Introduction to Categorical Logic

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Chapter 2

Propositional Logic

Propositional logic is the logic of propositional connectives like $p \wedge q$ and $p \Rightarrow q$. As was the case for algebraic theories, the general approach will be to determine suitable categorical structures to model the logical operations, and then use categories with such structure to represent (abstract) propositional theories. Adjoints will play a special role, as we will describe the basic logical operations as such. We again show that the semantics is "functorial", meaning that the models of a theory are functors that preserve the categorical structure. We will show that there are classifying categories for all propositional theories, as was the case for the algebraic theories that we have already met.

A more abstract, algebraic perspective will then relate the propositional case of syntax-semantics duality with classical Stone duality for Boolean algebras, and related results from lattice theory will provide an algebraic treatment of Kripke semantics for intuitionistic (and modal) propositional logic.

2.1 Propositional calculus

Before going into the details of the categorical approach, we first briefly review the propositional calculus from a conventional point of view, as we did for algebraic theories. We focus first on the *classical* propositional logic, before considering the intuitionistic case in Section 2.9.

In the style of Section **??**, we have the following (abstract) syntax for (propositional) formulas:

$$\begin{aligned}
\text{Propositional variable } p \ &::= \ \mathsf{p}_1 \mid \mathsf{p}_2 \mid \mathsf{p}_3 \mid \cdots \\
\text{Propositional formula } \phi \ &::= \ p \mid \top \mid \bot \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid \phi_1 \Rightarrow \phi_2 \mid \phi_1 \Leftrightarrow \phi_2
\end{aligned}$$

An example of a formula is therefore $\big(\mathsf{p}_3 \Leftrightarrow ((((\neg\mathsf{p}_1) \vee (\mathsf{p}_2 \wedge \bot)) \vee \mathsf{p}_1) \Rightarrow \mathsf{p}_3)\big)$. We will make use of the usual conventions for parenthesis, with binding order $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$. Thus e.g. the foregoing may also be written unambiguously as $\mathsf{p}_3 \Leftrightarrow \neg\mathsf{p}_1 \vee \mathsf{p}_2 \wedge \bot \vee \mathsf{p}_1 \Rightarrow \mathsf{p}_3$.

Natural deduction

The system of *natural deduction* for propositional logic has one form of judgement

$$\mathtt{p}_1, \ldots, \mathtt{p}_n \mid \phi_1, \ldots, \phi_m \vdash \phi$$

where $\mathtt{p}_1, \ldots, \mathtt{p}_n$ is a *context* consisting of distinct propositional variables, the formulas $\phi_1, \ldots, \phi_m$ are the *hypotheses* and ϕ is the *conclusion*. The variables in the hypotheses and the conclusion must occur among those listed in the context. The hypotheses are regarded as a (finite) set; so they are unordered, have no repetitions, and may be empty. We may abbreviate the context of variables by Γ, and we often omit it.

Deductive entailment (or *derivability*) $\Phi \vdash \phi$ is thus a relation between finite sets of formulas Φ and single formulas ϕ. It is defined as the smallest such relation satisfying the following rules:

1. Hypothesis:

$$\frac{}{\Phi \vdash \phi} \text{ if } \phi \text{ occurs in } \Phi$$

2. Truth:

$$\frac{}{\Phi \vdash \top}$$

3. Falsehood:

$$\frac{\Phi \vdash \bot}{\Phi \vdash \phi}$$

4. Conjunction:

$$\frac{\Phi \vdash \phi \qquad \Phi \vdash \psi}{\Phi \vdash \phi \wedge \psi} \qquad \frac{\Phi \vdash \phi \wedge \psi}{\Phi \vdash \phi} \qquad \frac{\Phi \vdash \phi \wedge \psi}{\Phi \vdash \psi}$$

5. Disjunction:

$$\frac{\Phi \vdash \phi}{\Phi \vdash \phi \vee \psi} \qquad \frac{\Phi \vdash \psi}{\Phi \vdash \phi \vee \psi} \qquad \frac{\Phi \vdash \phi \vee \psi \qquad \Phi, \phi \vdash \theta \qquad \Phi, \psi \vdash \theta}{\Phi \vdash \theta}$$

6. Implication:

$$\frac{\Phi, \phi \vdash \psi}{\Phi \vdash \phi \Rightarrow \psi} \qquad \frac{\Phi \vdash \phi \Rightarrow \psi \qquad \Phi \vdash \phi}{\Phi \vdash \psi}$$

For the purpose of deduction, we define $\neg\phi := \phi \Rightarrow \bot$ and $\phi \Leftrightarrow \psi := (\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi)$. To obtain *classical* logic we need only include one of the following additional rules.

7. Classical logic:

$$\frac{}{\Phi \vdash \phi \vee \neg\phi} \qquad \frac{\Phi \vdash \neg\neg\phi}{\Phi \vdash \phi}$$

A *proof* of a judgement $\Phi \vdash \phi$ is a *finite* tree built from the above inference rules whose root is $\Phi \vdash \phi$. For example, here is a proof of $\phi \vee \psi \vdash \psi \vee \phi$ using the disjunction rules:

$$\dfrac{\dfrac{}{\phi \vee \psi \vdash \phi \vee \psi} \qquad \dfrac{\dfrac{}{\phi \vee \psi, \phi \vdash \phi}}{\phi \vee \psi, \phi \vdash \psi \vee \phi} \qquad \dfrac{\dfrac{}{\phi \vee \psi, \psi \vdash \psi}}{\phi \vee \psi, \psi \vdash \psi \vee \phi}}{\phi \vee \psi \vdash \psi \vee \phi}$$

A judgment $\Phi \vdash \phi$ is *provable* if there exists a proof of it. Observe that every proof has at its leaves either the rule for $\top$ or an instance of the rule of hypothesis (or the Excluded Middle rule for classical logic).

Remark 2.1.1. An alternate form of presentation for proofs in natural deduction that is more, well, natural uses trees of formulas, rather than of judgements, with leaves labelled by *assumptions* ϑ that may also occur in *cancelled* form $[\vartheta]$. Thus for example the introduction and elimination rules for conjunction would be written in the form:

$$\dfrac{\begin{matrix}\Phi \\ \vdots \\ \phi\end{matrix} \qquad \begin{matrix}\Phi \\ \vdots \\ \psi\end{matrix}}{\phi \wedge \psi} \qquad\qquad \dfrac{\begin{matrix}\Phi \\ \vdots \\ \phi \wedge \psi\end{matrix}}{\phi} \qquad \dfrac{\begin{matrix}\Phi \\ \vdots \\ \phi \wedge \psi\end{matrix}}{\psi}$$

An example of a proof tree with cancelled assumptions is the one for disjunction elimination:

$$\dfrac{\begin{matrix}\Phi \\ \vdots \\ \phi \vee \psi\end{matrix} \qquad \begin{matrix}\Phi, [\phi] \\ \vdots \\ \vartheta\end{matrix} \qquad \begin{matrix}\Phi, [\psi] \\ \vdots \\ \vartheta\end{matrix}}{\vartheta}$$

And the above rule of implication introduction takes the form:

$$\dfrac{\begin{matrix}\Phi, [\phi] \\ \vdots \\ \psi\end{matrix}}{\phi \Rightarrow \psi}$$

In these examples, the cancellation occurred at the last step. In order to continue such a proof, we need a device to indicate *when* a cancellation occurs, *i.e.* at which step of the proof. This can be done as follows:

$$\dfrac{\dfrac{\begin{matrix}\Phi, [\alpha]^2 \\ \vdots \\ \phi \vee \psi\end{matrix} \qquad \begin{matrix}\Phi, [\phi]^1 \\ \vdots \\ \vartheta\end{matrix} \qquad \begin{matrix}\Phi, [\psi]^1 \\ \vdots \\ \vartheta\end{matrix}}{\vartheta}\,(1)}{\alpha \Rightarrow \vartheta}\,(2)$$

This proof tree represents a derivation of the judgement $\Phi \vdash \alpha \Rightarrow \vartheta$. A proof tree in which all the assumptions have been cancelled represents a derivation of an unconditional judgement such as $\vdash \phi$.

We will have a better way to record such proofs in Section **??**.

Exercise 2.1.2. Derive each of the two classical rules (2.1), called *Excluded Middle* and *Double Negation*, from the other.

2.2 Truth values

The idea of an axiomatic system of deductive, logical reasoning goes to back to Frege, who gave the first such system for propositional calculus (and more) in his *Begriffsschrift* of 1879. The question soon arose whether Frege's rules (or rather, their derivable consequences — it was clear that one could chose the primitive basis in different but equivalent ways) were correct, and if so, whether they were *all* the correct ones. An ingenious solution was proposed by Russell's student Wittgenstein, who came up with an entirely different way of singling out a set of "valid" propositional formulas in terms of assignments of truth values to the variables occurring in them. He interpreted this as showing that logical validity was really a matter of the logical structure of a proposition, rather than depending any particular system of derivations. The same idea seems to have been had independently by Post, who proved that the valid propositional formulas coincide with the ones derivable in Whitehead and Russell's *Principia Mathematica* (which is propositionally equivalent to Frege's system), a fact that we now refer to as the *soundness* and *completeness* of propositional logic.

In more detail, let a *valuation* v be an assignment of a "truth-value" $0, 1$ to each propositional variable, $v(\mathtt{p}_n) \in \{0, 1\}$. We can then extend the valuation to all propositional formulas $[\![\phi]\!]^v$ by the following recursion.

$$
\begin{aligned}
[\![\mathtt{p}_n]\!]^v &= v(\mathtt{p}_n) \\
[\![\top]\!]^v &= 1 \\
[\![\bot]\!]^v &= 0 \\
[\![\neg\phi]\!]^v &= 1 - [\![\phi]\!]^v \\
[\![\phi \wedge \psi]\!]^v &= \min([\![\phi]\!]^v, [\![\psi]\!]^v) \\
[\![\phi \vee \psi]\!]^v &= \max([\![\phi]\!]^v, [\![\psi]\!]^v) \\
[\![\phi \Rightarrow \psi]\!]^v &= 1 \text{ iff } [\![\phi]\!]^v \leq [\![\psi]\!]^v \\
[\![\phi \Leftrightarrow \psi]\!]^v &= 1 \text{ iff } [\![\phi]\!]^v = [\![\psi]\!]^v
\end{aligned}
$$

This is sometimes expressed using the "semantic consequence" notation $v \vDash \phi$ to mean that $[\![\phi]\!]^v = 1$. The above specification then takes the following form, in which the condition for the truth of a formula is given in terms of its informal “meaning”:

$$
\begin{array}{rll}
v \vDash \top & \text{always} & \\
v \vDash \bot & \text{never} & \\
v \vDash \neg\phi & \text{iff} & \text{not } v \vDash \phi \\
v \vDash \phi \wedge \psi & \text{iff} & v \vDash \phi \text{ and } v \vDash \psi \\
v \vDash \phi \vee \psi & \text{iff} & v \vDash \phi \text{ or } v \vDash \psi \\
v \vDash \phi \Rightarrow \psi & \text{iff} & v \vDash \phi \text{ implies } v \vDash \psi \\
v \vDash \phi \Leftrightarrow \psi & \text{iff} & v \vDash \phi \text{ iff } v \vDash \psi
\end{array}
$$

Finally, ϕ is *valid*, written $\vDash \phi$, is defined by,

$$
\begin{array}{rll}
\vDash \phi & \text{iff} & v \vDash \phi \text{ for all } v \\
 & \text{iff} & [\![\phi]\!]^v = 1 \text{ for all } v\,.
\end{array}
$$

And, more generally, we define $\phi_1, ..., \phi_n$ *semantically entails* ϕ, written

$$\phi_1, ..., \phi_n \vDash \ \phi, \tag{2.1}$$

to mean that for all valuations v such that $v \vDash \phi_k$ for all k, also $v \vDash \phi$.

Given a formula in context $\Gamma \mid \phi$ and a valuation v for the variables in Γ, one can check whether $v \vDash \phi$ using a *truth table*, which is a systematic way of calculating the value of $[\![\phi]\!]^v$. For example, under the assignment $v(\mathtt{p}_1) = 1, v(\mathtt{p}_2) = 0, v(\mathtt{p}_3) = 1$ we can calculate $[\![\phi]\!]^v$ for $\phi = \big(\mathtt{p}_3 \Leftrightarrow ((((\neg\mathtt{p}_1) \vee (\mathtt{p}_2 \wedge \bot)) \vee \mathtt{p}_1) \Rightarrow \mathtt{p}_3)\big)$ as follows.

$\mathtt{p}_1$	$\mathtt{p}_2$	$\mathtt{p}_3$	$\mathtt{p}_3$	$\Leftrightarrow$	$\neg$	$\mathtt{p}_1$	$\vee$	$\mathtt{p}_2$	$\wedge$	$\bot$	$\vee$	$\mathtt{p}_1$	$\Rightarrow$	$\mathtt{p}_3$
1	0	1	1	1	0	1	0	0	0	0	1	1	1	1

The value of the formula ϕ under the valuation v is then the value in the column under the main connective, in this case $\Leftrightarrow$, and thus $[\![\phi]\!]^v = 1$.

Displaying all 2^3 valuations for the context $\Gamma = \mathtt{p}_1, \mathtt{p}_2, \mathtt{p}_3$, therefore results in a table that checks for validity of ϕ,

$\mathtt{p}_1$	$\mathtt{p}_2$	$\mathtt{p}_3$	$\mathtt{p}_3$	$\Leftrightarrow$	$\neg$	$\mathtt{p}_1$	$\vee$	$\mathtt{p}_2$	$\wedge$	$\bot$	$\vee$	$\mathtt{p}_1$	$\Rightarrow$	$\mathtt{p}_3$
1	1	1	.	1	...									
1	1	0	.	1		...								
1	0	1	1	1	0	1	0	0	0	0	1	1	1	1
1	0	0	.	1				...						
0	1	1	.	1					...					
0	1	0	.	1						...				
0	0	1	.	1							...			
0	0	0	.	1								...		

In this case, working out the other rows shows that ϕ is indeed valid, thus $\vDash \phi$.

Theorem 2.2.1 (Soundness and Completeness of Propositional Calculus)**.** *Let Φ be any set of formulas and ϕ any formula, then*

$$\Phi \vdash \phi \iff \Phi \vDash \phi.$$

In particular, for any propositional formula ϕ we have

$$\vdash \phi \iff \vDash \phi.$$

Thus derivability and validity coincide.

Proof. Let us sketch the usual proof, for later reference.

(*Soundness*:) First assume $\Phi \vdash \phi$ is provable, meaning there is a finite derivation of $\Phi \vdash \phi$ by the rules of inference. We show by induction on the set of derivations that $\Phi \vDash \phi$, meaning that for any valuation v such that $v \vDash \Phi$ also $v \vDash \phi$. For this, observe that in each individual rule of inference, if $\Psi \vDash \psi$ for all the premisses of the rule, then $\Phi \vDash \phi$ for the conclusion (the set of premisses may change from the premisses to the conclusion if the rule involves a cancellation).

(*Competeness*:) Suppose that $\Phi \nvdash \phi$, then $\Phi, \neg\phi \nvdash \bot$ (using double negation elimination). By Lemma 2.2.2 below, there is a valuation v such that $v \vDash \{\Phi, \neg\phi\}$. Thus in particular $v \vDash \Phi$ and $v \nvDash \phi$, therefore $\Phi \nvDash \phi$. □

The key lemma is this:

Lemma 2.2.2 (Model Existence)**.** *If a set Φ of formulas is* consistent*, in the sense that $\Phi \nvdash \bot$, then it has a* model*, i.e. a valuation v such that $v \vDash \Phi$.*

Proof. Let Φ be any consistent set of formulas. We extend $\Phi \subseteq \Psi$ to one that is *maximally consistent*, meaning Ψ is consistent, and if $\Psi \subseteq \Psi'$ and Ψ' is consistent, then $\Psi = \Psi'$. Enumerate the formulas $\phi_0, \phi_1, ...$, and let,

$$\begin{aligned} \Phi_0 &= \Phi, \\ \Phi_{n+1} &= \Phi_n \cup \phi_n \text{ if consistent, else } \Phi_n, \\ \Psi &= \textstyle\bigcup_n \Phi_n. \end{aligned}$$

One can then show that Ψ is indeed maximally consistent, and for every formula ψ, either $\psi \in \Psi$ or $\neg\psi \in \Psi$ and not both (exercise!). Now for each propositional variable $\mathtt{p}$, define $v_\Psi(\mathtt{p}) = 1$ just if $\mathtt{p} \in \Psi$. Finally, one shows that $[\![\phi]\!]^{v_\Psi} = 1$ just if $\phi \in \Psi$, and therefore $v_\Psi \vDash \Psi \supseteq \Phi$. □

Exercise 2.2.3. Show that for any maximally consistent set Ψ of formulas, either $\psi \in \Psi$ or $\neg\psi \in \Psi$ and not both. Conclude from this that for the valuation v_Ψ defined by $v_\Psi(\mathtt{p}) = 1$ just if $\mathtt{p} \in \Psi$, we indeed have $[\![\phi]\!]^{v_\Psi} = 1$ just if $\phi \in \Psi$, as claimed in the proof of the Model Existence Lemma 2.2.2.

2.3 Boolean algebra

There is of course another approach to propositional logic, which also goes back to the 19th century, namely that of Boolean algebra, which draws on the analogy between the propositional operations and the arithmetical ones.

