Introduction to Categorical Logic

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Contents

Chapter 1

Algebraic Theories

Algebraic theories are descriptions of structures that are given entirely in terms of operations and equations. All such algebraic notions have in common some quite deep and general properties, from the existence of free algebras to Lawvere's duality theory. The most basic of these are presented in this chapter. The development also serves as a first example and template for the general scheme of *functorial semantics*, to be applied to other logical notions in later chapters.

1.1 Syntax and semantics

We begin with a general approach to algebraic structures such as groups, rings, and lattices. These are characterized by axiomatizations which involve only a single sort of variables and constants, operations, and equations. It is important that the operations are defined everywhere, which excludes two important examples: fields, because the inverse of 0 is undefined, and categories because composition is defined only for certain pairs of morphisms.

Let us start with the quintessential algebraic theory: the theory of groups. In first-order logic, a group can be described as a set G with a binary operation $\cdot : G \times G \to G$, satisfying the two first-order axioms:

$$\forall x, y, z \in G \,.\, (x \cdot y) \cdot z = x \cdot (y \cdot z)$$
$$\exists e \in G \,.\, \forall x \in G \,.\, \exists y \in G \,.\, (e \cdot x = x \cdot e = x \wedge x \cdot y = y \cdot x = e)$$

Taking a closer look at the logical form of these axioms, we see that the second one, which expresses the existence of a unit and inverse elements, is somewhat unsatisfactory because it involves nested quantifiers. Not only does this complicate the interpretation, but it is not really necessary, since the unit element and inverse operation in a group are uniquely determined. Thus we can add them to the structure and reformulate as follows. The unit is to be represented by a distinguished *constant* $e \in G$, and the inverse is to be a unary *operation* ${}^{-1} : G \to G$. We then obtain an equivalent formulation in which all axioms can be expressed as (universally quantified) *equations*:

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$
$$x \cdot e = x \qquad e \cdot x = x$$
$$x \cdot x^{-1} = e \qquad x^{-1} \cdot x = e$$

The universal quantifiers $\forall\, x \in G, \forall\, y \in G$, etc. are no longer needed in stating the axioms, since we can interpret all variables as ranging over all elements of G (because of our restriction to totally defined operations). Nor do we really need to explicitly mention the particular set G in the specification. Finally, since the constant e can be regarded as a nullary operation, i.e., a function $e : \mathbf{1} \to G$, the specification of the group concept consists solely of operations and equations. This leads to the following general definition of an algebraic theory.

Definition 1.1.1. A *signature* Σ for an algebraic theory consists of a family of sets $\{\Sigma_k\}_{k \in \mathbb{N}}$. The elements of Σ_k are called the *k-ary operations*. In particular, the elements of Σ_0 are the *nullary operations* or *constants*.

The *terms* of a signature Σ are the expressions constructed inductively by the following rules:

1. variables x, y, z, …, are terms,
2. if $t_1, \ldots, t_k$ are terms and $f \in \Sigma_k$ is a k-ary operation then $f(t_1, \ldots, t_k)$ is a term.

Definition 1.1.2 (cf. Definition 1.2.10)**.** An *algebraic theory* $\mathbb{T} = (\Sigma_{\mathbb{T}}, A_{\mathbb{T}})$ is given by a signature $\Sigma_{\mathbb{T}}$ and a set $A_{\mathbb{T}}$ of *axioms*, which are equations between terms (formally, pairs of terms).

Algebraic theories are also called *equational theories*. We do not assume that the sets Σ_k or $A_{\mathbb{T}}$ are finite, but the individual terms and equations always involve only finitely many variables.

Example 1.1.3. The theory of a commutative ring with unit is an algebraic theory. There are two nullary operations (constants) 0 and 1, a unary operation $-$, and two binary operations $+$ and $\cdot$. The equations are:

$$\begin{aligned} (x+y)+z &= x+(y+z) & \qquad (x \cdot y) \cdot z &= x \cdot (y \cdot z) \\ x + 0 &= x & x \cdot 1 &= x \\ 0 + x &= x & 1 \cdot x &= x \\ x + (-x) &= 0 & (x+y) \cdot z &= x \cdot z + y \cdot z \\ (-x) + x &= 0 & z \cdot (x+y) &= z \cdot x + z \cdot y \\ x + y &= y + x & x \cdot y &= y \cdot x \end{aligned}$$

Example 1.1.4. The "empty" or trivial theory $\mathbb{T}_0$ with no operations and no equations is the theory of a set.

Example 1.1.5. The theory with one constant and no equations is the theory of a *pointed set*, cf. Example **??**.

Example 1.1.6. Let R be a ring. There is an algebraic theory of left R-modules. It has one constant 0, a unary operation $-$, a binary operation $+$, and for each $a \in R$ a unary operation $\overline{a}$, called *scalar multiplication by* a. The following equations hold:

$$\begin{aligned} (x+y)+z &= x+(y+z)\ , & x+y &= y+x\ , \\ x+0 &= x\ , & 0+x &= x\ , \\ x+(-x) &= 0\ , & (-x)+x &= 0\ . \end{aligned}$$

For every $a, b \in R$ we also have the equations

$$\overline{a}(x+y) = \overline{a}\,x + \overline{a}\,y\ , \qquad \overline{a}(\overline{b}\,x) = \overline{(ab)}\,x\ , \qquad \overline{(a+b)}\,x = \overline{a}\,x + \overline{b}\,x\ .$$

Scalar multiplication by a is usually written as $a \cdot x$ instead of $\overline{a}\,x$. If we replace the ring R by a field $\mathbb{F}$ we obtain the algebraic theory of a vector space over $\mathbb{F}$ (even though the theory of fields is not algebraic!).

Example 1.1.7. In computer science, inductive datatypes are examples of algebraic theories. For example, the datatype of binary trees with leaves labeled by integers might be defined as follows in a programming language:

```
type tree = Leaf of int | Node of tree * tree
```

This corresponds to the algebraic theory with a constant `Leaf` n for each integer n and a binary operation `Node`. There are no equations. Actually, when computer scientists define a datatype like this, they have in mind a particular model of the theory, namely the *free* one.

Example 1.1.8. An obvious non-example is the theory of posets, formulated with a binary relation symbol $x \leq y$ and the usual axioms of reflexivity, transitivity and anti-symmetry, namely:

$$\begin{gathered} x \leq x \\ x \leq y\ ,\ y \leq z \Rightarrow x \leq z \\ x \leq y\ ,\ y \leq x \Rightarrow x = x \end{gathered}$$

On the other hand, using an operation of greatest lower bound or "meet" $x \wedge y$, one can make the equational theory of "$\wedge$-semilattices":

$$\begin{gathered} x \wedge x = x \\ x \wedge y = y \wedge x \\ x \wedge (y \wedge z) = (x \wedge y) \wedge z \end{gathered}$$

Then, defining a partial ordering by $x \leq y \Leftrightarrow x = (x \wedge y)$ we arrive at the notion of a "poset with meets", which *is* equational (of course, the same can be done with joins $x \vee y$ as well). We will show later (in section **??**) that there is no reformulation of the general theory of posets into an equivalent equational one by considering the *category of models* of the theory, i.e. the category of posets, and showing that it lacks a general property enjoyed by all categories of algebras.

Exercise 1.1.9. Let G be a group. Formulate the notion of a (left) G-set (i.e. a functor $G \to \mathsf{Set}$) as an algebraic theory.

1.1.1 Models of algebraic theories

Let us now consider *models* of an algebraic theory, i.e. *algebras*. Classically, a group can be given by a set G, an element $e \in G$, a function $m : G \times G \to G$ and a function $i : G \to G$, satisfying the group axioms:

$$\begin{aligned} m(x, m(y, z)) &= m(m(x, y), z) \\ m(x, i\,x) &= m(i\,x, x) = e \\ m(x, e) &= m(e, x) = x \end{aligned}$$

for any $x, y, z \in G$. Observe, however, that this notion can easily be generalized so that we can speak of models of group theory in categories other than Set. This is accomplished simply by translating the equations between arbitrary elements into equations between the operations themselves: thus a group is given, first, by an object $G \in \mathsf{Set}$ and three morphisms

$$e : \mathsf{1} \to G\,, \qquad m : G \times G \to G\,, \qquad i : G \to G\,.$$

The associativity axiom is then expressed by the commutativity of the following diagram:

$$\begin{tikzcd} G \times G \times G \arrow[r, "m \times \pi_2"] \arrow[d, "\pi_0 \times m"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd} \tag{1.1}$$

Note that we have omitted the canonical associativity function $G \times (G \times G) \cong (G \times G) \times G$, which should be inserted into the top left corner of the diagram. The equations for the unit and the inverse are similarly expressed by commutativity of the following diagrams:

$$
\begin{tikzcd}
G \times \mathsf{1} \arrow[r, "1_G \times e"] \arrow[dr, "\pi_0"'] & G \times G \arrow[d, "m"] & \mathsf{1} \times G \arrow[l, "e \times 1_G"'] \arrow[dl, "\pi_1"] \\
& G &
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "\langle 1_G, i \rangle"] \arrow[d, "!_G"'] & G \times G \arrow[d, "m"] & G \arrow[l, "\langle i, 1_G \rangle"'] \arrow[d, "!_G"] \\
\mathsf{1} \arrow[r, "e"'] & G & \mathsf{1} \arrow[l, "e"]
\end{tikzcd}
\tag{1.2}
$$

This formulation makes sense in any category $\mathcal{C}$ with finite products.

Definition 1.1.10. Let $\mathcal{C}$ be a category with finite products. A *group in* $\mathcal{C}$ consists of an object G equipped with arrows:

$$
\begin{tikzcd}
G \times G \arrow[r, "m"] & G & G \arrow[l, "i"'] \\
& \mathsf{1} \arrow[u, "e"'] &
\end{tikzcd}
$$

such that the above diagrams (1.1) and (1.2) expressing the group equations commute.

There is also an obvious corresponding generalization of a group homomorphism in Set to homomorphisms of groups in $\mathcal{C}$. Namely, an arrow in $\mathcal{C}$ between (the underlying objects of) groups, say $h : M \to N$, is a homomorphism if it commutes with the interpretations of the basic operations m, i, and e,

$$h \circ m^M = m^N \circ h^2 \qquad h \circ i^M = i^N \circ h \qquad h \circ e^M = e^N$$

as indicated in:

$$
\begin{tikzcd}
M^2 \arrow[r, "h^2"] \arrow[d, "m^M"'] & N^2 \arrow[d, "m^N"] \\
M \arrow[r, "h"'] & N
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, "h"] \arrow[d, "i^M"'] & N \arrow[d, "i^N"] \\
M \arrow[r, "h"'] & N
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{1} \arrow[r, "="] \arrow[d, "e^M"'] & \mathsf{1} \arrow[d, "e^N"] \\
M \arrow[r, "h"'] & N
\end{tikzcd}
$$

Together with the evident composition and identity arrows inherited from $\mathcal{C}$, this gives a category of groups in $\mathcal{C}$, which we denote:

$$\mathsf{Group}(\mathcal{C})$$

In general, we define an *interpretation* I of a theory $\mathbb{T}$ in a category $\mathcal{C}$ with finite products to consist of an object $I \in \mathcal{C}$ and, for each basic operation f of arity k, a morphism $f^I : I^k \to I$. (More formally, I is the tuple consisting of an underlying object $|I|$ and the interpretations f^I, but we shall write simply I for $|I|$.) In particular, basic constants are interpreted as morphisms $\mathsf{1} \to I$. The interpretation is then extended to all terms as follows: a general term t will be interpreted together with a *context of variables* $x_1, \dots, x_n$ (a list without repetitions), where the variables appearing in t are among those appearing in the context. We write

$$x_1, \dots, x_n \mid t \tag{1.3}$$

for a term t in context $x_1, \dots x_n$. The interpretation of such a term in context (1.3) is a morphism $t^I : I^n \to I$, determined by the following specification:

1. The interpretation of a variable x_i among the $x_1, \dots x_n$ is the i-th projection $\pi_i : I^n \to I$.

2. A term of the form $f\,(t_1, \dots, t_k)$ is interpreted as the composite:

$$I^n \xrightarrow{\left(t_1{}^I, \dots, t_k{}^I\right)} I^k \xrightarrow{\quad f^I \quad} I$$

   where $t_i{}^I : I^n \to I$ is the interpretation of the subterm t_i, for $i = 1, \dots, k$, and f^I is the interpretation of the basic operation f.

It is clear that the interpretation of a term really depends on the context, and when necessary we shall write $t^I = [\, x_1, \dots, x_n \mid t\,]^I$. For example, the term $f\,x_1$ is interpreted as a morphism $f^I : I \to I$ in context x_1, and as the morphism $f^I \circ \pi_1 : I^2 \to I$ in the context x_1, x_2.

Suppose u and v are terms in context $x_1, \dots, x_n$. Then we say that the equation in context $x_1, \dots, x_n \mid u = v$ is *satisfied* by the interpretation I if u^I and v^I are the same morphism in $\mathcal{C}$. In particular, if $u = v$ is an axiom of the theory, and $x_1, \dots, x_n$ are all the variables appearing in either u or v, we say that I *satisfies the axiom* $u = v$, written

$$I \models u = v,$$

if $[x_1, \dots, x_n \mid u]^I$ and $[x_1, \dots, x_n \mid v]^I$ are the same morphism,

$$\begin{tikzcd} I^n \arrow[r, shift left, "{[x_1, \dots, x_n \mid u]^I}"] \arrow[r, shift right, "{[x_1, \dots, x_n \mid v]^I}"'] & I\,. \end{tikzcd} \tag{1.4}$$

We can then define, as expected:

Definition 1.1.11 (cf. Definition 1.2.10)**.** A *model* M of an algebraic theory $\mathbb{T}$ in a category $\mathcal{C}$ with finite products (also called a $\mathbb{T}$-*algebra*) is an interpretation of the signature $\Sigma_{\mathbb{T}}$,

$$f^I : I^k \longrightarrow I \qquad \text{in } \mathcal{C},$$

for all $f \in \Sigma_{\mathbb{T}}$, that satisfies the axioms $A_{\mathbb{T}}$,

$$I \models u = v,$$

for all $(u = v) \in A_{\mathbb{T}}$.

A *homomorphism* of models $h : M \to N$ is an arrow in $\mathcal{C}$ that commutes with the interpretations of the basic operations,

$$h \circ f^M = f^N \circ h^k$$

for all $f \in \Sigma_{\mathbb{T}}$, as indicated in:

$$\begin{tikzcd} M^k \arrow[r, "h^k"] \arrow[d, "f^M"'] & N^k \arrow[d, "f^N"] \\ M \arrow[r, "h"'] & N \end{tikzcd}$$

The category of $\mathbb{T}$-models in $\mathcal{C}$ is written,

$$\mathsf{Mod}(\mathbb{T}, \mathcal{C}).$$

A model of the trivial theory $\mathbb{T}_0$ in $\mathcal{C}$ is therefore just an object A in $\mathcal{C}$, and a homomorphism is just a map, so

$$\mathsf{Mod}(\mathbb{T}_0, \mathcal{C}) = \mathcal{C}.$$

A model of the theory $\mathbb{T}_{\mathsf{Group}}$ of groups in $\mathcal{C}$ is a group in $\mathcal{C}$, in the above sense, and similarly for homomorphisms, so:

$$\mathsf{Mod}(\mathbb{T}_{\mathsf{Group}}, \mathcal{C}) = \mathsf{Group}(\mathcal{C})$$

as defined above. In particular, a model in Set is just a group in the usual sense, so we have:

$$\mathsf{Mod}(\mathbb{T}_{\mathsf{Group}}, \mathsf{Set}) = \mathsf{Group}(\mathsf{Set}) = \mathsf{Group}.$$

An example of a new kind is provided by the following.

Example 1.1.12. A model of the theory of groups in a functor category $\mathsf{Set}^{\mathbb{C}}$ is the same thing as a functor from $\mathbb{C}$ into the category groups,

$$\mathsf{Group}(\mathsf{Set}^{\mathbb{C}}) = \mathsf{Group}(\mathsf{Set})^{\mathbb{C}} \cong \mathsf{Group}^{\mathbb{C}}.$$

Indeed, for each object $C \in \mathbb{C}$ there is an evaluation functor,

$$\mathsf{eval}_C : \mathsf{Set}^{\mathbb{C}} \to \mathsf{Set}$$

with $\mathsf{eval}_C(F) = F(C)$, and evaluation preserves products since these are computed pointwise in the functor category. Moreover, every arrow $h : C \to D$ in $\mathbb{C}$ gives rise to an obvious natural transformation $h : \mathsf{eval}_C \to \mathsf{eval}_D$. Thus for any group G in $\mathsf{Set}^{\mathbb{C}}$, we have groups $\mathsf{eval}_C(G)$ for each $C \in \mathbb{C}$ and group homomorphisms $h_G : C(G) \to D(G)$, comprising a functor $G : \mathbb{C} \to \mathsf{Group}$. Conversely, it is clear that a functor $H : \mathbb{C} \to \mathsf{Group}$ determines a group H in $\mathsf{Set}^{\mathbb{C}}$ with underlying object $U \circ H$, where $U : \mathsf{Group} \to \mathsf{Set}$ is the forgetful functor, so that for each $C \in \mathbb{C}$ we have a group HC with underlying set $UHC = |HC|$. These constructions are clearly mutually inverse (up to canonical isomorphisms determined by the choice of products). Thus, briefly, *a group in the category of variable sets may be regarded as a variable group.*

Exercise 1.1.13. Verify the details of the isomorphism of categories

$$\mathsf{Mod}(\mathbb{T}, \mathsf{Set}^{\mathbb{C}}) \cong \mathsf{Mod}(\mathbb{T}, \mathsf{Set})^{\mathbb{C}},$$

as example 1.1.12, for an arbitrary algebraic theory $\mathbb{T}$.

Exercise 1.1.14. Determine what a group is in the following categories: the category of graphs Graph, the category of topological spaces Top, and the category of groups Group. (Hint: Only the last case is tricky. Before thinking too hard about it, prove the following lemma [Bor94, Lemma 3.11.6], known as the Eckmann-Hilton argument. Let G be a set provided with two binary operations $\cdot$ and $\star$ and a common unit e, so that $x \cdot e = e \cdot x = x \star e = e \star x = x$. Suppose the two operations commute, i.e., $(x \star y) \cdot (z \star w) = (x \cdot z) \star (y \cdot w)$. Then they coincide, and are *commutative* and associative.)

