

The Effective 2-Topos

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jww/

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The Effective 1·Topos

Recall the Effective topos $\mathcal{E}ff$:

$PAsm \rightarrow Asm \rightarrow \mathcal{E}ff$

Lex

Regular

Exact

$Asm = \frac{\text{reg}}{\text{lex}}(PAsm)$ free completion

$\mathcal{E}ff = \frac{\text{ex}}{\text{reg}}(Asm)$ "

$= \frac{\text{ex}}{\text{reg}} \frac{\text{reg}}{\text{lex}}(PAsm)$.

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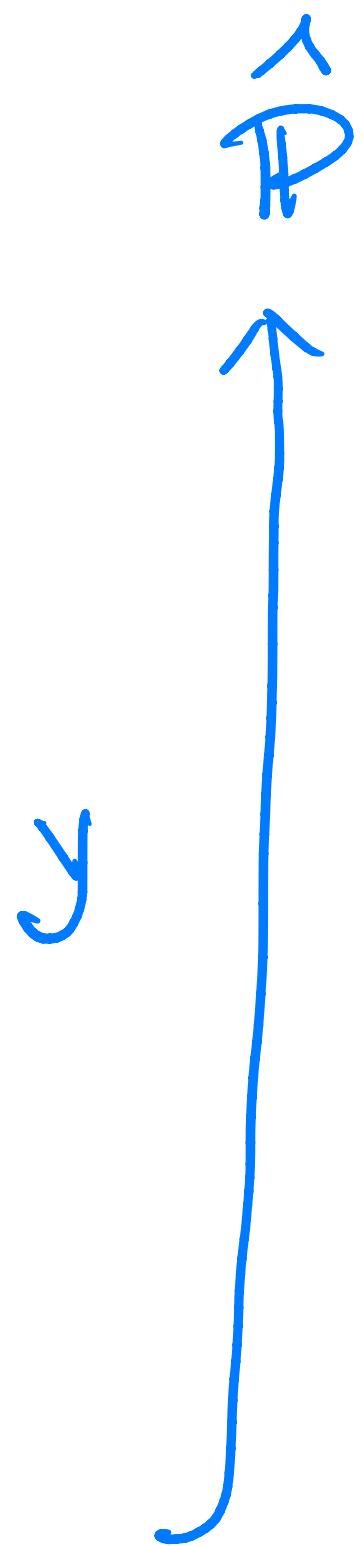
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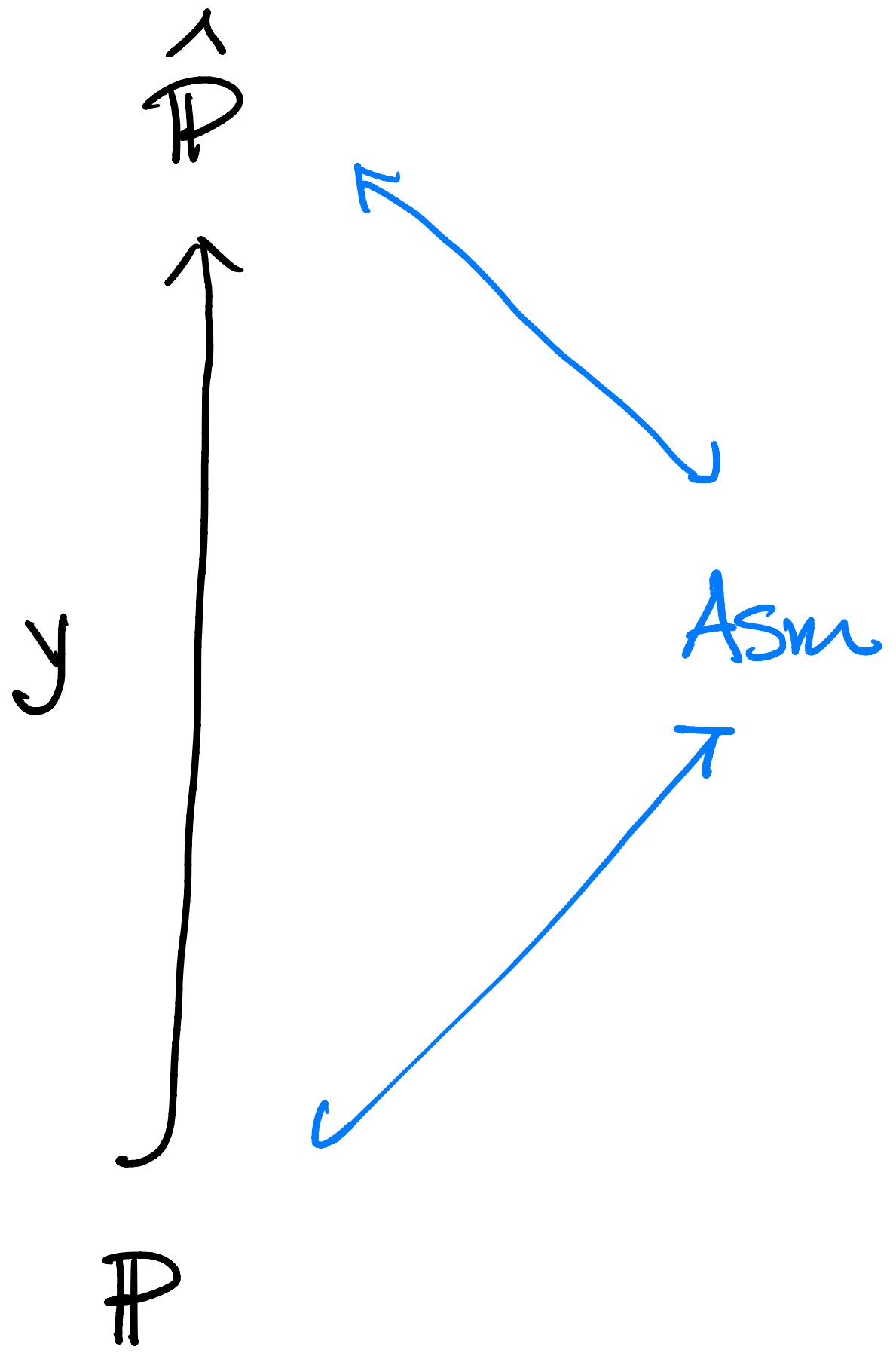
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free
Colin