Definition 2.3.1. A *Boolean algebra* is a set B equipped with the operations:

$$\begin{aligned} 0,1 : 1 &\to B \\ \neg : B &\to B \\ \wedge, \vee : B \times B &\to B \end{aligned}$$

satisfying the following equations:

$$\begin{array}{rl} x \vee x = x & x \wedge x = x \\ x \vee y = y \vee x & x \wedge y = y \wedge x \\ x \vee (y \vee z) = (x \vee y) \vee z & x \wedge (y \wedge z) = (x \wedge y) \wedge z \\ x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) & x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \\ 0 \vee x = x & 1 \wedge x = x \\ 1 \vee x = 1 & 0 \wedge x = 0 \\ \neg(x \vee y) = \neg x \wedge \neg y & \neg(x \wedge y) = \neg x \vee \neg y \\ x \vee \neg x = 1 & x \wedge \neg x = 0 \end{array}$$

This is of course an algebraic theory, like those considered in the previous chapter. Familiar examples of Boolean algebras are $2 = \{0, 1\}$, with the usual operations, and more generally, any powerset $\mathcal{P}X$, with the set-theoretic operations $A \vee B = A \cup B$, etc. (indeed, $2 = \mathcal{P}1$ is a special case.).

Exercise 2.3.2. Show that the free Boolean algebra $B(n)$ on n-many generators is the double powerset $\mathcal{P}\mathcal{P}(n)$, and determine the free functor on finite sets.

One can use equational reasoning in Boolean algebra as an alternative to the deductive propositional calculus as follows. For a propositional formula in context $\Gamma \mid \phi$, let us say that ϕ is *equationally provable* if we can prove $\phi = 1$ by equational reasoning (Section **??**), from the laws of Boolean algebras above. More generally, for a set of formulas Φ and a formula ψ let us define the (*ad hoc*) relation of *equational provability*,

$$\Phi \vdash_{\mathsf{eq}} \psi \tag{2.2}$$

to mean that $\psi = 1$ can be proven equationally from (the Boolean equations and) the set of all equations $\phi = 1$, for $\phi \in \Phi$. Since we don't have any laws for the connectives $\Rightarrow$ or $\Leftrightarrow$, let us replace them with their Boolean equivalents, by adding the equations:

$$\begin{aligned} \phi \Rightarrow \psi &= \neg\phi \vee \psi\,, \\ \phi \Leftrightarrow \psi &= (\neg\phi \vee \psi) \wedge (\neg\psi \vee \phi)\,. \end{aligned}$$

Here for example is an equational proof of $(\phi \Rightarrow \psi) \vee (\psi \Rightarrow \phi)$.

$$\begin{aligned}
(\phi \Rightarrow \psi) \vee (\psi \Rightarrow \phi) &= (\neg\phi \vee \psi) \vee (\neg\psi \vee \phi) \\
&= \neg\phi \vee (\psi \vee (\neg\psi \vee \phi)) \\
&= \neg\phi \vee ((\psi \vee \neg\psi) \vee \phi) \\
&= \neg\phi \vee (1 \vee \phi) \\
&= \neg\phi \vee 1 \\
&= 1 \vee \neg\phi \\
&= 1
\end{aligned}$$

Thus we have

$$\vdash_{\mathsf{eq}} (\phi \Rightarrow \psi) \vee (\psi \Rightarrow \phi)\,.$$

We now ask: *How is equational provability $\Phi \vdash_{\mathsf{eq}} \phi$ related to deductive entailment $\Phi \vdash \phi$ and semantic entailment $\Phi \vDash \phi$?*

Exercise 2.3.3. Using equational reasoning, show that every propositional formula ϕ has both a *conjunctive* $\phi^{\wedge}$ and a *disjunctive* $\phi^{\vee}$ *Boolean normal form* such that:

1. The formula $\phi^{\vee}$ is an n-fold disjunction of m-fold conjunctions of *positive* $\mathtt{p_i}$ or *negative* $\neg\mathtt{p_j}$ propositional variables,

   $$\phi^{\vee} \;=\; (\mathsf{q}_{11} \wedge \ldots \wedge \mathsf{q}_{1m_1}) \vee \ldots \vee (\mathsf{q}_{n1} \wedge \ldots \wedge \mathsf{q}_{nm_n})\,, \qquad \mathsf{q}_{ij} \in \{\mathsf{p}_{ij}, \neg\mathsf{p}_{ij}\}\,,$$

   and $\phi^{\wedge}$ is the same, but with the roles of $\vee$ and $\wedge$ reversed.

2. Both

   $$\vdash_{\mathsf{eq}} \phi \Leftrightarrow \phi^{\vee} \qquad \text{and} \qquad \vdash_{\mathsf{eq}} \phi \Leftrightarrow \phi^{\wedge}\,.$$

(*Hint:* Rewrite the formula in terms of just conjunction, disjunction, and negation, and then do both normal forms at the same time, by structural induction on the formula.)

Remark 2.3.4. We can already use Exercise 2.3.3 to show that equational provability is equivalent to semantic validity,

$$\vdash_{\mathsf{eq}} \phi \iff \vDash \phi\,.$$

To show this, we first put the formula ϕ into conjunctive normal form, and then read off a truth valuation that falsifies it, just if there is one. Indeed, the CNF is valued as 1 just if each conjunct is, and that holds just if each conjunct contains a propositional letter $\mathtt{p}$ in both positive and negative $\neg\mathtt{p}$ form. And in that case, the CNF clearly reduces to 1 by an equational calculation. Conversely, if the CNF does not so reduce, it must have a conjunct that does not satisfy the condition just stated – and so we can read off a valuation making all propositional letters in that conjunct 0.

Exercise 2.3.5. A Boolean algebra can be partially ordered by defining $x \leq y$ as

$$x \leq y \iff x \vee y = y \qquad \text{or equivalently} \qquad x \leq y \iff x \wedge y = x\,.$$

Thus a Boolean algebra is a (poset) category. Show that as a category, a Boolean algebra has all finite limits and colimits and is cartesian closed, with $x \Rightarrow y := \neg x \vee y$ as the exponential of x and y. Moreover, a finitely complete and cocomplete cartesian closed poset is a Boolean algebra just if it satisfies $x = (x \Rightarrow 0) \Rightarrow 0$. Finally, show that homomorphisms of Boolean algebras $f : B \to B'$ are the same thing as functors (i.e. monotone maps) that preserve all finite limits and colimits.

2.4 Lawvere duality for Boolean algebras

Let us apply the machinery of algebraic theories from Chapter **??** to the algebraic theory of Boolean algebras and see what we get. The algebraic theory $\mathbb{B}$ of Boolean algebras is a finite product (FP) category with objects $1, B, B^2, ...$, containing a Boolean algebra $\mathsf{U}_{\mathbb{B}}$, with underlying object $|\mathsf{U}_{\mathbb{B}}| = B$. By Theorem **??**, $\mathbb{B}$ has the universal property that finite product preserving (FP) functors from $\mathbb{B}$ into any FP-category $\mathcal{C}$ correspond (pseudo-)naturally to Boolean algebras in $\mathcal{C}$,

$$\mathsf{Hom}_{\mathsf{FP}}(\mathbb{B}, \mathcal{C}) \simeq \mathsf{BA}(\mathcal{C})\,. \tag{2.3}$$

The correspondence is mediated by evaluating an FP functor $F : \mathbb{B} \to \mathcal{C}$ at (the underlying structure of) the Boolean algebra $\mathsf{U}_{\mathbb{B}}$ to get a Boolean algebra $F(\mathsf{U}_{\mathbb{B}})$ in $\mathcal{C}$:

$$\frac{F : \mathbb{B} \longrightarrow \mathcal{C} \qquad \mathsf{FP}}{\overline{F(\mathsf{U}_{\mathbb{B}}) \qquad\quad \mathsf{BA}(\mathcal{C})}}$$

We call $\mathsf{U}_{\mathbb{B}}$ the *universal Boolean algebra*. Given a Boolean algebra B in $\mathcal{C}$, we write

$$\mathsf{B}^\sharp : \mathbb{B} \longrightarrow \mathcal{C}$$

for the associated *classifying functor*. By the equivalence of categories (2.3), we have isos,

$$\mathsf{B}^\sharp(\mathsf{U}_{\mathbb{B}}) \cong \mathsf{B}, \qquad F(\mathsf{B})^\sharp \cong F\,.$$

And in particular, $\mathsf{B}^\sharp \cong 1_{\mathbb{B}} : \mathbb{B} \to \mathbb{B}$.

By (the logical form of) Lawvere duality, Corollary **??**, we know that $\mathbb{B}^{\mathsf{op}}$ can be identified with a full subcategory $\mathsf{mod}(\mathbb{B})$ of $\mathbb{B}$-models in Set (i.e. Boolean algebras),

$$\mathbb{B}^{\mathsf{op}} = \mathsf{mod}(\mathbb{B}) \hookrightarrow \mathsf{Mod}(\mathbb{B}) = \mathsf{BA}(\mathsf{Set})\,, \tag{2.4}$$

namely, that consisting of the finitely generated free Boolean algebras $F(n) = PP([n])$ for $[n]$ an n-element set. Composing (2.4) and (2.3), we have an embedding of $\mathbb{B}^{\mathsf{op}}$ into the functor category,

$$\mathbb{B}^{\mathsf{op}} \hookrightarrow \mathsf{BA}(\mathsf{Set}) \simeq \mathsf{Hom}_{\mathsf{FP}}(\mathbb{B}, \mathsf{Set}) \hookrightarrow \mathsf{Set}^{\mathbb{B}}\,, \tag{2.5}$$

which, up to isomorphism, is just the (contravariant) Yoneda embedding, taking $B^n \in \mathbb{B}$ to the covariant representable functor $\mathsf{y}^{\mathbb{B}}(B^n) = \mathsf{Hom}_{\mathbb{B}}(B^n, -)$ (cf. Theorem **??**).

Now let us consider provability of equations between terms $\phi : B^n \to B$ in the theory $\mathbb{B}$, which are essentially the same as propositional formulas in context $(\mathsf{p}_1, ..., \mathsf{p}_n \mid \phi)$ modulo $\mathbb{B}$-provable equality. The universal Boolean algebra $\mathsf{U}_{\mathbb{B}}$ is logically generic, in the sense that for any such formulas ϕ, ψ, we have $\mathsf{U}_{\mathbb{B}} \vDash \phi = \psi$ just if $\mathbb{B} \vdash \phi = \psi$ (Proposition **??**). The latter condition is equational provability from the axioms for Boolean algebras, which was used in the definition of $\vdash_{\mathsf{eq}} \phi$ (cf. 2.2). So we have:

$$\vdash_{\mathsf{eq}} \phi \iff \mathbb{B} \vdash \phi = 1 \iff \mathsf{U}_{\mathbb{B}} \vDash \phi = 1\,.$$

As we showed in Proposition **??**, the image of the universal model $\mathsf{U}_{\mathbb{B}}$ under the (FP) *covariant* Yoneda embedding,

$$\mathsf{y}_{\mathbb{B}} : \mathbb{B} \to \mathsf{Set}^{\mathbb{B}^{\mathsf{op}}}$$

is also a logically generic model, with underlying object $|\mathsf{y}_{\mathbb{B}}(\mathsf{U}_{\mathbb{B}})| = \mathsf{Hom}_{\mathbb{B}}(-, B)$. By Proposition **??** we can use that fact to restrict attention to Boolean algebras in Set, and in particular, to the finitely generated free ones $F(n)$, when testing for equational provability. Specifically, using the (FP) evaluation functors $\mathsf{eval}_{B^n} : \mathsf{Set}^{\mathbb{B}^{\mathsf{op}}} \to \mathsf{Set}$ for all objects $B^n \in \mathbb{B}$, we can continue the above reasoning as follows:

$$\begin{aligned}
\vdash_{\mathsf{eq}} \phi &\iff \mathbb{B} \vdash \phi = 1 \\
&\iff \mathsf{U}_{\mathbb{B}} \vDash \phi = 1 \\
&\iff \mathsf{y}_{\mathbb{B}}(\mathsf{U}_{\mathbb{B}}) \vDash \phi = 1 \\
&\iff \mathsf{eval}_{B^n}\mathsf{y}_{\mathbb{B}}(\mathsf{U}_{\mathbb{B}}) \vDash \phi = 1 \quad \text{for all } B^n \in \mathbb{B} \\
&\iff F(n) \vDash \phi = 1 \quad \text{for all } n.
\end{aligned}$$

The last step holds because the image of $\mathsf{y}_{\mathbb{B}}(\mathsf{U}_{\mathbb{B}})$ under eval_{B^n} is exactly the free Boolean algebra $\mathsf{eval}_{B^n}\mathsf{y}_{\mathbb{B}}(\mathsf{U}_{\mathbb{B}}) = F(n)$ (cf. Exercise **??**). Indeed, for the underlying objects we have

$$\mathsf{eval}_{B^n}\mathsf{y}_{\mathbb{B}}(\mathsf{U}_{\mathbb{B}}) \cong \mathsf{Hom}_{\mathbb{B}}(B^n, B) \cong \mathsf{Hom}_{\mathsf{BA}^{\mathsf{op}}}(F(n), F(1)) \cong \mathsf{Hom}_{\mathsf{BA}}(F(1), F(n)) \cong |F(n)|\,.$$

Thus to test for equational provability it suffices to check the equations in the free algebras $F(n)$ (which makes sense, since $F(n)$ is usually *defined* in terms of equational provability). We have therefore shown:

Lemma 2.4.1. *A formula in context* $\mathsf{p}_1, ..., \mathsf{p}_k \mid \phi$ *is equationally provable* $\vdash_{\mathsf{eq}} \phi$ *just in case it holds in every finitely generated free Boolean algebra* $F(n)$, i.e. $F(n) \vDash \phi = 1$.

Recall that the condition $F(n) \vDash \phi = 1$ means that the equation $\phi = 1$ holds *generally* in $F(n)$, i.e. for any elements $f_1, ..., f_k \in F(n)$, we have $\phi[f_1/\mathsf{p}_1, ..., f_k/\mathsf{p}_k] = 1$, where the expression $\phi[f_1/\mathsf{p}_1, ..., f_k/\mathsf{p}_k]$ denotes the element of $F(n)$ resulting from interpreting the propositional variables p_i as the elements f_i and evaluating the resulting expression using the Boolean operations of $F(n)$. But now observe that the recipe:

for any elements $f_1, ..., f_k \in F(n)$, let the expression

$$\phi[f_1/\mathtt{p}_1, ..., f_k/\mathtt{p}_k] \tag{2.6}$$

denote the element of $F(n)$ resulting from interpreting the propositional variables $\mathtt{p}_i$ as the elements f_i and evaluating the resulting expression using the Boolean operations of $F(n)$

just describes the unique Boolean homomorphism

$$F(1) \xrightarrow{\overline{\phi}} F(k) \xrightarrow{\overline{(f_1, ..., f_k)}} F(n)\,,$$

where $\overline{(f_1, ..., f_k)} : F(k) \to F(n)$ is determined by the elements $f_1, ..., f_k \in F(n)$, and $\overline{\phi} : F(1) \to F(k)$ by the corresponding element $(\mathtt{p}_1, ..., \mathtt{p}_k \mid \phi) \in F(k)$. It is therefore equivalent to check the case $k = n$ and $f_i = \mathtt{p}_i$, i.e. the "universal case"

$$(\mathtt{p}_1, ..., \mathtt{p}_k \mid \phi) = 1 \quad \text{in } F(k)\,. \tag{2.7}$$

Finally, then, we have:

Proposition 2.4.2 (Boolean-valued completeness of the equational propositional calculus)**.** *Equational propositional calculus is sound and complete with respect to* boolean-valued models *in* Set*, in the sense that a propositional formula ϕ is equationally provable from the laws of Boolean algebra,*

$$\vdash_{\mathsf{eq}} \phi\,,$$

just if it holds generally in any Boolean algebra (in Set*), which we may denote*

$$\vDash_{\mathsf{BA}} \phi\,.$$

Proof. By "holding generally" is meant that it holds for all elements of the Boolean algebra B, in the sense displayed after the Lemma. But, as above, this is equivalent to the condition that for all $b_1, ..., b_k \in \mathsf{B}$, for $\overline{(b_1, ..., b_k)} : F(k) \to \mathsf{B}$ we have $\overline{(b_1, ..., b_k)}(\phi) = 1$ in B, which in turn is clearly equivalent to the previously determined "universal" condition (2.7) that $\phi = 1$ in $F(k)$. □

We leave the analogous statement for equational entailment $\Phi \vdash_{\mathsf{eq}} \phi$ and Boolean-valued entailment $\Phi \vDash_{\mathsf{BA}} \phi$ as an exercise.

Corollary 2.4.3. *Show that a propositional formula $\mathtt{p}_1, ..., \mathtt{p}_k \mid \phi$ is equationally provable $\vdash_{\mathsf{eq}} \phi$, just if it holds in the free Boolean algebra $F(\omega)$ on countably many generators $\omega = \{\mathtt{p}_1, \mathtt{p}_2, ...\}$, with the variables $\mathtt{p}_1, ..., \mathtt{p}_k$ interpreted as the corresponding generators of $F(\omega)$.*

Exercise 2.4.4. Prove this as an easy corollary of Proposition 2.4.2.

