Algebraic Type Theory

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1. Natural Models of DTT

Def. (A. 2012) A natural model consists of:

- a cat $\mathbb{C}$
- presheaves $T, \tilde{T}$
- a natural transf. $t : \tilde{T} \to T$ that is representable:

$$\forall c \in \mathbb{C} \ \forall x \in Tc \, : \quad \exists f : d \to c \ \exists y \in \tilde{T}d$$

$$\begin{tikzcd}
yd \arrow[r, "y"] \arrow[d, "yf"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{T} \arrow[d, "t"] \\
yc \arrow[r, "x"'] & T
\end{tikzcd} .$$

Remarks

$$C \vdash a : A \iff \begin{tikzcd} & \tilde{T} \arrow[d, "t"] \\ yC \arrow[ur, dashed, "a"] \arrow[r, "A"'] & T \end{tikzcd}$$

(1) • $\mathbb{C}$ cat of ctx's
• T presheaf of types
• $\tilde{T}$ presheaf of terms

(2) Representability is ctx. extension:

$$\begin{tikzcd} yC.A \arrow[r, "q_A"] \arrow[d, "p_A"'] \arrow[dr, phantom, "\lrcorner"] & \tilde{T} \arrow[d, "t"] \\ yC \arrow[r, "A"'] & T \end{tikzcd}$$

(3) This is equivalent to CwF.

(4) (A.2012) gives conditions on t equivalent to the CwF having
$1, \Sigma, \Pi, Eq, Id$

(5) Namely, e.g.:

$$
\begin{tikzcd}
1 \arrow[r] \arrow[d, "!"'] \arrow[dr, phantom, "\lrcorner\ (1)", very near start] & \tilde{T} \arrow[d, "t"] \\
1 \arrow[r] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
\tilde{T}^2 \arrow[r] \arrow[d, "t^2"'] \arrow[dr, phantom, "\lrcorner\ (\Sigma)", very near start] & \tilde{T} \arrow[d, "t"] \\
T^2 \arrow[r] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
\tilde{T}^* \arrow[r] \arrow[d, "t^*"'] \arrow[dr, phantom, "\lrcorner\ (\Pi)", very near start] & \tilde{T} \arrow[d, "t"] \\
T^* \arrow[r] & \tilde{T}
\end{tikzcd}
$$

unit type — dependent sum — dependent product

(6) We shall abstract this structure to form that of a "Martin-Löf algebra".

2. Polynomial Functors

Let $\mathcal{E}$ be LCCC.

Every $f: A \to B$ determines a polynomial functor

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "P_f"] \arrow[d, "A^*"'] & & \mathcal{E} \\ \mathcal{E}/A \arrow[r, "f_*"] & \mathcal{E}/B \arrow[ur, "B_!"'] & \end{tikzcd}$$

$$\begin{tikzcd} X & X \times A \arrow[l] \arrow[d] & & P_f X \arrow[d] \\ & A \arrow[rr, "f"'] & & B \end{tikzcd}$$

2. Polynomial Functors

(1) In the internal DTT of $\mathcal{E}$:

$$P_f(X) = B_! f_* A^*(X) = B_! f_* f^* B^*(X) = \sum_{b:B} X^{A_b}.$$

(2) UMP of $P_f X$ is:

$$(b,x) : Z \longrightarrow P_f X$$

$$\overline{\overline{\hspace{6cm}}}$$

$$\begin{tikzcd}
X & A_b \arrow[l, "x"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
 & Z \arrow[r, "b"'] & B
\end{tikzcd}$$

(3) The assignment $f \rightsquigarrow P_f$ is <u>functorial</u> on pullbacks:

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\
B \arrow[r] & D
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
\mathcal{E} \arrow[d, bend right=50, "P_f"'] \arrow[d, bend left=50, "P_g"] \arrow[d, phantom, "\overset{\cdot}{\Rightarrow}\tau"] \\
\mathcal{E}
\end{tikzcd}
$$

$$
\mathcal{E}^{\downarrow} \supset \mathcal{E}^{I} \xhookrightarrow[\text{f\&f}]{P} \mathrm{End}^{c}(\mathcal{E}) \subset [\mathcal{E}, \mathcal{E}]
$$

