Notes on Type Theory

[DRAFT: APRIL 22, 2025]

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Chapter 4

Homotopy Type Theory

The extensional dependent type theory of the previous chapter is in some ways a very natural system that admits an intuitively clear model in the locally cartesian closed category of sets and related categories. But for computational purposes, and specifically for the important application of type theory to proof checking in a computer proof assistant such as Agda or Lean, it has some serious defects: the equality relation between terms (or types) is not *decidable*: there is no algorithm that will determine whether two closed terms of a given type $s, t : A$ are (judgementally) equal $s \equiv t : A$. Indeed, there is no normalization procedure for reducing terms to normal forms—otherwise we could use it to decide whether two terms were equal by normalizing them and then comparing their normal forms. Relatedly, one cannot effectively decide whether a given type (e.g an equality type such as $\mathtt{Eq}_A(s,t)$) is inhabited (which would be a decision procedure for the *provability* of $s =_A t$), even given a candidate "proof term" $p : \mathtt{Eq}_A(s,t)$ (which would be a decision procedure for *being a proof*).

For this reason, the extensional system is often replaced in applications by a weaker one, called *intensional type theory*, which enjoys better computational behavior, such as decidability of equality and type-checking, and normalization. A good discussion of these and several related issues, such as canonicity and consistency can be found in Chapter 3 of the book [AG].

However, this is only one side of the story. The intensional theory was mainly a technical device for specialists in computational type theory (and a conceptual challenge from the semantic point of view) until around 2006, when it was discovered that this theory admitted a homotopical (and higher-categorical) interpretation, which led to the discovery of Homotopy type theory (HoTT) [Awo12]. This interpretation not only helped to clarify the intensional theory, and prove useful in investigating its computational properties, but also opened up a wide range of applications outside of the conventional areas of type theory, *vis.* computational and constructive mathematics. For, quite independently of such applications, the homotopical interpretation permits the use of intensional type theory as a powerful and expressive *internal language* for formal reasoning in homotopy theory and higher category theory, both highly abstract areas of mathematics, for which new and rigorous tools for calculation and proof are quite welcome. Moreover, the fortuitous fact that this system also has the good computational behavior that it does has led to the use of computational proof assistants in homotopy theory and higher category theory, even ahead of some more down-to-earth branches of mathematics, where such exotic semantics were not needed.

The homotopical interpretation was already anticipated by a 2-dimensional one in the category of groupoids, a special case of a higher categorical model that already suffices to make some of the essential features of such models clear. Thus we shall briefly review this model below, after introducing the intensional theory, and before considering the general homotopical semantics using weak factorization systems. Such "weak" interpretations also bring to a head the coherence issues that we deferred in the previous chapter, and we conclude with one approach to strictifying such interpretations using natural models, aka, categories with families.

4.1 Identity types

We begin by recalling from Section **??** the rules for *equality* types in the *extensional* system: The formation, introduction, elimination, and computation rules for equality types were as follows:

$$\frac{s : A \qquad t : A}{s =_A t \ \ \texttt{type}} \qquad\qquad \frac{a : A}{\texttt{refl}(a) : (a =_A a)}$$

$$\frac{p : s =_A t}{s \equiv t : A} \qquad\qquad \frac{p : s =_A t}{p \equiv \texttt{refl}(s) : (s =_A s)}$$

The *Identity types* in the intensional theory, also written $x =_A y$, or sometimes $\texttt{Id}_A(x, y)$, have the same formation and introduction rules as the Equality types, but the *elimination rule* of "equality reflection" is replaced by the following elimination rule:

$$\frac{x : A, y : A, z : \texttt{Id}_A(x, y) \vdash C(x, y, z) \ \texttt{type}, \qquad x : A \vdash c(x) : C(x, x, \texttt{refl}(x))}{x : A, y : A, z : \texttt{Id}_A(x, y) \vdash \texttt{J}(x, y, z, c) : C}$$

in which the variable x is bound in the occurance of c within the eliminator $\texttt{J}$. The associated *computation rule* then becomes:

$$x : A \vdash \texttt{J}(x, x, \texttt{refl}(x), c) \equiv c(x) : C(x, x, \texttt{refl}(x))$$

In HoTT, the elimination rule is called *path induction*, for reasons that will become clear.

To see how the elimination rule works, let us derive the basic laws of identity, namely reflexivity, symmetry, and transitivity, as well as Leibniz's Law the *indicernibility of identicals*, also known as the substitution of equals for equals.

- Reflexivity: states that $x =_A y$ is a reflexive relation, but this is just the $\texttt{Id}$-formation and intro rules:

$$x : A, y : A \vdash x =_A y \ \ \texttt{type}\ , \qquad x : A \vdash \texttt{refl}(x) : x =_A x$$

- Symmetry: can be stated as $x : A, y : A, u : x =_A y \vdash \,?\, : y =_A x$, which can be proved with an `Id`-elim as follows:

$$\frac{x : A \vdash \mathtt{refl}(x) : x =_A x}{x : A, y : A, u : x =_A y \vdash \mathtt{J}(x, y, u, \mathtt{refl}) : y =_A x}$$

- Transitivity: we wish to show

$$x : A, y : A, z : A, u : x =_A y, v : y =_A z \vdash \,?\, : x =_A z$$

regarding $z : A$ as a fixed parameter, which we can move to the front of the context, we want to apply an `Id`-elim with respect to the assumption $u : x =_A y$, so we can set x to y, and look for a premiss of the form:

$$z : A, y : A, v : y =_A z \vdash \,?\, : y =_A z$$

We cannot simply take v, however, since the order of the types in the context is still wrong for `Id`-elim, but we can move the assumption $v : y =_A z$ to the right with a λ-abstraction to obtain

$$z : A, y : A \vdash \ \lambda v\,.v : y =_A z \to y =_A z\,,$$

and now we *can* apply the planned `Id`-elim with respect to $u : x =_A y$ with the "motive" being $y =_A z \to x =_A z$ to obtain

$$z : A, y : A, x : A, u : x =_A y \vdash \mathtt{J}(x, y, u, \lambda v.v) : y =_A z \to x =_A z$$

from which follows the desired

$$x : A, y : A, z : A, u : x =_A y, v : y =_A z \vdash \mathtt{J}(x, y, u, \lambda v.v)\, v : x =_A z\,.$$

- Substitution: to show

$$\frac{x : A \vdash C(x)\ \ \mathtt{type}}{x : A, y : A, u : x =_A y \vdash \,?\, : C(x) \to C(y)}$$

it suffices to have a premiss of the form

$$x : A \vdash c(x) : C(x) \to C(x)$$

for this, we can take $c(x) = \lambda z : C(x).\, z : C(x) \to C(x)$ to obtain

$$x : A, y : A, u : x =_A y \vdash \mathtt{J}(x, y, u, x.\lambda z : C(x).z) : C(x) \to C(y)\,.$$

Note that the variable x is bound in the `J` term.

Many more properties of $\mathtt{Id}$-types and their associated $\mathtt{J}$-terms are shown in the introductory texts [Uni13, Rij25]. One key fact is that the higher identity types $\mathtt{Id}_{\mathtt{Id}_A(a,b)}(p,q)$ are no longer degenerate, but themselves may have terms that are non-identical, i.e. not propositionally equal, leading to so-called *higher types*. This "failure of UIP" (uniqueness of identity proofs) in the intensional system was first shown using the groupoid model, which sheds considerable light on the intensional system.

Exercise 4.1.1. Show that given any $a, b, c : A$ and $p : a =_A b$ and $q : b =_A c$, one can define a composite $p \cdot q : a =_A c$ (using the transitivity of $=_A$). Then show that, for any $p : a =_A b$, the symmetry term $\sigma(p) : b =_A a$ satisfies the (propositional) equation $\sigma(p) \cdot p = \mathtt{refl}$. Is either of $\sigma(p) \cdot p = \mathtt{refl}$ or $\mathtt{refl} \cdot \sigma(p) = \mathtt{refl}$ judgemental? What about associativity of $p \cdot q$

Exercise 4.1.2. Show that $p \cdot q$ from the previous exercise is (propositionally) associative.

Exercise 4.1.3. Show that any term $f : A \to B$ acts on identities $p : a =_A b$, in the sense that there is a term $\mathtt{ap}(f)(p) : fa =_B fb$. Is $\mathtt{ap}(f)$ "functorial" (in the evident sense)?

Exercise 4.1.4. Observe that the Substitution property means that the assignment

$$(a : A) \mapsto C(a)\ \mathtt{type}$$

is functorial (in some sense). Is it strictly functorial?

4.1.1 The naive interpretation

We can try to interpret the (intensional) identity types in the naive way, as we did for extensional dependent type theory. This would give the formation and introduction rules as a type family $\mathtt{Id}_A \to A \times A$ with a partial section over the diagonal substitution $\delta_A : (x : A) \to (x : A, y : A)$.

$$\begin{tikzcd} & \mathtt{Id}_A \arrow[d, "p"] \\ A \arrow[ur, "\mathtt{refl}"] \arrow[r, "\delta_A"'] & A \times A \end{tikzcd}$$

where we are writing $\mathtt{Id}_A$ for the extended context $(A, A, \mathtt{Id}_A)$ and p for the dependent family $(x : A, y : A \vdash \mathtt{Id}_A(x, y))$. The elimination rule then takes the form:

$$\begin{tikzcd} A \arrow[r, "c"] \arrow[d, "\mathtt{refl}"'] & C \arrow[d] \\ \mathtt{Id}_A \arrow[r, equal] \arrow[ur, dotted, "\mathtt{J}"'] & \mathtt{Id}_A \end{tikzcd} \tag{4.1}$$

for any type family $C \to \mathtt{Id}_A$, with the computation rule asserting that the top triangle commutes (the bottom triangle commutes by the assumption tht $\mathtt{J}$ is a section of $C \to \mathtt{Id}_A$).

But now recall that in extensional type theory, *any* map $f : B \to A$ can be regarded as a type family over A, namely by taking the *graph factorization*

$$B \cong \Sigma_{a:A}\Sigma_{b:B}\mathtt{Eq}_A(a, fb) \longrightarrow B \times A.$$

So we can take the family C in the elimination to be $\mathtt{refl} : A \to \mathtt{Id}_A$, to obtain:

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "\mathtt{refl}"'] & A \arrow[d, "\mathtt{refl}"] \\
\mathtt{Id}_A \arrow[r, equal] \arrow[ur, dotted, "\mathtt{J}"'] & \mathtt{Id}_A
\end{tikzcd}$$

We therefore get an iso $A \cong \mathtt{Id}_A$, making the identity type isomorphic to the extensional equality type $\mathtt{Eq}_A = A \to A \times A$.