Let us summarize what we know so far. By Exercise **??**, we already knew that equational provability in Boolean algebra is equivalent to semantic validity,

$$\vdash_{\mathsf{eq}} \phi \iff \vDash \phi\,.$$

This was based on a certain *decision procedure* for validity in classical propositional logic, originally due to Bernays [**?**], restated in terms of Boolean algebra. Like the classical proof of the Completeness Theorem 2.2.1,

$$\vdash \phi \iff \vDash \phi\,,$$

we would like to analyze this result, too, in general categorical terms, in order to be able to extend and generalize it to other systems of logic.

Our algebraic approach via Lawvere duality resulted in Proposition 2.4.2, which says that equational provability is equivalent to what we have called *Boolean-valued validity*,

$$\vdash_{\mathsf{eq}} \phi \iff \vDash_{\mathsf{BA}} \phi \quad\iff\quad \mathsf{B} \vDash \phi \quad \text{for all } \mathsf{B}\,. \tag{2.8}$$

This is essentially the Boolean algebra case of our Proposition **??**, the completeness of equational reasoning with respect to algebras in Set, originally proved by Birkhoff.

It still remains to relate equational provability $\vdash_{\mathsf{eq}} \phi$ with deduction $\vdash \phi$, and Boolean-valued validity $\vDash_{\mathsf{BA}} \phi$ with semantic validity $\vDash \phi$, which is just the special case $\mathbb{2} \vDash_{\mathsf{BA}} \phi$. We shall consider deduction $\vdash \phi$ via a different approach in the following section, one that regards Boolean algebras as special finite product categories, rather than special Lawvere algebraic theories.

Exercise 2.4.5. For a formula in context $\mathsf{p}_1, ..., \mathsf{p}_k \mid \vartheta$ and a Boolean algebra B, let the expression $\vartheta[b_1/\mathsf{p}_1, ..., b_k/\mathsf{p}_k]$ denote the element of B resulting from interpreting the propositional variables p_i in the context as the elements b_i of B, and evaluating the resulting expression using the Boolean operations of B. For any *finite* set of propositional formulas Φ and any formula ψ, let $\Gamma = \mathsf{p}_1, ..., \mathsf{p}_k$ be a context for (the formulas in) $\Phi \cup \{\psi\}$. Finally, recall that $\Phi \vdash_{\mathsf{eq}} \psi$ means that $\psi = 1$ is equationally provable from the set of equations $\{\phi = 1 \mid \phi \in \Phi\}$. Show that $\Phi \vdash_{\mathsf{eq}} \psi$ just if for all finitely generated free Boolean algebras $F(n)$, the following condition holds:

> For any elements $f_1, ..., f_k \in F(n)$, if $\phi[f_1/\mathsf{p}_1, ..., f_k/\mathsf{p}_k] = 1$ for all $\phi \in \Phi$, then $\psi[f_1/\mathsf{p}_1, ..., f_k/\mathsf{p}_k] = 1$.

Is it sufficient to just take $F(k)$ and its generators $\mathsf{p}_1, ..., \mathsf{p}_k$ as the $f_1, ..., f_k$? Is it equivalent to take all Boolean algebras B, rather than the finitely generated free ones $F(n)$? Determine a condition that is equivalent to $\Phi \vdash_{\mathsf{eq}} \psi$ for not necessarily finite sets Φ.

2.5 Functorial semantics for propositional logic

Considering the algebraic theory of Boolean algebras suggested the idea of a Boolean valuation of propositional logic, generalizing the truth valuations of section 2.2. This can be seen as applying the framework of functorial semantics to a different system of logic than that of equational theories, represented as finite product categories, namely that represented categorically by *poset* categories with finite products $\wedge$ *and coproducts* $\vee$ (each of these cases could, of course, also be considered separately, giving $\wedge$-semi-lattices and categories with finite products $\times$ and coproducts $+$, respectively). Thus we are moving from the top right corner to the bottom center position in the following Hasse diagram of structured categories:

$$\begin{tikzcd}
+ \arrow[dr, dash] \arrow[dd, dash] & & \times \arrow[dl, dash] \arrow[dd, dash] & \mathsf{Cat} \\
& +\,,\times \arrow[dd, dash] & & \\
\vee \arrow[dr, dash] & & \wedge \arrow[dl, dash] & \mathsf{Pos} \\
& \vee\,,\wedge & &
\end{tikzcd}$$

In Chapter **??** we shall see how first-order logic results categorically from these two cases by "indexing the lower one over the upper one", and in Chapters **??** and **??** we shall consider simple and dependent type theory as "categorified" versions of propositional and first-order logic. It is for this reason (rather than a dogmatic commitment to categorical methods!) that we continue our reformulation of the basic results of classical propositional logic in functorial terms.

Definition 2.5.1. A *propositional theory* $\mathbb{T}$ consists of a set $V_{\mathbb{T}}$ of propositional variables, called the *basic* or *atomic propositions*, and a set $A_{\mathbb{T}}$ of propositional formulas (over $V_{\mathbb{T}}$), called the *axioms*. The *consequences* $\Phi \vdash_{\mathbb{T}} \phi$ are those judgements that are derivable by natural deduction (as in Section 2.1), from the axioms $A_{\mathbb{T}}$.

Definition 2.5.2. Let $\mathbb{T} = (V_{\mathbb{T}}, A_{\mathbb{T}})$ be a propositional theory and $\mathcal{B}$ a Boolean algebra. A *model* of $\mathbb{T}$ in $\mathcal{B}$, also called a *Boolean valuation of* $\mathbb{T}$ is an *interpretation function* $v : V_{\mathbb{T}} \to |\mathcal{B}|$ such that, for every $\alpha \in A_{\mathbb{T}}$, we have $[\![\alpha]\!]^v = 1_{\mathcal{B}}$ in $\mathcal{B}$, where the extension

$[\![-]\!]^v$ of v from $V_{\mathbb{T}}$ to all formulas (over $V_{\mathbb{T}}$) is defined in the expected way, namely:

$$\begin{aligned}
[\![\mathsf{p}]\!]^v &= v(\mathsf{p}), \quad \mathsf{p} \in V_{\mathbb{T}} \\
[\![\top]\!]^v &= 1_{\mathcal{B}} \\
[\![\bot]\!]^v &= 0_{\mathcal{B}} \\
[\![\neg\phi]\!]^v &= \neg_{\mathcal{B}}[\![\phi]\!]^v \\
[\![\phi \wedge \psi]\!]^v &= [\![\phi]\!]^v \wedge_{\mathcal{B}} [\![\psi]\!]^v \\
[\![\phi \vee \psi]\!]^v &= [\![\phi]\!]^v \vee_{\mathcal{B}} [\![\psi]\!]^v \\
[\![\phi \Rightarrow \psi]\!]^v &= \neg_{\mathcal{B}}[\![\phi]\!]^v \vee_{\mathcal{B}} [\![\psi]\!]^v
\end{aligned}$$

Finally, let $\mathsf{Mod}(\mathbb{T}, \mathcal{B})$ be the set of all $\mathbb{T}$-models in $\mathcal{B}$. Given a Boolean homomorphism $f : \mathcal{B} \to \mathcal{B}'$, there is an induced mapping $\mathsf{Mod}(\mathbb{T}, f) : \mathsf{Mod}(\mathbb{T}, \mathcal{B}) \to \mathsf{Mod}(\mathbb{T}, \mathcal{B}')$, determined by setting $\mathsf{Mod}(\mathbb{T}, f)(v) = f \circ v$, which is clearly functorial.

Theorem 2.5.3. *The functor* $\mathsf{Mod}(\mathbb{T}) : \mathsf{BA} \to \mathsf{Set}$ *is representable, with representing Boolean algebra* $\mathcal{B}_{\mathbb{T}}$*, the* classifying Boolean algebra *of* $\mathbb{T}$.

The classifying Boolean algebra $\mathcal{B}_{\mathbb{T}}$ is closely related to the conventional *Lindenbaum-Tarski algebra* of $\mathbb{T}$.

Proof. We construct $\mathcal{B}_{\mathbb{T}}$ from the "syntax of $\mathbb{T}$" in two steps:

Step 1: Suppose first that $A_{\mathbb{T}}$ is empty, so $\mathbb{T}$ is just a set V of propositional variables. Then define the classifying Boolean algebra $\mathcal{B}[V]$ by

$$\mathcal{B}[V] \;=\; \{\phi \mid \phi \text{ is a formula in context } V\}/\!\sim$$

where the equivalence relation $\sim$ is *(deductively) provable bi-implication*,

$$\phi \sim \psi \iff \vdash \psi \Leftrightarrow \psi.$$

The operations are (well-)defined on equivalence classes by setting,

$$[\phi] \wedge [\psi] \;=\; [\phi \wedge \psi],$$

and so on. (The reader who has not seen this construction before should fill in the details!)

Step 2: In the general case $\mathbb{T} = (V_{\mathbb{T}}, A_{\mathbb{T}})$, let

$$\mathcal{B}_{\mathbb{T}} = \mathcal{B}[V_{\mathbb{T}}]/\!\sim_{\mathbb{T}},$$

where the equivalence relation $\sim_{\mathbb{T}}$ is now $A_{\mathbb{T}}$*-provable bi-implication*,

$$\phi \sim_{\mathbb{T}} \psi \iff A_{\mathbb{T}} \vdash \psi \Leftrightarrow \psi.$$

The operations are defined as before, but now on equivalence classes $[\phi]$ modulo $A_{\mathbb{T}}$.

Observe that the construction of $\mathcal{B}_{\mathbb{T}}$ is a variation on that of the *syntactic category* construction $\mathcal{C}_{\mathbb{T}} = \mathsf{Syn}(\mathbb{T})$ of the classifying category of an algebraic theory $\mathbb{T}$, in the sense of the previous chapter. Indeed, the statement of the theorem is just the universal property of $\mathcal{B}_{\mathbb{T}}$ as the classifying category of $\mathbb{T}$-models, namely

$$\mathsf{Hom}_{\mathsf{BA}}(\mathcal{B}_{\mathbb{T}}, \mathcal{B}) \;\cong\; \mathsf{Mod}(\mathbb{T}, \mathcal{B})\,, \tag{2.9}$$

naturally in $\mathcal{B}$. (Since $\mathsf{Mod}(\mathbb{T}, \mathcal{B})$ is now a *set* rather than a category, we can classify it up to *isomorphism* rather than equivalence of categories.) The proof of this fact is a variation on the proof of the corresponding theorem **??** from Chapter 1. Further details are given in the following Remark 2.5.5 for the interested reader. □

Remark 2.5.4. The Lindenbaum-Tarski algebra of a propositional theory is usually defined in semantic terms using (truth) valuations. Our definition of $\mathcal{B}_{\mathbb{T}}$ in terms of *provability* is more useful in the present setting, as it parallels that of the syntactic category $\mathcal{C}_{\mathbb{T}}$ of an algebraic theory, and will allow us to prove Theorem 2.2.1 by analogy to Theorem **??** for algebraic theories.

Remark 2.5.5 (Adjoint Rules for Propositional Calculus)**.** For the construction of the classifying algebra $\mathcal{B}_{\mathbb{T}}$, it is convenient to reformulate the rules of inference for the propositional calculus in the following equivalent *adjoint form*: Contexts Γ may be omitted, since the rules leave them unchanged (there is no variable binding). We may also omit hypotheses that remain unchanged. Thus e.g. the *hypothesis* rule may be written in any of the following equivalent ways.

$$\frac{}{\Gamma \mid \phi_1, \ldots, \phi_m \vdash \phi_i} \qquad \frac{}{\phi_1, \ldots, \phi_m \vdash \phi_i} \qquad \frac{}{\phi \vdash \phi}$$

The structural rules can then be stated as follows:

$$\frac{}{\phi \vdash \phi} \qquad \frac{\phi \vdash \psi \qquad \psi \vdash \vartheta}{\phi \vdash \vartheta}$$

$$\frac{\phi \vdash \vartheta}{\psi, \phi \vdash \vartheta} \qquad \frac{\phi, \phi \vdash \vartheta}{\phi \vdash \vartheta} \qquad \frac{\phi, \psi \vdash \vartheta}{\psi, \phi \vdash \vartheta}$$

The rules for the propositional connectives can be given in the following adjoint form, where the double line indicates a two-way rule (with the obvious two instances when there are two conclusions, in going from bottom to top).

$$\frac{}{\phi \vdash \top} \qquad \frac{}{\bot \vdash \phi}$$

$$\genfrac{}{}{1.5pt}{}{\vartheta \vdash \phi \quad \vartheta \vdash \psi}{\vartheta \vdash \phi \wedge \psi} \qquad \genfrac{}{}{1.5pt}{}{\phi \vdash \vartheta \quad \psi \vdash \vartheta}{\phi \vee \psi \vdash \vartheta} \qquad \genfrac{}{}{1.5pt}{}{\vartheta, \phi \vdash \psi}{\vartheta \vdash \phi \Rightarrow \psi}$$

For the purpose of deduction, negation $\neg\phi$ is again treated as defined by $\phi \Rightarrow \bot$ and bi-implication $\phi \Leftrightarrow \psi$ by $(\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi)$. For *classical* logic we also include the rule of *double negation*:

$$\frac{}{\neg\neg\phi \vdash \phi} \tag{2.10}$$

It is now obvious that the set of formulas is preordered by $\phi \vdash \psi$, and that the poset reflection agrees with the deducibility equivalence relation,

$$\phi \dashv\vdash \psi \iff \phi \sim \psi\,.$$

Moreover, $\mathcal{B}_{\mathbb{T}}$ clearly has all finite limits $\top, \wedge$ and colimits $\bot, \vee$, is cartesian closed $\wedge \dashv \Rightarrow$, and is therefore a *Heyting algebra* (see Section **??** below). The rule of double negation then makes it a Boolean algebra.

The proof of the universal property of $\mathcal{B}_{\mathbb{T}}$ is essentially the same as that for $\mathcal{C}_{\mathbb{T}}$.

Exercise 2.5.6. Fill in the details of the proof that $\mathcal{B}_{\mathbb{T}}$ is a well-defined Boolean algebra, with the universal property stated in (2.9). (*Hint:* The well-definedness of the operations $[\phi] \wedge [\psi]$, etc., just requires a few deductions, but the well-definedness of the Boolean homomorphism $v^\sharp : \mathcal{B}_{\mathbb{T}} \to \mathcal{B}$ classifying a model $v : V_{\mathbb{T}} \to |\mathcal{B}|$ requires the *soundness* of deduction with respect to Boolean-valued semantics. Just state this precisely and sketch a proof of it.)

Just as for the case of algebraic theories and FP categories, we now have the following corollary of the classifying theorem 2.5.3, which again follows from the fact that the classifying Boolean algebra $\mathcal{B}_{\mathbb{T}}$ is *logically generic*, in virtue of its syntactic construction.

Corollary 2.5.7. *For any formula ϕ, derivability from the axioms $A_{\mathbb{T}} \vdash \phi$ is equivalent to validity under all Boolean-valued models of $\mathbb{T}$,*

$$A_{\mathbb{T}} \vdash \phi \iff A_{\mathbb{T}} \vDash_{\mathsf{BA}} \phi\,.$$

Proof. We have

$$A_{\mathbb{T}} \vdash \phi \iff \mathcal{B}_{\mathbb{T}} \vDash_{\mathsf{BA}} \phi\,,$$

essentially by definition, where on the righthand side it suffices to check the canonical model $u : V_{\mathbb{T}} \to |\mathcal{B}_{\mathbb{T}}|$ associated to the identity $\mathcal{B}_{\mathbb{T}} \to \mathcal{B}_{\mathbb{T}}$. But if $u \vDash_{\mathsf{BA}} \phi$, then also $v \vDash_{\mathsf{BA}} \phi$ for any $v : V_{\mathbb{T}} \to |\mathcal{B}|$, since $v = v^\sharp u$, and the homomorphism $v^\sharp : \mathcal{B}_{\mathbb{T}} \to \mathcal{B}$ preserves models. Thus $\mathcal{B}_{\mathbb{T}} \vDash_{\mathsf{BA}} \phi \;\Rightarrow\; A_{\mathbb{T}} \vDash_{\mathsf{BA}} \phi$. The converse is immediate. □

Note that the recipe displayed at (2.6) for a Boolean valuation in $F(n)$ of a formula in context $\mathsf{p}_1, ..., \mathsf{p}_k \mid \phi$ is exactly the (canonical) *model* in $F(n)$, with underlying valuation $\{\mathsf{p}_1, ..., \mathsf{p}_k\} \to F(n)$, of the theory $\mathbb{T} = \{\mathsf{p}_1, ..., \mathsf{p}_k\}$. So

$$F(n) \vDash_{\mathsf{BA}} \phi \iff [\![\phi]\!] = 1 \text{ in } F(n)\,.$$

Inspecting the universal property (2.9) of $\mathcal{B}_{\mathbb{T}}$ for the case $\mathbb{T} = \{\mathsf{p}_1, ..., \mathsf{p}_n\}$, we obtain:

Corollary 2.5.8. *The classifying Boolean algebra for the theory* $\{\mathtt{p}_1, ..., \mathtt{p}_n\}$ *is the finitely generated, free Boolean algebra,*

$$\mathcal{B}[\mathtt{p}_1, ..., \mathtt{p}_n] \cong F(n)\,,$$

(which, recall, is the double powerset $PP[n]$*). And generally,* $\mathcal{B}[V]$ *is the free Boolean algebra on the set* V*, for any set* V*.*

Indeed, for any valuation (= arbitrary function) $v : \{\mathtt{p}_1, ..., \mathtt{p}_n\} \to |\mathcal{B}|$ we have a unique extension $[\![-]\!]^v : \mathcal{B}[\mathtt{p}_1, ..., \mathtt{p}_n] \to \mathcal{B}$, which upon inspection of Definition 2.5.2 we recognize as exactly a Boolean homomorphism.

$$\begin{tikzcd}
\mathcal{B}[\mathtt{p}_1, ..., \mathtt{p}_n] \arrow[r, "{[\![-]\!]^v}"] & \mathcal{B} \\
\{\mathtt{p}_1, ..., \mathtt{p}_n\} \arrow[u] \arrow[ur, "v"'] &
\end{tikzcd}$$

The isomorphism $\mathcal{B}[\mathtt{p}_1, ..., \mathtt{p}_n] \cong F(n)$ of Corollary 2.5.7 expresses the fact that the relations of derivability by natural deduction $\Phi \vdash \phi$ and equational provability $\Phi \vdash_{\mathsf{eq}} \phi$ agree,

$$\Phi \vdash \phi \iff \Phi \vdash_{\mathsf{eq}} \phi\,, \tag{2.11}$$

answering one of the two questions from the end of Section 2.4.

