{p}_3 \Leftrightarrow ((((\neg \mathtt{p}_1) \vee (\mathtt{p}_2 \wedge \bot)) \vee \mathtt{p}_1) \Rightarrow \mathtt{p}_3)\big)$. We will make use of the usual conventions for parenthesis, with binding order $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$. Thus e.g. the foregoing may also be written unambiguously as $\mathtt{p}_3 \Leftrightarrow \neg \mathtt{p}_1 \vee \mathtt{p}_2 \wedge \bot \vee \mathtt{p}_1 \Rightarrow \mathtt{p}_3$.

Natural deduction

The system of *natural deduction* for propositional logic has one form of judgement

$$\mathtt{p}_1, \ldots, \mathtt{p}_n \mid \phi_1, \ldots, \phi_m \vdash \phi$$

where $\mathtt{p}_1, \ldots, \mathtt{p}_n$ is a *context* consisting of distinct propositional variables, the formulas $\phi_1, \ldots, \phi_m$ are the *hypotheses* and ϕ is the *conclusion*. The variables in the hypotheses and the conclusion must occur among those listed in the context. The hypotheses are regarded as a (finite) set; so they are unordered, have no repetitions, and may be empty. We may abbreviate the context of variables by Γ, and we often omit it.

Deductive entailment (or *derivability*) $\Phi \vdash \phi$ is thus a relation between finite sets of formulas Φ and single formulas ϕ. It is defined as the smallest such relation satisfying the following rules:

1. Hypothesis:
$$\frac{}{\Phi \vdash \phi} \text{ if } \phi \text{ occurs in } \Phi$$
2. Truth:
$$\frac{}{\Phi \vdash \top}$$
3. Falsehood:
$$\frac{\Phi \vdash \bot}{\Phi \vdash \phi}$$
4. Conjunction:
$$\frac{\Phi \vdash \phi \qquad \Phi \vdash \psi}{\Phi \vdash \phi \wedge \psi} \qquad \frac{\Phi \vdash \phi \wedge \psi}{\Phi \vdash \phi} \qquad \frac{\Phi \vdash \phi \wedge \psi}{\Phi \vdash \psi}$$
5. Disjunction:
$$\frac{\Phi \vdash \phi}{\Phi \vdash \phi \vee \psi} \qquad \frac{\Phi \vdash \psi}{\Phi \vdash \phi \vee \psi} \qquad \frac{\Phi \vdash \phi \vee \psi \qquad \Phi, \phi \vdash \theta \qquad \Phi, \psi \vdash \theta}{\Phi \vdash \theta}$$
6. Implication:
$$\frac{\Phi, \phi \vdash \psi}{\Phi \vdash \phi \Rightarrow \psi} \qquad \frac{\Phi \vdash \phi \Rightarrow \psi \qquad \Phi \vdash \phi}{\Phi \vdash \psi}$$

For the purpose of deduction, we define $\neg\phi := \phi \Rightarrow \bot$ and $\phi \Leftrightarrow \psi := (\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi)$. To obtain *classical* logic we need only include one of the following additional rules.

7. Classical logic:
$$\frac{}{\Phi \vdash \phi \vee \neg\phi} \qquad \frac{\Phi \vdash \neg\neg\phi}{\Phi \vdash \phi}$$

A *proof* of $\Phi \vdash \phi$ is a *finite* tree built from the above inference rules whose root is $\Phi \vdash \phi$. For example, here is a proof of $\phi \vee \psi \vdash \psi \vee \phi$ using the disjunction rules:

$$\dfrac{\dfrac{}{\phi \vee \psi \vdash \phi \vee \psi} \qquad \dfrac{\dfrac{}{\phi \vee \psi, \phi \vdash \phi}}{\phi \vee \psi, \phi \vdash \psi \vee \phi} \qquad \dfrac{\dfrac{}{\phi \vee \psi, \psi \vdash \psi}}{\phi \vee \psi, \psi \vdash \psi \vee \phi}}{\phi \vee \psi \vdash \psi \vee \phi}$$

A judgment $\Phi \vdash \phi$ is *provable* if there exists a proof of it. Observe that every proof has at its leaves either the rule for $\top$ or a hypothesis.

Exercise 2.1.1. Derive each of the two classical rules (2.1), called *excluded middle* and *double negation*, from the other.

2.2 Truth values

The idea of an axiomatic system of deductive, logical reasoning goes to back to Frege, who gave the first such system for propositional calculus (and more) in his *Begriffsschrift* of 1879. The question soon arose whether Frege's rules (or rather, their derivable consequences – it was clear that one could chose the primitive basis in different but equivalent ways) were correct, and if so, whether they were *all* the correct ones. An ingenious solution was proposed by Russell's student Wittgenstein, who came up with an entirely different way of singling out a set of "valid" propositional formulas in terms of assignments of truth values to the variables occurring in them. He interpreted this as showing that logical validity was really a matter of the logical structure of a proposition, and not depedent on any particular system of derivations. The same idea seems to have been had independently by Post, who proved that the valid propositional formulas coincide with the ones derivable in Whitehead and Russell's *Principia Mathematica* (which is propositionally equivalent to Frege's system), a fact that we now refer to as the *soundness* and *completeness* of propositional logic.

In more detail, let a *valuation* v be an assignment of a "truth-value" $0, 1$ to each propositional variable, $v(\mathtt{p}_n) \in \{0, 1\}$. We can then extend the valuation to all propositional formulas $[\![\phi]\!]^v$ by the recursion,

$$\begin{aligned}
[\![\mathtt{p}_n]\!]^v &= v(\mathtt{p}_n) \\
[\![\top]\!]^v &= 1 \\
[\![\bot]\!]^v &= 0 \\
[\![\neg\phi]\!]^v &= 1 - [\![\phi]\!]^v \\
[\![\phi \wedge \psi]\!]^v &= \min([\![\phi]\!]^v, [\![\psi]\!]^v) \\
[\![\phi \vee \psi]\!]^v &= \max([\![\phi]\!]^v, [\![\psi]\!]^v) \\
[\![\phi \Rightarrow \psi]\!]^v &= 1 \text{ iff } [\![\phi]\!]^v \leq [\![\psi]\!]^v \\
[\![\phi \Leftrightarrow \psi]\!]^v &= 1 \text{ iff } [\![\phi]\!]^v = [\![\psi]\!]^v
\end{aligned}$$

This is sometimes expressed using the "semantic consequence" notation $v \vDash \phi$ to mean that $[\![\phi]\!]^v = 1$. Then the above specification takes the form:

$$
\begin{array}{rll}
v \vDash \top & \text{always} & \\
v \vDash \bot & \text{never} & \\
v \vDash \neg\phi & \text{iff} & v \nvDash \phi \\
v \vDash \phi \wedge \psi & \text{iff} & v \vDash \phi \text{ and } v \vDash \psi \\
v \vDash \phi \vee \psi & \text{iff} & v \vDash \phi \text{ or } v \vDash \psi \\
v \vDash \phi \Rightarrow \psi & \text{iff} & v \vDash \phi \text{ implies } v \vDash \psi \\
v \vDash \phi \Leftrightarrow \psi & \text{iff} & v \vDash \phi \text{ iff } v \vDash \psi
\end{array}
$$

Finally, ϕ is *valid*, written $\vDash \phi$, is defined by,

$$\vDash \phi \quad \text{iff} \quad v \vDash \phi \text{ for all } v.$$

And, more generally, we define $\phi_1, ..., \phi_n$ *semantically entails* ϕ, written

$$\phi_1, ..., \phi_n \vDash \ \phi, \tag{2.1}$$

to mean that for all valuations v such that $v \vDash \phi_k$ for all k, also $v \vDash \phi$.