$$P_{\text{Asm}} = \hat{P}$$

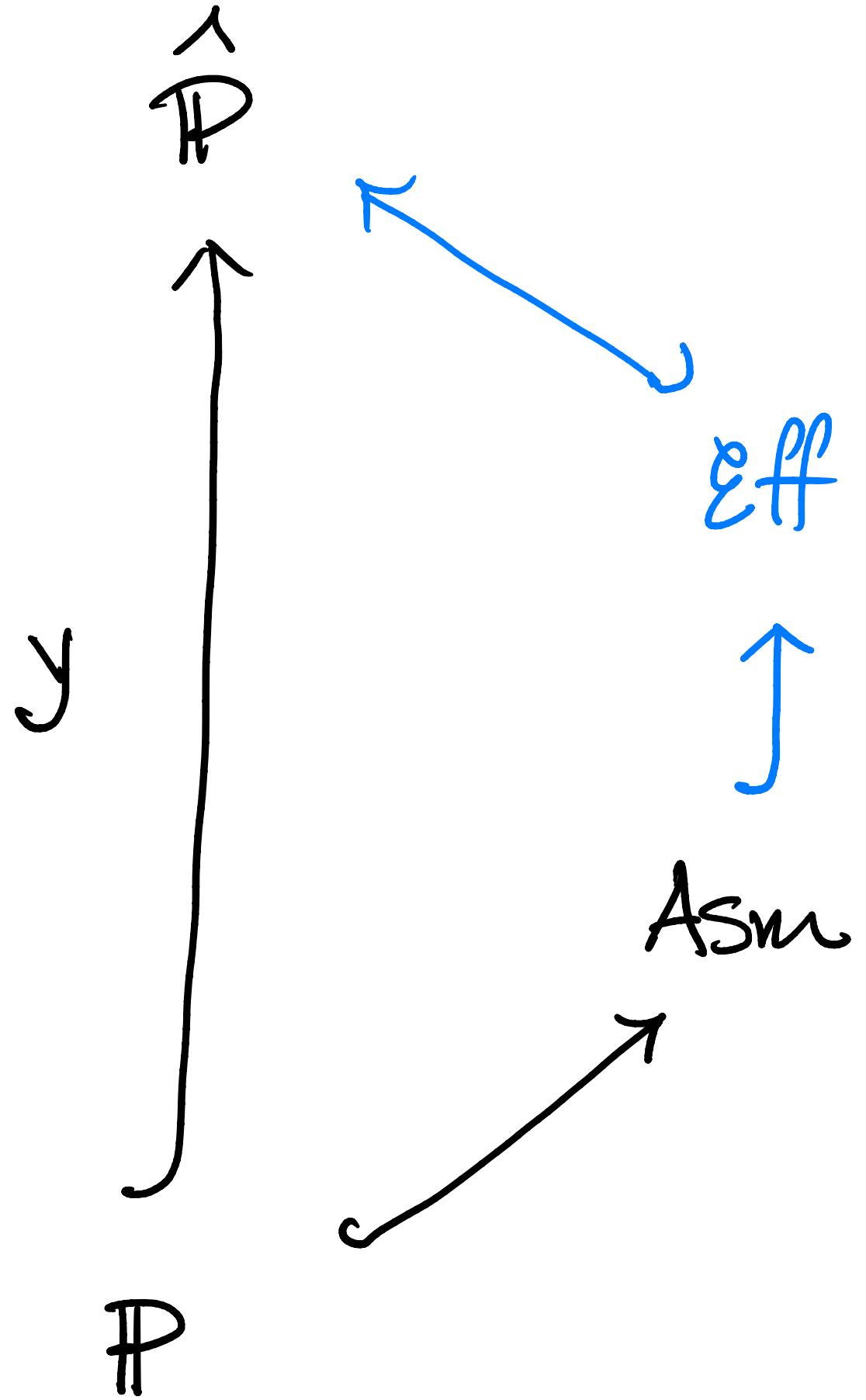


free kernel quotients

$$k \rightarrowtail P \xrightarrow{\quad} Q$$

$$\downarrow \pi$$

$$P/k \rightarrowtail P \xrightarrow{\quad} Q$$



free exact quotients

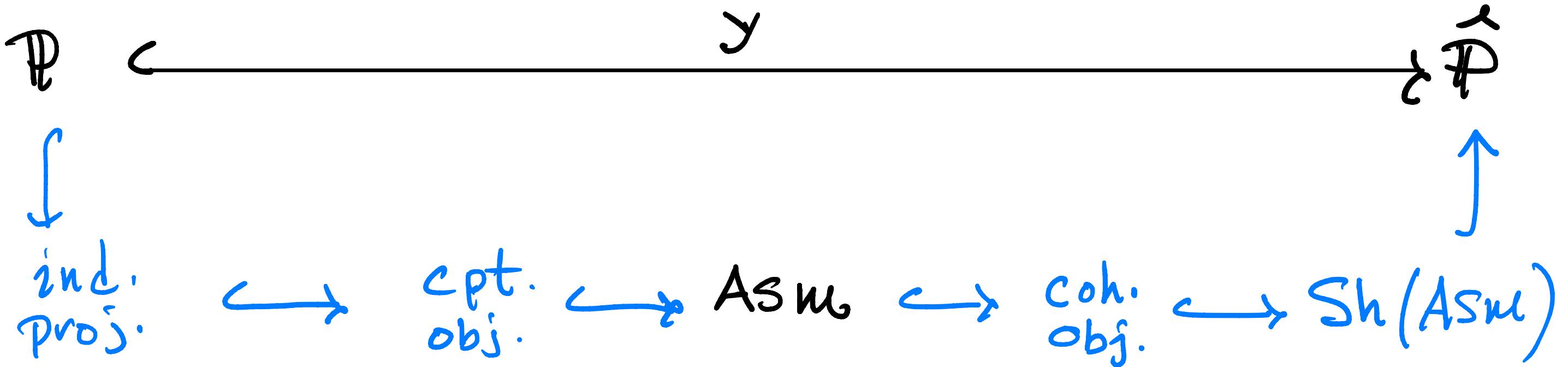
$$E \rightarrowtail P \twoheadrightarrow P/E$$

free Kernel quotients

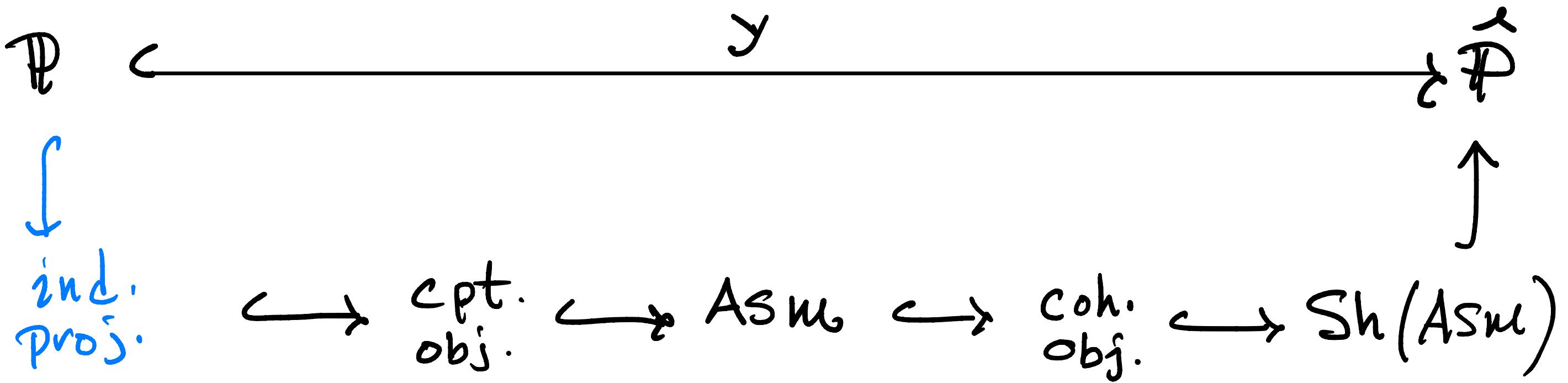
