# Introduction to Categorical Logic

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# Contents

# Chapter 2

# Propositional Logic

Propositional logic is the logic of propositional connectives like $p \wedge q$ and $p \Rightarrow q$. As was the case for algebraic theories, the general approach will be to determine suitable categorical structures to model the logical operations, and then use categories with such structure to represent (abstract) propositional theories. Adjoints will play a special role, as we will describe the basic logical operations as such. We again show that the semantics is "functorial", meaning that the models of a theory are functors that preserve the categorical structure. We will show that there are classifying categories for all propositional theories, as was the case for the algebraic theories that we have already met.

A more abstract, algebraic perspective will then relate the propositional case of syntax-semantics duality with classical Stone duality for Boolean algebras, and related results from lattice theory will provide an algebraic treatment of Kripke semantics for intuitionistic (and modal) propositional logic.

## 2.1 Propositional calculus

Before going into the details of the categorical approach, we first briefly review the propositional calculus from a conventional point of view, as we did for algebraic theories. We focus first on the *classical* propositional logic, before considering the intuitionistic case in Section **??**.

In the style of Section **??**, we have the following (abstract) syntax for (propositional) formulas:

$$
\begin{array}{rcl}
\text{Propositional variable } p & ::= & \mathsf{p}_1 \mid \mathsf{p}_2 \mid \mathsf{p}_3 \mid \cdots \\
\text{Propositional formula } \phi & ::= & p \mid \top \mid \bot \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid \phi_1 \Rightarrow \phi_2 \mid \phi_1 \Leftrightarrow \phi_2
\end{array}
$$

An example of a formula is therefore $\big(\mathsf{p}_3 \Leftrightarrow ((((\neg\mathsf{p}_1) \vee (\mathsf{p}_2 \wedge \bot)) \vee \mathsf{p}_1) \Rightarrow \mathsf{p}_3)\big)$. We will make use of the usual conventions for parenthesis, with binding order $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$. Thus e.g. the foregoing may also be written unambiguously as $\mathsf{p}_3 \Leftrightarrow \neg\mathsf{p}_1 \vee \mathsf{p}_2 \wedge \bot \vee \mathsf{p}_1 \Rightarrow \mathsf{p}_3$.

## Natural deduction

The system of *natural deduction* for propositional logic has one form of judgement

$$\mathtt{p}_1, \ldots, \mathtt{p}_n \mid \phi_1, \ldots, \phi_m \vdash \phi$$

where $\mathtt{p}_1, \ldots, \mathtt{p}_n$ is a *context* consisting of distinct propositional variables, the formulas $\phi_1, \ldots, \phi_m$ are the *hypotheses* and $\phi$ is the *conclusion*. The variables in the hypotheses and the conclusion must occur among those listed in the context. The hypotheses are regarded as a (finite) set; so they are unordered, have no repetitions, and may be empty. We may abbreviate the context of variables by $\Gamma$, and we often omit it.

*Deductive entailment* (or *derivability*) $\Phi \vdash \phi$ is thus a relation between finite sets of formulas $\Phi$ and single formulas $\phi$. It is defined as the smallest such relation satisfying the following rules:

1. Hypothesis:
$$\frac{}{\Phi \vdash \phi} \text{ if } \phi \text{ occurs in } \Phi$$

2. Truth:
$$\frac{}{\Phi \vdash \top}$$

3. Falsehood:
$$\frac{\Phi \vdash \bot}{\Phi \vdash \phi}$$

4. Conjunction:
$$\frac{\Phi \vdash \phi \qquad \Phi \vdash \psi}{\Phi \vdash \phi \wedge \psi} \qquad \frac{\Phi \vdash \phi \wedge \psi}{\Phi \vdash \phi} \qquad \frac{\Phi \vdash \phi \wedge \psi}{\Phi \vdash \psi}$$

5. Disjunction:
$$\frac{\Phi \vdash \phi}{\Phi \vdash \phi \vee \psi} \qquad \frac{\Phi \vdash \psi}{\Phi \vdash \phi \vee \psi} \qquad \frac{\Phi \vdash \phi \vee \psi \qquad \Phi, \phi \vdash \theta \qquad \Phi, \psi \vdash \theta}{\Phi \vdash \theta}$$

6. Implication:
$$\frac{\Phi, \phi \vdash \psi}{\Phi \vdash \phi \Rightarrow \psi} \qquad \frac{\Phi \vdash \phi \Rightarrow \psi \qquad \Phi \vdash \phi}{\Phi \vdash \psi}$$

For the purpose of deduction, we define $\neg\phi := \phi \Rightarrow \bot$ and $\phi \Leftrightarrow \psi := (\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi)$. To obtain *classical* logic we need only include one of the following additional rules.