Exercise 4.1.5. Prove that in the extensional theory, the graph factorization does indeed make any map $f : B \to A$ isomorphic to a family of types over its codomain. *Hint:* Consider the following two-pullback diagram.

$$\begin{tikzcd}
\Sigma_{a:A}\Sigma_{b:B}\mathtt{Eq}_A(a, fb) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\
A \times B \arrow[r, "A \times f"'] \arrow[d, "p_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \times A \arrow[d, "p_2"] \\
B \arrow[r, "f"'] & A
\end{tikzcd}$$

4.2 The groupoid model

Exercises 4.1.1 – 4.1.4 from the last section suggest an interpretation of the intensional version of dependent type theory, namely with types as *groupoids* and type families as *functors*. Such an interpretation was first given by [HS98] in order to show that the principle of Uniqueness of Identity Proofs (UIP) – which holds in the extensional theory – indeed fails in the intensional one. We shall briefly sketch this result here.

In order to give a model of intensional type theory we should define what it means to be *be* a model of (intensional) type theory. We will do this in section **??** below – for now we simple describe a single model in the category Gpd of groupoids, which will turn out to be an instance of the general notion. For the extensional theory, we defined a model simply to be an interpretation into an LCCC $\mathcal{E}$, with contexts Γ interpreted as objects of $\mathcal{E}$, substitutions $\sigma : \Delta \to \Gamma$ as arrows of $\mathcal{E}$, type families $\Gamma \vdash A$ as objects of $\mathcal{E}/_{\Gamma}$, and terms $\Gamma \vdash a : A$ as sections of the associated families. Substitution into families and terms was (weakly) interpreted' as pullback (in the sense that there was an unresolved coherence issue), and the Σ and Π type formers were adjoints to pullback. Finally, the equality type

$x : A, y : A \vdash \mathtt{Eq}_A(x, y)$ was interpreted as the diagonal $A \to A \times A$.

$$\begin{tikzcd}
(x : A, y : A, z : \mathtt{Eq}_A(x, y)) \arrow[r, equal] \arrow[d] & A \arrow[d] \\
(x : A, y : A) \arrow[r, equal] & A \times A
\end{tikzcd}$$

We may simplify the notation for category of contexts by using Σ-types, writing e.g. $(x : A, y : A, z : \mathtt{Eq}_A(x, y)) = \Sigma_{x:A}\Sigma_{y:A}\mathtt{Eq}_A(x, y)$ or even $\mathtt{Eq}_A$, and $(x : A, y : A) = A \times A$, etc.

The groupoid interpretation of the intensional theory is based on the idea that the identity type of a type (interpreted as a groupoid) G can be interpreted by the *path groupoid* of G, which we shall write as G^{I}.

Definition 4.2.1. If the groupoid $\mathsf{G} = G_0 \rightrightarrows G_1$ has objects G_0 and arrows G_1, the *path groupoid* $\mathsf{G}^{\mathsf{I}} = (|(\mathsf{G}^{\mathsf{I}})^{\mathsf{I}}| \rightrightarrows |\mathsf{G}^{\mathsf{I}}|)$ has as objects $|\mathsf{G}^{\mathsf{I}}| = G_1$, and as arrows $|(\mathsf{G}^{\mathsf{I}})^{\mathsf{I}}|$ the set of all commutative squares in G, with the obvious source and target maps.

In other words, the path groupoid is the arrow category $\mathsf{G}^{\downarrow}$. Recall that the category Gpd of (small) groupoids is a cartesian closed subcategory of Cat, and that there is a *walking arrow* groupoid I with exactly two objects and two (mutually inverse) non-identity arrows,

$$\mathsf{I} \qquad = \qquad \begin{tikzcd} (0 \arrow[r, bend right] & 1) \arrow[l, bend right] \end{tikzcd}$$

The notation G^{I} for the path groupoid is then correctly the exponential of G by I in Gpd (and in Cat). Observe that there are functors $\mathsf{dom}, \mathsf{cod} : \mathsf{G}^{\mathsf{I}} \rightrightarrows \mathsf{G}$, as well as one $\mathsf{id} : \mathsf{G} \to \mathsf{G}^{\mathsf{I}}$, making $\mathsf{G}^{\mathsf{I}} \rightrightarrows \mathsf{G}$ into an internal groupoid in Gpd, for any object G. This will be our interpretation of the identity type of the type interpreted by G.

More formally, we interpret:

- Contexts Γ: groupoids, i.e. objects of Gpd,
- Substitutions $\sigma : \Delta \to \Gamma$: homomorphisms of groupoids, i.e. arrows of Gpd,
- Types $\Gamma \vdash A$: functors $A : \mathsf{G} \to \mathsf{Gpd}$, where G interprets Γ,
- Terms $\Gamma \vdash a : A$: natural transformations $a : 1 \to A$ between functors, where 1 is the terminal functor in $\mathsf{Gpd}^{\mathsf{G}}$,
- Context extension $(\Gamma, A) \to \Gamma$: the Grothendieck construction $\int_{\mathsf{G}} A \to \mathsf{G}$.

In order to model the type formers Σ, Π, etc. of intensional type theory in Gpd, we must deal with the fact that Gpd is not locally cartesian closed, although it is cartesian closed. Recall that in order to model extensional type theory in presheaves we used the fact that $\mathsf{Set}^{\mathbb{C}^{\mathsf{op}}}$ is always a CCC and that for any $P \in \mathsf{Set}^{\mathbb{C}^{\mathsf{op}}}$ we have an equivalence

$$\mathsf{Set}^{\mathbb{C}^{\mathsf{op}}}/_P \;\simeq\; \mathsf{Set}^{\int P^{\mathsf{op}}},$$

and thus every slice is also a CCC. For groupoids, something similar is the case, but instead of the full slice category Gpd/G we use the subcategory of *fibrations* $\mathsf{Fib}_\mathsf{G} \hookrightarrow \mathsf{Gpd}/\mathsf{G}$, for which we have an equivalence

$$\mathsf{Fib}_\mathsf{G} \;\simeq\; \mathsf{Gpd}^\mathsf{G} \;\simeq\; \mathsf{Gpd}(\mathsf{Set}^{\mathsf{G}^{\mathsf{op}}})\,,$$

Since the proof that Gpd is a CCC doesn't depend on the classical logic of Set, the category of internal groupoids in a topos like $\mathsf{Set}^{\mathsf{G}^{\mathsf{op}}}$ is also a CCC. Thus we have that Fib_G is a CCC for any groupoid G.

Definition 4.2.2. A (split op-) fibration of groupoids $p : \mathsf{A} \to \mathsf{G}$ is a functor satisfying the condition: for every $a \in \mathsf{A}$ and $\gamma : pa \to g$ there is given a "lift" $\ell(a, \gamma) : a \to \tilde{g}$ with $p(\ell(a, \gamma)) = p$, and moreover,

1. $\ell(a, 1_{pa}) = 1_a : a \to a\,,$

2. for $\gamma' : g = p(\tilde{g}) \to h$, the lift of the composite is the composite of the lifts:

$$\ell(a, \gamma' \circ \gamma) = \ell(\tilde{g}, \gamma') \circ \ell(a, \gamma) : a \to \tilde{h}\,.$$

Proposition 4.2.3. *The category* Fib_G *of fibrations of groupoids and functors* $f : \mathsf{A} \to \mathsf{B}$ *over* G *that preserve the lifts is equivalent to the functor category* Gpd^G*.*

The interpretation of the context extension $(\Gamma, A) \to \Gamma$ is to be projection $\int_\mathsf{G} A \to \mathsf{G}$ given by the Grothendieck construction, and this is indeed a fibration of groupoids. Indeed, the functor taking $A : \mathsf{G} \to \mathsf{Gpd}$ to $\int_\mathsf{G} A \to \mathsf{G}$ mediates the equivalence $\int : \mathsf{Gpd}^\mathsf{G} \simeq \mathsf{Fib}_\mathsf{G}$.

For the base change functors along a fibration $p : \mathsf{A} \to \mathsf{G}$, we then have left and right adjoints as follows:

$$\begin{tikzcd} \mathsf{A} \arrow[d, "p"'] & \mathsf{Gpd}^\mathsf{A} \arrow[r, "\int", "\sim"'] & \mathsf{Fib}_\mathsf{A} \arrow[d, bend right, "\Sigma"'] \arrow[d, bend left, "\Pi"] \arrow[d, phantom, "\dashv\, p^* \,\dashv"] \\ \mathsf{G} & \mathsf{Gpd}^\mathsf{G} \arrow[u, "p^*"] \arrow[r, "\int", "\sim"'] & \mathsf{Fib}_\mathsf{G} \arrow[u] \end{tikzcd}$$

To show this, it needs to be shown that:

1. the pullback of a fibration is a fibration,

2. the composite of fibrations is a fibration,

3. there is a push-forward fibration of a fibration along a fibration, which is right adjoint to pullback.

The proof uses the CCC structure in the categories $\mathsf{Fib}_\mathsf{A} \simeq \mathsf{Gpd}^\mathsf{A} \simeq \mathsf{Gpd}(\mathsf{Set}^\mathsf{A})$ and is similar to the proof of the LCCC structure for a category of presheaves.

Identity types

To interpret the $\mathtt{Id}$-type of a type (interpreted as, say) $\mathsf{A} = (A_1 \rightrightarrows A_0)$ in Gpd (or indeed in any relative version $\mathsf{Gpd}(\mathsf{Set}^{\mathsf{G}})$), we shall use the path groupoid

$$\mathtt{Id}_{\mathsf{A}} = \mathsf{A}^{\mathsf{I}} \to \mathsf{A} \times \mathsf{A}\,,$$

which is easily seen to be a fibration, and therefore corresponds to a functor $\mathtt{Id}_A : \mathsf{A} \times \mathsf{A} \to \mathsf{Gpd}$, namely that with *discrete groupoids* as its values:

$$\mathtt{Id}_A(a,b) = \{p : a \to b\} \subseteq A_1\,.$$

Note that for two objects $a, b \in \mathsf{A}$ there may be many different arrows $f : a \to b$ in $\mathtt{Id}_A(a,b)$, but for two such parallel arrows $f, g : a \rightrightarrows b$ in A, regarded as objects in the path groupoid $\mathtt{Id}_A = \mathsf{A}^{\mathsf{I}}$, there need be no arrow between them in the (discrete) groupoid $\mathtt{Id}_A(a,b)$; and indeed, there will be one (which is then unique) just if $f = g$. Thus we will have the desired violation of UIP, once we have shown that this interpretation satisfies the rules for intensional $\mathtt{Id}$-types.