(4) The composite of polynomial functors is polynomial:

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "P_{g\cdot f}"] \arrow[dr, "P_f"'] & & \mathcal{E} \\ & \mathcal{E} \arrow[ur, "P_g"'] & \end{tikzcd}$$

$$\begin{matrix} A \\ f\downarrow \\ B \end{matrix} \quad \begin{matrix} C \\ g\downarrow \\ D \end{matrix} \quad \rightsquigarrow \quad \begin{matrix} E \\ \downarrow g\cdot f \\ F \end{matrix}$$

And $P\left(! \begin{smallmatrix} 1 \\ \downarrow \\ 1 \end{smallmatrix}\right) = 1_{\mathcal{E}}$, so there is an equivalence of monoids:

$$(\mathcal{E}^I, \cdot, !) \simeq (\mathrm{Poly}(\mathcal{E}), \circ, 1_{\mathcal{E}}) .$$

(5) Construction of $g \cdot f$:

$$
\begin{tikzcd}
A & P_2 \arrow[l] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[ddr, dashed, bend left] & & \\
B \arrow[from=u, "f"'] & P_1 \arrow[l, "p_2"] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\
& P_g B \arrow[r, "p_1"'] & D
\end{tikzcd}
$$

$g \cdot f = p_1^* g \circ p_2^* f$

(6) Polynomials preserve pullbacks, so they lift to $\mathcal{E}^I$:

$$
\begin{tikzcd}
\mathcal{E}^I \arrow[r, "P_f^I"] \arrow[d, shift right] \arrow[d, shift left] & \mathcal{E}^I \arrow[d, shift right] \arrow[d, shift left] \\
\mathcal{E} \arrow[r, "P_f"'] & \mathcal{E}
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[d, "g"'] & \overset{P_f^I}{\rightsquigarrow} & P_f C \arrow[d, "P_f g := f * g"] \\
D & & P_f D
\end{tikzcd}
$$

3. M-L Algebras

Def. A M-L algebra in a LCCC $\mathcal{E}$ is a map

$$t : \tilde{T} \longrightarrow T$$

with structure:

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, "!"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{T} \arrow[d, "t"] \\ 1 \arrow[r] & T \end{tikzcd} \qquad \begin{tikzcd} \tilde{T}^2 \arrow[r] \arrow[d, "t \cdot t"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{T} \arrow[d, "t"] \\ T^2 \arrow[r] & T \end{tikzcd} \qquad \begin{tikzcd} \tilde{T}^* \arrow[r] \arrow[d, "t * t"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{T} \arrow[d, "t"] \\ T^* \arrow[r] & T \end{tikzcd}$$

unit — multiplication — closure

dominance

the general pattern :

$$
\begin{tikzcd}
\tilde{T} \arrow[d, "t"'] & P_2 \arrow[l] \arrow[d] \arrow[dl, phantom, very near start, "\llcorner"] & \\
T & P_1 \arrow[l, "p_2"] \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \tilde{T} \arrow[d, "t"] \\
& P_t T \arrow[r, "p_1"'] & T
\end{tikzcd}
$$

$$
\begin{tikzcd}
\Sigma_{P_1} P_2 \arrow[r, "\tilde{m}", "\text{“pair”}"'] \arrow[d, "t \cdot t"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \tilde{T} \arrow[d, "t"] \\
P_t T \arrow[r, "\text{“}\Sigma\text{”}", "m"'] & T
\end{tikzcd}
$$

$$
\begin{tikzcd}
P_2 \arrow[d] & \Pi_{P_1} P_2 \arrow[r, "\tilde{c}", "\text{“}\lambda\text{”}"'] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \tilde{T} \arrow[d, "t"] \\
P_1 \arrow[r] & P_t T \arrow[r, "\text{“}\Pi\text{”}", "c"'] & T
\end{tikzcd}
$$