Toward answering the other question of the relation between Boolean-valued validity $\Phi \vDash_{\mathsf{BA}} \phi$ and truth-valued validity $\Phi \vDash \phi$, consider the *finitely presented* Boolean algebras, which can be described as those of the form

$$\mathcal{B}_{\mathbb{T}} = \mathcal{B}[\mathtt{p}_1, ..., \mathtt{p}_n]/\alpha\,,$$

for a finite theory $\mathbb{T} = (\mathtt{p}_1, ..., \mathtt{p}_n; \alpha_1, ..., \alpha_m)$, where the slice category of a Boolean algebra $\mathcal{B}$ over an element $\beta \in \mathcal{B}$ is the *downset* (or *principal ideal*)

$$\mathcal{B}/\beta = \downarrow(\beta) = \{b \in \mathcal{B} \mid b \leq \beta\}\,.$$

To see this, given $\mathbb{T} = (V_{\mathbb{T}}, A_{\mathbb{T}})$, if $A_{\mathbb{T}}$ is finite, then let

$$\alpha_{\mathbb{T}} := \bigwedge_{\alpha \in A_{\mathbb{T}}} \alpha\,,$$

so we clearly have

$$\mathcal{B}_{\mathbb{T}} = \mathcal{B}[V_{\mathbb{T}}]/\alpha_{\mathbb{T}}\,.$$

If $V_{\mathbb{T}} = \{\mathtt{p}_1, ..., \mathtt{p}_n\}$ is also finite, then we have

$$\mathcal{B}_{\mathbb{T}} \cong \mathcal{B}[\mathtt{p}_1, ..., \mathtt{p}_n]/\alpha_{\mathbb{T}}\,.$$

It is now easy to show that the finitely presented objects in the category of Boolean algebras are exactly those of the form $\mathcal{B}[\mathsf{p}_1, ..., \mathsf{p}_n]/\alpha_{\mathbb{T}}$, using the fact that a (Boolean) algebra A is finitely presented if and only if it has a presentation (by n-many generators and m-many equations) as a coequalizer of finitely generated free algebras,

$$F(m) \rightrightarrows F(n) \longrightarrow A\,. \tag{2.12}$$

Exercise 2.5.9. Show that the classifying Boolean algebras $\mathcal{B}_{\mathbb{T}}$, for *finite sets* $V_{\mathbb{T}}$ of variables and $A_{\mathbb{T}}$ of formulas, are exactly the *finitely presented* ones in the sense stated in (2.12). In general algebraic categories $\mathcal{A}$ such coequalizers of finitely generated free algebras are exactly those for which the representable functor $\mathsf{Hom}(A, -) : \mathcal{A} \to \mathsf{Set}$ preserves all filtered colimits. Show that the finitely presented Boolean algebras in the sense of (2.12) do indeed have this property.

The following is a special case of the universal property of the slice category

$$X^* : \mathbb{C} \to \mathbb{C}/_X\,,$$

for any $\mathbb{C}$ with finite limits. The reader not already familiar with this fact should definitely do the exercise!

Exercise 2.5.10. For any Boolean algebra $\mathcal{B}$ and any $\beta \in \mathcal{B}$, consider the map

$$\beta^* : \mathcal{B} \to \mathcal{B}/\beta\,,$$

with $\beta^*(x) = \beta \wedge x$.

(i) Show that $\mathcal{B}/\beta \cong {\downarrow}(\beta)$ is a Boolean algebra, and that β^* is a Boolean homomorphism with $\beta^*(\beta) = 1 \in \mathcal{B}/\beta$.

(ii) If $h : \mathcal{B} \to \mathcal{B}'$ is any homomorphism, then $h(\beta) = 1 \in \mathcal{B}'$ if and only if there is a factorization

$$\begin{tikzcd} \mathcal{B} \arrow[r, "h"] \arrow[d, "\beta^*"'] & \mathcal{B}' \\ \mathcal{B}/\beta \arrow[ur, "\overline{h}"'] & \end{tikzcd} \tag{2.13}$$

of h through β^*, and then $\overline{h}$ is unique with $\overline{h} \circ \beta^* = h$.

(iii) Show that if $\mathcal{B}_{\mathbb{T}} = \mathcal{B}[\mathsf{p}_1, ..., \mathsf{p}_n]/\alpha$ classifies (models of) the theory $\mathbb{T} = (\mathsf{p}_1...\mathsf{p}_n, \alpha)$ and $\mathsf{p}_1, ..., \mathsf{p}_n \mid \beta$, then $\mathcal{B}_{\mathbb{T}}/\beta$ classifies models of the extended theory $\mathbb{T}' = (\mathsf{p}_1...\mathsf{p}_n, \alpha, \beta)$.

Lemma 2.5.11. *Let* $\mathcal{B}[\mathsf{p}_1, ..., \mathsf{p}_n]/\alpha$ *be a* finitely presented *Boolean algebra in which* $0 \neq 1$. *Then there is a Boolean homomorphism*

$$h : \mathcal{B}[\mathsf{p}_1, ..., \mathsf{p}_n]/\alpha \to \mathbb{2}\,.$$

Proof. By Exercise 2.5.9, we can assume that $\mathcal{B}[\mathtt{p}_1,...,\mathtt{p}_n]/\alpha = \mathcal{B}_{\mathbb{T}}$ classifies (models of) $\mathbb{T} = (\mathtt{p}_1...\mathtt{p}_n, \alpha)$. By the assumption that $0 \neq 1$ in $\mathcal{B}[\mathtt{p}_1,...,\mathtt{p}_n]/\alpha$, we must have $\alpha \neq 0$ in the free Boolean algebra $\mathcal{B}[\mathtt{p}_1,...,\mathtt{p}_n]$. It then suffices to give a valuation $v : \{\mathtt{p}_1,...,\mathtt{p}_n\} \to 2$ such that $[\![\alpha]\!]^v = 1$, for then (by Exercise 2.5.10) we will have a factorization,

$$\begin{tikzcd} \mathcal{B}[\mathtt{p}_1,...,\mathtt{p}_n] \arrow[r, "m_v"] \arrow[d, "\alpha^*"'] & \mathbb{2} \\ \mathcal{B}[\mathtt{p}_1,...,\mathtt{p}_n]/\alpha \arrow[ur, "\overline{m_v}"'] & \end{tikzcd} \tag{2.14}$$

where $m_v = [\![-]\!]^v$ is the "model" associated to the valuation $v : \{\mathtt{p}_1,...,\mathtt{p}_n\} \to 2$, and $\alpha \wedge - : \mathcal{B}[\mathtt{p}_1,...,\mathtt{p}_n] \to \mathcal{B}[\mathtt{p}_1,...,\mathtt{p}_n]/\alpha$ is the canonical Boolean projection to the "quotient" Boolean algebra given by the slice category, and $\overline{m_v}$ is the extension of m_v along α^* resulting from the universal property of slicing a category with finite products. (Informally, α has a truth table with 2^n rows, corresponding to the valuations $v : \{\mathtt{p}_1,...,\mathtt{p}_n\} \to 2$, and we know that the main column for α is not all 0's, so we can find a row in which it is 1 and read off the corresponding valuation.) More formally, as in Remark 2.3.4, we can put α into a disjunctive normal form $\alpha = \alpha_1 \vee ... \vee \alpha_k$ and one of the disjuncts α_i must then also be non-zero. Since $\alpha_i = q_1 \wedge ... \wedge q_m$ with each q_j either positive $\mathtt{p}$ or negative $\neg\mathtt{p}$, if both $\mathtt{p}$ and $\neg\mathtt{p}$ occur, then $\alpha_i = 0$, so the $\mathtt{p}$ in each q_j must occur only once in α_i. We can then define v accordingly, with $v(\mathtt{p}) = 1$ iff $\mathtt{p}$ occurs positively in α_i, and we will have $[\![\alpha_i]\!]^v = 1$. This valuation $v : \{\mathtt{p}_1,...,\mathtt{p}_n\} \to \mathbb{2}$ then determines a Boolean homomorphism $[\![-]\!]^v : \mathcal{B}[\mathtt{p}_1...\mathtt{p}_n] \to \mathbb{2}$ with $[\![\alpha]\!]^v = 1$, as required for a homomorphism

$$\mathcal{B}[\mathtt{p}_1...\mathtt{p}_n]/\alpha \to \mathbb{2}\,.$$

□

Proposition 2.5.12. *For any formula ϕ, Boolean-valued validity and truth-valued validity are equivalent,*

$$\vDash_{\mathsf{BA}} \phi \iff \vDash \phi\,. \tag{2.15}$$

Proof. Since $\vDash_{\mathsf{BA}} \phi$ means that $\mathcal{B} \vDash_{\mathsf{BA}} \phi$ for all Boolean algebras $\mathcal{B}$, and $\vDash \phi$ means the same for valuations in $\mathbb{2}$, the implication from left to right is trivial. For the converse, let $(\mathtt{p}_1,...,\mathtt{p}_n \,|\, \phi)$, and consider $\phi \in \mathcal{B}[\mathtt{p}_1,...,\mathtt{p}_n]$. If $h(\phi) = 1$ for all homomorphisms $h : \mathcal{B}[\mathtt{p}_1,...,\mathtt{p}_n] \to \mathbb{2}$, then $\mathcal{B}[\mathtt{p}_1,...,\mathtt{p}_n]/\neg\phi$ can have no homomorphism $\overline{h} : \mathcal{B}[\mathtt{p}_1,...,\mathtt{p}_n]/\neg\phi \to \mathbb{2}$ (else $\overline{h}(\neg\phi) = 1$ would give $h(\neg\phi) = 1$ and so $h(\phi) = 0$). Therefore $0 = 1$ in $\mathcal{B}[\mathtt{p}_1,...,\mathtt{p}_n]/\neg\phi$ by Lemma 2.5.11. But then $0 = \neg\phi \wedge 1 = \neg\phi$ in $\mathcal{B}[\mathtt{p}_1,...,\mathtt{p}_n]$, whence $\phi = \neg\neg\phi = \neg 0 = 1$, so $h(\phi) = 1 \in \mathcal{B}$ for all $h : \mathcal{B}[\mathtt{p}_1,...,\mathtt{p}_n] \to \mathcal{B}$. □

Exercise 2.5.13. Extend Proposition 2.5.14 to entailment, for any finite set Φ,

$$\Phi \vDash_{\mathsf{BA}} \phi \iff \Phi \vDash \phi\,.$$

Combining this last result (2.15) with the previous one (2.11) and (2.8) from the last section, we arrive finally at our desired reconstruction of the classical completeness theorem:

Proposition 2.5.14. *For any formula ϕ, provability by deduction and truth-valued validity are equivalent,*

$$\vdash \phi \iff \vDash \phi\,. \tag{2.16}$$

And the same holds relative to a set Φ of premises.

Let us now unwind the foregoing "reproof" into a direct argument, from the present point of view: A formula ϕ in context $\mathsf{p}_1, ..., \mathsf{p}_n \mid \phi$ determines an element in the free Boolean algebra $\mathcal{B}[\mathsf{p}_1, ..., \mathsf{p}_n]$. If $\vdash \phi$ then $\phi = 1$ in $\mathcal{B}[\mathsf{p}_1, ..., \mathsf{p}_n]$, so clearly $h(\phi) = 1$ for every $h : \mathcal{B}[\mathsf{p}_1, ..., \mathsf{p}_n] \to 2$, which means exactly $\vDash \phi$. Conversely, if $\vDash \phi$ then $h(\phi) = 1$ for every $h : \mathcal{B}[\mathsf{p}_1, ..., \mathsf{p}_n] \to 2$, so $\neg\phi$ can have no model in 2. Thus $\mathcal{B}[\mathsf{p}_1, ..., \mathsf{p}_n]/\neg\phi$ must be degenerate, with $0 = 1$. So $[\bot] = [\neg\phi]$ and therefore $\neg\phi \vdash \bot$, so $\vdash \neg\neg\phi$, so $\vdash \phi$.

The main fact used here is that the finitely generated, free Boolean algebras $\mathcal{B}(n) = \mathcal{B}[\mathsf{p}_1, ..., \mathsf{p}_n]$ have enough Boolean homomorphisms $h : \mathcal{B}(n) \to 2$ to separate any non-zero element $\phi \neq 0$, in the sense that if $h(\phi) = 0$ for all such h then $\phi = 0$. In other words, the canonical homomorphism

$$\mathcal{B}(n) \longrightarrow \Pi_{h \in \mathcal{B}(n)^*} 2\,, \tag{2.17}$$

is injective, for $\mathcal{B}(n)^* = \mathsf{BA}(\mathcal{B}(n), 2)$. This is reminiscent of the proof of completeness for algebraic theories, which also used an embedding of the syntactic category $\mathcal{C}_{\mathbb{T}}$ into a power of Set by a "sufficient" set of models $\mathcal{C}_{\mathbb{T}} \to \mathsf{Set}$,

$$\mathcal{C}_{\mathbb{T}} \hookrightarrow \mathsf{Set}^{\mathsf{mod}(\mathbb{T})}$$

namely those of the form $\mathcal{C}_{\mathbb{T}}(U^n, -) \cong \mathsf{mod}(\mathbb{T})(-, F(n)) : \mathcal{C}_{\mathbb{T}} \to \mathsf{Set}$. For Boolean algebras, the embedding (2.17) is the main point of the Stone Representation Theorem.

2.6 Stone representation

Regarding a Boolean algebra $\mathcal{B}$ as a category with finite products, consider its Yoneda embedding $\mathsf{y} : \mathcal{B} \hookrightarrow \mathsf{Set}^{\mathcal{B}^{\mathsf{op}}}$. Since the hom-set $\mathcal{B}(x, y)$ is always 2-valued, we have a factorization,

$$\mathsf{y} : \mathcal{B} \hookrightarrow 2^{\mathcal{B}^{\mathsf{op}}} \hookrightarrow \mathsf{Set}^{\mathcal{B}^{\mathsf{op}}} \tag{2.18}$$

in which each factor still preserves the finite products (note that the products in 2 are preserved by the inclusion $2 \hookrightarrow \mathsf{Set}$, and the products in the functor categories $2^{\mathcal{B}^{\mathsf{op}}}$ and $\mathsf{Set}^{\mathcal{B}^{\mathsf{op}}}$ are taken pointwise). Indeed, this is an instance of a general fact. In the category $\mathsf{Cat}_\times$ of finite product categories (and $\times$-preserving functors), the inclusion of the full subcategory of posets with $\wedge$ (the $\wedge$-semilattices) has a *right adjoint* R, in addition to the left adjoint L of poset reflection.

$$\begin{tikzcd} \mathsf{Cat}_\times \arrow[d, bend right=50, "L"'] \arrow[d, bend left=50, "R"] \\ \mathsf{Pos}_\wedge \arrow[u, hook, "i"'] \end{tikzcd}$$

For a finite product category $\mathbb{C}$, the poset $R\mathbb{C}$ is the subcategory $\mathsf{Sub}(1) \hookrightarrow \mathbb{C}$ of subobjects of the terminal object 1 (equivalently, the category of monos $m : M \rightarrowtail 1$). The reason for this is that a $\times$-preserving functor $f : A \to \mathbb{C}$ from a poset A with meets takes every object $a \in A$ to a mono $f(a) \rightarrowtail 1$ in $\mathbb{C}$, since $a = a \wedge a$ implies the following is a product diagram in A.

$$\begin{tikzcd} a \arrow[r] & 1 \\ a \arrow[u] \arrow[r] & a \arrow[u] \end{tikzcd}$$

Exercise 2.6.1. Prove this, and use it to verify that $R = \mathsf{Sub}(1)$ is indeed a right adjoint to the inclusion of $\wedge$-semilattices into finite-product categories.