To show that, consider the diagram below, which we have already encountered as (4.3). We take any fibration $p : \mathsf{C} \to \mathtt{Id}_{\mathsf{A}}$ and any section $c : \mathsf{A} \to \mathsf{C}$ over the insertion of identity arrows into the path groupoid $\mathtt{refl} : \mathsf{A} \to \mathsf{A}^{\mathsf{I}} = \mathtt{Id}_A$, and we need a diagonal filler J.

$$\begin{tikzcd}
\mathsf{A} \arrow[r, "c"] \arrow[d, "\mathtt{refl}"'] & \mathsf{C} \arrow[d, "p"] \\
\mathtt{Id}_A \arrow[r, equal] \arrow[ur, dotted, "\mathsf{J}"'] & \mathtt{Id}_A
\end{tikzcd}$$

Since the diagram commutes by assumption, for any $a \in \mathsf{A}$ we have $pca = 1_a$. Let $\alpha : a \to b$ be any object in $\mathtt{Id}_A = \mathsf{A}^{\mathsf{I}}$ and observe that there is always an arrow $\chi_\alpha : 1_a \Rightarrow \alpha$ in $\mathtt{Id}_A = \mathsf{A}^{\mathsf{I}}$, namely $\chi_\alpha = (1_a, \alpha)$.

$$\begin{tikzcd}
a \arrow[r, "1_a"] \arrow[d, "1_a"'] \arrow[dr, phantom, "\chi_\alpha"] & a \arrow[d, "\alpha"] \\
a \arrow[r, "\alpha"'] & b
\end{tikzcd}$$

Since $p : \mathsf{C} \to \mathtt{Id}_{\mathsf{A}}$ is a fibration, there is a lift $\ell(ca, \chi_\alpha) : ca \to \tilde{\alpha}$. We then set

$$\mathsf{J}(\alpha) = \tilde{\alpha}$$

to obtain a functor $\mathsf{J} : \mathtt{Id}_{\mathsf{A}} \to \mathsf{A}$ making the two triangles in the diagram commute.

Exercise 4.2.4. Prove this!

Exercise 4.2.5. Show that the composition of fibrations $\mathsf{B} \to \mathsf{A}$ and $\mathsf{A} \to \mathsf{G}$ is a fibration. (This will be used for the interpretation of the type $\Gamma \vdash \Sigma_{\mathsf{A}}\mathsf{B}$, where $\Gamma \vdash \mathsf{A}$ and $\Gamma, \mathsf{A} \vdash \mathsf{B}$.)

4.3 Weak factorization systems

We can axiomatize the features of the groupoid model that allowed us to model intensional type theory using the notion of a *weak factorization system*, which is important in axiomatic homotopy theory. This is a weakening of the notion of an orthogonal factorization system from Definition **??**:

Definition 4.3.1. An *weak factorization system (wfs)* on a category $\mathcal{C}$ consists of two classes of arrows (L, R) such that:

1. Every map $f : A \to B$ factors $f = r \circ \ell$ into $\ell \in \mathsf{L}$ followed by $r \in \mathsf{R}$,

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\ell"'] & & B \\ & C \arrow[ur, "r"'] & \end{tikzcd}$$

2. Given any commutative square with an L-map on the left and an R-map on the right,

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\ell"'] & B \arrow[d, "r"] \\ C \arrow[r] \arrow[ur, dotted, "j"'] & D \end{tikzcd} \tag{4.2}$$

   there is a (not necessarily unique) *diagonal filler* j as indicated, making both triangles commute.

3. The classes L, R are closed under retracts in the arrow category $\mathcal{C}^{\downarrow}$.

Given such a wfs on a finitely complete category $\mathcal{C}$, we shall interpret the contexts and substitutions as the objects and arrows of $\mathcal{C}$, the type families as the right maps R, and the terms as sections of the right maps. The first part of the following is required for the interpretation of substitution, and the second part is used for context extension and Σ-types.

Lemma 4.3.2. *In a wfs* (L, R) *on a finitely complete category* $\mathcal{C}$*, the right maps are stable under pullback along all maps. (Dually, the left maps are stable under pushouts along all maps.) Moreover, both* L *and* R *are closed under composition.*

Before giving the proof, we develop an important aspect of wfs's: weak orthogonality. For any maps $f : A \to B$ and $g : C \to D$, let us write

$$f \pitchfork g$$

and say that f *is weakly orthogonal to* g if every commutative square with f on the left and g on the right has a diagonal filler j as in (4.2). We also say that "f has the left lifting property with respect to g" and "g has the right lifting property with respect to f". More generally, for any class of arrows S in $\mathcal{C}$, write

$$\begin{aligned} S \pitchfork f &= s \pitchfork f \text{ for all } s \in S \\ f \pitchfork S &= f \pitchfork s \text{ for all } s \in S \end{aligned}$$

and let

$$\begin{aligned} S^{\pitchfork} &= \{f \mid S \pitchfork f\} \\ {}^{\pitchfork}S &= \{f \mid f \pitchfork S\}. \end{aligned}$$

Finally, let $S \pitchfork T$ mean that $s \pitchfork t$ for all $s \in S$ and $\mathsf{t} \in T$. Then in a wfs (L, R) we clearly have

$$\mathsf{L} \pitchfork \mathsf{R},$$

by axiom 2, but in fact more is true:

Lemma 4.3.3. *Given two classes of maps* (L, R) *in a category* $\mathcal{C}$ *satisfying the factorization and diagonal filler axioms for a wfs above,* L *and* R *are also closed under retracts if and only if*

$$\mathsf{L}^{\pitchfork} = \mathsf{R} \quad \textit{and} \quad \mathsf{L} = {}^{\pitchfork}\mathsf{R}.$$

Proof. Suppose (L, R) is a wfs, so both classes are closed under retracts. We need to show that if $f : A \to B$ satisfies $\mathsf{L} \pitchfork \mathsf{f}$, then $f \in \mathsf{R}$ (the converse is already true by $\mathsf{L} \pitchfork \mathsf{R}$. Factor $f = r \circ \ell$ and consider the diagram

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "\ell"'] & A \arrow[d, "f"] \\ C \arrow[r, "r"'] \arrow[ur, dotted, "j"] & B \end{tikzcd} \tag{4.3}$$

which commutes and has a diagonal filler j, since $\ell \in \mathsf{L}$. We can rearrange (4.3) into a retract diagram as follows.

$$\begin{tikzcd} A \arrow[r, "\ell"] \arrow[d, "f"'] & C \arrow[r, "j"] \arrow[d, "r"] & A \arrow[d, "f"] \\ B \arrow[r, equal] & B \arrow[r, equal] & B \end{tikzcd} \tag{4.4}$$

Thus $f \in \mathsf{R}$, since R is closed under retracts. The argument for $\mathsf{L} = {}^{\pitchfork}\mathsf{R}$ is dual. We leave the converse as an exercise. □

Exercise 4.3.4. Show that (L, R) is a wfs if the factorization axiom holds and $\mathsf{L}^{\pitchfork} = \mathsf{R}$ and $\mathsf{L} = {}^{\pitchfork}\mathsf{R}$.

Proof. (of Lemma 4.3.2) Let $f : A \to B$ be in R and consider its pullback along any $B' \to B$:

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ B' \arrow[r] & B \end{tikzcd} \tag{4.5}$$

To show $f' \in \mathsf{R}$, it suffices by Lemma 4.3.3 to show that f' has the right lifting property with respect to L, but this follows easily from f' being a pullback of f, which does. We leave the rest of the proof as an exercise. $\square$

Exercise 4.3.5. Finish the proof of Lemma 4.3.2.

It now follows that we can interpret the structural rules of dependent type theory, as well as the context extension operation, using the R maps as the type families, just as we did using arbitrary maps in an lccc. The rules for Σ types will also be satisfied, since these essentially state that Σ is left adjoint to pullback along type families, and therefore closure of the right maps under composition means that they are closed under Σ-types.

Let us see that we can also interpret the rules for $\mathtt{Id}$-types. The formation rule for $\mathtt{Id}_A$ is interpreted by factoring the diagonal substitution $\delta : A \to A \times A$ into a left map followed by a right map:

$$\begin{tikzcd} & \mathtt{Id}_A \arrow[d, "p"] \\ A \arrow[ur, "\mathtt{refl}"] \arrow[r, "\delta"'] & A \times A \end{tikzcd}$$

This also interprets the introduction rule, using the left map in the factorization as the interpretation of the $\mathtt{refl}$ term. For the elimination rule, suppose we have a type family $p : \mathsf{C} \to \mathtt{Id}_\mathsf{A}$ and a section $c : \mathsf{A} \to \mathsf{C}$ over $\mathtt{refl} : \mathsf{A} \to \mathtt{Id}_A$; then we need a diagonal filler $\mathtt{J}$.

$$\begin{tikzcd} \mathsf{A} \arrow[r, "c"] \arrow[d, "\mathtt{refl}"'] & \mathsf{C} \arrow[d, "p"] \\ \mathtt{Id}_A \arrow[r, equal] \arrow[ur, dotted, "\mathtt{J}"'] & \mathtt{Id}_A \end{tikzcd}$$

But since $\mathtt{refl} : A \to \mathtt{Id}_A$ is a left map by the factorization, and $C \to \mathtt{Id}_A$ is a right map by the interpretation of type families as right maps, there is such a filler by the second axiom of wfs's. Thus we have already shown:

Proposition 4.3.6 ([AW09])**.** *In a fintely complete category $\mathcal{C}$ with a wfs, the rules of intensional identity types are soundly modeled by interpreting the type families as the right maps and the identity type $\mathtt{Id}_A$ as a factorization of the diagonal $\delta : A \to A \times A$ into a left map $\mathtt{refl} : A \to \mathtt{Id}_A$ followed by a right map $\mathtt{Id}_A \to A \times A$.*

This kind of interpretation includes many important "naturally occurring" examples involving *Quillen model categories*, which are categories equipped with two interlocking wfs's (see [AW09]). The Π-types can also be interpreted in this way, if the right maps of the wfs pushforward along right maps, as is the case in examples such as right-proper Quillen model categories and Π-tribes in the sense of [Joy17]. Indeed, the groupoid model from the previous section was an instance of such a wfs: as the right maps one can take the isofibrations, and the left maps are then the equivalences that are injective on objects [GG08].

Remark 4.3.7 (Coherence)**.** The coherence issue that we have been postponing is now even more pressing, however, because the factorizations in a wfs need not be stable under pullback, and so the following (slightly schematic) substitution rule for the `Id` type former will not be soundly modeled, even up to isomorphism.

$$\frac{\Gamma \vdash A\,, \quad \Gamma \vdash a : A, \quad \Gamma \vdash b : A, \quad \sigma : \Delta \to \Gamma}{\Delta \vdash \mathtt{Id}_A(a,b)[\sigma] \equiv \mathtt{Id}_{A[\sigma]}(a[\sigma], b[\sigma])}$$

A similar problem occurs with respect to the `J`-term, which is also required to respect (certain) substitutions, but need not do so under this interpretation. One solution to this problem makes use of an algebraic structure on the weak factorization system called an *algebraic weak factorization system*. This approach is discussed in [GL23]. We shall develop a different solution in the next section.