Note:

$$
\begin{tikzcd}
\Pi_{P_1} P_2 \arrow[d] \\
P_t T
\end{tikzcd}
=
\begin{tikzcd}
P_t \tilde{T} \arrow[d, "P_t t = t * t"] \\
P_t T
\end{tikzcd}
$$

- the unit determines a cart. nat. trans.

$$u : 1_{\mathcal{E}} \to P_t$$

- The mult. determines

$$m : P_t \circ P_t \longrightarrow P_t$$

- The closure determines an algebra structure

$$c : P_t(t) \longrightarrow t$$

- In terms of the monoidal cat $(\mathcal{E}^I, \cdot, !)$:
  - a monoid structure $! \to t \leftarrow t \cdot t$
  - a module structure $t \propto t \to t$

__Basic Example:__ A CwF $(\mathbb{C}, t : \tilde{T} \to T)$ is a ML-algebra in $\hat{\mathbb{C}}$ iff it has $1, \Sigma, \Pi$.

Conversely:

__Thm__ Let $t : \tilde{T} \longrightarrow T$ be a ML-algebra in $\mathcal{E}$. Define a CwF $\hat{t}$ on $\mathcal{E}$ by just "mapping in":

$$\begin{tikzcd} Tm(-) = \mathrm{Hom}_{\mathcal{E}}(-, \tilde{T}) \arrow[d, "\hat{t}"'] \arrow[d, shift left=20, "\mathrm{Hom}_{\mathcal{E}}(-, t)"] \\ Ty(-) = \mathrm{Hom}_{\mathcal{E}}(-, T) \end{tikzcd}$$

Then $\hat{t}$ has $1, \Sigma, \Pi$ as a CwF. (pf: Yoneda preserves ML alg.s.)

4. Examples

(i) Display maps Take any map $t : \tilde{T} \to T$ in $\mathcal{E}$, and define display maps $\mathcal{D}_t \subseteq \mathbb{C}_1$ by:

$$\begin{array}{c} D \\ d \downarrow \\ E \end{array} \in \mathcal{D}_t \quad \Longleftrightarrow \quad \begin{tikzcd} D \arrow[r] \arrow[d, "d"'] \arrow[dr, phantom, "\lrcorner\ \exists"] & \tilde{T} \arrow[d, "t"] \\ E \arrow[r] & T \end{tikzcd}.$$

Then $\mathcal{D}_t$ is closed under pullbacks, and:

- under isos & composition if t is a dominance,
- under pushforwards if t is closed.

So $(\mathbb{C}, \mathcal{D}_t)$ is a Π-clan* (Joyal) if t is a ML algebra.

Conversely:

Thm (A. 2012) Given a display map cat $(\mathbb{C}, \mathcal{D})$, there's a $d_{\mathcal{D}}: \tilde{D} \to D$ in $\hat{\mathbb{C}}$ that's a ML-algebra if $(\mathbb{C}, \mathcal{D})$ is closed under isos, composition, and pushforwards, i.e. if $(\mathbb{C}, \mathcal{D})$ is a Π-clan.

In fact:

$$\begin{array}{ccc} \tilde{D} & & \coprod y\,\mathrm{dom}(d) \\ \Big\downarrow d_{\mathcal{D}} := & \coprod\limits_{d \in \mathcal{D}} yd & \Big\downarrow \\ D & & \coprod y\,\mathrm{cod}(d) \end{array}$$

So $\mathcal{D}_{d_{\mathcal{D}}} = \mathcal{D}$.