$$K \rightarrowtail P \longrightarrow Q$$

$$\downarrow P/K$$

Factorization of Yoneda



Factorization of Yoneda



indecomposable projectives:

$$\cdot I = X + Y \Rightarrow I = X$$

$$\text{or } I = Y \quad \left. \begin{array}{c} \\ \end{array} \right\} \quad \iff$$

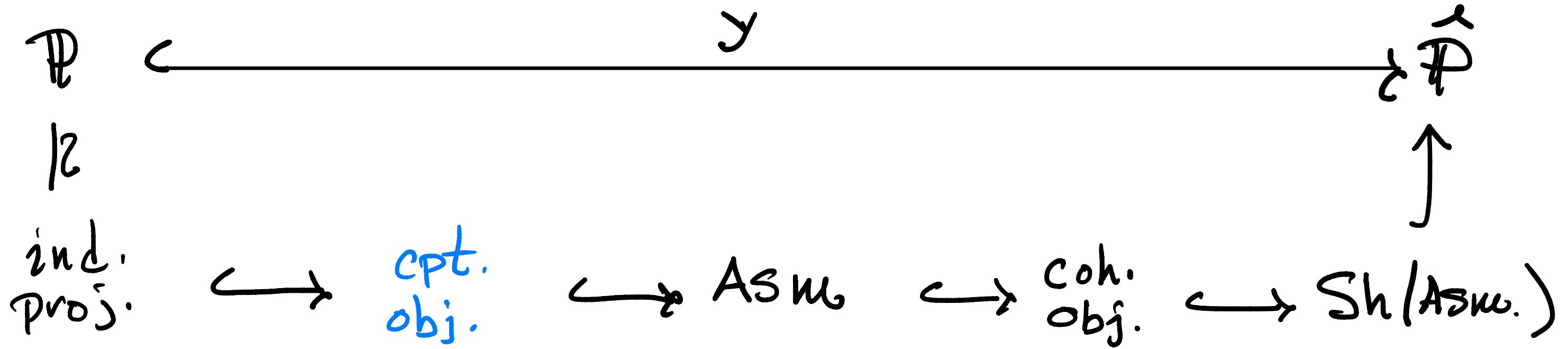
$$\cdot P \xrightarrow{\sim} E \xrightarrow{\sim} X$$

$$P = yP$$

f. some $P \in \mathbb{P}$

"representable".

