

## 2. Propositional Calculus

- Consider the positive propositional calculus:

$$\text{PPC} := P \mid T \mid \varphi \wedge \psi \mid \varphi \Rightarrow \psi$$

- The rules of inference are as usual:

$$\frac{\cdot}{\varphi \vdash T} \qquad \frac{\vartheta \vdash \varphi, \vartheta \vdash \psi}{\vartheta \vdash \varphi \wedge \psi} \qquad \frac{\vartheta, \varphi \vdash \psi}{\vartheta \vdash \varphi \Rightarrow \psi}$$

- A Kripke model  $(K, \models)$  is a poset  $K$ , with a relation  $\models \models_P$  s.t.  
 $i \leq j, j \models_P \varphi \Rightarrow i \models_P \varphi$ .
- Extend  $\models$  to all  $\varphi \in \text{PPC}$  by:
  - $j \models T$  always,
  - $j \models \varphi \wedge \psi$  if  $j \models \varphi$  &  $j \models \psi$ ,
  - $j \models \varphi \Rightarrow \psi$  iff  $i \models \varphi$  implies  $i \models \psi$ , f.a.  $i \leq j$
- Let  $K \models \varphi$  if  $j \models \varphi$  f.a.  $j \in K$ , and  $\models \varphi$  if  $K \models \varphi$  f.a.  $(K, \models)$ .  
"  $\varphi$  is Kripke valid"

## Prop (Kripke completeness of PPC)

$$\vdash \varphi \quad \text{iff} \quad \Vdash \varphi .$$

Pf. (i) Order (the formulas of) PPC by  $\varphi \vdash \psi$ ,  
 & identify  $\varphi = \psi$  iff  $\varphi \dashv \vdash \psi$ ,  
 Call the resulting poset  $\mathcal{C}_{\text{PPC}}$ .

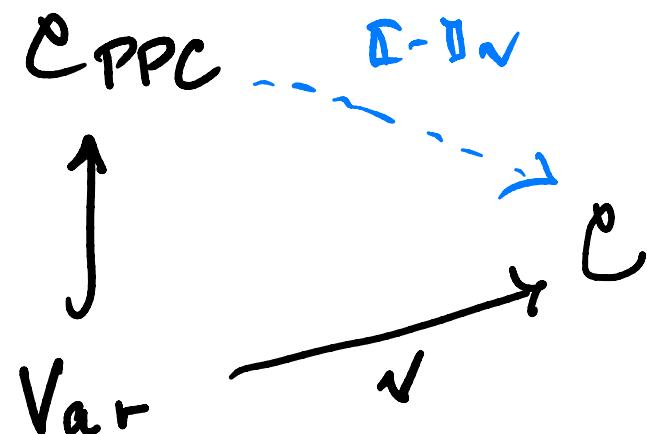
(ii)  $\mathcal{C}_{\text{PPC}}$  has:

- a terminal object  $T$
  - products  $\varphi \wedge \psi$
  - exponentials  $\varphi \Rightarrow \psi$
- } a cartesian  
closed poset  
=: CCP

(iii)  $C_{PPC}$  is the free CCP on the set

$$\{P, q, \dots\} := \text{Var},$$

meaning:



If  $C$  is any CCP &  $\nu$  is any function,  
then there's a unique CCP map  $\Gamma - \Pi_\nu$  s.th.

$$[\Gamma_P] = \nu P$$

$$[\Gamma_T] = T$$

$$[\Gamma_{\varphi \wedge \psi}] = [\Gamma_\varphi] \wedge [\Gamma_\psi]$$

$$[\Gamma_{\varphi \Rightarrow \psi}] = [\Gamma_\varphi] \Rightarrow [\Gamma_\psi],$$

briefly:

$$CCP(C_{PPC}, C) \cong \text{Set}(\text{VAR}, C).$$

Next, we need the following ...

