

Categorical Logic

- Outline :

- 1) Review of CT
- 2) Propositional Calculus
- 3) λ -Calculus

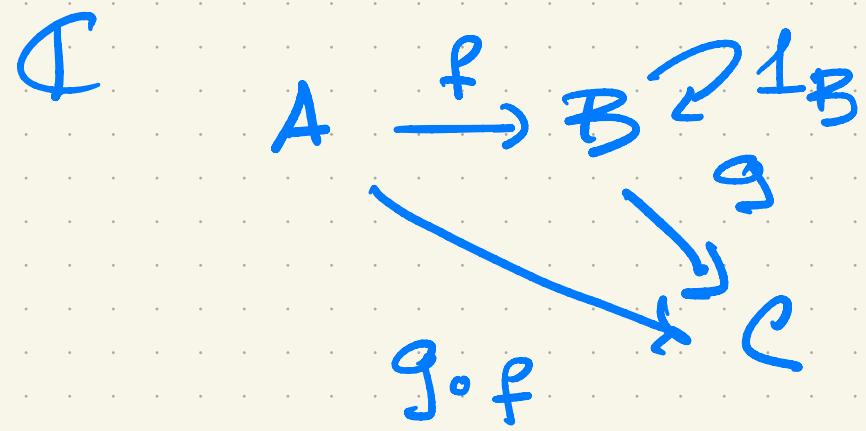
- Notes :

awodey.github.io

1. Review of CT

Basic Def.s :

Category

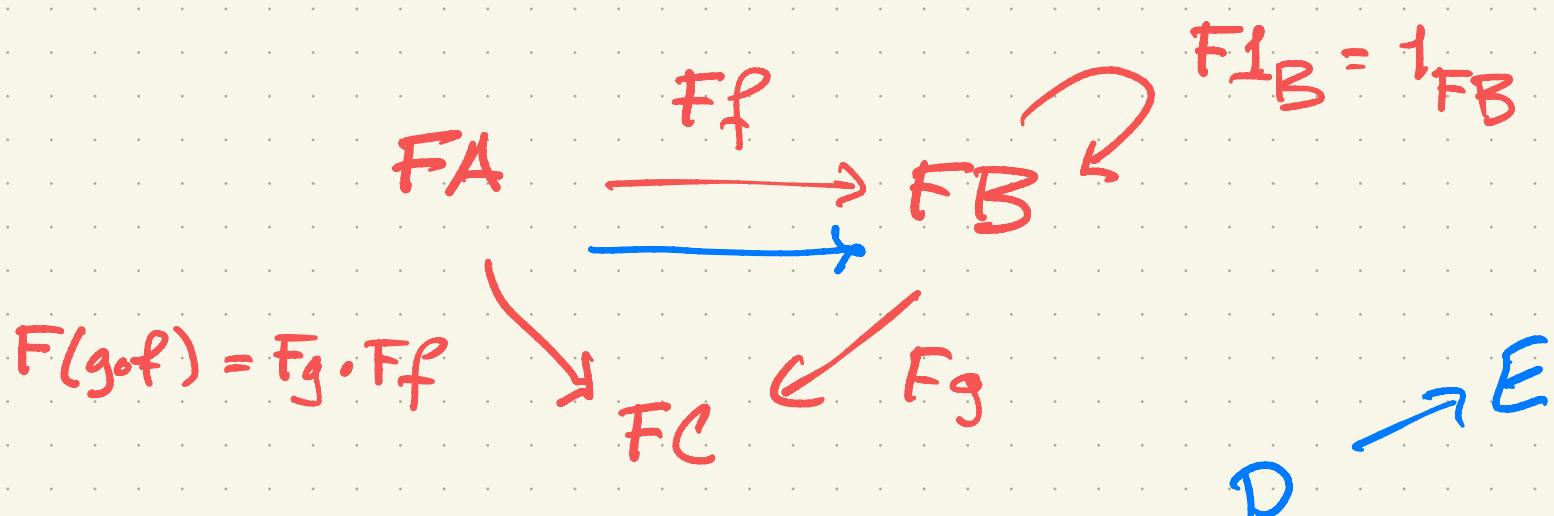
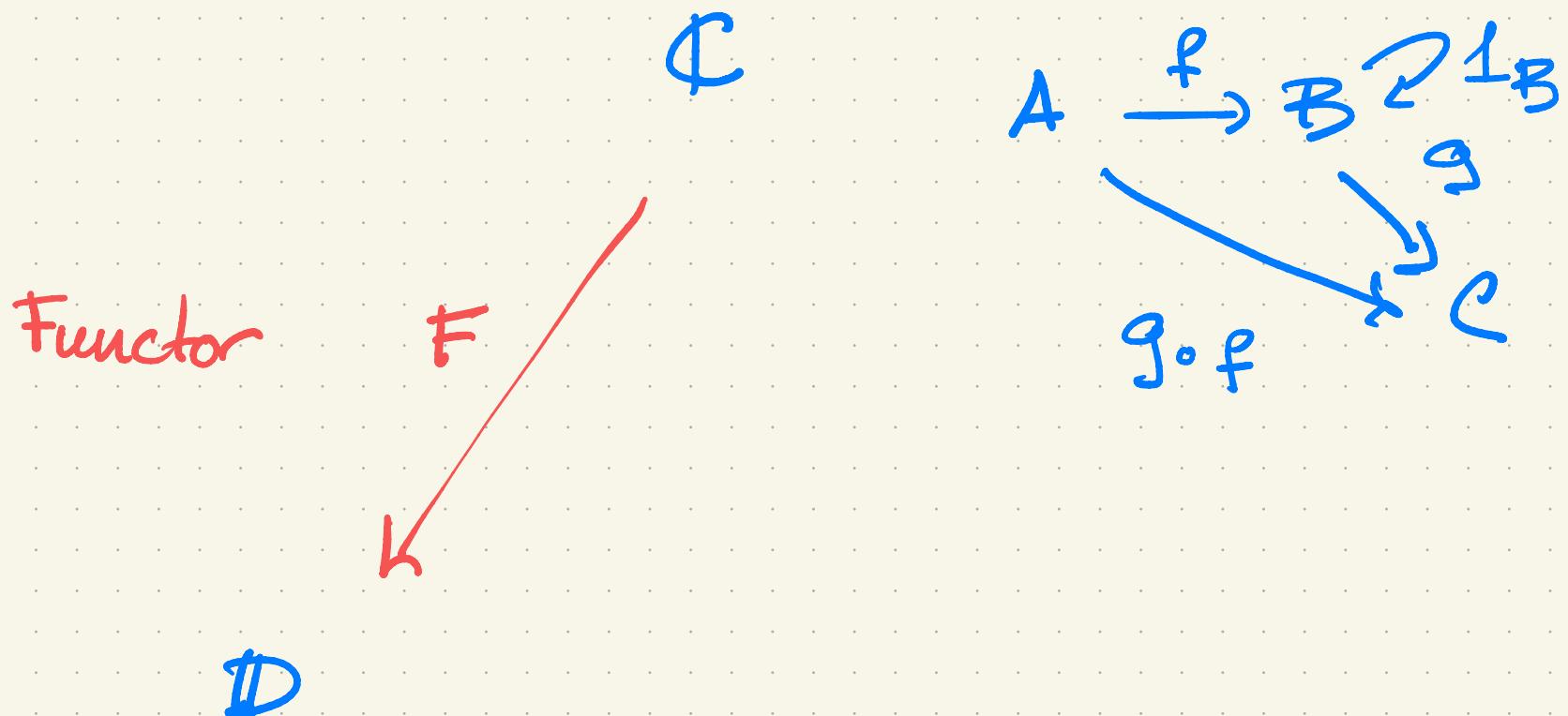