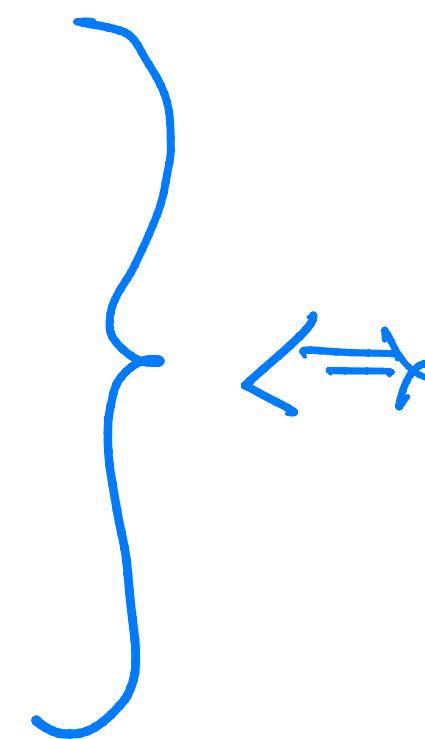
Factorization of Yoneda



K Compact :

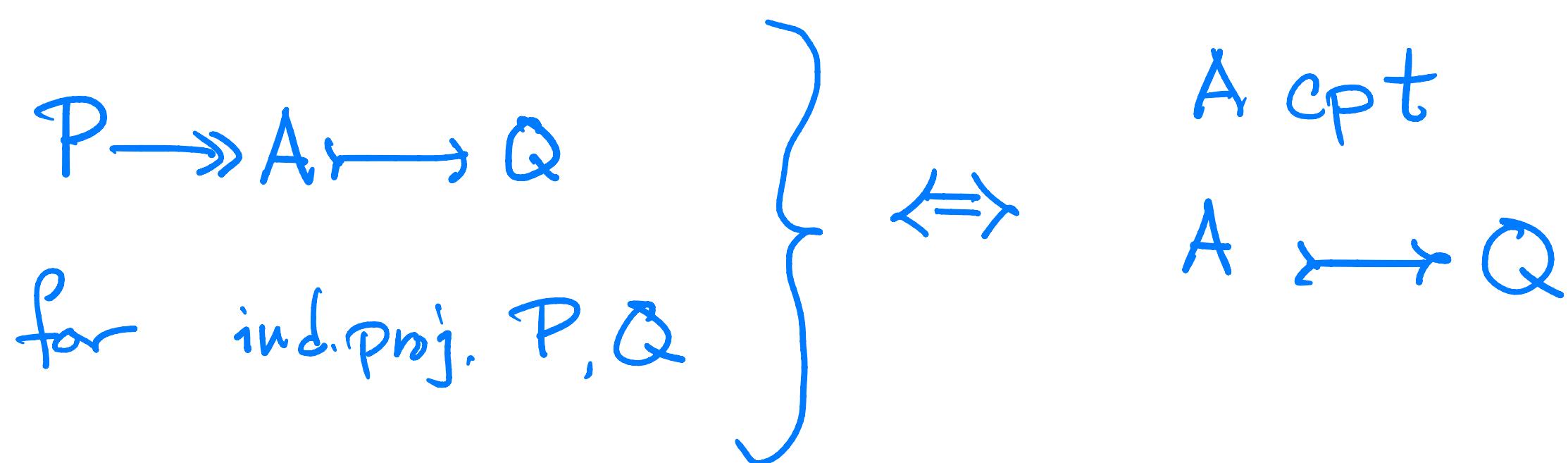
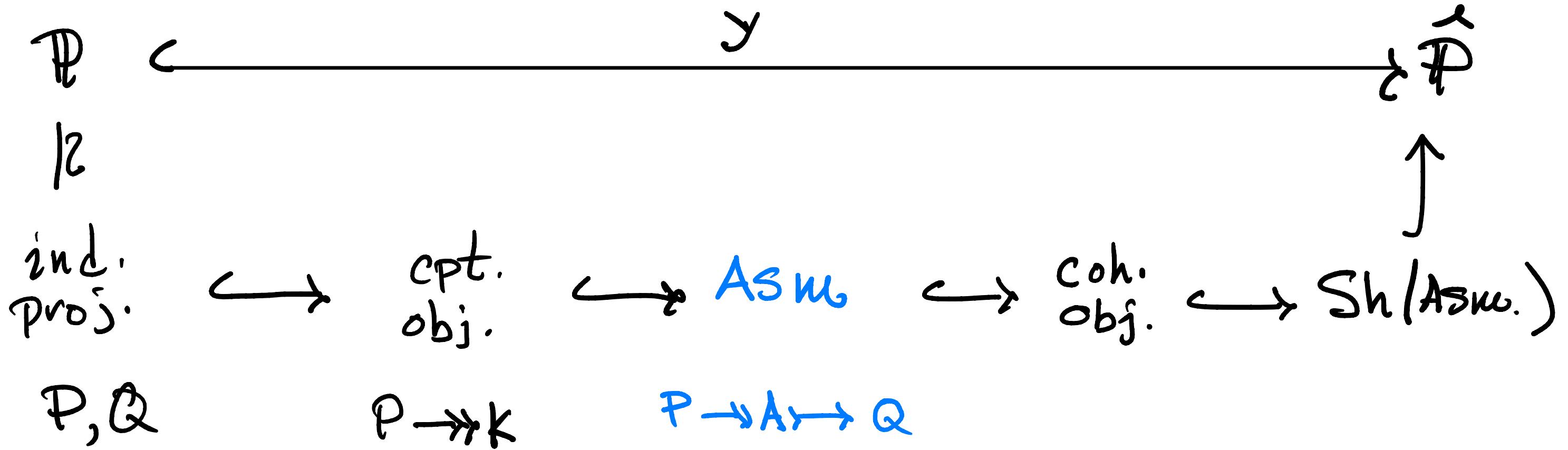
$(X_i \rightarrow K)_i$ covers