(iii) Finite sets In $\mathcal{E} = \mathrm{Set}$, let

$$\begin{array}{c} \tilde{\mathbb{N}} \\ {\scriptstyle nat}\big\downarrow \\ \mathbb{N} \end{array} \quad = \quad \begin{array}{c} \sum\limits_{n:\mathbb{N}} [n] \\ \big\downarrow \\ \mathbb{N} \end{array} \quad = \quad \begin{array}{c} < \\ \big\downarrow \\ \mathbb{N} \times \mathbb{N} \\ \big\downarrow{\scriptstyle p_2} \\ \mathbb{N} \end{array} .$$

Polynomial functor $P_{nat} : \mathrm{Set} \longrightarrow \mathrm{Set}$ is then

$$P_{nat}(X) = \sum_{n:\mathbb{N}} X^n$$

$$= 1 + X + X^2 + \ldots .$$

- <u>unit</u> : $u_X : X \longrightarrow 1 + X + \dots$ (+ inclusion)

- <u>multiplication</u> : $P_{nat}^2 \longrightarrow P_{nat}$

$$P_{nat}^2 = P_{nat} \circ nat X$$

$$\begin{tikzcd}
P_2 \arrow[r, "\tilde{m}"] \arrow[d, "nat^2"'] \arrow[dr, phantom, "\lrcorner"] & \tilde{\mathbb{N}} \arrow[d] \\
\sum_n \mathbb{N}^n = P_{nat}\mathbb{N} \arrow[r, "m"'] & \mathbb{N}
\end{tikzcd}$$

$$m_n : \mathbb{N}^n \longrightarrow \mathbb{N}$$
$$(k_1 \dots k_n) \longmapsto k_1 + \dots + k_n$$

- <u>closure</u> : $c_n(k_1 \dots k_n) = k_1 \cdot \dots \cdot k_n$.

(iii) Bool

$$\begin{array}{ccccc} \tilde{2} & & \{0\} & & \sum_{\delta:2} \delta \\ T \big\downarrow & = & \big\downarrow & = & \big\downarrow \\ 2 & & \{0,1\} & & 2 \end{array}.$$

$$P_T(X) = \sum_{\delta:2} X^{\delta} = 1 + X.$$

- **unit:** $X \longrightarrow 1 + X$ $+$-incl.

- **mult:** $1+(1+X) \longrightarrow 1+X$ ∇

- **closure:**

$$\begin{tikzcd} 1+1 \arrow[r] \arrow[d] \arrow[dr, phantom, "+"] & 1 \arrow[d] \\ 1+2 \arrow[r] & 2 \end{tikzcd}$$

Soc.

(iv) Groth. Universe

Take any cardinal α & do "the same thing":

$$\begin{array}{ll} \tilde{S}_\alpha & \text{"}\sum_{a \in S_\alpha} a\text{"} \\ \downarrow & \\ S_\alpha & \text{"sets of size} < \alpha\text{"} \end{array}$$

ML-algebra if α is inaccessible.

(v) Syntactic CwF of DTT w/ $1, \Sigma, \Pi$

$\widehat{\mathrm{Ctx}}$:

$$\begin{array}{l} \mathrm{Terms}(-) = \{ - \vdash a : A \} \\ \quad \downarrow \\ \mathrm{Types}(-) = \{ - \vdash A \ \mathrm{type} \} \end{array}$$

this should be the <u>initial</u> ML-algebra!

(vi) Hofmann-Streicher Universe

Given a cardinal $\alpha \in \mathrm{Set}$, for any $\mathbb{C}$, we have the H-S universe in $\hat{\mathbb{C}}$:

$$
\begin{array}{ll}
U : \mathbb{C}^{op} \longrightarrow \mathrm{Set}_\alpha & \tilde{U}_\alpha \\
\qquad\qquad\qquad\qquad \approx & \downarrow \\
E : \int_{\mathbb{C}} U^{op} \longrightarrow \mathrm{Set}_\alpha & U_\alpha \ .
\end{array}
$$

<u>Prop</u>. $\tilde{U}_\alpha \longrightarrow U_\alpha$ is a ML-algebra (for suitable α).

This follows directly from the 3 facts...