1. Review of CT

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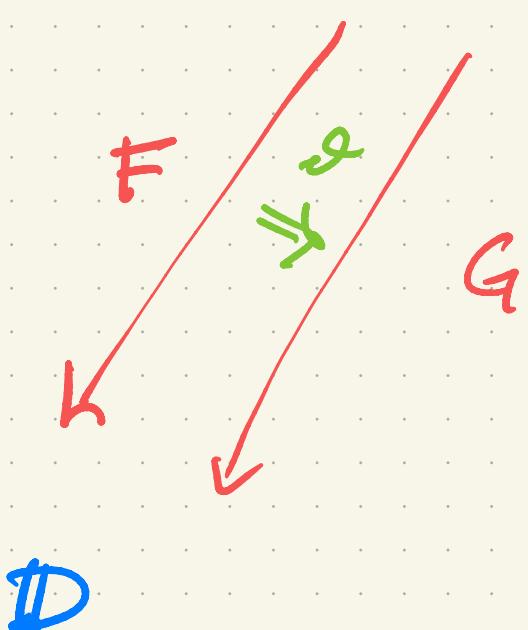
(1)



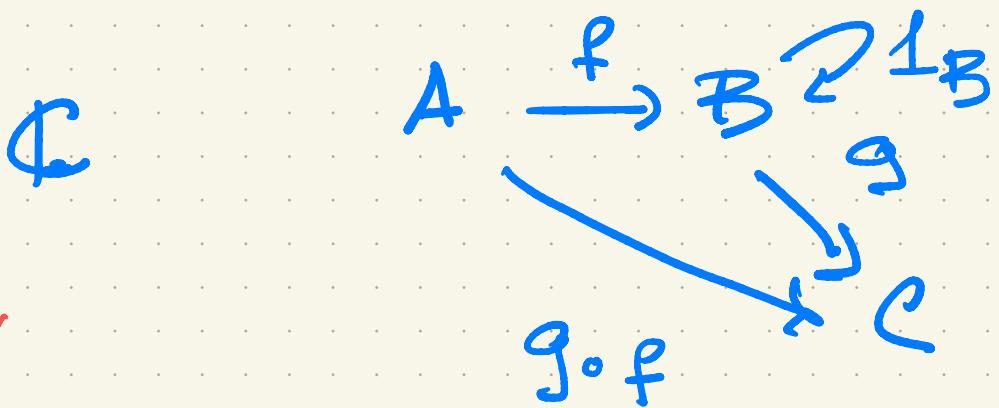
1. Review of CT

Basic Def.s :

