

# Categorical Logic

- Outline :

- 1) Review of CT
- 2) Propositional Calculus
- 3)  $\lambda$ -Calculus

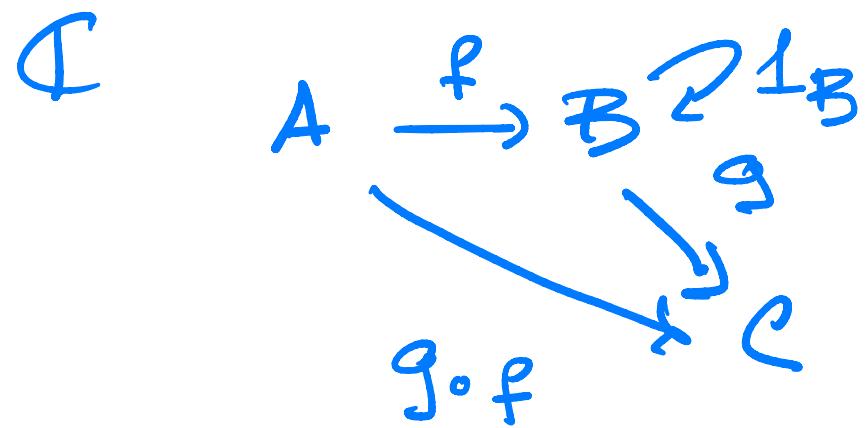
- Notes:

[awodey.github.io](https://awodey.github.io)

## 1. Review of CT

Basic Def.s :

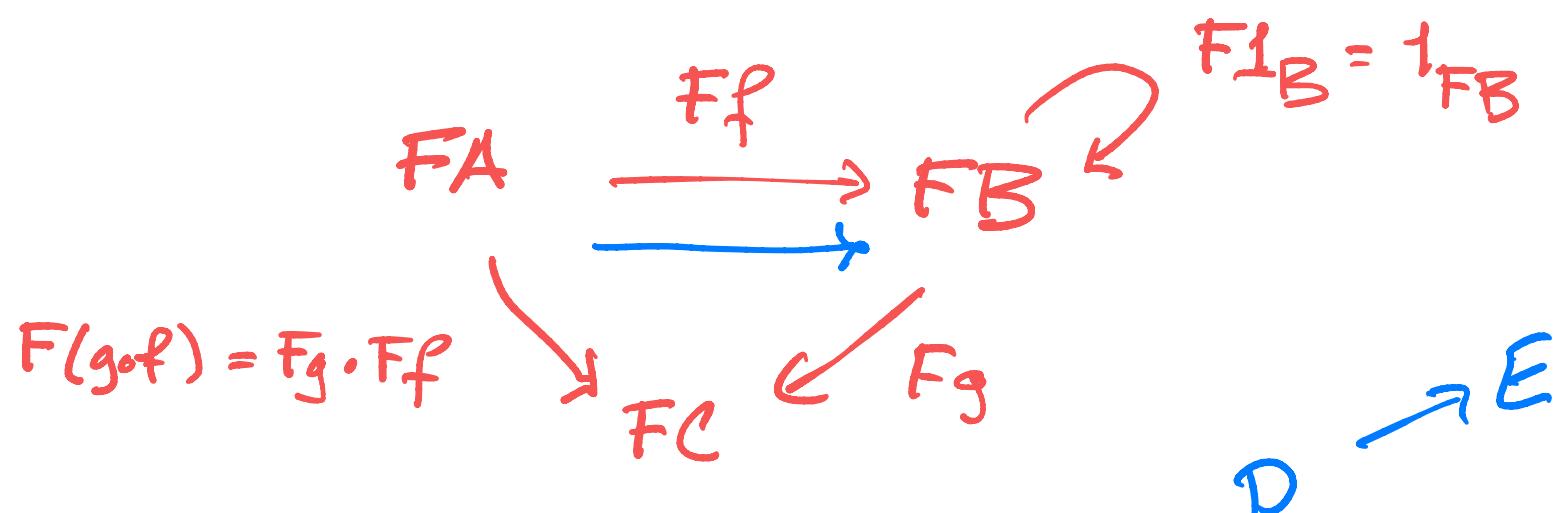
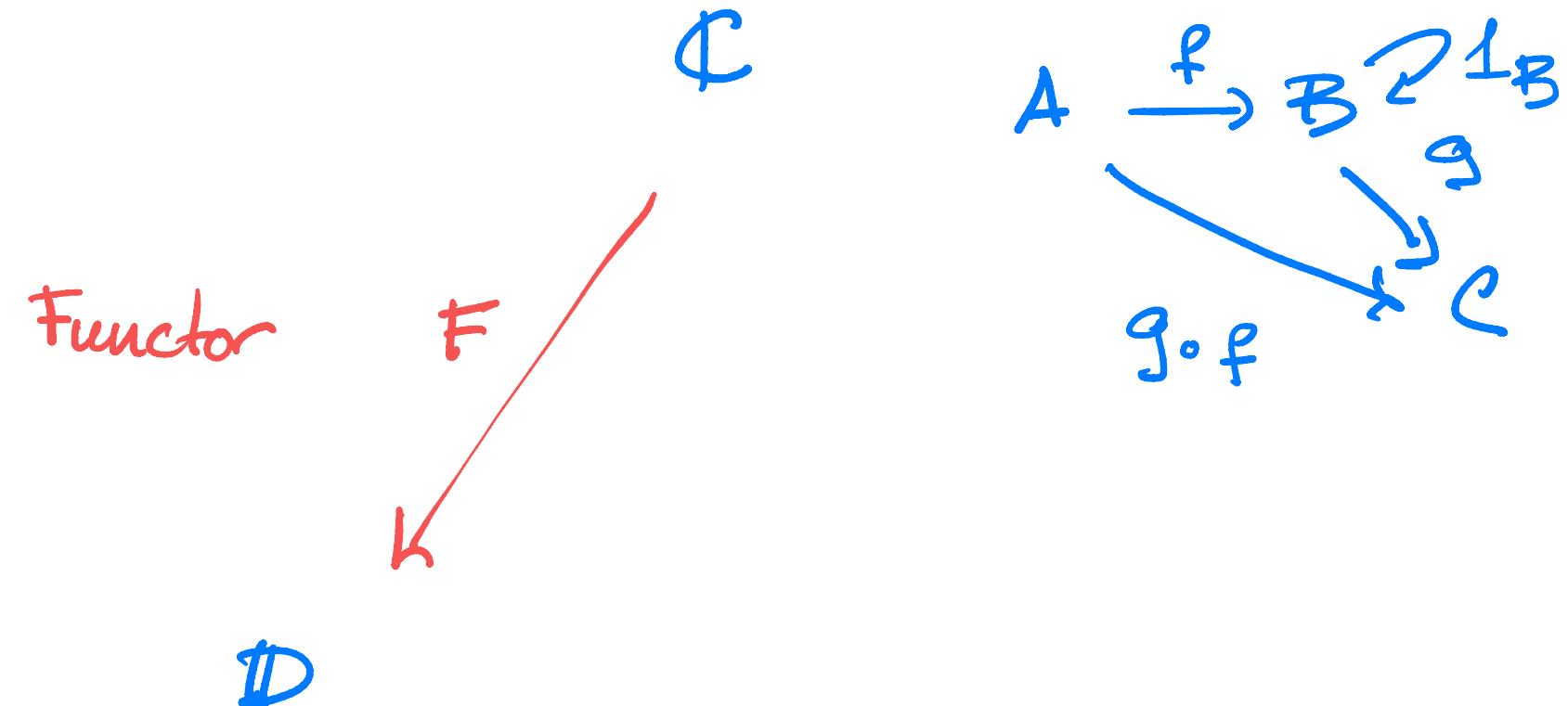
Category



