The Effective 2-Topos

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The Effective 1-Topos

Recall the Effective topos $\mathcal{E}ff$:

$$\mathrm{PAsm} \rightsquigarrow \mathrm{Asm} \rightsquigarrow \mathcal{E}ff$$

Lex Regular Exact

$$\mathrm{Asm} = {}^{reg}/_{lex}(\mathrm{PAsm}) \quad \text{free completion}$$

$$\mathcal{E}ff = {}^{ex}/_{reg}(\mathrm{Asm}) \quad \text{"} \qquad ,$$

$$\mathcal{E}ff = {}^{ex}/_{reg}\,{}^{reg}/_{lex}\left(\mathrm{PAsm}\right) \qquad .$$

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free colim

$$\begin{tikzcd} \hat{\mathbb{P}} \\ \arrow[u, hook, "y"] \end{tikzcd}$$

PAsm $= \mathbb{P}$

free colim

$$
\begin{tikzcd}
\hat{\mathbb{P}} \arrow[dr, leftrightarrow] & \\
& \mathrm{Asm} \\
\mathbb{P} \arrow[uu, "y"] \arrow[ur, leftrightarrow] &
\end{tikzcd}
$$

$$\begin{tikzcd}
\hat{\mathbb{P}} & \\
& \mathrm{Asm} \arrow[ul, hook] \\
\mathbb{P} \arrow[uu, hook, "y"] \arrow[ur, hook] &
\end{tikzcd}$$

free kernel quotients

$$\begin{tikzcd}
K \arrow[r, shift left] \arrow[r, shift right] & P \arrow[rr] \arrow[dr, two heads] & & Q \\
& & P/K \arrow[ur, hook] &
\end{tikzcd}$$

$$
\begin{tikzcd}
\hat{\mathbb{P}} & \\
& \mathcal{E}ff \arrow[ul, hook] \\
& \mathsf{Asm} \arrow[u, hook] \\
\mathbb{P} \arrow[uuu, hook, "y"] \arrow[ur, hook] &
\end{tikzcd}
$$

free exact quotients

$$
\begin{tikzcd}
E \arrow[r, shift left] \arrow[r, shift right] & P \arrow[r, two heads] & P/E
\end{tikzcd}
$$

free kernel quotients

$$
\begin{tikzcd}
K \arrow[r, shift left] \arrow[r, shift right] & P \arrow[rr] \arrow[dr, two heads] & & Q \\
& & P/K \arrow[ur, tail] &
\end{tikzcd}
$$

Factorization of Yoneda

$$\begin{tikzcd}
\mathbb{P} \arrow[rrrrrr, hook, "y"] \arrow[d, equal] & & & & & & \widehat{\mathbb{P}} \\
\mathrm{PAsm} \arrow[r, hook] & \text{cpt. obj.} \arrow[r, hook] & \mathrm{Asm} \arrow[r, hook] & \text{coh. obj.} \arrow[r, hook] & \mathrm{Sh}(\mathrm{Asm}) \arrow[urr] & &
\end{tikzcd}$$

Factorization of Yoneda

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\mathbb{P} \arrow[rrrrrr, hook, "y"] \arrow[d, equal] & & & & & & \hat{\mathbb{P}} \\
\mathrm{PAsm} \arrow[r, hook] & \text{cpt. obj.} \arrow[r, hook] & \mathrm{Asm} \arrow[r, hook] & \text{coh. obj.} \arrow[rrr, hook] & & & \mathrm{Sh}(\mathrm{Asm}) \arrow[u]
\end{tikzcd}$$

<u>indecomp. proj.</u> :

- $I = X + Y \Rightarrow I = X$ or $I = Y$
- $\begin{tikzcd} & E \arrow[d, two heads] \\ P \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$

$\iff P = yp$ f. some $p \in \mathbb{P}$.