Lemma 1: A Kripke model  $(K, \mathbb{H})$  of PPC

is the same thing as a CCP map

$$\mathbb{I}-\mathbb{J} : \mathcal{C}_{PPC} \longrightarrow \hat{K}$$

Pf: There's a bijection:

$$\mathbb{H} \subseteq K \times \text{Var} \quad \text{w/ } i \leq j \# p \Rightarrow i \# p$$


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$$K \times \text{Var} \longrightarrow \mathbb{2} = (0 \leq 1) \quad \text{in Pos}$$


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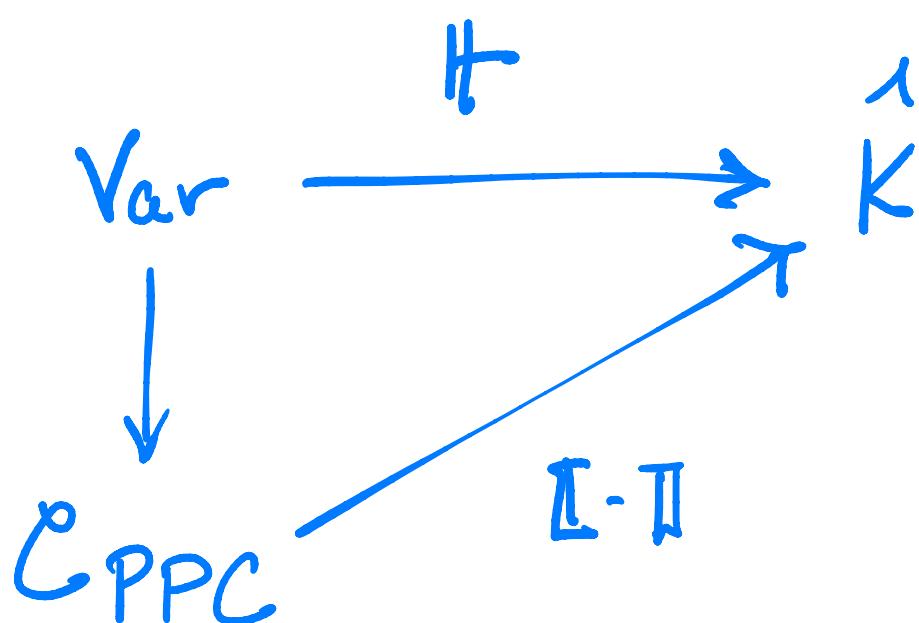
$$\text{Var} \longrightarrow \mathbb{2}^K = \hat{K} \quad \text{in Pos}$$


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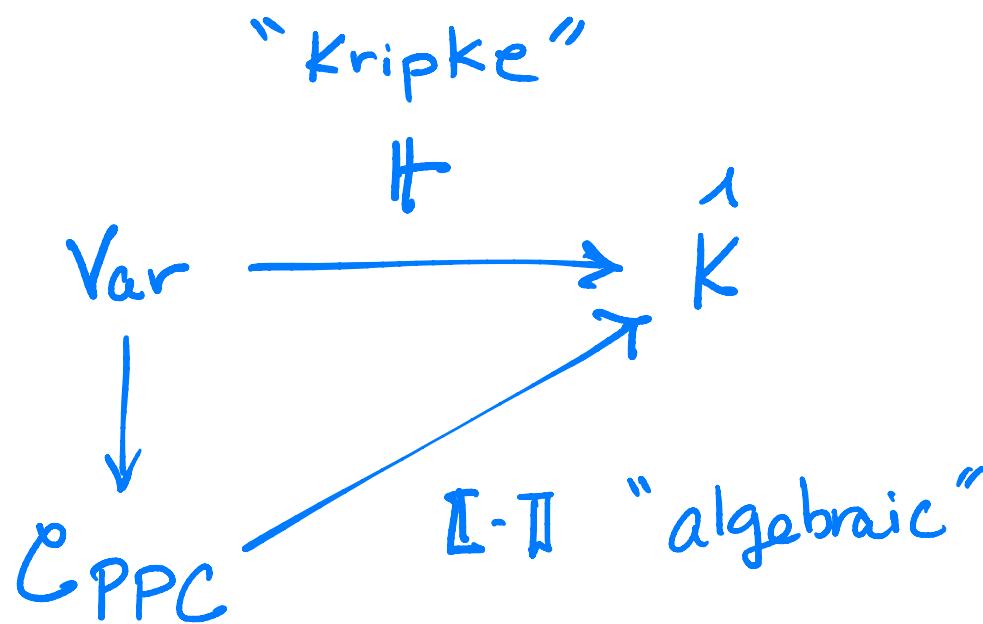
$$\mathbb{I}-\mathbb{J} : \mathcal{C}_{PPC} \longrightarrow \hat{K} \quad \text{in CCP}$$