$\nexists X_k \rightarrow K$ f. some k

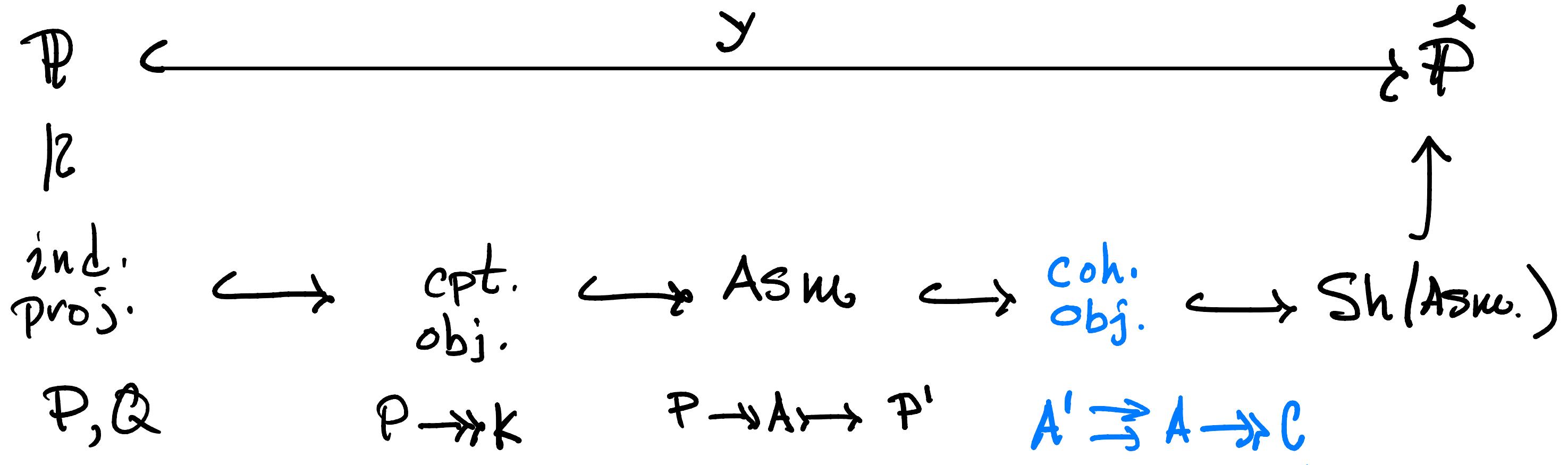


$P \rightarrow\!\!\! \rightarrow K$
f. some P

Factorization of Yoneda



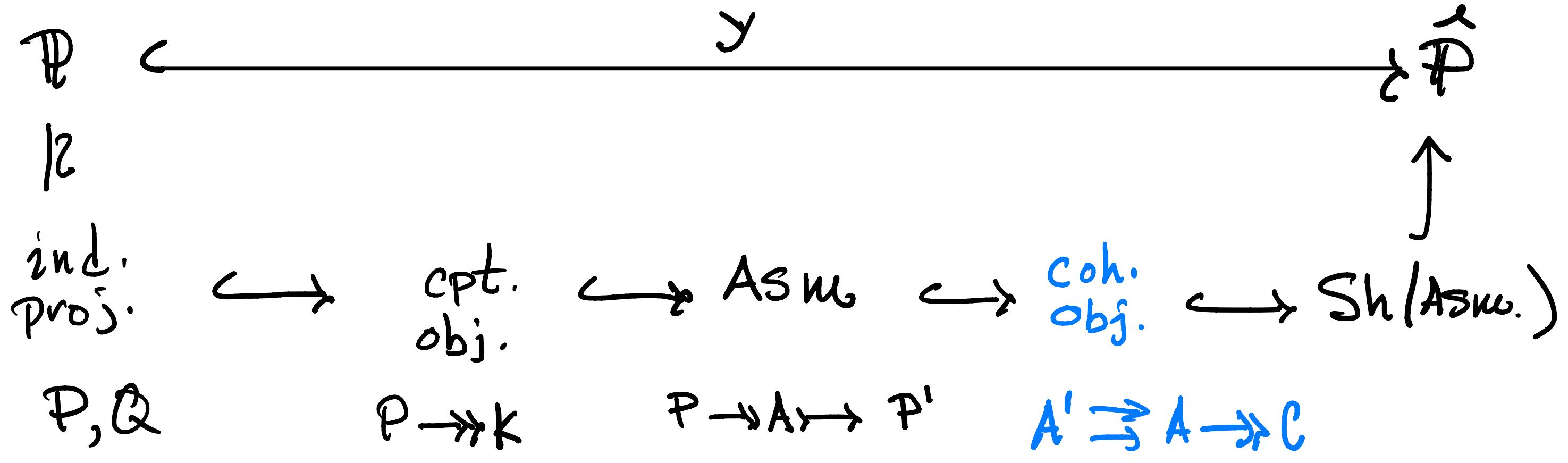
Factorization of Yoneda



C coherent :

- C is cpt.
- $C \xrightarrow{\Delta} C \times C$ cpt.

