Kripke-Joyal Forcing for Martin-Löf Type Theory

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Motivation

- Martin Löf type theory (MLTT) is common generalization of first-order logic (FOL) and the simply-typed lambda calculus, and is a powerful and expressive system of formal logic.
- It serves as the basis of Homotopy Type Theory, as well as several computer proof systems such as Agda, Coq, and Lean.
- It is a challenging problem to give semantics for MLTT that are both precise enough to strictly model the syntax and yet flexible enough to admit basic mathematical constructions.
- Kripke-Joyal forcing provides such semantics for both FOL and HOL and is here generalized to MLTT.

Kripke-Joyal forcing for FOL

Let $\mathbb{C}$ be a small category. For the topos of presheaves, write

$$\widehat{\mathbb{C}} = [\mathbb{C}^{\mathsf{op}}, \mathsf{Set}]\,.$$

We interpret a FOL formula $x{:}X \mid \phi$ over $X \in \widehat{\mathbb{C}}$ as a subobject,

$$\{x{:}X \mid \phi\} \rightarrowtail X\,.$$

Definition. Let $x : \mathsf{y}c \to X$. We say that x **forces** ϕ **at stage** c, if there is a factorization as on the right below.

$$c \Vdash \phi(x)$$

$$\begin{tikzcd}
 & \{x{:}X \mid \phi\} \arrow[d, tail] \\
\mathsf{y}c \arrow[ur, dotted, bend left] \arrow[r, "x"'] & X
\end{tikzcd}$$

Kripke-Joyal forcing for FOL

Remark

- If $c \Vdash \phi(x)$ for *all* elements $x : \mathsf{y}c \to X$ we then have

$$\{x{:}X \mid \phi\} \cong X,$$

- If ϕ is *closed* we then have

$$\{\phi\} \cong 1\,.$$

- We then say that ϕ **holds** on $\mathbb{C}$ and write

$$\mathbb{C} \Vdash \phi\,.$$

Kripke-Joyal forcing for FOL

Key fact: We can recursively unwind the condition $c \Vdash \phi(x)$ according to the structure of ϕ,

$$
\begin{array}{lll}
c \Vdash \phi(x) \vee \psi(x) & \text{iff} & c \Vdash \phi(x) \text{ or } c \Vdash \psi(x) \\
c \Vdash \phi(x) \wedge \psi(x) & \text{iff} & c \Vdash \phi(x) \text{ and } c \Vdash \psi(x) \\
c \Vdash \phi(x) \Rightarrow \psi(x) & \text{iff} & d \Vdash \phi(xf) \text{ implies } d \Vdash \psi(xf), \text{ for all } f : d \to c \\
c \Vdash \exists y.\vartheta(x, y) & \text{iff} & c \Vdash \vartheta(x, y) \text{ for some } y : \mathrm{y}c \to Y \\
c \Vdash \forall y.\vartheta(x, y) & \text{iff} & d \Vdash \vartheta(xf, y) \text{ for all } f : d \to c \text{ and } y : \mathrm{y}d \to Y
\end{array}
$$

This provides a quasi-mechanical procedure for determining whether a formula holds in a model.

Kripke-Joyal forcing for MLTT

For MLTT we instead need to force a *dependent type*.

$$x : X \vdash A$$

This is interpreted as a map $A \to X$ (an indexed family A_x), rather than a mere subobject $\{x : X \mid \phi\} \rightarrowtail X$.

We will also need to *force a term*.

$$x : X \vdash {\color{red} t : } A$$

This is interpreted as a section.

$$\begin{tikzcd} A \arrow[d] \\ X \arrow[u, bend left, dotted, "{\color{red} t}"] \end{tikzcd}$$

Kripke-Joyal forcing for MLTT

To force terms **in stages** we use **partial** sections.

$$c \Vdash t_x : A_x$$

$$\begin{tikzcd} & A \arrow[d] \\ \mathrm{y}c \arrow[ur, dotted, bend left, "t_x"] \arrow[r, "x"'] & X \end{tikzcd}$$

Changing stages along $f : d \to c$ results in a **coherence condition**.