Factorization of Yoneda

$$\begin{array}{ccccccccc}
\mathbb{P} & & & \xhookrightarrow{\quad y \quad} & & & & & \hat{\mathbb{P}} \\
\wr\!\mid & & & & & & & & \uparrow \\
\text{ind. proj.} & \hookrightarrow & \text{cpt. obj.} & \hookrightarrow & \mathrm{Asm} & \hookrightarrow & \text{coh. obj.} & \hookrightarrow & \mathrm{Sh}(\mathrm{Asm}) \\
P, Q & & P \twoheadrightarrow K & & & & & &
\end{array}$$

K cpt :

$(X_i \longrightarrow K)_i$ covers

$\Rightarrow \quad X_k \twoheadrightarrow K$ f. some k

$\Bigg\} \iff$ $P \twoheadrightarrow K$ f. some P

Factorization of Yoneda

$$\begin{tikzcd}
\mathbb{P} \arrow[rrrr, hook, "y"] \arrow[d, phantom, "\wr"] & & & & \widehat{\mathbb{P}} \\
\text{ind. proj.} \arrow[r, hook] & \text{cpt. obj.} \arrow[r, hook] & \mathrm{Asm} \arrow[r, hook] & \text{coh. obj.} \arrow[r, hook] & \mathrm{Sh}(\mathrm{Asm}.) \arrow[u]
\end{tikzcd}$$

P, Q $\qquad$ $P \twoheadrightarrow K$ $\qquad$ $P \twoheadrightarrow A \rightarrowtail Q$

$$\left.\begin{array}{l} P \twoheadrightarrow A \rightarrowtail Q \\ \text{for ind. proj. } P, Q \end{array}\right\} \iff \begin{array}{l} A \text{ cpt} \\ A \rightarrowtail Q \end{array}$$

Factorization of Yoneda

$$\begin{tikzcd}
\mathbb{P} \arrow[rrrr, hook, "y"] \arrow[d, phantom, "\cong" description] & & & & \hat{\mathbb{P}} \\
\text{ind. proj.} \arrow[r, hook] & \text{cpt. obj.} \arrow[r, hook] & \mathrm{Asm} \arrow[r, hook] & \text{coh. obj.} \arrow[r, hook] & \mathrm{Sh}(\mathrm{Asm}) \arrow[u]
\end{tikzcd}$$

P, Q $\qquad$ $P \twoheadrightarrow K$ $\qquad$ $P \twoheadrightarrow A \rightarrowtail P'$ $\qquad$ $A' \rightrightarrows A \twoheadrightarrow C$

C <u>coherent</u> :

- C is cpt.
- $C \xrightarrow{\Delta} C \times C$ cpt.

Factorization of Yoneda

$$\begin{array}{ccccccccc}
\mathbb{P} & & & & \overset{y}{\hookrightarrow} & & & & \hat{\mathbb{P}} \\
\wr\! \| & & & & & & & & \uparrow \\
\text{ind. proj.} & \hookrightarrow & \text{cpt. obj.} & \hookrightarrow & \mathrm{Asm} & \hookrightarrow & \text{coh. obj.} & \hookrightarrow & \mathrm{Sh}(\mathrm{Asm}) \\
P, Q & & P \twoheadrightarrow K & & P \twoheadrightarrow A \rightarrowtail P' & & A' \rightrightarrows A \twoheadrightarrow C. & &
\end{array}$$

C **coherent**:

- C is cpt.
- $C \xrightarrow{\Delta} C \times C$ cpt.

where

$X \to Y$ is **cpt**:

$$\begin{tikzcd}
K' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\
K \arrow[r] & Y
\end{tikzcd}.$$

Factorization of Yoneda

$$\begin{tikzcd}
\mathbb{P} \arrow[rrrr, hook, "y"] \arrow[d, phantom, "\cong"] & & & & \hat{\mathbb{P}} \\
\text{ind. proj.} \arrow[r, hook] & \text{cpt. obj.} \arrow[r, hook] & \mathrm{Asm} \arrow[r, hook] & \text{coh. obj.} \arrow[r, hook] & \mathrm{Sh}(\mathrm{Asm.}) \arrow[u]
\end{tikzcd}$$

P, Q $\qquad$ $P \twoheadrightarrow K$ $\qquad$ $P \rightarrowtail A \rightarrowtail P'$ $\qquad$ $A' \rightrightarrows A \twoheadrightarrow C$

C <u>coherent</u> :

- C is cpt.
- $C \xrightarrow{\Delta} C \times C$ cpt.