# 1. Review of CT

Basic Def.s :

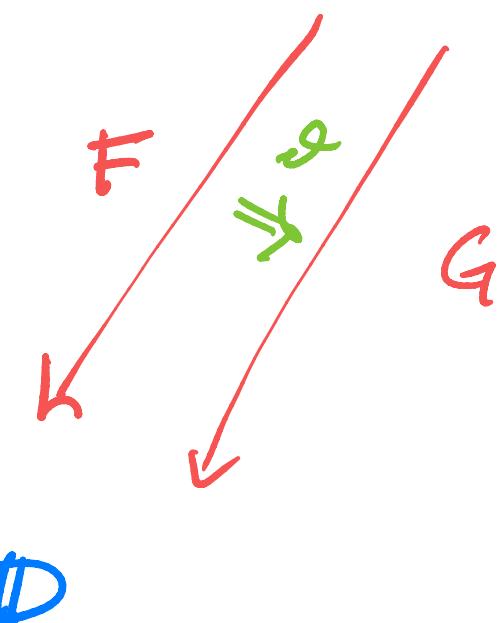
Category



# 1. Review of CT

Basic Def.s :

Functor

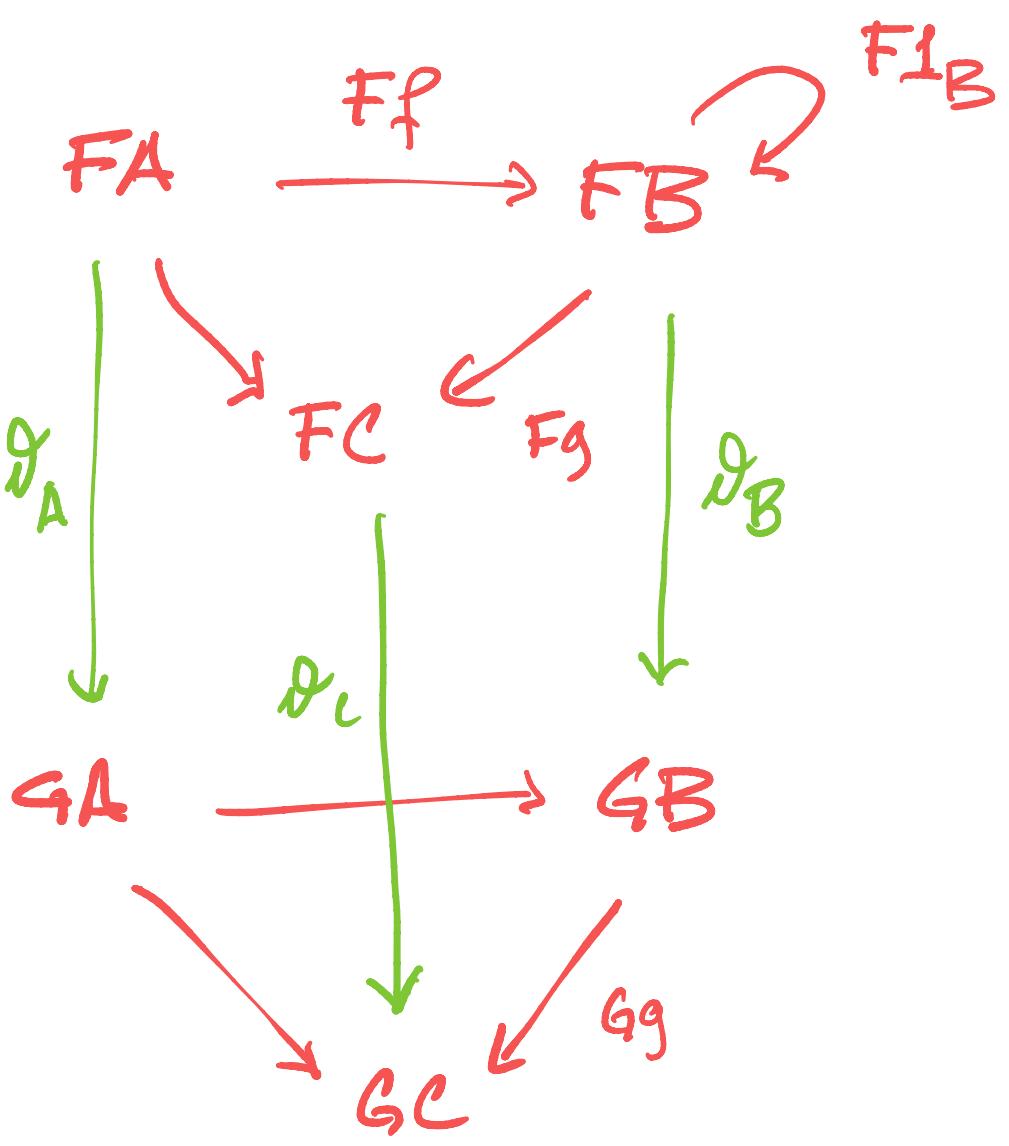
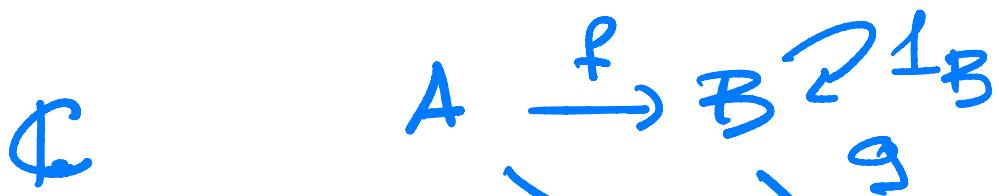


Natural Transformation

$$(\eta_c : FC \rightarrow GC)_{c \in \mathbb{C}}$$

$$\eta_c \circ Fg = Gg \circ \eta_B$$

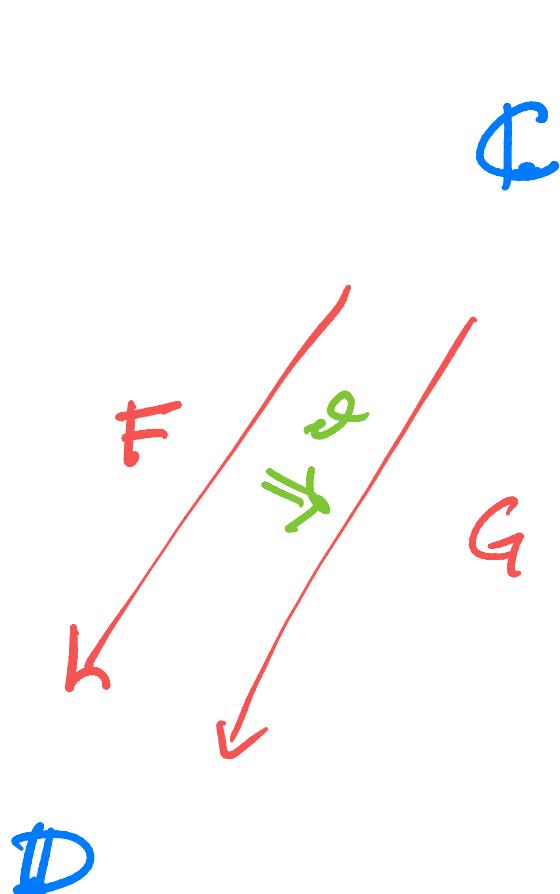
Category



# 1. Review of CT

Basic Def.s :