Fact 1: $\tilde{\mathcal{U}}_\alpha \to \mathcal{U}_\alpha$ is the "nerve" of the universal α-small discrete fibration $\dot{\mathrm{Set}}_\alpha^{op} \longrightarrow \mathrm{Set}_\alpha^{op}$ in Cat:

$$\begin{tikzcd} \hat{\mathbb{C}} \arrow[d, bend right, "\int_{\mathbb{C}}"'] \arrow[d, phantom, "\dashv"] \\ \mathrm{Cat} \arrow[u, bend right, "\nu_{\mathbb{C}}"'] \end{tikzcd}$$

$$\begin{tikzcd} \tilde{\mathcal{U}}_\alpha \arrow[d] & \nu_{\mathbb{C}} \dot{\mathrm{Set}}_\alpha^{op} \arrow[d] \\ \mathcal{U}_\alpha \arrow[ur, phantom, "="] & \nu_{\mathbb{C}} \mathrm{Set}_\alpha^{op} \end{tikzcd}$$

$$\begin{tikzcd} \int_{\mathbb{C}} D \cong \dot{D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathrm{Set}}_\alpha^{op} \arrow[d] \\ \mathbb{C} \arrow[r, "D"'] & \mathrm{Set}_\alpha^{op} \end{tikzcd}$$

Fact 2: The nerve $N_{\mathbb{C}} : \mathrm{Cat} \longrightarrow \hat{\mathbb{C}}$ preserves ML-algebras.

Fact 3: $\mathrm{Set}^{op}_{\dot{\alpha}} \longrightarrow \mathrm{Set}^{op}_{\alpha}$ is a ML-algebra in Cat *

(for suitable α).

So indeed:

Prop. $\tilde{\mathcal{U}}_\alpha \longrightarrow \mathcal{U}_\alpha$ is a ML-algebra in $\hat{\mathbb{C}}$

(for suitable α).

<u>Fun Corollary</u>: the SOC $1 \to \Omega$ in $\widehat{\mathbb{C}}$ is also an ML-algebra.

Because it's the nerve of $T : 1 \to 2$,

$$\begin{tikzcd} 1 \arrow[d, "t"'] \\ \Omega \end{tikzcd} = \begin{tikzcd} \nu 1 \arrow[d, "\nu T"] \\ \nu 2 \end{tikzcd}.$$

5. Free Completions

Let $(\mathbb{C}, t : \tilde{T} \to T)$ be a CwF in $\mathcal{E} = \hat{\mathbb{C}}$.

- We saw that TFAE:

ML-Alg	CwF	
$! \longrightarrow t$	unit type	$* : 1$
$t \cdot t \longrightarrow t$	sum type	$\Sigma_A B$
$t * t \longrightarrow t$	product type	$\Pi_A B$

- Given <u>any</u> CwF t, we can freely add these structures to make the <u>free ML-algebra on t</u>.

<u>Step 1</u> Using $P_f + P_g = P_{f+g}$ in $\mathrm{Poly}(\mathcal{E}) \simeq \mathcal{E}^I$, we have

in $\mathrm{Poly}(\mathcal{E})$ $$1_{\mathcal{E}} \longrightarrow 1_{\mathcal{E}} + P_t \longleftarrow P_t ,$$

in $\mathcal{E}^I$
$$\begin{tikzcd}
1 \arrow[r] \arrow[d, "!"'] \arrow[dr, phantom, very near start, "\lrcorner"] & 1 + \tilde{T} \arrow[d, "!+t"] & \tilde{T} \arrow[l] \arrow[d, "t"] \arrow[dl, phantom, very near start, "\llcorner"] \\
1 \arrow[r] & 1 + T & T \arrow[l]
\end{tikzcd}$$
.

Since $1_{\mathcal{E}} + P_t = P_{!+t}$ is the free pointed endofunctor on P_t, the map $t \dot{\to}\, !+t$ freely adds a <u>unit type</u> to the CwF t.