Given a formula in context $\Gamma \mid \phi$ and a valuation v for the variables in Γ, one can check whether $v \vDash \phi$ using a *truth table*, which is a systematic way of calculating the value of $[\![\phi]\!]^v$. For example, under the assignment $v(\mathtt{p}_1) = 1, v(\mathtt{p}_2) = 0, v(\mathtt{p}_3) = 1$ we can calculate $[\![\phi]\!]^v$ for $\phi = \big(\mathtt{p}_3 \Leftrightarrow ((((\neg\mathtt{p}_1) \vee (\mathtt{p}_2 \wedge \bot)) \vee \mathtt{p}_1) \Rightarrow \mathtt{p}_3)\big)$ as follows.

$\mathtt{p}_1$	$\mathtt{p}_2$	$\mathtt{p}_3$	$\mathtt{p}_3$	$\Leftrightarrow$	$\neg$	$\mathtt{p}_1$	$\vee$	$\mathtt{p}_2$	$\wedge$	$\bot$	$\vee$	$\mathtt{p}_1$	$\Rightarrow$	$\mathtt{p}_3$
1	0	1	1	1	0	1	0	0	0	0	1	1	1	1

The value of the formula ϕ under the valuation v is then the value in the column under the main connective, in this case $\Leftrightarrow$, and thus $[\![\phi]\!]^v = 1$.

Displaying all 2^3 valuations for the context $\Gamma = \mathtt{p}_1, \mathtt{p}_2, \mathtt{p}_3$, therefore results in a table that checks for validity of ϕ,

$\mathtt{p}_1$	$\mathtt{p}_2$	$\mathtt{p}_3$	$\mathtt{p}_3$	$\Leftrightarrow$	$\neg$	$\mathtt{p}_1$	$\vee$	$\mathtt{p}_2$	$\wedge$	$\bot$	$\vee$	$\mathtt{p}_1$	$\Rightarrow$	$\mathtt{p}_3$
1	1	1	.	1	...									
1	1	0	.	1		...								
1	0	1	1	1	0	1	0	0	0	0	1	1	1	1
1	0	0	.	1				...						
0	1	1	.	1					...					
0	1	0	.	1						...				
0	0	1	.	1							...			
0	0	0	.	1								...		

In this case, working out the other rows shows that ϕ is indeed valid, thus $\vDash \phi$.

Theorem 2.2.1 (Soundness and Completeness of Propositional Calculus)**.** *Let Φ be any set of formulas and ψ any formula, then*

$$\Phi \vdash \psi \iff \Phi \vDash \psi.$$

In particular, for any propositional formula ϕ we have

$$\vdash \phi \iff \vDash \phi.$$

Thus derivability and validity coincide.

Proof. Let us sketch the usual proof, for later reference.

(*Soundness*:) First assume $\Phi \vdash \psi$, meaning there is a finite derivation of ψ, all of the hypotheses of which are in the set Φ. Take a valuation v such that $v \vDash \Phi$, meaning that $v \vDash \phi$ for all $\phi \in \Phi$. Observe that for each rule of inference, for any valuation v, if $v \vDash \vartheta$ for all the hypotheses of the rule, then $v \vDash \gamma$ for the conclusion. By induction on the derivations therefore $v \vDash \phi$.

(*Competeness*:) Suppose that $\Phi \nvdash \psi$, then $\Phi, \neg\psi \nvdash \bot$ (using double negation elimination). By Lemma 2.2.2 below, there is a valuation v such that $v \vDash \{\Phi, \neg\psi\}$. Thus in particular $v \vDash \Phi$ and $v \nvDash \psi$, therefore $\Phi \nvDash \psi$. □

The key lemma is this:

Lemma 2.2.2 (Model Existence)**.** *A set Φ of formulas is* consistent*, $\Phi \nvdash \bot$, just if it has a* model*, i.e. a valuation v such that $v \vDash \Phi$.*

Proof. Let Φ be any consistent set of formulas. We extend $\Phi \subseteq \Psi$ to one that is *maximally consistent*, meaning that for every formula ψ, either $\psi \in \Psi$ or $\neg\psi \in \Psi$ and not both. Enumerate the formulas $\phi_0, \phi_1, ...$, and let,

$$\begin{aligned} \Phi_0 &= \Phi, \\ \Phi_{n+1} &= \Phi_n \cup \phi_n \text{ if consistent, else } \Phi_n, \\ \Psi &= \textstyle\bigcup_n \Phi_n. \end{aligned}$$

Now for each propositional variable $\mathtt{p}$, define $v(\mathtt{p}) = 1$ just if $\mathtt{p} \in \Psi$. □

2.3 Boolean algebra

There is of course another apporach to propositional logic, which also goes back to the 19th century, namely that of Boolean algebra, which draws on the analogy between the propositional operations and the arithmetical ones.

Definition 2.3.1. A *Boolean algebra* is a set B equipped with the operations:

$$\begin{aligned} 0, 1 &: 1 \to B \\ \neg &: B \to B \\ \wedge, \vee &: B \times B \to B \end{aligned}$$

satisfying the following equations:

$$\begin{aligned}
x \vee x &= x & x \wedge x &= x \\
x \vee y &= y \vee x & x \wedge y &= y \wedge x \\
x \vee (y \vee z) &= (x \vee y) \vee z & x \wedge (y \wedge z) &= (x \wedge y) \wedge z \\
x \wedge (y \vee z) &= (x \wedge y) \vee (x \wedge z) & x \vee (y \wedge z) &= (x \vee y) \wedge (x \vee z) \\
0 \vee x &= x & 1 \wedge x &= x \\
1 \vee x &= 1 & 0 \wedge x &= 0 \\
\neg(x \vee y) &= \neg x \wedge \neg y & \neg(x \wedge y) &= \neg x \vee \neg y \\
x \vee \neg x &= 1 & x \wedge \neg x &= 0
\end{aligned}$$

This is of course an algebraic theory, like those considered in the previous chapter. Familiar examples of Boolean algebras are $2 = \{0, 1\}$, with the usual operations, and more generally, any powerset $\mathcal{P}X$, with the set-theoretic operations $A \vee B = A \cup B$, etc. (indeed, $2 = \mathcal{P}1$ is a special case.).

Exercise 2.3.2. Show that the free Boolean algebra $B(n)$ on n-many generators is the double powerset $\mathcal{PP}(n)$, and determine the free functor on finite sets.

One can use equational reasoning in Boolean algebra as an alternative to the deductive propositional calculus as follows. For a propositional formula in context $\Gamma \mid \phi$, let us say that ϕ is *equationally provable* if we can prove $\phi = 1$ by equational reasoning (Section **??**), from the laws of Boolean algebras above. More generally, for a set of formulas Φ and a formula ψ let us define the *ad hoc* relation of *equational provability*,

$$\Phi \vdash^{=} \psi \tag{2.2}$$

to mean that $\psi = 1$ can be proven equationally from (the Boolean equations and) the set of all equations $\phi = 1$, for $\phi \in \Phi$. Since we don't have any laws for the connectives $\Rightarrow$ or $\Leftrightarrow$, let us replace them with their Boolean equivalents, by adding the equations:

$$\begin{aligned}
&\phi \Rightarrow \psi = \neg\phi \vee \psi\,, \\
&\phi \Leftrightarrow \psi = (\neg\phi \vee \psi) \wedge (\neg\psi \vee \phi)\,.
\end{aligned}$$

For example, here is an equational proof of $(\phi \Rightarrow \psi) \vee (\psi \Rightarrow \phi)$.