Functor



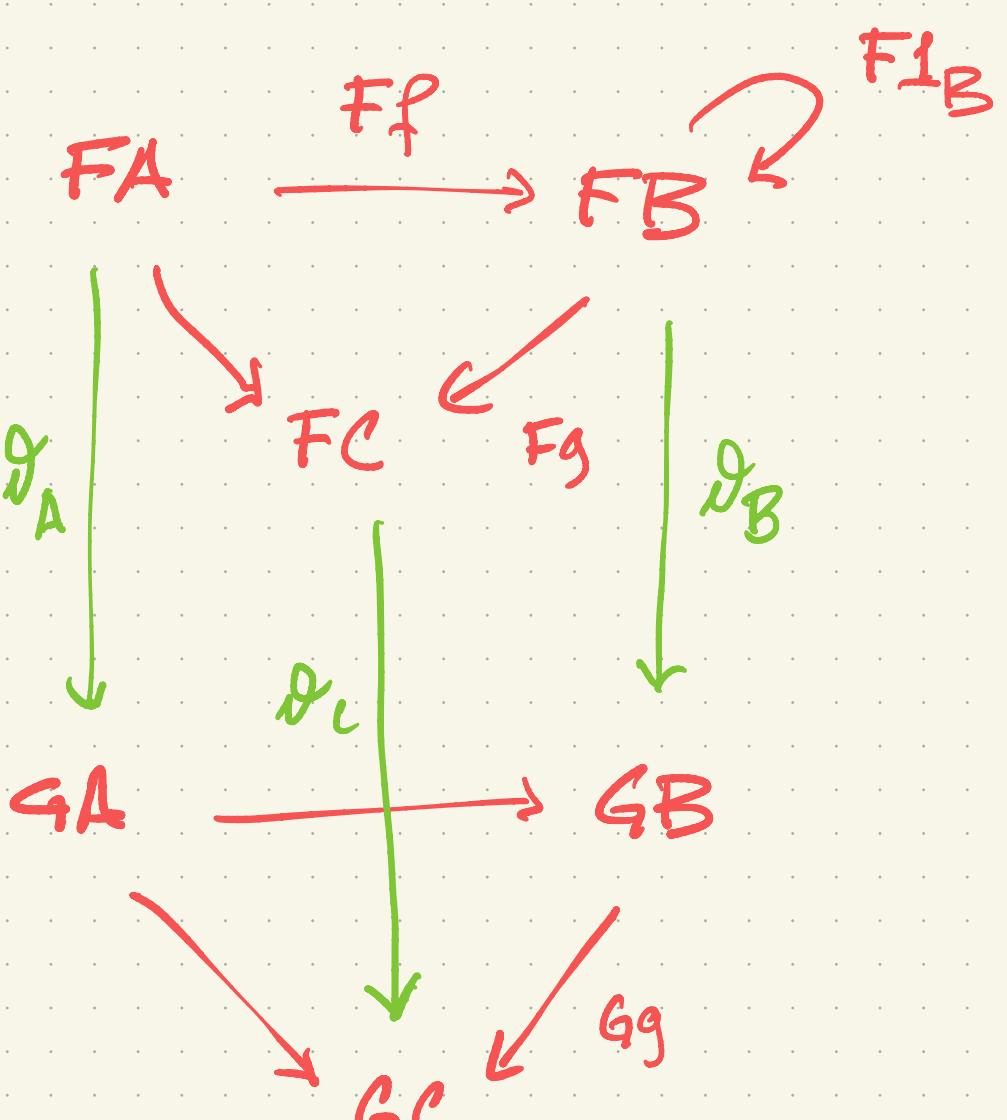
Category



Natural Transformation

$$(\delta_c : FC \rightarrow GC)_{c \in C}$$

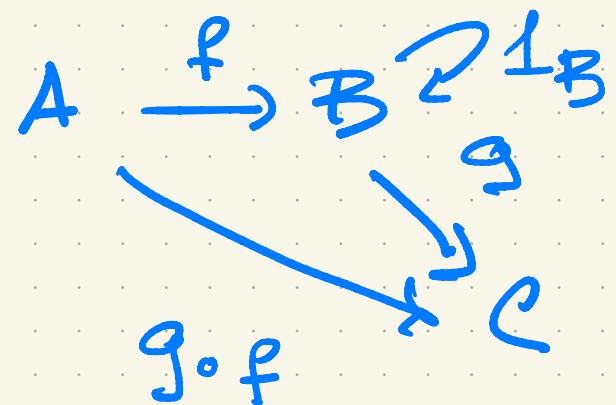
$$\delta_c \circ Fg = Gg \circ \delta_B$$



