Introduction to Categorical Logic

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Contents

Chapter 1

Algebraic Theories

Algebraic theories are descriptions of structures that are given entirely in terms of operations and equations. All such algebraic notions have in common some quite deep and general properties, from the existence of free algebras to Lawvere's duality theory. The most basic of these are presented in this chapter. The development also serves as a first example and template for the general scheme of *functorial semantics*, to be applied to other logical notions in later chapters.

1.1 Syntax and semantics

We begin with a general approach to algebraic structures such as groups, rings, and lattices. These are characterized by axiomatizations which involve only a single sort of variables and constants, operations, and equations. It is important that the operations are defined everywhere, which excludes two important examples: fields, because the inverse of 0 is undefined, and categories because composition is defined only for certain pairs of morphisms.

Let us start with the quintessential algebraic theory: the theory of groups. In first-order logic, a group can be described as a set G with a binary operation $\cdot : G \times G \to G$, satisfying the two first-order axioms:

$$\forall\, x, y, z \in G \,.\, (x \cdot y) \cdot z = x \cdot (y \cdot z)$$
$$\exists\, e \in G \,.\, \forall\, x \in G \,.\, \exists\, y \in G \,.\, (e \cdot x = x \cdot e = x \wedge x \cdot y = y \cdot x = e)$$

Taking a closer look at the logical form of these axioms, we see that the second one, which expresses the existence of a unit and inverse elements, is somewhat unsatisfactory because it involves nested quantifiers. Not only does this complicate the interpretation, but it is not really necessary, since the unit element and inverse operation in a group are uniquely determined. Thus we can add them to the structure and reformulate as follows. The unit is to be represented by a distinguished *constant* $e \in G$, and the inverse is to be a unary *operation* ${}^{-1} : G \to G$. We then obtain an equivalent formulation in which all axioms can be expressed as (universally quantified) *equations*:

$$
\begin{gathered}
x \cdot (y \cdot z) = (x \cdot y) \cdot z \\
x \cdot e = x \qquad\qquad e \cdot x = x \\
x \cdot x^{-1} = e \qquad\qquad x^{-1} \cdot x = e
\end{gathered}
$$

The universal quantifiers $\forall\, x \in G, \forall\, y \in G$, etc. are no longer needed in stating the axioms, since we can interpret all variables as ranging over all elements of G (because of our restriction to totally defined operations). Nor do we really need to explicitly mention the particular set G in the specification. Finally, since the constant e can be regarded as a nullary operation, i.e., a function $e : \mathbf{1} \to G$, the specification of the group concept consists solely of operations and equations. This leads to the following general definition of an algebraic theory.

Definition 1.1.1. A *signature* Σ for an algebraic theory consists of a family of sets $\{\Sigma_k\}_{k \in \mathbb{N}}$. The elements of Σ_k are called the *k-ary operations*. In particular, the elements of Σ_0 are the *nullary operations* or *constants*.

The *terms* of a signature Σ are the expressions constructed inductively by the following rules:

1. variables x, y, z, …, are terms,
2. if $t_1, \ldots, t_k$ are terms and $f \in \Sigma_k$ is a k-ary operation then $f(t_1, \ldots, t_k)$ is a term.

Definition 1.1.2 (cf. Definition **??**)**.** An *algebraic theory* $\mathbb{T} = (\Sigma_{\mathbb{T}}, A_{\mathbb{T}})$ is given by a signature $\Sigma_{\mathbb{T}}$ and a set $A_{\mathbb{T}}$ of *axioms*, which are equations between terms (formally, pairs of terms).

Algebraic theories are also called *equational theories*. We do not assume that the sets Σ_k or $A_{\mathbb{T}}$ are finite, but the terms and equations always involve only finitely many variables.

Example 1.1.3. The theory of a commutative ring with unit is an algebraic theory. There are two nullary operations (constants) 0 and 1, a unary operation $-$, and two binary operations $+$ and $\cdot$. The equations are:

$$
\begin{aligned}
(x + y) + z &= x + (y + z) & (x \cdot y) \cdot z &= x \cdot (y \cdot z) \\
x + 0 &= x & x \cdot 1 &= x \\
0 + x &= x & 1 \cdot x &= x \\
x + (-x) &= 0 & (x + y) \cdot z &= x \cdot z + y \cdot z \\
(-x) + x &= 0 & z \cdot (x + y) &= z \cdot x + z \cdot y \\
x + y &= y + x & x \cdot y &= y \cdot x
\end{aligned}
$$

Example 1.1.4. The "empty" theory with no operations and no equations is the theory of a set.

Example 1.1.5. The theory with one constant and no equations is the theory of a *pointed set*, cf. Example **??**.

Example 1.1.6. Let R be a ring. There is an algebraic theory of left R-modules. It has one constant 0, a unary operation $-$, a binary operation $+$, and for each $a \in R$ a unary operation $\overline{a}$, called *scalar multiplication by a*. The following equations hold:

$$\begin{aligned} (x+y)+z &= x+(y+z)\ , & x+y &= y+x\ , \\ x+0 &= x\ , & 0+x &= x\ , \\ x+(-x) &= 0\ , & (-x)+x &= 0\ . \end{aligned}$$

For every $a, b \in R$ we also have the equations

$$\overline{a}(x+y) = \overline{a}\,x + \overline{a}\,y\ , \qquad \overline{a}(\overline{b}\,x) = \overline{(ab)}\,x\ , \qquad \overline{(a+b)}\,x = \overline{a}\,x + \overline{b}\,x\ .$$

Scalar multiplication by a is usually written as $a \cdot x$ instead of $\overline{a}\,x$. If we replace the ring R by a field $\mathbb{F}$ we obtain the algebraic theory of a vector space over $\mathbb{F}$ (even though the theory of fields is not algebraic!).

Example 1.1.7. In computer science, inductive datatypes are examples of algebraic theories. For example, the datatype of binary trees with leaves labeled by integers might be defined as follows in a programming language:

```
type tree = Leaf of int | Node of tree * tree
```

This corresponds to the algebraic theory with a constant `Leaf` n for each integer n and a binary operation `Node`. There are no equations. Actually, when computer scientists define a datatype like this, they have in mind a particular model of the theory, namely the *free* one.