Now the functor category $\mathbb{2}^{\mathcal{B}^{\mathsf{op}}} = \mathsf{Pos}(\mathcal{B}^{\mathsf{op}}, \mathbb{2})$ occurring in (2.18), consists of all *contravariant*, monotone maps $\mathcal{B}^{\mathsf{op}} \to \mathbb{2}$ (which indeed is $\mathsf{Sub}(1) \hookrightarrow \mathsf{Set}^{\mathbb{C}^{\mathsf{op}}}$), and is easily seen to be isomorphic to the poset $\mathsf{Down}(\mathcal{B})$ of all *sieves* (or "downsets") in $\mathcal{B}$: subsets $S \subseteq \mathcal{B}$ that are downward closed, $x \leq y \in S \Rightarrow x \in S$, ordered by subset inclusion $S \subseteq T$. Explicitly, the isomorphism

$$\mathsf{Pos}(\mathcal{B}^{\mathsf{op}}, \mathbb{2}) \cong \mathsf{Down}(\mathcal{B}) \tag{2.19}$$

is given by taking $f : \mathcal{B}^{\mathsf{op}} \to \mathbb{2}$ to $f^{-1}(1)$ and $S \subseteq \mathcal{B}$ to the function $f_S : \mathcal{B}^{\mathsf{op}} \to \mathbb{2}$ with $f_S(b) = 1 \Leftrightarrow b \in S$. Under this isomorphism, the Yoneda embedding takes an element $b \in \mathcal{B}$ *covariantly* to the *principal downset* $\downarrow b \subseteq \mathcal{B}$ of all $x \leq b$.

Exercise 2.6.2. Show that (2.19) is indeed an isomorphism of posets, and that it sends the Yoneda embedding to the principal sieve mapping, as claimed.

For algebraic theories $\mathbb{A}$, we used the Yoneda embedding to give a completeness theorem for equational logic with respect to Set-valued models, by composing the (faithful functor) $\mathsf{y} : \mathbb{A} \hookrightarrow \mathsf{Set}^{\mathbb{A}^{\mathsf{op}}}$ with the (jointly faithful) evaluation functors $\mathsf{eval}_A : \mathsf{Set}^{\mathbb{A}^{\mathsf{op}}} \to \mathsf{Set}$, for all objects $A \in \mathbb{A}$. This amounts to considering all *covariant* representables $\mathsf{eval}_A \circ \mathsf{y} = \mathbb{A}(A, -) : \mathbb{A} \to \mathsf{Set}$, and observing that these are then (both $\times$-preserving and) jointly faithful.

We can do the same thing for a Boolean algebra $\mathcal{B}$ (which is, after all, a finite product category) to get a jointly faithful family of $\times$-preserving, monotone maps $\mathcal{B}(b, -) : \mathcal{B} \to \mathbb{2}$, i.e. $\wedge$-semilattice homomorphisms. By taking the preimages of $\{1\} \hookrightarrow \mathbb{2}$, such homomorphisms correspond to *filters* in $\mathcal{B}$: "upsets" that are also closed under $\wedge$. The representables then correspond to the *principal filters* $\uparrow b \subseteq \mathcal{B}$. The problem with using this approach for a completeness theorem for *propositional* logic is that such $\wedge$-homomorphisms $\mathcal{B} \to \mathbb{2}$ are not *models*, because they need not preserve the joins $\phi \vee \psi$ (nor the complements $\neg\phi$).

Lemma 2.6.3. *Let $\mathcal{B}, \mathcal{B}'$ be Boolean algebras and $f : \mathcal{B} \to \mathcal{B}'$ a distributive lattice homomorphism. Then f preserves negation, and so is Boolean. The category* Bool *of Boolean algebras is thus a* full *subcategory of the category* DLat *of distributive lattices.*

Proof. The complement $\neg b$ is the unique element of $\mathcal{B}$ such that both $b \vee \neg b = 1$ and $b \wedge \neg b = 0$. □

This suggests representing a Boolean algebra $\mathcal{B}$, not by its filters, but by its *prime* filters, which correspond bijectively to distributive lattice homomorphisms $\mathcal{B} \to \mathbb{2}$.

Definition 2.6.4. A filter $F \subseteq \mathcal{D}$ in a distributive lattice $\mathcal{D}$ is *prime* if $b \vee b' \in F$ implies $b \in F$ or $b' \in F$. Equivalently, just if the corresponding $\wedge$-semilattice homomorphism $f_F : \mathcal{B} \to \mathbb{2}$ is a lattice homomorphism.

Now if $\mathcal{B}$ is Boolean, it then follows that prime filters $F \subseteq \mathcal{B}$ are in bijection with Boolean homomorphisms $\mathcal{B} \to \mathbb{2}$, via the assignment $F \mapsto f_F : \mathcal{B} \to \mathbb{2}$ with $f_F(b) = 1 \Leftrightarrow b \in F$ and $(f : \mathcal{B} \to \mathbb{2}) \mapsto F_f := f^{-1}(1) \subseteq \mathcal{B}$. The prime filter F_f may be called the *(filter) kernel* of $f : \mathcal{B} \to \mathbb{2}$.

Proposition 2.6.5. *In a Boolean algebra $\mathcal{B}$, the following conditions on a subset $F \subseteq \mathcal{B}$ are equivalent.*

1. *F is a prime filter*
2. *the complement $\mathcal{B} \backslash F$ is a* prime ideal *(defined as a prime filter in $\mathcal{B}^{\mathsf{op}}$).*
3. *the complement $\mathcal{B} \backslash F$ is an* ideal *(defined as a filter in $\mathcal{B}^{\mathsf{op}}$).*
4. *F is a filter, and for each $b \in \mathcal{B}$, either $b \in F$ or $\neg b \in \mathcal{F}$ and not both.*
5. *F is a* maximal filter*: F is a filter and for all filters G, if $F \subseteq G$ then $F = G$ (also called an* ultrafilter*).*
6. *the map $f_F : \mathcal{B} \to \mathbb{2}$ given by $f_F(b) = 1 \Leftrightarrow b \in F$ (as in (2.19)) is a Boolean homomorphism.*

Proof. Exercise! □

The following lemma is sometimes referred to as the *(Boolean) prime ideal theorem.*

Lemma 2.6.6. *Let $\mathcal{B}$ be a Boolean algebra, $I \subseteq \mathcal{B}$ an ideal, and $F \subseteq \mathcal{B}$ a filter, with $I \cap F = \emptyset$. There is a prime filter $P \supseteq F$ with $I \cap P = \emptyset$.*

Proof. Suppose first that $I = \{0\}$ is the trivial ideal, and that $\mathcal{B}$ is countable, with $b_0, b_1, ...$ an enumeration of its elements. As in the proof of the Model Existence Lemma, we build an increasing sequence of filters $F_0 \subseteq F_1 \subseteq \dots$ as follows:

$$F_0 = F$$

$$F_{n+1} = \begin{cases} F_n & \text{if } \neg b_n \in F_n \\ \{f \wedge b \mid f \in F_n,\ b_n \leq b\} & \text{otherwise} \end{cases}$$

$$P = \bigcup_n F_n$$

One then shows that each F_n is a filter, that $I \cap F_n = \emptyset$ for all n and so $I \cap P = \emptyset$, and that for each b_n, either $b_n \in P$ or $\neg b_n \in P$, whence P is prime.

For $I \subseteq \mathcal{B}$ a nontrivial ideal we take the quotient Boolean algebra $\mathcal{B} \twoheadrightarrow \mathcal{B}/I$, defined as the algebra of equivalence classes $[b]$ where $a \sim_I b \Leftrightarrow a \vee i = b \vee j$ for some $i, j \in I$. One shows that this is indeed a Boolean algebra and that the projection onto equivalence classes $\pi_I : \mathcal{B} \twoheadrightarrow \mathcal{B}/I$ is a Boolean homomorphism with (ideal) kernel $\pi^{-1}([0]) = I$. Now apply the foregoing argument to obtain a prime filter $P : \mathcal{B}/I \to \mathbb{2}$. The composite $p_I = P \circ \pi_I : \mathcal{B} \to \mathbb{2}$ is then a Boolean homomorphism with (filter) kernel $p_I^{-1}(1)$ which is prime, contains F and is disjoint from I.

The case where $\mathcal{B}$ is uncountable is left as an exercise. □

Exercise 2.6.7. Finish the proof of Lemma 2.6.6 by (i) verifying the construction of the quotient Boolean algebra $\mathcal{B} \twoheadrightarrow \mathcal{B}/I$, and (ii) considering the case where $\mathcal{B}$ is uncountable (*Hint*: either use Zorn's lemma, or well-order $\mathcal{B}$.)

Theorem 2.6.8 (Stone representation theorem)**.** *Let $\mathcal{B}$ be a Boolean algebra. There is an injective Boolean homomorphism $\mathcal{B} \rightarrowtail \mathcal{P}X$ into a powerset.*

Proof. Let X be the set of prime filters in $\mathcal{B}$ and consider the map $h : \mathcal{B} \to \mathcal{P}X$ given by $h(b) = \{F \mid b \in F\}$. Clearly $h(0) = \emptyset$ and $h(1) = X$. Moreover, for *any* filter F, we have $b \in F$ and $b' \in F$ if and only if $b \wedge b' \in F$, so $h(b \wedge b') = h(b) \cap h(b')$. If F is prime, then $b \in F$ or $b' \in F$ if and only if $b \vee b' \in F$, so $h(b \vee b') = h(b) \cup h(b')$. Thus h is a Boolean homomorphism. Let $a \neq b \in \mathcal{B}$, and we want to show that $h(a) \neq h(b)$. It suffices to assume that $a < b$ (otherwise, consider $a \wedge b$, for which we cannot have both $a \wedge b = a$ and $a \wedge b = b$). We seek a prime filter $P \subseteq \mathcal{B}$ with $b \in P$ but $a \notin P$. Apply Lemma 2.6.6 to the ideal $\downarrow a$ and the filter $\uparrow b$. □

2.7 Stone duality

Note that in the Stone representation $\mathcal{B} \rightarrowtail \mathcal{P}(X_{\mathcal{B}})$ the powerset Boolean algebra

$$\mathcal{P}(X_{\mathcal{B}}) \cong \mathsf{Set}\big(\mathsf{Bool}(\mathcal{B}, \mathbb{2}), 2\big)$$

is evidently (covariantly) functorial in $\mathcal{B}$, and has an apparent "double-dual" form $\mathcal{B}^{**}$. This suggests a possible duality between the categories Bool and Set,

$$\begin{tikzcd} \mathsf{Bool}^{\mathsf{op}} \arrow[r, bend left, "*"] & \mathsf{Set} \arrow[l, bend left, "*"] \end{tikzcd} \tag{2.20}$$

with contravariant functors

$$\mathcal{B}^* = \mathsf{Bool}(\mathcal{B}, \mathbb{2}),$$

the set of prime filters, for a Boolean algebra $\mathcal{B}$, and

$$S^* = \mathsf{Set}(S, 2),$$

the powerset Boolean algebra, for a set S. This indeed gives a contravariant adjunction "on the right",

$$\frac{\mathcal{B} \to \mathcal{P}S \qquad \mathsf{Bool}}{S \to X_{\mathcal{B}} \qquad \mathsf{Set}} \tag{2.21}$$

by applying the contravariant functors

$$\begin{aligned} \mathcal{P}S &= \mathsf{Set}(S, 2), \\ X_{\mathcal{B}} &= \mathsf{Bool}(\mathcal{B}, \mathbb{2}), \end{aligned}$$

and then precomposing with the respective "evaluation" natural transformations,

$$\begin{aligned} \eta_{\mathcal{B}} : \mathcal{B} &\longrightarrow \mathcal{P}X_{\mathcal{B}} \cong \mathsf{Set}\big(\mathsf{Bool}(\mathcal{B}, \mathbb{2}), 2\big), \\ \varepsilon_S : S &\longrightarrow X_{\mathcal{P}S} \cong \mathsf{Bool}\big(\mathsf{Set}(S, 2), \mathbb{2}\big). \end{aligned}$$

The homomorphism $\eta_{\mathcal{B}}$ takes an element $b \in \mathcal{B}$ to the set of prime filters that contain it, and the function ε_S takes an element $s \in S$ to the principal filter $\uparrow\{s\} \subseteq \mathcal{P}S$, which is prime since the singleton set $\{s\}$ is an *atom* in $\mathcal{P}S$, i.e., a minimal, non-zero element.

Exercise 2.7.1. Verify the adjunction (2.23).

The adjunction (2.23) is not an equivalence, however, because neither of the units $\eta_{\mathcal{B}}$ nor ε_S is in general an isomorphism. We can improve it by *topologizing* the set $X_{\mathcal{B}}$ of prime filters, in order to be able to cut down the powerset $\mathcal{P}X_{\mathcal{B}} \cong \mathsf{Set}(X_{\mathcal{B}}, 2)$ to just the *continuous* functions into the discrete space 2, which then correspond to the *clopen* sets in $X_{\mathcal{B}}$. To do so, we take as *basic open sets* all those sets of the form:

$$B_b = \{P \in X_{\mathcal{B}} \mid b \in P\}, \qquad b \in \mathcal{B}. \tag{2.22}$$

These sets are closed under finite intersections, because $B_a \cap B_b = B_{a \wedge b}$. Indeed, if $P \in B_a \cap B_b$ then $a \in P$ and $b \in P$, whence $a \wedge b \in P$, and conversely.

Definition 2.7.2. For any Boolean algebra $\mathcal{B}$, the *prime spectrum* of $\mathcal{B}$ is a topological space $X_{\mathcal{B}}$ with the prime filters $P \subseteq \mathcal{B}$ as points, and the sets B_b of (2.22), for all $b \in \mathcal{B}$, as basic open sets. The prime spectrum $X_{\mathcal{B}}$ is also called the *Stone space* of $\mathcal{B}$.

Proposition 2.7.3. *The open sets $\mathcal{O}(X_{\mathcal{B}})$ of the Stone space are in order-preserving, bijective correspondence with the ideals $I \subseteq \mathcal{B}$ of the Boolean algebra, with the principal ideals $\downarrow b$ corresponding exactly to the clopen sets.*

Proof. Exercise! □

We now have an improved adjunction

$$\begin{tikzcd} \mathsf{Bool}^{\mathsf{op}} \arrow[r, bend left, "\mathsf{Spec}"] & \mathsf{Top} \arrow[l, bend left, "\mathsf{Clop}"] \end{tikzcd} \tag{2.23}$$

$$\begin{aligned}\mathsf{Spec}(\mathcal{B}) &= (X_{\mathcal{B}}, \mathcal{O}(X_{\mathcal{B}}))\\ \mathsf{Clop}(X) &= \mathsf{Top}(X, 2),\end{aligned}$$

for which, up to isomorphism, the space $\mathsf{Spec}(\mathcal{B})$ has the underlying set $\mathsf{Bool}(\mathcal{B}, \mathbb{2})$ given by "homming" into the Boolean algebra $\mathbb{2}$, and the Boolean algebra $\mathsf{Clop}(X) = \mathsf{Top}(X, 2)$ is similarly determined by mapping into the "topological Boolean algebra" given by the discrete topological space 2. Such an adjunction is said to be induced by a *dualizing object*: an object that can be regarded as "living in two different categories". Here the dualizing object 2 is acting both as a space and as a Boolean algebra. In the Lawvere duality of Chapter 1 (and others to be met later on), the role of dualizing object is played by the category Set of all sets.

Toward the goal of determining the image of the functor $\mathsf{Spec} : \mathsf{Bool}^{\mathsf{op}} \to \mathsf{Top}$, observe first that the Stone space $X_{\mathcal{B}}$ of a Boolean algebra $\mathcal{B}$ is a subspace of a product of finite discrete spaces,

$$X_{\mathcal{B}} \cong \mathsf{Bool}(\mathcal{B}, \mathbb{2}) \hookrightarrow \prod_{|\mathcal{B}|} 2,$$

and is therefore a compact Hausdorff space by Tychonoff's theorem. Indeed, the basis (2.22) is just the subspace topology on $X_{\mathcal{B}}$ with respect to the product topology on $\prod_{|\mathcal{B}|} 2$. The latter space is moreover *totally disconnected*, meaning that it has a subbasis of clopen subsets, namely all those of the form $f^{-1}(\delta) \subseteq |\mathcal{B}|$ for $f : |\mathcal{B}| \to 2$ and $\delta = 0, 1$.

Lemma 2.7.4. *The prime spectrum $X_{\mathcal{B}}$ of a Boolean algebra $\mathcal{B}$ is a totally disconnected, compact, Hausdorff space.*

Proof. Since $\prod_{|\mathcal{B}|} 2$ has just been shown to be a totally disconnected, compact Hausdorff space, we need only see that the subspace $X_{\mathcal{B}}$ is closed. Consider the subspaces

$$2^{|\mathcal{B}|}_{\wedge},\ 2^{|\mathcal{B}|}_{\vee},\ 2^{|\mathcal{B}|}_{1},\ 2^{|\mathcal{B}|}_{0} \subseteq 2^{|\mathcal{B}|}$$

consisting of the functions $f : |\mathcal{B}| \to 2$ that preserve $\wedge, \vee, 1, 0$ respectively. Since each of these is closed, so is their intersection $X_{\mathcal{B}}$. In more detail, the set of maps $f : |\mathcal{B}| \to 2$ that preserve e.g. $\wedge$ can be described as an equalizer

$$\begin{tikzcd} 2^{|\mathcal{B}|}_{\wedge} \arrow[r, tail] & 2^{|\mathcal{B}|} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & 2^{|\mathcal{B}| \times |\mathcal{B}|} \end{tikzcd}$$

where the maps s, t take an arrow $f : |\mathcal{B}| \to 2$ to the two different composites around the square

$$\begin{tikzcd} |\mathcal{B}| \times |\mathcal{B}| \arrow[r, "\wedge"] \arrow[d, "f \times f"'] & |\mathcal{B}| \arrow[d, "f"] \\ 2 \times 2 \arrow[r, "\wedge"'] & 2. \end{tikzcd}$$

But the equalizer $2^{|\mathcal{B}|}_{\wedge} \rightarrowtail 2^{|\mathcal{B}|}$ is the pullback of the diagonal on $2^{|\mathcal{B}| \times |\mathcal{B}|}$, which is closed since $2^{|\mathcal{B}| \times |\mathcal{B}|}$ is Hausdorff. The other cases are analogous. □

Definition 2.7.5. A topological space is called *Stone* if it is totally disconnected, compact, and Hausdorff. Let $\mathsf{Stone} \hookrightarrow \mathsf{Top}$ be the full subcategory of topological spaces consisting of Stone spaces and continuous functions between them.