7. Classical logic:
$$\frac{}{\Phi \vdash \phi \vee \neg\phi} \qquad \frac{\Phi \vdash \neg\neg\phi}{\Phi \vdash \phi}$$

A *proof* of a judgement $\Phi \vdash \phi$ is a *finite* tree built from the above inference rules whose root is $\Phi \vdash \phi$. For example, here is a proof of $\phi \vee \psi \vdash \psi \vee \phi$ using the disjunction rules:

$$\dfrac{\dfrac{}{\phi \vee \psi \vdash \phi \vee \psi} \qquad \dfrac{\dfrac{}{\phi \vee \psi, \phi \vdash \phi}}{\phi \vee \psi, \phi \vdash \psi \vee \phi} \qquad \dfrac{\dfrac{}{\phi \vee \psi, \psi \vdash \psi}}{\phi \vee \psi, \psi \vdash \psi \vee \phi}}{\phi \vee \psi \vdash \psi \vee \phi}$$

A judgment $\Phi \vdash \phi$ is *provable* if there exists a proof of it. Observe that every proof has at its leaves either the rule for $\top$ or an instance of the rule of hypothesis (or the Excluded Middle rule for classical logic).

**Remark 2.1.1.** An alternate form of presentation for proofs in natural deduction that is more, well, natural uses trees of formulas, rather than of judgements, with leaves labelled by *assumptions* $\vartheta$ that may also occur in *cancelled* form $[\vartheta]$. Thus for example the introduction and elimination rules for conjunction would be written in the form:

$$\dfrac{\begin{array}{c}\Phi\\ \vdots\\ \phi\end{array} \quad \begin{array}{c}\Phi\\ \vdots\\ \psi\end{array}}{\phi \wedge \psi} \qquad\qquad \dfrac{\begin{array}{c}\Phi\\ \vdots\\ \phi \wedge \psi\end{array}}{\phi} \qquad \dfrac{\begin{array}{c}\Phi\\ \vdots\\ \phi \wedge \psi\end{array}}{\psi}$$

An example of a proof tree with cancelled assumptions is the one for disjunction elimination:

$$\dfrac{\begin{array}{c}\Phi\\ \vdots\\ \phi \vee \psi\end{array} \qquad \begin{array}{c}\Phi, [\phi]\\ \vdots\\ \vartheta\end{array} \qquad \begin{array}{c}\Phi, [\psi]\\ \vdots\\ \vartheta\end{array}}{\vartheta}$$

And the above rule of implication introduction takes the form:

$$\dfrac{\begin{array}{c}\Phi, [\phi]\\ \vdots\\ \psi\end{array}}{\phi \Rightarrow \psi}$$

In these examples, the cancellation occurred at the last step. In order to continue such a proof, we need a device to indicate *when* a cancellation occurs, *i.e.* at which step of the proof. This can be done as follows:

$$\dfrac{\dfrac{\begin{array}{c}\Phi, [\alpha]^2\\ \vdots\\ \phi \vee \psi\end{array} \qquad \begin{array}{c}\Phi, [\phi]^1\\ \vdots\\ \vartheta\end{array} \qquad \begin{array}{c}\Phi, [\psi]^1\\ \vdots\\ \vartheta\end{array}}{\vartheta}\,(1)}{\alpha \Rightarrow \vartheta}\,(2)$$

This proof tree represents a derivation of the judgement $\Phi \vdash \alpha \Rightarrow \vartheta$. A proof tree in which all the assumptions have been cancelled represents a derivation of an unconditional judgement such as $\vdash \phi$.

We will have a better way to record such proofs in Section **??**.

**Exercise 2.1.2.** Derive each of the two classical rules (2.1), called *Excluded Middle* and *Double Negation*, from the other.

## 2.2 Truth values

The idea of an axiomatic system of deductive, logical reasoning goes to back to Frege, who gave the first such system for propositional calculus (and more) in his *Begriffsschrift* of 1879. The question soon arose whether Frege's rules (or rather, their derivable consequences — it was clear that one could chose the primitive basis in different but equivalent ways) were correct, and if so, whether they were *all* the correct ones. An ingenious solution was proposed by Russell's student Wittgenstein, who came up with an entirely different way of singling out a set of "valid" propositional formulas in terms of assignments of truth values to the variables occurring in them. He interpreted this as showing that logical validity was really a matter of the logical structure of a proposition, rather than depending any particular system of derivations. The same idea seems to have been had independently by Post, who proved that the valid propositional formulas coincide with the ones derivable in Whitehead and Russell's *Principia Mathematica* (which is propositionally equivalent to Frege's system), a fact that we now refer to as the *soundness* and *completeness* of propositional logic.