1.1.2 Theories as categories

The syntactically presented notion of an algebraic theory is a practical convenience, but as a specification of a mathematical concept, say that of a group, it has some defects. We would prefer a *presentation-free* notion that captures the group concept without tying it to a specific syntactic presentation (the example below indicates why). One such notion can be given by a category with a certain universal property, which determines it uniquely, up to equivalence of categories.

Let us consider group theory again. The algebraic axiomatization in terms of unit, multiplication and inverse is not the only possible one. For example, an alternative formulation uses the unit e and a binary operation $\odot$, called *double division*, along with a single axiom [McC93]:

$$(x \odot (((x \odot y) \odot z) \odot (y \odot e))) \odot (e \odot e) = z \ .$$

The usual group operations are related to double division as follows:

$$x \odot y = x^{-1} \cdot y^{-1} \ , \qquad x^{-1} = x \odot e \ , \qquad x \cdot y = (x \odot e) \odot (y \odot e) \ .$$

There may be practical reasons for prefering one formulation of group theory over another, but this should not determine what the general concept of a group is. For example, we would like to avoid particular choices of basic constants, operations, and axioms. This is akin to the situation where an algebra is presented by generators and relations: the algebra itself is regarded as independent of any particular such presentation. Similarly, one usually prefers a basis-free theory of vector spaces: it is better to formulate the general idea of a vector space without refering explicitly to a basis, even though every vector space has one.

As a first step, one could simply take *all* operations built from unit, multiplication, and inverse as basic, and *all* valid equations of group theory as axioms. But we can go a step further and collect all the operations into a category, thus forgetting about which ones were "basic", and which equalities were "axioms". We first describe this construction of a "syntactic category" $\mathsf{Syn}(\mathbb{T})$ for an algebraic theory $\mathbb{T}$, and then determine a universal characterization of it.

As objects of $\mathsf{Syn}(\mathbb{T})$ we take the *contexts*, i.e. sequences of distinct variables,

$$[x_1, \ldots, x_n] \,. \qquad (n \geq 0)$$

Actually, it will be more convenient to take equivalence classes under renaming of variables, so that $[x_1, x_3] = [x_2, x_1]$. That is to say, the objects are just natural numbers; but it will be useful to continue to write them as contexts.

A morphism from $[x_1, \ldots, x_m]$ to $[x_1, \ldots, x_n]$ is then an n-tuple $(t_1, \ldots, t_n)$, where each t_k is a term in the context $x_1, \ldots, x_m$, possibly after renaming the variables. Two such morphisms $(t_1, \ldots, t_n)$ and $(s_1, \ldots, s_n)$ are equal if, and only if, the axioms of the theory formally imply that $t_k = s_k$ for every $k = 1, \ldots, n$,

$$\mathbb{T} \vdash t_k = s_k \,.$$

Here we are using the usual notion of equational deduction $\mathbb{T} \vdash$ (see Section A.5). Strictly speaking, morphisms are thus *equivalence classes* of tuples of terms in context,

$$[x_1, \ldots, x_m \mid t_1, \ldots, t_n] : [x_1, \ldots, x_m] \longrightarrow [x_1, \ldots, x_n],$$

where two terms are equivalent when the theory proves them to be equal (after renaming the variables). Since it is rather cumbersome to work with such equivalence classes, we shall work with the terms directly, but keeping in mind that equality between them is this equivalence. Note also that the context of the morphism agrees with its domain, so we can omit it from the notation when that domain is clear. The composition of two morphisms

$$\begin{aligned}(t_1, \ldots, t_m) &: [x_1, \ldots, x_k] \longrightarrow [x_1, \ldots, x_m] \\ (s_1, \ldots, s_n) &: [x_1, \ldots, x_m] \longrightarrow [x_1, \ldots, x_n]\end{aligned}$$

is the morphism $(r_1, \ldots, r_n)$ whose i-th component is obtained by simultaneously substituting in s_i the terms $t_1, \ldots, t_m$ for the variables $x_1, \ldots, x_m$:

$$r_i = s_i[\, t_1/x_1, \ldots, t_m/x_m \,] \qquad (1 \leq i \leq n)$$

The identity morphism on the object $[x_1, \ldots, x_n]$ is the equivalence class of $(x_1, \ldots, x_n)$.

Using the usual rules of deduction for equational logic (Section A.5), it is easy to verify that these specifications are well-defined on equivalence classes, and therefore make $\mathsf{Syn}(\mathbb{T})$ a category.

Definition 1.1.15. The category $\mathsf{Syn}(\mathbb{T})$ just defined is called the *syntactic category* of the algebraic theory $\mathbb{T}$.

The syntactic category $\mathsf{Syn}(\mathbb{T})$ (which may be thought of as the "Lindenbaum-Tarski category" of $\mathbb{T}$, see **??**) contains the same "algebraic" information as the theory $\mathbb{T}$ from which it was built, but in a syntax-invariant way. Two different syntactic presentations of $\mathbb{T}$ — like the ones for groups mentioned above — will give rise to essentially the same category $\mathsf{Syn}(\mathbb{T})$ (i.e. up to isomorphism). In this sense, the category $\mathsf{Syn}(\mathbb{T})$ is the abstract, algebraic object presented by the "generators and relations" (the operations and equations) of $\mathbb{T}$. But there is another, more important, sense in which $\mathsf{Syn}(\mathbb{T})$ represents $\mathbb{T}$, as we next show.

Exercise 1.1.16. Show that the syntactic category $\mathsf{Syn}(\mathbb{T})$ has all finite products.

1.1.3 Models as functors

Having represented an algebraic theory $\mathbb{T}$ by the syntactic category $\mathsf{Syn}(\mathbb{T})$ constructed from it, we next show that $\mathsf{Syn}(\mathbb{T})$ has the universal property that models of $\mathbb{T}$ correspond uniquely to certain functors from $\mathsf{Syn}(\mathbb{T})$. More precisely, given any category with finite products $\mathcal{C}$ (which we shall call an *FP-category*), there is a natural (in $\mathcal{C}$) equivalence,

$$\frac{\mathcal{M} \in \mathsf{Mod}(\mathbb{T}, \mathcal{C})}{M : \mathsf{Syn}(\mathbb{T}) \to \mathcal{C}} \tag{1.5}$$

between models $\mathcal{M}$ of $\mathbb{T}$ in $\mathcal{C}$ and finite product preserving functors ("*FP-functors*") $M : \mathsf{Syn}(\mathbb{T}) \to \mathcal{C}$. The equivalence is mediated by a "universal model" $\mathcal{U}$ in $\mathsf{Syn}(\mathbb{T})$, corresponding to the identity functor $1_{\mathsf{Syn}(\mathbb{T})} : \mathsf{Syn}(\mathbb{T}) \to \mathsf{Syn}(\mathbb{T})$ under the above displayed equivalence. By naturality, every model $\mathcal{M}$ then arises as the functorial image $M(\mathcal{U}) \cong \mathcal{M}$ of $\mathcal{U}$ under an essentially unique FP-functor $M : \mathsf{Syn}(\mathbb{T}) \to \mathcal{C}$.

To give the details of the correspondence (1.5), let $\mathbb{T}$ be an arbitrary algebraic theory and $\mathsf{Syn}(\mathbb{T})$ the syntactic category constructed from $\mathbb{T}$ as in Definition 1.1.15. It is easy to show that the product in $\mathsf{Syn}(\mathbb{T})$ of two objects $[x_1, \ldots, x_n]$ and $[x_1, \ldots, x_m]$ is the object $[x_1, \ldots, x_{n+m}]$, and that $\mathsf{Syn}(\mathbb{T})$ has all finite products, including $1 = [-]$, the empty context (see Exercise 1.1.16). Moreover, there is a distinguished $\mathbb{T}$-model $\mathcal{U}$ in $\mathsf{Syn}(\mathbb{T})$ consisting of the signature $\Sigma_{\mathbb{T}}$ itself, which we call the *syntactic model*: the underlying object $U = |\mathcal{U}|$ is the context $[x_1]$ of length one, and each operation symbol f, of say arity k, is interpreted as "itself",

$$\begin{tikzcd}
U^k \arrow[r, "f^{\mathcal{U}}"] \arrow[d, "="'] & U \arrow[d, "="] \\
{[x_1, \ldots, x_k]} \arrow[r, "{[\, f(x_1, \ldots, x_k)\,]}"'] & {[x_1]}
\end{tikzcd} \tag{1.6}$$

The axioms are then satisfied, because the equivalence relation on terms is just $\mathbb{T}$-provable equality (see Section A.5). Explicitly, for all terms s, t we have:

$$\mathcal{U} \models s = t \quad \iff \quad \mathbb{T} \vdash s = t. \tag{1.7}$$

We record this fact as the following.

Proposition 1.1.17. *The syntactic model $\mathcal{U}$ in $\mathsf{Syn}(\mathbb{T})$ is "logically generic" in the sense that it satisfies all and only the $\mathbb{T}$-provable equations, as in* (1.7).

Proof. For the proof, one shows that every term t is interpreted in $\mathcal{U}$ by "itself", i.e. by its own equivalence class under $\mathbb{T}$-provable equality,

$$(\, x_1, \ldots, x_m \mid t\,)^{\mathcal{U}} = [\, x_1, \ldots, x_m \mid t\,]$$

This is a simple induction on the construction of t, where the base case is given by (1.6). □

Even more important than being logically generic, though, is the following *universal property* of the syntactic model $\mathcal{U}$ in $\mathsf{Syn}(\mathbb{T})$.

> Any model $\mathcal{M}$ in any finite product category $\mathcal{C}$ is the image of $\mathcal{U}$ under an essentially unique, finite product preserving functor $\mathcal{M}^\sharp : \mathsf{Syn}(\mathbb{T}) \to \mathcal{C}$,
>
> $$\mathcal{M}^\sharp(\mathcal{U}) \cong \mathcal{M}\,.$$

(See Definition 1.1.20 below for a more precise formulation.) In this sense, the syntactic category $\mathsf{Syn}(\mathbb{T})$ may be thought of as the "free finite product category with a model of $\mathbb{T}$". To show this formally, first observe that any FP-functor $F : \mathsf{Syn}(\mathbb{T}) \to \mathcal{C}$ takes the syntactic model $\mathcal{U}$ in $\mathsf{Syn}(\mathbb{T})$ to a model $F\mathcal{U}$ in $\mathcal{C}$, with interpretations

$$f^{F\mathcal{U}} = Ff^{\mathcal{U}} : FU^k \to FU \qquad \text{for each } f \in \Sigma_k.$$

Indeed, that is true for any FP-category $\mathcal{D}$ in place of $\mathsf{Syn}(\mathbb{T})$ and any model in $\mathcal{D}$. Similarly, any natural transformation $\vartheta : F \to G$ between FP-functors determines a homomorphism of models $h = \vartheta_{\mathcal{U}} : F\mathcal{U} \to G\mathcal{U}$. In more detail, suppose $f : U \times U \to U$ is a basic operation, then there is a commutative diagram,

$$\begin{tikzcd}
FU \times FU \arrow[r, "h \times h"] \arrow[d, "\cong"] \arrow[dd, bend right=60, "f^{F\mathcal{U}}"'] & GU \times GU \arrow[d, "\cong"'] \arrow[dd, bend left=60, "f^{G\mathcal{U}}"] \\
F(U \times U) \arrow[r, "\vartheta_{U\times U}"] \arrow[d, "Ff"] & G(U \times U) \arrow[d, "Gf"'] \\
FU \arrow[r, "h = \vartheta_U"'] & GU
\end{tikzcd}$$

where the upper square commutes by preservation of products, and the lower one by naturality. Thus the operation "evaluation at $\mathcal{U}$" always determines a functor,

$$\mathsf{eval}_{\mathcal{U}} : \mathsf{Hom}_{\mathrm{FP}}(\mathsf{Syn}(\mathbb{T}), \mathcal{C}) \longrightarrow \mathsf{Mod}(\mathbb{T}, \mathcal{C}) \tag{1.8}$$

from the category of finite product preserving functors $\mathsf{Syn}(\mathbb{T}) \to \mathcal{C}$, with natural transformations as arrows, into the category of $\mathbb{T}$-models in $\mathcal{C}$. Indeed, this much is also true for any model in any FP-category $\mathcal{D}$; what is special about $\mathcal{U}$ is the following.

Proposition 1.1.18. *The functor* (1.8) *is an equivalence of categories, natural in* $\mathcal{C}$.

Proof. Let $\mathcal{M}$ be any model in an FP-category $\mathcal{C}$. Then the underlying interpretation of $\mathcal{M}$ is an assignment $f \mapsto f^{\mathcal{M}}$ for $f \in \Sigma$, which determines a functor $\mathcal{M}^\sharp : \mathsf{Syn}(\mathbb{T}) \to \mathcal{C}$, defined on objects by

$$\mathcal{M}^\sharp[x_1, \ldots, x_k] = |\mathcal{M}|^k$$

and on morphisms by

$$\mathcal{M}^\sharp[\, t_1, \ldots, t_n \,] = \left(t_1{}^{\mathcal{M}}, \ldots, t_n{}^{\mathcal{M}}\right).$$

In more detail, $\mathcal{M}^\sharp$ is defined on a morphism

$$[x_1, \ldots, x_k \mid t\,] : [x_1, \ldots, x_k] \to [x_1, \ldots, x_n]$$

in $\mathsf{Syn}(\mathbb{T})$ by the following rules:

1. The morphism

$$[x_1, \ldots, x_k \mid x_i] : [x_1, \ldots, x_k] \to [x_1]$$

   is mapped to the i-th projection

$$\pi_i : M^k \to M.$$

2. The morphism

$$[x_1, \ldots, x_k \mid f(t_1, \ldots, t_m)] : [x_1, \ldots, x_k] \to [x_1]$$

   is mapped to the composite

$$M^k \xrightarrow{\left(\mathcal{M}^\sharp t_1, \ldots, \mathcal{M}^\sharp t_m\right)} M^m \xrightarrow{f^{\mathcal{M}}} M$$

   where the $\mathcal{M}^\sharp t_i : M^k \to M$ are the values of $\mathcal{M}^\sharp$ on the morphisms $[t_i] : [x_1, \ldots, x_k] \to [x_i]$, for $i = 1, \ldots, m$, and $f^{\mathcal{M}}$ is the interpretation of the basic operation f.

3. The morphism

$$[\, t_1, \ldots, t_n \,] : [x_1, \ldots, x_k] \to [x_1, \ldots, x_n]$$

   is mapped to the morphism $\left(\mathcal{M}^\sharp t_1, \ldots, \mathcal{M}^\sharp t_n\right)$ where the $\mathcal{M}^\sharp t_i$ are the values of $\mathcal{M}^\sharp$ on the morphisms $[t_i] : [x_1, \ldots, x_k] \to [x_i]$, and

$$\left(\mathcal{M}^\sharp t_1, \ldots, \mathcal{M}^\sharp t_n\right) : M^k \longrightarrow M^n$$

   is the evident n-tuple in the FP-category $\mathcal{C}$.

That $\mathcal{M}^\sharp : \mathsf{Syn}(\mathbb{T}) \to \mathcal{C}$ really is a functor follows from the assumption that the interpretation M is a model, meaning that all the equations of the theory are satisfied by it, so that these specifications are well-defined on equivalence classes. Here we use the *soundness* of equational deduction with respect to models in FP categories.

Note that the functor $\mathcal{M}^\sharp$ is defined in such a way that it obviously preserves finite products, and that there is an isomorphism of models,

$$\mathcal{M}^\sharp(\mathcal{U}) \cong \mathcal{M}.$$

Thus we have shown that the functor "evaluation at $\mathcal{U}$",

$$\mathsf{eval}_{\mathcal{U}} : \mathsf{Hom}_{\mathrm{FP}}(\mathsf{Syn}(\mathbb{T}), \mathcal{C}) \longrightarrow \mathsf{Mod}(\mathbb{T}, \mathcal{C}) \tag{1.9}$$

is essentially surjective on objects, since $\mathsf{eval}_{\mathcal{U}}(\mathcal{M}^\sharp) = \mathcal{M}^\sharp(\mathcal{U}) \cong \mathcal{M}$.

We leave the verification that it is full and faithful as an easy exercise.

Exercise 1.1.19. Verify this. (Hint: A homomorphism is entirely determined by what it does to the underlying object, and a natural transformation is similarly determined by its component at $[x_1]$.)

