Axiom of Choice and Excluded Middle in Categorical Logic

Steven Awodey
The University of Chicago

Spring 1995

Abstract

The axiom of choice is shown to hold in the predicative logic of any locally cartesian closed category. A predicative form of excluded middle is then shown to be equivalent to the usual form of choice in topoi.

The logic of topoi is a version of higher-order, intuitionistic logic (see [3]). In this setting, Diaconescu [2] has shown that the axiom of choice (AC) entails the law of excluded middle (EM). This result sits well with a certain conception of logical truth, according to which AC is neither a principle of logic, nor even compatible with reasoning that eschews EM.

According to some other conceptions, however, AC is a logical principle and EM is not. Notable examples are the type theories of Tait [7], [8] and Martin-Löf [5], as is—informally—the logic underlying Bishop's constructive analysis [1] (as noted in [5]). Such systems of logic evidently cannot be modeled in topoi in the standard way. However, Seely [6] has shown how to model a range of such type theories in locally cartesian closed (LCC) categories (the source of this idea is Lawvere [4]). The type theories considered by Seely, which are closely related to those of Tait and Martin-Löf, will here be called *predicative*. In addition to elementary logic, they include higher-order quantification over functions between types, functions of functions, etc., but not over propositional functions (there is no type of propositions). They also have more liberal type-forming operations than conventional higher-order logic; e.g. such expressions as $\exists_{y\in(\forall_{x\in X}\phi(x))}\psi(y)$ may be well-formed. Details of the syntax of predicative type theories can be found in the literature just cited. The equivalence between predicative type theories and LCC categories established in [6] allows us to derive results concerning the former by working with the latter. Below (Theorem 1) a purely category theoretical proof of AC is given for LCC categories. Thus AC is a theorem in any predicative type theory. Theorem 2 also applies this method.

As an aside, Theorem 1 supports the view—advanced by Tait in [8]—that AC follows from a constructive interpretation of the logical constants, for predicative type theories have such a constructive character. For example, a sentence of the form $\exists_{x \in X} \phi(x)$ is provable only if there is a closed term α of type X such that $\phi(\alpha)$ is provable. Such proof-theoretic considerations underlie the "propositions-as-types" interpretation of these type theories (also known as the Curry-Howard isomorphism), according to which a proposition is the type of its proofs. For details, see Tait [7], [8].

We recall in outline the interpretation of predicative logic in LCC categories, assuming familiarity with basic category theory; details are in [6]. Let $\mathcal{T}$ be an LCC category. Thus $\mathcal{T}$ has a terminal object 1, and for every arrow $f : X \to Y$ in $\mathcal{T}$ the functor $\Sigma_f : \mathcal{T}/X \to \mathcal{T}/Y$ given by composition with f has a right adjoint $f^* : \mathcal{T}/Y \to \mathcal{T}/X$ (pullback along f), which itself has a right adjoint $\Pi_f : \mathcal{T}/X \to \mathcal{T}/Y$. Here $\mathcal{T}/Z$ denotes the "slice" (or "comma") category over the object Z of $\mathcal{T}$; the objects of $\mathcal{T}/Z$ are the arrows $D \to Z$ in $\mathcal{T}$ with codomain Z (for all objects D), and the arrows of $\mathcal{T}/Z$ are commutative triangles in $\mathcal{T}$,

$$\begin{tikzcd} D \arrow[rr] \arrow[dr] & & D' \arrow[dl] \\ & Z. & \end{tikzcd}$$