In more detail, let a *valuation* $v$ be an assignment of a "truth-value" $0, 1$ to each propositional variable, $v(\mathtt{p}_n) \in \{0, 1\}$. We can then extend the valuation to all propositional formulas $[\![\phi]\!]^v$ by the following recursion.

$$
\begin{aligned}
[\![\mathtt{p}_n]\!]^v &= v(\mathtt{p}_n) \\
[\![\top]\!]^v &= 1 \\
[\![\bot]\!]^v &= 0 \\
[\![\neg\phi]\!]^v &= 1 - [\![\phi]\!]^v \\
[\![\phi \wedge \psi]\!]^v &= \min([\![\phi]\!]^v, [\![\psi]\!]^v) \\
[\![\phi \vee \psi]\!]^v &= \max([\![\phi]\!]^v, [\![\psi]\!]^v) \\
[\![\phi \Rightarrow \psi]\!]^v &= 1 \text{ iff } [\![\phi]\!]^v \leq [\![\psi]\!]^v \\
[\![\phi \Leftrightarrow \psi]\!]^v &= 1 \text{ iff } [\![\phi]\!]^v = [\![\psi]\!]^v
\end{aligned}
$$

This is sometimes expressed using the "semantic consequence" notation $v \vDash \phi$ to mean that $[\![\phi]\!]^v = 1$. The above specification then takes the following form, in which the condition for the truth of a formula is given in terms of its informal "meaning":

$$
\begin{array}{rll}
v \vDash \top & \text{always} \\
v \vDash \bot & \text{never} \\
v \vDash \neg\phi & \text{iff} & \text{not } v \vDash \phi \\
v \vDash \phi \wedge \psi & \text{iff} & v \vDash \phi \text{ and } v \vDash \psi \\
v \vDash \phi \vee \psi & \text{iff} & v \vDash \phi \text{ or } v \vDash \psi \\
v \vDash \phi \Rightarrow \psi & \text{iff} & v \vDash \phi \text{ implies } v \vDash \psi \\
v \vDash \phi \Leftrightarrow \psi & \text{iff} & v \vDash \phi \text{ iff } v \vDash \psi
\end{array}
$$

Finally, $\phi$ is *valid*, written $\vDash \phi$, is defined by,

$$
\begin{array}{rll}
\vDash \phi & \text{iff} & v \vDash \phi \text{ for all } v \\
 & \text{iff} & [\![\phi]\!]^v = 1 \text{ for all } v\,.
\end{array}
$$

And, more generally, we define $\phi_1, ..., \phi_n$ *semantically entails* $\phi$, written

$$\phi_1, ..., \phi_n \vDash \ \phi, \tag{2.1}$$

to mean that for all valuations $v$ such that $v \vDash \phi_k$ for all $k$, also $v \vDash \phi$.

Given a formula in context $\Gamma \mid \phi$ and a valuation $v$ for the variables in $\Gamma$, one can check whether $v \vDash \phi$ using a *truth table*, which is a systematic way of calculating the value of $[\![\phi]\!]^v$. For example, under the assignment $v(\mathtt{p}_1) = 1, v(\mathtt{p}_2) = 0, v(\mathtt{p}_3) = 1$ we can calculate $[\![\phi]\!]^v$ for $\phi = \big(\mathtt{p}_3 \Leftrightarrow ((((\neg\mathtt{p}_1) \vee (\mathtt{p}_2 \wedge \bot)) \vee \mathtt{p}_1) \Rightarrow \mathtt{p}_3)\big)$ as follows.

| $\mathtt{p}_1$ | $\mathtt{p}_2$ | $\mathtt{p}_3$ | $\mathtt{p}_3$ | $\Leftrightarrow$ | $\neg$ | $\mathtt{p}_1$ | $\vee$ | $\mathtt{p}_2$ | $\wedge$ | $\bot$ | $\vee$ | $\mathtt{p}_1$ | $\Rightarrow$ | $\mathtt{p}_3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |

The value of the formula $\phi$ under the valuation $v$ is then the value in the column under the main connective, in this case $\Leftrightarrow$, and thus $[\![\phi]\!]^v = 1$.

Displaying all $2^3$ valuations for the context $\Gamma = \mathtt{p}_1, \mathtt{p}_2, \mathtt{p}_3$, therefore results in a table that checks for validity of $\phi$,

| $\mathtt{p}_1$ | $\mathtt{p}_2$ | $\mathtt{p}_3$ | $\mathtt{p}_3$ | $\Leftrightarrow$ | $\neg$ | $\mathtt{p}_1$ | $\vee$ | $\mathtt{p}_2$ | $\wedge$ | $\bot$ | $\vee$ | $\mathtt{p}_1$ | $\Rightarrow$ | $\mathtt{p}_3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | . | 1 | ... | | | | | | | | | |
| 1 | 1 | 0 | . | 1 | | ... | | | | | | | | |
| 1 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | . | 1 | | | | ... | | | | | | |
| 0 | 1 | 1 | . | 1 | | | | | ... | | | | | |
| 0 | 1 | 0 | . | 1 | | | | | | ... | | | | |
| 0 | 0 | 1 | . | 1 | | | | | | | ... | | | |
| 0 | 0 | 0 | . | 1 | | | | | | | | ... | | |

In this case, working out the other rows shows that $\phi$ is indeed valid, thus $\vDash \phi$.

