Categorical Logic

Autumn School on Proof and Computation

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Chapter 1

Category Theory

1.1 Categories

Definition 1.1.1. A *category* $\mathcal{C}$ consists of classes

$\mathcal{C}_0$ of objects A, B, C, ...
$\mathcal{C}_1$ of morphisms f, g, h, ...

such that:

- Each morphism f has uniquely determined *domain* $\mathsf{dom}\, f$ and *codomain* $\mathsf{cod}\, f$, which are objects. This is written:
$$f : \mathsf{dom}\, f \to \mathsf{cod}\, f$$
- For any morphisms $f : A \to B$ and $g : B \to C$ there exists a uniquely determined *composition* $g \circ f : A \to C$. Composition is associative:
$$h \circ (g \circ f) = (h \circ g) \circ f \ ,$$
where domains are codomains are as follows:
$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$
- For every object A there exists the *identity* morphism $\mathsf{1}_A : A \to A$ which is a unit for composition,
$$\mathsf{1}_A \circ f = f \ , \qquad\qquad g \circ \mathsf{1}_A = g \ ,$$
where $f : B \to A$ and $g : A \to C$.

Morphisms are also called *arrows* or *maps*. Note that morphisms do not actually have to be functions, and objects need not be sets or spaces of any sort. We often write $\mathcal{C}$ instead of $\mathcal{C}_0$.

Definition 1.1.2. A category $\mathcal{C}$ is *small* when the objects $\mathcal{C}_0$ and the morphisms $\mathcal{C}_1$ are sets (as opposed to proper classes). A category is *locally small* when for all objects $A, B \in \mathcal{C}_0$ the class of morphisms with domain A and codomain B, written $\mathsf{Hom}(A, B)$ or $\mathcal{C}_0(A, B)$, is a set.

We normally restrict attention to locally small categories, so unless we specify otherwise all categories are taken to be locally small. Next we consider several examples of categories.

1.1.1 Examples

The empty category $\mathsf{0}$ The empty category has no objects and no arrows.

The unit category $\mathsf{1}$ The unit category, also called the terminal category, has one object $\star$ and one arrow $1_\star$:

$$\begin{tikzcd} \star \arrow[loop right, "1_\star"] \end{tikzcd}$$

Other finite categories There are other finite categories, for example the category with two objects and one (non-identity) arrow, and the category with two parallel arrows:

$$\begin{tikzcd} \star \arrow[r] & \bullet \end{tikzcd} \qquad\qquad \begin{tikzcd} \star \arrow[r, bend left] \arrow[r, bend right] & \bullet \end{tikzcd}$$

Groups as categories Every group $(G, \cdot)$, is a category with a single object $\star$ and each element of G as a morphism:

$$\begin{tikzcd} \star \arrow[loop above, "b"] \arrow[loop right, "a"] \arrow[loop below, "c"] \end{tikzcd} \qquad\qquad a, b, c, \ldots \in G$$

The composition of arrows is given by the group operation:

$$a \circ b = a \cdot b$$

The identity arrow is the group unit e. This is indeed a category because the group operation is associative and the group unit is the unit for the composition. In order to get a category, we do not actually need to know that every element in G has an inverse. It suffices to take a *monoid*, also known as *semigroup*, which is an algebraic structure with an associative operation and a unit.

We can turn things around and *define* a monoid to be a category with a single object. A group is then a category with a single object in which every arrow is an *isomorphism* (in the sense of definition 1.1.5 below).

Posets as categories Recall that a *partially ordered set*, or *poset* $(P, \leq)$, is a set with a reflexive, transitive, and antisymmetric relation:

$$x \leq x \qquad \text{(reflexive)}$$
$$x \leq y \,\&\, y \leq z \Rightarrow x \leq z \qquad \text{(transitive)}$$
$$x \leq y \,\&\, y \leq x \Rightarrow x = y \qquad \text{(antisymmetric)}$$

Each poset is a category whose objects are the elements of P, and there is a single arrow $p \to q$ between $p, q \in P$ if, and only if, $p \leq q$. Composition of $p \to q$ and $q \to r$ is the unique arrow $p \to r$, which exists by transitivity of $\leq$. The identity arrow on p is the unique arrow $p \to p$, which exists by reflexivity of $\leq$.

Antisymmetry tells us that any two isomorphic objects in P are equal.[1] We do not need antisymmetry in order to obtain a category, i.e., a *preorder* would suffice.

Again, we may *define* a preorder to be a category in which there is at most one arrow between any two objects. A poset is a skeletal preorder, i.e. one in which the only isomorphisms are the identity arrows. We allow for the possibility that a preorder or a poset is a proper class rather than a set.

A particularly important example of a poset category is the poset of open sets $\mathcal{O}X$ of a topological space X, ordered by inclusion.

Sets as categories Any set S is a category whose objects are the elements of S and whose only arrows are identity arrows. Such a category, in which the only arrows are the identity arrows, is called a *discrete category*.

1.1.2 Categories of structures

In general, structures like groups, topological spaces, posets, etc., determine categories in which the maps are structure-preserving functions, composition is composition of functions, and identity morphisms are identity functions:

- Group is the category whose objects are groups and whose morphisms are group homomorphisms.
- Top is the category whose objects are topological spaces and whose morphisms are continuous maps.
- Set is the category whose objects are sets and whose morphisms are functions.[2]
- Graph is the category of (directed) graphs an graph homomorphisms.
- Poset is the category of posets and monotone maps.

[1] A category in which isomorphic object are equal is a *skeletal* category.

[2] A function between sets A and B is a relation $f \subseteq A \times B$ such that for every $x \in A$ there exists a unique $y \in B$ for which $\langle x, y \rangle \in f$. A morphism in Set is a triple $\langle A, f, B \rangle$ such that $f \subseteq A \times B$ is a function.

Such categories of structures are generally *large*, but locally small. Note that it is not necessary to check the associative and unit laws for such categories of functions (why?), unlike the following example.

Exercise 1.1.3. The *category of relations* Rel has as objects all sets $A, B, C, \ldots$ and as arrows $A \to B$ the relations $R \subseteq A \times B$. The composite of $R \subseteq A \times B$ and $S \subseteq B \times C$, and the identity arrow on A, are defined by:

$$\begin{aligned} S \circ R &= \left\{ \langle x, z \rangle \in A \times C \;\middle|\; \exists\, y \in B \,.\, xRy \;\&\; ySz \right\}, \\ 1_A &= \left\{ \langle x, x \rangle \;\middle|\; x \in A \right\}. \end{aligned}$$

Show that this is indeed a category!

1.1.3 Basic notions

We recall some further basic notions from category theory.

Definition 1.1.4. A *subcategory* $\mathcal{C}'$ of a category $\mathcal{C}$ is given by a subclass of objects $\mathcal{C}'_0 \subseteq \mathcal{C}_0$ and a subclass of morphisms $\mathcal{C}'_1 \subseteq \mathcal{C}_1$ such that $f \in \mathcal{C}'_1$ implies $\mathsf{dom}\, f, \mathsf{cod}\, f \in \mathcal{C}'_0$, $1_A \in \mathcal{C}'_1$ for every $A \in \mathcal{C}'_0$, and $g \circ f \in \mathcal{C}'_1$ whenever $f, g \in \mathcal{C}'_1$ are composable.

A subcategory $\mathcal{C}'$ of $\mathcal{C}$ is *full* if for all $A, B \in \mathcal{C}'_0$, we have $\mathcal{C}'(A, B) = \mathcal{C}(A, B)$, i.e. every $f : A \to B$ in $\mathcal{C}_1$ is also in $\mathcal{C}'_1$.

Definition 1.1.5. An *inverse* of a morphism $f : A \to B$ is a morphism $f^{-1} : B \to A$ such that

$$f \circ f^{-1} = 1_B \qquad \text{and} \qquad f^{-1} \circ f = 1_A \,.$$

A morphism that has an inverse is an *isomorphism*, or *iso*. If there exists a pair of mutually inverse morphisms $f : A \to B$ and $f^{-1} : B \to A$ we say that the objects A and B are *isomorphic*, written $A \cong B$.

The notation f^{-1} is justified because an inverse, if it exists, is unique. A *left inverse* is a morphism $g : B \to A$ such that $g \circ f = 1_A$, and a *right inverse* is a morphism $g : B \to A$ such that $f \circ g = 1_B$. A left inverse is also called a *retraction*, whereas a right inverse is called a *section*.

Definition 1.1.6. A *monomorphism*, or *mono*, is a morphism $f : A \to B$ that can be cancelled on the left: for all $g : C \to A$, $h : C \to A$,

$$f \circ g = f \circ h \Rightarrow g = h \,.$$

An *epimorphism*, or *epi*, is a morphism $f : A \to B$ that can be cancelled on the right: for all $g : B \to C$, $h : B \to A$,

$$g \circ f = h \circ f \Rightarrow g = h \,.$$

In Set monomorphisms are the injective functions and epimorphisms are the surjective functions. Isomorphisms in Set are the bijective functions. Thus, in Set a morphism is iso if, and only if, it is both mono and epi. However, this example is misleading! In general, a morphism can be mono and epi without being an iso. For example, the non-identity morphism in the category consisting of two objects and one morphism between them is both epi and mono, but it has no inverse. A more interesting example of morphisms that are both epi and mono but are not iso occurs in the category Top of topological spaces and continuous maps, where not every continuous bijection is a homeomorphism.

A *diagram* of objects and morphisms is a directed graph whose vertices are objects of a category and edges are morphisms between them, for example:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[r, "h"] & C \arrow[dl, "j"] \\ D \arrow[ur, "m"] \arrow[r, "k"'] & E & \end{tikzcd}$$

Such a diagram is said to *commute* when the composition of morphisms along any two paths with the same beginning and end gives equal morphisms. Commutativity of the above diagram is equivalent to the following two equations:

$$f = m \circ g \,, \qquad\qquad k = j \circ h \circ m \,.$$

From these we can derive $k \circ g = j \circ h \circ f$ by a *diagram chase*.

1.2 Functors

Definition 1.2.1. A *functor* $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ consists of functions

$$F_0 : \mathcal{C}_0 \to \mathcal{D}_0 \qquad \text{and} \qquad F_1 : \mathcal{C}_1 \to \mathcal{D}_1$$

such that, for all $f : A \to B$ and $g : B \to C$ in $\mathcal{C}$:

$$\begin{aligned} F_1 f : F_0 A \to F_0 B \,, \\ F_1(g \circ f) = (F_1 g) \circ (F_1 f) \,, \\ F_1(1_A) = 1_{F_0 A} \,. \end{aligned}$$

We usually write F for both F_0 and F_1.

A functor is thus a homomorphism of the category structure; note that it maps commutative diagrams to commutative diagrams because it preserves composition.

We may form the "category of categories" Cat whose objects are small categories and whose morphisms are functors. Composition of functors is composition of the corresponding functions, and the identity functor is one that is identity on objects and on morphisms. The category Cat is large but locally small.

Definition 1.2.2. A functor $F : \mathcal{C} \to \mathcal{D}$ is *faithful* when it is "locally injective on morphisms", in the sense that for all $f, g : A \to B$, if $Ff = Fg$ then $f = g$.

A functor $F : \mathcal{C} \to \mathcal{D}$ is *full* when it is "locally surjective on morphisms": for every $g : FA \to FB$ there exists $f : A \to B$ such that $g = Ff$.

We consider several examples of functors.

1.2.1 Functors between sets, monoids and posets

When sets, monoids, groups, and posets are regarded as categories, the functors turn out to be the *usual morphisms*, for example:

- A functor between sets S and T is a function from S to T.
- A functor between groups G and H is a group homomorphism from G to H.
- A functor between posets P and Q is a monotone function from P to Q.

Exercise 1.2.3. Verify that the above claims are correct.

1.2.2 Forgetful functors

For categories of structures Group, Top, Graph, Poset, ..., there is a *forgetful* functor U which maps an object to the underlying set and a morphism to the underlying function. For example, the forgetful functor $U : \mathsf{Group} \to \mathsf{Set}$ maps a group $(G, \cdot)$ to the set G and a group homomorphism $f : (G, \cdot) \to (H, \star)$ to the function $f : G \to H$.

There are also forgetful functors that forget only part of the structure, for example the forgetful functor $U : \mathsf{Ring} \to \mathsf{Group}$ which maps a ring $(R, +, \times)$ to the additive group $(R, +)$ and a ring homomorphism $f : (R, +_R, \cdot_S) \to (S, +_S, \cdot_S)$ to the group homomorphism $f : (R, +_R) \to (S, +_S)$. Note that there is another forgetful functor $U' : \mathsf{Ring} \to \mathsf{Mon}$ from rings to monoids.

Exercise 1.2.4. Show that taking the graph $\Gamma(f) = \big\{\langle x, f(x)\rangle \bigm| x \in A\big\}$ of a function $f : A \to B$ determines a functor $\Gamma : \mathsf{Set} \to \mathsf{Rel}$, from sets and functions to sets and relations, which is the identity on objects. Is this a forgetful functor?

1.3 Constructions of Categories and Functors

1.3.1 Product of categories

Given categories $\mathcal{C}$ and $\mathcal{D}$, we form the *product category* $\mathcal{C} \times \mathcal{D}$ whose objects are pairs of objects $\langle C, D\rangle$ with $C \in \mathcal{C}$ and $D \in \mathcal{D}$, and whose morphisms are pairs of morphisms $\langle f, g\rangle : \langle C, D\rangle \to \langle C', D'\rangle$ with $f : C \to C'$ in $\mathcal{C}$ and $g : D \to D'$ in $\mathcal{D}$. Composition is given by $\langle f, g\rangle \circ \langle f', g'\rangle = \langle f \circ f', g \circ g'\rangle$.

There are evident *projection* functors

$$\begin{tikzcd} & \mathcal{C} \times \mathcal{D} \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & \\ \mathcal{C} & & \mathcal{D} \end{tikzcd}$$

which act as indicated in the following diagrams:

$$\begin{tikzcd} & \langle C, D \rangle \arrow[dl, mapsto, "\pi_0"'] \arrow[dr, mapsto, "\pi_1"] & \\ C & & D \end{tikzcd} \qquad \begin{tikzcd} & \langle f, g \rangle \arrow[dl, mapsto, "\pi_0"'] \arrow[dr, mapsto, "\pi_1"] & \\ f & & g \end{tikzcd}$$

Exercise 1.3.1. Show that, for any categories $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, there are distinguished isos:

$$\begin{gathered} \mathsf{1} \times \mathbb{C} \cong \mathbb{C} \\ \mathbb{B} \times \mathbb{C} \cong \mathbb{C} \times \mathbb{B} \\ \mathbb{A} \times (\mathbb{B} \times \mathbb{C}) \cong (\mathbb{A} \times \mathbb{B}) \times \mathbb{C} \end{gathered}$$

Does this make Cat a (commutative) monoid?

1.3.2 Slice categories

Given a category $\mathcal{C}$ and an object $A \in \mathcal{C}$, the *slice* category $\mathcal{C}/A$ has as objects, morphisms into A,

$$\begin{tikzcd} B \arrow[d, "f"] \\ A \end{tikzcd} \tag{1.1}$$

and as morphisms, commutative diagrams over A:

$$\begin{tikzcd} B \arrow[rr, "g"] \arrow[dr, "f"'] & & B' \arrow[dl, "f'"] \\ & A & \end{tikzcd} \tag{1.2}$$

That is, a morphism from $f : B \to A$ to $f' : B' \to A$ is a morphism $g : B \to B'$ such that $f = f' \circ g$. Composition of morphisms in $\mathcal{C}/A$ is composition of morphisms in $\mathcal{C}$.

There is a forgetful functor $U_A : \mathcal{C}/A \to \mathcal{C}$ which maps an object (1.1) to its domain B, and a morphism (1.2) to the morphism $g : B \to B'$.

Furthermore, for each morphism $h : A \to A'$ in $\mathcal{C}$ there is a functor "composition by h",

$$\mathcal{C}/h : \mathcal{C}/A \to \mathcal{C}/A'$$

which maps an object (1.1) to the object $h \circ f : B \to A'$ and a morphisms (1.2) to the morphism

$$\begin{tikzcd} B \arrow[rr, "g"] \arrow[dr, "h \circ f"'] & & B' \arrow[dl, "h \circ f'"] \\ & A' & \end{tikzcd}$$

The construction of slice categories is itself a functor

$$\mathcal{C}/- : \mathcal{C} \to \mathsf{Cat}$$

provided that $\mathcal{C}$ is small. This functor maps each $A \in \mathcal{C}$ to the category $\mathcal{C}/A$ and each morphism $h : A \to A'$ to the composition functor $\mathcal{C}/h : \mathcal{C}/A \to \mathcal{C}/A'$.

Since Cat is itself a category, we may form the slice category $\mathsf{Cat}/\mathcal{C}$ for any small category $\mathcal{C}$. The slice functor $\mathcal{C}/-$ then factors through the forgetful functor $U_{\mathcal{C}} : \mathsf{Cat}/\mathcal{C} \to \mathsf{Cat}$ via a functor $\overline{\mathcal{C}} : \mathcal{C} \to \mathsf{Cat}/\mathcal{C}$,

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\overline{\mathcal{C}}"] \arrow[dr, "\mathcal{C}/-"'] & \mathsf{Cat}/\mathcal{C} \arrow[d, "U_{\mathcal{C}}"] \\ & \mathsf{Cat} \end{tikzcd}$$

where for $A \in \mathcal{C}$, the object part $\overline{\mathcal{C}}A$ is

$$\begin{tikzcd} \mathcal{C}/A \arrow[d, "U_A"] \\ \mathcal{C} \end{tikzcd}$$

and for $h : A \to A'$ in $\mathcal{C}$, the morphism part $\overline{\mathcal{C}}h$ is

$$\begin{tikzcd} \mathcal{C}/A \arrow[rr, "\mathcal{C}/h"] \arrow[dr, "U_A"'] & & \mathcal{C}/A' \arrow[dl, "U_{A'}"] \\ & \mathcal{C} & \end{tikzcd}$$

1.3.3 Arrow categories

Similar to the slice categories, an arrow category has arrows as objects, but without a fixed codomain. Given a category $\mathcal{C}$, the *arrow* category $\mathcal{C}^{\to}$ has as objects the morphisms of $\mathcal{C}$,

$$\begin{tikzcd} A \arrow[d, "f"] \\ B \end{tikzcd} \tag{1.3}$$

and as morphisms $f \to f'$ the commutative squares,

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\ B \arrow[r, "g'"'] & B'. \end{tikzcd} \tag{1.4}$$

That is, a morphism from $f : A \to B$ to $f' : A' \to B'$ is a pair of morphisms $g : A \to A'$ and $g' : B \to B'$ such that $g' \circ f = f' \circ g$. Composition of morphisms in $\mathcal{C}^{\to}$ is just componentwise composition of morphisms in $\mathcal{C}$.

There are two evident forgetful functors $U_1, U_2 : \mathcal{C}^{\to} \to \mathcal{C}$, given by the domain and codomain operations. (Can you find a common section for these?)

1.3.4 Opposite categories

For a category $\mathcal{C}$ the *opposite category* $\mathcal{C}^{\mathsf{op}}$ has the same objects as $\mathcal{C}$, but all the morphisms are turned around, that is, a morphism $f : A \to B$ in $\mathcal{C}^{\mathsf{op}}$ is a morphism $f : B \to A$ in $\mathcal{C}$. The identity arrows in $\mathcal{C}^{\mathsf{op}}$ are the same as in $\mathcal{C}$, but the order of composition is reversed. The opposite of the opposite of a category is clearly the original category.

A functor $F : \mathcal{C}^{\mathsf{op}} \to \mathcal{D}$ is sometimes called a *contravariant functor* (from $\mathcal{C}$ to $\mathcal{D}$), and a functor $F : \mathcal{C} \to \mathcal{D}$ is a *covariant* functor.

For example, the opposite category of a preorder $(P, \leq)$ is the preorder P turned upside down, $(P, \geq)$.

Exercise 1.3.2. Given a functor $F : \mathcal{C} \to \mathcal{D}$, can you define a functor $F^{\mathsf{op}} : \mathcal{C}^{\mathsf{op}} \to \mathcal{D}^{\mathsf{op}}$ in such a way that $-^{\mathsf{op}}$ itself becomes a functor? On what category is it a functor?

1.3.5 Representable functors

Let $\mathcal{C}$ be a locally small category. Then for each pair of objects $A, B \in \mathcal{C}$ the collection of all morphisms $A \to B$ forms a set, written $\mathsf{Hom}_{\mathcal{C}}(A, B)$, $\mathsf{Hom}(A, B)$ or $\mathcal{C}(A, B)$. For every $A \in \mathcal{C}$ there is a functor

$$\mathcal{C}(A, -) : \mathcal{C} \to \mathsf{Set}$$

defined by

$$\begin{aligned} &\mathcal{C}(A, B) = \{ f \in \mathcal{C}_1 \mid f : A \to B \} \\ &\mathcal{C}(A, g) : f \mapsto g \circ f \end{aligned}$$

where $B \in \mathcal{C}$ and $g : B \to C$. In words, $\mathcal{C}(A, g)$ is composition by g. This is indeed a functor because, for any morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \tag{1.5}$$

we have
$$\mathcal{C}(A, h \circ g)f = (h \circ g) \circ f = h \circ (g \circ f) = \mathcal{C}(A, h)(\mathcal{C}(A, g)f) \;,$$
and $\mathcal{C}(A, 1_B)f = 1_A \circ f = f = 1_{\mathcal{C}(A,B)} f$.

We may also ask whether $\mathcal{C}(-, B)$ is a functor. If we define its action on morphisms to be precomposition,
$$\mathcal{C}(f, B) : g \mapsto g \circ f \;,$$
it becomes a *contravariant* functor,
$$\mathcal{C}(-, B) : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set} \;.$$
The contravariance is a consequence of precomposition; for morphisms (1.5) we have
$$\mathcal{C}(g \circ f, D)h = h \circ (g \circ f) = (h \circ g) \circ f = \mathcal{C}(f, D)(\mathcal{C}(g, D)h) \;.$$
A functor of the form $\mathcal{C}(A, -)$ is a *(covariant) representable functor*, and a functor of the form $\mathcal{C}(-, B)$ is a *(contravariant) representable functor*.

It follows that the hom-set is a functor
$$\mathcal{C}(-, -) : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Set}$$
which maps a pair of objects $A, B \in \mathcal{C}$ to the set $\mathcal{C}(A, B)$ of morphisms from A to B, and it maps a pair of morphisms $f : A' \to A$, $g : B \to B'$ in $\mathcal{C}$ to the function
$$\mathcal{C}(f, g) : \mathcal{C}(A, B) \to \mathcal{C}(A', B')$$
defined by
$$\mathcal{C}(f, g) : h \mapsto g \circ h \circ f \;.$$
(Why does it follow that this is a functor?)

1.3.6 Group actions

A group $(G, \cdot)$ is a category with one object $\star$ and elements of G as the morphisms. Thus, a functor $F : G \to \mathsf{Set}$ is given by a set $F\star = S$ and for each $a \in G$ a function $Fa : S \to S$ such that, for all $x \in S$, $a, b \in G$,
$$(Fe)x = x \;, \qquad\qquad (F(a \cdot b))x = (Fa)((Fb)x) \;.$$
Here e is the unit element of G. If we write $a \cdot x$ instead of $(Fa)x$, the above two equations become the familiar laws for a *left group action on the set S*:
$$e \cdot x = x \;, \qquad\qquad (a \cdot b) \cdot x = a \cdot (b \cdot x) \;.$$
Exercise 1.3.3. A *right group action* by a group $(G, \cdot)$ on a set S is an operation $\cdot : S \times G \to S$ that satisfies, for all $x \in S$, $a, b \in G$,
$$x \cdot e = x \;, \qquad\qquad x \cdot (a \cdot b) = (x \cdot a) \cdot b \;.$$
Exhibit right group actions as functors.

1.4 Natural Transformations and Functor Categories

Definition 1.4.1. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\eta : F \Longrightarrow G$ from F to G is a map $\eta : \mathcal{C}_0 \to \mathcal{D}_1$ which assigns to every object $A \in \mathcal{C}$ a morphism $\eta_A : FA \to GA$, called the *component of* η *at* A, such that for every $f : A \to B$ in $\mathcal{C}$ we have $\eta_B \circ Ff = Gf \circ \eta_A$, i.e., the following diagram in $\mathcal{D}$ commutes:

$$\begin{tikzcd} FA \arrow[r, "\eta_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\eta_B"'] & GB \end{tikzcd}$$

A simple example is given by the "twist" isomorphism $t : A \times B \to B \times A$ (in Set). Given any maps $f : A \to A'$ and $g : B \to B'$, there is a commutative square:

$$\begin{tikzcd} A \times B \arrow[r, "t_{A,B}"] \arrow[d, "f \times g"'] & B \times A \arrow[d, "g \times f"] \\ A' \times B' \arrow[r, "t_{A',B'}"'] & B' \times A' \end{tikzcd}$$

Thus naturality means that the two *functors* $F(X,Y) = X \times Y$ and $G(X,Y) = Y \times X$ are related to each other (by $t : F \to G$), and not simply their individual values $A \times B$ and $B \times A$. As a further example of a natural transformation, consider groups G and H as categories and two homomorphisms $f, g : G \to H$ as functors between them. A natural transformation $\eta : f \Longrightarrow g$ is given by a single element $\eta_\star = b \in H$ such that, for every $a \in G$, the following diagram commutes:

$$\begin{tikzcd} \star \arrow[r, "b"] \arrow[d, "fa"'] & \star \arrow[d, "ga"] \\ \star \arrow[r, "b"'] & \star \end{tikzcd}$$

This means that $b \cdot fa = (ga) \cdot b$, that is $ga = b \cdot (fa) \cdot b^{-1}$. In other words, a natural transformation $f \Longrightarrow g$ is a *conjugation* operation $b^{-1} \cdot - \cdot b$ which transforms f into g.

For every functor $F : \mathcal{C} \to \mathcal{D}$ there exists the *identity transformation* $1_F : F \Longrightarrow F$ defined by $(1_F)_A = 1_A$. If $\eta : F \Longrightarrow G$ and $\theta : G \Longrightarrow H$ are natural transformations, then their composition $\theta \circ \eta : F \Longrightarrow H$, defined by $(\theta \circ \eta)_A = \theta_A \circ \eta_A$ is also a natural transformation. Composition of natural transformations is associative because it is composition in the codomain category $\mathcal{D}$. This leads to the definition of functor categories.

Definition 1.4.2. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The *functor category* $\mathcal{D}^{\mathcal{C}}$ is the category whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$ and whose morphisms are natural transformations between them.

A functor category may be quite large, too large in fact. In order to avoid problems with size we normally require $\mathcal{C}$ to be a locally small category. The "hom-class" of all natural transformations $F \Longrightarrow G$ is usually written as

$$\mathsf{Nat}(F, G)$$

instead of the more awkward $\mathsf{Hom}_{\mathcal{D}^{\mathcal{C}}}(F, G)$.

Suppose we have functors F, G, and H with a natural transformation $\theta : G \Longrightarrow H$, as in the following diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "G"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathbb{E} \end{tikzcd}$$

Then we can form a natural transformation $\theta \circ F : G \circ F \Longrightarrow H \circ F$ whose component at $A \in \mathcal{C}$ is $(\theta \circ F)_A = \theta_{FA}$.

Similarly, if we have functors and a natural transformation

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "G"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{D} \arrow[r, "F"] & \mathbb{E} \end{tikzcd}$$

we can form a natural transformation $(F \circ \theta) : F \circ G \Longrightarrow F \circ H$ whose component at $A \in \mathcal{C}$ is $(F \circ \theta)_A = F\theta_A$. These operations are known as *whiskering*.

A *natural isomorphism* is an isomorphism in a functor category. Thus, if $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ are two functors, a natural isomorphism between them is a natural transformation $\eta : F \Longrightarrow G$ whose components are isomorphisms. In this case, the inverse natural transformation $\eta^{-1} : G \Longrightarrow F$ is given by $(\eta^{-1})_A = (\eta_A)^{-1}$. We write $F \cong G$ when F and G are naturally isomorphic.

The definition of natural transformations is motivated in part by the fact that, for any small categories $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, we have

$$\mathsf{Cat}(\mathbb{A} \times \mathbb{B}, \mathbb{C}) \cong \mathsf{Cat}(\mathbb{A}, \mathbb{C}^{\mathbb{B}}) \ . \tag{1.6}$$

The isomorphism takes a functor $F : \mathbb{A} \times \mathbb{B} \to \mathbb{C}$ to the functor $\widetilde{F} : \mathbb{A} \to \mathbb{C}^{\mathbb{B}}$ defined on objects $A \in \mathbb{A}$, $B \in \mathbb{B}$ by

$$(\widetilde{F}A)B = F\langle A, B\rangle$$

and on a morphism $f : A \to A'$ by

$$(\widetilde{F}f)_B = F\langle f, 1_B\rangle \ .$$

The functor $\widetilde{F}$ is called the *transpose* of F.

The inverse isomorphism takes a functor $G : \mathbb{A} \to \mathbb{C}^{\mathbb{B}}$ to the functor $\widetilde{G} : \mathbb{A} \times \mathbb{B} \to \mathbb{C}$, defined on objects by

$$\widetilde{G}\langle A, B\rangle = (GA)B$$

and on a morphism $\langle f, g\rangle : A \times B \to A' \times B'$ by

$$\widetilde{G}\langle f, g\rangle = (Gf)_{B'} \circ (GA)g = (GA')g \circ (Gf)_B \ ,$$

where the last equation holds by naturality of Gf:

$$\begin{tikzcd} (GA)B \arrow[r, "(Gf)_B"] \arrow[d, "(GA)g"'] & (GA')B \arrow[d, "(GA')g"] \\ (GA)B' \arrow[r, "(Gf)_{B'}"'] & (GA')B' \end{tikzcd}$$

1.4.1 Directed graphs as a functor category

Recall that a *directed graph* G is given by a set of vertices G_V and a set of edges G_E. Each edge $e \in G_E$ has a uniquely determined *source* $\mathsf{src}_G\, e \in G_V$ and *target* $\mathsf{trg}_G\, e \in G_V$. We write $e : a \to b$ when a is the source and b is the target of e. A *graph homomorphism* $\phi : G \to H$ is a pair of functions $\phi_0 : G_V \to H_V$ and $\phi_1 : G_E \to H_E$, where we usually write ϕ for both ϕ_0 and ϕ_1, such that whenever $e : a \to b$ then $\phi_1 e : \phi_0 a \to \phi_0 b$. The category of directed graphs and graph homomorphisms is denoted by Graph.

Now let $\cdot \rightrightarrows \cdot$ be the category with two objects and two parallel morphisms, depicted by the following "sketch":

$$\begin{tikzcd} E \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & V \end{tikzcd}$$

An object of the functor category $\mathsf{Set}^{\cdot \rightrightarrows \cdot}$ is a functor $G : (\cdot \rightrightarrows \cdot) \to \mathsf{Set}$, which consists of two sets GE and GV and two functions $Gs : GE \to GV$ and $Gt : GE \to GV$. But this is precisely a directed graph whose vertices are GV, the edges are GE, the source of $e \in GE$ is $(Gs)e$ and the target is $(Gt)e$. Conversely, any directed graph G is a functor $G : (\cdot \rightrightarrows \cdot) \to \mathsf{Set}$, defined by

$$GE = G_E \ , \qquad GV = G_V \ , \qquad Gs = \mathsf{src}_G \ , \qquad Gt = \mathsf{trg}_G \ .$$

Now category theory begins to show its worth, for the morphisms in $\mathsf{Set}^{\cdot \rightrightarrows \cdot}$ are precisely the graph homomorphisms. Indeed, a natural transformation $\phi : G \Longrightarrow H$ between graphs is a pair of functions,

$$\phi_E : G_E \to H_E \qquad \text{and} \qquad \phi_V : G_V \to H_V$$

whose naturality is expressed by the commutativity of the following two diagrams:

$$
\begin{tikzcd}
G_E \arrow[r, "\phi_E"] \arrow[d, "\mathsf{src}_G"'] & H_E \arrow[d, "\mathsf{src}_H"] \\
G_V \arrow[r, "\phi_V"'] & H_V
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
G_E \arrow[r, "\phi_E"] \arrow[d, "\mathsf{trg}_G"'] & H_E \arrow[d, "\mathsf{trg}_H"] \\
G_V \arrow[r, "\phi_V"'] & H_V
\end{tikzcd}
$$

This is precisely the requirement that $e : a \to b$ implies $\phi_E e : \phi_V a \to \phi_V b$. Thus, in sum, we have,

$$\mathsf{Graph} = \mathsf{Set}^{\cdot \rightrightarrows \cdot}.$$

Exercise 1.4.3. Exhibit the arrow category $\mathcal{C}^{\to}$ and the category of group actions $\mathsf{Set}(G)$ as functor categories.

1.4.2 The Yoneda embedding

The example $\mathsf{Graph} = \mathsf{Set}^{\cdot \rightrightarrows \cdot}$ leads one to wonder which categories $\mathcal{C}$ can be represented as functor categories $\mathsf{Set}^{\mathcal{D}}$ for a suitably chosen $\mathcal{D}$ or, when that is not possible, at least as full subcategories of $\mathsf{Set}^{\mathcal{D}}$.

For a locally small category $\mathcal{C}$, there is the hom-functor

$$\mathcal{C}(-,-) : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Set}\ .$$

By transposing as in (1.6) we obtain the functor

$$\mathsf{y} : \mathcal{C} \to \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$$

which maps an object $A \in \mathcal{C}$ to the representable functor

$$\mathsf{y}A = \mathcal{C}(-, A) : B \mapsto \mathcal{C}(B, A)$$

and a morphism $f : A \to A'$ in $\mathcal{C}$ to the natural transformation $\mathsf{y}f : \mathsf{y}A \Longrightarrow \mathsf{y}A'$ whose component at B is

$$(\mathsf{y}f)_B = \mathcal{C}(B, f) : g \mapsto f \circ g\ .$$

This functor y is called the *Yoneda embedding.*

Exercise 1.4.4. Show that this *is* a functor.

Theorem 1.4.5 (Yoneda embedding)**.** *For any locally small category $\mathcal{C}$ the Yoneda embedding*

$$\mathsf{y} : \mathcal{C} \to \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$$

is full and faithful and injective on objects. Therefore, $\mathcal{C}$ is a full subcategory of $\mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$.

The proof of the theorem uses the famous Yoneda Lemma.

Lemma 1.4.6 (Yoneda)**.** *Every functor* $F : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ *is naturally isomorphic to the functor* $\mathsf{Nat}(\mathsf{y}-, F)$*. That is, for every* $A \in \mathcal{C}$*,*

$$\mathsf{Nat}(\mathsf{y}A, F) \cong FA \ ,$$

and this isomorphism is natural in A*.*

Indeed, the displayed isomorphism is also natural in F.

Proof. The desired natural isomorphism θ_A maps a natural transformation $\eta \in \mathsf{Nat}(\mathsf{y}A, F)$ to $\eta_A \mathbf{1}_A$. The inverse ${\theta_A}^{-1}$ maps an element $x \in FA$ to the natural transformation $({\theta_A}^{-1}x)$ whose component at B maps $f \in \mathcal{C}(B, A)$ to $(Ff)x$. To summarize, for $\eta : \mathcal{C}(-, A) \Longrightarrow F$, $x \in FA$ and $f \in \mathcal{C}(B, A)$, we have

$$\begin{aligned} &\theta_A : \mathsf{Nat}(\mathsf{y}A, F) \to FA \ , && {\theta_A}^{-1} : FA \to \mathsf{Nat}(\mathsf{y}A, F) \ , \\ &\theta_A \eta = \eta_A \mathbf{1}_A \ , && ({\theta_A}^{-1}x)_B f = (Ff)x \ . \end{aligned}$$

To see that θ_A and ${\theta_A}^{-1}$ really are inverses of each other, observe that

$$\theta_A({\theta_A}^{-1}x) = ({\theta_A}^{-1}x)_A \mathbf{1}_A = (F\mathbf{1}_A)x = \mathbf{1}_{FA}x = x \ ,$$

and also

$$({\theta_A}^{-1}(\theta_A \eta))_B f = (Ff)(\theta_A \eta) = (Ff)(\eta_A \mathbf{1}_A) = \eta_B(\mathbf{1}_A \circ f) = \eta_B f \ ,$$

where the third equality holds by the following naturality square for η:

$$\begin{tikzcd} \mathcal{C}(A, A) \arrow[r, "\eta_A"] \arrow[d, "\mathcal{C}(f, A)"'] & FA \arrow[d, "Ff"] \\ \mathcal{C}(B, A) \arrow[r, "\eta_B"'] & FB \end{tikzcd}$$

It remains to check that θ is natural, which amounts to establishing the commutativity of the following diagram, with $g : A \to A'$:

$$\begin{tikzcd} \mathsf{Nat}(\mathsf{y}A, F) \arrow[r, "\theta_A"] & FA \\ \mathsf{Nat}(\mathsf{y}A', F) \arrow[r, "\theta_{A'}"'] \arrow[u, "\mathsf{Nat}(\mathsf{y}g, F)"] & FA' \arrow[u, "Fg"'] \end{tikzcd}$$

The diagram is commutative because, for any $\eta : \mathsf{y}A' \Longrightarrow F$,

$$(Fg)(\theta_{A'}\eta) = (Fg)(\eta_{A'}1_{A'}) = \eta_A(1_{A'} \circ g) = \\ \eta_A(g \circ 1_A) = (\mathsf{Nat}(\mathsf{y}g, F)\eta)_A 1_A = \theta_A(\mathsf{Nat}(\mathsf{y}g, F)\eta)\ ,$$

where the second equality is justified by naturality of η. $\square$

Proof of Theorem 1.4.5. That the Yoneda embedding is full and faithful means that for all $A, B \in \mathcal{C}$ the map

$$\mathsf{y} : \mathcal{C}(A, B) \to \mathsf{Nat}(\mathsf{y}A, \mathsf{y}B)$$

which maps $f : A \to B$ to $\mathsf{y}f : \mathsf{y}A \Longrightarrow \mathsf{y}B$ is an isomorphism. But this is just the Yoneda Lemma applied to the case $F = \mathsf{y}B$. Indeed, with notation as in the proof of the Yoneda Lemma and $g : C \to A$, we see that the isomorphism

$$\theta_A^{-1} : \mathcal{C}(A, B) = (\mathsf{y}B)A \to \mathsf{Nat}(\mathsf{y}A, \mathsf{y}B)$$

is in fact y:

$$(\theta_A{}^{-1}f)_C g = ((\mathsf{y}A)g)f = f \circ g = (\mathsf{y}f)_C g\ .$$

Furthermore, if $\mathsf{y}A = \mathsf{y}B$ then $1_A \in \mathcal{C}(A, A) = (\mathsf{y}A)A = (\mathsf{y}B)A = \mathcal{C}(B, A)$ which can only happen if $A = B$. Therefore, y is injective on objects. $\square$

The following corollary is often useful.

Corollary 1.4.7. *For $A, B \in \mathcal{C}$, $A \cong B$ if, and only if, $\mathsf{y}A \cong \mathsf{y}B$ in $\mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$.*

Proof. Every functor preserves isomorphisms, and a full and faithful one also reflects them. (A functor $F : \mathcal{C} \to \mathcal{D}$ is said to *reflect* isomorphisms when $Ff : FA \to FB$ being an isomorphisms implies that $f : A \to B$ is an isomorphism.) $\square$

Exercise 1.4.8. Prove that a full and faithful functor reflects isomorphisms.

Functor categories $\mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$ are important enough to deserve a name. They are called *presheaf categories*, and a functor $F : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ is called a *presheaf* on $\mathcal{C}$. We also use the notation $\widehat{\mathcal{C}} = \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$.

1.4.3 Equivalence of categories

An isomorphism of categories $\mathcal{C}$ and $\mathcal{D}$ in Cat consists of functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

such that $G \circ F = 1_{\mathcal{C}}$ and $F \circ G = 1_{\mathcal{D}}$. This is often too restrictive a notion. A more general notion which replaces the above identities with natural isomorphisms is more useful.

Definition 1.4.9. An *equivalence of categories* is a pair of functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

such that there are natural isomorphisms

$$G \circ F \cong 1_{\mathcal{C}} \qquad \text{and} \qquad F \circ G \cong 1_{\mathcal{D}} .$$

We say that $\mathcal{C}$ and $\mathcal{D}$ are *equivalent categories* and write $\mathcal{C} \simeq \mathcal{D}$.

A functor $F : \mathcal{C} \to \mathcal{D}$ is called an *equivalence functor* if there exists $G : \mathcal{D} \to \mathcal{C}$ such that F and G form an equivalence.

The point of equivalence of categories is that it preserves almost all categorical properties, but ignores those concepts that are not of interest from a categorical point of view, such as identity of objects.

The following proposition requires the Axiom of Choice as stated. However, in many specific cases a canonical choice can be made without appeal to that axiom.

Proposition 1.4.10. *A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence functor if, and only if, F is full and faithful, and* essentially surjective *on objects, meaning that for every $B \in \mathcal{D}$ there exists $A \in \mathcal{C}$ such that $FA \cong B$.*

Proof. It is easily seen that the conditions are necessary, so we only show they are sufficient. Suppose $F : \mathcal{C} \to \mathcal{D}$ is full and faithful, and essentially surjective on objects. For each $B \in \mathcal{D}$, *choose* an object $GB \in \mathcal{C}$ and an isomorphism $\eta_B : F(GB) \to B$. If $f : B \to C$ is a morphism in $\mathcal{D}$, let $Gf : GB \to GC$ be the unique morphism in $\mathcal{C}$ for which

$$F(Gf) = {\eta_C}^{-1} \circ f \circ \eta_B . \tag{1.7}$$

Such a unique morphism exists because F is full and faithful. This defines a functor $G : \mathcal{D} \to \mathcal{C}$, as can be easily checked. In addition, (1.7) ensures that η is a natural isomorphism $F \circ G \Longrightarrow 1_{\mathcal{D}}$.

It remains to show that $G \circ F \cong 1_{\mathcal{C}}$. For $A \in \mathcal{C}$, let $\theta_A : G(FA) \to A$ be the unique morphism such that $F\theta_A = \eta_{FA}$. Naturality of θ_A follows from functoriality of F and naturality of η. Because F reflects isomorphisms, θ_A is an isomorphism for every A. □

Example 1.4.11. As an example of equivalence of categories we consider the category of sets and partial functions and the category of pointed sets.

A *partial function* $f : A \rightharpoonup B$ is a function defined on a subset $\mathsf{supp}\, f \subseteq A$, called the *support*[3] of f, and taking values in B. Composition of partial functions $f : A \rightharpoonup B$ and $g : B \rightharpoonup C$ is the partial function $g \circ f : A \rightharpoonup C$ defined by

$$\begin{gathered} \mathsf{supp}\,(g \circ f) = \big\{ x \in A \bigm| x \in \mathsf{supp}\, f \wedge fx \in \mathsf{supp}\, g \big\} \\ (g \circ f)x = g(fx) \quad \text{for } x \in \mathsf{supp}\,(g \circ f) \end{gathered}$$

[3] The support of a partial function $f : A \rightharpoonup B$ is usually called its *domain*, but this terminology conflicts with A being the domain of f as a morphism.

Composition of partial functions is associative. This way we obtain a category Par of sets and partial functions.

A *pointed set* (A, a) is a set A together with an element $a \in A$. A *pointed function* $f : (A, a) \to (B, b)$ between pointed sets is a function $f : A \to B$ such that $fa = b$. The category $\mathsf{Set}_\bullet$ consists of pointed sets and pointed functions.

The categories Par and $\mathsf{Set}_\bullet$ are equivalent. The equivalence functor $F : \mathsf{Set}_\bullet \to \mathsf{Par}$ maps a pointed set (A, a) to the set $F(A, a) = A \setminus \{a\}$, and a pointed function $f : (A, a) \to (B, b)$ to the partial function $Ff : F(A, a) \rightharpoonup F(B, b)$ defined by

$$\mathsf{supp}\,(Ff) = \left\{x \in A \;\middle|\; fx \neq b\right\} , \qquad\qquad (Ff)x = fx .$$

The inverse equivalence functor $G : \mathsf{Par} \to \mathsf{Set}_\bullet$ maps a set $A \in \mathsf{Par}$ to the pointed set $GA = (A + \{\bot_A\}, \bot_A)$, where $\bot_A$ is an element that does not belong to A. A partial function $f : A \rightharpoonup B$ is mapped to the pointed function $Gf : GA \to GB$ defined by

$$(Gf)x = \begin{cases} fx & \text{if } x \in \mathsf{supp}\, f \\ \bot_B & \text{otherwise} . \end{cases}$$

A good way to think about the "bottom" point $\bot_A$ is as a special "undefined value". Let us look at the composition of F and G on objects:

$$\begin{aligned} G(F(A, a)) &= G(A \setminus \{a\}) = ((A \setminus \{a\}) + \bot_A, \bot_A) \cong (A, a) . \\ F(GA) &= F(A + \{\bot_A\}, \bot_A) = (A + \{\bot_A\}) \setminus \{\bot_A\} = A . \end{aligned}$$

The isomorphism $G(F(A, a)) \cong (A, a)$ is easily seen to be natural.

Example 1.4.12. Another example of an equivalence of categories arises when we take the poset reflection of a preorder. Let $(P, \leq)$ be a preorder, If we think of P as a category, then $a, b \in P$ are isomorphic, when $a \leq b$ and $b \leq a$. Isomorphism $\cong$ is an equivalence relation, therefore we may form the quotient set $P/{\cong}$. The set $P/{\cong}$ is a poset for the order relation $\sqsubseteq$ defined by

$$[a] \sqsubseteq [b] \iff a \leq b .$$

Here $[a]$ denotes the equivalence class of a. We call $(P/{\cong}, \sqsubseteq)$ the *poset reflection* of P. The quotient map $q : P \to P/{\cong}$ is a functor when P and $P/{\cong}$ are viewed as categories. By Proposition 1.4.10, q is an equivalence functor. Trivially, it is faithful and surjective on objects. It is also full because $qa \sqsubseteq qb$ in $P/{\cong}$ implies $a \leq b$ in P.

1.5 Adjoint Functors

The notion of adjunction is perhaps the most important concept revealed by category theory. It is a fundamental logical and mathematical concept that occurs everywhere and often marks an important and interesting connection between two constructions of interest. In logic, adjoint functors are pervasive, although this is only recognizable through the lens of category theory.

1.5.1 Adjoint maps between preorders

Let us begin with a simple situation. We have already seen that a preorder $(P, \leq)$ is a category in which there is at most one morphism between any two objects. A functor between preorders is a monotone map. Suppose we have preorders P and Q with monotone maps back and forth,

$$\begin{tikzcd} P \arrow[r, bend left, "f"] & Q \arrow[l, bend left, "g"] \end{tikzcd} .$$

We say that f and g are *adjoint*, and write $f \dashv g$, when for all $x \in P$, $y \in Q$,

$$fx \leq y \iff x \leq gy \; . \tag{1.8}$$

Note that adjointness is *not* a symmetric relation. The map f is the *left adjoint* and g is the *right adjoint* (note their positions with respect to $\leq$).

Equivalence (1.8) is more conveniently displayed as

$$\frac{fx \leq y}{\overline{x \leq gy}}$$

The double line indicates the fact that this is a two-way rule: the top line implies the bottom line, and vice versa.

Let us consider two examples.

Conjunction is adjoint to implication Consider a propositional calculus with logical operations of conjunction $\wedge$ and implication $\Rightarrow$ (perhaps among others). The formulas of this calculus are built from variables x_0, x_1, x_2, ..., the truth values $\bot$ and $\top$, and the logical connectives $\wedge, \Rightarrow, \ldots$. The logical rules are given in natural deduction style:

$$\frac{}{\top} \qquad \frac{\bot}{A} \qquad \frac{A \quad B}{A \wedge B} \qquad \frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B}$$

$$\frac{A \Rightarrow B \qquad A}{B} \qquad\qquad \begin{array}{c} [u : A] \\ \vdots \\ \dfrac{B}{A \Rightarrow B}\, u \end{array}$$

For example, we read the inference rules for $\Rightarrow$ as, respectively, "from $A \Rightarrow B$ and A we infer B" and "if from assumption A we infer B, then (without any assumptions) we infer $A \Rightarrow B$". Discharged assumptions are indicated by enclosing them in brackets, along with a label $[u : A]$ for the assumption, which is recorded along with the rule that discharges it, as above.

Logical entailment $\vdash$ between formulas of the propositional calculus is the relation $A \vdash B$ which holds if, and only if, from assuming A we can infer B (by using only the inference rules of the calculus). It is trivially the case that $A \vdash A$, and also

$$\text{if } A \vdash B \text{ and } B \vdash C \text{ then } A \vdash C \ .$$

In other words, $\vdash$ is a reflexive and transitive relation on the set P of all propositional formulas, so that $(\mathsf{P}, \vdash)$ is a preorder.

Let A be a propositional formula. Define $f : \mathsf{P} \to \mathsf{P}$ and $g : \mathsf{P} \to \mathsf{P}$ to be the maps

$$fB = (A \wedge B) \ , \qquad\qquad gB = (A \Rightarrow B) \ .$$

To see that the maps f and g are functors we need to show they respect entailment. Indeed, if $B \vdash B'$ then $A \wedge B \vdash A \wedge B'$ and $A \Rightarrow B \vdash A \Rightarrow B'$ by the following two derivations.

$$\dfrac{\dfrac{A \wedge B}{A} \qquad \begin{matrix}\dfrac{A \wedge B}{B} \\ \vdots \\ B'\end{matrix}}{A \wedge B'} \qquad\qquad \dfrac{\begin{matrix}\dfrac{A \Rightarrow B \qquad [u : A]}{B} \\ \vdots \\ B'\end{matrix}}{A \Rightarrow B'}\, u$$

We claim that $f \dashv g$. For this we need to prove that $A \wedge B \vdash C$ if, and only if, $B \vdash A \Rightarrow C$. The following two derivations establish the required equivalence.

$$\dfrac{\begin{matrix}\dfrac{[u : A] \qquad B}{A \wedge B} \\ \vdots \\ C\end{matrix}}{A \Rightarrow C}\, u \qquad\qquad \dfrac{\begin{matrix}\dfrac{A \wedge B}{B} \\ \vdots \\ A \Rightarrow C\end{matrix} \qquad \dfrac{A \wedge B}{A}}{C}$$

Therefore, *conjunction is left adjoint to implication.*

Topological interior as an adjoint Recall that a *topological space* $(X, \mathcal{O}X)$ is a set X together with a family $\mathcal{O}X \subseteq \mathcal{P}X$ of subsets of X which contains $\emptyset$ and X, and is closed under finite intersections and arbitrary unions. The elements of $\mathcal{O}X$ are called the *open sets.*

The *topological interior* of a subset $S \subseteq X$ is the largest open set contained in S, namely,

$$\mathsf{int}\, S = \bigcup \left\{ U \in \mathcal{O}X \;\middle|\; U \subseteq S \right\} \ .$$

Both $\mathcal{O}X$ and $\mathcal{P}X$ are posets ordered by subset inclusion. The inclusion $i : \mathcal{O}X \to \mathcal{P}X$ is thus a monotone map, and so indeed is the interior $\mathsf{int} : \mathcal{P}X \to \mathcal{O}X$, as follows immediately from its construction. So we have:

$$\begin{tikzcd} \mathcal{O}X \arrow[r, bend left, "i"] & \mathcal{P}X \arrow[l, bend left, "\mathsf{int}"] \end{tikzcd}$$

Moreover, for $U \in \mathcal{O}X$ and $S \in \mathcal{P}X$ we plainly also have

$$\frac{iU \subseteq S}{\overline{U \subseteq \mathsf{int}\, S}}$$

since $\mathsf{int}\, S$ is the largest open set contained in S. Thus *topological interior is right adjoint* to the inclusion of $\mathcal{O}X$ into $\mathcal{P}X$.

1.5.2 Adjoint functors

Let us now generalize the notion of adjoint monotone maps from posets to the situation

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

with arbitrary categories and functors. For monotone maps $f \dashv g$, the adjunction condition is a bijection

$$\frac{fx \to y}{\overline{x \to gy}}$$

between morphisms of the form $fx \to y$ and morphisms of the form $x \to gy$. This is the notion that generalizes the special case; for any $A \in \mathcal{C}$, $B \in \mathcal{D}$ we require a bijection between the sets $\mathcal{D}(FA, B)$ and $\mathcal{C}(A, GB)$:

$$\frac{FA \to B}{\overline{A \to GB}}$$

Definition 1.5.1. An *adjunction* $F \dashv G$ between the functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

is a natural isomorphism θ between functors

$$\mathcal{D}(F-, -) : \mathcal{C}^{\mathsf{op}} \times \mathcal{D} \to \mathsf{Set} \qquad \text{and} \qquad \mathcal{C}(-, G-) : \mathcal{C}^{\mathsf{op}} \times \mathcal{D} \to \mathsf{Set}\ .$$

This means that for every $A \in \mathcal{C}$ and $B \in \mathcal{D}$ there is a bijection

$$\theta_{A,B} : \mathcal{D}(FA, B) \cong \mathcal{C}(A, GB)\ ,$$

and naturality of θ means that for $f : A' \to A$ in $\mathcal{C}$ and $g : B \to B'$ in $\mathcal{D}$ the following diagram commutes:

$$\begin{tikzcd} \mathcal{D}(FA, B) \arrow[r, "\theta_{A,B}"] \arrow[d, "\mathcal{D}(Ff, g)"'] & \mathcal{D}(A, GB) \arrow[d, "\mathcal{C}(f, Gg)"] \\ \mathcal{D}(FA', B') \arrow[r, "\theta_{A',B'}"'] & \mathcal{C}(A', GB') \end{tikzcd}$$

Equivalently, for every $h : FA \to B$ in $\mathcal{D}$,

$$Gg \circ (\theta_{A,B}h) \circ f = \theta_{A',B'}(g \circ h \circ Ff) \ .$$

We say that F is the *left adjoint* and G is the *right adjoint*.

We have already seen examples of adjoint functors. For any category $\mathbb{B}$ we have functors $(-) \times \mathbb{B}$ and $(-)^{\mathbb{B}}$ from Cat to Cat. Recall the isomorphism (1.6),

$$\mathsf{Cat}(\mathbb{A} \times \mathbb{B}, \mathbb{C}) \cong \mathsf{Cat}(\mathbb{A}, \mathbb{C}^{\mathbb{B}}) \ .$$

This isomorphism is in fact natural in $\mathbb{A}$ and $\mathbb{C}$, so that

$$(-) \times \mathbb{B} \dashv (-)^{\mathbb{B}} \ .$$

Similarly, for any set $B \in \mathsf{Set}$ there are functors

$$(-) \times B : \mathsf{Set} \to \mathsf{Set} \ , \qquad\qquad (-)^B : \mathsf{Set} \to \mathsf{Set} \ ,$$

where $A \times B$ is the cartesian product of A and B, and C^B is the set of all functions from B to C. For morphisms, $f \times B = f \times 1_B$ and $f^B = f \circ (-)$. We then indeed have a natural isomorphism, for all $A, C \in \mathsf{Set}$,

$$\mathsf{Set}(A \times B, C) \cong \mathsf{Set}(A, C^B) \ ,$$

which maps a function $f : A \times B \to C$ to the function $(\widetilde{f}x)y = f\langle x, y\rangle$. Therefore,

$$(-) \times B \dashv (-)^B \ .$$

Exercise 1.5.2. Verify that the definition (1.8) of adjoint monotone maps between preorders is a special case of Definition 1.5.1. What happened to the naturality condition?

For another example, consider the forgetful functor

$$U : \mathsf{Cat} \to \mathsf{Graph} \ ,$$

which maps a category to the underlying directed graph. It has a left adjoint $P \dashv U$. The functor P is the *free* construction of a category from a graph; it maps a graph G to the *category of paths* $P(G)$. The objects of $P(G)$ are the vertices of G. The morphisms of $P(G)$ are the finite paths

$$v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \cdots \xrightarrow{e_n} v_n$$

of edges in G, composition is concatenation of paths, and the identity morphism on a vertex v is the empty path starting and ending at v.

By using the Yoneda Lemma we can easily prove that adjoints are unique up to natural isomorphism.

Proposition 1.5.3. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be adjoint functors, with $F \dashv G$. If also $G' : \mathcal{D} \to \mathcal{C}$ with $F \dashv G'$, then $G \cong G'$.*

Proof. Since the Yoneda embedding is full and faithful, we have $GB \cong G'B$ if, and only if, $\mathcal{C}(-, GB) \cong \mathcal{C}(-, G'B)$. But this indeed holds, because, for any $A \in \mathcal{C}$, we have

$$\mathcal{C}(A, GB) \cong \mathcal{D}(FA, B) \cong \mathcal{C}(A, G'B) \ ,$$

naturally in A. □

Left adjoints are of course also unique up to isomorphism, by duality.

1.5.3 The unit of an adjunction

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be adjoint functors, $F \dashv G$, and let $\theta : \mathcal{D}(F-, -) \to \mathcal{C}(-, G-)$ be the natural isomorphism witnessing the adjunction. For any object $A \in \mathcal{C}$ there is a distinguished morphism $\eta_A = \theta_{A,FA} 1_{FA} : A \to G(FA)$,

$$\frac{1_{FA} : FA \to FA}{\eta_A : A \to G(FA)}$$

Since θ is natural in A, we have a natural transformation $\eta : 1_{\mathcal{C}} \Longrightarrow G \circ F$, which is called the *unit of the adjunction* $F \dashv G$. In fact, we can recover θ from η as follows. For $f : FA \to B$, we have

$$\theta_{A,B} f = \theta_{A,B}(f \circ 1_{FA}) = Gf \circ \theta_{A,FA}(1_{FA}) = Gf \circ \eta_A \ ,$$

where we used naturality of θ in the second step. Schematically, given any $f : FA \to B$, the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "\theta_{A,B} f"'] & G(FA) \arrow[d, "Gf"] \\ & GB \end{tikzcd}$$

Since $\theta_{A,B}$ is a bijection, it follows that *every* morphism $g : A \to GB$ has the form $g = Gf \circ \eta_A$ for a *unique* $f : FA \to B$. We say that $\eta_A : A \to G(FA)$ is a *universal* morphism to G, or that η has the following *universal mapping property*: for every $A \in \mathcal{C}$, $B \in \mathcal{D}$, and $g : A \to GB$, there exists a *unique* $f : FA \to B$ such that $g = Gf \circ \eta_A$:

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "g"'] & G(FA) \arrow[d, "Gf"] & & FA \arrow[d, dotted, "f"] \\ & GB & & B \end{tikzcd}$$

This means that an adjunction can be given in terms of its unit. The isomorphism $\theta : \mathcal{D}(F-,-) \to \mathcal{C}(-,G-)$ is then recovered by

$$\theta_{A,B} f = Gf \circ \eta_A \,.$$

Proposition 1.5.4. *A functor $F : \mathcal{C} \to \mathcal{D}$ is left adjoint to a functor $G : \mathcal{D} \to \mathcal{C}$ if, and only if, there exists a natural transformation*

$$\eta : 1_{\mathcal{C}} \Longrightarrow G \circ F \,,$$

called the unit *of the adjunction, such that, for all $A \in \mathcal{C}$ and $B \in \mathcal{D}$ the map $\theta_{A,B} : \mathcal{D}(FA, B) \to \mathcal{C}(A, GB)$, defined by*

$$\theta_{A,B} f = Gf \circ \eta_A \,,$$

is an isomorphism.

Let us demonstrate how the universal mapping property of the unit of an adjunction appears as a well known construction in algebra. Consider the forgetful functor from monoids to sets,

$$U : \mathsf{Mon} \to \mathsf{Set} \,.$$

Does it have a left adjoint $F : \mathsf{Set} \to \mathsf{Mon}$? In order to obtain one, we need a "most economical" way of making a monoid FX from a given set X. Such a construction readily suggests itself, namely the *free monoid* on X, consisting of finite sequences of elements of X,

$$FX = \left\{ x_1 \ldots x_n \;\middle|\; n \geq 0 \;\&\; x_1, \ldots, x_n \in X \right\} \,.$$

The monoid operation is concatenation of sequences

$$x_1 \ldots x_m \cdot y_1 \ldots y_n = x_1 \ldots x_m y_1 \ldots y_n \,,$$

and the empty sequence is the unit of the monoid. In order for F to be a functor, it should also map morphisms to morphisms. If $f : X \to Y$ is a function, define $Ff : FX \to FY$ by

$$Ff : x_1 \ldots x_n \mapsto (fx_1) \ldots (fx_n) \,.$$

There is an inclusion $\eta_X : X \to U(FX)$ which maps every element $x \in X$ to the singleton sequence x. This gives a natural transformation $\eta : 1_{\mathsf{Set}} \Longrightarrow U \circ F$.

The monoid FX is "free" in the sense that it "satisfies only the equations required by the monoid laws"; we make this precise as follows. For every monoid M and function $f : X \to UM$ there exists a unique monoid homomorphism $\overline{f} : FX \to M$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & U(FX) \arrow[d, "U\overline{f}"] \\
& UM
\end{tikzcd}$$

This is precisely the condition required by Proposition 1.5.4 for η to be the unit of the adjunction $F \dashv U$. In this case, the universal mapping property of η is just the usual characterization of the free monoid FX generated by the set X: a homomorphism from FX is uniquely determined by its values on the generators.

1.5.4 The counit of an adjunction

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be adjoint functors with $F \dashv G$, and let $\theta : \mathcal{D}(F-, -) \to \mathcal{C}(-, G-)$ be the natural isomorphism witnessing the adjunction. For any object $B \in \mathcal{D}$ we have a distinguished morphism $\varepsilon_B = \theta^{-1}_{GB,B} 1_{GB} : F(GB) \to B$ by:

$$\frac{1_{GB} : GB \to GB}{\varepsilon_B : F(GB) \to B}$$

The natural transformation $\varepsilon : F \circ G \Longrightarrow 1_{\mathcal{D}}$ is called the *counit* of the adjunction $F \dashv G$. It is the dual notion to the unit of an adjunction. We state briefly the basic properties of the counit, which are easily obtained by "turning around" all the morphisms in the previous section and exchanging the roles of the left and right adjoints.

The bijection $\theta^{-1}_{A,B}$ can be recovered from the counit. For $g : A \to GB$ in $\mathcal{C}$, we have

$$\theta^{-1}_{A,B} g = \theta^{-1}_{A,B}(1_{GB} \circ g) = \theta^{-1}_{A,B} 1_{GB} \circ Fg = \varepsilon_B \circ Fg \ .$$

The universal mapping property of the counit is this: for every $A \in \mathcal{C}$, $B \in \mathcal{D}$, and $f : FA \to B$, there exists a *unique* $g : A \to GB$ such that $f = \varepsilon_B \circ Fg$:

$$\begin{tikzcd}
B & F(GB) \arrow[l, "\varepsilon_B"'] & & GB \\
& FA \arrow[ul, "f"] \arrow[u, "Fg"'] & & A \arrow[u, dotted, "g"']
\end{tikzcd}$$

The following is the dual of Proposition 1.5.4.

Proposition 1.5.5. *A functor $F : \mathcal{C} \to \mathcal{D}$ is left adjoint to a functor $G : \mathcal{D} \to \mathcal{C}$ if, and only if, there exists a natural transformation*

$$\varepsilon : F \circ G \Longrightarrow 1_{\mathcal{D}} \ ,$$

called the counit *of the adjunction, such that, for all $A \in \mathcal{C}$ and $B \in \mathcal{D}$ the map $\theta^{-1}_{A,B} : \mathcal{C}(A, GB) \to \mathcal{D}(FA, B)$, defined by*

$$\theta^{-1}_{A,B} g = \varepsilon_B \circ Fg \ ,$$

is an isomorphism.

Let us consider again the forgetful functor $U : \mathsf{Mon} \to \mathsf{Set}$ and its left adjoint $F : \mathsf{Set} \to \mathsf{Mon}$, the free monoid construction. For a monoid $(M, \star) \in \mathsf{Mon}$, the counit of the adjunction $F \dashv U$ is a monoid homomorphism $\varepsilon_M : F(UM) \to M$, defined by

$$\varepsilon_M(x_1 x_2 \dots x_n) = x_1 \star x_2 \star \dots \star x_n \ .$$

It has the following universal mapping property: for $X \in \mathsf{Set}$, $(M, \star) \in \mathsf{Mon}$, and a homomorphism $f : FX \to M$ there exists a unique function $\overline{f} : X \to UM$ such that $f = \varepsilon_M \circ F\overline{f}$, namely

$$\overline{f}x = fx \ ,$$

where in the above definition $x \in X$ is viewed as an element of the set X on the left-hand side, and as an element of the free monoid FX on the right-hand side. To summarize, the universal mapping property of the counit ε is the familiar piece of wisdom that a homomorphism $f : FX \to M$ from a free monoid is already determined by its values on the generators.

1.6 Limits and Colimits

The following limits and colimits are all special cases of adjoint functors, as we shall see.

1.6.1 Binary products

In a category $\mathcal{C}$, the *(binary) product* of objects A and B is an object $A \times B$ together with *projections* $\pi_0 : A \times B \to A$ and $\pi_1 : A \times B \to B$ such that, for every object $C \in \mathcal{C}$ and every pair of morphisms $f : C \to A$, $g : C \to B$ there exists a *unique* morphism $h : C \to A \times B$ for which the following diagram commutes:

$$\begin{tikzcd}
 & C \arrow[dl, "f"'] \arrow[d, dotted, "h"] \arrow[dr, "g"] & \\
A & A \times B \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & B
\end{tikzcd}$$

We normally refer to the product $(A \times B, \pi_0, \pi_1)$ just by its object $A \times B$, but you should keep in mind that a product is given by an object *and* two projections. The arrow $h : C \to A \times B$ is denoted by $\langle f, g \rangle$. The property

$$\begin{aligned}
&\text{for all } C, \text{ for all } f : C \to A, \text{ for all } g : C \to B, \\
&\text{there is a unique } h : C \to A \times B, \\
&\text{with } \pi_0 \circ h = f \ \&\ \pi_1 \circ h = g
\end{aligned}$$

is the *universal mapping property* of the product $A \times B$. It characterizes the product of A and B uniquely up to isomorphism in the sense that if $(P, p_0 : P \to A, p_1 : P \to B)$ is another product of A and B, then there is a unique isomorphism $r : P \xrightarrow{\sim} A \times B$ such that $p_0 = \pi_0 \circ r$ and $p_1 = \pi_1 \circ r$.