Functor

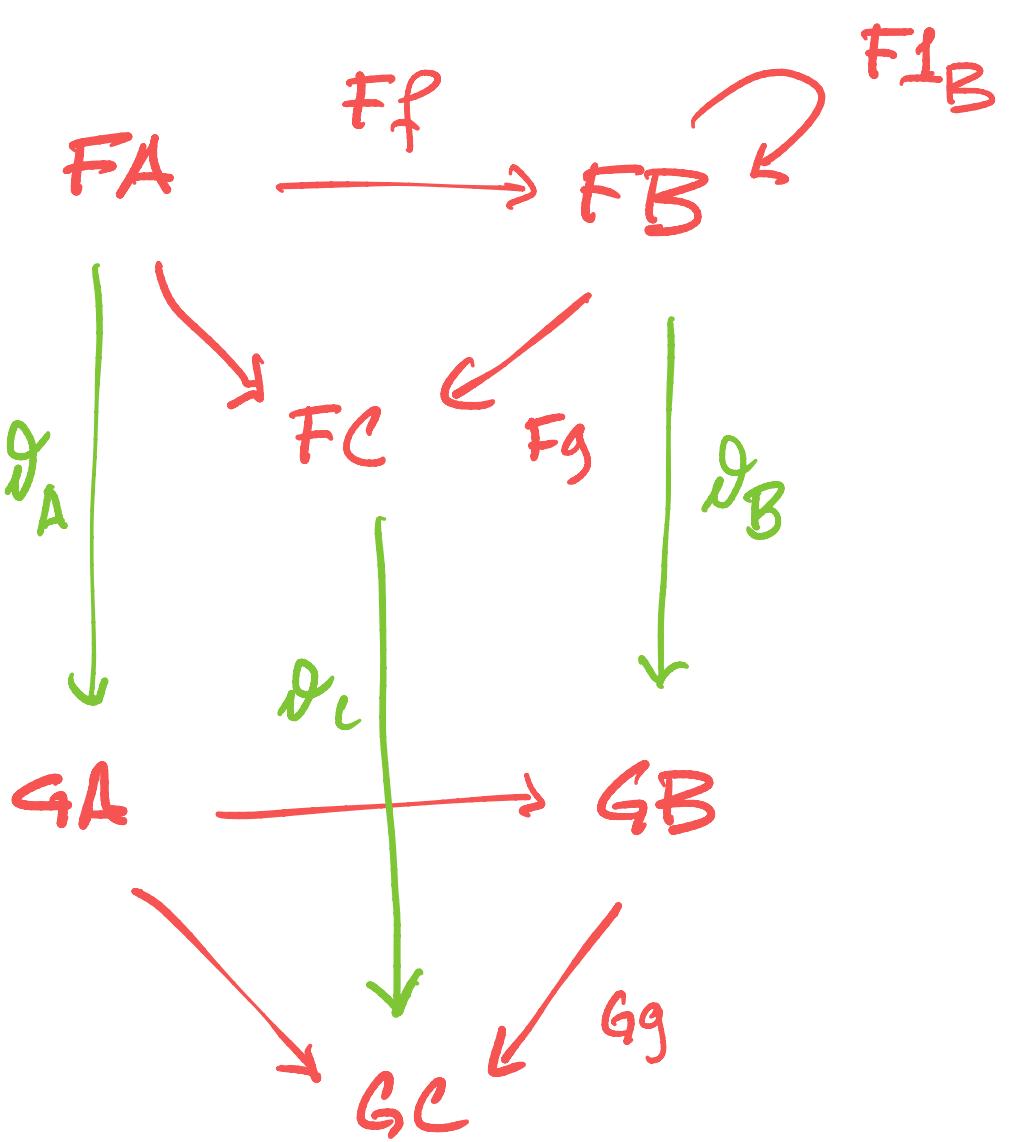
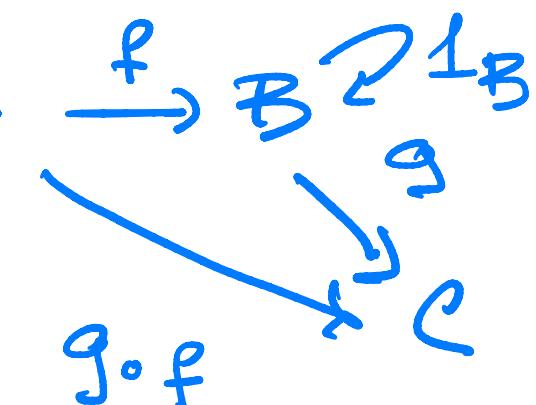


Natural Transformation

$$(\vartheta_c : FC \rightarrow GC)_{c \in C}$$

$$\vartheta_c \circ Fg = Gg \circ \vartheta_B$$

Category



Idea : "  $\vartheta : F \rightarrow G$  maps

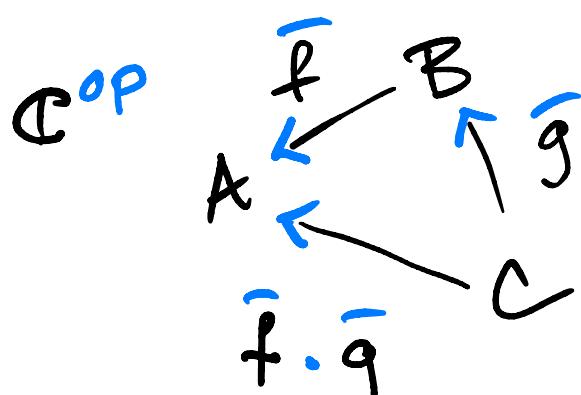
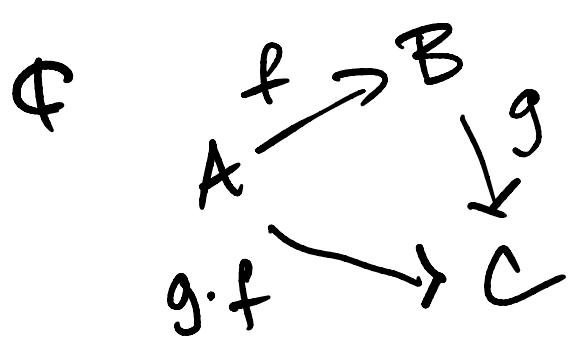
the structure  $F$

to the structure  $G$  . "

