

# On Hofmann-Streicher universes

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*to the memory of Erik Palmgren*

## Abstract

We have another look at the construction by Hofmann and Streicher of a semantic universe  $\mathbf{El} \rightarrow U$  for the interpretation of Martin-Löf type theory in a presheaf category  $[\mathbb{C}^{\text{op}}, \mathbf{Set}]$ . It turns out that  $\mathbf{El} \rightarrow U$  can be described as the *categorical nerve* of the classifier  $\mathbf{Set}^{\text{op}} \rightarrow \mathbf{Set}^{\text{op}}$  for discrete fibrations in  $\mathbf{Cat}$ , where the nerve is right adjoint to the well-known “Grothendieck construction” taking a presheaf  $P : \mathbb{C}^{\text{op}} \rightarrow \mathbf{Set}$  to its category of elements  $\int_{\mathbb{C}} P$ .

Let  $\widehat{\mathbb{C}} = [\mathbb{C}^{\text{op}}, \mathbf{Set}]$  be the category of presheaves on a small category  $\mathbb{C}$ .

## 1. The Hofmann-Streicher universe

In [HS97] the authors define a (type-theoretic) *universe*

$$E \longrightarrow U \tag{1}$$

in  $\widehat{\mathbb{C}}$  as follows. For  $I \in \mathbb{C}$ , set

$$U(I) = \mathbf{ob}(\widehat{\mathbb{C}/_I}) \tag{2}$$

$$E(I) = \coprod_{A \in U(I)} \mathbf{El}(\langle I, A \rangle) \tag{3}$$

$$\mathbf{El}(\langle I, A \rangle) = A(id_I) \tag{4}$$

with an evident associated action on morphisms, which need not concern us for the moment. A few comments are required:

1. In (2), we have taken the underlying set of objects of the category  $\widehat{\mathbb{C}/_I} = [\mathbb{C}/_I^{\text{op}}, \mathbf{Set}]$ .

2. In (4), and throughout, the authors steadfastly adopt the “categories with families” point of view in describing the morphism  $E \rightarrow U$  in  $\widehat{\mathbb{C}}$  as an object in

$$\widehat{\int_{\mathbb{C}} U} \simeq \widehat{\mathbb{C}}/U, \quad (5)$$

and thus as a presheaf on the *category of elements*  $\int_{\mathbb{C}} U$ , rather than specifying the object  $E$  in  $\widehat{\mathbb{C}}$ . Thus the argument  $\langle I, A \rangle \in \int_{\mathbb{C}} U$  in (4) consists of an object  $I \in \mathbb{C}$  and an element  $A \in U(I)$ .

3. In order to account for size issues, the authors assume a Grothendieck universe  $\mathcal{U}$  in **Set**, the elements of which are called *small*. The category  $\mathbb{C}$  is then assumed to be small, as are the values of the presheaves (unless otherwise stated).

The presheaf  $U$ , which is not small, is regarded as the Grothendieck universe  $\mathcal{U}$  “lifted” from **Set** to  $[\mathbb{C}^{\text{op}}, \mathbf{Set}]$ . We will analyse the construction of (1) from a slightly different perspective in order to arrive at its basic property as a classifier for small families in  $\widehat{\mathbb{C}}$ .

## 2. An unused adjunction

For a presheaf  $X$  on  $\mathbb{C}$ , recall that the category of elements is the comma category,

$$\int_{\mathbb{C}} X = \mathbf{y}_{\mathbb{C}}/X,$$

where  $\mathbf{y}_{\mathbb{C}} : \mathbb{C} \rightarrow [\mathbb{C}^{\text{op}}, \mathbf{Set}]$  is the Yoneda embedding, which we may suppress and write simply  $\mathbb{C}/X$ . While the category of elements  $\int_{\mathbb{C}} X$  is used in the specification of the Hofmann-Streicher universe  $\mathbf{El} \rightarrow U$  at the point (5), the authors seem to have missed a trick which would have simplified things:

**Proposition 1** ([Gro83], §28). *The category of elements functor  $\int_{\mathbb{C}} : \widehat{\mathbb{C}} \rightarrow \mathbf{Cat}$  has a right adjoint, which we denote*

$$\nu_{\mathbb{C}} : \mathbf{Cat} \rightarrow \widehat{\mathbb{C}}.$$

*For a small category  $\mathbb{A}$ , we call the presheaf  $\nu_{\mathbb{C}}(\mathbb{A})$  the  $\mathbb{C}$ -nerve of  $\mathbb{A}$ .*