4.4 Natural models

The semantics of DTT in LCCCs employed in the previous sections uses the "slice category" hyperdoctrine of an LCCC to interpret the dependent types. Thus the contexts Γ and substitutions $\sigma : \Delta \to \Gamma$ are interpreted as the objects and arrows of an LCCC $\mathcal{C}$, and the dependent types $\Gamma \vdash A$ and terms $\Gamma \vdash a : A$ are interpreted as objects $A \to \Gamma$ in the slice category $\mathcal{C}/\Gamma$ and their global sections $a : \Gamma \to A$ (over Γ). As we mentioned in Remark **??**, however, there is a problem with this kind of semantics (as first pointed out by [Hof95]): as a hyperdoctrine, this interpretation is a *pseudo*functor $\mathcal{C}/ : \mathcal{C}^{\mathsf{op}} \to \mathsf{Cat}$, but the syntax of DTT produces an actual *presheaf* of types in context $\mathsf{Ty} : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$, since substitution into dependent types is *strictly* functorial with respect to composition of substitutions, in the sense that for a type in context $\Gamma \vdash A$ and substitutions $\sigma : \Delta \to \Gamma$ and $\tau : \Phi \to \Delta$ we have a *judgemental equality of types in context*,

$$\Phi \vdash (A[\sigma])[\tau] \equiv A[\sigma \circ \tau]\,,$$

rather than the (canonical) isomorphism $\cong$ fitting into the two-pullbacks diagram of the hyperdoctrine, namely:

$$\begin{tikzcd}
(\sigma\circ\tau)^*A \arrow[r, "\cong"'] \arrow[rrr, bend left] \arrow[dr] & \tau^*\sigma^*A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \sigma^*A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\
 & \Phi \arrow[r, "\tau"] \arrow[rr, bend right, "\sigma\circ\tau"'] & \Delta \arrow[r, "\sigma"] & \Gamma
\end{tikzcd}$$

A similar problem occurs in the Beck-Chavalley conditions, where the hyperdoctrine structure has only canonical isos, rather than the strict equalities that obtain in the syntax, such as

$$(\Pi_{x:A}B)[\sigma] \equiv (\Pi_{x:A[\sigma]}B[\sigma])\,.$$

Exactly the same problem occurs, of course, if we use only the right maps in a wfs, rather than all maps in the slice category of an LCC. And for the `Id`-type former, we have an even more acute probelm, as noted in Remark 4.3.7, since the factorization of the diagonal is not even determined up to isomorphism, and need not be stable under pullback

There are various different solutions to these "coherence problems" in the literature, some involving *strictifications* of the LCC slice-category (or wfs-pullback) hyperdoctrine (including both left- and right-adjoint strictifications [?, ?]), as well as other semantics altogether, such as categories-with-families [Dyb96], categories-with-attributes [?], and comprehension categories [?]. A solution based on the notion of a *universe* $\widetilde{U} \to U$ was first proposed by Voevodsky [?]; in [Awo16], the universe approach is combined with the notion of a *representable natural transformation* to determine the semantic notion of a *natural model*, as follows.

Definition 4.4.1. For a small category $\mathbb{C}$, a natural transformation $f : Y \to X$ of presheaves on $\mathbb{C}$ is called *representable* if for every $C \in \mathbb{C}$ and $x \in X(C)$, there is given a $p : D \to C$ and a $y \in Y(D)$ such that the following square is a pullback.

$$\begin{tikzcd} \mathsf{y}D \arrow[r, "y"] \arrow[d, "\mathsf{y}p"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\ \mathsf{y}C \arrow[r, "x"'] & X \end{tikzcd} \tag{4.6}$$

We will show that a representable natural transformation is essentially the same thing as a *category with families* in the sense of Dybjer [Dyb96]. Indeed, let us write the objects of $\mathbb{C}$ as $\Gamma, \Delta, \dots$ and the arrows as $\sigma : \Delta \to \Gamma, \dots$, thinking of $\mathbb{C}$ as a "category of contexts". Let $\mathsf{t} : \dot{\mathsf{T}} \to \mathsf{T}$ be a representable map of presheaves, and interpret the elements as

$$\begin{aligned} \mathsf{T}(\Gamma) &= \{A \mid \Gamma \vdash A\} \\ \dot{\mathsf{T}}(\Gamma) &= \{a \mid \Gamma \vdash a : A,\ \text{for some } A\}, \end{aligned}$$

so that under Yoneda we have:

$$\begin{aligned} A \in \mathsf{T}(\Gamma) &\quad \text{iff} \quad \Gamma \vdash A \\ a \in \dot{\mathsf{T}}(\Gamma) &\quad \text{iff} \quad \Gamma \vdash a : A,\ \text{where } \mathsf{t} \circ a = A \end{aligned}$$

as indicated in:

$$\begin{tikzcd} & \dot{\mathsf{T}} \arrow[d, "\mathsf{t}"] \\ \mathsf{y}\Gamma \arrow[ur, "a"] \arrow[r, "A"'] & \mathsf{T} \end{tikzcd}$$

Thus we are regarding T as the *presheaf of types*, with $\mathsf{T}(\Gamma)$ the set of all types in context Γ, and $\dot{\mathsf{T}}$ as the *presheaf of terms*, with $\dot{\mathsf{T}}(\Gamma)$ the set of all terms in context Γ, while the component $\mathsf{t}_\Gamma : \dot{\mathsf{T}}(\Gamma) \to \mathsf{T}(\Gamma)$ is the typing of the terms in context Γ.

The naturality of $\mathsf{t} : \dot{\mathsf{T}} \to \mathsf{T}$ just means that for any substitution $\sigma : \Delta \to \Gamma$, we have an action on types and terms:

$$\begin{aligned} \Gamma \vdash A \quad &\mapsto \quad \Delta \vdash A\sigma \\ \Gamma \vdash a : A \quad &\mapsto \quad \Delta \vdash a\sigma : A\sigma\,. \end{aligned}$$

While, by functoriality, given any further $\tau : \Phi \to \Delta$, we have

$$(A\sigma)\tau = A(\sigma \circ \tau) \qquad (a\sigma)\tau = a(\sigma \circ \tau),$$

as well as

$$A1 = A \qquad a1 = a$$

for the identity substitution $1 : \Gamma \to \Gamma$.

Finally, the representability of the natural transformation $\mathsf{t} : \dot{\mathsf{T}} \to \mathsf{T}$ is exactly the operation of *context extension*: given any $\Gamma \vdash A$, by Yoneda we have the corresponding map $A : \mathsf{y}\Gamma \to \mathsf{T}$, and we let $p_A : \Gamma.A \to \Gamma$ be (the map representing) the pullback of t along A, as in (4.6). We therefore have a pullback square:

$$\begin{tikzcd} \mathsf{y}\Gamma.A \arrow[r, "q_A"] \arrow[d, "\mathsf{y}p_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{T}} \arrow[d, "\mathsf{t}"] \\ \mathsf{y}\Gamma \arrow[r, "A"'] & \mathsf{T} \end{tikzcd} \tag{4.7}$$

where the map $q_A : \Gamma.A \to \dot{\mathsf{T}}$ now determines a term

$$\Gamma.A \vdash q_A : Ap_A.$$

In type theory, the term $q_A : \Gamma.A \to \dot{\mathsf{T}}$ corresponds to the rule of "assumption":

$$\frac{\Gamma \vdash A \ \mathtt{type}}{\Gamma, x : A \vdash x : A}$$

We may hereafter omit the y for the Yoneda embedding, letting the Greek letters serve to distinguish representable presheaves and their maps.

Exercise 4.4.2. Show that the fact that (4.4.3) is a pullback means that given any $\sigma : \Delta \to \Gamma$ and $\Delta \vdash a : A\sigma$, there is a map

$$(\sigma, a) : \Delta \to \Gamma.A,$$

and this operation satisfies the equations

$$\begin{aligned} p_A \circ (\sigma, a) &= \sigma \\ q_A(\sigma, a) &= a, \end{aligned}$$

as indicated in the following diagram.

$$
\begin{tikzcd}
\Delta \arrow[drr, bend left, "a"] \arrow[dr, dotted, "{(\sigma, a)}"] \arrow[ddr, bend right, "\sigma"'] & & \\
& \Gamma.A \arrow[r, "q_A"'] \arrow[d, "p_A"] \arrow[dr, phantom, very near start, "\lrcorner"] & \dot{\mathsf{T}} \arrow[d, "\mathsf{t}"] \\
& \Gamma \arrow[r, "A"'] & \mathsf{T}
\end{tikzcd}
$$

Show moreover that the uniqueness of (σ, a) means that for any $\tau : \Delta' \to \Delta$ we also have:

$$
\begin{aligned}
(\sigma, a) \circ \tau &= (\sigma \circ \tau, a\tau) \\
(p_A, q_A) &= 1.
\end{aligned}
$$

Comparing the foregoing with the definition of a category with families in [Dyb96], we have shown:

Proposition 4.4.3. *Let* $\mathsf{t} : \dot{\mathsf{T}} \to \mathsf{T}$ *be a representable natural transformation of presheaves on a small category* $\mathbb{C}$ *with a terminal object. Then* t *determines a category with families, with* $\mathbb{C}$ *as the contexts and substitutions,* $\mathsf{T}(\Gamma)$ *as the types in context* Γ*, and* $\dot{\mathsf{T}}(\Gamma)$ *as the terms in context* Γ*.*

Remark 4.4.4. A category with families is usually defined in terms of a presheaf

$$\mathsf{Ty} : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$$

of types on the category $\mathcal{C}$ of contexts, together with a presheaf

$$\mathsf{Tm}' : (\textstyle\int_{\mathcal{C}} \mathsf{Ty})^{\mathsf{op}} \to \mathsf{Set}$$

of typed-terms on the category $\int_{\mathcal{C}} \mathsf{Ty}$ of types-in-context. We are using the equivalence of categories, valid for any category of presheaves $\mathsf{Set}^{\mathbb{C}^{\mathsf{op}}}$,

$$\mathsf{Set}^{\mathbb{C}^{\mathsf{op}}}/_{P} \;\simeq\; \mathsf{Set}^{(\int_{\mathbb{C}} P)^{\mathsf{op}}}$$

between the slice category over a presheaf P and the presheaves on its category of elements $\int_{\mathbb{C}} P$, to turn the presheaf $\mathsf{Tm}' : (\int_{\mathcal{C}} \mathsf{Ty})^{\mathsf{op}} \to \mathsf{Set}$ into one $\mathsf{Tm} : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ together with a map $\mathsf{Tm} \to \mathsf{Ty}$ in $\mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$.

We think of a representable map of presheaves on an arbitrary category $\mathbb{C}$ as a "type theory over $\mathbb{C}$", with $\mathbb{C}$ as the category of contexts and substitutions. We will show in Section 4.4.3 that such a map of presheaves is essentially determined by a class of maps in $\mathbb{C}$ that is closed under all pullbacks, corresponding to the "incoherent" interpretation of types in context as maps $A \to \Gamma$.

Definition 4.4.5. A *natural model of type theory* on a small category $\mathbb{C}$ is a representable map of presheaves $\mathsf{t} : \dot{\mathsf{T}} \to \mathsf{T}$.

Exercise 4.4.6 (The natural model of syntax)**.** Let $\mathbb{T}$ be a dependent type theory and $\mathcal{C}_{\mathbb{T}}$ its category of contexts and substitutions. Define the presheaves $\mathsf{Ty} : \mathcal{C}_{\mathbb{T}}{}^{\mathsf{op}} \to \mathsf{Set}$ of types-in-context and $\mathsf{Tm} : \mathcal{C}_{\mathbb{T}}{}^{\mathsf{op}} \to \mathsf{Set}$ of terms-in-context, along with a natural transformation

$$\mathsf{tp} : \mathsf{Tm} \to \mathsf{Ty}$$

that takes a term to its type. Show that $\mathsf{tp} : \mathsf{Tm} \to \mathsf{Ty}$ is a natural model of type theory.