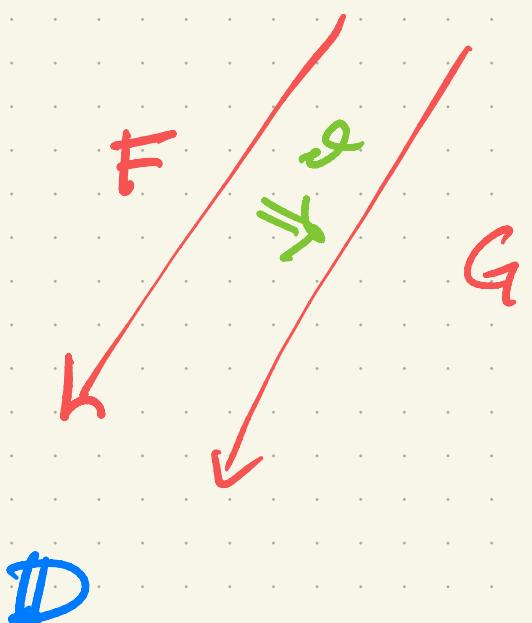
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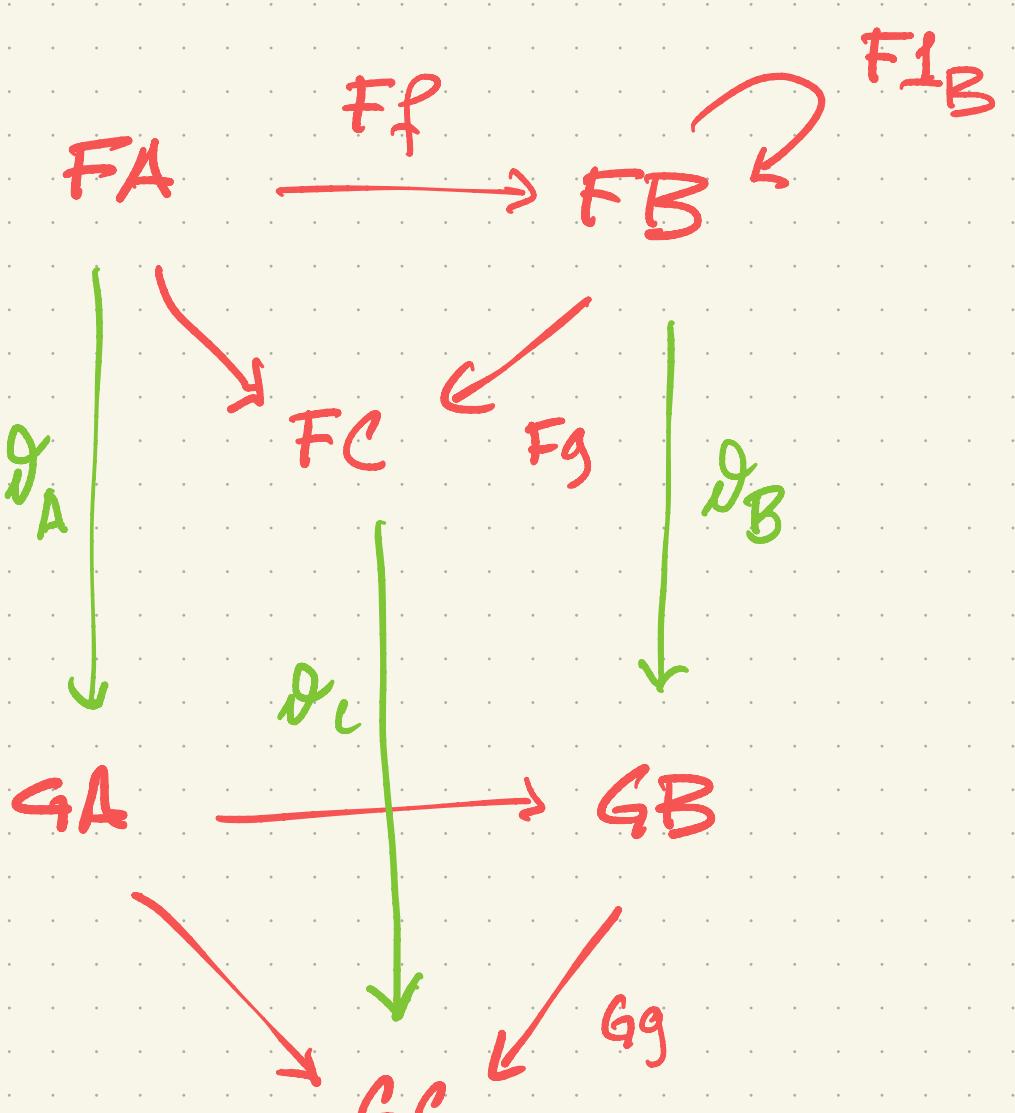
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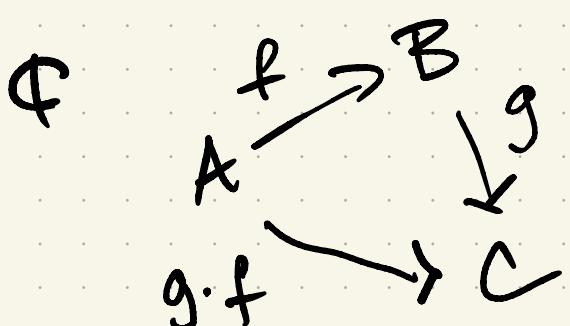
Idea : " $\delta: F \rightarrow G$ maps

the structure F

to the structure G . "