## Basic Examples:

(2)

- Set = Sets & functions
- Pos = Posets & monotone maps
- Top = Spaces & cts. maps
- Grp = Groups & homomorphisms
- Cat = Categories & functors
- P a poset : objects:  $P, q, \dots$   
arrows:  $P \leq q$
- X a space:  $C(X) = \text{Top}(X, \mathbb{R})$   
 $f \leq g := f_x \leq g_x \quad \forall x \in X.$
- $\mathcal{C}, \mathcal{D}$  cats:  $\mathcal{D}^{\mathcal{C}} = \text{Cat}(\mathcal{C}, \mathcal{D})$  functor cat  
objects: functors  $F: \mathcal{C} \rightarrow \mathcal{D}$   
arrows: nat. transf.  $\delta: F \Rightarrow G$
- $\mathcal{C}$  cat :  $\hat{\mathcal{C}} := \text{Set}^{\mathcal{C}^{\text{op}}}$  presheaves

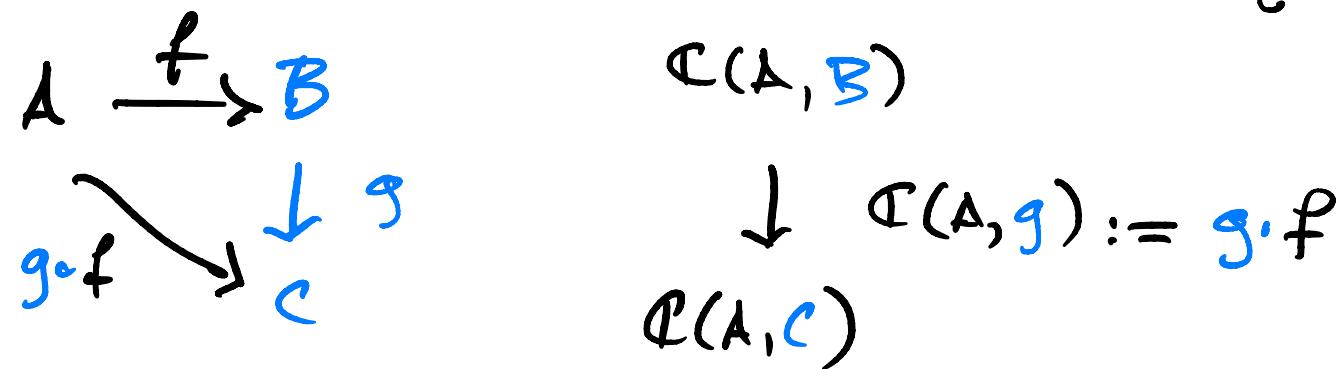


(3)

## Representable functor

$\text{Set}^{\mathcal{C}} \ni \mathcal{C}(A, -) : \mathcal{C} \rightarrow \text{Set}$

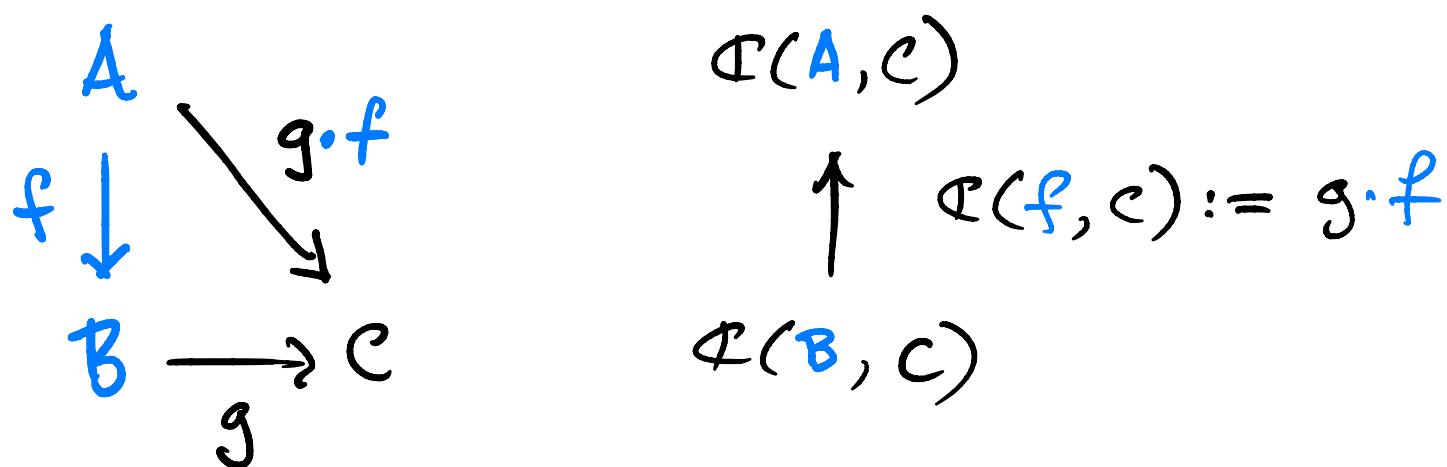
$$\begin{aligned} A \in \mathcal{C} \quad & B \mapsto \mathcal{C}(A, B) = \text{Hom}_{\mathcal{C}}(A, B) \\ & = \{ f : A \rightarrow B \} \end{aligned}$$