Finally, naturality in $\mathcal{C}$ means the following. Suppose $\mathcal{M}$ is a model of $\mathbb{T}$ in any FP-category $\mathcal{C}$. Any FP-functor $F : \mathcal{C} \to \mathcal{D}$ to another FP-category $\mathcal{D}$ then takes $\mathcal{M}$ to a model $F(\mathcal{M})$ in $\mathcal{D}$. Just as for the special case of $\mathcal{U}$, the interpretation is given by setting $f^{F(\mathcal{M})} = F(f^{\mathcal{M}})$ for the basic operations f (and composing with the canonical isos coming from preservation of products, $F(M) \times F(M) \cong F(M \times M)$, etc.). Since equations are described by commuting diagrams, F takes a model to a model, and the same is true for homomorphisms. Thus $F : \mathcal{C} \to \mathcal{D}$ induces a functor on $\mathbb{T}$-models,

$$\mathsf{Mod}(\mathbb{T}, F) : \mathsf{Mod}(\mathbb{T}, \mathcal{C}) \longrightarrow \mathsf{Mod}(\mathbb{T}, \mathcal{D}).$$

By naturality of (1.8), we mean that the following square commutes up to natural isomorphism:

$$\begin{tikzcd}
\mathsf{Hom}_{\mathrm{FP}}(\mathsf{Syn}(\mathbb{T}), \mathcal{C}) \arrow[r, "\mathsf{eval}_{\mathcal{U}}"] \arrow[d, "\mathsf{Hom}_{\mathrm{FP}}(\mathsf{Syn}(\mathbb{T}), F)"'] & \mathsf{Mod}(\mathbb{T}, \mathcal{C}) \arrow[d, "\mathsf{Mod}(\mathbb{T}, F)"] \\
\mathsf{Hom}_{\mathrm{FP}}(\mathsf{Syn}(\mathbb{T}), \mathcal{D}) \arrow[r, "\mathsf{eval}_{\mathcal{U}}"'] & \mathsf{Mod}(\mathbb{T}, \mathcal{C})
\end{tikzcd} \tag{1.10}$$

But this is clear, since for any FP-functor $M : \mathsf{Syn}(\mathbb{T}) \to \mathcal{C}$ we have:

$$\begin{aligned}
\mathsf{eval}_{\mathcal{U}} \circ \mathsf{Hom}_{\mathrm{FP}}(\mathsf{Syn}(\mathbb{T}), F)(M) &= (\mathsf{Hom}_{\mathrm{FP}}(\mathsf{Syn}(\mathbb{T}), F)(M))(\mathcal{U}) \\
&= (F \circ M)(\mathcal{U}) \\
&= F(M(\mathcal{U})) \\
&= F(\mathsf{eval}_{\mathcal{U}}(M)) \\
&\cong \mathsf{Mod}(\mathbb{T}, F)(\mathsf{eval}_{\mathcal{U}}(M)) \\
&= \mathsf{Mod}(\mathbb{T}, F) \circ \mathsf{eval}_{\mathcal{U}}(M).
\end{aligned}$$

□

The equivalence of categories

$$\mathsf{Hom}_{\mathrm{FP}}(\mathsf{Syn}(\mathbb{T}), \mathcal{C}) \simeq \mathsf{Mod}(\mathbb{T}, \mathcal{C}) \tag{1.11}$$

actually determines $\mathsf{Syn}(\mathbb{T})$ and the universal model $\mathcal{U}$ uniquely, up to equivalence of categories and isomorphism of models. Indeed, to recover $\mathcal{U}$, just put $\mathsf{Syn}(\mathbb{T})$ for $\mathcal{C}$ and the identity functor $1_{\mathsf{Syn}(\mathbb{T})}$ on the left, to get $\mathcal{U}$ in $\mathsf{Mod}(\mathbb{T}, \mathsf{Syn}(\mathbb{T}))$ on the right! To see that $\mathsf{Syn}(\mathbb{T})$ itself is also determined, observe that (1.11) says that the functor $\mathsf{Mod}(\mathbb{T}, \mathcal{C})$ is *representable*, with representing object $\mathsf{Syn}(\mathbb{T})$, in an appropriate (i.e. bicategorical) sense. As usual, this fact can also be formulated in elementary terms as a universal mapping property of $\mathsf{Syn}(\mathbb{T})$, as follows:

Definition 1.1.20. The *classifying category* of an algebraic theory $\mathbb{T}$ is an FP-category $\mathcal{C}_{\mathbb{T}}$ with a distinguished model $\mathcal{U}$, called the *universal model*, such that:

(i) for any model $\mathcal{M}$ in any FP-category $\mathcal{C}$, there is an FP-functor

$$\mathcal{M}^{\sharp} : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}$$

and an isomorphism of models $\mathcal{M} \cong \mathcal{M}^{\sharp}(\mathcal{U})$.

(ii) for any FP-functors $F, G : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}$ and model homomorphism $h : F(\mathcal{U}) \to G(\mathcal{U})$, there is a unique natural transformation $\vartheta : F \to G$ with

$$\vartheta_{\mathcal{U}} = h.$$

Observe that (i) says that the evaluation functor (1.8) is essentially surjective, and (ii) that it is full and faithful. The category $\mathcal{C}_{\mathbb{T}}$ is then determined, up to equivalence, by this universal mapping property. Specifically, if $(\mathcal{C}, \mathcal{U})$ and $(\mathcal{D}, \mathcal{V})$ are both classifying categories for the same theory, then there are classifying functors,

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\mathcal{V}^{\sharp}"] & \mathcal{D} \arrow[l, bend left, "\mathcal{U}^{\sharp}"] \end{tikzcd}$$

the composites of which are necessarily isomorphic to the respective identity functors, since e.g. $\mathcal{U}^{\sharp}(\mathcal{V}^{\sharp}(\mathcal{U})) \cong \mathcal{U}^{\sharp}(\mathcal{V}) \cong \mathcal{U}$.

We have now shown not only that every algebraic theory has a classifying category $\mathcal{C}_{\mathbb{T}}$, but also that the syntactic category $\mathsf{Syn}(\mathbb{T})$ is such a classifying category, and that it is essentially determined by that property. We record this as the following.

Theorem 1.1.21. *Every algebraic theory $\mathbb{T}$ has a classifying category $\mathcal{C}_{\mathbb{T}}$, which can be constructed as the syntactic category $\mathsf{Syn}(\mathbb{T})$ of $\mathbb{T}$, in the sense of Definition 1.1.15.*

Example 1.1.22. Let us see explicitly what the foregoing definitions give us in the case of the theory of groups $\mathbb{T}_{\mathsf{Group}}$. Let us write $\mathbb{G} = \mathcal{C}_{\mathbb{T}_{\mathsf{Group}}}$ for the classifying category, which has contexts $[x_1, \ldots, x_n]$ as objects, and terms built from variables and the group operations (modulo renaming of variables and provability from the group laws) as arrows. A finite product preserving functor $G : \mathbb{G} \to \mathsf{Set}$ is determined uniquely, up to natural isomorphism, by its action on the context $[x_1]$ and the terms representing the basic operations. If we set

$$\begin{aligned} |\mathcal{G}| &:= G[x_1]\ , & u^{\mathcal{G}} &:= G(\cdot \mid e)\ , \\ i^{\mathcal{G}} &:= G(x_1 \mid {x_1}^{-1})\ , & m_G &= G(x_1, x_2 \mid x_1 \cdot x_2)\ , \end{aligned}$$

then $\mathcal{G} = (|\mathcal{G}|, u^{\mathcal{G}}, i^{\mathcal{G}}, m^{\mathcal{G}})$ is just a group, with unit $u^{\mathcal{G}}$, inverse $i^{\mathcal{G}}$, and multiplication $m^{\mathcal{G}}$. That $\mathcal{G}$ satisfies the axioms for groups follows from the functoriality of G and preservation of finite products, which implies preservation of the corresponding commutative diagrams. Conversely, any group $\mathcal{G} = (G, u, i, m)$ determines a finite product preserving functor $\mathcal{G}^\sharp : \mathbb{G} \to \mathsf{Set}$, by setting $\mathcal{G}^\sharp[x_1] = G$, etc. Thus $\mathsf{Mod}(\mathbb{G}, \mathsf{Set})$ will indeed be equivalent to Group once we show that both categories have the same notion of morphisms. This is shown just as in the general case above.

Example 1.1.23. Recall from 1.1.12 that a group G in the functor category $\mathsf{Set}^{\mathbb{C}}$ is essentially the same thing as a functor $G : \mathbb{C} \to \mathsf{Group}$. From the point of view of algebras as functors, this amounts to the observation that product-preserving functors $\mathbb{G} \to \mathsf{Hom}(\mathbb{C}, \mathsf{Set})$ correspond (by exponential transposition) to functors $\mathbb{C} \to \mathsf{Hom}_{\mathsf{FP}}(\mathbb{G}, \mathsf{Set})$, where the latter Hom-set consists just of product-preserving functors. The correspondence extends to natural transformations, giving the previously observed (Example 1.1.12) equivalance of categories,

$$\mathsf{Group}(\mathsf{Set}^{\mathbb{C}}) \simeq \mathsf{Group}(\mathsf{Set})^{\mathbb{C}} = \mathsf{Group}^{\mathbb{C}}.$$

1.1.4 Soundness and completeness

Consider an algebraic theory $\mathbb{T}$ and an equation $s = t$ between terms of the theory. If the equation can be proved from the axioms of the theory, $\mathbb{T} \vdash s = t$, then every model $\mathcal{M}$ of the theory in any FP-category satisfies the equation, $\mathcal{M} \models s = t$. This is called the *soundness* of the equational calculus with respect to categorical models, and it can be shown by a straightforward induction on the equational proof that establishes $\mathbb{T} \vdash s = t$. The converse statement reads:

$$\mathcal{M} \models s = t, \text{ for all } \mathcal{M} \quad \Rightarrow \quad \mathbb{T} \vdash s = t\ .$$

This is called *completeness*, and (together with soundness) it says that the equational calculus suffices for proving all (and only) the equations that hold generally in the semantics. For functorial semantics, this condition holds in an especially strong way: by Proposition 1.1.17, we already know that the syntactic model $\mathcal{U}$ in $\mathsf{Syn}(\mathbb{T})$ is logically generic, in the sense that satisfaction by $\mathcal{U}$ is equivalent to provability in $\mathbb{T}$,

$$\mathcal{U} \models s = t \quad \iff \quad \mathbb{T} \vdash s = t.$$

But since $\mathsf{Syn}(\mathbb{T})$ is a classifying category for $\mathbb{T}$ and $\mathcal{U}$ is universal in the sense of Definition 1.1.20 it follows that we also have completeness:

Theorem 1.1.24 (Soundness and completeness of equational logic)**.** *For any terms s, t we have $\mathbb{T} \vdash s = t$ if and only if every model $\mathcal{M}$ in every FP-category $\mathcal{C}$ satisfies $s = t$.*

Proof. We have a classifying category $\mathcal{C}_{\mathbb{T}} \simeq \mathsf{Syn}(\mathbb{T})$ with universal model $\mathcal{U}$. If $\mathbb{T} \vdash s = t$, then by Proposition 1.1.17 we have $\mathcal{U} \models s = t$, meaning that $s^{\mathcal{U}} = t^{\mathcal{U}}$. But then for any model $\mathcal{M}$ in an FP-category $\mathcal{C}$, we obtain $\mathcal{M} \models s = t$ by applying the classifying functor $\mathcal{M}^\sharp : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}$, which preserves the interpretations of s and t,

$$\mathcal{M}^\sharp(s^{\mathcal{U}}) = s^{\mathcal{M}^\sharp(\mathcal{U})} = s^{\mathcal{M}}$$

and so from $s^{\mathcal{U}} = t^{\mathcal{U}}$ we get $s^{\mathcal{M}} = t^{\mathcal{M}}$.

Conversely, if $\mathcal{M} \models s = t$ for every model $\mathcal{M}$, then in particular $\mathcal{U} \models s = t$, and so $\mathbb{T} \vdash s = t$, since $\mathcal{U}$ is generic. □

Classically, it is seldom the case that there exists a generic model; instead, one usually considers completeness with respect to a class of special models, say, those in Set. Completeness with respect to a restricted class of models is of course a stronger statement than completeness with respect to all models in all categories; indeed, one need only test an equation in the restricted class to know that it can be proved, and therefore holds in all models. Toward the classical result, we can first consider completeness with respect to just "variable models" in Set, i.e. in arbitrary functor categories $\mathsf{Set}^{\mathbb{C}}$. That result follows immediately from the next lemma.

Lemma 1.1.25. *Let $\mathbb{T}$ be an algebraic theory. The Yoneda embedding*

$$\mathsf{y} : \mathcal{C}_{\mathbb{T}} \to \widehat{\mathcal{C}_{\mathbb{T}}} = \mathsf{Set}^{\mathcal{C}_{\mathbb{T}}^{\mathsf{op}}}$$

is a generic model for $\mathbb{T}$.

Proof. The Yoneda embedding $\mathsf{y} : \mathcal{C}_{\mathbb{T}} \to \widehat{\mathcal{C}_{\mathbb{T}}}$ preserves all limits, and in particular finite products, hence it determines a model $\mathcal{Y} = \mathsf{y}(\mathcal{U})$ in the category of presheaves $\widehat{\mathcal{C}_{\mathbb{T}}}$. Like all models, $\mathcal{Y}$ satisfies all the equations that hold in $\mathcal{U}$, simply because y is an FP functor. But because y is also faithful, any equation that holds in $\mathcal{Y}$ must already hold in $\mathcal{U}$, and is therefore provable, since $\mathcal{U}$ is generic. □

Example 1.1.26. We consider group theory one more time. We again write simply $\mathbb{G}$ for the syntactic (classifying) category of the theory $\mathbb{T}_{\mathsf{Group}}$ of groups. As a presheaf on $\mathbb{G}$, the generic group $\mathcal{Y} \in \widehat{\mathbb{G}}$ satisfies every equation that is satisfied by all groups, and no others. Let us describe its underlying object $Y = |\mathcal{Y}|$ explicitly as a "variable set". By definition, the presheaf Y is represented by the underlying object $U = |\mathcal{U}|$ of the universal group in $\mathbb{G}$, which in syntactic terms is the context with one variable,

$$Y = \mathsf{y}[x_1] = \mathbb{G}(-, [x_1]) .$$

The values of this functor thus comprise a family of sets parametrized by the objects $[x_1, \dots, x_n]$ of $\mathbb{G}$; namely, for every $n \in \mathbb{N}$, we have the set

$$Y_n = \mathbb{G}([x_1, \dots, x_n], [x_1])$$

consisting of all (equivalence classes of) terms $[\, x_1, \dots, x_n \mid t\,]$ in n variables (modulo the equations of group theory); but this is just the set of elements of the *free group* $F(n)$ on n generators! Thus we have

$$Y_n = \mathbb{G}([x_1, \dots, x_n], [x_1]) \cong |F(n)| \cong \mathsf{Set}(1, |F(n)|) \cong \mathsf{Group}(F(1), F(n)).$$

Moreover, the action of the functor Y on a map

$$s : [x_1, \dots, x_m] \longrightarrow [x_1, \dots, x_n] \qquad \text{in } \mathbb{G}$$

can be described by substitution of the terms $s = (s_1, \dots, s_n)$ into the elements $t \in Y_n$,

$$Y(s)(t) = \mathbb{G}(s, [x_1])(t) = t[\, s_1/x_1, \dots, s_n/x_n\,]\,.$$

In terms of the free groups $F(n)$, the terms $s_1, \dots, s_n$ in the context $x_1, \dots, x_m$ are elements of the free group $F(m)$, and so they determine a unique homomorphism

$$\overline{s} : F(n) \cong F(1) + ... + F(1) \longrightarrow F(m)$$

such that $\overline{s}(x_i) = s_i$ for $i = 1, ..., n$. Composition with $\overline{s} : F(n) \to F(m)$ then encodes the corresponding substitution, in the sense that the following diagram commutes (as the reader should verify!).

$$\begin{tikzcd}
{[\, x_1, \dots, x_n\,]} & \mathbb{G}([\, x_1, \dots, x_n\,], [x_1]) \arrow[r, "\cong"] \arrow[d, "\mathbb{G}(s, [x_1])"'] & \mathsf{Group}(F(1), F(n)) \arrow[d, "\mathsf{Group}(F(1), \overline{s})"] & F(n) \arrow[d, "\overline{s}"] \\
{[\, x_1, \dots, x_m\,]} \arrow[u, "s"] & \mathbb{G}([\, x_1, \dots, x_m\,], [x_1]) \arrow[r, "\cong"'] & \mathsf{Group}(F(1), F(m)) & F(m)
\end{tikzcd} \tag{1.12}$$

Finally, the unit, inverse, and multiplication operations of the internal group $\mathcal{Y}$ are determined at each stage Y_n by the corresponding operations on the free group $F(n)$ (as the reader should verify!). We will discover a deeper reason for this in Section 1.2.1.

Finally, we can consider the completeness of equational logic with respect to all Set-valued models $\mathcal{M} : \mathcal{C}_{\mathbb{T}} \to \mathsf{Set}$, which of course correspond to classical $\mathbb{T}$-algebras. We need the following:

Lemma 1.1.27. *For any small category $\mathbb{C}$, there is a jointly faithful family $(E_i)_{i \in I}$ of FP-functors $E_i : \mathsf{Set}^{\mathbb{C}} \to \mathsf{Set}$, with I a set. That is, for any maps $f, g : A \to B$ in $\mathsf{Set}^{\mathbb{C}}$, if $E_i(f) = E_i(g)$ for all $i \in I$, then $f = g$.*

Proof. We can take $I = \mathbb{C}_0$, the set of objects of $\mathbb{C}$, and the evaluation functors

$$E_c = \mathsf{eval}_c : \mathsf{Set}^{\mathbb{C}} \to \mathsf{Set}\,,$$

for all $c \in \mathbb{C}$. These are clearly jointly faithful. Note that they also preserve all limits and colimits, since these are constructed pointwise in functor categories. □

Proposition 1.1.28. *Suppose $\mathbb{T}$ is an algebraic theory. For any terms s, t,*

$$\mathcal{M} \models s = t \quad \textit{for all models } \mathcal{M} \textit{ in } \mathsf{Set} \qquad \Longleftrightarrow \qquad \mathbb{T} \vdash s = t.$$

Thus the equational logic of algebraic theories is sound and complete with respect to Set*-valued semantics.*

Proof. Combine the foregoing lemma with the fact, from Lemma 1.1.25, that the Yondea embedding is a generic model. □

The completeness of equational reasoning was originally proved by Birkhoff [Bir35]. The proof is not particularly difficult; we have chosen to redo it in this way because the method will generalize to other systems of logic in later chapters.

Exercise 1.1.29. We described the functor $Y = \mathsf{y}U : \mathbb{G}^{\mathsf{op}} \to \mathsf{Set}$ represented by the underlying object $U = [x_1]$ of the universal group $\mathcal{U}$ in terms of the free groups $F(n)$. Verify that the action of Y on the arrows of $\mathbb{G}$ is indeed given by substitution of terms by checking that diagram (1.12) commutes. Also describe the group structure on Y in $\widehat{\mathbb{G}}$ explicitly in terms of that on the free groups.

Exercise 1.1.30. Let $t = t(x_1, \dots, x_n)$ be a term of group theory in the variables $x_1, \dots, x_n$. On the one hand we can think of t as an element of the free group $F(n)$, and on the other we can consider the interpretation of t with respect to the representable group $\mathcal{Y}$ in $\widehat{\mathbb{G}}$, namely as a natural transformation $t^{\mathcal{Y}} : Y^n \to Y$. Suppose $s = s(x_1, \dots, x_n)$ is another such term in the same variables $x_1, \dots, x_n$. Show that $s^{\mathcal{Y}} = t^{\mathcal{Y}}$ if, and only if, $s = t$ in the free group $F(n)$.

1.1.5 Functorial semantics

Let us summarize our treatment of algebraic theories so far. We have reformulated certain traditional *logical* notions in terms of *categorical* ones. The traditional approach may be described as involving the following four different parts:

Terms

There is an underlying *type theory* consisting of types and terms. For algebraic theories there is only one type, which is not even explicitly mentioned. The terms are built from variables and a *signature* consisting of some basic operation symbols.