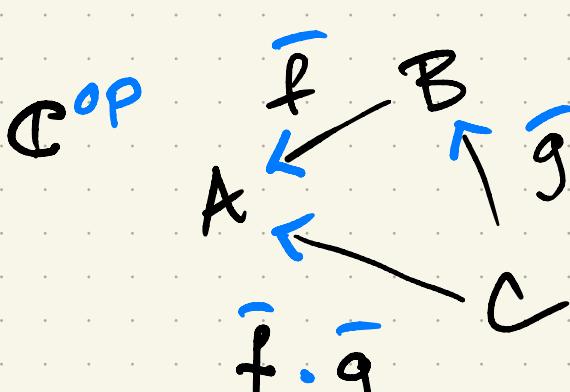
Basic Examples:

(2)

- Set = Sets & functions
- Pos = Posets & monotone maps
- Top = Spaces & cts. maps
- Grp = Groups & homomorphisms
- Cat = Categories & functors
- P a poset : objects: P, q, \dots
arrows: $P \leq q$
- X a space : $C(X) = \text{Top}(X, \mathbb{R})$
 $f \leq g := f_x \leq g_x \quad \forall x \in X.$
- C, D cats : $D^C = \text{Cat}(C, D)$ functor cat
objects: functors $F: C \rightarrow D$
arrows: nat. transf. $\delta: F \Rightarrow G$
- C cat : $\hat{C} := \text{Set}^{C^{\text{op}}}$ presheaves


$$f: A \xrightarrow{\quad} B$$

$$g: B \xrightarrow{\quad} C$$

$$g \circ f: A \xrightarrow{\quad} C$$
- C^{op} cat : $\hat{C}^{\text{op}} := \text{Set}^{C}$ presheaves


$$f: A \xleftarrow{\quad} B$$

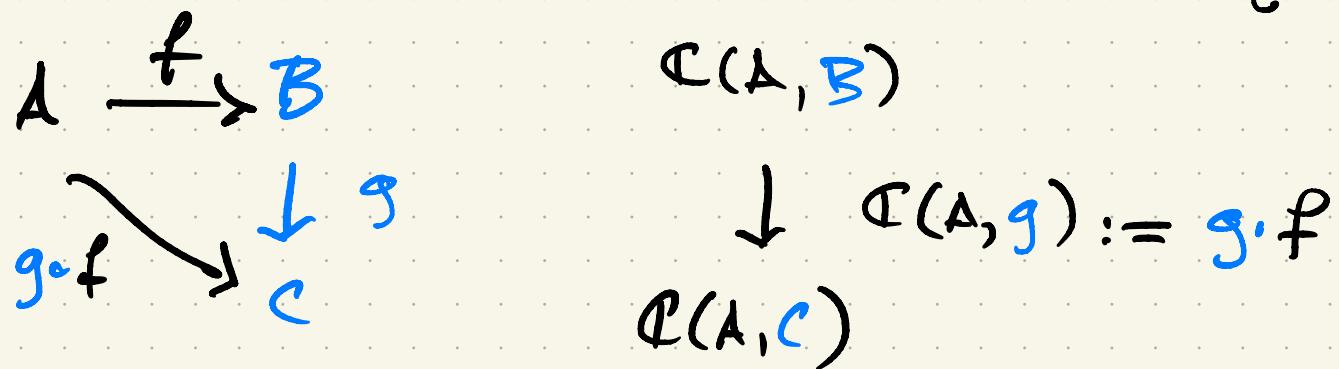
$$g: B \xrightarrow{\quad} C$$

$$f \circ g: A \xrightarrow{\quad} C$$

Representable functor

$\text{Set}^{\mathbb{C}} \ni \mathbb{C}(A, -) : \mathbb{C} \rightarrow \text{Set}$