## Remark

A Kripke model  $(K, \mathbb{H})$  thus corresponds to an algebraic model  $\mathbb{I}-\mathbb{J}$  in  $\hat{K}$  via

$$j \models \varphi \quad \text{iff} \quad j \in \llbracket \varphi \rrbracket .$$



The Kripke conditions correspond to algebraic ones:

$$i \leq j \models P \Rightarrow i \models P$$

$$\llbracket P \rrbracket \in \hat{K}$$

$$j \models T \quad \text{f.a. } j$$

$$\llbracket T \rrbracket = k$$

$$j \models \varphi \wedge \psi \quad \text{iff} \quad j \models \varphi \wedge j \models \psi$$

$$\llbracket \varphi \wedge \psi \rrbracket = \llbracket \varphi \rrbracket \wedge \llbracket \psi \rrbracket$$

$$j \models \varphi \Rightarrow \psi \quad \text{iff f.a. } i \leq j ,$$

$$\llbracket \varphi \Rightarrow \psi \rrbracket = \llbracket \varphi \rrbracket \Rightarrow \llbracket \psi \rrbracket$$

$$i \models \varphi \text{ implies } i \models \psi$$

(iv) The syntactic CCP  $\mathcal{C}_{PPC}$  has a  
canonical algebraic model, namely

$$\downarrow : \mathcal{C}_{CCP} \rightarrow \mathcal{C}_{CCP}^*,$$

it corresponds to a canonical Kripke model

$$(\mathcal{C}_{PPC}, \vdash),$$

with

$$\varphi \vdash \psi \text{ iff } \varphi \in \llbracket \psi \rrbracket \text{ iff } \varphi \vdash \psi.$$

So we have:

$$(*) \quad \mathcal{C}_{PPC} \vdash \varphi \text{ iff } \varphi \vdash \varphi \text{ f.a. } \varphi \in \mathcal{C}_{PPC}$$

$$\qquad \qquad \qquad \text{iff} \qquad \vdash \varphi.$$

(v) Thus we have :

$$\begin{aligned} \vdash \varphi &= K \vdash \varphi \text{ f.a. } (K, \vdash) \\ &\Rightarrow \mathcal{C}_{PPC} \vdash \varphi \\ &\Leftrightarrow \vdash \varphi. \quad \text{Completeness } \checkmark \end{aligned}$$

Conversely :

$$\begin{aligned} \vdash \varphi &\stackrel{*}{\Leftrightarrow} \mathcal{C}_{PPC} \vdash \varphi \quad \text{Soundness} \\ &\Rightarrow K \vdash \varphi \text{ f.a. } (K, \vdash) \end{aligned}$$

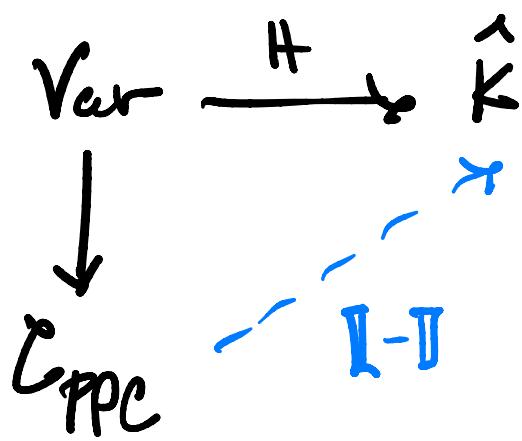
by the following ...