Equations

Algebraic theories have a particularly simple *logic* that involves only equations between terms and equational reasoning, which is basically *substitution* of equals for equals and the laws of an *equivalence relation*.

Theories

An *algebraic theory* then consists of a signature and a set of *axioms*, which are just equations between terms. Such theories are regarded as *logical syntax*: sets of uninterpreted, formal expressions, generated inductively by rules of inference.

Models

An algebraic theory can be modeled by a *set* equipped with some *operations* interpreting the signature. Such an interpretation is a *model* if it satisfies the axioms of the theory, meaning that the functions interpreting the terms that occur in the equational axioms are actually equal.

The alternative approach of *functorial semantics* may be summarized as follows:

Theories are categories

From a given theory we construct a structured category that captures the same information in a way that is independent of a particular presentation by basic operations and axioms.

Models are functors

A model is a structure-preserving functor from the theory to a category with the same structure. For algebraic theories, a model is a functor that preserves finite products, which ensures that all valid equations of the theory are preserved, and the axioms are therefore satisfied.

Homomorphisms are natural transformations

We obtain the notion of a homomorphism of models for free: since models are functors, the homomorphisms between them are just the natural transformations. Such homomorphisms agree with the usual notion, consisting of a function on the underlying sets that "respects" the algebraic structure.

Universal models

By allowing for models in categories other than Set, functorial semantics admits *universal models*: a model $\mathcal{U}$ in the classifying category $\mathcal{C}_{\mathbb{T}}$, such that any model anywhere is a functorial image of $\mathcal{U}$ by an essentially unique, structure-preserving functor. Thus $\mathcal{U}$ has all and only those logical properties that are had by all models, since such properties are preserved by the functors in question.

Logical completeness

The construction of the classifying category $\mathcal{C}_{\mathbb{T}}$ from the syntax of the theory $\mathbb{T}$ shows that the universal model is also *generic*: it has exactly those properties that are provable in the theory $\mathbb{T}$. This implies the *soundness and completeness* of the logic with respect to general categorical semantics. Completeness with respect to a restricted class of models, such as those in Set, then results from an embedding theorem for the classifying category.

1.2 Lawvere duality

The scheme of functorial semantics outlined in the previous section applies to many other systems of logic than algebraic theories, some of which will be considered in later chapters. A further aspect of this approach is especially transparent in the case of algebraic theories; namely, a deep and fascinating *duality* relating syntax and semantics. We devote the rest of this chapter to its investigation.

1.2.1 Logical duality

There is a remarkable and far-reaching duality in logic of the form

$$\text{Syntax} \;\simeq\; \text{Semantics}^{\mathsf{op}}.$$

It was discovered by F.W. Lawvere in the 1960s and presented in some early papers, [Law63a, Law63b, Law65, Law69], but it has still hardly been noticed in conventional logic—perhaps because its recognition requires the tools of category theory.

We can see this duality quite clearly in the case of algebraic theories. Let $\mathcal{C}_{\mathbb{T}}$ be the classifying category for an equational theory $\mathbb{T}$, like the theory of groups, constructed syntactically as in section 1.1.2 above. So the objects of $\mathcal{C}_{\mathbb{T}}$ are contexts of variables $[x_1 \dots, x_n]$, up to renaming, and the arrows $(t_1, ..., t_n) : [x_1 \dots, x_m] \to [x_1 \dots, x_n]$ are n-tuples of terms in context $[x_1 \dots, x_m \mid t_i]$, up to $\mathbb{T}$-provable equality. We will see that this syntactic category $\mathcal{C}_{\mathbb{T}}$ is in fact dual to a certain subcategory of *models* of $\mathbb{T}$ (in Set). Specifically, there is a small, full subcategory $\mathsf{mod}(\mathbb{T}) \hookrightarrow \mathsf{Mod}(\mathbb{T})$ and an equivalence of categories,

$$\mathcal{C}_{\mathbb{T}} \simeq \mathsf{mod}(\mathbb{T})^{\mathsf{op}},$$

making the *syntactic* category $\mathcal{C}_{\mathbb{T}}$ dual to a subcategory of the *semantic* category $\mathsf{Mod}(\mathbb{T})$. Thus, in particular, there is an invariant representation of the syntax of the theory $\mathbb{T}$ "hidden" inside the category of models of $\mathbb{T}$. Indeed, it is quite easy to specify $\mathsf{mod}(\mathbb{T})$ explicitly: it is the full subcategory on the *finitely generated free models* $F(n)$ of $\mathbb{T}$,

$$\mathsf{mod}(\mathbb{T})_0 = \left\{F(n) \;\middle|\; n \in \mathbb{N}\right\}.$$

We will have an even more intrinsic characterization below.

Theorem 1.2.1. *Let* $\mathbb{T}$ *be an algebraic theory, and let*

$$\mathsf{mod}(\mathbb{T}) \hookrightarrow \mathsf{Mod}(\mathbb{T})$$

be the full subcategory of finitely generated free models of $\mathbb{T}$*. Then* $\mathsf{mod}(\mathbb{T})^{\mathsf{op}}$ *classifies* $\mathbb{T}$ *models. That is to say, for any FP-category* $\mathcal{C}$*, there is an equivalence of categories,*

$$\mathsf{Hom}_{\mathrm{FP}}(\mathsf{mod}(\mathbb{T})^{\mathsf{op}}, \mathcal{C}) \simeq \mathsf{Mod}(\mathbb{T}, \mathcal{C}), \tag{1.13}$$

which is natural in $\mathcal{C}$.

Before giving the (somewhat lengthy, but straightforward) proof of the theorem, let us observe that the claimed syntax-semantics duality follows immediately. Indeed, given (1.13), there is then an equivalence,

$$\mathcal{C}_{\mathbb{T}} \simeq \mathsf{mod}(\mathbb{T})^{\mathsf{op}} \tag{1.14}$$

between the syntactic category $\mathcal{C}_{\mathbb{T}}$ and the opposite of the category $\mathsf{mod}(\mathbb{T})$ of finitely generated free models, because by Proposition 1.1.18, both categories $\mathcal{C}_{\mathbb{T}}$ and $\mathsf{mod}(\mathbb{T})^{\mathsf{op}}$ represent the same "semantics" functor $\mathsf{Mod}(\mathbb{T}, \mathcal{C})$.

Proof of Theorem 1.2.1. First, observe that $\mathsf{mod}(\mathbb{T})^{\mathsf{op}}$ has all finite products, since $\mathsf{mod}(\mathbb{T})$ has all finite coproducts. Indeed, we have

$$\begin{aligned} F(n) + F(m) &\cong F(n+m), \\ 0 &\cong F(0), \end{aligned}$$

for the finitely generated free algebras $F(n)$, since the left adjoint F preserves all colimits.

To determine the universal $\mathbb{T}$-algebra $\mathcal{U}$ in $\mathsf{mod}(\mathbb{T})^{\mathsf{op}}$ let

$$U = F(1),$$

so that every object in $\mathsf{mod}(\mathbb{T})^{\mathsf{op}}$ is a power of U,

$$F(n) \cong U^n.$$

We next interpret the signature $\Sigma_{\mathbb{T}}$. For each basic operation symbol $f \in \Sigma_{\mathbb{T}}$, with arity k, there is an element of the free algebra $F(k)$ built from the operation $f^{F(k)} : F(k)^k \to F(k)$ and the k generators $x_1, \dots, x_k \in F(k)$, namely

$$f^{F(k)}(x_1, \dots, x_k).$$

E.g. in the theory of groups, there is the element $x \cdot y$ in the free group on the two generators x, y. By freeness of $F(1)$, each element $t \in F(k)$ determines a unique homomorphism $\overline{t} : F(1) \to F(k)$ in $\mathsf{mod}(\mathbb{T})$ taking the generator $x \in F(1)$ to $t = \overline{t}(x)$. Thus there is a homomorphism

$$\overline{f^{F(k)}(x_1, \dots, x_k)} : F(1) \to F(k) \qquad \text{in } \mathsf{mod}(\mathbb{T})$$

associated to the element $f^{F(k)}(x_1, \dots, x_k) \in F(k)$. We take this map as the $\mathcal{U}$-interpretation of the basic operation symbol f,

$$f^{\mathcal{U}} = \overline{f^{F(k)}(x_1, \dots, x_k)} : U^k \to U \qquad \text{in } \mathsf{mod}(\mathbb{T})^{\mathsf{op}}.$$

It follows that for any term in context $x_1 \dots, x_k \mid t$, the interpretation

$$[x_1 \dots, x_k \mid t]^{\mathcal{U}} : U^k \to U$$

will be the unique homomorphism $\overline{t^{F(k)}} : F(1) \to F(k)$ corresponding to the element $t^{F(k)} \in F(k)$ (proof by induction!).

Moreover, it then follows that for every axiom $(s = t)$ of $\mathbb{T}$, we have $\mathcal{U} \models s = t$. Indeed,

$$[x_1 \ldots, x_k \mid s]^{\mathcal{U}} = [x_1 \ldots, x_k \mid t]^{\mathcal{U}} : U^k \to U$$

if, and only if, the corresponding homomorphisms $\bar{s}, \bar{t} : F(1) \to F(k)$ agree, which they do just if the associated elements of the free algebra $F(k)$ agree, by the freeness of $F(1)$. And the latter holds, in turn, simply because $F(k)$ is a $\mathbb{T}$-algebra. Indeed, consider the example of the two generators x, y of the free *abelian* group $F(2)$, for which we have $x \cdot y = y \cdot x$ simply because $F(2)$ is abelian. Thus we indeed have a $\mathbb{T}$-model $\mathcal{U}$ in $\mathsf{mod}(\mathbb{T})^{\mathsf{op}}$, consisting of the free algebras.

We next show that $\mathcal{U}$ has the required universal property, in three steps:

Step 1. Let $\mathcal{A}$ be any $\mathbb{T}$-algebra in Set. Then there is a product-preserving functor,

$$\mathcal{A}^\sharp : \mathsf{mod}(\mathbb{T})^{\mathsf{op}} \to \mathsf{Set}$$

with $\mathcal{A}^\sharp(\mathcal{U}) \cong \mathcal{A}$ (as $\mathbb{T}$-models), namely:

$$\mathcal{A}^\sharp(-) = \mathsf{Hom}_{\mathsf{Mod}(\mathbb{T})}(-, \mathcal{A}),$$

where we of course restrict the representable functor $\mathsf{Hom}_{\mathsf{Mod}(\mathbb{T})}(-, \mathcal{A}) : \mathsf{Mod}(\mathbb{T})^{\mathsf{op}} \to \mathsf{Set}$ along the (full) inclusion

$$\mathsf{mod}(\mathbb{T}) \hookrightarrow \mathsf{Mod}(\mathbb{T})$$

of the finitely generated, free algebras. The functor

$$\mathcal{A}^\sharp : \mathsf{mod}(\mathbb{T})^{\mathsf{op}} \to \mathsf{Set}$$

clearly preserves products: for each object $U^n \in \mathsf{mod}(\mathbb{T})^{\mathsf{op}}$, we have

$$\mathcal{A}^\sharp(U^n) = \mathsf{Hom}_{\mathsf{Mod}(\mathbb{T})}(F(n), \mathcal{A}) \cong \mathsf{Hom}_{\mathsf{Set}}(n, V(\mathcal{A})) \cong A^n.$$

And in particular $\mathcal{A}^\sharp(U) \cong A$.

Finally, let us show that for any basic operation f, we have $\mathcal{A}^\sharp(f^{\mathcal{U}}) = f^{\mathcal{A}}$, up to isomorphism. Indeed, given any algebra $\mathcal{A}$ and operation $f^{\mathcal{A}} : A^n \to A$, we have a commutative diagram,

$$\begin{tikzcd}
A^n \arrow[r, "f^{\mathcal{A}}"] \arrow[d, "\cong"'] & A \arrow[d, "\cong"] \\
\mathsf{Hom}(F(n), \mathcal{A}) \arrow[r, "f^*"'] & \mathsf{Hom}(F(1), \mathcal{A})
\end{tikzcd} \tag{1.15}$$

where f^* is precomposition with the homomorphism

$$F(n) \xleftarrow{\overline{f^{F(n)}(x_1,\ldots,x_n)}} F(1)$$

To see that (1.15) commutes, take any $(a_1,\ldots,a_n) \in A^n$ with associated homomorphism $\overline{(a_1,\ldots,a_n)} : F(n) \to \mathcal{A}$ and precompose with $\overline{f^{F(n)}(x_1,\ldots,x_n)}$ to get a map $F(1) \to \mathcal{A}$, picking out the element

$$\begin{aligned}
\overline{(a_1,\ldots,a_n)} \circ \overline{f^{F(n)}(x_1,\ldots,x_n)}(x) &= \overline{(a_1,\ldots,a_n)}(f^{F(n)}(x_1,\ldots,x_n)) \\
&= \overline{(a_1,\ldots,a_n)} \circ f^{F(n)}(x_1,\ldots,x_n) \\
&= f^{\mathcal{A}} \circ \overline{(a_1,\ldots,a_n)}(x_1,\ldots,x_n) \\
&= f^{\mathcal{A}}(a_1,\ldots,a_n)
\end{aligned}$$

where x is the generator of $F(1)$, and using the fact that $\overline{(a_1,\ldots,a_n)}$ is a homomorphism and therefore commutes with the respective interpretations of f.

But now note that

$$F(n) \xleftarrow{\overline{f^{F(n)}(x_1,\ldots,x_n)}} F(1)$$

in $\mathsf{mod}(\mathbb{T})$ is

$$U^n \xrightarrow[f^U]{} U$$

in $\mathsf{mod}(\mathbb{T})^{\mathsf{op}}$, and that $\mathsf{Hom}(F(n), \mathcal{A}) = \mathcal{A}^\sharp(U^n)$ and $f^* = \mathcal{A}^\sharp(f^U)$. Thus (1.15) shows that indeed $\mathcal{A}^\sharp(f^{\mathcal{U}}) = f^{\mathcal{A}}$, up to isomorphism. Thus we indeed have $\mathcal{A}^\sharp(\mathcal{U}) \cong \mathcal{A}$ as algebras, as required.

We leave it to the reader to verify that any homomorphism $h : F(\mathcal{U}) \to G(\mathcal{U})$ of $\mathbb{T}$-algebras $F(\mathcal{U}), G(\mathcal{U})$ arising from FP-functors $F, G : \mathsf{mod}(\mathbb{T})^{\mathsf{op}} \to \mathsf{Set}$ is of the form $h = \vartheta_{\mathcal{U}}$ for a unique natural transformation $\vartheta : F \to G$.

Exercise 1.2.2. Show this.

Step 2. Let $\mathbb{C}$ be any (locally small) category, and $\mathcal{A}$ a $\mathbb{T}$-algebra in $\mathsf{Set}^{\mathbb{C}}$. Using the isomorphism

$$\mathsf{Mod}(\mathbb{T}, \mathsf{Set}^{\mathbb{C}}) \cong \mathsf{Mod}(\mathbb{T})^{\mathbb{C}},$$

each $\mathcal{A}(C)$ is a $\mathbb{T}$-algebra (in Set), which by Step 1 has a classifying functor,

$$\mathcal{A}(C)^\sharp : \mathsf{mod}(\mathbb{T})^{\mathsf{op}} \to \mathsf{Set}.$$

Together, these determine a single functor $\mathcal{A}^\sharp : \mathsf{mod}(\mathbb{T})^{\mathsf{op}} \to \mathsf{Set}^{\mathbb{C}}$, defined on any $U^n \in \mathsf{mod}(\mathbb{T})^{\mathsf{op}}$ by

$$(\mathcal{A}^\sharp(U^n))(C) \cong \mathcal{A}(C)^\sharp(U^n) \cong (AC)^n.$$

The action on arrows $U^n \to U^m$ in $\mathsf{mod}(\mathbb{T})^{\mathsf{op}}$ is similarly determined pointwise by the components

$$(\mathcal{A}^\sharp(U^n))(C) \cong \mathcal{A}(C)^\sharp(U^n) \to \mathcal{A}(C)^\sharp(U^m) = (\mathcal{A}^\sharp(U^m))(C),$$

for all $C \in \mathbb{C}$.

In this way, we have an FP-functor $\mathcal{A}^\sharp : \mathsf{mod}(\mathbb{T})^{\mathsf{op}} \to \mathsf{Set}^{\mathbb{C}}$, and an isomorphism of models $\mathcal{A}^\sharp(\mathcal{U}) \cong \mathcal{A}$ in $\mathsf{Set}^{\mathbb{C}}$. It is then clear that any natural transformation $\mathcal{A}^\sharp \to \mathcal{B}^\sharp$ gives rise to a homomorphism $\mathcal{A}^\sharp(\mathcal{U}) \to \mathcal{B}^\sharp(\mathcal{U})$, and that the resulting functor

$$\mathsf{Hom}_{\mathrm{FP}}(\mathsf{mod}(\mathbb{T})^{\mathsf{op}}, \mathsf{Set}^{\mathbb{C}}) \to \mathsf{Mod}(\mathbb{T}, \mathsf{Set}^{\mathbb{C}})$$

is an equivalence.

Step 3. For the general case, let $\mathcal{C}$ be any (locally small) FP-category, and A a $\mathbb{T}$-algebra in $\mathcal{C}$. Use the Yoneda embedding

$$\mathsf{y} : \mathcal{C} \hookrightarrow \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$$

to send A to an algebra $\mathcal{A} = \mathsf{y}(\mathsf{A})$ in $\mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$ (since y preserves finite products). Now apply Step 2 to get a classifying functor,

$$\mathcal{A}^\sharp : \mathsf{mod}(\mathbb{T})^{\mathsf{op}} \to \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}.$$

We claim that $\mathcal{A}^\sharp$ factors through the Yoneda embedding by an FP-functor $\mathsf{A}^\sharp$,

$$\begin{tikzcd}
& \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}} \\
\mathsf{mod}(\mathbb{T})^{\mathsf{op}} \arrow[ur, "\mathcal{A}^\sharp"] \arrow[r, dotted, "\mathsf{A}^\sharp"'] & \mathcal{C}. \arrow[u, hook, "\mathsf{y}"']
\end{tikzcd}$$

Indeed, we know that the objects of $\mathsf{mod}(\mathbb{T})^{\mathsf{op}}$ all have the form U^n, and for their images we have

$$\mathcal{A}^\sharp(U^n) \cong \mathcal{A}^\sharp(U)^n \cong \mathsf{y}(\mathsf{A})^n \cong \mathsf{y}(\mathsf{A}^n)\,.$$

Thus the images of the objects of $\mathsf{mod}(\mathbb{T})^{\mathsf{op}}$ are all representable. Since y is full and faithful, the claim is established, and the resulting functor $\mathsf{A}^\sharp : \mathsf{mod}(\mathbb{T})^{\mathsf{op}} \to \mathcal{C}$ preserves finite products because $\mathcal{A}^\sharp$ does so, and y creates them. Clearly,

$$\mathsf{A}^\sharp(\mathcal{U}) \cong \mathsf{A},$$

since y reflects isos.