In order to further cut down the adjunction on the topological side, we can now restrict it to just the Stone spaces, since we know this subcategory will contain the image of the functor Spec. In fact, up to isomorphism, this is exactly the image:

Theorem 2.7.6. *There is a contravariant equivalence of categories between* Bool *and* Stone*,*

$$\begin{tikzcd} \mathsf{Bool}^{\mathsf{op}} \arrow[r, bend left, "*"] & \mathsf{Stone} \arrow[l, bend left, "*"] \end{tikzcd},$$

with contravariant functors $\mathcal{B}^* = X_{\mathcal{B}}$ the Stone space of a Boolean algebra $\mathcal{B}$, as in Definition 2.7.2, and $X^* = \mathsf{clopen}(X)$, the Boolean algebra of all clopen sets in the Stone space X.

Proof. We just need to show that the two units of the adjunction

$$\eta_{\mathcal{B}} : \mathcal{B} \to \mathsf{Top}\big(\mathsf{Bool}(\mathcal{B}, \mathbb{2}), 2\big),$$
$$\varepsilon_S : S \to \mathsf{Bool}\big(\mathsf{Top}(S, 2), \mathbb{2}\big).$$

are isomorphisms, the second assuming S is a Stone space.

We know by the Stone representation theorem 2.6.8 that $\eta_{\mathcal{B}}$ is an injective Boolean homomorphism, so its image, say

$$\mathcal{B}' \subseteq \mathsf{Top}\big(\mathsf{Bool}(\mathcal{B}, \mathbb{2}), 2\big) \cong \mathsf{Clop}(X_{\mathcal{B}}),$$

is a sub-Boolean algebra of the clopen sets of $X_{\mathcal{B}}$. It suffices to show that every clopen set of $X_{\mathcal{B}}$ is in $\mathcal{B}'$. Thus let $K \subseteq X_{\mathcal{B}}$ be clopen, and take $K = \bigcup_i B_i$ a cover by basic opens B_i, all of which, note, are of the form (2.22), and so are in $\mathcal{B}'$. Since K is closed and $X_{\mathcal{B}}$ compact, K is also compact, so there is a finite subcover, each element of which is in $\mathcal{B}'$. Thus their finite union K is also in $\mathcal{B}'$.

Let S be a Stone space and consider the continuous function

$$\varepsilon_S : S \to \mathsf{Bool}\big(\mathsf{Top}(S, 2), \mathbb{2}\big) \cong X_{\mathsf{Clop}(S)}$$

which takes $s \in S$ to the prime filter $F_s = \{K \in \mathsf{Clop}(S) \mid s \in K\}$ of all clopen sets containing it. Since S is Hausdorff, ε_S is a bijection on points, and it is continuous by construction. To see that it is open, let $K \subseteq S$ be a basic clopen set. The complement $S - K$ is therefore closed, and thus compact, and so is its image $\varepsilon_S(S - K)$, which is therefore closed. But since ε_S is a bijection, $\varepsilon_S(S - K)$ is the complement of $\varepsilon_S(K)$, which is therefore open. $\square$

Remark 2.7.7. Another way to cut down the adjunction (2.23),

$$\begin{tikzcd} \mathsf{Bool}^{\mathsf{op}} \arrow[r, bend left, "*"] & \mathsf{Set} \arrow[l, bend left, "*"] \end{tikzcd}$$

to an equivalence is to restrict the Boolean algebra side to *complete, atomic* Boolean algebras CABool and continuous (i.e. $\bigvee$-preserving) homomorphisms between them. One then obtains a duality

$$\mathsf{CABool}^{\mathsf{op}} \simeq \mathsf{Set},$$

between complete, atomic Boolean algebras and sets (see Johnstone [**?**]).

Remark 2.7.8. See Johnstone [**?**] for a more detailed presentation of the material in this section (and much more). Also see [**?**] for a generalization to distributive lattices and Heyting algebras, as well as to "Boolean algebras with operators", i.e. algebraic models of modal logic. For more on logical duality see [**?**]

2.8 Cartesian closed posets

Positive propositional calculus Next, let us consider cartesian closed posets as a categorical approach to propostitional logic.

Example 2.8.1. Consider the *positive propositional calculus* PPC with conjunction and implication, as in Section 2.1. Recall that PPC is the set of all propositional formulas ϕ constructed from propositional variables $p_1, p_2, \ldots$, a constant $\top$ for truth, and binary connectives for conjunction $\phi \wedge \psi$, and implication $\phi \Rightarrow \psi$.

As a category, PPC is a preorder under the relation $\phi \vdash \psi$ of logical entailment, determined for instance by the natural deduction system **??** of section **??**. As usual, it will be convenient to pass to the poset reflection of the preorder, which we shall denote by

$$\mathcal{C}_{\mathsf{PPC}}$$

by identifying ϕ and ψ when $\phi \dashv\vdash \psi$. (This is just the usual *Lindenbaum-Tarski* algebra of the system of propositional logic, as in Section 2.5.)

The conjunction $\phi \wedge \psi$ is a greatest lower bound of ϕ and ψ in $\mathcal{C}_{\mathsf{PPC}}$, because we have $\phi \wedge \psi \vdash \phi$ and $\phi \wedge \psi \vdash \psi$ and for all ϑ, if $\vartheta \vdash \phi$ and $\vartheta \vdash \psi$ then $\vartheta \vdash \phi \wedge \psi$. Since binary products in a poset are the same thing as greatest lower bounds, we see that $\mathcal{C}_{\mathsf{PPC}}$ has all binary products; and of course $\top$ is a terminal object.

We have already remarked that implication is right adjoint to conjunction in propositional calculus,

$$(-) \wedge \phi \;\dashv\; \phi \Rightarrow (-)\ . \tag{2.24}$$

Therefore $\phi \Rightarrow \psi$ is an exponential in $\mathcal{C}_{\mathsf{PPC}}$. The counit of the adjunction (the "evaluation" arrow) is the entailment

$$(\phi \Rightarrow \psi) \wedge \phi \vdash \psi\ ,$$

i.e. the familiar logical rule of *modus ponens.*

We have now shown:

Proposition 2.8.2. *The poset $\mathcal{C}_{\mathsf{PPC}}$ of positive propositional calculus is cartesian closed.*

Let us now use this fact to show that the positive propositional calculus is *deductively complete* with respect to the following notion of *Kripke semantics* [].

Definition 2.8.3 (Kripke model)**.** Let K be a poset. Suppose we have a relation

$$k \Vdash p$$

between elements $k \in K$ and propositional variables p, such that

$$j \leq k,\ k \Vdash p \quad \text{implies} \quad j \Vdash p. \tag{2.25}$$

Extend $\Vdash$ to all formulas ϕ in PPC by defining

$$\begin{array}{lll} k \Vdash \top & \text{always}, & \\ k \Vdash \phi \wedge \psi & \text{iff} & k \Vdash \phi \text{ and } k \Vdash \psi\,, \\ k \Vdash \phi \Rightarrow \psi & \text{iff} & \text{for all } j \leq k, \text{ if } j \Vdash \phi, \text{ then } j \Vdash \psi\,. \end{array} \tag{2.26}$$

Finally, say that ϕ *holds on* K, written

$$K \Vdash \phi$$

if $k \Vdash \phi$ for all $k \in K$ (for all such relations $\Vdash$).

Theorem 2.8.4 (Kripke completeness for PPC)**.** *A propositional formulas ϕ is provable from the rules of deduction for PPC if, and only if, $K \Vdash \phi$ for all posets K. Briefly:*

$$\mathsf{PPC} \vdash \phi \quad \textit{iff} \quad K \Vdash \phi \textit{ for all } K.$$

We will require the following (which extends the discussion in Section 2.6).

Lemma 2.8.5. *For any poset P, the poset $\downarrow\! P$ of all downsets in P, ordered by inclusion, is cartesian closed. Moreover, the downset embedding,*

$$\downarrow\!(-) : P \to \downarrow\! P$$

preserves any CCC structure that exists in P.

Proof. The total downset P is obviously terminal, and for any downsets $S, T \in \downarrow\! P$, the intersection $S \cap T$ is also closed down, so we have the products $S \wedge T = S \cap T$. For the exponential, set

$$S \Rightarrow T = \{p \in P \mid \downarrow\!(p) \cap S \subseteq T\}.$$

Then for any downset Q we have

$$Q \subseteq S \Rightarrow T \quad \text{iff} \quad \downarrow\!(q) \cap S \subseteq T, \text{ for all } q \in Q. \tag{2.27}$$

But that means that

$$\textstyle\bigcup_{q\in Q}(\downarrow(q)\cap S)\subseteq T\,,$$

which is equivalent to $Q\cap S\subseteq T$, since $\bigcup_{q\in Q}(\downarrow(q)\cap S)=(\bigcup_{q\in Q}\downarrow(q))\cap S=Q\cap S$.

The preservation of CCC structure by $\downarrow(-):P\to\downarrow P$ follows from its preservation by the Yoneda embedding, of which $\downarrow(-)$ is a factor,

$$\begin{tikzcd} & \mathsf{Set}^{P^{\mathsf{op}}} \\ P \arrow[ur, hook, "\mathsf{y}"] \arrow[r, "\downarrow(-)"'] & \downarrow P \arrow[u, hook] \end{tikzcd}$$

But it is also easy enough to check directly: preservation of any limits 1, $p\wedge q$ that exist in P are clear. Suppose $p\Rightarrow q$ is an exponential; then for any downset D we have:

$$\begin{array}{lll} D\subseteq\downarrow(p\Rightarrow q) & \text{iff} & \downarrow(d)\subseteq\downarrow(p\Rightarrow q)\ ,\ \text{for all } d\in D \\ & \text{iff} & d\leq p\Rightarrow q\ ,\ \text{for all } d\in D \\ & \text{iff} & d\wedge p\leq q\ ,\ \text{for all } d\in D \\ & \text{iff} & \downarrow(d\wedge p)\subseteq\downarrow(q)\ ,\ \text{for all } d\in D \\ & \text{iff} & \downarrow(d)\cap\downarrow(p)\subseteq\downarrow(q)\ ,\ \text{for all } d\in D \\ & \text{iff} & D\subseteq\downarrow(p)\Rightarrow\downarrow(q) \end{array}$$

where the last line is by (2.27). (Note that in line (3) we assumed that $d\wedge p$ exists for all $d\in D$; this can be avoided by a slightly more complicated argument.) $\square$

Proof. (of Theorem 2.8.4) The proof follows a now-familiar pattern, which we only sketch:

1. The syntactic category $\mathcal{C}_{\mathsf{PPC}}$ is a CCC, with $\top=1$, $\phi\times\psi=\phi\wedge\psi$, and $\psi^\phi=\phi\Rightarrow\psi$. In fact, it is the free cartesian closed poset on the generating set $\mathsf{Var}=\{p_1,p_2,\dots\}$ of propositional variables.

2. A (Kripke) model $(K,\Vdash)$ is the same thing as a CCC functor $\mathcal{C}_{\mathsf{PPC}}\to\downarrow K$, which by Step 1 is just an arbitrary map $\mathsf{Var}\to\downarrow K$, as in (2.25). To see this, observe that we have a bijective correspondence between CCC functors $[\![-]\!]$ and Kripke relations $\Vdash$; indeed, by the exponential adjunction in the cartesian closed category Pos, there is a natural bijection,

$$\frac{[\![-]\!]:\mathcal{C}_{\mathsf{PPC}}\longrightarrow\downarrow K\cong 2^{K^{\mathsf{op}}}}{\Vdash\,:K^{\mathsf{op}}\times\mathcal{C}_{\mathsf{PPC}}\longrightarrow 2}$$

where we use the poset 2 to classify downsets in a poset P (via upsets in P^{op}),

$$\downarrow P\cong 2^{P^{\mathsf{op}}}\cong\mathsf{Pos}(P^{\mathsf{op}},2)\,,$$

by taking the 1-kernel $f^{-1}(1) \subseteq P$ of a monotone map $f : P^{\mathsf{op}} \to \mathbb{2}$. (The contravariance will be convenient in Step 3). Note that the monotonicity of $\Vdash$ yields the conditions

$$p \leq q\,,\ q \Vdash \phi \implies p \Vdash \phi$$

and

$$p \Vdash \phi\,,\ \phi \vdash \psi \implies p \Vdash \psi\,.$$

and the CCC preservation of the transpose $[\![-]\!]$ yields the Kripke forcing conditions (2.26) (exercise!).

3. For any model $(K, \Vdash)$, by the adjunction in (2) we then have $K \Vdash \phi$ iff $[\![\phi]\!] = K$, the total downset.

4. Because the downset/Yoneda embedding $\downarrow$ preserves the CCC structure (by Lemma 2.8.5), $\mathcal{C}_{\mathsf{PPC}}$ has a *canonical model*, namely $(\mathcal{C}_{\mathsf{PPC}}, \Vdash)$, where:

$$\frac{\downarrow(-)\ :\ \mathcal{C}_{\mathsf{PPC}} \longrightarrow \downarrow\mathcal{C}_{\mathsf{PPC}} \cong \mathbb{2}^{\mathcal{C}_{\mathsf{PPC}}{}^{\mathsf{op}}} \hookrightarrow \mathsf{Set}^{\mathcal{C}_{\mathsf{PPC}}{}^{\mathsf{op}}}}{\Vdash\ :\ \mathcal{C}_{\mathsf{PPC}}{}^{\mathsf{op}} \times \mathcal{C}_{\mathsf{PPC}} \longrightarrow \mathbb{2} \hookrightarrow \mathsf{Set}}$$

5. Now note that for the Kripke relation $\Vdash$ in (4), we have $\Vdash\, = \,\vdash$, since it's essentially the transpose of the Yoneda embedding. Thus the model is logically generic, in the sense that $\mathcal{C}_{\mathsf{PPC}} \Vdash \phi$ iff $\mathsf{PPC} \vdash \phi$.

$\square$

Exercise 2.8.6. Verify the claim that CCC preservation of the transpose $[\![-]\!]$ of $\Vdash$ yields the Kripke forcing conditions (2.26).

Exercise 2.8.7. Give a countermodel to show that $\mathsf{PPC} \nvdash (\phi \Rightarrow \psi) \Rightarrow \phi$

2.9 Heyting algebras

We now extend the positive propositional calculus to the full intuitionistic propositional calculus. This involves adding the finite coproducts 0 and $p \vee q$ to notion of a cartesian closed poset, to arrive at the general notion of a Heyting algebra. Heyting algebras are to intuitionistic logic as Boolean algebras are to classical logic: each is an algebraic description of the corresponding logical calculus. We shall review both the algebraic and the logical points of view; as we shall see, many aspects of the theory of Boolean algebras carry over to Heyting algebras. For instance, in order to prove the Kripke completeness of the full system of intuitionistic propositional calculus, we will need an alternative to Lemma 2.8.5, because the Yoneda embedding does not in general preserve coproducts. For that we will again use a version of the Stone representation theorem, this time in a generalized form due to Joyal.

Distributive lattices

Recall first that a (bounded) *lattice* is a poset that has finite limits and colimits. In other words, a lattice $(L, \leq, \wedge, \vee, 1, 0)$ is a poset $(L, \leq)$ with distinguished elements $1, 0 \in L$, and binary operations meet $\wedge$ and join $\vee$, satisfying for all $x, y, z \in L$,

$$0 \leq x \leq 1 \qquad \frac{z \leq x \qquad z \leq y}{\overline{\overline{z \leq x \wedge y}}} \qquad \frac{x \leq z \qquad x \leq y}{\overline{\overline{x \vee y \leq z}}}$$

A *lattice homomorphism* is a function $f : L \to K$ between lattices which preserves finite limits and colimits, i.e., $f0 = 0$, $f1 = 1$, $f(x \wedge y) = fx \wedge fy$, and $f(x \vee y) = fx \vee fy$. The category of lattices and lattice homomorphisms is denoted by Lat.

A lattice can be axiomatized equationally as a set with two distinguished elements 0 and 1 and two binary operations $\wedge$ and $\vee$, satisfying the following equations:

$$\begin{aligned} (x \wedge y) \wedge z &= x \wedge (y \wedge z) , & (x \vee y) \vee z &= x \vee (y \vee z) , \\ x \wedge y &= y \wedge x , & x \vee y &= y \vee x , \\ x \wedge x &= x , & x \vee x &= x , \\ 1 \wedge x &= x , & 0 \vee x &= x , \end{aligned}$$
$$x \wedge (y \vee x) = x = (x \wedge y) \vee x . \tag{2.28}$$

The partial order on L is then determined by

$$x \leq y \iff x \wedge y = x .$$

Exercise 2.9.1. Show that in a lattice $x \leq y$ if, and only if, $x \wedge y = x$ if, and only if, $x \vee y = y$.