If in a category $\mathcal{C}$ every two objects have a product, we can turn binary products into an operation[4] by *choosing* a product $A \times B$ for each pair of objects $A, B \in \mathcal{C}$. In general this requires the Axiom of Choice, but in many specific cases a particular choice of products can be made without appeal to that axiom. When we view binary products as an operation, we say that "$\mathcal{C}$ has chosen products". The same holds for other instances of limits and colimits.

For example, in Set the usual cartesian product of sets is a product. In categories of structures, products are the usual construction: the product of topological spaces in Top is their topological product, the product of directed graphs in Graph is their cartesian product, the product of categories in Cat is their product category, and so on.

1.6.2 Terminal objects

A *terminal object* in a category $\mathcal{C}$ is an object $\mathsf{1} \in \mathcal{C}$ such that for every $A \in \mathcal{C}$ there exists a *unique* morphism $!_A : A \to \mathsf{1}$.

For example, in Set an object is terminal if, and only if, it is a singleton. The terminal object in Cat is the unit category $\mathsf{1}$ consisting of one object and one morphism.

Exercise 1.6.1. Prove that if $\mathsf{1}$ and $\mathsf{1}'$ are terminal objects in a category then they are isomorphic.

Exercise 1.6.2. Let Field be the category whose objects are fields and morphisms are field homomorphisms.[5] Does Field have a terminal object? What about the category Ring of rings?

1.6.3 Equalizers

Given objects and morphisms

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

we say that e *equalizes* f and g when $f \circ e = g \circ e$.[6] An *equalizer* of f and g is a *universal* equalizing morphism; thus $e : E \to A$ is an equalizer of f and g when it equalizes them and, for all $k : K \to A$, if $f \circ k = g \circ k$ then there exists a unique morphism $m : K \to E$

[4] More precisely, binary product is a functor from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$, cf. Section 1.6.11.

[5] A field $(F, +, \cdot, {}^{-1}, 0, 1)$ is a ring with a unit in which all non-zero elements have inverses. We also require that $0 \neq 1$. A homomorphism of fields preserves addition and multiplication, and consequently also 0, 1 and inverses.

[6] Note that this does *not* mean the diagram involving f, g and e is commutative!

such that $k = e \circ m$:

$$\begin{tikzcd}
E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
K \arrow[u, dotted, "m"] \arrow[ur, "k"'] & &
\end{tikzcd}$$

In Set the equalizer of parallel functions $f : A \to B$ and $g : A \to B$ is the set

$$E = \{x \in A \mid fx = gx\}$$

with $e : E \to A$ being the subset inclusion $E \subseteq A$, $ex = x$. In general, equalizers can be thought of as those subobjects (subsets, subgroups, subspaces, ...) that can be defined by an equation.

Exercise 1.6.3. Show that an equalizer is a monomorphism, i.e., if $e : E \to A$ is an equalizer of f and g, then, for all $r, s : C \to E$, $e \circ r = e \circ s$ implies $r = s$.

Definition 1.6.4. A morphism is a *regular mono* if it is an equalizer.

The difference between monos and regular monos is best illustrated in the category Top: a continuous map $f : X \to Y$ is mono when it is injective, whereas it is a regular mono when it is a topological embedding.[7]

1.6.4 Pullbacks

A *pullback* of $f : A \to C$ and $g : B \to C$ is an object P with morphisms $p_0 : P \to A$ and $p_1 : P \to B$ such that $f \circ p_0 = g \circ p_1$, and whenever Q, $q_0 : Q \to A$, and $q_1 : Q \to B$ are such that $f \circ q_0 = g \circ q_1$, there then exists a unique $h : Q \to P$ such that $q_0 = p_0 \circ h$ and $q_1 = p_1 \circ h$:

$$\begin{tikzcd}
Q \arrow[dr, dotted, "h"] \arrow[drr, bend left, "q_1"] \arrow[ddr, bend right, "q_0"'] & & \\
& P \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

We indicate that P is a pullback by drawing a square corner next to it, as in the above diagram. The pullback is sometimes written $A \times_C B$, since it is indeed a product in the slice category over C.

[7] A continuous map $f : X \to Y$ is a topological embedding when, for every $U \in \mathcal{O}X$, the image $f[U]$ is an open subset of the image $\mathsf{im}(f)$; this means that there exists $V \in \mathcal{O}Y$ such that $f[U] = V \cap \mathsf{im}(f)$.

In Set, the pullback of $f : A \to C$ and $g : B \to C$ is the set

$$P = \{\langle x, y\rangle \in A \times B \mid fx = gy\}$$

and the functions $p_0 : P \to A$, $p_1 : P \to B$ are the projections, $p_0\langle x, y\rangle = x$, $p_1\langle x, y\rangle = y$.

When we form the pullback of $f : A \to C$ and $g : B \to C$ we may also say that we *pull g back along f* and draw the diagram

$$\begin{tikzcd}
f^*B \arrow[r] \arrow[d, "f^*g"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}$$

We think of $f^*g : f^*B \to A$ as the inverse image of B along f. This terminology is explained by looking at the pullback of a subset inclusion $u : U \hookrightarrow C$ along a function $f : A \to C$ in the category Set:

$$\begin{tikzcd}
f^*U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d, hook, "u"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}$$

In this case the pullback is $\{\langle x, y\rangle \in A \times U \mid fx = y\} \cong \{x \in A \mid fx \in U\} = f^*U$, the inverse image of U along f.

Exercise 1.6.5. Prove that in a category $\mathcal{C}$, a morphism $f : A \to B$ is mono if, and only if, the following diagram is a pullback:

$$\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[d, "1_A"'] & A \arrow[d, "f"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

1.6.5 Limits

Let us now define the general notion of a limit.

A *diagram of shape $\mathcal{I}$* in a category $\mathcal{C}$ is a functor $D : \mathcal{I} \to \mathcal{C}$, where the category $\mathcal{I}$ is called the *index category*. We use letters i, j, k, … for objects of an index category $\mathcal{I}$, call them *indices*, and write D_i, D_j, D_k, … instead of Di, Dj, Dk, …

For example, if $\mathcal{I}$ is the category with three objects and three morphisms

$$\begin{tikzcd} & 1 \arrow[dl, "12"'] \arrow[d, "13"] \\ 2 \arrow[r, "23"'] & 3 \end{tikzcd}$$

where $13 = 23 \circ 12$ then a diagram of shape $\mathcal{I}$ is a commutative diagram

$$\begin{tikzcd} & D_1 \arrow[dl, "d_{12}"'] \arrow[d, "d_{13}"] \\ D_2 \arrow[r, "d_{23}"'] & D_3 \end{tikzcd} \tag{1.9}$$

For each object $A \in \mathcal{C}$, the *constant A-valued diagram* of shape $\mathcal{I}$ is given by the constant functor $\Delta_A : \mathcal{I} \to \mathcal{C}$, which maps every object to A and every morphism to $\mathsf{1}_A$.

Let $D : \mathcal{I} \to \mathcal{C}$ be a diagram of shape $\mathcal{I}$. A *cone* on D from an object $A \in \mathcal{C}$ is a natural transformation $\alpha : \Delta_A \Longrightarrow D$. This means that for every index $i \in \mathcal{I}$ there is a morphism $\alpha_i : A \to D_i$ such that whenever $u : i \to j$ in $\mathcal{I}$ then $\alpha_j = Du \circ \alpha_i$.

For a given diagram $D : \mathcal{I} \to \mathcal{C}$, we can collect all cones on D into a category $\mathsf{Cone}(D)$ whose objects are cones on D. A morphism between cones $f : (A, \alpha) \to (B, \beta)$ is a morphism $f : A \to B$ in $\mathcal{C}$ such that $\alpha_i = \beta_i \circ f$ for all $i \in \mathcal{I}$. Morphisms in $\mathsf{Cone}(D)$ are composed as morphisms in $\mathcal{C}$. A morphism $f : (A, \alpha) \to (B, \beta)$ is also called a factorization of the cone (A, α) through the cone (B, β).

A *limit* of a diagram $D : \mathcal{I} \to \mathcal{C}$ is a terminal object in $\mathsf{Cone}(D)$. Explicitly, a limit of D is given by a cone (L, λ) such that for every other cone (A, α) there exists a *unique* morphism $f : A \to L$ such that $\alpha_i = \lambda_i \circ f$ for all $i \in \mathcal{I}$. We denote (the object part of) a limit of D by one of the following:

$$\lim D \qquad \lim_{i \in \mathcal{I}} D_i \qquad \varprojlim_{i \in \mathcal{I}} D_i \,.$$

Limits are also called *projective limits*. We say that a category *has limits of shape* $\mathcal{I}$ when every diagram of shape $\mathcal{I}$ in $\mathcal{C}$ has a limit.

Products, terminal objects, equalizers, and pullbacks are all special cases of limits:

- a product $A \times B$ is the limit of the functor $D : \mathsf{2} \to \mathcal{C}$ where $\mathsf{2}$ is the discrete category on two objects 0 and 1, and $D_0 = A$, $D_1 = B$.
- a terminal object $\mathsf{1}$ is the limit of the (unique) functor $D : \mathsf{0} \to \mathcal{C}$ from the empty category.
- an equalizer of $f, g : A \to B$ is the limit of the functor $D : (\cdot \rightrightarrows \cdot) \to \mathcal{C}$ which maps one morphism to f and the other one to g.

- the pullback of $f : A \to C$ and $g : B \to C$ is the limit of the functor $D : \mathcal{I} \to \mathcal{C}$ where $\mathcal{I}$ is the category

$$\begin{tikzcd} & \bullet \arrow[d, "2"] \\ \bullet \arrow[r, "1"'] & \bullet \end{tikzcd}$$

with $D1 = f$ and $D2 = g$.

It is clear how to define the product of an arbitrary family of objects

$$\left\{A_i \in \mathcal{C} \;\middle|\; i \in I\right\} .$$

Such a family is a diagram of shape I, where I is viewed as a discrete category. A *product* $\prod_{i\in I} A_i$ is then given by an object $P \in \mathcal{C}$ and morphisms $\pi_i : P \to A_i$ such that, whenever we have a family of morphisms $\left\{f_i : B \to A_i \;\middle|\; i \in I\right\}$ there exists a *unique* morphism $\langle f_i\rangle_{i\in I} : B \to P$ such that $f_i = \pi_i \circ f$ for all $i \in I$.

A *finite product* is a product of a finite family. As a special case we see that a terminal object is the product of an empty family. It is not hard to show that a category has finite products precisely when it has a terminal object and binary products.

A diagram $D : \mathcal{I} \to \mathcal{C}$ is *small* when $\mathcal{I}$ is a small category. A *small limit* is a limit of a small diagram. A *finite limit* is a limit of a diagram whose index category is finite.

Exercise 1.6.6. Prove that a limit, when it exists, is unique up to isomorphism.

The following proposition and its proof tell us how to compute arbitrary limits from simpler ones. We omit detailed proofs as they can be found in any standard textbook on category theory.

Proposition 1.6.7. *The following are equivalent for a category $\mathcal{C}$:*

1. *$\mathcal{C}$ has a terminal object and all pullbacks.*
2. *$\mathcal{C}$ has equalizers and all finite products.*
3. *$\mathcal{C}$ has all finite limits.*

Proof. We only show how to get binary products from pullbacks and a terminal object. For objects A and B, let P be the pullback of $!_A$ and $!_B$:

$$\begin{tikzcd} P \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "!_B"] \\ A \arrow[r, "!_A"'] & \mathsf{1} \end{tikzcd}$$

Then (P, π_0, π_1) is a product of A and B because, for all $f : X \to A$ and $g : X \to B$, it is trivially the case that $!_A \circ f = !_B \circ g$. $\square$

Proposition 1.6.8. *The following are equivalent for a category $\mathcal{C}$:*

1. *$\mathcal{C}$ has equalizers and all small products.*
2. *$\mathcal{C}$ has all small limits.*

Proof. We indicate how to construct an arbitrary limit from a product and an equalizer. Let $D : \mathcal{I} \to \mathcal{C}$ be a small diagram of an arbitrary shape $\mathcal{I}$. First form an $\mathcal{I}_0$-indexed product P and an $\mathcal{I}_1$-indexed product Q

$$P = \prod_{i \in \mathcal{I}_0} D_i \,, \qquad\qquad Q = \prod_{u \in \mathcal{I}_1} D_{\mathsf{cod}\, u} \,.$$

By the universal property of products, there are unique morphisms $f : P \to Q$ and $g : P \to Q$ such that, for all morphisms $u \in \mathcal{I}_1$,

$$\pi^Q_u \circ f = Du \circ \pi^P_{\mathsf{dom}\, u} \,, \qquad\qquad \pi^Q_u \circ g = \pi^P_{\mathsf{cod}\, u} \,.$$

Let E be the equalizer of f and g,

$$\begin{tikzcd} E \arrow[r, "e"] & P \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Q \end{tikzcd}$$

For every $i \in \mathcal{I}$ there is a morphism $\varepsilon_i : E \to D_i$, namely $\varepsilon_i = \pi^P_i \circ e$. We claim that (E, ε) is a limit of D. First, (E, ε) is a cone on D because, for all $u : i \to j$ in $\mathcal{I}$,

$$Du \circ \varepsilon_i = Du \circ \pi^P_i \circ e = \pi^Q_u \circ f \circ e = \pi^Q_u \circ g \circ e = \pi^P_j \circ e = \varepsilon_j \,.$$

If (A, α) is any cone on D there exists a unique $t : A \to P$ such that $\alpha_i = \pi^P_i \circ t$ for all $i \in \mathcal{I}$. For every $u : i \to j$ in $\mathcal{I}$ we have

$$\pi^Q_u \circ g \circ t = \pi^P_j \circ t = t_j = Du \circ t_i = Du \circ \pi^P_i \circ t = \pi^Q_u \circ f \circ t \,,$$

therefore $g \circ t = f \circ t$. This implies that there is a unique factorization $k : A \to E$ such that $t = e \circ k$. Now for every $i \in \mathcal{I}$

$$\varepsilon_i \circ k = \pi^P_i \circ e \circ k = \pi^P_i \circ t = \alpha_i$$

so that $k : A \to E$ is the required factorization of the cone (A, α) through the cone (E, ε). To see that k is unique, suppose $m : A \to E$ is another factorization such that $\alpha_i = \varepsilon_i \circ m$ for all $i \in \mathcal{I}$. Since e is mono it suffices to show that $e \circ m = e \circ k$, which is equivalent to proving $\pi^P_i \circ e \circ m = \pi^P_i \circ e \circ k$ for all $i \in \mathcal{I}$. This last equality holds because

$$\pi^P_i \circ e \circ k = \pi^P_i \circ t = \alpha_i = \varepsilon_i \circ m = \pi^P_i \circ e \circ m \,.$$

□

A category is *(small) complete* when it has all small limits, and it is *finitely complete* (or *left exact*, briefly *lex*) when it has finite limits.

Limits of presheaves Let $\mathcal{C}$ be a locally small category. Then the presheaf category $\widehat{\mathcal{C}} = \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$ has all small limits and they are computed pointwise, e.g., $(P \times Q)A = PA \times QA$ for $P, Q \in \widehat{\mathcal{C}}$, $A \in \mathcal{C}$. To see that this is really so, let $\mathcal{I}$ be a small index category and $D : \mathcal{I} \to \widehat{\mathcal{C}}$ a diagram of presheaves. Then for every $A \in \mathcal{C}$ the diagram D can be instantiated at A to give a diagram $DA : \mathcal{I} \to \mathsf{Set}$, $(DA)_i = D_i A$. Because Set is small complete, we can define a presheaf L by computing the limit of DA:

$$LA = \lim DA = \varprojlim_{i \in \mathcal{I}} D_i A \; .$$

We should keep in mind that $\lim DA$ is actually given by an object $(\lim DA)$ and a natural transformation $\delta A : \Delta_{(\lim DA)} \Longrightarrow DA$. The value of LA is supposed to be just the object part of $\lim DA$. From a morphism $f : A \to B$ we obtain for each $i \in \mathcal{I}$ a function $D_i f \circ (\delta A)_i : LA \to D_i B$, and thus a cone $(LA, Df \circ \delta A)$ on DB. Presheaf L maps the morphism $f : A \to B$ to the unique factorization $Lf : LA \Longrightarrow LB$ of the cone $(LA, Df \circ \delta A)$ on DB through the limit cone LB on DB.

For every $i \in \mathcal{I}$, there is a function $\Lambda_i = (\delta A)_i : LA \to D_i A$. The family $\{\Lambda_i\}_{i \in \mathcal{I}}$ is a natural transformation from Δ_{LA} to DA. This gives us a cone (L, Λ) on D, which is in fact a limit cone. Indeed, if (S, Σ) is another cone on D then for every $A \in \mathcal{C}$ there exists a unique function $\phi_A : SA \to LA$ because SA is a cone on DA and LA is a limit cone on DA. The family $\{\phi_A\}_{A \in \mathcal{C}}$ is the unique natural transformation $\phi : S \Longrightarrow L$ for which $\Sigma = \phi \circ \Lambda$.

1.6.6 Colimits

Colimits are the dual notion of limits. Thus, a *colimit* of a diagram $D : \mathcal{I} \to \mathcal{C}$ is a limit of the dual diagram $D^{\mathsf{op}} : \mathcal{I}^{\mathsf{op}} \to \mathcal{C}^{\mathsf{op}}$ in the dual (i.e., opposite) category $\mathcal{C}^{\mathsf{op}}$:

$$\operatorname{colim}(D : \mathcal{I} \to \mathcal{C}) = \lim(D^{\mathsf{op}} : \mathcal{I}^{\mathsf{op}} \to \mathcal{C}^{\mathsf{op}}) \; .$$

Explicitly, the colimit of a diagram $D : \mathcal{I} \to \mathcal{C}$ is the initial object in the category of *cocones* $\mathsf{Cocone}(D)$ on D. A cocone (A, α) on D is a natural transformation $\alpha : D \Longrightarrow \Delta_A$. It is given by an object $A \in \mathcal{C}$ and, for each $i \in \mathcal{I}$, a morphism $\alpha_i : D_i \to A$, such that $\alpha_i = \alpha_j \circ Du$ whenever $u : i \to j$ in $\mathcal{I}$. A morphism between cocones $f : (A, \alpha) \to (B, \beta)$ is a morphism $f : A \to B$ in $\mathcal{C}$ such that $\beta_i = f \circ \alpha_i$ for all $i \in \mathcal{I}$.

A colimit of $D : \mathcal{I} \to \mathcal{C}$ is then given by a cocone (C, ζ) on D such that, for every cocone (A, α) on D there exists a unique morphism $f : C \to A$ such that $\alpha_i = f \circ \zeta_i$ for all $i \in D$. We denote a colimit of D by one of the following:

$$\operatorname{colim} D \qquad\qquad \operatorname{colim}_{i \in \mathcal{I}} D_i \qquad\qquad \varinjlim_{i \in \mathcal{I}} D_i \; .$$

Colimits are also called *inductive limits*.

Exercise 1.6.9. Formulate the dual of Proposition 1.6.7 and Proposition 1.6.8 for colimits (coequalizers are defined in Section 1.6.9).

1.6.7 Binary coproducts

In a category $\mathcal{C}$, the *(binary) coproduct* of objects A and B is an object $A+B$ together with *injections* $\iota_0 : A \to A+B$ and $\iota_1 : B \to A+B$ such that, for every object $C \in \mathcal{C}$ and all morphisms $f : A \to C$, $g : B \to C$ there exists a *unique* morphism $h : A+B \to C$ for which the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\iota_0"] \arrow[dr, "f"'] & A+B \arrow[d, dotted, "h"] & B \arrow[l, "\iota_1"'] \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

The arrow $h : A+B \to C$ is denoted by $[f,g]$.

The coproduct $A+B$ is the colimit of the diagram $D : \mathsf{2} \to \mathcal{C}$, where $\mathcal{I}$ is the discrete category on two objects 0 and 1, and $D_0 = A$, $D_1 = B$.

In Set the coproduct is the disjoint union, defined by

$$X + Y = \left\{\langle 0, x\rangle \;\middle|\; x \in X\right\} \cup \left\{\langle 1, y\rangle \;\middle|\; x \in Y\right\} \ ,$$

where 0 and 1 are distinct sets, for example $\emptyset$ and $\{\emptyset\}$. Given functions $f : X \to Z$ and $g : Y \to Z$, the unique function $[f,g] : X+Y \to Z$ is the usual *definition by cases*:

$$[f,g]u = \begin{cases} fx & \text{if } u = \langle 0, x\rangle \\ gx & \text{if } u = \langle 1, x\rangle \ . \end{cases}$$

Exercise 1.6.10. Show that the categories of posets and of topological spaces both have coproducts.

1.6.8 Initial objects

An *initial object* in a category $\mathcal{C}$ is an object $\mathsf{0} \in \mathcal{C}$ such that for every $A \in \mathcal{C}$ there exists a *unique* morphism $\mathsf{o}_A : \mathsf{0} \to A$.

An initial object is the colimit of the empty diagram.

In Set, the initial object is the empty set.

Exercise 1.6.11. What is the initial and what is the terminal object in the category of groups?

A *zero object* is an object that is both initial and terminal.

Exercise 1.6.12. Show that in the category of Abelian[8] groups finite products and coproducts agree, that is $\mathsf{0} \cong \mathsf{1}$ and $A \times B \cong A + B$.

Exercise 1.6.13. Suppose A and B are Abelian groups. Is there a difference between their coproduct in the category Group of groups, and their coproduct in the category $\mathsf{AbGroup}$ of Abelian groups?

[8] An Abelian group is one that satisfies the commutative law $x \cdot y = y \cdot x$.

1.6.9 Coequalizers

Given objects and morphisms

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "q"] & Q \end{tikzcd}$$

we say that q *coequalizes* f and g when $e \circ f = e \circ g$. A *coequalizer* of f and g is a *universal* coequalizing morphism; thus $q : B \to Q$ is a coequalizer of f and g when it coequalizes them and, for all $s : B \to S$, if $s \circ f = s \circ g$ then there exists a *unique* morphism $r : Q \to S$ such that $s = r \circ q$:

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "q"] \arrow[dr, "s"'] & Q \arrow[d, dotted, "r"] \\ & & S \end{tikzcd}$$

In Set the coequalizer of parallel functions $f : A \to B$ and $g : A \to B$ is the quotient set $Q = B/{\sim}$ where $\sim$ is the least equivalence relation on B satisfying

$$fx = gy \Rightarrow x \sim y \ .$$

The function $q : B \to Q$ is the canonical quotient map which assigns to each element $x \in B$ its equivalence class $[x] \in B/{\sim}$. In general, a coequalizer can be thought of as the quotient by the equivalence relation generated by the corresponding equation.

Exercise 1.6.14. Show that a coequalizer is an epimorphism, i.e., if $q : B \to Q$ is a coequalizer of f and g, then, for all $u, v : Q \to T$, $u \circ q = v \circ q$ implies $u = v$. [Hint: use the duality between limits and colimits and Exercise 1.6.3.]

Definition 1.6.15. A morphism is a *regular epi* if it is a coequalizer.

The difference between epis and regular epis is also illustrated in the category Top: a continuous map $f : X \to Y$ is epi when it is surjective, whereas it is a regular epi when it is a topological quotient map.[9]

1.6.10 Pushouts

A *pushout* of $f : A \to B$ and $g : A \to C$ is an object Q with morphisms $q_0 : B \to Q$ and $q_1 : C \to Q$ such that $q_0 \circ f = q_1 \circ g$, and whenever $r_0 : B \to R$, $r_1 : C \to R$ are such that

[9] A continuous map $f : X \to Y$ is a topological quotient map when it is surjective and, for every $U \subseteq Y$, U is open if, and only if, f^*U is open.

$r_0 \circ f = r_1 \circ g$, then there exists a unique $s : Q \to R$ such that $r_0 = s \circ q_0$ and $r_1 = s \circ q_1$:

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "q_1"] \arrow[ddr, bend left, "r_1"] & \\
B \arrow[r, "q_0"'] \arrow[drr, bend right, "r_0"'] & Q \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dotted, "s"] & \\
 & & R
\end{tikzcd}$$

We indicate that Q is a pushout by drawing a square corner next to it, as in the above diagram. The above pushout Q is sometimes denoted by $B +_A C$.

A pushout as above is a colimit of the diagram $D : \mathcal{I} \to \mathcal{C}$ where the index category $\mathcal{I}$ is

$$\begin{tikzcd}
\bullet \arrow[r, "2"] \arrow[d, "1"'] & \bullet \\
\bullet &
\end{tikzcd}$$

and $D1 = f$, $D2 = g$.

In Set, the pushout of $f : A \to C$ and $g : B \to C$ is the quotient set

$$Q = (B + C)/{\sim}$$

where $B + C$ is the disjoint union of B and C, and $\sim$ is the least equivalence relation on $B + C$ such that, for all $x \in A$,

$$fx \sim gx \; .$$

The functions $q_0 : B \to Q$, $q_1 : C \to Q$ are the injections, $q_0 x = [x]$, $q_1 y = [y]$, where $[x]$ is the equivalence class of x.

1.6.11 Limits as adjoints

Limits and colimits can be defined as adjoints to certain very simple functors.

First, observe that an object $A \in \mathcal{C}$ can be viewed as a functor from the terminal category $\mathsf{1}$ to $\mathcal{C}$, namely the functor which maps the only object $\star$ of $\mathsf{1}$ to A. Since $\mathsf{1}$ is the terminal object in Cat, there exists a unique functor $!_{\mathcal{C}} : \mathcal{C} \to \mathsf{1}$, which maps every object of $\mathcal{C}$ to $\star$.

Now we can ask whether this simple functor $!_{\mathcal{C}} : \mathcal{C} \to \mathsf{1}$ has any adjoints. Indeed, it has a right adjoint just if $\mathcal{C}$ has a terminal object $1_{\mathcal{C}}$, for the corresponding functor $1_{\mathcal{C}} : \mathsf{1} \to \mathcal{C}$ has the property that, for every $A \in \mathcal{C}$ we have a (trivially natural) bijective correspondence:

$$\frac{!_A : A \to 1_{\mathcal{C}}}{\overline{\overline{\;1_\star : \,!_{\mathcal{C}} A \to \star\;}}}$$

Similarly, an initial object is a left adjoint to $!_{\mathcal{C}}$:

$$\mathsf{0}_{\mathcal{C}} \dashv\, !_{\mathcal{C}} \dashv \mathsf{1}_{\mathcal{C}} \,.$$

Now consider the diagonal functor,

$$\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C},$$

defined by $\Delta A = \langle A, A \rangle$, $\Delta f = \langle f, f \rangle$. When does this have adjoints?

If $\mathcal{C}$ has all binary products, then they determine a functor

$$- \times - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

which maps $\langle A, B \rangle$ to $A \times B$ and a pair of morphisms $\langle f : A \to A', g : B \to B' \rangle$ to the unique morphism $f \times g : A \times B \to A' \times B'$ for which $\pi_0 \circ (f \times g) = f \circ \pi_0$ and $\pi_1 \circ (f \times g) = g \circ \pi_1$,

$$\begin{tikzcd}
A \arrow[d, "f"'] & A \times B \arrow[l, "\pi_0"'] \arrow[r, "\pi_1"] \arrow[d, "f \times g"] & B \arrow[d, "g"] \\
A' & A' \times B' \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & B'
\end{tikzcd}$$

The product functor $\times$ is right adjoint to the diagonal functor Δ. Indeed, there is a natural bijective correspondence:

$$\frac{\langle f, g \rangle : \langle A, A \rangle \to \langle B, C \rangle}{f \times g : A \to B \times C}$$

Similarly, binary coproducts are easily seen to be left adjoint to the diagonal functor,

$$+ \dashv \Delta \dashv \times \,.$$

Now in general, consider limits of shape $\mathcal{I}$ in a category $\mathcal{C}$. There is the constant diagram functor

$$\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{I}}$$

that maps $A \in \mathcal{C}$ to the constant diagram $\Delta_A : \mathcal{I} \to \mathcal{C}$. The limit construction is a functor

$$\varprojlim : \mathcal{C}^{\mathcal{I}} \to \mathcal{C}$$

that maps each diagram $D \in \mathcal{C}^{\mathcal{I}}$ to its limit $\varprojlim D$. These two are adjoint, $\Delta \dashv \varprojlim$, because there is a natural bijective correspondence between cones $\alpha : \Delta_A \Longrightarrow D$ on D, and their factorizations through the limit of D,

$$\frac{\alpha : \Delta_A \Longrightarrow D}{A \to \varprojlim D}$$

An analogous correspondence holds for colimits, so that we obtain a pair of adjunctions,

$$\varinjlim \dashv \Delta \dashv \varprojlim \,,$$

which, of course, subsume all the previously mentioned cases.

Exercise 1.6.16. How are the functors $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{I}}$, $\varinjlim : \mathcal{C}^{\mathcal{I}} \to \mathcal{C}$, and $\varprojlim : \mathcal{C}^{\mathcal{I}} \to \mathcal{C}$ defined on morphisms?

1.6.12 Preservation of limits

We say that a functor $F : \mathcal{C} \to \mathcal{D}$ *preserves products* when, given a product

$$A \xleftarrow{\pi_0} A \times B \xrightarrow{\pi_1} B$$

its image in $\mathcal{D}$,

$$FA \xleftarrow{F\pi_0} F(A \times B) \xrightarrow{F\pi_1} FB$$

is a product of FA and FB. If $\mathcal{D}$ has chosen binary products, F preserves binary products if, and only if, the unique morphism $f : F(A \times B) \to FA \times FB$ which makes the following diagram commutative is an isomorphism: [10]

$$\begin{tikzcd}
 & F(A \times B) \arrow[dl, "F\pi_0"'] \arrow[d, "f"] \arrow[dr, "F\pi_1"] & \\
FA & FA \times FB \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & FB
\end{tikzcd}$$

In general, a functor $F : \mathcal{C} \to \mathcal{D}$ is said to *preserve limits* of shape $\mathcal{I}$ when it maps limit cones to limit cones: if (L, λ) is a limit of $D : \mathcal{I} \to \mathcal{C}$ then $(FL, F \circ \lambda)$ is a limit of $F \circ D : \mathcal{I} \to \mathcal{D}$.

Analogously, a functor $F : \mathcal{C} \to \mathcal{D}$ is said to *preserve colimits* of shape $\mathcal{I}$ when it maps colimit cocones to colimit cocones: if (C, ζ) is a colimit of $D : \mathcal{I} \to \mathcal{C}$ then $(FC, F \circ \zeta)$ is a colimit of $F \circ D : \mathcal{I} \to \mathcal{D}$.

Proposition 1.6.17. *(a) A functor preserves finite (small) limits if, and only if, it preserves equalizers and finite (small) products. (b) A functor preserves finite (small) colimits if, and only if, it preserves coequalizers and finite (small) coproducts.*

Proof. This follows from the fact that limits are constructed from equalizers and products, cf. Proposition 1.6.8, and that colimits are constructed from coequalizers and coproducts, cf. Exercise 1.6.9. $\square$

Proposition 1.6.18. *For a locally small category $\mathcal{C}$, the Yoneda embedding $\mathsf{y} : \mathcal{C} \to \widehat{\mathcal{C}}$ preserves all limits that exist in $\mathcal{C}$.*

[10] Products are determined up to isomorphism only, so it would be too restrictive to require $F(A \times B) = FA \times FB$. When that is the case, however, we say that the functor F *strictly* preserves products.

Proof. Suppose (L, λ) is a limit of $D : \mathcal{I} \to \mathcal{C}$. The Yoneda embedding maps D to the diagram $\mathsf{y} \circ D : \mathcal{I} \to \widehat{\mathcal{C}}$, defined by

$$(\mathsf{y} \circ D)_i = \mathsf{y}D_i = \mathcal{C}(-, D_i) \; .$$

and it maps the limit cone (L, λ) to the cone $(\mathsf{y}L, \mathsf{y} \circ \lambda)$ on $\mathsf{y} \circ D$, defined by

$$(\mathsf{y} \circ \lambda)_i = \mathsf{y}\lambda_i = \mathcal{C}(-, \lambda_i) \; .$$

To see that $(\mathsf{y}L, \mathsf{y} \circ \lambda)$ is a limit cone on $\mathsf{y} \circ D$, consider a cone (M, μ) on $\mathsf{y} \circ D$. Then $\mu : \Delta_M \Longrightarrow D$ consists of a family of functions, one for each $i \in \mathcal{I}$ and $A \in \mathcal{C}$,

$$(\mu_i)_A : MA \to \mathcal{C}(A, D_i) \; .$$

For every $A \in \mathcal{C}$ and $m \in MA$ we get a cone on D consisting of morphisms

$$(\mu_i)_A m : A \to D_i \; . \qquad (i \in \mathcal{I})$$

There exists a unique morphism $\phi_A m : A \to L$ such that $(\mu_i)_A m = \lambda_i \circ \phi_A m$. The family of functions

$$\phi_A : MA \to \mathcal{C}(A, L) = (\mathsf{y} \circ L)A \qquad (A \in \mathcal{C})$$

forms a factorization $\phi : M \Longrightarrow \mathsf{y}L$ of the cone (M, μ) through the cone (L, λ). This factorization is unique because each $\phi_A m$ is unique. □

In effect we showed that a covariant representable functor $\mathcal{C}(A, -) : \mathcal{C} \to \mathsf{Set}$ preserves existing limits,

$$\mathcal{C}(A, \varprojlim_{i \in \mathcal{I}} D_i) \cong \varprojlim_{i \in \mathcal{I}} \mathcal{C}(A, D_i) \; .$$

By duality, the contravariant representable functor $\mathcal{C}(-, A) : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ maps existing colimits to limits,

$$\mathcal{C}(\varinjlim_{i \in \mathcal{I}} D_i, A) \cong \varprojlim_{i \in \mathcal{I}} \mathcal{C}(D_i, A) \; .$$

Exercise 1.6.19. Prove the above claim that a contravariant representable functor $\mathcal{C}(-, A) : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ maps existing colimits to limits. Use duality between limits and colimits. Does it also follow *by a simple duality argument* that a contravariant representable functor $\mathcal{C}(-, A)$ maps existing limits to colimits? How about a covariant representable functor $\mathcal{C}(A, -)$ mapping existing colimits to limits?

Exercise 1.6.20. Prove that a functor $F : \mathcal{C} \to \mathcal{D}$ preserves monos if it preserves limits. In particular, the Yoneda embedding preserves monos. Hint: Exercise 1.6.5.

Proposition 1.6.21. *Right adjoints preserve limits, and left adjoints preserve colimits.*

Proof. Suppose we have adjoint functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

and a diagram $D : \mathcal{I} \to \mathcal{D}$ whose limit exists in $\mathcal{D}$. We would like to use the following slick application of Yoneda Lemma to show that G preserves limits: for every $A \in \mathcal{C}$,

$$\begin{aligned} \mathcal{C}(A, G(\varprojlim D)) \cong \mathcal{D}(FA, \varprojlim D) \cong \varprojlim_{i \in \mathcal{I}} \mathcal{D}(FA, D_i) & \\ \cong \varprojlim_{i \in \mathcal{I}} \mathcal{C}(A, GD_i) \cong \mathcal{C}(A, \varprojlim (G \circ D)) \,. \end{aligned}$$

Therefore $G(\lim D) \cong \lim(G \circ D)$. However, this argument only works if we already know that the limit of $G \circ D$ exists.

We can also prove the stronger claim that whenever the limit of $D : \mathcal{I} \to \mathcal{D}$ exists then the limit of $G \circ D$ exists in $\mathcal{C}$ and its limit is $G(\lim D)$. So suppose (L, λ) is a limit cone of D. Then $(GL, G \circ \lambda)$ is a cone on $G \circ D$. If (A, α) is another cone on $G \circ D$, we have by adjunction a cone (FA, γ) on D,

$$\frac{\alpha_i : A \to GD_i}{\gamma_i : FA \to D_i}$$

There exists a unique factorization $f : FA \to L$ of this cone through (L, λ). Again by adjunction, we obtain a unique factorization $g : A \to GL$ of the cone (A, α) through the cone $(GL, G \circ \lambda)$:

$$\frac{f : FA \to L}{g : A \to GL}$$

The factorization g is unique because γ is uniquely determined from α, f uniquely from α, and g uniquely from f.

By a dual argument, a left adjoint preserves colimits. $\square$

Chapter 2

Propositional Logic

Propositional logic is the logic of propositional connectives like $p \wedge q$ and $p \Rightarrow q$. As was the case for algebraic theories, the general approach will be to determine suitable categorical structures to model the logical operations, and then use categories with such structure to represent (abstract) propositional theories. Adjoints will play a special role, as we will describe the basic logical operations as such. We again show that the semantics is "functorial", meaning that the models of a theory are functors that preserve the categorical structure. We will show that there are classifying categories for all propositional theories, as was the case for the algebraic theories that we have already met.

A more abstract, algebraic perspective will then relate the propositional case of syntax-semantics duality with classical Stone duality for Boolean algebras, and related results from lattice theory will provide an algebraic treatment of Kripke semantics for intuitionistic (and modal) propositional logic.

2.1 Propositional calculus

Before going into the details of the categorical approach, we first briefly review the propositional calculus from a conventional point of view, as we did for algebraic theories. We focus first on the *classical* propositional logic, before considering the intuitionistic case in Section 3.4.

In the style of Section **??**, we have the following (abstract) syntax for (propositional) formulas:

$$\begin{array}{rcl} \text{Propositional variable } p & ::= & \mathtt{p}_1 \mid \mathtt{p}_2 \mid \mathtt{p}_3 \mid \cdots \\ \text{Propositional formula } \phi & ::= & p \mid \top \mid \bot \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid \phi_1 \Rightarrow \phi_2 \mid \phi_1 \Leftrightarrow \phi_2 \end{array}$$

An example of a formula is therefore $\big(\mathtt{p}_3 \Leftrightarrow ((((\neg\mathtt{p}_1) \vee (\mathtt{p}_2 \wedge \bot)) \vee \mathtt{p}_1) \Rightarrow \mathtt{p}_3)\big)$. We will make use of the usual conventions for parenthesis, with binding order $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$. Thus e.g. the foregoing may also be written unambiguously as $\mathtt{p}_3 \Leftrightarrow \neg\mathtt{p}_1 \vee \mathtt{p}_2 \wedge \bot \vee \mathtt{p}_1 \Rightarrow \mathtt{p}_3$.

Natural deduction

The system of *natural deduction* for propositional logic has one form of judgement

$$\mathtt{p}_1, \ldots, \mathtt{p}_n \mid \phi_1, \ldots, \phi_m \vdash \phi$$

where $\mathtt{p}_1, \ldots, \mathtt{p}_n$ is a *context* consisting of distinct propositional variables, the formulas $\phi_1, \ldots, \phi_m$ are the *hypotheses* and ϕ is the *conclusion*. The variables in the hypotheses and the conclusion must occur among those listed in the context. The hypotheses are regarded as a (finite) set; so they are unordered, have no repetitions, and may be empty. We may abbreviate the context of variables by Γ, and we often omit it.

Deductive entailment (or *derivability*) $\Phi \vdash \phi$ is thus a relation between finite sets of formulas Φ and single formulas ϕ. It is defined as the smallest such relation satisfying the following rules:

1. Hypothesis:
$$\frac{}{\Phi \vdash \phi} \text{ if } \phi \text{ occurs in } \Phi$$

2. Truth:
$$\frac{}{\Phi \vdash \top}$$

3. Falsehood:
$$\frac{\Phi \vdash \bot}{\Phi \vdash \phi}$$

4. Conjunction:
$$\frac{\Phi \vdash \phi \qquad \Phi \vdash \psi}{\Phi \vdash \phi \wedge \psi} \qquad \frac{\Phi \vdash \phi \wedge \psi}{\Phi \vdash \phi} \qquad \frac{\Phi \vdash \phi \wedge \psi}{\Phi \vdash \psi}$$

5. Disjunction:
$$\frac{\Phi \vdash \phi}{\Phi \vdash \phi \vee \psi} \qquad \frac{\Phi \vdash \psi}{\Phi \vdash \phi \vee \psi} \qquad \frac{\Phi \vdash \phi \vee \psi \qquad \Phi, \phi \vdash \theta \qquad \Phi, \psi \vdash \theta}{\Phi \vdash \theta}$$

6. Implication:
$$\frac{\Phi, \phi \vdash \psi}{\Phi \vdash \phi \Rightarrow \psi} \qquad \frac{\Phi \vdash \phi \Rightarrow \psi \qquad \Phi \vdash \phi}{\Phi \vdash \psi}$$

For the purpose of deduction, we define $\neg\phi := \phi \Rightarrow \bot$ and $\phi \Leftrightarrow \psi := (\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi)$. To obtain *classical* logic we need only include one of the following additional rules.

7. Classical logic:
$$\frac{}{\Phi \vdash \phi \vee \neg\phi} \qquad \frac{\Phi \vdash \neg\neg\phi}{\Phi \vdash \phi}$$

A *proof* of $\Phi \vdash \phi$ is a *finite* tree built from the above inference rules whose root is $\Phi \vdash \phi$. For example, here is a proof of $\phi \vee \psi \vdash \psi \vee \phi$ using the disjunction rules:

$$\dfrac{\dfrac{}{\phi \vee \psi \vdash \phi \vee \psi} \qquad \dfrac{\dfrac{}{\phi \vee \psi, \phi \vdash \phi}}{\phi \vee \psi, \phi \vdash \psi \vee \phi} \qquad \dfrac{\dfrac{}{\phi \vee \psi, \psi \vdash \psi}}{\phi \vee \psi, \psi \vdash \psi \vee \phi}}{\phi \vee \psi \vdash \psi \vee \phi}$$

A judgment $\Phi \vdash \phi$ is *provable* if there exists a proof of it. Observe that every proof has at its leaves either the rule for $\top$ or a hypothesis.

Exercise 2.1.1. Derive each of the two classical rules (2.1), called *excluded middle* and *double negation*, from the other.

2.2 Truth values

The idea of an axiomatic system of deductive, logical reasoning goes to back to Frege, who gave the first such system for propositional calculus (and more) in his *Begriffsschrift* of 1879. The question soon arose whether Frege's rules (or rather, their derivable consequences – it was clear that one could chose the primitive basis in different but equivalent ways) were correct, and if so, whether they were *all* the correct ones. An ingenious solution was proposed by Russell's student Wittgenstein, who came up with an entirely different way of singling out a set of "valid" propositional formulas in terms of assignments of truth values to the variables occurring in them. He interpreted this as showing that logical validity was really a matter of the logical structure of a proposition, and not depedent on any particular system of derivations. The same idea seems to have been had independently by Post, who proved that the valid propositional formulas coincide with the ones derivable in Whitehead and Russell's *Principia Mathematica* (which is propositionally equivalent to Frege's system), a fact that we now refer to as the *soundness* and *completeness* of propositional logic.

In more detail, let a *valuation* v be an assignment of a "truth-value" $0, 1$ to each propositional variable, $v(\mathtt{p}_n) \in \{0, 1\}$. We can then extend the valuation to all propositional formulas $[\![\phi]\!]^v$ by the recursion,

$$\begin{aligned}
[\![\mathtt{p}_n]\!]^v &= v(\mathtt{p}_n) \\
[\![\top]\!]^v &= 1 \\
[\![\bot]\!]^v &= 0 \\
[\![\neg\phi]\!]^v &= 1 - [\![\phi]\!]^v \\
[\![\phi \wedge \psi]\!]^v &= \min([\![\phi]\!]^v, [\![\psi]\!]^v) \\
[\![\phi \vee \psi]\!]^v &= \max([\![\phi]\!]^v, [\![\psi]\!]^v) \\
[\![\phi \Rightarrow \psi]\!]^v &= 1 \text{ iff } [\![\phi]\!]^v \leq [\![\psi]\!]^v \\
[\![\phi \Leftrightarrow \psi]\!]^v &= 1 \text{ iff } [\![\phi]\!]^v = [\![\psi]\!]^v
\end{aligned}$$

This is sometimes expressed using the "semantic consequence" notation $v \vDash \phi$ to mean that $[\![\phi]\!]^v = 1$. Then the above specification takes the form:

$$
\begin{array}{rll}
v \vDash \top & \text{always} & \\
v \vDash \bot & \text{never} & \\
v \vDash \neg\phi & \text{iff} & v \nvDash \phi \\
v \vDash \phi \wedge \psi & \text{iff} & v \vDash \phi \text{ and } v \vDash \psi \\
v \vDash \phi \vee \psi & \text{iff} & v \vDash \phi \text{ or } v \vDash \psi \\
v \vDash \phi \Rightarrow \psi & \text{iff} & v \vDash \phi \text{ implies } v \vDash \psi \\
v \vDash \phi \Leftrightarrow \psi & \text{iff} & v \vDash \phi \text{ iff } v \vDash \psi
\end{array}
$$

Finally, ϕ is *valid*, written $\vDash \phi$, is defined by,

$$\vDash \phi \quad \text{iff} \quad v \vDash \phi \text{ for all } v.$$

And, more generally, we define $\phi_1, ..., \phi_n$ *semantically entails* ϕ, written

$$\phi_1, ..., \phi_n \vDash \ \phi, \tag{2.1}$$

to mean that for all valuations v such that $v \vDash \phi_k$ for all k, also $v \vDash \phi$.

Given a formula in context $\Gamma \mid \phi$ and a valuation v for the variables in Γ, one can check whether $v \vDash \phi$ using a *truth table*, which is a systematic way of calculating the value of $[\![\phi]\!]^v$. For example, under the assignment $v(\mathtt{p}_1) = 1, v(\mathtt{p}_2) = 0, v(\mathtt{p}_3) = 1$ we can calculate $[\![\phi]\!]^v$ for $\phi = \big(\mathtt{p}_3 \Leftrightarrow ((((\neg\mathtt{p}_1) \vee (\mathtt{p}_2 \wedge \bot)) \vee \mathtt{p}_1) \Rightarrow \mathtt{p}_3)\big)$ as follows.

$\mathtt{p}_1$	$\mathtt{p}_2$	$\mathtt{p}_3$	$\mathtt{p}_3$	$\Leftrightarrow$	$\neg$	$\mathtt{p}_1$	$\vee$	$\mathtt{p}_2$	$\wedge$	$\bot$	$\vee$	$\mathtt{p}_1$	$\Rightarrow$	$\mathtt{p}_3$
1	0	1	1	1	0	1	0	0	0	0	1	1	1	1

The value of the formula ϕ under the valuation v is then the value in the column under the main connective, in this case $\Leftrightarrow$, and thus $[\![\phi]\!]^v = 1$.

Displaying all 2^3 valuations for the context $\Gamma = \mathtt{p}_1, \mathtt{p}_2, \mathtt{p}_3$, therefore results in a table that checks for validity of ϕ,

$\mathtt{p}_1$	$\mathtt{p}_2$	$\mathtt{p}_3$	$\mathtt{p}_3$	$\Leftrightarrow$	$\neg$	$\mathtt{p}_1$	$\vee$	$\mathtt{p}_2$	$\wedge$	$\bot$	$\vee$	$\mathtt{p}_1$	$\Rightarrow$	$\mathtt{p}_3$
1	1	1	.	1	...									
1	1	0	.	1		...								
1	0	1	1	1	0	1	0	0	0	0	1	1	1	1
1	0	0	.	1				...						
0	1	1	.	1					...					
0	1	0	.	1						...				
0	0	1	.	1							...			
0	0	0	.	1								...		

In this case, working out the other rows shows that ϕ is indeed valid, thus $\vDash \phi$.

Theorem 2.2.1 (Soundness and Completeness of Propositional Calculus)**.** *Let Φ be any set of formulas and ψ any formula, then*

$$\Phi \vdash \psi \iff \Phi \vDash \psi.$$

In particular, for any propositional formula ϕ we have

$$\vdash \phi \iff \vDash \phi.$$

Thus derivability and validity coincide.

Proof. Let us sketch the usual proof, for later reference.

(*Soundness*:) First assume $\Phi \vdash \psi$, meaning there is a finite derivation of ψ, all of the hypotheses of which are in the set Φ. Take a valuation v such that $v \vDash \Phi$, meaning that $v \vDash \phi$ for all $\phi \in \Phi$. Observe that for each rule of inference, for any valuation v, if $v \vDash \vartheta$ for all the hypotheses of the rule, then $v \vDash \gamma$ for the conclusion. By induction on the derivations therefore $v \vDash \phi$.

(*Competeness*:) Suppose that $\Phi \nvdash \psi$, then $\Phi, \neg\psi \nvdash \bot$ (using double negation elimination). By Lemma 2.2.2 below, there is a valuation v such that $v \vDash \{\Phi, \neg\psi\}$. Thus in particular $v \vDash \Phi$ and $v \nvDash \psi$, therefore $\Phi \nvDash \psi$. □

The key lemma is this:

Lemma 2.2.2 (Model Existence)**.** *A set Φ of formulas is* consistent*, $\Phi \nvdash \bot$, just if it has a* model*, i.e. a valuation v such that $v \vDash \Phi$.*

Proof. Let Φ be any consistent set of formulas. We extend $\Phi \subseteq \Psi$ to one that is *maximally consistent*, meaning that for every formula ψ, either $\psi \in \Psi$ or $\neg\psi \in \Psi$ and not both. Enumerate the formulas $\phi_0, \phi_1, ...$, and let,

$$\begin{aligned} \Phi_0 &= \Phi, \\ \Phi_{n+1} &= \Phi_n \cup \phi_n \text{ if consistent, else } \Phi_n, \\ \Psi &= \textstyle\bigcup_n \Phi_n. \end{aligned}$$

Now for each propositional variable $\mathtt{p}$, define $v(\mathtt{p}) = 1$ just if $\mathtt{p} \in \Psi$. □

2.3 Boolean algebra

There is of course another apporach to propositional logic, which also goes back to the 19th century, namely that of Boolean algebra, which draws on the analogy between the propositional operations and the arithmetical ones.

Definition 2.3.1. A *Boolean algebra* is a set B equipped with the operations:

$$\begin{aligned} 0, 1 &: 1 \to B \\ \neg &: B \to B \\ \wedge, \vee &: B \times B \to B \end{aligned}$$

satisfying the following equations:

$$\begin{aligned}
x \vee x &= x & x \wedge x &= x \\
x \vee y &= y \vee x & x \wedge y &= y \wedge x \\
x \vee (y \vee z) &= (x \vee y) \vee z & x \wedge (y \wedge z) &= (x \wedge y) \wedge z \\
x \wedge (y \vee z) &= (x \wedge y) \vee (x \wedge z) & x \vee (y \wedge z) &= (x \vee y) \wedge (x \vee z) \\
0 \vee x &= x & 1 \wedge x &= x \\
1 \vee x &= 1 & 0 \wedge x &= 0 \\
\neg(x \vee y) &= \neg x \wedge \neg y & \neg(x \wedge y) &= \neg x \vee \neg y \\
x \vee \neg x &= 1 & x \wedge \neg x &= 0
\end{aligned}$$

This is of course an algebraic theory, like those considered in the previous chapter. Familiar examples of Boolean algebras are $2 = \{0, 1\}$, with the usual operations, and more generally, any powerset $\mathcal{P}X$, with the set-theoretic operations $A \vee B = A \cup B$, etc. (indeed, $2 = \mathcal{P}1$ is a special case.).

Exercise 2.3.2. Show that the free Boolean algebra $B(n)$ on n-many generators is the double powerset $\mathcal{P}\mathcal{P}(n)$, and determine the free functor on finite sets.

One can use equational reasoning in Boolean algebra as an alternative to the deductive propositional calculus as follows. For a propositional formula in context $\Gamma \mid \phi$, let us say that ϕ is *equationally provable* if we can prove $\phi = 1$ by equational reasoning (Section **??**), from the laws of Boolean algebras above. More generally, for a set of formulas Φ and a formula ψ let us define the *ad hoc* relation of *equational provability*,

$$\Phi \vdash^{=} \psi \tag{2.2}$$

to mean that $\psi = 1$ can be proven equationally from (the Boolean equations and) the set of all equations $\phi = 1$, for $\phi \in \Phi$. Since we don't have any laws for the connectives $\Rightarrow$ or $\Leftrightarrow$, let us replace them with their Boolean equivalents, by adding the equations:

$$\begin{aligned}
&\phi \Rightarrow \psi = \neg\phi \vee \psi\,, \\
&\phi \Leftrightarrow \psi = (\neg\phi \vee \psi) \wedge (\neg\psi \vee \phi)\,.
\end{aligned}$$

For example, here is an equational proof of $(\phi \Rightarrow \psi) \vee (\psi \Rightarrow \phi)$.

$$\begin{aligned}
(\phi \Rightarrow \psi) \vee (\psi \Rightarrow \phi) &= (\neg\phi \vee \psi) \vee (\neg\psi \vee \phi) \\
&= \neg\phi \vee (\psi \vee (\neg\psi \vee \phi)) \\
&= \neg\phi \vee ((\psi \vee \neg\psi) \vee \phi) \\
&= \neg\phi \vee (1 \vee \phi) \\
&= \neg\phi \vee 1 \\
&= 1 \vee \neg\phi \\
&= 1
\end{aligned}$$

Thus,

$$\vdash^{=} (\phi \Rightarrow \psi) \vee (\psi \Rightarrow \phi)\,.$$

We now ask: *What is the relationship between equational provability $\Phi \vdash^{=} \phi$, deductive entailment $\Phi \vdash \phi$, and semantic entailment $\Phi \vDash \phi$?*

Exercise 2.3.3. Using equational reasoning, show that every propositional formula ϕ has both a *conjunctive* $\phi^{\wedge}$ and a *disjunctive* $\phi^{\vee}$ *Boolean normal form* such that:

1. The formula $\phi^{\vee}$ is an n-fold disjunction of m-fold conjunctions of *positive* $\mathtt{p_i}$ or *negative* $\neg\mathtt{p_j}$ propositional variables,

$$\phi^{\vee} \;=\; (\mathtt{q}_{11} \wedge ... \wedge \mathtt{q}_{1m_1}) \vee ... \vee (\mathtt{q}_{n1} \wedge ... \wedge \mathtt{q}_{nm_n})\,, \qquad \mathtt{q}_{ij} \in \{\mathtt{p}_{ij}, \neg\mathtt{p}_{ij}\}\,,$$

   and $\phi^{\wedge}$ is the same, but with the roles of $\vee$ and $\wedge$ reversed.

2. Both

$$\vdash^{=} \phi \Leftrightarrow \phi^{\vee} \qquad \text{and} \qquad \vdash^{=} \phi \Leftrightarrow \phi^{\wedge}\,.$$

Exercise 2.3.4. Using Exercise 2.3.3, show that for every propositional formula ϕ, equational provability is equivalent to semantic validity,

$$\vdash^{=} \phi \iff \vDash \phi\,.$$

Hint: Put ϕ into conjunctive normal form and read off a truth valuation that falsifies it, if there is one.

Exercise 2.3.5. A Boolean algebra can be partially ordered by defining $x \leq y$ as

$$x \leq y \iff x \vee y = y \qquad \text{or equivalently} \qquad x \leq y \iff x \wedge y = x\,.$$

Thus a Boolean algebra is a (poset) category. Show that as a category, a Boolean algebra has all finite limits and colimits and is cartesian closed, and that a finitely complete and cocomplete cartesian closed poset is a Boolean algebra just if it satisfies $x = (x \Rightarrow 0) \Rightarrow 0$, where, as before, we define $x \Rightarrow y := \neg x \vee y$. Finally, show that homomorphisms of Boolean algebras $f : B \to B'$ are the same thing as functors (i.e. monotone maps) that preserve all finite limits and colimits.

2.4 Lawvere duality for Boolean algebras

Let us apply the machinery of algebraic theories from Chapter **??** to the algebraic theory of Boolean algebras and see what we get. The algebraic theory $\mathbb{B}$ of Boolean algebras is a finite product (FP) category with objects $1, B, B^2, ...$, containing a Boolean algebra $\mathcal{B}$, with underlying object $|\mathcal{B}| = B$. By Theorem **??**, $\mathbb{B}$ has the universal property that finite product preserving (FP) functors from $\mathbb{B}$ into any FP-category $\mathbb{C}$ correspond (pseudo-)naturally to Boolean algebras in $\mathbb{C}$,

$$\mathsf{Hom_{FP}}(\mathbb{B}, \mathbb{C}) \simeq \mathsf{BA}(\mathbb{C})\,. \tag{2.3}$$

The correspondence is mediated by evaluating an FP functor $F : \mathbb{B} \to \mathbb{C}$ at (the underlying structure of) the Boolean algebra $\mathcal{B}$ to get a Boolean algebra $F(\mathcal{B}) = \mathsf{BA}(F)(\mathcal{B})$ in $\mathbb{C}$:

$$\begin{array}{cc} F : \mathbb{B} \longrightarrow \mathbb{C} & \mathsf{FP} \\ \hline\hline F(\mathcal{B}) & \mathsf{BA}(\mathbb{C}) \end{array}$$

We call $\mathcal{B}$ the *universal Boolean algebra.* Given a Boolean algebra $\mathcal{A}$ in $\mathbb{C}$, we write

$$\mathcal{A}^\sharp : \mathbb{B} \longrightarrow \mathbb{C}$$

for the associated *classifying functor*. By the equivalence of categories (2.3), we have isos,

$$\mathcal{A}^\sharp(\mathcal{B}) \cong \mathcal{A}, \qquad F(\mathcal{B})^\sharp \cong F\,.$$

And in particular, $\mathcal{B}^\sharp \cong 1_{\mathbb{B}} : \mathbb{B} \to \mathbb{B}$.

By Lawvere duality, Corollary **??**, we know that $\mathbb{B}^{\mathsf{op}}$ can be identified with a full subcategory $\mathsf{mod}(\mathbb{B})$ of $\mathbb{B}$-models in Set (i.e. Boolean algebras),

$$\mathbb{B}^{\mathsf{op}} = \mathsf{mod}(\mathbb{B}) \hookrightarrow \mathsf{Mod}(\mathbb{B}) = \mathsf{BA}(\mathsf{Set})\,, \tag{2.4}$$

namely, that consisting of the finitely generated free Boolean algebras $F(n)$. Composing (2.4) and (2.3), we have an embedding of $\mathbb{B}^{\mathsf{op}}$ into the functor category,

$$\mathbb{B}^{\mathsf{op}} \hookrightarrow \mathsf{BA}(\mathsf{Set}) \simeq \mathsf{Hom_{FP}}(\mathbb{B}, \mathsf{Set}) \hookrightarrow \mathsf{Set}^{\mathbb{B}}\,, \tag{2.5}$$

which, up to isomorphism, is just the (contravariant) Yoneda embedding, taking $B^n \in \mathbb{B}$ to the covariant representable functor $\mathsf{y}^{\mathbb{B}}(B^n) = \mathsf{Hom}_{\mathbb{B}}(B^n, -)$ (cf. Theorem **??**).

Now consider provability of equations between terms $\phi : B^k \to B$ in the theory $\mathbb{B}$, which are essentially the same as propositional formulas in context $(\mathsf{p}_1, ..., \mathsf{p}_k \mid \phi)$ modulo $\mathbb{B}$-provable equality. The universal Boolean algebra $\mathcal{B}$ is logically generic, in the sense that for any such formulas ϕ, ψ, we have $\mathcal{B} \vDash \phi = \psi$ just if $\mathbb{B} \vdash \phi = \psi$ (Proposition **??**). The latter condition is equational provability from the axioms for Boolean algebras, which is just what was used in the definition of $\vdash^{=} \phi$ (cf. 2.2). Thus, in particular,

$$\vdash^{=} \phi \iff \mathbb{B} \vdash \phi = 1 \iff \mathcal{B} \vDash \phi = 1\,.$$

As we showed in Proposition **??**, the image of the universal model $\mathcal{B}$ under the (FP) *covariant* Yoneda embedding,

$$\mathsf{y}_{\mathbb{B}} : \mathbb{B} \to \mathsf{Set}^{\mathbb{B}^{\mathsf{op}}}$$

is also a logically generic model, with underlying object $|\mathsf{y}_{\mathbb{B}}(\mathcal{B})| = \mathsf{Hom}_{\mathbb{B}}(-, B)$. By Proposition **??** we can use that fact to restrict attention to Boolean algebras in Set, and in particular, to the finitely generated free ones $F(n)$, when testing for equational provability. Specifically, using the (FP) evaluation functors $\mathsf{eval}_{B^n} : \mathsf{Set}^{\mathbb{B}^{\mathsf{op}}} \to \mathsf{Set}$ for all objects $B^n \in \mathbb{B}$, we can extend the above reasoning as follows:

$$\begin{aligned} \vdash^{=} \phi \iff & \mathbb{B} \vdash \phi = 1 \\ \iff & \mathcal{B} \vDash \phi = 1 \\ \iff & \mathsf{y}_{\mathbb{B}}(\mathcal{B}) \vDash \phi = 1 \\ \iff & \mathsf{eval}_{B^n}\mathsf{y}_{\mathbb{B}}(\mathcal{B}) \vDash \phi = 1 \quad \text{for all } B^n \in \mathbb{B} \\ \iff & F(n) \vDash \phi = 1 \quad \text{for all } n. \end{aligned}$$

The last step holds because the image of $\mathsf{y}_{\mathbb{B}}(\mathcal{B})$ under eval_{B^n} is the free Boolean algebra $F(n)$ (cf. Exercise **??**). Indeed, for the underlying objects we have

$$\begin{aligned} |\mathsf{eval}_{B^n}\mathsf{y}_{\mathbb{B}}(\mathcal{B})| &\cong \mathsf{eval}_{B^n}|\mathsf{y}_{\mathbb{B}}(\mathcal{B})| \cong \mathsf{eval}_{B^n}\mathsf{y}_{\mathbb{B}}(|\mathcal{B}|) \cong \mathsf{eval}_{B^n}\mathsf{y}_{\mathbb{B}}(B) \cong \mathsf{y}_{\mathbb{B}}(B)(B^n) \\ &\cong \mathsf{Hom}_{\mathbb{B}}(B^n, B) \cong \mathsf{Hom}_{\mathsf{BA}^{\mathsf{op}}}(F(n), F(1)) \cong \mathsf{Hom}_{\mathsf{BA}}(F(1), F(n)) \cong |F(n)|\,. \end{aligned}$$

Thus to test for equational provability it suffices to check the equations in the free algebras $F(n)$ (which makes sense, since these are usually *defined* in terms of equational provability). We have therefore shown:

Lemma 2.4.1. *A formula in context $\mathsf{p}_1, ..., \mathsf{p}_k \mid \phi$ is equationally provable $\vdash_{=} \phi$ just in case, for every free Boolean algebra $F(n)$, we have $F(n) \vDash \phi = 1$.*

The condition $F(n) \vDash \phi = 1$ means that the equation $\phi = 1$ holds *generally* in $F(n)$, i.e. for any elements $f_1, ..., f_k \in F(n)$, we have $\phi[f_1/\mathsf{p}_1, ..., f_k/\mathsf{p}_k] = 1$, where the expression $\phi[f_1/\mathsf{p}_1, ..., f_k/\mathsf{p}_k]$ denotes the element of $F(n)$ resulting from interpreting the propositional variables p_i as the elements f_i and evaluating the resulting expression using the Boolean operations of $F(n)$. But now observe that the recipe:

> for any elements $f_1, ..., f_k \in F(n)$, let the expression
>
> $$\phi[f_1/\mathsf{p}_1, ..., f_k/\mathsf{p}_k] \tag{2.6}$$
>
> denote the element of $F(n)$ resulting from interpreting the propositional variables p_i as the elements f_i and evaluating the resulting expression using the Boolean operations of $F(n)$

describes the unique Boolean homomorphism

$$F(1) \xrightarrow{\overline{\phi}} F(k) \xrightarrow{\overline{(f_1, ..., f_k)}} F(n)\,,$$

where $\overline{(f_1, ..., f_k)} : F(k) \to F(n)$ is determined by the elements $f_1, ..., f_k \in F(n)$, and $\overline{\phi} : F(1) \to F(k)$ by the corresponding element $(\mathsf{p}_1, ..., \mathsf{p}_k \mid \phi) \in F(k)$. It is therefore equivalent to check the case $k = n$ and $f_i = \mathsf{p}_i$, i.e. the "universal case"

$$(\mathsf{p}_1, ..., \mathsf{p}_k \mid \phi) = 1 \quad \text{in } F(k)\,. \tag{2.7}$$

Finally, we then have:

Proposition 2.4.2 (Completeness of the equational propositional calculus)**.** *Equational propositional calculus is sound and complete with respect to* boolean-valued models *in* Set*, in the sense that a propositional formula ϕ is equationally provable from the laws of Boolean algebra,*

$$\vdash^{=} \phi\,,$$

just if it holds generally in any Boolean algebra (in Set*).*

Proof. By "holding generally" is meant the universal quantification of the equation over elements of a given Boolean algebra B, which is of course equivalent to saying that it holds for all elements of B, in the sense stated after the Lemma. But, as above, this is equivalent to the condition that for all $b_1, ..., b_k \in B$, for $\overline{(b_1, ..., b_k)} : F(k) \to B$ we have $\overline{(b_1, ..., b_k)}(\phi) = 1$ in B, which in turn is clearly equivalent to the previously determined "universal" condition (2.7) that $\phi = 1$ in $F(k)$. □

The analogous statement for equational entailment $\Phi \vdash_{=} \phi$ is left as an exercise.