$$\begin{aligned}
(\phi \Rightarrow \psi) \vee (\psi \Rightarrow \phi) &= (\neg\phi \vee \psi) \vee (\neg\psi \vee \phi) \\
&= \neg\phi \vee (\psi \vee (\neg\psi \vee \phi)) \\
&= \neg\phi \vee ((\psi \vee \neg\psi) \vee \phi) \\
&= \neg\phi \vee (1 \vee \phi) \\
&= \neg\phi \vee 1 \\
&= 1 \vee \neg\phi \\
&= 1
\end{aligned}$$

Thus,

$$\vdash^{=} (\phi \Rightarrow \psi) \vee (\psi \Rightarrow \phi)\,.$$

We now ask: *What is the relationship between equational provability* $\Phi \vdash^{=} \phi$*, deductive entailment* $\Phi \vdash \phi$*, and semantic entailment* $\Phi \vDash \phi$*?*

Exercise 2.3.3. Using equational reasoning, show that every propositional formula ϕ has both a *conjunctive* $\phi^{\wedge}$ and a *disjunctive* $\phi^{\vee}$ *Boolean normal form* such that:

1. The formula $\phi^{\vee}$ is an n-fold disjunction of m-fold conjunctions of *positive* $\mathtt{p_i}$ or *negative* $\neg\mathtt{p_j}$ propositional variables,

$$\phi^{\vee} \;=\; (\mathtt{q}_{11} \wedge ... \wedge \mathtt{q}_{1m_1}) \vee ... \vee (\mathtt{q}_{n1} \wedge ... \wedge \mathtt{q}_{nm_n})\,, \qquad \mathtt{q}_{ij} \in \{\mathtt{p}_{ij}, \neg\mathtt{p}_{ij}\}\,,$$

   and $\phi^{\wedge}$ is the same, but with the roles of $\vee$ and $\wedge$ reversed.

2. Both

$$\vdash^{=} \phi \Leftrightarrow \phi^{\vee} \qquad \text{and} \qquad \vdash^{=} \phi \Leftrightarrow \phi^{\wedge}\,.$$

Exercise 2.3.4. Using Exercise 2.3.3, show that for every propositional formula ϕ, equational provability is equivalent to semantic validity,

$$\vdash^{=} \phi \iff \vDash \phi\,.$$

Hint: Put ϕ into conjunctive normal form and read off a truth valuation that falsifies it, if there is one.

Exercise 2.3.5. A Boolean algebra can be partially ordered by defining $x \leq y$ as

$$x \leq y \iff x \vee y = y \qquad \text{or equivalently} \qquad x \leq y \iff x \wedge y = x\,.$$

Thus a Boolean algebra is a (poset) category. Show that as a category, a Boolean algebra has all finite limits and colimits and is cartesian closed, and that a finitely complete and cocomplete cartesian closed poset is a Boolean algebra just if it satisfies $x = (x \Rightarrow 0) \Rightarrow 0$, where, as before, we define $x \Rightarrow y := \neg x \vee y$. Finally, show that homomorphisms of Boolean algebras $f : B \to B'$ are the same thing as functors (i.e. monotone maps) that preserve all finite limits and colimits.

2.4 Lawvere duality for Boolean algebras

Let us apply the machinery of algebraic theories from Chapter **??** to the algebraic theory of Boolean algebras and see what we get. The algebraic theory $\mathbb{B}$ of Boolean algebras is a finite product (FP) category with objects $1, B, B^2, ...$, containing a Boolean algebra $\mathcal{B}$, with underlying object $|\mathcal{B}| = B$. By Theorem **??**, $\mathbb{B}$ has the universal property that finite product preserving (FP) functors from $\mathbb{B}$ into any FP-category $\mathbb{C}$ correspond (pseudo-)naturally to Boolean algebras in $\mathbb{C}$,

$$\mathsf{Hom}_{\mathsf{FP}}(\mathbb{B}, \mathbb{C}) \simeq \mathsf{BA}(\mathbb{C}) . \tag{2.3}$$

The correspondence is mediated by evaluating an FP functor $F : \mathbb{B} \to \mathbb{C}$ at (the underlying structure of) the Boolean algebra $\mathcal{B}$ to get a Boolean algebra $F(\mathcal{B}) = \mathsf{BA}(F)(\mathcal{B})$ in $\mathbb{C}$:

$$\frac{F : \mathbb{B} \longrightarrow \mathbb{C} \qquad \mathsf{FP}}{\overline{F(\mathcal{B}) \qquad\qquad \mathsf{BA}(\mathbb{C})}}$$

We call $\mathcal{B}$ the *universal Boolean algebra*. Given a Boolean algebra $\mathcal{A}$ in $\mathbb{C}$, we write

$$\mathcal{A}^\sharp : \mathbb{B} \longrightarrow \mathbb{C}$$

for the associated *classifying functor*. By the equivalence of categories (2.3), we have isos,

$$\mathcal{A}^\sharp(\mathcal{B}) \cong \mathcal{A}, \qquad F(\mathcal{B})^\sharp \cong F .$$

And in particular, $\mathcal{B}^\sharp \cong 1_{\mathbb{B}} : \mathbb{B} \to \mathbb{B}$.

By Lawvere duality, Corollary **??**, we know that $\mathbb{B}^{\mathsf{op}}$ can be identified with a full subcategory $\mathsf{mod}(\mathbb{B})$ of $\mathbb{B}$-models in Set (i.e. Boolean algebras),

$$\mathbb{B}^{\mathsf{op}} = \mathsf{mod}(\mathbb{B}) \hookrightarrow \mathsf{Mod}(\mathbb{B}) = \mathsf{BA}(\mathsf{Set}) , \tag{2.4}$$

namely, that consisting of the finitely generated free Boolean algebras $F(n)$. Composing (2.4) and (2.3), we have an embedding of $\mathbb{B}^{\mathsf{op}}$ into the functor category,

$$\mathbb{B}^{\mathsf{op}} \hookrightarrow \mathsf{BA}(\mathsf{Set}) \simeq \mathsf{Hom}_{\mathsf{FP}}(\mathbb{B}, \mathsf{Set}) \hookrightarrow \mathsf{Set}^{\mathbb{B}} , \tag{2.5}$$

which, up to isomorphism, is just the (contravariant) Yoneda embedding, taking $B^n \in \mathbb{B}$ to the covariant representable functor $\mathsf{y}^{\mathbb{B}}(B^n) = \mathsf{Hom}_{\mathbb{B}}(B^n, -)$ (cf. Theorem **??**).

Now consider provability of equations between terms $\phi : B^k \to B$ in the theory $\mathbb{B}$, which are essentially the same as propositional formulas in context $(\mathsf{p}_1, ..., \mathsf{p}_k \mid \phi)$ modulo $\mathbb{B}$-provable equality. The universal Boolean algebra $\mathcal{B}$ is logically generic, in the sense that for any such formulas ϕ, ψ, we have $\mathcal{B} \vDash \phi = \psi$ just if $\mathbb{B} \vdash \phi = \psi$ (Proposition **??**). The latter condition is equational provability from the axioms for Boolean algebras, which is just what was used in the definition of $\vdash^{=} \phi$ (cf. 2.2). Thus, in particular,

$$\vdash^{=} \phi \iff \mathbb{B} \vdash \phi = 1 \iff \mathcal{B} \vDash \phi = 1 .$$