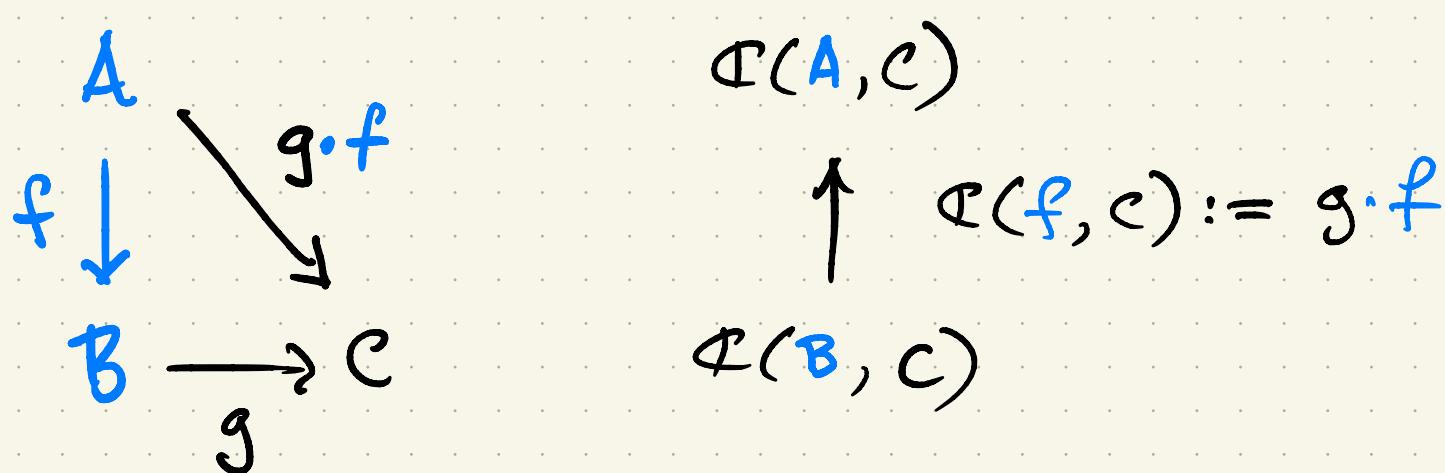
$$\begin{aligned} A \in \mathbb{C} \quad & B \mapsto \mathbb{C}(A, B) = \text{Hom}_{\mathbb{C}}(A, B) \\ & = \{ f : A \rightarrow B \} \end{aligned}$$



Contravariant version

$\text{Set}^{\mathbb{C}^{\text{op}}} \ni \mathbb{C}(-, C) : \mathbb{C}^{\text{op}} \rightarrow \text{Set}$

$$C \in \mathbb{C} \quad B \mapsto \mathbb{C}(B, C)$$



The contravariant representable is written:

$$y\mathbb{C} := \mathbb{C}(-, C) : \mathbb{C}^{\text{op}} \rightarrow \text{Set}$$

Def. The Yoneda embedding is the functor

$$y : \mathcal{C} \longrightarrow \text{Set}^{\mathcal{C}^{\text{op}}}$$

$$c \longmapsto \mathcal{C}(-, c)$$

with action on arrows in \mathcal{C} given by composition:

$$\begin{array}{ccc} c & & yc \\ h \downarrow & \rightsquigarrow & \Downarrow & yh \\ D & & & yD \end{array},$$

$$\begin{array}{ccc} B \xrightarrow{g} C & & \mathcal{C}(B, C) \\ \downarrow h & & \downarrow \mathcal{C}(B, h) =: (yh)_B \\ h \circ g \rightarrow D & & \mathcal{C}(B, D) \end{array},$$

Lemma (Yoneda) For $c \in \mathcal{C}$, $F \in \text{Set}^{\mathcal{C}^{\text{op}}}$,

$$Fc \cong \text{Hom}(yc, F) \quad \text{nat. in } C \& F.$$

So natural transf.s $\alpha : yc \rightarrow F$

Correspond to elements $a \in Fc$

Prop. The Yoneda embedding is full & faithful:

$$y : \mathcal{C}(C, D) \xrightarrow{\cong} \text{Set}^{\mathcal{C}^{\text{op}}}(yc, yD)$$

$$c \xrightarrow{h} D$$

$$yc \xrightarrow{yh} yD$$

Pf. $\text{Hom}(yc, yD) \cong (yD)c = \mathcal{C}(C, D)$

Adjoints Let \mathcal{C}, \mathcal{D} cats. (5)

Def. An adjunction consists of:

• functors

• a bijection

$$\mathcal{D}(FC, D) \cong \mathcal{C}(C, UD)$$

$$\begin{array}{ccc} & \mathcal{D} & \\ F \uparrow & \dashv & \downarrow U \\ C & & \end{array} \quad \begin{array}{c} FC \rightarrow D \\ \hline \hline \\ C \rightarrow UD \end{array} \quad \text{nat. in } \mathcal{C}, \mathcal{D}$$

Examples:

1) Posets P, Q & monotone $P \xrightleftharpoons[f]{g} Q$ s.t. $f^{-1}g :$

$$P \leq g q \quad \text{iff} \quad f_p \leq q$$

E.g. take $Q = P \times P$ and $P \xrightarrow[\Delta]{} P \times P$.

$$P \mapsto (p, p)$$

• What is Δ^{-1} ?

$$(p, p) \leq (a, b) \Leftrightarrow p \leq a \text{ & } p \leq b$$

$$\Leftrightarrow p \leq a \wedge b$$

$$P \times P$$

$$\Delta \uparrow -1 \quad \nwarrow$$

Adjoints Let \mathcal{C}, \mathcal{D} cats. (5)