Corollary 2.4.2 is a (very) special case of the Gödel completeness theorem for first-order logic, for *just* the equational fragment of *just* the specific theory of Boolean algebras (although, an analogous result of course holds for any other algebraic theory, and many other systems of logic can be reduced to the algebraic case). Nonetheless, it suggests another approach to the semantics of propositional logic based upon the idea of a *Boolean valuation*, generalizing the traditional truth-value semantics from Section 2.2. We pursue this idea systematically in the following section.

Exercise 2.4.3. For a formula in context $\mathtt{p}_1, ..., \mathtt{p}_k \mid \vartheta$ and a Boolean algebra $\mathcal{A}$, let the expression $\vartheta[a_1/\mathtt{p}_1, ..., a_k/\mathtt{p}_k]$ denote the element of $\mathcal{A}$ resulting from interpreting the propositional variables $\mathtt{p}_i$ in the context as the elements a_i of $\mathcal{A}$, and evaluating the resulting expression using the Boolean operations of $\mathcal{A}$. For any *finite* set of propositional formulas Φ and any formula ψ, let $\Gamma = \mathtt{p}_1, ..., \mathtt{p}_k$ be a context for (the formulas in) $\Phi \cup \{\psi\}$. Finally, recall that $\Phi \vdash^{=} \psi$ means that $\psi = 1$ is equationally provable from the set of equations $\{\phi = 1 \mid \phi \in \Phi\}$. Show that $\Phi \vdash^{=} \psi$ just if for all finitely generated free Boolean algebras $F(n)$, the following condition holds:

> For any elements $f_1, ..., f_k \in F(n)$, if $\phi[f_1/\mathtt{p}_1, ..., f_k/\mathtt{p}_k] = 1$ for all $\phi \in \Phi$, then $\psi[f_1/\mathtt{p}_1, ..., f_k/\mathtt{p}_k] = 1$.

Is it sufficient to just take $F(k)$ and its generators $\mathtt{p}_1, ..., \mathtt{p}_k$ as the $f_1, ..., f_k$? Is it equivalent to take all Boolean algebras B, rather than the finitely generated free ones $F(n)$? Determine a condition that is equivalent to $\Phi \vdash^{=} \psi$ for not necessarily finite sets Φ.

2.5 Functorial semantics for propositional logic

Considering the algebraic theory of Boolean algebras suggests the idea of a Boolean valuation of propositional logic, generalizing the truth valuations of section 2.2. This can be seen as applying the framework of functorial semantics to a different system of logic than that of finite product categories, namely that represented categorically by *poset* categories with finite products $\wedge$ *and coproducts* $\vee$ (each of these specializations could, of course, also be considered separately, giving $\wedge$-semi-lattices and categories with finite products $\times$ and coproducts $+$, respectively). Thus we are moving from the top right corner to the bottom center position in the following Hasse diagram of structured categories:

$$\begin{tikzcd}
+ \arrow[dr, dash] \arrow[dd, dash] & & \times \arrow[dl, dash] \arrow[dd, dash] \\
& +\,,\times \arrow[dd, dash] & \\
\vee \arrow[dr, dash] & & \wedge \arrow[dl, dash] \\
& \vee\,,\wedge &
\end{tikzcd}$$

In Chapter **??** we shall see how first-order logic results categorically from these two cases by "indexing the lower one over the upper one".

Definition 2.5.1. A *propositional theory* $\mathbb{T}$ consists of a set $V_{\mathbb{T}}$ of propositional variables, called the *basic* or *atomic propositions*, and a set $A_{\mathbb{T}}$ of propositional formulas (over $V_{\mathbb{T}}$), called the *axioms*. The *consequences* $\Phi \vdash_{\mathbb{T}} \phi$ are those judgements that are derivable by natural deduction (as in Section 2.1), from the axioms $A_{\mathbb{T}}$.

Definition 2.5.2. Let $\mathbb{T} = (V_{\mathbb{T}}, A_{\mathbb{T}})$ be a propositional theory and $\mathcal{B}$ a Boolean algebra. A *model* of $\mathbb{T}$ in $\mathcal{B}$, also called a *Boolean valuation of* $\mathbb{T}$ is an *interpretation function* $v : V_{\mathbb{T}} \to |\mathcal{B}|$ such that, for every $\alpha \in A_{\mathbb{T}}$, we have $[\![\alpha]\!]^v = 1_{\mathcal{B}}$ in $\mathcal{B}$, where the extension $[\![-]\!]^v$ of v from $V_{\mathbb{T}}$ to all formulas (over $V_{\mathbb{T}}$) is defined in the expected way, namely:

$$\begin{aligned}
[\![\mathtt{p}]\!]^v &= v(\mathtt{p}), \quad \mathtt{p} \in V_{\mathbb{T}} \\
[\![\top]\!]^v &= 1_{\mathcal{B}} \\
[\![\bot]\!]^v &= 0_{\mathcal{B}} \\
[\![\neg\phi]\!]^v &= \neg_{\mathcal{B}}[\![\phi]\!]^v \\
[\![\phi \wedge \psi]\!]^v &= [\![\phi]\!]^v \wedge_{\mathcal{B}} [\![\psi]\!]^v \\
[\![\phi \vee \psi]\!]^v &= [\![\phi]\!]^v \vee_{\mathcal{B}} [\![\psi]\!]^v \\
[\![\phi \Rightarrow \psi]\!]^v &= \neg_{\mathcal{B}}[\![\phi]\!]^v \vee_{\mathcal{B}} [\![\psi]\!]^v
\end{aligned}$$

Finally, let $\mathsf{Mod}(\mathbb{T}, \mathcal{B})$ be the set of all $\mathbb{T}$-models in $\mathcal{B}$. Given a Boolean homomorphism $f : \mathcal{B} \to \mathcal{B}'$, there is an induced mapping $\mathsf{Mod}(\mathbb{T}, f) : \mathsf{Mod}(\mathbb{T}, \mathcal{B}) \to \mathsf{Mod}(\mathbb{T}, \mathcal{B}')$, determined by setting $\mathsf{Mod}(\mathbb{T}, f)(v) = f \circ v$, which is clearly functorial.

Theorem 2.5.3. *The functor* $\mathsf{Mod}(\mathbb{T}) : \mathsf{BA} \to \mathsf{Set}$ *is representable, with representing Boolean algebra* $\mathcal{B}_{\mathbb{T}}$*, called the* Lindenbaum-Tarski algebra *of* $\mathbb{T}$*.*

Proof. We construct $\mathcal{B}_{\mathbb{T}}$ in two steps:

Step 1: Suppose first that $A_{\mathbb{T}}$ is empty, so $\mathbb{T}$ is just a set V of propositional variables. Define the *Lindenbaum-Tarski algebra* $\mathcal{B}[V]$ by

$$\mathcal{B}[V] \;=\; \{\phi \mid \phi \text{ is a formula in context } V\}/\sim$$

where the equivalence relation $\sim$ is *(deductively) provable bi-implication*,

$$\phi \sim \psi \iff \vdash \psi \Leftrightarrow \psi.$$

The operations are (well-)defined on equivalence classes by setting,

$$[\phi] \wedge [\psi] \;=\; [\phi \wedge \psi],$$

and so on. (The reader who has not seen this construction before should fill in the details!)

Step 2: In the general case $\mathbb{T} = (V_{\mathbb{T}}, A_{\mathbb{T}})$, let

$$\mathcal{B}_{\mathbb{T}} = \mathcal{B}[V_{\mathbb{T}}]/\sim_{\mathbb{T}}\,,$$

where the equivalence relation $\sim_{\mathbb{T}}$ is now $A_{\mathbb{T}}$*-provable bi-implication*,

$$\phi \sim_{\mathbb{T}} \psi \iff A_{\mathbb{T}} \vdash \psi \Leftrightarrow \psi.$$

The operations are defined as before, but now on equivalence classes $[\phi]$ modulo $A_{\mathbb{T}}$.

Now observe that the construction of $\mathcal{B}_{\mathbb{T}}$ is a variation on that of the *syntactic category* $\mathcal{C}_{\mathbb{T}}$ of the algebraic theory $\mathbb{T}$ in the sense of the previous chapter, and the statement of the theorem is its universal property as the classifying category of $\mathbb{T}$-models, namely

$$\mathsf{Mod}(\mathbb{T}, \mathcal{B}) \cong \mathsf{Hom}_{\mathsf{BA}}(\mathcal{B}_{\mathbb{T}}, \mathcal{B})\,, \tag{2.8}$$

naturally in $\mathcal{B}$. (Indeed, since $\mathsf{Mod}(\mathbb{T}, \mathcal{B})$ is now a *set* rather than a category, we can classify it up to *isomorphism* rather than equivalence of categories.) The proof of this fact is a variation on the proof of the corresponding theorem **??** from Chapter 1. Further details are given in the following Remark 2.5.4 for the interested reader. □

Remark 2.5.4 (Adjoint Rules for Propositional Calculus)**.** For the construction of the Lindenbaum-Tarski algebra $\mathcal{B}_{\mathbb{T}}$, it is convenient to reformulate the rules of inference for the propositional calculus in the following equivalent *adjoint form*:

Contexts Γ may be omitted, since the rules leave them unchanged (there is no variable binding). We may also omit hypotheses that remain unchanged. Thus e.g. the *hypothesis* rule may be written in any of the following equivalent ways.

$$\frac{}{\Gamma \mid \phi_1, \ldots, \phi_m \vdash \phi_i} \qquad \frac{}{\phi_1, \ldots, \phi_m \vdash \phi_i} \qquad \frac{}{\phi \vdash \phi}$$

The structural rules can then be stated as follows:

$$\frac{}{\phi \vdash \phi} \qquad \frac{\phi \vdash \psi \qquad \psi \vdash \vartheta}{\phi \vdash \vartheta}$$

$$\frac{\phi \vdash \vartheta}{\psi, \phi \vdash \vartheta} \qquad \frac{\phi, \phi \vdash \vartheta}{\phi \vdash \vartheta} \qquad \frac{\phi, \psi \vdash \vartheta}{\psi, \phi \vdash \vartheta}$$

The rules for the propositional connectives can be given in the following adjoint form, where the double line indicates a two-way rule (with the obvious two instances when there are two conclusions).

$$\frac{}{\phi \vdash \top} \qquad \frac{}{\bot \vdash \phi}$$

$$\frac{\vartheta \vdash \phi \quad \vartheta \vdash \psi}{\vartheta \vdash \phi \wedge \psi} \qquad \frac{\phi \vdash \vartheta \quad \psi \vdash \vartheta}{\phi \vee \psi \vdash \vartheta} \qquad \frac{\vartheta, \phi \vdash \psi}{\vartheta \vdash \phi \Rightarrow \psi}$$

For the purpose of deduction, negation $\neg\phi$ is again treated as defined by $\phi \Rightarrow \bot$ and bi-implication $\phi \Leftrightarrow \psi$ by $(\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi)$. For *classical* logic we also include the rule of *double negation*:

$$\frac{}{\neg\neg\phi \vdash \phi} \tag{2.9}$$

It is now obvious that the set of formulas is preordered by $\phi \vdash \psi$, and that the poset reflection agrees with the deducibility equivalence relation,

$$\phi \dashv\vdash \psi \iff \phi \sim \psi\,.$$

Moreover, $\mathcal{B}_{\mathbb{T}}$ clearly has all finite limits $\top, \wedge$ and colimits $\bot, \vee$, is cartesian closed $\wedge \dashv \Rightarrow$, and is therefore a *Heyting algebra* (see Section **??** below). The rule of double negation then makes it a Boolean algebra.

The proof of the universal property of $\mathcal{B}_{\mathbb{T}}$ is essentially the same as that for $\mathcal{C}_{\mathbb{T}}$.

Exercise 2.5.5. Fill in the details of the proof that $\mathcal{B}_{\mathbb{T}}$ is a well-defined Boolean algebra, with the universal property stated in (2.8).

Just as for the case of algebraic theories and FP categories, we now have the following corollary of the classifying theorem 2.5.3. (Note that the recipe at (2.6) for a Boolean valuation in $F(n)$ of the formula in context $\mathtt{p}_1, ..., \mathtt{p}_k \mid \phi$ is exactly a *model* in $F(n)$ of the theory $\mathbb{T} = \{\mathtt{p}_1, ..., \mathtt{p}_k\}$.)

Corollary 2.5.6. *For any set of formulas Φ and formula ϕ, derivability $\Phi \vdash \phi$ is equivalent to validity under all Boolean valuations. Therefore by Proposition 2.4.2 (and Exercise 2.4.3), we also have*

$$\Phi \vdash \phi \iff \Phi \vdash^{=} \phi\,.$$

Remark 2.5.7. If $A_{\mathbb{T}}$ is non-empty, but finite, then let

$$\alpha_{\mathbb{T}} := \bigwedge_{\alpha \in A_{\mathbb{T}}} \alpha \,.$$

We then have

$$\mathcal{B}_{\mathbb{T}} = \mathcal{B}[V_{\mathbb{T}}]/\alpha_{\mathbb{T}} \,,$$

where as usual $\mathcal{B}/b$ denotes the slice category of the Boolean algebra $\mathcal{B}$ over an element $b \in \mathcal{B}$.

Remark 2.5.8. Our definition of the Lindenbaum-Tarski algebra is given in terms of *provability*, rather than the more familiar semantic definition using (truth) valuations. The two are, of course, equivalent in light of Theorem 2.2.1, but since we intend to prove that theorem, this definition will be more useful, as it parallels that of the syntactic category $\mathcal{C}_{\mathbb{T}}$ of an algebraic theory.

Inspecting the universal property (2.8) of $\mathcal{B}_{\mathbb{T}}$ for the case $\mathcal{B}[V]$ where there are no axioms, we now have the following.

Corollary 2.5.9. *The Lindenbaum-Tarski algebra $\mathcal{B}[V]$ is the free Boolean algebra on the set V. In particular, $\mathcal{B}[\mathtt{p}_1, ..., \mathtt{p}_n]$ is the finitely generated, free Boolean algebra $F(n)$.*

The isomorphism $\mathcal{B}[\mathtt{p}_1, ..., \mathtt{p}_n] \cong F(n)$ expresses the fact recorded in Corollary 2.5.6 that the relations of derivability by natural deduction $\Phi \vdash \phi$ and equational provability $\Phi \vdash^{=} \phi$ agree — answering part of the question at the end of Section **??**.

Exercise 2.5.10. Show that the Boolean algebras $\mathcal{B}_{\mathbb{T}}$ for *finite sets* $V_{\mathbb{T}}$ of variables and $A_{\mathbb{T}}$ of formulas are exactly the *finitely presented* ones.

Finally, we can use the following to finish the comparison of $\vdash \phi$ and $\vDash \phi$.

Lemma 2.5.11. *Let $\mathcal{B}$ be a* finitely presented *Boolean algebra in which $0 \neq 1$. Then there is a Boolean homomorphism*

$$h : \mathcal{B} \to \mathbb{2} \,.$$

Proof. By Exercise 2.5.10, we can assume that $\mathcal{B} = \mathcal{B}[\mathtt{p}_1 ... \mathtt{p}_n]/\alpha$ classifying the theory $\mathbb{T} = (\mathtt{p}_1 ... \mathtt{p}_n, \alpha)$. By the assumption that $0 \neq 1$ in $\mathbb{B} = \mathcal{B}[\mathtt{p}_1 ... \mathtt{p}_n]/\alpha$, we have $\alpha \neq 0$ in the free Boolean algebra $F(n) \cong \mathcal{B}[\mathtt{p}_1 ... \mathtt{p}_n]$, whence $\alpha \nvdash \bot$. Since $F(n) \cong \mathcal{P}\mathcal{P}(n)$, there is a valuation $\vartheta : \{\mathtt{p}_1 ... \mathtt{p}_n\} \to \mathbb{2}$ such that $[\![\alpha]\!]^{\vartheta} = 1$. This is exactly a Boolean homomorphism $\mathcal{B}[\mathtt{p}_1 ... \mathtt{p}_n]/\alpha \to \mathbb{2}$, as required. □

Corollary 2.5.12. *For any set of formulas Φ and formula ϕ, derivability $\Phi \vdash \phi$ is equivalent to semantic entailment,*

$$\Phi \vDash \phi \iff \Phi \vdash \phi \,.$$

Proof. By 2.5.6, it suffices to show that $\Phi \vDash \phi$ is equivalent to $\Phi \vdash^{=} \phi$, but the latter we know to be equivalent to holding in all Boolean valuations in free Boolean algebras $F(n)$, and the former to holding in all *truth* valuations, i.e. Boolean valuations in $\mathbb{2}$. Thus it will suffice to embed $F(n)$ as a Boolean algebra into a powerset $\mathcal{P}X = \mathbb{2}^X$, for a set X. By Lemma 2.5.11 we can take $X = 2^n$. □

2.6 Stone representation

Regarding a Boolean algebra $\mathcal{B}$ as a category with finite products, consider its Yoneda embedding $\mathsf{y} : \mathcal{B} \hookrightarrow \mathsf{Set}^{\mathcal{B}^{\mathsf{op}}}$. Since the hom-set $\mathcal{B}(x, y)$ is $\mathbb{2}$-valued, we have a factorization,

$$\mathcal{B} \hookrightarrow \mathbb{2}^{\mathcal{B}^{\mathsf{op}}} \hookrightarrow \mathsf{Set}^{\mathcal{B}^{\mathsf{op}}}$$

in which each factor still preserves the finite products (note that the products in $\mathbb{2}$ are preserved by the inclusion $\mathbb{2} \hookrightarrow \mathsf{Set}$, and the products in the functor categories are taken pointwise). Indeed, this is an instance of a general fact. In the category $\mathsf{Cat}_\times$ of finite product categories (and $\times$-preserving functors), the inclusion of the full subcategory of posets with $\wedge$ (the $\wedge$-semilattices) has a *right adjoint* R, in addition to the left adjoint L of poset reflection.

$$\begin{tikzcd} \mathsf{Cat}_\times \arrow[d, bend right, "L"'] \arrow[d, bend left, "R"] \\ \mathsf{Pos}_\wedge \arrow[u, hook, "i"] \end{tikzcd}$$

For a finite product category $\mathbb{C}$, the poset $R\mathbb{C}$ is the subcategory $\mathsf{Sub}(1) \hookrightarrow \mathbb{C}$ of subobjects of the terminal object 1 (equivalently, the category of monos $m : M \rightarrowtail 1$). The reason for this is that a $\times$-preserving functor $f : A \to \mathbb{C}$ from a poset A with meets takes every object $a \in A$ to a mono $f(a) \rightarrowtail 1$ in $\mathbb{C}$, since the following is a product diagram in A.

$$\begin{tikzcd} a \arrow[r] & 1 \\ a \arrow[u] \arrow[r] & a \arrow[u] \end{tikzcd}$$

Exercise 2.6.1. Prove this, and use it to verify that $R = \mathsf{Sub}(1)$ is indeed a right adjoint to the inclusion of $\wedge$-semilattices into finite-product categories.

Now the functor category $\mathbb{2}^{\mathcal{B}^{\mathsf{op}}} = \mathsf{Pos}(\mathcal{B}^{\mathsf{op}}, \mathbb{2})$ of all *contravariant*, monotone maps $\mathcal{B}^{\mathsf{op}} \to \mathbb{2}$ (which indeed is $\mathsf{Sub}(1) \hookrightarrow \mathsf{Set}^{\mathbb{C}^{\mathsf{op}}}$) is easily seen to be isomorphic to the poset $\downarrow\!\mathcal{B}$ of all *sieves* (or "downsets") in $\mathcal{B}$: subsets $S \subseteq \mathcal{B}$ that are downward closed, $x \leq y \in S \Rightarrow x \in S$, ordered by subset inclusion $S \subseteq T$. Explicitly, the isomorphism

$$\mathsf{Pos}(\mathcal{B}^{\mathsf{op}}, \mathbb{2}) \cong \downarrow\!\mathcal{B} \tag{2.10}$$

is given by taking $f : \mathcal{B}^{\mathsf{op}} \to \mathbb{2}$ to $f^{-1}(1)$ and $S \subseteq \mathcal{B}$ to the function $f_S : \mathcal{B}^{\mathsf{op}} \to \mathbb{2}$ with $f_S(b) = 1 \Leftrightarrow b \in S$. Under this isomorphism, the Yoneda embedding takes an element $b \in \mathcal{B}$ *covariantly* to the *principal downset* $\downarrow\! b \subseteq \mathcal{B}$ of all $x \leq b$.

Exercise 2.6.2. Show that (2.10) is indeed an isomorphism of posets, and that it takes the Yoneda embedding to the principal sieve mapping, as claimed.

For algebraic theories $\mathbb{A}$, we used the Yoneda embedding to give a completeness theorem for equational logic with respect to Set-valued models, by composing the (faithful functor)

$\mathsf{y} : \mathbb{A} \hookrightarrow \mathsf{Set}^{\mathbb{A}^{\mathsf{op}}}$ with the (jointly faithful) evaluation functors $\mathsf{eval}_A : \mathsf{Set}^{\mathbb{A}^{\mathsf{op}}} \to \mathsf{Set}$, for all objects $A \in \mathbb{A}$. This amounts to considering all *covariant* representables $\mathsf{eval}_A \circ \mathsf{y} = \mathbb{A}(A, -) : \mathbb{A} \to \mathsf{Set}$, and observing that these are then (both $\times$-preserving and) jointly faithful.

We can do the same thing for a Boolean algebra $\mathcal{B}$ (which is, after all, a finite product category) to get a jointly faithful family of $\times$-preserving, monotone maps $\mathcal{B}(b, -) : \mathcal{B} \to \mathbb{2}$, i.e. $\wedge$-semilattice homomorphisms. By taking the preimages of $\{1\} \hookrightarrow \mathbb{2}$, such homomorphisms correspond to *filters* in $\mathcal{B}$: "upsets" that are also closed under $\wedge$. The representables then correspond to the *principal filters* $\uparrow b \subseteq \mathcal{B}$. The problem with using this approach for a completeness theorem for *propositional* logic is that such $\wedge$-homomorphisms $\mathcal{B} \to \mathbb{2}$ are not *models*, because they need not preserve the joins $\phi \vee \psi$ (nor the complements $\neg\phi$).

Lemma 2.6.3. *Let $\mathcal{B}, \mathcal{B}'$ be Boolean algebras and $f : \mathcal{B} \to \mathcal{B}'$ a distributive lattice homomorphism. Then f preserves negation, and so is Boolean. The category* Bool *of Boolean algebras is thus a full subcategory of the category* DLat *of distributive lattices.*

Proof. The complement $\neg b$ is the unique element of $\mathcal{B}$ such that both $b \vee \neg b = 1$ and $b \wedge \neg b = 0$. $\square$

This suggests representing a Boolean algebra $\mathcal{B}$, not by its filters, but by its *prime* filters, which correspond bijectively to distributive lattice homomorphisms $\mathcal{B} \to \mathbb{2}$.

Definition 2.6.4. A filter $F \subseteq \mathcal{D}$ in a distributive lattice $\mathcal{D}$ is *prime* if $b \vee b' \in F$ implies $b \in F$ or $b' \in F$. Equivalently, just if the corresponding $\wedge$-semilattice homomorphism $f_F : \mathcal{B} \to \mathbb{2}$ is a lattice homomorphism.

If $\mathcal{B}$ is Boolean, it then follows that prime filters $F \subseteq \mathcal{B}$ are in bijection with Boolean homomorphisms $\mathcal{B} \to \mathbb{2}$, via the assignment $F \mapsto f_F : \mathcal{B} \to \mathbb{2}$ with $f_F(b) = 1 \Leftrightarrow b \in F$ and $(f : \mathcal{B} \to \mathbb{2}) \mapsto F_f := f^{-1}(1) \subseteq \mathcal{B}$. The prime filter F_f may be called the *(filter) kernel* of $f : \mathcal{B} \to \mathbb{2}$.

Proposition 2.6.5. *In a Boolean algebra $\mathcal{B}$, the following conditions on a subset $F \subseteq \mathcal{B}$ are equivalent.*

1. *F is a prime filter*
2. *the complement $\mathcal{B}\backslash F$ is a* prime ideal *(defined as a prime filter in $\mathcal{B}^{\mathsf{op}}$).*
3. *the complement $\mathcal{B}\backslash F$ is an* ideal *(defined as a filter in $\mathcal{B}^{\mathsf{op}}$).*
4. *F is a filter, and for each $b \in \mathcal{B}$, either $b \in F$ or $\neg b \in \mathcal{F}$ and not both.*
5. *F is a* maximal filter*: F is a filter and for all filters G, if $F \subseteq G$ then $F = G$ (also called an* ultrafilter*).*
6. *the map $f_F : \mathcal{B} \to \mathbb{2}$ given by $f_F(b) = 1 \Leftrightarrow b \in F$ (as in (2.10)) is a Boolean homomorphism.*

Proof. Exercise! $\square$

The following lemma is sometimes referred to as the *(Boolean) prime ideal theorem.*

Lemma 2.6.6. *Let $\mathcal{B}$ be a Boolean algebra, $I \subseteq \mathcal{B}$ an ideal, and $F \subseteq \mathcal{B}$ a filter, with $I \cap F = \emptyset$. There is a prime filter $P \supseteq F$ with $I \cap P = \emptyset$.*

Proof. Suppose first that $I = \{0\}$ is the trivial ideal, and that $\mathcal{B}$ is countable, with $b_0, b_1, ...$ an enumeration of its elements. As in the proof of the Model Existence Lemma, we build an increasing sequence of filters $F_0 \subseteq F_1 \subseteq \dots$ as follows:

$$\begin{aligned} F_0 &= F \\ F_{n+1} &= \begin{cases} F_n & \text{if } \neg b_n \in F_n \\ \{f \wedge b \,|\, f \in F_n,\ b_n \leq b\} & \text{otherwise} \end{cases} \\ P &= \bigcup_n F_n \end{aligned}$$

One then shows that each F_n is a filter, that $I \cap F_n = \emptyset$ for all n and so $I \cap P = \emptyset$, and that for each b_n, either $b_n \in P$ or $\neg b_n \in P$, whence P is prime.

For $I \subseteq \mathcal{B}$ a nontrivial ideal we take the quotient Boolean algebra $\mathcal{B} \twoheadrightarrow \mathcal{B}/I$, defined as the algebra of equivalence classes $[b]$ where $a \sim_I b \Leftrightarrow a \vee i = b \vee j$ for some $i, j \in I$. One shows that this is indeed a Boolean algebra and that the projection onto equivalence classes $\pi_I : \mathcal{B} \twoheadrightarrow \mathcal{B}/I$ is a Boolean homomorphism with (ideal) kernel $\pi^{-1}([0]) = I$. Now apply the foregoing argument to obtain a prime filter $P : \mathcal{B}/I \to \mathbb{2}$. The composite $p_I = P \circ \pi_I : \mathcal{B} \to \mathbb{2}$ is then a Boolean homomorphism with (filter) kernel $p_I^{-1}(1)$ which is prime, contains F and is disjoint from I.

The case where $\mathcal{B}$ is uncountable is left as an exercise. $\square$

Exercise 2.6.7. Finish the proof by (i) verifying the construction of the quotient Boolean algebra $\mathcal{B} \twoheadrightarrow \mathcal{B}/I$, and (ii) considering the case where $\mathcal{B}$ is uncountable (*Hint*: either use Zorn's lemma, or well-order $\mathcal{B}$.)

Theorem 2.6.8 (Stone representation theorem)**.** *Let $\mathcal{B}$ be a Boolean algebra. There is an injective Boolean homomorphism $\mathcal{B} \rightarrowtail \mathcal{P}X$ into a powerset.*

Proof. Let X be the set of prime filters in $\mathcal{B}$ and consider the map $h : \mathcal{B} \to \mathcal{P}X$ given by $h(b) = \{F \,|\, b \in F\}$. Clearly $h(0) = \emptyset$ and $h(1) = X$. Moreover, for *any* filter F, we have $b \in F$ and $b' \in F$ if and only if $b \wedge b' \in F$, so $h(b \wedge b') = h(b) \cap h(b')$. If F is prime, then $b \in F$ or $b' \in F$ if and only if $b \vee b' \in F$, so $h(b \vee b') = h(b) \cup h(b')$. Thus h is a Boolean homomorphism. Let $a \neq b \in \mathcal{B}$, and we want to show that $h(a) \neq h(b)$. It suffices to assume that $a < b$ (otherwise, consider $a \wedge b$, for which we cannot have both $a \wedge b = a$ and $a \wedge b = b$). We seek a prime filter $P \subseteq \mathcal{B}$ with $b \in P$ but $a \notin P$. Apply Lemma 2.6.6 to the ideal $\downarrow a$ and the filter $\uparrow b$. $\square$

2.7 Stone duality

Note that in the Stone representation $\mathcal{B} \rightarrowtail \mathcal{P}(X_{\mathcal{B}})$ the powerset Boolean algebra

$$\mathcal{P}(X_{\mathcal{B}}) \cong \mathsf{Set}\big(\mathsf{Bool}(\mathcal{B}, \mathbb{2}), 2\big)$$

is evidently (covariantly) functorial in $\mathcal{B}$, and has an apparent "double-dual" form $\mathcal{B}^{**}$. This suggests a possible duality between the categories Bool and Set,

$$\begin{tikzcd} \mathsf{Bool}^{\mathsf{op}} \arrow[r, bend left, "*"] & \mathsf{Set} \arrow[l, bend left, "*"] \end{tikzcd} \tag{2.11}$$

with contravariant functors $\mathcal{B}^* = \mathsf{Bool}(\mathcal{B}, \mathbb{2})$, the set of prime filters, for a Boolean algebra $\mathcal{B}$, and $S^* = \mathsf{Set}(S, 2)$, the powerset Boolean algebra, for a set S. This indeed gives a contravariant adjunction "on the right",

$$\frac{\mathcal{B} \to \mathcal{P}S \qquad \mathsf{Bool}}{S \to X_{\mathcal{B}} \qquad \mathsf{Set}} \tag{2.12}$$

by applying the contravariant functors

$$\begin{aligned} \mathcal{P}S &= \mathsf{Set}(S, 2), \\ X_{\mathcal{B}} &= \mathsf{Bool}(\mathcal{B}, \mathbb{2}), \end{aligned}$$

and then precomposing with the respective "evaluation" natural transformations,

$$\begin{aligned} \eta_{\mathcal{B}} &: \mathcal{B} \to \mathcal{P}X_{\mathcal{B}} \cong \mathsf{Set}\big(\mathsf{Bool}(\mathcal{B}, \mathbb{2}), 2\big), \\ \varepsilon_S &: S \to X_{\mathcal{P}S} \cong \mathsf{Bool}\big(\mathsf{Set}(S, 2), \mathbb{2}\big). \end{aligned}$$

The homomorphism $\eta_{\mathcal{B}}$ takes an element $b \in \mathcal{B}$ to the set of prime filters that contain it, and the function ε_S takes an element $s \in S$ to the principal filter $\uparrow\{s\} \subseteq \mathcal{P}S$, which is prime since the singleton set $\{s\}$ is an *atom* in $\mathcal{P}S$, i.e., a minimal, non-zero element.

Exercise 2.7.1. Verify the adjunction (2.14).

The adjunction (2.14) is not an equivalence, however, because neither of the units $\eta_{\mathcal{B}}$ nor ε_S is in general an isomorphism. We can do better by topologizing the set $X_{\mathcal{B}}$ of prime filters, in order to be able to cut down the powerset $\mathcal{P}X_{\mathcal{B}} \cong \mathsf{Set}(X_{\mathcal{B}}, 2)$ to just the *continuous* functions into the discrete space 2, which then correspond to the clopen sets in $X_{\mathcal{B}}$. To do so, we take as *basic open sets* all those sets of the form:

$$B_b = \{P \in X_{\mathcal{B}} \mid b \in P\}, \qquad b \in \mathcal{B}. \tag{2.13}$$

These sets are closed under finite intersections, because $B_a \cap B_b = B_{a \wedge b}$. Indeed, if $P \in B_a \cap B_b$ then $a \in P$ and $b \in P$, whence $a \wedge b \in P$, and conversely.

Definition 2.7.2. For any Boolean algebra $\mathcal{B}$, the *prime spectrum* of $\mathcal{B}$ is a topological space $X_{\mathcal{B}}$ with the prime filters $P \subseteq \mathcal{B}$ as points, and the sets B_b of (2.13), for all $b \in \mathcal{B}$, as basic open sets. The prime spectrum $X_{\mathcal{B}}$ is also called the *Stone space* of $\mathcal{B}$.

Proposition 2.7.3. *The open sets $\mathcal{O}(X_{\mathcal{B}})$ of the Stone space are in order-preserving, bijective correspondence with the ideals $I \subseteq \mathcal{B}$ of the Boolean algebra, with the principal ideals $\downarrow b$ corresponding exactly to the clopen sets.*

Proof. Exercise! □

We now have an improved adjunction

$$\begin{tikzcd} \mathsf{Bool}^{\mathsf{op}} \arrow[r, bend left, "\mathsf{Spec}"] & \mathsf{Top} \arrow[l, bend left, "\mathsf{Clop}"] \end{tikzcd} \tag{2.14}$$

$$\begin{aligned} \mathsf{Spec}(\mathcal{B}) &= (X_{\mathcal{B}}, \mathcal{O}(X_{\mathcal{B}})) \\ \mathsf{Clop}(X) &= \mathsf{Top}(X, 2), \end{aligned}$$

for which, up to isomorphism, the space $\mathsf{Spec}(\mathcal{B})$ has the underlying set $\mathsf{Bool}(\mathcal{B}, \mathbb{2})$ given by "homming" into the Boolean algebra $\mathbb{2}$, and the Boolean algebra $\mathsf{Clop}(X) = \mathsf{Top}(X, 2)$ is similarly determined by mapping into the "topological Boolean algebra" given by the discrete topological space 2. Such an adjunction is said to be induced by a *dualizing object*: an object that can be regarded as "living in two different categories". Here the dualizing object 2 is acting both as a space and as a Boolean algebra. In the Lawvere duality of Chapter 1 (and others to be met later on), the role of dualizing object is played by the category Set of all sets.

Toward the goal of determining the image of the functor $\mathsf{Spec} : \mathsf{Bool}^{\mathsf{op}} \to \mathsf{Top}$, observe first that the Stone space $X_{\mathcal{B}}$ of a Boolean algebra $\mathcal{B}$ is a subspace of a product of finite discrete spaces,

$$X_{\mathcal{B}} \cong \mathsf{Bool}(\mathcal{B}, \mathbb{2}) \hookrightarrow \prod_{|\mathcal{B}|} 2,$$

and is therefore a compact Hausdorff space by Tychonoff's theorem. Indeed, the basis (2.13) is just the subspace topology on $X_{\mathcal{B}}$ with respect to the product topology on $\prod_{|\mathcal{B}|} 2$. The latter space is moreover *totally disconnected*, meaning that it has a subbasis of clopen subsets, namely all those of the form $f^{-1}(\delta) \subseteq |\mathcal{B}|$ for $f : |\mathcal{B}| \to 2$ and $\delta = 0, 1$.

Lemma 2.7.4. *The prime spectrum $X_{\mathcal{B}}$ of a Boolean algebra $\mathcal{B}$ is a totally disconnected, compact, Hausdorff space.*

Proof. Since $\prod_{|\mathcal{B}|} 2$ has just been shown to be a totally disconnected, compact Hausdorff space, we need only see that the subspace $X_{\mathcal{B}}$ is closed. Consider the subspaces

$$2^{|\mathcal{B}|}_{\wedge},\ 2^{|\mathcal{B}|}_{\vee},\ 2^{|\mathcal{B}|}_{1},\ 2^{|\mathcal{B}|}_{0} \subseteq 2^{|\mathcal{B}|}$$

consisting of the functions $f : |\mathcal{B}| \to 2$ that preserve $\wedge, \vee, 1, 0$ respectively. Since each of these is closed, so is their intersection $X_{\mathcal{B}}$. In more detail, the set of maps $f : |\mathcal{B}| \to 2$ that preserve e.g. $\wedge$ can be described as an equalizer

$$\begin{tikzcd} 2^{|\mathcal{B}|}_{\wedge} \arrow[r, tail] & 2^{|\mathcal{B}|} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & 2^{|\mathcal{B}| \times |\mathcal{B}|} \end{tikzcd}$$

where the maps s, t take an arrow $f : |\mathcal{B}| \to 2$ to the two different composites around the square

$$\begin{tikzcd} |\mathcal{B}| \times |\mathcal{B}| \arrow[r, "\wedge"] \arrow[d, "f \times f"'] & |\mathcal{B}| \arrow[d, "f"] \\ 2 \times 2 \arrow[r, "\wedge"'] & 2. \end{tikzcd}$$

But the equalizer $2^{|\mathcal{B}|}_{\wedge} \rightarrowtail 2^{|\mathcal{B}|}$ is the pullback of the diagonal on $2^{|\mathcal{B}| \times |\mathcal{B}|}$, which is closed since $2^{|\mathcal{B}| \times |\mathcal{B}|}$ is Hausdorff. The other cases are analogous . $\square$

Definition 2.7.5. A topological space is called *Stone* if it is totally disconnected, compact, and Hausdorff. Let $\mathsf{Stone} \hookrightarrow \mathsf{Top}$ be the full subcategory of topological spaces consisting of Stone spaces and continuous functions between them.

In order to further cut down the adjunction on the topological side, we can now restrict it to just the Stone spaces, since we know this subcategory will contain the image of the functor Spec. In fact, up to isomorphism, this is exactly the image:

Theorem 2.7.6. *There is a contravariant equivalence of categories between* Bool *and* Stone*,*

$$\begin{tikzcd} \mathsf{Bool}^{\mathsf{op}} \arrow[r, bend left, "*"] & \mathsf{Stone}\,, \arrow[l, bend left, "*"] \end{tikzcd}$$

with contravariant functors $\mathcal{B}^* = X_{\mathcal{B}}$ the Stone space of a Boolean algebra $\mathcal{B}$, as in Definition 2.7.2, and $X^* = \mathsf{clopen}(X)$, the Boolean algebra of all clopen sets in the Stone space X.

Proof. We just need to show that the two units of the adjunction

$$\eta_{\mathcal{B}} : \mathcal{B} \to \mathsf{Top}\big(\mathsf{Bool}(\mathcal{B}, \mathbb{2}), 2\big),$$
$$\varepsilon_S : S \to \mathsf{Bool}\big(\mathsf{Top}(S, 2), \mathbb{2}\big).$$

are isomorphisms, the second assuming S is a Stone space.

We know by the Stone representation theorem 2.6.8 that $\eta_{\mathcal{B}}$ is an injective Boolean homomorphism, so its image, say

$$\mathcal{B}' \subseteq \mathsf{Top}\big(\mathsf{Bool}(\mathcal{B}, \mathbb{2}), 2\big) \cong \mathsf{Clop}(X_{\mathcal{B}})\,,$$

is a sub-Boolean algebra of the clopen sets of $X_{\mathcal{B}}$. It suffices to show that every clopen set of $X_{\mathcal{B}}$ is in $\mathcal{B}'$. Thus let $K \subseteq X_{\mathcal{B}}$ be clopen, and take $K = \bigcup_i B_i$ a cover by basic opens B_i, all of which, note, are of the form (2.13), and so are in $\mathcal{B}'$. Since K is closed and $X_{\mathcal{B}}$ compact, K is also compact, so there is a finite subcover, each element of which is in $\mathcal{B}'$. Thus their finite union K is also in $\mathcal{B}'$.

Let S be a Stone space and consider the continuous function

$$\varepsilon_S : S \to \mathsf{Bool}\big(\mathsf{Top}(S, 2), 2\big) \cong X_{\mathsf{Clop}(S)}$$

which takes $s \in S$ to the prime filter $F_s = \{K \in \mathsf{Clop}(S) \mid s \in K\}$ of all clopen sets containing it. Since S is Hausdorff, ε_S is a bijection on points, and it is continuous by construction. To see that it is open, let $K \subseteq S$ be a basic clopen set. The complement $S - K$ is therefore closed, and thus compact, and so is its image $\varepsilon_S(S - K)$, which is therefore closed. But since ε_S is a bijection, $\varepsilon_S(S - K)$ is the complement of $\varepsilon_S(K)$, which is therefore open. □

Remark 2.7.7. Another way to cut down the adjunction (2.14),

$$\begin{tikzcd} \mathsf{Bool}^{\mathsf{op}} \arrow[r, bend left, "*"] & \mathsf{Set} \arrow[l, bend left, "*"] \end{tikzcd}$$

to an equivalence is to restrict the Boolean algebra side to *complete, atomic* Boolean algebras CABool and continuous (i.e. $\bigvee$-preserving) homomorphisms between them. One then obtains a duality

$$\mathsf{CABool}^{\mathsf{op}} \simeq \mathsf{Set},$$

between complete, atomic Boolean algebras and sets (see Johnstone [**?**]).

Remark 2.7.8. See Johnstone [**?**] for a more detailed presentation of the material in this section (and much more). Also see [**?**] for a generalization to distributive lattices and Heyting algebras, as well as to "Boolean algebras with operators", i.e. algebraic models of modal logic. For more on logical duality see [**?**]

Chapter 3

Cartesian Closed Categories and λ-Calculus

3.1 Categorification and the Curry-Howard correspondence

Consider the following natural deduction proof in propositional calculus.

$$\dfrac{\dfrac{\dfrac{\dfrac{[(A\wedge B)\wedge(A\Rightarrow B)]^1}{A\wedge B}}{A} \qquad \dfrac{[(A\wedge B)\wedge(A\Rightarrow B)]^1}{A\Rightarrow B}}{B}}{(A\wedge B)\wedge(A\Rightarrow B)\Rightarrow B}\,{\scriptstyle (1)}$$

This deduction shows that

$$\vdash (A\wedge B)\wedge(A\Rightarrow B)\Rightarrow B.$$

But so does the following:

$$\dfrac{\dfrac{\dfrac{[(A\wedge B)\wedge(A\Rightarrow B)]^1}{A\Rightarrow B} \qquad \dfrac{\dfrac{[(A\wedge B)\wedge(A\Rightarrow B)]^1}{A\wedge B}}{A}}{B}}{(A\wedge B)\wedge(A\Rightarrow B)\Rightarrow B}\,{\scriptstyle (1)}$$

As does:

$$\dfrac{\dfrac{\dfrac{[(A\wedge B)\wedge(A\Rightarrow B)]^1}{A\wedge B}}{B}}{(A\wedge B)\wedge(A\Rightarrow B)\Rightarrow B}\,{\scriptstyle (1)}$$

There is a sense in which the first two proofs are "equivalent", but not the first and the third. The relation (or property) of *provability* in propositional calculus $\vdash \phi$ discards such differences in the proofs that witness it. According to the "proof-relevant" point of view, sometimes called *propositions as types*, one retains as relevant some information about the way in which a proposition is proved. This is effected by annotating the proofs with *proof-terms* as they are constructed, as follows:

$$\dfrac{\dfrac{\dfrac{[x:(A\wedge B)\wedge(A\Rightarrow B)]^1}{\pi_2(x):A\Rightarrow B} \qquad \dfrac{\dfrac{[x:(A\wedge B)\wedge(A\Rightarrow B)]^1}{\pi_1(x):A\wedge B}}{\pi_1(\pi_1(x)):A}}{\pi_2(x)(\pi_1(\pi_1(x))):B}}{\lambda x.\pi_2(x)(\pi_1(\pi_1(x))):(A\wedge B)\wedge(A\Rightarrow B)\Rightarrow B}\,(1)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[x:(A\wedge B)\wedge(A\Rightarrow B)]^1}{\pi_1(x):A\wedge B}}{\pi_1(\pi_1(x)):A} \qquad \dfrac{[x:(A\wedge B)\wedge(A\Rightarrow B)]^1}{\pi_2(x):A\Rightarrow B}}{\pi_2(x)(\pi_1(\pi_1(x))):B}}{\lambda x.\pi_2(x)(\pi_1(\pi_1(x))):(A\wedge B)\wedge(A\Rightarrow B)\Rightarrow B}\,(1)$$

$$\dfrac{\dfrac{\dfrac{[x:(A\wedge B)\wedge(A\Rightarrow B)]^1}{\pi_1(x):A\wedge B}}{\pi_2(\pi_1(x)):B}}{\lambda x.\pi_2(\pi_1(x)):(A\wedge B)\wedge(A\Rightarrow B)\Rightarrow B}\,(1)$$

The proof terms for the first two proofs are the same, namely $\lambda x.\pi_2(x)(\pi_1(\pi_1(x)))$, but the term for the third one is $\lambda x.\pi_2(\pi_1(x))$, reflecting the difference in the proofs. The assignment works by labelling assumptions as variables, and then associating term-constructors to the different rules of inference: pairing and projection to conjunction introduction and elimination, function application and λ-abstraction to implication elimination (*modus ponens*) and introduction. The use of variable binding to represent cancellation of premisses is a particularly effective device.

From the categorical point of view, the relation of deducibility $\phi \vdash \psi$ is a mere preorder. The addition of proof terms $x : \phi \vdash t : \psi$ results in a *categorification* of this preorder, in the sense that it is a "proper" category, the preordered reflection of which is the deducibility preorder. And now the following remarkable fact emerges: it is hardly surprising that the deducibility preorder has, say, finite products $\phi \wedge \psi$ or even exponentials $\phi \Rightarrow \psi$; but it is *amazing* that the category with proof terms $x : \phi \vdash t : \psi$ as arrows, also turns out to be a cartesian closed category, and indeed a proper one, with distinct parallel arrows, such as

$$\begin{aligned} \pi_2(x)(\pi_1(\pi_1(x))) &: (A\wedge B)\wedge(A\Rightarrow B) \longrightarrow B, \\ \pi_2(\pi_1(x)) &: (A\wedge B)\wedge(A\Rightarrow B) \longrightarrow B. \end{aligned}$$

This *category of proofs* contains information about the "proof theory" of the propositional calculus, as opposed to its mere relation of deducibility. The calculus of proof terms can be presented formally in a system of *simple type theory*, with an alternate interpretation as a formal system of function application and abstraction. This dual interpretation—as the proof theory of propositional logic, and as a system of type theory for the specification of functions—is called the *Curry-Howard correspondence* []. From the categorical point of view, it expresses the structural equivalence between the cartesian closed categories of proofs in propositional logic and terms in simple type theory. Both of these can be seen as categorifications of their preorder reflection, the deducibility preorder of propositional logic (cf. [**?**]).

In the following sections, we shall consider this remarkable correspondence in detail, as well as some extensions of the basic case represented by cartesian closed categories: categories with coproducts, cocomplete categories, and categories equipped with modal operators. In the next chapter, it will be seen that this correspondence even extends to proofs in quantified predicate logic and terms in dependent type theory, and beyond.

3.2 Cartesian closed categories

Exponentials

We begin with the notion of an exponential B^A of two objects A, B in a category, motivated by a couple of important examples. Consider first the category Pos of posets and monotone functions. For posets P and Q the set $\mathsf{Hom}(P, Q)$ of all monotone functions between them is again a poset, with the pointwise order:

$$f \leq g \iff fx \leq gx \quad \text{for all } x \in P . \qquad (f, g : P \to Q)$$

Thus $\mathsf{Hom}(P, Q)$ is again an object of Pos, when equipped with a suitable order.

Similarly, given monoids $K, M \in \mathsf{Mon}$, there is a natural monoid structure on the set $\mathsf{Hom}(K, M)$, defined pointwise by

$$(f \cdot g)x = fx \cdot gx . \qquad (f, g : K \to M,\, x \in K)$$

Thus the category Mon also admits such "internal Hom"s. The same thing works in the category Group of groups and group homomophisms, where the set $\mathsf{Hom}(G, H)$ of all homomorphisms between groups G and H can be given a pointwise group structure.

These examples suggest a general notion of "internal Hom" in a category: an "object of morphisms $A \to B$" which corresponds to the hom-set $\mathsf{Hom}(A, B)$. The other ingredient needed is an "evaluation" operation $\epsilon : B^A \times A \to B$ which evaluates a morphism $f \in B^A$ at an argument $x \in A$ to give a value $\epsilon \circ \langle f, x \rangle \in B$. This is always going to be present for the underlying functions if we're starting from a set of functions $\mathsf{Hom}(A, B)$, but it needs to be an actual morphism in the category. Finally, we need an operation of "transposition", taking a morphism $f : C \times A \to B$ to one $\widetilde{f} : C \to A^B$. We shall see that this in fact separates the previous two examples.

Definition 3.2.1. In a category $\mathcal{C}$ with binary products, an *exponential* (B^A, ϵ) of objects A and B is an object B^A together with a morphism $\epsilon : B^A \times A \to B$, called the *evaluation* morphism, such that for every $f : C \times A \to B$ there exists a *unique* morphism $\widetilde{f} : C \to B^A$, called the *transpose*[1] of f, for which the following diagram commutes.

$$\begin{tikzcd} B^A & B^A \times A \arrow[r, "\epsilon"] & B \\ C \arrow[u, dotted, "\widetilde{f}"] & C \times A \arrow[u, "\widetilde{f} \times 1_A"] \arrow[ur, "f"'] & \end{tikzcd}$$

Commutativity of the diagram of course means that $f = \epsilon \circ (\widetilde{f} \times 1_A)$.

Definition 3.2.1 is called the *universal property of the exponential.* It is just the category-theoretic way of saying that a function $f : C \times A \to B$ of two variables can be viewed as a function $\widetilde{f} : C \to B^A$ of one variable that maps $z \in C$ to a function $\widetilde{f}z = f\langle z, -\rangle : A \to B$ that maps $x \in A$ to $f\langle z, x\rangle$. The relationship between f and $\widetilde{f}$ is then

$$f\langle z, x\rangle = (\widetilde{f}z)x \ .$$

That is all there is to it, really, except that variables and elements never need to be mentioned. The benefit of this is that the definition is applicable also in categories whose objects are not *sets* and whose morphisms are not *functions*—even though some of the basic examples are of that sort.

In Poset the exponential Q^P of posets P and Q is the set of all monotone maps $P \to Q$, ordered pointwise, as above. The evaluation map $\epsilon : Q^P \times P \to Q$ is just the usual evaluation of a function at an argument. The transpose of a monotone map $f : R \times P \to Q$ is the map $\widetilde{f} : R \to Q^P$, defined by, $(\widetilde{f}z)x = f\langle z, x\rangle$, i.e. the transposed *function.* We say that the category Pos *has all exponentials.*

Definition 3.2.2. Suppose $\mathcal{C}$ has all finite products. An object $A \in \mathcal{C}$ is *exponentiable* when the exponential B^A exists for every $B \in \mathcal{C}$. We say that $\mathcal{C}$ *has exponentials* if every object is exponentiable. A *cartesian closed category (ccc)* is a category that has all finite products and exponentials.

Example 3.2.3. Consider again the example of the set $\mathsf{Hom}(M, N)$ of homomorphisms between two monoids M, N, equipped with the pointwise monoid structure. To be a monoid homomorphism. the transpose $\widetilde{h} : 1 \to \mathsf{Hom}(M, N)$ of a homomorphism $h : 1 \times M \to N$ would have to take the unit element $u \in 1$ to the unit homomorphism $u : M \to N$, which is the constant function at the unit $u \in N$. Since $1 \times M \cong M$, that would mean that *all* homomorphisms $h : M \to N$ would have the same transpose $\widetilde{h} = u : 1 \to \mathsf{Hom}(M, N)$. So Mon cannot be cartesian closed. The same argument works in the category Group, and in many related ones. (But see **??** below on one way of embedding Group into a CCC.)

Exercise 3.2.4. Is the evaluation function $\mathsf{eval} : \mathsf{Hom}(M, N) \times M \to N$ a homomorphism of monoids?

[1]Also, f is called the transpose of $\widetilde{f}$, so that f and $\widetilde{f}$ are each other's transpose.

Two characterizations of CCCs

Proposition 3.2.5. *In a category $\mathcal{C}$ with binary products an object A is exponentiable if, and only if, the functor*

$$- \times A : \mathcal{C} \to \mathcal{C}$$

has a right adjoint

$$-^A : \mathcal{C} \to \mathcal{C} \; .$$

Proof. If such a right adjoint exists then the exponential of A and B is (B^A, ϵ_B), where $\epsilon : -^A \times A \Longrightarrow \mathsf{1}_{\mathcal{C}}$ is the counit of the adjunction. The universal property of the exponential is precisely the universal property of the counit ϵ.

Conversely, suppose for every B there is an exponential (B^A, ϵ_B). As the object part of the right adjoint we then take B^A. For the morphism part, given $g : B \to C$, we can define $g^A : B^A \to C^A$ to be the transpose of $g \circ \epsilon_B$,

$$g^A = (g \circ \epsilon_B)^{\sim}$$

as indicated below.

$$\begin{tikzcd} B^A \times A \arrow[r, "\epsilon_B"] \arrow[d, "g^A \times \mathsf{1}_A"'] & B \arrow[d, "g"] \\ C^A \times A \arrow[r, "\epsilon_C"'] & C \end{tikzcd} \tag{3.1}$$

The counit $\epsilon : -^A \times A \Longrightarrow \mathsf{1}_{\mathcal{C}}$ at B is then ϵ_B itself, and the naturality square for ϵ is then exactly (3.1), i.e. the defining property of $(f \circ \epsilon_B)^{\sim}$:

$$\epsilon_C \circ (g^A \times \mathsf{1}_A) = \epsilon_C \circ ((g \circ \epsilon_B)^{\sim} \times \mathsf{1}_A) = g \circ \epsilon_B \; .$$

The universal property of the counit ϵ is precisely the universal property of the exponential (B^A, ϵ_B) □

Note that because exponentials can be expressed as right adjoints to binary products, they are determined uniquely up to isomorphism. Moreover, the definition of a cartesian closed category can then be phrased entirely in terms of adjoint functors: we just need to require the existence of the terminal object, binary products, and exponentials.

Proposition 3.2.6. *A category $\mathcal{C}$ is cartesian closed if, and only if, the following functors have* right *adjoints:*

$$\begin{aligned} !_{\mathcal{C}} &: \mathcal{C} \to \mathsf{1} \; , \\ \Delta &: \mathcal{C} \to \mathcal{C} \times \mathcal{C} \; , \\ (- \times A) &: \mathcal{C} \to \mathcal{C} \; . \end{aligned} \qquad (A \in \mathcal{C})$$

Here $!_{\mathcal{C}}$ is the unique functor from $\mathcal{C}$ to the terminal category $\mathsf{1}$ and Δ is the diagonal functor $\Delta A = \langle A, A \rangle$, and the right adjoint of $- \times A$ is exponentiation by A.

$\square$

The significance of the adjoint formulation is that it implies the possibility of a purely *equational* specification (adjoint structure on a category is "equational" in a sense that can be made precise; see [**?**]). We can therefore give an explicit, equational formulation of cartesian closed categories.

Proposition 3.2.7 (Equational version of CCC)**.** *A category $\mathcal{C}$ is cartesian closed if, and only if, it has the following structure:*

1. *An object $\mathsf{1} \in \mathcal{C}$ and a morphism $!_A : A \to \mathsf{1}$ for every $A \in \mathcal{C}$.*
2. *An object $A \times B$ for all $A, B \in \mathcal{C}$ together with morphisms $\pi_0 : A \times B \to A$ and $\pi_1 : A \times B \to B$, and for every pair of morphisms $f : C \to A$, $g : C \to B$ a morphism $\langle f, g\rangle : C \to A \times B$.*
3. *An object B^A for all $A, B \in \mathcal{C}$ together with a morphism $\epsilon : B^A \times A \to B$, and a morphism $\widetilde{f} : C \to B^A$ for every morphism $f : C \times A \to B$.*

These new objects and morphisms are required to satisfy the following equations:

1. *For every $f : A \to \mathsf{1}$,*
$$f = !_A \,.$$
2. *For all $f : C \to A$, $g : C \to B$, $h : C \to A \times B$,*
$$\pi_0 \circ \langle f, g\rangle = f \,, \qquad \pi_1 \circ \langle f, g\rangle = g \,, \qquad \langle \pi_0 \circ h, \pi_1 \circ h\rangle = h \,.$$
3. *For all $f : C \times A \to B$, $g : C \to B^A$,*
$$\epsilon \circ (\widetilde{f} \times \mathsf{1}_A) = f \,, \qquad (\epsilon \circ (g \times \mathsf{1}_A))^{\sim} = g \,.$$
where for $e : E \to E'$ and $f : F \to F'$ we define
$$e \times f := \langle e\pi_0, f\pi_1\rangle : E \times F \to E' \times F'.$$

These equations ensure that certain diagrams commute and that the morphisms that are required to exist are unique. For example, let us prove that $(A \times B, \pi_0, \pi_1)$ is the product of A and B. For $f : C \to A$ and $g : C \to B$ there exists a morphism $\langle f, g\rangle : C \to A \times B$. Equations

$$\pi_0 \circ \langle f, g\rangle = f \qquad \text{and} \qquad \pi_1 \circ \langle f, g\rangle = g$$

enforce the commutativity of the two triangles in the following diagram:

$$\begin{tikzcd}
 & C \arrow[dl, "g"'] \arrow[d, "\langle f,g\rangle"] \arrow[dr, "f"] & \\
A & A \times B \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & B
\end{tikzcd}$$

Suppose $h : C \to A \times B$ is another morphism such that $f = \pi_0 \circ h$ and $g = \pi_1 \circ h$. Then by the third equation for products we get

$$h = \langle \pi_0 \circ h, \pi_1 \circ h \rangle = \langle f, g \rangle \ ,$$

and so $\langle f, g \rangle$ is unique.

Exercise 3.2.8. Use the equational characterization of CCCs, Proposition 3.2.7, to show that the category Pos of posets and monotone functions *is* cartesian closed, as claimed. Also verify that that Mon is not. Which parts of the definition fail in Mon?

3.3 Positive propositional calculus

We begin with the example of a cartesian closed poset and a first application to propositional logic.

Example 3.3.1. Consider the *positive propositional calculus* PPC with conjunction and implication, as in Section 2.1. Recall that PPC is the set of all propositional formulas ϕ constructed from propositional variables $p_1, p_2, \ldots$, a constant $\top$ for truth, and binary connectives for conjunction $\phi \wedge \psi$, and implication $\phi \Rightarrow \psi$.

As a category, PPC is a preorder under the relation $\phi \vdash \psi$ of logical entailment, determined for instance by the natural deduction system **??** of section **??**. As usual, it will be convenient to pass to the poset reflection of the preorder, which we shall denote by

$$\mathcal{C}_{\mathsf{PPC}}$$

by identifying ϕ and ψ when $\phi \dashv\vdash \psi$. (This is just the usual *Lindenbaum-Tarski* algebra of the system of propositional logic, as in Section 2.5.)

The conjunction $\phi \wedge \psi$ is a greatest lower bound of ϕ and ψ in $\mathcal{C}_{\mathsf{PPC}}$, because we have $\phi \wedge \psi \vdash \phi$ and $\phi \wedge \psi \vdash \psi$ and for all ϑ, if $\vartheta \vdash \phi$ and $\vartheta \vdash \psi$ then $\vartheta \vdash \phi \wedge \psi$. Since binary products in a poset are the same thing as greatest lower bounds, we see that $\mathcal{C}_{\mathsf{PPC}}$ has all binary products; and of course $\top$ is a terminal object.

We have already remarked that implication is right adjoint to conjunction in propositional calculus,

$$(-) \wedge \phi \ \dashv \ \phi \Rightarrow (-) \ . \tag{3.2}$$

Therefore $\phi \Rightarrow \psi$ is an exponential in $\mathcal{C}_{\mathsf{PPC}}$. The counit of the adjunction (the "evaluation" arrow) is the entailment

$$(\phi \Rightarrow \psi) \wedge \phi \vdash \psi \ ,$$

i.e. the familiar logical rule of *modus ponens*.

We have now shown:

Proposition 3.3.2. *The poset $\mathcal{C}_{\mathsf{PPC}}$ of positive propositional calculus is cartesian closed.*

Let us now use this fact to show that the positive propositional calculus is *deductively complete* with respect to the following notion of *Kripke semantics* [].

Definition 3.3.3 (Kripke model)**.** Let K be a poset. Suppose we have a relation

$$k \Vdash p$$

between elements $k \in K$ and propositional variables p, such that

$$j \leq k,\ k \Vdash p \quad \text{implies} \quad j \Vdash p. \tag{3.3}$$

Extend $\Vdash$ to all formulas ϕ in PPC by defining

$$\begin{array}{lll} k \Vdash \top & \text{always,} & \\ k \Vdash \phi \wedge \psi & \text{iff} & k \Vdash \phi \text{ and } k \Vdash \psi\,, \\ k \Vdash \phi \Rightarrow \psi & \text{iff} & \text{for all } j \leq k, \text{ if } j \Vdash \phi, \text{ then } j \Vdash \psi\,. \end{array} \tag{3.4}$$

Finally, say that ϕ *holds on* K, written

$$K \Vdash \phi$$

if $k \Vdash \phi$ for all $k \in K$ (for all such relations $\Vdash$).

Theorem 3.3.4 (Kripke completeness for PPC)**.** *A propositional formulas ϕ is provable from the rules of deduction for PPC if, and only if, $K \Vdash \phi$ for all posets K. Briefly:*

$$\mathsf{PPC} \vdash \phi \quad \textit{iff} \quad K \Vdash \phi \textit{ for all } K.$$

We will require the following (which extends the discussion in Section 2.6).

Lemma 3.3.5. *For any poset P, the poset $\downarrow\! P$ of all downsets in P, ordered by inclusion, is cartesian closed. Moreover, the downset embedding,*

$$\downarrow\!(-) : P \to \downarrow\! P$$

preserves any CCC structure that exists in P.

Proof. The total downset P is obviously terminal, and for any downsets $S, T \in \downarrow\! P$, the intersection $S \cap T$ is also closed down, so we have the products $S \wedge T = S \cap T$. For the exponential, set

$$S \Rightarrow T = \{p \in P \mid \ \downarrow\!(p) \cap S \subseteq T\}.$$

Then for any downset Q we have

$$Q \subseteq S \Rightarrow T \quad \text{iff} \quad \downarrow\!(q) \cap S \subseteq T, \text{ for all } q \in Q. \tag{3.5}$$

But that means that

$$\textstyle\bigcup_{q \in Q} (\downarrow\!(q) \cap S) \subseteq T\,,$$

which is equivalent to $Q \cap S \subseteq T$, since $\bigcup_{q \in Q}(\downarrow(q) \cap S) = (\bigcup_{q \in Q} \downarrow(q)) \cap S = Q \cap S$.

The preservation of CCC structure by $\downarrow(-) : P \to \downarrow P$ follows from its preservation by the Yoneda embedding, of which $\downarrow(-)$ is a factor,

$$\begin{tikzcd} & \mathsf{Set}^{P^{\mathsf{op}}} \\ P \arrow[ur, hook, "\mathsf{y}"] \arrow[r, "\downarrow(-)"'] & \downarrow P \arrow[u, hook] \end{tikzcd}$$

But it is also easy enough to check directly: preservation of any limits 1, $p \wedge q$ that exist in P are clear. Suppose $p \Rightarrow q$ is an exponential; then for any downset D we have:

$$\begin{array}{lll} D \subseteq \downarrow(p \Rightarrow q) & \text{iff} & \downarrow(d) \subseteq \downarrow(p \Rightarrow q)\ , \text{ for all } d \in D \\ & \text{iff} & d \leq p \Rightarrow q\ , \text{ for all } d \in D \\ & \text{iff} & d \wedge p \leq q\ , \text{ for all } d \in D \\ & \text{iff} & \downarrow(d \wedge p) \subseteq \downarrow(q)\ , \text{ for all } d \in D \\ & \text{iff} & \downarrow(d) \cap \downarrow(p) \subseteq \downarrow(q)\ , \text{ for all } d \in D \\ & \text{iff} & D \subseteq \downarrow(p) \Rightarrow \downarrow(q) \end{array}$$

where the last line is by (3.5). (Note that in line (3) we assumed that $d \wedge p$ exists for all $d \in D$; this can be avoided by a slightly more complicated argument.) □

Proof. (of Theorem 3.3.4) The proof follows a now-familiar pattern, which we only sketch:

1. The syntactic category $\mathcal{C}_{\mathsf{PPC}}$ is a CCC, with $\top = 1$, $\phi \times \psi = \phi \wedge \psi$, and $\psi^{\phi} = \phi \Rightarrow \psi$. In fact, it is the free cartesian closed poset on the generating set $\mathsf{Var} = \{p_1, p_2, \dots\}$ of propositional variables.

2. A (Kripke) model $(K, \Vdash)$ is the same thing as a CCC functor $\mathcal{C}_{\mathsf{PPC}} \to \downarrow K$, which by Step 1 is just an arbitrary map $\mathsf{Var} \to \downarrow K$, as in (3.3). To see this, observe that we have a bijective correspondence between CCC functors $[\![-]\!]$ and Kripke relations $\Vdash$; indeed, by the exponential adjunction in the cartesian closed category Pos, there is a natural bijection,
$$\frac{[\![-]\!] : \mathcal{C}_{\mathsf{PPC}} \longrightarrow \downarrow K \cong 2^{K^{\mathsf{op}}}}{\Vdash : K^{\mathsf{op}} \times \mathcal{C}_{\mathsf{PPC}} \longrightarrow 2}$$
where we use the poset 2 to classify downsets in a poset P (via upsets in P^{op}),
$$\downarrow P \cong 2^{P^{\mathsf{op}}} \cong \mathsf{Pos}(P^{\mathsf{op}}, 2)\,,$$
by taking the 1-kernel $f^{-1}(1) \subseteq P$ of a monotone map $f : P^{\mathsf{op}} \to 2$. (The contravariance will be convenient in Step 3). Note that the monotonicity of $\Vdash$ yields the conditions
$$p \leq q\,,\ q \Vdash \phi \implies p \Vdash \phi$$

and

$$p \Vdash \phi\,,\ \phi \vdash \psi \implies p \Vdash \psi\,.$$

and the CCC preservation of the transpose $[\![-]\!]$ yields the Kripke forcing conditions (3.4) (exercise!).