$$\frac{c \Vdash t_x : A_x}{d \Vdash t_{xf} : A_{xf}}$$

$$\begin{tikzcd} & & A \arrow[d] \\ \mathrm{y}d \arrow[urr, dotted, bend left, "t_{xf}"] \arrow[r, "f"'] & \mathrm{y}c \arrow[ur, dotted, bend left, "t_x"'] \arrow[r, "x"'] & X \end{tikzcd}$$

Kripke-Joyal forcing for MLTT

This requires a stricter interpretation than in the propositional case:

$$\frac{c \Vdash \phi(x)}{d \Vdash \phi(xf)}$$

$$\begin{tikzcd}
& & \{x : X \mid \phi\} \arrow[d, tail] \\
\mathsf{y}d \arrow[r, "f"'] \arrow[urr, dotted, bend left] & \mathsf{y}c \arrow[r, "x"'] \arrow[ur, dotted, bend left] & X
\end{tikzcd}$$

Kripke-Joyal forcing for MLTT

We will use a *universe* to ensure coherence.

$c \Vdash t_x : \alpha(x)$

$$\begin{tikzcd} & X.\alpha \arrow[r] \arrow[d] & \dot{\mathsf{U}} \arrow[d] \\ \mathsf{y}c \arrow[ur, dotted, bend left, "t_x"] \arrow[r, "x"'] & X \arrow[r, "\alpha"'] & \mathsf{U} \end{tikzcd}$$

This is like using the *subobject classifier* to interpret FOL.

$c \Vdash \phi(x)$

$$\begin{tikzcd} & \{x : X \mid \phi\} \arrow[r] \arrow[d, tail] & 1 \arrow[d] \\ \mathsf{y}c \arrow[ur, dotted, bend left] \arrow[r, "x"'] & X \arrow[r, "\phi"'] & \Omega \end{tikzcd}$$

Kripke-Joyal forcing for MLTT

Proposition (Forcing terms)

For any type in context $X \vdash \alpha$ the following are equivalent.

- *there is a term t such that*

$$X \vdash t : \alpha$$

- *for all $x : \mathrm{y}c \to X$ there is given coherently t_x such that*

$$c \Vdash t_x : \alpha(x)\,.$$

Kripke-Joyal forcing for MLTT

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Kripke-Joyal forcing for MLTT

Proof. Coherence means that $t_{xf} = t_x \circ f$.

$$
\begin{tikzcd}
 & & X.\alpha \arrow[r] \arrow[d] & \dot{\mathsf{U}} \arrow[d] \\
\mathsf{y}d \arrow[r, "f"'] \arrow[urr, dotted, bend left, "t_{xf}"] & \mathsf{y}c \arrow[r, "x"'] \arrow[ur, dotted, bend left, "t_x"'] & X \arrow[r, "\alpha"'] & \mathsf{U}
\end{tikzcd}
$$

But these partial sections correspond to partial lifts of α,

$$
\begin{tikzcd}
 & & X.\alpha \arrow[r] \arrow[d] & \dot{\mathsf{U}} \arrow[d] \\
\mathsf{y}d \arrow[r, "f"'] \arrow[urrr, dotted, "{\color{red}t_{xf}}"] & \mathsf{y}c \arrow[r, "x"'] \arrow[urr, dotted, "{\color{red}t_x}"'] & X \arrow[r, "\alpha"'] & \mathsf{U}
\end{tikzcd}
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Kripke-Joyal forcing for MLTT

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\begin{tikzcd}
 & & X.\alpha \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{U}} \arrow[d] \\
\mathsf{y}d \arrow[r, "f"'] \arrow[urr, dotted, hook, bend left, "t_{xf}"] & \mathsf{y}c \arrow[r, "x"'] \arrow[ur, dotted, hook, bend left, "t_x"'] & X \arrow[r, "\alpha"'] & \mathsf{U}
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\end{tikzcd}
$$

So the proof that $X \vdash t : \alpha$ is complete by Yoneda.

Outline

1. The universe $\dot{\mathsf{U}} \to \mathsf{U}$

For κ sufficiently large, define small categories:

$$
\begin{array}{lll}
\mathsf{Set}_\kappa \hookrightarrow \mathsf{Set} & \qquad & \textit{small sets} \\
\dot{\mathsf{Set}}_\kappa \hookrightarrow \dot{\mathsf{Set}} & \qquad & \textit{small pointed sets}
\end{array}
$$

These can be lifted to presheaves:

$$
\begin{aligned}
\dot{\mathsf{U}} &= \mathsf{Cat}\big(\mathbb{C}/_{-}^{\mathsf{op}},\ \dot{\mathsf{Set}}_\kappa\big) \\
\mathsf{U} &= \mathsf{Cat}\big(\mathbb{C}/_{-}^{\mathsf{op}},\ \mathsf{Set}_\kappa\big)
\end{aligned}
$$

with the evident natural map $\dot{\mathsf{U}} \to \mathsf{U}$ that “forgets the point”.