As we showed in Proposition **??**, the image of the universal model $\mathcal{B}$ under the (FP) *covariant* Yoneda embedding,

$$\mathsf{y}_{\mathbb{B}} : \mathbb{B} \to \mathsf{Set}^{\mathbb{B}^{\mathsf{op}}}$$

is also a logically generic model, with underlying object $|\mathsf{y}_{\mathbb{B}}(\mathcal{B})| = \mathsf{Hom}_{\mathbb{B}}(-, B)$. By Proposition **??** we can use that fact to restrict attention to Boolean algebras in Set, and in particular, to the finitely generated free ones $F(n)$, when testing for equational provability. Specifically, using the (FP) evaluation functors $\mathsf{eval}_{B^n} : \mathsf{Set}^{\mathbb{B}^{\mathsf{op}}} \to \mathsf{Set}$ for all objects $B^n \in \mathbb{B}$, we can extend the above reasoning as follows:

$$\begin{aligned} \vdash^{=} \phi &\iff \mathbb{B} \vdash \phi = 1 \\ &\iff \mathcal{B} \vDash \phi = 1 \\ &\iff \mathsf{y}_{\mathbb{B}}(\mathcal{B}) \vDash \phi = 1 \\ &\iff \mathsf{eval}_{B^n}\mathsf{y}_{\mathbb{B}}(\mathcal{B}) \vDash \phi = 1 \quad \text{for all } B^n \in \mathbb{B} \\ &\iff F(n) \vDash \phi = 1 \quad \text{for all } n. \end{aligned}$$

The last step holds because the image of $\mathsf{y}_{\mathbb{B}}(\mathcal{B})$ under eval_{B^n} is the free Boolean algebra $F(n)$ (cf. Exercise **??**). Indeed, for the underlying objects we have

$$\begin{aligned} |\mathsf{eval}_{B^n}\mathsf{y}_{\mathbb{B}}(\mathcal{B})| &\cong \mathsf{eval}_{B^n}|\mathsf{y}_{\mathbb{B}}(\mathcal{B})| \cong \mathsf{eval}_{B^n}\mathsf{y}_{\mathbb{B}}(|\mathcal{B}|) \cong \mathsf{eval}_{B^n}\mathsf{y}_{\mathbb{B}}(B) \cong \mathsf{y}_{\mathbb{B}}(B)(B^n) \\ &\cong \mathsf{Hom}_{\mathbb{B}}(B^n, B) \cong \mathsf{Hom}_{\mathsf{BA}^{\mathsf{op}}}(F(n), F(1)) \cong \mathsf{Hom}_{\mathsf{BA}}(F(1), F(n)) \cong |F(n)|\,. \end{aligned}$$

Thus to test for equational provability it suffices to check the equations in the free algebras $F(n)$ (which makes sense, since these are usually *defined* in terms of equational provability). We have therefore shown:

Lemma 2.4.1. *A formula in context* $\mathsf{p}_1, ..., \mathsf{p}_k \mid \phi$ *is equationally provable* $\vdash_{=} \phi$ *just in case, for every free Boolean algebra* $F(n)$*, we have* $F(n) \vDash \phi = 1$.

The condition $F(n) \vDash \phi = 1$ means that the equation $\phi = 1$ holds *generally* in $F(n)$, i.e. for any elements $f_1, ..., f_k \in F(n)$, we have $\phi[f_1/\mathsf{p}_1, ..., f_k/\mathsf{p}_k] = 1$, where the expression $\phi[f_1/\mathsf{p}_1, ..., f_k/\mathsf{p}_k]$ denotes the element of $F(n)$ resulting from interpreting the propositional variables p_i as the elements f_i and evaluating the resulting expression using the Boolean operations of $F(n)$. But now observe that the recipe:

> for any elements $f_1, ..., f_k \in F(n)$, let the expression
>
> $$\phi[f_1/\mathsf{p}_1, ..., f_k/\mathsf{p}_k] \tag{2.6}$$
>
> denote the element of $F(n)$ resulting from interpreting the propositional variables p_i as the elements f_i and evaluating the resulting expression using the Boolean operations of $F(n)$

describes the unique Boolean homomorphism

$$F(1) \xrightarrow{\overline{\phi}} F(k) \xrightarrow{\overline{(f_1, ..., f_k)}} F(n)\,,$$

where $\overline{(f_1, ..., f_k)} : F(k) \to F(n)$ is determined by the elements $f_1, ..., f_k \in F(n)$, and $\overline{\phi} : F(1) \to F(k)$ by the corresponding element $(\mathsf{p}_1, ..., \mathsf{p}_k \mid \phi) \in F(k)$. It is therefore equivalent to check the case $k = n$ and $f_i = \mathsf{p}_i$, i.e. the "universal case"

$$(\mathsf{p}_1, ..., \mathsf{p}_k \mid \phi) = 1 \quad \text{in } F(k)\,. \tag{2.7}$$

Finally, we then have:

Proposition 2.4.2 (Completeness of the equational propositional calculus)**.** *Equational propositional calculus is sound and complete with respect to* boolean-valued models *in* Set*, in the sense that a propositional formula* ϕ *is equationally provable from the laws of Boolean algebra,*

$$\vdash^{=} \phi \,,$$

just if it holds generally in any Boolean algebra (in Set*).*

Proof. By "holding generally" is meant the universal quantification of the equation over elements of a given Boolean algebra B, which is of course equivalent to saying that it holds for all elements of B, in the sense stated after the Lemma. But, as above, this is equivalent to the condition that for all $b_1, ..., b_k \in B$, for $\overline{(b_1, ..., b_k)} : F(k) \to B$ we have $\overline{(b_1, ..., b_k)}(\phi) = 1$ in B, which in turn is clearly equivalent to the previously determined "universal" condition (2.7) that $\phi = 1$ in $F(k)$. □

The analogous statement for equational entailment $\Phi \vdash_{=} \phi$ is left as an exercise.

Corollary 2.4.2 is a (very) special case of the Gödel completeness theorem for first-order logic, for *just* the equational fragment of *just* the specific theory of Boolean algebras (although, an analogous result of course holds for any other algebraic theory, and many other systems of logic can be reduced to the algebraic case). Nonetheless, it suggests another approach to the semantics of propositional logic based upon the idea of a *Boolean valuation*, generalizing the traditional truth-value semantics from Section 2.2. We pursue this idea systematically in the following section.

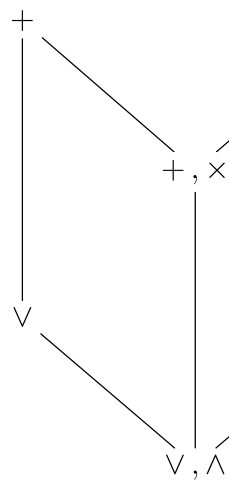
Exercise 2.4.3. For a formula in context $\mathtt{p}_1, ..., \mathtt{p}_k \mid \vartheta$ and a Boolean algebra $\mathcal{A}$, let the expression $\vartheta[a_1/\mathtt{p}_1, ..., a_k/\mathtt{p}_k]$ denote the element of $\mathcal{A}$ resulting from interpreting the propositional variables $\mathtt{p}_i$ in the context as the elements a_i of $\mathcal{A}$, and evaluating the resulting expression using the Boolean operations of $\mathcal{A}$. For any *finite* set of propositional formulas Φ and any formula ψ, let $\Gamma = \mathtt{p}_1, ..., \mathtt{p}_k$ be a context for (the formulas in) $\Phi \cup \{\psi\}$. Finally, recall that $\Phi \vdash^{=} \psi$ means that $\psi = 1$ is equationally provable from the set of equations $\{\phi = 1 \mid \phi \in \Phi\}$. Show that $\Phi \vdash^{=} \psi$ just if for all finitely generated free Boolean algebras $F(n)$, the following condition holds:

> For any elements $f_1, ..., f_k \in F(n)$, if $\phi[f_1/\mathtt{p}_1, ..., f_k/\mathtt{p}_k] = 1$ for all $\phi \in \Phi$, then $\psi[f_1/\mathtt{p}_1, ..., f_k/\mathtt{p}_k] = 1$.