3. For any model $(K, \Vdash)$, by the adjunction in (2) we then have $K \Vdash \phi$ iff $[\![\phi]\!] = K$, the total downset.

4. Because the downset/Yoneda embedding $\downarrow$ preserves the CCC structure (by Lemma 3.3.5), $\mathcal{C}_{\mathsf{PPC}}$ has a *canonical model*, namely $(\mathcal{C}_{\mathsf{PPC}}, \Vdash)$, where:

$$\frac{\downarrow(-)\ :\ \mathcal{C}_{\mathsf{PPC}} \longrightarrow {\downarrow}\mathcal{C}_{\mathsf{PPC}} \cong 2^{\mathcal{C}_{\mathsf{PPC}}{}^{\mathsf{op}}} \hookrightarrow \mathsf{Set}^{\mathcal{C}_{\mathsf{PPC}}{}^{\mathsf{op}}}}{\Vdash\ :\ \mathcal{C}_{\mathsf{PPC}}{}^{\mathsf{op}} \times \mathcal{C}_{\mathsf{PPC}} \longrightarrow 2 \hookrightarrow \mathsf{Set}}$$

5. Now note that for the Kripke relation $\Vdash$ in (4), we have $\Vdash\, = \,\vdash$, since it's essentially the transpose of the Yoneda embedding. Thus the model is logically generic, in the sense that $\mathcal{C}_{\mathsf{PPC}} \Vdash \phi$ iff $\mathsf{PPC} \vdash \phi$.

$\square$

Exercise 3.3.6. Verify the claim that CCC preservation of the transpose $[\![-]\!]$ of $\Vdash$ yields the Kripke forcing conditions (3.4).

Exercise 3.3.7. Give a countermodel to show that $\mathsf{PPC} \nvdash (\phi \Rightarrow \psi) \Rightarrow \phi$

3.4 Heyting algebras

We now extend the positive propositional calculus to the full intuitionistic propositional calculus. This involves adding the finite coproducts 0 and $p \vee q$ to notion of a cartesian closed poset, to arrive at the general notion of a Heyting algebra. Heyting algebras are to intuitionistic logic as Boolean algebras are to classical logic: each is an algebraic description of the corresponding logical calculus. We shall review both the algebraic and the logical points of view; as we shall see, many aspects of the theory of Boolean algebras carry over to Heyting algebras. For instance, in order to prove the Kripke completeness of the full system of intuitionistic propositional calculus, we will need an alternative to Lemma 3.3.5, because the Yoneda embedding does not in general preserve coproducts. For that we will again use a version of the Stone representation theorem, this time in a generalized form due to Joyal.

Distributive lattices

Recall first that a (bounded) *lattice* is a poset that has finite limits and colimits. In other words, a lattice $(L, \leq, \wedge, \vee, 1, 0)$ is a poset $(L, \leq)$ with distinguished elements $1, 0 \in L$, and binary operations meet $\wedge$ and join $\vee$, satisfying for all $x, y, z \in L$,

$$0 \leq x \leq 1 \qquad \frac{z \leq x \qquad z \leq y}{\overline{z \leq x \wedge y}} \qquad \frac{x \leq z \qquad x \leq y}{\overline{x \vee y \leq z}}$$

A *lattice homomorphism* is a function $f : L \to K$ between lattices which preserves finite limits and colimits, i.e., $f0 = 0$, $f1 = 1$, $f(x \wedge y) = fx \wedge fy$, and $f(x \vee y) = fx \vee fy$. The category of lattices and lattice homomorphisms is denoted by Lat.

A lattice can be axiomatized equationally as a set with two distinguished elements 0 and 1 and two binary operations $\wedge$ and $\vee$, satisfying the following equations:

$$\begin{aligned}
(x \wedge y) \wedge z &= x \wedge (y \wedge z) \;, & (x \vee y) \vee z &= x \vee (y \vee z) \;, \\
x \wedge y &= y \wedge x \;, & x \vee y &= y \vee x \;, \\
x \wedge x &= x \;, & x \vee x &= x \;, \\
1 \wedge x &= x \;, & 0 \vee x &= x \;, \\
\end{aligned}$$
$$x \wedge (y \vee x) = x = (x \wedge y) \vee x \;. \tag{3.6}$$

The partial order on L is then determined by

$$x \leq y \iff x \wedge y = x \;.$$

Exercise 3.4.1. Show that in a lattice $x \leq y$ if, and only if, $x \wedge y = x$ if, and only if, $x \vee y = y$.

A lattice is *distributive* if the following distributive laws hold in it:

$$\begin{aligned}
(x \vee y) \wedge z &= (x \wedge z) \vee (y \wedge z) \;, \\
(x \wedge y) \vee z &= (x \vee z) \wedge (y \vee z) \;.
\end{aligned} \tag{3.7}$$

It turns out that if one distributive law holds then so does the other [**?**, I.1.5].

A *Heyting algebra* is a cartesian closed lattice H. This means that it has an operation $\Rightarrow$, satisfying for all $x, y, z \in H$

$$\frac{z \wedge x \leq y}{\overline{z \leq x \Rightarrow y}}$$

A *Heyting algebra homomorphism* is a lattice homomorphism $f : K \to H$ between Heyting algebras that preserves implication, i.e., $f(x \Rightarrow y) = (fx \Rightarrow fy)$. The category of Heyting algebras and their homomorphisms is denoted by Heyt.

Heyting algebras can be axiomatized equationally as a set H with two distinguished elements 0 and 1 and three binary operations $\wedge$, $\vee$ and $\Rightarrow$. The equations for a Heyting algebra are the ones listed in (3.6), as well as the following ones for $\Rightarrow$.

$$\begin{aligned}(x \Rightarrow x) &= 1 \;, \\ x \wedge (x \Rightarrow y) &= x \wedge y \;, \\ y \wedge (x \Rightarrow y) &= y \;, \\ (x \Rightarrow (y \wedge z)) &= (x \Rightarrow y) \wedge (x \Rightarrow z) \;.\end{aligned} \tag{3.8}$$

For a proof, see [**?**, I.1], where one can also find a proof that every Heyting algebra is distributive (exercise!).

Example 3.4.2. We know from Lemma 3.3.5 that for any poset P, the poset $\downarrow P$ of all downsets in P, ordered by inclusion, is cartesian closed. Moreover, we know that $\downarrow P \cong \mathbb{2}^{P^{\mathsf{op}}}$, as a poset, with the reverse pointwise ordering on monotone maps $P^{\mathsf{op}} \to \mathbb{2}$, or equivalently, $\downarrow P \cong \mathbb{2}^{P}$, with the functions ordered pointwise. Since $\mathbb{2}$ is a lattice, we can also take joins $f \vee g$ pointwise, in order to get joins in $\mathbb{2}^P$, which then correspond to finite unions of the corresponding downsets $f^{-1}\{0\} \cup g^{-1}\{0\}$. Thus, in sum, for any poset P, the lattice $\downarrow P \cong \mathbb{2}^P$ is a Heyting algebra, with the downsets ordered by inclusion, and the functions ordered pointwise.

Intuitionistic propositional calculus

There is a forgetful functor $U : \mathsf{Heyt} \to \mathsf{Set}$ which maps a Heyting algebra to its underlying set, and a homomorphism of Heyting algebras to the underlying function. Because Heyting algebras are models of an equational theory, there is a left adjoint $H \dashv U$, which is the usual "free" construction mapping a set S to the free Heyting algebra HS generated by it. As for all algebraic strictures, the construction of HS can be performed in two steps: first, define a set HS of formal expressions, and then quotient it by an equivalence relation generated by the axioms for Heyting algebras.

Thus let HS be the set of formal expressions generated inductively by the following rules:

1. Generators: if $x \in S$ then $x \in HS$.

2. Constants: $\bot, \top \in HS$.

3. Connectives: if $\phi, \psi \in HS$ then $(\phi \wedge \psi), (\phi \vee \psi), (\phi \Rightarrow \psi) \in HS$.

We impose an equivalence relation on HS, which we write as equality $=$ and think of as such; it is defined as the smallest equivalence relation satisfying axioms (3.6) and (3.8). This forces HS to be a Heyting algebra. We define the action of the functor H on morphisms as usual: a function $f : S \to T$ is mapped to the Heyting algebra morphism $Hf : HS \to HT$ defined by

$$(Hf)\bot = \bot \;, \qquad (Hf)\bot = \bot \;, \qquad (Hf)x = fx \;,$$
$$(Hf)(\phi \star \psi) = ((Hf)\phi) \star ((Hf)\psi) \;,$$

where $\star$ stands for $\wedge$, $\vee$ or $\Rightarrow$.

The inclusion $\eta_S : S \to U(HS)$ of generators into the underlying set of the free Heyting algebra HS is then the component at S of a natural transformation $\eta : 1_{\mathsf{Set}} \Longrightarrow U \circ H$, which is of course the unit of the adjunction $H \dashv U$. To see this, consider a Heyting algebra K and an arbitrary function $f : S \to UK$. Then the Heyting algebra homomorphism $\overline{f} : HS \to K$ defined by

$$\overline{f}\bot = \bot\ , \qquad \overline{f}\bot = \bot\ , \qquad \overline{f}x = fx\ ,$$
$$\overline{f}(\phi \star \psi) = (\overline{f}\phi) \star (\overline{f}\psi)\ ,$$

where $\star$ stands for $\wedge$, $\vee$ or $\Rightarrow$, makes the following triangle commute:

$$\begin{tikzcd}
S \arrow[r, "\eta_S"] \arrow[dr, "f"'] & U(HS) \arrow[d, "U\overline{f}"] \\
& K
\end{tikzcd}$$

It is the unique such morphism because any two homomorphisms from HS which agree on generators must be equal. This is proved by induction on the structure of the formal expressions in HS.

We may now define the *intuitionistic propositional calculus* IPC to be the free Heyting algebra IPC on countably many generators p_0, p_1, $\ldots$, called *atomic propositions* or *propositional variables*. This is a somewhat unorthodox definition from a logical point of view—normally we would start from a *calculus* consisting of a formal language, judgements, and rules of inference—but of course, by now, we realize that the two approaches are essentially equivalent.

Having said that, let us also describe IPC in the conventional way. The formulas of IPC are built inductively from propositional variables p_0, p_1, $\ldots$, constants falsehood $\bot$ and truth $\top$, and binary operations conjunction $\wedge$, disjunction $\vee$ and implication $\Rightarrow$. The basic judgment of IPC is *logical entailment*

$$u_1 : A_1, \ldots, u_k : A_k \vdash B$$

which means "hypotheses A_1, $\ldots$, A_k entail proposition B". The hypotheses are labeled with distinct labels u_1, $\ldots$, u_k so that we can distinguish them, which is important when the same hypothesis appears more than once. Because the hypotheses are labeled it is irrelevant in what order they are listed, as long as the labels are not getting mixed up. Thus, the hypotheses $u_1 : A \vee B, u_2 : B$ are the same as the hypotheses $u_2 : B, u_1 : A \vee B$, but different from the hypotheses $u_1 : B, u_2 : A \vee B$. Sometimes we do not bother to label the hypotheses.

The left-hand side of a logical entailment is called the *context* and the right-hand side is the *conclusion*. Thus logical entailment is a relation between contexts and conclusions. The context may be empty. If Γ is a context, u is a label which does not occur in Γ, and A is a formula, then we write $\Gamma, u : A$ for the context Γ extended by the hypothesis $u : A$.

Definition 3.4.3. *Deductive entailment* is the smallest relation satisfying the following rules:

1. Conclusion from a hypothesis:
$$\frac{}{\Gamma \vdash A} \text{ if } u : A \text{ occurs in } \Gamma$$

2. Truth:
$$\frac{}{\Gamma \vdash \top}$$

3. Falsehood:
$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A}$$

4. Conjunction:
$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}$$

5. Disjunction:
$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \qquad \frac{\Gamma \vdash A \vee B \qquad \Gamma, u : A \vdash C \qquad \Gamma, v : B \vdash C}{\Gamma \vdash C}$$

6. Implication:
$$\frac{\Gamma, u : A \vdash B}{\Gamma \vdash A \Rightarrow B} \qquad \frac{\Gamma \vdash A \Rightarrow B \qquad \Gamma \vdash A}{\Gamma \vdash B}$$

A *proof* of $\Gamma \vdash A$ is a *finite* tree built from the above inference rules whose root is $\Gamma \vdash A$. A judgment $\Gamma \vdash A$ is *provable* if there exists a proof of it. Observe that every proof has at its leaves either the rule for $\top$ or a conclusion from a hypothesis.

You may wonder what happened to negation. In intuitionistic propositional calculus, negation is defined in terms of implication and falsehood as

$$\neg A \;\equiv\; A \Rightarrow \bot \,.$$

Properties of negation are then derived from the rules for implication and falsehood, see Exercise 3.4.7

Let P be the set of all formulas of IPC, preordered by the relation

$$A \vdash B \,, \qquad (A, B \in P)$$

where we did not bother to label the hypothesis A. Clearly, it is the case that $A \vdash A$. To see that $\vdash$ is transitive, suppose Π_1 is a proof of $A \vdash B$ and Π_2 is a proof of $B \vdash C$. Then we can obtain a proof of $A \vdash C$ from a proof Π_2 of $B \vdash C$ by replacing in it each use of the hypothesis B by the proof Π_1 of $A \vdash B$. This is worked out in detail in the next two exercises.

Exercise 3.4.4. Prove the following statement by induction on the structure of the proof Π: if Π is a proof of $\Gamma, u : A, v : A \vdash B$ then there is a proof of $\Gamma, u : A \vdash B$.

Exercise 3.4.5. Prove the following statement by induction on the structure of the proof Π_2: if Π_1 is a proof of $\Gamma \vdash A$ and Π_2 is a proof of $\Gamma, u : A \vdash B$, then there is a proof of $\Gamma \vdash B$.

Let IPC be the poset reflection of the preorder $(P, \vdash)$. The elements of IPC are equivalence classes $[A]$ of formulas, where two formulas A and B are equivalent if both $A \vdash B$ and $B \vdash A$ are provable. The poset IPC is just the free Heyting algebra on countably many generators p_0, p_1, ...

Classical propositional calculus

Another look:

An element $x \in L$ of a lattice L is said to be *complemented* when there exists $y \in L$ such that

$$x \vee y = 1 , \qquad\qquad x \wedge y = 0 .$$

We say that y is the *complement* of x.

In a distributive lattice, the complement of x is unique if it exists. Indeed, if both y and z are complements of x then

$$y \wedge z = (y \wedge z) \vee 0 = (y \wedge z) \vee (y \wedge x) = y \wedge (z \vee x) = y \wedge 1 = y ,$$

hence $y \leq z$. A symmetric argument shows that $z \leq y$, therefore $y = z$. The complement of x, if it exists, is denoted by $\neg x$.

A *Boolean algebra* is a distributive lattice in which every element is complemented. In other words, a Boolean algebra B has the *complementation* operation $\neg$ which satisfies, for all $x \in B$,

$$x \wedge \neg x = 0 , \qquad\qquad x \vee \neg x = 1 . \tag{3.9}$$

The full subcategory of Lat consisting of Boolean algebras is denoted by Bool.

Exercise 3.4.6. Prove that every Boolean algebra is a Heyting algebra. Hint: how is implication encoded in terms of negation and disjunction in classical logic?

In a Heyting algebra not every element is complemented. However, we can still define a *pseudo complement* or *negation* operation $\neg$ by

$$\neg x = (x \Rightarrow 0) ,$$

Then $\neg x$ is the largest element for which $x \wedge \neg x = 0$. While in a Boolean algebra $\neg\neg x = x$, in a Heyting algebra we only have $\neg\neg x \leq x$ in general. An element x of a Heyting algebra for which $\neg\neg x = x$ is called a *regular* element.

Exercise 3.4.7. Derive the following properties of negation in a *Heyting* algebra:

$$\begin{aligned} x &\leq \neg\neg x \ , \\ \neg x &= \neg\neg\neg x \ , \\ x \leq y &\Rightarrow \neg y \leq \neg x \ , \\ \neg\neg(x \wedge y) &= \neg\neg x \wedge \neg\neg y \ . \end{aligned}$$

Exercise 3.4.8. Prove that the topology $\mathcal{O}X$ of any topological space X is a Heyting algebra. Describe in topological language the implication $U \Rightarrow V$, the negation $\neg U$, and the regular elements $U = \neg\neg U$ in $\mathcal{O}X$.

Exercise 3.4.9. Show that for a Heyting algebra H, the regular elements of H form a Boolean algebra $H_{\neg\neg} = \{x \in H \mid x = \neg\neg x\}$. Here $H_{\neg\neg}$ is viewed as a subposet of H. Hint: negation $\neg'$, conjunction $\wedge'$, and disjunction $\vee'$ in $H_{\neg\neg}$ are expressed as follows in terms of negation, conjunction and disjunction in H, for $x, y \in H_{\neg\neg}$:

$$\neg' x = \neg x \ , \qquad x \wedge' y = \neg\neg(x \wedge y) \ , \qquad x \vee' y = \neg\neg(x \vee y) \ .$$

The *classical propositional calculus (CPC)* is obtained from the intuitionistic propositional calculus by the addition of the logical rule known as *tertium non datur*, or the *law of excluded middle*:

$$\frac{}{\Gamma \vdash A \vee \neg A}$$

Alternatively, we could add the law known as *reductio ad absurdum*, or *proof by contradiction*:

$$\frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A} \ .$$

Identifying logically equivalent formulas of CPC, we obtain a poset CPC ordered by logical entailment. This poset is the *free Boolean algebra* on countably many generators. The construction of a free Boolean algebra can be performed just like described for the free Heyting algebra above. The equational axioms for a Boolean algebra are the axioms for a lattice (3.6), the distributive laws (3.7), and the complement laws (3.9).

Exercise* 3.4.10. Is CPC isomorphic to the Boolean algebra $\mathsf{IPC}_{\neg\neg}$ of the regular elements of IPC?

Exercise 3.4.11. Show that in a Heyting algebra H, one has $\neg\neg x = x$ for all $x \in H$ if, and only if, $y \vee \neg y = 1$ for all $y \in H$. Hint: half of the equivalence is easy. For the other half, observe that the assumption $\neg\neg x = x$ means that negation is an order-reversing bijection $H \to H$. It therefore transforms joins into meets and vice versa, and so the *De Morgan laws* hold:

$$\neg(x \wedge y) = \neg x \vee \neg y \ , \qquad \neg(x \vee y) = \neg x \wedge \neg y \ .$$

Together with $y \wedge \neg y = 0$, the De Morgan laws easily imply $y \vee \neg y = 1$. See [**?**, I.1.11].

Kripke semantics for IPC

We now prove the Kripke completeness of IPC, extending Theorem 3.3.4, namely:

Theorem 3.4.12 (Kripke completeness for IPC)**.** *Let K be a poset equipped with a* forcing relation $k \Vdash p$ *between elements $k \in K$ and propositional variables p, satisfying*

$$j \leq k,\ k \Vdash p \quad \textit{implies} \quad j \Vdash p. \tag{3.10}$$

Extend $\Vdash$ to all formulas ϕ in IPC by defining

$$\begin{array}{lll}
k \Vdash \top & \textit{always,} & \\
k \Vdash \bot & \textit{never,} & \\
k \Vdash \phi \wedge \psi & \textit{iff} & k \Vdash \phi \textit{ and } k \Vdash \psi\,, \qquad (3.11)\\
k \Vdash \phi \vee \psi & \textit{iff} & k \Vdash \phi \textit{ or } k \Vdash \psi\,, \qquad (3.12)\\
k \Vdash \phi \Rightarrow \psi & \textit{iff} & \textit{for all } j \leq k, \textit{ if } j \Vdash \phi, \textit{ then } j \Vdash \psi\,.
\end{array}$$

Finally, write $K \Vdash \phi$ if $k \Vdash \phi$ for all $k \in K$ (for all such relations $\Vdash$).

Then a propositional formulas ϕ is provable from the rules of deduction for IPC *(Definition 3.4.3) if, and only if, $K \Vdash \phi$ for all posets K. Briefly:*

$$\mathsf{PPC} \vdash \phi \quad \textit{iff} \quad K \Vdash \phi \textit{ for all } K.$$

Let us first see that we cannot simply reuse the proof from that theorem, because the downset (Yoneda) embedding that we used there

$$\downarrow\ : \mathsf{IPC} \hookrightarrow\ \downarrow\!(\mathsf{IPC}) \tag{3.13}$$

would not preserve the coproducts $\bot$ and $\phi \vee \psi$. Indeed, $\downarrow(\bot) \neq \emptyset$, because it contains $\bot$ itself! And in general $\downarrow(\phi \vee \psi) \neq\downarrow(\phi) \cup \downarrow(\psi)$, because the righthand side need not contain, e.g., $\phi \vee \psi$.

Instead, we will generalize the Stone Representation theorem 2.6.8 from Boolean algebras to Heyting algebras, using a theorem due to Joyal (cf. [**?**, **?**]). First, recall that the Stone representation provided, for any Boolean algebra $\mathcal{B}$, an injective Boolean homomorphism into a powerset,

$$\mathcal{B} \rightarrowtail \mathcal{P}X\,.$$

For X we took the set of prime filters $\mathsf{Bool}(\mathcal{B}, \mathbb{2})$, and the map $h : \mathcal{B} \rightarrowtail \mathcal{P}\mathsf{Bool}(\mathcal{B}, \mathbb{2})$ was given by $h(b) = \{F \mid b \in F\}$. Transposing $\mathcal{P}\mathsf{Bool}(\mathcal{B}, \mathbb{2}) \cong \mathbb{2}^{\mathsf{Bool}(\mathcal{B},\mathbb{2})}$ in the cartesian closed category Pos, we arrive at the (monotone) evaluation map

$$\mathsf{eval} : \mathsf{Bool}(\mathcal{B}, \mathbb{2}) \times \mathcal{B} \to \mathbb{2}. \tag{3.14}$$

Now recall that the category of Boolean algebras is full in the category DLat of distributive lattices,

$$\mathsf{Bool}(\mathcal{B}, \mathbb{2}) = \mathsf{DLat}(\mathcal{B}, \mathbb{2})\,.$$

For any Heyting algebra $\mathcal{H}$ (or indeed any distributive lattice), the Homset $\mathsf{DLat}(\mathcal{H}, \mathbb{2})$, ordered pointwise, is isomorphic to the *poset* of all prime filters in $\mathcal{H}$ ordered by inclusion, by taking $f : \mathcal{H} \to \mathbb{2}$ to its (filter) kernel $f^{-1}\{1\} \subseteq \mathcal{H}$. In particular, the poset $\mathsf{DLat}(\mathcal{H}, \mathbb{2})$ is no longer discrete when $\mathcal{H}$ is not Boolean, since a prime ideal in a Heyting algebra need not be maximal.

The transpose of the (monotone) evaluation map,

$$\mathsf{eval} : \mathsf{DLat}(\mathcal{H}, \mathbb{2}) \times \mathcal{H} \to \mathbb{2}. \tag{3.15}$$

will then be the (monotone) map

$$\epsilon : \mathcal{H} \longrightarrow \mathbb{2}^{\mathsf{DLat}(\mathcal{H}, \mathbb{2})}, \tag{3.16}$$

which takes $p \in \mathcal{H}$ to the "evaluation at p" map $f \mapsto f(p) \in \mathbb{2}$, i.e.,

$$\epsilon_p(f) = f(p) \qquad \text{for } p \in \mathcal{H} \text{ and } f : \mathcal{H} \to \mathbb{2}\,.$$

As before, the poset $\mathbb{2}^{\mathsf{DLat}(\mathcal{H}, \mathbb{2})}$ (ordered pointwise) may be identified with the upsets in the poset $\mathsf{DLat}(\mathcal{H}, \mathbb{2})$, ordered by inclusion, which recall from Example 3.4.2 is always a Heyting algebra. Thus, in sum, we have a monotone map,

$$\mathcal{H} \longrightarrow {\uparrow}\mathsf{DLat}(\mathcal{H}, \mathbb{2})\,, \tag{3.17}$$

which generalizes the Stone representation from Boolean to Heyting algebras.

Theorem 3.4.13 (Joyal)**.** *Let $\mathcal{H}$ be a Heyting algebra. There is an injective Heyting homomorphism*

$$\mathcal{H} \rightarrowtail {\uparrow}J$$

into a Heyting algebra of upsets in a poset J.

Note that in this form, the theorem literally generalizes the Stone representation theorem, because when $\mathcal{H}$ is Boolean we can take J to be discrete, and then ${\uparrow}J \cong \mathsf{Pos}(J, \mathbb{2}) \cong \mathcal{P}J$ is Boolean, whence the Heyting embedding is also Boolean. The proof will again use the transposed evaluation map,

$$\epsilon : \mathcal{H} \longrightarrow {\uparrow}\mathsf{DLat}(\mathcal{H}, \mathbb{2}) \cong \mathbb{2}^{\mathsf{DLat}(\mathcal{H}, \mathbb{2})}$$

which, as before, is injective, by the Prime Ideal Theorem (see Lemma 2.6.6). We will use it in the following form due to Birkhoff.

Lemma 3.4.14 (Birkhoff's Prime Ideal Theorem)**.** *Let D be a distributive lattice, $I \subseteq D$ an ideal, and $x \in D$ with $x \notin I$. There is a prime ideal $I \subseteq P \subset D$ with $x \notin P$.*

Proof. As in the proof of Lemma 2.6.6, it suffice to prove it for the case $I = (0)$. This time, we use Zorn's Lemma: a poset in which every chain has an upper bound has maximal elements. Consider the poset $\mathcal{I}\backslash x$ of "ideals I without x", $x \notin I$, ordered by inclusion.

The union of any chain $I_0 \subseteq I_1 \subseteq \dots$ in $\mathcal{I}\backslash x$ is clearly also in $\mathcal{I}\backslash x$, so we have (at least one) maximal element $M \in \mathcal{I}\backslash x$. We claim that $M \subseteq D$ is prime. To that end, take $a, b \in D$ with $a \wedge b \in M$. If $a, b \notin M$, let $M[a] = \{n \leq m \vee a \mid m \in M\}$, the ideal join of M and $\downarrow(a)$, and similarly for $M[b]$. Since M is maximal without x, we therefore have $x \in M[a]$ and $x \in M[b]$. Thus let $x \leq m \vee a$ and $x \leq m' \vee b$ for some $m, m' \in M$. Then $x \vee m' \leq m \vee m' \vee a$ and $x \vee m \leq m \vee m' \vee b$, so taking meets on both sides gives

$$(x \vee m') \wedge (x \vee m) \leq (m \vee m' \vee a) \wedge (m \vee m' \vee b) = (m \vee m') \vee (a \wedge b).$$

Since the righthand side is in the ideal M, so is the left. But then $x \leq x \vee (m \wedge m')$ is also in M, contrary to our assumption that $M \in \mathcal{I}\backslash x$. □

Proof of Theorem 3.4.13. As in (3.17), let $J = \mathsf{DLat}(\mathcal{H}, \mathbb{2})$ be the poset of prime filters in $\mathcal{H}$, and consider the "evaluation" map (3.17),

$$\epsilon : \mathcal{H} \longrightarrow \mathbb{2}^{\mathsf{DLat}(\mathcal{H},\mathbb{2})} \cong \uparrow \mathsf{DLat}(\mathcal{H}, \mathbb{2})$$

given by $\epsilon(p) = \{F \mid p \in F \text{ prime}\}$.

Clearly $\epsilon(0) = \emptyset$ and $\epsilon(1) = \mathsf{DLat}(\mathcal{H}, \mathbb{2})$, and similarly for the other meets and joins, so ϵ is a lattice homomorphism. Moreover, if $p \neq q \in \mathcal{H}$ then, as in the proof of 2.6.8, we have that $\epsilon(p) \neq \epsilon(q)$, by the Prime Ideal Theorem (Lemma 3.4.14). Thus it just remains to show that

$$\epsilon(p \Rightarrow q) \;=\; \epsilon(p) \Rightarrow \epsilon(q)\,.$$

Unwinding the definitions, it suffices to show that, for all $f \in \mathsf{DLat}(\mathcal{H}, \mathbb{2})$,

$$f(p \Rightarrow q) = 1 \quad \text{iff} \quad \text{for all } g \geq f,\ g(p) = 1 \text{ implies } g(q) = 1. \tag{3.18}$$

Equivalently, for all prime filters $F \subseteq \mathcal{H}$,

$$p \Rightarrow q \in F \quad \text{iff} \quad \text{for all prime } G \supseteq F,\ p \in G \text{ implies } q \in G. \tag{3.19}$$

Now if $p \Rightarrow q \in F$, then for all (prime) filters $G \supseteq F$, also $p \Rightarrow q \in G$, and so $p \in G$ implies $q \in G$, since $(p \Rightarrow q) \wedge p \leq q$.

Conversely, suppose $p \Rightarrow q \notin F$, and we seek a prime filter $G \supseteq F$ with $p \in G$ but $q \notin G$. Consider the filter

$$F[p] = \{x \wedge p \leq h \in \mathcal{H} \mid x \in F\}\,,$$

which is the join of F and $\uparrow(p)$ in the poset of filters. If $q \in F[p]$, then $x \wedge p \leq q$ for some $x \in F$, whence $x \leq p \Rightarrow q$, and so $p \Rightarrow q \in F$, contrary to assumption; thus $q \notin F[p]$. By the Prime Ideal Theorem again (applied to the distributive lattice $\mathcal{H}^{\mathsf{op}}$) there is a prime filter $G \supseteq F[p]$ with $q \notin G$. □

Exercise 3.4.15. Give a Kripke countermodel to show that the Law of Excluded Middle $\phi \vee \neg\phi$ is not provable in IPC.

3.5 Frames and spaces

A poset $(P, \leq)$, viewed as a category, is *cocomplete* when it has suprema (least upper bounds) of arbitrary subsets. This is so because coequalizers in a poset always exist, and coproducts are precisely least upper bounds. Recall that the supremum of $S \subseteq P$ is an element $\bigvee S \in P$ such that, for all $y \in S$,

$$\bigvee S \leq y \iff \forall\, x : S \,.\, x \leq y \;.$$

In particular, $\bigvee \emptyset$ is the least element of P and $\bigvee P$ is the greatest element of P. Similarly, a poset is *complete* when it has infima (greatest lower bounds) of arbitrary subsets; the infimum of $S \subseteq P$ is an element $\bigwedge S \in P$ such that, for all $y \in S$,

$$y \leq \bigwedge S \iff \forall\, x : S \,.\, y \leq x \;.$$

Proposition 3.5.1. *A poset is complete if, and only if, it is cocomplete.*

Proof. Infima and suprema are expressed in terms of each other as follows:

$$\begin{aligned} \bigwedge S &= \bigvee \left\{ y \in P \;\middle|\; \forall\, x : S \,.\, y \leq x \right\} \;, \\ \bigvee S &= \bigwedge \left\{ y \in P \;\middle|\; \forall\, x : S \,.\, x \leq y \right\} \;. \end{aligned}$$

□

Thus, we usually speak of *complete* posets only, even when we work with arbitrary suprema.

Suppose P is a complete poset. When is it cartesian closed? Being a complete poset, it has the terminal object, namely the greatest element $1 \in P$, and it has binary products which are binary infima. If P is cartesian closed then for all $x, y \in P$ there exists an exponential $(x \Rightarrow y) \in P$, which satisfies, for all $z \in P$,

$$\frac{z \wedge x \leq y}{z \leq x \Rightarrow y} \;.$$

With the help of this adjunction we derive the *infinite distributive law*, for an arbitrary family $\left\{ y_i \in P \;\middle|\; i \in I \right\}$,

$$x \wedge \textstyle\bigvee_{i \in I} y_i = \bigvee_{i \in I} (x \wedge y_i) \tag{3.20}$$

as follows:

$$\begin{array}{c} x \wedge \bigvee_{i \in I} y_i \leq z \\ \hline\hline \bigvee_{i \in I} y_i \leq (x \Rightarrow z) \\ \hline\hline \forall\, i : I \,.\, (y_i \leq (x \Rightarrow z)) \\ \hline\hline \forall\, i : I \,.\, (x \wedge y_i \leq z) \\ \hline\hline \bigvee_{i \in I} (x \wedge y_i) \leq z \end{array}$$

Now since $x \wedge \bigvee_{i\in I} y_i$ and $\bigvee_{i\in I}(x \wedge y_i)$ have the same upper bounds they must be equal.

Conversely, suppose the distributive law (3.20) holds. Then we can *define* $x \Rightarrow y$ to be

$$(x \Rightarrow y) = \bigvee \left\{ z \in P \;\middle|\; x \wedge z \leq y \right\} \;. \tag{3.21}$$

The best way to show that $x \Rightarrow y$ is the exponential of x and y is to use the characterization of adjoints by counit, as in Proposition 1.5.5. In the case of $\wedge$ and $\Rightarrow$ this amounts to showing that, for all $x, y \in P$,

$$x \wedge (x \Rightarrow y) \leq y \;, \tag{3.22}$$

and that, for $z \in P$,

$$(x \wedge z \leq y) \Rightarrow (z \leq x \Rightarrow y) \;.$$

This implication follows directly from (3.5.7), and (3.22) follows from the distributive law:

$$x \wedge (x \Rightarrow y) = x \wedge \bigvee \left\{ z \in P \;\middle|\; x \wedge z \leq y \right\} = \bigvee \left\{ x \wedge z \;\middle|\; x \wedge z \leq y \right\} \leq y \;.$$

Complete cartesian closed posets are called *frames*.

Definition 3.5.2. A *frame* is a poset that is complete and cartesian closed, thus a frame is a complete Heyting algebra. Equivalently, a frame is a complete poset satisfying the (infinite) distributive law

$$x \wedge \bigvee_{i\in I} y_i = \bigvee_{i\in I}(x \wedge y_i) \;.$$

A *frame morphism* is a function $f : L \to M$ between frames that preserves finite infima and arbitrary suprema. The category of frames and frame morphisms is denoted by Frame.

Warning: a frame morphism need not preserve exponentials!

Example 3.5.3. Given a poset P, the downsets $\downarrow P$ form a complete lattice under the inclusion order $S \subseteq T$, and with the set theoretic operations $\bigcup$ and $\bigcap$ as $\bigvee$ and $\bigwedge$. Since $\downarrow P$ is already known to be a Heyting algebra (Example 3.4.2), it is therefore also a frame. (Alternately, we can show that it is a frame by noting that the operations $\bigcup$ and $\bigcap$ satisfy the infinite distributive law, and then infer that it is a Heyting algebra.)

A monotone map $f : P \to Q$ between posets gives rise to a frame map

$$\downarrow f : \downarrow Q \longrightarrow \downarrow P \,,$$

as can be seen by recalling that $\downarrow P \cong 2^P$ as posets. Note that as a (co)limit preserving functor on complete posets, $2^f : 2^Q \longrightarrow 2^P$ has both left and right adjoints. These functors are usually written $f_! \dashv f^* \dashv f_*$. Although it does not in general preserve Heyting implications $S \Rightarrow T$, the monotone map $\downarrow f : \downarrow Q \longrightarrow \downarrow P$ is indeed a morphism of frames. We therefore have a contravariant functor

$$\downarrow(-) : \; \mathsf{Pos} \to \mathsf{Frame}^{\mathsf{op}}. \tag{3.23}$$

Example 3.5.4. The topology $\mathcal{O}X$ of a topological space X, ordered by inclusion, is a frame because finite intersections and arbitrary unions of open sets are open. The distributive law holds because intersections distribute over unions. If $f : X \to Y$ is a continuous map between topological spaces, the inverse image map $f^* : \mathcal{O}Y \to \mathcal{O}X$ is a frame homomorphism. Thus, there is a functor

$$\mathcal{O} : \mathsf{Top} \to \mathsf{Frame}^{\mathsf{op}}$$

which maps a space X to its topology $\mathcal{O}X$ and a continuous map $f : X \to Y$ to the inverse image map $f^* : \mathcal{O}Y \to \mathcal{O}X$.

The category $\mathsf{Frame}^{\mathsf{op}}$ is called the category of *locales* and is denoted by Loc. When we think of a frame as an object of Loc we call it a locale.

Example 3.5.5. Let P be a poset and define a topology on the elements of P by defining the opens to be the upsets,

$$\mathcal{O}P = \uparrow P \cong \mathsf{Pos}(P, \mathbb{2}).$$

These open sets are not only closed under arbitrary unions and finite intersections, but also under *arbitrary* intersections. Such a topological space is said to be an *Alexandrov space*.

Exercise* 3.5.6. This exercise is meant for students with some background in topology. For a topological space X and a point $x \in X$, let $N(x)$ be the neighborhood filter of x,

$$N(x) = \left\{ U \in \mathcal{O}X \;\middle|\; x \in U \right\} .$$

Recall that a T_0-*space* is a topological space X in which points are determined by their neighborhood filters,

$$N(x) = N(y) \Rightarrow x = y . \qquad (x, y \in X)$$

Let Top_0 be the full subcategory of Top on T_0-spaces. The functor $\mathcal{O} : \mathsf{Top} \to \mathsf{Loc}$ restricts to a functor $\mathcal{O} : \mathsf{Top}_0 \to \mathsf{Loc}$. Prove that $\mathcal{O} : \mathsf{Top}_0 \to \mathsf{Loc}$ is a faithful functor. Is it full?

Topological semantics for IPC

It should now be clear how to interpret IPC into a topological space X: each formula ϕ is assigned to an open set $[\![\phi]\!] \in \mathcal{O}X$ in such a way that $[\![-]\!]$ is a homomorphism of Heyting algebras.

Definition 3.5.7. A *topological model* of IPC is a space X and an interpretation of formulas,

$$[\![-]\!] : \mathsf{IPC} \to \mathcal{O}X ,$$

satisfying the conditions:

$$\begin{aligned}
[\![\top]\!] &= X \\
[\![\bot]\!] &= \emptyset \\
[\![\phi \wedge \psi]\!] &= [\![\phi]\!] \cap [\![\psi]\!] \\
[\![\phi \vee \psi]\!] &= [\![\phi]\!] \cup [\![\psi]\!] \\
[\![\phi \Rightarrow \psi]\!] &= [\![\phi]\!] \Rightarrow [\![\psi]\!]
\end{aligned}$$

The Heyting implication $[\![\phi]\!] \Rightarrow [\![\psi]\!]$ in $\mathcal{O}X$, is defined in (3.5.7) as

$$[\![\phi]\!] \Rightarrow [\![\psi]\!] \; = \; \bigcup \left\{ U \in \mathcal{O}X \;\middle|\; U \wedge [\![\phi]\!] \leq [\![\psi]\!] \right\} .$$

Joyal's representation theorem 3.4.13 easily implies that IPC is sound and complete with respect to topological semantics.

Corollary 3.5.8. *A formula ϕ is provable in* IPC *if, and only if, it holds in every topological interpretation $[\![-]\!]$ into a space X, briefly:*

$$\mathsf{IPC} \vdash \phi \qquad \textit{iff} \qquad [\![\phi]\!] = X \textit{ for all spaces } X .$$

Proof. Put the Alexandrov topology on the upsets of prime ideals in the Heyting algebra IPC □

Exercise 3.5.9. Give a topological countermodel to show that the Law of Double Negation $\neg\neg\phi \Rightarrow \phi$ is not provable in IPC.

3.6 Proper CCCs

We begin by reviewing some important examples of cartesian closed categories that are *not posets*, most of which have already been discussed.

Example 3.6.1. The first example is the category Set. We already know that the terminal object is a singleton set and that binary products are cartesian products. The exponential of X and Y in Set is just the set of all functions from X to Y,

$$Y^X = \left\{ f \subseteq X \times Y \;\middle|\; \forall\, x : X \,.\, \exists!\, y : Y \,.\, \langle x, y \rangle \in f \right\} .$$

The evaluation morphism $\mathsf{eval} : Y^X \times X \to Y$ is the usual evaluation of a function at an argument, i.e., $\mathsf{eval}\langle f, x \rangle$ is the unique $y \in Y$ for which $\langle x, y \rangle \in f$.

Example 3.6.2. The category Cat of all small categories is cartesian closed. The exponential of small categories $\mathcal{C}$ and $\mathcal{D}$ is the category $\mathcal{D}^{\mathcal{C}}$ of functors, with natural transformations as arrows (see 1.6). Note that if $\mathcal{D}$ is a groupoid (all arrows are isos), then so is $\mathcal{D}^{\mathcal{C}}$. It follows that the category of groupoids is full (even as a 2-category) in Cat. Since limits of groupoids in Cat are also groupoids, the inclusion of the full subcategory $\mathsf{Grpd} \hookrightarrow \mathsf{Cat}$ preserves limits, too, and is therefore a full inclusion of CCCs.

Example 3.6.3. The same reasoning as in the previous example shows that the full subcategory $\mathsf{Pos} \hookrightarrow \mathsf{Cat}$ of all small posets and monotone maps is also cartesian closed. It is worth noting that, unlike the previous cases, the (limit preserving) forgetful functor $U : \mathsf{Poset} \to \mathsf{Set}$ does *not* preserve exponentials; in general $U(Q^P) \subseteq (UQ)^{UP}$ is a *proper* subset.

Exercise 3.6.4. There is a full and faithful functor $I : \mathsf{Set} \to \mathsf{Poset}$ that preserves finite limits as well as exponentials. How is this related to the example $\mathsf{Grpd} \hookrightarrow \mathsf{Cat}$?

The foregoing examples are instances of the following general situation.

Proposition 3.6.5. *Let $\mathcal{E}$ be a CCC and $i : \mathcal{S} \hookrightarrow \mathcal{E}$ a full subcategory with finite products and a left adjoint reflection $L : \mathcal{E} \to \mathcal{S}$ preserving finite products. Suppose moreover that for any two objects A, B in $\mathcal{S}$, the exponential iB^{iA} is again in $\mathcal{S}$. Then $\mathcal{S}$ has all exponentials, and these are preserved by i.*

Proof. By assumption, we have $L \dashv i$ with isomorphic counit $LiS \cong S$ for all $S \in \mathcal{S}$. Let us identify $\mathcal{S}$ with the subcategory of $\mathcal{E}$ that is its image under $i : \mathcal{S} \hookrightarrow \mathcal{E}$. The assumption that B^A is again in $\mathcal{S}$ for all $A, B \in \mathcal{S}$, along with the fullness of $\mathcal{S}$ in $\mathcal{E}$, gives the exponentials, and the closure of $\mathcal{S}$ under finite products in $\mathcal{E}$ ensures that the required transposes will also be in $\mathcal{S}$.

Alternately, for any $A, B \in \mathcal{S}$ set $B^A = L(iB^{iA})$. Then for any $C \in \mathcal{S}$, we have natural isos:

$$\begin{aligned}
\mathcal{S}(C \times A, B) &\cong \mathcal{E}(i(C \times A), iB) \\
&\cong \mathcal{E}(iC \times iA, iB) \\
&\cong \mathcal{E}\big(iC, iB^{iA}\big) \\
&\cong \mathcal{E}\big(iC, iL(iB^{iA})\big) \\
&\cong \mathcal{S}\big(C, L(iB^{iA})\big) \\
&\cong \mathcal{S}\big(C, B^A\big)
\end{aligned}$$

where in the fifth line we used the assumption that iB^{iA} is again in $\mathcal{S}$, in the form $iB^{iA} \cong iE$ for some $E \in \mathcal{S}$, which is then necessarily $L(iB^{iA}) = LiE \cong E$. ☐

A related general situation that covers some (but not all) of the above examples is this:

Proposition 3.6.6. *Let $\mathcal{E}$ be a CCC and $i : \mathcal{S} \hookrightarrow \mathcal{E}$ a full subcategory with finite products and a right adjoint reflection $R : \mathcal{E} \to \mathcal{S}$. If i preserves finite products, then $\mathcal{S}$ also has all exponentials, and these are computed first in $\mathcal{E}$, and then reflected by R into $\mathcal{S}$.*

Proof. For any $A, B \in \mathcal{S}$ set $B^A = R(iB^{iA})$ as described. Now for any $C \in \mathcal{S}$, we have natural isos:

$$\begin{aligned}
\mathcal{S}(C \times A, B) &\cong \mathcal{E}(i(C \times A), iB) \\
&\cong \mathcal{E}(iC \times iA, iB) \\
&\cong \mathcal{E}\big(iC, iB^{iA}\big) \\
&\cong \mathcal{S}\big(C, R(iB^{iA})\big) \\
&\cong \mathcal{S}\big(C, B^{A}\big) .
\end{aligned}$$

□

An example of the foregoing is the inclusion of the opens into the powerset of points of a space X,

$$\mathcal{O}X \hookrightarrow \mathcal{P}X$$

This frame homomorphism is associated to the map $|X| \to X$ of locales (or in this case, spaces) from the discrete space on the set of points of X.

Exercise 3.6.7. Which of the examples follows from which proposition?

Example 3.6.8. A presheaf category $\widehat{\mathbb{C}}$ is cartesian closed, provided the index category $\mathbb{C}$ is small. To see what the exponential of presheaves P and Q ought to be, we use the Yoneda Lemma. If Q^P exists, then by Yoneda Lemma and the adjunction $(- \times P) \dashv (-^P)$, we have for all $A \in \mathbb{C}$,

$$Q^P(A) \cong \mathsf{Nat}(\mathsf{y}A, Q^P) \cong \mathsf{Nat}(\mathsf{y}A \times P, Q) \ .$$

Because $\mathcal{C}$ is small $\mathsf{Nat}(\mathsf{y}A \times P, Q)$ is a set, so we can *define* Q^P to be the presheaf

$$Q^P = \mathsf{Nat}(\mathsf{y}{-} \times P, Q) \ .$$

The evaluation morphism $E : Q^P \times P \Longrightarrow Q$ is the natural transformation whose component at A is

$$\begin{aligned}
&E_A : \mathsf{Nat}(\mathsf{y}A \times P, Q) \times PA \to QA \ , \\
&E_A : \langle \eta, x \rangle \mapsto \eta_A \langle 1_A, x \rangle \ .
\end{aligned}$$

The transpose of a natural transformation $\phi : R \times P \Longrightarrow Q$ is the natural transformation $\widetilde{\phi} : R \Longrightarrow Q^P$ whose component at A is the function that maps $z \in RA$ to the natural transformation $\widetilde{\phi}_A z : \mathsf{y}A \times P \Longrightarrow Q$, whose component at $B \in \mathcal{C}$ is

$$\begin{aligned}
&(\widetilde{\phi}_A z)_B : \mathcal{C}(B, A) \times PB \to QB \ , \\
&(\widetilde{\phi}_A z)_B : \langle f, y \rangle \mapsto \phi_B \langle (Rf) z, y \rangle \ .
\end{aligned}$$

Exercise 3.6.9. Verify that the above definition of Q^P really gives an exponential of presheaves P and Q.

It follows immediately that the category of graphs Graph is cartesian closed because it is the presheaf category $\mathsf{Set}^{\cdot \rightrightarrows \cdot}$. The same is of course true for the "category of functions", i.e. the arrow category $\mathsf{Set}^{\rightarrow}$, as well as the category of simplicial sets $\mathsf{Set}^{\Delta^{\mathsf{op}}}$ from topology.

Exercise 3.6.10. This exercise is for students with some background in linear algebra. Let Vec be the category of real vector spaces and linear maps between them. Given vector spaces X and Y, the linear maps $\mathcal{L}(X, Y)$ between them form a vector space. So define $\mathcal{L}(X, -) : \mathsf{Vec} \to \mathsf{Vec}$ to be the functor which maps a vector space Y to the vector space $\mathcal{L}(X, Y)$, and it maps a linear map $f : Y \to Z$ to the linear map $\mathcal{L}(X, f) : \mathcal{L}(X, Y) \to \mathcal{L}(X, Z)$ defined by $h \mapsto f \circ h$. Show that $\mathcal{L}(X, -)$ has a left adjoint $- \otimes X$, but also show that this adjoint is *not* the binary product in Vec.

A few other instructive examples that can be explored by the interested reader are the following.

- Etale spaces over a base space X. This category can be described as consisting of *local homeomorphisms* $f : Y \to X$ and commutative triangles over X between such maps. It is also equivalent to the category $\mathsf{Sh}(X)$ of *sheaves* on X. See [**?**, ch.n].
- Various subcategories of topological spaces (sequential spaces, compactly-generated spaces). Cf. [**?**].
- Dana Scott's category Equ of equilogical spaces [**?**].

3.7 Simply typed λ-calculus

The λ-calculus is the abstract theory of functions, just like group theory is the abstract theory of symmetries. There are two basic operations that can be performed with functions. The first one is the *application* of a function to an argument: if f is a function and a is an argument, then fa is the application of f to a, also called the *value* of f at a. The second operation is *abstraction*: if x is a variable and t is an expression in which x may appear, then there is a function f defined by the equation

$$fx = t \;.$$

Here we gave the name f to the newly formed function. But we could have expressed the same function without giving it a name; this is usually written as

$$x \mapsto t \;,$$

and it means "x is mapped to t". In λ-calculus we use a different notation, which is more convenient when abstractions are nested:

$$\lambda x.\, t \;.$$

This operation is called *λ-abstraction*. For example, $\lambda x.\, \lambda y.\, (x + y)$ is the function which maps an argument a to the function $\lambda y.\, (a + y)$, which maps an argument b ... .

In an expression $\lambda x.\, t$ the variable x is said to be *bound* in t.

Remark 3.7.1. It may seem strange that in specifying the abstraction of a function, we switched from talking about objects (functions, arguments, values) to talking about *expressions*: variables, names, equations. This "syntactic" point of view seems to have been part of the notion of a function since its beginnings, in the theory of algebraic equations. It is the reason that the λ-calculus is part of *logic*, unlike the theory of cartesian closed categories, which remains thoroughly semantical (and "variable-free"). The relation between the two different points of view will occupy the remainder of this chapter.

Remark 3.7.2. There are two kinds of λ-calculus, the *typed* and the *untyped* one. In the untyped version there are no restrictions on how application is formed, so that an expression such as

$$\lambda x.\,(xx)$$

is valid, whatever it may mean. In typed λ-calculus every expression has a *type*, and there are rules for forming valid expressions and types. For example, we can only form an application f, a when a has a type A and f has a type $A \to B$, which indicates a function taking arguments of type A and giving results of type B. The judgment that expression t has a type A is written as

$$t : A\,.$$

To computer scientists the idea of expressions having types is familiar from programming languages, whereas mathematicians can think of types as sets and read $t : A$ as $t \in A$. We will concentrate on the typed λ-calculus.

We now give a precise definition of what constitutes a *simply-typed λ-calculus*. First, we are given a set of *simple types*, which are generated from *basic types* by formation of products and function types:

$$\begin{aligned} \text{Basic type } \mathtt{B} &::= \mathtt{B}_0 \mid \mathtt{B}_1 \mid \mathtt{B}_2 \cdots \\ \text{Simple type } A &::= \mathtt{B} \mid A_1 \times A_2 \mid A_1 \to A_2. \end{aligned}$$

Function types associate to the right:

$$A \to B \to C \equiv A \to (B \to C)\,.$$

We assume there is a countable set of *variables* x, y, u, ... We are also given a set of *basic constants*. The set of *terms* is generated from variables and basic constants by the following grammar:

$$\begin{aligned} \text{Variable } v &::= x \mid y \mid z \mid \cdots \\ \text{Constant } c &::= \mathtt{c}_1 \mid \mathtt{c}_2 \mid \cdots \\ \text{Term } t &::= v \mid c \mid * \mid \langle t_1, t_2 \rangle \mid \mathtt{fst}\, t \mid \mathtt{snd}\, t \mid t_1\, t_2 \mid \lambda x : A\,.\,t \end{aligned}$$

In words, this means:

1. a variable is a term,

2. each basic constant is a term,
3. the constant $*$ is a term, called the *unit*,
4. if u and t are terms then $\langle u, t\rangle$ is a term, called a *pair*,
5. if t is a term then $\mathtt{fst}\, t$ and $\mathtt{snd}\, t$ are terms,
6. if u and t are terms then $u\, t$ is a term, called an *application*
7. if x is a variable, A is a type, and t is a term, then $\lambda x : A \,.\, t$ is a term, called a λ-*abstraction*.

The variable x is *bound* in $\lambda x : A \,.\, t$. Application associates to the left, thus $s\, t\, u = (s\, t)\, u$. The *free variables* $\mathsf{FV}(t)$ of a term t are computed as follows:

$$
\begin{aligned}
\mathsf{FV}(x) &= \{x\} \qquad \text{if } x \text{ is a variable} \\
\mathsf{FV}(a) &= \emptyset \qquad \text{if } a \text{ is a basic constant} \\
\mathsf{FV}(\langle u, t\rangle) &= \mathsf{FV}(u) \cup \mathsf{FV}(t) \\
\mathsf{FV}(\mathtt{fst}\, t) &= \mathsf{FV}(t) \\
\mathsf{FV}(\mathtt{snd}\, t) &= \mathsf{FV}(t) \\
\mathsf{FV}(u\, t) &= \mathsf{FV}(u) \cup \mathsf{FV}(t) \\
\mathsf{FV}(\lambda x.\, t) &= \mathsf{FV}(t) \setminus \{x\} \ .
\end{aligned}
$$

If $x_1, \ldots, x_n$ are *distinct* variables and $A_1, \ldots, A_n$ are types then the sequence

$$x_1 : A_1, \ldots, x_n : A_n$$

is a *typing context*, or just *context*. The empty sequence is sometimes denoted by a dot $\cdot$, and it is a valid context. Context are denoted by capital Greek letters Γ, Δ, …

A *typing judgment* is a judgment of the form

$$\Gamma \mid t : A$$

where Γ is a context, t is a term, and A is a type. In addition the free variables of t must occur in Γ, but Γ may contain other variables as well. We read the above judgment as "in context Γ the term t has type A". Next we describe the rules for deriving typing judgments.

Each basic constant c_i has a uniquely determined type C_i,

$$\frac{}{\Gamma \mid \mathsf{c}_i : C_i}$$

The type of a variable is determined by the context:

$$\frac{}{x_1 : A_1, \ldots, x_i : A_i, \ldots, x_n : A_n \mid x_i : A_i}\ (1 \leq i \leq n)$$

The constant $*$ has type $\mathtt{1}$:

$$\frac{}{\Gamma \mid * : \mathtt{1}}$$

The typing rules for pairs and projections are:

$$\frac{\Gamma \mid u : A \qquad \Gamma \mid t : B}{\Gamma \mid \langle u, t \rangle : A \times B} \qquad \frac{\Gamma \mid t : A \times B}{\Gamma \mid \mathtt{fst}\, t : A} \qquad \frac{\Gamma \mid t : A \times B}{\Gamma \mid \mathtt{snd}\, t : B}$$

The typing rules for application and λ-abstraction are:

$$\frac{\Gamma \mid t : A \to B \qquad \Gamma \mid u : A}{\Gamma \mid t\, u : B} \qquad \frac{\Gamma, x : A \mid t : B}{\Gamma \mid (\lambda x : A \,.\, t) : A \to B}$$

Lastly, we have *equations* between terms; for terms of type A in context Γ,

$$\Gamma \mid u : A\,, \qquad \Gamma \mid t : B\,,$$

the judgment that they are equal is written as

$$\Gamma \mid u = t : A\,.$$

Note that u and t necessarily have the same type; it does *not* make sense to compare terms of different types. We have the following rules for equations:

1. Equality is an equivalence relation:

$$\frac{}{\Gamma \mid t = t : A} \qquad \frac{\Gamma \mid t = u : A}{\Gamma \mid u = t : A} \qquad \frac{\Gamma \mid t = u : A \qquad \Gamma \mid u = v : A}{\Gamma \mid t = v : A}$$

2. The weakening rule:

$$\frac{\Gamma \mid u = t : A}{\Gamma, x : B \mid u = t : A}$$

3. Unit type:

$$\frac{}{\Gamma \mid t = * : \mathtt{1}}$$

4. Equations for product types:

$$\frac{\Gamma \mid u = v : A \qquad \Gamma \mid s = t : B}{\Gamma \mid \langle u, s \rangle = \langle v, t \rangle : A \times B}$$

$$\frac{\Gamma \mid s = t : A \times B}{\Gamma \mid \mathtt{fst}\, s = \mathtt{fst}\, t : A} \qquad \frac{\Gamma \mid s = t : A \times B}{\Gamma \mid \mathtt{snd}\, s = \mathtt{snd}\, t : A}$$

$$\frac{}{\Gamma \mid t = \langle \mathtt{fst}\, t, \mathtt{snd}\, t \rangle : A \times B}$$

$$\frac{}{\Gamma \mid \mathtt{fst}\, \langle u, t \rangle = u : A} \qquad \frac{}{\Gamma \mid \mathtt{snd}\, \langle u, t \rangle = t : A}$$

5. Equations for function types:

$$\frac{\Gamma \mid s = t : A \to B \qquad \Gamma \mid u = v : A}{\Gamma \mid s\,u = t\,v : B}$$

$$\frac{\Gamma, x : A \mid t = u : B}{\Gamma \mid (\lambda x : A \,.\, t) = (\lambda x : A \,.\, u) : A \to B}$$

$$\frac{}{\Gamma \mid (\lambda x : A \,.\, t)u = t[u/x] : A} \qquad (\beta\text{-rule})$$

$$\frac{}{\Gamma \mid \lambda x : A \,.\, (t\,x) = t : A \to B} \text{ if } x \notin \mathsf{FV}(t) \qquad (\eta\text{-rule})$$

This completes the description of a simply-typed λ-calculus.

Apart from the above rules for equality we might want to impose additional equations. In this case we do not speak of a λ-calculus but rather of a λ-*theory*. Thus, a λ-theory $\mathbb{T}$ is given by a set of basic types, a set of basic constants, and a set of *equations* of the form

$$\Gamma \mid u = t : A \,.$$

We summarize the preceding definitions.

Definition 3.7.3. A *simply-typed λ-calculus* is given by a set of *basic types* and a set of *basic constants* together with their types. *A simply-typed λ-theory* is a simply-typed λ-calculus together with a set of equations.

We use letters $\mathbb{S}$, $\mathbb{T}$, $\mathbb{U}$, ... to denote theories.

Example 3.7.4. The theory of a group is a simply-typed λ-theory. It has one basic type $\mathtt{G}$ and three basic constant, the unit $\mathtt{e}$, the inverse $\mathtt{i}$, and the group operation $\mathtt{m}$,

$$\mathtt{e} : \mathtt{G}\,, \qquad\qquad \mathtt{i} : \mathtt{G} \to \mathtt{G}\,, \qquad\qquad \mathtt{m} : \mathtt{G} \times \mathtt{G} \to \mathtt{G}\,,$$

with the following equations:

$$\begin{aligned} x : \mathtt{G} &\mid \mathtt{m}\langle x, \mathtt{e}\rangle = x : \mathtt{G} \\ x : \mathtt{G} &\mid \mathtt{m}\langle \mathtt{e}, x\rangle = x : \mathtt{G} \\ x : \mathtt{G} &\mid \mathtt{m}\langle x, \mathtt{i}\,x\rangle = \mathtt{e} : \mathtt{G} \\ x : \mathtt{G} &\mid \mathtt{m}\langle \mathtt{i}\,x, x\rangle = \mathtt{e} : \mathtt{G} \\ x : \mathtt{G}, y : \mathtt{G}, z : \mathtt{G} &\mid \mathtt{m}\langle x, \mathtt{m}\langle y, z\rangle\rangle = \mathtt{m}\langle \mathtt{m}\langle x, y\rangle, z\rangle : \mathtt{G} \end{aligned}$$

These are just the familiar axioms for a group.

Example 3.7.5. In general, any algebraic theory $\mathbb{A}$ determines a λ-theory $\mathbb{A}_\lambda$. There is one basic type $\mathtt{A}$ and for each operation f of arity k there is a basic constant $\mathtt{f} : \mathtt{A}^k \to \mathtt{A}$, where $\mathtt{A}^k$ is the k-fold product $\mathtt{A} \times \cdots \times \mathtt{A}$. It is understood that $\mathtt{A}^0 = \mathtt{1}$. The terms of $\mathbb{A}$ are translated to the terms of the corresponding λ-theory in a straightforward manner. For every axiom $t = u$ of $\mathbb{A}$ the corresponding axiom in the λ-theory is

$$x_1 : \mathtt{A}, \ldots, x_n : \mathtt{A} \mid t = u : \mathtt{A}$$

where $x_1, \ldots, x_n$ are the variables occurring in t and u.

Example 3.7.6. The theory of a directed graph is a simply-typed theory with two basic types, $\mathtt{V}$ for vertices and $\mathtt{E}$ for edges, and two basic constant, source $\mathtt{src}$ and target $\mathtt{trg}$,

$$\mathtt{src} : \mathtt{E} \to \mathtt{V}\,, \qquad\qquad \mathtt{trg} : \mathtt{E} \to \mathtt{V}\,.$$

There are no equations.

Example 3.7.7. An example of a λ-theory is readily found in the theory of programming languages. The mini-programming language PCF is a simply-typed λ-calculus with a basic type $\mathtt{nat}$ for natural numbers, and a basic type $\mathtt{bool}$ of Boolean values,

$$\text{Basic type } \mathtt{B} ::= \mathtt{nat\ type} \mid \mathtt{bool\ type}.$$

There are basic constants zero $\mathtt{0}$, successor $\mathtt{succ}$, the Boolean constants $\mathtt{true}$ and $\mathtt{false}$, comparison with zero $\mathtt{iszero}$, and for each type A the *conditional* $\mathtt{cond}_A$ and the *fixpoint* operator $\mathtt{fix}_A$. They have the following types:

$$\begin{aligned} \mathtt{0} &: \mathtt{nat} \\ \mathtt{succ} &: \mathtt{nat} \to \mathtt{nat} \\ \mathtt{true} &: \mathtt{bool} \\ \mathtt{false} &: \mathtt{bool} \\ \mathtt{iszero} &: \mathtt{nat} \to \mathtt{bool} \\ \mathtt{cond}_A &: \mathtt{bool} \to A \to A \\ \mathtt{fix}_A &: (A \to A) \to A \end{aligned}$$

The equational axioms of PCF are:

$$\begin{aligned} \cdot &\mid \mathtt{iszero}\,\mathtt{0} = \mathtt{true} : \mathtt{bool} \\ x : \mathtt{nat} &\mid \mathtt{iszero}\,(\mathtt{succ}\,x) = \mathtt{false} : \mathtt{bool} \\ u : A, t : A &\mid \mathtt{cond}_A\,\mathtt{true}\,u\,t = u : A \\ u : A, t : A &\mid \mathtt{cond}_A\,\mathtt{false}\,u\,t = t : A \\ t : A \to A &\mid \mathtt{fix}_A\,t = t\,(\mathtt{fix}_A\,t) : A \end{aligned}$$

Example 3.7.8. Another example of a λ-theory is the *theory of a reflexive type*. This theory has one basic type $\mathtt{D}$ and two constants

$$\mathtt{r} : \mathtt{D} \to \mathtt{D} \to \mathtt{D} \qquad\qquad \mathtt{s} : (\mathtt{D} \to \mathtt{D}) \to \mathtt{D}$$

satisfying the equation

$$f : \mathtt{D} \to \mathtt{D} \mid \mathtt{r}\,(\mathtt{s}\,f) = f : \mathtt{D} \to \mathtt{D} \tag{3.24}$$

which says that $\mathtt{s}$ is a section and $\mathtt{r}$ is a retraction, so that the function type $\mathtt{D} \to \mathtt{D}$ is a subspace (even a retract) of $\mathtt{D}$. A type with this property is said to be *reflexive*. We may additionally stipulate the axiom

$$x : \mathtt{D} \mid \mathtt{s}\,(\mathtt{r}\,x) = x : \mathtt{D} \tag{3.25}$$

which implies that $\mathtt{D}$ is isomorphic to $\mathtt{D} \to \mathtt{D}$.

Untyped λ-calculus

We briefly describe the *untyped λ-calculus*. It is a theory whose terms are generated by the following grammar:

$$t ::= v \mid t_! \, t_2 \mid \lambda x .\, t \;.$$

In words, a variable is a term, an application $t\,t'$ is a term, for any terms t and t', and a λ-abstraction $\lambda x.\, t$ is a term, for any term t. Variable x is bound in $\lambda x.\, t$. A *context* is a list of distinct variables,

$$x_1, \dots, x_n \;.$$

We say that a term t is valid in context Γ if the free variables of t are listed in Γ. The judgment that two terms u and t are equal is written as

$$\Gamma \mid u = t \;,$$

where it is assumed that u and t are both valid in Γ. The context Γ is not really necessary but we include it because it is always good practice to list the free variables.

The rules of equality are as follows:

1. Equality is an equivalence relation:

$$\frac{}{\Gamma \mid t = t} \qquad\qquad \frac{\Gamma \mid t = u}{\Gamma \mid u = t} \qquad\qquad \frac{\Gamma \mid t = u \qquad \Gamma \mid u = v}{\Gamma \mid t = v}$$

2. The weakening rule:

$$\frac{\Gamma \mid u = t}{\Gamma, x \mid u = t}$$

3. Equations for application and λ-abstraction:

$$\frac{\Gamma \mid s = t \qquad \Gamma \mid u = v}{\Gamma \mid s\,u = t\,v} \qquad\qquad \frac{\Gamma, x \mid t = u}{\Gamma \mid \lambda x .\, t = \lambda x .\, u}$$

$$\frac{}{\Gamma \mid (\lambda x .\, t) u = t[u/x]} \qquad (\beta\text{-rule})$$

$$\frac{}{\Gamma \mid \lambda x .\, (t\,x) = t} \quad \text{if } x \notin \mathsf{FV}(t) \qquad (\eta\text{-rule})$$

The untyped λ-calculus can be translated into the theory of a reflexive type from Example 3.7.8. An untyped context Γ is translated to a typed context Γ^* by typing each variable in Γ with the reflexive type $\mathtt{D}$, i.e., a context $x_1, \dots, x_k$ is translated to $x_1 : \mathtt{D}, \dots, x_k : \mathtt{D}$. An untyped term t is translated to a typed term t^* as follows:

$$\begin{aligned} x^* &= x \qquad \text{if } x \text{ is a variable} \;, \\ (u\,t)^* &= (\mathtt{r}\, u^*) t^* \;, \\ (\lambda x .\, t)^* &= \mathtt{s}\, (\lambda x : \mathtt{D} .\, t^*) \;. \end{aligned}$$

For example, the term $\lambda x.\,(x\,x)$ translates to $\mathtt{s}\,(\lambda x : \mathtt{D}\,.\,((\mathtt{r}\,x)\,x))$. A judgment

$$\Gamma \mid u = t \tag{3.26}$$

is translated to the judgment

$$\Gamma^* \mid u^* = t^* : \mathtt{D}\,. \tag{3.27}$$

Exercise* **3.7.9.** Prove that if equation (3.26) is provable then equation (3.27) is provable as well. Identify precisely at which point in your proof you need to use equations (3.24) and (3.25). Does provability of (3.27) imply provability of (3.26)?