Naturality of the equivalence

$$\mathsf{Hom}_{\mathrm{FP}}(\mathsf{mod}(\mathbb{T})^{\mathsf{op}}, \mathcal{C}) \simeq \mathsf{Mod}(\mathbb{T}, \mathcal{C}),$$

in $\mathcal{C}$ is essentially automatic, using the fact that it is induced by evaluating an FP functor $F : \mathsf{mod}(\mathbb{T})^{\mathsf{op}} \to \mathcal{C}$ at the universal model $\mathcal{U}$ in $\mathsf{mod}(\mathbb{T})^{\mathsf{op}}$. $\square$

As already mentioned, since the classifying category is uniquely determined, up to equivalence, by its universal property, combining the foregoing theorem with the syntactic construction of $\mathcal{C}_{\mathbb{T}}$ given in theorem 1.1.18 yields the following:

Corollary 1.2.3 (Logical duality for algebraic theories)**.** *For any algebraic theory $\mathbb{T}$, there is an equivalence of categories,*

$$\mathcal{C}_{\mathbb{T}} \simeq \mathsf{mod}(\mathbb{T})^{\mathsf{op}} \tag{1.16}$$

between the syntactic category $\mathcal{C}_{\mathbb{T}}$ and the opposite of the category $\mathsf{mod}(\mathbb{T})$ *of finitely generated, free models.*

Thus the construction of the classifying category $\mathcal{C}_{\mathbb{T}}$ from the syntax of $\mathbb{T}$, on the one hand, and its semantic construction as $\mathsf{mod}(\mathbb{T})$, taken together, imply that there is an invariant representation of the *syntax* of $\mathbb{T}$ hidden, as it were, in the opposite of the *semantics*, namely the category $\mathsf{Mod}(\mathbb{T})$ of all $\mathbb{T}$-models. The reader may wish to reflect on the importance of (i) considering the *category* of all models, rather than the mere collection of them, and (ii) generalizing from set-theoretic to categorical models, in arriving at the fundamental logical duality expressed by (1.16).

In section 1.2.5 below, we shall consider how to actually *recover* this syntactic category $\mathcal{C}_{\mathbb{T}}$ from the semantic category $\mathsf{Mod}(\mathbb{T})$ by identifying the subcategory $\mathsf{mod}(\mathbb{T})$ intrinsically—indeed, it will be seen to consist of certain continuous functors $\mathsf{Mod}(\mathbb{T}) \to \mathsf{Set}$. Before doing so, however, let us examine the fundamental equivalence (1.16) explicitly in a few special cases.

Example 1.2.4. Consider the trivial theory $\mathbb{T}_0$ of Example 1.1.4, with no basic operations or equations. A model of $\mathbb{T}_0$ in Set is just a set X, equipped with no operations, and satisfying no further conditions (and similarly in any other FP category). All $\mathbb{T}_0$-algebras are free, and the finitely generated ones are just the finite sets, thus

$$\mathsf{mod}(\mathbb{T}_0) = \mathsf{Set}_{\mathrm{fin}}$$

is the category of finite sets (to be specific, let us take a skeleton, with one n-element set $[n]$ for each $n \in \mathbb{N}$). Theorem 1.2.1 tells us that, for any FP category $\mathcal{C}$, there is an equivalence

$$\mathsf{Hom}_{\mathrm{FP}}(\mathsf{Set}_{\mathrm{fin}}^{\mathsf{op}}, \mathcal{C}) \simeq \mathsf{Mod}(\mathbb{T}_0, \mathcal{C}) \simeq \mathcal{C}.$$

This simply says that $\mathsf{Set}_{\mathrm{fin}}^{\mathsf{op}}$ is the free FP category on one object. Equivalently, $\mathsf{Set}_{\mathrm{fin}}$ is the free finite *coproduct* category on one object. And this is indeed the case, as one can easily see directly (the objects are the finite cardinal numbers $[0], [1], [2] = [1] + [1], \dots$).

The logical duality of corollary 1.2.3 now tells us that the dual of the category of finite sets is the *syntactic category* of $\mathbb{T}_0$,

$$\mathcal{C}_{\mathbb{T}_0} \ \simeq \ \mathsf{Set}_{\mathrm{fin}}^{\mathsf{op}}.$$

Thus the syntax of the pure theory of equality $\mathbb{T}_0$ is "hidden" in the opposite of the category of finite sets. The terms are simply tuples of variables $(x_1, \dots, x_n)$, and the valid equations are those that are true of them *as terms*. Our corollary tells us that this is the category of finite sets, which we can see by reading the contexts $[x_1, \ldots, x_n]$ as coproducts $1+\cdots+1$ and a tuple such as $(x_2, x_5) : [x_1, \ldots, x_5] \to [x_1, x_2]$ as a *cotuple* like $[i_2, i_5] : 1+1 \to 1_1 + \ldots + 1_5$.

Example 1.2.5. For a less trivial example, consider the theory $\mathbb{T}_{\mathsf{Ab}}$ of abelian groups. Duality tells us that the syntactic category $\mathcal{C}_{\mathbb{T}_{\mathsf{Ab}}}$ is dual to the category of finitely generated, free abelian groups $\mathsf{Ab}_{\mathrm{fg}}$,

$$\mathcal{C}_{\mathbb{T}_{\mathsf{Ab}}} \simeq \mathsf{Ab}_{\mathrm{fg}}^{\mathsf{op}}.$$

This gives us a representation of the syntax of (abelian) group theory in the category of abelian groups, which can be described concretely as follows:

- the basic types of variables $[-] = 1,\ [x_1] = U,\ [x_1, x_2] = U \times U, \ldots$ are represented by the abelian groups $\{0\},\ \mathbb{Z},\ \mathbb{Z} + \mathbb{Z} = \mathbb{Z}^2, \ldots,$

- the group unit $u : 1 \to U$ is the zero homomorphism $0 : \mathbb{Z} \to \{0\}$,

- the inverse $i : U \to U$ is the homomorphism $\mathbb{Z} \to \mathbb{Z}$ taking 1 to -1,

- the group operation $m : U \times U \to U$ is the homomorphism $+ : \mathbb{Z} \to \mathbb{Z} + \mathbb{Z}$ taking 1 to $\langle 1, 1\rangle = \langle 1, 0\rangle + \langle 0, 1\rangle$ (using $\mathbb{Z} + \mathbb{Z} \cong \mathbb{Z} \times \mathbb{Z}$),

- the laws of abelian groups (and no further ones!) hold under this interpretation, because the group structure on any abelian group A is induced by precomposing with these "co-operations", by (1.15), as indicated in the following diagram for the sum $a + b$ of elements $a, b \in A$.

$$\begin{tikzcd}
\mathbb{Z}+\mathbb{Z} \arrow[r, "{(a,b)}"] & A \\
\mathbb{Z} \arrow[u, "+"] \arrow[ur, "a+b"'] &
\end{tikzcd}$$

Example 1.2.6. The category of *affine schemes* is, by definition, the dual of the category of commutative rings with unit,

$$\mathsf{Scheme}_{\mathrm{aff}} = \mathsf{Ring}^{\mathsf{op}}$$

There is therefore a ring object in affine schemes – called the *affine line* – based on the finitely generated free algebra $F(1) = \mathbb{Z}[x]$, the ring of polynomials in one variable x with integer coefficients. The "co-operations" of $+$ and $\cdot$ are given in rings by the homomorphisms $\mathbb{Z}[x] \to \mathbb{Z}[x, y]$ taking the generator x to the elements $x + y$ and $x \cdot y$.

Exercise 1.2.7. Prove directly that $\mathsf{Set}_{\mathrm{fin}}$ is the free finite coproduct category on one object.

Exercise 1.2.8. Show that for any algebraic theory $\mathbb{T}$, the forgetful functor $V : \mathsf{Mod}(\mathbb{T}) \to \mathsf{Set}$ underlies an algebra $\mathcal{V}$ in the functor category $\mathsf{Set}^{\mathsf{Mod}(\mathbb{T})}$. In more detail, each n-ary operation symbol f determines a natural transformation $f^{\mathcal{V}} : V^n \to V$, since the homomorphisms in $\mathsf{Mod}(\mathbb{T})$ commute with the various operations interpreting f. Indeed, given any algebra $\mathcal{A}$ we have the underlying set $V(\mathcal{A}) = A$ and an operation $f^{\mathcal{A}} : A^n \to A$, and for every homomorphism $h : \mathcal{A} \to \mathcal{B}$ to another algebra $\mathcal{B}$, there is a commutative square,

$$
\begin{tikzcd}
A^n \arrow[r, "h^n"] \arrow[d, "f^{\mathcal{A}}"'] & B^n \arrow[d, "f^{\mathcal{B}}"] \\
A \arrow[r, "h"'] & B.
\end{tikzcd}
\tag{1.17}
$$

So we can set $(f^{\mathcal{V}})_{\mathcal{A}} = f^{\mathcal{A}}$ to get a natural transformation $f^{\mathcal{V}} : V^n \to V$. Now check that this really is an algebra $\mathcal{V}$ in $\mathsf{Set}^{\mathsf{Mod}(\mathbb{T})}$.

Exercise 1.2.9. * Show that the algebra described in the previous exercise is represented by the universal one $\mathcal{U}$ in $\mathsf{mod}(\mathbb{T})^{\mathsf{op}} \hookrightarrow \mathsf{Mod}(\mathbb{T})^{\mathsf{op}}$ under the (covariant) Yoneda embedding,

$$\mathsf{y} : \mathsf{Mod}(\mathbb{T})^{\mathsf{op}} \longrightarrow \mathsf{Set}^{\mathsf{Mod}(\mathbb{T})}.$$

1.2.2 Lawvere algebraic theories

Nothing in the foregoing account of the functorial semantics for algebraic theories really depended on the primarily syntactic nature of such theories, i.e. their specification in terms of operations and equations. We can thus immediately generalize it to “abstract” algebraic theories, which can be regarded as providing a *presentation-free* notion of an algebraic theory.

Definition 1.2.10 (cf. Definition 1.1.2)**.** A *Lawvere algebraic theory* $\mathbb{A}$ is a small category with finite products whose objects form a sequence $A^0, A^1, A^2, \ldots$ such that $A^m \times A^n = A^{m+n}$ for all $m, n \in \mathbb{N}$. In particular, $1 = A^0$ is the terminal object, and every object is a product of finitely many copies of the *generating object* $A = A^1$.

A *model* of a Lawvere algebraic theory $\mathbb{A}$ in any category $\mathcal{C}$ with finite products is a finite-product-preserving functor $M : \mathbb{A} \to \mathcal{C}$, and a *homomorphism of models* is a natural transformation $\vartheta : M \to M'$ between such functors.

We could just as well have taken the natural numbers $0, 1, 2, \ldots$ themselves as the objects of a Lawvere algebraic theory $\mathbb{A}$, but the notation A^n is more suggestive. A Lawvere algebraic theory $\mathbb{A}$ in the sense of the above definition determines an algebraic theory in the sense of Definition 1.1.2 as follows. As basic operations with arity k we take all of the morphisms $A^k \to A$:

$$\Sigma(\mathbb{A})_k = \mathsf{Hom}_{\mathbb{A}}(A^k, A) \tag{1.18}$$

There is a canonical interpretation in $\mathbb{A}$ of terms built from variables and morphisms $A^k \to A$, namely each morphism is interpreted by itself, and variables are interpreted as product projects, as usual. An equation $u = v$ is taken as an axiom of the theory $\mathbb{A}$ if the canonical interpretations of u and v coincide. Of course, the conventional logical notions of a model 1.1.11 and a homomorphism of models then also correspond to the new, functorial ones in an obvious way.

This more abstract view of algebraic theories immediately suggests some interesting examples.

Example 1.2.11. The algebraic theory $\mathcal{C}^\infty$ of smooth maps is the category whose objects are n-dimensional Euclidean spaces 1, $\mathbb{R}$, $\mathbb{R}^2$, ..., and whose morphisms are $\mathcal{C}^\infty$-maps between them. Recall that a $\mathcal{C}^\infty$-map $f : \mathbb{R}^n \to \mathbb{R}$ is a function which has all higher partial derivatives, and that a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is a $\mathcal{C}^\infty$-map exactly when all of its composites $\pi_k \circ f : \mathbb{R}^n \to \mathbb{R}$ with the projections $\pi_k : \mathbb{R}^m \to \mathbb{R}$ are $\mathcal{C}^\infty$-maps.

A model of this theory in Set is a finite product preserving functor $A : \mathcal{C}^\infty \to \mathsf{Set}$. Up to natural isomorphism, such a model can be described as follows. A $\mathcal{C}^\infty$-model is given by a set A and for every smooth map $f : \mathbb{R}^n \to \mathbb{R}$ a function $Af : A^n \to A$ such that if $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^m \to \mathbb{R}$, $i = 1, \ldots, n$, are smooth maps then, for all $a_1, \ldots, a_m \in A$,

$$Af\big((Ag_1)\langle a_1, \ldots, a_m\rangle, \ldots, (Ag_n)\langle a_1, \ldots, a_m\rangle\big) = A(f \circ \langle g_1, \ldots, g_n\rangle)\langle a_1, \ldots, a_m\rangle \ .$$

In particular, since multiplication and addition are smooth maps, A is a commutative ring with unit. Such structures are known as *$\mathcal{C}^\infty$-rings*. Therefore, the models in Set of the theory of smooth maps are the $\mathcal{C}^\infty$-rings (cf. [MR91]).

Example 1.2.12. Recall that a *(total) recursive function* $f : \mathbb{N}^m \to \mathbb{N}^n$ is one that can be computed by a Turing machine. This means that there exists a Turing machine which on input $\langle a_1, \ldots, a_m\rangle$ outputs the value of $f\langle a_1, \ldots, a_m\rangle$. The algebraic theory Rec of recursive functions is the category whose objects are finite powers of the natural numbers 1, $\mathbb{N}$, $\mathbb{N}^2$, ..., and whose morphisms are recursive functions between them. The models of this theory in a category $\mathcal{C}$ with finite products give a notion of computability in $\mathcal{C}$.

Indeed, let us consider the category of all set-theoretic models $\mathcal{R} = \mathsf{Mod}(\mathsf{Rec})$. First, there is the "identity" model $I \in \mathcal{R}$, defined by $I\mathbb{N}^k = \mathbb{N}^k$ and $If = f$. Given any model $S \in \mathcal{R}$, its object part is determined by $S_1 = S\mathbb{N}$ since $S\mathbb{N}^k = S_1^k$. For every $n \in \mathbb{N}$ there is a morphism $1 \to \mathbb{N}$ in Rec defined by $\star \mapsto n$. Thus we have for each $n \in \mathbb{N}$ an element $s_n = S(\star \mapsto n) : 1 \to S_1$. This defines a function $s : \mathbb{N} \to S_1$ which in turn determines a natural transformation $\sigma : I \Longrightarrow S$ whose component at $\mathbb{N}^k$ is $s \times \cdots \times s : \mathbb{N}^k \to S_1^k$.

Example 1.2.13. In a category $\mathcal{C}$ with finite products every object $A \in \mathcal{C}$ determines a full subcategory consisting of the finite powers $1, A, A^2, A^3, \ldots$ and all morphisms between them. This is the *total theory* $\mathbb{T}(A)$ of the object A in $\mathcal{C}$.

Free algebras

In order to extend the logical duality of the foregoing section to the abstract case, we will require the notion of a *free model* of an abstract algebraic theory. Of course, we already have the conventional notion of free models determined in terms of the associated conventional algebraic theory given by (1.18). But we can also determine free models directly in terms of the abstract theory, in a way which then agrees with the conventional case.

Let $\mathbb{A}$ be a Lawvere algebraic theory, with objects $1, A, A^2, \ldots$. We have the category of models,

$$\mathsf{Mod}(\mathbb{A}) = \mathsf{Hom}_{\mathsf{FP}}(\mathbb{A}, \mathsf{Set}).$$

Let us first define the *forgetful functor* by evaluating at the generating object $A \in \mathbb{A}$,

$$U : \mathsf{Mod}(\mathbb{A}) \to \mathsf{Set} \tag{1.19}$$

$$(M : \mathbb{A} \to \mathsf{Set}) \mapsto M(A). \tag{1.20}$$

We shall also write

$$|M| = U(M) = M(A). \tag{1.21}$$

Now for the (finitary) free functor $F : \mathsf{Set}_{\mathrm{fin}} \to \mathsf{Mod}(\mathbb{A})$, we set:

$$\begin{aligned} F(0) &= \mathsf{Hom}_{\mathbb{A}}(1, -) \\ F(1) &= \mathsf{Hom}_{\mathbb{A}}(A, -) \\ &\vdots \\ F(n) &= \mathsf{Hom}_{\mathbb{A}}(A^n, -). \end{aligned}$$

Note that this is a composite of the two (contravariant) functors,

$$\mathsf{Set}_{\mathrm{fin}} \to \mathbb{A}^{\mathsf{op}} \to \mathsf{Mod}(\mathbb{A}),$$

given by $n \mapsto A^n$ and $X \mapsto \mathsf{Hom}_{\mathbb{A}}(X, -)$, and is therefore (covariantly) functorial. Note also that the representables $\mathsf{Hom}_{\mathbb{A}}(A^n, -)$ do indeed preserve finite products, and are therefore in the full subcategory $\mathsf{Mod}(\mathbb{A}) \hookrightarrow \mathsf{Set}^{\mathbb{A}}$.