## Contravariant version

$\text{Set}^{\mathcal{C}^{\text{op}}} \ni \mathcal{C}(-, C) : \mathcal{C}^{\text{op}} \rightarrow \text{Set}$

$$C \in \mathcal{C} \quad B \mapsto \mathcal{C}(B, C)$$



The contravariant representable is written:

$y \mathcal{C} := \mathcal{C}(-, C) : \mathcal{C}^{\text{op}} \rightarrow \text{Set}$

Def. The Yoneda embedding is the functor (4)

$$y : \mathcal{C} \longrightarrow \text{Set}^{\mathcal{C}^{\text{op}}}$$

$$c \longmapsto \mathcal{C}(-, c)$$

with action on arrows in  $\mathcal{C}$  given by composition :

$$\begin{array}{ccc} c & & y^c \\ h \downarrow & \rightsquigarrow & \Downarrow yh \\ D & & yD \end{array},$$

$$\begin{array}{ccc} B & \xrightarrow{f} & C & \mathcal{C}(B, C) \\ & \downarrow h & & \downarrow \mathcal{C}(B, h) =: (yh)_B \\ h \cdot g & \downarrow f & & \mathcal{C}(B, D) \end{array},$$

Lemma (Yoneda) For  $c \in \mathcal{C}$ ,  $F \in \text{Set}^{\mathcal{C}^{\text{op}}}$ ,

$$Fc \cong \text{Hom}(yc, F) \quad \text{nat. in } C \& F.$$

So natural transf.s

$$\alpha : yc \rightarrow F$$

Correspond to elements

$$\alpha \in Fc$$

Prop. The Yoneda embedding is full & faithful:

$$y : \mathcal{C}(C, D) \xrightarrow{\cong} \text{Set}^{\mathcal{C}^{\text{op}}}(yc, yD)$$

$$c \xrightarrow{h} d$$

$$yc \xrightarrow{yh} yD$$

Pf.  $\text{Hom}(yc, yD) \cong (yD)c = \mathcal{C}(C, D)$

Adjoints Let  $\mathcal{C}, \mathcal{D}$  cats. (5)

Def. An adjunction consists of:

• functors

• a bijection

$$\mathcal{D}(FC, D) \cong \mathcal{C}(C, UD)$$

$$\begin{array}{ccc} & \mathcal{D} & \\ F \uparrow - \downarrow U & & \frac{FC \longrightarrow D}{C \longrightarrow UD} \\ & \mathcal{C} & \end{array} \quad \text{nat. in } \mathcal{C}, \mathcal{D}$$

Examples:

1) Posets  $P, Q$  & monotone  $P \xrightleftharpoons[f]{g} Q$  s.t.  $f^{-1}g :$

$$P \leq g q \quad \text{iff} \quad f_p \leq q$$

E.g. take  $Q = P \times P$  and  $P \xrightarrow[\Delta]{} P \times P$ .  
 $P \mapsto (p, p)$

• What is  $\Delta^{-1}$ ?

$$(p, p) \leq (a, b) \Leftrightarrow p \leq a \text{ & } p \leq b$$

$$\Leftrightarrow p \leq a \wedge b$$

$$\begin{array}{c} P \times P \\ \Delta \uparrow - \downarrow \\ P \end{array}$$

Adjoints Let  $\mathcal{C}, \mathcal{D}$  cats. (5)

Def. An adjunction consists of:

- functors

- a bijection

$$\mathcal{D}(FC, D) \cong \mathcal{C}(C, UD)$$

$$\begin{array}{ccc} & \mathcal{D} & \\ F \uparrow & \dashv & \downarrow U \\ \mathcal{C} & & \end{array} \quad \frac{FC \longrightarrow D}{C \longrightarrow UD} \quad \text{nat. in } \mathcal{C}, \mathcal{D}$$

Examples:

1) Posets  $P, Q$  & monotone  $P \xrightarrow{\quad f \quad} Q \xleftarrow{\quad g \quad} Q$  s.t.  $f \dashv g$ :

$$p \leq gq \quad \text{iff} \quad fp \leq q$$

E.g. take  $Q = P \times P$  and  $P \xrightarrow{\Delta} P \times P$ .  
 $P \mapsto (P, P)$

• What is  $\Delta \dashv$ ?

$$(P, P) \leq (a, b) \Leftrightarrow p \leq a \& p \leq b$$

$$\Leftrightarrow p \leq a \wedge b$$

• What about  $? \dashv \Delta$

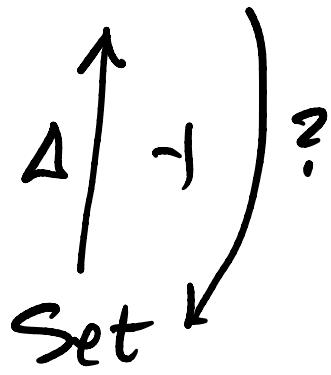
$$\checkmark \left( \begin{matrix} P \times P \\ \dashv \Delta \uparrow \dashv \\ P \end{matrix} \right) \wedge$$

$$\begin{array}{c} \Delta(P) \\ (a, b) \leq (P, P) \\ \hline a \leq P \& b \leq P \\ \hline a \vee b \leq P \end{array}$$