Step I: We seek u in $\mathcal{E}^I$ with

$$t \xrightarrow{\bullet} u \xleftarrow{\bullet} u \bullet u \qquad \text{(universal)}$$

- As in the theory of endofunctors as "data types", this is given by solving the "domain equation"

$$u \cong 1 + u \bullet t \quad .$$

This also adds a point $1 \to u$.

- The solution is the $\varinjlim$ of the sequence

$$0 \to T0 \to T^2 0 \to \cdots \quad ,$$

for the endofunctor

$$T(x) = ! + x \bullet t$$

on $\mathcal{E}^I \simeq \mathrm{Poly}(\mathcal{E})$.

The colimit in $\mathcal{E}^I$ is seen to be the object

$$u = 1 + t + t \cdot t + t^{3\cdot} + \dots$$
$$= \sum_n t^{n\cdot} \quad .$$

For this, one uses the following important

Fact: If $t : \tilde{T} \longrightarrow T$ in $\hat{\mathbb{C}}$ is representable, then the polynomial functor $P_t : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ has a right adjoint, and therefore preserves all colimits. So in particular:

$$t \cdot (a + b) \cong t \cdot a + t \cdot b , \qquad a, b \in \mathcal{E}^I .$$

- Let us consider this free completion under Σ-types in terms of the "type theory" $t : \tilde{T} \longrightarrow T$.

- We have the objects in $\mathcal{E}^I$:

$$\begin{array}{ccccccccc} 1 & & \tilde{T} & & \tilde{T}^2 & & \tilde{T}^3 & & \\ {\scriptstyle !}\downarrow & & {\scriptstyle t}\downarrow & & {\scriptstyle t\cdot t}\downarrow & & {\scriptstyle t^{3\cdot}}\downarrow & & \cdots \\ 1 & & T & & T^2 & & T^3 & & \end{array}$$

- Mapping into these classifies ...

Objects:

$$\begin{tikzcd}
1 \arrow[d, "!"'] & \tilde{T} \arrow[d, "t"'] & \tilde{T}^2 \arrow[d, "t \cdot t"'] & \tilde{T}^3 \arrow[d, "t^3 \cdot"'] & \cdots \\
1 & T & T^2 & T^3 &
\end{tikzcd}$$

classify:

$$\begin{tikzcd}
 & & & D \arrow[d] & \\
 & & C \arrow[d] & C \arrow[d] & \\
 & B \arrow[d] & B \arrow[d] & B \arrow[d] & \cdots \\
A & A & A & A &
\end{tikzcd}$$

ctx's:

$A \qquad A.B \qquad A.B.C \qquad A.B.C.D \quad \ldots$

- Thus $u : \tilde{\mathcal{U}} \to \mathcal{U}$ classifies the ctx's of t:

$\quad = (1 + t + t^2 + \cdots)$

$$\begin{tikzcd} C_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{\mathcal{U}} \arrow[d, "u"] \\ \vdots \arrow[d] & \\ C_0 \arrow[r] & \mathcal{U} \end{tikzcd}$$

$C_0 . C_1 . \cdots . C_n$

- This agrees with the idea that the cat. of ctx's of a theory "freely adds Σ-types".

- Can be used to give a "base change" of CwFs:

$$(\mathbb{C}, t) \longrightarrow (\mathbb{C}^{\Sigma}, t^{\Sigma}).$$

Work in Progress

- Π-types $\qquad \mathcal{U} = t + \mathcal{U} \propto \mathcal{U}$?

- Eq $\qquad \Delta u \longrightarrow u$

$$\begin{tikzcd}
\tilde{\mathcal{U}} \arrow[r] \arrow[d, "\Delta u"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{\mathcal{U}} \arrow[d, "u"] \\
\tilde{\mathcal{U}} \times_{\mathcal{U}} \tilde{\mathcal{U}} \arrow[r] & \mathcal{U}
\end{tikzcd}$$

- Id $\qquad \mathcal{U}^{I} \longrightarrow \mathcal{U}$

- $\mathcal{U}$ $\qquad$?