Example 1.1.8. An obvious non-example is the theory of posets, formulated with a binary relation symbol $x \leq y$ and the usual axioms of reflexivity, transitivity and anti-symmetry, namely:

$$\begin{gathered} x \leq x \\ x \leq y\ ,\ y \leq z \Rightarrow x \leq z \\ x \leq y\ ,\ y \leq x \Rightarrow x = x \end{gathered}$$

On the other hand, using an operation of greatest lower bound or "meet" $x \wedge y$, one can make the equational theory of "$\wedge$-semilattices":

$$\begin{gathered} x \wedge x = x \\ x \wedge y = y \wedge x \\ x \wedge (y \wedge z) = (x \wedge y) \wedge z \end{gathered}$$

Then, defining a partial ordering by $x \leq y \iff x \wedge y = x$ we arrive at the notion of a "poset with meets", which *is* equational (of course, the same can be done with joins $x \vee y$ as well). We will show later (in section **??**) that there is no reformulation of the general theory of posets into an equivalent equational one.

Exercise 1.1.9. Let G be a group. Formulate the notion of a (left) G-set (i.e. a functor $G \to \mathsf{Set}$) as an algebraic theory.

1.1.1 Models of algebraic theories

Let us now consider what a *model* of an algebraic theory is. In classical algebra, a group is given by a set G, an element $e \in G$, a function $m : G \times G \to G$ and a function $i : G \to G$, satisfying the group axioms:

$$
\begin{aligned}
m\,(x, m\,(y, z)) &= m\,(m\,(x, y)\,, z) \\
m\,(x, i\,x) &= m\,(i\,x, x) = e \\
m\,(x, e) &= m\,(e, x) = x
\end{aligned}
$$

for any $x, y, z \in G$. Observe that this notion can easily be generalized so that we can speak of models of group theory in categories other than Set. This is accomplished simply by translating the equations between arbitrary elements into equations between the operations themselves: thus a group is given, first, by an object $G \in \mathsf{Set}$ and three morphisms

$$
e : \mathsf{1} \to G\,, \qquad m : G \times G \to G\,, \qquad i : G \to G\,.
$$

The associativity axiom is then expressed by the commutativity of the following diagram:

$$
\begin{tikzcd}
G \times G \times G \arrow[r, "m \times \pi_2"] \arrow[d, "\pi_0 \times m"'] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"'] & G
\end{tikzcd}
\tag{1.1}
$$

Note that we have omitted the canonical associativity function $G \times (G \times G) \cong (G \times G) \times G$, which should be inserted into the top left corner of the diagram. The equations for the unit and the inverse are similarly expressed by commutativity of the following diagrams:

$$
\begin{tikzcd}
G \times \mathsf{1} \arrow[r, "1_G \times e"] \arrow[dr, "\pi_0"'] & G \times G \arrow[d, "m"] & \mathsf{1} \times G \arrow[l, "e \times 1_G"'] \arrow[dl, "\pi_1"] \\
& G &
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "\langle 1_G, i \rangle"] \arrow[d, "!_G"'] & G \times G \arrow[d, "m"] & G \arrow[l, "\langle i, 1_G \rangle"'] \arrow[d, "!_G"] \\
\mathsf{1} \arrow[r, "e"'] & G & \mathsf{1} \arrow[l, "e"]
\end{tikzcd}
\tag{1.2}
$$

This formulation makes sense in any category $\mathcal{C}$ with finite products.

Definition 1.1.10. Let $\mathcal{C}$ be a category with finite products. A *group in* $\mathcal{C}$ consists of an object G equipped with arrows:

$$\begin{tikzcd} G \times G \arrow[r, "m"] & G & G \arrow[l, "i"'] \\ & 1 \arrow[u, "e"'] & \end{tikzcd}$$

such that the above diagrams (1.1) and (1.2) expressing the group equations commute.

There is also an obvious corresponding generalization of a group homomorphism in Set to homomorphisms of groups in $\mathcal{C}$. Namely, an arrow in $\mathcal{C}$ between (the underlying objects of) groups, say $h : M \to N$, is a homomorphism if it commutes with the interpretations of the basic operations m, i, and e,

$$h \circ m^M = m^N \circ h^2 \qquad h \circ i^M = i^N \circ h \qquad h \circ e^M = e^N$$

as indicated in:

$$\begin{tikzcd} M^2 \arrow[r, "h^2"] \arrow[d, "m^M"'] & N^2 \arrow[d, "m^N"] \\ M \arrow[r, "h"'] & N \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "h"] \arrow[d, "i^M"'] & N \arrow[d, "i^N"] \\ M \arrow[r, "h"'] & N \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "="] \arrow[d, "e^M"'] & 1 \arrow[d, "e^N"] \\ M \arrow[r, "h"'] & N \end{tikzcd}$$

Together with the evident composition and identity arrows inherited from $\mathcal{C}$, this gives a category of groups in $\mathcal{C}$, which we denote:

$$\mathsf{Group}(\mathcal{C})$$

In general, we define an *interpretation* I of a theory $\mathbb{T}$ in a category $\mathcal{C}$ with finite products to consist of an object $I \in \mathcal{C}$ and, for each basic operation f of arity k, a morphism $f^I : I^k \to I$. (More formally, I is the tuple consisting of an underlying object $|I|$ and the interpretations f^I, but we shall write simply I for $|I|$.) In particular, basic constants are interpreted as morphisms $1 \to I$. The interpretation is then extended to all terms as follows: a general term t will be interpreted together with a *context of variables* $x_1, \ldots, x_n$ (a list without repetitions), where the variables appearing in t are among those appearing in the context. We write

$$x_1, \ldots, x_n \mid t \tag{1.3}$$

for a term t in context $x_1, \ldots x_n$. The interpretation of such a term in context (1.3) is a morphism $t^I : I^n \to I$, determined by the following specification:

1. The interpretation of a variable x_i among the $x_1, \ldots x_n$ is the i-th projection $\pi_i : I^n \to I$.

2. A term of the form $f\,(t_1, \ldots, t_k)$ is interpreted as the composite:

$$I^n \xrightarrow{\left(t_1{}^I, \ldots, t_k{}^I\right)} I^k \xrightarrow{\qquad f^I \qquad} I$$

where $t_i{}^I : I^n \to I$ is the interpretation of the subterm t_i, for $i = 1, \ldots, k$, and f^I is the interpretation of the basic operation f.