*Proof.* As suggested by the name, the adjunction  $\int_{\mathbb{C}} \dashv \nu_{\mathbb{C}}$  can be seen as the familiar “realization  $\dashv$  nerve” construction with respect to the covariant

functor  $\mathbb{C}/- : \mathbb{C} \rightarrow \mathbf{Cat}$ , as indicated below.

$$\begin{array}{ccc}
 \widehat{\mathbb{C}} & \begin{array}{c} \xleftarrow{\nu_{\mathbb{C}}} \\ \xrightarrow{f_{\mathbb{C}}} \end{array} & \mathbf{Cat} \\
 \uparrow y & \nearrow \mathbb{C}/- & \\
 \mathbb{C} & & 
 \end{array} \tag{6}$$

In detail, for  $\mathbb{A} \in \mathbf{Cat}$  and  $c \in \mathbb{C}$ , let  $\nu_{\mathbb{C}}(\mathbb{A})(c)$  be the Hom-set of functors,

$$\nu_{\mathbb{C}}(\mathbb{A})(c) = \mathbf{Cat}(\mathbb{C}/_c, \mathbb{A}),$$

with contravariant action on  $h : d \rightarrow c$  given by precomposing a functor  $P : \mathbb{C}/_c \rightarrow \mathbb{A}$  with the postcomposition functor

$$\mathbb{C}/_h : \mathbb{C}/_d \longrightarrow \mathbb{C}/_c.$$

For the adjunction, observe that the slice category  $\mathbb{C}/_c$  is the category of elements of the representable functor  $y_c$ ,

$$\int_{\mathbb{C}} y_c \cong \mathbb{C}/_c.$$

Thus for representables  $y_c$ , we have the required natural isomorphism

$$\widehat{\mathbb{C}}(y_c, \nu_{\mathbb{C}}(\mathbb{A})) \cong \nu_{\mathbb{C}}(\mathbb{A})(c) = \mathbf{Cat}(\mathbb{C}/_c, \mathbb{A}) \cong \mathbf{Cat}(\int_{\mathbb{C}} y_c, \mathbb{A}).$$

For arbitrary presheaves  $X$ , one uses the presentation of  $X$  as a colimit of representables over the index category  $\int_{\mathbb{C}} X$ , and the easy to prove fact that  $\int_{\mathbb{C}}$  itself preserves colimits. Indeed, for any category  $\mathbb{D}$ , we have an isomorphism in  $\mathbf{Cat}$ ,

$$\varinjlim_{d \in \mathbb{D}} \mathbb{D}/_d \cong \mathbb{D}.$$

□

When  $\mathbb{C}$  is fixed, as here, we may omit the subscript from the notation  $y_{\mathbb{C}}$  and  $\int_{\mathbb{C}}$  and  $\nu_{\mathbb{C}}$ . The unit and counit maps of the adjunction  $\int \dashv \nu$ , vis.

$$\begin{aligned}
 \eta : X &\longrightarrow \nu \int X, \\
 \epsilon : \int \nu \mathbb{A} &\longrightarrow \mathbb{A},
 \end{aligned}$$

are as follows. At  $c \in \mathbb{C}$ , for  $x : y_c \rightarrow X$ , the functor  $(\eta_X)_c(x) : \mathbb{C}/_c \rightarrow \mathbb{C}/_X$  is just composition with  $x$ ,

$$(\eta_X)_c(x) = \mathbb{C}/_x : \mathbb{C}/_c \longrightarrow \mathbb{C}/_X. \tag{7}$$

For  $\mathbb{A} \in \mathbf{Cat}$ , the functor  $\epsilon : \int \nu \mathbb{A} \rightarrow \mathbb{A}$  takes a pair  $(c \in \mathbb{C}, f : \mathbb{C}/_c \rightarrow \mathbb{A})$  to the object  $f(1_c) \in \mathbb{A}$ ,

$$\epsilon(c, f) = f(1_c).$$

**Lemma 2.** *For any  $f : Y \rightarrow X$ , the naturality square below is a pullback.*

$$\begin{array}{ccc} Y & \xrightarrow{\eta_Y} & \nu \int Y \\ f \downarrow & & \downarrow \nu f \\ X & \xrightarrow{\eta_X} & \nu \int X. \end{array} \quad (8)$$