1. The universe $\dot{\mathsf{U}} \to \mathsf{U}$

Definition (Small presheaves)

A presheaf A is *small* if all its values are small.
A map $A \to X$ is *small* if all its fibers A_x are small.

$$\begin{tikzcd} A_x \arrow[r] \arrow[d] & A \arrow[d] \\ \mathsf{y}c \arrow[r, "x"'] & X \end{tikzcd}$$

Lemma ($\dot{\mathsf{U}} \to \mathsf{U}$ classifies small maps)

For small $A \to X$ there is an $\alpha : X \to \mathsf{U}$ and a pullback

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \dot{\mathsf{U}} \arrow[d] \\ X \arrow[r, "\alpha"'] & \mathsf{U} \end{tikzcd}$$

1. The universe $\dot{\mathsf{U}} \to \mathsf{U}$

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Lemma ($\dot{\mathsf{U}} \to \mathsf{U}$ classifies small maps)

For small $A \to X$ there is a canonical $\alpha : X \to \mathsf{U}$ and a chosen pullback

$$\begin{tikzcd} A \arrow[r, "\cong"] \arrow[d] & X.\alpha \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{U}} \arrow[d] \\ X \arrow[r, "="'] & X \arrow[r, "\alpha"'] & \mathsf{U} \end{tikzcd}$$

2. The natural model of MLTT

Let $f : Y \to X$ and consider the two-pullbacks diagram arising from substitution.

$$\frac{X \vdash \alpha}{Y \vdash \alpha f} \qquad \begin{tikzcd} Y.\alpha f \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X.\alpha \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{U}} \arrow[d] \\ Y \arrow[r, "f"'] & X \arrow[r, "\alpha"'] & \mathsf{U} \end{tikzcd}$$

The pullback functor f^* is modeled by precomposition of classifying maps into U.

$$\begin{tikzcd} Y \arrow[d, "f"] & \\ X \arrow[r, "\alpha"'] & \mathsf{U} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Hom}(Y, \mathsf{U}) \arrow[r, "\sim"] & \mathcal{S}/_Y \arrow[r, hook] & \mathcal{E}/_Y \\ \mathrm{Hom}(X, \mathsf{U}) \arrow[r, "\sim"'] \arrow[u, "- \circ f"] & \mathcal{S}/_X \arrow[r, hook] \arrow[u, "f^*"] & \mathcal{E}/_X \arrow[u, "f^*"'] \end{tikzcd}$$

2. The natural model of MLTT

For small $A \to X$ the adjoint functors

$$\Sigma_A B \dashv A^* \dashv \Pi_A B$$

$$\begin{tikzcd}
& B \arrow[d] & \\
\Sigma_A B \arrow[dr] & A \arrow[d] & \Pi_A B \arrow[dl] \\
& X &
\end{tikzcd}$$

all preserve the small maps,

$$\begin{tikzcd}
\mathcal{S}/_A \arrow[r, hook] \arrow[d, bend right, "\Sigma_A"'] \arrow[d, bend left, "\Pi_A"] & \mathcal{E}/_A \arrow[d, bend right, "\Sigma_A"'] \arrow[d, bend left, "\Pi_A"] \\
\mathcal{S}/_X \arrow[r, hook] \arrow[u, "A^*"] & \mathcal{E}/_X \arrow[u, "A^*"]
\end{tikzcd}$$

2. The natural model of MLTT

The type formers Σ, Π are then induced by structure on $\dot{\mathsf{U}} \to \mathsf{U}$, just as the quantifiers $\exists, \forall$ are induced by structure on $1 \to \Omega$.