Is it sufficient to just take $F(k)$ and its generators $\mathtt{p}_1, ..., \mathtt{p}_k$ as the $f_1, ..., f_k$? Is it equivalent to take all Boolean algebras B, rather than the finitely generated free ones $F(n)$? Determine a condition that is equivalent to $\Phi \vdash^{=} \psi$ for not necessarily finite sets Φ.

2.5 Functorial semantics for propositional logic

Considering the algebraic theory of Boolean algebras suggests the idea of a Boolean valuation of propositional logic, generalizing the truth valuations of section 2.2. This can be seen as applying the framework of functorial semantics to a different system of logic than that of finite product categories, namely that represented categorically by *poset* categories with finite products $\wedge$ *and coproducts* $\vee$ (each of these specializations could, of course, also be considered separately, giving $\wedge$-semi-lattices and categories with finite products $\times$ and coproducts $+$, respectively). Thus we are moving from the top right corner to the bottom center position in the following Hasse diagram of structured categories:



In Chapter **??** we shall see how first-order logic results categorically from these two cases by "indexing the lower one over the upper one".

Definition 2.5.1. A *propositional theory* $\mathbb{T}$ consists of a set $V_{\mathbb{T}}$ of propositional variables, called the *basic* or *atomic propositions*, and a set $A_{\mathbb{T}}$ of propositional formulas (over $V_{\mathbb{T}}$), called the *axioms*. The *consequences* $\Phi \vdash_{\mathbb{T}} \phi$ are those judgements that are derivable by natural deduction (as in Section 2.1), from the axioms $A_{\mathbb{T}}$.

Definition 2.5.2. Let $\mathbb{T} = (V_{\mathbb{T}}, A_{\mathbb{T}})$ be a propositional theory and $\mathcal{B}$ a Boolean algebra. A *model* of $\mathbb{T}$ in $\mathcal{B}$, also called a *Boolean valuation of* $\mathbb{T}$ is an *interpretation function* $v : V_{\mathbb{T}} \to |\mathcal{B}|$ such that, for every $\alpha \in A_{\mathbb{T}}$, we have $[\![\alpha]\!]^v = 1_{\mathcal{B}}$ in $\mathcal{B}$, where the extension $[\![-]\!]^v$ of v from $V_{\mathbb{T}}$ to all formulas (over $V_{\mathbb{T}}$) is defined in the expected way, namely:

$$\begin{aligned}
[\![\mathtt{p}]\!]^v &= v(\mathtt{p}), \quad \mathtt{p} \in V_{\mathbb{T}} \\
[\![\top]\!]^v &= 1_{\mathcal{B}} \\
[\![\bot]\!]^v &= 0_{\mathcal{B}} \\
[\![\neg\phi]\!]^v &= \neg_{\mathcal{B}}[\![\phi]\!]^v \\
[\![\phi \wedge \psi]\!]^v &= [\![\phi]\!]^v \wedge_{\mathcal{B}} [\![\psi]\!]^v \\
[\![\phi \vee \psi]\!]^v &= [\![\phi]\!]^v \vee_{\mathcal{B}} [\![\psi]\!]^v \\
[\![\phi \Rightarrow \psi]\!]^v &= \neg_{\mathcal{B}}[\![\phi]\!]^v \vee_{\mathcal{B}} [\![\psi]\!]^v
\end{aligned}$$

Finally, let $\mathsf{Mod}(\mathbb{T}, \mathcal{B})$ be the set of all $\mathbb{T}$-models in $\mathcal{B}$. Given a Boolean homomorphism $f : \mathcal{B} \to \mathcal{B}'$, there is an induced mapping $\mathsf{Mod}(\mathbb{T}, f) : \mathsf{Mod}(\mathbb{T}, \mathcal{B}) \to \mathsf{Mod}(\mathbb{T}, \mathcal{B}')$, determined by setting $\mathsf{Mod}(\mathbb{T}, f)(v) = f \circ v$, which is clearly functorial.

Theorem 2.5.3. *The functor* $\mathsf{Mod}(\mathbb{T}) : \mathsf{BA} \to \mathsf{Set}$ *is representable, with representing Boolean algebra* $\mathcal{B}_{\mathbb{T}}$*, called the* Lindenbaum-Tarski algebra *of* $\mathbb{T}$*.*

Proof. We construct $\mathcal{B}_{\mathbb{T}}$ in two steps:

Step 1: Suppose first that $A_{\mathbb{T}}$ is empty, so $\mathbb{T}$ is just a set V of propositional variables. Define the *Lindenbaum-Tarski algebra* $\mathcal{B}[V]$ by

$$\mathcal{B}[V] \;=\; \{\phi \mid \phi \text{ is a formula in context } V\}/\sim$$

where the equivalence relation $\sim$ is *(deductively) provable bi-implication*,

$$\phi \sim \psi \iff \vdash \psi \Leftrightarrow \psi.$$

The operations are (well-)defined on equivalence classes by setting,

$$[\phi] \wedge [\psi] \;=\; [\phi \wedge \psi],$$

and so on. (The reader who has not seen this construction before should fill in the details!)

Step 2: In the general case $\mathbb{T} = (V_{\mathbb{T}}, A_{\mathbb{T}})$, let

$$\mathcal{B}_{\mathbb{T}} = \mathcal{B}[V_{\mathbb{T}}]/\sim_{\mathbb{T}},$$

where the equivalence relation $\sim_{\mathbb{T}}$ is now $A_{\mathbb{T}}$*-provable bi-implication*,

$$\phi \sim_{\mathbb{T}} \psi \iff A_{\mathbb{T}} \vdash \psi \Leftrightarrow \psi.$$

The operations are defined as before, but now on equivalence classes $[\phi]$ modulo $A_{\mathbb{T}}$.

Now observe that the construction of $\mathcal{B}_{\mathbb{T}}$ is a variation on that of the *syntactic category* $\mathcal{C}_{\mathbb{T}}$ of the algebraic theory $\mathbb{T}$ in the sense of the previous chapter, and the statement of the theorem is its universal property as the classifying category of $\mathbb{T}$-models, namely

$$\mathsf{Mod}(\mathbb{T}, \mathcal{B}) \cong \mathsf{Hom}_{\mathsf{BA}}(\mathcal{B}_{\mathbb{T}}, \mathcal{B}), \tag{2.8}$$

naturally in $\mathcal{B}$. (Indeed, since $\mathsf{Mod}(\mathbb{T}, \mathcal{B})$ is now a *set* rather than a category, we can classify it up to *isomorphism* rather than equivalence of categories.) The proof of this fact is a variation on the proof of the corresponding theorem **??** from Chapter 1. Further details are given in the following Remark 2.5.4 for the interested reader. □

Remark 2.5.4 (Adjoint Rules for Propositional Calculus)**.** For the construction of the Lindenbaum-Tarski algebra $\mathcal{B}_{\mathbb{T}}$, it is convenient to reformulate the rules of inference for the propositional calculus in the following equivalent *adjoint form*:

Contexts Γ may be omitted, since the rules leave them unchanged (there is no variable binding). We may also omit hypotheses that remain unchanged. Thus e.g. the *hypothesis* rule may be written in any of the following equivalent ways.

$$\frac{}{\Gamma \mid \phi_1, \ldots, \phi_m \vdash \phi_i} \qquad \frac{}{\phi_1, \ldots, \phi_m \vdash \phi_i} \qquad \frac{}{\phi \vdash \phi}$$

