Categorical Logic

- Outline:

  1) Review of CT
  2) Propositional Calculus
  3) λ-Calculus

- Notes:

  awodey.github.io

1. Review of CT

Basic Def.s :

Category

$\mathcal{C}$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, bend right, "g \circ f"'] & B \arrow[loop right, "1_B"] \arrow[d, "g"] \\
 & C
\end{tikzcd}$$

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\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{C} \arrow[d, "F"'] \\
\mathbb{D}
\end{tikzcd}
$$

Functor F

$$
\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[r, shift right=2] \arrow[dr, bend right, "F(g \circ f) = Fg \circ Ff"'] & FB \arrow[loop right, "F1_B = 1_{FB}"] \arrow[d, "Fg"] \\
& FC
\end{tikzcd}
$$

$$
\begin{tikzcd}
D \arrow[r] & E
\end{tikzcd}
$$

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Functor

$$\begin{tikzcd} \mathbb{C} \arrow[d, bend right, "F"'{name=F}] \arrow[d, bend left, "G"{name=G}] \arrow[from=F, to=G, Rightarrow, "\vartheta"] \\ \mathbb{D} \end{tikzcd}$$

Natural Transformation

$$\left(\vartheta_C : FC \to GC\right)_{C \in \mathbb{C}}$$

$$\begin{tikzcd} FA \arrow[rr, "Ff"] \arrow[dr] \arrow[dd, "\vartheta_A"'] & & FB \arrow[loop right, "F1_B"] \arrow[dl, "Fg"] \arrow[dd, "\vartheta_B"] \\ & FC \arrow[dd, "\vartheta_C"] & \\ GA \arrow[rr] \arrow[dr] & & GB \arrow[dl, "Gg"] \\ & GC & \end{tikzcd}$$

$$\vartheta_C \circ Fg = Gg \circ \vartheta_B$$

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Functor

$$\begin{tikzcd} \mathbb{C} \arrow[d, bend right, "F"'] \arrow[d, bend left, "G"] \arrow[d, phantom, "\overset{\vartheta}{\Rightarrow}"] \\ \mathbb{D} \end{tikzcd}$$

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$$(\vartheta_C : FC \to GC)_{C \in \mathbb{C}}$$

$$\vartheta_C \circ Fg = Gg \circ \vartheta_B$$

Idea: "$\vartheta : F \to G$ maps the structure F to the structure G."

Basic Examples:

- **Set** = Sets & functions
- **Pos** = Posets & monotone maps
- **Top** = Spaces & cts. maps
- **Grp** = Groups & homomorphisms
- **Cat** = Categories & functors
- P a poset : objects: $p, q, \dots$
  arrows: $p \leq q$
- X a space : $\mathcal{C}(X) = \mathbf{Top}(X, \mathbb{R})$
  $f \leq g \; := \; fx \leq gx$ f. $x \in X$.
- $\mathbb{C}, \mathbb{D}$ cats : $\mathbb{D}^{\mathbb{C}} = \mathbf{Cat}(\mathbb{C}, \mathbb{D})$ functor cat
  objects: functors $F : \mathbb{C} \to \mathbb{D}$
  arrows: nat. transf. $\vartheta : F \Rightarrow G$
- $\mathbb{C}$ cat : $\hat{\mathbb{C}} := \mathbf{Set}^{\mathbb{C}^{op}}$ presheaves

$$\mathbb{C} \qquad \begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[ur, "f"] \arrow[r, "g \cdot f"'] & C \end{tikzcd} \qquad\qquad \mathbb{C}^{op} \qquad \begin{tikzcd} & B \arrow[dl, "\bar{f}"'] \\ A & C \arrow[u, "\bar{g}"'] \arrow[l, "\bar{f} \cdot \bar{g}"] \end{tikzcd}$$

.

Representable functor

$\mathrm{Set}^{\mathbb{C}} \ni \mathbb{C}(A,-) : \mathbb{C} \longrightarrow \mathrm{Set}$

$A \in \mathbb{C}$

$$B \longmapsto \mathbb{C}(A,B) = \mathrm{Hom}_{\mathbb{C}}(A,B) = \{ f : A \to B \}$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C}(A,B) \arrow[d, "{\mathbb{C}(A,g) := g\circ f}"] \\ \mathbb{C}(A,C) \end{tikzcd}$$

Contravariant version

$\mathrm{Set}^{\mathbb{C}^{op}} \ni \mathbb{C}(-,C) : \mathbb{C}^{op} \longrightarrow \mathrm{Set}$

$C \in \mathbb{C}$

$$B \longmapsto \mathbb{C}(B,C)$$

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "g\cdot f"] & \\ B \arrow[r, "g"'] & C \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C}(A,C) \\ \mathbb{C}(B,C) \arrow[u, "{\mathbb{C}(f,C) := g\cdot f}"'] \end{tikzcd}$$