For subobjects:

$$
\begin{tikzcd}
A \arrow[d] & \Omega^A \arrow[d, bend right, "\exists_A"'] \arrow[d, bend left, "\forall_A"] & \mathrm{Hom}(1, \Omega^A) \arrow[r, "\cong"] \arrow[d, bend right, "\exists_A"'] \arrow[d, bend left, "\forall_A"] & \mathrm{Sub}(A) \arrow[d, bend right, "\exists_A"'] \arrow[d, bend left, "\forall_A"] \\
1 & \Omega \arrow[u, "*"] & \mathrm{Hom}(1, \Omega) \arrow[u] \arrow[r, "\cong"'] & \mathrm{Sub}(1) \arrow[u, "*"]
\end{tikzcd}
$$

For types we require some preliminaries.

2. The natural model of MLTT

The polynomial object

$$P\mathsf{U} = \sum_{A:\mathsf{U}} \mathsf{U}^{[A]}$$

classifies *types in context*:

$$\frac{(A, B) : \Gamma \longrightarrow P\mathsf{U}}{\Gamma.A \vdash B}$$

Similarly, the object

$$P\dot{\mathsf{U}} = \sum_{A:\mathsf{U}} \dot{\mathsf{U}}^{[A]}$$

classifies *terms in context* $\Gamma.A \vdash b : B$.

2. The natural model of MLTT

Proposition

The universe $\dot{\mathsf{U}} \to \mathsf{U}$ models the rules for products just if there are maps λ, Π making a pullback diagram.

$$\begin{tikzcd}
P\dot{\mathsf{U}} \arrow[r, "\lambda"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{U}} \arrow[d] \\
P\mathsf{U} \arrow[r, "\Pi"'] & \mathsf{U}
\end{tikzcd}$$

2. The natural model of MLTT

The right adjoint $A^* \dashv \Pi_A B$ is then induced by composing classifying maps with $\Pi : P\mathsf{U} \to \mathsf{U}$.

$$\begin{tikzcd}
B \arrow[d] & \\
A \arrow[d] & \Pi_A B \arrow[dl] \\
X &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Hom}(X, P\mathsf{U}) \arrow[r] \arrow[d, bend left, "\Pi"] & \mathcal{S}^2/_X \arrow[d, bend left, "\Pi"] \\
\mathrm{Hom}(X, \mathsf{U}) \arrow[r] \arrow[u, "*"] & \mathcal{S}/_X \arrow[u, "*"]
\end{tikzcd}$$

2. The natural model of MLTT

There is a similar structure $\Sigma : P\mathsf{U} \to \mathsf{U}$ inducing the left adjoint $\Sigma_A \dashv A^*$.

$$
\begin{tikzcd}
 & B \arrow[d] \\
\Sigma_A B \arrow[dr] & A \arrow[d] \\
 & X
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Hom}(X, P\mathsf{U}) \arrow[r] \arrow[d, bend right, "\Sigma"'] & \mathcal{S}^2/_X \arrow[d, bend right, "\Sigma"'] \\
\mathrm{Hom}(X, \mathsf{U}) \arrow[r] \arrow[u, "*"'] & \mathcal{S}/_X \arrow[u, "*"']
\end{tikzcd}
$$

2. The natural model of MLTT

Proposition

The natural model structure on the universe provides a strict interpretation of MLTT.

$$
\begin{tikzcd}
P\mathsf{U} \arrow[d, bend right=60, "\Sigma"'] \arrow[d, bend left=60, "\Pi"] \\
\mathsf{U} \arrow[u, "*"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Hom}(X, P\mathsf{U}) \arrow[r, "\cong"] \arrow[d, bend right=60, "\Sigma"'] \arrow[d, bend left=60, "\Pi"] & \mathcal{S}^2/_X \arrow[d, bend right=60, "\Sigma"'] \arrow[d, bend left=60, "\Pi"] \\
\mathrm{Hom}(X, \mathsf{U}) \arrow[u, "*"'] \arrow[r, "\cong"'] & \mathcal{S}/_X \arrow[u, "*"']
\end{tikzcd}
$$

We use this structure to give forcing conditions for Σ and Π at $x : \mathsf{y}c \to X$, as in

$$c \Vdash t : \Sigma_{y:\alpha(x)}\beta(x, y)$$

$$c \Vdash t : \Pi_{y:\alpha(x)}\beta(x, y)$$

3. The Kripke-Joyal forcing rules

Theorem (A-Gambino-Hazratpour)