The structural rules can then be stated as follows:

$$\frac{}{\phi \vdash \phi} \qquad \frac{\phi \vdash \psi \qquad \psi \vdash \vartheta}{\phi \vdash \vartheta}$$

$$\frac{\phi \vdash \vartheta}{\psi, \phi \vdash \vartheta} \qquad \frac{\phi, \phi \vdash \vartheta}{\phi \vdash \vartheta} \qquad \frac{\phi, \psi \vdash \vartheta}{\psi, \phi \vdash \vartheta}$$

The rules for the propositional connectives can be given in the following adjoint form, where the double line indicates a two-way rule (with the obvious two instances when there are two conclusions).

$$\frac{}{\phi \vdash \top} \qquad \frac{}{\bot \vdash \phi}$$

$$\frac{\vartheta \vdash \phi \quad \vartheta \vdash \psi}{\vartheta \vdash \phi \wedge \psi} \qquad \frac{\phi \vdash \vartheta \quad \psi \vdash \vartheta}{\phi \vee \psi \vdash \vartheta} \qquad \frac{\vartheta, \phi \vdash \psi}{\vartheta \vdash \phi \Rightarrow \psi}$$

For the purpose of deduction, negation $\neg\phi$ is again treated as defined by $\phi \Rightarrow \bot$ and bi-implication $\phi \Leftrightarrow \psi$ by $(\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi)$. For *classical* logic we also include the rule of *double negation*:

$$\frac{}{\neg\neg\phi \vdash \phi} \tag{2.9}$$

It is now obvious that the set of formulas is preordered by $\phi \vdash \psi$, and that the poset reflection agrees with the deducibility equivalence relation,

$$\phi \dashv\vdash \psi \iff \phi \sim \psi\,.$$

Moreover, $\mathcal{B}_{\mathbb{T}}$ clearly has all finite limits $\top, \wedge$ and colimits $\bot, \vee$, is cartesian closed $\wedge \dashv \Rightarrow$, and is therefore a *Heyting algebra* (see Section **??** below). The rule of double negation then makes it a Boolean algebra.

The proof of the universal property of $\mathcal{B}_{\mathbb{T}}$ is essentially the same as that for $\mathcal{C}_{\mathbb{T}}$.

Exercise 2.5.5. Fill in the details of the proof that $\mathcal{B}_{\mathbb{T}}$ is a well-defined Boolean algebra, with the universal property stated in (2.8).

Just as for the case of algebraic theories and FP categories, we now have the following corollary of the classifying theorem 2.5.3. (Note that the recipe at (2.6) for a Boolean valuation in $F(n)$ of the formula in context $\mathtt{p}_1, ..., \mathtt{p}_k \mid \phi$ is exactly a *model* in $F(n)$ of the theory $\mathbb{T} = \{\mathtt{p}_1, ..., \mathtt{p}_k\}$.)

Corollary 2.5.6. *For any set of formulas Φ and formula ϕ, derivability $\Phi \vdash \phi$ is equivalent to validity under all Boolean valuations. Therefore by Proposition 2.4.2 (and Exercise 2.4.3), we also have*

$$\Phi \vdash \phi \iff \Phi \vdash^{=} \phi\,.$$

Remark 2.5.7. If $A_{\mathbb{T}}$ is non-empty, but finite, then let

$$\alpha_{\mathbb{T}} \;:=\; \bigwedge_{\alpha \in A_{\mathbb{T}}} \alpha\,.$$

We then have

$$\mathcal{B}_{\mathbb{T}} = \mathcal{B}[V_{\mathbb{T}}]/\alpha_{\mathbb{T}}\,,$$

where as usual $\mathcal{B}/b$ denotes the slice category of the Boolean algebra $\mathcal{B}$ over an element $b \in \mathcal{B}$.

Remark 2.5.8. Our definition of the Lindenbaum-Tarski algebra is given in terms of *provability*, rather than the more familiar semantic definition using (truth) valuations. The two are, of course, equivalent in light of Theorem 2.2.1, but since we intend to prove that theorem, this definition will be more useful, as it parallels that of the syntactic category $\mathcal{C}_{\mathbb{T}}$ of an algebraic theory.

Inspecting the universal property (2.8) of $\mathcal{B}_{\mathbb{T}}$ for the case $\mathcal{B}[V]$ where there are no axioms, we now have the following.

Corollary 2.5.9. *The Lindenbaum-Tarski algebra $\mathcal{B}[V]$ is the free Boolean algebra on the set V. In particular, $\mathcal{B}[\mathtt{p}_1, ..., \mathtt{p}_n]$ is the finitely generated, free Boolean algebra $F(n)$.*

The isomorphism $\mathcal{B}[\mathtt{p}_1, ..., \mathtt{p}_n] \cong F(n)$ expresses the fact recorded in Corollary 2.5.6 that the relations of derivability by natural deduction $\Phi \vdash \phi$ and equational provability $\Phi \vdash^{=} \phi$ agree — answering part of the question at the end of Section **??**.

Exercise 2.5.10. Show that the Boolean algebras $\mathcal{B}_{\mathbb{T}}$ for *finite sets* $V_{\mathbb{T}}$ of variables and $A_{\mathbb{T}}$ of formulas are exactly the *finitely presented* ones.

Finally, we can use the following to finish the comparison of $\vdash \phi$ and $\vDash \phi$.

Lemma 2.5.11. *Let $\mathcal{B}$ be a* finitely presented *Boolean algebra in which $0 \neq 1$. Then there is a Boolean homomorphism*

$$h : \mathcal{B} \to \mathbb{2}\,.$$

Proof. By Exercise 2.5.10, we can assume that $\mathcal{B} = \mathcal{B}[\mathtt{p}_1 ... \mathtt{p}_n]/\alpha$ classifying the theory $\mathbb{T} = (\mathtt{p}_1 ... \mathtt{p}_n, \alpha)$. By the assumption that $0 \neq 1$ in $\mathbb{B} = \mathcal{B}[\mathtt{p}_1 ... \mathtt{p}_n]/\alpha$, we have $\alpha \neq 0$ in the free Boolean algebra $F(n) \cong \mathcal{B}[\mathtt{p}_1 ... \mathtt{p}_n]$, whence $\alpha \nvdash \bot$. Since $F(n) \cong \mathcal{P}\mathcal{P}(n)$, there is a valuation $\vartheta : \{\mathtt{p}_1 ... \mathtt{p}_n\} \to \mathbb{2}$ such that $[\![\alpha]\!]^{\vartheta} = 1$. This is exactly a Boolean homomorphism $\mathcal{B}[\mathtt{p}_1 ... \mathtt{p}_n]/\alpha \to \mathbb{2}$, as required. □

Corollary 2.5.12. *For any set of formulas Φ and formula ϕ, derivability $\Phi \vdash \phi$ is equivalent to semantic entailment,*

$$\Phi \vDash \phi \iff \Phi \vdash \phi\,.$$

Proof. By 2.5.6, it suffices to show that $\Phi \vDash \phi$ is equivalent to $\Phi \vdash^{=} \phi$, but the latter we know to be equivalent to holding in all Boolean valuations in free Boolean algebras $F(n)$, and the former to holding in all *truth* valuations, i.e. Boolean valuations in $\mathbb{2}$. Thus it will suffice to embed $F(n)$ as a Boolean algebra into a powerset $\mathcal{P}X = \mathbb{2}^X$, for a set X. By Lemma 2.5.11 we can take $X = 2^n$. □