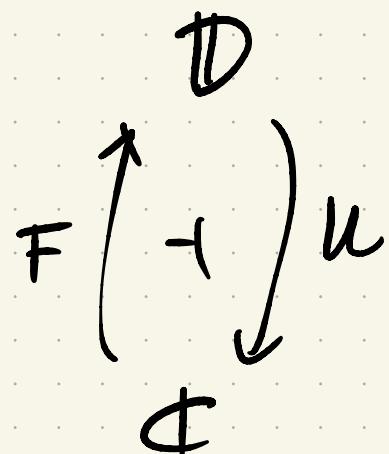
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$$\frac{FC \rightarrow D}{C \rightarrow UD}$$

nat. in C, D

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$$\Leftrightarrow P \leq a \wedge b$$

* What about $? - \Delta$

$$\checkmark \left(\begin{array}{c} P \times P \\ \dashv \Delta \dashv - \\ P \end{array} \right)_\wedge$$

$$(a, b) \leq (P, P)$$

$$P \times P$$

$$a \leq P \wedge b \leq P$$

$$P$$

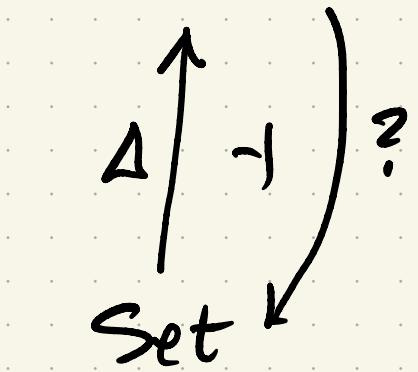
$$a \vee b \leq P$$

$$P$$

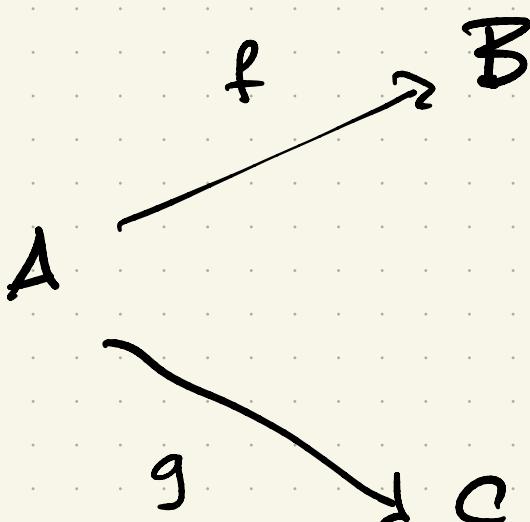
2) Replace P by Set :

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Set \times Set



$(A, A) \rightarrow (B, C)$



so $\Delta \vdash x$.

$A \xrightarrow{(f,g)} B \times C$

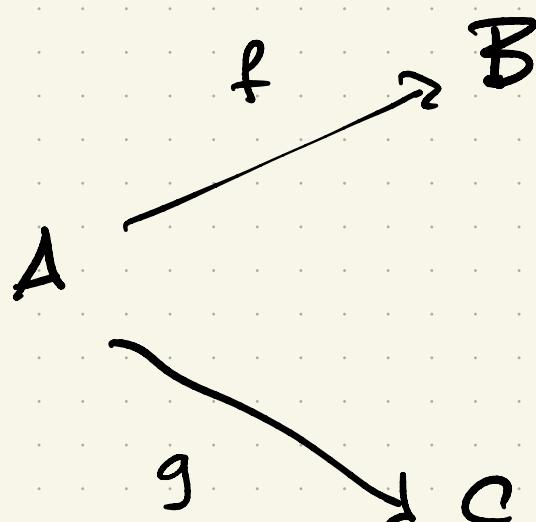
2) Replace P by Set : (6)

Set \times Set

$$\begin{array}{c} \Delta \uparrow \dashv \\ + \end{array} \quad ?$$

Set

$$(A, A) \longrightarrow (B, C)$$



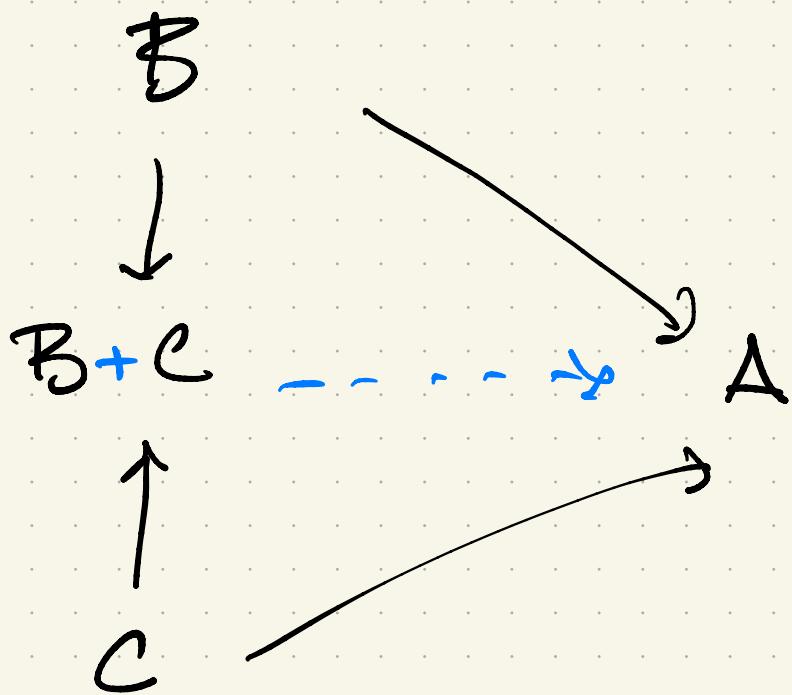
so $\Delta \dashv +$.