3.8 Interpretation of λ-calculus in CCCs

We now consider semantic aspects of λ-calculus and λ-theories. Suppose $\mathbb{T}$ is a λ-calculus and $\mathcal{C}$ is a cartesian closed category. An *interpretation* $[\![-]\!]$ of $\mathbb{T}$ in $\mathcal{C}$ is given by the following data:

1. For every basic type A in $\mathbb{T}$ an object $[\![A]\!] \in \mathcal{C}$. The interpretation is extended to all types by

$$[\![\mathtt{1}]\!] = \mathsf{1}\,, \qquad [\![A \times B]\!] = [\![A]\!] \times [\![B]\!]\,, \qquad [\![A \to B]\!] = [\![B]\!]^{[\![A]\!]}\,.$$

2. For every basic constant c of type A a morphism $[\![c]\!] : \mathsf{1} \to [\![A]\!]$.

The interpretation is extended to all terms in context as follows. A context $\Gamma = x_1 : A_1, \cdots, x_n : A_n$ is interpreted as the object

$$[\![A_1]\!] \times \cdots \times [\![A_n]\!]\,,$$

and the empty context is interpreted as the terminal object $\mathsf{1}$. A typing judgment

$$\Gamma \mid t : A$$

is interpreted as a morphism

$$[\![\Gamma \mid t : A]\!] : [\![\Gamma]\!] \to [\![A]\!]\,.$$

The interpretation is defined inductively by the following rules:

1. The i-th variable is interpreted as the i-th projection,

$$[\![x_0 : A_0, \ldots, x_n : A_n \mid x_i : A_i]\!] = \pi_i : [\![\Gamma]\!] \to [\![A_i]\!]\,.$$

2. A basic constant $c : A$ in context Γ is interpreted as the composition

$$[\![\Gamma]\!] \xrightarrow{!_{[\![\Gamma]\!]}} \mathsf{1} \xrightarrow{[\![c]\!]} [\![A]\!]$$

3. The interpretation of projections and pairs is

$$\begin{aligned}
[\![\Gamma \mid \langle t, u\rangle : A \times B]\!] &= \langle [\![\Gamma \mid t : A]\!], [\![\Gamma \mid u : B]\!]\rangle : [\![\Gamma]\!] \to [\![A]\!] \times [\![B]\!] \\
[\![\Gamma \mid \mathtt{fst}\, t : A]\!] &= \pi_0 \circ [\![\Gamma \mid t : A \times B]\!] : [\![\Gamma]\!] \to [\![A]\!] \\
[\![\Gamma \mid \mathtt{snd}\, t : A]\!] &= \pi_1 \circ [\![\Gamma \mid t : A \times B]\!] : [\![\Gamma]\!] \to [\![B]\!] \,.
\end{aligned}$$

4. The interpretation of application and λ-abstraction is

$$\begin{aligned}
[\![\Gamma \mid t\, u : B]\!] &= e \circ \langle [\![\Gamma \mid t : A \to B]\!], [\![\Gamma \mid u : A]\!]\rangle : [\![\Gamma]\!] \to [\![B]\!] \\
[\![\Gamma \mid \lambda x : A \,.\, t : A \to B]\!] &= ([\![\Gamma, x : A \mid t : B]\!])^{\sim} : [\![\Gamma]\!] \to [\![B]\!]^{[\![A]\!]}
\end{aligned}$$

   where $e : [\![A \to B]\!] \times [\![A]\!] \to [\![B]\!]$ is the evaluation morphism for $[\![B]\!]^{[\![A]\!]}$ and $([\![\Gamma, x : A \mid t : B]\!])^{\sim}$ is the transpose of the morphism

$$[\![\Gamma, x : A \mid t : B]\!] : [\![\Gamma]\!] \times [\![A]\!] \to [\![B]\!] \,.$$

An interpretation of the λ-calculus of a theory $\mathbb{T}$ is a *model* of the theory if it satisfies all axioms of $\mathbb{T}$. This means that, for every axiom $\Gamma \mid t = u : A$, the interpretations of u and t coincide as arrows in $\mathcal{C}$,

$$[\![\Gamma \mid u : A]\!] = [\![\Gamma \mid t : A]\!] : [\![\Gamma]\!] \longrightarrow [\![A]\!].$$

It follows that all equations provable in $\mathbb{T}$ are satisfied in the model, by the following fact.

Proposition 3.8.1 (Soundness). *If $\mathbb{T}$ is a λ-theory and $[\![-]\!]$ a model of $\mathbb{T}$ in a cartesian closed category $\mathcal{C}$, then for every equation in context $\Gamma \mid t = u : A$ that is provable from the axioms of $\mathbb{T}$, we have*

$$[\![\Gamma \mid u : A]\!] = [\![\Gamma \mid t : A]\!] : [\![\Gamma]\!] \longrightarrow [\![A]\!] \,.$$

Briefly, for all $\mathbb{T}$-models $[\![-]\!]$,

$$\mathbb{T} \vdash (\Gamma \mid t = u : A) \quad \textit{implies} \quad [\![-]\!] \models (\Gamma \mid t = u : A) \,.$$

Remark 3.8.2 (Inhabitation). There is another notion of provability for the λ-calculus, related to the Curry-Howard correspondence of section 3.1, relating it to propositional logic. If we regard types as "propositions" rather than structures, and terms as "proofs" rather than operations, then it is more natural to ask whether there even *is* a term $a : A$ of some type, than whether two terms of the same type are equal $s = t : A$. This only makes sense when A is considered in the empty context $\cdot \vdash A$, rather than $\Gamma \vdash A$ for non-empty Γ (consider the case where $\Gamma = x : A, \dots$). We say that a type A is *inhabited* (by a closed term) when there is some $\vdash a : A$, and regard an inhabited type A a *provable.* In this sense, there is another notion of soundness as follows.

Proposition 3.8.3 (Inhabitation soundness)**.** *If $\mathbb{T}$ is a λ-theory and $[\![-]\!]$ a model of $\mathbb{T}$ in a cartesian closed category $\mathcal{C}$, then for every closed type A that is inhabited in $\mathbb{T}$ by a closed term, $\vdash a : A$, there is a corresponding point*

$$[\![a]\!] : 1 \to [\![A]\!]\,,$$

in $\mathcal{C}$. Briefly, for all $\mathbb{T}$-models $[\![-]\!]$,

$$\vdash a : A \quad \textit{implies} \quad [\![a]\!] : 1 \to [\![A]\!]\,.$$

This follows immediately from the fact that $[\![\cdot]\!] = 1$ for $\Gamma = \cdot$ the empty context.

Example 3.8.4.

1. A model of an algebraic theory $\mathbb{A}$, extended to a λ-theory $\mathbb{A}_\lambda$ as in Example 3.7.5, taken in a CCC $\mathcal{C}$, is just a model of the algebraic theory $\mathbb{A}$ in the underlying finite product category $|\mathcal{C}|$ of $\mathcal{C}$. A difference, however, is that in defining the *category of models*
$$\mathsf{Mod}_\times(\mathbb{A}, |\mathcal{C}|)$$
we can take all homomorphism of models the algebraic theory as arrows, while the arrows in the category
$$\mathsf{Mod}_\lambda(\mathbb{A}_\lambda, \mathcal{C})$$
of λ-models are best taken to be isomorphisms, for which one has an obvious way to deal with the contravariance of the function type $[\![A \to B]\!] = [\![B]\!]^{[\![A]\!]}$.

2. A model of the theory of a reflexive type, Example 3.7.8, in Set must be the one-element 1 (prove this!). Fortunately, the exponentials in categories of presheaves are *not* computed pointwise - otherwise it would follow that there would be no non-trivial models at all in small categories! That there are such non-trivial models is an important fact in the semantics of programming languages and the subject called *domain theory*. A basic paper in which this is shown is [**?**].

3. A model of a propositional theory $\mathbb{T}$, regarded as a λ-theory, in a CCC poset P is the same thing as before: an interpretation of the atomic propositions $p_1, p_2, ...$ of $\mathbb{T}$ as elements $[\![p_1]\!], [\![p_2]\!], ... \in P$, such that the axioms $\phi_1, \phi_2, ...$ of $\mathbb{T}$ are all sent to $1 \in P$ by the extension of $[\![-]\!]$ to all formulas, i.e. $[\![\phi_1]\!] = [\![\phi_2]\!] = ... = 1 \in P$.

3.9 Functorial semantics

In Section **??** we saw that algebraic theories can be viewed as categories, cf. Definition **??**, and models as functors, cf. Definition **??**, and we arranged this categorical analysis of the traditional relationship between syntax and sematics into the framework that we called *functorial semantics*. The same can be done with λ-theories and their models in CCCs. The first step is to build a *syntactic category* $\mathcal{C}_\mathbb{T}$ from a λ-theory $\mathbb{T}$. This is done as follows:

- The objects of $\mathcal{C}_\mathbb{T}$ are the types of $\mathbb{T}$.

- Morphisms $A \to B$ are terms in context
$$[x : A \mid t : B] ,$$
where two such terms $x : A \mid t : B$ and $x : A \mid u : B$ represent the same morphism when $\mathbb{T}$ proves $x : A \mid t = u : B$.

- Composition of the terms
$$[x : A \mid t : B] : A \longrightarrow B \qquad \text{and} \qquad [y : B \mid u : C] : B \longrightarrow C$$
is the term obtained by substituting t for y in u:
$$[x : A \mid u[t/y] : C] : A \longrightarrow C .$$

- The identity morphism on A is the term $[x : A \mid x : A]$.

Proposition 3.9.1. *The syntactic category $\mathcal{C}_{\mathbb{T}}$ built from a λ-theory is cartesian closed.*

Proof. We omit the equivalence classes brackets $[x : A \mid t : B]$ and treat equivalent terms as equal.

- The terminal object is the unit type $\mathtt{1}$. For any type A the unique morphism $!_A : A \to \mathtt{1}$ is
$$x : A \mid * : \mathtt{1} .$$
This morphism is unique because
$$\Gamma \mid t = \star : \mathtt{1}$$
is an axiom for the terms of unit type $\mathtt{1}$.

- The product of objects A and B is the type $A \times B$. The first and the second projections are the terms
$$p : A \times B \mid \mathtt{fst}\, p : A , \qquad\qquad p : A \times B \mid \mathtt{snd}\, p : B .$$
Given morphisms
$$z : C \mid t : A , \qquad\qquad z : C \mid u : B ,$$
the term
$$z : C \mid \langle t, u \rangle : A \times B$$
represents the unique morphism satisfying
$$z : C \mid \mathtt{fst}\, \langle t, u \rangle = t : A , \qquad\qquad z : C \mid \mathtt{snd}\, \langle t, u \rangle = u : B .$$
Indeed, if $\mathtt{fst}\, s = t$ and $\mathtt{snd}\, s = u$ then
$$s = \langle \mathtt{fst}\, s, \mathtt{snd}\, s \rangle = \langle t, u \rangle .$$

- The exponential of objects A and B is the type $A \to B$ with the evaluation morphism

$$p : (A \to B) \times A \mid (\mathtt{fst}\, p)(\mathtt{snd}\, p) : B\ .$$

The transpose of the morphism $p : C \times A \mid t : B$ is

$$z : C \mid \lambda x : A\,.\,(t[\langle z, x\rangle/p]) : A \to B\ .$$

Showing that this is the transpose of t amounts to

$$(\lambda x : A\,.\,(t[\langle \mathtt{fst}\, p, x\rangle/p]))(\mathtt{snd}\, p) = t[\langle \mathtt{fst}\, p, \mathtt{snd}\, p\rangle/p] = t[p/p] = t\ ,$$

which is a valid chain of equations in λ-calculus. The transpose is unique, because any morphism $z : C \mid s : A \to B$ that satisfies

$$(s[\mathtt{fst}\, p/z])(\mathtt{snd}\, p) = t$$

is equal to $\lambda x : A\,.\,(t[\langle z, x\rangle/p])$. First observe that

$$t[\langle z, x\rangle/p] = (s[\mathtt{fst}\, p/z])(\mathtt{snd}\, p)[\langle z, x\rangle/p] = \\ (s[\mathtt{fst}\,\langle z, x\rangle/z])(\mathtt{fst}\,\langle z, x\rangle) = (s[z/z])\, x = s\, x\ .$$

Therefore,

$$\lambda x : A\,.\,(t[\langle z, x\rangle/p]) = \lambda x : A\,.\,(s\, x) = s\ ,$$

as required.

□

The syntactic category allows us to "redefine" models as functors. More precisely, we have the following.

Lemma 3.9.2. *A* model $[\![-]\!]$ *of a λ-theory $\mathbb{T}$ in a cartesian closed category $\mathcal{C}$ determines a cartesian closed functor $M : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}$ with*

$$M(A) = [\![A]\!], \quad M(c) = [\![c]\!], \tag{3.28}$$

for all basic types A and basic constants c. Moreover, M is unique up to a unique isomorphism of CCC functors, in the sense that given another model N satisfying (3.28)*, there is a unique natural iso $M \cong N$ determined inductively by the comparison maps $M(1) \cong N(1)$,*

$$M(A \times B) \;\cong\; MA \times MB \;\cong\; NA \times NB \;\cong\; N(A \times B)\,,$$

and similarly for $M(B^A)$.

Proof. Straightforward. □

We now have the usual functorial semantics theorem:

Theorem 3.9.3. *For any λ-theory $\mathbb{T}$, the syntactic category $\mathcal{C}_{\mathbb{T}}$ classifies $\mathbb{T}$-models, in the sense that for any cartesian closed category $\mathcal{C}$ there is an equivalence of categories*

$$\mathsf{Mod}_{\lambda}\big(\mathbb{T}, \mathcal{C}\big) \;\simeq\; \mathsf{CCC}\big(\mathcal{C}_{\mathbb{T}}, \mathcal{C}\big)\,, \tag{3.29}$$

naturally in $\mathcal{C}$.

Proof. Note that the categories involved in (3.29) are actually groupoids, as discussed in example 3.8.4(1). The only thing remaining to show is that given a model $[\![-]\!]^M$ in a CCC $\mathcal{C}$ and a CCC functor $f : \mathcal{C} \to \mathcal{D}$, there is an induced model $[\![-]\!]^{fM}$ in $\mathcal{D}$, given by the interpretation $[\![A]\!]^{fM} = f[\![A]\!]^M$. This is straigtforward, just as for algebraic theories. $\square$

We can now proceed just as we did in the case of algebraic theories and prove that the semantics of λ-theories in cartesian closed categories is *complete*, in virtue of the syntactic construction of the classifying category $\mathcal{C}_{\mathbb{T}}$. Specifically, a λ-theory $\mathbb{T}$ has a canonical interpretation $[-]$ in the syntactic category $\mathcal{C}_{\mathbb{T}}$, which interprets a basic type A as itself, and a basic constant c of type A as the morphism $[x : \mathtt{1} \mid c : A]$. The canonical interpretation is a model of $\mathbb{T}$, also known as the *syntactic model*, in virtue of the definition of the equivalence relation $[-]$ on terms. In fact, it is a *logically generic* model of $\mathbb{T}$, because by the construction of $\mathcal{C}_{\mathbb{T}}$, for any terms $\Gamma \mid u : A$ and $\Gamma \mid t : A$, we have

$$\begin{aligned} \mathbb{T} \vdash (\Gamma \mid u = t : A) \quad &\Longleftrightarrow \quad [\Gamma \mid u : A] = [\Gamma \mid t : A] \\ &\Longleftrightarrow \quad [-] \models \Gamma \mid u = t : A\,. \end{aligned}$$

For the record, we therefore have shown:

Proposition 3.9.4. *For any λ-theory $\mathbb{T}$,*

$$\mathbb{T} \vdash (\Gamma \mid t = u : A) \quad \textit{if, and only if,} \quad [-] \models (\Gamma \mid t = u : A) \textit{ for the syntactic model } [-].$$

Of course, the syntactic model $[-]$ is the one associated under (3.29) to the identity functor $\mathcal{C}_{\mathbb{T}} \to \mathcal{C}_{\mathbb{T}}$, i.e. it is the *universal* one. It therefore satisfies an equation just in case the equation holds in all models, by the classifying property of $\mathcal{C}_{\mathbb{T}}$, and the preservation of satisfaction of equations by CCC functors (Proposition 3.8.1).

Corollary 3.9.5. *For any λ-theory $\mathbb{T}$,*

$$\mathbb{T} \vdash (\Gamma \mid t = u : A) \quad \textit{if, and only if,} \quad M \models (\Gamma \mid t = u : A) \textit{ for every CCC model } M.$$

Moreover, a closed type A is inhabited $\vdash a : A$ if, and only if, there is a point $1 \to [\![A]\!]$ in every model M.

3.10 The internal language of a CCC

We can take the correspondence between λ-theories and CCCs one step further and organize the former into a category, which is then equivalent to that of the latter. For this we first need to define a suitable notion of *morphism of theories*. A *translation* $\tau : \mathbb{T} \to \mathbb{U}$ of a λ-theory $\mathbb{T}$ into a λ-theory $\mathbb{U}$ is given by the following data:

1. For each basic type A in $\mathbb{T}$ a type τA in $\mathbb{U}$. The translation is then extended to all types by the rules

$$\tau\mathsf{1} = \mathsf{1} , \qquad \tau(A \times B) = \tau A \times \tau B , \qquad \tau(A \to B) = \tau A \to \tau B .$$

2. For each basic constant c of type A in $\mathbb{A}$ a term τc of type τA in $\mathbb{U}$. The translation of terms is then extended to all terms by the rules

$$\begin{aligned} \tau(\mathtt{fst}\, t) &= \mathtt{fst}\,(\tau t) , & \tau(\mathtt{snd}\, t) &= \mathtt{snd}\,(\tau t) , \\ \tau\langle t, u\rangle &= \langle \tau t, \tau u\rangle , & \tau(\lambda x : A \,.\, t) &= \lambda x : \tau A \,.\, \tau t , \\ \tau(t\, u) &= (\tau t)(\tau u) , & \tau x &= x \quad \text{(if x is a variable)} . \end{aligned}$$

A context $\Gamma = x_1 : A_1, \ldots, x_n : A_n$ is translated by τ to the context

$$\tau\Gamma = x_1 : \tau A_1, \ldots, x_n : \tau A_n .$$

Furthermore, a translation is required to preserve the axioms of $\mathbb{T}$: if $\Gamma \mid t = u : A$ is an axiom of $\mathbb{T}$ then $\mathbb{U}$ proves $\tau\Gamma \mid \tau t = \tau u : \tau A$. It then follows that all equations proved by $\mathbb{T}$ are translated to valid equations in $\mathbb{U}$.

A moment's consideration shows that a translation $\tau : \mathbb{T} \to \mathbb{U}$ is the same thing as a model of $\mathbb{T}$ in $\mathcal{C}_{\mathbb{U}}$, despite being specified entirely syntactically. Clearly, λ-theories and translations between them form a category. Translations compose as functions, therefore composition is associative. The identity translation $\iota_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}$ translates every type to itself and every constant to itself. It corresponds to the canonical interpretation of $\mathbb{T}$ in $\mathcal{C}_{\mathbb{T}}$.

Definition 3.10.1. $\lambda\mathsf{Thr}$ is the category whose objects are λ-theories and morphisms are translations between them.

Let $\mathcal{C}$ be a small cartesian closed category. There is a λ-theory $\mathbb{L}(\mathcal{C})$ that corresponds to $\mathcal{C}$, called the *internal language of* $\mathcal{C}$, defined as follows:

1. For every object $A \in \mathcal{C}$ there is a basic type $\ulcorner A \urcorner$.

2. For every morphism $f : A \to B$ there is a basic constant $\ulcorner f \urcorner$ whose type is $\ulcorner A \urcorner \to \ulcorner B \urcorner$.

3. For every $A \in \mathcal{C}$ there is an axiom

$$x : \ulcorner A \urcorner \mid \ulcorner \mathsf{1}_A \urcorner x = x : \ulcorner A \urcorner .$$

4. For all morphisms $f : A \to B$, $g : B \to C$, and $h : A \to C$ such that $h = g \circ f$, there is an axiom
$$x : \ulcorner A \urcorner \mid \ulcorner h \urcorner x = \ulcorner g \urcorner (\ulcorner f \urcorner x) : \ulcorner C \urcorner \,.$$

5. There is a constant
$$\mathtt{T} : \mathtt{1} \to \ulcorner 1 \urcorner \,,$$
and for all $A, B \in \mathcal{C}$ there are constants
$$\mathtt{P}_{A,B} : \ulcorner A \urcorner \times \ulcorner B \urcorner \to \ulcorner A \times B \urcorner \,, \qquad \mathtt{E}_{A,B} : (\ulcorner A \urcorner \to \ulcorner B \urcorner) \to \ulcorner B^A \urcorner \,.$$
They satisfy the following axioms:
$$\begin{aligned}
u : \ulcorner 1 \urcorner &\mid \mathtt{T}* = u : \ulcorner 1 \urcorner \\
z : \ulcorner A \times B \urcorner &\mid \mathtt{P}_{A,B} \langle \ulcorner \pi_0 \urcorner z, \ulcorner \pi_1 \urcorner z \rangle = z : \ulcorner A \times B \urcorner \\
w : \ulcorner A \urcorner \times \ulcorner B \urcorner &\mid \langle \ulcorner \pi_0 \urcorner (\mathtt{P}_{A,B} w), \ulcorner \pi_1 \urcorner (\mathtt{P}_{A,B} w) \rangle = w : \ulcorner A \urcorner \times \ulcorner B \urcorner \\
f : \ulcorner B^A \urcorner &\mid \mathtt{E}_{A,B}(\lambda x : \ulcorner A \urcorner . (\ulcorner \mathsf{ev}_{A,B} \urcorner (\mathtt{P}_{A,B} \langle f, x \rangle))) = f : \ulcorner B^A \urcorner \\
f : \ulcorner A \urcorner \to \ulcorner B \urcorner &\mid \lambda x : \ulcorner A \urcorner . (\ulcorner \mathsf{ev}_{A,B} \urcorner (\mathtt{P}_{A,B} \langle (\mathtt{E}_{A,B} f), x \rangle)) = f : \ulcorner A \urcorner \to \ulcorner B \urcorner
\end{aligned}$$

The purpose of the constants $\mathtt{T}$, $\mathtt{P}_{A,B}$, $\mathtt{E}_{A,B}$, and the axioms for them is to ensure the isomorphisms $\ulcorner 1 \urcorner \cong \mathtt{1}$, $\ulcorner A \times B \urcorner \cong \ulcorner A \urcorner \times \ulcorner B \urcorner$, and $\ulcorner B^A \urcorner \cong \ulcorner A \urcorner \to \ulcorner B \urcorner$. Types A and B are said to be *isomorphic* if there are terms
$$x : A \mid t : B \,, \qquad y : B \mid u : A \,,$$
such that $\mathbb{T}$ proves
$$x : A \mid u[t/y] = x : A \,, \qquad y : B \mid t[u/x] = y : B \,.$$
Furthermore, an *equivalence of theories* $\mathbb{T}$ and $\mathbb{U}$ is a pair of translations
$$\begin{tikzcd} \mathbb{T} \arrow[r, bend left, "\tau"] & \mathbb{U} \arrow[l, bend left, "\sigma"] \end{tikzcd}$$
such that, for any type A in $\mathbb{T}$ and any type B in $\mathbb{U}$,
$$\sigma(\tau A) \cong A \,, \qquad \tau(\sigma B) \cong B \,.$$

The assignment $\mathcal{C} \mapsto \mathbb{L}(\mathcal{C})$ extends to a functor
$$\mathbb{L} : \mathsf{CCC} \to \lambda\mathsf{Thr} \,,$$
where CCC is the category of small cartesian closed categories and functors between them that preserve finite products and exponentials. Such functors are also called *cartesian closed functors* or *ccc functors*. If $F : \mathcal{C} \to \mathcal{D}$ is a cartesian closed functor then $\mathbb{L}(F) : \mathbb{L}(\mathcal{C}) \to \mathbb{L}(\mathcal{D})$ is the translation given by:

1. A basic type $\ulcorner A \urcorner$ is translated to $\ulcorner FA \urcorner$.

2. A basic constant $\ulcorner f \urcorner$ is translated to $\ulcorner Ff \urcorner$.

3. The basic constants $\mathtt{T}$, $\mathtt{P}_{A,B}$ and $\mathtt{E}_{A,B}$ are translated to $\mathtt{T}$, $\mathtt{P}_{FA,BA}$ and $\mathtt{E}_{FA,FB}$, respectively.

We now have a functor $\mathbb{L} : \mathsf{CCC} \to \lambda\mathsf{Thr}$. How about the other direction? We already have the construction of syntactic category which maps a λ-theory $\mathbb{T}$ to a small cartesian closed category $\mathcal{C}_{\mathbb{T}}$. This extends to a functor

$$\mathcal{C} : \lambda\mathsf{Thr} \to \mathsf{CCC} ,$$

because a translation $\tau : \mathbb{T} \to \mathbb{U}$ induces a functor $\mathcal{C}_\tau : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}_{\mathbb{U}}$ in an obvious way: a basic type $A \in \mathcal{C}_{\mathbb{T}}$ is mapped to the object $\tau A \in \mathcal{C}_{\mathbb{U}}$, and a basic constant $x : \mathtt{1} \mid c : A$ is mapped to the morphism $x : \mathtt{1} \mid \tau c : A$. The rest of $\mathcal{C}_\tau$ is defined inductively on the structure of types and terms.

Theorem 3.10.2. *The functors $\mathbb{L} : \mathsf{CCC} \to \lambda\mathsf{Thr}$ and $\mathcal{C} : \lambda\mathsf{Thr} \to \mathsf{CCC}$ constitute an equivalence of categories, "up to equivalence". This means that for any $\mathcal{C} \in \mathsf{CCC}$ there is an equivalence of catgories*

$$\mathcal{C} \simeq \mathcal{C}_{\mathbb{L}(\mathcal{C})} ,$$

and for any $\mathbb{T} \in \lambda\mathsf{Thr}$ there is an equivalence of theories

$$\mathbb{T} \simeq \mathbb{L}(\mathcal{C}_{\mathbb{T}}) .$$

Proof. For a small cartesian closed category $\mathcal{C}$, consider the functor $\eta_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}_{\mathbb{L}(\mathcal{C})}$, defined for an object $A \in \mathcal{C}$ and $f : A \to B$ in $\mathcal{C}$ by

$$\eta_{\mathcal{C}} A = \ulcorner A \urcorner , \qquad \eta_{\mathcal{C}} f = (x : \ulcorner A \urcorner \mid \ulcorner f \urcorner x : \ulcorner B \urcorner) .$$

To see that $\eta_{\mathcal{C}}$ is a functor, observe that $\mathbb{L}(\mathcal{C})$ proves, for all $A \in \mathcal{C}$,

$$x : \ulcorner A \urcorner \mid \ulcorner \mathtt{1}_A \urcorner x = x : \ulcorner A \urcorner$$

and for all $f : A \to B$ and $g : B \to C$,

$$x : \ulcorner A \urcorner \mid \ulcorner g \circ f \urcorner x = \ulcorner g \urcorner (\ulcorner f \urcorner x) : \ulcorner C \urcorner .$$

To see that $\eta_{\mathcal{C}}$ is an equivalence of categories, it suffices to show that for every object $X \in \mathcal{C}_{\mathbb{L}(\mathcal{C})}$ there exists an object $\theta_{\mathcal{C}} X \in \mathcal{C}$ such that $\eta_{\mathcal{C}}(\theta_{\mathcal{C}} X) \cong X$. The choice map $\theta_{\mathcal{C}}$ is defined inductively by

$$\theta_{\mathcal{C}} \mathtt{1} = \mathtt{1} , \qquad \theta_{\mathcal{C}} \ulcorner A \urcorner = A ,$$
$$\theta_{\mathcal{C}}(Y \times Z) = \theta_{\mathcal{C}} X \times \theta_{\mathcal{C}} Y , \qquad \theta_{\mathcal{C}}(Y \to Z) = (\theta_{\mathcal{C}} Z)^{\theta_{\mathcal{C}} Y} .$$

We skip the verification that $\eta_{\mathcal{C}}(\theta_{\mathcal{C}} X) \cong X$. In fact, $\theta_{\mathcal{C}}$ can be extended to a functor $\theta_{\mathcal{C}} : \mathcal{C}_{\mathbb{L}(\mathcal{C})} \to \mathcal{C}$ so that $\theta_{\mathcal{C}} \circ \eta_{\mathcal{C}} \cong 1_{\mathcal{C}}$ and $\eta_{\mathcal{C}} \circ \theta_{\mathcal{C}} \cong 1_{\mathcal{C}_{\mathbb{L}(\mathcal{C})}}$.

Given a λ-theory $\mathbb{T}$, we define a translation $\tau_{\mathbb{T}} : \mathbb{T} \to \mathbb{L}(\mathcal{C}_{\mathbb{T}})$. For a basic type A let

$$\tau_{\mathbb{T}} A = \ulcorner A \urcorner \,.$$

The translation $\tau_{\mathbb{T}} c$ of a basic constant c of type A is

$$\tau_{\mathbb{T}} c = \ulcorner x : \mathtt{1} \mid c : \tau_{\mathbb{T}} A \urcorner \,.$$

In the other direction we define a translaton $\sigma_{\mathbb{T}} : \mathbb{L}(\mathcal{C}_{\mathbb{T}}) \to \mathbb{T}$ as follows. If $\ulcorner A \urcorner$ is a basic type in $\mathbb{L}(\mathcal{C}_{\mathbb{T}})$ then

$$\sigma_{\mathbb{T}} \ulcorner A \urcorner = A \,,$$

and if $\ulcorner x : A \mid t : B \urcorner$ is a basic constant of type $\ulcorner A \urcorner \to \ulcorner B \urcorner$ then

$$\sigma_{\mathbb{T}} \ulcorner x : A \mid t : B \urcorner = \lambda x : A \,.\, t \,.$$

The basic constants $\mathtt{T}$, $\mathtt{P}_{A,B}$ and $\mathtt{E}_{A,B}$ are translated by $\sigma_{\mathbb{T}}$ into

$$\begin{aligned} \sigma_{\mathbb{T}} \, \mathtt{T} &= \lambda x : \mathtt{1} \,.\, x \,, \\ \sigma_{\mathbb{T}} \, \mathtt{P}_{A,B} &= \lambda p : A \times B \,.\, p \,, \\ \sigma_{\mathbb{T}} \, \mathtt{E}_{A,B} &= \lambda f : A \to B \,.\, f \,. \end{aligned}$$

If A is a type in $\mathbb{T}$ then $\sigma_{\mathbb{T}}(\tau_{\mathbb{T}} A) = A$. For the other direction, we would like to show, for any type X in $\mathbb{L}(\mathcal{C}_{\mathbb{T}})$, that $\tau_{\mathbb{T}}(\sigma_{\mathbb{T}} X) \cong X$. We prove this by induction on the structure of type X:

1. If $X = \mathtt{1}$ then $\tau_{\mathbb{T}}(\sigma_{\mathbb{T}} \mathtt{1}) = \mathtt{1}$.
2. If $X = \ulcorner A \urcorner$ is a basic type then A is a type in $\mathbb{T}$. We proceed by induction on the structure of A:
   (a) If $A = \mathtt{1}$ then $\tau_{\mathbb{T}}(\sigma_{\mathbb{T}} \ulcorner \mathtt{1} \urcorner) = \mathtt{1}$. The types $\mathtt{1}$ and $\ulcorner \mathtt{1} \urcorner$ are isomorphic via the constant $\mathtt{T} : \mathtt{1} \to \ulcorner \mathtt{1} \urcorner$.
   (b) If A is a basic type then $\tau_{\mathbb{T}}(\sigma_{\mathbb{T}} \ulcorner A \urcorner) = \ulcorner A \urcorner$.
   (c) If $A = B \times C$ then $\tau_{\mathbb{T}}(\sigma_{\mathbb{T}} \ulcorner B \times C \urcorner) = \ulcorner B \urcorner \times \ulcorner C \urcorner$. But we know $\ulcorner B \urcorner \times \ulcorner C \urcorner \cong \ulcorner B \times C \urcorner$ via the constant $\mathtt{P}_{A,B}$.
   (d) The case $A = B \to C$ is similar.
3. If $X = Y \times Z$ then $\tau_{\mathbb{T}}(\sigma_{\mathbb{T}}(Y \times Z)) = \tau_{\mathbb{T}}(\sigma_{\mathbb{T}} Y) \times \tau_{\mathbb{T}}(\sigma_{\mathbb{T}} Z)$. By induction hypothesis, $\tau_{\mathbb{T}}(\sigma_{\mathbb{T}} Y) \cong Y$ and $\tau_{\mathbb{T}}(\sigma_{\mathbb{T}} Z) \cong Z$, from which we easily obtain
$$\tau_{\mathbb{T}}(\sigma_{\mathbb{T}} Y) \times \tau_{\mathbb{T}}(\sigma_{\mathbb{T}} Z) \cong Y \times Z \,.$$
4. The case $X = Y \to Z$ is similar.

□

Exercise 3.10.3. In the previous proof we defined, for each $\mathcal{C} \in \mathsf{CCC}$, a functor $\eta_C : \mathcal{C} \to \mathcal{C}_{\mathbb{L}(\mathcal{C})}$. Verify that this determines a natural transformation $\eta : 1_{\mathsf{CCC}} \Longrightarrow \mathcal{C} \circ \mathbb{L}$. Can you say anything about naturality of the translations $\tau_{\mathbb{T}}$ and $\sigma_{\mathbb{T}}$? What would it even mean for a translation to be natural?

Remark 3.10.4. Discussion of untyped λ-calculus: we do not know that the syntactic construction is non-trivial. But existence of non-trivial models tells us that it is not (which implies a suitable notion of consistency of untyped λ-calculus).

Give an untyped model satisfying β-reduction. Refer to literature for $\beta\eta$-models.

Remark 3.10.5. Adding coproducts $0, A + B$, also for presheaf models.

3.11 Embedding and completeness theorems

We have considered the λ-calculus as a common generalization of both propositional logic, modelled by poset CCCs such as Boolean and Heyting algebras, and equational logic, modelled by finite product categories. Accordingly, there are then two different notions of "provability", as discussied in Remark 3.8.2; namely the derivability of a closed term $\vdash a : A$, and the derivability of an equation between two (not necessarily closed) terms of the same type $\Gamma \vdash s = t : A$. With respect to the semantics, there are then two different corresponding notions of soundness and completeness: for "inhabitation" of types, and for equality of terms. We consider special cases of these notions in more detail below.

Conservativity

With regard to the former notion, inhabitation, one can also consider the question of how it compares with simple provability in *propositional logic*: e.g. a positive propositional formula ϕ in the variables $p_1, p_2, ..., p_n$ obviously determines a type Φ in the corresponding λ-theory $\mathbb{T}(X_1, X_2, ..., X_n)$ over n basic type symbols. What is the relationship between provability in positive propositional logic, $\mathsf{PPL} \vdash \phi$, and inhabitation in the associated λ-theory, $\mathbb{T}(X_1, X_2, ..., X_n) \vdash t : \Phi$? Let us call this the question of *conservativity* of λ-calculus over PPL. According to the basic idea of the Curry-Howard correspondence from Section 3.1, the λ-calculus is essentially the "proof theory of PPL". So one should expect that starting from an inhabited type Φ, a derivation of a term $\mathbb{T}(X_1, X_2, ..., X_n) \vdash t : \Phi$ should result in a corresponding proof of ϕ in PPL just by "rubbing out the proof terms". Conversely, given a provable formula $\vdash \phi$, one should be able to annotate a proof of it in PPL to obtain a derivation of a term $\mathbb{T}(X_1, X_2, ..., X_n) \vdash t : \Phi$ in the λ-calculus (although perhaps not the same term that one started with, if the proof was obtained from rubbing out a term).

We can make this idea precise semantically as follows. Write $|\mathcal{C}|$ for the poset reflection of a category $\mathcal{C}$, that is, the left adjoint to the inclusion $i : \mathsf{Pos} \hookrightarrow \mathsf{Cat}$, and let $\eta : \mathcal{C} \to |\mathcal{C}|$ be the unit of the adjunction.

Lemma 3.11.1. *If $\mathcal{C}$ is cartesian closed, then so is $|\mathcal{C}|$, and $\eta : \mathcal{C} \to |\mathcal{C}|$ preserves the CCC structure.*

Proof. Exercise! □

Exercise 3.11.2. Prove Lemma 3.11.1.

Corollary 3.11.3. *The syntactic category $\mathsf{PPC}(p_1, p_2, ..., p_n)$ of the positive propositional calculus on n propositional variables is the poset reflection the syntactic category $\mathcal{C}_{\mathbb{T}(X_1,X_2,...,X_n)}$ of the λ-theory $\mathbb{T}(X_1, X_2, ..., X_n)$,*

$$|\mathcal{C}_{\mathbb{T}(X_1,X_2,...,X_n)}| \cong \mathsf{PPC}(p_1, p_2, ..., p_n)\,.$$

Proof. We already know that $\mathcal{C}_{\mathbb{T}(X_1,X_2,...,X_n)}$ is the free cartesian closed category on n generating objects, and that $\mathsf{PPC}(p_1, p_2, ..., p_n)$ is the free cartesian closed poset on n generating elements. We have an obvious CCC map

$$\mathcal{C}_{\mathbb{T}(X_1,X_2,...,X_n)} \longrightarrow \mathsf{PPC}(p_1, p_2, ..., p_n)$$

taking generators to generators, and it extends along the quotient map to $|\mathcal{C}_{\mathbb{T}(X_1,X_2,...,X_n)}|$ by the universal property of the poset reflection. Thus it suffices to show that the quotient map preserves, and indeed creates, the CCC structure on $|\mathcal{C}_{\mathbb{T}(X_1,X_2,...,X_n)}|$, which follows from the Lemma 3.11.1. □

Remark 3.11.4. Corollary 3.11.3 can be extended to other systems of type theory and logic, with further operations such as CCCs with sums $0, A + B$ ("bicartesian closed categories"), and the full intuitionistic propositional calculus IPC with the logical operations $\bot$ and $p \vee q$. We leave this as a topic for the interested student.

Completeness

As was the case for algebraic and propositional logics, the fact that there is a generic model (Proposition 3.9.4) allows the general completeness theorem stated in Corollary 3.9.5 to be specialized to various classes of special models, via embedding (or "representation") theorems, this time for CCCs, rather than for finite product categories or Boolean/Heyting algebras. We shall consider three such cases: "variable" models, topological models, and Kripke models. Note that this follows that same pattern that we saw for the "proof irrelevant" case of propostional logic, but in some cases, the proofs require much more sophisticated methods.

Variable models

By a *variable model* of the λ-calculus we mean one in a ccc of the form $\widehat{\mathbb{C}} = \mathsf{Set}^{\mathbb{C}^{\mathsf{op}}}$, i.e. presheaves on a category $\mathbb{C}$. We regard such a model as "varying over $\mathbb{C}$", just as a presheaf of groups on the simplex category Δ may be seen both as a simplicial group – a simplicial object in the category of groups – and as a group in the category $\mathsf{Set}^{\Delta^{\mathsf{op}}}$ of simplicial sets. The basic fact that we use in specializing Proposition 3.9.4 to such variable models is the following, which is one of the fundamental facts of categorical semantics.

Lemma 3.11.5. *For any small category $\mathbb{C}$, the Yoneda embedding*

$$\mathsf{y} : \mathbb{C} \hookrightarrow \mathsf{Set}^{\mathbb{C}^{\mathsf{op}}}$$

preserves cartesian closed structure.

Proof. We just evaluate $\mathsf{y}A(X) = \mathbb{C}(X, A)$. It is clear that $\mathsf{y}1(X) = \mathbb{C}(X, 1) \cong 1$ naturally in X, and that $\mathsf{y}(A \times B)(X) = \mathbb{C}(X, A \times B) \cong \mathbb{C}(X, A) \times \mathbb{C}(X, B) \cong (\mathsf{y}A \times \mathsf{y}B)(X)$ for all A, B, X, naturally in all three arguments. For $B^A \in \mathbb{C}$, we then have

$$\mathsf{y}(B^A)(X) = \mathbb{C}(X, B^A) \cong \mathbb{C}(X \times A, B) \cong \widehat{\mathbb{C}}(\mathsf{y}(X \times A), \mathsf{y}B) \cong \widehat{\mathbb{C}}(\mathsf{y}X \times \mathsf{y}A, \mathsf{y}B),$$

since y is full and faithful and preserves $\times$. But now recall that the exponential Q^P of presheaves P, Q is defined at X by the specification

$$Q^P(X) \;=\; \widehat{\mathbb{C}}(\mathsf{y}X \times P, Q)\,.$$

So $\widehat{\mathbb{C}}(\mathsf{y}X \times \mathsf{y}A, \mathsf{y}B) = \mathsf{y}B^{\mathsf{y}A}(X)$, and we're done. □

Proposition 3.11.6. *For any λ-theory $\mathbb{T}$, we have the following:*

1. *A type A is inhabited,*
$$\mathbb{T} \vdash a : A$$
if, and only if, there is a point
$$1 \to [\![A]\!]$$
in every model $[\![-]\!]$ in a CCC of presheaves $\mathsf{Set}^{\mathbb{C}^{\mathsf{op}}}$ on a small category $\mathbb{C}$.

2. *For any terms $\Gamma \mid s, t : A$,*
$$\mathbb{T} \vdash (\Gamma \mid s = t : A)$$
if, and only if,
$$[\![\Gamma \vdash s : A]\!] = [\![\Gamma \vdash t : A]\!] : [\![\Gamma]\!] \longrightarrow [\![A]\!]$$
for every such presheaf model.

Proof. We can specialize the general completeness statement of Corollary 3.9.5 to CCCs of the form $\widehat{\mathbb{C}}$ using Lemma 3.11.5, together with the fact that the Yoneda embedding is full (and therefore reflects inhabitation) and faithful (and therefore reflects equations). □

Topological models

See [**?**]

Kripke models

See [**?**]

Models based on computability and continuity

See [?]

3.12 Modal operators and monads

See [?]

Chapter 4

First-Order Logic

Having considered equational theories, we now move on to first-order logic. This is the usual predicate logic with propositional connectives like $\wedge$ and $\Rightarrow$, and quantifiers $\forall$ and $\exists$. The general approach to studying logic via category theory is to determine categorical structures that model the first-order logical operations, or a suitable fragment of it, and then consider categories with these structures and functors that preserve them. Here adjoint functors play an imporant role, as the basic logical operations are recognized as adjoints. We again show that the semantics is "functorial", meaning that models of a theory are functors that preserve suitable categorical structure. We again construct classifying categories representing theories, which are the counterparts of the algebraic theories that we have already met.

Let us demonstrate our approach informally with an example. In section **??** we considered models of algebraic theories in categories with finite products. Recall that e.g. a group is a structure of the form:

$$e : \mathbf{1} \to G\,, \qquad m : G \times G \to G\,, \qquad i : G \to G\,.$$

for which, moreover, certain diagrams built from these basic arrows must commute. We can express some properties of groups in terms of further equations, for example commutativity

$$x \cdot y = y \cdot x\,.$$

As we saw, such equations can be interpreted in any category with finite products. This provides a large scope for categorical semantics of algebraic theories.

However, there are also many significant properties of algebraic structures which cannot be expressed with equations. Consider the statement that a group (G, e, m, i) has no non-trivial square roots of unity,

$$\forall\, x : G\,.\,(x \cdot x = e \Rightarrow x = e)\,. \tag{4.1}$$

This is a first-order logical statement which cannot be rewritten as a system of equations (proof!). To see what its categorical interpretation ought to be, we look at its usual set-theoretic interpretation. Each of subformulas, $x \cdot x = e$ and $x = e$, determines a subset of G:

$$\begin{tikzcd} \{x \in G \mid x \cdot x = e\} \arrow[rr, tail, dotted] \arrow[dr, "i"'] & & \{x \in G \mid x = e\} \arrow[dl, "j"] \\ & G & \end{tikzcd}$$

The implication $x \cdot x = e \Rightarrow x = e$ holds when $\{x \in G \mid x \cdot x = e\}$ is contained in $\{x \in G \mid x = e\}$. In categorical language we can say that the inclusion i factors through the inclusion j. Observe also that such a factorization is necessarily a mono and is unique, if it exists. The defining formulas of the subsets $\{x \in G \mid x \cdot x = e\}$ and $\{x \in G \mid x = e\}$ are equations, and so the subsets themselves can be constructed as equalizers (as above, interpreting $\cdot$ as m):

$$\begin{tikzcd} \{x \in G \mid x \cdot x = e\} \arrow[r, tail] & G \arrow[r, "\langle 1_G, 1_G \rangle"] \arrow[rr, bend right, "e \circ !_G"'] & G \times G \arrow[r, "m"] & G \end{tikzcd}$$

$$\begin{tikzcd} \{x \in G \mid x = e\} \arrow[r, tail] & G \arrow[r, shift left, "1_G"] \arrow[r, shift right, "e \circ !_G"'] & G \end{tikzcd}$$

In sum, we can interpret condition (4.1) in any category with products and equalizers, i.e. in any category with finite limits.[1] This allows us to define the notion of a group without square roots of unity in any category $\mathcal{C}$ with finite limits as an object G with morphisms $e : 1 \to G$, $m : G \times G \to G$ and $i : G \to G$ such that (G, e, m, i) is a group in $\mathcal{C}$, and the equalizer of $m \circ \langle 1_G, 1_G \rangle$ and $e \circ !_G$ factors through $e : 1 \to G$.

The aim of this chapter is to analyze how such examples can be treated in general. We want to relate first-order logic and fragments of it to categorical structures that are suitable for the interpretation of the logic. The general outline will be as follows:

1. A language $\mathcal{L}$ for a first-order theory consists, as usual, of some basic relation, function, and constant symbols, say $\mathcal{L} = (R, f, c)$.

2. An $\mathcal{L}$-structure in a category $\mathcal{C}$ with finite limits is an interpretation of $\mathcal{L}$ in $\mathcal{C}$ as an object M equipped with corresponding relations and operations (of appropriate arities), e.g.

$$R^M \rightarrowtail M \times \cdots \times M$$
$$f^M : M \times \cdots \times M \to M$$
$$c^M : 1 \to M.$$

[1] We are *not* claiming that finite limits suffice for an interpretation of arbitrary formulas built from universal quantifiers and implications. The formula at hand has a very special form $\forall x \,.\, (\varphi(x) \Longrightarrow \psi(x))$, where $\varphi(x)$ and $\psi(x)$ do not contain further $\forall$ or $\Longrightarrow$.

3. Formulas $\varphi(x_1, \dots, x_n)$ in (some fragment of) first-order logic will be interpreted as "generalized subsets", i.e. subobjects,

$$[\![\varphi(x_1, \dots, x_n)]\!] \rightarrowtail M \times \cdots \times M.$$

The interpretation makes use of categorical operations in $\mathcal{C}$ corresponding to the logical ones appearing in the formula $\varphi(x_1, \dots, x_n)$.

4. A theory $\mathbb{T}$ in (a fragment of) first-order logic will consist of a set of (binary) sequents,

$$\varphi(x_1, \dots, x_n) \vdash \psi(x_1, \dots, x_n).$$

5. A model of the theory is then an interpretation M in which the corresponding subobjects satisfy all the sequents of $\mathbb{T}$, in the sense that

$$[\![\varphi(x_1, \dots, x_n)]\!] \leq [\![\psi(x_1, \dots, x_n)]\!] \qquad \text{in } \mathsf{Sub}(M^n).$$

6. We shall give a deductive calculus for such sequents, prove that it is sound with respect to categorical models, and then use it to construct a classifying category $\mathcal{C}_{\mathbb{T}}$, with the expected universal property: models of $\mathbb{T}$ correspond to (structure-preserving) functors on $\mathcal{C}_{\mathbb{T}}$.

7. Completeness of the calculus in general follows from classification, and more specialized completeness results from embedding theorems applied to the classifying category.

4.1 Theories

A *first-order theory* $\mathbb{T}$ consists of an underlying *type theory* and a set of formulas in a *fragment of first-order logic.* Anticipating Chapter **??**, the type theory is given by a set of basic types, a set of basic constants together with their types, rules for forming types, and rules and axioms for deriving typing judgments

$$x_1 : A_1, \dots, x_n : A_n \mid t : B \ ,$$

expressing that term t has type B in typing context $x_1 : A_1, \dots, x_n : A_n$, and a set of axioms and rules of inference which tell us which equations between terms

$$x_1 : A_1, \dots, x_n : A_n \mid t = u : B \ ,$$

are valid. This part of the theory $\mathbb{T}$ may be regarded as providing the underlying structure, on top of which the logical formulas are defined. For first-order logic, the underlying type theory will be essentially the same as the equational logic that we already met in Chapter **??**.

A fragment of first-order logic is then given by a set of *basic relation symbols* together with a specification of which first-order operations are being considered in building formulas. Each basic relation symbol has a *signature* $(A_1, \ldots, A_n)$, which specifies the types of its arguments. The *arity* of a relation symbol is the number of arguments it takes. The judgment[2]

$$x_1 : A_1, \ldots, x_n : A_n \mid \phi \ \mathtt{pred}$$

states that ϕ is a well-formed formula in typing context $x_1 : A_1, \ldots, x_n : A_n$. For each basic relation symbol R with signature $(A_1, \ldots, A_n)$ there is an inference rule

$$\frac{\Gamma \mid t_1 : A_1 \qquad \cdots \qquad \Gamma \mid t_n : A_n}{\Gamma \mid R(t_1, \ldots, t_n) \ \mathtt{pred}}$$

Depending on what fragment of first-order logic is involved, there may be other rules for forming logical formulas. For example, if equality is present, then for each type A there is a rule

$$\frac{\Gamma \mid t : A \qquad \Gamma \mid u : A}{\Gamma \mid t =_A u \ \mathtt{pred}}$$

and if conjunction is present, then there is a rule

$$\frac{\Gamma \mid \varphi \ \mathtt{pred} \qquad \Gamma \mid \psi \ \mathtt{pred}}{\Gamma \mid \varphi \wedge \psi \ \mathtt{pred}}$$

Other such rules will be given when we come to the study of particular logical operations.

The basic logical judgment of a first-order theory is *logical entailment* between formulas,

$$x_1 : A_1, \ldots, x_n : A_n \mid \varphi_1, \ldots, \varphi_m \vdash \psi$$

which states that in the typing context $x_1 : A_1, \ldots, x_n : A_n$, the hypotheses $\varphi_1, \ldots, \varphi_m$ entail ψ. It is understood that the terms appearing in the formulas are well-typed in the typing context, and that formulas $\varphi_1, \ldots, \varphi_m, \psi$ are part of the fragment of the logic of $\mathbb{T}$. When the fragment contains conjunction $\wedge$ it is convenient to restrict attention to *binary* sequents,

$$x_1 : A_1, \ldots, x_n : A_n \mid \varphi \vdash \psi,$$

by replacing $\varphi_1, \ldots, \varphi_m$ with $\varphi_1 \wedge \ldots \wedge \varphi_m$. When the fragment contains equality, we may replace the type-theoretic equality judgments

$$x_1 : A_1, \ldots, x_n : A_n \mid t = u : B$$

with the logical statements

$$x_1 : A_1, \ldots, x_n : A_n \mid \cdot \vdash t =_B u \ .$$

[2] We follow type-theoretic practice here by adding the tag `pred` to turn what would otherwise be an exhibited formula in context into a judgement concerning the formula.

The subscript at the equality sign indicates the type at which the equality is taken. In a theory $\mathbb{T}$ there are basic entailments, or axioms, which together with the inference rules for the opertations involved can be used for deriving valid judgments, as usual.

We shall consider several standard fragments of first-order logic, determined by selecting a subset of logical connectives and quantifiers. These are as follows:

1. *Full first-order logic* is built from logical operations
$$= \quad \top \quad \bot \quad \neg \quad \wedge \quad \vee \quad \Rightarrow \quad \forall \quad \exists\,.$$
2. *Cartesian logic* is the fragment built from
$$= \quad \top \quad \wedge\,.$$
3. *Regular logic* is the fragment built from
$$= \quad \top \quad \wedge \quad \exists\,.$$
4. *Coherent logic* is the fragment built from
$$= \quad \top \quad \wedge \quad \exists \quad \bot \quad \vee\,.$$
5. A *geometric formula* is a formula of the form
$$\forall\, x : A\,.\,(\varphi \Longrightarrow \psi)\ ,$$
where φ and ψ are coherent formulas.

The names for these fragments come from the names of various categorical structures in which they are interpreted.

The well-formed terms and formulas of a first-order theory $\mathbb{T}$ constitute its *language*. It may seem that we are doing things backwards, because we should have spoken of first-order languages before we spoke of first-order theories. While this is possible for simple theories, it becomes difficult to do when we consider more complicated theories in which types and logical formulas are intertwined. In such cases the typing judgments and logical entailments may be given by a mutual recursive definition. In order to find out whether a given term is well-formed, we might have to prove a logical statement. In everyday mathematics this occurs all the time, for example, to show that the term $\int_0^\infty f$ denotes a real number, it may be necessary to prove that $f : \mathbb{R} \to \mathbb{R}$ is an integrable function and that the integral has a finite value. This is why it does not always make sense to strictly differentiate a language from a theory.[3]

In order to focus on the logical part of first-order theories, we are going to limit attention to only two very simple kinds of type theory. A *single-sorted* first-order theory has as its underlying type theory a single type A, and for each $k \in \mathbb{N}$ a set of basic k-ary function symbols. The rules for typing judgments are:

[3] However, it *does* make sense to distinguish syntax from theory. Rules of substitution and the behaviour of free and bound variables are syntactic considerations, for example.

1. Variables in contexts:
$$\frac{}{x_1 : A, \ldots, x_n : A \mid x_i : A}$$

2. For each basic function symbol f of arity k, there is an inference rule
$$\frac{\Gamma \mid t_1 : A \qquad \cdots \qquad \Gamma \mid t_n : A}{\Gamma \mid f(t_1, \ldots, t_n) : A}$$

This much is essentially an algebraic theory. In addition, however, a single-sorted first-order theory may contain relation symbols, formulas, axioms, and rules of inference which an algebraic theory does not.

A slight generalization of a single-sorted theory is a *multi-sorted* one. Its underlying type theory is given by a set of types, and a set of basic function symbols. Each function symbol f has a *signature* $(A_1, \ldots, A_n; B)$, where n is the arity of f and $A_1, \ldots, A_n, B$ are types. The rules for typing judgments are:

1. Variables in contexts:
$$\frac{}{x_1 : A_1, \ldots, x_n : A_n \mid x_i : A_i}$$

2. For each basic function symbol f with signature $(A_1, \ldots, A_n; B)$, there is an inference rule
$$\frac{\Gamma \mid t_1 : A_1 \qquad \cdots \qquad \Gamma \mid t_n : A_n}{\Gamma \mid f(t_1, \ldots, t_n) : B}$$

We often write suggestively $f : A_1 \times \cdots \times A_n \to B$ to indicate that $(A_1, \ldots, A_n; B)$ is the signature of f. However, this does not mean that $A_1 \times \cdots \times A_n \to B$ is a type! A multi-sorted first-order theory does *not* have any type forming operations, such as $\times$ and $\to$.

4.2 Predicates as subobjects

Formulas of first-order logic will be interpreted as "generalized subsets", i.e. subobjects. We therefore need to review some of the basic theory of these.

Let A be an object in a category $\mathcal{C}$. If $i : I \rightarrowtail A$ and $j : J \rightarrowtail A$ are monos into A, we say that i is smaller than j, and write $i \leq j$, when there exists a morphism $k : I \to J$ such that the following diagram commutes:

$$\begin{tikzcd}
I \arrow[rr, dotted, "k"] \arrow[dr, tail, "i"'] & & J \arrow[dl, tail, "j"] \\
& A &
\end{tikzcd}$$

If such a k exists then it, too, is monic, since i is, and it is unique, since j is monic. The class $\mathsf{Mono}(A)$ of all monos into A is this preordered by this relation $\leq$, it is the same as the slice category $\mathsf{Mono}(\mathcal{C})/A$ of all monos in $\mathcal{C}$, sliced over the object A. Let $\mathsf{Sub}(A)$ be the poset reflection of this preorder. Thus the elements of $\mathsf{Sub}(A)$ are equivalence classes of monos into A, where monos $i : I \rightarrowtail A$ and $j : J \rightarrowtail A$ are equivalent when $i \leq j$ and $j \leq i$ (note that then $I \cong J$). The induced relation $\leq$ on $\mathsf{Sub}(A)$ is then a partial order.

We have to be a bit careful with the formation of $\mathsf{Sub}(A)$, since it is defined as a quotient of a *class* $\mathsf{Mono}(A)$. In many particular cases the general construction by quotients can be avoided. If we can demonstrate that the preorder $\mathsf{Mono}(A)$ is equivalent, as a category, to a poset P then we can simply take $\mathsf{Sub}(A) = P$. At any rate, we usually require that $\mathsf{Sub}(A)$ is small.

Definition 4.2.1. A category $\mathcal{C}$ is *well-powered* when, for all $A \in \mathcal{C}$, there is at most a *set* of subobjects of A, so that the category $\mathsf{Mono}(A)$ is equivalent to a small poset. In other words, for every $A \in \mathcal{C}$, $\mathsf{Sub}(A)$ is a small category.

We shall often speak of subobjects as if they were monos rather than equivalence classes of monos. It is understood that we mean the subobjects represented by monos and not the monos themselves. Sometimes we refer to a mono $i : I \rightarrowtail A$ by its domain I only, even though the object I itself does not determine the morphism i. Hopefully this will not cause confusion, as it is always going to be clear which mono is meant to go along with the object I.

In a category $\mathcal{C}$ with finite limits the assignment $A \mapsto \mathsf{Sub}(A)$ is the object part of the *subobject functor*

$$\mathsf{Sub} : \mathcal{C}^{\mathsf{op}} \to \mathsf{Poset}\ .$$

The morphism part of Sub is pullback. More precisely, given a morphism $f : A \to B$, let $\mathsf{Sub}(f) = f^* : \mathsf{Sub}(B) \to \mathsf{Sub}(A)$ be the monotone map which maps the subobject $[i : I \rightarrowtail B]$ to the subobject $[f^*i : f^*I \rightarrowtail A]$, where $f^*i : f^*I \rightarrowtail A$ is a pullback of i along f:

$$\begin{tikzcd}
f^*I \arrow[r] \arrow[d, tail, "f^*i"'] \arrow[dr, phantom, "\lrcorner", very near start] & I \arrow[d, tail, "i"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

Recall that a pullback of a mono is again mono, so this definition makes sense. We need to verify (1) that if two monos $i : I \rightarrowtail A$ and $j : J \rightarrowtail A$ are equivalent, then their pullbacks are so as well; and (2) that $\mathsf{Sub}(1_A) = 1_{\mathsf{Sub}(A)}$ and $\mathsf{Sub}(g \circ f) = \mathsf{Sub}(f) \circ \mathsf{Sub}(g)$. These all follow easily from the fact that pullback is a functor $\mathcal{C}/B \to \mathcal{C}/A$, which reduces to the familiar "2-pullbacks" lemma:

Lemma 4.2.2. *Suppose both squares in the following diagram are pullbacks:*

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

Then the outer rectangle is a pullback diagram as well. Moreover, if the outer rectangle and the right square are pullbacks, then so is the left square.

Proof. This is left as an exercise in diagram chasing. □

Of course, pullbacks are really only determined up to isomorphism, but this does not cause any problems because isomorphic monos represent the same subobject.

In the semantics to be given below, a formula

$$x : A \mid \varphi \ \texttt{pred}$$

will be interpreted as a subobject

$$[\![x : A \mid \varphi]\!] \rightarrowtail [\![A]\!].$$

Thus $\mathsf{Sub}(A)$ can be regarded as the poset of "predicates" on A, generalizing the powerset of a set A. Logical operations on formulas then correspond to operations on $\mathsf{Sub}(A)$. The structure of $\mathsf{Sub}(A)$ therefore determines which logical connectives can be interpreted. If $\mathsf{Sub}(A)$ is a Heyting algebra, then we can interpret the full intuitionistic propositional calculus (cf. Subsection 3.4), but if $\mathsf{Sub}(A)$ only has binary meets then all that can be interpreted are $\top$ and $\wedge$. We will work out details of different operations in the following sections, but one common aspect that we require is the "stability" of the interpretation of the logical operations, in a sense that we now make clear.

Substitution and stability

Let us consider the interpretation of substitution of terms for variables. There are two kinds of substitution, into a term, and into a formula. We may substitute a term $x : A \mid t : B$ for a variable y in a term $y : B \mid u : C$ to obtain a new term $x : A \mid u[t/y] : C$. If t and u are interpreted as morphisms

$$[\![A]\!] \xrightarrow{[\![t]\!]} [\![B]\!] \xrightarrow{[\![u]\!]} [\![C]\!]$$

then $u[t/y]$ is interpreted as their composition:

$$[\![x : A \mid u[t/y] : C]\!] = [\![y : B \mid u : C]\!] \circ [\![x : A \mid t : B]\!] \ .$$

Thus, *substitution into a term is composition.*

The second kind of substitution occurs when we substitute a term $x : A \mid t : B$ for a variable y in a formula $y : B \mid \varphi$ to obtain a new formula $x : A \mid \varphi[t/y]$. If t is interpreted as a morphism $[\![t]\!] : [\![A]\!] \to [\![B]\!]$ and φ is interpreted as a subobject $[\![\varphi]\!] \rightarrowtail [\![B]\!]$ then the interpretation of $\varphi[t/y]$ is the pullback of $[\![\varphi]\!]$ along $[\![t]\!]$:

$$\begin{tikzcd}
{[\![\varphi[t/y]]\!] = [\![t]\!]^*[\![\varphi]\!]} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, very near start, "\lrcorner"] & {[\![\varphi]\!]} \arrow[d, tail] \\
{[\![A]\!]} \arrow[r, "{[\![t]\!]}"'] & {[\![B]\!]}
\end{tikzcd}$$

Thus, *substitution into a formula is pullback*,

$$[\![x : A \mid \varphi[t/y]\,]\!] = [\![x : A \mid t : B]\!]^* [\![y : B \mid \varphi]\!].$$

Now, because substitution respects the syntactical, logical operations, e.g.

$$(\varphi \wedge \psi)[t/x] = \varphi[t/x] \wedge \psi[t/x],$$

the categorical structures used to interpret the various logical operations such as $\wedge$ must also behave well with respect to the interpretation of substitution, i.e. pullback. We say that a categorical property or structure is *stable (under pullbacks)* if it is preserved by pullbacks.

For example, a category $\mathcal{C}$ has *stable meets* if each poset $\mathsf{Sub}(A)$ has binary meets, and the pullback of a meet $I \wedge J \rightarrowtail A$ along any map $f : B \to A$ is the meet $f^*I \wedge f^*J \rightarrowtail A$ of the respective pullbacks,

$$f^*(I \wedge J) = f^*I \wedge f^*J.$$

This means that the meet operation,

$$\wedge : \mathsf{Sub}(A) \times \mathsf{Sub}(A) \longrightarrow \mathsf{Sub}(A)$$

is natural in A, in the sense that for any map $f : B \to A$ the following diagram commutes.

$$\begin{tikzcd}
\mathsf{Sub}(A) \times \mathsf{Sub}(A) \arrow[r, "\wedge_A"] \arrow[d, "f^* \times f^*"'] & \mathsf{Sub}(A) \arrow[d, "f^*"] \\
\mathsf{Sub}(B) \times \mathsf{Sub}(B) \arrow[r, "\wedge_B"'] & \mathsf{Sub}(B)
\end{tikzcd}$$

Exercise 4.2.3. Show that any category $\mathcal{C}$ with finite limits has stable meets in the foregoing sense: each poset $\mathsf{Sub}(A)$ has all finite meets (i.e. including the "empty meet" 1), and these are stable under pullbacks. Conclude that $\mathsf{Sub} : \mathcal{C}^{\mathsf{op}} \longrightarrow \mathsf{Posets}$ factors through the subcategory of $\wedge$-semi-lattices.

Generalized elements

In any category, we sometimes consider arbitrary arrows $x : X \to C$ as *generalized elements* of C, thinking thereby of variable elements or parameters. With respect to a subobject $S \rightarrowtail C$, such an element is said to be *in the subobject*, writtten

$$x \in_C S,$$

if it factors through (any mono representing) the subobject,

$$\begin{tikzcd}
 & S \arrow[d, tail] \\
X \arrow[ur, dotted] \arrow[r, "x"'] & C
\end{tikzcd}$$

which, observe, it then does uniquely. The following "generalized element semantics" can be a useful technique for "externalizing" the operations on subobjects into statements about generalized elements.