**Theorem 2.2.1** (Soundness and Completeness of Propositional Calculus)**.** *Let $\Phi$ be any set of formulas and $\phi$ any formula, then*

$$\Phi \vdash \phi \iff \Phi \vDash \phi.$$

*In particular, for any propositional formula $\phi$ we have*

$$\vdash \phi \iff \vDash \phi.$$

*Thus derivability and validity coincide.*

*Proof.* Let us sketch the usual proof, for later reference.

(*Soundness*:) First assume $\Phi \vdash \phi$ is provable, meaning there is a finite derivation of $\Phi \vdash \phi$ by the rules of inference. We show by induction on the set of derivations that $\Phi \vDash \phi$, meaning that for any valuation $v$ such that $v \vDash \Phi$ also $v \vDash \phi$. For this, observe that in each individual rule of inference, if $\Psi \vDash \psi$ for all the premisses of the rule, then $\Phi \vDash \phi$ for the conclusion (the set of premisses may change from the premisses to the conclusion if the rule involves a cancellation).

(*Competeness*:) Suppose that $\Phi \nvdash \phi$, then $\Phi, \neg\phi \nvdash \bot$ (using double negation elimination). By Lemma 2.2.2 below, there is a valuation $v$ such that $v \vDash \{\Phi, \neg\phi\}$. Thus in particular $v \vDash \Phi$ and $v \nvDash \phi$, therefore $\Phi \nvDash \phi$. □

The key lemma is this:

**Lemma 2.2.2** (Model Existence)**.** *If a set $\Phi$ of formulas is* consistent*, in the sense that $\Phi \nvdash \bot$, then it has a* model*, i.e. a valuation $v$ such that $v \vDash \Phi$.*

*Proof.* Let $\Phi$ be any consistent set of formulas. We extend $\Phi \subseteq \Psi$ to one that is *maximally consistent*, meaning $\Psi$ is consistent, and if $\Psi \subseteq \Psi'$ and $\Psi'$ is consistent, then $\Psi = \Psi'$. Enumerate the formulas $\phi_0, \phi_1, ...,$ and let,

$$\begin{aligned} \Phi_0 &= \Phi, \\ \Phi_{n+1} &= \Phi_n \cup \phi_n \text{ if consistent, else } \Phi_n, \\ \Psi &= \textstyle\bigcup_n \Phi_n. \end{aligned}$$

One can then show that $\Psi$ is indeed maximally consistent, and for every formula $\psi$, either $\psi \in \Psi$ or $\neg\psi \in \Psi$ and not both (exercise!). Now for each propositional variable $\mathtt{p}$, define $v_\Psi(\mathtt{p}) = 1$ just if $\mathtt{p} \in \Psi$. Finally, one shows that $[\![\phi]\!]^{v_\Psi} = 1$ just if $\phi \in \Psi$, and therefore $v_\Psi \vDash \Psi \supseteq \Phi$. □

**Exercise 2.2.3.** Show that for any maximally consistent set $\Psi$ of formulas, either $\psi \in \Psi$ or $\neg\psi \in \Psi$ and not both. Conclude from this that for the valuation $v_\Psi$ defined by $v_\Psi(\mathtt{p}) = 1$ just if $\mathtt{p} \in \Psi$, we indeed have $[\![\phi]\!]^{v_\Psi} = 1$ just if $\phi \in \Psi$, as claimed in the proof of the Model Existence Lemma 2.2.2.

## 2.3 Boolean algebra

There is of course another apporach to propositional logic, which also goes back to the 19th century, namely that of Boolean algebra, which draws on the analogy between the propositional operations and the arithmetical ones.

**Definition 2.3.1.** A *Boolean algebra* is a set $B$ equipped with the operations:

$$\begin{aligned} 0, 1 : 1 &\to B \\ \neg : B &\to B \\ \wedge, \vee : B \times B &\to B \end{aligned}$$

satisfying the following equations:

$$\begin{aligned} x \vee x &= x & x \wedge x &= x \\ x \vee y &= y \vee x & x \wedge y &= y \wedge x \\ x \vee (y \vee z) &= (x \vee y) \vee z & x \wedge (y \wedge z) &= (x \wedge y) \wedge z \\ x \wedge (y \vee z) &= (x \wedge y) \vee (x \wedge z) & x \vee (y \wedge z) &= (x \vee y) \wedge (x \vee z) \\ 0 \vee x &= x & 1 \wedge x &= x \\ 1 \vee x &= 1 & 0 \wedge x &= 0 \\ \neg(x \vee y) &= \neg x \wedge \neg y & \neg(x \wedge y) &= \neg x \vee \neg y \\ x \vee \neg x &= 1 & x \wedge \neg x &= 0 \end{aligned}$$

This is of course an algebraic theory, like those considered in the previous chapter. Familiar examples of Boolean algebras are $2 = \{0, 1\}$, with the usual operations, and more generally, any powerset $\mathcal{P}X$, with the set-theoretic operations $A \vee B = A \cup B$, etc. (indeed, $2 = \mathcal{P}1$ is a special case.).