4.4.1 Modeling $1, \Sigma, \Pi$

Given a natural model $\mathsf{t} : \dot{\mathsf{T}} \to \mathsf{T}$, we will make extensive use of the associated *polynomial endofunctor* $\mathsf{P_t} : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ (cf. [?]), defined by

$$\mathsf{P_t} = \mathsf{T}_! \circ \mathsf{t}_* \circ \dot{\mathsf{T}}^* : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}\,,$$

$$\begin{tikzcd}
\mathsf{Set}^{\mathbb{C}^{\mathsf{op}}} \arrow[r, "\mathsf{P_t}"] \arrow[d, "\dot{\mathsf{T}}^*"'] & \mathsf{Set}^{\mathbb{C}^{\mathsf{op}}} \\
\mathsf{Set}^{\mathbb{C}^{\mathsf{op}}}/_{\dot{\mathsf{T}}} \arrow[r, "\mathsf{t}_*"'] & \mathsf{Set}^{\mathbb{C}^{\mathsf{op}}}/_{\mathsf{T}} \arrow[u, "\mathsf{T}_!"']
\end{tikzcd}$$

The action of $\mathsf{P_t}$ on an object X may be depicted:

$$\begin{tikzcd}
X & X \times \dot{\mathsf{T}} \arrow[l] \arrow[d] & & \mathsf{P_t}(X) \arrow[d] \\
& \dot{\mathsf{T}} \arrow[rr, "\mathsf{t}"'] & & \mathsf{T}
\end{tikzcd}$$

We call $\mathsf{t} : \dot{\mathsf{T}} \to \mathsf{T}$ the *signature* of $\mathsf{P_t}$ and briefly recall the following *universal mapping property* from [?].

Lemma 4.4.7. *For any $p : E \to B$ in a locally cartesian closed category $\mathcal{E}$, the polynomial functor $\mathsf{P}_p : \mathcal{E} \to \mathcal{E}$ has the following universal property: for any objects $X, Z \in \mathcal{E}$, maps $f : Z \to \mathsf{P}_p(X)$ correspond bijectively to pairs of maps $f_1 : Z \to B$ and $f_2 : Z \times_B E \to Z$, as indicated below.*

$$\frac{Z \xrightarrow{\quad f \quad} \mathsf{P}_p(X)}{\begin{tikzcd}
X & Z \times_B E \arrow[l, "f_2"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
& Z \arrow[r, "f_1"'] & B
\end{tikzcd}} \tag{4.8}$$

The correspondence is natural in both X and Z, in the expected sense.

This universal property is also suggested by the conventional type theoretic notation, namely:

$$\mathsf{P}_p(X) = \Sigma_{b:B} X^{E_b}$$

The lemma can be used to determine the signature $p \cdot q$ for the composite $\mathsf{P}_p \circ \mathsf{P}_q$ of two polynomial functors, which is again polynomial, and for which we therefore have

$$\mathsf{P}_{p \cdot q} \;\cong\; \mathsf{P}_p \circ \mathsf{P}_q \,. \tag{4.9}$$

Indeed, let $p : B \to A$ and $q : D \to C$, and consider the following diagram resulting from applying the correspondence (4.10) to the identity arrow,

$$\langle a, c \rangle = 1_{\mathsf{P}_p(C)} : \mathsf{P}_p(C) \to \mathsf{P}_p(C) \,,$$

and taking Q to be the indicated pullback.

$$\begin{tikzcd}
D \arrow[d, "q"'] & Q \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner"] \arrow[dd, dotted, bend left, "p\cdot q"] & \\
C & \pi^* B \arrow[l, "c"] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d, "p"] \\
 & \mathsf{P}_p(C) \arrow[r, "a"'] & A
\end{tikzcd} \tag{4.10}$$

The map $p \cdot q$ is then defined to be the indicated composite,

$$p \cdot q \;=\; a^* p \circ c^* q \,.$$

The condition (4.9) can then be checked using the correspondence (4.10) (also see [**?**]).

Definition 4.4.8. A natural model $\mathsf{t} : \dot{\mathsf{T}} \to \mathsf{T}$ over $\mathbb{C}$ will be said to *model* the type formers $1, \Sigma, \Pi$ if there are pullback squares in $\hat{\mathbb{C}}$ of the following form,

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{T}} \arrow[d, "\mathsf{t}"] & \dot{\mathsf{T}}_2 \arrow[r] \arrow[d, "\mathsf{t}\cdot\mathsf{t}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{T}} \arrow[d, "\mathsf{t}"] & \mathsf{P}_\mathsf{t}(\dot{\mathsf{T}}) \arrow[r] \arrow[d, "\mathsf{P}_\mathsf{t}(\mathsf{t})"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{T}} \arrow[d, "\mathsf{t}"] \\
1 \arrow[r] & \mathsf{T} & \mathbb{T}_2 \arrow[r] & \mathsf{T} & \mathsf{P}_\mathsf{t}(\mathsf{T}) \arrow[r] & \mathsf{T}
\end{tikzcd} \tag{4.11}$$

where $\mathsf{t} \cdot \mathsf{t} : \dot{\mathsf{T}}_2 \to \mathsf{T}_2$ is determined by $\mathsf{P}_{\mathsf{t}\cdot\mathsf{t}} \cong \mathsf{P}_\mathsf{t} \circ \mathsf{P}_\mathsf{t}$ as in (4.9).

The terminology is justified by the following result from [**?**].

Theorem 4.4.9 ([Awo16] Theorem XXX)**.** *Let* $\mathsf{t} : \dot{\mathsf{T}} \to \mathsf{T}$ *be a natural model. The associated category with families satisfies the usual rules for the type-formers* $1, \Sigma, \Pi$ *just if* $\mathsf{t} : \dot{\mathsf{T}} \to \mathsf{T}$ *models the same in the sense of Definition 4.4.8.*

We only sketch the case of Π-types, but the other type formers will be treated in detail in Section **??**.

Proposition 4.4.10. *The natural model* $\mathsf{t} : \dot{\mathsf{T}} \to \mathsf{T}$ *models* Π*-types just if there are maps* λ *and* Π *making the following a pullback diagram.*

$$\begin{tikzcd} \mathsf{P_t}(\dot{\mathsf{T}}) \arrow[r, "\lambda"] \arrow[d, "\mathsf{P_t(t)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{T}} \arrow[d, "\mathsf{t}"] \\ \mathsf{P_t}(\mathsf{T}) \arrow[r, "\Pi"'] & \mathsf{T} \end{tikzcd} \tag{4.12}$$

Proof. Unpacking the definitions, we have $\mathsf{P_t}(\mathsf{T}) = \Sigma_{A:\mathsf{T}}\mathsf{T}^A$, etc., so diagram (4.14) becomes:

$$\begin{tikzcd} \Sigma_{A:\mathsf{T}}\dot{\mathsf{T}}^A \arrow[r, "\lambda"] \arrow[d, "\Sigma_{A:\mathsf{T}}\mathsf{t}^A"'] & \dot{\mathsf{T}} \arrow[d, "\mathsf{t}"] \\ \Sigma_{A:\mathsf{T}}\mathsf{T}^A \arrow[r, "\Pi"'] & \mathsf{T} \end{tikzcd}$$

For $\Gamma \in \mathbb{C}$, maps $\Gamma \to \Sigma_{A:\mathsf{T}}\mathsf{T}^A$ correspond to pairs (A, B) with $A : \Gamma \to \mathsf{T}$ and $B : \Gamma, A \to \mathsf{T}$, and thus to $\Gamma \vdash A$ and $\Gamma, A \vdash B$. Similarly, a map $\Gamma \to \Sigma_{A:\mathsf{T}}\dot{\mathsf{T}}^A$ corresponds to a pair (A, b) with $\Gamma \vdash A$ and $\Gamma, A \vdash b : B$, the typing of b resulting from composing with the map

$$\Sigma_{A:\mathsf{T}}\mathsf{t}^A : \Sigma_{A:\mathsf{T}}\dot{\mathsf{T}}^A \to \Sigma_{A:\mathsf{T}}\mathsf{T}^A\,.$$

$$\begin{tikzcd} & \Sigma_{A:\mathsf{T}}\dot{\mathsf{T}}^A \arrow[r, "\lambda"] \arrow[dd] & \dot{\mathsf{T}} \arrow[dd, "\mathsf{t}"] \\ \Gamma \arrow[ur, "(A,b)"] \arrow[urr, "\lambda_A b"] \arrow[dr, "(A,B)"'] \arrow[drr, "\Pi_A B"] & & \\ & \Sigma_{A:\mathsf{T}}\mathsf{T}^A \arrow[r, "\Pi"'] & \mathsf{T} \end{tikzcd}$$

The composition across the top is then the term $\Gamma \vdash \lambda_{x:A}b$, the type of which is determined by composing with t and comparing with the composition across the bottom, namely $\Gamma \vdash \Pi_{x:A}B$. In this way, the lower horizontal arrow in the diagram models the Π-*formation rule*:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash \Pi_{x:A}B}$$

and the upper horizontal arrow, along with the commutativity of the diagram, models the Π-*introduction rule*:

$$\frac{\Gamma, A \vdash b : B}{\Gamma \vdash \lambda_{x:A}b : \Pi_{x:A}B}$$

The square (4.14) is a pullback just if, for every $(A, B) : \Gamma \to \Sigma_{A:\mathsf{T}}\mathsf{T}^A$ and every $t : \Gamma \to \dot{\mathsf{T}}$ with $\mathsf{t} \circ t = \Pi_A B$, there is a unique $(A, b) : \Gamma \to \Sigma_{A:\mathsf{T}}\dot{\mathsf{T}}^A$ with $b : B$ and $\lambda_A b = t$. In terms of the interpretation, given $\Gamma, A \vdash B$ and $\Gamma \vdash t : \Pi_{x:A}B$, there is a term $\Gamma, A \vdash t' : B$ with $\lambda_{x:A}t' = t$, and t' is unique with this property. This is just what is provided by the Π-*elimination rule*:

$$\frac{\Gamma, A \vdash B \qquad \Gamma \vdash t : \Pi_{x:A}B \qquad \Gamma \vdash x : A}{\Gamma, A \vdash t\,x : B}$$

in conjunction with the Π-*computation rules*:

$$\lambda_{x:A}(t\,x) = t : \Pi_A B$$
$$(\lambda_{x:A} b)\,x = b : B$$

□

4.4.2 Identity types

A natural model $\mathsf{t} : \dot{\mathsf{T}} \to \mathsf{T}$ in a presheaf category $\widehat{\mathbb{C}}$ was defined in [Awo16] to model both the extensional and intensional identity types of Martin-Löf type theory in terms of the existence of certain additional structures, which we briefly review. Condition (1) below captures the extensional equality types of Martin-Löf type theory. The condition given in *op.cit.* for the intensional case is replaced below by a simplification suggested by R. Garner.