For adjointness we need to check that:

$$\mathsf{Hom}_{\mathsf{Mod}(\mathbb{A})}(F(n), M) \cong \mathsf{Hom}_{\mathsf{Set}}(n, U(M)) \tag{1.22}$$

(naturally in both arguments, of course). The right-hand side is plainly just $|M|^n$. For the left-hand side we have:

$$\begin{aligned} \mathsf{Hom}_{\mathsf{Mod}(\mathbb{A})}(F(n), M) &= \mathsf{Hom}_{\mathsf{Mod}(\mathbb{A})}(\mathsf{Hom}_{\mathbb{A}}(A^n, -), M) \\ &= \mathsf{Hom}_{\mathsf{Set}^{\mathbb{A}}}(\mathsf{Hom}_{\mathbb{A}}(A^n, -), M) \\ &\cong M(A^n) && \text{(by Yoneda)} \\ &\cong M(A)^n && (M \text{ is FP}) \\ &= |M|^n && (1.21). \end{aligned}$$

The full definition of the free functor

$$F : \mathsf{Set} \to \mathsf{Mod}(\mathbb{A})$$

is then given by writing an arbitrary set X as a (filtered) colimit of its finite subsets $X_i \subseteq X$, and setting $F(X) = \mathrm{colim}_i\, F(X_i)$ in the category $\mathsf{Set}^{\mathbb{A}}$. Since filtered colimits commute with finite products, these colimits taken in $\mathsf{Set}^{\mathbb{A}}$ and will remain in $\mathsf{Mod}(\mathbb{A})$.

Theorem 1.2.14. *For any set X with free algebra $F(X)$ as just defined, and any $\mathbb{A}$-model M, there is a natural isomorphism,*

$$\mathsf{Hom}_{\mathsf{Mod}(\mathbb{A})}(F(X), M) \cong \mathsf{Hom}_{\mathsf{Set}}(X, U(M))\,. \tag{1.23}$$

Proof. The proof is now an easy exercise. □

By definition, the finitely generated free models $F(n)$ are just the representables $\mathsf{Hom}_{\mathbb{A}}(A^n, -)$; therefore as the "semantic dual" $\mathsf{mod}(\mathbb{A}) \hookrightarrow \mathsf{Mod}(\mathbb{A})$ of the theory $\mathbb{A}$, in the sense of corollary 1.2.3, we simply have (the full subcategory of $\mathsf{Hom}_{\mathsf{FP}}(\mathbb{A}, \mathsf{Set})$ on) the image of the Yoneda embedding,

$$\begin{tikzcd}
\mathsf{mod}(\mathbb{A}) \arrow[r, hook] & \mathsf{Mod}(\mathbb{A}) = \mathsf{Hom}_{\mathrm{FP}}(\mathbb{A}, \mathsf{Set}) \arrow[r, hook] & \mathsf{Set}^{\mathbb{A}} \\
\mathbb{A}^{\mathsf{op}} \arrow[u, "\simeq"] \arrow[rr, "="'] & & \mathbb{A}^{\mathsf{op}}. \arrow[u, hook, "\mathsf{y}"']
\end{tikzcd}$$

In the abstract case, then, the logical duality

$$\mathbb{A} \simeq \mathsf{mod}(\mathbb{A})^{\mathsf{op}}$$

comes down to the fact that the (contravariant) Yoneda embedding

$$\mathbb{A}^{\mathsf{op}} \hookrightarrow \mathsf{Set}^{\mathbb{A}}$$

presents $\mathbb{A}$ as (the dual of) a full subcategory of (product-preserving!) functors. Thus we have shown:

Theorem 1.2.15. *For any Lawvere algebraic theory $\mathbb{A}$, there is an equivalence,*

$$\mathbb{A} \simeq \mathsf{mod}(\mathbb{A})^{\mathsf{op}}$$

between $\mathbb{A}$ and the full subcategory of finitely generated free models.

Exercise 1.2.16. Prove theorem 1.2.14.

1.2.3 Algebraic categories

Given an *arbitrary* category $\mathcal{A}$, we may ask: *When is $\mathcal{A}$ the category of models for some algebraic theory?* Such categories are sometimes called *varieties*, at least in universal algebra, and there are well-known *recognition theorems* such as Birkhoff's famous HSP-theorem, which says that a class of interpretations for some fixed signature are all those satisfying a set of equations if the class is closed under Products, Subalgebras, and Homomorphic images (i.e. quotients by an algebra congruence). Toward the goal of "recognizing" a *category* of algebras (without being given the signature!), let us define:

Definition 1.2.17. An *algebraic category* $\mathcal{A}$ is a (locally small) category equivalent to one of the form

$$\mathsf{Hom}_{\mathsf{FP}}(\mathbb{A}, \mathsf{Set}) \hookrightarrow \mathsf{Set}^{\mathbb{A}}$$

where $\mathbb{A}$ is any small finite product category and $\mathsf{Hom}_{\mathsf{FP}}(\mathbb{A}, \mathsf{Set})$ is the full subcategory of finite product preserving functors. If $\mathbb{A}$ is a Lawvere algebraic theory (i.e. the objects are generated under finite products by a single object), then we will say that $\mathcal{A}$ is a *Lawvere algebraic category.*

If $\mathcal{A} \simeq \mathsf{Hom}_{\mathsf{FP}}(\mathbb{A}, \mathsf{Set})$ is a Lawvere algebraic category, then in particular there will be a forgetful functor, determined by evaluation at the generating object A of $\mathbb{A}$,

$$U = \mathsf{eval}_A : \mathcal{A} \to \mathsf{Set}. \tag{1.24}$$

It follows immediately that U preserves all limits, and one can show without difficulty that it also preserves all filtered colimits (cf. exercise 1.2.22). We require only one further condition to "recognize" $\mathcal{A}$ as algebraic, namely creation of "U-absolute coequalizers".

Definition 1.2.18. In any category $\mathcal{C}$, a coequalizer $c : Y \to Z$ of maps $a, b : X \rightrightarrows Y$ is *absolute* if, for every category $\mathcal{D}$ and functor $F : \mathcal{C} \to \mathcal{D}$, the map $Fc : FY \to FZ$ is a coequalizer of the maps $Fa, Fb : FX \rightrightarrows FY$. A functor $F : \mathcal{C} \to \mathcal{D}$ *creates F-absolute coequalizers* if for every parallel pair of maps $a, b : X \rightrightarrows Y$ in $\mathcal{C}$ and absolute coequalizer $q : FY \to Q$ of $Fa, Fb : FX \rightrightarrows FY$ in $\mathcal{D}$, there is a unique object Z and map $c : Y \to Z$ in $\mathcal{C}$ with $FZ = Q$ and $Fc = q$, which, moreover, is a coequalizer in $\mathcal{C}$.

$$\begin{tikzcd}
\mathcal{C} \arrow[d, "F"'] & X \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & Y \arrow[r, dotted, "c"] & Z \\
\mathcal{D} & FX \arrow[r, shift left, "Fa"] \arrow[r, shift right, "Fb"'] & FY \arrow[r, "q"'] & Q
\end{tikzcd} \tag{1.25}$$

Theorem 1.2.19. *Given a category $\mathcal{A}$ equipped with a functor $U : \mathcal{A} \to \mathsf{Set}$, the following conditions are equivalent.*

1. *$\mathcal{A}$ is a Lawvere algebraic category; i.e. there is a Lawvere algebraic theory $\mathbb{A}$, and an equivalence,*

$$\mathcal{A} \simeq \mathsf{Hom}_{\mathsf{FP}}(\mathbb{A}, \mathsf{Set}) \hookrightarrow \mathsf{Set}^{\mathbb{A}}$$

   between $\mathcal{A}$ and the full subcategory of finite product preserving functors on $\mathbb{A}$, associating $U : \mathcal{A} \to \mathsf{Set}$ to the evaluation at the generating object of $\mathbb{A}$.

2. *$U : \mathcal{A} \to \mathsf{Set}$ has a left adjoint $F : \mathsf{Set} \to \mathcal{A}$, preserves all filtered colimits, and creates U-absolute coequalizers.*

3. *$\mathcal{A}$ is monadic over* Set *(via $U : \mathcal{A} \to \mathsf{Set}$),*

$$\mathcal{A} \simeq \mathsf{Set}^T$$

for a finitary monad $T : \mathsf{Set} \to \mathsf{Set}$.

Proof. (1⇒2) Suppose first that $\mathcal{A}$ is Lawvere algebraic, so

$$\mathcal{A} \simeq \mathsf{Mod}(\mathbb{A}) = \mathsf{Hom}_{\mathsf{FP}}(\mathbb{A}, \mathsf{Set}) \hookrightarrow \mathsf{Set}^{\mathbb{A}}$$

for a Lawvere algebraic theory $\mathbb{A}$. By theorem 1.2.14 we know that U has a left adjoint $F : \mathsf{Set} \to \mathcal{A}$. It also preserves filtered colimits because they commute with finite products, and so a filtered colimit of FP functors, calculated in $\mathsf{Set}^{\mathbb{A}}$, is again an FP functor. This suffices, since colimits are computed pointwise in $\mathsf{Set}^{\mathbb{A}}$ and U is an evaluation functor.

For creation of U-absolute coequalizers, suppose we have maps $f, g : A \rightrightarrows B$ in $\mathcal{A}$ and an absolute coequalizer $c : UB \to C$ for $Uf, Ug : UA \rightrightarrows UB$ in Set; we want to put an algebra structure on C making c a homomorphism $c : B \to C$ in $\mathcal{A}$, and a coequalizer of f and g.

$$\begin{tikzcd}
UA^n \arrow[r, shift left, "Uf^n"] \arrow[r, shift right, "Ug^n"'] \arrow[d, "\sigma^A"'] & UB^n \arrow[r, "c^n"] \arrow[d, "\sigma^B"] & C^n \arrow[d, dotted, "\sigma^C"] \\
UA \arrow[r, shift left, "Uf"] \arrow[r, shift right, "Ug"'] & UB \arrow[r, "c"'] & C
\end{tikzcd} \tag{1.26}$$

For each function symbol $\sigma \in \Sigma$ we have commutative squares as on the left in the above diagram, because f and g are homomorphisms. It follows by a simple diagram chase that $c \circ \sigma^B$ coequalizes the pair $Uf^n, Ug^n : UA^n \rightrightarrows UB^n$. Since $c : UB \to C$ is absolute, it is preserved by the functor $(-)^n$, and therefore $c^n : UB^n \to C^n$ is a coequalizer of Uf^n, Ug^n. There is therefore a unique map $\sigma^C : C^n \to C$ as indicated, making the right hand square commute. Doing this for each $\sigma \in \Sigma$ gives an interpretation of Σ on C. This is seen to be an algebra structure because the maps c^n are surjections. Thus $c : B \to C$ is a homomorphism, which is easily seen to be a coequalizer in $\mathcal{A}$.

(2⇒3) Taking the standard monad (T, η, μ) on Set with underlying functor $T = U \circ F$, we want to show that the canonical comparison map

$$\mathcal{A} \to \mathsf{Set}^T$$

to the category of T-algebras is an isomorphism. This follows from the condition that U creates absolute coequalizers by Beck's theorem; see [Lan71, VI.7]. Moreover, T preserves filtered colimits (i.e. is "finitary") because each of F and U do so.

(3⇒1) Let (T, η, μ) be a finitary monad on Set and $U : \mathsf{Set}^T \to \mathsf{Set}$ the forgetful functor from the category of T-algebras. We want an algebraic theory $\mathbb{A}$ and an equivalence

$$\mathsf{Set}^T \simeq \mathsf{Mod}(\mathbb{A})$$

over U and evaluation at the generator of $\mathbb{A}$, where recall $\mathsf{Mod}(\mathbb{A}) = \mathsf{Hom}_{\mathsf{FP}}(\mathbb{A}, \mathsf{Set})$. Let

$$\mathbb{A} = \mathsf{FGF}(\mathsf{Set}^T)^{\mathsf{op}} \tag{1.27}$$

be the dual of the full subcategory of finitely generated free T-algebras. The objects of $\mathbb{A}$ are of the form $T_0, T_1, T_2, \ldots$ where $T_n = T(n)$, equipped with the multiplication $\mu_n : T^2(n) \to T(n)$ as algebra structure map. Since, as free algebras, $T(n+m) \cong T(n) + T(m)$ we indeed have $T_n \times T_m \cong T_{n+m}$ as objects of $\mathbb{A}$, and T_1 as the generating object.
By the first two steps of this proof, we know that the algebraic category $\mathsf{Mod}(\mathbb{A})$ is also (finitary) monadic,

$$\mathsf{Mod}(\mathbb{A}) \simeq \mathsf{Set}^M,$$

with monad $M = U_M \circ F_M$, where $F_M \dashv U_M$ is the free-forgetful adjunction for $\mathsf{Mod}(\mathbb{A}) = \mathsf{Hom}_{\mathsf{FP}}(\mathbb{A}, \mathsf{Set})$, and $U_M \cong \mathsf{eval}_{T_1}$. Thus it will suffice to show that $M \cong T$, as monads, in order to conclude that

$$\mathsf{Mod}(\mathbb{A}) \simeq \mathsf{Set}^M \simeq \mathsf{Set}^T.$$

Moreover, since both M and T are finitary, it suffices to show that their respective restrictions to the dense subcategory $\mathsf{Set}_{\text{fin}} \hookrightarrow \mathsf{Set}$ are isomorphic. By (1.22), we know that the finite free functor $F_M(n)$ has the form

$$F_M(n) = \mathsf{Hom}_{\mathbb{A}}(T_n, -) = \mathsf{Hom}_{\mathsf{FGF}(\mathsf{Set}^T)}(-, \langle T(n), \mu_n \rangle)$$

thus using the fact that $U_M \cong \mathsf{eval}_{T_1}$ we see that

$$\begin{aligned} M(n) = U_M(F_M(n)) &= U_M\big(\mathsf{Hom}_{\mathsf{FGF}(\mathsf{Set}^T)}(-, \langle T(n), \mu_n \rangle)\big) \\ &\cong \mathsf{Hom}_{\mathsf{FGF}(\mathsf{Set}^T)}(\langle T(1), \mu_1 \rangle, \langle T(n), \mu_n \rangle) \\ &\cong \mathsf{Hom}_{\mathsf{Set}}(1, T(n)) \cong T(n). \end{aligned}$$

□

Remark 1.2.20. Another "recognition theorem" that can be found in [Bor94] is the following:

Theorem (Borceux II.3.9)**.** *Given a category $\mathcal{A}$, equipped with a functor $U : \mathcal{A} \to \mathsf{Set}$, the following conditions are equivalent.*

1. *$\mathcal{A}$ is equivalent to the category of models of some Lawvere algebraic theory $\mathbb{T}$,*

$$\mathcal{A} \simeq \mathsf{Mod}(\mathbb{T})$$

   with $U : \mathcal{A} \to \mathsf{Set}$ the corresponding forgetful functor.

2. *$\mathcal{A}$ has coequalizers and kernel pairs, and $U : \mathcal{A} \to \mathsf{Set}$ has a left adjoint $F : \mathsf{Set} \to \mathcal{A}$, preserves all filtered colimits and regular epimorphisms, and reflects isomorphisms.*

Exercise 1.2.21. A *split coequalizer* for maps $f, g : A \rightrightarrows B$ is a map $e : B \to C$ together with s and t as indicated below,

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "e"] \arrow[l, bend left=60, "t"] & C \arrow[l, bend left=60, "s"] \end{tikzcd} \tag{1.28}$$

satisfying the equations

$$ef = eg, \quad ft = 1_B, \quad gt = se, \quad es = 1_C.$$

Show that a split coequalizer is an absolute coequalizer.

Exercise 1.2.22. A filtered colimit of algebras can be described directly as follows: First consider the case of sets. Let the index category $\mathbb{J}$ be filtered and $D : \mathbb{J} \to \mathsf{Set}$ a diagram. The colimiting set $\mathrm{colim}_j D_j$ can be described as the quotient of the coproduct $(\coprod_j Dj)/\sim$, where the equivalence relation $\sim$ is defined by:

$$(d_i \in D_i) \sim (d_j \in D_j) \;\Leftrightarrow\; t_{ik}(d_i) = t_{jk}(d_j) \text{ for some } t_{ik} : i \to k \text{ and } t_{jk} : j \to k \text{ in } \mathbb{J}.$$

1. Show that this is an equivalence relation using the filteredness of $\mathbb{J}$.
2. Now assume that the D_j all have an algebra structure and that all the transition maps $t_{ik} : D_i \to D_k$ are homomorphisms. Show that the colimit set $D_\infty = \mathrm{colim}_j D_j$ is also an algebra of the same kind by defining each of the operations $\sigma_\infty : D_\infty \times ... \times D_\infty \to D_\infty$ on equivalence classes as
$$\sigma_\infty\langle [d_i], ..., [d'_j]\rangle = [\sigma_k\langle t_{ik}(d_i), ..., t_{jk}(d'_j)\rangle]$$
for suitable k. Show that this is well-defined, and that D_∞, so equipped, also satisfies the equations satisfied by the D_j.

Example 1.2.23. A field is a ring in which every non-zero element has a multiplicative inverse. The theory of fields is (apparently) not algebraic, because the axiom

$$x \neq 0 \;\Rightarrow\; \exists y(x \cdot y = 1)$$

is not simply an equation. But in principle there could be an equivalent algebraic formulation of the theory which would somehow circumvent this problem. We can show that this is not the case by proving that the category Field of fields and field homomorphisms is not algebraic.

First observe that a category of models $\mathsf{Mod}(\mathbb{A})$ always has a terminal object because Set has a terminal object 1, and the constant functor $\Delta_1 : \mathbb{A} \to \mathsf{Set}$ which maps everything to 1 is a model. The functor Δ_1 is the terminal object in $\mathsf{Mod}(\mathbb{A})$ because it is the terminal functor in the functor category $\mathsf{Set}^{\mathbb{A}}$. In order to see that Field is not algebraic it thus suffices to show that there is no terminal field.

Exercise 1.2.24. Show that the category Field does not have a terminal object. (Hint: suppose that T is the terminal field and use the unique homomorphism $\mathbb{Z}_2 \to T$ to see that $1 + 1 = 0$ in T, then reason similarly using the unique homomorphism $\mathbb{Z}_3 \to T$.)