**Exercise 2.3.2.** Show that the free Boolean algebra $B(n)$ on $n$-many generators is the double powerset $\mathcal{P}\mathcal{P}(n)$, and determine the free functor on finite sets.

One can use equational reasoning in Boolean algebra as an alternative to the deductive propositional calculus as follows. For a propositional formula in context $\Gamma \mid \phi$, let us say that $\phi$ is *equationally provable* if we can prove $\phi = 1$ by equational reasoning (Section **??**), from the laws of Boolean algebras above. More generally, for a set of formulas $\Phi$ and a formula $\psi$ let us define the (*ad hoc*) relation of *equational provability*,

$$\Phi \vdash_{\mathsf{eq}} \psi \tag{2.2}$$

to mean that $\psi = 1$ can be proven equationally from (the Boolean equations and) the set of all equations $\phi = 1$, for $\phi \in \Phi$. Since we don't have any laws for the connectives $\Rightarrow$ or $\Leftrightarrow$, let us replace them with their Boolean equivalents, by adding the equations:

$$\begin{aligned} \phi \Rightarrow \psi &= \neg\phi \vee \psi \,, \\ \phi \Leftrightarrow \psi &= (\neg\phi \vee \psi) \wedge (\neg\psi \vee \phi) \,. \end{aligned}$$

Here for example is an equational proof of $(\phi \Rightarrow \psi) \vee (\psi \Rightarrow \phi)$.

$$\begin{aligned}(\phi \Rightarrow \psi) \vee (\psi \Rightarrow \phi) &= (\neg\phi \vee \psi) \vee (\neg\psi \vee \phi) \\ &= \neg\phi \vee (\psi \vee (\neg\psi \vee \phi)) \\ &= \neg\phi \vee ((\psi \vee \neg\psi) \vee \phi) \\ &= \neg\phi \vee (1 \vee \phi) \\ &= \neg\phi \vee 1 \\ &= 1 \vee \neg\phi \\ &= 1\end{aligned}$$

Thus we have

$$\vdash_{\mathsf{eq}} (\phi \Rightarrow \psi) \vee (\psi \Rightarrow \phi)\,.$$

We now ask: *How is equational provability* $\Phi \vdash_{\mathsf{eq}} \phi$ *related to deductive entailment* $\Phi \vdash \phi$ *and semantic entailment* $\Phi \vDash \phi$*?*

**Exercise 2.3.3.** Using equational reasoning, show that every propositional formula $\phi$ has both a *conjunctive* $\phi^\wedge$ and a *disjunctive* $\phi^\vee$ *Boolean normal form* such that:

1. The formula $\phi^\vee$ is an $n$-fold disjunction of $m$-fold conjunctions of *positive* $\mathtt{p_i}$ or *negative* $\neg\mathtt{p_j}$ propositional variables,

$$\phi^\vee \;=\; (\mathsf{q}_{11} \wedge \ldots \wedge \mathsf{q}_{1m_1}) \vee \ldots \vee (\mathsf{q}_{n1} \wedge \ldots \wedge \mathsf{q}_{nm_n})\,, \qquad \mathsf{q}_{ij} \in \{\mathsf{p}_{ij}, \neg\mathsf{p}_{ij}\}\,,$$

   and $\phi^\wedge$ is the same, but with the roles of $\vee$ and $\wedge$ reversed.

2. Both

$$\vdash_{\mathsf{eq}} \phi \Leftrightarrow \phi^\vee \qquad \text{and} \qquad \vdash_{\mathsf{eq}} \phi \Leftrightarrow \phi^\wedge\,.$$

(*Hint:* Rewrite the formula in terms of just conjunction, disjunction, and negation, and then do both normal forms at the same time, by structural induction on the formula.)

**Remark 2.3.4.** We can already use Exercise 2.3.3 to show that equational provability is equivalent to semantic validity,

$$\vdash_{\mathsf{eq}} \phi \iff \vDash \phi\,.$$

To show this, we first put the formula $\phi$ into conjunctive normal form, and then read off a truth valuation that falsifies it, just if there is one. Indeed, the CNF is valued as 1 just if each conjunct is, and that holds just if each conjunct contains a propositional letter $\mathtt{p}$ in both positive and negative $\neg\mathtt{p}$ form. And in that case, the CNF clearly reduces to 1 by an equational calculation. Conversely, if the CNF does not so reduce, it must have a conjunct that does not satisfy the condition just stated – and so we can read off a valuation making all propositional letters in that conjunct 0.

**Exercise 2.3.5.** A Boolean algebra can be partially ordered by defining $x \leq y$ as

$$x \leq y \iff x \vee y = y \qquad \text{or equivalently} \qquad x \leq y \iff x \wedge y = x\,.$$

Thus a Boolean algebra is a (poset) category. Show that as a category, a Boolean algebra has all finite limits and colimits and is cartesian closed, with $x \Rightarrow y := \neg x \vee y$ as the exponential of $x$ and $y$. Moreover, a finitely complete and cocomplete cartesian closed poset is a Boolean algebra just if it satisfies $x = (x \Rightarrow 0) \Rightarrow 0$. Finally, show that homomorphisms of Boolean algebras $f : B \to B'$ are the same thing as functors (i.e. monotone maps) that preserve all finite limits and colimits.