Proposition 4.2.4. *Let C be any object in a category $\mathcal{C}$ with finite limits.*

1. *for the top element $1 \in \mathsf{Sub}(C)$, and for all $x : X \to C$,*
$$x \in_C 1.$$

2. *for any $S, T \in \mathsf{Sub}(C)$,*
$$S \leq T \iff x \in_C S \text{ implies } x \in_C T, \text{ for all } x : X \to C.$$

3. *for any $S, T \in \mathsf{Sub}(C)$, and for all $x : X \to C$,*
$$x \in_C S \wedge T \iff x \in_C S \text{ and } x \in_C T.$$

4. *for the subobject $\Delta = [\langle 1_C, 1_C \rangle] \in \mathsf{Sub}(C \times C)$, and for all $x, y : X \to C$,*
$$\langle x, y \rangle \in \Delta \iff x = y.$$

5. *for the equalizer $E_{(f,g)} \rightarrowtail A$ of a pair of arrows $f, g : A \rightrightarrows B$, and for all $x : X \to A$,*
$$x \in_A E_{(f,g)} \iff fx = gx.$$

6. *for the pullback $f^*S \rightarrowtail A$ of a subobject $S \rightarrowtail B$ along any arrow $f : A \to B$, and for all $x : X \to A$,*
$$x \in_A f^*S \iff fx \in_B S.$$

Exercise 4.2.5. Prove the proposition.

4.3 Cartesian logic

As a first example we look at the logic of *cartesian categories*, which are categories with finite limits, to be called *cartesian logic*. This is a logic of formulas over a multi-sorted type theory with unit type $\mathtt{1}$. (See section **??** for multi-sorted type theories and the axioms for the unit type). The logical operations are $=$, $\top$, and $\wedge$.

Formation rules for cartesian logic

Given a basic language consisting of a stock of relation and function symbols (with arities), the terms are built up as usual from the basic function symbols and variables (we take "constants" to be 0-ary function symbols). The rules for constructing formulas are as follows:

1. The 0-ary relation symbol $\top$ is a formula:
$$\frac{}{\Gamma \mid \top \ \texttt{pred}}$$

2. For each basic relation symbol R with signature $(A_1, \ldots, A_n)$ there is a rule
$$\frac{\Gamma \mid t_1 : A_1 \qquad \cdots \qquad \Gamma \mid t_n : A_n}{\Gamma \mid R(t_1, \ldots, t_n) \ \texttt{pred}}$$

3. For each type A, there is a rule
$$\frac{\Gamma \mid s : A \qquad \Gamma \mid t : A}{\Gamma \mid s =_A t \ \texttt{pred}}$$

4. Conjunction:
$$\frac{\Gamma \mid \varphi \ \texttt{pred} \qquad \Gamma \mid \psi \ \texttt{pred}}{\Gamma \mid \varphi \wedge \psi \ \texttt{pred}}$$

5. Weakening:
$$\frac{\Gamma \mid \varphi \ \texttt{pred}}{\Gamma, x : A \mid \varphi \ \texttt{pred}}$$

Observe that, as usual, there is then a derived operation of substitution of terms for variables into formulas, defined by structural recursion on the above specification of formulas:

Substitution:
$$\frac{\Gamma \mid t : A \qquad \Gamma, x : A \mid \varphi \ \texttt{pred}}{\Gamma \mid \varphi[t/x] \ \texttt{pred}}$$

Inference rules for cartesian logic

Although we do not yet need them, we state the rules of inference here, too, for the convenience of having the entire specification of cartesian logic in one place. As already mentioned, we can conveniently state this deductive calculus entirely in terms of *binary* sequents,
$$\Gamma \mid \psi \vdash \varphi.$$

We omit mention of the context Γ when it is the same in the premisses and conclusion of a rule.

1. Weakening:
$$\frac{\Gamma \mid \psi \vdash \varphi}{\Gamma, x : A \mid \psi \vdash \varphi}$$
2. Substitution:
$$\frac{\Gamma \mid t : A \qquad \Gamma, x : A \mid \psi \vdash \varphi}{\Gamma \mid \psi[t/x] \vdash \varphi[t/x]}$$
3. Identity:
$$\frac{}{\varphi \vdash \varphi}$$
4. Cut:
$$\frac{\psi \vdash \theta \qquad \theta \vdash \varphi}{\psi \vdash \varphi}$$
5. Equality:
$$\frac{}{\psi \vdash t =_A t} \qquad \frac{\psi \vdash t =_A u \qquad \psi \vdash \varphi[t/z]}{\psi \vdash \varphi[u/z]}$$
6. Truth:
$$\frac{}{\psi \vdash \top}$$
7. Conjunction:
$$\frac{\vartheta \vdash \varphi \qquad \vartheta \vdash \psi}{\vartheta \vdash \varphi \wedge \psi} \qquad \frac{\vartheta \vdash \varphi \wedge \psi}{\vartheta \vdash \psi} \qquad \frac{\vartheta \vdash \varphi \wedge \psi}{\vartheta \vdash \varphi}$$

Exercise 4.3.1. Derive symmetry and transitivity of equality:
$$\frac{\Gamma \mid \psi \vdash t =_A u}{\Gamma \mid \psi \vdash u =_A t} \qquad \frac{\Gamma \mid \psi \vdash t =_A u \qquad \Gamma \mid \psi \vdash u =_A v}{\Gamma \mid \psi \vdash t =_A v}$$

Example 4.3.2. The theory of a poset is a cartesian theory. There is one basic sort $\mathtt{P}$ and one binary relation symbol $\leq$ with signature $(\mathtt{P}, \mathtt{P})$. The axioms are the familiar axioms for reflexivity, transitivity, and antisymmetry:
$$\begin{aligned} x : \mathtt{P} &\mid \cdot \vdash x \leq x \\ x : \mathtt{P}, y : \mathtt{P}, z : \mathtt{P} &\mid x \leq y,\ y \leq z \vdash x \leq z \\ x : \mathtt{P}, y : \mathtt{P} &\mid x \leq y,\ y \leq x \vdash x =_{\mathtt{P}} y \end{aligned}$$
There are also many examples, such as ordered groups, ordered fields, etc., that are posets with further algebraic structure.

Example 4.3.3. An *equivalence relation* in a cartesian category is a model of the corresponding theory with one basic sort $\mathtt{A}$ and one binary relation symbol $\sim$ with signature $(\mathtt{A}, \mathtt{A})$. The axioms are the familiar axioms for reflexivity, symmetry, and transitivity:
$$\begin{aligned} x : \mathtt{A} &\mid \cdot \vdash x \sim x \\ x : \mathtt{A}, y : \mathtt{A} &\mid x \sim y \vdash y \sim x \\ x : \mathtt{A}, y : \mathtt{A}, z : \mathtt{A} &\mid x \sim y\ \wedge\ y \sim z \vdash x \sim z \end{aligned}$$

Semantics for cartesian logic

In order to give the semantics of cartesian logic, we require a couple of useful propositions regarding cartesian categories.

Proposition 4.3.4. *If a category $\mathcal{C}$ has pullbacks then, for every $A \in \mathcal{C}$, $\mathsf{Sub}(A)$ has finite limits. Moreover, these are stable under pullback.*

Proof. The poset $\mathsf{Sub}(A)$ has finite limits if it has a top object and binary meets. The top object of $\mathsf{Sub}(A)$ is the subobject $[\mathtt{1}_A : A \to A]$. The meet of subobjects $i : I \rightarrowtail A$ and $j : J \rightarrowtail A$ is the subobject $i \wedge j = i \circ (i^* j) = j \circ (j^* i) : I \wedge J \rightarrowtail A$ obtained by pullback, as in the following diagram:

$$\begin{tikzcd}
I \wedge J \arrow[r, tail, "j^* i"] \arrow[d, tail, "i^* j"'] \arrow[dr, phantom, "\lrcorner", very near start] & J \arrow[d, tail, "j"] \\
I \arrow[r, tail, "i"'] & A
\end{tikzcd}$$

It is easy to verify that $I \wedge J$ is the infimum of I and J. Finally, stability follows from a familiar diagram chase on a cube of pullbacks. $\square$

Proposition 4.3.5. *If a category has finite products and pullbacks of monos along monos then it has all finite limits.*

Proof. It is sufficient to show that the category has equalizers. To construct the equalizer of parallel arrows $f : A \to B$ and $g : A \to B$, first observe that the arrows

$$A \xrightarrow{\langle \mathtt{1}_A, f \rangle} A \times B \qquad\qquad A \xrightarrow{\langle \mathtt{1}_A, g \rangle} A \times B$$

are monos because the projection $\pi_0 : A \times B \to A$ is their left inverse. Therefore, we may construct the pullback

$$\begin{tikzcd}
P \arrow[r, tail, "p"] \arrow[d, tail, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, tail, "\langle \mathtt{1}_A, f \rangle"] \\
A \arrow[r, tail, "\langle \mathtt{1}_A, g \rangle"'] & A \times B
\end{tikzcd}$$

The morphisms p and q coincide because $\langle \mathtt{1}_A, f \rangle$ and $\langle \mathtt{1}_A, g \rangle$ have a common left inverse π_0:

$$p = \mathtt{1}_A \circ p = \pi_0 \circ \langle \mathtt{1}_A, f \rangle \circ p = \pi_0 \circ \langle \mathtt{1}_A, f \rangle \circ q = \mathtt{1}_A \circ q = q \ .$$

Let us show that $p : P \to A$ is the equalizer of f and g. First, p equalizes f and g,

$$f \circ p = \pi_1 \circ \langle \mathtt{1}_A, f \rangle \circ p = \pi_1 \circ \langle \mathtt{1}_A, g \rangle \circ q = g \circ q = g \circ p \ .$$

If $k : K \to A$ also equalizes f and g then

$$\langle 1_A, f\rangle \circ k = \langle k, f \circ k\rangle = \langle k, g \circ k\rangle = \langle 1_A, g\rangle \circ k \;,$$

therefore by the universal property of the constructed pullback there exists a unique factorization $\overline{k} : K \to P$ such that $k = p \circ \overline{k}$, as required. □

We now explain how cartesian logic is interpreted in a cartesian category $\mathcal{C}$ (i.e. $\mathcal{C}$ is finitely complete). Let $\mathbb{T}$ be a multi-sorted cartesian theory. Recall that the type theory of $\mathbb{T}$ is specified by a set of sorts (types) $A, ...$ and a set of basic function symbols $f, ...$ together with their signatures, while the logic is given by a set of basic relation symbols $R, ...$ with their signatures, and a set of axioms in the form of logical entailments between formulas in context,

$$\Gamma \mid \psi \vdash \varphi.$$

Definition 4.3.6. An *interpretation* of $\mathbb{T}$ in $\mathcal{C}$ is given by the following data, where Γ stands for a typing context $x_1 : A_1, \ldots, x_n : A_n$, and ψ and φ are formulas:

1. Each sort A is interpreted as an object $[\![A]\!]$, with the unit sort $\mathtt{1}$ being interpreted as the terminal object 1.

2. A typing context $x_1 : A_1, \ldots, x_n : A_n$ is interpreted as the product $[\![A_1]\!] \times \cdots \times [\![A_n]\!]$. The empty context is interpreted as the terminal object 1.

3. A basic function symbol f with signature $(A_1, \ldots, A_m; B)$ is interpreted as a morphism $[\![f]\!] : [\![A_1]\!] \times \cdots [\![A_m]\!] \to [\![B]\!]$.

4. A basic relation symbol R with signature $(A_1, \ldots, A_n)$ is interpreted as a subobject $[\![R]\!] \in \mathsf{Sub}([\![A_1]\!] \times \cdots \times [\![A_n]\!])$.

We then extend the interpretation to all terms and formulas as follows:

1. A term in context $\Gamma \mid t : B$ is interpreted as a morphism

   $$[\![\Gamma \mid t : B]\!] : [\![\Gamma]\!] \to [\![B]\!]$$

   according to the following specification.

   - A variable $x_0 : A_1, \ldots, x_n : A_n \mid x_i : A_i$ is interpreted as the i-th projection $\pi_i : [\![A_1]\!] \times \cdots \times [\![A_n]\!] \to [\![A_i]\!]$.
   - The interpretation of $\Gamma \mid * : \mathtt{1}$ is the unique morphism $!_{[\![\Gamma]\!]} : [\![\Gamma]\!] \to 1$.
   - A composite term $\Gamma \mid f(t_1, \ldots, t_m) : B$, where f is a basic function symbol with signature $(A_1, \ldots, A_m; B)$, is interpreted as the composition

     $$[\![\Gamma]\!] \xrightarrow{\langle [\![t_1]\!], \ldots, [\![t_m]\!]\rangle} [\![A_1]\!] \times \cdots \times [\![A_m]\!] \xrightarrow{[\![f]\!]} [\![B]\!]$$

     Here $[\![t_i]\!]$ is shorthand for $[\![\Gamma \mid t_i : A_i]\!]$.

2. A formula in a context $\Gamma \mid \varphi$ is interpreted as a subobject $[\![\Gamma \mid \varphi]\!] \in \mathsf{Sub}([\![\Gamma]\!])$ according to the following specification.

   - The logical constant $\top$ is interpreted as the maximal subobject, represented by the identity arrow:
   $$[\![\Gamma \mid \top]\!] = [\, \mathsf{1}_{[\![\Gamma]\!]} : [\![\Gamma]\!] \to [\![\Gamma]\!] \,]$$
   - An atomic formula $\Gamma \mid R(t_1, \ldots, t_m)$, where R is a basic relation symbol with signature $(A_1, \ldots, A_m)$ is interpreted as the left-hand side of the pullback:
   $$\begin{tikzcd} {[\![\Gamma \mid R(t_1, \ldots, t_m)]\!]} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & {[\![R]\!]} \arrow[d, tail] \\ {[\![\Gamma]\!]} \arrow[r, "{\langle [\![t_1]\!], \ldots, [\![t_m]\!] \rangle}"'] & {[\![A_1]\!] \times \cdots \times [\![A_m]\!]} \end{tikzcd}$$
   - An equation $\Gamma \mid t =_A u$ `pred` is interpreted as the subobject represented by the equalizer of $[\![\Gamma \mid t : A]\!]$ and $[\![\Gamma \mid u : A]\!]$:
   $$\begin{tikzcd} {[\![\Gamma \mid t =_A u]\!]} \arrow[r, tail] & {[\![\Gamma]\!]} \arrow[r, shift left, "{[\![t]\!]}"] \arrow[r, shift right, "{[\![u]\!]}"'] & {[\![A]\!]} \end{tikzcd}$$
   - By Proposition 4.3.4, each $\mathsf{Sub}(A)$ is a poset with binary meets. Thus we interpret a conjunction $\Gamma \mid \varphi \wedge \psi$ `pred` as the meet of subobjects
   $$[\![\Gamma \mid \varphi \wedge \psi]\!] = [\![\Gamma \mid \varphi]\!] \wedge [\![\Gamma \mid \psi]\!] \ .$$
   - A formula formed by weakening is interpreted as pullback along a projection:
   $$\begin{tikzcd} {[\![\Gamma, x : A \mid \varphi]\!]} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & {[\![\Gamma \mid \varphi]\!]} \arrow[d, tail, "i"] \\ {[\![\Gamma]\!] \times [\![A]\!]} \arrow[r, "\pi"'] & {[\![\Gamma]\!]} \end{tikzcd}$$
   Computing this pullback one sees that the interpretation of $[\![\Gamma, x : A \mid \varphi]\!]$ turns out to be the subobject
   $$\begin{tikzcd} {[\![\Gamma \mid \varphi]\!] \times [\![A]\!]} \arrow[r, tail, "i \times \mathsf{1}_A"] & {[\![\Gamma]\!] \times [\![A]\!]} \end{tikzcd}$$

This concludes the definition of an interpretation of a cartesian theory $\mathbb{T}$ in a cartesian category $\mathcal{C}$.

As was explained in the previous section, the operation of substitution of terms into formulas is interpreted as pullback:

Lemma 4.3.7. *Let the formula $\Gamma, x : A \mid \varphi$ and the term $\Gamma \mid t : A$ be given. Then the substituted formula $\Gamma \mid \varphi[t/x]$ is interpreted as the pullback indicated in the following diagram:*

$$\begin{tikzcd}
\llbracket \Gamma \mid \varphi[t/x] \rrbracket \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \llbracket \Gamma, x : A \mid \varphi \rrbracket \arrow[d, tail] \\
\llbracket \Gamma \rrbracket \arrow[r, "\langle \mathbf{1}_{\llbracket \Gamma \rrbracket}, \llbracket t \rrbracket \rangle"'] & \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket
\end{tikzcd}$$

Proof. A simple induction on the structure of φ. We do the case where φ is an atomic formula $R(t_1, \ldots, t_m)$. Let $\Gamma = x_1 : A_1, \ldots, x_n : A_n$ and $\Gamma, x : A \mid t_i : B_i$ for $i = 1, \ldots, m$, where $(B_1, \ldots, B_m)$ is the signature of R. For the interpretation of $\Gamma, x : A \mid R(t_1, \ldots, t_m)$, by Definition 4.3.6 we have a pullback diagram:

$$\begin{tikzcd}
\llbracket \Gamma \mid R(t_1, \ldots, t_m) \rrbracket \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \llbracket R \rrbracket \arrow[d, tail] \\
\llbracket \Gamma, x : A \rrbracket \arrow[r, "\langle \llbracket t_1 \rrbracket, \ldots, \llbracket t_m \rrbracket \rangle"'] & \llbracket B_1 \rrbracket \times \cdots \times \llbracket B_m \rrbracket
\end{tikzcd}$$

Now suppose $\Gamma \mid t : A$, and consider the substitution

$$\Gamma \mid R(t_1, \ldots, t_m)[t/x] = \Gamma \mid R(t_1[t/x], \ldots, t_m[t/x])$$

For the interpretations of the substituted terms $t_i[t/x]$ we have the composites

$$\llbracket t_i[t/x] \rrbracket = \llbracket t_i \rrbracket \circ \langle \mathbf{1}_{\llbracket \Gamma \rrbracket}, \llbracket t \rrbracket \rangle : \llbracket \Gamma \rrbracket \longrightarrow \llbracket \Gamma, x : A \rrbracket \longrightarrow \llbracket B_i \rrbracket$$

by (associativity of composition and) the definition of the interpretation of terms. Thus for the interpretation of $\Gamma \mid R(t_1, \ldots, t_m)[t/x]$ we have the outer pullback rectangle below.

$$\begin{tikzcd}
\llbracket \Gamma \mid R(t_1, \ldots, t_m)[t/x] \rrbracket \arrow[r, dotted] \arrow[d, tail] \arrow[rr, bend left] & \llbracket \Gamma, x : A \mid R(t_1, \ldots, t_m) \rrbracket \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \llbracket R \rrbracket \arrow[d, tail] \\
\llbracket \Gamma \rrbracket \arrow[r, "\langle \mathbf{1}_{\llbracket \Gamma \rrbracket}, \llbracket t \rrbracket \rangle"'] \arrow[rr, bend right, "\langle \llbracket t_1[t/x] \rrbracket, \ldots, \llbracket t_m[t/x] \rrbracket \rangle"'] & \llbracket \Gamma, x : A \rrbracket \arrow[r, "\langle \llbracket t_1 \rrbracket, \ldots, \llbracket t_m \rrbracket \rangle"'] & \llbracket B_1 \rrbracket \times \cdots \times \llbracket B_m \rrbracket
\end{tikzcd}$$

But since the righthand square is a pullback, there is a unique dotted arrow as indicated. By the 2-pullbacks lemma, the lefthand square is then also a pullback, as required. □

Exercise 4.3.8. Complete the proof.

When we deal with several different interpretations at once we may name them M, N, $\ldots$, and subscript the semantic brackets accordingly, $[\![\Gamma]\!]_M$, $[\![\Gamma]\!]_N$, $\ldots$

Definition 4.3.9. If $\Gamma \mid \psi \vdash \psi$ is one of the logical entailment axioms of $\mathbb{T}$ and

$$[\![\Gamma \mid \psi]\!]_M \leq [\![\Gamma \mid \varphi]\!]_M$$

holds in an interpretation M, then we say that M *satisfies* or *models* $\Gamma \mid \psi \vdash \varphi$ and write

$$M \models \ \Gamma \mid \psi \vdash \varphi \,.$$

An interpretation M is a *model* of $\mathbb{T}$ if it satisfies all the axioms of $\mathbb{T}$.

Theorem 4.3.10 (Soundness of cartesian logic)**.** *If a cartesian theory $\mathbb{T}$ proves an entailment*

$$\Gamma \mid \psi \vdash \varphi$$

then every model M of $\mathbb{T}$ satisfies the entailment:

$$M \models \ \Gamma \mid \psi \vdash \varphi \,.$$

Proof. The proof proceeds by induction on the proof of the entailment. In the following we often omit the typing context Γ to simplify notation, and all inequalities are interpreted in $\mathsf{Sub}([\![\Gamma]\!])$. We consider all possible last steps in the proof of the entailment:

1. Weakening: if $[\![\Gamma \mid \psi]\!] \leq [\![\Gamma \mid \varphi]\!]$ in $\mathsf{Sub}([\![\Gamma]\!])$ then
$$[\![\Gamma, x : A \mid \psi]\!] = [\![\Gamma \mid \psi]\!] \times A \leq [\![\Gamma \mid \varphi]\!] \times A = [\![\Gamma, x : A \mid \varphi]\!] \quad \text{in } \mathsf{Sub}([\![\Gamma, x : A]\!]).$$

2. Substitution: by lemma 4.3.7, substitution is interpreted by pullback so that $[\![\varphi[t/x]]\!] = \langle \mathsf{1}_{[\![\psi]\!]}, [\![t]\!]\rangle^* [\![\varphi]\!]$ and $[\![\psi[t/x]]\!] = \langle \mathsf{1}_{[\![\psi]\!]}, [\![t]\!]\rangle^* [\![\psi]\!]$. Because
$$\langle \mathsf{1}_{[\![\psi]\!]}, [\![t]\!]\rangle^* : \mathsf{Sub}([\![\psi]\!]) \to \mathsf{Sub}([\![\psi]\!] \times [\![A]\!])$$
is a functor it is a monotone map, therefore $[\![\psi]\!] \leq [\![\varphi]\!]$ implies
$$\langle \mathsf{1}_{[\![\psi]\!]}, [\![t]\!]\rangle^* [\![\psi]\!] \leq \langle \mathsf{1}_{[\![\psi]\!]}, [\![t]\!]\rangle^* [\![\varphi]\!] \,.$$

3. Identity: trivially
$$[\![\varphi]\!] \leq [\![\varphi]\!] \,.$$

4. Cut: if $[\![\psi]\!] \leq [\![\theta]\!]$ and $[\![\theta]\!] \leq [\![\varphi]\!]$ then clearly $[\![\psi]\!] \leq [\![\varphi]\!]$, since $\mathsf{Sub}([\![\Gamma, x : A]\!])$ is a poset.

5. Truth: trivially $[\![\psi]\!] \leq [\![\top]\!]$.

6. The rules for conjunction clearly hold because by the definition of infimum $[\![\vartheta]\!] \leq [\![\varphi \wedge \psi]\!]$ if, and only if, $[\![\vartheta]\!] \leq [\![\varphi]\!]$ and $[\![\vartheta]\!] \leq [\![\psi]\!]$.

7. Equality: the axiom $t =_A t$ is satisfied because an equalizer of $[\![t]\!]$ with itself is the maximal subobject:
$$[\![\psi]\!] \leq [\mathsf{1}_{[\![\Gamma]\!]} : [\![\Gamma]\!] \to [\![\Gamma]\!]] = [\![t =_A t]\!] \, .$$
For the other axiom, suppose $[\![\psi]\!] \leq [\![t =_A u]\!]$ and $[\![\psi]\!] \leq [\![\varphi[t/z]]\!]$. It suffices to show $[\![t =_A u]\!] \wedge [\![\varphi[t/z]]\!] \leq [\![\varphi[u/z]]\!]$ for then
$$[\![\psi]\!] \leq [\![t =_A u]\!] \wedge [\![\varphi[t/z]]\!] \leq [\![\varphi[u/z]]\!] \, .$$
The interpretation of $P = [\![t =_A u]\!] \wedge [\![\varphi[t/z]]\!]$ is obtained by two successive pullbacks, as in the following diagram:
$$\begin{tikzcd}
P \arrow[r] \arrow[d, tail] \arrow[dr, phantom, very near start, "\lrcorner"] & {[\![\varphi[t/z]]\!]} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, very near start, "\lrcorner"] & {[\![\varphi]\!]} \arrow[d, tail] \\
{[\![t =_A u]\!]} \arrow[r, tail, "e"'] & {[\![\Gamma]\!]} \arrow[r, "{\langle \mathsf{1}_\Gamma, [\![t]\!] \rangle}"'] & {[\![\Gamma]\!] \times [\![A]\!]}
\end{tikzcd}$$
Here e is the equalizer of $[\![u]\!]$ and $[\![t]\!]$. Observe that e equalizes $\langle \mathsf{1}_{[\![\Gamma]\!]}, [\![t]\!] \rangle$ and $\langle \mathsf{1}_{[\![\Gamma]\!]}, [\![u]\!] \rangle$ as well:
$$\langle \mathsf{1}_{[\![\Gamma]\!]}, [\![t]\!] \rangle \circ e = \langle e, [\![t]\!] \circ e \rangle = \langle e, [\![u]\!] \circ e \rangle = \langle \mathsf{1}_{[\![\Gamma]\!]}, [\![u]\!] \rangle \circ e \, .$$
Therefore, if we replace $\langle \mathsf{1}_{[\![\Gamma]\!]}, [\![t]\!] \rangle$ with $\langle \mathsf{1}_{[\![\Gamma]\!]}, [\![u]\!] \rangle$ in the above diagram, the outer rectangle still commutes. By the universal property of the pullback
$$\begin{tikzcd}
{[\![\varphi[u/z]]\!]} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, very near start, "\lrcorner"] & {[\![\varphi]\!]} \arrow[d, tail] \\
{[\![\Gamma]\!]} \arrow[r, "{\langle \mathsf{1}_\Gamma, [\![u]\!] \rangle}"'] & {[\![\Gamma]\!] \times [\![A]\!]}
\end{tikzcd}$$
it follows that P also factors through $[\![\varphi[u/z]]\!]$, as required.

□

Example 4.3.11. Recall the cartesian theory of posets (example 4.3.2). There is one basic sort `P` and one binary relation symbol $\leq$ with signature $(\mathtt{P}, \mathtt{P})$ and the axioms of reflexivity, transitivity, and antisymmetry. A poset in a cartesian category $\mathcal{C}$ is thus given by an object P, which is the interpretation of the sort `P`, and a subobject $r : R \rightarrowtail P \times P$, which the interpretation of $\leq$, such that the axioms are satisfied. As an example we spell out when the reflexivity axiom is satisfied. The interpretation of $x : P \mid x \leq x$ is obtained by the following pullback:

$$\begin{tikzcd}
[\![x \leq x]\!] \arrow[r] \arrow[d, tail] & R \arrow[d, tail, "r"] \\
P \arrow[r, "\Delta"'] \arrow[ur, dotted, "\rho"] & P \times P
\end{tikzcd}$$

where $\Delta = \langle 1_P, 1_P \rangle$ is the diagonal. The first axiom is satisfied when $[\![x \leq x]\!] = 1_P$, which happens if, and only if, Δ factors through r, as indicated. Therefore, reflexivity can be expressed as follows: there exists a "reflexivity" morphism $\rho : P \to R$ such that $r \circ \rho = \Delta$. Equivalently, the morphisms $\pi_0 \circ r$ and $\pi_1 \circ r$ have a common right inverse ρ.

As an example, of a poset in a cartesian category other than Set, observe that since the definition is stated entirely in terms of finite limits, and these are computed pointwise in functor categories $\mathsf{Set}^{\mathbb{C}}$, it follows that a poset P in $\mathsf{Set}^{\mathbb{C}}$ is the same thing as a functor $P : \mathbb{C} \to \mathsf{Poset}$. Indeed, as was the case for algebraic theories, we have an equivalence (an isomorphism, actually) of categories,

$$\mathsf{Poset}(\mathsf{Set}^{\mathbb{C}}) \;\cong\; \mathsf{Poset}(\mathsf{Set})^{\mathbb{C}} \;\cong\; \mathsf{Poset}^{\mathbb{C}}.$$

4.3.1 Subtypes

Let us consider whether the theory of a category is a cartesian theory. We begin by expressing the definition of a category so that it can be interpreted in any cartesian category $\mathcal{C}$. An *internal category* in $\mathcal{C}$ consists of an *object of morphisms* C_1, an *object of objects* C_0, and *domain*, *codomain*, and *identity* morphisms,

$$\mathsf{dom} : C_1 \to C_0 \,, \qquad \mathsf{cod} : C_1 \to C_0 \,, \qquad \mathsf{id} : C_0 \to C_1 \,.$$

There is also a *composition* morphism $c : C_2 \to C_1$, where C_2 is obtained by the pullback

$$\begin{tikzcd}
C_2 \arrow[r, "p_1"] \arrow[d, "p_0"'] & C_1 \arrow[d, "\mathsf{dom}"] \\
C_1 \arrow[r, "\mathsf{cod}"'] & C_0
\end{tikzcd}$$

The following equations must hold:

$$\mathsf{dom} \circ i = 1_{C_0} = \mathsf{cod} \circ i \,,$$
$$\mathsf{cod} \circ p_1 = \mathsf{cod} \circ c \,, \qquad \mathsf{dom} \circ p_0 = \mathsf{dom} \circ c \,.$$
$$c \circ \langle 1_{C_1}, i \circ \mathsf{dom} \rangle = 1_{C_1} = c \circ \langle i \circ \mathsf{cod}, 1_{C_1} \rangle \,,$$

The first two equations state that the domain and codomain of an identity morphism 1_A are both A. The second equation states that $\mathsf{cod}\,(f \circ g) = \mathsf{cod}\,f$ and the third one that

$\mathsf{dom}\,(f \circ g) = \mathsf{dom}\,g$. The fourth equation states that $f \circ 1_{\mathsf{dom}\,f} = f = 1_{\mathsf{cod}\,f} \circ f$. It remains to express associativity of composition. For this purpose we construct the pullback

$$\begin{tikzcd} C_3 \arrow[r, "q_2"] \arrow[d, "q_{01}"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \arrow[d, "\mathsf{dom}"] \\ C_2 \arrow[r, "\mathsf{cod} \circ p_1"'] & C_0 \end{tikzcd}$$

The object C_3 can be thought of as the set of triples of morphisms (f, g, h) such that $\mathsf{cod}\,f = \mathsf{dom}\,g$ and $\mathsf{cod}\,g = \mathsf{dom}\,h$. We denote $q_0 = p_0 \circ q_{01}$ and $q_1 = p_1 \circ q_{01}$. The morphisms $q_0, q_1, q_2 : C_3 \to C_1$ are like three projections which select the first, second, and third element of a triple, respectively. With this notation we can write $q_{01} = \langle q_0, q_1 \rangle_{C_2}$ because q_{01} is the unique morphism such that $p_0 \circ q_{01} = q_0$ and $p_1 \circ q_{01} = q_1$. The subscript C_2 reminds us that the "pair" $\langle q_0, q_1 \rangle_{C_2}$ is obtained by the universal property of the pullback C_2.

Morphisms $c \circ q_{01} : C_3 \to C_1$ and $q_2 : C_3 \to C_1$ factor through the pullback C_2 because

$$\mathsf{cod} \circ c \circ q_{01} = \mathsf{cod} \circ p_1 \circ q_0 = \mathsf{dom} \circ q_2 \ .$$

Thus let $r : C_3 \to C_2$ be the unique factorization for which $p_0 \circ r = c \circ q_{01}$ and $p_1 \circ r = q_2$. Because p_0 and p_1 are like projections from C_2 to C_1, morphism r can be thought of as a pair of morphisms, so we write $r = \langle c \circ q_{01}, q_2 \rangle_{C_2}$. Morphism $c \circ \langle c \circ q_{01}, q_2 \rangle_{C_2} : C_3 \to C_1$ corresponds to the operations $\langle f, g, h \rangle \mapsto (f, g) \circ h$, whereas the morphism corresponding to $\langle f, g, h \rangle \mapsto f \circ (g \circ h)$ is obtained in a similar way and is equal to

$$c \circ \langle q_0, c \circ \langle q_1, q_2 \rangle_{C_2} \rangle_{C_2} : C_3 \to C_1 \ .$$

Thus associativity is expressed by the equation

$$c \circ \langle c \circ \langle q_0, q_1 \rangle_{C_2}, q_2 \rangle_{C_2} = c \circ \langle q_0, c \circ \langle q_1, q_2 \rangle_{C_2} \rangle_{C_2} \ .$$

Example 4.3.12. An internal category in Set is a small category.

Example 4.3.13. An internal category in $\mathsf{Set}^{\mathbb{C}}$ is a functor $\mathbb{C} \to \mathsf{Cat}$. Indeed, as in previous examples of cartesian theories we have an equivalence of categories,

$$\mathsf{Cat}(\mathsf{Set}^{\mathbb{C}}) \ \cong \ \mathsf{Cat}(\mathsf{Set})^{\mathbb{C}} \ \cong \ \mathsf{Cat}^{\mathbb{C}}.$$

We have successfully formulated the theory of a category so that it makes sense in any cartesian category. In fact, the definition of an internal category refers only to certain pullbacks, hence the notion of an internal category makes sense in any category with pullbacks. However, if we try to formulate it as a multi-sorted cartesian theory, there is a problem. Obviously, there ought to be a basic sort of objects $\mathtt{C}_0$ and a basic sort of morphisms $\mathtt{C}_1$. There are also basic function symbols with signatures

$$\mathtt{dom} : (\mathtt{C}_1; \mathtt{C}_0) \qquad \mathtt{cod} : (\mathtt{C}_1; \mathtt{C}_0) \qquad \mathtt{id} : (\mathtt{C}_0, \mathtt{C}_1) .$$

However, it is not clear what the signature for composition should be. It is not $(\mathtt{C}_1, \mathtt{C}_1; \mathtt{C}_1)$ because composition is undefined for non-composable pairs of morphisms. We might be tempted to postulate another basic sort $\mathtt{C}_2$ but then we would have no way of stating that $\mathtt{C}_2$ is the pullback of `dom` and `cod`. And even if we somehow axiomatized the fact that $\mathtt{C}_2$ is a pullback, we would then still have to formalize the object $\mathtt{C}_3$ of composable triples, $\mathtt{C}_4$ of composable quadruples, and so on. What we lack is the ability to define the type $\mathtt{C}_2$ as a *subtype* of $\mathtt{C}_1 \times \mathtt{C}_1$.

One way to remedy the situation is to use a richer underlying type theory; in Chapter **??** we will consider the system of *dependent type theory*, which provides the means to capture such notions as the theory of categories (and much more). Here we consider a small step in that direction, namely *simple subtypes*. The formation rule for simple subtypes is

$$\frac{x : A \mid \varphi \ \mathtt{pred}}{\{x : A \mid \varphi\} \ \mathtt{type}}$$

We can think of $\{x : A \mid \varphi\}$ as the subobject of all those $x : A$ that satisfy φ. Note that we did not allow an arbitrary context Γ to be present. This means that we cannot define subtypes that depend on parameters, which why they are called "simple".

Inference rules for subtypes are as follows:

$$\frac{\Gamma \mid t : \{x : A \mid \varphi\}}{\Gamma \mid \mathtt{in}_\varphi\, t : A} \qquad \frac{\Gamma \mid t : \{x : A \mid \varphi\}}{\Gamma \mid \cdot \vdash \varphi[\mathtt{in}_\varphi\, t/x]} \qquad \frac{\Gamma \mid t : A \qquad \Gamma \mid \cdot \vdash \varphi[t/x]}{\Gamma \mid \mathtt{rs}_\varphi\, t : \{x : A \mid \varphi\}}$$

$$\frac{\Gamma, x : A \mid \varphi, \psi \vdash \theta}{\overline{\Gamma, y : \{x : A \mid \varphi\} \mid \psi[\mathtt{in}_\varphi\, y/x] \vdash \theta[\mathtt{in}_\varphi\, y/x]}}$$

The first rule states that a term t of subtype $\{x : A \mid \varphi\}$ can be converted to a term $\mathtt{in}_\varphi\, t$ of type A. We can think of the constant $\mathtt{in}_\varphi$ as the *inclusion* $\mathtt{in}_\varphi : \{x : A \mid \varphi\} \to A$. The second rule states that every term of a subtype $\{x : A \mid \varphi\}$ satisfies the defining predicate φ. The third rule states that a term t of type A which satisfies φ can be converted to a term $\mathtt{rs}_\varphi\, t$ of type $\{x : A \mid \varphi\}$. A good way to think of the constant $\mathtt{rs}_\varphi$ is as a partially defined *restriction*, or a type-casting operations, $\mathtt{rs}_\varphi : A \rightharpoonup \{x : A \mid \varphi\}$.[4] The last rule tells us how to replace a variable x of type A and an assumption φ about it with a variable y of type $\{x : A \mid \varphi\}$ and remove the assumption. Note that this is a two-way rule.

There are two more axioms that relate inclusions and restrictions:

$$\frac{\Gamma \mid t : \{x : A \mid \varphi\}}{\Gamma \mid \cdot \vdash \mathtt{rs}_\varphi\, (\mathtt{in}_\varphi\, t) = t} \qquad \frac{\Gamma \mid t : A \qquad \Gamma \mid \cdot \vdash \varphi[t/x]}{\Gamma \mid \cdot \vdash \mathtt{in}_\varphi\, (\mathtt{rs}_\varphi\, t) = t\ .}$$

[4] Inclusions and restrictions are like type-casting operations in some programming languages. For example in Java, an inclusion corresponds to an (implicit) type cast from a class to its superclass, whereas a restriction corresponds to a type cast from a class to a subclass. Must I write that Java is a registered trademark of Sun Microsystems?

In an informal discussion it is customary for the inclusions and restrictions to be omitted, or at least for the subscript φ to be missing.[5]

Exercise 4.3.14. Suppose $x : A \mid \psi$ and $x : A \mid \varphi$ are formulas. Show that

$$x : A \mid \psi \vdash \varphi$$

is provable if, and only if, $\{x : A \mid \psi\}$ factors through $\{x : A \mid \varphi\}$, which means that there exists a term k,

$$y : \{x : A \mid \psi\} \mid k : \{x : A \mid \varphi\} \;,$$

such that

$$y : \{x : A \mid \psi\} \mid \cdot \vdash \mathtt{in}_\psi\, y =_A \mathtt{in}_\varphi\, k$$

is provable. Show also that k is determined uniquely up to provable equality.

Example 4.3.15. We are now able to formulate the theory of a category as a cartesian theory whose underlying type theory has product types and subset types. The basic types are the type of objects $\mathtt{C}_0$ and the type of morphisms $\mathtt{C}_1$. We define the type $\mathtt{C}_2$ to be

$$\mathtt{C}_2 \equiv \{p : C_1 \times C_1 \mid \mathtt{cod}(\mathtt{fst}\, p) = \mathtt{dom}(\mathtt{snd}\, p)\} \;.$$

The basic function symbols and their signatures are:

$$\mathtt{dom} : \mathtt{C}_1 \to \mathtt{C}_0 \;, \qquad \mathtt{cod} : \mathtt{C}_1 \to \mathtt{C}_0 \;, \qquad \mathtt{id} : \mathtt{C}_0 \to \mathtt{C}_1 \;, \qquad \mathtt{c} : \mathtt{C}_2 \to \mathtt{C}_1 \;.$$

The axioms are:

$$\begin{aligned}
a : \mathtt{C}_0 \mid \cdot &\vdash \mathtt{dom}(\mathtt{id}(a)) = a \\
a : \mathtt{C}_0 \mid \cdot &\vdash \mathtt{cod}(\mathtt{id}(a)) = a \\
f : \mathtt{C}_1, g : \mathtt{C}_1 \mid \mathtt{cod}(f) = \mathtt{dom}(g) &\vdash \mathtt{dom}(\mathtt{c}(\mathtt{rs}\,\langle f, g\rangle)) = f \\
f : \mathtt{C}_1, g : \mathtt{C}_1 \mid \mathtt{cod}(f) = \mathtt{dom}(g) &\vdash \mathtt{cod}(\mathtt{c}(\mathtt{rs}\,\langle f, g\rangle)) = g \\
f : \mathtt{C}_1 \mid \cdot &\vdash \mathtt{c}(\mathtt{rs}\,\langle \mathtt{id}(\mathtt{dom}(f)), f\rangle) = f \\
f : \mathtt{C}_1 \mid \cdot &\vdash \mathtt{c}(\mathtt{rs}\,\langle f, \mathtt{id}(\mathtt{cod}(f))\rangle) = f
\end{aligned}$$

Lastly, the associativity axiom is

$$\begin{aligned}
&f : \mathtt{C}_1, g : \mathtt{C}_1, h : \mathtt{C}_1 \mid \mathtt{cod}(f) = \mathtt{dom}(g), \mathtt{cod}(g) = \mathtt{dom}(h) \vdash \\
&\qquad\qquad \mathtt{c}(\mathtt{rs}\,\langle \mathtt{c}(\mathtt{rs}\,\langle f, g\rangle), h\rangle) = \mathtt{c}(\mathtt{rs}\,\langle f, \mathtt{c}(\mathtt{rs}\,\langle g, h\rangle)\rangle) \;.
\end{aligned}$$

This notation is quite unreadable. If we write $g \circ f$ instead of $\mathtt{c}(\mathtt{rs}\,\langle f, g\rangle)$ then the axioms take on a more familiar form. For example, associativity is just $h \circ (g \circ f) = (h \circ g) \circ f$. However, we need to remember that we may form the term $g \circ f$ only if we first prove $\mathtt{dom}(g) = \mathtt{cod}(f)$.

[5] Strictly speaking, even the notation $\mathtt{in}_\varphi\, t$ is imprecise because it does not indiciate that ϕ stands in the context $x : A$. The correct notation would be $\mathtt{in}_{(x:A|\varphi)}\, t$, where x is bound in the subscript. A similar remark holds for $\mathtt{rs}_\varphi\, t$.

A subtype $\{x : A \mid \varphi\}$ is interpreted as the domain of a monomorphism representing $x : A \mid \varphi$:

$$\llbracket \{x : A \mid \varphi\} \rrbracket \rightarrowtail^{\llbracket x : A \mid \varphi \rrbracket} \llbracket A \rrbracket$$

Some care must be taken here because monos representing a given subobject are only determined up to isomorphism. We assume that a suitable canonical choice of monos can be made.

An inclusion $\Gamma \mid \mathtt{in}_\varphi\, t : A$ is interpreted as the composition

$$\llbracket \Gamma \rrbracket \xrightarrow{\llbracket t \rrbracket} \llbracket \{x : A \mid \varphi\} \rrbracket \rightarrowtail^{\llbracket x : A \mid \varphi \rrbracket} \llbracket A \rrbracket$$

A restriction $\Gamma \mid \mathtt{rs}_\varphi\, t : \{x : A \mid \varphi\}$ is interpreted as the unique $\overline{\llbracket t \rrbracket}$ which makes the following diagram commute:

$$\begin{tikzcd} \llbracket \Gamma \rrbracket \arrow[r, "\overline{\llbracket t \rrbracket}"] \arrow[dr, "\llbracket t \rrbracket"'] & \llbracket x : A \mid \varphi \rrbracket \arrow[d, tail] \\ & \llbracket A \rrbracket \end{tikzcd}$$

Exercise 4.3.16. Formulate and prove a soundness theorem for subtypes. Pay attention to the interpretation of restrictions, where you need to show unique existence of $\overline{\llbracket t \rrbracket}$.

Remark 4.3.17. Another approach to the logic of cartesian categories that captures the theory of categories and related notions involving partial operations is that of *essentially algebraic theories*, due to P. Freyd; see [?, ?]. A third approach is that of *dependent type theory* to be developed in **??** below. Finally, we will see in Section 4.5.3 that the theory of categories can be formulated as a *regular theory*.

4.4 Quantifiers as adjoints

The categorical semantics of quantification is one of the central features of the subject, and quite possibly one of the nicest contributions of categorical logic to the field of logic. You might expect that the quantifiers ∀ and ∃ are "just a big conjunction and disjunction", respectively. In fact the Polish school of algebraic logicians worked to realize this point of view—but categorical logic shows how quantifiers are treated algebraically as adjoint functors to give a much more satisfactory theory. The original treatment can be found in the classic paper [**?**].

Let us first recall the rules of inference for quantifiers. The formation rules are:

$$\frac{\Gamma, x : A \mid \varphi \ \mathtt{pred}}{\Gamma \mid (\forall\, x : A \,.\, \varphi)\ \mathtt{pred}} \qquad \frac{\Gamma, x : A \mid \varphi \ \mathtt{pred}}{\Gamma \mid (\exists\, x : A \,.\, \varphi)\ \mathtt{pred}}$$

The variable x is bound in $\forall\, x : A \,.\, \varphi$ and $\exists\, x : A \,.\, \varphi$. If x and y are distinct variables and x does not occur freely in the term t then substitution of t for y commutes with quantification over x:

$$
\begin{aligned}
(\exists\, x : A \,.\, \varphi)[t/y] &= \exists\, x : A \,.\, (\varphi[t/y]) \ , \\
(\forall\, x : A \,.\, \varphi)[t/y] &= \forall\, x : A \,.\, (\varphi[t/y]) \ .
\end{aligned}
$$

For each quantifier we have a two-way rule of inference:

$$
\frac{\Gamma, x : A \mid \psi \vdash \varphi}{\Gamma \mid \psi \vdash \forall\, x : A \,.\, \varphi}
\qquad\qquad
\frac{\Gamma, x : A \mid \varphi \vdash \vartheta}{\Gamma \mid (\exists\, x : A \,.\, \varphi) \vdash \vartheta}
$$

Note that these rules implicitly impose the usual condition that x must not occur freely in ψ and ϑ, because ψ and ϑ are supposed to be well formed in context Γ, which does not contain x.

Exercise 4.4.1. A common way of stating the inference rules for quantifiers is as follows. For the universal quantifier, the introduction and elimination rules are

$$
\frac{\Gamma, x : A \mid \psi \vdash \varphi}{\Gamma \mid \psi \vdash \forall\, x : A \,.\, \varphi}
\qquad\qquad
\frac{\Gamma \mid t : A \qquad \Gamma \mid \psi \vdash \forall\, x : A \,.\, \varphi}{\Gamma \mid \psi \vdash \varphi[t/x]}
$$

The introduction rule for existential quantifier is

$$
\frac{\Gamma \mid t : A \qquad \Gamma \mid \psi \vdash \varphi[t/x]}{\Gamma \mid \psi \vdash \exists\, x : A \,.\, \varphi}
$$

and the elimination rule is

$$
\frac{\Gamma \mid \psi \vdash \exists\, x : A \,.\, \varphi \qquad \Gamma, x : A \mid \varphi \vdash \vartheta}{\Gamma \mid \psi \vdash \vartheta}
$$

Note that these rules implicitly impose a requirement that x does not occur in Γ and that it does not occur freely in ψ because the context $\Gamma, x : A$ must be well formed and the hypotheses ψ must be well formed in context Γ. Show that these rules can be derived from the ones above, and vice versa. Of course, you may also use the inference rules for cartesian logic, cf. page 125.

In order to discover what the semantics of existential quantifier ought to be, we look at the following instance of the two-way rule for quantifiers:

$$
\frac{y : B, x : A \mid \varphi \vdash \vartheta}{y : B \mid \exists\, x : A \,.\, \varphi \vdash \vartheta} \tag{4.2}
$$

First observe that this rule implicitly requires

$$
y : B, x : A \mid \varphi \ \mathtt{pred} \qquad\quad y : B \mid \vartheta \ \mathtt{pred} \qquad\quad y : B \mid (\exists\, x : A \,.\, \varphi) \ \mathtt{pred}
$$

This is required for the entailments to be well-formed. The fourth judgement

$$y : B, x : A \mid \vartheta \;\texttt{pred}$$

follows from the second one above by weakening,

$$\frac{y : B \mid \vartheta \;\texttt{pred}}{y : B, x : A \mid \vartheta \;\texttt{pred}}$$

The interpretations of φ, ϑ, and $\exists\, x : A \,.\, \varphi$ are therefore subobjects

$$\begin{aligned}
[\![y : B, x : A \mid \varphi]\!] &\in \mathsf{Sub}([\![B]\!] \times [\![A]\!]) \;, \\
[\![y : B \mid \vartheta]\!] &\in \mathsf{Sub}([\![B]\!]) \;, \\
[\![y : B \mid \exists\, x : A \,.\, \varphi]\!] &\in \mathsf{Sub}([\![B]\!]) \;.
\end{aligned}$$

And the weakened instance of ϑ in the context $y : B, x : A$ is interpreted by pullback along a projection, cf. page 129, as in the following pullback diagram:

$$\begin{tikzcd}
{[\![y : B, x : A \mid \vartheta]\!]} \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[\![B]\!] \times [\![A]\!]} \arrow[d, "\pi"] \\
{[\![y : B \mid \vartheta]\!]} \arrow[r, tail] & {[\![B]\!]}
\end{tikzcd}$$

Thus we have

$$[\![y : B, x : A \mid \vartheta]\!] = \pi^* [\![y : B \mid \vartheta]\!] \,,$$

with weakening interpreted as the pullback functor

$$\pi^* : \mathsf{Sub}([\![B]\!]) \to \mathsf{Sub}([\![B]\!] \times [\![A]\!]) \,.$$

We will interpret existential quantification $\exists x : A$ as a suitable functor

$$\exists_A : \mathsf{Sub}([\![B]\!] \times [\![A]\!]) \to \mathsf{Sub}([\![B]\!])$$

so that

$$[\![y : B \mid \exists\, x : A \,.\, \varphi]\!] = \exists_A [\![y : B, x : A \mid \varphi]\!] \;.$$

The interpretation of the two-way rule (4.2) then becomes a two-way inequality rule

$$\frac{[\![y : B, x : A \mid \varphi]\!] \leq \pi^* [\![y : B \mid \vartheta]\!]}{\exists_A [\![y : B, x : A \mid \varphi]\!] \leq [\![y : B \mid \vartheta]\!]}$$

Replacing the interpretations of φ and ϑ by general subobjects $S \in \mathsf{Sub}([\![B]\!] \times [\![A]\!])$ and $T \in \mathsf{Sub}([\![B]\!])$, we obtain the more suggestive formulation

$$\frac{S \leq \pi^* T}{\exists_A S \leq T} \tag{4.3}$$

This is nothing but an adjunction between $\exists_A$ and π^*! Indeed, the operations $\exists_A$ and π^* are functors on the subjects posets $\mathsf{Sub}(\llbracket B \rrbracket \times \llbracket A \rrbracket)$ and $\mathsf{Sub}(\llbracket B \rrbracket)$, and the bijection of hom-sets (4.3) is exactly the statement of an adjunction between them. Thus *existential quantification is left-adjoint to weakening*:

$$\exists_A \dashv \pi^*$$

An exactly dual argument shows that *universal quantification is right-adjoint to weakening*:

$$\pi^* \dashv \forall_A$$

Thus, in sum, we have shown that the rules of inference require the quantifiers to be interpreted as operations that are adjoints to the interpretation of weakening, i.e. pullback π^* along the projection $\pi : \llbracket B \rrbracket \times \llbracket A \rrbracket \to \llbracket B \rrbracket$.

$$\begin{tikzcd}
\mathsf{Sub}(\llbracket B \rrbracket \times \llbracket A \rrbracket) \arrow[d, bend right=60, "\exists"'] \arrow[d, bend left=60, "\forall"] \\
\mathsf{Sub}(\llbracket B \rrbracket) \arrow[u, "\pi^*"{description}]
\end{tikzcd}
\qquad \exists \dashv \pi^* \dashv \forall$$

Note that the familiar side-conditions on the conventional rules for the quantifiers, to the effect that "x cannot occur freely in ψ", etc., which may seem like tiresome bookkeeping, are actually of the essence, since they express the weakening operation to which the quantifiers themselves are adjoints.

Let us see how this works for the usual interpretation in Set. A predicate $y : B, x : A \mid \varphi$ corresponds to a subset $\Phi \subseteq B \times A$, and $y : B \mid \vartheta$ corresponds to a subset $\Theta \subseteq B$. Weakening of Θ is the subset $\pi^*\Theta = \Theta \times A \subseteq B \times A$. Then we have

$$\begin{aligned}
\exists_A \Phi &= \{ y \in B \mid \exists\, x : A \,.\, \langle x, y \rangle \in \Phi \} \subseteq B \;, \\
\forall_A \Phi &= \{ y \in B \mid \forall\, x : A \,.\, \langle x, y \rangle \in \Phi \} \subseteq B \;.
\end{aligned}$$

A moment's thought convinces us that with this interpretation we do indeed have

$$\frac{\Phi \subseteq \Theta \times A}{\exists_A \Phi \subseteq \Theta} \qquad\qquad \frac{\Theta \times A \subseteq \Phi}{\Theta \subseteq \forall_A \Phi}$$

The unit of the adjunction $\exists_A \dashv \pi^*$ amounts to the inequality

$$\Phi \subseteq (\exists_A \Phi) \times A \;, \tag{4.4}$$

and the universal property of the unit says that $\exists_A \Phi$ is the smallest set satisfying (4.4). Similarly, the counit of the adjunction $\pi^* \dashv \forall_A$ is just the inequality

$$(\forall_A \Phi) \times A \subseteq \Phi\ , \tag{4.5}$$

and the universal property of the counit says that $\forall_A \Phi$ is the largest set satisfying (4.5). Figure 4.1 shows the geometric meaning of existential and universal quantification.

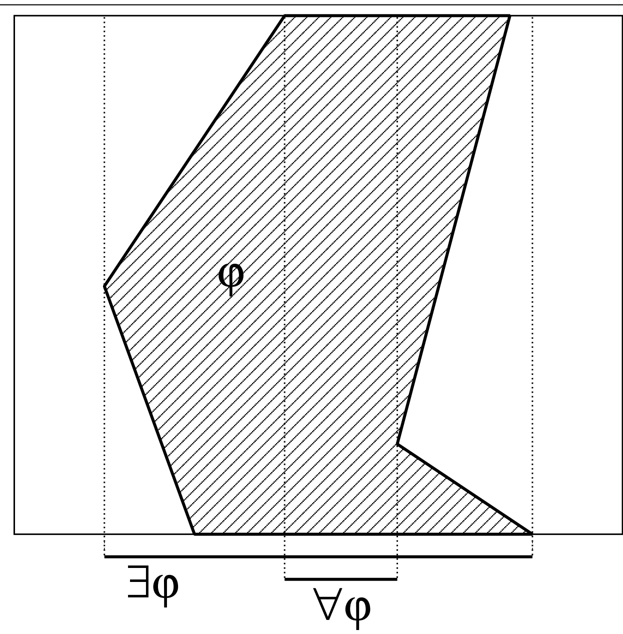

Figure 4.1: $\exists\varphi$ and $\forall\varphi$

Exercise 4.4.2. What do the universal properties of the counit of $\exists_A \dashv \pi^*$ and the unit of $\pi^* \dashv \forall_A$ say?

The weakening functor π^* is a special case of a pullback functor $f^* : \mathsf{Sub}(B) \to \mathsf{Sub}(A)$ for a morphism $f : B \to A$. This gives us the idea that we may regard the left and the right adjoint to f^* as a kind of generalized existential and universal quantifier.

We may also be tempted to *define* quantifiers as left and right adjoints to pullback functors. However there is a bit more to quantifiers than that—we are still missing the important *Beck-Chevalley condition*.

4.4.1 The Beck-Chevalley condition

Recall that quantification commutes with substitution, as long as no variables are captured by the quantifier. Thus if $\Gamma \mid t : B$ and $\Gamma, y : B, x : A \mid \varphi$ `pred` then

$$(\exists\, x : A \,.\, \varphi)[t/y] = \exists\, x : A \,.\, (\varphi[t/y])\ .$$
$$(\forall\, x : A \,.\, \varphi)[t/y] = \forall\, x : A \,.\, (\varphi[t/y])\ .$$

If semantics of quantifiers is to be sound, the interpretation of these equations must be valid. Because substitution of a term in a formula is interpreted as pullback this means that quantifiers must be *stable* under pullbacks. This is known as the *Beck-Chevalley condition*.

Definition 4.4.3. A family of functors $F_f : \mathsf{Sub}(A) \to \mathsf{Sub}(B)$ parametrized by morphisms $f : A \to B$ is said to satisfy the *Beck-Chevalley condition* when for every pullback on the left-hand side, the right-hand square commutes:

$$
\begin{tikzcd}
C \arrow[r, "h"] \arrow[d, "k"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
D \arrow[r, "g"'] & B
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathsf{Sub}(C) \arrow[d, "F_k"'] & \mathsf{Sub}(A) \arrow[l, "h^*"'] \arrow[d, "F_f"] \\
\mathsf{Sub}(D) & \mathsf{Sub}(B) \arrow[l, "g^*"]
\end{tikzcd}
$$

To convince ourselves that Beck-Chevalley condition is what we want, we spell it out explicitely in the case of a substitution into an existentially quantified formula. In order to keep the notation simple we omit the semantic brackets $[\![-]\!]$. Suppose we have a term $\Gamma \mid t : B$ and a formula $\Gamma, y : B, x : A \mid \varphi$ `pred`. The diagram

$$
\begin{tikzcd}
\Gamma \times A \arrow[r, "\langle \pi_0, t \circ \pi_0, \pi_1 \rangle"] \arrow[d, "\pi_0^{\Gamma,A}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma \times B \times A \arrow[d, "\pi_0^{\Gamma,B,A}"] \\
\Gamma \arrow[r, "\langle 1_\Gamma, t \rangle"'] & \Gamma \times B
\end{tikzcd}
$$

is a pullback. By Beck-Chevalley condition for $\exists$, the following square commutes:

$$
\begin{tikzcd}
\mathsf{Sub}(\Gamma \times A) \arrow[d, "\exists_A^{\Gamma,A}"'] & \mathsf{Sub}(\Gamma \times B \times A) \arrow[l, "\langle \pi_0, t \circ \pi_0, \pi_1 \rangle^*"'] \arrow[d, "\exists_A^{\Gamma,B,A}"] \\
\mathsf{Sub}(\Gamma) & \mathsf{Sub}(\Gamma \times B) \arrow[l, "\langle 1_\Gamma, t \rangle^*"]
\end{tikzcd}
$$

Therefore, for $\Gamma, y : B, x : A \mid \varphi$ `pred`,

$$
[\![(\exists\, x : A \,.\, \varphi)[t/y]]\!] = \langle 1_\Gamma, t \rangle^* (\exists_A^{\Gamma,B,A} [\![\varphi]\!]) = \\
\exists_A^{\Gamma,A} (\langle \pi_0, t \circ \pi_0, \pi_1 \rangle^* [\![\varphi]\!]) = [\![\exists\, x : A \,.\, (\varphi[t/y])]\!] \;.
$$

This is precisely the equation we wanted. The Beck-Chevalley condition says that (interpretations of) the quantifiers commute with pullbacks, in just the way that the syntactic operations of applying quantifiers to formulas commute with substitutions of terms, which are interpreted as pullbacks.

Definition 4.4.4. A cartesian category $\mathcal{C}$ has *existential quantifiers* if, for every $f : A \to B$, the left adjoint $\exists_f \dashv f^*$ exists and it satisfies the Beck-Chevalley condition. Similarly, $\mathcal{C}$ has *universal quantifiers* if the right adjoints $f^* \dashv \forall_f$ exist and they satisfy the Beck-Chevalley condition.

Given both adjoints $\exists_f \dashv f^* \dashv \forall_f$, it actually suffices to have the Beck-Chevalley condition for either one in order to infer it for both:

Proposition 4.4.5. *If for every $f : A \to B$, both the left and right adjoints exist*

$$\exists_f \dashv f^* \dashv \forall_f$$

then the left adjoint satisfies the Beck-Chevalley condition iff the right adjoint does.

Proof. Suppose we have the Beck-Chevalley condition for the left adjoints $\exists$, and that we are given a pullback square as on the left below. We want to check the Beck-Chevalley square for the right adjoints $\forall$, as indicated on the right below.

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[d, "k"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ D \arrow[r, "g"'] & B \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Sub}(C) \arrow[d, "\forall_k"'] & \mathsf{Sub}(A) \arrow[l, "h^*"'] \arrow[d, "\forall_f"] \\ \mathsf{Sub}(D) & \mathsf{Sub}(B) \arrow[l, "g^*"] \end{tikzcd}$$

Swapping all the functors in the righthand diagram for their left adjoints we obtain the following.

$$\begin{tikzcd} \mathsf{Sub}(C) \arrow[r, "\exists_h"] & \mathsf{Sub}(A) \\ \mathsf{Sub}(D) \arrow[u, "k^*"] \arrow[r, "\exists_g"'] & \mathsf{Sub}(B) \arrow[u, "f^*"'] \end{tikzcd}$$

But this is a Beck-Chevalley square for (the "transpose" of) the original pullback diagram, and therefore commutes by the Beck-Chevalley condition for the left adjoints $\exists$. The original diagram of right adjoints therefore also commutes, by uniqueness of adjoints.

The argument for the dual case is, well, dual. □

Exercise 4.4.6. In Set we can identify $\mathsf{Sub}(-)$ with powersets because $\mathsf{Sub}(X) \cong \mathcal{P}X$. Then quantifiers along a function $f : A \to B$ are functions

$$\exists_f : \mathcal{P}A \to \mathcal{P}B \,, \qquad\qquad \forall_f : \mathcal{P}A \to \mathcal{P}B \,.$$

Verify that

$$\begin{aligned} \exists_f U &= \big\{ b \in B \;\big|\; \exists\, a : A \,.\, (fa = b \wedge a \in U) \big\} \;, \\ \forall_f U &= \big\{ b \in B \;\big|\; \forall\, a : A \,.\, (fa = b \Rightarrow a \in U) \big\} \;. \end{aligned}$$

Thus $\exists_f U$ is just the usual direct image of U by f, sometimes written $f_!(U)$, or simply $f(U)$. But have you seen $\forall_f U$ before? It can also be written as $\forall_f U = \{b \in B \mid f^*\{b\} \subseteq U\}$. What is the meaning of $\exists_q$ and $\forall_q$ when $q : A \to A/{\sim}$ is a canonical quotient map that maps an element $x \in A$ to its equivalence class $qx = [x]$ under an equivalence relation $\sim$ on A?

4.5 Regular logic

We next consider the question of when a cartesian category has existential quantifiers. It turns out that this is closely related to the notion of a *regular category*, a concept which first arose in the context of abelian categories and axiomatic homology theory, quite independently of categorical logic. We will see for instance that all algebraic categories, in the sense of Chapter **??**, are regular.

4.5.1 Regular categories

Throughout this section we work in a cartesian category $\mathcal{C}$. We begin with some general definitions. The *kernel pair* of a morphism $f : A \to B$ is the pair of morphisms $k_1, k_2 : K \rightrightarrows A$ obtained as in the following pullback

$$\begin{tikzcd} K \arrow[r, "k_2"] \arrow[d, "k_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Note that a kernel pair determines an equivalence relation $\langle k_1, k_2 \rangle : K \rightarrowtail A \times A$, in the sense that the map $\langle k_1, k_2 \rangle$ is a mono that satisfies the reflexivity, symmetry and transitivity conditions. In Set the mono $\langle k_1, k_2 \rangle : K \rightarrowtail A \times A$ is the equivalence relation $\sim$ on A defined by

$$x \sim y \iff fx = fy\ .$$

Indeed, a kernel pair in a general cartesian category is a model of the cartesian theory of an equivalence relation, in the sense of example 4.3.3.

Exercise 4.5.1. Prove this.

In general, the *quotient* by the equivalence relation determined by the kernel pair k_1, k_2 is their coequalizer $q : A \to Q$, if it exists,

$$\begin{tikzcd} K \arrow[r, shift left, "k_1"] \arrow[r, shift right, "k_2"'] & A \arrow[r, "q"] & Q \end{tikzcd}$$

Such a coequalizer is called a *kernel quotient*.

Because $f \circ k_1 = f \circ k_2$, we see that f factors through q by a unique morphism $m : Q \to A$,

$$\begin{tikzcd}
K \arrow[r, shift left, "k_1"] \arrow[r, shift right, "k_2"'] & A \arrow[rr, "f"] \arrow[dr, "q"'] & & B \\
& & Q \arrow[ur, "m"'] &
\end{tikzcd} \tag{4.6}$$

As a coequalizer, $q : A \to Q$ is always epic; indeed, epis that are coequalizers will be called *regular epimorphisms* and will be denoted by arrows with triangular heads:

$$e : A \twoheadrightarrow B$$

It is of some interest to know when the second factor $m : Q \to B$ in (4.6) is guaranteed to be a mono. For example, in Set the function $m : Q \to B$ is defined by $m[x] = fx$, where $Q = A/{\sim}$ as above. In this case m is indeed injective, because $m[x] = m[y]$ implies $fx = fy$, hence $x \sim y$ and $[x] = [y]$.

Definition 4.5.2. A category with finite limits is *regular* when it has kernel quotients, and regular epis are stable under pullback. Thus, in detail:

1. the kernel pair of any map has a coequalizer, and
2. any pullback of a regular epi is a regular epi.

Exercise 4.5.3. Suppose $e : A \twoheadrightarrow B$ is a regular epi. Prove that it is the coequalizer of its own kernel pair.

Let us return to (4.6) and show that m is monic in any regular category. Consider the following diagram, in which h_1, h_2 are constructed as the kernel pair of m, and the other three squares are constructed as pullbacks:

$$\begin{tikzcd}
K \arrow[r, two heads, "p_2"] \arrow[d, two heads, "p_1"'] \arrow[dr, two heads, "r"] & \cdot \arrow[r] \arrow[d, two heads, "s_2"] & A \arrow[d, two heads, "q"] \\
\cdot \arrow[r, two heads, "s_1"'] \arrow[d] & H \arrow[r, "h_2"] \arrow[d, "h_1"'] & Q \arrow[d, "m"] \\
A \arrow[r, two heads, "q"'] & Q \arrow[r, "m"'] & B
\end{tikzcd}$$

Because all the smaller squares are pullbacks the large square is a pullback as well, therefore the left-hand vertical morphism is $k_1 : K \to A$, and the morphism across the top is $k_2 : K \to A$, and we have the kernel pair $k_1, k_2 : K \rightrightarrows A$ of $f = m \circ q$. The morphisms s_1,

s_2, p_1, and p_2 are all regular epis because they are pullbacks of the regular epi q. The morphism $r = s_2 \circ p_2 = s_1 \circ p_1$ is epic because it is a composition of regular epis. Observe that

$$h_1 \circ r = q \circ k_1 = q \circ k_2 = h_2 \circ r \ ,$$

and so, because r is epic, $h_1 = h_2$. But this means that m is monic, since the maps in its kernel pair are equal; indeed, given any $u, v : U \to Q$ with $m \circ u = m \circ v$, there exists a $w : U \to H$ such that $u = w \circ h_1 = w \circ h_2 = v$.