$\Bigg\} \iff A' \rightrightarrows A \twoheadrightarrow C$ exact.

Factorization of Yoneda

$$\begin{tikzcd}
\mathbb{P} \arrow[r, hook, "y"] \arrow[d, phantom, "\cong" rotate=90] & & & & \hat{\mathbb{P}} \\
\text{ind. proj.} \arrow[r, hook] & \text{cpt. obj.} \arrow[r, hook] & \mathrm{Asm} \arrow[r, hook] & \text{coh. obj.} \arrow[r, hook] & \mathrm{Sh}(\mathrm{Asm.}) \arrow[u]
\end{tikzcd}$$

$P, Q \qquad P \twoheadrightarrow K \qquad P \twoheadrightarrow A \rightarrowtail P' \qquad A' \rightrightarrows A \twoheadrightarrow C$

F <u>sheaf</u> on Asm (for the reg. epi. top.) $\iff$

$$\begin{tikzcd}
A \times_B A \arrow[r, shift left] \arrow[r, shift right] & A \arrow[r] \arrow[d, two heads] & F \\
& B \arrow[ur, dashed] &
\end{tikzcd}$$

Factorization of Yoneda

$$\begin{tikzcd}
\mathbb{P} \arrow[rrrr, hook, "y"] \arrow[d, phantom, "\wr"] & & & & \widehat{\mathbb{P}} \arrow[d, phantom, "\wr"] \\
\text{ind. proj.} \arrow[r, hook] & \text{cpt. obj.} \arrow[r, hook] & \mathrm{Asm} \arrow[r, hook] & \text{coh. obj.} \arrow[r, hook] & \mathrm{Sh}(\mathrm{Asm.})
\end{tikzcd}$$

P, Q $\qquad$ $P \twoheadrightarrow K$ $\qquad$ $P \twoheadrightarrow A \rightarrowtail P'$ $\qquad$ $A' \rightrightarrows A \twoheadrightarrow C$

F sheaf on Asm (for the reg. epi. top.) $\iff$ $F \in \widehat{\mathbb{P}} = [\mathbb{P}^{op}, \mathrm{Set}]$.

Thm (Lack) For a reg. cat. $\mathcal{R}$,

$$\mathcal{R}_{ex/reg} \subseteq \mathrm{Sh}(\mathcal{R}, \text{reg epi})$$

$$= \langle E \mid yA' \rightrightarrows yA \twoheadrightarrow E \rangle$$

$$\text{f. } A, A' \in \mathcal{R}$$

.

Thm (Lack) For a reg. cat. $\mathcal{R}$,

$$\mathcal{R}_{ex/reg} \subseteq Sh(\mathcal{R}, \text{reg epi})$$

$$= \langle E \mid yA' \rightrightarrows yA \twoheadrightarrow E \rangle$$

f. $A, A' \in \mathcal{R}$.

Cor. $Sh(Asm)_{coh} = Asm_{ex/reg}$.

Thm (Lack) For a reg. cat. R,

$$R_{ex/reg} \subseteq Sh(R, \text{reg epi})$$
$$= \langle E \mid yA' \rightrightarrows yA \twoheadrightarrow E \rangle$$

f. $A, A' \in R$

Cor. $Sh(R)_{coh} = Asm_{ex/reg} = \mathcal{E}ff$!

Summary

$$
\begin{array}{ccccccc}
\mathrm{PAsm} & \subset & \mathrm{Asm} & \subset & \mathrm{Eff} & \subset & \widehat{\mathrm{PAsm}} \\
 & & & & \wr & & \wr \\
 & & & & \mathrm{CohSh}(\mathrm{Asm}) & \subset & \mathrm{Sh}(\mathrm{Asm}) \\
 & & & & & & \text{Groth. topos}
\end{array}
$$