Let $X \in \widehat{\mathbb{C}}$ *and* $\alpha : X \to \mathsf{U}$ *and* $\beta : X.\alpha \to \mathsf{U}$*.*
For all $x : \mathsf{y}c \to X$*, we have*

$$
\begin{array}{llll}
c \Vdash & t : 0 & \textit{iff} & t \neq t \\
c \Vdash & t : 1 & \textit{iff} & t = * \\
c \Vdash & t : (\alpha + \beta)(x) & \textit{iff} & c \Vdash a : \alpha(x) \quad \textit{or} \quad c \Vdash b : \beta(x) \\
c \Vdash & t : (\alpha \times \beta)(x) & \textit{iff} & c \Vdash a : \alpha(x) \quad \textit{and} \quad c \Vdash b : \beta(x) \\
c \Vdash & t : (\Sigma_\alpha \beta)(x) & \textit{iff} & c \Vdash a : \alpha(x) \quad \textit{and} \quad c \Vdash b : \beta(x, a) \\
c \Vdash & t : (\alpha \to \beta)(x) & \textit{iff} & \textit{for all } f : d \to c \textit{ and } d \Vdash a : \alpha(xf) \\
 & & & \textit{there's } d \Vdash b_{f,a} : \beta(xf) \textit{ coherently} \\
c \Vdash & t : (\Pi_\alpha \beta)(x) & \textit{iff} & \textit{for all } f : d \to c \textit{ and } d \Vdash a : \alpha(xf) \\
 & & & \textit{there's } d \Vdash b_{f,a} : \beta(xf, a) \textit{ coherently}
\end{array}
$$

3. Kripke-Joyal forcing rules

Finally, we set:

Definition. Let $X \in \widehat{\mathbb{C}}$ and $\alpha : X \to \mathsf{U}$ a type over X.
We say that $\mathbb{C}$ **forces a term of type** α,

$$\mathbb{C} \Vdash X \vdash t : \alpha$$

if for all $c \in \mathbb{C}$ and all $x : \mathsf{y}c \to X$, there is given coherently

$$c \Vdash t_x : \alpha(x)\,.$$

4. The completeness theorem

Theorem (A-Gambino-Hazratpour)

Let C be a closed type in MLTT *with the type forming operations*

$$0,\ 1,\ X,\ A+B,\ A\times B,\ A\to B,\ \Sigma_A B,\ \Pi_A B,\ s=_A t\,.$$

There is a closed term $\vdash t : C$ *if, and only if, for all categories* $\mathbb{C}$ *and all presheaves* X *on* $\mathbb{C}$*, one has* $\mathbb{C} \Vdash t : C$*. Briefly,*

$$\mathsf{MLTT} \vdash t : C \qquad \textit{iff} \qquad \mathbb{C} \Vdash t : C \quad \textit{for all } \mathbb{C} \textit{ and } X.$$

Moreover, it suffices to assume that $\mathbb{C}$ *is a poset.*

4. The completeness theorem

Proof. Let $\mathbb{T}$ be the classifying category of MLTT and $p : \mathrm{Sh}(X_{\mathbb{T}}) \twoheadrightarrow \widehat{\mathbb{T}}$ the spatial cover. Take $P = \mathcal{O}X_{\mathbb{T}}$.

There are LCCC embeddings:

$$\mathbb{T} \xhookrightarrow{\mathsf{y}} \widehat{\mathbb{T}} \xhookrightarrow{p^*} \mathrm{Sh}(X_{\mathbb{T}}) \hookrightarrow \widehat{\mathcal{O}X_{\mathbb{T}}}.$$

So we have:

$$\begin{array}{rcll}
\mathrm{MLTT} \vdash t : C & \iff & 1 \xrightarrow{t} C & \mathbb{T} \\
& \iff & 1 \cong \mathsf{y}1 \xrightarrow{\mathsf{y}t} \mathsf{y}C \cong [\![C]\!]^{\mathbb{T}} & \widehat{\mathbb{T}} \\
& \iff & 1 \cong p^*\mathsf{y}1 \xrightarrow{p^*\mathsf{y}t} p^*\mathsf{y}C \cong [\![C]\!]^{X_{\mathbb{T}}} & \mathrm{Sh}(X_{\mathbb{T}}) \\
& \iff & \mathcal{O}X_{\mathbb{T}} \Vdash t : C & \square
\end{array}$$

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