*Proof.* It suffices to prove it for the case  $f : X \rightarrow 1$ . Thus consider the square

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \nu \int X \\ \downarrow & & \downarrow \\ 1 & \xrightarrow{\eta_1} & \nu \int 1. \end{array} \quad (9)$$

Evaluating at  $c \in \mathbb{C}$  and applying (7) then gives the following square in  $\mathbf{Set}$ .

$$\begin{array}{ccc} Xc & \xrightarrow{\mathbb{C}/_-} & \mathbf{Cat}(\mathbb{C}/_c, \mathbb{C}/_X) \\ \downarrow & & \downarrow \\ 1c & \xrightarrow{\mathbb{C}/_-} & \mathbf{Cat}(\mathbb{C}/_c, \mathbb{C}/_1) \end{array} \quad (10)$$

The image of  $* \in 1c$  along the bottom is the forgetful functor  $U_c : \mathbb{C}/_c \rightarrow \mathbb{C}$ , and its fiber under the map on the right is therefore the set of functors  $F : \mathbb{C}/_c \rightarrow \mathbb{C}/_X$  such that  $U_X \circ F = U_c$ , where  $U_X : \mathbb{C}/_X \rightarrow \mathbb{C}$  is also a forgetful functor. But any such  $F$  is easily seen to be uniquely of the form  $\mathbb{C}/_x$  for  $x = F(1_c) : yc \rightarrow X$ .  $\square$

### 3. Classifying families

For the terminal presheaf  $1 \in \widehat{\mathbb{C}}$ , we have  $\int 1 \cong \mathbb{C}$ , so for every  $X \in \widehat{\mathbb{C}}$  there is a canonical projection  $\int X \rightarrow \mathbb{C}$ , which is easily seen to be a discrete fibration. It follows that for any map  $Y \rightarrow X$  of presheaves, the associated map  $\int Y \rightarrow \int X$  is also a discrete fibration. Ignoring size issues for the moment, recall that discrete fibrations in  $\mathbf{Cat}$  are classified by the forgetful functor  $\mathbf{Set}^{\text{op}} \rightarrow \mathbf{Set}^{\text{op}}$  from (the opposites of) the category of pointed sets

to that of sets (cf. [Web07]). For every presheaf  $X \in \widehat{\mathbb{C}}$ , we therefore have a pullback diagram in  $\mathbf{Cat}$ ,

$$\begin{array}{ccc} \int X & \longrightarrow & \mathbf{Set}^{\mathbf{op}} \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{C} & \xrightarrow{X} & \mathbf{Set}^{\mathbf{op}}. \end{array} \quad (11)$$

Transposing by the adjunction  $\int \dashv \nu$  then gives a commutative square in  $\widehat{\mathbb{C}}$ ,

$$\begin{array}{ccc} X & \longrightarrow & \nu \mathbf{Set}^{\mathbf{op}} \\ \downarrow & & \downarrow \\ 1 & \xrightarrow{\bar{X}} & \nu \mathbf{Set}^{\mathbf{op}}. \end{array} \quad (12)$$

**Lemma 3.** *The square (12) is a pullback in  $\widehat{\mathbb{C}}$ . More generally, for any map  $Y \rightarrow X$  in  $\widehat{\mathbb{C}}$ , there is a pullback square*

$$\begin{array}{ccc} Y & \longrightarrow & \nu \mathbf{Set}^{\mathbf{op}} \\ \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & \nu \mathbf{Set}^{\mathbf{op}}. \end{array} \quad (13)$$

*Proof.* Apply the right adjoint  $\nu$  to the pullback square (11) and paste the naturality square (8) from Lemma 2 on the left, to obtain the transposed square (13) as a pasting of two pullbacks.  $\square$

Let us write  $\dot{\mathcal{V}} \rightarrow \mathcal{V}$  for the vertical map on the right in (13), so that

$$\begin{aligned} \dot{\mathcal{V}} &= \nu \mathbf{Set}^{\mathbf{op}} \\ \mathcal{V} &= \nu \mathbf{Set}^{\mathbf{op}}. \end{aligned} \quad (14)$$

We can summarize our results so far as follows.