A lattice is *distributive* if the following distributive laws hold in it:

$$\begin{aligned} (x \vee y) \wedge z &= (x \wedge z) \vee (y \wedge z) , \\ (x \wedge y) \vee z &= (x \vee z) \wedge (y \vee z) . \end{aligned} \tag{2.29}$$

It turns out that if one distributive law holds then so does the other [**?**, I.1.5].

A *Heyting algebra* is a cartesian closed lattice H. This means that it has an operation $\Rightarrow$, satisfying for all $x, y, z \in H$

$$\frac{z \wedge x \leq y}{\overline{\overline{z \leq x \Rightarrow y}}}$$

A *Heyting algebra homomorphism* is a lattice homomorphism $f : K \to H$ between Heyting algebras that preserves implication, i.e., $f(x \Rightarrow y) = (fx \Rightarrow fy)$. The category of Heyting algebras and their homomorphisms is denoted by Heyt.

Heyting algebras can be axiomatized equationally as a set H with two distinguished elements 0 and 1 and three binary operations $\wedge$, $\vee$ and $\Rightarrow$. The equations for a Heyting algebra are the ones listed in (2.28), as well as the following ones for $\Rightarrow$.

$$\begin{aligned} (x \Rightarrow x) &= 1 \;, \\ x \wedge (x \Rightarrow y) &= x \wedge y \;, \\ y \wedge (x \Rightarrow y) &= y \;, \\ (x \Rightarrow (y \wedge z)) &= (x \Rightarrow y) \wedge (x \Rightarrow z) \;. \end{aligned} \tag{2.30}$$

For a proof, see [**?**, I.1], where one can also find a proof that every Heyting algebra is distributive (exercise!).

Example 2.9.2. We know from Lemma 2.8.5 that for any poset P, the poset $\downarrow P$ of all downsets in P, ordered by inclusion, is cartesian closed. Moreover, we know that $\downarrow P \cong \mathbb{2}^{P^{\mathsf{op}}}$, as a poset, with the reverse pointwise ordering on monotone maps $P^{\mathsf{op}} \to \mathbb{2}$, or equivalently, $\downarrow P \cong \mathbb{2}^{P}$, with the functions ordered pointwise. Since $\mathbb{2}$ is a lattice, we can also take joins $f \vee g$ pointwise, in order to get joins in $\mathbb{2}^P$, which then correspond to finite unions of the corresponding downsets $f^{-1}\{0\} \cup g^{-1}\{0\}$. Thus, in sum, for any poset P, the lattice $\downarrow P \cong \mathbb{2}^P$ is a Heyting algebra, with the downsets ordered by inclusion, and the functions ordered pointwise.

Intuitionistic propositional calculus

There is a forgetful functor $U : \mathsf{Heyt} \to \mathsf{Set}$ which maps a Heyting algebra to its underlying set, and a homomorphism of Heyting algebras to the underlying function. Because Heyting algebras are models of an equational theory, there is a left adjoint $H \dashv U$, which is the usual "free" construction mapping a set S to the free Heyting algebra HS generated by it. As for all algebraic strictures, the construction of HS can be performed in two steps: first, define a set HS of formal expressions, and then quotient it by an equivalence relation generated by the axioms for Heyting algebras.

Thus let HS be the set of formal expressions generated inductively by the following rules:

1. Generators: if $x \in S$ then $x \in HS$.

2. Constants: $\bot, \top \in HS$.

3. Connectives: if $\phi, \psi \in HS$ then $(\phi \wedge \psi), (\phi \vee \psi), (\phi \Rightarrow \psi) \in HS$.

We impose an equivalence relation on HS, which we write as equality $=$ and think of as such; it is defined as the smallest equivalence relation satisfying axioms (2.28) and (2.30). This forces HS to be a Heyting algebra. We define the action of the functor H on morphisms as usual: a function $f : S \to T$ is mapped to the Heyting algebra morphism $Hf : HS \to HT$ defined by

$$(Hf)\bot = \bot \;, \qquad (Hf)\bot = \bot \;, \qquad (Hf)x = fx \;,$$
$$(Hf)(\phi \star \psi) = ((Hf)\phi) \star ((Hf)\psi) \;,$$

where $\star$ stands for $\wedge$, $\vee$ or $\Rightarrow$.

The inclusion $\eta_S : S \to U(HS)$ of generators into the underlying set of the free Heyting algebra HS is then the component at S of a natural transformation $\eta : 1_{\mathsf{Set}} \Longrightarrow U \circ H$, which is of course the unit of the adjunction $H \dashv U$. To see this, consider a Heyting algebra K and an arbitrary function $f : S \to UK$. Then the Heyting algebra homomorphism $\overline{f} : HS \to K$ defined by

$$\overline{f}\bot = \bot\ , \qquad \overline{f}\bot = \bot\ , \qquad \overline{f}x = fx\ ,$$
$$\overline{f}(\phi \star \psi) = (\overline{f}\phi) \star (\overline{f}\psi)\ ,$$

where $\star$ stands for $\wedge$, $\vee$ or $\Rightarrow$, makes the following triangle commute:

$$\begin{tikzcd} S \arrow[r, "\eta_S"] \arrow[dr, "f"'] & U(HS) \arrow[d, "U\overline{f}"] \\ & K \end{tikzcd}$$

It is the unique such morphism because any two homomorphisms from HS which agree on generators must be equal. This is proved by induction on the structure of the formal expressions in HS.

We may now define the *intuitionistic propositional calculus* IPC to be the free Heyting algebra IPC on countably many generators p_0, p_1, ..., called *atomic propositions* or *propositional variables*. This is a somewhat unorthodox definition from a logical point of view—normally we would start from a *calculus* consisting of a formal language, judgements, and rules of inference—but of course, by now, we realize that the two approaches are essentially equivalent.

Having said that, let us also describe IPC in the conventional way. The formulas of IPC are built inductively from propositional variables p_0, p_1, ..., constants falsehood $\bot$ and truth $\top$, and binary operations conjunction $\wedge$, disjunction $\vee$ and implication $\Rightarrow$. The basic judgment of IPC is *logical entailment*

$$u_1 : A_1, \ldots, u_k : A_k \vdash B$$

which means "hypotheses A_1, ..., A_k entail proposition B". The hypotheses are labeled with distinct labels u_1, ..., u_k so that we can distinguish them, which is important when the same hypothesis appears more than once. Because the hypotheses are labeled it is irrelevant in what order they are listed, as long as the labels are not getting mixed up. Thus, the hypotheses $u_1 : A \vee B, u_2 : B$ are the same as the hypotheses $u_2 : B, u_1 : A \vee B$, but different from the hypotheses $u_1 : B, u_2 : A \vee B$. Sometimes we do not bother to label the hypotheses.

The left-hand side of a logical entailment is called the *context* and the right-hand side is the *conclusion*. Thus logical entailment is a relation between contexts and conclusions. The context may be empty. If Γ is a context, u is a label which does not occur in Γ, and A is a formula, then we write $\Gamma, u : A$ for the context Γ extended by the hypothesis $u : A$.

Definition 2.9.3. *Deductive entailment* is the smallest relation satisfying the following rules:

1. Conclusion from a hypothesis:
$$\frac{}{\Gamma \vdash A} \text{ if } u : A \text{ occurs in } \Gamma$$
2. Truth:
$$\frac{}{\Gamma \vdash \top}$$
3. Falsehood:
$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A}$$
4. Conjunction:
$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}$$
5. Disjunction:
$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \qquad \frac{\Gamma \vdash A \vee B \qquad \Gamma, u : A \vdash C \qquad \Gamma, v : B \vdash C}{\Gamma \vdash C}$$
6. Implication:
$$\frac{\Gamma, u : A \vdash B}{\Gamma \vdash A \Rightarrow B} \qquad \frac{\Gamma \vdash A \Rightarrow B \qquad \Gamma \vdash A}{\Gamma \vdash B}$$

A *proof* of $\Gamma \vdash A$ is a *finite* tree built from the above inference rules whose root is $\Gamma \vdash A$. A judgment $\Gamma \vdash A$ is *provable* if there exists a proof of it. Observe that every proof has at its leaves either the rule for $\top$ or a conclusion from a hypothesis.

You may wonder what happened to negation. In intuitionistic propositional calculus, negation is defined in terms of implication and falsehood as

$$\neg A \equiv A \Rightarrow \bot .$$

Properties of negation are then derived from the rules for implication and falsehood, see Exercise 2.9.7

Let P be the set of all formulas of IPC, preordered by the relation

$$A \vdash B , \qquad (A, B \in P)$$

where we did not bother to label the hypothesis A. Clearly, it is the case that $A \vdash A$. To see that $\vdash$ is transitive, suppose Π_1 is a proof of $A \vdash B$ and Π_2 is a proof of $B \vdash C$. Then we can obtain a proof of $A \vdash C$ from a proof Π_2 of $B \vdash C$ by replacing in it each use of the hypothesis B by the proof Π_1 of $A \vdash B$. This is worked out in detail in the next two exercises.

Exercise 2.9.4. Prove the following statement by induction on the structure of the proof Π: if Π is a proof of $\Gamma, u : A, v : A \vdash B$ then there is a proof of $\Gamma, u : A \vdash B$.

Exercise 2.9.5. Prove the following statement by induction on the structure of the proof Π_2: if Π_1 is a proof of $\Gamma \vdash A$ and Π_2 is a proof of $\Gamma, u : A \vdash B$, then there is a proof of $\Gamma \vdash B$.

Let IPC be the poset reflection of the preorder $(P, \vdash)$. The elements of IPC are equivalence classes $[A]$ of formulas, where two formulas A and B are equivalent if both $A \vdash B$ and $B \vdash A$ are provable. The poset IPC is just the free Heyting algebra on countably many generators p_0, p_1, ...

Classical propositional calculus

Another look:

An element $x \in L$ of a lattice L is said to be *complemented* when there exists $y \in L$ such that

$$x \vee y = 1 , \qquad\qquad x \wedge y = 0 .$$

We say that y is the *complement* of x.

In a distributive lattice, the complement of x is unique if it exists. Indeed, if both y and z are complements of x then

$$y \wedge z = (y \wedge z) \vee 0 = (y \wedge z) \vee (y \wedge x) = y \wedge (z \vee x) = y \wedge 1 = y ,$$

hence $y \leq z$. A symmetric argument shows that $z \leq y$, therefore $y = z$. The complement of x, if it exists, is denoted by $\neg x$.

A *Boolean algebra* is a distributive lattice in which every element is complemented. In other words, a Boolean algebra B has the *complementation* operation $\neg$ which satisfies, for all $x \in B$,

$$x \wedge \neg x = 0 , \qquad\qquad x \vee \neg x = 1 . \tag{2.31}$$

The full subcategory of Lat consisting of Boolean algebras is denoted by Bool.

Exercise 2.9.6. Prove that every Boolean algebra is a Heyting algebra. Hint: how is implication encoded in terms of negation and disjunction in classical logic?

In a Heyting algebra not every element is complemented. However, we can still define a *pseudo complement* or *negation* operation $\neg$ by

$$\neg x = (x \Rightarrow 0) ,$$

Then $\neg x$ is the largest element for which $x \wedge \neg x = 0$. While in a Boolean algebra $\neg\neg x = x$, in a Heyting algebra we only have $\neg\neg x \leq x$ in general. An element x of a Heyting algebra for which $\neg\neg x = x$ is called a *regular* element.

Exercise 2.9.7. Derive the following properties of negation in a *Heyting* algebra:

$$x \leq \neg\neg x \;,$$
$$\neg x = \neg\neg\neg x \;,$$
$$x \leq y \Rightarrow \neg y \leq \neg x \;,$$
$$\neg\neg(x \wedge y) = \neg\neg x \wedge \neg\neg y \;.$$

Exercise 2.9.8. Prove that the topology $\mathcal{O}X$ of any topological space X is a Heyting algebra. Describe in topological language the implication $U \Rightarrow V$, the negation $\neg U$, and the regular elements $U = \neg\neg U$ in $\mathcal{O}X$.

Exercise 2.9.9. Show that for a Heyting algebra H, the regular elements of H form a Boolean algebra $H_{\neg\neg} = \{x \in H \mid x = \neg\neg x\}$. Here $H_{\neg\neg}$ is viewed as a subposet of H. Hint: negation $\neg'$, conjunction $\wedge'$, and disjunction $\vee'$ in $H_{\neg\neg}$ are expressed as follows in terms of negation, conjunction and disjunction in H, for $x, y \in H_{\neg\neg}$:

$$\neg' x = \neg x \;, \qquad x \wedge' y = \neg\neg(x \wedge y) \;, \qquad x \vee' y = \neg\neg(x \vee y) \;.$$

The *classical propositional calculus (CPC)* is obtained from the intuitionistic propositional calculus by the addition of the logical rule known as *tertium non datur*, or the *law of excluded middle*:

$$\frac{}{\Gamma \vdash A \vee \neg A}$$

Alternatively, we could add the law known as *reductio ad absurdum*, or *proof by contradiction*:

$$\frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A} \;.$$

Identifying logically equivalent formulas of CPC, we obtain a poset CPC ordered by logical entailment. This poset is the *free Boolean algebra* on countably many generators. The construction of a free Boolean algebra can be performed just like described for the free Heyting algebra above. The equational axioms for a Boolean algebra are the axioms for a lattice (2.28), the distributive laws (2.29), and the complement laws (2.31).

Exercise* 2.9.10. Is CPC isomorphic to the Boolean algebra $\mathsf{IPC}_{\neg\neg}$ of the regular elements of IPC?

Exercise 2.9.11. Show that in a Heyting algebra H, one has $\neg\neg x = x$ for all $x \in H$ if, and only if, $y \vee \neg y = 1$ for all $y \in H$. Hint: half of the equivalence is easy. For the other half, observe that the assumption $\neg\neg x = x$ means that negation is an order-reversing bijection $H \to H$. It therefore transforms joins into meets and vice versa, and so the *De Morgan laws* hold:

$$\neg(x \wedge y) = \neg x \vee \neg y \;, \qquad \neg(x \vee y) = \neg x \wedge \neg y \;.$$

Together with $y \wedge \neg y = 0$, the De Morgan laws easily imply $y \vee \neg y = 1$. See [**?**, I.1.11].

Kripke semantics for IPC

We now prove the Kripke completeness of IPC, extending Theorem 2.8.4, namely:

Theorem 2.9.12 (Kripke completeness for IPC)**.** *Let K be a poset equipped with a* forcing relation *$k \Vdash p$ between elements $k \in K$ and propositional variables p, satisfying*

$$j \leq k,\ k \Vdash p \quad \textit{implies} \quad j \Vdash p. \tag{2.32}$$

Extend $\Vdash$ to all formulas ϕ in IPC by defining

$$\begin{array}{lll} k \Vdash \top & \textit{always}, & \\ k \Vdash \bot & \textit{never}, & \\ k \Vdash \phi \wedge \psi & \textit{iff} & k \Vdash \phi \textit{ and } k \Vdash \psi\,, \qquad (2.33) \\ k \Vdash \phi \vee \psi & \textit{iff} & k \Vdash \phi \textit{ or } k \Vdash \psi\,, \qquad (2.34) \\ k \Vdash \phi \Rightarrow \psi & \textit{iff} & \textit{for all } j \leq k, \textit{ if } j \Vdash \phi, \textit{ then } j \Vdash \psi\,. \end{array}$$

Finally, write $K \Vdash \phi$ if $k \Vdash \phi$ for all $k \in K$ (for all such relations $\Vdash$).

Then a propositional formulas ϕ is provable from the rules of deduction for IPC *(Definition 2.9.3) if, and only if, $K \Vdash \phi$ for all posets K. Briefly:*

$$\mathsf{PPC} \vdash \phi \quad \textit{iff} \quad K \Vdash \phi \textit{ for all } K.$$

Let us first see that we cannot simply reuse the proof from that theorem, because the downset (Yoneda) embedding that we used there

$$\downarrow\ : \mathsf{IPC} \hookrightarrow\ \downarrow\!(\mathsf{IPC}) \tag{2.35}$$

would not preserve the coproducts $\bot$ and $\phi \vee \psi$. Indeed, $\downarrow(\bot) \neq \emptyset$, because it contains $\bot$ itself! And in general $\downarrow(\phi \vee \psi) \neq \downarrow(\phi) \cup \downarrow(\psi)$, because the righthand side need not contain, e.g., $\phi \vee \psi$.

Instead, we will generalize the Stone Representation theorem 2.6.8 from Boolean algebras to Heyting algebras, using a theorem due to Joyal (cf. [**?**, **?**]). First, recall that the Stone representation provided, for any Boolean algebra $\mathcal{B}$, an injective Boolean homomorphism into a powerset,

$$\mathcal{B} \rightarrowtail \mathcal{P}X\,.$$

For X we took the set of prime filters $\mathsf{Bool}(\mathcal{B}, \mathbb{2})$, and the map $h : \mathcal{B} \rightarrowtail \mathcal{P}\mathsf{Bool}(\mathcal{B}, \mathbb{2})$ was given by $h(b) = \{F \,|\, b \in F\}$. Transposing $\mathcal{P}\mathsf{Bool}(\mathcal{B}, \mathbb{2}) \cong \mathbb{2}^{\mathsf{Bool}(\mathcal{B}, \mathbb{2})}$ in the cartesian closed category Pos, we arrive at the (monotone) evaluation map

$$\mathsf{eval} : \mathsf{Bool}(\mathcal{B}, \mathbb{2}) \times \mathcal{B} \to \mathbb{2}. \tag{2.36}$$

Now recall that the category of Boolean algebras is full in the category DLat of distributive lattices,

$$\mathsf{Bool}(\mathcal{B}, \mathbb{2}) = \mathsf{DLat}(\mathcal{B}, \mathbb{2})\,.$$

For any Heyting algebra $\mathcal{H}$ (or indeed any distributive lattice), the Homset $\mathsf{DLat}(\mathcal{H}, \mathbb{2})$, ordered pointwise, is isomorphic to the *poset* of all prime filters in $\mathcal{H}$ ordered by inclusion, by taking $f : \mathcal{H} \to \mathbb{2}$ to its (filter) kernel $f^{-1}\{1\} \subseteq \mathcal{H}$. In particular, the poset $\mathsf{DLat}(\mathcal{H}, \mathbb{2})$ is no longer discrete when $\mathcal{H}$ is not Boolean, since a prime ideal in a Heyting algebra need not be maximal.