Factorization of Yoneda



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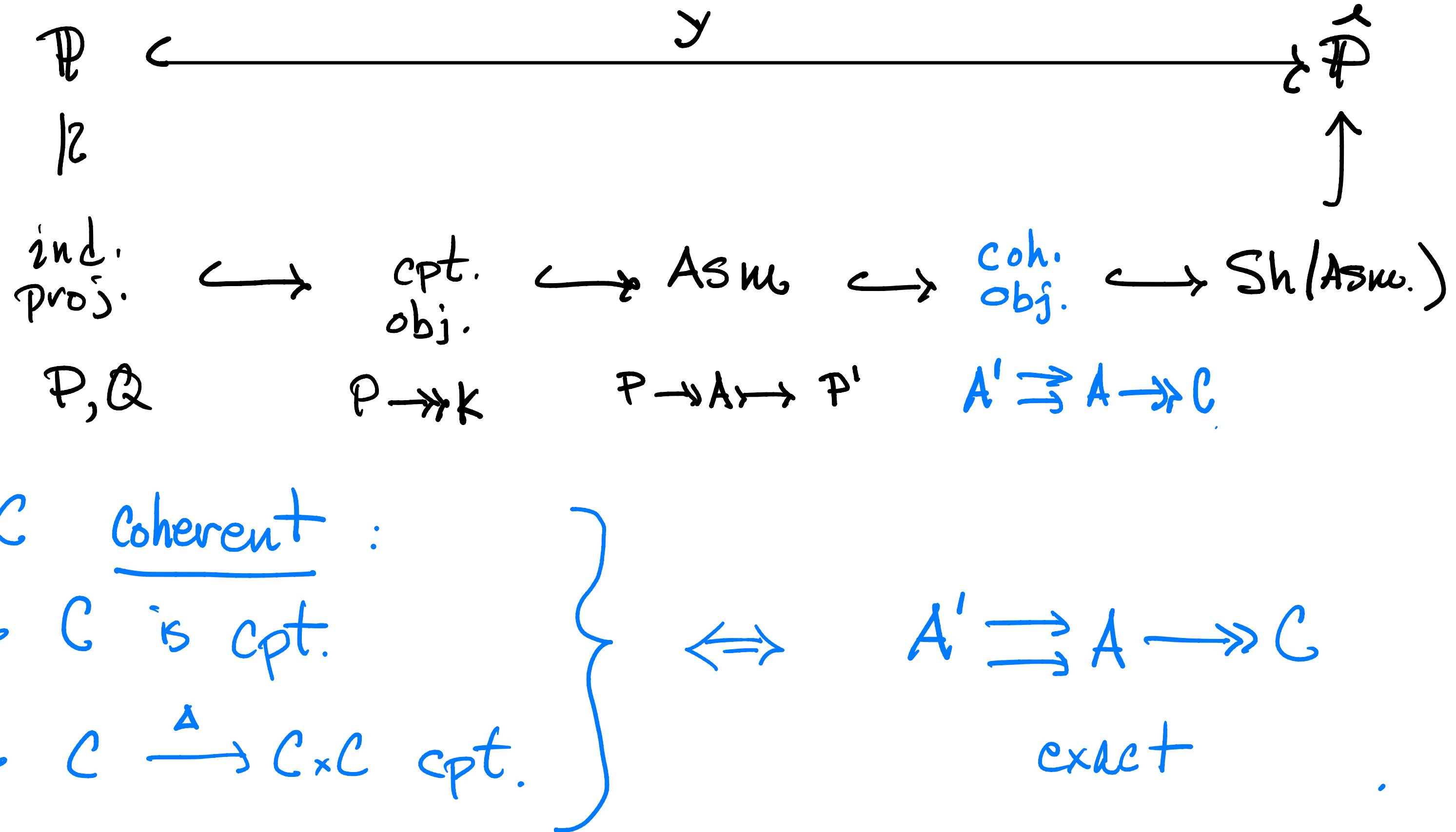
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where

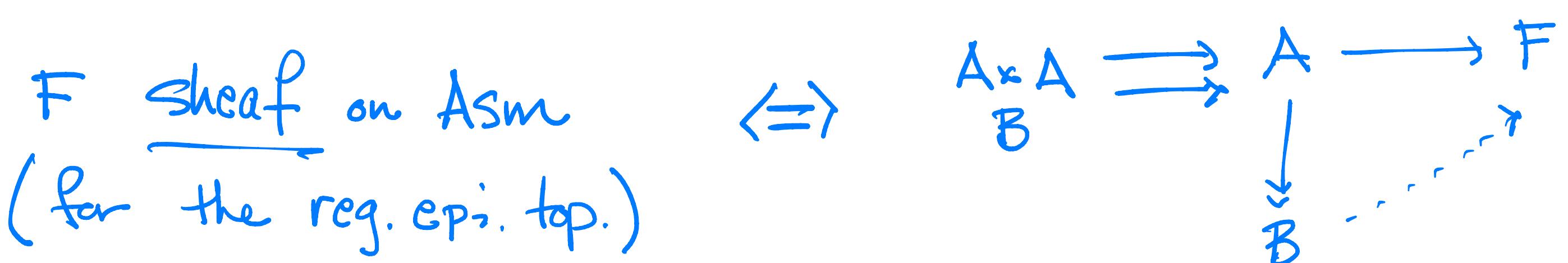
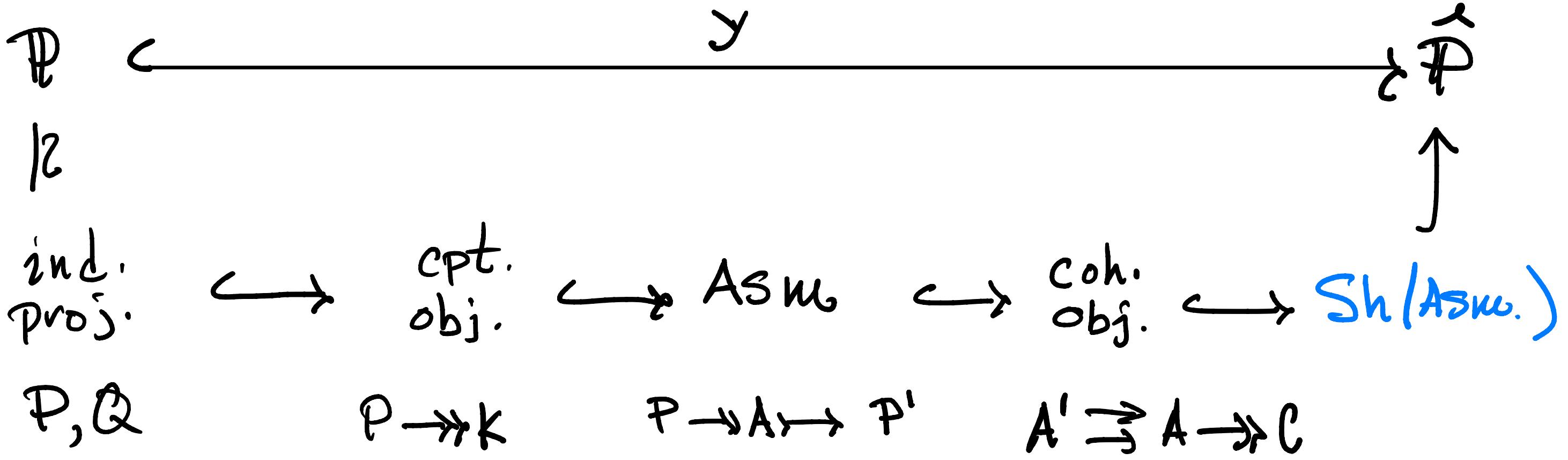
$X \rightarrowtail Y$ is cpt :