It is clear that the interpretation of a term really depends on the context, and when necessary we shall write $t^I = [\,x_1, \ldots, x_n \mid t\,]^I$. For example, the term $f\,x_1$ is interpreted as a morphism $f^I : I \to I$ in context x_1, and as the morphism $f^I \circ \pi_1 : I^2 \to I$ in the context x_1, x_2.

Suppose u and v are terms in context $x_1, \ldots, x_n$. Then we say that the equation in context $x_1, \ldots, x_n \mid u = v$ is *satisfied* by the interpretation I if u^I and v^I are the same morphism in $\mathcal{C}$. In particular, if $u = v$ is an axiom of the theory, and $x_1, \ldots, x_n$ are all the variables appearing in either u or v, we say that I *satisfies the axiom* $u = v$, written

$$I \models u = v,$$

if $[x_1, \ldots, x_n \mid u]^I$ and $[x_1, \ldots, x_n \mid v]^I$ are the same morphism,

$$\begin{tikzcd} I^n \arrow[r, shift left, "{[x_1, \ldots, x_n \mid u]^I}"] \arrow[r, shift right, "{[x_1, \ldots, x_n \mid v]^I}"'] & I \end{tikzcd} \tag{1.4}$$

We can then define, as expected:

Definition 1.1.11 (cf. Definition **??**)**.** A *model* M of an algebraic theory $\mathbb{T}$ in a category $\mathcal{C}$ with finite products is an interpretation I of the theory that satisfies the axioms of $\mathbb{T}$,

$$I \models u = v,$$

for all $(u = v) \in A_{\mathbb{T}}$.

A *homomorphism* of models $h : M \to N$ is an arrow in $\mathcal{C}$ that commutes with the interpretations of the basic operations,

$$h \circ f^M = f^N \circ h^k$$

for all $f \in \Sigma_{\mathbb{T}}$, as indicated in:

$$\begin{tikzcd} M^k \arrow[r, "h^k"] \arrow[d, "f^M"'] & N^k \arrow[d, "f^N"] \\ M \arrow[r, "h"'] & N \end{tikzcd}$$

The category of $\mathbb{T}$-models in $\mathcal{C}$ is written,

$$\mathsf{Mod}(\mathbb{T}, \mathcal{C}).$$

A model of the empty theory $\mathbb{T}_0$ in a category $\mathcal{C}$ with finite products is just an object $A \in \mathcal{C}$, and a homomorphism is just a map in $\mathcal{C}$, so

$$\mathsf{Mod}(\mathbb{T}_0, \mathcal{C}) = \mathcal{C}.$$

A model of the theory $\mathbb{T}_{\mathsf{Group}}$ of groups in $\mathcal{C}$ is a group in $\mathcal{C}$, in the above sense, and similarly for homomorphisms, so:

$$\mathsf{Mod}(\mathbb{T}_{\mathsf{Group}}, \mathcal{C}) = \mathsf{Group}(\mathcal{C})$$

as defined above. In particular, a model in Set is just a group in the usual sense, so we have:

$$\mathsf{Mod}(\mathbb{T}_{\mathsf{Group}}, \mathsf{Set}) = \mathsf{Group}(\mathsf{Set}) = \mathsf{Group}.$$

An example of a new kind is provided the following.

Example 1.1.12. A model of the theory of groups in a functor category $\mathsf{Set}^{\mathbb{C}}$ is the same thing as a functor from $\mathbb{C}$ into the category groups,

$$\mathsf{Group}(\mathsf{Set}^{\mathbb{C}}) \cong \mathsf{Group}^{\mathbb{C}}.$$

Indeed, for each object $C \in \mathbb{C}$ there is an evaluation functor,

$$\mathsf{eval}_C : \mathsf{Set}^{\mathbb{C}} \to \mathsf{Set}$$

with $\mathsf{eval}_C(F) = F(C)$, and evaluation preserves products since these are computed pointwise in the functor category. Moreover, every arrow $h : C \to D$ in $\mathbb{C}$ gives rise to an obvious natural transformation $h : \mathsf{eval}_C \to \mathsf{eval}_D$. Thus for any group G in $\mathsf{Set}^{\mathbb{C}}$, we have groups $\mathsf{eval}_C(G)$ for each $C \in \mathbb{C}$ and group homomorphisms $h_G : C(G) \to D(G)$, comprising a functor $G : \mathbb{C} \to \mathsf{Group}$. Conversely, it is clear that a functor $H : \mathbb{C} \to \mathsf{Group}$ determines a group H in $\mathsf{Set}^{\mathbb{C}}$ with underlying object $|HC|$, where $| - | : \mathsf{Group} \to \mathsf{Set}$ is the forgetful functor. These constructions are clearly mutually inverse (up to canonical isomorphisms determined by the choice of products). In this way, *a group in the category of variable sets may be regarded as a variable group.*

Exercise 1.1.13. Verify the details of the isomorphism of categories

$$\mathsf{Mod}(\mathbb{T}, \mathsf{Set}^{\mathbb{C}}) \cong \mathsf{Mod}(\mathbb{T}, \mathsf{Set})^{\mathbb{C}}$$

discussed in example 1.1.12 for an arbitrary algebraic theory $\mathbb{T}$.

Exercise 1.1.14. Determine what a group is in the following categories: the category of graphs Graph, the category of topological spaces Top, and the category of groups Group. (Hint: Only the last case is tricky. Before thinking too hard about it, prove the following lemma [Bor94, Lemma 3.11.6], known as the Eckmann-Hilton argument. Let G be a set provided with two binary operations $\cdot$ and $\star$ and a common unit e, so that $x \cdot e = e \cdot x = x \star e = e \star x = x$. Suppose the two operations commute, i.e., $(x \star y) \cdot (z \star w) = (x \cdot z) \star (y \cdot w)$. Then they coincide, are *commutative* and associative.)

1.1.2 Theories as categories

The syntactically presented notion of an algebraic theory is a notational convenience, but as a specification of a mathematical concept, say that of a group, it has some defects. We would prefer a *presentation-free* notion that captures the group concept without tying it to a specific syntactic presentation (the example below indicates why). One such notion can be given by a category with a certain universal property, which determines it uniquely, up to equivalence of categories.