1.2.4 Algebraic functors

Classically, a *syntactic translation* of one algebraic theory into another is an assignment of types to types and terms to terms, respecting the tupling operations and substitutions of terms for variables. Such a translation can be described abstractly as a finite product preserving functor,

$$T : \mathbb{A} \to \mathbb{B}$$

between the associated (Lawvere) algebraic theories. Every such translation induces a "definable" functor on the semantics, just by precomposition:

$$T^*(M) = M \circ T.$$

$$\mathsf{Mod}(\mathbb{A}) \xleftarrow{T^*} \mathsf{Mod}(\mathbb{B}) \tag{1.29}$$

$$\begin{tikzcd} \mathbb{A} \arrow[r, "T"] \arrow[dr, "T^*(M)"'] & \mathbb{B} \arrow[d, "M"] \\ & \mathsf{Set} \end{tikzcd}$$

For instance, let $\mathbb{A}_0 = (\mathsf{Set}_{\mathrm{fin}})^{\mathsf{op}}$ the trivial theory ($\mathbb{A}_0 \simeq \mathcal{C}_{\mathbb{T}_0}$), so that $\mathsf{Mod}(\mathbb{A}_0) \simeq \mathsf{Set}$. Then for any Lawere algebraic theory $\mathbb{A}$, the generating object $A \in \mathbb{A}$ has a classifying functor

$$A : \mathbb{A}_0 \to \mathbb{A}$$

which induces the forgetful functor by precomposition:

$$\mathsf{Set} \simeq \mathsf{Mod}(\mathbb{A}_0) \xleftarrow{A^*} \mathsf{Mod}(\mathbb{A})$$

$$\begin{tikzcd} \mathbb{A}_0 \arrow[r, "A"] \arrow[dr, "A^*M"'] & \mathbb{A} \arrow[d, "M"] \\ & \mathsf{Set} \end{tikzcd}$$

More generally, by the universal property of $\mathbb{A}$, every translation $T : \mathbb{A} \to \mathbb{B}$ corresponds to a "model of $\mathbb{A}$ in the syntax of $\mathbb{B}$":

$$\frac{T : \mathbb{A} \to \mathbb{B}}{\hat{T} \in \mathsf{Mod}(\mathbb{A}, \mathbb{B})}$$

For instance, since every ring R has an underlying group $\mathsf{Grp}(R)$, the universal ring $\mathcal{U}_{\mathbb{R}}$ in the theory of rings $\mathbb{R}$ has one $\mathsf{Grp}(\mathcal{U}_{\mathbb{R}})$, which is therefore classified by a unique functor from the theory of groups,

$$\mathsf{Grp}(\mathcal{U}_{\mathbb{R}})^{\sharp} : \mathbb{G} \to \mathbb{R}.$$

This translation induces a functor on the corresponding categories of models,

$$\mathsf{Grp}(\mathcal{U}_{\mathbb{R}})^{*} : \mathsf{Ring} \to \mathsf{Group}, \tag{1.30}$$

which of course is just the underlying group functor $\mathsf{Grp} : \mathsf{Ring} \to \mathsf{Group}$.

We can now ask: *Which functors $f : \mathsf{Mod}(\mathbb{B}) \to \mathsf{Mod}(\mathbb{A})$ between algebraic categories are of the form $f = T^*$ for a translation $T : \mathbb{A} \to \mathbb{B}$ of theories?* Let us call these *algebraic functors*. We consider first the case where ($\mathbb{A}$ and $\mathbb{B}$ are Lawvere algebraic and) T takes the generator A of $\mathbb{A}$ to the generator B of $\mathbb{B}$,

$$T(A) \cong B.$$

Then T^* commutes with the forgetful functors, which, recall, are evaluation at the generators, $U_{\mathbb{A}}(M) = M(A)$:

$$\begin{tikzcd}
\mathsf{Mod}(\mathbb{B}) \arrow[rr, "T^*"] \arrow[dr, "U_{\mathbb{B}}"'] & & \mathsf{Mod}(\mathbb{A}) \arrow[dl, "U_{\mathbb{A}}"] \\
& \mathsf{Set} &
\end{tikzcd}$$

This is simply because

$$(U_{\mathbb{A}} \circ T^*)(M) = U_{\mathbb{A}}(M \circ T) = (M \circ T)(A) \cong M(T(A)) \cong M(B) = U_{\mathbb{B}}(M).$$

In fact, we shall see that this condition is already sufficient! We first require the following.

Lemma 1.2.25. *Let $\mathcal{A}$ be Lawvere algebraic. The forgetful functor $U : \mathcal{A} \to \mathsf{Set}$ not only preserves, but also* creates *all small limits, filtered colimits, and regular epimorphisms.*

Proof. This is a standard fact, and not difficult to prove; the reader can either prove it as an exercise, or look it up in [ALR03]. □

Proposition 1.2.26. *For Lawvere algebraic theories $\mathbb{A}$ and $\mathbb{B}$, given any functor $f : \mathsf{Mod}(\mathbb{B}) \to \mathsf{Mod}(\mathbb{A})$ with*

$$U_{\mathbb{B}} \cong U_{\mathbb{A}} \circ f,$$

there is a unique (up to iso) FP-functor $T : \mathbb{A} \to \mathbb{B}$ such that

$$f \cong T^*.$$

Proof 1. Consider the diagram:

$$\begin{tikzcd}
\mathsf{Mod}(\mathbb{B}) \arrow[rr, "f"] \arrow[dr, "U_{\mathbb{B}}"] & & \mathsf{Mod}(\mathbb{A}) \arrow[dl, "U_{\mathbb{A}}"'] \\
& \mathsf{Set} \arrow[ul, "F_{\mathbb{B}}"] \arrow[ur, "F_{\mathbb{A}}"'] &
\end{tikzcd}$$

where in each pair we have an adjunction $F \dashv U$ by Theorem 1.2.19, and the central triangle commutes up to iso. We seek an FP-functor $T : \mathbb{A} \to \mathbb{B}$ such that $f \cong T^*$.

Since by Lemma 1.2.25, $U_{\mathbb{A}}$ creates limits, and $U_{\mathbb{A}} \circ f \cong U_{\mathbb{B}}$ preserves them, it follows that f also preserves them. The same argument applies to filtered colimits (and regular epis). Now $\mathsf{Mod}(\mathbb{A})$ is locally finitely presentable, as a reflective subcategory of the functor category $\mathsf{Set}^{\mathbb{A}}$ with a filtered-colimit preserving inclusion (cf. [AR94]). Thus since $f : \mathsf{Mod}(\mathbb{B}) \to \mathsf{Mod}(\mathbb{A})$ preserves (small) limits and filtered colimits, it has a left adjoint

$$f_! : \mathsf{Mod}(\mathbb{A}) \to \mathsf{Mod}(\mathbb{B})$$

by the Adjoint Functor Theorem. Indeed, one can check the solution set condition directly (see [Lan71, AR94]). From $U_{\mathbb{B}} \cong U_{\mathbb{A}} \circ f$, it then follows that $F_{\mathbb{B}} \cong f_! \circ F_{\mathbb{A}}$. In particular, for the generators $A = F_{\mathbb{A}}(1)$ and $B = F_{\mathbb{B}}(1)$ we have $f_!(A) = f_! F_{\mathbb{A}}(1) \cong F_{\mathbb{B}}(1) = B$, and then $f_!(\coprod_n A) = \coprod_n B$ since $f_!$ preserves coproducts.

Since $\mathbb{A} \simeq \mathsf{mod}(\mathbb{A})^{\mathsf{op}}$, the dual of the subcategory of finitely generated free models, by Theorem 1.2.15, and the same holds for $\mathbb{B}$, the functor $f_! : \mathsf{Mod}(\mathbb{A}) \to \mathsf{Mod}(\mathbb{B})$ restricts and dualizes to an FP "translation of theories" $T : \mathbb{A} \to \mathbb{B}$ as in,

$$\begin{tikzcd}
\mathbb{A} \arrow[r, dotted, "T"] \arrow[d, hook, "\mathsf{y}^{\mathsf{op}}"'] & \mathbb{B} \arrow[d, hook, "\mathsf{y}^{\mathsf{op}}"] \\
\mathsf{Mod}(\mathbb{A})^{\mathsf{op}} \arrow[r, "f_!^{\mathsf{op}}"'] & \mathsf{Mod}(\mathbb{B})^{\mathsf{op}}
\end{tikzcd}$$

such that

$$T(A^n) = T(F_{\mathbb{A}}(n)) = f_!^{\mathsf{op}}(F_{\mathbb{A}}(n)) \cong F_{\mathbb{B}}(n) = B^n. \tag{1.31}$$

It remains to see that $f \cong T^* : \mathsf{Mod}(\mathbb{B}) \to \mathsf{Mod}(\mathbb{A})$. Indeed, for any model $M : \mathbb{B} \to \mathsf{Set}$, we have

$$\begin{aligned}
f(M)(A) &\cong \mathsf{Set}^{\mathbb{A}}(\mathsf{y}A, f(M)) && \text{Yoneda} \\
&\cong \mathsf{Mod}(\mathbb{A})(\mathsf{y}A, f(M)) && \mathsf{Mod}(\mathbb{A}) \hookrightarrow \mathsf{Set}^{\mathbb{A}} \\
&\cong \mathsf{Mod}(\mathbb{B})(f_!(\mathsf{y}A), M) && f_! \dashv f \\
&\cong \mathsf{Mod}(\mathbb{B})(\mathsf{y}(TA), M) && (1.31) \\
&\cong \mathsf{Set}^{\mathbb{B}}(\mathsf{y}(TA), M) && \mathsf{Mod}(\mathbb{A}) \hookrightarrow \mathsf{Set}^{\mathbb{A}} \\
&\cong M(TA) && \text{Yoneda} \\
&\cong T^*(M)(A),
\end{aligned}$$

naturally in M. The case of an arbitrary object $A^n \in \mathbb{A}$ follows, since the models $f(M)$ and $T^*(M)$ preserve products. $\square$

Corollary 1.2.27. *For a functor $f : \mathsf{Mod}(\mathbb{B}) \to \mathsf{Mod}(\mathbb{A})$ between Lawvere algebraic categories, the following are equivalent.*

1. *f commutes with the forgetful functors, $U_{\mathbb{A}} \circ f \cong U_{\mathbb{B}}$.*
2. *f is algebraic, $f = T^*$, for an FP functor $T : \mathbb{A} \to \mathbb{B}$ which preserves the generator, $T(A) \cong B$.*

$\square$

In fact, more is true: there is a *bi*equivalence of categories

$$\mathsf{LAlgCat}/\mathsf{Set} \;\simeq\; \left(\mathsf{Set}_{\mathrm{fin}}^{\mathsf{op}}/\mathsf{LAlgTh}\right)^{\mathsf{op}}, \tag{1.32}$$

where on the left we have the category of Lawvere algebraic categories $\mathcal{A}$, equipped with their canonical forgetful functors $U_{\mathbb{A}} : \mathcal{A} \to \mathsf{Set}$, and algebraic functors between them that commute up to natural isomorphism over the base, and on the right (the dual of) the category of Lawvere algebraic theories and FP functors that preserve the generator. A "biequivalence" is like an equivalence, but only up to equivalence! Observe that the left-hand side of (1.32) is not even locally small, while the entire category on the right is small. (See e.g. [ARV10] for details.) This "global" Syntax-Semantics duality can be extended even further, as sketched in the following exercises.

Exercise 1.2.28. Show that for any algebraic theory $\mathbb{A}$, the full inclusion $\mathsf{Mod}(\mathbb{A}) \hookrightarrow \mathsf{Set}^{\mathbb{A}}$ has a left adjoint. (*Hint:* use the Adjoint Functor Theorem.)

Exercise 1.2.29. Assuming the result of the previous exercise, show that the precomposition functor $T^* : \mathsf{Mod}(\mathbb{B}) \longrightarrow \mathsf{Mod}(\mathbb{A})$ induced by *any* translation $T : \mathbb{A} \to \mathbb{B}$ (not necessarily preserving the generator) always has a left adjoint $T_! : \mathsf{Mod}(\mathbb{A}) \longrightarrow \mathsf{Mod}(\mathbb{B})$.

Exercise 1.2.30. Assuming the results of the previous two exercises, show that an algebraic functor $f : \mathsf{Mod}(\mathbb{B}) \longrightarrow \mathsf{Mod}(\mathbb{A})$, induced by a translation $T : \mathbb{A} \to \mathbb{B}$ as $f = T^*$, satisfies the following conditions:

(i) f preserves limits.

(ii) f preserves filtered colimits.

(iii) f preserves regular epimorphisms.

Hint: Since f has a left adjoint $f_! : \mathsf{Mod}(\mathbb{A}) \longrightarrow \mathsf{Mod}(\mathbb{B})$, we know that $f_!(A) \cong B^n$ for some $0 \leq n$. Now use $\mathbb{B} \simeq \mathsf{mod}(\mathbb{B})^{\mathsf{op}}$.

Remark 1.2.31. The converse of Exercise 1.2.30 holds as well, under a certain condition on the syntactic categories, to be explained below. We then obtain a duality of the form

$$\mathsf{LAlgCat} \;\simeq\; \mathsf{LAlgTh}^{\mathsf{op}}\,, \tag{1.33}$$

generalizing (1.32) by eliminating the "base point". This generalizes even further to "many-sorted" algebraic theories $\mathbb{A}$ not assumed to be generated by a single object, and thus given simply by small FP categories. The corresonding semantic category $\mathsf{Mod}(\mathbb{A})$ still consists of all FP-functors $\mathbb{A} \to \mathsf{Set}$. The question, when is a "semantic functor"

$$f : \mathsf{Mod}(\mathbb{B}) \longrightarrow \mathsf{Mod}(\mathbb{A})$$

between such algebraic categories induced by a "syntactic translation" $T : \mathbb{A} \to \mathbb{B}$ of such algebraic theories can also be answered in the abstract setting, providing a characterization of a (general) *algebraic functor*: it is again one that preserves all limits, filtered colimits, and regular epimorphisms. The resulting duality

$$\mathsf{AlgCat} \;\simeq\; \mathsf{AlgTh}^{\mathsf{op}}\,, \tag{1.34}$$

requires the condition on the syntactic side, however, that the algebraic theories are closed under retracts—as does (1.33). See [ALR03] for details.

1.2.5 Dualities for algebraic theories

We summarize and generalize the different dualities for algebraic theories the arose in this chapter. See the references [ALR03, ARV10] for details.

In Section 1.2.1, we had the *logical duality* relating an individual algebraic theory $\mathbb{T}$ and its category of models $\mathsf{Mod}(\mathbb{T})$, given by an equivalence of categories

$$\mathcal{C}_{\mathbb{T}}{}^{\mathsf{op}} \simeq \mathsf{mod}(\mathbb{T}) \hookrightarrow \mathsf{Mod}(\mathbb{T}),$$

between the syntactic category $\mathcal{C}_{\mathbb{T}}$ and a full subcategory of the semantics $\mathsf{mod}(\mathbb{T}) \hookrightarrow \mathsf{Mod}(\mathbb{T})$ consisting of the finitely generated free models.

The "semantics" functor Mod is represented by assigning to each model M an essentially unique FP functor $M^\sharp : \mathcal{C}_{\mathbb{T}} \to \mathsf{Set}$, providing an equivalence of categories,

$$\mathsf{Mod}(\mathbb{T}) \;\simeq\; \mathsf{Hom}_{\mathsf{FP}}(\mathcal{C}_{\mathbb{T}}, \mathsf{Set}). \tag{1.35}$$

By (1.35), the assignment $\mathbb{T} \mapsto \mathsf{Mod}(\mathbb{T})$ is *contravariantly functorial in the theory* $\mathbb{T}$, leading to a global semantic representation of the syntax of algebraic theories,

$$\mathsf{Mod} : \mathsf{AlgTh}^{\mathsf{op}} \to \mathsf{Cat}.$$

Regarding a theory abstractly as a small finite product category $\mathbb{A}$, and a syntactic translation as an FP functor $T : \mathbb{A} \to \mathbb{B}$, we can recognize the essential image of the semantics functor Mod as consisting of certain *locally presentable* categories $\mathcal{A} = \mathsf{Mod}(\mathbb{A})$, which may be called *algebraic categories*.[1]

An algebraic theory $\mathbb{A}$ can then be recovered functorially from its algebraic category of models $\mathcal{A} = \mathsf{Mod}(\mathbb{A})$ as the finitely generated free models, but also intrinsically, by considering the category of all *algebraic functors*

$$f : \mathcal{A} \to \mathsf{Set}\,,$$

defined as functors preserving limits, filtered colimits, and regular epimorphisms. Indeed, this follows from the duality

$$\mathsf{AlgTh}^{\mathsf{op}} \;\simeq\; \mathsf{AlgCat}\,, \tag{1.36}$$

by taking $\mathsf{Set} = \mathsf{Mod}(\mathbb{A}_0)$, where $\mathbb{A}_0 = \mathsf{Set}_{\mathsf{fin}}^{\mathsf{op}}$ is the free FP category on an object (which we recognized in Example 1.2.4 as the classifying category of the trivial theory $\mathbb{T}_0$), so that we have a sequence of equivalences:

$$\frac{\dfrac{\dfrac{\mathcal{A} \longrightarrow \mathsf{Set}}{\mathsf{Mod}(\mathbb{A}) \longrightarrow \mathsf{Mod}(\mathbb{A}_0)}}{\mathbb{A}_0 \longrightarrow \mathbb{A}}}{\mathbb{A}}$$

In this way, the category Set serves as a "dualizing object", representing both the semantics Mod and the syntax Theory as contrvariant functors,

$$\begin{tikzcd} \mathsf{AlgTh}^{\mathsf{op}} \arrow[r, bend left, "\mathsf{Mod}"] & \mathsf{AlgCat} \arrow[l, bend left, "\mathsf{Theory}"] \end{tikzcd} \tag{1.37}$$

using the two different structures on Set,

$$\begin{aligned} \mathsf{Mod}(\mathbb{A}) &\simeq \mathsf{Hom}_{\mathsf{FP}}(\mathbb{A}, \mathsf{Set}), \\ \mathsf{Theory}(\mathcal{A}) &\simeq \mathsf{Hom}_{\mathsf{Alg}}(\mathcal{A}, \mathsf{Set}). \end{aligned}$$

The functors in (1.37) do *not* yet form a biequivalence, however, but only an adjunction. The syntax-semantics duality of (1.36) results from cutting down the syntax side AlgTh to just those theories in the image of the Theory functor, which can be described as those $\mathbb{A}$ that are closed under retracts. The unit of the adjunction

$$\eta : \mathbb{A} \to \mathsf{Theory}(\mathsf{Mod}(\mathbb{A}))$$

is then the closure of the finite limit category $\mathbb{A}$ under retracts—the so-called *Cauchy completion* of $\mathbb{A}$.