$$\begin{array}{ccc} K' & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow \\ K & \longrightarrow & Y \end{array}$$

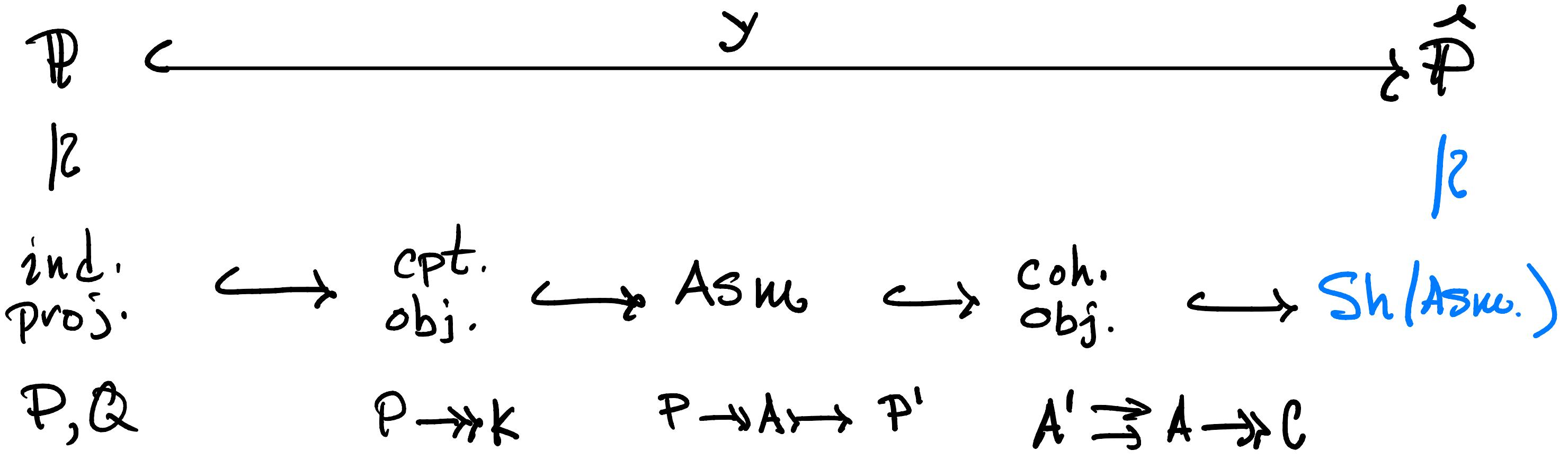
Factorization of Yoneda



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Factorization of Yoneda



F sheaf on ASM
 (for the reg. epis. top.)

$$\begin{array}{ccc}
 A \times_A & \xrightarrow{\quad} & A \longrightarrow F \\
 \downarrow_B & & \downarrow \pi \\
 B & \dashrightarrow & B
 \end{array}$$

$$\Leftrightarrow \quad F \in \hat{P} = [P^{\text{op}}, \text{Set}]$$

Thm (Lack)

For a reg. cat. \mathcal{R} ,

$$\mathcal{R}_{\text{ex/reg}} \subseteq \text{Sh}(\mathcal{R}, \text{reg epi})$$

$$= \langle E \mid yA' \xrightarrow{\quad} yA \rightarrowtail E \rangle$$

f. $A, A' \in \mathcal{R}$

Thm (Lack)

For a reg. cat. R ,

$$R_{\text{ex/reg}} \subseteq \text{Sh}(R, \text{reg epi})$$

$$= \left\langle E \mid yA' \xrightarrow{\quad} yA \rightarrowtail E \right\rangle$$

f. $A, A' \in R$

Cor. $\text{Sh}(\text{Asm})_{\text{coh}} = \text{ex/reg}(\text{Asm}).$

Thm (Lack)

For a reg. Cat. R ,

$$\begin{aligned} R_{\text{ex/reg}} &\subseteq \text{Sh}(R, \text{reg epi}) \\ &= \left\langle E \mid yA' \xrightarrow{\exists} yA \rightarrowtail E \right\rangle \\ &\quad \text{f. } A, A' \in R \end{aligned}$$

Cor. $\text{Sh}(R)_{\text{coh}} = \text{ex/reg(Asm)} = \text{eff!}$

Summary

$\text{PAsm} \subset \text{Asm} \subset \text{Eff} \subset \overset{\wedge}{\text{PAsm}}$

↓ ↓

$\text{CohSh}(\text{Asm}) \subset \text{Sh}(\text{Asm})$

Groth. topos !