Proposition 4.5.4. *In a regular category every morphism $f : A \to B$ factors as a composition of a regular epi q followed by a mono m,*

$$\begin{tikzcd} A \arrow[r, twoheadrightarrow, "q"'] \arrow[rr, bend left, "f"] & Q \arrow[r, rightarrowtail, "m"'] & B \end{tikzcd}$$

The factorization is unique up to isomorphism.

Proof. By uniqueness of the factorization we mean that if

$$\begin{tikzcd} A \arrow[r, twoheadrightarrow, "q'"'] \arrow[rr, bend left, "f"] & Q' \arrow[r, rightarrowtail, "m'"'] & B \end{tikzcd}$$

is another such factorization, then there exists an isomorphism $i : Q \to Q'$ such that $q' = i \circ q$ and $m = m' \circ i$.

$$\begin{tikzcd} A \arrow[r, twoheadrightarrow, "q'"] \arrow[d, twoheadrightarrow, "q"'] & Q' \arrow[d, rightarrowtail, "m'"] \\ Q \arrow[r, rightarrowtail, "m"'] \arrow[ur, dotted, "i"'] & B \end{tikzcd}$$

As the factorization of f we take the one constructed in (4.6). Then q is a regular epi by construction, and we have just shown that m is monic. So it only remains to show that the factorization is unique. Suppose f also factors as $f = m' \circ q'$ where q' is a regular epi and m' is monic. Consider the following diagram, in which k_1, k_2 is the kernel pair of f, q is the coequalizer of k_1 and k_2, and h_1, h_2 is the kernel pair of q' so that q' is the coequalizer of h_1 and h_2:

$$\begin{tikzcd}
 & H \arrow[d, shift right, "h_1"'] \arrow[d, shift left, "h_2"] & \\
K \arrow[r, shift left, "k_1"] \arrow[r, shift right, "k_2"'] & A \arrow[r, two heads, "q"] \arrow[d, two heads, "q'"'] & Q \arrow[d, tail, "m"] \arrow[dl, dotted, shift right, "i"'] \\
 & Q' \arrow[r, tail, "m'"'] \arrow[ur, dotted, shift right, "j"'] & B
\end{tikzcd}$$

Because $m' \circ q' \circ k_1 = m \circ q \circ k_1 = m \circ q \circ k_2 = m' \circ q' \circ k_2$ and m' is monic, $q' \circ k_1 = q' \circ k_2$. So there exists a unique $i : Q \to Q'$ such that $q' = i \circ q$. But then $m' \circ i \circ q = m' \circ q' = f = m \circ q$ and because q is epi, $m' \circ i = m$.

We prove that i is iso by constructing its inverse j. Because $m \circ q \circ h_1 = m' \circ q \circ h_1 = m' \circ q \circ h_2 = m \circ q \circ h_2$ and m is monic, $q \circ h_1 = q \circ h_2$. So there exists a unique $j : Q' \to Q$ such that $q = j \circ q'$. Now we have $i \circ j \circ q' = i \circ q = \mathsf{1}_{Q'} \circ q'$, from which we conclude that $i \circ j = \mathsf{1}_{Q'}$ because q' is epi. Similarly, $j \circ i \circ q = j \circ q' = \mathsf{1}_Q \circ q$, therefore $j \circ i = \mathsf{1}_Q$. □

Corollary 4.5.5. *A map $f : A \to B$ that is both a regular epi and a mono is an iso.*

Proof. Consider the following outer square, regarded as two different reg-epi/mono factorizations.

$$\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[d, "f"'] & A \arrow[d, "f"] \\
B \arrow[r, "1_B"'] \arrow[ur, dotted, "d"'] & B
\end{tikzcd}$$

A diagonal d is then an inverse of f. □

A factorization $f = m \circ q$ as in Proposition 4.5.4 determines a subobject

$$\mathsf{im}(f) = [m : Q \rightarrowtail B] \in \mathsf{Sub}(B) \,,$$

called the *image of f*. It is characterized as the least subobject of B through which f factors.

Proposition 4.5.6. *For a morphism $f : A \to B$ in a regular category $\mathcal{C}$, the image $\mathsf{im}(f) \rightarrowtail B$ is the least subobject $U \rightarrowtail B$ of B through which f factors.*

Proof. Suppose f factors through $v : V \rightarrowtail B$ as

$$\begin{tikzcd}
A \arrow[r, "g"'] \arrow[rr, bend left, "f"] & V \arrow[r, tail, "v"'] & B
\end{tikzcd}$$

and consider the factorization of f, as in (4.6). Since $v \circ g \circ k_1 = f \circ k_1 = f \circ k_2 = v \circ g \circ k_2$ and v is mono, $g \circ k_1 = g \circ k_2$, therefore there exists a unique $\overline{g} : Q \to V$ such that $g = \overline{g} \circ q$. Now $v \circ \overline{g} \circ q = v \circ g = f = m \circ q$ and because q is epic, $v \circ \overline{g} = m$ as required. (The reader should draw the corresponding diagram.) □

Definition 4.5.7. A functor $F : \mathcal{C} \to \mathcal{D}$ is *regular* if it preserves finite limits and regular epis. It follows that F preserves image factorizations. The category of regular functors $\mathcal{C} \to \mathcal{D}$ and natural transformations is denoted by $\mathsf{Reg}(\mathcal{C}, \mathcal{D})$.

Examples of regular categories

Let us consider some examples of regular categories.

1. The category Set is regular. It is complete and cocomplete, so it has in particular all finite limits and coequalizers. To show that the pullback of a regular epi is again a regular epi, note that in Set the epis are exactly the surjections, and a surjection is a quotient of its kernel pair, and thus a regular epi. It therefore it suffices to show that the pullback of a surjection is a surjection, which is easy.

2. More generally, any presheaf category $\widehat{\mathcal{C}}$ is also regular, because it is complete and cocomplete, with (co)limits computed pointwise. Thus, again, every epi is regular, and epis are stable under pullbacks.

3. ("Fuzzy logic") Let H be a complete Heyting algebra; thus H is a cartesian closed poset with all small joins $\bigvee_i p_i$. The category of *H-presets* has as objects all pairs $(X, e_X : X \to H)$ where X is a set and e_X is a function, called the *existence predicate of X*. For $x \in X$, $e_X(x)$ can be thought of as "the amount by which x exists". A *morphism of presets* is a function $f : X \to Y$ satisfying, for all $x \in X$,
$$e_X(x) \leq e_Y(fx).$$
This is a regular category, with the following structure.
   - the terminal object is $\top : 1 \to H$,
   - the product of $e_A : A \to H$ and $e_B : B \to H$ is
   $$e_A \wedge e_B : A \times B \to H,$$
   where $(e_a \wedge e_B)(a, b) = e_A(a) \wedge e_B(b)$,
   - the equalizer of two maps $f, g : A \to B$ is their equalizer as functions, $A' = \{a \mid f(a) = g(a)\} \hookrightarrow A$, with the restriction of $e_A : A \to H$ to $A' \subseteq A$.
   - a map $f : A \to B$ is a regular epi if and only if it is a surjective function and for all $b \in B$:
   $$e_B(b) = \bigvee_{f(a)=b} e_A(a)$$

Exercise 4.5.8. Verify that H-presets form a regular category, and compute the regular epi-mono factorization of a map.

The next example deserves to be a proposition.

Proposition 4.5.9. *The category* $\mathsf{Mod}(\mathbb{A}, \mathsf{Set})$ *of set-theoretic models of an algebraic theory* $\mathbb{A}$ *is regular.*

Proof. We sketch a proof, for details see [**?**, Theorem 3.5.4]. Recall that the objects of $\mathsf{Mod}(\mathbb{A}) = \mathsf{Mod}(\mathbb{A}, \mathsf{Set})$ are $\mathbb{A}$-algebras, which are structures $A = (|A|, f_1, f_2, \ldots)$ where $|A|$ is the underlying set and $f_1, f_2, \ldots$ are the basic operations on $|A|$. Every such $\mathbb{A}$-algebra is also required to satisfy the equational axioms of $\mathbb{A}$. A morphism $h : A \to B$ is a function $h : |A| \to |B|$ that preserves the basic operations.

The category $\mathsf{Mod}(\mathbb{A})$ of $\mathbb{A}$-algebras has small limits, which are created by the forgetful functor $U : \mathsf{Mod}(\mathbb{A}) \to \mathsf{Set}$. Thus the product of $\mathbb{A}$-algebras A and B has as its underlying set $|A \times B| = |A| \times |B|$, and the basic operations of $A \times B$ are computed separately on each factor, and similarly for products of arbitrary (small) families $\prod_i A_i$. An equalizer of morphisms $g, h : A \to B$ has as its underlying set the equalizer of $g, h : |A| \to |B|$, and the basic operations inherited from A.

To see that coequalizers of kernel pairs exist, consider a morphism $h : A \to B$. We can form the quotient $\mathbb{A}$-algebra Q whose underlying set is $|Q| = |A|/{\sim}$, where $\sim$ is the relation defined by

$$x \sim y \iff hx = hy \;,$$

which is just the kernel quotient of the underlying function h. A basic operation $f_Q : |Q|^k \to |Q|$ is induced by the basic operation $f_A : |A|^k \to |A|$ by

$$f_Q\langle [x_1], \ldots, [x_k] \rangle = [f_A\langle x_1, \ldots, x_k \rangle] \;.$$

It is easily verified that this is well-defined, that Q is an $\mathbb{A}$-algebra, and that the canonical quotient map $q : A \to Q$ is the coequalizer of the kernel pair of h.

Lastly regular epis in $\mathsf{Mod}(\mathbb{A})$ are stable because pullbacks and kernel pairs are computed as in Set, and a morphism $h : A \to B$ is a regular epi in $\mathsf{Mod}(\mathbb{A})$ if, and only if, the underlying function $h : |A| \to |B|$ is a regular epi in Set, which is therefore stable under pullback. □

We now know that categories of groups, rings, modules, $\mathcal{C}^\infty$-rings and other algebraic categories are regular. The preceding proposition is useful also for showing that certain structures can*not* be axiomatized by algebraic theories. The category of posets is an example of a category that is not regular; therefore the theory of partial orders cannot be axiomatized solely by equations.

Exercise 4.5.10. Show that Poset is not regular. (Hint: find a regular epi that is not stable under pullback.) Conclude that there is no purely equational reformulation of the cartesian theory of posets.

Exercise* 4.5.11. Is Top regular? Hint: is there is a topological quotient map $q : X \twoheadrightarrow X'$ and a space Y such that $q \times 1_Z : X \times Y \twoheadrightarrow X' \times Y$ is not a quotient map?

Remark 4.5.12 (Exactness)**.** A regular category $\mathcal{C}$ is said to be *exact* [**?**] if *every* equivalence relation (not just those arising as kernel pairs) has a quotient. It can be shown fairly easily that categories of algebras are not just regular but also exact: an equivalence relation in such a category is a congruence relation with respect to the algebraic operations, and its (underlying set) quotient is then necessarily also a homomorphism, and thus a coequalizer of algebras.

Exercise 4.5.13. Prove that the regular epis and monos in a regular category $\mathcal{C}$ form the two classes $(\mathcal{L}, \mathcal{R})$, respectively, of an *orthogonal factorization system* in the following sense:

1. every arrow $f : A \to B$ factors as $f = r \circ l$ with $l \in \mathcal{L}$ and $r \in \mathcal{R}$,

2. $\mathcal{L}$ is the class of all arrows left-orthogonal to all maps in $\mathcal{R}$, and $\mathcal{R}$ is the class of all arrows right-orthogonal to all maps in $\mathcal{L}$, where $l : A \to B$ is said to be *left-orthogonal* to $r : X \to Y$, and r is said to be *right-orthogonal* to l, if for every commutative square as on the outside below,

$$\begin{tikzcd} A \arrow[r] \arrow[d, "l"'] & X \arrow[d, "r"] \\ B \arrow[r] \arrow[ur, dotted, "d"'] & Y, \end{tikzcd}$$

   there is a unique diagonal arrow d as indicated making both triangles commute.

4.5.2 Images and existential quantifiers

Recall that the poset $\mathsf{Sub}(A)$ is equivalent to the preordered category $\mathsf{Mono}(A)$ of monos into A. If we compose an equivalence functor $\mathsf{Sub}(A) \to \mathsf{Mono}(A)$ with the inclusion $\mathsf{Mono}(A) \to \mathcal{C}/A$ we obtain a (full and faithful) inclusion functor

$$I : \mathsf{Sub}(A) \hookrightarrow \mathcal{C}/A\,. \tag{4.7}$$

In the other direction we have the "image functor" $\mathsf{im} : \mathcal{C}/A \to \mathsf{Sub}(A)$, which maps an object $f : B \to A$ in $\mathcal{C}/A$ to the subobject $\mathsf{im}(f) \rightarrowtail A$.

Exercise 4.5.14. In order to show that im is in fact a functor, prove that $f = g \circ h$ implies $\mathsf{im}(f) \leq \mathsf{im}(g)$.

Proposition 4.5.6 says that the image functor is left adjoint to the inclusion functor (4.7),

$$\mathsf{im} \dashv I\,.$$

Furthermore, images are stable in the sense that the following diagram commutes for all $f : A \to B$ (as does the corresponding one with the inclusion I in place of im).

$$\begin{tikzcd}
\mathcal{C}/A \arrow[d, "\mathsf{im}_A"'] & \mathcal{C}/B \arrow[l, "f^*"'] \arrow[d, "\mathsf{im}_B"] \\
\mathsf{Sub}(A) & \mathsf{Sub}(B) \arrow[l, "f^*"]
\end{tikzcd} \tag{4.8}$$

The functor f^* on the top is the "change of base" functor given by pullback of an arbitrary map, and the functor f^* on the bottom is the pullback functor acting on subjects. To see that (4.8) commutes, consider $g : C \to B$ and the following diagram:

$$\begin{tikzcd}
f^*C \arrow[r] \arrow[d, two heads] \arrow[dd, bend right=60, "f^*g"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, two heads] \arrow[dd, bend left=60, "g"] \\
\cdot \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{im}(g) \arrow[d, tail] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

On the right-hand side we have the factorization of g, which is then pulled back along f. Because monos and regular epis are both stable, this gives a factorization of the pullback f^*g, hence (by the uniqueness of factorizations, Proposition 4.5.4) the claimed equality

$$\mathsf{im}(f^*g) = f^*(\mathsf{im}(g)) \,.$$

Proposition 4.5.15. *A regular category has existential quantifiers. The existential quantifier along $f : A \to B$,*

$$\exists_f : \mathsf{Sub}(A) \longrightarrow \mathsf{Sub}(B),$$

is given by

$$\exists_f[m : M \rightarrowtail A] = \mathsf{im}(f \circ m) \,,$$

as indicated below.

$$\begin{tikzcd}
M \arrow[r, two heads] \arrow[d, tail, "m"'] & \mathsf{im}(f \circ m) \arrow[d, tail] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

Proof. Recall that composition

$$\Sigma_f : \mathcal{C}/A \longrightarrow \mathcal{C}/B$$

by a map $f : A \to B$ is left adjoint to pullback f^* along f. Thus we are defining $\exists_f = \mathsf{im} \circ \Sigma_f \circ I$ as shown below.

$$\begin{tikzcd}
\mathsf{Sub}(A) \arrow[r, dotted, "\exists_f"] \arrow[d, hook, "I"'] & \mathsf{Sub}(B) \\
\mathcal{C}/A \arrow[r, "\Sigma_f"'] & \mathcal{C}/B \arrow[u, "\mathsf{im}"']
\end{tikzcd}$$

First we verify that $\exists_f \dashv f^*$ on subobjects. For $U \rightarrowtail A$ and $V \rightarrowtail B$:

$$\begin{array}{ll}
\exists_f U \le V & \text{in } \mathsf{Sub}(B) \\
\hline\hline
\mathsf{im} \circ \Sigma_f \circ I(U) \le V & \text{in } \mathsf{Sub}(B) \\
\hline\hline
\Sigma_f \circ I(U) \le I(V) & \text{in } \mathcal{C}/B \\
\hline\hline
I(U) \to f^* I(V) & \text{in } \mathcal{C}/A \\
\hline\hline
I(U) \to I(f^* V) & \text{in } \mathcal{C}/A \\
\hline\hline
U \le f^* V & \text{in } \mathsf{Sub}(A)
\end{array}$$

In the second step in the above derivation we used the adjunction between $\mathsf{im} : \mathcal{C}/B \to \mathsf{Sub}(B)$ and the inclusion $\mathsf{Sub}(B) \to \mathcal{C}/B$.

The Beck-Chevalley condition follows from stability of image factorizations. Indeed, given a pullback

$$\begin{tikzcd}
D \arrow[r, "h"] \arrow[d, "k"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

and a subobject $U \rightarrowtail C$, (4.8) gives

$$\begin{aligned}
f^*(\exists_g U) = f^* \circ \mathsf{im} \circ \Sigma_g \circ I(U) = \mathsf{im} \circ f^* \circ \Sigma_g \circ I(U) &= \mathsf{im} \circ \Sigma_k \circ h^* \circ I(U) \\
&= \mathsf{im} \circ \Sigma_k \circ I \circ h^*(U) = \exists_k(h^* U)
\end{aligned}$$

$$
\begin{tikzcd}
 & f^*U \arrow[r] \arrow[dl, two heads] \arrow[d, tail] \arrow[dr, phantom, very near start, "\lrcorner"] & U \arrow[d, tail] \arrow[dr, two heads] & \\
\exists_k f^*U \arrow[dr, tail] & D \arrow[r, "h"] \arrow[d, "k"'] \arrow[dr, phantom, very near start, "\lrcorner"] & C \arrow[d, "g"] & \exists_g U \arrow[dl, tail] \\
 & A \arrow[r, "f"'] & B &
\end{tikzcd}
$$

as required. ☐

Summarizing the results of this section, we have the following.

Proposition 4.5.16. *In any regular category, for every map $f : A \to B$ we have the following situation, where f^* is pullback:*

$$
\begin{tikzcd}
\mathsf{Sub}(A) \arrow[r, shift right, "\exists_f"'] \arrow[d, hook, shift left, "I"] & \mathsf{Sub}(B) \arrow[l, shift right, "f^*"'] \arrow[d, hook, shift left, "I"] \\
\mathcal{C}/A \arrow[u, shift left, "\mathsf{im}"] \arrow[r, shift right, "\Sigma_f"'] & \mathcal{C}/B \arrow[l, shift right, "f^*"'] \arrow[u, shift left, "\mathsf{im}"]
\end{tikzcd}
$$

with adjunctions

$$\exists_f \dashv f^*, \quad \mathsf{im} \dashv I, \quad \Sigma_f \dashv f^*$$

and natural isos

$$f^* \circ \mathsf{im} \cong \mathsf{im} \circ f^*, \quad f^* \circ I \cong I \circ f^*.$$

Note, moreover, that

$$\exists_f \circ \mathsf{im} \cong \mathsf{im} \circ \Sigma_f$$

then follows.

Finally, we call attention to the following special fact.

Proposition 4.5.17 (Frobenius Reciprocity)**.** *Given a map $f : A \to B$ and subobjects $U \leq A$ and $V \leq B$, the following equation holds in $\mathsf{Sub}(B)$.*

$$\exists_f(U \wedge f^*V) = \exists_f U \wedge V$$

Exercise 4.5.18. Prove Frobenius reciprocity, using the following diagram.

$$\begin{tikzcd}
 & U \wedge f^*V \arrow[rr] \arrow[d] \arrow[ddl] & & \exists_f U \wedge V \arrow[d] \arrow[ddl] \\
 & f^*V \arrow[rr] \arrow[ddl] & & V \arrow[ddl, tail] \\
U \arrow[rr] \arrow[d, tail] & & \exists_f U \arrow[d] & \\
A \arrow[rr, "f"'] & & B &
\end{tikzcd}$$

4.5.3 Regular theories

A regular category has finite limits and image factorizations, therefore it allows us to interpret a type theory with the terminal type and binary products, and a logic with equality, conjunction, and existential quantifiers. This system is called *regular logic*.

Definition 4.5.19. A (many-sorted) *regular theory* $\mathbb{T}$ is a (many-sorted) type theory together with a set of axioms expressed in the fragment of logic built from $=$, $\top$, $\wedge$, and $\exists$.

In more detail, a regular theory consists of the following data, extending the notion of cartesian theory from section 4.3.

- basic type symbols $A_1, \ldots, A_k$,
- basic function symbols $f, \ldots$ (with signature) $(A_1, \cdots, A_m; B)$,
- basic relation symbols $R, \ldots$ (with signature) $(A_1, \cdots, A_n)$.

We then define by induction the set of terms in context,

$$\Gamma \mid t : A \,,$$

as well as the formulas in context,

$$\Gamma \mid \varphi \ \texttt{pred}\,.$$

Here is the first place where things differ from cartesian logic; we extend the formation rules for cartesian formulas (section 4.3) by the further clause:

6. Existential Quantifier:

$$\frac{\Gamma, x : A \mid \varphi \ \texttt{pred}}{\Gamma \mid \exists x : A.\, \varphi \ \texttt{pred}}$$

(We also add the evident additional clause for sustitution of terms into existentially quantified formulas, namely $(\exists x : A.\, \varphi)[t/y] = \exists x : A.\, (\varphi[t/y])$.) This defines the notion of a *regular formula*, i.e. ones built from the atomic formulas $s = t$ and $R(t_1, \ldots, t_n)$ using the logical operations $\top$, $\wedge$, and $\exists$.

A regular theory then includes, finally, a set of axioms of the form

$$\Gamma \mid \varphi \vdash \psi$$

where φ, ψ are *regular formulas.*

Example 4.5.20.

1. A ring A (with unit 1) is called *von Neumann regular* if for every element a there is at least one element x for which $a = a \cdot x \cdot a$. Such an x may be thought of as a "weak inverse" of a. The theory of *von Neumann regular rings* is thus an extension of the usual theory of rings with unit by adding the single axiom

$$a : A \mid \top \vdash \exists x : A \,.\, a = a \cdot x \cdot a$$

2. A perhaps more familiar example is the theory of categories, with two basic types A, O for arrows and objects, 3 basic function symbols $\mathrm{dom}, \mathrm{cod} : (A; O)$ and $\mathrm{id} : (O; A)$ and one basic relation symbol $C : (A, A, A)$, where the latter is for the relation $C(x, y, z) =$ "z is the composite of x and y". The axioms for C are as follows (with abbreviated notation for the context):

$$\begin{aligned} x, y, z : A \mid C(x, y, z) &\vdash \mathrm{cod}(x) = \mathrm{dom}(y) \wedge \mathrm{dom}(z) = \mathrm{dom}(x) \wedge \mathrm{cod}(z) = \mathrm{cod}(y) \\ x, y : A \mid \mathrm{cod}(x) = \mathrm{dom}(y) &\vdash \exists z.\, C(x, y, z) \\ x, y, z, z' : A \mid C(x, y, z) \wedge C(x, y, z') &\vdash z = z' \end{aligned}$$

Recall the previous versions of the theory of categories as cartesian theories in 4.3.17. Are the homomorphisms of categories, as models of a regular theory, the same thing as functors?

3. The theory of an *inhabited object* has a single type A, no function or relation symbols, and the single axiom:

$$\cdot \mid \top \vdash \exists x : A.\, x = x$$

A model is an object that is "inhabited" by at least one (unnamed) element, but the homomorphisms need not preserve anything – in this sense being inhabited is a *property*, not a *structure.*

The *rules of inference* of regular logic are those of cartesian logic (section 4.3), with an additional rule for the existential quantifier:

8. Existential Quantifier:

$$\frac{y : B, x : A \mid \varphi \vdash \vartheta}{\overline{y : B \mid \exists x : A \,.\, \varphi \vdash \vartheta}}$$

Note that the lower judgement is well-formed only if $x : A$ does not occur freely in ϑ.

We also add a rule coresponding to Frobenius reciprocity, Proposition 4.5.17, in the form

9. Frobenius:
$$x : A \mid (\exists y : B.\varphi) \wedge \psi \;\vdash\; \exists y : B.(\varphi \wedge \psi)$$
provided the variable $y : B$ does not occur freely in ψ.

Note that the converse of Frobenius is easily derivable, so we have the interderivability of $(\exists y : B.\varphi) \wedge \psi$ and $\exists y : B.(\varphi \wedge \psi)$ when $y : B$ is not free in ψ. The Frobenius rule will be derivable in the extended system of Heyting logic (see Proposition 4.6.15), and could be made derivable in a suitably formulated system of regular logic using multi-sequents $\Gamma \mid \varphi_1, \ldots, \varphi_n \vdash \psi$.

Semantics of regular theories

Turning to semantics, an *interpretation* of a regular theory $\mathbb{T}$ in a regular category $\mathcal{C}$ extends the notion for cartesian logic (section 4.3), and is given by the following data:

1. Each basic sort A is interpreted as an object $[\![A]\!]$.
2. Each basic constant f with signature $(A_1, \ldots, A_n; B)$ is interpreted as a morphism $[\![f]\!] : [\![A_1]\!] \times \cdots \times [\![A_n]\!] \to [\![B]\!]$.
3. Each basic relation symbol R with signature $(A_1, \ldots, A_n)$ is interpreted as a subobject $[\![R]\!] \in \mathsf{Sub}([\![A_1]\!] \times \cdots \times [\![A_1]\!])$.

This is the same as for cartesian logic, as is the extension of the interpretation to all terms,
$$[\![\Gamma \mid t : A]\!] : [\![\Gamma]\!] \longrightarrow [\![A]\!]$$
For the formulas, we extended the interpretation to cartesian formulas as before (section **??**),
$$[\![\Gamma \mid \varphi]\!] \rightarrowtail [\![\Gamma]\!] \,.$$
Finally, existential formulas $\exists\, x : A \,.\, \varphi$ are interpreted by the existential quantifiers in the regular category,
$$[\![\Gamma \mid \exists\, x : A \,.\, \varphi]\!] = \exists_A [\![\Gamma, x : A \;\big|\; \varphi]\!] \,,$$
where
$$\exists_A = \exists_\pi : \mathsf{Sub}([\![\Gamma]\!] \times [\![A]\!]) \to \mathsf{Sub}([\![\Gamma]\!])$$
is the existential quantifier along the projection $\pi : [\![\Gamma]\!] \times [\![A]\!] \to [\![\Gamma]\!]$.

The following is immediate from these definitions, and the considerations in section 4.4.

Proposition 4.5.21. *The rules of regular logic are sound with respect to the interpretation in regular categories.*

Exercise 4.5.22. Prove this.

If all the axioms of $\mathbb{T}$ hold in a given interpretation, then we again say that the interpretation is a *model* of the theory $\mathbb{T}$. Morphisms of models are just morphisms of the underlying cartesian structures. Thus for any regular theory $\mathbb{T}$ and regular category $\mathcal{C}$, there is a *category of models*,

$$\mathsf{Mod}(\mathbb{T}, \mathcal{C}) .$$

Moreover, this semantic category is functorial in $\mathcal{C}$ with respect to regular functors $\mathcal{C} \to \mathcal{D}$, which, recall, preserve finite limits and regular epis. Indeed, if $F : \mathcal{C} \to \mathcal{D}$ is regular then given a model M in $\mathcal{C}$ with underlying cartesian structure $[\![A]\!]_M, [\![f]\!]_M, [\![R]\!]_M$, etc., we can determine an interpretation FM in $\mathcal{D}$ by setting:

$$[\![A]\!]_{FM} = F([\![A]\!]_M),\ [\![f]\!]_{FM} = F([\![f]\!]_M),\ [\![R]\!]_{FM} = F([\![f]\!]_M)$$

etc., and these will have the correct types (up to isomorphism). To show that FM is a $\mathbb{T}$-model, if M is one and F is regular, consider an axiom of $\mathbb{T}$ of the form $\Gamma \mid \varphi \vdash \psi$. Satisfaction by M means that $[\![\Gamma \mid \varphi]\!]_M \leq [\![\Gamma \mid \psi]\!]_M$ in $\mathsf{Sub}([\![\Gamma]\!]_M)$, which in turn means that there is a (necessarily unique) factorization,

$$\begin{tikzcd}
{[\![\Gamma \mid \varphi]\!]_M} \arrow[rr, dotted] \arrow[dr] & & {[\![\Gamma \mid \psi]\!]_M} \arrow[dl] \\
& {[\![\Gamma]\!]_M} &
\end{tikzcd} ,$$

Applying the cartesian functor F will result in an inclusion of subobjects $F[\![\Gamma \mid \varphi]\!]_M \leq F[\![\Gamma \mid \psi]\!]_M$ in $\mathsf{Sub}(F[\![\Gamma]\!]_M) = \mathsf{Sub}([\![\Gamma]\!]_{FM})$. Thus is clearly suffices to show that for any regular formula φ,

$$F[\![\Gamma \mid \varphi]\!]_M = [\![\Gamma \mid \varphi]\!]_{FM} .$$

This is an easy induction on φ, using the regularity of F.

Proposition 4.5.23. *Given a regular functor $F : \mathcal{C} \to \mathcal{D}$, taking images determines a functor*

$$F_* : \mathsf{Mod}(\mathbb{T}, \mathcal{C}) \longrightarrow \mathsf{Mod}(\mathbb{T}, \mathcal{D}) .$$

Proof. It only remains show the effect of F_* on morphisms of models. But these are just homomorphisms of the underlying cartesian structure, so they are clearly preserved by the cartesian functor F. $\square$

An associated result, which we will need, is the following.

Proposition 4.5.24. *Given regular categories $\mathcal{C}$ and $\mathcal{D}$ and a model M in $\mathcal{C}$, evaluation at M determines a functor*

$$\mathsf{eval}_M : \mathsf{Reg}(\mathcal{C}, \mathcal{D}) \longrightarrow \mathsf{Mod}(\mathbb{T}, \mathcal{D}) ,$$

which is natural in $\mathcal{D}$.

The proof is straightforward and can be left as an exercise. The naturality means that for any a regular functor $G : \mathcal{D} \longrightarrow \mathcal{D}'$, the following commutes (up to natural isomorphism, as usual):

$$\begin{tikzcd}
\mathsf{Reg}(\mathcal{C}, \mathcal{D}) \arrow[r, "\mathsf{eval}_M"] \arrow[d, "\mathsf{Reg}(\mathcal{C}, G)"'] & \mathsf{Mod}(\mathbb{T}, \mathcal{D}) \arrow[d, "G_*"] \\
\mathsf{Reg}(\mathcal{C}, \mathcal{D}') \arrow[r, "\mathsf{eval}_M"'] & \mathsf{Mod}(\mathbb{T}, \mathcal{D}')
\end{tikzcd}$$

Exercise 4.5.25. Prove this.

Exercise 4.5.26. Show that for any small category $\mathbb{C}$ and regular theory $\mathbb{T}$, there is an equivalence between models in the functor category and functors into the category of models,

$$\mathsf{Mod}(\mathbb{T}, \mathsf{Set}^{\mathbb{C}}) \ \simeq \ \mathsf{Mod}(\mathbb{T})^{\mathbb{C}} .$$

Hint: this is just as for the algebraic and cartesian cases.

4.5.4 Classifying category of a regular theory

We will next show that the framework of *functorial semantics* applies to regular logic and regular categories: there is a *classifying category* $\mathcal{C}_{\mathbb{T}}$ for $\mathbb{T}$-models, for which there is an equivalence, natural in $\mathcal{C}$,

$$\mathsf{Mod}(\mathbb{T}, \mathcal{C}) \ \simeq \ \mathsf{Reg}(\mathcal{C}_{\mathbb{T}}, \mathcal{C}) ,$$

where $\mathsf{Reg}(-, -)$ is the category of regular functors and natural transformations.

Remark 4.5.27. The construction of $\mathcal{C}_{\mathbb{T}}$, and the corollary completeness theorem, are analogous to a perhaps familiar way of proving the completeness theorem for classical propositional logic: one first constructs the *Lindenbaum-Tarski algebra* of propositional logic with respect to a propositional theory $\mathbb{T}$ (a set of formulas) as the set $\mathsf{PL} = \{\varphi \mid \varphi \text{ a propositional formula}\}$, quotiented by $\mathbb{T}$-provable logical equivalence, $\varphi \sim_{\mathbb{T}} \psi$ iff $\mathbb{T} \vdash \varphi \leftrightarrow \psi$,

$$\mathcal{L}_{\mathbb{T}} = \mathsf{PL}/\sim_{\mathbb{T}} \ .$$

The quotient set $\mathcal{L}_{\mathbb{T}}$ becomes a Boolean algebra by defining the Boolean operations in terms of the expected propositional logical analogues,

$$[\varphi] \wedge [\psi] = [\varphi \wedge \psi] , \quad \neg[\varphi] = [\neg\varphi] , \quad [\top] = 1 , \quad \text{etc.}.$$

One then has a Boolean-valuation of PL in $\mathcal{L}_{\mathbb{T}}$, namely $[-]$, for which

$$[\varphi] = [\psi] \quad \text{iff} \quad \mathbb{T} \vdash \varphi \leftrightarrow \psi .$$

In particular, then, $[\varphi] = 1$ iff $\mathbb{T} \vdash \varphi$. Classical completeness with respect to valuations in the Boolean algebra $\mathbf{2} = \{1, 0\}$ then follows e.g. from Stone's representation theorem, which embeds the Boolean algebra $\mathcal{L}_{\mathbb{T}}$ into a powerset $\mathcal{P}(X) \cong \mathbf{2}^X$, where X is the set of prime ideals in $\mathcal{L}_{\mathbb{T}}$, corresponding to Boolean valuations $\mathcal{L}_{\mathbb{T}} \to \mathbf{2}$ ("rows of a truth table").

Our syntactic construction of the classifying category $\mathcal{C}_{\mathbb{T}}$ can be regarded as a generalization of this method, with $\mathcal{C}_{\mathbb{T}}$ as the "Lindenbaum-Tarski category" of the (regular) theory $\mathbb{T}$. This will give a completeness theorem with respect to models in regular categories, which can in turn be specialized to Set-valued completeness by embedding $\mathcal{C}_{\mathbb{T}}$ into a "power of Set", i.e. Set^X for a set X of classical Set-valued models, i.e. regular functors $\mathcal{C}_{\mathbb{T}} \to \mathsf{Set}$. See Section **??** below for the second step.

We first sketch the construction of the classifying category $\mathcal{C}_{\mathbb{T}}$ of an arbitrary regular theory $\mathbb{T}$ (a more detailed account can be found in [**?**, **?**]). An object of $\mathcal{C}_{\mathbb{T}}$ is represented by a formula in context,

$$[\Gamma \mid \varphi],$$

where $\Gamma \mid \varphi$ `pred`. Two such objects $[\Gamma \mid \varphi]$ and $[\Gamma \mid \psi]$ are equal if $\mathbb{T}$ proves both

$$\Gamma \mid \varphi \vdash \psi\ , \qquad\qquad \Gamma \mid \psi \vdash \varphi\ .$$

Objects which differ only in the names of free variables are also considered equal:

$$[x : A \mid \varphi] = [y : A \mid \varphi[y/x]] \qquad (\text{no } y \text{ in } \varphi)$$

A morphism

$$[x : A \mid \varphi] \xrightarrow{\ \rho\ } [y : B \mid \psi]$$

is represented by a formula $x : A, y : B \mid \rho$ such that $\mathbb{T}$ proves that ρ is a *functional relation* from φ to ψ:

$$\begin{aligned} x : A \mid \varphi &\vdash \exists y : B \,.\, \rho && \text{(total)} \\ x : A, y : B, z : B \mid \rho \wedge \rho[z/y] &\vdash y = z && \text{(single-valued)} \\ x : A, y : B \mid \rho &\vdash \varphi \wedge \psi && \text{(well-typed)} \end{aligned}$$

Two functional relations ρ and σ represent the same morphism if $\mathbb{T}$ proves both

$$x : A, y : B \mid \rho \vdash \sigma\ , \qquad\qquad x : A, y : B \mid \sigma \vdash \rho\ .$$

Relations which only differ in the names of free variables are also considered equal.

(Strictly speaking, a morphism

$$[x : A, y : B \mid \rho] : [x : A \mid \varphi] \to [y : B \mid \psi]$$

should be taken to be the triple

$$\big([x : A, y : B \mid \rho],\ [x : A \mid \varphi],\ [y : B \mid \psi]\big)$$

so that one knows what the domain and codomain are, but we shall often write simply

$$\rho : [x : A \mid \varphi] \to [y : B \mid \psi]$$

since the rest can be recovered from that much data.)

The identity morphism on $[x : A \mid \varphi]$ is

$$1_{[x:A|\varphi]} = [x : A, x' : A \mid (x = x') \wedge \varphi] : [x : A \mid \varphi] \to [x' : A \mid \varphi[x'/x]] \; .$$

Note that we used the variable substitution $\varphi[x'/x]$ and the identification $[x : A \mid \varphi] = [x' : A \mid \varphi[x'/x]]$ in order to make this definition.

Composition of morphisms

$$[x : A \mid \varphi] \xrightarrow{\;\rho\;} [y : B \mid \psi] \xrightarrow{\;\tau\;} [z : C \mid \theta]$$

is given by the relational product,

$$\tau \circ \rho = (\exists\, y : B \,.\, (\rho \wedge \tau)) \; .$$

Of course, one needs to check that this *is* a morphism from φ to ϑ, i.e. that it is total, single-valued, and well-typed. We leave the detailed proof that $\mathcal{C}_{\mathbb{T}}$ is a category as an exercise; let us just show how to prove that composition of morphisms is associative. Given morphisms

$$[x : A \mid \varphi] \xrightarrow{\;\rho\;} [y : B \mid \psi] \xrightarrow{\;\tau\;} [z : C \mid \theta] \xrightarrow{\;\sigma\;} [u : D \mid \zeta]$$

we need to derive in context $x : A, u : D$

$$\exists\, z : C \,.\, ((\exists\, y : B \,.\, (\rho \wedge \tau)) \wedge \sigma) \;\dashv\vdash\; \exists\, y : B \,.\, (\rho \wedge (\exists\, z : C \,.\, (\tau \wedge \sigma)))$$

This follows easily with repeated application of the Frobenius rule (Section 4.5.3).

Exercise 4.5.28. Extend the definition of $\mathcal{C}_{\mathbb{T}}$ to morphisms between objects with arbitrary contexts,

$$[\Gamma \mid \varphi] \xrightarrow{\;\rho\;} [\Delta \mid \psi]$$

(use relations $\Gamma, \Delta \mid \rho$), and provide a proof that $\mathcal{C}_{\mathbb{T}}$ is a category.

Proposition 4.5.29. *The category $\mathcal{C}_{\mathbb{T}}$ is regular.*

Proof. We sketch the constructions required for regularity.

- The terminal object is $[\cdot \mid \top]$.
- The product of $[x : A \mid \varphi]$ and $[y : B \mid \psi]$, where x and y are distinct variables, is the object
$$[x : A, y : B \mid \varphi \wedge \psi] \; .$$
The first projection from the product is
$$x : A, y : B, x' : A \mid x = x' \wedge \varphi \wedge \psi \, ,$$
and the second projection is
$$x : A, y : B, y' : B \mid y = y' \wedge \varphi \wedge \psi \, ,$$
where we rename the codomains of the projections $[x : A \mid \varphi] = [x' : A \mid \varphi[x'/x]]$, etc., to make the context variables distinct.

- An equalizer of morphisms

$$[x : A \mid \varphi] \underset{\tau}{\overset{\rho}{\rightrightarrows}} [y : B \mid \psi]$$

is

$$[x : A \mid \exists\, y : B\,.\,(\rho \wedge \tau)] \xrightarrow{\ \varepsilon\ } [x' : A \mid \varphi[x'/x]]$$

where ε is the morphism

$$x : A, x' : A \mid (x = x') \wedge \exists\, y : B\,.\,(\rho \wedge \tau)\ .$$

- Finally, let us consider coequalizers of kernel pairs. The kernel pair of a map

$$\rho : [x : A \mid \varphi] \longrightarrow [y : B \mid \psi]$$

is

$$K \underset{\kappa_2}{\overset{\kappa_1}{\rightrightarrows}} [x : A \mid \varphi]$$

where K is the object

$$[u : A, v : A \mid \exists\, y : B\,.\,(\rho[u/x] \wedge \rho[v/x])]\ ,$$

the morphism κ_1 is

$$u : A, v : A, x : A \mid (u = x) \wedge \exists\, y : B\,.\,(\rho[u/x] \wedge \rho[v/x])\ ,$$

and κ_2 is

$$u : A, v : A, x : A \mid (v = x) \wedge \exists\, y : B\,.\,(\rho[u/x] \wedge \rho[v/x])\ .$$

Now the coequalizer of κ_1 and κ_2 can be shown to be the morphism

$$[x : A \mid \varphi] \xrightarrow{\ \rho\ } [y : B \mid \exists x : A.\,\rho]\ ,$$

where $[y : B \mid \exists\, x : A\,.\,\rho]$ is the image of ρ, as a subobject of $[y : B \mid \psi]$.

The following lemma shows that regular epis are stable under pullback. □

Lemma 4.5.30. *1. A map $\rho : [x : A \mid \varphi] \longrightarrow [y : B \mid \psi]$ is a regular epi if and only if*

$$y : B \mid \psi \vdash \exists x : A.\,\rho$$

2. Regular epis are stable under pullback in $\mathcal{C}_{\mathbb{T}}$.

Proof. For (1), suppose $\rho : [x : A \mid \varphi] \to [y : B \mid \psi]$ is a regular epi. We claim first that if ρ factors through some subobject $U \rightarrowtail [y : B \mid \psi]$ then $U = [y : B \mid \psi]$ is the maximal suboject. Indeed, since ρ is regular epi it is a coequalizer of its kernel pair. But if ρ factors through a subobject $U \rightarrowtail [y : B \mid \psi]$, say by $r : [x : A \mid \varphi] \to U$, then r is also a coequalizer of the kernel pair of ρ, as one can easily check. Thus $U \rightarrowtail [y : B \mid \psi]$ must be iso.

Now, up to iso, every $U \rightarrowtail [y : B \mid \psi]$ is of the form $U = [y : B \mid \vartheta]$ with $y \mid \vartheta \vdash \psi$, and ρ factors through $[y : B \mid \vartheta]$ iff

$$y : B \mid \exists x : A \,.\rho \vdash \vartheta \,.$$

Thus for all ϑ we have that:

$$(y : B \mid \exists x : A \,.\rho \vdash \vartheta) \;\Rightarrow\; (y : B \mid \psi \vdash \vartheta)\,.$$

Whence $y : B \mid \psi \vdash \exists x : A \,.\rho$. The convere is immediate from the specification of the kernel quotient above.

For (2), suppose we have a pullback diagram, which has the form indicated below.

$$\begin{tikzcd}
{[x : A, y : B \mid \varphi \wedge \psi \wedge \exists z : C.\,(\sigma \wedge \rho)]} \arrow[r, "\rho^*\sigma"] \arrow[d, "\sigma^*\rho"'] & {[y : B \mid \psi]} \arrow[d, "\rho"] \\
{[x : A \mid \varphi]} \arrow[r, "\sigma"'] & {[z : C \mid \vartheta]}
\end{tikzcd}$$

The maps $\sigma^*\rho$ and $\rho^*\sigma$ are represented by the relations:

$$\begin{aligned}
\sigma^*\rho &= \big(x : A,\, y : B,\, x' : A \mid x = x' \wedge \varphi \wedge \psi \wedge \exists z : C.\,(\sigma \wedge \rho)\big) \\
\rho^*\sigma &= \big(x : A,\, y : B,\, y' : B \mid y = y' \wedge \varphi \wedge \psi \wedge \exists z : C.\,(\sigma \wedge \rho)\big)
\end{aligned}$$

If ρ is regular epi, then by (1) we have

$$z : C \mid \vartheta \vdash \exists y : B.\, \rho\,. \tag{4.9}$$

To show that the pullback $\sigma^*\rho$ is regular epi, again by (1) we need to show

$$x' : A \mid \varphi[x'/x] \vdash \exists x : A\, \exists y : B.\, \big(x = x' \wedge \varphi \wedge \psi \wedge \exists z : C.\,(\sigma \wedge \rho)\big)\,. \tag{4.10}$$

We can make use thereby of the functionality of σ and ρ, specifically we have

$$x : A,\, z : C \mid \sigma \vdash \varphi \wedge \vartheta \qquad \text{and} \qquad x : A \mid \varphi \vdash \exists z : C.\, \sigma\,. \tag{4.11}$$

The result now follows by a simple deduction. □

Exercise 4.5.31. Show that in $\mathcal{C}_\mathbb{T}$ the regular-epi mono factorization of a morphism $\rho : [x : A \mid \varphi] \to [y : B \mid \psi]$ is given by

$$[x : A \mid \varphi] \xrightarrow{\ \rho\ } [y : B \mid \exists\, x : A \,.\, \rho] \xrightarrow{\ \iota\ } [z : B \mid \psi[z/y]]$$

where ι is the morphism

$$y : B, z : B \mid (y = z) \wedge (\exists\, x : A \,.\, \rho)\ .$$

Theorem 4.5.32 (Functorial semantics for regular logic)**.** *For any regular theory $\mathbb{T}$, the syntactic category $\mathcal{C}_\mathbb{T}$ classifies $\mathbb{T}$-models in regular categories. Specifically, for any regular category $\mathcal{C}$, there is an equivalence of categories*

$$\mathsf{Reg}(\mathcal{C}_\mathbb{T}, \mathcal{C}) \simeq \mathsf{Mod}(\mathbb{T}, \mathcal{C}) \tag{4.12}$$

which is natural in $\mathcal{C}$. In particular, there is a universal model U in $\mathcal{C}_\mathbb{T}$.

Proof. We have just constructed $\mathcal{C}_\mathbb{T}$ and shown that it is regular.

The universal model U, corresponding to the identity functor $\mathcal{C}_\mathbb{T} \to \mathcal{C}_\mathbb{T}$ under (4.12), is determined as follows:

- Each sort A is interpreted by the object $[x : A \mid \top]$
- A basic constant f with signature $(A_1, \ldots, A_n; B)$ is interpreted by the formula

  $$x_1 : A_1, \ldots, x_n : A_n, y : B \mid f(x_1, \ldots, x_n) = y\ .$$

  which is plainly a functional relation and thus a morphism $[\![A_1]\!] \times \cdots \times [\![A_n]\!] \longrightarrow [\![B]\!]$.
- A relation symbol R with signature $(A_1, \ldots, A_n)$ is interpreted by the subobject represented by the morphism

  $$\rho : [x_1 : A_1, \ldots, x_n : A_n \mid R(x_1, \ldots, x_n)] \longrightarrow [y_1 : A_1, \ldots, y_n : A_n \mid \top]$$

  where ρ is the formula

  $$x_1 : A_1, \ldots, x_n : A_n, y_1 : A_1, \ldots, y_n : A_n \mid R(x_1, \ldots, x_n) \wedge x_1 = y_1 \wedge \cdots \wedge x_n = y_n\ ,$$

  which is easily shown to be monic.

It is now straightforward to show that with respect to this structure, a formula $\Gamma \mid \varphi$ is interpreted as (the subobject determined by) the map

$$\iota : [\Gamma \mid \varphi] \longrightarrow [\Gamma \mid \top]$$

where ι is the formula

$$\Gamma, \Gamma' \mid \Gamma = \Gamma' \wedge \varphi\ ,$$

(with obvious abbreviations) which, again, is easily shown to be monic. Moreover, for any formulas $\Gamma \mid \varphi$ and $\Gamma \mid \psi$ we then have

$$U \models \Gamma \mid \varphi \vdash \psi \qquad \Longleftrightarrow \qquad \mathbb{T} \text{ proves } \Gamma \mid \varphi \vdash \psi\,.$$

Thus in particular U is indeed a $\mathbb{T}$-model.

We next construct a functor $\mathsf{Reg}(\mathcal{C}_{\mathbb{T}}, \mathcal{C}) \to \mathsf{Mod}(\mathbb{T}, \mathcal{C})$. Suppose $\mathcal{C}$ is regular and $F : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{C}$ a regular functor, then by Proposition 4.5.24, applying F to U determines a model FU in $\mathcal{C}$ with

$$[\![A]\!]_{FU} = F([\![A]\!]_U)\,,$$

and similarly for the other parts of the structure f, R, etc. Satisfaction of an entailment $\Gamma \mid \varphi \vdash \psi$ is preserved, because the interpretation of the logical operations is determined by the regular structure: pullbacks, images, etc., so that $[\![\varphi]\!]_U \leq [\![\psi]\!]_U$ in $\mathsf{Sub}([\![\Gamma]\!])$ implies

$$[\![\varphi]\!]_{FU} = F([\![\varphi]\!]_U) \leq F([\![\psi]\!]_U) = [\![\psi]\!]_{FU}$$

in $\mathsf{Sub}([\![\Gamma]\!]_{FU})$.

Moreover, just as for algebraic structures, every natural transformation between regular functors $\vartheta : F \Rightarrow G$ determines a homomorphism of the evaluated models by taking components $\vartheta_U : FU \to GU$. In this way, as in Proposition 4.5.24, evaluation at U is a functor

$$\mathsf{eval}_U : \mathsf{Reg}(\mathcal{C}_{\mathbb{T}}, \mathcal{C}) \longrightarrow \mathsf{Mod}(\mathbb{T}, \mathcal{C})\,.$$

We claim that this functor, which is the one mentioned in (4.12), is full and faithful and essentially surjective. The naturality in $\mathcal{C}$ of the equivalence then follows directly from its determination by evaluation at U and Proposition 4.5.24.

To see that eval_U is essentially surjective, let M be a model in $\mathcal{C}$. We will define a regular functor

$$M^\sharp : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{C}$$

with $M^\sharp(U) \cong M$. Since M is a model, there are objects $[\![A]\!]_M$ interpreting each type A, as well as interpretations

$$[\![\Gamma \mid \varphi]\!] \rightarrowtail [\![\Gamma]\!]$$

for all formulas and

$$[\![\Gamma \mid t : B]\!] : [\![\Gamma]\!] \longrightarrow [\![B]\!]$$

for all terms. Using these, we determine the functor $M^\sharp : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}$ by taking an object $[\Gamma \mid \varphi]$ to $[\![\Gamma \mid \varphi]\!]_M$, i.e. the domain of a mono representing the subobject $[\![\Gamma \mid \varphi]\!]_M \rightarrowtail [\![\Gamma]\!]_M$. Thus, for the record,

$$M^\sharp[\Gamma \mid \varphi] \;=\; [\![\Gamma \mid \varphi]\!]_M\,.$$

In the verification that those formulas in context $[\Gamma \mid \varphi]$ that are identified in $\mathcal{C}_{\mathbb{T}}$ are also identified in $\mathcal{C}$, we use the fact that the rules of inference for regular logic are sound in the regular category $\mathcal{C}$. Note in particular that for each basic type A, we then have

$$M^\sharp([\![A]\!]_U) = M^\sharp([x : A \mid \top]) \cong [\![x : A \mid \top]\!]_M \cong [\![A]\!]_M\,,$$

so that $M^\sharp(U) \cong M$ as required.

Functional relations in $\mathcal{C}_\mathbb{T}$ determine functional relations in $\mathcal{C}$, again by soundness, which determines the action of $M^\sharp$ on arrows, as well as the functoriality of these assignments.

Finally, to show that eval_U is full and faithful, let $F, G : \mathcal{C}_\mathbb{T} \longrightarrow \mathcal{C}$ be regular functors classifying models FU and GU, and let $h : FU \to GU$ be a model homomorphism. We then have maps

$$h_{[x:A|\top]} : F([x : A \mid \top]) \longrightarrow G([x : A \mid \top])$$

for all basic types A, and these commute with the interpretations of the function symbols f, and preserve the basic relations R, in the obvious sense, because h is a homomorphism. It only remains to determine the components

$$h_{[\Gamma|\varphi]} : F([\Gamma \mid \varphi]) \to G([\Gamma \mid \varphi])\,, \tag{4.13}$$

and to show that they commute with all maps $\rho : [\Gamma \mid \varphi] \to [\Delta \mid \psi]$. Define

$$h_{[\Gamma|\varphi]} : F[\Gamma \mid \varphi] = [\![\Gamma \mid \varphi]\!]_{FU} \longrightarrow [\![\Gamma \mid \varphi]\!]_{GU} = G[\Gamma \mid \varphi]$$

by induction on the structure of φ. The base cases involving the primitive relations $R, \ldots$ and equality of terms are given by the assumption that $h : FU \to GU$ is a model homomorphism, so we just need to check that for every definable subobject

$$[\![\Gamma \mid \varphi]\!]_{FU} \rightarrowtail [\![\Gamma \mid \top]\!]_{FU}$$

the following diagram can be filled in as indicated.

$$\begin{tikzcd}
{[\![\Gamma \mid \varphi]\!]_{FU}} \arrow[r, tail] \arrow[d, dotted, "h_{[\Gamma|\varphi]}"'] & {[\![\Gamma \mid \top]\!]_{FU}} \arrow[d, "h_{[\Gamma|\top]}"] \\
{[\![\Gamma \mid \varphi]\!]_{GU}} \arrow[r, tail] & {[\![\Gamma \mid \top]\!]_{GU}}
\end{tikzcd} \tag{4.14}$$

Suppose we have e.g. $\varphi = \exists x : A.\, \psi$, and we have already determined

$$h_{[\Gamma,x:A|\psi]} : [\![\Gamma, x : A \mid \psi]\!]_{FU} \longrightarrow [\![\Gamma, x : A \mid \psi]\!]_{GU}\,.$$

An easy diagram chase shows that there is a unique $h_{[\Gamma|\exists x:A.\,\psi]}$ determined by the image factorizations indicated below.

$$\begin{tikzcd}
{[\![\Gamma, x : A \mid \psi]\!]_{FU}} \arrow[r, two heads] \arrow[d, "h_{[\Gamma,x:A|\psi]}"'] & {[\![\Gamma \mid \varphi]\!]_{FU}} \arrow[r, tail] \arrow[d, dotted, "h_{[\Gamma|\exists x:A.\,\psi]}"'] & {[\![\Gamma \mid \top]\!]_{FU}} \arrow[d, "h_{[\Gamma|\top]}"] \\
{[\![\Gamma, x : A \mid \psi]\!]_{GU}} \arrow[r, two heads] & {[\![\Gamma \mid \varphi]\!]_{GU}} \arrow[r, tail] & {[\![\Gamma \mid \top]\!]_{GU}}
\end{tikzcd}$$

The other cases are even more direct. Thus we have defined the components (4.13); we leave the required naturality with respect to all maps $\rho : [\Gamma \mid \varphi] \to [\Delta \mid \psi]$ as an exercise. $\square$

Exercise 4.5.33. Prove the naturality of the maps (4.13), using the following trick. In any category with finite products, suppose we have objects and arrows

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "g"'] & D \end{tikzcd} \tag{4.15}$$

Let $\hat{f} = \langle 1_A, f\rangle : A \rightarrowtail A \times B$ be the graph of f, and similarly for $\hat{g} : C \rightarrowtail C \times D$. Then the diagram (4.15) commutes iff the following one does.

$$\begin{tikzcd} A \arrow[r, tail, "\hat{f}"] \arrow[d, "\alpha"'] & A \times B \arrow[d, "\alpha \times \beta"] \\ C \arrow[r, tail, "\hat{g}"'] & C \times D \end{tikzcd}$$

Corollary 4.5.34. *The rules of regular logic are sound and complete with respect to semantics in regular categories: a regular theory $\mathbb{T}$ proves an entailment*

$$\Gamma \mid \varphi \vdash \psi \tag{4.16}$$

if, and only if, every model of $\mathbb{T}$ satisfies it.

Proof. As for algebraic logic, soundness follows from classification (although we have of course already proved it separately in Proposition 4.6.7, and made use of it in the proof of the theorem!): if (4.16) is provable from $\mathbb{T}$, then it holds in the universal model U in $\mathcal{C}_{\mathbb{T}}$ by the construction of U,

$$U \models \Gamma \mid \varphi \vdash \psi\,.$$

But since regular functors preserve the interpretations of regular formulas $[\![\Gamma \mid \varphi]\!], [\![\Gamma \mid \psi]\!]$ (as well as entailments between them), the entailment (4.16) then holds also in any model M in any regular $\mathcal{C}$, since there is a classifying functor $M^\sharp : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}$ taking U to M, for which

$$M^\sharp([\![\Gamma \mid \varphi]\!]_U) \cong [\![\Gamma \mid \varphi]\!]_M\,.$$

Completeness follows from the syntactic construction of the universal model U in $\mathcal{C}_{\mathbb{T}}$. The model U is logically generic, in the sense that

$$U \models \Gamma \mid \varphi \vdash \psi \quad \Longleftrightarrow \quad \mathbb{T} \text{ proves } \Gamma \mid \varphi \vdash \psi\,.$$

Thus if $\Gamma \mid \varphi \vdash \psi$ holds in all models, then it holds in particular in U, and is therefore provable. $\square$

4.5.5 Coherent logic

A regular category is coherent if all the subobject posets are distributive lattices, and that structure is stable under pullback. We add rules to regular logic to describe this further structure, show that the rules are sound in coherent categories, and extend the results on functorial semantics of the previous section to the coherent case, including the completeness theorem.

Definition 4.5.35. A cartesian category $\mathcal{C}$ is *coherent* if:

1. $\mathcal{C}$ is regular, i.e. it has coequalizers of kernel pairs, and regular epimorphisms are stable under pullback,

2. each subobject poset $\mathsf{Sub}(A)$ has all finite joins, in particular 0 and $U \vee V$,

3. for each map $f : A \to B$, the pullback functor $f^* : \mathsf{Sub}(B) \longrightarrow \mathsf{Sub}(A)$ preserves the joins:
$$f^* 0_B = 0_A, \qquad f^*(U \vee V) = f^*U \vee f^*V\,.$$

Note that since joins are stable under pullback in a coherent category, the meets distribute over the joins,
$$U \wedge (V \vee W) = (U \wedge V) \vee (U \wedge W)\,, \tag{4.17}$$
so that the posets $\mathsf{Sub}(A)$ are distributive lattices. Indeed, this follows from the fact that $U \wedge V$ may be written as
$$U \wedge V = \Sigma_U \circ U^*(V) \tag{4.18}$$
where $\Sigma_U : \mathsf{Sub}(U) \to \mathsf{Sub}(A)$ is the left adjoint (composition) of the pullback functor $U^* : \mathsf{Sub}(A) \to \mathsf{Sub}(U)$ along the inclusion $U \rightarrowtail A$. Since left adjoints preserve colimits, and thus joins, we therefore have
$$U \wedge (V \vee W) = \Sigma_U \circ U^*(V \vee W) = \Sigma_U \circ U^*(V) \vee \Sigma_U \circ U^*(W) = (U \wedge V) \vee (U \wedge W)\,.$$

A category is said to have have *stable sums* if it has all finite coproducts, in particular an initial object 0 and binary coproducts $A+B$, and these are stable under pullback, in the expected sense. The following simple observation provides plenty of examples of coherent categories.

Proposition 4.5.36. *Regular categories with stable sums are coherent.*

Proof. Given subobjects $U, V \rightarrowtail A$, let $U \vee V$ be the image of the canonical map $U+V \to A$ as indicated below.

$$\begin{tikzcd}
& U+V \arrow[d, two heads] & \\
U \arrow[ur] \arrow[dr, tail] & U \vee V \arrow[d, tail] & V \arrow[ul] \arrow[dl, tail] \\
& A &
\end{tikzcd}$$

This is easily seem to be the supremum of U and V in $\mathsf{Sub}(A)$. Since the unique map $0 \to A$ is always monic, it determines the subobject $0 \rightarrowtail A$. Thus $\mathsf{Sub}(A)$ has all finite joins, and they are stable by stability of the coproducts and image factorizations. $\square$

As examples of coherent categories we thus have Set and $\mathsf{Set}_{\mathrm{fin}}$, as well as all functor categories $\mathsf{Set}^{\mathbb{C}}$ since limits and colimits (and thus image factorizations) there are computed pointwise.

Exercise 4.5.37. Is the category of H-presets for a heyting algebra H from Section 4.5.1 coherent?

Coherent logic is the extension of regular logic by adding rules corresponding to joins.

Definition 4.5.38. A *coherent theory* $\mathbb{T}$ is (a type theory together with) a set of axioms expressed in the fragment of logic built from $=$, $\top$, $\bot$, $\wedge$, $\vee$, and $\exists$.

We thus extend the formation rules for formulas in context by two additional clauses:

7. The 0-ary relation symbol $\bot$ (pronounced "false") is a formula :

$$\frac{\cdot}{\Gamma \mid \bot \ \mathtt{pred}}$$

8. Disjunction:

$$\frac{\Gamma \mid \varphi \ \mathtt{pred} \qquad \Gamma \mid \psi \ \mathtt{pred}}{\Gamma \mid \varphi \vee \psi \ \mathtt{pred}}$$

(We also again add the evident additional clauses for substitution of terms into formulas.)

A *coherent theory* then consists of axioms of the form

$$\Gamma \mid \varphi \vdash \psi$$

where φ, ψ are *coherent formulas*. Coherent logic not only allows for disjunctions $\varphi \vee \psi$ on both side of the $\vdash$, but the presence of the symbol $\bot$ allows for a certain amount of negation, in the form $\varphi \vdash \bot$, as the following classical example illustrates.

Example 4.5.39.

1. A ring A (with unit 1) is called *local* if it has a unique maximal ideal. This can be captured with two coherent axioms of the form $0 = 1 \vdash \bot$ (to ensure that $0 \neq 1$), and

$$x : A, y : A \mid \exists z : A.\, z(x + y) = 1 \vdash (\exists z : A.\, zx = 1) \vee (\exists z : A.\, zy = 1)$$

2. Another example is the theory of *fields*, which can be axiomatized by again adding to the theory of rings the law $0 = 1 \vdash \bot$, together with the following:

$$x : A \mid \top \vdash x = 0 \vee (\exists y : A.\, xy = 1)$$

which is a clever way of saying that every non-zero element has a multiplicative inverse.

3. An order example is the notion of a *linear order*, which adds to the cartesian theory of posets the *totality* axiom:

$$x : P, y : P \mid x \leq y \vee y \leq q\,.$$

4. For another example of how we can make use of the constant *false* $\bot$ to get the effect of negation, at least for entire axioms, even though the coherent fragment does not include negation, consider the theory of graphs, with two basic sorts E for edges and V for verticies, and two operations $s, t : (E; V)$ for source and target. A graph $G = (E_G, V_G, s_G, t_G)$ is *acyclic* if it satisfies all the finitely many axioms

$$\exists e_1 \ldots e_n : E.\, \big(t(e_1) = s(e_2) \wedge \ldots \wedge t(e_n) = s(e_1)\big) \vdash \bot\,.$$

The *rules of inference of coherent logic* are those of regular logic (Section 4.5.3), with additional rules for falshood the disjunctions:

10. Falsehood:

$$\frac{}{\bot \vdash \psi}$$

11. Disjunction:

$$\frac{\varphi \vdash \vartheta \qquad \psi \vdash \vartheta}{\varphi \vee \psi \vdash \vartheta} \qquad \frac{\varphi \vee \psi \vdash \vartheta}{\varphi \vdash \vartheta} \qquad \frac{\varphi \vee \psi \vdash \vartheta}{\psi \vdash \vartheta}$$

12. Distributivity:

$$\varphi \wedge (\psi \vee \vartheta) \vdash\ (\varphi \wedge \psi) \vee (\varphi \wedge \vartheta)$$

The latter of course coresponds to the distributive law (4.17); note that the converse can be derived. Like the Frobenius rule, this will be derivable in the extended system of Heyting logic (see Proposition 4.6.14), and could also be made derivable in a suitably formulated system of coherent logic using multi-sequents $\Gamma \mid \varphi_1, \ldots, \varphi_n \vdash \psi$.

The *semantics for coherent logic* extends that for regular logic in the expected way: the disjunctive formulas are interpreted as the corresponding joins in the subobject lattices,

$$[\![\Gamma \mid \bot]\!] = 0\,, \qquad\qquad [\![\Gamma \mid \varphi \vee \psi]\!] = [\![\Gamma \mid \varphi]\!] \vee [\![\Gamma \mid \psi]\!]\,.$$

The additional clauses in the proof of soundness are routine. We can then extend the syntactic construction of the regular classifying category $\mathcal{C}_{\mathbb{T}}$ to include all coherent formulas and prove the following extended functorial semantics theorem for models in coherent categories and *coherent functors*, which are defined to be regular functors that preserve all finite joins of subobjects.

Theorem 4.5.40 (Functorial semantics for coherent logic)**.** *For any coherent theory $\mathbb{T}$, the syntactic category $\mathcal{C}_{\mathbb{T}}$ classifies $\mathbb{T}$-models in coherent categories. Specifically, for any coherent category $\mathcal{C}$, there is an equivalence of categories, natural in $\mathcal{C}$,*

$$\mathsf{Coh}(\mathcal{C}_{\mathbb{T}}, \mathcal{C}) \simeq \mathsf{Mod}(\mathbb{T}, \mathcal{C})\,, \tag{4.19}$$

where $\mathsf{Coh}(\mathcal{C}_{\mathbb{T}}, \mathcal{C})$ is the category of coherent functors and natural transformations. In particular, there is a universal model U in $\mathcal{C}_{\mathbb{T}}$.

The corresponding completeness theorem 4.5.34 then holds as well. We leave the routine details to the reader.