Let us consider group theory again. The algebraic axiomatization in terms of unit, multiplication and inverse is not the only possible one. For example, an alternative formulation uses the unit e and a binary operation $\odot$, called *double division*, along with a single axiom [McC93]:

$$(x \odot (((x \odot y) \odot z) \odot (y \odot e))) \odot (e \odot e) = z \;.$$

The usual group operations are related to double division as follows:

$$x \odot y = x^{-1} \cdot y^{-1} \;, \qquad x^{-1} = x \odot e \;, \qquad x \cdot y = (x \odot e) \odot (y \odot e) \;.$$

There may be practical reasons for prefering one formulation of group theory over another, but this should not determine what the general concept of a group is. For example, we would like to avoid particular choices of basic constants, operations, and axioms. This is akin to the situation where an algebra is presented by generators and relations: the algebra itself is regarded as independent of any particular such presentation. Similarly, one usually prefers a basis-free theory of vector spaces: it is better to formulate the general idea of a vector space without refering explicitly to a basis, even though every vector space has one.

As a first step, we could simply take *all* operations built from unit, multiplication, and inverse as basic, and *all* valid equations of group theory as axioms. But we can go a step further and collect all the operations into a category, thus forgetting about which ones were "basic" and which ones "derived", and which equalities were "axioms". We first describe this construction of a category $\mathcal{C}_{\mathbb{T}}$ for a general algebraic theory $\mathbb{T}$, and then determine another characterization of it.

As objects of $\mathcal{C}_{\mathbb{T}}$ we take *contexts*, i.e. sequences of distinct variables,

$$[x_1, \ldots, x_n] \;. \qquad (n \geq 0)$$

Actually, it will be convenient to take equivalence classes under renaming of variables, so that $[x_1, x_3] = [x_2, x_1]$. That is to say, the objects are just natural numbers.

A morphism from $[x_1, \ldots, x_m]$ to $[x_1, \ldots, x_n]$ is an n-tuple $(t_1, \ldots, t_n)$, where each t_k is a term in the context $x_1, \ldots, x_m$, possibly after renaming the variables. Two such morphisms $(t_1, \ldots, t_n)$ and $(s_1, \ldots, s_n)$ are equal if, and only if, the axioms of the theory imply that $t_k = s_k$ for every $k = 1, \ldots, n$,

$$\mathbb{T} \vdash t_k = s_k$$

Strictly speaking, morphisms are thus (tuples of) *equivalence classes* of terms in context

$$[x_1, \ldots, x_m \mid t_1, \ldots, t_n\,] : [x_1, \ldots, x_m] \longrightarrow [x_1, \ldots, x_n],$$

where two terms are equivalent when the theory proves them to be equal (after renaming the variables). Since it is rather cumbersome to work with such equivalence classes, we shall work with the terms directly, but keeping in mind that equality between them is this equivalence. Note also that the context of the morphism agrees with its domain, so we can omit it from the notation when that domain is clear. The composition of morphisms

$$(t_1, \ldots, t_m) : [x_1, \ldots, x_k] \to [x_1, \ldots, x_m]$$
$$(s_1, \ldots, s_n) : [x_1, \ldots, x_m] \to [x_1, \ldots, x_n]$$

is the morphism $(r_1, \ldots, r_n)$ whose i-th component is obtained by simultaneously substituting in s_i the terms $t_1, \ldots, t_m$ for the variables $x_1, \ldots, x_m$:

$$r_i = s_i[t_1, \ldots, t_m/x_1, \ldots, x_m] \qquad (1 \le i \le n)$$

The identity morphism on $[x_1, \ldots, x_n]$ is $(x_1, \ldots, x_n)$. Using the usual rules of deduction for equational logic, it is easy to verify that these specifications are well-defined on equivalence classes, and therefore make $\mathcal{C}_{\mathbb{T}}$ a category.

Definition 1.1.15. The category $\mathcal{C}_{\mathbb{T}}$ just defined is called the *syntactic category* of the theory $\mathbb{T}$.

The syntactic category $\mathcal{C}_{\mathbb{T}}$ (which may be thought of as the "Lindenbaum-Tarski category" of $\mathbb{T}$, see **??**) contains the same "algebraic" information as the theory $\mathbb{T}$ from which it was built, but in a syntax-invariant way. Two different syntactic presentations of $\mathbb{T}$ — like the ones for groups mentioned above — will give rise to essentially the same category $\mathcal{C}_{\mathbb{T}}$ (i.e. up to isomorphism). In this sense, the category $\mathcal{C}_{\mathbb{T}}$ is the abstract, algebraic object presented by the "generators and relations" (the operations and equations) of $\mathbb{T}$. But there is another, still more important, sense in which $\mathcal{C}_{\mathbb{T}}$ represents $\mathbb{T}$, as we next show.

Exercise 1.1.16. Show that the syntactic category $\mathcal{C}_{\mathbb{T}}$ has all finite products.

1.1.3 Models as functors

Having represented an algebraic theory $\mathbb{T}$ by the syntactic category $\mathcal{C}_{\mathbb{T}}$ constructed from it, we next show that $\mathcal{C}_{\mathbb{T}}$ has the universal property that models of $\mathbb{T}$ correspond uniquely to certain functors from $\mathcal{C}_{\mathbb{T}}$. More precisely, given any category with finite products $\mathcal{C}$ (which we shall call an *FP-category*), there is a natural (in $\mathcal{C}$) equivalence,

$$\frac{\mathcal{M} \in \mathsf{Mod}(\mathbb{T}, \mathcal{C})}{M : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}} \tag{1.5}$$

between models $\mathcal{M}$ of $\mathbb{T}$ in $\mathcal{C}$ and finite product preserving functors ("*FP-functors*") $M : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}$. The equivalence is mediated by a "universal model" $\mathcal{U}$ in $\mathcal{C}_{\mathbb{T}}$, corresponding to the identity functor $1_{\mathcal{C}_{\mathbb{T}}} : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}_{\mathbb{T}}$ in the above displayed correspondence, so that every model $\mathcal{M}$ arises as the functorial image $M(\mathcal{U}) \cong \mathcal{M}$ of $\mathcal{U}$ under an essentially unique FP-functor

$M : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}$. The possibility of such universal models is an advantage of the generalized notion of a model in a category other than Set.