2.6 Stone representation

Regarding a Boolean algebra $\mathcal{B}$ as a category with finite products, consider its Yoneda embedding $\mathsf{y} : \mathcal{B} \hookrightarrow \mathsf{Set}^{\mathcal{B}^{\mathsf{op}}}$. Since the hom-set $\mathcal{B}(x, y)$ is $\mathbb{2}$-valued, we have a factorization,

$$\mathcal{B} \hookrightarrow \mathbb{2}^{\mathcal{B}^{\mathsf{op}}} \hookrightarrow \mathsf{Set}^{\mathcal{B}^{\mathsf{op}}}$$

in which each factor still preserves the finite products (note that the products in $\mathbb{2}$ are preserved by the inclusion $\mathbb{2} \hookrightarrow \mathsf{Set}$, and the products in the functor categories are taken pointwise). Indeed, this is an instance of a general fact. In the category $\mathsf{Cat}_\times$ of finite product categories (and $\times$-preserving functors), the inclusion of the full subcategory of posets with $\wedge$ (the $\wedge$-semilattices) has a *right adjoint* R, in addition to the left adjoint L of poset reflection.

$$\begin{tikzcd} \mathsf{Cat}_\times \arrow[d, bend right=60, "L"'] \arrow[d, bend left=60, "R"] \\ \mathsf{Pos}_\wedge \arrow[u, hook, "i"'] \end{tikzcd}$$

For a finite product category $\mathbb{C}$, the poset $R\mathbb{C}$ is the subcategory $\mathsf{Sub}(1) \hookrightarrow \mathbb{C}$ of subobjects of the terminal object 1 (equivalently, the category of monos $m : M \rightarrowtail 1$). The reason for this is that a $\times$-preserving functor $f : A \to \mathbb{C}$ from a poset A with meets takes every object $a \in A$ to a mono $f(a) \rightarrowtail 1$ in $\mathbb{C}$, since the following is a product diagram in A.

$$\begin{tikzcd} a \arrow[r] & 1 \\ a \arrow[u] \arrow[r] & a \arrow[u] \end{tikzcd}$$

Exercise 2.6.1. Prove this, and use it to verify that $R = \mathsf{Sub}(1)$ is indeed a right adjoint to the inclusion of $\wedge$-semilattices into finite-product categories.

Now the functor category $\mathbb{2}^{\mathcal{B}^{\mathsf{op}}} = \mathsf{Pos}(\mathcal{B}^{\mathsf{op}}, \mathbb{2})$ of all *contravariant*, monotone maps $\mathcal{B}^{\mathsf{op}} \to \mathbb{2}$ (which indeed is $\mathsf{Sub}(1) \hookrightarrow \mathsf{Set}^{\mathbb{C}^{\mathsf{op}}}$) is easily seen to be isomorphic to the poset $\downarrow\!\mathcal{B}$ of all *sieves* (or "downsets") in $\mathcal{B}$: subsets $S \subseteq \mathcal{B}$ that are downward closed, $x \leq y \in S \Rightarrow x \in S$, ordered by subset inclusion $S \subseteq T$. Explicitly, the isomorphism

$$\mathsf{Pos}(\mathcal{B}^{\mathsf{op}}, \mathbb{2}) \;\cong \downarrow\!\mathcal{B} \tag{2.10}$$

is given by taking $f : \mathcal{B}^{\mathsf{op}} \to \mathbb{2}$ to $f^{-1}(1)$ and $S \subseteq \mathcal{B}$ to the function $f_S : \mathcal{B}^{\mathsf{op}} \to \mathbb{2}$ with $f_S(b) = 1 \Leftrightarrow b \in S$. Under this isomorphism, the Yoneda embedding takes an element $b \in \mathcal{B}$ *covariantly* to the *principal sieve* $\downarrow\! b \subseteq \mathcal{B}$ of all $x \leq b$.

Exercise 2.6.2. Show that (2.10) is indeed an isomorphism of posets, and that it takes the Yoneda embedding to the principal sieve mapping, as claimed.

For algebraic theories $\mathbb{A}$, we used the Yoneda embedding to give a completeness theorem for equational logic with respect to Set-valued models, by composing the (faithful functor)

$\mathsf{y} : \mathbb{A} \hookrightarrow \mathsf{Set}^{\mathbb{A}^{\mathsf{op}}}$ with the (jointly faithful) evaluation functors $\mathsf{eval}_A : \mathsf{Set}^{\mathbb{A}^{\mathsf{op}}} \to \mathsf{Set}$, for all objects $A \in \mathbb{A}$. This amounts to considering all *covariant* representables $\mathsf{eval}_A \circ \mathsf{y} = \mathbb{A}(A, -) : \mathbb{A} \to \mathsf{Set}$, and observing that these are then (both $\times$-preserving and) jointly faithful.

We can do the same thing for a Boolean algebra $\mathcal{B}$ (which is, after all, a finite product category) to get a jointly faithful family of $\times$-preserving, monotone maps $\mathcal{B}(b, -) : \mathcal{B} \to \mathbb{2}$, i.e. $\wedge$-semilattice homomorphisms. By taking the primages of $\{1\} \hookrightarrow \mathbb{2}$, such homomorphisms correspond to *filters* in $\mathcal{B}$: "upsets" that are also closed under $\wedge$. The representables then correspond to the *principal filters* $\uparrow b \subseteq \mathcal{B}$. The problem with using this approach for a completeness theorem for *propositional* logic is that such $\wedge$-homomorphisms $\mathcal{B} \to \mathbb{2}$ are not *models*, because they need not preserve the joins $\phi \vee \psi$ (nor the complements $\neg\phi$).

Lemma 2.6.3. *Let $\mathcal{B}, \mathcal{B}'$ be Boolean algebras and $f : \mathcal{B} \to \mathcal{B}'$ a distributive lattice homomorphism. Then f preserves negation, and so is Boolean. The category* Bool *of Boolean algebras is thus a full subcategory of the category* DLat *of distributive lattices.*

Proof. The complement $\neg b$ is the unique element of $\mathcal{B}$ such that both $b \vee \neg b = 1$ and $b \wedge \neg b = 0$. □

This suggests representing a Boolean algebra $\mathcal{B}$, not by its filters, but by its *prime* filters, which correspond bijectively to distributive lattice homomorphisms $\mathcal{B} \to \mathbb{2}$.

Definition 2.6.4. A filter $F \subseteq \mathcal{D}$ in a distributive lattice $\mathcal{D}$ is *prime* if $b \vee b' \in F$ implies $b \in F$ or $b' \in F$. Equivalently, just if the corresponding $\wedge$-semilattice homomorphism $f_F : \mathcal{B} \to \mathbb{2}$ is a lattice homomorphism.

If $\mathcal{B}$ is Boolean, it then follows that prime filters $F \subseteq \mathcal{B}$ are in bijection with Boolean homomorphisms $\mathcal{B} \to \mathbb{2}$, via the assignment $F \mapsto f_F : \mathcal{B} \to \mathbb{2}$ with $f_F(b) = 1 \Leftrightarrow b \in F$ and $(f : \mathcal{B} \to \mathbb{2}) \mapsto F_f := f^{-1}(1) \subseteq \mathcal{B}$. The prime filter F_f may be called the *(filter) kernel* of $f : \mathcal{B} \to \mathbb{2}$.