2) Replace  $P$  by Set :

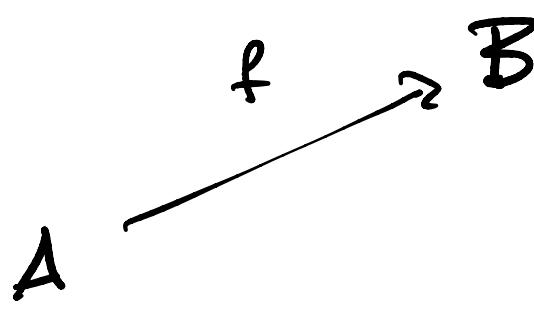
(6)

Set  $\times$  Set



$\Delta(A)$

$(A, A) \rightarrow (B, C)$



so  $\Delta \dashv x$ .

$A \longrightarrow B \times C$

$(f, g)$

2) Replace  $P$  by Set : (6)

Set  $\times$  Set

$$\Delta \begin{pmatrix} \dashv \\ \vdash \end{pmatrix} ?$$

Set

$$(A, A) \longrightarrow (B, C)$$


---

$$\begin{array}{ccc} & f & \rightarrow B \\ A & \downarrow & \\ & g & \searrow C \end{array}$$


---

so  $\Delta \dashv x$ .

$$A \longrightarrow B \times C$$

(f, g)

---

and of course :

$$\begin{array}{c} \text{Set} \times \text{Set} \\ + \begin{pmatrix} \dashv \Delta \dashv \end{pmatrix} \times \\ \text{Set} \end{array}$$

$$\begin{array}{ccc} B & \downarrow & \\ B+C & \dashrightarrow & A \\ \uparrow & & \\ C & \nearrow & \end{array}$$

BTW : An adjunction

$$F \dashv U$$

is always mediated by  
two distinguished maps :

$$\begin{array}{c} \stackrel{\imath}{\longrightarrow} \\ F C \longrightarrow F C \\ \hline C \xrightarrow{\eta} U F C \text{ "unit"} \end{array}$$

E.g. the unit of  $+ \dashv \Delta$  is

$$B \xrightarrow{i_1} B+C \xleftarrow{i_2} C$$

$$\begin{array}{c} \stackrel{\imath}{\longrightarrow} \\ U D \longrightarrow U D \\ \hline F U D \xrightarrow{\varepsilon} D \text{ "counit"} \end{array}$$

3) We also have  $\Delta \dashv \times$  in Pos, Cat, Grp, Top, ...

- In Pos, fix  $P$  and consider the functor

$$(-) \times P : \text{Pos} \rightarrow \text{Pos}$$

- This has a right adjoint:

$$\begin{array}{ccc} A \times P & \longrightarrow & B \\ \hline \hline A & \longrightarrow & B^P = \text{Pos}(P, B) \end{array} \quad f \leq g := f_P \leq g_P \forall_P$$

- We also have such right adjoints

$$(-) \times A \dashv (-)^A \quad \text{called } \underline{\text{exponentials}}$$

in Set, Cat, Set $^\mathbb{C}$ , ...

Def. A cat  $\mathcal{C}$  is cartesian closed

if there are right adjoints:

$$\begin{array}{c} \bullet \quad \begin{array}{c} \mathbb{1} \\ \uparrow - \\ \mathcal{C} \end{array} \end{array} \quad \begin{array}{c} \mathcal{C} \times \mathcal{C} \\ \uparrow - \\ \mathcal{C} \end{array} \quad \begin{array}{c} \mathbb{C} \\ \uparrow - \\ \mathcal{C} \end{array} \quad (A \in \mathcal{C})$$

4)  $P$  poset has  $\vee$  if for all  $S \subseteq P$ , (8)

there's  $\bigvee S \in P$  s.th.

$$\bigvee S \leq_P \Leftrightarrow s \leq_P \text{ for all } s \in S .$$

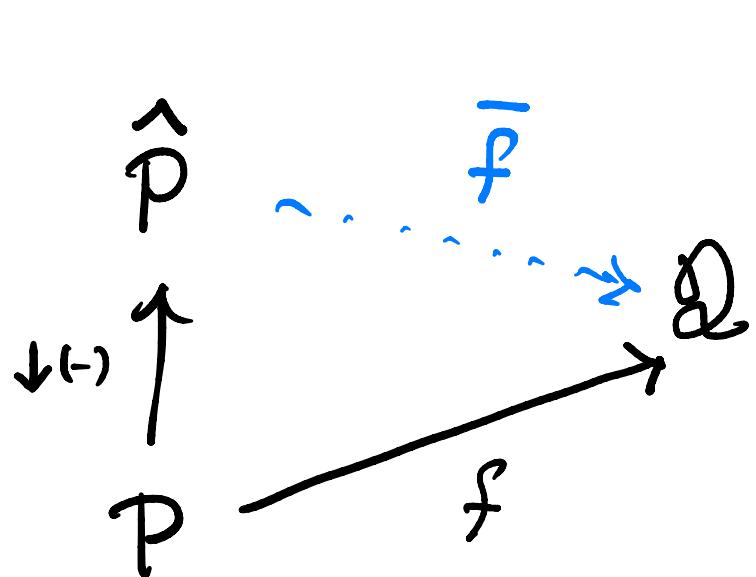
This says that  $\vee \vdash \downarrow$  :

$$\begin{array}{ccc} \partial P & & S \subseteq \downarrow P = \{x \in P\} \\ \downarrow \vee \quad \uparrow \downarrow \downarrow \downarrow & & \hline \hline \\ P & & \bigvee S \leq P \end{array}$$