To give the details of the correspondence (1.5), let $\mathbb{T}$ be an arbitrary algebraic theory and $\mathcal{C}_{\mathbb{T}}$ the syntactic category constructed from $\mathbb{T}$ as in Definition 1.1.15. It is easy to show that the product in $\mathcal{C}_{\mathbb{T}}$ of two objects $[x_1, \dots, x_n]$ and $[x_1, \dots, x_m]$ is the object $[x_1, \dots, x_{n+m}]$, and that $\mathcal{C}_{\mathbb{T}}$ has all finite products (including $\mathbf{1} = [-]$, the empty context). Moreover, there is a distinguished $\mathbb{T}$-model U in $\mathcal{C}_{\mathbb{T}}$ consisting of "the language itself", which we call the *syntactic model*: the underlying object is the context $U = [x_1]$ of length one, and each operation symbol f of, say, arity k is interpreted as itself,

$$f^U = [\, x_1, \dots, x_k \mid f(x_1, \dots, x_k)\,] : U^k = [x_1, \dots, x_k] \longrightarrow [x_1] = U.$$

The axioms are all satisfied, because the equivalence relation on terms is determined by $\mathbb{T}$-provability. Explicitly, for terms s, t, we have:

$$U \models s = t \iff s^U = t^U \iff \mathbb{T} \vdash s = t. \tag{1.6}$$

We record this fact as the following.

Proposition 1.1.17. *The syntactic model U in $\mathcal{C}_{\mathbb{T}}$ is "generic" in the sense that it satisfies all and only the $\mathbb{T}$-provable equations.*

Even more importantly, though, the syntactic model U in $\mathcal{C}_{\mathbb{T}}$ has the following *universal property*:

Proposition 1.1.18. *Any model M in any finite product category $\mathcal{C}$ is the image of U under an essentially unique, finite product preserving functor $\mathcal{C}_{\mathbb{T}} \to \mathcal{C}$.*

In this sense, the syntactic category $\mathcal{C}_{\mathbb{T}}$ is the "free finite product category with a model of $\mathbb{T}$". The precise meaning of the proposition is given by the following proof. First, observe that any FP-functor $F : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}$ takes the syntactic model U in $\mathcal{C}_{\mathbb{T}}$ (or indeed any model) to a model FU in $\mathcal{C}$, with interpretations

$$f^{FU} = Ff^U : FU^k \to FU \qquad \text{for each } f \in \Sigma_k.$$

Moreover, any natural transformation $\vartheta : F \to G$ between FP-functors determines a homomorphism of models $h = \vartheta_U : FU \to GU$. In more detail, suppose $f : U \times U \to U$ is a basic operation, then there is a commutative diagram,

$$\begin{tikzcd} FU \times FU \arrow[r, "h \times h"] \arrow[d, "\cong"] \arrow[dd, bend right, "f^{FU}"'] & GU \times GU \arrow[d, "\cong"'] \arrow[dd, bend left, "f^{GU}"] \\ F(U \times U) \arrow[r, "\vartheta_{U\times U}"] \arrow[d, "Ff"] & G(U \times U) \arrow[d, "Gf"'] \\ FU \arrow[r, "h = \vartheta_U"'] & GU \end{tikzcd}$$

where the upper square commutes by preservation of products, and the lower one by naturality. Thus the operation "evaluation at U" determines a functor,

$$\mathsf{eval}_U : \mathsf{Hom}_{\mathrm{FP}}(\mathcal{C}_{\mathbb{T}}, \mathcal{C}) \longrightarrow \mathsf{Mod}(\mathbb{T}, \mathcal{C}) \tag{1.7}$$

from the category of finite product preserving functors $\mathcal{C}_{\mathbb{T}} \to \mathcal{C}$, with natural transformations as arrows, into the category of $\mathbb{T}$-models in $\mathcal{C}$.

Proposition 1.1.19. *The functor* (1.7) *is an equivalence of categories, natural in* $\mathcal{C}$.

Proof. Let M be any model in an FP-category $\mathcal{C}$. Then the assignment $f \mapsto f^M$ given by the interpretation part of M determines a functor $M^\sharp : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}$, defined on objects by

$$M^\sharp[x_1, \ldots, x_k] = M^k$$

and on morphisms by

$$M^\sharp(t_1, \ldots, t_n) = \langle t_1{}^M, \ldots, t_n{}^M \rangle.$$

In more detail, $M^\sharp$ is defined on morphisms

$$[x_1, \ldots, x_k \mid t\,] : [x_1, \ldots, x_k] \to [x_1, \ldots, x_n]$$

in $\mathcal{C}_{\mathbb{T}}$ by the following rules:

1. The morphism

$$(x_i) : [x_1, \ldots, x_k] \to [x_1]$$

is mapped to the i-th projection

$$\pi_i : M^k \to M.$$

2. The morphism
$$(f(t_1,\dots,t_m)) : [x_1,\dots,x_k] \to [x_1]$$
is mapped to the composite
$$M^k \xrightarrow{(M^\sharp t_1,\dots,M^\sharp t_m)} M^m \xrightarrow{\quad M^\sharp f \quad} M$$
where $M^\sharp t_i : M^k \to M$ is the value of $M^\sharp$ on the morphisms $(t_i) : [x_1,\dots,x_k] \to [x_1]$, for $i = 1,\dots,m$, and $M^\sharp f = f^M$ is the interpretation of the basic operation f.

3. The morphism
$$(t_1,\dots,t_n) : [x_1,\dots,x_k] \to [x_1,\dots,x_n]$$
is mapped to the morphism $\langle M^\sharp t_1,\dots,M^\sharp t_n\rangle$ where $M^\sharp t_i$ is the value of $M^\sharp$ on the morphism $(t_i) : [x_1,\dots,x_k] \to [x_1]$, and
$$\langle M^\sharp t_1,\dots,M^\sharp t_n\rangle : M^k \longrightarrow M^n$$
is the evident n-tuple in the FP-category $\mathcal{C}$.