Definition 4.4.11. Let $\mathsf{t} : \dot{\mathsf{T}} \to \mathsf{T}$ be a map in an lccc $\mathcal{E}$.

1. The map $\mathsf{t} : \dot{\mathsf{T}} \to \mathsf{T}$ is said to model the (extensional) *equality type former* if there are structure maps $(\mathtt{refl}, \mathsf{Eq})$ making a pullback square:

$$\begin{tikzcd}
\dot{\mathsf{T}} \arrow[r, "\mathtt{refl}"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{T}} \arrow[d, "\mathsf{t}"] \\
\dot{\mathsf{T}} \times_{\mathsf{T}} \dot{\mathsf{T}} \arrow[r, "\mathsf{Eq}"'] & \mathsf{T}
\end{tikzcd}$$

2. The map $\mathsf{t} : \dot{\mathsf{T}} \to \mathsf{T}$ is said to model the (intensional) *identity type former* if there are structure maps $(\mathsf{i}, \mathsf{Id})$ making a commutative square,

$$\begin{tikzcd}
\dot{\mathsf{T}} \arrow[r, "\mathsf{i}"] \arrow[d, "\delta"'] & \dot{\mathsf{T}} \arrow[d, "\mathsf{t}"] \\
\dot{\mathsf{T}} \times_{\mathsf{T}} \dot{\mathsf{T}} \arrow[r, "\mathsf{Id}"'] & \mathsf{T}
\end{tikzcd} \tag{4.13}$$

together with a weak pullback structure J for the resulting comparison square, in the sense of (4.15) below.

To describe the map J, let us see how (2) models identity types. Under the interpretation already described in Section **??** the maps $\mathtt{Id}$ and i in

$$\begin{tikzcd}
\dot{\mathsf{T}} \arrow[r, "\mathsf{i}"] \arrow[d, "\delta"'] & \dot{\mathsf{T}} \arrow[d, "\mathsf{t}"] \\
\dot{\mathsf{T}} \times_{\mathsf{T}} \dot{\mathsf{T}} \arrow[r, "\mathsf{Id}"'] & \mathsf{T}
\end{tikzcd}$$

respectively, directly model the formation and introduction rules.

$$x, y : A \vdash \mathtt{Id}_A(x, y)$$
$$x : A \vdash \mathsf{i}(x) : \mathtt{Id}_A(x, x)$$

Next, pull t back along Id to get an object I and a map $\rho : \dot{\mathsf{T}} \to \mathsf{I}$,

$$\begin{tikzcd}
\dot{\mathsf{T}} \arrow[r, "\rho"] \arrow[rr, bend left, "\mathsf{i}"] \arrow[dr, "\delta"'] & \mathsf{I} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{T}} \arrow[d, "\mathsf{t}"] \\
 & \dot{\mathsf{T}} \times_{\mathsf{T}} \dot{\mathsf{T}} \arrow[r, "\mathsf{Id}"'] & \mathsf{T}
\end{tikzcd}$$

which commutes with the compositions to T as indicated below.

$$\begin{tikzcd}
\dot{\mathsf{T}} \arrow[r, "\rho"] \arrow[dr, "\mathsf{t}"'] & I \arrow[d, "q"] \\
 & \mathsf{T}
\end{tikzcd}$$

The map $\rho : \dot{\mathsf{T}} \to \mathsf{I}$, which can be interpreted as the substitution $(x) \mapsto (x, x, \mathsf{i}x)$, gives rise to a "restriction" natural transformation of polynomial endofunctors ([?]),

$$\rho^* : P_q \to P_{\mathsf{t}}\,,$$

evaluation of which at $\mathsf{t} : \dot{\mathsf{T}} \to \mathsf{T}$ results in the following commutative naturality square.

$$\begin{tikzcd}
P_q\dot{\mathsf{T}} \arrow[r, "\rho^*_{\dot{\mathsf{T}}}"] \arrow[d, "P_q\mathsf{t}"'] & P_{\mathsf{t}}\dot{\mathsf{T}} \arrow[d, "P_{\mathsf{t}}\mathsf{t}"] \\
P_q\mathsf{T} \arrow[r, "\rho^*_{\mathsf{T}}"'] & P_{\mathsf{t}}\mathsf{T}
\end{tikzcd} \tag{4.14}$$

A *weak pullback structure* J is a section of the resulting comparison map.

$$\begin{tikzcd}
P_q\dot{\mathsf{T}} \arrow[r] & P_q\mathsf{T} \times_{P_{\mathsf{t}}\mathsf{T}} P_{\mathsf{t}}\dot{\mathsf{T}} \arrow[l, dotted, bend right, "\mathsf{J}"']
\end{tikzcd} \tag{4.15}$$

To show that this models the standard elimination rule

$$\frac{x : A \vdash c(x) : C(\rho x)}{x, y : A, z : \mathtt{Id}_A(x, y) \vdash \mathsf{J}_c(x, y, z) : C(x, y, z)}$$

take any object $\Gamma \in \mathcal{C}$ and maps $(A, A, \mathtt{Id}_A \vdash C) : \Gamma \to P_q\mathsf{T}$ and $(A \vdash c) : \Gamma \to P_{\mathsf{t}}\dot{\mathsf{T}}$ with equal composites to $P_{\mathsf{t}}\mathsf{T}$, meaning that $A \vdash c : C(\rho x)$. Composing the resulting map

$$(A \vdash c(x) : C(\rho x)) : \Gamma \longrightarrow P_q\mathsf{T} \times_{P_{\mathsf{t}}\mathsf{T}} P_{\mathsf{t}}\dot{\mathsf{T}}$$

with $\mathsf{J} : P_q\mathsf{T} \times_{P_\mathsf{t}\mathsf{T}} P_\mathsf{t}\dot{\mathsf{T}} \to P_q\dot{\mathsf{T}}$ then indeed provides a term

$$x : A, y : A, z : \mathtt{Id}_A(x,y) \vdash \mathsf{J}_c(x,y,z) : C(x,y,z)\,.$$

The computation rule

$$x : A \vdash \mathsf{J}_c(\rho x) = c(x) : C(\rho x)$$

then says exactly that J is indeed a section of the comparison map (4.15).

Proposition 4.4.12 (R. Garner). *The map* $\mathsf{t} : \dot{\mathsf{T}} \to \mathsf{T}$ *models intensional* identity types *just if there are maps* $(\mathsf{i}, \mathtt{Id})$ *making the diagram* (4.13) *commute, together with a weak pullback structure* J *for the resulting comparison square* (4.14).

4.4.3 Strictification

Let $\mathsf{t} : \dot{\mathsf{T}} \to \mathsf{T}$ be a natural model over a (small) finitely complete category $\mathbb{C}$, and consider the set

$$\mathcal{D}_\mathsf{t} \subseteq \mathbb{C}_1$$

of all $d : D' \to D$ in $\mathbb{C}$ that occur as pullbacks of t, as in

$$\begin{tikzcd}
\mathsf{y}D' \arrow[r] \arrow[d, "\mathsf{y}d"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{T}} \arrow[d, "\mathsf{t}"] \\
\mathsf{y}D \arrow[r] & \mathsf{T}
\end{tikzcd}$$

Clearly $\mathcal{D}_\mathsf{t}$ is closed under pullbacks in $\mathbb{C}$ and isos in the arrow category $\mathbb{C}^\downarrow$, and so $(\mathbb{C}, \mathcal{D}_\mathsf{t})$ is a *category with display maps* in the sense of [Tay99](§8.3). Such a pair $(\mathbb{C}, \mathcal{D})$ consisting of a finitely complete category $\mathbb{C}$ with a "stable" class of maps $\mathcal{D}$ can be used to basic model dependent type theory as a common generalization of semantics in an LCCC and in a category with an wfs. Additional conditions on $(\mathbb{C}, \mathcal{D})$ are of course needed to model the different type formers, leading to such notions as *clans* and *tribes* [Joy17].

Given a category with display maps $(\mathbb{C}, \mathcal{D})$ we can form the natural transformation $\mathsf{t}_\mathcal{D}$ over $\mathbb{C}$ simply by taking the coproduct of all the display maps $d : D' \to D$ in $\mathcal{D}$:

$$\begin{tikzcd}
\dot{\mathsf{T}}_\mathcal{D} \arrow[r, equal] \arrow[d, "\mathsf{t}_\mathcal{D}"'] & \coprod_{d\in\mathcal{D}} D' \arrow[d, "\coprod_{d\in\mathcal{D}} d"] \\
\mathsf{T}_\mathcal{D} \arrow[r, equal] & \coprod_{d\in\mathcal{D}} D
\end{tikzcd} \tag{4.16}$$

The proof of following is an easy exercise.

Lemma 4.4.13. *The natural transformation* $\mathsf{t}_\mathcal{D} : \dot{\mathsf{T}}_\mathcal{D} \to \mathsf{T}_\mathcal{D}$ *is representable.*

Exercise 4.4.14. Give the proof!

We now have constructions going back and forth:

$$\mathsf{NaturalModels} \underset{\mathcal{D}_{\mathsf{t}}}{\overset{\mathsf{t}_{\mathcal{D}}}{\rightleftarrows}} \mathsf{Cats\ w/DisplayMaps}$$

By defining suitable morphisms on both sides, these operations can be made functorial, and the constructions become adjoint, $\mathsf{t}_{\mathcal{D}} \dashv \mathcal{D}_{\mathsf{t}}$. We can leave the details to the reader.

Finally, by considering the appropriate additional structures on a category with display maps in order to model the type formers (for example local cartesian closure or a right proper weak factorization system), one can refine the adjunction to give a proof of the following "strictification theorem" (cf. [LW15]):

Theorem 4.4.15 ([Awo16]). *If* $(\mathbb{C}, \mathcal{D})$ *is a* Π*-tribe in the sense of Joyal [Joy17] then* $\mathsf{t}_{\mathcal{D}} : \dot{\mathsf{T}}_{\mathcal{D}} \to \mathsf{T}_{\mathcal{D}}$ *is a natural model with the type constructors* $1, \Sigma, \Pi, \mathtt{Id}$*.*

4.5 Universes

We recall the notion of a *Hofmann-Streicher universe*, as reformulated in [Awo24]. For any presheaf X on a small category $\mathbb{C}$, recall that the category of elements is the comma category,

$$\textstyle\int_{\mathbb{C}} X \ = \ \mathsf{y}_{\mathbb{C}}/_X \, ,$$

where $\mathsf{y}_{\mathbb{C}} : \mathbb{C} \to \mathsf{Set}^{\mathbb{C}^{\mathsf{op}}}$ is the Yoneda embedding, which we may supress and write simply as $\mathbb{C}/_X$. The following can be found already in [Gro83](§28).