Exercise 4.5.41. Extend the functorial semantics theorem 4.5.32 from regular to coherent logic. Specifically, one must determine the components (4.13) of a natural transformation for the extended language of coherent logic.

4.6 Heyting categories

In this section we consider coherent categories that also model the universal quantifier $\forall$, in the sense of Section 4.4; such categories will be seen to model full first-order logic. One could also consider *cartesian* categories modeling $\forall$, without being coherent, and thus modeling the fragment of logic consisting of $u = v, \top, \wedge, \forall$, but we will not do so separately.

Definition 4.6.1. A *Heyting category* is a coherent category with universal quantifiers in the sense of Section 4.4. Thus for every map $f : A \to B$, the pullback functor $f^* : \mathsf{Sub}(B) \to \mathsf{Sub}(A)$ has a right adjoint,

$$\forall_f : \mathsf{Sub}(A) \to \mathsf{Sub}(B)\,,$$

in addition to the left adjoint $\exists_f : \mathsf{Sub}(A) \to \mathsf{Sub}(B)$ given by taking images.

Note that in a Heyting category, one therefore has both adjoints to pullback along any map $f : A \to B$,

$$\begin{tikzcd} \mathsf{Sub}(A) \arrow[r, shift left=2, "\exists_f"] \arrow[r, shift right=2, "\forall_f"'] & \mathsf{Sub}(B) \arrow[l, "f^*" description] \end{tikzcd} \qquad \exists_f \dashv f^* \dashv \forall_f\,. \tag{4.20}$$

Moreover, the Beck-Chevalley conditions from Section 4.4.1 are satisfied for both $\exists_f$ (by Proposition 4.5.15) and $\forall_f$ (by Proposition 4.4.5).

One way to get a Heyting structure on a category $\mathcal{C}$ is when the operations on the subobject lattices $\mathsf{Sub}(A)$ are inherited from related ones on the slice categories $\mathcal{C}/A$; this happens when $\mathcal{C}$ is *locally cartesian closed*. Recall that a cartesian closed category is a category that has products and exponentials. A category of locally cartesian closed when every slice is cartesian closed.

Definition 4.6.2. A category $\mathcal{C}$ is *locally cartesian closed (lccc)* when it has a terminal object and every slice $\mathcal{C}/A$ is cartesian closed.

Note that every slice category $\mathcal{C}/A$ has a terminal object, namely the identity morphism $1_A : A \to A$, and all $\mathcal{C}/A$ have binary products if, and only if, $\mathcal{C}$ has pullbacks. Thus a locally cartesian closed category has all finite limits because it has a terminal object and pullbacks. In addition, a locally cartesian closed category is cartesian closed because $\mathcal{C} \cong \mathcal{C}/1$. We describe how exponentials in a slice $\mathcal{C}/A$ can be computed in terms of *change* *of base functors* and *dependent products*. Given a morphism $f : A \to B$ in $\mathcal{C}$, the "change of base along f" is the pullback functor

$$f^* : \mathcal{C}/B \to \mathcal{C}/A \; .$$

A right adjoint to f^*, when it exists, is called a *dependent product along* f, denoted

$$\Pi_f : \mathcal{C}/A \to \mathcal{C}/B \; .$$

Now an exponential of $b : B \to A$ and $c : C \to A$ in $\mathcal{C}/A$ can be computed in terms of Π_b and b^*. For any $d : D \to A$, we have $b \times_A d = (b^* d) \circ b = \Sigma_b(b^* d)$, hence

$$\begin{array}{c} b \times_A d \to c \\ \hline\hline \Sigma_b(b^* d) \to c \\ \hline\hline b^* d \to b^* c \\ \hline\hline d \to \Pi_b(b^* c) \end{array}$$

Therefore, $c^b = \Pi_b(b^* c)$.

We have proved that if a cartesian category $\mathcal{C}$ has dependent product $\Pi_f : \mathcal{C}/A \to \mathcal{C}/B$ along every morphism $f : A \to B$ then it is locally cartesian closed. The converse holds as well, that is every lccc has dependent products. For a proof see Section **??** or [**?**, 9.20].

Proposition 4.6.3. *A category $\mathcal{C}$ with a terminal object is locally cartesian closed if, and only if, for any $f : A \to B$ the change of base functor $f^* : \mathcal{C}/B \to \mathcal{C}/A$ has a right adjoint $\Pi_f : \mathcal{C}/A \to \mathcal{C}/B$.*

Proposition 4.6.4. *In an lccc $\mathcal{C}$, for any $f : A \to B$ the change of base functor $f^* : \mathcal{C}/B \to \mathcal{C}/A$ preserves the ccc structure.*

Proof. We need to show that f^* preserves terminal objects, binary products, and exponentials in slices. Because f^* is a right adjoint it preserves limits, hence it preserves terminal objects and binary products. To see that it preserves exponentials we first show that $f^* \circ \Pi_g \cong \Pi_{f^* g} \circ (g^* f)^*$ for $g : C \to B$. Given any $d : D \to C$, and $e : E \to A$:

$$\begin{array}{c} e \to f^*(\Pi_g d) \\ \hline\hline \Sigma_f e \to \Pi_g d \\ \hline\hline g^*(\Sigma_f e) \to d \\ \hline\hline g^*(f \circ e) \to d \\ \hline\hline (g^* f) \circ ((f^* g)^* e) \to d \\ \hline\hline (f^* g)^* e \to (g^* f)^* d \\ \hline\hline e \to \Pi_{f^* g}((g^* f)^* d) \end{array}$$

By the Yoneda Lemma it follows that $f^*(\Pi_g d) \cong \Pi_{f^*g}((g^*f)^*d)$. Now we have, for any $c : C \to B$ and $d : D \to B$,

$$f^*c^d = f^*(\Pi_d(d^*c)) = \Pi_{f^*d}((d^*f)^*(d^*c)) = \Pi_{f^*d}((f^*d)^*(f^*c)) = (f^*c)^{(f^*d)} .$$

□

Exercise 4.6.5. In the preceding proof we used the fact that $(d^*f)^*(d^*c) \cong (f^*d)^*(f^*c)$ and $g^*(f \circ e) \cong (g^*f) \circ ((f^*g)^*e)$. Prove that this is really so.

Locally cartesian closed categories are an important example of categories with universal quantifiers.

Proposition 4.6.6. *A locally cartesian closed category has universal quantifiers.*

Proof. Suppose $\mathcal{C}$ is locally cartesian closed. First observe that a morphism $m : M \to A$ is mono if, and only if, the morphism

$$\begin{tikzcd} M \arrow[rr, "m"] \arrow[dr, "m"'] & & A \arrow[dl, "1_A"] \\ & A & \end{tikzcd}$$

is mono in $\mathcal{C}/A$. Because right adjoints preserve monos, $\Pi_f : \mathcal{C}/A \to \mathcal{C}/B$ preserve monos for any $f : A \to B$, that is, if $m : M \rightarrowtail A$ is mono then $\Pi_f m : \Pi_f M \to B$ is mono in $\mathcal{C}$. Therefore, we may define $\forall_f$ as the restriction of Π_f to $\mathsf{Sub}(A)$. To be more precise, a subobject $[m : M \rightarrowtail A]$ is mapped by $\forall_f$ to the subobject $[\Pi_f m : \Pi_f M \rightarrowtail B]$. This works because for any monos $m : M \rightarrowtail A$ and $n : N \rightarrowtail B$ we have

$$\begin{array}{rr} f^*[m : M \rightarrowtail A] \leq [n : N \rightarrowtail B] & \text{in } \mathsf{Sub}(B) \\ \hline\hline f^*m \to n & \text{in } \mathcal{C}/B \\ \hline\hline m \to \Pi_f n & \text{in } \mathcal{C}/A \\ \hline\hline [m] \leq \forall_f[n] & \text{in } \mathsf{Sub}(A) \end{array}$$

The Beck-Chevalley condition for $\forall_f$ follows from Proposition 4.6.4. Indeed, if $g : C \to B$ and $m : M \rightarrowtail C$ then

$$f^*(\Pi_g m) \cong \Pi_{f^*g}((g^*f)^*m) ,$$

therefore

$$f^*(\forall_g[m : M \rightarrowtail C]) = \forall_{f^*g}((g^*f)^*[m : M \rightarrowtail C]) ,$$

as required. □

Summarizing, diagram (4.21), which may be called *Lawvere's hyperdoctrine diagram*, displays the relation between the quantifiers and the change of base functors.

$$
\begin{tikzcd}
\mathcal{C}/A \arrow[rr, shift left=8, "\Sigma_f"] \arrow[rr, shift right=8, "\Pi_f"'] \arrow[ddd, shift right, "\mathsf{im}"'] & & \mathcal{C}/B \arrow[ll, "f^*" description] \arrow[ddd, shift right, "\mathsf{im}"'] \\
\\
\\
\mathsf{Sub}(A) \arrow[rr, shift left=8, "\exists_f"] \arrow[rr, shift right=8, "\forall_f"'] \arrow[uuu, hook, shift right, "I"'] & & \mathsf{Sub}(B) \arrow[ll, "f^*" description] \arrow[uuu, hook, shift right, "I"']
\end{tikzcd}
\tag{4.21}
$$

In Section 4.6.3 below we shall see that all presheaf categories $\mathsf{Set}^{\mathbb{C}^{\mathsf{op}}}$ are Heyting, and therefore have universal quantifiers, which we will compute explicitly (they are *not* pointwise!).

4.6.1 Heyting logic

We can now extend the *formation rules* for the logical language to include universally quantified formulas in the expected way:

$$
\frac{\Gamma, x : A \mid \varphi \ \texttt{pred}}{\Gamma \mid \forall x : A.\, \varphi \ \texttt{pred}}
$$

The corresponding additional *rule of inference* for the universal quantifier is:

$$
\frac{y : B, x : A \mid \vartheta \vdash \varphi}{y : B \mid \vartheta \vdash \forall x : A.\, \varphi}
$$

Note that the lower judgement is well-formed only if $x : A$ does not occur freely in ϑ.

Finally, we extend the *interpretation* from coherent formulas from (Section 4.5.5) to formulas including universal quantifiers by the additional clause for $\forall x : A.\, \varphi$ using the universal quantifiers in the Heyting category,

$$
[\![\Gamma \mid \forall x : A.\, \varphi]\!] = \forall_A [\![\Gamma, x : A \bigm| \varphi]\!] \ ,
$$

where

$$
\forall_A = \forall_\pi : \mathsf{Sub}([\![\Gamma]\!] \times [\![A]\!]) \to \mathsf{Sub}([\![\Gamma]\!])
$$

is the universal quantifier along the projection $\pi : [\![\Gamma]\!] \times [\![A]\!] \to [\![\Gamma]\!]$.

The following is then immediate from the results of section 4.4.

Proposition 4.6.7. *The rules for the universal quantifier are sound with respect to the interpretation in Heyting categories.*

Implication

Recall that the rules of inference for implication state that $\Rightarrow$ is right adjoint to $\wedge$:

$$\frac{\Gamma \mid \vartheta \ \texttt{pred} \qquad \Gamma \mid \varphi \ \texttt{pred}}{\Gamma \mid (\vartheta \Rightarrow \varphi) \ \texttt{pred}} \qquad\qquad \frac{\Gamma \mid \psi \wedge \vartheta \vdash \varphi}{\Gamma \mid \psi \vdash \vartheta \Rightarrow \varphi}$$

Exercise 4.6.8. Show that the above two-way rule can be replaced by the following introduction and elimination rules:

$$\frac{\Gamma \mid \psi \wedge \vartheta \vdash \varphi}{\Gamma \mid \psi \vdash \vartheta \Rightarrow \varphi} \qquad\qquad \frac{\Gamma \mid \psi \vdash \vartheta \Rightarrow \varphi \qquad \Gamma \mid \psi \vdash \vartheta}{\Gamma \mid \psi \vdash \varphi}$$

We expect that in order to interpret implication in a cartesian category $\mathcal{C}$ we require $\mathsf{Sub}(A)$ to be a Heyting algebra for every $A \in \mathcal{C}$. However, we must not forget that implication interacts with substitution by the rule

$$(\vartheta \Rightarrow \varphi)[t/x] \ = \ \vartheta[t/x] \Rightarrow \varphi[t/x] \ .$$

Semantically this means that implication is *stable* under pullbacks.

Definition 4.6.9. A cartesian category $\mathcal{C}$ has *implications* when, for every $A \in \mathcal{C}$, the poset $\mathsf{Sub}(A)$ is a Heyting algebra with stable implication $\Rightarrow$. This means that for $U, V \in \mathsf{Sub}(A)$ and $f : B \to A$,

$$f^*(U \Rightarrow V) = (f^*U \Rightarrow f^*V) \ .$$

Proposition 4.6.10. *If a cartesian category has universal quantifiers then it has implications.*

Proof. Let $[u : U \rightarrowtail A]$ and $[v : V \rightarrowtail A]$ be subobjects of A. Define

$$([u] \Rightarrow [v]) = \forall_u(u^*[v]) \ ,$$

as indicated below

$$\begin{tikzcd}
u^*([v]) \arrow[r, tail] \arrow[d, tail] & V \arrow[d, tail, "v"] & \forall_u u^*([v]) \arrow[dl, tail] \\
U \arrow[r, tail, "u"'] & A &
\end{tikzcd}$$

Then for any subobject $[w : W \rightarrowtail A]$ we have:

$$\begin{array}{ll}
[w] \leq [u] \Rightarrow [v] & \text{in } \mathsf{Sub}(A) \\
\hline\hline
[w] \leq \forall_u(u^*[v]) & \text{in } \mathsf{Sub}(A) \\
\hline\hline
u^*[w] \leq u^*[v] & \text{in } \mathsf{Sub}(U) \\
\hline\hline
\exists_u(u^*w) \leq v & \text{in } \mathsf{Sub}(A) \\
\hline\hline
[u] \wedge [w] \leq [v] & \text{in } \mathsf{Sub}(A)
\end{array}$$

Note that we used the decomposition of $[u] \wedge [w]$ as $\exists_u(u^*w)$ from (4.18).

Finally, stability of $\Rightarrow$ follows from Beck-Chevalley condition for $\forall$. $\square$

Exercise 4.6.11. Prove the last claim of the proof.

Corollary 4.6.12. *Any LCCC has universal quantifiers and implications.*

Negation

Now that we have Heyting implication $U \Rightarrow V$ making each $\mathsf{Sub}(A)$ a Heyting algebra, we can also define *negation* $\neg U$ as usual in a Heyting algebra, namely:

$$\neg U \;=\; (U \Rightarrow 0)\,, \tag{4.22}$$

where 0 is the bottom element $[0 \rightarrowtail A]$ of $\mathsf{Sub}(A)$. These negations are stable under pullback because the Heyting implications and the bottom element 0 are stable.

We can therefore add *formulas* with negation to the logical language, along with the evident two-way *rule of inference*:

$$\frac{\Gamma \mid \varphi\ \mathtt{pred}}{\Gamma \mid \neg\,\varphi\ \mathtt{pred}} \qquad\qquad \frac{\Gamma \mid \vartheta \vdash \neg\,\varphi}{\Gamma \mid \vartheta \wedge \varphi \vdash \bot}$$

We give negated formulas the obvious *interpretation*: given $[\![\varphi]\!]$ in $\mathsf{Sub}(A)$, we set

$$[\![\neg\varphi]\!] \;=\; \neg[\![\varphi]\!] \;=\; [\![\varphi]\!] \Rightarrow 0\,.$$

using the Heyting implication $\Rightarrow$ and bottom element 0 in $\mathsf{Sub}(A)$. The following is then immediate.

Proposition 4.6.13. *The rules for negation are sound in any Heyting category.*

Given Heyting implication, we can prove the distributivity rule from Section 4.5.5 for conjunction and disjunction.

Proposition 4.6.14. *The distributivity rule is provable in Heyting logic:*

$$\varphi \wedge (\psi \vee \vartheta) \;\vdash\; (\varphi \wedge \psi) \vee (\varphi \wedge \vartheta)$$

Proof.

$$\frac{\dfrac{\dfrac{\dfrac{(\varphi \wedge \psi) \vee (\varphi \wedge \vartheta) \vdash \zeta}{(\varphi \wedge \psi) \vdash \zeta \qquad (\varphi \wedge \vartheta) \vdash \zeta}}{\psi \vdash \varphi \Rightarrow \zeta \qquad \vartheta \vdash \varphi \Rightarrow \zeta}}{\psi \vee \vartheta \vdash \varphi \Rightarrow \zeta}}{\varphi \wedge (\psi \vee \vartheta) \vdash \zeta}$$

Thus, in fact,

$$\varphi \wedge (\psi \vee \vartheta) \;\dashv\vdash\; (\varphi \wedge \psi) \vee (\varphi \wedge \vartheta).$$

□

Perhaps more surprisingly, given universal quantifiers, we can actually prove the Frobenius rule from Section 4.5.3 for existential quantifiers.

Proposition 4.6.15. *The Frobenius rule is provable in Heyting logic:*

$$(\exists y : B.\, \varphi) \wedge \psi \;\vdash\; \exists y : B.\, (\varphi \wedge \psi)$$

provided the variable $y : B$ does not occur freely in ψ.

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\exists y : B.\, (\varphi \wedge \psi) \vdash \zeta}{y : B \mid \varphi \wedge \psi \vdash \zeta}}{y : B \mid \varphi \vdash \psi \Rightarrow \zeta}}{(\exists y : B.\, \varphi) \vdash \psi \Rightarrow \zeta}}{(\exists y : B.\, \varphi) \wedge \psi \;\vdash \zeta}$$

Thus, in fact,

$$(\exists y : B.\varphi) \wedge \psi \;\dashv\vdash\; \exists y : B.(\varphi \wedge \psi).$$

□

Exercise 4.6.16. In classical logic, one has the *de Morgen laws* for negation,

$$\neg(\varphi \wedge \psi) \dashv\vdash \neg\varphi \vee \neg\psi$$
$$\neg(\varphi \vee \psi) \dashv\vdash \neg\varphi \wedge \neg\psi$$

Which of these four entailments can you prove in Heyting logic?

Adjoint rules of Heyting logic

Figure 4.2 collects the rules of inference for Heyting logic, also known as *intuitionistic first-order logic*. These are stated as two-way rules to emphasize the respective underlying adjunctions. The rules for disjunction and conjunction in the bottom-up direction are, of course, to be understood a two separate rules, left and right. The contexts are omitted where there is no change between the top and bottom, thus e.g. the rule for existential quantifier can be stated in full as:

$$\dfrac{\Gamma, x : A \mid \varphi \vdash \vartheta}{\Gamma \mid \exists\, x : A\,.\, \varphi \vdash \vartheta}$$

Negation $\neg\,\varphi$ is treated as a defined by

$$\neg\,\varphi \;:=\; \varphi \Rightarrow \bot\,.$$

$$\frac{}{\bot \vdash \varphi} \qquad\qquad \frac{}{\varphi \vdash \top}$$

$$\frac{\varphi \vdash \vartheta \qquad \psi \vdash \vartheta}{\varphi \vee \psi \vdash \vartheta} \qquad\qquad \frac{\vartheta \vdash \varphi \qquad \vartheta \vdash \psi}{\vartheta \vdash \varphi \wedge \psi}$$

$$\frac{\vartheta \wedge \varphi \vdash \psi}{\vartheta \vdash \varphi \Rightarrow \psi}$$

$$\frac{x : A \mid \varphi \vdash \vartheta}{\exists\, x : A\,.\,\varphi \vdash \vartheta} \qquad\qquad \frac{x : A \mid \vartheta \vdash \varphi}{\vartheta \vdash \forall\, x : A\,.\,\varphi}$$

Figure 4.2: Adjoint rules of inference for Heyting logic

It therefore satisfies the derived rule:

$$\frac{\vartheta \wedge \varphi \vdash \bot}{\vartheta \vdash \neg\, \varphi}$$

The rules for *equality*, recall from Section 4.3, were:

$$\frac{}{\psi \vdash t =_A t} \qquad\qquad \frac{\psi \vdash t =_A u \qquad \psi \vdash \varphi[t/z]}{\psi \vdash \varphi[u/z]} \tag{4.23}$$

Lawvere [?] observed that equality can also be seen as an adjoint, namely to the operation of pullback along the diagonal $\Delta : A \to A \times A$ in any cartesian category. Indeed, we have an adjunction

$$\begin{tikzcd} \mathsf{Sub}(A) \arrow[d, shift right, "\exists_\Delta"'] \\ \mathsf{Sub}(A \times A) \arrow[u, shift right, "\Delta^*"'] \end{tikzcd} \qquad\qquad \frac{x : A \mid \vartheta(x) \vdash \varphi(x, x)}{x : A, y : A \mid (x = y) \wedge \vartheta(x) \vdash \varphi(x, y)} \tag{4.24}$$

where we have displayed the variables in the style $\varphi(x, y)$ in order to emphasize the effect of Δ^* as a "contraction of variables",

$$\Delta^*(\varphi(x, y)) = \varphi(x, x)\,.$$

The effect of the left adjoint $\exists_\Delta$ (which is simply composition with Δ, because it is monic) is given by

$$\exists_\Delta(\vartheta(x)) \;=\; \big(x = y \wedge \vartheta(x)\big)\,.$$

The adjoint rule (4.24) may be called *Lawvere's Law.* It is equivalent to the standard rules (4.23).

Exercise 4.6.17. Prove the equivalence of (4.23) and (4.24).

We state the following for the record as a summary of the foregoing discussion.

Proposition 4.6.18 (Soundness)**.** *The adjoint rules of inference for Heyting logic, as stated in Figure 4.2 and including Lawvere's Law* (4.24)*, are sound in any Heyting category.*

We will show in Section **??** that these rules are also complete.

4.6.2 Intuitionistic first-order logic

Heyting logic with equality is often called *intuitionistic first-order logic*. It lacks the classical laws of excluded middle $\varphi \vee \neg\varphi$ and double negation elimination $\neg\neg\varphi \Rightarrow \varphi$, but adding either one of these implies the other (proof!), and gives a system equivalent to standard first-order logic – with one exception: one still cannot prove the classical law

$$\forall x : A.\, \varphi \vdash \exists x : A\,.\varphi\,. \tag{4.25}$$

The latter law, which is satisfied only in non-empty domains, is considered by many to be a defect of the conventional formulation of first-order logic. It would follow if we were to forget about the contexts, essentially permitting inferences of the form

$$\frac{x : A \mid \varphi \vdash \psi}{\cdot \mid \varphi \vdash \psi} \tag{4.26}$$

when $x : A$ does not occur freely in φ or ψ.

Exercise 4.6.19. Assume the rule (4.26) and prove the entailment (4.25).

Any conventional first-order theory can be formulated in Heyting logic, often in more than one way, since classical logic may collapse differences between concepts that are intuitionistically distinct (like, most simply, φ and $\neg\neg\varphi$). Our interest in intuitionistic logic does not arise from any philosophical scruples about the validity of the classical laws of excluded middle or double negation, but rather the fact that the logic of variable structures is naturally intuitionistic, as we will see in Section **??**.

Example 4.6.20. An example of a first-order theory that is not (immediately) coherent is the theory of dense linear orders. In addition to the poset axioms, and the totality axiom $x, y : P \mid \top \vdash (x \leq y \vee y \leq x)$, one adds density e.g. in the form

$$x, y : P \mid (x \leq y \wedge x \neq y) \vdash (\exists z : P.\, x \leq z \wedge x \neq z \wedge z \leq y \wedge z \neq y)\,.$$

Classifying category for a Heyting theory

Given a theory in first-order intuitionistic logic $\mathbb{T}$, we can build the syntactic category $\mathcal{C}_{\mathbb{T}}$ from the formulas over $\mathbb{T}$, as was done for coherent logic in Section 4.5.4. The objects again have the form $[\Gamma \mid \varphi]$, but now using the Heyting formulas φ, including the logical operations $\forall$, and $\Rightarrow$. The result will then be a coherent category with universal quantifiers, and thus a Heyting category in the sense of Definition 4.6.1. Given another Heyting category $\mathcal{C}$ with a $\mathbb{T}$-model $M \in \mathsf{Mod}(\mathbb{T}, \mathcal{C})$, the interpretation $[\![-]\!]_M$ associated to the model M determines a Heyting functor,

$$M^\sharp : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{C} \tag{4.27}$$

$$[\Gamma \mid \varphi] \longmapsto [\![\Gamma \mid \varphi]\!]_M \tag{4.28}$$

We would like to show that $\mathcal{C}_{\mathbb{T}}$ classifies $\mathbb{T}$-models, in the sense that this assignment determines an equivalence of categories, associating homomorphisms of $\mathbb{T}$-models $h : M \to N$ in the category $\mathsf{Mod}(\mathbb{T}, \mathcal{C})$, and natural transformations of the associated classifying Heyting functors $M^\sharp \to N^\sharp$ in $\mathcal{C}_{\mathbb{T}} \to \mathcal{C}$.

However, there is a problem. Reviewing the proof of Theorem 4.5.32, we needed to show that definable subobjects are natural in model homomorphisms, in the following sense: let $F, G : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{C}$ be functors classifying models FU and GU, and let $h : FU \to GU$ be a model homomorphism. We have maps $h_A : F(A) \longrightarrow G(A)$ for all basic types $A = [x : A \mid \top]$, commuting with the interpretations of the function symbols f and the basic relations R. For each object $[x : A \mid \varphi]$, say, the components

$$h_{[x:A|\varphi]} : F[x : A \mid \varphi] = [\![x : A \mid \varphi]\!]_{FU} \longrightarrow [\![x : A \mid \varphi]\!]_{GU} = G[x : A \mid \varphi]$$

were then defined on definable subobject $[\![x : A \mid \varphi]\!]_{FU} \rightarrowtail [\![A]\!]_{FU} = FA$, in such a way that the following diagram commutes as indicated.

$$\begin{tikzcd}
{[\![x : A \mid \varphi]\!]_{FU}} \arrow[r, tail] \arrow[d, dotted, "h_{[x:A|\varphi]}"'] & {[\![A]\!]_{FU}} \arrow[d, "h_A"] \\
{[\![x : A \mid \varphi]\!]_{GU}} \arrow[r, tail] & {[\![A]\!]_{GU}}
\end{tikzcd} \tag{4.29}$$

This we could do for all *coherent* formulas φ, as was shown by induction on the structure of φ. However, this is no longer possible when φ is Heyting. Most simply, if $\varphi = \neg\psi$ for coherent ψ, there is no need for the following to commute on the left.

$$\begin{tikzcd}
{[\![x : A \mid \neg\psi]\!]_{FU}} \arrow[r, tail] \arrow[d, dotted, "h_{[x:A|\neg\psi]}"'] & {[\![A]\!]_{FU}} \arrow[d, "h_A"] & {[\![x : A \mid \psi]\!]_{FU}} \arrow[l, tail] \arrow[d, "h_{[x:A|\psi]}"] \\
{[\![x : A \mid \neg\psi]\!]_{GU}} \arrow[r, tail] & {[\![A]\!]_{GU}} & {[\![x : A \mid \psi]\!]_{GU}} \arrow[l, tail]
\end{tikzcd} \tag{4.30}$$

Very concretely, let $\mathbb{T}$ be the theory of groups, FU and GU groups in Set and $h_A : [\![A]\!]_{FU} \to [\![A]\!]_{GU}$ the trivial homomorphism that takes everything $a \in [\![A]\!]_{FU}$ to the unit $e_{GU} \in [\![A]\!]_{GU}$, and ψ the formula $x : A \mid x = e$. Then $[\![x : A \mid \psi]\!]_{GU} = \{e_{GU}\}$ and so $[\![x : A \mid \neg\psi]\!]_{GU} = \{y \in [\![A]\!]_{GU} \mid y \neq e_{GU}\}$, so there is a factorization $h_{[x:A|\neg\psi]} : [\![x : A \mid \neg\psi]\!]_{FU} \to [\![x : A \mid \neg\psi]\!]_{GU}$ only if FU is trivial.

The same holds, of course, for subobjects defined by the other Heyting operations, such as $[x : A \mid \vartheta \Rightarrow \psi]$ and $[x : A \mid \forall y : B.\psi]$; there need not be any factorizations $h_{[x:A|\varphi]}$ as indicted in (4.29).

Our solution (although not the only possible one) is to consider only *isomophisms of models* $h : M \cong N$ and *natural isomorphisms* between the classifying functors.

Lemma 4.6.21. *In the situation of diagram* (4.29)*, if the model homomorphism* $h : FU \to GU$ *is an isomorphism, then for any Heyting formula* $[\Gamma \mid \varphi]$ *there is a unique factorization*

$$h_{[\Gamma|\varphi]} : F[\Gamma \mid \varphi] = [\![x : A \mid \varphi]\!]_{FU} \longrightarrow [\![x : A \mid \varphi]\!]_{GU} = G[x : A \mid \varphi]$$

making the corresponding diagram (4.29) *commute.*

Proof. Induction on φ. □

Now for every Heyting category $\mathcal{C}$, let us define $\mathsf{Mod}(\mathbb{T}, \mathcal{C})^i$ to be the category of $\mathbb{T}$-models in $\mathcal{C}$, and their isomorphisms; thus $\mathsf{Mod}(\mathbb{T}, \mathcal{C})^i$ is a groupoid. Accordingly we let $\mathsf{Heyt}(\mathcal{C}_{\mathbb{T}}, \mathcal{C})^i$ to be the category of all Heyting functors $\mathcal{C}_{\mathbb{T}} \to \mathcal{C}$ and natural *iso*morphisms between them – thus also a groupoid. Then just as in previous cases we can show:

Theorem 4.6.22 (Functorial semantics for intuitionistic first-order logic)**.** *For any theory* $\mathbb{T}$ *in (intuitionistic) first-order logic, the syntactic category* $\mathcal{C}_{\mathbb{T}}$ *classifies* $\mathbb{T}$*-models in Heyting categories. Specifically, for any Heyting category* $\mathcal{C}$*, there is an equivalence of categories, natural in* $\mathcal{C}$*,*

$$\mathsf{Heyt}(\mathcal{C}_{\mathbb{T}}, \mathcal{C})^i \simeq \mathsf{Mod}(\mathbb{T}, \mathcal{C})^i \,, \tag{4.31}$$

where $\mathsf{Heyt}(\mathcal{C}_{\mathbb{T}}, \mathcal{C})^i$ *is the groupoid of Heyting functors and natural isomorphisms, and* $\mathsf{Mod}(\mathbb{T}, \mathcal{C})^i$ *is the groupoid of* $\mathbb{T}$*-models in* $\mathcal{C}$*. In particular, there is a universal model* U *in* $\mathcal{C}_{\mathbb{T}}$*.*

The corresponding completeness theorem 4.5.34 for intuitionistic first-order logic with respect to models in Heyting categories then holds as well. We leave the routine details to the reader.

Boolean categories

A Boolean category may be defined as a coherent category in which every subobject $U \rightarrowtail A$ is *complemented*, in the sense that it there is some (necessarily unique) $V \rightarrowtail A$ such that $U \wedge V \leq 0$ and $1 \leq U \vee V$ in $\mathsf{Sub}(A)$. One can then introduce the Boolean negation $\neg U = V$, and show that each $\mathsf{Sub}(A)$ is a Boolean algebra. Indeed one can then show that every Boolean category is Heyting, using the familiar definitions $\forall\varphi = \neg\exists\neg\varphi$ and $\varphi \Rightarrow \psi = \neg\varphi \vee \psi$.

This definition, however, leads to the wrong notion of a "Boolean classifying category", for the reasons just discussed with respect to Heyting categories: although every coherent functor between Boolean categories is Boollean, the natural transformations between classifying functors will not be simply the homomorphisms. (They will be something interesting, namely elementary embeddings, but we shall not pursue this further here; see [**?**].) Thus it seems preferable for our purposed to define a Boolean category to be a Heyting category with complemented subobjects:

Definition 4.6.23. A Heyting catgeory $\mathcal{C}$ is *Boolean* if every subobject lattice $\mathsf{Sub}(A)$ is a Booean algebra. Thus for all subobjects $U \rightarrowtail A$, the Heyting complement $\neg U$ satisfies $U \vee \neg U = 1$ in $\mathsf{Sub}(A)$.

Of course, the category Set is Boolean. A presheaf category $\mathsf{Set}^{\mathbb{C}}$ is in general *not* Boolean, but an important special case always is, namely when $\mathbb{C}$ is a groupoid. (Set^G is called the *category of G-sets*.)

Exercise 4.6.24. Regard a group G as a category with one object. Show that in the functor category Set^G, every subobject lattice $\mathsf{Sub}(A)$ is a Boolean algebra.

The classifying category theorem 4.6.22 for Heyting categories, and indeed the entire framework of functorial semantics, applies *mutatis mutandis* to classical first-order logic and Boolean categories. We will not spell out the details, which do not differ in any unexpected way from the more general Heyting case.

Exercise 4.6.25. Assume that $\mathcal{C}$ is coherent and has complemented subobjects in the sense just defined. Prove that then each $\mathsf{Sub}(A)$ is a Boolean algebra, and that $\mathcal{C}$ is a Heyting category.

Exercise 4.6.26. Show that a Heyting category $\mathcal{C}$ is Boolean if, and only if, in each $\mathsf{Sub}(A)$ the Heyting complement $\neg U$ always satisfies $\neg\neg U = U$.

4.6.3 Examples of Heyting categories

Sets. The category Set is of course complete and cocomplete. It is cartesian closed, with function sets $B^A = \{f : A \to B\}$ as exponentials. It is also locally cartesian closed, because the slice category Set/I is equivalent to the category Set^I of I-indexed families of sets $(A_i)_{i\in I}$, for which the exponentials can be computed pointwise: for $A = (A_i)_{i\in I}$ and $B = (B_i)_{i\in I}$ we can set $B^A = ({B_i}^{A_i})_{i\in I}$. Since pullback is therefore a left adjoint, regular epis are stable and so Set is coherent. It is then Heyting by Proposition 4.6.6.

In order to compute the Heyting structure explicitly, consider any map $f : A \to B$ and the resulting adjunctions from (4.20),

$$\begin{tikzcd} \mathsf{Sub}(A) \arrow[r, bend left, "\exists_f"] \arrow[r, bend right, "\forall_f"'] & \mathsf{Sub}(B) \arrow[l, "f^*"'] \end{tikzcd} \qquad \exists_f \dashv f^* \dashv \forall_f\,.$$

For $U \in \mathsf{Sub}(A)$ and $V \in \mathsf{Sub}(B)$ we then have:

$$\begin{aligned} f^*(V) &= f^{-1}(V) = \{a \in A \mid f(a) \in V\} \\ \exists_f(U) &= \{b \in B \mid \text{for some } a \in f^{-1}\{b\}, a \in U\} \\ \forall_f(U) &= \{b \in B \mid \text{for all } a \in f^{-1}\{b\}, a \in U\} \end{aligned} \tag{4.32}$$

It follows that in Set the implications $U \Rightarrow V$ for $U, V \in \mathsf{Sub}(A)$ have the form

$$\begin{aligned} (U \Rightarrow V) &= \{a \in A \mid a \in U \text{ implies } a \in V\} \\ &= (A - U) \cup V\,. \end{aligned}$$

For negation, we then have

$$\begin{aligned} \neg U &= \{a \in A \mid a \notin U\} \\ &= (A - U)\,, \end{aligned}$$

as expected. Of course, Set is Boolean.

Exercise 4.6.27. In Set consider the dependent sum and product along the unique function $I \to 1$. Show that for $a : A \to I$ the set $\Pi_I A$ is the set of right inverses of a:

$$\Pi_I A = \left\{ s : I \to A \;\middle|\; a \circ s = 1_I \right\} \ .$$

If $(A_i)_{i \in I}$ is a family of sets indexed by I and we take

$$A = \coprod_{i \in I} A_i = \left\{ \langle i, x\rangle \in I \times \bigcup_{i \in I} A_i \;\middle|\; i \in I \;\&\; x \in A_i \right\}$$

with $a = \pi_0 : \langle i, x\rangle \mapsto i$ then $\Pi_{!_I} A$ is precisely the cartesian product $\Pi_{i \in I} A_i$. Calculate what Π_f is in Set for a general $f : J \to I$, and conclude that Set is locally cartesian closed.

Presheaves. For a small category $\mathbb{C}$, the presheaf category $\widehat{\mathbb{C}} = \mathsf{Set}^{\mathbb{C}^{\mathsf{op}}}$ has pointwise limits and colimits and is cartesian closed with the exponential of presheaves P, Q calculated using Yoneda as,

$$Q^P(C) \cong \mathsf{Hom}(\mathsf{y}C, Q^P) \cong \mathsf{Hom}(\mathsf{y}C \times P, Q)\,, \qquad \text{for } C \in \mathbb{C}.$$

But then $\widehat{\mathbb{C}}$ is also LCC, because for any presheaf P, the slice category $\widehat{\mathbb{C}}/P$ is equivalent to presheaves on the *category of elements* $\int_{\mathbb{C}} P$,

$$\widehat{\mathbb{C}}/P \;=\; (\mathsf{Set}^{\mathbb{C}^{\mathsf{op}}})/P \;\simeq\; \mathsf{Set}^{(\int_{\mathbb{C}} P)^{\mathsf{op}}}\,.$$

See [**?**, 9.23].

We first consider the poset $\mathsf{Sub}(P)$ for any presheaf P on $\mathbb{C}$. Let $U \rightarrowtail P$ be any subobject, then since monos in are pointwise in $\widehat{\mathbb{C}}$, and they are represented by subsets in Set, we can represent U by a family $UC \subseteq PC$ of subsets. If $f : P \to Q$ is a natural transformation, the inverse image of $V \rightarrowtail Q$ can then be calculated pointwise from $f_C : PC \to QC$ as

$$f^*(V)(C) = f_C^{-1}(VC) = \{x \in PC \mid f_C(x) \in VC\}\,.$$

The image $\exists_f(U)$, as a coequalizer, is also pointwise, therefore

$$\exists_f(U)(C) = \{y \in QC \mid \text{for some } x \in f_C^{-1}\{y\},\, x \in UC\}\,.$$

The direct image $\forall_f(U)$ is however *not pointwise*, so we must determine it directly. The problem with the obvious attempt

$$\forall_f(U)(C) \stackrel{?}{=} \{y \in QC \mid \text{for all } x \in f_C^{-1}\{y\},\, x \in UC\}\,.$$

is that it is not functorial in C! In order to correct this, have to modify it by taking instead

$$\forall_f(U)(C) = \{y \in QC \mid \text{for all } h : D \to C, \text{for all } x \in f_D^{-1}\{y.h\},\, x \in UD\}\,, \tag{4.33}$$

where we have written $y.h$ for the action of Q on $y \in QC$, i.e. $Q(h)(y) \in QD$.

Lemma 4.6.28. *The specification* (4.33) *is the universal quantifier* $\forall_f$ *in presheaves.*

Proof. Consider the diagram

$$\begin{tikzcd}
\mathsf{y}C \times_Q P \arrow[rr, dotted] \arrow[dr, "y'"] \arrow[dddd] & & U \arrow[dl, tail] \\
& P \arrow[d, "f"] & \\
& Q & \\
& & \\
\mathsf{y}C \arrow[uur, "y"] \arrow[rr, dotted] & & \forall_f U \arrow[uul, tail]
\end{tikzcd}$$

For all $y \in QC$, we have $y \in \forall_f U$ iff the pullback $y' = f^*y$ factors through $U \rightarrowtail P$, as indicated. Replacing the pullback $\mathsf{y}C \times_Q P$ by its generalized elements, the latter condition is equivalent to saying that for all $\mathsf{y}D$ and $\mathsf{y}h : \mathsf{y}D \to \mathsf{y}C$ and $x \in PD$, if $f \circ x = y \circ \mathsf{y}h$, then $x \in UD$, as shown below.

$$
\begin{tikzcd}
\mathsf{y}D \arrow[rrrd, dotted, bend left] \arrow[rrdd, bend left, "x"] \arrow[rdddd, bend right, "\mathsf{y}h"'] & & & \\
& \mathsf{y}C \times_Q P \arrow[rr, dotted] \arrow[rd, "y'"'] \arrow[ddd] & & U \arrow[ld, tail] \\
& & P \arrow[d, "f"] & \\
& & Q & \\
& \mathsf{y}C \arrow[ru, "y"'] \arrow[rr, dotted] & & \forall_f U \arrow[lu, tail]
\end{tikzcd}
$$

But the last condition is equivalent to saying for all D and all $h : D \to C$ and all $x \in PD$, if $x \in f_D^{-1}\{y.h\}$, then $x \in UD$, which is the righthand side of (4.33). □

Proposition 4.6.29. *For any natural transformation $f : P \to Q$, there are adjoints*

$$
\begin{tikzcd}
\mathsf{Sub}(P) \arrow[r, bend left, "\exists_f"] \arrow[r, bend right, "\forall_f"'] & \mathsf{Sub}(Q) \arrow[l, "f^*"']
\end{tikzcd}
\qquad \exists_f \dashv f^* \dashv \forall_f \,.
$$

These are determined by the following formulas, where $U \rightarrowtail P$ and $V \rightarrowtail Q$ and $C \in \mathbb{C}$:

$$
\begin{aligned}
f^*(V)(C) &= \{x \in PC \mid f_C(x) \in VC\} \\
\exists_f(U)(C) &= \{y \in QC \mid \text{for some } x \in PC,\ f_C(x) = y \ \&\ x \in UC\} \\
\forall_f(U)(C) &= \{y \in QC \mid \text{for all } h : D \to C, \text{ for all } x \in PD,\ f_D(x) = y.h \text{ implies } x \in UD\}
\end{aligned}
\tag{4.34}
$$

□

The implication $U \Rightarrow V$ for $U, V \in \mathsf{Sub}(P)$ therefore has the form, for each $C \in \mathbb{C}$,

$$(U \Rightarrow V)(C) \;=\; \{x \in PC \mid \text{for all } h : D \to C,\ x.h \in UD \text{ implies } x.h \in VD\}\,.$$

And the negation $\neg U \in \mathsf{Sub}(P)$ is then, for each $C \in \mathbb{C}$,

$$(\neg U)(C) \;=\; \{x \in PC \mid \text{for all } h : D \to C,\ x.h \notin UD\}\,.$$

Exercise 4.6.30. Prove the last two statements, computing $U \Rightarrow V$ and $\neg U$.

Sets through time. For presheaves on a poset K, the foregoing description of the Heyting structure becomes a bit simpler. Let us consider "covariant presheaves", i.e. functors $A : K \to \mathsf{Set}$. We can regard such a functor as a "set developing through (branching) time", with each later time $i \leq j$ giving rise to a transition map $A_i \to A_j$, which we may denote by

$$A_i \ni a \longmapsto a_j \in A_j \, .$$

For any map $f : A \to B$ (a family of functions $f_i : A_i \to B_i$ compatible with the development over time), we again have the adjunctions

$$\begin{tikzcd} \mathsf{Sub}(A) \arrow[r, bend left, "\exists_f"] \arrow[r, bend right, "\forall_f"'] & \mathsf{Sub}(B) \arrow[l, "f^*"'] \end{tikzcd} \qquad \exists_f \dashv f^* \dashv \forall_f \, .$$

These can now be described by the following formulas, where $U \in \mathsf{Sub}(A)$ and $V \in \mathsf{Sub}(B)$ and $i \in K$:

$$\begin{aligned} f^*(V)_i &= \{x \in A_i \mid f_i(x) \in V_i\} \\ \exists_f(U)_i &= \{y \in B_i \mid \text{for some } x \in A_i,\ f_i(x) = y \ \&\ x \in U_i\} \\ \forall_f(U)_i &= \{y \in B_i \mid \text{for all } j \geq i, \text{ for all } x \in A_j,\ f_j(x) = y_j \text{ implies } x \in U_j\} \end{aligned} \tag{4.35}$$

The implication $U \Rightarrow V$ for $U, V \in \mathsf{Sub}(A)$ then has the form, for each $i \in K$,

$$(U \Rightarrow V)_i \ = \ \{x \in A_i \mid \text{for all } j \geq i,\ x_j \in U_j \text{ implies } x_j \in V_j\} \, .$$

And the negation $\neg U \in \mathsf{Sub}(A)$ is then, for each $i \in K$,

$$(\neg U)_i \ = \ \{x \in A_i \mid \text{for all } j \geq i,\ x_j \notin U_j\} \, .$$

Exercise 4.6.31. Show that for the arrow category $\mathbf{2} = \cdot \to \cdot$ the functor category $\mathsf{Set}^{\to}$ is *not* Boolean.

Remark 4.6.32 (Bi-Heyting categories)**.** We know by Proposition 4.6.29 that in presheaf categories $\mathsf{Set}^{\mathbb{C}^{\mathsf{op}}}$, each subobject lattice $\mathsf{Sub}(P)$ is a Heyting algebra. Define a *bi-Heyting category* to be a Heyting category in which each $\mathsf{Sub}(P)$ is a *bi-Heyting algebra*, meaning that both $\mathsf{Sub}(P)$ and its opposite $\mathsf{Sub}(P)^{\mathsf{op}}$ are Heyting algebras. One can show that any presheaf category is also bi-Heyting (this follows from the fact that limits and colimits in presheaves are computed pointwise, but see also Exercise 4.6.33 below). See [**?**, **?**, **?**] for more on bi-Heyting categories.

Exercise 4.6.33. Complete the following sketch to show that any presheaf category $\mathsf{Set}^{\mathbb{C}^{\mathsf{op}}}$ is bi-Heyting.

1. Every presheaf P is covered by a coproduct of representables,

$$\coprod_{C \in \mathbb{C},\, x \in PC} \mathsf{y}C \twoheadrightarrow P \, .$$

2. There is therefore an injective lattice homomorphism
$$\mathsf{Sub}(P) \rightarrowtail \prod_{C \in \mathbb{C}, x \in PC} \mathsf{Sub}(\mathsf{y}C)\,.$$
3. It thus suffices to show that all $\mathsf{Sub}(\mathsf{y}C)$ are bi-Heyting.
4. The poset $\mathsf{Sub}(\mathsf{y}C)$ is isomorphic to the poset of *sieves* on C in $\mathbb{C}$: sets S of arrows with codomain C, closed under precomposition by arbitrary arrows, i.e. $(s : C' \to C) \in S$ and $t : C'' \to C'$ implies $s \circ t \in S$.
5. Writing $|\mathbb{D}|$ for the poset reflection of an arbitrary category $\mathbb{D}$, the sieves on C are the same as lower sets in the poset reflection of the slice category $|\mathbb{C}/C|$, thus $\mathsf{Sub}(\mathsf{y}C) \cong {\downarrow}|\mathbb{C}/C|$.
6. For *any* poset P, the poset of lower sets ${\downarrow}P$, ordered by inclusion, form a Heyting algebra.
7. The opposite category of ${\downarrow}P$ is isomorphic to the upper sets ${\uparrow}P$.
8. But since ${\uparrow}P = {\downarrow}(P^{\mathsf{op}})$, by (6) the poset $({\downarrow}P)^{\mathsf{op}}$ is also a Heyting algebra.
9. Thus $\mathsf{Sub}(\mathsf{y}C)$ is a bi-Heyting algebra.

4.7 Kripke-Joyal semantics

In section 4.2, we introduced the idea of using "generalized elements" $z : Z \to C$ as a way of externalizing the interpretation of the logical language. With respect to a subobject $S \rightarrowtail C$, such an element is said to be *in the subobject*, writtten $z \in_C S$, if it factors through $S \rightarrowtail C$.

$$\begin{tikzcd} & S \arrow[d, tail] \\ Z \arrow[ur, dotted] \arrow[r, "z"'] & C \end{tikzcd}$$

Generalized elements provide a way of *testing for satisfaction* of a formula $(x : A \mid \varphi)$ by a model M, as follows. Let A_M be the interpretion of the type A in the model M, so that the formula determines a subobject $[\![x : A \mid \varphi]\!]_M \rightarrowtail A_M$. Note that in Heyting logic, with $\forall$ and $\Rightarrow$, we can consider satisfaction of individual formulas $(x : A \mid \varphi)$ rather than entailments $(x : A \mid \varphi \vdash \psi)$, by turning the latter into $(\top \vdash \forall x : A.\, \varphi \Rightarrow \psi)$.

Definition 4.7.1. For a theory $\mathbb{T}$ in first-order logic we say that a model M *satisfies* a formula $(x : A \mid \varphi)$, written $M \models (x : A \mid \varphi)$, if the subobject $[\![x : A \mid \varphi]\!]_M \rightarrowtail A_M$ is the maximal one 1_{A_M}.

Note that this notion of satisfaction of a formula agrees with our previous notion of satisfaction for the entailment $x : A \mid \top \vdash \varphi$,

$$\begin{aligned} M \models (x : A \mid \varphi) \quad &\text{iff} \quad [\![x : A \mid \varphi]\!]_M = 1_{A_M} \\ &\text{iff} \quad M \models (x : A \mid \top \vdash \varphi)\,. \end{aligned} \tag{4.36}$$

Now observe that the condition $[\![x : A \mid \varphi]\!]_M = 1_{A_M}$ holds just in case every element $z : Z \to A_M$ factors through the subobject $[\![x : A \mid \varphi]\!]_M \rightarrowtail A_M$. It is convenient to use the *forcing* notation $\Vdash$ for this condition, writing

$$Z \Vdash \varphi(z) \quad \text{for} \quad z \in_{A_M} [\![x : A \mid \varphi]\!]_M\,.$$

We can then use forcing to test for satisfaction, by asking whether *all* generalized elements $z : Z \to A_M$ factor through $[\![x : A \mid \varphi]\!]_M \rightarrowtail A_M$, and thus "force" the formula $(x : A \mid \varphi)$:

$$M \models (x : A \mid \varphi) \qquad \text{iff} \qquad \text{for all } z : Z \to A_M,\ Z \Vdash \varphi(z)\,.$$

We summarize these conventions in the following Definition and Lemma.

Definition 4.7.2 (Kripke-Joyal Forcing)**.** In any Heyting category $\mathcal{C}$, define the *forcing relation* $\Vdash$ as follows: for a formula $(x : A \mid \varphi)$ in the langage of a theory $\mathbb{T}$, and a $\mathbb{T}$-model M, let A_M interpret the type symbol A; then for any $z : Z \to A_M$, we define the relation "z forces φ" by

$$\begin{aligned} Z \Vdash \varphi(z) \quad &\text{iff} \quad z \in_{A_M} [\![x : A \mid \varphi]\!]_M \\ &\text{iff} \quad z : Z \to A_M \text{ factors as} \quad \begin{tikzcd} & {[\![x : A \mid \varphi]\!]_M} \arrow[d, tail] \\ Z \arrow[ur, dotted] \arrow[r, "z"'] & A_M \end{tikzcd} . \end{aligned} \tag{4.37}$$

Lemma 4.7.3. *For any model M, we have:*

$$M \models (x : A \mid \varphi) \qquad \textit{iff} \qquad \textit{for all } z : Z \to A_M,\ Z \Vdash \varphi(z)\,. \tag{4.38}$$

□

Of course, we also define forcing for formulas with a context of variables $\Gamma = x_1 : A_1, \ldots x_n : A_n$, and then we have

$$M \models (\Gamma \mid \varphi) \qquad \text{iff} \qquad \text{for all } z : Z \to \Gamma_M,\ Z \Vdash \varphi(z)\,.$$

where $\Gamma_M = (A_1)_M \times \ldots \times (A_1)_M$, and $\varphi(z) = \varphi(z_1, \ldots, z_n)$ where $z_i = \pi_i z : Z \to \Gamma_M \to (A_i)_M$. In the extremal case, we have a formula $\cdot \mid \varphi$ with no free variables (a *closed* *formula* or *sentence*), for which the interpretation $[\![\cdot \mid \varphi]\!] \rightarrowtail 1$ is in $\mathsf{Sub}(1)$. For such a closed formula, we have

$$\begin{aligned} M \models (\cdot \mid \varphi) \qquad & \text{iff} \qquad \text{for all } z : Z \to 1,\ Z \Vdash \varphi \\ & \text{iff} \qquad [\![\cdot \mid \varphi]\!] = 1 \,. \end{aligned} \tag{4.39}$$

In this sense, the Heyting algebra $\mathsf{Sub}(1)$ contains the *truth-values* of statements $(\cdot \mid \varphi)$ in the internal logic, which hold if and only if $[\![\cdot \mid \varphi]\!] = 1$.

The forcing relation $Z \Vdash \varphi(z)$ defined in (4.37) allows us to turn an internal statement $[\![x : A \mid \varphi]\!]_M$, i.e. a formula interpreted as an object of $\mathcal{C}$, into an external one, i.e. an ordinary statement that makes reference to objects an arrows of $\mathcal{C}$. We first restrict attention to categories of presheaves $\widehat{\mathbb{C}} = \mathsf{Set}^{\mathbb{C}^{\mathsf{op}}}$, for the sake of simplicity (but see Remark 4.7.5 below.) In this case, we can restrict to generalized elements $z : Z \to A_M$ of the special form $c : \mathsf{y}C \to A_M$, i.e. with representable domains, because Lemma 4.7.3 clearly still holds when so restricted: $M \models (x : A \mid \varphi)$ iff for all $c : \mathsf{y}C \to A_M$, we have $\mathsf{y}C \Vdash \varphi(c)$. Moreover, we then write simply $C \Vdash \varphi(c)$ for $\mathsf{y}C \Vdash \varphi(c)$. Observe that because (by Yoneda) $c : \mathsf{y}C \to A_M$ corresponds to $c \in A_M(C)$ in Set, with subset $([\![x : A \mid \varphi]\!]_M)(C) \subseteq A_M(C)$, we have, finally, the equivalence

$$C \Vdash \varphi(c) \quad \text{iff} \quad c \in [\![x : A \mid \varphi]\!]_M(C) \,. \tag{4.40}$$

Theorem 4.7.4 (Kripke-Joyal Semantics)**.** *For any presheaf category $\widehat{\mathbb{C}}$ and model M of a theory $\mathbb{T}$ in first-order logic, let $(x : A \mid \varphi)$, $(x : A \mid \psi)$, and $(x : A, y : B \mid \vartheta)$ be formulas in the language of $\mathbb{T}$, and let $C \in \mathbb{C}$ and $c, c_1, c_2 : \mathsf{y}C \to A_M$ be any maps. Then we have*

1. $C \Vdash \top(c)$ *always.*
2. $C \Vdash \bot(c)$ *never.*
3. $C \Vdash c_1 = c_2$ *iff* $c_1 = c_2$ *as arrows* $\mathsf{y}C \to A_M$.
4. $C \Vdash \varphi(c) \wedge \psi(c)$ *iff* $C \Vdash \varphi(c)$ *and* $C \Vdash \psi(c)$.
5. $C \Vdash \varphi(c) \vee \psi(c)$ *iff* $C \Vdash \varphi(c)$ *or* $C \Vdash \psi(c)$.
6. $C \Vdash \varphi(c) \Rightarrow \psi(c)$ *iff for all* $d : D \to C$, $D \Vdash \varphi(c.d)$ *implies* $D \Vdash \psi(c.d)$.
7. $C \Vdash \neg\varphi(c)$ *iff for no* $d : D \to C$, $D \Vdash \varphi(c.d)$.
8. $C \Vdash \exists y : B.\, \vartheta(c, y)$ *iff for some* $c' : C \to B_M$, $C \Vdash \vartheta(c, c')$.
9. $C \Vdash \forall y : B.\, \vartheta(c, y)$ *iff for all* $d : D \to C$, *for all* $d' : D \to B_M$, $D \Vdash \vartheta(c.d, d')$.

Proof. We just do a few cases and leave the rest to the reader.

...

Use (4.34) for the non-obvious cases. □

Examples: LEM, DN, a map is epic, monic, iso. Constant domains.

Remark 4.7.5. There are several variations on Kripke-Joyal semantics for various special kinds of categories: presheaves on a poset P, sheaves on a topological space or a complete Heyting algebra, G-sets for a group or groupoid G, sheaves on a Grothendieck site (i.e. a Grothendieck topos), as well as a general case for arbitrary Heyting categories. Many of these are discussed in [**?**]. In the case of sheaves, the clauses for falsehood $\bot$, disjunction $\vee$, and the existential quantifier $\exists$ typically become more involved. The result is then akin to what is known in constructive logic as Beth semantics.

We next consider another case that is even simpler than presheaves, namely covariant Set-valued functors on a poset P, which may be called "Kripke models".

Exercise 4.7.6. Show that for a group G, regarded as a category with one object, the functor category Set^G is Boolean.

Exercise 4.7.7. Prove Lemma 4.7.3 in the restricted case of presheaves and generalized elements with representable domains, $a : \mathsf{y}C \to A_M$.

4.7.1 Kripke models

As already mentioned, we can regard covariant functors $A : K \to \mathsf{Set}$ on a *poset* K as "sets developing through time". A model in such a category Set^K is a parametrized family of models, $(M_i)_{i \in I}$, or a *variable model*, which can be thought of as changing through space or (non-linearly ordered) time, represented by K. The satisfaction of a formula by such a variable structure can be tested by forcing, as a special case of Theorem 4.7.4. The result becomes simplified somewhat in the clauses for $\forall$ and $\Rightarrow$, in a way that agrees with the original semantics of Kripke [**?**].

Theorem 4.7.8 (Kripke Semantics)**.** *For any first-order theory $\mathbb{T}$ and poset K and model M in the functor category Set^K, let $(x : A \mid \varphi)$, $(x : A \mid \psi)$, and $(x : A, y : B \mid \vartheta)$ be formulas in the language of $\mathbb{T}$, and let $i \in K$ and $a, a_1, a_2 : \mathsf{y}i \to A_M$ be any maps (respectively elements $a, a_1, a_2 \in (A_M)_i$. Then for each $i \in K$ we write $i \Vdash \varphi(a)$ for the relation $a \in ([\![x : A \mid \varphi]\!]_M)_i$. We can then calculate:*

1. *$i \Vdash \top(a)$ always.*
2. *$i \Vdash \bot(a)$ never.*
3. *$i \Vdash a_1 = a_2$ iff $a_1 = a_2$ as elements of the set $(A_M)_i$.*
4. *$i \Vdash \varphi(a) \wedge \psi(a)$ iff $i \Vdash \varphi(a)$ and $i \Vdash \psi(a)$.*
5. *$i \Vdash \varphi(a) \vee \psi(a)$ iff $i \Vdash \varphi(a)$ or $i \Vdash \psi(a)$.*
6. *$i \Vdash \varphi(a) \Rightarrow \psi(a)$ iff for all $j \geq i$, $j \Vdash \varphi(a_j)$ implies $j \Vdash \psi(a_j)$.*
7. *$i \Vdash \neg\varphi(a)$ iff for no $j \geq i$, $j \Vdash \varphi(a_j)$.*

8. $i \Vdash \exists y : B.\, \vartheta(a, y)$ *iff* *for some* $b : \mathsf{y}i \to B_M$, $i \Vdash \vartheta(a, b)$.

9. $i \Vdash \forall y : B.\, \vartheta(a, y)$ *iff* *for all* $j \geq i$, *for all* $b : \mathsf{y}j \to B_M$, $j \Vdash \vartheta(a_j, b)$.

Proof. Use (4.35) for the non-obvious cases. □

Examples: LEM, DN, a map is epic, monic, iso. Constant domain, increasing domain, individuals and trans-world identity. Presheaf of real-valued functions on a space is an ordered ring.

4.7.2 Completeness

We know by Theorem 4.6.22 that intuitionstic first-order logic is complete with respect to models in Heyting categories, and moreover, that for every theory $\mathbb{T}$, there is a *generic model*, namely the universal one U in the classifying category $\mathcal{C}_{\mathbb{T}}$. The model U is logically generic in the sense that, for any formula $(x : A \mid \varphi)$, there is an equivalence

$$U \models (x : A \mid \varphi) \quad \text{iff} \quad \mathbb{T} \vdash (x : A \mid \varphi)\,.$$

(The symbol $\vdash$ is once again available for provability from a set of formulas, the axioms of $\mathbb{T}$, now that we can consider single formulas rather than entailments $\varphi \vdash \psi$; see Definition 4.7.1.)

Lemma 4.7.9. *A functor* $F : \mathcal{C} \to \mathcal{D}$ *is said to be* conservative *if it reflects isomorphisms. A conservative Heyting functor between Heyting categories induces an injective homomorphism on the Heyting algebras* $\mathsf{Sub}(A)$ *for all* $A \in \mathcal{C}$. *Such a functor is always faithful.*

Proof. Let $F : \mathcal{C} \to \mathcal{D}$ be Heyting and conservative. The induced functor $\mathsf{Sub}(F) : \mathsf{Sub}(A) \to \mathsf{Sub}(FA)$, taking $U \rightarrowtail A$ to $FU \rightarrowtail FA$, is easily seen to preserve the Heyting operations, because F is Heyting. As for groups, a homomorphism of Heyting algebras is injective iff it has a trivial kernel $\mathsf{Sub}(F)^{-1}(1)$. Let $U \rightarrowtail A$ be in the kernel, i.e. $FU \rightarrowtail FA$ is iso. Then $U \rightarrowtail A$ is iso since F is conservative. To see that F is faithful consider the equalizer of a parallel pair of maps. □

By the foregoing lemma, in order to show completeness of first-order intuitionistic logic with respect to the Kripke-Joyal semantics of Theorem 4.7.4, it will suffice if we can embed $\mathcal{C}_{\mathbb{T}}$ by a conservative Heyting functor into a functor category $\widehat{\mathbb{C}} = \mathsf{Set}^{\mathbb{C}^{\mathsf{op}}}$ for some suitable (small) category $\mathbb{C}$,

$$F : \mathcal{C}_{\mathbb{T}} \rightarrowtail \widehat{\mathbb{C}}\,.$$

For then, if $FU \models (x : A \mid \varphi)$ in $\widehat{\mathbb{C}}$, then $U \models (x : A \mid \varphi)$ in $\mathcal{C}_{\mathbb{T}}$, since

$$\begin{aligned} FU \models (x : A \mid \varphi) \ &\text{iff } 1 = [\![x : A \mid \varphi]\!]_{FU} = F([\![x : A \mid \varphi]\!]_U) \\ &\text{iff } 1 = [\![x : A \mid \varphi]\!]_U \\ &\text{iff } U \models (x : A \mid \varphi)\,. \end{aligned}$$

Such an embedding suffices, therefore, to prove completeness with respect to models in categories of the form $\widehat{\mathbb{C}}$, for which we have Kripke-Joyal semantics. The following representation theorem from [?] is originally due to Joyal.

Theorem 4.7.10 (Joyal)**.** *For any small Heyting category $\mathcal{H}$ there is a regular category $\mathcal{R}$ and a conservative Heyting functor*

$$\mathcal{H} \rightarrowtail \mathsf{Set}^{\mathcal{R}} . \tag{4.41}$$

The proof of Joyal's theorem is beyond the scope of these notes, but we will mention that the category $\mathcal{R}$ is itself a subcategory of a functor category $\mathcal{R} \hookrightarrow \mathsf{Reg}(\mathcal{H}, \mathsf{Set})$ where $\mathsf{Reg}(\mathcal{H}, \mathsf{Set})$ is the category of all *regular* (not Heyting!) functors $\mathcal{H} \to \mathsf{Set}$. Here we obtain a glimpse of a generalization of Lawvere duality (as well as Stone duality, as emphasized in [?]) to regular categories, as developed by Makkai in [?]. The remarkable fact here is that the "double dual" embedding 4.41 is not just regular, but Heyting.

Theorem 4.7.11. *Intuitionistic first-order logic is sound and complete with respect to the Kripke-Joyal semantics of 4.7.4. Specifically, for every theory $\mathbb{T}$, there is a model M in a presheaf category $\widehat{\mathbb{C}}$ with the property that, for every closed formula φ,*

$$\mathbb{T} \vdash \varphi \quad \textit{iff} \quad M \models \varphi \quad \textit{iff} \quad \mathbb{C} \Vdash \varphi ,$$

where by $\mathbb{C} \Vdash \varphi$ we mean $C \Vdash \varphi$ for all $C \in \mathbb{C}$.

Finally, in order to specialize even further to the case of a Kripke model Set^K for a poset K, we can use the following "covering theorem".

Theorem 4.7.12 (Diaconescu)**.** *For any small category $\mathbb{C}$ there is a poset K and a conservative Heyting functor*

$$\mathsf{Set}^{\mathbb{C}} \rightarrowtail \mathsf{Set}^{K} . \tag{4.42}$$

For a sketch of the proof (see [?, IX.9] for details), the poset K may be taken to be $\mathsf{String}(\mathbb{C})$, consisting of finite strings of arrows in $\mathbb{C}$,

$$s = (C_n \xrightarrow{s_n} C_{n-1} \longrightarrow \ldots \longrightarrow C_1 \xrightarrow{s_1} C_0)$$

ordered by $t \leq s$ iff t extends s to the left, i.e. $s_i = t_i$ for all s_i in the string s. There is an evident functor

$$\pi : \mathsf{String}(\mathbb{C}) \longrightarrow \mathbb{C}$$

taking $s = (s_0, \ldots, s_n)$ to the "first" object C_n and $t \leq s$ to the evident composite of the extra initial t's. The functor π induces one on the functor categories by precomposition

$$\pi^* : \mathsf{Set}^{\mathbb{C}} \longrightarrow \mathsf{Set}^{\mathsf{String}(\mathbb{C})} .$$

One can show by a direct calculation that π^* is Heyting and that it is conservative, using the fact that π is surjective on both arrows and objects.

Corollary 4.7.13. *Intuitionistic first-order logic is sound and complete with respect to the Kripke semantics of Theorem 4.7.8. Specifically, for every theory* $\mathbb{T}$*, there is a poset* K *and a model* M *in* Set^K *with the property that, for every closed formula* φ*,*

$$\mathbb{T} \vdash \varphi \quad \textit{iff} \quad M \models \varphi \quad \textit{iff} \quad K \Vdash \varphi\,,$$

where by $K \Vdash \varphi$ *we mean* $k \Vdash \varphi$ *for all* $k \in K$*.*

Chapter 5

Dependent Type Theory