That $M^\sharp : \mathcal{C}_\mathbb{T} \to \mathcal{C}$ really is a functor now follows from the assumption that the interpretation M is a model, meaning that all the equations of the theory are satisfied by it, so that the above specification is well-defined on equivalence classes. Observe that the functor $M^\sharp$ is defined in such a way that it obviously preserves finite products, and that, moreover, there is an isomorphism of models,
$$M^\sharp(U) \cong M.$$
Thus we have shown that the functor "evaluation at U",
$$\mathsf{eval}_U : \mathsf{Hom}_{\mathrm{FP}}(\mathcal{C}_\mathbb{T},\mathcal{C}) \longrightarrow \mathsf{Mod}(\mathbb{T},\mathcal{C}) \tag{1.8}$$
is essentially surjective on objects, since $\mathsf{eval}_U(M^\sharp) = M^\sharp(U) \cong M$.

We leave the verification that it is full and faithful as an easy exercise.

Exercise 1.1.20. Verify this.

Finally, naturality in $\mathcal{C}$ means the following. Suppose M is a model of $\mathbb{T}$ in any FP-category $\mathcal{C}$. Any FP-functor $F : \mathcal{C} \to \mathcal{D}$ to another FP-category $\mathcal{D}$ then takes M to a model $F(M)$ in $\mathcal{D}$. As for the special case of U, the interpretation is given by setting $f^{F(M)} = F(f^M)$ for the basic operations f (and composing with the canonical isos coming from preservation of products, $F(M) \times F(M) \cong F(M \times M)$, etc.). Since equations are described by commuting diagrams, F takes a model to a model, and the same is true for homomorphisms. Thus $F : \mathcal{C} \to \mathcal{D}$ induces a functor on $\mathbb{T}$-models,
$$\mathsf{Mod}(\mathbb{T},F) : \mathsf{Mod}(\mathbb{T},\mathcal{C}) \longrightarrow \mathsf{Mod}(\mathbb{T},\mathcal{D}).$$

By naturality of (1.7), we mean that the following square commutes up to natural isomorphism:

$$\begin{tikzcd} \mathsf{Hom}_{\mathrm{FP}}(\mathcal{C}_{\mathbb{T}}, \mathcal{C}) \arrow[r, "\mathsf{eval}_U"] \arrow[d, "\mathsf{Hom}_{\mathrm{FP}}(\mathcal{C}_{\mathbb{T}}, F)"'] & \mathsf{Mod}(\mathbb{T}, \mathcal{C}) \arrow[d, "\mathsf{Mod}(\mathbb{T}, F)"] \\ \mathsf{Hom}_{\mathrm{FP}}(\mathcal{C}_{\mathbb{T}}, \mathcal{D}) \arrow[r, "\mathsf{eval}_U"'] & \mathsf{Mod}(\mathbb{T}, \mathcal{C}) \end{tikzcd} \tag{1.9}$$

But this is clear, since for any FP-functor $M : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}$ we have:

$$\begin{aligned} \mathsf{eval}_U \circ \mathsf{Hom}_{\mathrm{FP}}(\mathcal{C}_{\mathbb{T}}, F)(M) &= (\mathsf{Hom}_{\mathrm{FP}}(\mathcal{C}_{\mathbb{T}}, F)(M))(U) \\ &= (F \circ M)(U) \\ &= F(M(U)) \\ &= F(\mathsf{eval}_U(M)) \\ &\cong \mathsf{Mod}(\mathbb{T}, F)(\mathsf{eval}_U(M)) \\ &= \mathsf{Mod}(\mathbb{T}, F) \circ \mathsf{eval}_U(M). \end{aligned}$$

□

The equivalence of categories

$$\mathsf{Hom}_{\mathrm{FP}}(\mathcal{C}_{\mathbb{T}}, \mathcal{C}) \simeq \mathsf{Mod}(\mathbb{T}, \mathcal{C}) \tag{1.10}$$

actually determines $\mathcal{C}_{\mathbb{T}}$ and the universal model U uniquely, up to equivalence of categories and isomorphism of models. Indeed, to recover U, just put $\mathcal{C}_{\mathbb{T}}$ for $\mathcal{C}$ and the identity functor $1_{\mathcal{C}_{\mathbb{T}}}$ on the left, to get U in $\mathsf{Mod}(\mathbb{T}, \mathcal{C}_{\mathbb{T}})$ on the right! To see that $\mathcal{C}_{\mathbb{T}}$ itself is also determined, observe that (1.10) says that the functor $\mathsf{Mod}(\mathbb{T}, \mathcal{C})$ is representable, with representing object $\mathcal{C}_{\mathbb{T}}$, in an appropriate (i.e. bicategorical) sense. As usual, this fact can also be formulated in elementary terms as a universal mapping property of $\mathcal{C}_{\mathbb{T}}$, as follows:

Definition 1.1.21. The *classifying category* of an algebraic theory $\mathbb{T}$ is an FP-category $\mathcal{C}_{\mathbb{T}}$ with a distinguished model U, called the *universal model*, such that:

(i) for any model M in any FP-category $\mathcal{C}$, there is an FP-functor

$$M^{\sharp} : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}$$

and an isomorphism of models $M \cong M^{\sharp}(U)$.

(ii) for any FP-functors $F, G : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}$ and model homomorphism $h : F(U) \to G(U)$, there is a unique natural transformation $\vartheta : F \to G$ with

$$\vartheta_U = h.$$

Observe that (i) says that the evaluation functor (1.7) is essentially surjective, and (ii) that it is full and faithful. The category $\mathcal{C}_{\mathbb{T}}$ is then determined, up to equivalence, by this universal mapping property. Specifically, if $(\mathcal{C}, U)$ and $(\mathcal{D}, V)$ are both classifying categories for the same theory, then there are classifying functors,

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "V^\sharp"] & \mathcal{D} \arrow[l, bend left, "U^\sharp"] \end{tikzcd}$$

the composites of which are necessarily isomorphic to the respective identity functors, since e.g. $U^\sharp(V^\sharp(U)) \cong U^\sharp(V) \cong U$.

We have now shown not only that every algebraic theory has a classifying category, but also that the syntactic category is essentially determined by that distinguishing property. We record this as the following.