**Proposition 4.** *The nerve  $\dot{\mathcal{V}} \rightarrow \mathcal{V}$  of the classifier for discrete fibrations  $\mathbf{Set}^{\mathbf{op}} \rightarrow \mathbf{Set}^{\mathbf{op}}$ , as defined in (14), classifies natural transformations  $Y \rightarrow X$  in  $\widehat{\mathbb{C}}$ , in the sense that there is always a pullback square,*

$$\begin{array}{ccc} Y & \longrightarrow & \dot{\mathcal{V}} \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{\tilde{Y}} & \mathcal{V}. \end{array} \quad (15)$$

The classifying map  $\tilde{Y} : X \rightarrow \mathcal{V}$  is determined by the adjunction  $\int \dashv \nu$  as the transpose of the classifying map of the discrete fibration  $\int X \rightarrow \int Y$ .

Of course,  $\dot{\mathcal{V}} \rightarrow \mathcal{V}$  itself cannot be a map in  $\widehat{\mathbb{C}}$ , for reasons of size.

#### 4. Small maps

Let  $\alpha$  be a cardinal number and call the sets that are strictly smaller  $\alpha$ -small. Let  $\mathbf{Set}_\alpha \hookrightarrow \mathbf{Set}$  be the full subcategory of  $\alpha$ -small sets. Call a presheaf  $X : \mathbb{C}^{\text{op}} \rightarrow \mathbf{Set}$   $\alpha$ -small if all of its values are  $\alpha$ -small sets, and thus if, and only if, it factors through  $\mathbf{Set}_\alpha \hookrightarrow \mathbf{Set}$ . Call a map  $f : Y \rightarrow X$  of presheaves  $\alpha$ -small if all of the fibers  $f_c^{-1}\{x\} \subseteq Yc$  are  $\alpha$ -small sets (for all  $c \in \mathbb{C}$  and  $x \in Xc$ ). The latter condition is of course equivalent to saying that, in the pullback square over the element  $x : yc \rightarrow X$ ,

$$\begin{array}{ccc} Y_x & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow f \\ yc & \xrightarrow{x} & X, \end{array} \quad (16)$$

the presheaf  $Y_x$  is  $\alpha$ -small.

Now let us restrict the specification (14) of  $\dot{\mathcal{V}} \rightarrow \mathcal{V}$  to the  $\alpha$ -small sets:

$$\begin{aligned} \dot{\mathcal{V}}_\alpha &= \nu \mathbf{Set}_\alpha^{\text{op}} \\ \mathcal{V}_\alpha &= \nu \mathbf{Set}_\alpha^{\text{op}}. \end{aligned} \quad (17)$$

Then the evident forgetful map  $\dot{\mathcal{V}}_\alpha \rightarrow \mathcal{V}_\alpha$  is a map in the category  $\widehat{\mathbb{C}}$  of presheaves, and it is in fact  $\alpha$ -small. Moreover, it has the following basic property, which is just a restriction of the basic property of  $\dot{\mathcal{V}} \rightarrow \mathcal{V}$  stated in Proposition 4.

**Proposition 5.** *The map  $\dot{\mathcal{V}}_\alpha \rightarrow \mathcal{V}_\alpha$  classifies  $\alpha$ -small maps  $f : Y \rightarrow X$  in  $\widehat{\mathbb{C}}$ , in the sense that there is always a pullback square,*

$$\begin{array}{ccc} Y & \longrightarrow & \dot{\mathcal{V}}_\alpha \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{\tilde{Y}} & \mathcal{V}_\alpha. \end{array} \quad (18)$$

The classifying map  $\tilde{Y} : X \rightarrow \mathcal{V}_\alpha$  is determined by the adjunction  $\int \dashv \nu$  as (the factorization of) the transpose of the classifying map of the discrete fibration  $\int X \rightarrow \int Y$ .

*Proof.* If  $Y \rightarrow X$  is  $\alpha$ -small, its classifying map  $\tilde{Y} : X \rightarrow \mathcal{V}$  factors through  $\mathcal{V}_\alpha \hookrightarrow \mathcal{V}$ , as indicated below,

$$\begin{array}{ccccc}
 Y & \xrightarrow{\quad} & \nu \mathbf{Set}_\alpha^{\text{op}} & \hookrightarrow & \nu \mathbf{Set}^{\text{op}} \\
 \downarrow & & \downarrow & & \downarrow \\
 X & \xrightarrow{\quad} & \nu \mathbf{Set}_\alpha^{\text{op}} & \hookrightarrow & \nu \mathbf{Set}^{\text{op}}, \\
 & \searrow & & \nearrow & \\
 & & \tilde{Y} & & 
 \end{array} \tag{19}$$

in virtue of the following adjoint transposition,

$$\begin{array}{ccccc}
 \int Y & \xrightarrow{\quad} & \mathbf{Set}_\alpha^{\text{op}} & \hookrightarrow & \mathbf{Set}^{\text{op}} \\
 \downarrow & & \downarrow & & \downarrow \\
 \int X & \xrightarrow{\quad} & \mathbf{Set}_\alpha^{\text{op}} & \hookrightarrow & \mathbf{Set}^{\text{op}}. \\
 & \searrow & & \nearrow & \\
 & & & & 
 \end{array} \tag{20}$$