## 2.4 Lawvere duality for Boolean algebras

Let us apply the machinery of algebraic theories from Chapter **??** to the algebraic theory of Boolean algebras and see what we get. The algebraic theory $\mathbb{B}$ of Boolean algebras is a finite product (FP) category with objects $1, B, B^2, ...$, containing a Boolean algebra $\mathsf{U}_{\mathbb{B}}$, with underlying object $|\mathsf{U}_{\mathbb{B}}| = B$. By Theorem **??**, $\mathbb{B}$ has the universal property that finite product preserving (FP) functors from $\mathbb{B}$ into any FP-category $\mathcal{C}$ correspond (pseudo-)naturally to Boolean algebras in $\mathcal{C}$,

$$\mathsf{Hom}_{\mathsf{FP}}(\mathbb{B}, \mathcal{C}) \simeq \mathsf{BA}(\mathcal{C})\,. \tag{2.3}$$

The correspondence is mediated by evaluating an FP functor $F : \mathbb{B} \to \mathcal{C}$ at (the underlying structure of) the Boolean algebra $\mathsf{U}_{\mathbb{B}}$ to get a Boolean algebra $F(\mathsf{U}_{\mathbb{B}})$ in $\mathcal{C}$:

$$\begin{array}{cc} F : \mathbb{B} \longrightarrow \mathcal{C} & \mathsf{FP} \\ \hline\hline F(\mathsf{U}_{\mathbb{B}}) & \mathsf{BA}(\mathcal{C}) \end{array}$$

We call $\mathsf{U}_{\mathbb{B}}$ the *universal Boolean algebra*. Given a Boolean algebra $\mathsf{B}$ in $\mathcal{C}$, we write

$$\mathsf{B}^{\sharp} : \mathbb{B} \longrightarrow \mathcal{C}$$

for the associated *classifying functor*. By the equivalence of categories (2.3), we have isos,

$$\mathsf{B}^{\sharp}(\mathsf{U}_{\mathbb{B}}) \cong \mathsf{B}, \qquad F(\mathsf{B})^{\sharp} \cong F\,.$$

And in particular, $\mathsf{B}^{\sharp} \cong 1_{\mathbb{B}} : \mathbb{B} \to \mathbb{B}$.

By (the logical form of) Lawvere duality, Corollary **??**, we know that $\mathbb{B}^{\mathsf{op}}$ can be identified with a full subcategory $\mathsf{mod}(\mathbb{B})$ of $\mathbb{B}$-models in $\mathsf{Set}$ (i.e. Boolean algebras),

$$\mathbb{B}^{\mathsf{op}} = \mathsf{mod}(\mathbb{B}) \hookrightarrow \mathsf{Mod}(\mathbb{B}) = \mathsf{BA}(\mathsf{Set})\,, \tag{2.4}$$

namely, that consisting of the finitely generated free Boolean algebras $F(n) = PP([n])$ for $[n]$ an $n$-element set. Composing (2.4) and (2.3), we have an embedding of $\mathbb{B}^{\mathsf{op}}$ into the functor category,

$$\mathbb{B}^{\mathsf{op}} \hookrightarrow \mathsf{BA}(\mathsf{Set}) \simeq \mathsf{Hom}_{\mathsf{FP}}(\mathbb{B}, \mathsf{Set}) \hookrightarrow \mathsf{Set}^{\mathbb{B}}\,, \tag{2.5}$$

which, up to isomorphism, is just the (contravariant) Yoneda embedding, taking $B^n \in \mathbb{B}$ to the covariant representable functor $\mathsf{y}^{\mathbb{B}}(B^n) = \mathsf{Hom}_{\mathbb{B}}(B^n, -)$ (cf. Theorem **??**).

Now let us consider provability of equations between terms $\phi : B^n \to B$ in the theory $\mathbb{B}$, which are essentially the same as propositional formulas in context $(\mathsf{p}_1, ..., \mathsf{p}_n \mid \phi)$ modulo $\mathbb{B}$-provable equality. The universal Boolean algebra $\mathsf{U}_{\mathbb{B}}$ is logically generic, in the sense that for any such formulas $\phi, \psi$, we have $\mathsf{U}_{\mathbb{B}} \vDash \phi = \psi$ just if $\mathbb{B} \vdash \phi = \psi$ (Proposition **??**). The latter condition is equational provability from the axioms for Boolean algebras, which was used in the definition of $\vdash_{\mathsf{eq}} \phi$ (cf. 2.2). So we have:

$$\vdash_{\mathsf{eq}} \phi \iff \mathbb{B} \vdash \phi = 1 \iff \mathsf{U}_{\mathbb{B}} \vDash \phi = 1 \,.$$