Theorem 1.1.22. *Every algebraic theory $\mathbb{T}$ has the syntactic category $\mathcal{C}_{\mathbb{T}}$ as a classifying category.*

Example 1.1.23. Let us see explicitly what the foregoing definitions give us in the case of the theory of groups $\mathbb{T}_{\mathsf{Group}}$. Let us write $\mathbb{G} = \mathcal{C}_{\mathbb{T}_{\mathsf{Group}}}$ for the classifying category, which consists of contexts $[x_1, \ldots, x_n]$ as objects, and terms built from variables and the group operations (modulo renaming and the group laws) as arrows. A finite product preserving functor $M : \mathbb{G} \to \mathsf{Set}$ is then determined uniquely, up to natural isomorphism, by its action on the context $[x_1]$ and the terms representing the basic operations. If we set

$$\begin{aligned} G &= M[x_1]\,, & e &= M(\cdot \mid e)\,, \\ i &= M(x_1 \mid {x_1}^{-1})\,, & m &= M(x_1, x_2 \mid x_1 \cdot x_2)\,, \end{aligned}$$

then (G, e, i, m) is just a group, with unit e, inverse i, and multiplication m. That G satisfies the axioms for groups follows from the functoriality of M and preservation of finite products, which implies preservation of the corresponding commutative diagrams. Conversely, any group (G, e, i, m) determines a finite product preserving functor $M_G : \mathbb{G} \to \mathsf{Set}$, by setting $M_G[x_1] = G$, etc. Thus $\mathsf{Mod}_{\mathsf{Set}}(\mathbb{G})$ will indeed be equivalent to Group once we show that both categories have the same notion of morphisms. This is shown just as in the general case above.

Example 1.1.24. Recall from 1.1.12 that a group G in the functor category $\mathsf{Set}^{\mathbb{C}}$ is essentially the same thing as a functor $G : \mathbb{C} \to \mathsf{Group}$. From the point of view of algebras as functors, this amounts to the observation that product-preserving functors $\mathbb{G} \to \mathsf{Hom}(\mathbb{C}, \mathsf{Set})$ correspond (by exponential transposition) to functors $\mathbb{C} \to \mathsf{Hom}_{\mathsf{FP}}(\mathbb{G}, \mathsf{Set})$, where the latter Hom-set consists just of product-preserving functors. Indeed, the correspondence extends to natural transformations to give the previously observed equivalance of categories,

$$\mathsf{Group}(\mathsf{Set}^{\mathbb{C}}) \simeq (\mathsf{Group}(\mathsf{Set}))^{\mathbb{C}} \simeq \mathsf{Group}^{\mathbb{C}}.$$

1.1.4 Completeness

Consider an algebraic theory $\mathbb{T}$ and an equation $s = t$ between terms of the theory. If the equation can be proved from the axioms of the theory, $\mathbb{T} \vdash s = t$, then every model M of the theory satisfies the equation, $M \models s = t$; this is the *soundness* of the equational calculus with respect to models in categories. The converse statement is:

$$M \models s = t \text{ for all } M \quad \Rightarrow \quad \mathbb{T} \vdash s = t\ .$$

This is called *completeness*, and (together with soundness) it says that the calculus of equations suffices for proving all (and only) the ones that hold in the semantics. For functorial semantics, this holds in an especially strong way: by Proposition 1.1.17, we already know that the syntactic model $U \in \mathsf{Mod}_{\mathbb{C}}(\mathbb{T})$ is generic, in the sense that satisfaction by U is equivalent to provability in $\mathbb{T}$,

$$U \models s = t \iff \mathbb{T} \vdash s = t.$$

But since U is also universal, it follows immediately that we also have soundness and completeness:

Theorem 1.1.25 (Soundness and Completeness)**.** *For any equation $s = t$, we have $\mathbb{T} \vdash s = t$ if and only if every model M in every FP-category $\mathcal{C}$ satisfies $s = t$.*

Proof. If $\mathbb{T} \vdash s = t$, then by Proposition 1.1.17 (the syntactic construction of $\mathcal{C}_{\mathbb{T}}$) we have $U \models s = t$, meaning that $s^U = t^U$. But then for any model M in an FP-category $\mathcal{C}$, we obtain $M \models s = t$ by applying the classifying functor $M^\sharp : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}$, which preserves the interpretations of s and t,

$$M^\sharp(s^U) = s^{M^\sharp(U)} = s^M$$

and so from $s^U = t^U$ we can infer $s^M = t^M$.

Conversely, if $M \models s = t$ for every model M, then in particular $U \models s = t$, and so $\mathbb{T} \vdash s = t$, since U is generic. □

Classically, it is seldom the case that there exists a generic model; instead, we consider the class of all models in Set. Completeness with respect to such a restricted class of models is of course a stronger statement than completeness with respect to all models in all categories. Toward the classical result, we first consider completeness with respect to "variable models" in Set, i.e. in presheaf categories $\mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$.

Proposition 1.1.26. *Let $\mathbb{T}$ be an algebraic theory. The Yoneda embedding*

$$\mathsf{y} : \mathcal{C}_{\mathbb{T}} \to \widehat{\mathcal{C}_{\mathbb{T}}}$$

is a generic model for $\mathbb{T}$.

Proof. The Yoneda embedding $\mathsf{y} : \mathcal{C}_{\mathbb{T}} \to \widehat{\mathcal{C}_{\mathbb{T}}}$ preserves limits, and in particular finite products, hence it determines a model $\mathsf{y}U$ in the category of presheaves $\widehat{\mathcal{C}_{\mathbb{T}}}$. Like all models, $\mathsf{y}U$ satisfies all the equations that hold in U, simply because y is an FP functor, but because it is also faithful, any equation that holds in $\mathsf{y}U$ must already hold in U. □

Example 1.1.27. We consider group theory one last time. As a presheaf on (the classifying category of) the theory of groups $\mathbb{G}$, the generic group $\mathsf{y}U$ satisfies every equation that is satisfied by all groups, and no others. Let us describe its underlying object explicitly as a variable set. The presheaf $\mathsf{y}U$ is represented by the context with one variable, $U = [x_1]$,

$$\mathsf{y}U = \mathsf{y}[x_1] = \mathbb{G}(-, [x_1]) .$$

The values of this functor are thus a family of sets, parametrized by the objects of $\mathbb{G}$; namely, for every $n \in \mathbb{N}$, we have the set

$$\mathsf{y}U_n = \mathbb{G}([x_1, \dots, x_n], [x_1])$$

consisting of all terms in n variables, modulo the equations of group theory; but this is just the set of elements of the *free group* $F(n)$ on n generators! Thus we have

$$\mathsf{y}U_n = \mathbb{G}([x_1, \dots, x_n], [x_1]) \cong |F(n)| = \mathsf{Set}(1, |F(n)|) \cong \mathsf{Group}(F(1), F(n)).$$

Moreover, the unit, inverse, and multiplication operations on $\mathsf{y}U$ agree at each stage $\mathsf{y}U_n$ with the operations on the free group $F(n)$ (as the reader should verify).