The transpose of the (monotone) evaluation map,

$$\mathsf{eval} : \mathsf{DLat}(\mathcal{H}, \mathbb{2}) \times \mathcal{H} \to \mathbb{2}. \tag{2.37}$$

will then be the (monotone) map

$$\epsilon : \mathcal{H} \longrightarrow \mathbb{2}^{\mathsf{DLat}(\mathcal{H},\mathbb{2})}, \tag{2.38}$$

which takes $p \in \mathcal{H}$ to the "evaluation at p" map $f \mapsto f(p) \in \mathbb{2}$, i.e.,

$$\epsilon_p(f) = f(p) \qquad \text{for } p \in \mathcal{H} \text{ and } f : \mathcal{H} \to \mathbb{2}\,.$$

As before, the poset $\mathbb{2}^{\mathsf{DLat}(\mathcal{H},\mathbb{2})}$ (ordered pointwise) may be identified with the upsets in the poset $\mathsf{DLat}(\mathcal{H}, \mathbb{2})$, ordered by inclusion, which recall from Example 2.9.2 is always a Heyting algebra. Thus, in sum, we have a monotone map,

$$\mathcal{H} \longrightarrow {\uparrow}\mathsf{DLat}(\mathcal{H}, \mathbb{2})\,, \tag{2.39}$$

which generalizes the Stone representation from Boolean to Heyting algebras.

Theorem 2.9.13 (Joyal)**.** *Let $\mathcal{H}$ be a Heyting algebra. There is an injective Heyting homomorphism*

$$\mathcal{H} \rightarrowtail {\uparrow}J$$

into a Heyting algebra of upsets in a poset J.

Note that in this form, the theorem literally generalizes the Stone representation theorem, because when $\mathcal{H}$ is Boolean we can take J to be discrete, and then ${\uparrow}J \cong \mathsf{Pos}(J, \mathbb{2}) \cong \mathcal{P}J$ is Boolean, whence the Heyting embedding is also Boolean. The proof will again use the transposed evaluation map,

$$\epsilon : \mathcal{H} \longrightarrow {\uparrow}\mathsf{DLat}(\mathcal{H}, \mathbb{2}) \cong \mathbb{2}^{\mathsf{DLat}(\mathcal{H},\mathbb{2})}$$

which, as before, is injective, by the Prime Ideal Theorem (see Lemma 2.6.6). We will use it in the following form due to Birkhoff.

Lemma 2.9.14 (Birkhoff's Prime Ideal Theorem)**.** *Let D be a distributive lattice, $I \subseteq D$ an ideal, and $x \in D$ with $x \notin I$. There is a prime ideal $I \subseteq P \subset D$ with $x \notin P$.*

Proof. As in the proof of Lemma 2.6.6, it suffice to prove it for the case $I = (0)$. This time, we use Zorn's Lemma: a poset in which every chain has an upper bound has maximal elements. Consider the poset $\mathcal{I}\backslash x$ of "ideals I without x", $x \notin I$, ordered by inclusion.

The union of any chain $I_0 \subseteq I_1 \subseteq ...$ in $\mathcal{I}\backslash x$ is clearly also in $\mathcal{I}\backslash x$, so we have (at least one) maximal element $M \in \mathcal{I}\backslash x$. We claim that $M \subseteq D$ is prime. To that end, take $a, b \in D$ with $a \wedge b \in M$. If $a, b \notin M$, let $M[a] = \{n \leq m \vee a \mid m \in M\}$, the ideal join of M and $\downarrow(a)$, and similarly for $M[b]$. Since M is maximal without x, we therefore have $x \in M[a]$ and $x \in M[b]$. Thus let $x \leq m \vee a$ and $x \leq m' \vee b$ for some $m, m' \in M$. Then $x \vee m' \leq m \vee m' \vee a$ and $x \vee m \leq m \vee m' \vee b$, so taking meets on both sides gives

$$(x \vee m') \wedge (x \vee m) \leq (m \vee m' \vee a) \wedge (m \vee m' \vee b) = (m \vee m') \vee (a \wedge b).$$

Since the righthand side is in the ideal M, so is the left. But then $x \leq x \vee (m \wedge m')$ is also in M, contrary to our assumption that $M \in \mathcal{I}\backslash x$. □

Proof of Theorem 2.9.13. As in (2.39), let $J = \mathsf{DLat}(\mathcal{H}, \mathbb{2})$ be the poset of prime filters in $\mathcal{H}$, and consider the "evaluation" map (2.39),

$$\epsilon : \mathcal{H} \longrightarrow \mathbb{2}^{\mathsf{DLat}(\mathcal{H},\mathbb{2})} \cong \uparrow\mathsf{DLat}(\mathcal{H}, \mathbb{2})$$

given by $\epsilon(p) = \{F \mid p \in F \text{ prime}\}$.

Clearly $\epsilon(0) = \emptyset$ and $\epsilon(1) = \mathsf{DLat}(\mathcal{H}, \mathbb{2})$, and similarly for the other meets and joins, so ϵ is a lattice homomorphism. Moreover, if $p \neq q \in \mathcal{H}$ then, as in the proof of 2.6.8, we have that $\epsilon(p) \neq \epsilon(q)$, by the Prime Ideal Theorem (Lemma 2.9.14). Thus it just remains to show that

$$\epsilon(p \Rightarrow q) \;=\; \epsilon(p) \Rightarrow \epsilon(q)\,.$$

Unwinding the definitions, it suffices to show that, for all $f \in \mathsf{DLat}(\mathcal{H}, \mathbb{2})$,

$$f(p \Rightarrow q) = 1 \quad \text{iff} \quad \text{for all } g \geq f,\ g(p) = 1 \text{ implies } g(q) = 1. \tag{2.40}$$

Equivalently, for all prime filters $F \subseteq \mathcal{H}$,

$$p \Rightarrow q \in F \quad \text{iff} \quad \text{for all prime } G \supseteq F,\ p \in G \text{ implies } q \in G. \tag{2.41}$$

Now if $p \Rightarrow q \in F$, then for all (prime) filters $G \supseteq F$, also $p \Rightarrow q \in G$, and so $p \in G$ implies $q \in G$, since $(p \Rightarrow q) \wedge p \leq q$.

Conversely, suppose $p \Rightarrow q \notin F$, and we seek a prime filter $G \supseteq F$ with $p \in G$ but $q \notin G$. Consider the filter

$$F[p] = \{x \wedge p \leq h \in \mathcal{H} \mid x \in F\}\,,$$

which is the join of F and $\uparrow(p)$ in the poset of filters. If $q \in F[p]$, then $x \wedge p \leq q$ for some $x \in F$, whence $x \leq p \Rightarrow q$, and so $p \Rightarrow q \in F$, contrary to assumption; thus $q \notin F[p]$. By the Prime Ideal Theorem again (applied to the distributive lattice $\mathcal{H}^{\mathsf{op}}$) there is a prime filter $G \supseteq F[p]$ with $q \notin G$. □

Exercise 2.9.15. Give a Kripke countermodel to show that the Law of Excluded Middle $\phi \vee \neg\phi$ is not provable in IPC.

2.10 Frames and spaces

A poset $(P, \leq)$, viewed as a category, is *cocomplete* when it has suprema (least upper bounds) of arbitrary subsets. This is so because coequalizers in a poset always exist, and coproducts are precisely least upper bounds. Recall that the supremum of $S \subseteq P$ is an element $\bigvee S \in P$ such that, for all $y \in S$,

$$\bigvee S \leq y \iff \forall x : S \,.\, x \leq y \,.$$

In particular, $\bigvee \emptyset$ is the least element of P and $\bigvee P$ is the greatest element of P. Similarly, a poset is *complete* when it has infima (greatest lower bounds) of arbitrary subsets; the infimum of $S \subseteq P$ is an element $\bigwedge S \in P$ such that, for all $y \in S$,

$$y \leq \bigwedge S \iff \forall x : S \,.\, y \leq x \,.$$

Proposition 2.10.1. *A poset is complete if, and only if, it is cocomplete.*

Proof. Infima and suprema are expressed in terms of each other as follows:

$$\begin{aligned} \bigwedge S &= \bigvee \left\{ y \in P \,\middle|\, \forall x : S \,.\, y \leq x \right\} \,, \\ \bigvee S &= \bigwedge \left\{ y \in P \,\middle|\, \forall x : S \,.\, x \leq y \right\} \,. \end{aligned}$$

□

Thus, we usually speak of *complete* posets only, even when we work with arbitrary suprema.

Suppose P is a complete poset. When is it cartesian closed? Being a complete poset, it has the terminal object, namely the greatest element $1 \in P$, and it has binary products which are binary infima. If P is cartesian closed then for all $x, y \in P$ there exists an exponential $(x \Rightarrow y) \in P$, which satisfies, for all $z \in P$,

$$\frac{z \wedge x \leq y}{z \leq x \Rightarrow y} \,.$$

With the help of this adjunction we derive the *infinite distributive law*, for an arbitrary family $\left\{ y_i \in P \,\middle|\, i \in I \right\}$,

$$x \wedge \textstyle\bigvee_{i \in I} y_i = \bigvee_{i \in I} (x \wedge y_i) \tag{2.42}$$

as follows:

$$\begin{array}{c} x \wedge \bigvee_{i \in I} y_i \leq z \\ \hline\hline \bigvee_{i \in I} y_i \leq (x \Rightarrow z) \\ \hline\hline \forall i : I \,.\, (y_i \leq (x \Rightarrow z)) \\ \hline\hline \forall i : I \,.\, (x \wedge y_i \leq z) \\ \hline\hline \bigvee_{i \in I} (x \wedge y_i) \leq z \end{array}$$

Now since $x \wedge \bigvee_{i \in I} y_i$ and $\bigvee_{i \in I}(x \wedge y_i)$ have the same upper bounds they must be equal.

Conversely, suppose the distributive law (2.42) holds. Then we can *define* $x \Rightarrow y$ to be

$$(x \Rightarrow y) = \bigvee \left\{ z \in P \;\middle|\; x \wedge z \leq y \right\} \; . \tag{2.43}$$

The best way to show that $x \Rightarrow y$ is the exponential of x and y is to use the characterization of adjoints by counit, as in Proposition **??**. In the case of $\wedge$ and $\Rightarrow$ this amounts to showing that, for all $x, y \in P$,

$$x \wedge (x \Rightarrow y) \leq y \; , \tag{2.44}$$

and that, for $z \in P$,

$$(x \wedge z \leq y) \Rightarrow (z \leq x \Rightarrow y) \; .$$

This implication follows directly from (2.10.7), and (2.44) follows from the distributive law:

$$x \wedge (x \Rightarrow y) = x \wedge \bigvee \left\{ z \in P \;\middle|\; x \wedge z \leq y \right\} = \bigvee \left\{ x \wedge z \;\middle|\; x \wedge z \leq y \right\} \leq y \; .$$

Complete cartesian closed posets are called *frames*.

Definition 2.10.2. A *frame* is a poset that is complete and cartesian closed, thus a frame is a complete Heyting algebra. Equivalently, a frame is a complete poset satisfying the (infinite) distributive law

$$x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I}(x \wedge y_i) \; .$$

A *frame morphism* is a function $f : L \to M$ between frames that preserves finite infima and arbitrary suprema. The category of frames and frame morphisms is denoted by Frame.

Warning: a frame morphism need not preserve exponentials!

Example 2.10.3. Given a poset P, the downsets $\downarrow P$ form a complete lattice under the inclusion order $S \subseteq T$, and with the set theoretic operations $\bigcup$ and $\bigcap$ as $\bigvee$ and $\bigwedge$. Since $\downarrow P$ is already known to be a Heyting algebra (Example 2.9.2), it is therefore also a frame. (Alternately, we can show that it is a frame by noting that the operations $\bigcup$ and $\bigcap$ satisfy the infinite distributive law, and then infer that it is a Heyting algebra.)

A monotone map $f : P \to Q$ between posets gives rise to a frame map

$$\downarrow f : \downarrow Q \longrightarrow \downarrow P \, ,$$

as can be seen by recalling that $\downarrow P \cong 2^P$ as posets. Note that as a (co)limit preserving functor on complete posets, $2^f : 2^Q \longrightarrow 2^P$ has both left and right adjoints. These functors are usually written $f_! \dashv f^* \dashv f_*$. Although it does not in general preserve Heyting implications $S \Rightarrow T$, the monotone map $\downarrow f : \downarrow Q \longrightarrow \downarrow P$ is indeed a morphism of frames. We therefore have a contravariant functor

$$\downarrow(-) : \; \mathsf{Pos} \to \mathsf{Frame}^{\mathsf{op}}. \tag{2.45}$$

Example 2.10.4. The topology $\mathcal{O}X$ of a topological space X, ordered by inclusion, is a frame because finite intersections and arbitrary unions of open sets are open. The distributive law holds because intersections distribute over unions. If $f : X \to Y$ is a continuous map between topological spaces, the inverse image map $f^* : \mathcal{O}Y \to \mathcal{O}X$ is a frame homomorphism. Thus, there is a functor

$$\mathcal{O} : \mathsf{Top} \to \mathsf{Frame}^{\mathsf{op}}$$

which maps a space X to its topology $\mathcal{O}X$ and a continuous map $f : X \to Y$ to the inverse image map $f^* : \mathcal{O}Y \to \mathcal{O}X$.

The category $\mathsf{Frame}^{\mathsf{op}}$ is called the category of *locales* and is denoted by Loc. When we think of a frame as an object of Loc we call it a locale.

Example 2.10.5. Let P be a poset and define a topology on the elements of P by defining the opens to be the upsets,

$$\mathcal{O}P = \uparrow P \cong \mathsf{Pos}(P, \mathbb{2}).$$

These open sets are not only closed under arbitrary unions and finite intersections, but also under *arbitrary* intersections. Such a topological space is said to be an *Alexandrov space.*

Exercise* 2.10.6. This exercise is meant for students with some background in topology. For a topological space X and a point $x \in X$, let $N(x)$ be the neighborhood filter of x,

$$N(x) = \left\{ U \in \mathcal{O}X \;\middle|\; x \in U \right\} .$$

Recall that a T_0-*space* is a topological space X in which points are determined by their neighborhood filters,

$$N(x) = N(y) \Rightarrow x = y \; . \qquad (x, y \in X)$$

Let Top_0 be the full subcategory of Top on T_0-spaces. The functor $\mathcal{O} : \mathsf{Top} \to \mathsf{Loc}$ restricts to a functor $\mathcal{O} : \mathsf{Top}_0 \to \mathsf{Loc}$. Prove that $\mathcal{O} : \mathsf{Top}_0 \to \mathsf{Loc}$ is a faithful functor. Is it full?

Topological semantics for IPC

It should now be clear how to interpret IPC into a topological space X: each formula ϕ is assigned to an open set $[\![\phi]\!] \in \mathcal{O}X$ in such a way that $[\![-]\!]$ is a homomorphism of Heyting algebras.

Definition 2.10.7. A *topological model* of IPC is a space X and an interpretation of formulas,

$$[\![-]\!] : \mathsf{IPC} \to \mathcal{O}X \, ,$$

satisfying the conditions:

$$\begin{aligned}
[\![\top]\!] &= X \\
[\![\bot]\!] &= \emptyset \\
[\![\phi \wedge \psi]\!] &= [\![\phi]\!] \cap [\![\psi]\!] \\
[\![\phi \vee \psi]\!] &= [\![\phi]\!] \cup [\![\psi]\!] \\
[\![\phi \Rightarrow \psi]\!] &= [\![\phi]\!] \Rightarrow [\![\psi]\!]
\end{aligned}$$

The Heyting implication $[\![\phi]\!] \Rightarrow [\![\psi]\!]$ in $\mathcal{O}X$, is defined in (2.10.7) as

$$[\![\phi]\!] \Rightarrow [\![\psi]\!] \;=\; \bigcup \left\{ U \in \mathcal{O}X \;\middle|\; U \wedge [\![\phi]\!] \leq [\![\psi]\!] \right\} .$$

Joyal's representation theorem 2.9.13 easily implies that IPC is sound and complete with respect to topological semantics.

Corollary 2.10.8. *A formula ϕ is provable in IPC if, and only if, it holds in every topological interpretation $[\![-]\!]$ into a space X, briefly:*

$$\mathsf{IPC} \vdash \phi \qquad \textit{iff} \qquad [\![\phi]\!] = X \textit{ for all spaces } X .$$

Proof. Put the Alexandrov topology on the upsets of prime ideals in the Heyting algebra IPC □

Exercise 2.10.9. Give a topological countermodel to show that the Law of Double Negation $\neg\neg\phi \Rightarrow \phi$ is not provable in IPC.

Modal logic