Proposition 2.6.5. *In a Boolean algebra $\mathcal{B}$, the following conditions on a subset $F \subseteq \mathcal{B}$ are equivalent.*

1. *F is a prime filter*
2. *the complement $\mathcal{B} \backslash F$ is a* prime ideal *(defined as a prime filter in $\mathcal{B}^{\mathsf{op}}$).*
3. *the complement $\mathcal{B} \backslash F$ is an* ideal *(defined as a filter in $\mathcal{B}^{\mathsf{op}}$).*
4. *F is a filter, and for each $b \in \mathcal{B}$, either $b \in F$ or $\neg b \in \mathcal{F}$ and not both.*
5. *F is a* maximal filter*: F is a filter and for all filters G, if $F \subseteq G$ then $F = G$ (also called an* ultrafilter*).*
6. *the map $f_F : \mathcal{B} \to \mathbb{2}$ given by $f_F(b) = 1 \Leftrightarrow b \in F$ (as in (2.10)) is a Boolean homomorphism.*

Proof. Exercise! □

Lemma 2.6.6. *Let $\mathcal{B}$ be a Boolean algebra, $I \subseteq \mathcal{B}$ an ideal, and $F \subseteq \mathcal{B}$ a filter, with $I \cap F = \emptyset$. There is a prime filter $P \supseteq F$ with $I \cap P = \emptyset$.*

Proof. Suppose first that $I = \{0\}$ is the trivial ideal, and that $\mathcal{B}$ is countable, with $b_0, b_1, ...$ an enumeration of its elements. As in the proof of the Model Existence Lemma, we build an increasing sequence of filters $F_0 \subseteq F_1 \subseteq \dots$ as follows:

$$F_0 = F$$
$$F_{n+1} = \begin{cases} F_n & \text{if } \neg b_n \in F_n \\ \{f \wedge b \,|\, f \in F_n,\ b_n \leq b\} & \text{otherwise} \end{cases}$$
$$P = \bigcup_n F_n$$

One then shows that each F_n is a filter, that $I \cap F_n = \emptyset$ for all n and so $I \cap P = \emptyset$, and that for each b_n, either $b_n \in P$ or $\neg b_n \in P$, whence P is prime.

For $I \subseteq \mathcal{B}$ a nontrivial ideal we take the quotient Boolean algebra $\mathcal{B} \twoheadrightarrow \mathcal{B}/I$, defined as the algebra of equivalence classes $[b]$ where $a \sim_I b \Leftrightarrow a \vee i = b \vee j$ for some $i, j \in I$. One shows that this is indeed a Boolean algebra and that the projection onto equivalence classes $\pi_I : \mathcal{B} \twoheadrightarrow \mathcal{B}/I$ is a Boolean homomorphism with (ideal) kernel $\pi^{-1}([0]) = I$. Now apply the foregoing argument to obtain a prime filter $P : \mathcal{B}/I \to \mathbb{2}$. The composite $p_I = P \circ \pi_I : \mathcal{B} \to \mathbb{2}$ is then a Boolean homomorphism with (filter) kernel $p_I^{-1}(1)$ which is prime, contains F and is disjoint from I.

The case where $\mathcal{B}$ is uncountable is left as an exercise. □

Exercise 2.6.7. Finish the proof by (i) verifying the construction of the quotient Boolean algebra $\mathcal{B} \twoheadrightarrow \mathcal{B}/I$, and (ii) considering the case where $\mathcal{B}$ is uncountable (*Hint*: either use Zorn's lemma, or well-order $\mathcal{B}$.)

Theorem 2.6.8 (Stone representation theorem)**.** *Let $\mathcal{B}$ be a Boolean algebra. There is an injective Boolean homomorphism $\mathcal{B} \rightarrowtail \mathcal{P}X$ into a powerset.*

Proof. Let X be the set of prime filters in $\mathcal{B}$ and consider the map $h : \mathcal{B} \to X$ given by $h(x) = \{F \,|\, x \in F\}$. Clearly $h(0) = \emptyset$ and $h(1) = X$. Moreover, for any filter F, we have $b \in F$ and $b' \in F$ if and only if $b \wedge b' \in F$, so $h(b \wedge b') = h(b) \cap h(b')$. If F is prime, then $b \in F$ or $b' \in F$ if and only if $b \vee b' \in F$, so $h(b \vee b') = h(b) \cup h(b')$. Thus h is a Boolean homomorphism. Let $a \neq b \in \mathcal{B}$, and we want to show that $h(a) \neq h(b)$. It suffices to assume that $a < b$ (otherwise, consider $a \wedge b$, for which we cannot have both $a \wedge b = a$ and $a \wedge b = b$). We seek a prime filter $P \subseteq \mathcal{B}$ with $b \in P$ but $a \notin P$. Apply Lemma 2.6.6 to the ideal $\downarrow a$ and the filter $\uparrow b$. □

2.7 Stone duality

Note that in the Stone representation $\mathcal{B} \rightarrowtail \mathcal{P}(X)$ the powerset Boolean algebra

$$X_{\mathcal{B}} = \mathsf{Set}\big(\mathsf{Bool}(\mathcal{B}, 2), 2\big)$$

is evidently (covariantly) functorial in $\mathcal{B}$, and has an apparent "double-dual" form $\mathcal{B}^{**}$. This suggests a "duality" between the categories Bool and Set,

$$\begin{tikzcd} \mathsf{Bool}^{\mathsf{op}} \arrow[r, bend left, "*"] & \mathsf{Set} \arrow[l, bend left, "*"] \end{tikzcd}$$

with contravariant functors $\mathcal{B}^* = X_{\mathcal{B}} = \mathsf{Bool}(\mathcal{B}, 2)$ for a Boolean algebra $\mathcal{B}$, and $S^* = \mathcal{P}S$, the powerset Boolean algebra, for a set S. In fact, we can do even better by topologizing each set $X_{\mathcal{B}}$ of prime filters $P \subseteq \mathcal{B}$ with basic open sets of the form

$$B_b = \{P \in X_{\mathcal{B}} \mid b \in P\}, \qquad b \in \mathcal{B} \tag{2.11}$$

These sets are closed under finite intersections, because $B_a \cap B_b = B_{a \wedge b}$. Indeed, if $P \in B_a \cap B_b$ then $a \in P$ and $b \in P$, whence $a \wedge b \in P$, and conversely.

Definition 2.7.1. For any Boolean algebra $\mathcal{B}$, the *Stone space* $X_{\mathcal{B}}$ is a topological space with the prime filters $P \subseteq \mathcal{B}$ as points and as basic open sets the B_b of (2.11), for all $b \in \mathcal{B}$.

Proposition 2.7.2. *For any Boolean algebra $\mathcal{B}$, the open sets $\mathcal{O}(X_{\mathcal{B}})$ of the Stone space $X_{\mathcal{B}}$ are in bijective correspondence with ideals $I \subseteq \mathcal{B}$, and with principal ideals $\downarrow b$ corresponding exactly to clopen sets.*

Corollary 2.7.3. *The Stone space $X_{\mathcal{B}}$ of a Boolean algebra $\mathcal{B}$ is a totally disconnected, compact, Hausdorff space.*

Definition 2.7.4. A topological space is called *Stone* if it is totally disconnected, compact, and Hausdorff. Let $\mathsf{Stone} \hookrightarrow \mathsf{Top}$ be the full subcategory of topological spaces consisting of Stone spaces and continuous functions between them.

Theorem 2.7.5. *There is a contravariant equivalence of categories between* Bool *and* Stone*,*

$$\begin{tikzcd} \mathsf{Bool}^{\mathsf{op}} \arrow[r, bend left, "*"] & \mathsf{Stone}\,, \arrow[l, bend left, "*"] \end{tikzcd}$$

with contravariant functors $\mathcal{B}^* = X_{\mathcal{B}}$ the Stone space of a Boolean algebra $\mathcal{B}$, as in Definition 2.7.1, and $X^* = \mathsf{clopen}(X)$, the Boolean algebra of all clopen sets in the Stone space X.

Proof. □

Remark 2.7.6. See Johnstone [Joh82] for a more thorough presentation of the material in this section, and much more. Also see [MR95] for a generalization to Heyting algebras.

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