The contravariant representable is written:

$$yC := \mathbb{C}(-,C) : \mathbb{C}^{op} \longrightarrow \mathrm{Set} .$$

Def. The Yoneda embedding is the functor

$$y : \mathbb{C} \longrightarrow \mathbf{Set}^{\mathbb{C}^{op}}$$
$$C \longmapsto \mathbb{C}(-,C)$$

with action on arrows in $\mathbb{C}$ given by composition:

$$\begin{tikzcd} C \arrow[d, "h"'] \\ D \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} yC \arrow[d, Rightarrow, "yh"] \\ yD \end{tikzcd},$$

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[dr, dashed, "h\cdot g"'] & C \arrow[d, "h"] \\ & D \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C}(B,C) \arrow[d, "\mathbb{C}(B,h) =: (yh)_B"] \\ \mathbb{C}(B,D) \end{tikzcd}.$$

Lemma (Yoneda) For $C \in \mathbb{C}$, $F \in \mathbf{Set}^{\mathbb{C}^{op}}$,

$$FC \cong \mathrm{Hom}(yC, F) \qquad \text{nat. in } C \,\&\, F.$$

So natural transf.s $\quad \alpha : yC \longrightarrow F$

correspond to elements $\quad a \in FC$.

Prop. The Yoneda embedding is full & faithful:

$$y : \mathbb{C}(C,D) \xrightarrow[\cong]{} \mathbf{Set}^{\mathbb{C}^{op}}(yC, yD)$$
$$C \xrightarrow[h]{} D \qquad\qquad yC \xrightarrow[yh]{} yD$$

Pf. $$\mathrm{Hom}(yC, yD) \underset{\text{Yoneda}}{\cong} (yD)C = \mathbb{C}(C,D).$$

Adjoints

Let $\mathbb{C}, \mathbb{D}$ cats.

Def. An <u>adjunction</u> consists of:

- functors
- a bijection

$$\mathbb{D}(FC, D) \cong \mathbb{C}(C, UD)$$

$$\begin{tikzcd} \mathbb{D} \arrow[d, bend left, "U"] \arrow[d, phantom, "\dashv"] \\ \mathbb{C} \arrow[u, bend left, "F"] \end{tikzcd} \qquad \frac{FC \longrightarrow D}{C \longrightarrow UD} \qquad \text{nat. in } C, D.$$

Examples:

1) Posets P, Q & monotone $\begin{tikzcd} P \arrow[r, bend left, "f"] & Q \arrow[l, bend left, "g"] \end{tikzcd}$ s.th. $f \dashv g$:

$$p \leq gq \quad \text{iff} \quad fp \leq q .$$

E.g. take $Q = P \times P$ and $P \xrightarrow{\Delta} P \times P$.
$\qquad\qquad\qquad\qquad\qquad\qquad\quad p \longmapsto (p, p)$

- What is $\Delta \dashv$?

$$(p, p) \leq (a, b) \iff p \leq a \;\&\; p \leq b$$
$$\iff p \leq a \wedge b .$$

$$\begin{tikzcd} P \times P \arrow[d, bend left, "\wedge"] \arrow[d, phantom, "\dashv"] \\ P \arrow[u, bend left, "\Delta"] \end{tikzcd}$$

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- that is

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Examples:

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- What about $? \dashv \Delta$

$$\begin{tikzcd} P \times P \arrow[d, bend left, "\wedge"] \\ P \arrow[u, "\Delta"] \arrow[u, bend left=60, "\vee"] \end{tikzcd}$$

$$\frac{\dfrac{(a,b) \leq (p,p) \quad P \times P}{a \leq p \ \& \ b \leq p \quad P}}{a \vee b \leq p \quad P}$$

2) Replace P by Set :

$$
\begin{tikzcd}
\mathrm{Set} \times \mathrm{Set} \arrow[d, bend left=40, "?"] \\
\mathrm{Set} \arrow[u, "\Delta"] \arrow[u, phantom, shift left=4, "\dashv"]
\end{tikzcd}
$$

$$
\frac{\dfrac{(A,A) \longrightarrow (B,C)}{\begin{tikzcd}[row sep=small] & B \\ A \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & C \end{tikzcd}}}{A \xrightarrow[(f,g)]{} B \times C}
$$

So $\Delta \dashv \times$.

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$$(A,A) \longrightarrow (B,C)$$
$$\overline{\overline{\begin{tikzcd} & B \\ A \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & C \end{tikzcd}}}$$
$$A \xrightarrow[(f,g)]{} B\times C$$

So $\Delta \dashv \times$.

and of course :

$$\begin{tikzcd} \mathbf{Set}\times\mathbf{Set} \arrow[d, bend left, "\times"] \arrow[d, bend right, "+"'] \\ \mathbf{Set} \arrow[u, "\Delta"] \arrow[u, phantom, "\dashv \quad \dashv"] \end{tikzcd}$$

$$\begin{tikzcd} B \arrow[d] \arrow[dr] & \\ B+C \arrow[r, dashed] & A \\ C \arrow[u] \arrow[ur] & \end{tikzcd}$$

BTW : An adjunction

$F \dashv U$

is always mediated by two distinguished maps :

$$\frac{FC \xrightarrow{1} FC}{C \xrightarrow[\eta]{} UFC} \quad \text{"unit"}$$

E.g. the unit of $+ \dashv \Delta$ is

$$B \xrightarrow[i_1]{} B+C \xleftarrow[i_2]{} C$$

$$\frac{UD \xrightarrow{1} UD}{FUD \xrightarrow[\varepsilon]{} D} \quad \text{"counit"}$$

3) We also have $\Delta \dashv \times$ in Pos, Cat, Grp, Top, ...