Lemma 2: Given  $\varphi \in \text{PPC}$ ,

$\mathcal{L}_{\text{PPC}} \Vdash \varphi$  implies  $K \Vdash \varphi$  f.a.  $(K, \Vdash)$ .

pf.  $\mathcal{L}_{\text{PPC}} \Vdash \varphi \Rightarrow T \Vdash \varphi$   
 $\Rightarrow T \models \varphi$   
 $\Rightarrow T \leq \varphi \quad \text{in } \mathcal{L}_{\text{PPS}}$  .

Given any  $(K, \Vdash)$  we get a PPC map  $\llbracket - \rrbracket$ :



Thus:

$$\llbracket T \rrbracket \leq \llbracket \varphi \rrbracket$$

so:

$$K \Vdash \varphi$$

Kripke completeness for PPC ✓

Next we can extend this result to :

- (1)  $\text{IPC} = \text{PPC} \wedge \perp, \vee$  (Kripke)
- (2) Topological semantics (Tarski)
- (3) A translation (Gödel)

$$\text{IPC} \hookrightarrow \text{CPC} \wedge \#$$

(Discuss each one briefly)

# 1. Extension from PPC to IPC

(9)

Def. A Heyting algebra is a CCP with all

finite joins:  $\perp, P \vee q, \neg\neg$

Equivalently, a bdd. lattice w/  $a \Rightarrow b$ .

## Examples

i. any Boolean algebra  $B$  is a HA,

$$P \Rightarrow q := \neg P \vee q$$

ii. for any topological space  $(X, \mathcal{O}_X)$ , the open sets form a HA,

$$U \Rightarrow V = \bigcup_{W \cap U \subseteq V} W$$

iii. any  $V$ -complete distributive lattice,  
e.g.  $\hat{P}$  f. any poset  $P$ .

iv. any complete linear order,

e.g.  $[0, 1] \subseteq \mathbb{R}$ . "fuzzy logic"

By (iii) the canonical model of PPC

$$\downarrow : \mathcal{E}_{PPC} \longrightarrow \widehat{\mathcal{E}}_{PPC}$$

is therefore valued in a HA !

Since being a HA is algebraic, there's a unique extension to the V-Completion, from CCPs to HAs:

$$\begin{array}{ccc} \mathcal{E}_{PPC} & \xrightarrow{\quad \downarrow \quad} & \widehat{\mathcal{E}}_{PPC} \\ & \searrow & \nearrow \tau \\ & \mathcal{E}_{PPC}^V & \end{array}$$

By (iii) the canonical model of PPC

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Since being a HA is algebraic, there's a unique extension to the V-Completion, from CCPs to HAs:

$$\begin{array}{ccc} \mathcal{E}_{PPC} & \xrightarrow{\quad \downarrow \quad} & \widehat{\mathcal{E}}_{PPC} \\ & \searrow & \nearrow \\ & \mathcal{E}_{PPC}^V = \mathcal{E}_{IPC} & \end{array}$$

Now:

syntactic poset of IPC

And for  $\varphi, \psi$  in IPC, we again have:

$$\varphi \vdash_{IPC} \psi \Leftrightarrow \downarrow \varphi \leq \downarrow \psi \text{ in } \widehat{\mathcal{E}}_{IPC}.$$

But unfortunately the CCP embedding

$$\downarrow : \mathcal{E}_{IPC} \longrightarrow \widehat{\mathcal{E}}_{IPC}$$

is not Heyting (it does not preserve  $\perp, \vee$ ).