Summary

$$\mathrm{PAsm} \subset \mathrm{Asm} \subset \mathcal{E}ff \subset \widehat{\mathrm{PAsm}}$$

$$\begin{array}{ccc} & \mathcal{E}ff & \subset & \widehat{\mathrm{PAsm}} \\ & \wr\!\wr & & \wr\!\wr \\ & \mathrm{CohSh}(\mathrm{Asm}) & \subset & \mathrm{Sh}(\mathrm{Asm}) \end{array}$$

Now in order to get a 2-topos we take <u>internal groupoids</u> in the Groth. topos $\mathrm{Sh}(\mathrm{Asm})$,

$$\mathrm{Gpd}(\mathrm{Sh}(\mathrm{Asm})).$$

Summary

$$\mathrm{PAsm} \subset \mathrm{Asm} \subset \mathcal{E}\!f\!f \subset \widehat{\mathrm{PAsm}}$$

$$\mathcal{E}\!f\!f \simeq \mathrm{CohSh}(\mathrm{Asm}) \subset \widehat{\mathrm{PAsm}} \simeq \mathrm{Sh}(\mathrm{Asm})$$

Now in order to get a 2-topos we take <u>internal groupoids</u> in the Groth. topos $\mathrm{Sh}(\mathrm{Asm})$.

$$\mathrm{Gpd}(\mathrm{Sh}(\mathrm{Asm})) = \mathrm{Gpd}([\mathbb{P}^{op}, \mathrm{Set}]) = [\mathbb{P}^{op}, \mathrm{Gpd}].$$

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Finally, we shall restrict to a subcat of "coherent groupoids" in such a way that

$$\begin{tikzcd}
\mathrm{CohGpd} \arrow[r, hook] & \mathrm{Gpd} \\
\mathcal{E}ff \arrow[u, hook] \arrow[r, hook] \arrow[ur, phantom, "\urcorner"] & \mathrm{Gpd}_0 \arrow[u, hook]
\end{tikzcd}$$

in $\mathrm{Sh}(\mathrm{Asm})$

$= [\mathbb{P}^{op}, \mathrm{Gpd}]$.

QMS on $\mathcal{G}pd(\mathcal{E})$

There are many ways to put a QMS on $\mathcal{G}pd(\mathcal{E})$ for Groth. topos $\mathcal{E}$. We choose one that:

1) is constructive,
2) works well with the groupoid model of HoTT,
3) gives $\mathcal{E}ff$ as the coherent o.types:

$$\mathcal{E}ff = \mathcal{C}oh\mathcal{G}pd(\mathcal{E})_0 \subseteq \mathcal{G}pd(\mathcal{E})_0 \subseteq \mathcal{G}pd(\mathcal{E}).$$

Using results of Everaert, Kieboom, vd Linden, we can show:

<u>Thm</u>. There's a model stucture on $\mathrm{Gpd}(\hat{\mathbb{P}})$ with:

- Weak equivalences = equivalences of cats $\mathbb{G} \simeq \mathbb{H}$,
- Cofibrations = functors $c : \mathbb{A} \rightarrowtail \mathbb{B}$ s.th. c_0 is a retract of $A_0 \longrightarrow A_0 + C$, f.s. C in $\hat{\mathbb{P}}$,
- Fibrations = f with "internal path lifting":

$$\begin{tikzcd} \mathbb{1} \arrow[r] \arrow[d] & \mathbb{E} \arrow[d, "f"] \\ \mathbb{I} \arrow[r] \arrow[ur, dashed] & \mathbb{B} \end{tikzcd}$$

Coherent Groupoid

Def. A groupoid $\mathcal{G} = (G_1 \rightrightarrows G_0)$ in $\widehat{\mathbb{P}}$ is **coherent** if:

- G_0 is cpt: $P \twoheadrightarrow G_0$ f. some $P \in \mathbb{P}$,
- $G_1 \longrightarrow G_0 \times G_0$ is a cpt map.

Prop. Let $\mathbb{G} = (G_1 \rightrightarrows G_0)$ be a coherent gpd that is discrete, $\mathbb{G}^I \rightarrowtail \mathbb{G}$. Then $\pi_0 \mathbb{G}$ is a coherent object.

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Then $\pi_0 \mathbb{G}$ is a coherent object.

Pf. Since G_0 is cpt, there's a cover $P \twoheadrightarrow G_0$.