$$A \longrightarrow B \times C$$

(f, g)

and of course :

$$\begin{array}{c} \text{Set} \times \text{Set} \\ + \quad \Delta \uparrow \dashv \\ \text{Set} \end{array}$$



BTW : An adjunction

$$F \dashv U$$

is always mediated by
two distinguished maps :

$$\begin{array}{c} \frac{F \dashv U}{\text{unit}} \\ \eta: F(C) \longrightarrow UC \\ \text{counit} \\ \epsilon: CU \longrightarrow D \end{array}$$

E.g. the unit of $+ \dashv \Delta$ is

$$B \xrightarrow{i_1} B+C \xleftarrow{i_2} C$$

$$\frac{\text{unit}}{UD \xrightarrow{\eta} UD}$$

$$\frac{\text{counit}}{FUD \xrightarrow{\epsilon} D}$$

"counit"

3) We also have $\Delta \dashv \times$ in Pos, Cat, Grp, Top, ...

- In Pos, fix P and consider the functor

$$(-) \times P : \text{Pos} \rightarrow \text{Pos}$$

- This has a right adjoint:

$$A \times P \longrightarrow B$$

$$\hline \hline$$

$$A \longrightarrow B^P = \text{Pos}(P, B)$$

$$f \leq g :=$$

$$f_P \leq g_P \forall_P$$

- We also have such right adjoints

$$(-) \times A \dashv (-)^A \quad \text{Called } \underline{\text{exponentials}}$$

in Set, Cat, Set^C , ...

Def. A cat C is cartesian closed

if there are right adjoints:

$$\begin{array}{c} \bullet \quad 1 \\ \vdash \uparrow \dashv \\ \text{C} \end{array}$$

$$\begin{array}{c} C \times C \\ \Delta \uparrow \dashv \\ C \end{array}$$

$$\begin{array}{c} C \\ (-) \times A \uparrow \dashv \\ C \end{array}$$

$$(A \in C)$$

4) P poset has V if for all $S \subseteq P$, (8)

there's $\bigvee S \in P$ s.th.

$$\bigvee S \leq_P \Leftrightarrow s \leq_P \text{ for all } s \in S .$$

This says that $V \vdash \downarrow$:

$$\begin{array}{ccc} \partial P & & S \subseteq \downarrow p = \{x \in P\} \\ \downarrow V \quad \uparrow \downarrow (-) & & \hline \hline \\ P & & \bigvee S \leq p \end{array}$$

Prop. The poset \hat{P} of downsets,

$$\downarrow: P \rightarrow \hat{P} := \{D \subseteq P \mid x \leq d \in D \Rightarrow x \in D\}$$

is the free completion of P under V :

$$\begin{array}{ccc} \hat{P} & & \text{(i)} \hat{P} \text{ has V} \\ \downarrow (-) \quad \uparrow \bar{f} & & \\ P & \xrightarrow{f} & Q \end{array}$$

(ii) for any Q w/V &
any $f: P \rightarrow Q$

there's a unique \bar{f}
that preserves V &

$$\bar{f} \downarrow P = f_P .$$

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- (ii) for any Q w/ V & any $f: P \rightarrow Q$
there's a unique \bar{f}
that preserves \bigvee &

So:

$$\bar{f}|_P = f_P .$$

$$V\text{-Pos}(\hat{P}, Q) \cong \text{Pos}(P, Q) .$$

Prop. For any poset P , the poset \hat{P}
is cartesian closed & the map

$$\downarrow(-) : P \rightarrow \hat{P}$$

preserves any CC structure in P .

Pf. $T \& \lambda$ in \hat{P} as in $\wp P$.

$$\begin{aligned} A \Rightarrow B &= \bigvee \{D \mid D \wedge A \subseteq B\} \\ &= \{p \in P \mid \downarrow p \cap A \subseteq B\}. \end{aligned}$$

For any downset C ,

$$C \leq A \Rightarrow B \text{ iff } \downarrow C \cap A \subseteq B \text{ f.a. } c \in C.$$

So:

$$C \cap A = (\bigcup_{c \in C} \downarrow c) \cap A = \bigcup_{c \in C} (\downarrow c \cap A) \subseteq B.$$

Then:

$$\downarrow T = P$$

$$\downarrow(a \wedge b) = \downarrow a \wedge \downarrow b$$

$$\downarrow(a \Rightarrow b) = \downarrow a \Rightarrow \downarrow b$$

Note: the map $\downarrow(-) : P \rightarrow \hat{P}$ reflects \leq :

$$\downarrow p \leq \downarrow q = p \in \downarrow q = p \leq q.$$

In particular \downarrow is injective.