To summarize, the presheaf of groups $\mathsf{y}U$ on the theory $\mathbb{G}$ is naturally isomorphic to the presheaf of free groups $F(x_1, \dots, x_n)$ on n-generators. We will see below in more detail why this is so.

Finally, we consider completeness with respect to Set-valued models $M : \mathcal{C}_{\mathbb{T}} \to \mathsf{Set}$, which of course correspond to classical groups. We need the following:

Lemma 1.1.28. *For any small category $\mathbb{C}$, there is a jointly faithful family $(E_i)_{i \in I}$ of FP-functors $E_i : \mathsf{Set}^{\mathbb{C}^{\mathsf{op}}} \to \mathsf{Set}$, with I a set. That is, for any maps $f, g : A \to B$ in $\mathsf{Set}^{\mathbb{C}^{\mathsf{op}}}$, if $E_i(f) = E_i(g)$ for all $i \in I$, then $f = g$.*

Proof. We can take the evaluation functors $\mathrm{ev}_C : \mathsf{Set}^{\mathbb{C}^{\mathsf{op}}} \to \mathsf{Set}$ for all $C \in \mathbb{C}$. These are clearly jointly faithful, and they preserve all limits and colimits, which are constructed pointwise in presheaves. □

Proposition 1.1.29. *Suppose $\mathbb{T}$ is an algebraic theory. For every equation $s = t$ between terms of the theory $\mathbb{T}$,*

$$M \models s = t \quad \textit{for all models } M \textit{ in } \mathsf{Set} \iff \mathbb{T} \vdash s = t.$$

Thus the equational calculus of algebraic theories is complete with respect to Set-valued semantics.

Proof. Combine the foregoing lemma with the fact, from Proposition 1.1.26, that the Yondea embedding is a generic model. □

Exercise 1.1.30. We described the object part of the representable functor $\mathsf{y}U$ given by the universal group U as a functor $\mathbb{G}^{\mathsf{op}} \to \mathsf{Set}$, in terms of the free groups $F(n)$. What is the action of $\mathsf{y}U$ on morphisms in $\mathbb{G}$, in these terms? Also describe the group structure on $\mathsf{y}U$ in $\widehat{\mathbb{G}}$ explicitly.

Exercise 1.1.31. Let s be a term of group theory with variables $x_1, \ldots, x_n$. On the one hand we can think of s as an element of the free group $F(n)$, and on the other we can consider the interpretation of s with respect to the universal group U, namely as a natural transformation $s^U : U^n \Longrightarrow U$. Suppose t is another term of group theory with variables $x_1, \ldots, x_n$. Show that $s^U = t^U$ if, and only if, $s = t$ in the free group $F(n)$.

1.1.5 Functorial semantics

Let us summarize our treatment of algebraic theories thus far. We have reformulated certain traditional *logical* notions in terms of *categorical* ones. The traditional approach may be described as involving the four different parts:

Type theory
: There is an underlying type theory consisting of types and terms. For algebraic theories there is only one type, which is not even explicitly mentioned. The terms are built from variables and basic operations.

Logic
: Algebraic theories have a very simple kind of logic that involves only equations between terms and equational reasoning, i.e. substitution of equals for equals.

Theory
: A theory is given by basic terms and axioms. The terms are expressed in the type theory, and the axioms are expressed in the logic.

Semantics
: The type theory and logic can be interpreted in categories of the appropriate kind. For algebraic theories we used categories with finite products. The interpretation is determined by induction on the structure of the types, terms, and formulas (the equations). It is *denotational* in the sense that the types and terms of the theory "denote" the objects and morphisms. An interpretation of a theory is a *model* if it satisfies all the axioms of the theory, where in the present case the notion of satisfaction just means that the arrows interpreting the terms occurring in the equations are actually equal.

The alternative approach of *functorial semantics* developed here may be summarized as follows:

Theories are categories
: From a given theory we constructed a category which expresses essentially the same information, but is syntax-invariant, in the sense that it does not depend on a particular presentation by (basic) operations and axioms. The structure of the category reflects the underlying type theory and logic. For example, single sorted algebraic theories give rise to categories with finite products.

Models are functors

A model is a (structure-preserving) functor from the category representing the theory to a category with appropriate structure to interpret the logic. The requirement that all axioms of the theory must be satisfied by a model translates to the requirement that the model is a functor, and that it preserves the structure of the category representing the theory. For models of algebraic theories, we only required that they preserve finite products, which along with functoriality ensures that all valid equations of the theory are preserved, and the axioms are therefore satisfied.

Homomorphisms are natural transformations

We obtain the notion of a homomorphisms of models for free: since models are functors, homomorphisms between them are natural transformations. Homomorphisms between models of algebraic theories turned out to be the usual notion of homomorphisms that respect the algebraic structure.

Universal model

By admitting models in categories other than Set, functorial semantics admits *universal models*: a model U in the classifying category $\mathcal{C}_{\mathbb{T}}$, such that any model anywhere is a functorial image of U by an essentially unique, structure-preserving functor. Such a universal model is then "logically generic", in the sense that it has all and only those logical properties had by all models, since such properties are preserved by the functors in question.

Logical completeness

The construction of the classifying category from the logical syntax of the theory shows the *soundness and completeness* of the theory with respect to general categorical semantics. Completeness with respect to a restricted class of models (e.g. Set-valued ones) results from an embedding theorem for the classifying category.

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