Take the p.b. K and its cover P'.

$$\begin{tikzcd}
P' \arrow[r, two heads] \arrow[dr] & K \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & G_1 \arrow[d] \\
& P_0 \times P_0 \arrow[r, two heads] & G_0 \times G_0
\end{tikzcd}$$

Prop. Let $\mathbb{G} = (G_1 \rightrightarrows G_0)$ be a coherent gpd that is discrete $\mathbb{G}^I \rightarrowtail \mathbb{G}$.

Then $\pi_0 \mathbb{G}$ is a coherent object.

Pf. Since G_0 is cpt, there's a cover $P \twoheadrightarrow G_0$.

Take the p.b. K and its cover P'.

Since $\mathbb{G}$ is discrete,

$G_1 \longrightarrow G_0 \times G_0$ is monic.

$$\begin{tikzcd}
P' \arrow[r, two heads] \arrow[dr] & K \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & G_1 \arrow[d, tail] \\
& P_0 \times P_0 \arrow[r, two heads] & G_0 \times G_0
\end{tikzcd}$$

<u>Prop.</u> Let $\mathbb{G} = (G_1 \rightrightarrows G_0)$ be a coherent gpd that is discrete $\mathbb{G}^I \rightarrowtail \mathbb{G}$.
then $\pi_0 \mathbb{G}$ is a coherent object.

Pf. Since G_0 is cpt, there's a cover $P \twoheadrightarrow G_0$.

Take the p.b. K and its cover P'.

$$\begin{tikzcd}
P' \arrow[r, two heads] \arrow[dr] & K \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & G_1 \arrow[d, tail] \\
& P \times P \arrow[r, two heads] & G_0 \times G_0
\end{tikzcd}$$

Since $\mathbb{G}$ is discrete,
$G_1 \longrightarrow G_0 \times G_0$ is monic.
Hence so is $K \rightarrowtail P \times P$,
so K is an assembly.

Prop. Let $\mathbb{G} = (G_1 \rightrightarrows G_0)$ be a coherent gpd that is discrete $\mathbb{G}^I \rightarrowtail \mathbb{G}$. Then $\pi_0 \mathbb{G}$ is a coherent object.

Pf. Since G_0 is cpt, there's a cover $P \twoheadrightarrow G_0$.

Take the p.b. K and its cover P'.

$$\begin{tikzcd}
P' \arrow[r, two heads] \arrow[dr] & K \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & G_1 \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \bar{G} \arrow[d, tail] \\
& P \times P \arrow[r, two heads] & G_0 \times G_0 \arrow[r, two heads] & \bar{G} \times \bar{G}
\end{tikzcd}$$

Since $\mathbb{G}$ is discrete,

$G_1 \longrightarrow G_0 \times G_0$ is monic.

Hence so is $K \rightarrowtail P \times P$.

So K is an assembly.

Write $\bar{G} = G_0/G_1 = \pi_0 \mathbb{G}$.

We have $K \rightrightarrows P \twoheadrightarrow \bar{G}$ exact.

So $\bar{G} = \pi_0 \mathbb{G}$ is coherent. $\square$

Thm In $\mathrm{Sh}(\mathrm{Asm}) = \hat{\mathbb{P}}$, we have

$$
\begin{tikzcd}
\mathcal{E}\mathit{ff} \simeq \mathrm{CohGpd}_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Gpd}_0 \arrow[d] \\
\mathrm{CohGpd} \arrow[r] & \mathrm{Gpd}
\end{tikzcd}
$$

Thm In $Sh(Asm) = \hat{\mathbb{P}}$, we have

$$
\begin{tikzcd}
\mathcal{E}ff \simeq \mathrm{CohGpd}_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Gpd}_0 \arrow[d] \\
\mathcal{E}ff^2 := \mathrm{CohGpd} \arrow[r] & \mathrm{Gpd}
\end{tikzcd}
$$

So we propose to take $\mathcal{E}ff^2 = \mathrm{CohGpd}(Sh(Asm))$, a 2-topos with $\mathcal{E}ff^2_0 = \mathcal{E}ff$.

FINITO !