Proposition 4.5.1. *The category of elements functor* $\int_{\mathbb{C}} : \widehat{\mathbb{C}} \longrightarrow \mathsf{Cat}$ *has a right adjoint, which we denote*

$$\nu_{\mathbb{C}} : \mathsf{Cat} \longrightarrow \widehat{\mathbb{C}} \, .$$

For a small category $\mathbb{A}$*, we call the presheaf* $\nu_{\mathbb{C}}(\mathbb{A})$ *the* $\mathbb{C}$-nerve *of* $\mathbb{A}$*.*

Proof. As suggested by the name, the adjunction $\int_{\mathbb{C}} \dashv \nu_{\mathbb{C}}$ can be seen as the familiar "realization $\dashv$ nerve" construction with respect to the covariant post-composition functor $\mathbb{C}/_{-} : \mathbb{C} \to \mathsf{Cat}$, as indicated below.

$$\begin{tikzcd} \widehat{\mathbb{C}} \arrow[r, shift right, "\int_{\mathbb{C}}"'] & \mathsf{Cat} \arrow[l, shift right, "\nu_{\mathbb{C}}"'] \\ \mathbb{C} \arrow[u, "\mathsf{y}"] \arrow[ur, "\mathbb{C}/_{-}"'] & \end{tikzcd} \tag{4.17}$$

In detail, for $\mathbb{A} \in \mathsf{Cat}$ and $c \in \mathbb{C}$, let $\nu_{\mathbb{C}}(\mathbb{A})(c)$ be the Hom-set of functors,

$$\nu_{\mathbb{C}}(\mathbb{A})(c) = \mathsf{Cat}\big(\mathbb{C}/_c \, , \, \mathbb{A}\big) \, ,$$

with contravariant action on $h : d \to c$ given by pre-composing a functor $P : \mathbb{C}/_c \to \mathbb{A}$ with the post-composition functor

$$\mathbb{C}/_h : \mathbb{C}/_d \longrightarrow \mathbb{C}/_c \,.$$

For the adjunction, observe that the slice category $\mathbb{C}/_c$ is the category of elements of the representable functor $\mathsf{y}c$,

$$\textstyle\int_{\mathbb{C}} \mathsf{y}c \;\cong\; \mathbb{C}/_c \,.$$

Thus for representables $\mathsf{y}c$, we indeed have the required natural isomorphism

$$\widehat{\mathbb{C}}\big(\mathsf{y}c\,,\, \nu_{\mathbb{C}}(\mathbb{A})\big) \;\cong\; \nu_{\mathbb{C}}(\mathbb{A})(c) \;=\; \mathsf{Cat}\big(\mathbb{C}/_c\,,\, \mathbb{A}\big) \;\cong\; \mathsf{Cat}\big(\textstyle\int_{\mathbb{C}} \mathsf{y}c\,,\, \mathbb{A}\big)\,.$$

For arbitrary presheaves X, one uses the presentation of X as a colimit of representables over the index category $\int_{\mathbb{C}} X$, and the easy to prove fact that $\int_{\mathbb{C}}$ itself preserves colimits. Indeed, for any category $\mathbb{D}$, we have an isomorphism in Cat,

$$\varinjlim_{d \in \mathbb{D}} \mathbb{D}/_d \;\cong\; \mathbb{D}\,.$$

$\square$

When $\mathbb{C}$ is fixed, as here, we may omit the subscript from the expressions $\mathsf{y}_{\mathbb{C}}$ and $\int_{\mathbb{C}}$ and $\nu_{\mathbb{C}}$. The unit and counit maps of the adjunction $\int \dashv \nu$,

$$\begin{aligned} \eta : X &\longrightarrow \nu \textstyle\int X\,, \\ \epsilon : \textstyle\int \nu\mathbb{A} &\longrightarrow \mathbb{A}\,, \end{aligned}$$

are as follows. At $c \in \mathbb{C}$, for $x : \mathsf{y}c \to X$, the functor $(\eta_X)_c(x) : \mathbb{C}/_c \to \mathbb{C}/_X$ is just composition with x,

$$(\eta_X)_c(x) = \mathbb{C}/_x : \mathbb{C}/_c \longrightarrow \mathbb{C}/_X\,. \tag{4.18}$$

For $\mathbb{A} \in \mathsf{Cat}$, the functor $\epsilon : \int \nu\mathbb{A} \to \mathbb{A}$ takes a pair $(c \in \mathbb{C}, f : \mathbb{C}/_c \to \mathbb{A})$ to the object $f(1_c) \in \mathbb{A}$,

$$\epsilon(c, f) = f(1_c).$$

Lemma 4.5.2. *For any $f : Y \to X$, the naturality square below is a pullback.*

$$\begin{tikzcd} Y \arrow[r, "\eta_Y"] \arrow[d, "f"'] & \nu\textstyle\int Y \arrow[d, "\nu\int f"] \\ X \arrow[r, "\eta_X"'] & \nu\textstyle\int X. \end{tikzcd} \tag{4.19}$$

Proof. It suffices to prove this for the case $f : X \to 1$. Thus consider the square

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d] & \nu\textstyle\int X \arrow[d] \\ 1 \arrow[r, "\eta_1"'] & \nu\textstyle\int 1. \end{tikzcd} \tag{4.20}$$

Evaluating at $c \in \mathbb{C}$ and applying (4.18) gives the following square in Set.

$$\begin{tikzcd} Xc \arrow[r, "\mathbb{C}/_-"] \arrow[d] & \mathsf{Cat}\big(\mathbb{C}/_c\,, \mathbb{C}/_X\big) \arrow[d] \\ 1c \arrow[r, "\mathbb{C}/_-"'] & \mathsf{Cat}\big(\mathbb{C}/_c\,, \mathbb{C}/_1\big) \end{tikzcd} \tag{4.21}$$

The image of $* \in 1c$ along the bottom is the forgetful functor $U_c : \mathbb{C}/_c \to \mathbb{C}$, and its fiber under the map on the right is therefore the set of functors $F : \mathbb{C}/_c \to \mathbb{C}/_X$ such that $U_X \circ F = U_c$, where $U_X : \mathbb{C}/_X \to \mathbb{C}$ is also a forgetful functor. But any such F is easily seen to be uniquely of the form $\mathbb{C}/_x$ for $x = F(1_c) : \mathsf{y}c \to X$. $\square$

Remark 4.5.3. For the category of elements of the terminal presheaf 1 we have $\int 1 \cong \mathbb{C}$. So for every presheaf X there is a canonical projection $\int X \to \mathbb{C}$, and the functor $\int : \widehat{\mathbb{C}} \to \mathsf{Cat}$ thus factors through the slice category $\mathsf{Cat}/_{\mathbb{C}}$.

$$\begin{tikzcd} \widehat{\mathbb{C}} \arrow[r, "\int/_1"] \arrow[dr, "\int"'] & \mathsf{Cat}/_{\mathbb{C}} \arrow[d, "\mathbb{C}_!"] \\ & \mathsf{Cat} \end{tikzcd} \tag{4.22}$$

The adjunction $\int \dashv \nu : \mathsf{Cat} \to \widehat{\mathbb{C}}$ factors as well, but it is the unfactored adjunction that is more useful for the present purpose.

4.5.1 Classifying families

For every presheaf X the canonical projection $\int X \to \mathbb{C}$ of Remark 4.5.3 is easily seen to be a discrete fibration. It follows that for any natural transformation $Y \to X$ the associated functor $\int Y \to \int X$ is also a discrete fibration. Ignoring size issues (dealt with in [Awo24]), recall that discrete fibrations in Cat are classified by the forgetful functor $\dot{\mathsf{Set}}^{\mathsf{op}} \to \mathsf{Set}^{\mathsf{op}}$ from (the opposites of) the category of pointed sets to that of sets (cf. [?]). For every presheaf $X \in \widehat{\mathbb{C}}$, we therefore have a pullback diagram as follows in Cat.

$$\begin{tikzcd} \int X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{Set}}^{\mathsf{op}} \arrow[d] \\ \mathbb{C} \arrow[r, "X"'] & \mathsf{Set}^{\mathsf{op}} \end{tikzcd} \tag{4.23}$$

Using $\int 1 \cong \mathbb{C}$ and transposing by the adjunction $\int \dashv \nu$ then gives a commutative square in $\widehat{\mathbb{C}}$,

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \nu\dot{\mathsf{Set}}^{\mathsf{op}} \arrow[d] \\ 1 \arrow[r, "\tilde{X}"'] & \nu\mathsf{Set}^{\mathsf{op}}. \end{tikzcd} \tag{4.24}$$

Lemma 4.5.4. *The square* (4.24) *is a pullback in* $\widehat{\mathbb{C}}$*. More generally, for any map* $Y \to X$ *in* $\widehat{\mathbb{C}}$*, there is a pullback square*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \nu\dot{\mathsf{Set}}^{\mathsf{op}} \arrow[d] \\ X \arrow[r] & \nu\mathsf{Set}^{\mathsf{op}}. \end{tikzcd} \tag{4.25}$$

Proof. Apply the right adjoint ν to the pullback square (4.23) and paste the naturality square (4.19) from Lemma 4.5.2 on the left, to obtain the transposed square (4.25) as a pasting of two pullbacks. □

Let us write $\dot{\mathsf{V}} \to \mathsf{V}$ for the vertical map on the right in (4.25); that is, let:

$$\begin{aligned} \dot{\mathsf{V}} &:= \nu\dot{\mathsf{Set}}^{\mathsf{op}} \\ \mathsf{V} &:= \nu\mathsf{Set}^{\mathsf{op}} \end{aligned} \tag{4.26}$$

We can then summarize our results so far as follows.

Proposition 4.5.5. *The* $\mathbb{C}$*-nerve* $\dot{\mathsf{V}} \to \mathsf{V}$ *of the classifier* $\dot{\mathsf{Set}}^{\mathsf{op}} \to \mathsf{Set}^{\mathsf{op}}$ *for discrete fibrations in* Cat*, as defined in* (4.26)*, (weakly) classifies natural transformations* $Y \to X$ *in presheaves* $\widehat{\mathbb{C}}$*, in the sense that there is always a pullback square as follows.*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{V}} \arrow[d] \\ X \arrow[r, "\tilde{Y}"'] & \mathsf{V} \end{tikzcd} \tag{4.27}$$

The classifying map $\tilde{Y} : X \to \mathsf{V}$ *is determined by the adjunction* $\int \dashv \nu$ *as the transpose of the classifying map of the discrete fibration* $\int Y \to \int X$*.*

Of course, as defined in (4.26), the classifier $\dot{\mathsf{V}} \to \mathsf{V}$ cannot be a map in $\widehat{\mathbb{C}}$, for reasons of size; this is addressed in [Awo24].