Summary

$$\text{PAsm} \subset \text{Asm} \subset \text{Eff} \subset \overset{1}{\text{PAsm}}$$

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Now in order to get a 2-topos we take
internal groupoids in the Groth. topos $\text{Sh}(\text{Asm})$,

 $\text{Gpd}(\text{Sh}(\text{Asm}))$.

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Now in order to get a 2-topos we take
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$$\text{Gpd}(\text{Sh}(\text{Asm})) = \text{Gpd}([\mathbb{P}^{\text{op}}, \text{Set}]) = [\mathbb{P}^{\text{op}}, \text{Gpd}]$$

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Finally, we shall restrict to a subcat of
 "Coherent groupoids" in such a way that

$$\begin{array}{ccc} \mathrm{Coh}\mathrm{Gpd} & \hookrightarrow & \mathrm{Gpd} \\ \downarrow & \lrcorner & \uparrow \\ \mathrm{Eff} & \hookrightarrow & \mathrm{Gpd}_0 \end{array}$$

$$\begin{aligned} &\text{in } \mathrm{Sh}(\mathrm{Asm}) \\ &= [\mathbb{P}^{\mathrm{op}}, \mathrm{Gpd}] . \end{aligned}$$

QMS on $\text{Gpd}(\mathcal{E})$

There are many ways to put a QMS on $\text{Gpd}(\mathcal{E})$ for Groth. bpos \mathcal{E} . We choose one that:

- 1) is constructive,
- 2) works well with the groupoid model of HoTT,
- 3) gives \mathcal{E}^{ff} as the coherent o-types:

$$\mathcal{E}^{\text{ff}} = \text{Coh } \text{Gpd}(\mathcal{E})_o \subseteq \text{Gpd}(\mathcal{E})_o \subseteq \text{Gpd}(\mathcal{E}).$$

Using results of Everaert, Kieboom, vd Linden, we can show:

Thm. There's a **model structure** on $\text{Gpd}(\hat{\mathcal{P}})$ with:

- **Weak equivalences** = equivalences of cats $\mathcal{G} \simeq \mathcal{H}$,
- **Cofibrations** = functors $c: A \rightarrow B$ s.t.
 c_0 is a retract of $A_0 \rightarrow A_0 + C$, f.s. $C \in \hat{\mathcal{P}}$,
- **Fibrations** = f with
"internal path lifting":
$$\begin{array}{ccc} I & \xrightarrow{\quad} & E \\ \downarrow & \nearrow & \downarrow f \\ I' & \xrightarrow{\quad} & B \end{array}$$

Cohesive Groupoid

Def. A groupoid $G = (G, \rightrightarrows G_0)$ in $\hat{\mathcal{P}}$ is
Cohesive if :

- G_0 is cpt: $P \rightarrowtail G_0$ f. some $P \in \mathcal{P}$,
- $G_1 \rightarrow G_0 \times G_0$ is a cpt map.
- ... tbd

Prop. Let $\mathbb{G} = (G_1 \rightrightarrows G_0)$ be a coherent gpd and a 0-type $\mathbb{G}^I \rightarrow \mathbb{G}$.

then $\pi_0 \mathbb{G}$ is a coherent object.

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Pf. Since G_0 is cpt it has a cover $P \rightarrow G_0$.

Take the p.b. K and its cover P' , since Δ cpt.

$$\begin{array}{ccccc}
 P' & \twoheadrightarrow & K & \longrightarrow & G_1 \\
 & \searrow & \downarrow & & \downarrow \Delta \\
 & & P \times P & \longrightarrow & G_0 \times G_0
 \end{array}$$

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$\Delta: G_1 \rightarrow G_0 \times G_0$ is monic.

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Hence so is $K \rightarrowtail P \times P$,

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 & & & & \xrightarrow{\quad} & \bar{G} \times \bar{G}
 \end{array}$$

Write $\bar{\mathbb{G}} = G_0/G_1 = \pi_0 \mathbb{G}$.

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We then have $K \rightarrow P \rightarrow \bar{\mathbb{G}}$ exact.

$$\begin{array}{ccccccc}
 P' & \longrightarrow & K & \longrightarrow & G_1 & \longrightarrow & \bar{\mathbb{G}} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 P \times P & \longrightarrow & G_0 \times G_0 & \longrightarrow & \bar{\mathbb{G}} \times \bar{\mathbb{G}}
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So $\bar{\mathbb{G}} = \pi_0 \mathbb{G}$ is coherent. \square

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 \end{array}$$

Then In $\text{Sh}(\text{AS}_K) = \hat{P}$, we have

$$\begin{array}{ccc} \mathcal{E}\mathbf{ff} & \cong & \text{CohGpd}_0 \longrightarrow \text{Gpd}_0 \\ & & \downarrow \\ & & \text{CohGpd} \longrightarrow \text{Gpd} \end{array} .$$

Then In $\text{Sh}(\text{As}_\mathcal{U}) = \hat{\mathbb{P}}$, we have

$$\begin{array}{ccc}
 \mathfrak{Eff} & \cong & \text{CohGpd}_0 \longrightarrow \text{Gpd}_0 \\
 & & \downarrow \\
 \mathfrak{Eff}^2 := & \text{CohGpd} & \longrightarrow \text{Gpd} \\
 & & \downarrow
 \end{array}.$$

Proposal Take $\mathfrak{Eff}^2 = \text{CohGpd}(\text{Sh}(\text{As}_\mathcal{U}))$,

a 2-topos with $\mathfrak{Eff}_0 = \mathfrak{Eff}$.

FINITOS !