Note that the square on the right is evidently a pullback, and the one on the left therefore is, too, because the outer rectangle is the classifying pullback of the discrete fibration  $\int Y \rightarrow \int X$ , as stated. Thus the left square in (19) is a pullback.  $\square$

## 5. Examples

1. Let  $\alpha = \kappa$  a strongly inaccessible cardinal, so that  $\mathbf{ob}(\mathbf{Set}_\kappa)$  is a Grothendieck universe. Then the Hofmann-Streicher universe of (1) is recovered in the present setting as the  $\kappa$ -small map classifier

$$E \cong \dot{\mathcal{V}}_\kappa \longrightarrow \mathcal{V}_\kappa \cong U$$

in the sense of Proposition 5. Indeed, for  $c \in \mathbb{C}$ , we have

$$\mathcal{V}_\kappa c = \nu(\mathbf{Set}_\kappa^{\text{op}})(c) = \mathbf{Cat}(\mathbb{C}/_c, \mathbf{Set}_\kappa^{\text{op}}) = \mathbf{ob}(\widehat{\mathbb{C}/_c}) = U c. \tag{21}$$

For  $\dot{\mathcal{V}}_\kappa$  we then have,

$$\begin{aligned}
 \dot{\mathcal{V}}_\kappa c &= \nu(\dot{\mathbf{Set}}_\kappa^{\text{op}})(c) = \mathbf{Cat}(\mathbb{C}/_c, \dot{\mathbf{Set}}_\kappa^{\text{op}}) \\
 &\cong \coprod_{A \in \mathcal{V}_\kappa c} \mathbf{Cat}_{\mathbb{C}/_c}(\mathbb{C}/_c, A^* \mathbf{Set}_\kappa^{\text{op}})
 \end{aligned} \tag{22}$$

where the  $A$ -summand in (22) is defined by taking sections of the pullback indicated below.

$$\begin{array}{ccc}
A^* \mathbf{Set}_\kappa^{\text{op}} & \longrightarrow & \dot{\mathbf{Set}}_\kappa^{\text{op}} \\
\downarrow \lrcorner & \nearrow & \downarrow \\
\mathbb{C}/_c & \xrightarrow{A} & \mathbf{Set}_\kappa^{\text{op}}
\end{array} \tag{23}$$

But  $A^* \mathbf{Set}_\kappa^{\text{op}} \cong \int_{\mathbb{C}/_c} A$  over  $\mathbb{C}/_c$ , and sections of this discrete fibration in  $\mathbf{Cat}$  correspond uniquely to natural maps  $1 \rightarrow A$  in  $\widehat{\mathbb{C}/_c}$ . Since 1 is representable in  $\widehat{\mathbb{C}/_c}$  we can continue (22) by

$$\begin{aligned}
\dot{\mathcal{V}}_\kappa c &\cong \coprod_{A \in \mathcal{V}_\kappa c} \mathbf{Cat}_{\mathbb{C}/_c}(\mathbb{C}/_c, A^* \mathbf{Set}_\kappa^{\text{op}}) \\
&\cong \coprod_{A \in \mathcal{V}_\kappa c} \widehat{\mathbb{C}/_c}(1, A) \\
&\cong \coprod_{A \in \mathcal{V}_\kappa c} A(1_c) \\
&= \coprod_{A \in \mathcal{V}_\kappa c} \mathbf{El}(\langle c, A \rangle) \\
&= Ec.
\end{aligned}$$