Prop. The poset  $\hat{P}$  of downsets,

$$\downarrow: P \rightarrow \hat{P} := \{D \subseteq P \mid x \leq d \in D \Rightarrow x \in D\}$$

is the free completion of  $P$  under  $\vee$  :



(i)  $\hat{P}$  has  $\vee$

(ii) for any  $Q$  w/ $\vee$  &  
any  $f: P \rightarrow Q$

there's a unique  $\bar{f}$   
that preserves  $\vee$  &

$$\bar{f} \downarrow_P = f_P .$$

4)  $P$  poset has  $\vee$  if for all  $S \subseteq P$ , (8)

there's  $\bigvee S \in P$  s.th.

$$\bigvee S \leq_P \Leftrightarrow s \leq_P \text{ for all } s \in S .$$

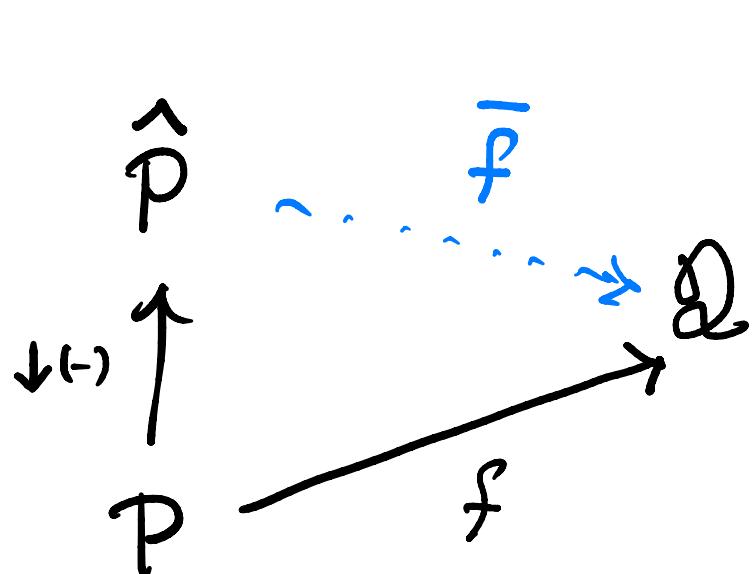
This says that  $\bigvee + \downarrow$  :

$$\begin{array}{ccc} \partial P & & S \subseteq \downarrow P = \{x \in P\} \\ \bigvee \uparrow \downarrow \rightarrow & \equiv & \bigvee S \leq P \end{array}$$

Prop. The poset  $\hat{P}$  of downsets,

$$\downarrow: P \rightarrow \hat{P} := \{D \subseteq P \mid x \leq d \in D \Rightarrow x \in D\}$$

is the free completion of  $P$  under  $\bigvee$  :



(i)  $\hat{P}$  has  $\bigvee$

(ii) for any  $Q$  w/ $\bigvee$  &  
any  $f: P \rightarrow Q$

there's a unique  $\bar{f}$   
that preserves  $\bigvee$  &

so:

$$\bar{f} \downarrow_P = f_P .$$

$$V\text{-Pos}(\hat{P}, Q) \cong \text{Pos}(P, Q) .$$

Prop. For any poset  $P$ , the poset  $\hat{P}$  is cartesian closed & the map

$$\downarrow(-) : P \rightarrow \hat{P}$$

preserves any cc structure in  $P$ .

Pf.  $T \& \wedge$  in  $\hat{P}$  as in  $\wp P$ .

$$\begin{aligned} A \Rightarrow B &= \bigvee \{D \mid D \wedge A \subseteq B\} \\ &= \{p \in P \mid \downarrow p \cap A \subseteq B\}. \end{aligned}$$

For any downset  $C$ ,

$$C \leq A \Rightarrow B \text{ iff } \downarrow C \cap A \subseteq B \text{ f.a. } c \in C.$$

So:

$$C \cap A = (\bigcup_{c \in C} \downarrow c) \cap A = \bigcup_{c \in C} (\downarrow c \cap A) \subseteq B.$$

Then:

$$\begin{aligned} \downarrow T &= P \\ \downarrow(a \wedge b) &= \downarrow a \wedge \downarrow b \\ \downarrow(a \Rightarrow b) &= \downarrow a \Rightarrow \downarrow b. \end{aligned}$$

Note: the map  $\downarrow(-) : P \rightarrow \hat{P}$  reflects  $\leq$ :

$$\downarrow p \leq \downarrow q = p \in \downarrow q = p \leq q.$$

In particular  $\downarrow$  is injective.