From the logical point of view, the objects of $\mathcal{T}$ are regarded simultaneously as propositions and as types. An arrow $f : X \to Y$ of $\mathcal{T}$ is regarded both as a proof of Y from the premise X, and as a term of type Y with a single free variable of type X. *Qua* proposition, an object Y is true in $\mathcal{T}$ iff it has a proof, i.e. an arrow $1 \to Y$, from the terminal object 1, which itself is regarded as a true proposition. A propositional function on Y *qua* type is then a proposition-valued function on Y, hence a Y-indexed family of objects of $\mathcal{T}$, hence an object of the slice category $\mathcal{T}/Y$. If $\psi(y)$ is such a propositional function on Y and $\alpha : 1 \to Y$ is a closed term, then the substitution $\psi(\alpha)$ of α for y in $\psi(y)$ is given by $\psi(\alpha) = \alpha^*(\psi(y))$ (the pullback of $\psi(y)$ along α), which is an object of $\mathcal{T}/1 \cong \mathcal{T}$ and hence a "proposition". More generally, if $\tau : X \to Y$ is any term of type Y, then $\psi(\tau) = \tau^*(\psi(y))$ is an object of $\mathcal{T}/X$, thus a propositional function on X. Let $\phi(x, y)$ be a propositional function on $X \times Y$ and $\pi : X \times Y \to Y$ the second projection; the quantifiers are interpreted by setting $\exists_{x \in X} \phi(x, y) = \Sigma_\pi(\phi(x, y))$ and $\forall_{x \in X} \phi(x, y) = \Pi_\pi(\phi(x, y))$. The adjointness conditions for Σ_π, π^*, and Π_π then become the two-way rules of inference:

$$\frac{\exists_{x\in X}\phi(x,y) \to \psi(y)}{\phi(x,y) \to \pi^*\psi(y)} \qquad\qquad \frac{\pi^*\psi(y) \to \phi(x,y)}{\psi(y) \to \forall_{x\in X}\phi(x,y)}$$

where the propositional function $\pi^*\psi(y) = \psi(\pi)$ on $X \times Y$ is just $\psi(y)$ with a dummy variable over X. Finally, for any object Z of $\mathcal{T}$, the slice $\mathcal{T}/Z$ has products and exponentials; then for any objects ϕ and ψ in $\mathcal{T}/Z$, let $\phi \wedge \psi = \phi \times \psi$ and $\phi \Rightarrow \psi = \psi^{\phi}$. The product/exponential adjunction becomes the two-way rule, for any objects ϕ, ψ, ϑ in $\mathcal{T}/Z$:

$$\frac{\phi \wedge \psi \to \vartheta}{\phi \to \psi \Rightarrow \vartheta}$$

Now consider

$$\text{(AC)} \qquad \forall_{x\in X}\exists_{y\in Y}\phi(x,y) \Rightarrow \exists_{f\in Y^X}\forall_{x\in X}\phi(x,f(x))$$

in the logic of an LCC category $\mathcal{T}$. Here ϕ is a propositional function on $X \times Y$, for objects X and Y of $\mathcal{T}$. Thus the schema AC holds in $\mathcal{T}$ iff, for any objects X, Y in $\mathcal{T}$ and ϕ in $\mathcal{T}/X \times Y$, there exists in $\mathcal{T}$ an arrow

$$1 \longrightarrow [\,\forall_{x\in X}\exists_{y\in Y}\phi(x,y) \Rightarrow \exists_{f\in Y^X}\forall_{x\in X}\phi(x,f(x))\,],$$

hence iff there exists at least one arrow

$$\forall_{x\in X}\exists_{y\in Y}\phi(x,y) \longrightarrow \exists_{f\in Y^X}\forall_{x\in X}\phi(x,f(x)).$$

In fact, something much stronger is true:

Theorem 1 *For any LCC category $\mathcal{T}$, and any objects X, Y in $\mathcal{T}$ and $\phi(x,y)$ in $\mathcal{T}/X \times Y$, there is an isomorphism:*

$$\forall_{x\in X}\exists_{y\in Y}\phi(x,y) \cong \exists_{f\in Y^X}\forall_{x\in X}\phi(x,f(x)).$$