4.5.2 Type universes

A universe such as $\dot{\mathsf{V}} \to \mathsf{V}$ in a category of presheaves (or indeed a natural model $\mathsf{t} : \dot{\mathsf{T}} \to \mathsf{T}$) can be used to model the following *rules for type universes* in Martin-Löf type theory.

$$\frac{}{\Gamma \vdash \mathsf{U}\ \mathtt{type}} \qquad \frac{\Gamma \vdash a : \mathsf{U}}{\Gamma \vdash \mathsf{El}(a)\ \mathtt{type}}$$

Of course, we interpret the "decoding family" $x : \mathsf{U} \vdash \mathsf{El}(x)$ as the display map $\dot{\mathsf{V}} \to \mathsf{V}$. In practice, one often simply writes

$$\frac{\Gamma \vdash A : \mathsf{U}}{\Gamma \vdash A\ \mathtt{type}}$$

leaving the "decoding" El implicit. The above are the formation and elimination rules for a universe; one might also expect to see an introduction rule such as

$$\frac{\Gamma \vdash A\ \mathtt{type}}{\Gamma \vdash \mathsf{code}(A) : \mathsf{U}}$$

along with computation rules like

$$\mathsf{El}(\mathsf{code}(A)) \equiv A\,, \quad \mathsf{code}(\mathsf{El}(a)) \equiv a\,.$$

Unfortunately, this would imply $\mathsf{U} : \mathsf{U}$, which is inconsistent. Instead of the introduction and computation rules, there are other ways of populating a universe; see [AG](§2.6.2) for a good discussion.

A type universe U is convenient, because it can be used to replace a dependent family of types $\Gamma, x : X \vdash A\ \mathtt{type}$ by a term $\Gamma \vdash A : X \to \mathsf{U}$. It can also used in further constructions, such as the polynomials $\mathsf{P_t}(\mathsf{T})$ from section 4.4.1. See [AG](§2.6.2) for some other important consequences of adding type universes to the type theory, such as proving $\mathtt{true} \neq \mathtt{false} : \mathtt{Bool}$.

Given a universe U, we may regard the types $A : \mathsf{U}$ as *small*. A universe of types interpreted as $\dot{\mathsf{V}} \to \mathsf{V}$ then acts as a "small type classifier," because a display map $A \to X$ interpreting the small type family $X \vdash A\ \mathtt{type}$, is classified by a map $\tilde{A} : X \to \mathsf{V}$, interpreting the term $\vdash A : X \to \mathsf{V}$, as in (4.27).

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{V}} \arrow[d] \\ X \arrow[r, "\tilde{A}"'] & \mathsf{V} \end{tikzcd} \tag{4.28}$$

Unlike with subobject classifiers, however, the classifying map $\tilde{A} : X \to \mathsf{V}$ of a given family $A \to X$ cannot be expected to be unique, essentially because pullbacks are determined only up to isomorphism. That such maps can be chosen uniquely "up to homotopy," in a certain precise sense, is the content of the celebrated *univalence axiom*, to which we now turn.

4.6 Univalence

Let $\dot{\mathsf{U}} \to \mathsf{U}$ be a universe in presheaves, e.g., a natural model as in Section 4.4, or a Hofmann-Streicher universe as in Section 4.5, and suppose that $\dot{\mathsf{U}} \to \mathsf{U}$ supports the structures for $1, \Sigma, \Pi, \mathtt{Id}$ in the sense of Sections 4.4.1–4.4.2. It follows that the maps $A \to X$ that are classified by $\dot{\mathsf{U}} \to \mathsf{U}$, in the sense of (4.28), are closed under those same type formers, in the sense of the "incoherent" interpretation in LCCCs, wfs's, or categories with display maps, regarding objects X as contexts and classified maps $A \to X$ (together with a chosen classifier $\tilde{A} : X \to \mathsf{U}$) as families of types in context X.

An *equivalence* between types $A \simeq B$ (possibly over X) is like an isomorphism of sets, but formulated in a way that is more suitable for types with "higher structure" arising from non-degenerate $\mathtt{Id}$-types. Formally:

Definition 4.6.1. A map $e : A \to B$ is a (type) *equivalence* if it has both right and left inverses, in the sense that there are maps $f, g : B \rightrightarrows A$ such that

$$e \circ f =_{B^B} 1_B \qquad \text{and} \qquad g \circ e =_{A^A} 1_A\,.$$

Formally, given $e : A \to B$ we define

$$\mathsf{isEquiv}(e) \;:\equiv\; \Sigma_{f,g:B\to A}\, \mathtt{Id}_{B^B}(e \circ f, 1_B) \times \mathtt{Id}_{A^A}(g \circ e, 1_A)\,.$$

and then for $A, B : \mathsf{U}$, we define the notation $A \simeq B$ by

$$A \simeq B \;=\; \mathsf{Equiv}(A, B) \;:\equiv\; \Sigma_{e:A\to B}\, \mathsf{isEquiv}(e)\,.$$

We construct the interpretation of the type family $A, B : \mathsf{U} \vdash A \simeq B$ as a map $\mathsf{Equiv} \to \mathsf{U} \times \mathsf{U}$ as follows.

First, pull $\dot{\mathsf{U}} \to \mathsf{U}$ back along the two different projections $\mathsf{U} \times \mathsf{U} \rightrightarrows \mathsf{U}$ to obtain two different objects $\dot{\mathsf{U}}_1, \dot{\mathsf{U}}_2$ over $\mathsf{U} \times \mathsf{U}$, and then take their exponential $[\dot{\mathsf{U}}_1, \dot{\mathsf{U}}_2]$ in the slice category over $\mathsf{U} \times \mathsf{U}$, which interprets the type family $A, B : \mathsf{U} \vdash A \to B$.

$$\begin{tikzcd}
\dot{\mathsf{U}} \arrow[d] & \dot{\mathsf{U}}_1 \arrow[l] \arrow[dr] \arrow[dl, phantom, "\llcorner", very near start] & {[\dot{\mathsf{U}}_1, \dot{\mathsf{U}}_2]} \arrow[d] & \dot{\mathsf{U}}_2 \arrow[r] \arrow[dl] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{U}} \arrow[d] \\
\mathsf{U} & & \mathsf{U} \times \mathsf{U} \arrow[ll] \arrow[rr] & & \mathsf{U}
\end{tikzcd}$$

Indeed, after pulling $\dot{\mathsf{U}}_1, \dot{\mathsf{U}}_2$ back to $[\dot{\mathsf{U}}_1, \dot{\mathsf{U}}_2]$ there is a (universal) arrow $\varepsilon : \dot{\mathsf{U}}_1' \to \dot{\mathsf{U}}_2'$ over $[\dot{\mathsf{U}}_1, \dot{\mathsf{U}}_2]$, as indicated below.

$$\begin{tikzcd}
& \dot{\mathsf{U}}_1' \arrow[rr, "\varepsilon"] \arrow[d] \arrow[dr] \arrow[dr, phantom, "\lrcorner", very near start] & & \dot{\mathsf{U}}_2' \arrow[d] \arrow[dl] \arrow[dl, phantom, "\llcorner", very near start] & \\
\dot{\mathsf{U}} \arrow[d] & \dot{\mathsf{U}}_1 \arrow[l] \arrow[dr] \arrow[dl, phantom, "\llcorner", very near start] & {[\dot{\mathsf{U}}_1, \dot{\mathsf{U}}_2]} \arrow[d] & \dot{\mathsf{U}}_2 \arrow[r] \arrow[dl] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{U}} \arrow[d] \\
\mathsf{U} & & \mathsf{U} \times \mathsf{U} \arrow[ll] \arrow[rr] & & \mathsf{U}
\end{tikzcd} \tag{4.29}$$

As $\mathsf{Equiv} \to \mathsf{U} \times \mathsf{U}$ we then take the composite of the canonical projection $\mathsf{isEquiv}(\varepsilon) \to [\dot{\mathsf{U}}_1, \dot{\mathsf{U}}_2]$ with the map $[\dot{\mathsf{U}}_1, \dot{\mathsf{U}}_2] \to \mathsf{U} \times \mathsf{U}$.

$$
\begin{tikzcd}
\Sigma_{e:A\to B}\mathsf{isEquiv}(e) \arrow[r] \arrow[d, equal] & {[\dot{\mathsf{U}}_1, \dot{\mathsf{U}}_2]} \arrow[d] \\
\mathsf{Equiv} \arrow[r] & \mathsf{U} \times \mathsf{U}
\end{tikzcd}
$$

Given $\tilde{A}, \tilde{B} : X \to \mathsf{U}$ classifying types $A \to X$ and $B \to X$, factorizations of the map $(\tilde{A}, \tilde{B}) : X \to \mathsf{U} \times \mathsf{U}$ through $\mathsf{Equiv} \to \mathsf{U} \times \mathsf{U}$ can then be seen using (4.29) to correspond to equivalences $A \simeq B$ over X, as indicated in the following, in which the indicated map $\mathsf{E}_1 \simeq \mathsf{E}_2$ is the pullback of $\varepsilon : \dot{\mathsf{U}}_1' \to \dot{\mathsf{U}}_2'$ along $\mathsf{Equiv} \to [\dot{\mathsf{U}}_1, \dot{\mathsf{U}}_2]$.

$$
\begin{tikzcd}
 & & \mathsf{E}_1 \arrow[r, "\simeq"] \arrow[dr] & \mathsf{E}_2 \arrow[d] \\
A \arrow[r, "\simeq"] \arrow[urr, dotted] \arrow[dr] & B \arrow[d] \arrow[urr, dotted] & & \mathsf{Equiv} \arrow[d] \\
 & X \arrow[rr, "(\alpha,\beta)"'] \arrow[urr, dotted] & & \mathsf{U} \times \mathsf{U}
\end{tikzcd}
$$

The correspondence is natural in X, because it is mediated by pulling back a universal instance.

If U is itself a type in a "larger" model $\dot{\mathsf{U}}_1 \to \mathsf{U}_1$ with $\mathtt{Id}$-types, then there is also the type family $A, B : \mathsf{U} \vdash \mathtt{Id}_{\mathsf{U}}(A, B)$, which is interpreted as a map $\mathtt{Id}_{\mathsf{U}} \to \mathsf{U} \times \mathsf{U}$. Since for every $A : \mathsf{U}$ the map $1_A : A \to A$ is an equivalence, by $\mathtt{Id}_{\mathsf{U}}$-elim we obtain a distinguished map $\mathtt{Id}_{\mathsf{U}}(A, B) \to \mathsf{Equiv}(A, B)$ over $\mathsf{U} \times \mathsf{U}$. The *univalence axiom* is the statement that this map is itself an equivalence:

$$
\begin{tikzcd}
\mathtt{Id}_{\mathsf{U}} \arrow[r, "\simeq"] \arrow[d] & \mathsf{Equiv} \arrow[d] \\
\mathsf{U} \times \mathsf{U} \arrow[r, equal] & \mathsf{U} \times \mathsf{U}
\end{tikzcd}
\tag{4.30}
$$

Pulling the equivalence back along any $(A, B) : X \to \mathsf{U} \times \mathsf{U}$ results in the more familiar formulation:

$$
(A =_{\mathsf{U}} B) \simeq (A \simeq B). \tag{UA}
$$

The interpretation of univalence thus requires at least a universe U with $1, \Sigma, \Pi, \mathtt{Id}$ (in order to define $A \simeq B$), inside a larger universe $\mathsf{U} : \mathsf{U}_1$ also with $1, \Sigma, \Pi, \mathtt{Id}$ (in order to have the family $A =_{\mathsf{U}} B$ and to state the central equivalence in UA). The operations on the larger universe are usually required to restrict to the corresponding ones on the smaller universe. See [AG] for a detailed presentation.

4.6.1 Function extensionality

4.6.2 The h-levels

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