2. By functoriality of  $\nu : \mathbf{Cat} \rightarrow \widehat{\mathbb{C}}$ , a sequence of Grothendieck universes

$$\mathcal{U} \subseteq \mathcal{U}' \subseteq \dots$$

in  $\mathbf{Set}$  gives rise to a (cumulative) sequence of type-theoretic universes

$$\mathcal{V} \rightarrow \mathcal{V}' \rightarrow \dots$$

in  $\widehat{\mathbb{C}}$ . More precisely, there is a sequence of cartesian squares,

$$\begin{array}{ccccc}
\dot{\mathcal{V}} & \rightarrow & \dot{\mathcal{V}}' & \rightarrow & \dots \\
\downarrow \lrcorner & & \downarrow \lrcorner & & \\
\mathcal{V} & \rightarrow & \mathcal{V}' & \rightarrow & \dots,
\end{array} \tag{24}$$

in the image of  $\nu : \mathbf{Cat} \rightarrow \widehat{\mathbb{C}}$ , classifying small maps in  $\widehat{\mathbb{C}}$  of increasing size, in the sense of Proposition 5.

3. Let  $\alpha = 2$  so that  $1 \rightarrow 2$  is the subobject classifier of  $\mathbf{Set}$ , and

$$\mathbb{1} = \dot{\mathbf{Set}}_2^{\text{op}} \rightarrow \mathbf{Set}_2^{\text{op}} = \mathbb{2}$$



is then a classifier in  $\mathbf{Cat}$  for sieves, i.e. full subcategories  $\mathbb{S} \hookrightarrow \mathbb{A}$  closed under the domains of arrows  $a \rightarrow s$  for  $s \in \mathbb{S}$ . The nerve  $\dot{\mathcal{V}}_2 \rightarrow \mathcal{V}_2$  is then exactly the subobject classifier  $1 \rightarrow \Omega$  of  $\widehat{\mathbb{C}}$ ,

$$1 = \nu \mathbb{1} = \dot{\mathcal{V}}_2 \longrightarrow \mathcal{V}_2 = \nu \mathbb{2} = \Omega.$$

4. Let  $i : \mathbb{2} \hookrightarrow \mathbf{Set}_\kappa$  and  $p : \mathbf{Set}_\kappa \rightarrow \mathbb{2}$  be the embedding-retraction pair with  $i : \mathbb{2} \hookrightarrow \mathbf{Set}_\kappa$  the inclusion of the full subcategory on the sets  $\{0, 1\}$  and  $p : \mathbf{Set}_\kappa \rightarrow \mathbb{2}$  the retraction that takes  $0 = \emptyset$  to itself, and everything else (i.e. the non-empty sets) to  $1 = \{\emptyset\}$ . There is a retraction (of arrows) in  $\mathbf{Cat}$ ,

$$\begin{array}{ccccc} \mathbb{1} & \hookrightarrow & \mathbf{Set}_\kappa & \longrightarrow & \mathbb{1} \\ \downarrow & \lrcorner & \downarrow & & \downarrow \\ \mathbb{2} & \xhookrightarrow{i} & \mathbf{Set}_\kappa & \twoheadrightarrow[p] & \mathbb{2} \end{array} \quad (25)$$

where the left square is a pullback.

By the functoriality of  $(-)^{\text{op}}$  and  $\nu : \mathbf{Cat} \rightarrow \widehat{\mathbb{C}}$  we then have a retract diagram in  $\widehat{\mathbb{C}}$ , again with a pullback on the left,

$$\begin{array}{ccccc} 1 & \hookrightarrow & \dot{\mathcal{V}}_\kappa & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow & & \downarrow \\ \Omega & \xhookrightarrow{\{-\}} & \mathcal{V}_\kappa & \twoheadrightarrow[-] & \Omega \end{array} \quad (26)$$

where for any  $\phi : X \rightarrow \Omega$  the subobject  $\{\phi\} \rightarrowtail X$  is classified as a small map by the composite  $\{\phi\} : X \rightarrow \mathcal{V}_\kappa$ , and for any small map  $A \rightarrow X$ , the image  $[A] \rightarrowtail X$  is classified as a subobject by the composite  $[\alpha] : X \rightarrow \mathcal{V}_\kappa \rightarrow \Omega$ , where  $\alpha : X \rightarrow \mathcal{V}_\kappa$  classifies  $A \rightarrow X$ . The idempotent composite

$$\|-\| = \{[-]\} : \mathcal{V}_\kappa \longrightarrow \mathcal{V}_\kappa$$

is the *propositional truncation modality* in the natural model of type theory given by  $\dot{\mathcal{V}}_\kappa \rightarrow \mathcal{V}_\kappa$  (see [AGH21]).

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