[1] These can be characterized as cocomplete categories $\mathcal{A}$ having a set G of objects such that every $A \in \mathcal{A}$ is a sifted colimit of objects $G \in \mathsf{G}$, for each of which $\mathsf{Hom}(G, -)$ preserves sifted colimits. See [ARV10].

Appendix A

Logic

A.1 Concrete and abstract syntax

By *syntax* we generally mean manipulation of finite strings of symbols according to given *grammatical rules*. For instance, the strings “$7)6 + /(8$” and “$(6 + 8)/7$” both consist of the same symbols but you will recognize one as junk and the other as *well formed* because you have (implicitly) applied the grammatical rules for arithmetical expressions.

Grammatical rules are usually quite complicated, as they need to prescribe associativity of operators (does “$5 + 6 + 7$” mean “$(5 + 6) + 7$” or “$5 + (6 + 7)$”?) and their precedence (does “$6 + 8/7$” mean “$(6 + 8)/7$” or “$6 + (8/7)$”?), the role of *white space* (empty space between symbols and line breaks), rules for nesting and balancing parentheses, etc. It is not our intention to dwell on such details, but rather to focus on the mathematical nature of well-formed expressions, namely that they represent inductively generated finite trees.[1] Under this view the string “$(6+8)/7$” is just a concrete representation of the tree depicted in Figure A.1.

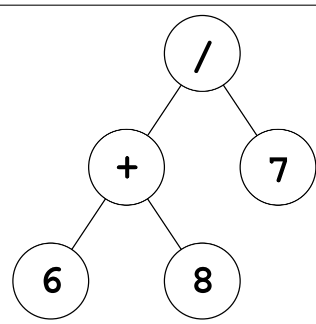

Figure A.1: The tree represented by $(6 + 8)/7$

Concrete representation of expressions as finite strings of symbols is called *concrete syntax*, while in *abstract syntax* we view expressions as finite trees. The passage from the

[1] We are limiting attention to the so-called *context-free* grammar, which are sufficient for our purposes. More complicated grammars are rarely used to describe formal languages in logic and computer science.

former to the latter is called *parsing* and is beyond the scope of this book. We will always specify only abstract syntax and assume that the corresponding concrete syntax follows the customary rules for parentheses, associativity and precedence of operators.

As an illustration we give rules for the (abstract) syntax of propositional calculus in *Backus-Naur* form:

$$\text{Propositional variable } p ::= \mathtt{p}_1 \mid \mathtt{p}_2 \mid \mathtt{p}_3 \mid \cdots$$
$$\text{Propositional formula } \phi ::= p \mid \bot \mid \top \mid \phi_1 \land \phi_2 \mid \phi_1 \lor \phi_2 \mid \phi_1 \Rightarrow \phi_2 \mid \neg\phi$$

The vertical bars should be read as "or". The first rule says that a propositional variable is the constant $\mathtt{p}_1$, or the constant $\mathtt{p}_2$, or the constant $\mathtt{p}_3$, etc.[2] The second rule tells us that there are seven inductive rules for building a propositional formula:

- a propositional variable is a formula,
- the constants $\bot$ and $\top$ are formulas,
- if ϕ_1, ϕ_2, and ϕ are formulas, then so are $\phi_1 \land \phi_2$, $\phi_1 \lor \phi_2$, $\phi_1 \Rightarrow \phi_2$, and $\neg\phi$.

Even though abstract syntax rules say nothing about parentheses or operator associativity and precedence, we shall rely on established conventions for mathematical notation and write down concrete representations of propositional formulas, e.g., $\mathtt{p}_4 \land (\mathtt{p}_1 \lor \neg \mathtt{p}_1) \land \mathtt{p}_4 \lor \mathtt{p}_2$.

A word of warning: operator associativity in syntax is not to be confused with the usual notion of associativity in mathematics. We say that an operator $\star$ is *left associative* when an expression $x \star y \star z$ represents the left-hand tree in Figure A.2, and *right associative* when it represents the right-hand tree. Thus the usual operation of subtraction $-$ is left

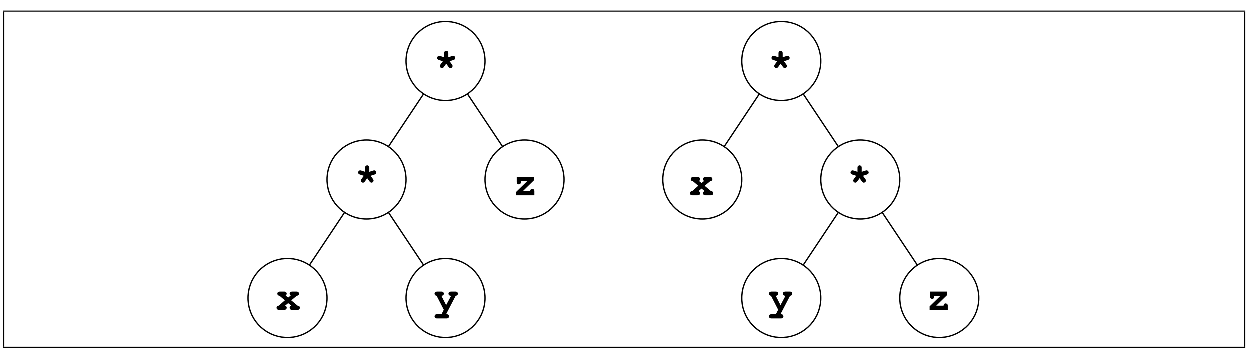

Figure A.2: Left and right associativity of $x \star y \star z$

associative, but is not associative in the usual mathematical sense.

[2] In an actual computer implementation we would allow arbitrary finite strings of letters as propositional variables. In logic we only care about the fact that we can never run out of fresh variables, i.e., that there are countably infinitely many of them.

A.2 Free and bound variables

Variables appearing in an expression may be *free* or *bound*. For example, in expressions

$$\int_0^1 \sin(a \cdot x)\, dx, \qquad x \mapsto ax^2 + bx + c, \qquad \forall\, x\,.\,(x < a \lor x > b)$$

the variables a, b and c are free, while x is bound by the integral operator $\int$, the function formation $\mapsto$, and the universal quantifier $\forall$, respectively. To be quite precise, it is an *occurrence* of a variable that is free or bound. For example, in expression $\phi(x) \lor \exists\, x\,.\, A\psi(x, x)$ the first occurrence of x is free and the remaining ones are bound.

In this book the following operators bind variables:

- quantifiers $\exists$ and $\forall$, cf. **??**,
- λ-abstraction, cf. **??**,
- search for others **??**.

When a variable is bound we may always rename it, provided the renaming does not confuse it with another variable. In the integral above we could rename x to y, but not to a because the binding operation would *capture* the free variable a to produce the unintended $\int_0^1 \sin(a^2)\, da$. Renaming of bound variables is called *α-renaming*.

We consider two expressions *equal* if they only differ in the names of bound variables, i.e., if one can be obtained from the other by α-renaming. Furthermore, we adhere to *Barendregt's variable convention [**?**, p. 2]*, which says that bound variables are always chosen so as to differ from free variables. Thus we would never write $\phi(x) \lor \exists\, x\,.\, A\psi(x, x)$ but rather $\phi(x) \lor \exists\, y\,.\, A\psi(y, y)$. By doing so we need not worry about capturing or otherwise confusing free and bound variables.

In logic we need to be more careful about variables than is customary in traditional mathematics. Specifically, we always specify which free variables may appear in an expression.[3] We write

$$x_1 : A_1, \ldots, x_n : A_n \mid t$$

to indicate that expression t may contain only free variables $x_1, \ldots, x_n$ of types $A_1, \ldots, A_n$. The list

$$x_1 : A_1, \ldots, x_n : A_n$$

is called a *context* in which t appears. To see why this is important consider the different meaning that the expression $x^2 + y^2 \leq 1$ recevieves in different contexts:

- $x : \mathbb{Z}, y : \mathbb{Z} \mid x^2 + y^2 \leq 1$ denotes the set of tuples $\{(-1, 0), (0, 1), (1, 0), (0, -1)\}$,
- $x : \mathbb{R}, y : \mathbb{R} \mid x^2 + y^2 \leq 1$ denotes the closed unit disc in the plane, and

[3]This is akin to one of the guiding principles of good programming language design, namely, that all variables should be *declared* before they are used.

- $x : \mathbb{R}, y : \mathbb{R}, z : \mathbb{R} \mid x^2 + y^2 \leq 1$ denotes the infinite cylinder in space whose base is the closed unit disc.

In single-sorted theories there is only one type or sort A. In this case we abbreviate a context by listing just the variables, $x_1, \ldots, x_n$.

A.3 Substitution

Substitution is a basic syntactic operation which replaces (free occurrences of) distinct variables $x_1, \ldots, x_n$ in an expression t with expressions $t_1, \ldots, t_n$, which is written as

$$t[t_1/x_1, \ldots, t_n/x_n].$$

We sometimes abbreviate this as $t[\vec{t}/\vec{x}]$ where $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{t} = (t_1, \ldots, t_n)$. Here are several examples:

$$\begin{aligned}(x^2 + x + y)[(2+3)/x] &= (2+3)^2 + (2+3) + y \\ (x^2 + y)[y/x, x/y] &= y^2 + x \\ \big(\forall x \,.\, \big(x^2 < y + x^3\big)\big)\,[x + y/y] &= \forall z \,.\, \big(z^2 < (x+y) + z^3\big)\,.\end{aligned}$$

Notice that in the third example we first renamed the bound variable x to z in order to avoid a capture by $\forall$.

Substitution is simple to explain in terms of trees. Assuming Barendregt's convention, the substitution $t[u/x]$ means that in the tree t we replace the leaves labeled x by copies of the tree u. Thus a substitution never changes the structure of the tree–it only "grows" new subtrees in places where the substituted variables occur as leaves.

Substitution satisfies the distributive law

$$(t[u/x])[v/y] = (t[v/y])[u[v/y]/x],$$

provided x and y are distinct variables. There is also a corresponding multivariate version which is written the same way with a slight abuse of vector notation:

$$(t[\vec{u}/\vec{x}])[\vec{v}/\vec{y}] = (t[\vec{v}/\vec{y}])[\vec{u}[\vec{v}/\vec{y}]/\vec{x}].$$

A.4 Judgments and deductive systems

A formal system, such as first-order logic or type theory, concerns itself with *judgments*. There are many kinds of judgments, such as:

- The most common judgments are equations and other logical statements. We distinguish a formula ϕ and the judgment "ϕ holds" by writing the latter as

$$\vdash \phi\,.$$

  The symbol $\vdash$ is generally used to indicate judgments.

- Typing judgments
$$\vdash t : A$$
expressing the fact that a term t has type A. This is not to be confused with the set-theoretic statement $t \in u$ which says that individuals t and u (of type "set") are in relation "element of" $\in$.

- Judgments expressing the fact that a certain entity is well formed. A typical example is a judgment
$$\vdash x_1 : A_1, \dots, x_n : A_n \quad \mathtt{ctx}$$
which states that $x_1 : A_1, \dots, x_n : A_n$ is a well-formed context. This means that $x_1, \dots, x_n$ are distinct variables and that $A_1, \dots, A_n$ are well-formed types. This kind of judgement is often omitted and it is tacitly assumed that whatever entities we deal with are in fact well-formed.

A *hypothetical judgement* has the form
$$H_1, \dots, H_n \vdash C$$
and means that hypotheses $H_1, \dots, H_n$ entail consequence C (with respect to a given decuctive system). We may also add a typing context to get a general form of judgment
$$x_1 : A_1, \dots, x_n : A_n \mid H_1, \dots, H_m \vdash C.$$
This should be read as: "if $x_1, \dots, x_n$ are variables of types $A_1, \dots, A_n$, respectively, then hypotheses $H_1, \dots, H_m$ entail conclusion C." For our purposes such contexts will suffice, but you should not be surprised to see other kinds of judgments in logic.

A *deductive system* is a set of inference rules for deriving judgments. A typical inference rule has the form
$$\frac{J_1 \quad J_2 \quad \cdots \quad J_n}{J}\, C$$
This means that we can infer judgment J if we have already derived judgments $J_1, \dots, J_n$, provided that the optional side-condition C is satisfied. An *axiom* is an inference rule of the form
$$\frac{}{J}$$
A *two-way rule*
$$\frac{\dfrac{J_1 \quad J_2 \quad \cdots \quad J_n}{}}{K_1 \quad K_2 \quad \cdots \quad K_m}$$
is a combination of $n+m$ inference rules stating that we may infer each K_i from $J_1, \dots, J_n$ and each J_i from $K_1, \dots, K_m$.

A *derivation* of a judgment J is a finite tree whose root is J, the nodes are inference rules, and the leaves are axioms. An example is presented in the next subsection.

The set of all judgments that hold in a given deductive system is generated inductively by starting with the axioms and applying inference rules.

A.5 Example: Equational reasoning

Equational reasoning is so straightforward that one almost doesn't notice it, consisting mainly, as it does, of "substituting equals for equals". The only judgements are equations between terms, $s = t$, which consist of function symbols, constants, and variables. The inference rules are just the usual ones making $s = t$ a congruence relation on the terms. More formally, we have the following specification of what may be called the *equational calculus.*

$$
\begin{aligned}
\text{Variable } v &::= x \mid y \mid z \mid \cdots \\
\text{Constant symbol } c &::= \mathtt{c}_1 \mid \mathtt{c}_2 \mid \cdots \\
\text{Function symbol } f^k &::= \mathtt{f}_1^{k_1} \mid \mathtt{f}_2^{k_2} \mid \cdots \\
\text{Term } t &::= v \mid c \mid f^k(t_1, \dots, t_k)
\end{aligned}
$$

The superscript on the function symbol f^k indicates the arity.

The equational calculus has just one form of judgement

$$x_1, \dots, x_n \mid t_1 = t_2$$

where $x_1, \dots, x_n$ is a *context* consisting of distinct variables, and the variables in the equation must occur among the ones listed in the context.

There are four inference rules for the equational calculus. They may be assumed to leave the contexts unchanged, which may therefore be omitted.

$$
\frac{}{t = t} \qquad \frac{t_1 = t_2}{t_2 = t_1} \qquad \frac{t_1 = t_2,\ t_2 = t_3}{t_1 = t_3} \qquad \frac{t_1 = t_2,\ t_3 = t_4}{t_1[t_3/x] = t_2[t_4/x]}
$$

An *equational theory* $\mathbb{T}$ consists of a set of constant and function symbols (with arities), and a set of equations, called *axioms.* We write

$$\mathbb{T} \vdash t_1 = t_2$$

to mean that the equation $t_1 = t_2$ has a derivation from the axioms of $\mathbb{T}$ using the equational calculus.

A.6 Example: Predicate calculus

We spell out the details of single-sorted predicate calculus and first-order theories. This is the most common deductive system taught in classical courses on logic.

The predicate calculus has the following syntax:

$$
\begin{aligned}
\text{Variable } v &::= x \mid y \mid z \mid \cdots \\
\text{Constant symbol } c &::= \mathtt{c}_1 \mid \mathtt{c}_2 \mid \cdots \\
\text{Function symbol}^4\ f^k &::= \mathtt{f}_1^{k_1} \mid \mathtt{f}_2^{k_2} \mid \cdots \\
\text{Term } t &::= v \mid c \mid f^k(t_1, \ldots, t_k) \\
\text{Relation symbol } R^m &::= \mathtt{R}_1^{m_1} \mid \mathtt{R}_2^{m_2} \mid \cdots \\
\text{Formula } \phi &::= \bot \mid \top \mid R^m(t_1, \ldots, t_m) \mid t_1 = t_2 \mid \\
&\qquad\qquad \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid \phi_1 \Rightarrow \phi_2 \mid \neg\phi \mid \forall x \,.\, \phi \mid \exists x \,.\, \phi.
\end{aligned}
$$

The variable x is bound in $\forall x \,.\, \phi$ and $\exists x \,.\, \phi$.

The predicate calculus has one form of judgement

$$x_1, \ldots, x_n \mid \phi_1, \ldots, \phi_m \vdash \phi$$

where $x_1, \ldots, x_n$ is a *context* consisting of distinct variables, $\phi_1, \ldots, \phi_m$ are *hypotheses* and ϕ is the *conclusion*. The free variables in the hypotheses and the conclusion must occur among the ones listed in the context. We abbreviate the context with Γ and Φ with hypotheses. Because most rules leave the context unchanged, we omit the context unless something interesting happens with it.

The following inference rules are given in the form of adjunctions. See Appendix **??** for the more usual formulation in terms of introduction an elimination rules.

$$\frac{}{\phi_1, \ldots, \phi_m \vdash \phi_i} \qquad \frac{}{\Phi \vdash \top} \qquad \frac{}{\Phi, \bot \vdash \phi}$$

$$\frac{\Phi \vdash \phi_1 \quad \Phi \vdash \phi_2}{\Phi \vdash \phi_1 \wedge \phi_2} \qquad \frac{\Phi, \phi_1 \vdash \psi \quad \Phi, \phi_2 \vdash \psi}{\Phi, \phi_1 \vee \phi_2 \vdash \psi} \qquad \frac{\Phi, \phi_1 \vdash \phi_2}{\Phi \vdash \phi_1 \Rightarrow \phi_2}$$

$$\frac{\Gamma, x, y \mid \Phi, x = y \vdash \phi}{\Gamma, x \mid \Phi \vdash \phi[x/y]} \qquad \frac{\Gamma, x \mid \Phi, \phi \vdash \psi}{\Gamma \mid \Phi, \exists x \,.\, \phi \vdash \psi} \qquad \frac{\Gamma, x \mid \Phi \vdash \phi}{\Gamma \mid \Phi \vdash \forall x \,.\, \phi}$$

The equality rule implicitly requires that y does not appear in Φ, and the quantifier rules implicitly require that x does not occur freely in Φ and ψ because the judgments below the lines are supposed to be well formed.

Negation $\neg\phi$ is defined to be $\phi \Rightarrow \bot$. To obtain *classical* logic we also need the law of excluded middle,

$$\frac{}{\Phi \vdash \phi \vee \neg\phi}$$

Comment on the fact that contraction and weakening are admissible.

Give an example of a derivation.

A *first-order theory* $\mathbb{T}$ consists of a set of constant, function and relation symbols with corresponding arities, and a set of formulas, called *axioms*.

Give examples of a first-order theories.

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