Proof: Given $\phi = \phi(x,y)$ in $\mathcal{T}/X \times Y$, $\phi(x,f(x))$ in $\mathcal{T}/X$ is the pullback of ϕ along the (variable) graph $g :=< p, ev >: Y^X \times X \to X \times Y$, where p is the second projection and $ev : Y^X \times X \to Y$ is the canonical evaluation arrow. So (with obvious notation) we're showing

$$\forall_X \circ \exists_Y \cong \exists_{Y^X} \circ \forall_X \circ g^* : \mathcal{T}/X \times Y \longrightarrow \mathcal{T},$$

i.e. that the following diagram commutes up to isomorphism.

$$\begin{array}{ccc} & g^* & \\ \mathcal{T}/(X\times Y) & \longrightarrow & \mathcal{T}/(Y^X\times X) \\ \exists_Y \downarrow & & \downarrow \forall_X \\ \mathcal{T}/X & & \mathcal{T}/Y^X \\ \forall_X \searrow & & \swarrow \exists_{Y^X} \\ & \mathcal{T} & \end{array} \tag{1}$$

Take $\phi : D \to X\times Y$ in the upper left-hand corner of (1) . Then $\exists_Y.\phi = q\circ\phi$ where $q : X\times Y \to X$ is the first projection. So $\forall_X\exists_Y.\phi$ can be calculated as the outer pullback in the following diagram,

$$\begin{array}{ccc} \forall_X\exists_Y.\phi & \longrightarrow & D^X \\ \psi \downarrow & & \downarrow \phi^X \\ Z & \longrightarrow & (X\times Y)^X \\ & h & \\ !_Z \downarrow & & \downarrow q^X \\ 1 & \longrightarrow & X^X, \\ & \lambda_X.1_X & \end{array} \tag{2}$$

where ψ, h, and Z make the two squares pullbacks. But then

$$Z \cong \forall_X.(q : X\times Y \to X) \cong Y^X,$$

so $\forall_X\exists_Y.\phi \cong \exists_{Y^X}.\psi$. Furthermore, $h = \lambda_X.g$, i.e. the X-transpose of g. So $\psi \cong (\lambda_X.g)^*.\phi^X$, and we just need $(\lambda_X.g)^*.\phi^X \cong \forall_X \circ g^*.\phi$. Taking any $\xi : D' \to Y^X$ in $\mathcal{T}/Y^X$, there are successive adjunctions:

$$\begin{array}{lll} \xi \longrightarrow (\lambda_X.g)^*.\phi^X & \mathcal{T}/Y^X & \\ \hline \Sigma_{(\lambda_X.g)}.\xi \longrightarrow \phi^X & \mathcal{T}/(Y\times X)^X & \\ \hline & & \text{by transposition} \\ \Sigma_g.(\xi\times 1_X) \longrightarrow \phi & \mathcal{T}/(Y\times X) & \\ \hline \xi\times 1_X \longrightarrow g^*.\phi & \mathcal{T}/(Y^X\times X) & \\ \hline & & \pi : Y^X\times X \to Y^X \text{ projection} \\ \pi^*.\xi \longrightarrow g^*.\phi & \mathcal{T}/(Y^X\times X) & \\ \hline \xi \longrightarrow \forall_X g^*.\phi & \mathcal{T}/Y^X & \end{array}$$

So the proof is complete by the Yoneda lemma.

Since topoi are LCC categories, it may be asked how Theorem 1 relates to Diaconescu's result that choice entails excluded middle in topoi. We shall show that the usual form of choice for topoi, viz. epis split, is equivalent to a predicative form of excluded middle. To this end, we consider predicative type theories with negation and disjunction, such as [5] and [8]. Observe that for any LCC category $\mathcal{T}$, the Yoneda embedding $\mathcal{T} \to \mathcal{S}et^{\mathcal{T}^{\rm op}}$ preserves all of the LCC structure, and $\mathcal{S}et^{\mathcal{T}^{\rm op}}$ is a topos. Since the Yoneda embedding is full and faithful, one may restrict attention to models of predicative type theories in topoi and still obtain the complete semantics of [6] . Colimits in topoi can then be used to interpret negation and disjunction as follows.