Instead, we use the following :

Thm (Joyal) For  $H$  any HA the map

$$j : H \longrightarrow \widehat{H^*},$$

where  $H^* = \text{Prime}(H)$

and  $j(h) = \{p \mid h \in p\}$

is both Heyting and conservative.

$$jx \leq jy \Rightarrow x \leq y$$

Generalizes Stone Representation

Theorem from BAs to HAS.

Pf: Uses Birkhoff's prime ideal theorem.

Now we proceed as for  $\text{PPC} \xleftrightarrow{\downarrow} \overset{\wedge}{\text{PPC}}$ ,  
 but using  $\text{IPC} \xleftrightarrow{j} \overset{\wedge}{\text{IPC}}^*$  instead: (12)

Thm. (Completeness of IPC, Kripke 1965)

Let  $\text{IPC} = \text{PPC}$  extended by:

$$\frac{\bullet}{\perp \vdash \varphi} \quad \frac{\varphi \vdash \delta \quad \psi \vdash \delta}{\varphi \vee \psi \vdash \delta}$$

Then:

$\vdash \varphi$  iff  $K \Vdash \varphi$  f.a. Kripke  
 models  $(K, \Vdash)$

Here  $K$  is a poset and again

$$\Vdash \subseteq \overset{\wedge}{K} \times \text{Var},$$

extended to  $\text{IPC} \supseteq \text{Var}$  by:

- $j \Vdash T$  f.a.  $j$
- $j \not\Vdash \perp$  f.a.  $j$
- $j \Vdash \varphi \wedge \psi$  iff  $j \Vdash \varphi$  &  $j \Vdash \psi$
- $j \Vdash \varphi \vee \psi$  iff  $j \Vdash \varphi$  or  $j \Vdash \psi$
- $j \Vdash \varphi \Rightarrow \psi$  iff  $i \Vdash \varphi$  implies  $i \Vdash \psi$  for  $i \leq j$

## 2. Extension to Topological Semantics

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Def. A topological model of IPC  
consists of a space  $X$  and  
a HA homomorphism

$$\mathbb{I} - \mathbb{I} : \mathcal{C}_{IPC} \longrightarrow \mathcal{O}X .$$

Note: this can be unwound in the expected  
way:

$$\mathbb{I} \tau \mathbb{I} = X$$

$$\mathbb{I} \varphi \wedge \psi \mathbb{I} = [\varphi \mathbb{I} \cap \psi \mathbb{I}] \\ \text{etc.}$$

Say a formula  $\varphi$  is (topologically) valid if

$$\mathbb{I} \varphi \mathbb{I} = X \quad \text{f.a. } (X, \mathbb{I} - \mathbb{I}).$$

Thm. (Tarski 1938)

A formula  $\varphi$  is topologically valid iff it  
is provable in IPC.

Pf. This follows directly from Joyal's thm:  
Take the space of prime ideals in  $\mathcal{C}_{IPC}$ ,

$$\text{Spec}(IPC) = \mathcal{C}_{IPC}^*,$$

topologized with the "Zariski" basic opens:

$$B_q = \{ p \mid q \in p \}.$$

### 3. Extension to Modal Logic

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We have an embedding of HA's,

$$\mathbb{I}-\mathbb{I} : \mathcal{E}_{IPC} \hookrightarrow \Theta^{\text{Spec}(IPC)}$$

For any space  $X$ , the interior operation

$$o : \mathcal{P}X \longrightarrow \mathcal{P}X$$

provides a (topological) model of the modal logic  $\square CPC$ :

$$\frac{\varphi \vdash \psi \quad \cdot \square \varphi \vdash \varphi \quad \cdot \square \varphi \vdash \square \square \varphi}{\square \varphi \vdash \square \psi \quad \cdot T \vdash \square T \quad \cdot \square \varphi \wedge \square \psi \vdash \square(\varphi \wedge \psi)}$$

Thm. (McKinsey-Tarski 1944)

$CPC^\square$  is complete with respect to topological models.

Remark : One can also use this to prove the completeness of Gödel's translation

$$IPC \longrightarrow \square CPC$$