As we showed in Proposition **??**, the image of the universal model $\mathsf{U}_{\mathbb{B}}$ under the (FP) *covariant* Yoneda embedding,

$$\mathsf{y}_{\mathbb{B}} : \mathbb{B} \to \mathsf{Set}^{\mathbb{B}^{\mathsf{op}}}$$

is also a logically generic model, with underlying object $|\mathsf{y}_{\mathbb{B}}(\mathsf{U}_{\mathbb{B}})| = \mathsf{Hom}_{\mathbb{B}}(-, B)$. By Proposition **??** we can use that fact to restrict attention to Boolean algebras in $\mathsf{Set}$, and in particular, to the finitely generated free ones $F(n)$, when testing for equational provability. Specifically, using the (FP) evaluation functors $\mathsf{eval}_{B^n} : \mathsf{Set}^{\mathbb{B}^{\mathsf{op}}} \to \mathsf{Set}$ for all objects $B^n \in \mathbb{B}$, we can continue the above reasoning as follows:

$$\begin{aligned}
\vdash_{\mathsf{eq}} \phi &\iff \mathbb{B} \vdash \phi = 1 \\
&\iff \mathsf{U}_{\mathbb{B}} \vDash \phi = 1 \\
&\iff \mathsf{y}_{\mathbb{B}}(\mathsf{U}_{\mathbb{B}}) \vDash \phi = 1 \\
&\iff \mathsf{eval}_{B^n}\mathsf{y}_{\mathbb{B}}(\mathsf{U}_{\mathbb{B}}) \vDash \phi = 1 \quad \text{for all } B^n \in \mathbb{B} \\
&\iff F(n) \vDash \phi = 1 \quad \text{for all } n.
\end{aligned}$$

The last step holds because the image of $\mathsf{y}_{\mathbb{B}}(\mathsf{U}_{\mathbb{B}})$ under $\mathsf{eval}_{B^n}$ is exactly the free Boolean algebra $\mathsf{eval}_{B^n}\mathsf{y}_{\mathbb{B}}(\mathsf{U}_{\mathbb{B}}) = F(n)$ (cf. Exercise **??**). Indeed, for the underlying objects we have

$$\mathsf{eval}_{B^n}\mathsf{y}_{\mathbb{B}}(\mathsf{U}_{\mathbb{B}}) \cong \mathsf{Hom}_{\mathbb{B}}(B^n, B) \cong \mathsf{Hom}_{\mathsf{BA}^{\mathsf{op}}}(F(n), F(1)) \cong \mathsf{Hom}_{\mathsf{BA}}(F(1), F(n)) \cong |F(n)| \,.$$

Thus to test for equational provability it suffices to check the equations in the free algebras $F(n)$ (which makes sense, since $F(n)$ is usually *defined* in terms of equational provability). We have therefore shown:

**Lemma 2.4.1.** *A formula in context* $\mathsf{p}_1, ..., \mathsf{p}_k \mid \phi$ *is equationally provable* $\vdash_{\mathsf{eq}} \phi$ *just in case it holds in every finitely generated free Boolean algebra* $F(n)$*,* i.e. $F(n) \vDash \phi = 1$.

Recall that the condition $F(n) \vDash \phi = 1$ means that the equation $\phi = 1$ holds *generally* in $F(n)$, i.e. for any elements $f_1, ..., f_k \in F(n)$, we have $\phi[f_1/\mathsf{p}_1, ..., f_k/\mathsf{p}_k] = 1$, where the expression $\phi[f_1/\mathsf{p}_1, ..., f_k/\mathsf{p}_k]$ denotes the element of $F(n)$ resulting from interpreting the propositional variables $\mathsf{p}_i$ as the elements $f_i$ and evaluating the resulting expression using the Boolean operations of $F(n)$. But now observe that the recipe:

for any elements $f_1, ..., f_k \in F(n)$, let the expression

$$\phi[f_1/\mathtt{p}_1, ..., f_k/\mathtt{p}_k] \tag{2.6}$$

denote the element of $F(n)$ resulting from interpreting the propositional variables $\mathtt{p}_i$ as the elements $f_i$ and evaluating the resulting expression using the Boolean operations of $F(n)$

just describes the unique Boolean homomorphism

$$F(1) \xrightarrow{\overline{\phi}} F(k) \xrightarrow{\overline{(f_1, ..., f_k)}} F(n)\,,$$

where $\overline{(f_1, ..., f_k)} : F(k) \to F(n)$ is determined by the elements $f_1, ..., f_k \in F(n)$, and $\overline{\phi} : F(1) \to F(k)$ by the corresponding element $(\mathtt{p}_1, ..., \mathtt{p}_k \mid \phi) \in F(k)$. It is therefore equivalent to check the case $k = n$ and $f_i = \mathtt{p}_i$, i.e. the "universal case"

$$(\mathtt{p}_1, ..., \mathtt{p}_k \mid \phi) = 1 \quad \text{in } F(k)\,. \tag{2.7}$$

Finally, then, we have:

**Proposition 2.4.2** (Completeness of the equational propositional calculus)**.** *Equational propositional calculus is sound and complete with respect to* boolean-valued models *in* $\mathsf{Set}$*, in the sense that a propositional formula* $\phi$ *is equationally provable from the laws of Boolean algebra,*

$$\vdash_{\mathsf{eq}} \phi\,,$$

*just if it holds generally in any Boolean algebra (in* $\mathsf{Set}$*).*

*Proof.* By "holding generally" is meant that it holds for all elements of the Boolean algebra $\mathsf{B}$, in the sense stated after the Lemma. But, as above, this is equivalent to the condition that for all $b_1, ..., b_k \in \mathsf{B}$, for $\overline{(b_1, ..., b_k)} : F(k) \to \mathsf{B}$ we have $\overline{(b_1, ..., b_k)}(\phi) = 1$ in $\mathsf{B}$, which in turn is clearly equivalent to the previously determined "universal" condition (2.7) that $\phi = 1$ in $F(k)$. □

We leave the analogous statement for equational entailment $\Phi \vdash_{\mathsf{eq}} \phi$ as an exercise.

**Corollary 2.4.3.** *Show that a propositional formula* $\mathtt{p}_1, ..., \mathtt{p}_k \mid \phi$ *is equationally provable* $\vdash_{\mathsf{eq}} \phi$*, just if it holds in the free Boolean algebra* $F(\omega)$ *on countably many generators* $\omega = \{\mathtt{p}_1, \mathtt{p}_2, ...\}$ *(with the variables* $\mathtt{p}_1, ..., \mathtt{p}_k$ *interpreted as the corresponding generators of* $F(\omega)$*).*

**Exercise 2.4.4.** Prove this as an easy corollary of Proposition 2.4.2.

Let us summarize what we know so far. By Exercise **??**, we already knew that equational provability in Boolean algebra is equivalent to semantic validity,

$$\vdash_{\mathsf{eq}} \phi \iff \vDash \phi\,.$$

This was based on a certain *decision procedure* for validity in classical propositional logic, originally due to Bernays [**?**], restated in terms of Boolean algebra. Then by Proposition 2.4.2 we know that it is also equivalent to what may be called *Boolean-valued validity*,

$$\vdash_{\mathsf{eq}} \phi \iff \mathsf{B} \vDash \phi \quad \text{for all } \mathsf{B}\,.$$

This is essentially our general Proposition **??**, *i.e.* the completeness of equational reasoning with respect to algebras in $\mathsf{Set}$, originally proved by Birkhoff.

It still remains to relate equational provability $\vdash_{\mathsf{eq}} \phi$ with deduction $\vdash \phi$, in order to have an algebraic proof of the traditional completeness theorem for propositional logic, Proposition 2.2.1. Indeed, we now just need to show that derivability $\vdash \phi$ is equivalent to any one of these three conditions: Boolean-valued validity $\mathsf{B} \vDash \phi$ for all $\mathsf{B}$, semantic validity $\vDash \phi$, algebraic provability $\vdash_{\mathsf{eq}} \phi$. This will follow from systematic considerations in the following section.

**Exercise 2.4.5.** For a formula in context $\mathtt{p}_1, ..., \mathtt{p}_k \mid \vartheta$ and a Boolean algebra $\mathsf{B}$, let the expression $\vartheta[b_1/\mathtt{p}_1, ..., b_k/\mathtt{p}_k]$ denote the element of $\mathsf{B}$ resulting from interpreting the propositional variables $\mathtt{p}_i$ in the context as the elements $b_i$ of $\mathsf{B}$, and evaluating the resulting expression using the Boolean operations of $\mathsf{B}$. For any *finite* set of propositional formulas $\Phi$ and any formula $\psi$, let $\Gamma = \mathtt{p}_1, ..., \mathtt{p}_k$ be a context for (the formulas in) $\Phi \cup \{\psi\}$. Finally, recall that $\Phi \vdash_{\mathsf{eq}} \psi$ means that $\psi = 1$ is equationally provable from the set of equations $\{\phi = 1 \mid \phi \in \Phi\}$. Show that $\Phi \vdash_{\mathsf{eq}} \psi$ just if for all finitely generated free Boolean algebras $F(n)$, the following condition holds:

> For any elements $f_1, ..., f_k \in F(n)$, if $\phi[f_1/\mathtt{p}_1, ..., f_k/\mathtt{p}_k] = 1$ for all $\phi \in \Phi$, then $\psi[f_1/\mathtt{p}_1, ..., f_k/\mathtt{p}_k] = 1$.

Is it sufficient to just take $F(k)$ and its generators $\mathtt{p}_1, ..., \mathtt{p}_k$ as the $f_1, ..., f_k$? Is it equivalent to take all Boolean algebras $\mathsf{B}$, rather than the finitely generated free ones $F(n)$? Determine a condition that is equivalent to $\Phi \vdash_{\mathsf{eq}} \psi$ for not necessarily finite sets $\Phi$.