Let $\mathcal{T}$ be a topos and X an object of $\mathcal{T}$. The slice $\mathcal{T}/X$ is then also a topos, so it has an initial object 0 and coproducts. For any objects ϕ, ψ in $\mathcal{T}/X$, put $\neg\phi = \phi \Rightarrow 0$ and $\phi \vee \psi = \phi + \psi$ (coproduct). For any ϑ in $\mathcal{T}/X$, there is a unique arrow $0 \to \vartheta$; so $\neg\vartheta$ is true in $\mathcal{T}/X$ iff $\vartheta \cong 0$. For disjunction one has, for any ϕ, ψ, ϑ in $\mathcal{T}/X$, the two-way rule:

$$\frac{\phi \to \vartheta,\ \ \psi \to \vartheta}{\phi \vee \psi \to \vartheta}$$

Like any contravariant exponential functor, $\neg : \mathcal{T}/X \to \mathcal{T}/X$ is self-adjoint on the right; so $\phi \Rightarrow \neg\neg\phi$ is always true. In general, $\neg\neg\phi \Rightarrow \phi$ is not, but "three nots is one" by adjointness. Now $\neg\phi$ is always open in $\mathcal{T}/X$, i.e. there is at most one arrow to $\neg\phi$ from any ψ in $\mathcal{T}/X$; so $\neg\phi$ is always a monomorphism into X. Since $\mathcal{T}$ is a topos, every ϕ in $\mathcal{T}/X$ has a support $\sigma.\phi = image(\phi)$ in $\mathcal{T}/X$, and on such subobjects the above defined negation agrees with the usual, topos-theoretic negation. Applying $\neg$ to the commutative triangle

$$\begin{tikzcd} \phi \arrow[rr] \arrow[dr] & & \neg\neg\phi \\ & \sigma.\phi \arrow[ur] & \end{tikzcd}$$

in $\mathcal{T}/X$ then shows $\neg\phi = \neg\sigma.\phi$. So $\neg\neg\phi$ is the $\neg\neg$-closure of the support of ϕ. Using this fact and the result of Diaconescu mentioned above, the proof of the following is by direct verification.

Theorem 2 *For any topos $\mathcal{T}$, the following are equivalent:*

(i) *For any object ϕ in any slice $\mathcal{T}/X$, $\neg\phi \vee \phi$ is true.*

(ii) *For any object ϕ in any slice $\mathcal{T}/X$, $\neg\neg\phi \Rightarrow \phi$ is true.*

(iii) *$\mathcal{T}$ has choice, i.e. every epimorphism in $\mathcal{T}$ splits.*

In a predicative type theory with negation and disjunction rules that can be modeled in topoi as indicated above, the laws of excluded middle and *duplex negatio affirmat* are thus equivalent to the usual, topos theoretic version of the axiom of choice.

References

[1] Bishop, E.: 1967, *Foundations of Constructive Analysis*, McGraw-Hill, New York.

[2] Diaconescu, R.: 1975, "Axiom of Choice and Complementation," *Proc. A.M.S.* 51, 175–8.

[3] Lambek, J. & Scott, P.: 1986, *Introduction to Higher-order Categorical Logic*, Cambridge University Press, Cambridge.

[4] Lawvere, F. W.: 1969, "Adjointness in Foundations," *Dialectica* 23, 281–96.

[5] Martin-Löf, P.: 1973, "An Intuitionistic Theory of Types: Predicative Part," *Logic Colloquium '73, Bristol*, ed. H. E. Rose & J. C. Sheperdson, North-Holland, Amsterdam, 73–118.

[6] Seely, R.A.G.: 1984, "Locally Cartesian Closed Categories and Type Theory," *Math. Proc. Camb. Phil. Soc.* 95, 33–48.

[7] Tait, W. W.: 1986, "Truth and Proof: The Platonism of Mathematics," *Synthese* 69, 341–70.

[8] Tait, W. W.: 1994, "The Law of Excluded Middle and the Axiom of Choice," *Mathematics and Mind*, ed. A. George, Oxford University Press, Oxford, 45–70.