- In Pos, fix P and consider the functor

$$(-)\times P : \mathrm{Pos} \longrightarrow \mathrm{Pos} .$$

- This has a right adjoint:

$$\frac{A \times P \longrightarrow B}{A \longrightarrow B^P} \qquad B^P = \mathrm{Pos}(P, B)$$

$f \leq g :=$ $f_p \leq g_p \ \forall p$

- We also have such right adjoints

$$(-)\times A \dashv (-)^A$$ called exponentials

in Set, Cat, $\mathrm{Set}^{\mathbb{C}}$, ...

Def. A cat $\mathbb{C}$ is cartesian closed if there are right adjoints:

$$\begin{tikzcd} \mathbb{1} \arrow[d, bend left=60, "1"] \\ \mathbb{C} \arrow[u, "!"] \arrow[u, phantom, shift left=3, "\dashv"] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C}\times\mathbb{C} \arrow[d, bend left=60, "\times"] \\ \mathbb{C} \arrow[u, "\Delta"] \arrow[u, phantom, shift left=3, "\dashv"] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C} \arrow[d, bend left=60, "(-)^A"] \\ \mathbb{C} \arrow[u, "(-)\times A"] \arrow[u, phantom, shift left=3, "\dashv"] \end{tikzcd} .$$

$(A \in \mathbb{C})$

4) P poset has $\bigvee$ if for all $S \subseteq P$, there's $\bigvee S \in P$ s.th.

$$\bigvee S \leq p \iff s \leq p \text{ f.all } s \in S .$$

This says that $\bigvee \dashv \downarrow$:

$$\begin{tikzcd} \wp P \arrow[d, bend right=40, "\bigvee"'] \arrow[d, phantom, "\dashv"] \\ P \arrow[u, "\downarrow(-)"'] \end{tikzcd} \qquad \frac{S \subseteq \downarrow p = \{x \leq p\}}{\bigvee S \leq p}$$

Prop. The poset $\hat{P}$ of downsets,

$$\downarrow : P \to \hat{P} := \{ D \subseteq P \mid x \leq d \in D \Rightarrow x \in D \}$$

is the free completion of P under $\bigvee$:

$$\begin{tikzcd} \hat{P} \arrow[r, dashed, "\bar{f}"] & Q \\ P \arrow[u, "\downarrow(-)"] \arrow[ur, "f"'] & \end{tikzcd}$$

(i) $\hat{P}$ has $\bigvee$

(ii) for any Q w/ $\bigvee$ & any $f : P \to Q$ there's a unique $\bar{f}$ that preserves $\bigvee$ & $\bar{f}\downarrow p = fp$.

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(ii) for any Q w/ $\bigvee$ & any $f : P \to Q$ there's a unique $\bar{f}$ that preserves $\bigvee$ & $\bar{f}{\downarrow}p = fp$.

So:

$$\bigvee\text{-}\mathrm{Pos}(\hat{P}, Q) \cong \mathrm{Pos}(P, Q).$$

Prop. For any poset P, the poset $\hat{P}$ is cartesian closed & the map

$$\downarrow(-) : P \longrightarrow \hat{P}$$

preserves any cc structure in P.

Pf. $\top$ & $\wedge$ in $\hat{P}$ as in $\mathcal{P}P$.

$$A \Rightarrow B = \bigvee \{D \mid D \wedge A \subseteq B\}$$
$$= \{p \in P \mid \downarrow p \cap A \subseteq B\}.$$

For any downset C,

$$C \leq A \Rightarrow B \quad \text{iff} \quad \downarrow c \cap A \subseteq B \quad \text{f.a. } c \in C.$$

So:

$$C \cap A = \left(\bigcup_{c \in C} \downarrow c\right) \cap A = \bigcup_{c \in C} (\downarrow c \cap A) \subseteq B.$$

Then:

$$\downarrow \top = P$$
$$\downarrow (a \wedge b) = \downarrow a \wedge \downarrow b$$
$$\downarrow (a \Rightarrow b) = \downarrow a \Rightarrow \downarrow b.$$

Note: the map $\downarrow(-) : P \to \hat{P}$ reflects $\leq$:

$$\downarrow p \leq \downarrow q = p \in \downarrow q = p \leq q.$$

In particular $\downarrow$ is injective.